COHOMOLOGICAL PROPERTIES OF THE PUNCTURED MAPPING CLASS GROUPS

DISSERTATION

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Let $\Gamma_{g,r}^i$ be the mapping class group of an orientable surface $S_{g,r}$ of genus $g$ with $r$ boundary components and $i$ punctures. Over the past two or three decades, many group-theoretical and topological properties of these mapping class groups, including properties about the computation of the homology and cohomology of these groups, have been established. Most of these results can be found in the surveys [Bil], [Mis] and the references therein. However, there are still some problems which haven't been resolved. One of these important problems concerns the periodicity of the cohomology $\Gamma_{g,r}^i$. The case $\Gamma_{g,0}^0$ was studied by Glover, Mislin and Xia in [G-M-XI]. This thesis is concerned with the periodicity of the cohomology of $\Gamma_{g,r}^i$ for $i \geq 1$. The main result is that the mapping class group $\Gamma_{g,0}^i(i \geq 1)$ ($\Gamma_g^i$ for short, which we will refer to as punctured mapping class group) has periodic cohomology; furthermore, the period is always 2. In Chapter 1, we present a proof which involves the Yagita invariant and the Chern class of the representation of the group $\Gamma_g^i$. As a byproduct of the main result, we know that $\Gamma_{g,r}^i$ is torsion free when $r > 0$.

It is known from [G-M-XI] that the mapping class group $\Gamma_{g,0}^0$ ($\Gamma_g^0$ for short, which we will refer to as unpunctured mapping class group) does not have periodic cohomology in general. In fact, $\Gamma_g^0$ is never 2-periodic. For an odd prime $p$, $\Gamma_g^i$ is $p$-periodic if and only if $g$ and $p$ satisfy certain relations. Moreover, the $p$-period depends on the
genus $g$. Hence it is somewhat surprising that the punctured mapping class group has such a nice periodicity property. Using this result, the calculation of the Farrell cohomology of $\Gamma^i_g$ ($i \geq 1$) becomes more feasible; specifically, we can calculate the $p$-torsion of the Farrell cohomology for some special values of $g$ and $i$. In Chapters 2, 3 and 4, we extend the definition and related properties of the fixed point data known for $\Gamma_g$ to $\Gamma^i_g$ ($i \geq 1$). We realize that above generalization of the definition and related results play an important role in the computation of $p$-torsion of the Farrell cohomology. In Chapters 5, 6 and 7, using these results and our main result, we compute the $p$-torsion of the Farrell cohomology of $\Gamma^i_g$ when $g$ is equal to $n(p - 1)/2$, $I \in \{n + 2, n + 1, n, n - 1\}$, where $n \in \{1, 2, 3\}$. This computational method can be generalized to some other cases. In Chapter 8, we discuss the $p$-torsion of the Farrell cohomology of $\Gamma^i_g$ when $g$ is equal to $n(p - 1)/2$, where $n$ is a positive integer, and $i \geq I$. This gives us a clearer picture of the cohomology for the general case.

Since the punctured mapping class group $\Gamma^i_g$ is closely related to the unpunctured mapping class group $\Gamma_g$, we obtain more information on the unpunctured mapping class group. All these groups are of intrinsic interest in topology and geometry, particularly in research concerning lower dimensional manifolds. Some of the related applications can be found in the references at the end of this thesis.
To my husband Ji Li
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INTRODUCTION

The introduction consists of four sections. The first two sections contain the algebraic topology background which we need for our research on the mapping class group. In the first section, we will review some properties of cohomology of groups which are related to the thesis. Other basic material, which can be found in the standard reference book “Cohomology of groups” by K. Brown [Brow1], is omitted. In the second section, we will state the basic facts on Riemann Surfaces which are contained in the book “Riemann Surfaces” by H. Farkas and I. Kra [Fa-kr]. The last two sections discuss the well-known results of the mapping class group, as well as the structure of the thesis. In the third section, we will recall one of the equivalent definitions of the mapping class group, some background information and an overview of previous research. Most of them can be found in the survey on the mapping class groups by G. Mislin [Mis]. In the fourth section, we will present the purpose and the structure of the thesis.
0.1 Cohomology of groups

The cohomology of groups is one of the most important subjects in Algebraic Topology. Many people are working in this area. Here we only present a few results which can be found in [Brow1] and are related to the thesis. In our notation, $\Gamma$ stands for a discrete group.

**Definition 0.1.1.** If $\mathbb{Z}$ which is regarded as a trivial $\mathbb{Z}[\Gamma]$ module, admits a projective resolution $P$, then the minimal length of $P$ is called the cohomological dimension of $\Gamma$, which is denoted by $cd(\Gamma)$.

Cohomological dimension measures the vanishing of the homology (cohomology) of $\Gamma$. We have the following propositions about cohomological dimension.

**Proposition 0.1.1.** (a) $\text{cd}(\Gamma) = \inf\{n : \mathbb{Z} \text{ admits a projective resolution of length } n\}$

$$= \inf\{n : H^i(\Gamma, -) = 0 \text{ for } i > n\};$$

(b) $\text{cd}(\Gamma) \leq \text{geometric dimension i.e., the minimal dimension of a CW-space } K(\Gamma, 1)$. If $\text{cd}(\Gamma) \geq 3$, then the equality holds;

(c) If $\text{cd}(\Gamma) < \infty$, then $\Gamma$ is torsion free;

(d) If $\Gamma$ is a torsion free finitely generated nilpotent group, then $\text{cd}(\Gamma) < \infty$;

(e) $\Gamma' \leq \Gamma$ implies $\text{cd}(\Gamma') \leq \text{cd}(\Gamma)$;

(f) If $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ is a short exact sequence, then $\text{cd}(\Gamma) \leq \text{cd}(\Gamma') + \text{cd}(\Gamma'')$;
(g) If $\Gamma = \Gamma_1 *_A \Gamma_2$, then $cd(\Gamma) \leq \max\{cd(\Gamma_1), cd(\Gamma_2), 1 + cd(A)\}$;

(h) (Serre’s Theorem) If $[\Gamma : \Gamma'] < \infty$ and $\Gamma$ is torsion free, then $cd(\Gamma') = cd(\Gamma)$.

We know that $cd(\Gamma) = 0$ if and only if $\Gamma$ is trivial. Moreover, $cd(\Gamma) = 1$ if and only if $\Gamma$ is free and non trivial.

The generalization of cohomological dimension is the virtual cohomological dimension which is defined as follows:

**Definition 0.1.2.** The virtual cohomological dimension of $\Gamma$ is defined to be the cohomological dimension of some torsion free subgroup of finite index, which is denoted by $vcd(\Gamma)$. If $\Gamma$ has no torsion free subgroup of finite index, then $vcd(\Gamma) = \infty$.

The $vcd(\Gamma)$ is well defined for the following reason. Suppose $\Gamma'$ and $\Gamma''$ are two torsion free subgroups of $\Gamma$ which have finite index. We need to show that $cd(\Gamma') = cd(\Gamma'')$. In fact, by Serre’s Theorem, we see that $cd(\Gamma') = cd(\Gamma' \cap \Gamma'') = cd(\Gamma'')$.

Obviously, $vcd(\Gamma) = 0$ if and only if $\Gamma$ is finite. We also have the following proposition.

**Proposition 0.1.2.** If $\Gamma$ is torsion free and $vcd(\Gamma) < \infty$, then $cd(\Gamma) < \infty$.

This follows from Serre’s Theorem.

Some groups which have finite virtual cohomological dimension also have a nice property of periodic cohomology. The following is the definition.
Definition 0.1.3. A group of finite virtual cohomology dimension is said to have periodic cohomology if for some \( d \neq 0 \) there is an element \( u \in \hat{H}^d(\Gamma, \mathbb{Z}) \) which is invertible in the ring \( \hat{H}^*(\Gamma, \mathbb{Z}) \). (Here and after \( \hat{H}^* \) stands for the Farrell cohomology).

Cup product with \( u \) gives a periodicity isomorphism

\[
\hat{H}^i(\Gamma, M) \cong \hat{H}^{i+d}(\Gamma, M)
\]

for any \( \Gamma \)-module \( M \) and any \( i \in \mathbb{Z} \). The smallest such \( d > 0 \) is called the period of \( \Gamma \).

Some groups do not have periodic cohomology, however, they have a weaker property, namely \( p \)-periodic cohomology.

Definition 0.1.4. \( \Gamma \) has \( p \)-periodic cohomology (where \( p \) is a prime) if the \( p \)-primary component \( \hat{H}^*(\Gamma, \mathbb{Z})_p \), which is itself a ring, contains an invertible element of non-zero degree \( d \). We then have

\[
\hat{H}^i(\Gamma, M)_p \cong \hat{H}^{i+d}(\Gamma, M)_p.
\]

Because the Farrell cohomology groups are torsion groups involving only finitely many primes, it is clear that \( \Gamma \) has periodic cohomology if and only if \( \Gamma \) has \( p \)-periodic cohomology for every prime \( p \). In fact, we have

\[
\hat{H}^*(\Gamma, \mathbb{Z}) \cong \prod_p \hat{H}^*(\Gamma, \mathbb{Z})_p,
\]

where \( p \) ranges over the primes such that \( \Gamma \) has \( p \)-torsion.

The equivalent conditions for the \((p-)\)periodic cohomology are very important in applications. Hence, we state them in the following three propositions.
Proposition 0.1.3. The following conditions are equivalent for a finite group $\Gamma$:

(i) $\Gamma$ has periodic cohomology.

(ii) For some $d \neq 0$, $\check{H}^d(\Gamma, \mathbb{Z}) \cong \mathbb{Z}/|\Gamma|\mathbb{Z}$.

(iii) For some $d \neq 0$, $\check{H}^d(\Gamma, \mathbb{Z})$ contains an element of order $|\Gamma|$.

(iv) Every abelian subgroup of $\Gamma$ is cyclic.

(v) Every elementary abelian $p$-subgroup of $\Gamma$ has rank $\leq 1$, where an elementary abelian $p$-group of rank $r \geq 0$ is a group isomorphic to $(\mathbb{Z}/p)^r = \mathbb{Z}/p \times \ldots \times \mathbb{Z}/p$ ($r$ factors).

(vi) The Sylow subgroups of $\Gamma$ are cyclic or generalized quaternion groups.

Proposition 0.1.4. For any finite group $\Gamma$ and prime $p$ dividing $|\Gamma|$, the following conditions are equivalent:

(i) $\Gamma$ has $p$-periodic cohomology.

(ii) The ring $\check{H}^*(\Gamma, \mathbb{Z}/p)$ contains an invertible element of non-zero degree.

(iii) Every elementary abelian $p$-subgroup of $\Gamma$ has rank $\leq 1$.

(iv) The $p$-Sylow subgroups of $\Gamma$ are cyclic or generalized quaternion groups.
Proposition 0.1.5. For a group $\Gamma$ of finite vcd, not necessarily finite, the following conditions are equivalent:

(i) $\Gamma$ has $p$-periodic cohomology.

(ii) There exists integer $d \neq 0$ such that for all $i \in \mathbb{Z}$, $\check{H}^i(\Gamma, M)_p \cong \check{H}^{i+d}(\Gamma, M)_p$ for all $\Gamma$-modules $M$.

(iii) Every elementary abelian $p$-subgroup of $\Gamma$ has rank $\leq 1$.

(iv) Every finite subgroup of $\Gamma$ has $p$-periodic cohomology.

If $\Gamma$ has $p$-periodic cohomology then so does any subgroup $\Gamma'$. The minimal $d > 0$ with property (ii) is called the $p$-period of $\Gamma$ and we write $p(\Gamma)$ for it.

The computation of the $p$-period of a group is very important. For it enables us to compute the Farrell cohomology of the group. The calculation of the $p$-period of a finite group is determined by the following two theorems:

Theorem 0.1.1. [Swan] If the 2-Sylow subgroup of a finite group $\Gamma$ is cyclic, the 2-period is 2. If the 2-Sylow subgroup of a finite group $\Gamma$ is generalized quaternion group, the 2-period is 4.

Theorem 0.1.2. [Swan] Suppose $p$ is an odd prime and the finite group $\Gamma$ has $p$-periodic cohomology. Let $P$ be a $p$-Sylow subgroup. Then the $p$-period of $\Gamma$ is twice the order of $N(P)/C(P)$ where $N$ and $C$ denote the normalizer and centralizer of $P$ in $\Gamma$, respectively.
There are also theorems to calculate the $p$-period of an infinite group with finite vcd.

**Theorem 0.1.3.** [G-M-X1] Let $p$ be a prime and $N$ a group of finite virtual cohomological dimension which is $p$-periodic. Suppose $N$ contains a normal subgroup $\mathbb{Z}/p < N$ of order $p$. Then the $p$-period of $N$ has the form $2[N : C(\mathbb{Z}/p)]^p$, where $C(\mathbb{Z}/p) < N$ denotes the centralizer of $\mathbb{Z}/p$ in $N$ and $l \geq 0$ an integer.

**Theorem 0.1.4.** [Brow1] Suppose $\Gamma$ has finite vcd and $p$-periodic cohomology. Then

$$\hat{H}^*(\Gamma, M)_{(p)} \cong \prod_{\mathbb{Z}/p \in S} \hat{H}^*(N(\mathbb{Z}/p), M)_{(p)},$$

where $S$ is a set of representatives of the conjugacy classes of subgroups of $\Gamma$ of order $p$.

**Theorem 0.1.5.** Suppose $\Gamma$ has finite vcd and $p$-periodic cohomology. Moreover, assume that $\Gamma$ contains only finitely many conjugacy classes of subgroups of order $p$. Then the $p$-period of $\Gamma$ is given by

$$p(\Gamma) = 2lcm_{\mathbb{Z}/p \in S}([N(\mathbb{Z}/p) : C(\mathbb{Z}/p)])^p$$

for some integer $k \geq 0$, where $S$ is a set of representatives of the conjugacy classes of subgroups of $\Gamma$ of order $p$.

Another way to find the $p$-period of an infinite group with finite vcd is by calculating the Yagita invariant. ([G-M-X2]) The Yagita invariant is defined as follows:
Let $\Gamma$ be a group of finite virtual cohomological dimension and $\pi < \Gamma$ any subgroup of prime order $p$. Because $\pi$ injects into any finite quotient of the form $\Gamma / \Delta$, where $\Delta$ is a torsion free normal subgroup of finite index in $\Gamma$, the image $\text{Im}(H^k(\Gamma; \mathbb{Z}) \to H^k(\pi; \mathbb{Z}))$ of the restriction map in cohomology is non-zero for some degree $k > 0$. Reduction mod-$p$ maps $H^*(\pi; \mathbb{Z})$ onto $F_p[u] \subset H^*(\pi; F_p)$ with $u$ a generator in $H^2(\pi; F_p)$. Thus, there exists a maximum value $m = m(\pi, \Gamma)$ such that

$$
\text{Im}(H^*(\Gamma; \mathbb{Z}) \to H^*(\pi; F_p)) \subset F_p[u^m] \subset H^*(\pi; F_p).
$$

Note that $m(\pi, \Gamma)$ is bounded by $m(\pi, \Gamma / \Delta)$, where $\Delta$ denotes as before a torsion free normal subgroup of finite index. Since $\Gamma / \Delta$ is finite, we conclude that $m(\pi, \Gamma)$ is bounded by a bound depending on $\Gamma$ only ([Ya]). The Yagita invariant of $\Gamma$ with respect to the prime $p$ is then defined to be the least common multiple of the values $2m(\pi, \Gamma)$, where $\pi$ ranges over all subgroups of order $p$ of $\Gamma$. It is denoted by $Y(\Gamma, p)$. The Yagita invariant is a generalization of the $p$-period of a $p$-periodic group to an arbitrary group which may not have $p$-periodic cohomology. For the $p$-periodic group, we have the following theorem:

**Theorem 0.1.6.** [G-M-X2] The Yagita invariant agrees with the $p$-period of a $p$-periodic group $\Gamma$ of finite vcd: $Y(\Gamma, p) = p(\Gamma)$.

**Proof.** The proof can be found in Xia Yi Ning’s PhD. thesis of 1990. ([Xi5])

**Q.E.D.**

**Remark.** (1) If $\Gamma$ is $p$-torsion free, then $\Gamma$ is $p$-periodic of period 1.

(2) The ordinary period of cohomology of $\Gamma$ is, of course, the least common multiple of all the $p$-periods where $p$ ranges over the primes such that $\Gamma$ has $p$-torsion.
It is worth of mentioning that Theorem 0.1.4 is not only useful in the calculation of the $p$ period, but also in the calculation of the $p$-torsion of the Farrell cohomology.

Another important tool in cohomology of groups is the Serre spectral sequence. We omit the basic material which can be found in any book on cohomology of groups, or some books about Algebraic Topology. We only state the following important theorem.

**Theorem 0.1.7.** For any group extension $1 \rightarrow \Gamma' \rightarrow \Gamma \rightarrow \Gamma'' \rightarrow 1$ and any $\Gamma$-module $M$, there is a spectral sequence of the form

$$E_2^{pq} = H^p(\Gamma'', H^q(\Gamma', M)) \Rightarrow H^{p+q}(\Gamma, M).$$

This theorem gives us a relationship of the group cohomology among the subgroup, the quotient group and the group itself.
0.2 Basic facts on Riemann Surfaces

The mapping class group is the class of mappings between Riemann surfaces. Hence, we need to refer some basic facts on Riemann surfaces. For several decades, Riemann surfaces have been a hot topic in topology. For our purpose, we only mention things we need and which are not trivial.

Definition 0.2.1. \(S_g\) stands for a closed (i.e. compact and connected with empty boundary) Riemann surface of genus \(g\).

The following theorems are about the mappings of Riemann surfaces.

Theorem 0.2.1. Suppose \(f : S_g \to S_g\) is a non-constant holomorphic mapping of order \(p\), a prime. Then the fixed points set of \(f\) is finite. The mapping \(f\) induces a projection mapping \(\pi : S_g \to S_h\) with \(S_h\) the orbit space of \(f\), i.e., the space \(S_g/(x \sim f(x))\). Note that \(\pi^{-1}(Q)\) has cardinality \(p\) for almost all \(Q \in S_h\). Let us denote by \(t\) the number of fixed points of \(f\). Then we have the Riemann-Hurwitz equation:

\[2g - 2 = p(2h - 2) + t(p - 1).\]

Theorem 0.2.2. Let \(\tau\) be a complex structure on \(S_g\) compatible with the smooth structure. Then the group of holomorphic automorphisms \(\text{Aut}(S_g, \tau)\) is a subgroup of \(\text{Diffeo}(S_g)\), the group of diffeomorphisms of \(S_g\), and the group \(\text{Aut}(S_g, \tau)\) is finite in case \(g > 1\).
0.3 Mapping class groups and related results

Precisely when mathematicians first began to consider the mapping class groups is unclear. From the 1920’s until the present, especially since the 1960’s, these groups became one of the central topics in contemporary mathematics since they are closely related to algebraic topology, algebraic geometry, the theory of combinatorial groups, the theory of Riemann surfaces, the theory of three-dimensional manifolds and physics. Many results have been established from the different points of view. However, some properties remain mysteries for the time being.

The symbol $S_{g,r}^i$ will be used to denote an orientable surface of genus $g$ which is obtained from a closed surface by removing $r$ open discs and $i$ points. Let $Diffeo^+(S_{g,r}^i)$ be the group of orientation preserving diffeomorphisms, and let $Iso(S_{g,r}^i)$ be the subgroup of diffeomorphisms which are isotopic to the identity, the restriction of the isotopy to $\partial(S_{g,r}^i)$ being the identity. The mapping class group $\Gamma_{g,r}^i$ is the quotient $Diffeo^+(S_{g,r}^i)/Iso(S_{g,r}^i)$. Of special interest among these groups are the classical braid groups $B_n = \Gamma_{0,n}^0$, $n = 1, 2, 3..., and the unpunctured mapping class groups $\Gamma_g = \Gamma_{g,0}^0$, $g = 0, 1, 2, 3...$ In this thesis, we will focus on $\Gamma_g^i = \Gamma_{g,0}^i$, $i \geq 1$, the punctured mapping class groups.

Another equivalent definition widely used is the following:

**Definition 0.3.1.** (i) $\Gamma_g = \{\text{the group of connected components of the group of orientation preserving diffeomorphisms of } S_g\}.$

(ii) $\Gamma_g^i = \pi_0 Diffeo^+(S_g, P_1, P_2, ... P_i)$, where $Diffeo^+(S_g, P_1, P_2, ... P_i) = \{\text{the group}$
of orientation preserving diffeomorphisms of $S_g$ which fix the points $P_j$ individually}. \\

(iii) Let $\tilde{S}_g$ be a Riemann surface with connected boundary components $\partial_1, \partial_2, ... \partial_r$, \\

$$\Gamma^{\partial_1, \partial_2, ..., \partial_r}_{\partial_1, \partial_2, ..., \partial_r} = \pi_0 \text{Diff}eo^+(\tilde{S}_g, P_1, P_2, ..., P_i; \partial_1, \partial_2, ..., \partial_r),$$

where $\text{Diff}eo^+(\tilde{S}_g, P_1, P_2, ..., P_i; \partial_1, \partial_2, ..., \partial_r) = \{ \text{the group of orientation preserving diffeomorphisms of } \tilde{S}_g \text{ which restrict to the identity on all } \partial_k \text{'s, fix the points } P_j \text{ individually} \}$. \\

The classical results about the mapping class groups are: \\

(i) Finite presentation: \\

The unpunctured mapping class group $\Gamma_g$ is finitely generated by Dehn Twist [De]. Actually, $\Gamma_g$ is finitely presented. However, it took 55 years (and a great deal of effort), until finally, Wajnryb [W] gave a simple finite presentation for $\Gamma_{g,1}$ and $\Gamma_g$. \\

(ii) Nielsen Realization Theorem: \\

By the definition of the group $\Gamma_g$, there is a surjective homomorphism $\phi : \text{Diff}eo^+(S_g) \rightarrow \Gamma_g$ with kernel the subgroup of diffeomorphisms of $S_g$ which are isotopic to the identity. The Nielsen Realization problem is the question whether a finite group in $\Gamma_g$ can be realized by a finite group of mappings. The answer is positive, however it took many years to get the correct proof. The related discussions and the generalization to the punctured mapping class group can be found in [Ni2], [Ke], [Zi], [Bi1], [MH].
(iii) Representation

There is a natural group homomorphism from \( \Gamma_g \) to the induced group of automorphisms of \( H_1(S_g) \), i.e., the integral symplectic modular group \( Sp(2g, \mathbb{Z}) \). The Torelli group \( I_g \), is the kernel of this homomorphism. Details can be found in [Bi1].

The (co)homological properties about the mapping class groups are:

(i) \( vcd(\Gamma_g) = 4g - 5 \) for \( g > 1 \). [Ha3].

(ii) \( \Gamma_g \) is of type WFL. (i.e., all torsion free subgroups of finite index admit finitely generated free resolution of finite length.) [Mis]

(iii) Let \( g > 1 \). Then

a) \( \Gamma_g \) is never 2-periodic;

b) for an odd prime \( p \), \( \Gamma_g \) is \( p \)-periodic if and only if one of the following two conditions holds:

b1) \( g \not\equiv 1 \mod(p) \);

b2) \( g \) is of the form \( kp + 1 \) with \( k \not\equiv 0, -1 \mod(p) \) and the interval \([ (2k + 3)/p, (2k + 2)/(p - 1) ] \) does not contain any integer.

The \( p \)-period of \( \Gamma_g \) depends on \( g \), and it can be found in [Mis].

(iv) There are some characteristic classes of group representations of the mapping class group which are very useful. The definition of the characteristic classes of
a group representation can be found in [Mis]. We use this tool to prove that $\Gamma_g^i$ has period 2.

(v) There are some stable properties in computing homology. Important contributions are due to Harer in [Ha2]. For example: Harer proved that $H_k(\Gamma_{g,r}^i) \to H_k(\Gamma_{g+1,r-2}^i)$ is an isomorphism for $g \geq 3k$, $r \geq 2$.

(vi) Another important tool for studying the cohomology of $\Gamma_g^i$ is its action on the Teichmüller space $T_g^i$ which is a smooth manifold homeomorphic to $(\mathbb{R}^+ \times \mathbb{R})^{3g-3+i}$. The material related to Teichmüller space can be found in [Ha5].

(vii) There are only finitely many conjugacy classes of finite subgroups of $\Gamma_g$. [Harv2]
0.4 The structure of the thesis

An interesting phenomenon which appears in the cohomology of some groups is the periodicity we have discussed in previous sections. This is very important because, if a group has periodic cohomology, then all the information about the cohomology groups of the group is given by finitely many of them. For the mapping class group $\Gamma^i_{g,r}$, the cohomology (homology) groups $H^*(\Gamma_{g,r}^i, \mathbb{Z})$ depend on the genus $g$ and the puncture $i$, as well as the boundary $r$. For general $g$, $i$ and $r$, no result about these cohomology groups has been established. Important contributions—namely the stable properties—are due to Harer, whom we have cited in Section 0.3. He proves that the homology group of a certain degree does not depend on $g$, $i$ and $r$ if the genus is large enough compared with the degree of the homology group. However, we still do not know much about the general case. The purpose of this thesis is to present more information about the cohomology groups of the mapping class group. We will show that the period of $\Gamma^i$ is 2 for $i \geq 1$; therefore, in order to understand the Farrell cohomology groups, we only need to know $\hat{H}^0(\Gamma_{g}^i, \mathbb{Z})$ and $\hat{H}^1(\Gamma_{g}^i, \mathbb{Z})$. Moreover, the thesis extends the useful tool—fixed point data—to the case of element of finite order in punctured mapping class group. As an application, for some special values of $g$ and $i$, we will present the computation of the $p$-torsion of the Farrell cohomology of $\Gamma_{g}^i$.

"Introduction" provides the background information, overview of previous research, problem, purpose, and structure of the thesis. Chapter 1 presents a proof for the main result, namely, $\Gamma^i_{g}$ has period 2 for $i \geq 1$. Chapter 2 generalizes the fixed point data to the case of element of finite order in punctured mapping class group.
Chapter 3 demonstrates the necessary and sufficient conditions for $\Gamma_g^i$ to contain a subgroup of order $p$. Chapter 4 sets up the relationship between conjugacy class of subgroups of order $p$ in $\Gamma_g^i$ and the fixed point data of some generator in the subgroup of order $p$. Chapter 5 calculates the $p$-torsion of the Farrell cohomology of $\Gamma_{(p-1)/2}^n$. Chapter 6 calculates the $p$-torsion of the Farrell cohomology of $\Gamma_{p-1}^n$. Chapter 7 calculates the $p$-torsion of the Farrell cohomology of $\Gamma_{3(p-1)/2}^n$. Chapter 8 discusses the $p$-torsion of the Farrell cohomology of $\Gamma_{n(p-1)/2}^i$. Appendix A offers some problems which have not been resolved.
CHAPTER 1

THE PERIOD OF $\Gamma^i_G$, $I \geq 1$

In this chapter, we establish the main result that all punctured mapping class groups $\Gamma^i_g$ have period 2. The method we use to obtain this is the following: The Yagita invariant $Y(\Gamma^i_g, p)$ is regarded as a generalization of the $p$-period of $\Gamma^i_g$; also, since we can prove that $\Gamma^i_g$ is $p$-periodic for $i \geq 1$, the Yagita invariant $Y(\Gamma^i_g, p)$ coincides with the $p$-period of $\Gamma^i_g$ by Theorem 0.1.6. Then in order to prove that the $p$-period of $\Gamma^i_g$ is 2, we just need to show that $Y(\Gamma^i_g, p)$ is 2.

This chapter is divided into two sections. In Section 1.1, we deal with the case $\Gamma^1_g$. First, we prove that $\Gamma^1_g$ has $p$-periodic cohomology for every $g$ and every prime $p$. The key observation is that, for any subgroup $H$ in $\Gamma^1_g$, we can get a lift $\widetilde{H}$ in $Diff e o^+(S_g, \ast)$; furthermore, the lift induces a representation $\tilde{\rho} : \widetilde{H} \to Gl_2^+(\mathbb{R})$ by letting $\widetilde{H}$ act on the tangent space of $S_g$ at $\ast$. In fact, this two dimensional real representation distinguishes $\Gamma^1_g$ from $\Gamma_g$, and provides us the periodicity. Second, we prove that if $\Gamma^1_g$ contains $p$-torsion, then $Y(\Gamma^1_g, p)$ is 2. In fact, as before, we have a faithful two dimensional real representation $\tilde{\rho} : \tilde{\pi} \to Gl_2^+(\mathbb{R})$, where $\tilde{\pi} \subset Diff e o^+(S_g, \ast)$ is the lifting of $\pi \subset \Gamma^1_g$ which is isomorphic to $\mathbb{Z}/p$. It is easy to get an induced mapping $B\tilde{\rho} : B\tilde{\pi} \to BGl_2^+(\mathbb{R})$. There exists a class $c'_1 \in H^2(BGl_2^+(\mathbb{R}), \mathbb{Z})$ which is related to the first Chern class $\tilde{c}_1 \in H^2(BU(1), \mathbb{Z})$, such that $(B\tilde{\rho})^*(c'_1) \neq 0$. 

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By diagram chasing, we obtain a class in $H^2(B\Gamma^1_g, \mathbb{Z})$ which can be restricted to a non-zero element in $H^2(B\pi, \mathbb{Z})$. Hence we obtain $Y(\Gamma^i_g, p) = 2$ by the definition of the Yagita invariant. The main result for $\Gamma^1_g$ then follows. Finally, some nice properties of period 2 groups are mentioned in this section.

In Section 1.2, using the short exact sequence $1 \to \pi_1S^r_{g,r} \to \Gamma^{i+1}_{g,r} \to \Gamma^i_{g,r} \to 1$, we get that the period of $\Gamma^i_g$ is 2 inductively. As a byproduct, by using the Gysin sequence and another short exact sequence $1 \to \mathbb{Z} \to \Gamma^{i+1}_{g,r+1} \to \Gamma^i_{g,r} \to 1$, we can show inductively that $\Gamma^i_{g,r}$ is torsion free for $g \geq 1$ and $r \geq 1$. 
1.1 The period of $\Gamma_9^1$

The following lemma sets up a relation between the $p$-period of a group and that of its quotient group.

**Lemma 1.1.1.** Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a short exact sequence of groups, with $cdA < \infty$ and $vcd(C) < \infty$. Then

(a) $vcd(B) < \infty$.

(b) If $C$ has $p$-periodic cohomology, then $B$ has $p$-periodic cohomology. Furthermore, $p(B)|p(C)$, where $p(.)$ stands for the $p$ period.

**Proof.** (a) If $vcd(C) < \infty$, then there exists $C' \triangleleft C$ such that $cd(C') < \infty$ and $[C : C'] < \infty$. By the basic properties of groups, one can find $j^{-1}(C') \triangleleft B$ and $B/j^{-1}(C') \cong C/C'$; therefore, $[B : j^{-1}(C')] < \infty$. In order to prove $vcd(B) < \infty$, it suffices to show that $cd(j^{-1}(C')) < \infty$. Note that $0 \to A \xrightarrow{i} j^{-1}(C') \xrightarrow{j} C \to 0$ is again a short exact sequence with $cd(A) < \infty$ and $cd(C') < \infty$, thus by Proposition 0.1.1.(f), we obtain $cd(j^{-1}(C')) < \infty$. The result then follows.

(b) Suppose $B$ does not have $p$-periodic cohomology, then there exists $\mathbb{Z}/p \times \mathbb{Z}/p \subset B$. It is not hard to derive that $j(\mathbb{Z}/p \times \mathbb{Z}/p) \cong \mathbb{Z}/p \times \mathbb{Z}/p$. In fact, $cdA < \infty$ implies that $A$ is torsion free, therefore $\text{im}(i)$ is torsion free, and so is $\ker(j)$. Hence, $j(\mathbb{Z}/p \times \mathbb{Z}/p)$ can not be isomorphic to $\mathbb{Z}/p$ or $0$. Therefore, we have $\mathbb{Z}/p \times \mathbb{Z}/p \cong j(\mathbb{Z}/p \times \mathbb{Z}/p) \subset C$. This contradicts the assumption that $C$ has $p$-periodic cohomology. So, $B$ has $p$-periodic cohomology.
By Theorem 0.1.6, \( p(B) = Y(B, p) \), \( p(C) = Y(C, p) \), where \( Y(., p) \) stands for the Yagita invariant of the group with respect to \( p \). By Lemma 1.3 of [Ya], it easily follows that:

\[ Y(B, p) \mid Y(C, p). \]

Therefore, \( p(B) \mid p(C) \). \( \text{Q.E.D.} \)

Our first aim is to prove that \( \Gamma_g^1 \) is \( p \)-periodic for every \( g \) and every prime \( p \) which is the following theorem. Notice that the theorem does not hold for any unpunctured mapping class group because we do not have such a nice two dimensional real representation in the case of unpunctured mapping class groups as in the case of punctured mapping class groups.

Theorem 1.1.1. \( \Gamma_g^1 \) has the following properties:

(a) \( \text{vcd}(\Gamma_g^1) < \infty \)

(b) \( \Gamma_g^1 \) has \( p \)-periodic cohomology for every \( g \) and every prime \( p \).

Proof. (a) We have a well-known short exact sequence \( 1 \to \pi_1 S_g \to \Gamma_g^1 \to \Gamma_g^0 \to 1 \) for \( g > 0 \), with \( \text{vcd}(\Gamma_g^0) < \infty \), and \( \text{cd}(\pi_1 S_g) < \infty \). By Lemma 1.1.1, the result follows.

(b) Suppose there exists some \( g \) and some prime \( p \), such that \( \Gamma_g^1 \) is not \( p \)-periodic. Then there exists \( \mathbb{Z}/p \times \mathbb{Z}/p \subset \Gamma_g^1 \), i.e., \( \sigma \times \tau \subset \Gamma_g^1 \). By the argument in Lemma 3 of [Ha-Za], we know that we can find \( f \) representing \( \sigma \) and \( f_0 \) representing \( \tau \), where \( f, f_0 \in \text{Diff}^+(S_g, \star) \) satisfy the followings:
\[ f^p = 1, \]
\[ f^p_0 = 1, \]
\[ f_0 f = f f_0. \]

Hence, \( < f_0 > \times < f > \) acts on \( S_g \) as diffeomorphisms, fixing \( * \). It induces an action of \( < f_0 > \times < f > \) on \( T_0 S_g \), the oriented tangent space of \( S_g \) at \( * \). We obtain thus a representation \( \rho : < f_0 > \times < f > \rightarrow GL_2^+(\mathbb{R}) \). Claim: \( \rho \) is faithful. Suppose not; then there exists \( \phi \in < f_0 > \times < f > \), where \( \phi \neq id \), such that \( \rho \phi \circ v = v \) for every \( v \in T_0 S_g \). This implies that \( \rho \phi \) fixes \( T_0 S_g \). We thus obtain that \( \phi \) fixes a neighborhood of \( * \) in \( S_g \) because \( \rho \phi \) acts by a rotation with respect to some complex invariant structure on \( S_g \). We then infer that \( \phi \) fixes all of \( S_g \), i.e., \( \phi = id \). This is contrary to our assumption. Hence, the claim follows. We now obtain that \( \rho(< f_0 > \times < f >) \cong \mathbb{Z}/p \times \mathbb{Z}/p \subseteq GL_2^+(\mathbb{R}) \), which is impossible because the maximal compact subgroup of \( GL_2^+(\mathbb{R}) \) is \( S^1 \), and all finite subgroups of \( S^1 \) are cyclic. Therefore, we conclude that \( \Gamma^1_g \) is \( p \)-periodic for every prime \( p \) and every \( g \). Q.E.D.

Since \( \Gamma^1_g \) is \( p \)-periodic, the natural question is to ask: What is the \( p \)-period of \( \Gamma^1_g \)? We prove that the \( p \)-period of \( \Gamma^1_g \) is 2, which need the following lemma.

**Lemma 1.1.2.** Let \( p_1 : Diffeo^+(S_g, *) \rightarrow \Gamma^1_g \) be the natural projection, and let \( Bp_1 : B\text{Diffeo}^+(S_g, *) \rightarrow B\Gamma^1_g \) be the map of classifying spaces induced by \( p_1 \). Then \( Bp_1 \) is a homotopy equivalence for \( g > 1 \).
PROOF. Fix a base point $*$ in $S_g$. Consider the evaluation map

$$r : \text{Diff}eo^+(S_g) \to S_g, \quad f \mapsto f(*)\text{.}$$

Then $r$ is a fibration with fiber $\text{Diff}eo^+(S_g,*)$, i.e., there is a fibration sequence,

$$\text{Diff}eo^+(S_g,*) \xrightarrow{i} \text{Diff}eo^+(S_g) \xrightarrow{r} S_g\text{.}$$

Hence we have a long exact sequence of homotopy groups:

$$\cdots \to \pi_n(\text{Diff}eo^+(S_g,*) , id) \to \pi_n(\text{Diff}eo^+(S_g), id) \to \pi_n(S_g,*) \to$$

$$\to \pi_{n-1}(\text{Diff}eo^+(S_g,*) , id) \to \cdots \to$$

$$\to \pi_1(\text{Diff}eo^+(S_g,*) , id) \to \pi_1(\text{Diff}eo^+(S_g), id) \to \pi_1(S_g,*) \to$$

$$\to \pi_0(\text{Diff}eo^+(S_g,*) ) \to \pi_0(\text{Diff}eo^+(S_g)) \to \pi_0(S_g)\text{.}$$

Since $\text{Diff}eo^0(S_g)$ (the connected component of the identity of $\text{Diff}eo^+(S_g)$) is contractible for $g > 1$ by [Ea-Ee], we have $\pi_i(\text{Diff}eo^+(S_g), id) = 0$ for $i \geq 1$ and $g > 1$. Also note that $S_g$ is connected, so $\pi_0(S_g) = 0$, $\pi_i S_g = 0$ for $i \geq 2$ and $g \geq 1$. Thus we obtain that $\pi_i(\text{Diff}eo^+(S_g,*), id) = 0$, i.e., $\pi_i(\text{Diff}eo^0(S_g,*)) = 0$ for all $i \geq 1$ and $g > 1$, where $\text{Diff}eo^0(S_g,*)$ is the connected component of the identity of $\text{Diff}eo^+(S_g,*).$ It is well known that $\text{Diff}eo^0(S_g,*)$ can be viewed as a CW-complex, by [Bred] Corollary 11.14, we infer that $\text{Diff}eo^0(S_g,*)$ is contractible. Hence the natural projection $p_1 : \text{Diff}eo^+(S_g,*) \to \Gamma_g^1$ is a homotopy equivalence.

For an arbitrary group $G$, one has $\pi_i BG \cong \pi_{i-1} G$ for $i \geq 1$. Therefore, $Bp_1 : B\text{Diff}eo^+(S_g,*) \to B\Gamma_g^1$ induces isomorphism

$$(Bp_1)_* : \pi_i(\text{Diff}eo^+(S_g,*)) \to \pi_i(\Gamma_g^1), i \geq 1\text{.}$$
Again by [Bred] Corollary 11.14, $BP_1 : BDiff^+(S_g, \ast) \to B\Gamma^1_g$ is a homotopy equivalence for $g > 1$. Q.E.D.

We now prove our main result.

**Theorem 1.1.2.** Let $g > 1$, and $p$ any prime. If $\Gamma^1_g$ contains $p$-torsion, then the $p$-period of $\Gamma^1_g$ is 2, i.e., $p(\Gamma^1_g) = 2$.

**Proof.** It suffices to show that $Y(\Gamma^1_g, p) = 2$. Pick an arbitrary $\pi \leq \Gamma^1_g$ such that $\pi \cong \mathbb{Z}/p$. By the argument in Lemma 3 of [Ha-Za], we know that $\pi$ can be realized as $\tilde{\pi} < Diff^+(S_g, \ast)$ with $\tilde{\pi} \cong \pi$. Thus, there is a commutative square

$$
\begin{array}{ccc}
\tilde{\pi} & \overset{i}{\longrightarrow} & Diff^+(S_g, \ast) \\
\rho_1 \downarrow & & \downarrow p_1 \\
\pi & \overset{i}{\longrightarrow} & \Gamma^1_g,
\end{array}
$$

On the other hand, as before, we obtain the following representation, which arises from letting a diffeomorphism of $(S_g, \ast)$ act on the tangent space of $S_g$ at $\ast$:

$$
\rho : Diff^+(S_g, \ast) \to Gl^+_2(\mathbb{R}),
$$

$$
\rho : f \mapsto df(\cdot : T_\ast S_g \to T_\ast S_g).
$$

Hence,

$$
\tilde{\rho} : \tilde{\pi} \overset{i}{\longrightarrow} Diff^+(S_g, \ast) \overset{\rho}{\longrightarrow} Gl^+_2(\mathbb{R})
$$

is a representation for $\tilde{\pi}$. Write $\tilde{\rho} = \tilde{\rho} : \tilde{\pi} \to Gl^+_2(\mathbb{R})$. Consider this representation $\tilde{\rho} : \tilde{\pi} \to Gl^+_2(\mathbb{R})$. For an arbitrary $g \in \tilde{\pi}$, the differential map $dg$ is a rotation with respect to some inner product on $T_\ast S_g$. If $dg$ fixes $T_\ast S_g$, then $g$ fixes a neighborhood
of $\ast$, therefore $g$ fixes all of $S_g$, and it follows that $g = \text{id}$. Thus, $\tilde{\rho}$ is a faithful representation. By a classical result, $\tilde{\rho}$ factors through the maximal compact subgroup $S^1 = SO(2)$, and we have the following diagram, with $\tilde{\rho}$ conjugate to $i'\rho'$:

$$
\tilde{\pi} \overset{\tilde{\rho}}{\longrightarrow} GL_2^+(\mathbb{R})
$$

\hspace{1cm} \overset{\star}{\longrightarrow} \overset{i'}{\longrightarrow}

$$
SO(2) \overset{\cong}{\longrightarrow} U(1) \longrightarrow GL_1(\mathbb{C}).
$$

Since $\tilde{\rho}$ is a faithful real representation, it follows that $f'\rho'$ is a 1-dimensional faithful complex representation. By a result of [Mis] on 1-dimension faithful complex representations of $\mathbb{Z}/p$, there exists $\tilde{c}_1 \in H^2(BU(1), \mathbb{Z})$ such that $(B(f'\rho'))^*(\tilde{c}_1) \neq 0$, i.e., $(Bf \circ B\rho')^*(\tilde{c}_1) \neq 0$ in $H^2(B\tilde{\pi}, \mathbb{Z})$. It implies that, $(B\rho')^*((Bf)^*(\tilde{c}_1)) \neq 0$, where $(Bf)^*(\tilde{c}_1) \in H^2(BSO(2), \mathbb{Z})$. Since $BGL_2^+(\mathbb{R}) \simeq BSO(2)$, there exists $c'_1 \in H^2(BGL_2^+(\mathbb{R}), \mathbb{Z})$ such that $(B\tilde{i'})^*(c'_1) = (Bf)^*(\tilde{c}_1)$. Hence,

$$(B\rho')^*((Bf)^*(\tilde{c}_1)) \neq 0,$$

implies

$$(B\rho')^*(B\tilde{i'})^*(c'_1) \neq 0,$$

and

$$(B\tilde{\rho})^*(c'_1) \neq 0.$$ 

Note that $\tilde{\rho} = \rho_2$, hence $(B\tilde{\rho})^* = (B\tilde{i^2})^*(Bf)^*$, so we have

$$(B\tilde{i^2})(B\rho)^*(c'_1) \neq 0. \quad (1)$$

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We now consider the following commutative diagram of classifying spaces:

\[
\begin{array}{ccc}
B\pi & \xrightarrow{Bi} & BDiffeo^+(S_g,*) \\
\cong & & \downarrow Bp_1 \\
B\pi & \xrightarrow{Bi} & B\Gamma_g^1
\end{array}
\]

Recall that by Lemma 1.1.2, \(Bp_1\) is a homotopy equivalence. Hence, by (1) and diagram chasing, there exists a class \(u \in H^2(B\Gamma_g^1,\mathbb{Z})\), which corresponds to the class \((B\rho)^*(c'_1) \in H^2(BDiffeo^+(S_g,*),\mathbb{Z})\), such that \(u' = (Bi)^*u \neq 0\).

Consider now the following commutative diagram:

\[
\begin{array}{ccc}
H^2(B\Gamma_g^1,\mathbb{Z}) & \xrightarrow{(Bi)^*} & H^2(B\pi,\mathbb{Z}) \\
\cong & & \cong \\
H^2(\Gamma_g^1,\mathbb{Z}) & \xrightarrow{i^*} & H^2(\pi,\mathbb{Z}).
\end{array}
\]

If we identify \(u\) with its image in \(H^2(\Gamma_g^1,\mathbb{Z})\), and \(u'\) with its image in \(H^2(\pi,\mathbb{Z})\), we have

\[i^*(u) = u' \neq 0.\]

Since \(\pi\) can be any subgroup of prime order \(p\), by the definition of the Yagita invariant, we conclude that \(Y(\Gamma_g^1,p) = 2\). Hence \(p(\Gamma_g^1) = 2\). It follows that \(\Gamma_g^1\) has period 2.

Q.E.D.

A group with period 2 has interesting properties as follows:

**Corollary 1.1.3.** All finite subgroups of \(\Gamma_g^1\) are cyclic.

**Proof.** Since \(\Gamma_g^1\) has period 2, any of its finite subgroup has period 2. By [Brow1], we know that a finite group has period 2 if and only if it is cyclic. Q.E.D.

Another interesting consequence of Theorem 1.1.2 is the following.
Theorem 1.1.4. For $\mathbb{Z}/p \subset \Gamma_g^1$, let $N(\mathbb{Z}/p)$ be the normalizer and $C(\mathbb{Z}/p)$ be the centralizer in $\Gamma_g^1$. Then $N(\mathbb{Z}/p) = C(\mathbb{Z}/p)$.

Proof. By Theorem 0.1.3, the $p$-period satisfies $p(N(\mathbb{Z}/p)) = 2|N(\mathbb{Z}/p)/C(\mathbb{Z}/p)|p^l$, for some $l \geq 0$. Since $p(N(\mathbb{Z}/p)) = 2$, The result follows. Q.E.D.
1.2 The period of $\Gamma^i_g$, $i > 1$

The following is a generalization of Theorem 1.1.1. and Theorem 1.1.2.

**Theorem 1.2.1.** (a) $vcd(\Gamma^i_g) < \infty$ for $g \geq 1$ and $i \geq 0$.

(b) $\Gamma^i_g$ has $p$-periodic cohomology for $g \geq 1$ and $i \geq 1$. Furthermore, if $\Gamma^i_g$ contains $p$-torsion, then the $p$-period of $\Gamma^i_g$ is 2, i.e., $p(\Gamma^i_g) = 2$ for $g \geq 1$ and $i \geq 1$.

**Proof.** (a) and (b): By Lemma 1.1 of [Ha 4], we have for $2g + r + i > 2$ the following short exact sequence:

$$1 \rightarrow \pi_1 S_{g,r}^i \rightarrow \Gamma^{i+1}_{g,r} \rightarrow \Gamma^i_{g,r} \rightarrow 1,$$

where $S_{g,r}^i$ denotes a smooth surface of genus $g$ with $r$ boundary components and $i$ punctures. If $r = 0$, then the short exact sequence takes the form:

$$1 \rightarrow \pi_1 S_g^i \rightarrow \Gamma^{i+1}_g \rightarrow \Gamma^i_g \rightarrow 1$$

for $2g + i > 2$ and $S_g^i$ stands for the surface $S_g$ with $i$ punctures.

Case (1): If $g > 1$, we have

$$1 \rightarrow \pi_1 S_g^i \rightarrow \Gamma^{i+1}_g \rightarrow \Gamma^i_g \rightarrow 1.$$

... 

$$1 \rightarrow \pi_1 S_g \rightarrow \Gamma^1_g \rightarrow \Gamma^0_g \rightarrow 1.$$

Since $cd(\pi_1 S_g) < \infty$ and $vcd(\Gamma^0_g) < \infty$, by Lemma 1.1.1, $vcd(\Gamma^1_g) < \infty$. By induction, we may assume that $vcd(\Gamma^i_g) < \infty$ for some $i \geq 0$. It is easy to see that $cd(\pi_1 S_g^i) < \infty$ for $i \geq 0$, again by Lemma 1.1.1, it follows that $vcd(\Gamma^{i+1}_g) < \infty$. Moreover, since
\( \Gamma_g^1 \) has \( p \)-periodic cohomology and \( p(\Gamma_g^1) \leq 2 \) for \( g > 1 \), by Lemma 1.1.1, \( \Gamma_g^1 \) has \( p \)-periodic cohomology and \( p(\Gamma_g^i) \leq 2 \) for \( g > 1 \) and \( i \geq 1 \).

Case (2): If \( g = 1 \), we have the short exact sequences:

\[
1 \rightarrow \pi_1 S_1^i \rightarrow \Gamma_1^{i+1} \rightarrow \Gamma_1^i \rightarrow 1.
\]

\[
\ldots
\]

\[
1 \rightarrow \pi_1 S_1 \rightarrow \Gamma_1^2 \rightarrow \Gamma_1^1 \rightarrow 1.
\]

Since \( cd(\pi_1 S_1) < \infty \), \( \Gamma_1^1 = \Gamma_1^0 \), and \( vcd(\Gamma_1^0) < \infty \), by Lemma 1.1.1, \( vcd(\Gamma_1^2) < \infty \). Similarly, we can get \( vcd(\Gamma_1^i) < \infty \) for \( i \geq 0 \) by induction. Moreover, since \( \Gamma_1^1 = \Gamma_1^0 \) has \( p \)-periodic cohomology and \( p(\Gamma_1^0) \leq 2 \), again by Lemma 1.1.1, \( \Gamma_1^i \) has \( p \)-periodic cohomology and \( p(\Gamma_1^i) \leq 2 \) for \( i \geq 1 \). Q.E.D.

As before, we can obtain the followings.

**Corollary 1.2.2.** All finite subgroups of \( \Gamma_g^i \) for \( i \geq 1 \) are cyclic.

**Proof.** It is similar to the proof of Corollary 1.1.3. Q.E.D.

**Theorem 1.2.3.** For \( \mathbb{Z}/p \subset \Gamma_g^i \) for \( i \geq 1 \), let \( N(\mathbb{Z}/p) \) be the normalizer and \( C(\mathbb{Z}/p) \) be the centralizer in \( \Gamma_g^i \). Then \( N(\mathbb{Z}/p) = C(\mathbb{Z}/p) \).

**Proof.** It is similar to the proof of Theorem 1.1.4. Q.E.D.

Recall that \( \Gamma_{g,r}^i \) denotes the mapping class group of \( S_{g,r}^i \), a surface with \( i \) punctures and \( r \) boundary components. We have the following theorem about \( \Gamma_{g,r}^i \).

**Theorem 1.2.4.** \( vcd(\Gamma_{g,r}^i) < \infty \) for \( g \geq 1 \) and \( r \geq 1 \). Furthermore, \( \Gamma_{g,r}^i \) is torsion free for \( g \geq 1 \) and \( r \geq 1 \) and moreover \( cd(\Gamma_{g,r}^i) < \infty \).
PROOF. By Lemma 1.1 of [Ha 4], we have a short exact sequence:

\[ 1 \rightarrow \mathbb{Z} \rightarrow \Gamma_{g,r}^i \rightarrow \Gamma_{g,r+1}^{i+1} \rightarrow 1 \]

for \( g \geq 1 \). We then have the following family of short exact sequences:

\[ 1 \rightarrow \mathbb{Z} \rightarrow \Gamma_{g,r}^i \rightarrow \Gamma_{g,r-1}^{i+1} \rightarrow 1 \quad \text{for} \quad g \geq 1. \]

\[ 1 \rightarrow \mathbb{Z} \rightarrow \Gamma_{g,r-1}^{i+1} \rightarrow \Gamma_{g,r-2}^{i+2} \rightarrow 1 \quad \text{for} \quad g \geq 1. \]

\[ \quad \ldots \]

\[ 1 \rightarrow \mathbb{Z} \rightarrow \Gamma_{g,1}^{i+r-1} \rightarrow \Gamma_{g,0}^{i+r} \rightarrow 1 \quad \text{for} \quad g \geq 1. \]

For \( r \geq 1, g \geq 1 \), by Theorem 1.2.1, \( vcd(\Gamma_{g,0}^{i+r}) < \infty \). By Lemma 1.1.1, \( vcd(\Gamma_{g,1}^{i+r-1}) < \infty \). Therefore by induction, \( vcd(\Gamma_{g,r}^i) < \infty \). Moreover, for the short exact sequence:

\[ 1 \rightarrow \mathbb{Z} \rightarrow \Gamma_{g,1}^{i+r-1} \rightarrow \Gamma_{g,0}^{i+r} \rightarrow 1 \quad \text{for} \quad g \geq 1, \]

using the Gysin sequence, we obtain a long exact sequence:

\[ \ldots \rightarrow H^j(\Gamma_{g,0}^{i+r}) \xrightarrow{\tau} H^{j+2}(\Gamma_{g,0}^{i+r}) \rightarrow H^{j+2}(\Gamma_{g,1}^{i+r-1}) \]

\[ 
\rightarrow H^{j+1}(\Gamma_{g,0}^{i+r}) \xrightarrow{\tau} H^{j+3}(\Gamma_{g,0}^{i+r}) \rightarrow H^{j+3}(\Gamma_{g,1}^{i+r-1}) \rightarrow \ldots \]

for cohomology with arbitrary coefficients. Since \( p(\Gamma_{g,0}^{i+r}) \leq 2 \) with \( \tau \) eventually inducing the periodicity, there exists a sufficiently large \( j \) such that \( H^N(\Gamma_{g,1}^{i+r-1}) = 0 \) for \( N \geq j + 2 \). So, \( cd(\Gamma_{g,1}^{i+r-1}) < \infty \). This implies that \( \Gamma_{g,1}^{i+r-1} \) is torsion free. Hence \( cd(\Gamma_{g,r}^i) < \infty \) and therefore \( \Gamma_{g,r}^i \) is torsion free for \( g \geq 1 \) and \( r \geq 1 \). \textbf{Q.E.D.}
CHAPTER 2
THE FIXED POINT DATA

This chapter consists of two sections. In Section 2.1, we review the definition of fixed point data of an orientation-preserving diffeomorphism of $S_g$ with finite order, which was introduced by Nielsen in [Ni2]. The notation used is different from Nielsen’s. Based on Nielsen’s work, Peter Symonds extended the concept of fixed point data to any element of $\Gamma_g$ with finite order. This can be done for two reasons. First, any element of $\Gamma_g$ with finite order can be realized by a diffeomorphism of $S_g$ with the same order. This is the “Nielsen’s Realization Theorem”, which was mentioned in Section 0.3. Second, Symonds proved that the fixed point data depends on the isotopy class of the diffeomorphisms of $S_g$ with finite order only. Fixed point data is one of the most important invariants of finite-order elements. Several applications can be found in [Sy], [Mis], [Xi3], [Xi4].

In Section 2.2, we extend the fixed point data to an element of finite order in $\Gamma_g$. Recall from Section 0.3 that $\Gamma_g = \pi_0 \text{Diff}^+(S_g, P_1, ..., P_i)$. We need the generalized form of Nielsen’s Realization Theorem and a generalization of Symonds’ Theorem. The second follows immediately from the original version of Symonds’ Theorem. The first, however, is hard. We only prove it for the case where the element of $\Gamma_g$ has prime order $p$, which is all we will need in the subsequent chapters. The general
proof can be found in [MH]; therefore, the fixed point data is well defined for any finite-order element of $\Gamma^i_g$. In Chapters 3-4, we will see that the fixed point data of an element of prime order $p$ in $\Gamma^i_g$ is closely related to the number of conjugacy classes of subgroups of order $p$ in $\Gamma^i_g$, which is necessary to calculate the $p$-torsion of the Farrell cohomology.
2.1 Definition of the fixed point data for an element $x \in \Gamma_g$ of finite order

(a) The fixed point data of an orientation preserving diffeomorphism $\phi$ of $S_g$ of order $n$.

As a basic fact on Riemann surfaces, we know that the set of points of $S_g$ at which $\mathbb{Z}/n = \langle \phi \rangle$ does not act freely is a finite set. Let us denote this finite set by $\text{Sing}(\langle \phi \rangle)$, the singular set of $\phi$. Let $\{x_i\}$ be a set of representatives of the orbits of $\text{Sing}(\langle \phi \rangle)$ under the $\langle \phi \rangle$ action. For each $x_i$, consider the stabilizer of the $\phi$ action at $x_i$ and call it $\text{Stab}_{\langle \phi \rangle}(x_i)$. If the order of $\text{Stab}_{\langle \phi \rangle}(x_i)$ is $\alpha_i$, then it is obvious that $\text{Stab}_{\langle \phi \rangle}(x_i)$ is generated by $\phi^{n/\alpha_i}$. Hence, with respect to a fixed Riemannian structure, $\phi^{n/\alpha_i}$ acts faithfully by rotation on the tangent space at $x_i$. Let $\beta_i$ be an integer such that $\phi^{\beta_i n/\alpha_i}$ acts by rotation through $2\pi/\alpha_i$ in the counterclockwise direction (clockwise defined in terms of the orientation of $S_g$), i.e., if $S_g$ is given a $\phi$ invariant complex structure, $\phi^{\beta_i n/\alpha_i}$ acts as multiplication by $e^{2\pi i/\alpha_i}$. Since $\phi^{\beta_i n/\alpha_i}$ generates $\text{Stab}_{\langle \phi \rangle}(x_i)$, $\beta_i$ is relatively prime to $\alpha_i$, so there is no loss of information in considering just $\beta_i/\alpha_i$ instead of $\alpha_i$ and $\beta_i$. By the "fixed point data" of $\phi$, we shall mean the collection:

$$\delta(\phi) = (n, g|\beta_1/\alpha_1, \beta_2/\alpha_2, ..., \beta_q/\alpha_q),$$

where $n$ is the order of $\phi$, $g$ is the genus of $S_g$, $q$ is the number of singular orbits of the $\phi$ action; the numbers $\beta_i/\alpha_i$ are not ordered. Furthermore, if $n$, $g$ are understood in the context, we shall omit them.
Special Case: If \( f \) is an element of order \( n \) in \( Diffeo^+(S_g) \), and if \( \text{Sing}(< f >) \) is empty, then we define \( \delta(f) \) to be empty.

In the later context, we pay more attention to the elements of prime order \( p \). Assume \( \phi \) is an element of order \( p \) in \( Diffeo^+(S_g) \). Now the singular orbits are the fixed points of the \( \phi \) action; therefore, \( \alpha_i = p \). We can omit \( \alpha_i \), also fix the \( \beta_i \)'s by assuming that \( 0 < \beta_i < p \), for all \( i \). If \( \phi \) acts on \( S_g \) with \( q \) fixed points, then we can write

\[
\delta(\phi) = (\beta_1, \beta_2, ..., \beta_q),
\]

where \( (\beta_1, \beta_2, ..., \beta_q) \) are unordered integers and \( 0 < \beta_i < p \), for all \( i \).

(b) The fixed point data for \( x \in \Gamma_g \) of finite order.

Let \( x \) be an element of order \( n \) in \( \Gamma_g \), by the "Nielsen Realization Theorem", every element of finite order of \( \Gamma_g \) can be represented by a diffeomorphism of the same order. So, we can choose a representative \( f \), an element of order \( n \) in \( Diffeo^+(S_g) \). We can thus define the fixed point data \( \delta(x) = \delta(f) \). The only problem here is that we can choose different representatives \( f \) for \( x \). However, Symonds [Sy] proved that the fixed point data of a diffeomorphism of finite order depends only upon its isotopy class. We now know that the definition does not rely on the choice of \( f \).

Let \( x \) be an element of order \( p \) in \( \Gamma_g \). We can choose \( f \), an element of order \( p \) in \( Diffeo^+(S_g) \) which represents \( x \). If \( f \) acts on \( S_g \) with \( q \) fixed points, then

\[
\delta(x) = (\beta_1, \beta_2, ..., \beta_q),
\]

where \( (\beta_1, \beta_2, ..., \beta_q) \) are well defined unordered integers with \( 0 < \beta_i < p \), for all \( i \).
2.2 Definition of the fixed point data for an element $x \in \Gamma^i_g$, 

$i \geq 1$, of finite order

(a) The fixed point data for $x \in \Gamma^1_g$ of finite order.

Let $x$ be an element of order $n$ in $\Gamma^1_g$, we will define $\delta_1(x)$, the fixed point data of $x$. By [Ha-Za] Lemma 3, we know that we can find $f$, an element of order $n$ in $\text{Diff}^{+}(S_g, P_1)$, which represents $x$. As in the previous section, we put $\delta(f) = (n, g|\beta_1/\alpha_1, \beta_2/\alpha_2, ... \beta_q/\alpha_q)$. Since $f$ fixes $P_1$, there exists $\beta_i/\alpha_i$ for some $i \in \{1, ..., q\}$ corresponding to the fixed point $P_1$. Without loss of generality, we may assume that $\beta_1/\alpha_1$ corresponds to $P_1$. In this case it is clear that $\alpha_1 = |\text{Stab}_x(f)(P_1)| = n$, so we may omit $\alpha_1$. If the order of $f$ and the genus $g$ are well understood in the context, we can now denote the fixed point data of $f$ by $\delta_1(f) = (\beta_1, \beta_2/\alpha_2, ... \beta_q/\alpha_q)$, where $\beta_1$ corresponds to the fixed point $P_1$, and $(\beta_2/\alpha_2, ... \beta_q/\alpha_q)$ is an unordered tuple. Note that we use $\delta_1(f)$ instead of $\delta(f)$ to distinguish the fixed point $P_1$ from other singular orbits representatives. Now let us define $\delta_1(x) = \delta_1(f)$. We need show that it is well defined. Suppose we have $\hat{f}$, another element of order $n$ in $\text{Diff}^{+}(S_g, P_1)$, which also represents $x$. Let $\delta_1(\hat{f}) = (\beta'_1, \beta'_2/\alpha'_2, ... \beta'_q/\alpha'_q)$, where $\beta'_1$ corresponds to $P_1$. Since $f$ is isotopic to $\hat{f}$ rel $P_1$, by $[Sy]$, the fixed point data only depends on the isotopy class, so $(\beta_1/\alpha_1, \beta_2/\alpha_2, ... \beta_q/\alpha_q) = (\beta'_1/\alpha'_1, \beta'_2/\alpha'_2, ... \beta'_q/\alpha'_q)$ as unordered tuples. Moreover, since the isotopy fixes $P_1$, we obtain $\beta_1/\alpha_1 = \beta'_1/\alpha'_1$, i.e., $\beta_1 = \beta'_1$ using the convention $0 < \beta_1, \beta'_1 < n$. Hence $\delta_1(x)$ is well defined.

We now consider the case that $x$ is an element of prime order $p$ in $\Gamma^1_g$. We can choose $f$, an element of order $p$ in $\text{Diff}^{+}(S_g, P_1)$ which represents $x$. If $f$ acts on

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$S_g$ with $q$ fixed points, then the (pointed) fixed point data of $x$ is

$$
\delta_1(x) = (\beta_1|\beta_2, \ldots, \beta_q),
$$

where $\beta_1$ corresponds to $P_1$, $(\beta_2, \ldots, \beta_q)$ are unordered integers and $0 < \beta_i < p$, for all $i$.

(b) The fixed point data for an element $x \in \Gamma_g^i, i > 1$, of finite order.

We now generalize the definition to the case $\Gamma_g^i (= \pi_0 \text{Diff}eo^+(S_g, P_1, \ldots P_i))$.

Let $x$ be an element of order $n$ in $\Gamma_g^i$. If we can find $f$, an element of order $n$ in $\text{Diff}eo^+(S_g, P_1, \ldots P_i)$, which represents $x$, then we may define $\delta_1(x) = \delta_1(f) = (\beta_1, \ldots, \beta_1|\beta_{i+1}/\alpha_{i+1}, \ldots, \beta_q/\alpha_q)$, where $\beta_1, \ldots, \beta_i$ are ordered, corresponding to $P_1, \ldots P_i$ respectively, and $\beta_{i+1}/\alpha_{i+1}, \ldots, \beta_q/\alpha_q$ are unordered. As before, it is easy to see that the definition does not rely on the choice of $f$. The only problem left is whether such $f$ exists. The following propositions of Realization Theorem answer this question.

**Proposition 2.2.1.** Let $\Gamma_g^i = \pi_0 \text{Diff}eo^+(S_g, P_1, \ldots, P_i)$. If $<x>$ is a subgroup of order $p$ in $\Gamma_g^i$, then we can find $f$, an element of order $p$ in $\text{Diff}eo^+(S_g, P_1, \ldots, P_i)$, which represents $x$.

**Proof.** The Teichmüller space $T_g^i$ of $\Gamma_g^i$ is homeomorphic to $(\mathbb{R}^+ \times \mathbb{R})^{3g-3+i}$ according to [Ha 5]. The mapping class group $\Gamma_g^i$ acts on $T_g^i$, and the quotient space is the moduli space $M_g^i$. For $<x> \subset \Gamma_g^i$, consider the action of $<x>$ on $T_g^i$. Based on Smith's Theory, every $\mathbb{Z}/p$ action on a finite dimensional contractible CW-complex has a fixed point. Therefore, $(T_g^i)^{<x>} \neq \emptyset$. If $t \in T_g^i$ is fixed by $<x> \subset \Gamma_g^i$, then we can think of $<x>$ as a group of holomorphic automorphisms of a complex
structure on $S_g$, the holomorphic automorphisms fixing $P_1, ..., P_i$. Thus we can find $f$, an element of order $p$ in $\text{Diff}eo^+(S_g, P_1, ..., P_i)$, which represents $x$. \textbf{Q.E.D.}

\textbf{Proposition 2.2.2.} Let $\Gamma_g^i = \pi_0 \text{Diff}eo^+(S_g, P_1, ..., P_i)$. If $<x>$ is a subgroup of order $n$ in $\Gamma_g^i$, then we can find $f$, an element of order $n$ in $\text{Diff}eo^+(S_g, P_1, ..., P_i)$, which represents $x$.

\textbf{Proof.} It can be found in [MH]. \textbf{Q.E.D.}

As before, we focus on the case of $x \in \Gamma_g^i$ of prime order $p$. We can choose $f$, an element of order $p$ in $\text{Diff}eo^+(S_g, P_1, ..., P_i)$ which represents $x$. If $f$ acts on $S_g$ with $q$ fixed points, then

$$\delta_t(x) = (\beta_1, \beta_2, ..., \beta_i|\beta_{i+1}, ..., \beta_q),$$

where $\beta_1$ corresponds to $P_1$, ... , $\beta_i$ corresponds to $P_i$, and $(\beta_{i+1}, ..., \beta_q)$ are unordered integers and $0 < \beta_i < p$, for all $i$. 

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CHAPTER 3

THE SUBGROUPS OF ORDER $P$ IN $\Gamma_G^T$

In this chapter, we discuss the existence of elements of order $p$ in a mapping class group. Recall that in Chapter 1, we obtained the main result: If $\Gamma_g^i$ has $p$-torsion, then the $p$-period of $\Gamma_g^i$ is 2, i.e., the $p$-torsion of the Farrell cohomology has period 2. On the other hand, if $\Gamma_g^i$ is $p$-torsion free, then by convention the $p$-period of $\Gamma_g^i$ is 1, i.e., the $p$-torsion of the Farrell cohomology vanishes. Therefore, the existence of elements of order $p$ give rise to the $p$-torsion of the Farrell cohomology. We will illustrate this relationship again in Chapters 5-8. In Chapter 2, we defined fixed point data for the elements of order $p$ in a mapping class group. However, we did not verify that such elements exist. Now we will deal with that question.

This chapter consists of two sections. In the first section, we present the necessary and sufficient conditions for an unpunctured mapping class group to contain subgroups of order $p$ in Propositions 3.1.1 and 3.1.2. In the second section, we extend the propositions to include punctured mapping class groups; and thus obtain Propositions 3.2.1 and 3.2.2. The tools we use to do this are the Riemann-Hurwitz equation and Proposition 1 in [E-E]. Finally, a number of examples are given.
3.1 The subgroups of order \( p \) in \( \Gamma_g \), the unpunctured mapping class group

It is known that \( \Gamma_0 = \langle e \rangle \), and \( \Gamma_1 = SL_2(\mathbb{Z}) = \mathbb{Z}/4 \times \mathbb{Z}/2 \mathbb{Z}/6 \). The group \( SL_2(\mathbb{Z}) \) is well understood and therefore, we only need consider the case \( g > 1 \).

If there is a subgroup of order \( p \) in \( \Gamma_g \), which we denote by \( \langle \alpha \rangle \), then by Nielsen’s Realization Theorem, we can find an element of order \( p \) in \( Diff^+(S_g) \), denoted by \( f \), which represents \( \alpha \) in \( \Gamma_g \). Now let us consider \( \langle f \rangle \) acting on \( S_g \). Denote the orbit space by \( S_h \), and the number of the fixed points by \( t \). We have the Riemann-Hurwitz equation: \( 2g - 2 = p(2h - 2) + t(p - 1) \) for some \( h \geq 0, \ t \geq 0 \). It is now easy to conclude the following proposition.

**Proposition 3.1.1.** If \( \Gamma_g \) contains a subgroup of order \( p \), then \( 2g - 2 = p(2h - 2) + t(p - 1) \) has a non-negative integer solution \((h, t)\).

From this Proposition, we obtain the following corollary.

**Corollary 3.1.1.** If the genus \( g \) satisfies \((k - 1)\frac{(p - 1)}{2} + (k - 1) < g < k\frac{(p - 1)}{2}\), for some \( k \in \{1, 2, 3...\} \), then \( \Gamma_g \) does not contain any subgroup of order \( p \).

**Proof.** First, we consider the case \( k = 1 \), i.e., \( 1 < g < \frac{(p - 1)}{2} \), thus \( p > 3 \) (recall that we assume \( g > 1 \)). We need to show for \( 1 < g < \frac{(p - 1)}{2} \), that \( 2g - 2 = p(2h - 2) + t(p - 1) \) has no non-negative integer solution \((h, t)\). Suppose \((h, t)\) is a non-negative integer solution. Since \( g < \frac{(p - 1)}{2} \), the lefthand side of the equation is less than \( p - 3 \), while the righthand side of the equation is always greater than or equal to \(-2p + t(p - 1)\). Hence we have the inequality \(-2p + t(p - 1) < p - 3 \), which
implies $t < 3$. We now discuss the three cases of $t$. If $t = 0$ is inserted back into the equation, we get $2g - 2 = p(2h - 2)$. We know that the left hand side is less than $p - 3$, so $p - 3 > p(2h - 2)$, i.e., $1 - \frac{3}{p} > 2h - 2$. Therefore $h = 0$ or $h = 1$. However, if $h = 0$, then $g = 1 - p$, which is impossible. If $h = 1$, then $g = 1$, contrary to our basic assumption $g > 1$. For case $t = 1$ or $t = 2$, a similar argument holds. Therefore $2g - 2 = p(2h - 2) + t(p - 1)$ has no non-negative integer solution $(h, t)$.

Second, we consider the case $k = 2$, i.e., $\frac{(p - 1)}{2} + 1 < g < p - 1$. Suppose $(h, t)$ is a non-negative integer solution, as above, we have $t < 4$. We have now four cases to discuss. If $t = 0$ is put back into the equation, we get $2g - 2 = p(2h - 2)$. Since $\frac{(p - 1)}{2} + 1 < g < p - 1$, we have $p - 1 < 2g - 2 < 2p - 4$. Hence $p - 1 < p(2h - 2) < 2p - 4$. This implies $1 - \frac{1}{p} < 2h - 2 < 2 - \frac{4}{p}$, a contradiction. For the case $t = 1$ or $t = 2$ or $t = 3$, we argue in a similar way. So $2g - 2 = p(2h - 2) + t(p - 1)$ has no non-negative solution $(h, t)$. We conclude that for $\frac{(p - 1)}{2} + 1 < g < p - 1$, there is no subgroup of order $p$ in $\Gamma_g$.

Finally, consider the case $k \geq 3$, i.e., $(k - 1)\frac{(p - 1)}{2} + (k - 1) < g < k\frac{(p - 1)}{2}$. It is then clear that $(k - 1)p + k - 3 < 2g - 2 < kp - k - 2$. Since $2g - 2 = p(2h - 2) + t(p - 1)$, and $p(2h - 2) + t(p - 1) \geq -2p + t(p - 1)$, we have $-2p + t(p - 1) < kp - k - 2$, thus we have $t < k + 2$. Also, the inequality $(k - 1)p + k - 3 < p(2h - 2) + t(p - 1) < kp - k - 2$ implies $(k - 1 - t) + \frac{t + k - 3}{p} < 2h - 2 < (k - t) + \frac{t - k - 2}{p}$, since $t < k + 2$, $\frac{t - k - 2}{p} < 0$. But since $k \geq 3$, $\frac{t + k - 3}{p} \geq 0$, there is no solution for the inequality, the proof then follows.

Q.E.D.

We can describe the above result as follows:
(1) The mapping class group $\Gamma_g$ has $p$-torsion only in a certain range around genus values of the form $k\frac{(p-1)}{2}$.

(2) For $g > 1$, the smallest $g$ such that $\Gamma_g$ may contain a subgroup of order $p$ is $g = (p - 1)/2$.

(3) The table of $g$ for which $\Gamma_g$ may contain a subgroup of order $p$ is the following:

\[
g - \text{table} \\
\frac{p-1}{2}, \frac{p+1}{2} \\
p - 1, p, p + 1 \\
3\frac{p-1}{2}, 3\frac{p-1}{2} + 1, 3\frac{p-1}{2} + 2, 3\frac{p-1}{2} + 3 \\
4\frac{p-1}{2}, 4\frac{p-1}{2} + 1, 4\frac{p-1}{2} + 2, 4\frac{p-1}{2} + 3, 4\frac{p-1}{2} + 4 \\
\vdots \\
(k-1)\frac{p-1}{2}, (k-1)\frac{p-1}{2} + 1, \ldots, (k-1)\frac{p-1}{2} + (k-1) \\
\vdots
\]

It is natural to ask the converse of Proposition 3.1.1:

If the equation $2g - 2 = p(2h - 2) + t(p - 1)$ has a non-negative integer solution of $(h, t)$, does $\Gamma_g$ contain a subgroup of order $p$? The answer is yes, partially. First, we need an elementary Lemma.

**Lemma 3.1.1.** There exists a sequence $\beta_1, \ldots, \beta_t$ of nonzero elements of $\mathbb{Z}/p$ such that $\sum_{i=1}^{t} \beta_i = 0 \pmod{p}$ if and only if $t \neq 1$.

**Proof.**
(\leq): t \geq 2. If t - 1 \neq 0( \text{ mod } p), then sp < t - 1 < (s + 1)p for some s \geq 0. Let \( \beta_1 = \beta_2 = \ldots = \beta_{t-1} = 1 \) and \( \beta_t = (s + 1)p - (t - 1) \). Hence, 0 < \beta_i < p and \( \sum_1^t \beta_i = 0( \text{ mod } p) \). If t - 1 = 0( \text{ mod } p) and t > 1, then t - 1 = kp for some k > 0. Let \( \beta_1 = \beta_2 = \ldots = \beta_{t-2} = 1, \beta_{t-1} = 2 \) and \( \beta_t = p - 1 \). Hence, 0 < \beta_i < p for \( i \in \{1, 2, \ldots, t\} \) and \( \sum_1^t \beta_i = 0( \text{ mod } p) \).

(\Rightarrow): If t = 1, we can not find a nonzero \( \beta_1 \) satisfying the equation. \textbf{Q.E.D.}

\textbf{Proposition 3.1.2.} If \( 2g - 2 = p(2h - 2) + t(p - 1) \) has a non-negative integer solution \( (h, t) \) and also \( t \neq 1 \), then \( \Gamma_g \) contains a subgroup of order \( p \).

\textbf{Proof.} By Proposition 1 of [E-E], there exists a sequence \( \beta_1, \ldots, \beta_t \) of nonzero elements of \( \mathbb{Z}/p \) such that \( \sum_1^t \beta_i = 0( \text{ mod } p) \), where \( t \) satisfies \( 2g - 2 = p(2h - 2) + t(p - 1) \) for some \( h \geq 0 \), if and only if there is an orientation preserving map of period \( p \) on \( S_g \) with the fixed point data \( (\beta_1, \ldots, \beta_t) \). Therefore, the result follows from Lemma 3.1.1. \textbf{Q.E.D.}

It is now clear that for fixed \( g \), we can know whether \( \Gamma_g \) contains a subgroup of order \( p \) or not. Let us consider some examples. We always assume \( g > 1 \) in this section.

Example 1. \( g = \frac{p-1}{2} \):

\( 2g - 2 = p(2h - 2) + t(p - 1) \) has a unique solution \( (h, t) = (0, 3) \). So \( \Gamma_{\frac{p-1}{2}} \) contains a subgroup of order \( p \). It follows that every diffeomorphism of order \( p \) in \( S_{\frac{p-1}{2}} \) regarded as an action on \( S_{\frac{p-1}{2}} \), has three fixed points, and the orbit space of the action is \( S^2 \).

Example 2. \( g = \frac{p+1}{2} \):

\( 2g - 2 = p(2h - 2) + t(p - 1) \) has a unique solution \( (h, t) = (1, 1) \) for \( g > 2 \);
$2g - 2 = p(2h - 2) + t(p - 1)$ has two solutions $(h, t) = (1, 1)$ and $(h, t) = (0, 4)$ for $g = 2$. So $\Gamma_{p+1}^{\frac{(p-1)}{2}}$ does not contain a subgroup of order $p$ for $g > 2$. $\Gamma_2$ contains a subgroup of order 3.

Example 3. $g = p - 1$:

$2g - 2 = p(2h - 2) + t(p - 1)$ has a unique solution $(h, t) = (0, 4)$ for $g > 2$; 
$2g - 2 = p(2h - 2) + t(p - 1)$ has solutions $(h, t) = (1, 1)$ and $(h, t) = (0, 4)$ for $g = 2$. 
So $\Gamma_{p-1}$ contains a subgroup of order $p$. Also, every diffeomorphism of order $p$ on $S_{p-1}$ regarded as an action on $S_{p-1}$, has four fixed points, and the orbit space is $S^2$.

Example 4. $g = p$:

$2g - 2 = p(2h - 2) + t(p - 1)$ has a unique solution $(h, t) = (1, 2)$ for $g > 3$; 
$2g - 2 = p(2h - 2) + t(p - 1)$ has solutions $(h, t) = (1, 2)$ and $(h, t) = (0, 6)$ for $g = 2$; 
$2g - 2 = p(2h - 2) + t(p - 1)$ has solutions $(h, t) = (1, 2)$ and $(h, t) = (0, 5)$ for $g = 3$. 
So $\Gamma_{p}$ contains a subgroup of order $p$. Also, for $g > 3$, every diffeomorphism of order $p$ on $S_{p}$ has two fixed points and the orbit space is $S_1$.

Example 5. $g = p + 1$:

$2g - 2 = p(2h - 2) + t(p - 1)$ has a unique solution $(h, t) = (2, 0)$ for $g > 6$; 
$2g - 2 = p(2h - 2) + t(p - 1)$ has solutions $(h, t) = (2, 0)$ and $(h, t) = (1, 4)$ and $(h, t) = (0, 8)$ for $g = 2$; $2g - 2 = p(2h - 2) + t(p - 1)$ has solutions $(h, t) = (2, 0)$ and $(h, t) = (0, 6)$ and $(h, t) = (1, 3)$ for $g = 4$; $2g - 2 = p(2h - 2) + t(p - 1)$ has solutions $(h, t) = (2, 0)$ and $(h, t) = (0, 5)$ for $g = 6$. So $\Gamma_{p+1}$ contains a subgroup of order $p$. Also, for $g > 6$, every diffeomorphism of order $p$ on $S_{p+1}$ has no fixed point and the orbit space is $S_2$.

Example 6. $g = k\frac{(p-1)}{2}$ for $k \geq 1$:

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\[ 2g - 2 = p(2h - 2) + t(p - 1) \] has a solution \((h, t) = (0, k+2)\). So \(\Gamma_{k(x-1)}\) contains a subgroup of order \(p\). Actually, all solutions of \(2g - 2 = p(2h - 2) + t(p - 1)\) are of the form \((h, t) = (\frac{(p-1)}{2}, k + 2 - p)\), \((h, t) = (2 \frac{(p-1)}{2}, k + 2 - 2p)\), ..., \((h, t) = (s \frac{(p-1)}{2}, k + 2 - sp)\), for some \(s \geq 0\) and \(k + 2 - sp \geq 0\). So, \(\Gamma_{k(x-1)}\) may contain \(\mathbb{Z}/p\)'s with different number of fixed points if \(k \geq 3\). However, if we assume \(p > k + 2\), then every diffeomorphism of order \(p\) on \(S_{k(x-1)/2}\) has \(k + 2\) fixed points and the orbit space is \(S^2\).

Example 7. \(g = 3 \frac{(p-1)}{2} + 2\):

\[ 2g - 2 = p(2h - 2) + t(p - 1) \] has solutions \((h, t) = (2, 1)\), \((h, t) = (\frac{5-p}{2}, 1 + p)\) and \((h, t) = (3 - p, 1 + 2p)\). So, for \(p > 5\), there is no subgroup of order \(p\) in \(\Gamma_{3(\frac{p-1}{2})+2}\).

Remark.

(1) Note that two integer solutions \((h_1, t_1)\) and \((h_2, t_2)\) of \(2g - 2 = p(2h - 2) + t(p - 1)\) satisfy \(t_1 - t_2 = 0(\mod p)\).

(2) The \(g\)-table are the possible \(g\)'s such that \(\Gamma_g\) may contain a subgroup of order \(p\). But there exists some \(g\) in the \(g\)-table such that \(\Gamma_g\) does not contain a subgroup of order \(p\).
3.2 The subgroups of order $p$ in $\Gamma^i_g$, the punctured mapping class group

Recall that we have the following exact sequences:

$$1 \rightarrow \pi_1 S^i_g \rightarrow \Gamma^{i+1}_g \rightarrow \Gamma^i_g \rightarrow 1.$$  

...  

$$1 \rightarrow \pi_1 S^i_g \rightarrow \Gamma^i_g \rightarrow \Gamma^0_g \rightarrow 1.$$  

We know that $\pi_1 S^i_g$ is torsion free. It follows that if $\Gamma^0_g$ does not contain any subgroup of order $p$, then $\Gamma^i_g$ does not contain any subgroup of order $p$. So, we have:

**Proposition 3.2.1.** If $\Gamma^i_g$ contains a subgroup of order $p$, then $2g - 2 = p(2h - 2) + t(p - 1)$ has a non-negative integer solution with $t \geq i$, $t \neq 1$.

From Section 3.1, we know the following facts:

If $(k - 1)\frac{(p-1)}{2} + (k - 1) < g < k\frac{(p-1)}{2}$, for some $k \in \{1, 2, 3...\}$, then $\Gamma^i_g$ does not contain any subgroup of order $p$ for $g > 1$.

The following Proposition is the converse of Proposition 3.2.1.

**Proposition 3.2.2.** If $2g - 2 = p(2h - 2) + t(p - 1)$ has a non-negative integer solution $(h, t)$, with $t \neq 1$ and $t \geq i$, then $\Gamma^i_g$ contains a subgroup of order $p$.

The proof is similar to the proof of Proposition 3.1.2.
CHAPTER 4
COUNTING CONJUGACY CLASSES OF SUBGROUPS
OF ORDER \( P \) IN \( \Gamma_G^I \)

In the previous chapter, we proved the necessary and sufficient conditions for the existence of subgroups of order \( p \) in a mapping class group. This chapter considers to count the number of conjugacy classes of subgroups of order \( p \). In the first section, we review Symonds' Conjugation Theorem on elements of order \( p \) in an unpunctured mapping class group, which is a generalization of Nielsen's Conjugation Theorem on diffeomorphisms of order \( p \) in \( S_g \). This theorem sets up a one-to-one correspondence between the conjugacy classes of the elements of order \( p \) and different values of fixed point data. Therefore, in order to get the number of conjugacy classes of the elements of order \( p \), it suffices to count the number of different values of fixed point data, which is equal to the number of solutions of a certain linear equation.

In the second section, we generalize Symonds' Conjugation Theorem to the punctured mapping class group \( \Gamma_g^i \). As an application, we obtain the number of conjugacy classes of the subgroups of order \( p \) (as opposed to the first section where we calculate the number of conjugacy classes of the elements of order \( p \)) in terms of the number of solutions of a certain linear equation. In the third section, we extend these results to the punctured mapping class group \( \Gamma_g^i \).
For a group with finite virtual cohomological dimension which also has $p$-periodic cohomology, the number of conjugacy classes of subgroups of order $p$ is very useful in calculating the cohomology. Recall that we have the following fundamental theorem which we cited as Theorem 0.1.4:

[Brown] If $\Gamma$ has finite $vcd$ and $p$-periodic cohomology, then

$$\hat{H}^i(\Gamma, M)_{(p)} \cong \prod_p \hat{H}^i(N(P), M)_{(p)},$$

where $P$ ranges over the conjugacy classes of subgroups of order $p$ in $\Gamma$.

As a result, this chapter has applications later on.
4.1 Counting conjugacy classes of subgroups of order $p$ in $\Gamma_g$

**Theorem 4.1.1. [Nielsen]** Two elements of order $p$ are conjugate in $\text{Diff}eo^+(S_g)$ if and only if they have the same fixed point data.

As we have seen in Chapter 2, [Sy] proves that the fixed point data of a diffeomorphism of finite order depends only upon its isotopy class, that is, its image in $\Gamma_g$. So we have:

**Theorem 4.1.2. [Symonds]** Two elements of order $p$ in $\Gamma_g$ are conjugate if and only if they have the same fixed point data.

Symonds' conjugation theorem presents us a method to count the number of conjugacy classes of elements of order $p$ in $\Gamma_g$, which is the following proposition.

**Proposition 4.1.1.** If a non-negative integer $t$ satisfies the Riemann Hurwitz equation $2g - 2 = p(2h - 2) + t(p - 1)$ and $t \neq 1$, then the number of conjugacy classes of elements of order $p$ in $\Gamma_g$ which act on $S_g$ with $t$ fixed points is the same as the number of different unordered integer tuples $(\beta_1, ..., \beta_t)$ such that $\beta_1 + ... + \beta_t = 0 \pmod{p}$ and $0 < \beta_i < p$ for all $i$.

**Proof.** By Theorem 4.1.2, we set up the one-to-one correspondence between the conjugacy classes of elements of order $p$ in $\Gamma_g$ which act on $S_g$ with $t$ fixed points and the fixed point data $(\beta_1, ..., \beta_t)$ of elements of order $p$ in $\Gamma_g$ which act on $S_g$ with $t$ fixed points. On the other hand, by Proposition 1 in [E-E], if $t$ satisfies the Riemann Hurwitz equation $2g - 2 = p(2h - 2) + t(p - 1)$, where $h$ is a non-negative integer, then every unordered integer tuple $(\beta_1, ..., \beta_t)$ which satisfies $\beta_1 + ... + \beta_t = 0 \pmod{p}$...
and $0 < \beta_i < p$ for all $i$ corresponds to the fixed point data $(\beta_1, ..., \beta_t)$ of an element of order $p$ in $\Gamma_g$ which act on $S_g$ with $t$ fixed points. So, counting the number of conjugacy classes of elements of order $p$ which act on $S_g$ with $t$ fixed points becomes the same as counting the number of different unordered integer tuples $(\beta_1, ..., \beta_t)$ such that $\beta_1 + ... + \beta_t = 0 \pmod{p}$, where $0 < \beta_i < p$, for all $i$. \textbf{Q.E.D.}

\textbf{Remarks.}

(1) We mentioned the number of fixed points of the element $x \in \Gamma_g$ of order $p$ in the above proof. This is meaningful, since [Sy] proved that the fixed point data of a diffeomorphism of finite order depends only upon its isotopy class, so isotopic diffeomorphisms of finite order have the same number of fixed points. Also, Nielsen proved that the fixed point data depends only upon the conjugacy class, so we can discuss the number of fixed points of a conjugacy class.

(2) The above proposition presents us a number-theoretical way to calculate the number of conjugacy classes of elements of order $p$ which act on $S_g$ with $t$ fixed points. In the subsequent chapters, what concerns us most is not the number of the conjugacy classes of elements of order $p$, but the number of conjugacy classes of subgroups of order $p$ in $\Gamma_g$. Of course, this involves more work. In fact, in some subgroups of order $p$, there are no elements conjugate to each other, while in other subgroups of order $p$, there are several elements conjugate to each other. No specific algorithm can be applied to this question. However, in the next section, when we discuss the punctured mapping class group $\Gamma_g^i$ for $i \geq 1$, we can have an easy approach. For we can prove that there are no elements in the subgroup of order $p$ in $\Gamma_g^i$ conjugate to each other for $i \geq 1$. 

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4.2 Counting conjugacy classes of subgroups of order $p$ in $\Gamma_g^1$

We deduce a conjugation theorem for the punctured mapping class group $\Gamma_g^1$.

**Theorem 4.2.1.** Let $\Gamma_g^1$ be $\pi_0(Diff^o(S_g, P_1))$. Let $\alpha, \alpha'$ be elements of order $p$ in $\Gamma_g^1$ which have fixed point data $\delta_1(\alpha) = (\beta_1 | \beta_2, ..., \beta_t)$, $\delta_1(\alpha') = (\beta'_1 | \beta'_2, ..., \beta'_t)$. Then, the following holds:

The element $\alpha$ is conjugate to $\alpha'$ in $\Gamma_g^1$ if and only if $\beta_1 = \beta'_1$, and $(\beta_2, ..., \beta_t) = (\beta'_2, ..., \beta'_t)$ as unordered tuples. In other words, two elements of order $p$ in $\Gamma_g^1$ are conjugate if and only if they have the same fixed point data.

**Proof.**

$(\Leftarrow)$: Let $f$ respectively $f'$, be an element of order $p$ in $Diff^o(S_g, P_1)$ which represents $\alpha$, respectively $\alpha'$. In a complex local coordinate system, the function $f$ can be viewed as the multiplication by $e^{(2\pi i)/p\beta_1^{-1}}$ near the fixed point $P_1$, the function $f'$ can be viewed as the multiplication by $e^{(2\pi i)/p\beta'_1^{-1}}$ near the fixed point $P_1$. Assume now $\beta_1 = \beta'_1$. Then we may choose a small enough disk $D$, centered at $P_1$, such that $f|_D = f'|_D =$multiplication by $e^{(2\pi i)/p\beta_1^{-1}}$. Indeed assume the radius of $D$ is $s_0$. Consider now the manifold $M = S_g - D^o$, $S_g$ with the open disk $D^o$ removed. Let $\phi = f|_M : M \to M$, $\phi' = f'|_M : M \to M$ be the restriction maps. We assume $(\beta_2, ..., \beta_t) = (\beta'_2, ..., \beta'_t)$, by the conjugation theorem in [Ni2], there exists $\psi \in Diff(S_g)$, such that $\psi^{-1}\phi'\psi = \phi$. Since $\psi$ sends the boundary of $D$ to itself, we can assume $\psi$ is defined on $\partial D$ as follows: $\psi(s_0e^{i\theta}) = s_0e^{ix(\theta)}$, where $x(\theta)$ is a differentiable function of $\theta$. Therefore, we can extend $\psi$ to $S_g$ by $\psi(re^{i\theta}) = re^{ix(\theta)}$, where $re^{i\theta} \in D$, and $r \leq s_0$. Denote the extension map by $h$. We have $h(P_1) = P_1$. 49
Because we assume $f = f' = \text{multiplication by } e^{(2\pi i)/p\beta_1}$ on $D$, we know that $h^{-1}f'h = f$. Hence $\alpha'$ is conjugate to $\alpha$ in $\Gamma_g^1$.

($\Rightarrow$):

If $\alpha$ is conjugate to $\alpha'$ in $\Gamma_g^1$, there exists $\gamma \in \Gamma_g^1$ such that $\gamma \alpha \gamma^{-1} = \alpha'$. Let $h$ represent $\gamma$, $h \in Diffeo^+(S_g, P_1)$. We have $hf^{-1}$ isotopic to $f'$ relative $P_1$. By the theorem in [Sy], the fixed point data only depend on the isotopy class, so $\delta_1(hfh^{-1}) = \delta_1(f')$. On the other hand, by Nielsen’s Conjugation Theorem, $\delta(hfh^{-1}) = \delta(f)$. Also since $d_{P_1}(hfh^{-1}) = d_{P_1}(f)$, where $d_{P_1}$ takes $f : S_g \to S_g$ to $d_{P_1}(f) : T_{P_1}S_g \to T_{P_1}S_g$, and $T_{P_1}S_g$ is the tangent space of $S_g$ at $P_1$. So, $\delta_1(hfh^{-1}) = \delta_1(f')$, and therefore $\delta_1(f) = \delta_1(f')$, i.e., $\beta_1 = \beta'_1$, and $(\beta_2, \ldots, \beta_t) = (\beta'_2, \ldots, \beta'_t)$. Q.E.D.

As an application, we have the following proposition to count the number of conjugacy classes of subgroups of order $p$ in $\Gamma_g^1$ which act on $S_g$ with $t$ fixed points.

**Proposition 4.2.1.** Let $t$ be a non-negative integer such that it satisfies the Riemann Hurwitz equation $2g - 2 = p(2h - 2) + t(p - 1)$ and $t > 1$. Then the number of different unordered integer tuples $(1|\beta_2, \ldots, \beta_t)$ such that $1 + \beta_2 + \ldots + \beta_t = 0(\mod p)$, where $0 < \beta_i < p$ for all $i$, is the same as the number of conjugacy classes of subgroups of order $p$ in $\Gamma_g^1$ which act on $S_g$ with $t$ fixed points.

**Proof.**

We notice the following two facts. First, in every subgroup of order $p$, there always exists a generator $x$, such that $\delta_1(x) = (1|\beta_2, \ldots, \beta_t)$. In fact, suppose we choose an arbitrary generator $y$, $\delta(y) = (\beta_1|\beta_2, \ldots, \beta_t)$ and $\beta_1 \neq 1$, then we can find $m$, such that $m\beta_1 = 1(\mod p)$. Therefore, $y^{\beta_1}$ generates the same subgroup as $y$ and $\delta_1(y^{\beta_1}) = (m\beta_1|m\beta_2, \ldots, m\beta_t) = (1|m\beta_2, \ldots, m\beta_t)$. Second, we claim that if
$x \in \Gamma^1_g$, then for $1 < k < p$, $x$ is not conjugate to $x^k$. In fact, if $\delta_1(x) = (1|\beta_2, ..., \beta_t)$, $\delta_1(x^k) = (l|l|\beta_2, ..., l\beta_t)$, where $lk = 1 \pmod{p}$. By Theorem 4.2.1, if $x$ is conjugate to $x^k$, then $1 = l \pmod{p}$, which implies $k = 1$. Therefore, no two elements in the subgroup of order $p$ are conjugate. By these facts as well as Theorem 4.2.1, we can set up the one to one correspondence between all conjugacy classes of subgroups of order $p$ in $\Gamma^1_g$ and all fixed point data $(1|\beta_2, ..., \beta_t)$ of elements of order $p$ in $\Gamma^1_g$. On the other hand, as before, we know that if $(h, t)$ satisfies $2g - 2 = p(2h - 2) + t(p - 1)$, where $h$ is a non-negative integer and $t > 1$, then every unordered integer tuple $(1|\beta_2, ..., \beta_t)$ such that $1 + \beta_2 + ... + \beta_t = 0 \pmod{p}$, where $0 < \beta_i < p$ for all $i$ can be realized as the fixed point data of a generator of some subgroup of order $p$ in $\Gamma^1_g$. So, in order to count the number of different conjugacy classes of subgroups of order $p$, it is necessary to count the number of different unordered integer tuples $(1|\beta_2, ..., \beta_t)$ such that $1 + \beta_2 + ... + \beta_t = 0 \pmod{p}$, where $0 < \beta_i < p$ for all $i$. \quad \textbf{Q.E.D.}

\textbf{Remark.}

Counting the number of conjugacy classes of subgroups of order $p$ is a number-theoretical problem of counting the number of different unordered integer tuples $(1|\beta_2, ..., \beta_t)$ such that $1 + \beta_2 + ... + \beta_t = 0 \pmod{p}$, where $0 < \beta_i < p$ for all $i$, and where $t$ is a non-negative integer which satisfies the Riemann Hurwitz equation $2g - 2 = p(2h - 2) + t(p - 1)$ and $t > 1$.

Now for fixed $g$ and $p$, the algorithm for counting the number of conjugacy classes of subgroups of order $p$ in $\Gamma^1_g$ is as follows:

(1) Find all non-negative integer solutions $(h, t)$ satisfying $2g - 2 = p(2h - 2) + t(p - 1)$ and $t > 1$;
(2) For each $t$, find the number of different unordered integer tuples $(1, \beta_2, \ldots, \beta_t)$ such that $1 + \beta_2 + \ldots + \beta_t = 0 \pmod{p}$, where $0 < \beta_i < p$ for all $i$;

(3) Add all solutions in step 2 where $t$ ranges over all non-negative integer solutions $(h, t)$ satisfying $2g - 2 = p(2h - 2) + t(p - 1)$ and $t > 1$. 
4.3 Counting conjugacy classes of subgroups of order \( p \) in \( \Gamma_g^i \) for \( i > 1 \)

This section is a generalization of the previous section.

**Theorem 4.3.1.** Let \( \Gamma_g^i = \pi_{0}(\text{Diff}eo^+(S_g, P_1, ..., P_i)) \), and let \( \alpha, \alpha' \) be elements of order \( p \) in \( \Gamma_g^i \), with \( \delta_i(\alpha) = (\beta_1, ..., \beta_i|\beta_{i+1}, ..., \beta_i) \), \( \delta_i(\alpha') = (\beta_1', ..., \beta_i'|\beta_{i+1}', ..., \beta_i') \). Then, the following holds:

The element \( \alpha \) is conjugate to \( \alpha' \) in \( \Gamma_g^i \) if and only if \( \beta_1 = \beta'_1, ..., \beta_i = \beta'_i \), and \( (\beta_{i+1}, ..., \beta_i) = (\beta'_{i+1}, ..., \beta'_i) \) as unordered integer tuples; i.e., two elements of order \( p \) in \( \Gamma_g^i \) are conjugate if and only if they have the same fixed point data.

**Proof.** It is similar to the proof of Theorem 4.2.1. \( \text{Q.E.D.} \)

**Proposition 4.3.1.** Let \( t \) be a non-negative integer which satisfies Riemann Hurwitz equation \( 2g - 2 = p(2h - 2) + t(p - 1) \) and \( t \geq i \). Then the number of different integer tuples \( (1, \beta_2, ..., \beta_i|\beta_{i+1}, ..., \beta_i) \) such that \( (1, \beta_2, ..., \beta_i) \) is ordered, \( (\beta_{i+1}, ..., \beta_i) \) is unordered, and \( 1 + \beta_2 + ... + \beta_i = 0 \text{ (mod } p) \), where \( 0 < \beta_i < p \) for all \( i \), is the same as the number of conjugacy classes of subgroups of order \( p \) in \( \Gamma_g^i \) which act on \( S_g \) with \( t \) fixed points.

**Proof.** It is similar to the proof of Proposition 4.2.1. \( \text{Q.E.D.} \)

Now for fixed \( g \) and \( p \), the algorithms for counting the number of conjugacy classes of subgroups of order \( p \) in \( \Gamma_g^i \) are as follows:

1. Find all non-negative integer solutions \((h, t)\) satisfying \( 2g - 2 = p(2h - 2) + t(p - 1) \) and \( t \geq i \) and \( t \neq 1 \);
(2) For each $t$, find the number of different integer tuples $(1, \beta_2, ..., \beta_t | \beta_{t+1}, ..., \beta_t)$ such that $(1, \beta_2, ..., \beta_t)$ is ordered, $(\beta_{t+1}, ..., \beta_t)$ is unordered, and $1 + \beta_2 + ... + \beta_t = 0 \pmod{p}$, where $0 < \beta_i < p$ for all $i$;

(3) Add all solutions in step 2 where $t$ ranges over all non-negative integer solutions $(h, t)$ satisfying $2g - 2 = p(2h - 2) + t(p - 1)$ and $t \geq i$ and $t \neq 1$. 
CHAPTER 5
THE P-TORSION OF THE FARRELL COHOMOLOGY
OF \( \Gamma^N_{(p-1)/2} \)

In this chapter we prove the following main theorem:

**Theorem 5.0.2.** Let \( p \) denote an odd prime number. Then

\[
\hat{H}^i(\Gamma_{(p-1)/2}^1, \mathbb{Z})_{(p)} = \begin{cases} 
\prod_{1}^{(p-1)/2} \mathbb{Z}/p & i = 0 \text{ mod}(2) \\
0 & i = 1 \text{ mod}(2)
\end{cases}
\]

\[
\hat{H}^i(\Gamma_{(p-1)/2}^2, \mathbb{Z})_{(p)} = \begin{cases} 
\prod_{1}^{p-2} \mathbb{Z}/p & i = 0 \text{ mod}(2) \\
0 & i = 1 \text{ mod}(2)
\end{cases}
\]

\[
\hat{H}^i(\Gamma_{(p-1)/2}^3, \mathbb{Z})_{(p)} = \begin{cases} 
\prod_{1}^{p-2} \mathbb{Z}/p & i = 0 \text{ mod}(2) \\
0 & i = 1 \text{ mod}(2)
\end{cases}
\]

\[\hat{H}^i(\Gamma_{(p-1)/2}^n, \mathbb{Z})_{(p)} = 0, \text{ for } n > 3.\]

As we observed in Chapter 1, \( \Gamma_{(p-1)/2}^n \) is \( p \)-periodic for \( n \geq 1 \). Thus by Theorem 0.1.4,

\[
\hat{H}^i(\Gamma_{(p-1)/2}^n, \mathbb{Z})_{(p)} \rightarrow \prod_{\mathbb{Z}/p \in \Omega} \hat{H}^i(N(\mathbb{Z}/p), \mathbb{Z})_{(p)}
\]

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is an isomorphism, where \( \Omega \) is a set of representatives for the conjugacy classes of subgroups of order \( p \) in \( \Gamma_{(p-1)/2}^n \), and \( N(\mathbb{Z}/p) \) stands for the normalizer of \( \mathbb{Z}/p \) in \( \Gamma_{(p-1)/2}^n \). In the first section, we calculate \( H^i(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} \) for \( \mathbb{Z}/p < \Gamma_{(p-1)/2}^n \). In fact, we prove that the normalizer \( N(\mathbb{Z}/p) \) is finite with order \( p \) or \( 2p \) for any \( \mathbb{Z}/p < \Gamma_{(p-1)/2}^n \). Hence, by Corollary 1.2.2, it is cyclic. In the second section, we count the conjugacy classes of subgroups of order \( p \) in \( \Gamma_{(p-1)/2}^n \). In fact, by Chapter 4, we know that the number of conjugacy classes of subgroups of order \( p \) is the number of solutions of a certain linear equation. Then the cohomology as stated follows.

Recall that Proposition 3.2.2 says that \( \Gamma_{(p-1)/2}^n \) contains a subgroup of order \( p \) if \( n \leq 3 \); furthermore, every element of order \( p \) acting on \( S_{(p-1)/2} \) has three fixed points.
5.1 Calculation of the $p$-torsion of the Farrell cohomology of the normalizer of $\mathbb{Z}/p$ in $\Gamma_{(p-1)/2}^n$

As we will see, the Farrell cohomology of the normalizer $N(\mathbb{Z}/p)$ of $\mathbb{Z}/p < \Gamma_{(p-1)/2}^1$ is the same for all conjugacy classes of subgroups of order $p$ in $\Gamma_{(p-1)/2}^1$.

**Lemma 5.1.1.** Let $\mathbb{Z}/p < \Gamma_{(p-1)/2}^1$. Then

$$\check{H}^i(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} = \begin{cases} \mathbb{Z}/p & i = 0 \mod(2) \\ 0 & i = 1 \mod(2) \end{cases}.$$

**Proof.** Fix $\mathbb{Z}/p = \langle \alpha > < \Gamma_{(p-1)/2}^1$. For $\theta \in N(\langle \alpha >) < \Gamma_{(p-1)/2}^1$, we have $\theta \alpha \theta^{-1} = \alpha^r$ for some $r \in \{1, 2, ..., p-1\}$. Then by [Ha-Za] Lemma 3, we can find $y$ (respectively $h$), orientation preserving diffeomorphisms of $S_{(p-1)/2}$ with fixed point $P_1$, which represent $\alpha$ (respectively $\theta$), such that $y^p = 1$, $h_y h^{-1} = y^r$ for the same $r \in \{1, 2, ..., p-1\}$. Consider now the $\langle y >$ action on $S_{(p-1)/2}$. Since $y$ has order $p$, using the Riemann-Hurwitz equation, we conclude that the action has three fixed points, $P_1, P_2, P_3$. Also, the orbit space is $S^2$. Thus we have the following diagram:

$$S_{(p-1)/2} \xrightarrow{h} S_{(p-1)/2}$$

$$\pi \downarrow \quad \pi \downarrow$$

$$S^2 \xrightarrow{f} S^2$$

where $\pi$ is the projection onto the orbit space of the $\langle y >$ action.

Claim: (a) $h$ is fiber-preserving, so it induces $f$. Indeed, if

$$\pi(x_1) = \pi(x_2),$$

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then

\[ x_1 = y^k(x_2), \quad 1 \leq k \leq p. \]

We have therefore

\[ h(x_1) = hy^k(x_2); \]
\[ h(x_1) = y^k h(x_2); \]
\[ \pi h(x_1) = \pi h(x_2). \]

Hence \( h \) induces \( f \).

(b) Since \( h \) is an orientation-preserving diffeomorphism, so is \( f \).

(c) \( h \) permutes \( P_2, P_3 \) and fixes \( P_1 \), so \( f \) permutes the corresponding points \( \hat{P}_2, \hat{P}_3 \) and fixes the corresponding point \( \hat{P}_1 \) in the orbit space \( S^2 \).

Since

\[ h y h^{-1} = y^r, \]
\[ hy(P_2) = y^r h(P_2), \]

which implies

\[ h(P_2) = y^r h(P_2). \]

Therefore, \( h(P_2) \) is in the fixed points set of \( y^r \), which is \( \{P_1, P_2, P_3\} \). Since \( h(P_1) = P_1, \) \( h \) is one to one and onto; therefore, we know \( h(P_2) \in \{P_2, P_3\} \). Similarly, \( h(P_3) \in \{P_2, P_3\} \).

By the above arguments and commuting diagram, we have

\[ f \in Diff_{eo}^+(S^2, \text{fixing } \hat{P}_1, \text{permuting } \{\hat{P}_2, \hat{P}_3\}). \]
Put $\pi_0Diff^+(S^2, \text{fixing } \hat{P}_1, \text{permuting } \{\hat{P}_2, \hat{P}_3\}) = \Gamma^{1,2}$. We now prove the following results:

1. We can define $I : N(\mathbb{Z}/p)/(\mathbb{Z}/p) \to \Gamma^{1,2}$, a one to one homomorphism.
2. $\Gamma^{1,2} = \mathbb{Z}/2$.
3. $N(\mathbb{Z}/p) = \mathbb{Z}/p$, or $N(\mathbb{Z}/p) = \mathbb{Z}/2p$.

From this, the lemma follows immediately.

Proof of (1): First, we define $\hat{I} : N(\mathbb{Z}/p) \to \Gamma^{1,2}$. As before, we may assume $\mathbb{Z}/p = \langle \alpha \rangle$; for an arbitrary $\theta$ in $N(\mathbb{Z}/p)$, we can define $\hat{I}(\theta) = [f]$, where $f$--a diffeomorphism of $S^2$--is induced by $h$, a diffeomorphism of $S_{(p-1)/2}$ which represents $\theta$, and $[\ ]$ stands for the isotopy class which fixes $P_1$ and preserves $\{P_2, P_3\}$. We need to show that $\hat{I}$ is well defined. In fact, if $\hat{h}$ is also representing $\theta$, then $h$ is isotopic to $\hat{h}$ rel $P_1$. Since $\theta \alpha \theta^{-1} = \alpha^r$ for some $r \in \{1, ..., p-1\}$, by the discussion above, we know that $hy\hat{h}^{-1} = y^r$, and $\hat{h}y\hat{h}^{-1} = y^r$, where $y$ is an element of order $p$ which represents $\alpha$. Moreover, by the argument of [Bi-Hi2], there exists an isotopy $H_s$ between $h$ and $\hat{h}$ such that $H_s(P_1) = P_1$ and $H_s y H_s^{-1} = y^r$. Hence, similar to claim (a), (b), (c), it is known that $H_s$ induces $F_s$. Then it is easy to get the following: $f$ is isotopic to $\hat{f}$ rel $\hat{P}_1$, rel $\{\hat{P}_2, \hat{P}_3\}$ by the isotopy $F_s$. In fact, $H_0 = h$, $H_1 = \hat{h}$, so we get $F_0 = f, F_1 = \hat{f}$. Therefore, $[f] = [\hat{f}]$. Hence, $\hat{I}$ is well defined. For $\alpha^i \in \mathbb{Z}/p < N(\mathbb{Z}/p)$, choose $y^i$ representing $\alpha^i$. Since $y^i$ induces id on $S^2$, $\hat{I}(\alpha^i) = [id]$. Therefore, $\hat{I}$ induces a homomorphism $I : N(\mathbb{Z}/p)/(\mathbb{Z}/p) \to \Gamma^{1,2}$. Now, we will prove that $I$ is one to one. Suppose $f$ is isotopic to $id$ rel $\hat{P}_1$, rel $\{\hat{P}_2, \hat{P}_3\}$, and $f$ is induced by $h$. The Homotopy Lifting Theorem shows that $h$ is isotopy to $y^i$ rel $P_1$, rel$\{P_2, P_3\}$, for some $1 \leq i \leq p$. So $I$ is one to one.
Proof of (2): Define \( \lambda : \text{Diff}^+(S^2, \text{fixing } \hat{P}_1, \text{permuting } \{\hat{P}_2, \hat{P}_3\}) \to \Sigma_2 \) as follows: \( \lambda(f) \) is the permutation of \( f \) induced on \( \{\hat{P}_2, \hat{P}_3\} \). Therefore, if we write \( \text{Diff}^+(S^2, \text{fixing } \hat{P}_1, \hat{P}_2, \hat{P}_3) =: D_0, \) \( \text{Diff}^+(S^2, \text{fixing } \hat{P}_1, \text{permuting } \{\hat{P}_2, \hat{P}_3\}) =: D, \) we have a fibration \( D_0 \to D \to \Sigma_2 \); note that \( D \to \Sigma_2 \) is onto. So, there is a long exact sequence:

\[
\ldots \to \pi_1(\Sigma_2) \to \pi_0D_0 \to \pi_0D \to \pi_0(\Sigma_2) \to 0.
\]

It is well known that \( \pi_0D_0 = K_3 = 0 \), where \( K_3 \) is the pure mapping class group of the sphere with 3 punctures in Birman's notation. Recall the definition \( \Gamma^{1,2} = \pi_0D \). So, \( \Gamma^{1,2} = \pi_0D = \Sigma_2 = \mathbb{Z}/2 \).

Proof of (3): By the above discussion, we know that \( I : N(\mathbb{Z}/p)/\langle \mathbb{Z}/p \rangle \to \Gamma^{1,2} = \mathbb{Z}/2 \) is one to one. So \( |N(\mathbb{Z}/p)/\langle \mathbb{Z}/p \rangle| \leq 2 \), i.e., \( |N(\mathbb{Z}/p)| = p \) or \( 2p \). It remains to show that \( N(\mathbb{Z}/p) = \mathbb{Z}/p \) or \( \mathbb{Z}/2p \). By Corollary 1.1.3, we know that all finite subgroups of \( \Gamma^{1}_{(p-1)/2} \) are cyclic, the result then follows.

Q.E.D.

Remark.

Proof (1) is an application of the proof of Lemma 2.1 in [Xi3].

Lemma 5.1.2. Let \( \mathbb{Z}/p < \Gamma^n_{(p-1)/2}, 2 \leq n \leq 3 \). Then

\[
\hat{H}^i(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} = \begin{cases} 
\mathbb{Z}/p & i = 0 \mod(2) \\
0 & i = 1 \mod(2)
\end{cases}.
\]

Proof. The argument is similar to the proof of Lemma 5.1.1, but easier. Q.E.D.

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5.2 Counting conjugacy classes of subgroups of order $p$ in $\Gamma_{(p-1)/2}^n$

We are now able to determine the number of conjugacy classes of subgroups of order $p$ in $\Gamma_{(p-1)/2}^n$.

Lemma 5.2.1. Let $p$ be an odd prime number. Then

(a) The number of conjugacy classes of subgroups of order $p$ in $\Gamma_{(p-1)/2}^1$ is $(p-1)/2$.

(b) The number of conjugacy classes of subgroups of order $p$ in $\Gamma_{(p-1)/2}^2$ is $p - 2$.

(c) The number of conjugacy classes of subgroups of order $p$ in $\Gamma_{(p-1)/2}^3$ is $p - 2$.

(d) There are no subgroups of order $p$ in $\Gamma_{(p-1)/2}^n$, $n > 3$.

Proof. As seen in chapter 3, we know that $t = 3$, $h = 0$ is the unique solution for the Riemann Hurwitz equation. By Proposition 3.2.2, $\Gamma_{(p-1)/2}^1$ contains a subgroup of order $p$. Furthermore, every $\mathbb{Z}/p$ acts on $S_{(p-1)/2}$ with three fixed points; the fixed point data of each generator in $\mathbb{Z}/p$ has the form $(\beta_1, \beta_2, \beta_3)$. By Proposition 4.2.1, the number of conjugacy classes of subgroups of order $p$ in $\Gamma_{(p-1)/2}^1$ is the number of different unordered integer tuples $(1|\beta_2, \beta_3)$ such that $1 + \beta_2 + \beta_3 = 0 \pmod{p}$, where $0 < \beta_i < p$ for all $i$, i.e., $\beta_2 + \beta_3 = p - 1$, where $0 < \beta_i < p$. Hence, we have $(\beta_2, \beta_3) = (1, p - 2)$ or $(2, p - 3)$ or ... $(p - 2, 1)$. As unordered tuples, we have $(p - 2 + 1)/2$ different choices. Therefore, (a) is proved.

By Proposition 4.3.1, the number of conjugacy classes of subgroups of order $p$ in $\Gamma_{(p-1)/2}^2$ is the number of different ordered integer tuples $(1, \beta_2|\beta_3)$ such that $1 + \beta_2 +
\[ \beta_3 \equiv 0 \pmod{p}, \text{ where } 0 < \beta_i < p \text{ for all } i, \text{ i.e., } \beta_2 + \beta_3 = p - 1, \text{ where } 0 < \beta_i < p. \]

Hence, we have \((\beta_2, \beta_3) = (1, p - 2)\) or \((2, p - 3)\) or \(\ldots (p - 2, 1)\). As ordered tuples, we have \((p - 2)\) different choices. Therefore, (b) is proved. By a similar argument, (c) is proved, and (d) follows from Proposition 3.2.1. \[ \text{Q.E.D.} \]

**Proof of Theorem 5.0.1.**

It is immediately from Lemmas 5.1.1, 5.1.2 and 5.2.1.
CHAPTER 6

THE $P$-TORSION OF THE FARRELL COHOMOLOGY

OF $\Gamma^N_{p-1}$

In this chapter we prove the following main theorem:

**Theorem 6.0.1.** Let $p$ denote an odd prime number. Then for $p > 3$,

$$\hat{H}^i(\Gamma^1_{p-1}, \mathbb{Z})_{(p)} = \begin{cases} \prod_{1}^{(p^2-1)/6} \mathbb{Z}/p & i = 0 \mod(2) \\ \prod_{1}^{(p^2-3p+2)/3} \mathbb{Z}/p & i = 1 \mod(2) \end{cases}$$

$$\hat{H}^i(\Gamma^2_{p-1}, \mathbb{Z})_{(p)} = \begin{cases} \prod_{1}^{(p-1)^2/2} \mathbb{Z}/p & i = 0 \mod(2) \\ \prod_{1}^{p^2-3p+3} \mathbb{Z}/p & i = 1 \mod(2) \end{cases}$$

$$\hat{H}^i(\Gamma^3_{p-1}, \mathbb{Z})_{(p)} = \begin{cases} \prod_{1}^{p^2-3p+3} \mathbb{Z}/p & i = 0 \mod(2) \\ \prod_{1}^{2(p^2-3p+3)} \mathbb{Z}/p & i = 1 \mod(2) \end{cases}$$

$$\hat{H}^i(\Gamma^4_{p-1}, \mathbb{Z})_{(p)} = \begin{cases} \prod_{1}^{p^2-3p+3} \mathbb{Z}/p & i = 0 \mod(2) \\ \prod_{1}^{2(p^2-3p+3)} \mathbb{Z}/p & i = 1 \mod(2) \end{cases}$$
Moreover,

\[
\hat{H}^i(\Gamma_{p-1}^n, \mathbb{Z})_{(p)} = 0, \quad \text{for} \quad n > 4.
\]

The computation relies again on the result that \( \Gamma_{p-1}^n \) is \( p \)-periodic for \( n \geq 1 \), so that by Theorem 0.1.4,

\[
\hat{H}^i(\Gamma_{p-1}^n, \mathbb{Z})_{(p)} \rightarrow \prod_{\mathbb{Z}/p \in \Omega} \hat{H}^i(N(\mathbb{Z}/p), \mathbb{Z})_{(p)}
\]

is an isomorphism, where \( \Omega \) is a set of representatives for the conjugacy classes of subgroups of order \( p \) in \( \Gamma_{p-1}^n \), and \( N(\mathbb{Z}/p) \) stands for the normalizer of \( \mathbb{Z}/p \) in \( \Gamma_{p-1}^n \). In the first section, we calculate \( \hat{H}^i(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} \) for \( \mathbb{Z}/p < \Gamma_{p-1}^n \). In fact, we prove that \( N(\mathbb{Z}/p)/\mathbb{Z}/p \) is an extension of \( K_4 \), the pure mapping class group of the sphere with four punctures, by a subgroup of \( \sum_3 \). The type of the subgroup is determined by the fixed point data of a generator of \( \mathbb{Z}/p \). Hence, using the Serre spectral sequence, we obtain the \( p \)-torsion of the Farrell cohomology of \( N(\mathbb{Z}/p)/\mathbb{Z}/p \). Also, we prove that the short exact sequence \( 0 \rightarrow \mathbb{Z}/p \rightarrow N(\mathbb{Z}/p) \rightarrow N(\mathbb{Z}/p)/\mathbb{Z}/p \rightarrow 0 \) is central, moreover, it splits. Using Serre spectral sequence or Künneth Theorem, the result follows. In the second section, we count conjugacy classes of subgroups of order \( p \) in \( \Gamma_{p-1}^n \).

In this chapter, we will focus on the case \( \Gamma_{p-1}^1 \), in fact, all other three cases (\( \Gamma_{p-1}^2 \), \( \Gamma_{p-1}^3 \), \( \Gamma_{p-1}^4 \)) are easier than this case, and they follow along the same lines. Recall that Proposition 3.2.2 says that \( \Gamma_{p-1}^n \) contains a subgroup of order \( p \) if \( n \leq 4 \); furthermore, every element of order \( p \) in \( \Gamma_{p-1}^n \) acting on \( S_{p-1} \) has four fixed points when \( p > 3 \). To avoid overcomplexity, we will assume \( p > 3 \) throughout this chapter.
6.1 Calculation of the \( p \)-torsion of the Farrell cohomology
of the normalizer of \( \mathbb{Z}/p \) in \( \Gamma^{1}_{p-1} \).

**Lemma 6.1.1.** Let \( \mathbb{Z}/p < \Gamma^{1}_{p-1} \), and \( N(\mathbb{Z}/p) \) be its normalizer in \( \Gamma^{1}_{p-1} \). There is an injective homomorphism \( I : N(\mathbb{Z}/p)/\mathbb{Z}/p \to \Gamma^{1,3} \), where \( \Gamma^{1,3} = \pi_0\text{Diffeo}^+(S^2, \text{fixing } \hat{P}_1, \text{permuting } \{\hat{P}_2, \hat{P}_3, \hat{P}_4\}) \).

**Proof.** The map \( I \) is defined as in Lemma 5.1.1, and the result follows. \textbf{Q.E.D.}

Now, by Lemma 6.1.1, we know that if we understand more about the image \( I(N(\mathbb{Z}/p)/\mathbb{Z}/p)) \), then we can determine \( N(\mathbb{Z}/p)/\mathbb{Z}/p \). Recall that every element of order \( p \) in \( \Gamma^{1}_{p-1} \) acting on \( S_{p-1} \) has four fixed points for \( p > 3 \).

**Lemma 6.1.2.** Let us write \( \mathbb{Z}/p = \langle \alpha \rangle < \Gamma^{1}_{p-1} \), and let \( y \) be an orientation preserving diffeomorphism of \( S_{p-1} \) of order \( p > 3 \) with the fixed point \( P_1 \), representing \( \alpha \). Now consider the \( \langle y \rangle \) action on \( S_{p-1} \), with four fixed points, \( P_1, P_2, P_3, P_4 \), and the orbit space \( S^2 \). Consider the following diagram:

\[
\begin{array}{ccc}
S_{p-1} - \{P_1, P_2, P_3, P_4\} & \longrightarrow & S_{p-1} - \{P_1, P_2, P_3, P_4\} \\
\pi & & \pi \\
S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4\} & \longrightarrow & S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4\}
\end{array}
\]

where \( \pi \) is the projection induced by the \( \langle y \rangle \) action, and \( S^2 \) is the orbit space.

Then

(a) \( \text{Im}(I : N(\mathbb{Z}/p)/\mathbb{Z}/p \to \Gamma^{1,3}) = \{[w] \in \Gamma^{1,3} | w \text{ lifts}\} \).

(b) Let \( [w] \in \Gamma^{1,3} \). Then \( w \) lifts to a diffeomorphism \( h : S_{p-1} - \{P_1, P_2, P_3, P_4\} \to S_{p-1} - \{P_1, P_2, P_3, P_4\} \) if and only if every closed curve of \( S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4\} \)
which lifts to a closed curve of $S_{p-1} - \{P_1, P_2, P_3, P_4\}$ maps (via $w$) to a closed curve of $S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4\}$ which lifts to a closed curve of $S_{p-1} - \{P_1, P_2, P_3, P_4\}$.

(c) Let $\gamma$ be a closed curve in $S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4\}$, and $[\gamma] = \prod x_i^{n_i} \prod x_i^{m_i} ... \prod x_i^{k_i} \in \pi_1(S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4\})$, where $x_i$ is represented by a simple closed curve around $P_i$. Then, $\gamma$ lifts to a closed curve if and only if

$$\sum n_i \beta_i + m_i \beta_i + ... + k_i \beta_i = 0 \pmod{p},$$

where $(\beta_1|\beta_2, \beta_3, \beta_4)$ is the fixed point data of $y$.

**Proof.** (a) If $[w] \in Im(I : N(\mathbb{Z}/p) / \mathbb{Z}/p \rightarrow \Gamma^{1,3})$, by definition of $I$ in Lemma 5.1.1, we know that $w$ lifts. On the other hand, suppose $[w] \in \Gamma^{1,3}$ and $w$ lifts to $h \in Diff^{eq}_+(S_{p-1}, fixing \ P_1, permuting \ \{P_2, P_3, P_4\})$. It suffices to show that $hyh^{-1}(s) = y^i(s)$ for $0 < i < p$ and $s \in S_{p-1}$. Assume $h(t) = s$, where $t \in S_{p-1}$. By the commutative diagram, we have

$$w^{-1} \pi h(t) = \pi(t).$$

This implies

$$t \in \text{preimage of } w^{-1} \pi h(t),$$

and therefore

$$h^{-1}(s) \in \text{preimage of } w^{-1} \pi h(t).$$

It follows that,

$$yh^{-1}(s) \in \text{preimage of } w^{-1} \pi h(t),$$

and we have

$$\pi yh^{-1}(s) = w^{-1} \pi h(t).$$
On the other hand, by the commutative diagram, we know that $\pi h(x) = w\pi(x)$ for $x \in S_{p-1}$. So, $\pi h y h^{-1}(s) = w\pi y h^{-1}(s)$.

Therefore,

$$\pi h y h^{-1}(s) = w w^{-1} \pi h(t),$$

and

$$\pi h y h^{-1}(s) = \pi h(t) = \pi(s).$$

This implies

$$h y h^{-1}(s) = y^i(s) \text{ for } 0 < i < p.$$  

Thus $[h][y][h^{-1}] = [y^i]$, and therefore $[h] \in N(\mathbb{Z}/p)$. Clearly, $\hat{I}([h]) = [w]$, and we have $[w]$ is in $Im(I : N(\mathbb{Z}/p)/\mathbb{Z}/p \to \Gamma^{1,3})$.

(b) $\Rightarrow$ This is obvious by the commutative diagram.

($\Leftarrow$) We need to show $w\pi(\pi_1(S_{p-1} - \{P_1, P_2, P_3, P_4\})) \subset \pi(\pi_1(S_{p-1} - \{P_1, P_2, P_3, P_4\}))$.

(We abuse the notation $w\pi$ (respectively $\pi$) and its induced mapping in fundamental group.) For an arbitrary $[\tilde{\gamma}] \in \pi_1(S_{p-1} - \{P_1, P_2, P_3, P_4\})$ where $\tilde{\gamma}$ is a closed curve in $S_{p-1} - \{P_1, P_2, P_3, P_4\}$, $\pi(\tilde{\gamma})$ is a closed curve in $S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4\}$ which lifts to $\tilde{\gamma}$. By the condition, we know that $w\pi(\tilde{\gamma})$ is a closed curve in $S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4\}$ which lifts to a closed curve $\beta \in S_{p-1} - \{P_1, P_2, P_3, P_4\}$. Thus $w\pi([\tilde{\gamma}]) = \pi([\beta])$, i.e., the image of the mapping $w\pi$ is contained in the image of the mapping $\pi$. Therefore, we conclude that $w$ lifts.

(c) We have a short exact sequence:

$$0 \to \pi_1(S_{p-1} - \{P_1, P_2, P_3, P_4\}) \xrightarrow{i} \pi_1(S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4\}) \xrightarrow{j} \mathbb{Z}/p \to 0.$$
We know that \( j \) maps \( x_i \in \pi_1(S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4\}) \) to \( \beta_i \in \mathbb{Z}/p \), where \( (\beta_1, \beta_2, \beta_3, \beta_4) \) is the fixed point data of \( y \). Then \( \gamma \) lifts to a closed curve if and only if \([\gamma]\) is in the image of \( i \). So, it is in the kernel of \( j \). Therefore, \( j([\gamma]) = 0 \), i.e.,

\[
\sum n_i \beta_i + m_i \beta_i + ... + k_i \beta_i = 0 \pmod{p}.
\]

Q.E.D.

Therefore, in order to understand \( I(N(\mathbb{Z}/p)/\mathbb{Z}/p) \), we also need to know more about \( \Gamma^{1,3} \). Consider the following natural short exact sequence:

\[
0 \to Diff_{eo}^+(S^2, fixing \ \hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4) \to
Diff_{eo}^+(S^2, fixing \ \hat{P}_1, permuting\{\hat{P}_2, \hat{P}_3, \hat{P}_4\}) \to \Sigma_3 \to 0.
\]

It induces a long exact sequence: \( \cdots \to 0 \to \Gamma^{4,0} \to \Gamma^{1,3} \to \Sigma_3 \to 0 \). By Birman's notation in [Bi], it is easy to see that \( \Gamma^{4,0} = K_4 \), where \( K_4 \) is the pure mapping class group of the sphere with four punctures which actually is a free group with two generators.

So we have a group extension:

\[
0 \to K_4 \to \Gamma^{1,3} \to \Sigma_3 \to 0.
\]

Let us write \( \phi : K_4 \to \Gamma^{1,3} \) and \( \lambda : \Gamma^{1,3} \to \Sigma_3 \), for the two maps in the above short exact sequence. To determine \( N(\mathbb{Z}/p)/\mathbb{Z}/p \), which is isomorphic to \( I(N(\mathbb{Z}/p)/\mathbb{Z}/p) \), we need to check two things:

Is \( \phi(K_4) \) contained in \( I(N(\mathbb{Z}/p)/\mathbb{Z}/p) \)? What is \( \lambda I(N(\mathbb{Z}/p)/\mathbb{Z}/p) \) in \( \Sigma_3 \)?

The following lemmas answer these questions.

**Lemma 6.1.3.** Let \([w] \in \Gamma^{1,3}, \) where \( \lambda([w]) = \delta \in \Sigma_3 = <(23),(34)> \). Consider the same commutative diagram as in Lemma 6.1.2, where \( \pi \) is the projection induced by \( <y> \) action, \( y \in Diff_{eo}^+(S_{p-1}, P_1) \) and \( y^p = id \). Suppose the fixed point data of
$y$ is $\delta_1(y)$, and $\delta_1(y) = (\beta_1|\beta_2, \beta_3, \beta_4)$, where $0 < \beta_i < p$, $\sum_{i=1}^{4} (\beta_i) = 0 \pmod{p}$. Then $w$ lifts if and only if the following condition is true: if $\sum m_i \beta_i = 0 \pmod{p}$, the $m_i$'s are all integers, then $\sum m_i \beta_{q(i)} = 0 \pmod{p}$.

**Proof.** This follows from Lemma 6.1.2 (b) and (c). \[ \text{Q.E.D.} \]

**Remark.** Lemma 6.1.2 and Lemma 6.1.3 are corollaries of the Lemma in Xia's paper [Xi4].

**Lemma 6.1.4.** Let $\mathbb{Z}/p < \Gamma_{p-1}^{1}$ with normalizer $N(\mathbb{Z}/p)$. Let $I : N(\mathbb{Z}/p) / \mathbb{Z}/p \to \Gamma_{1,3}$ be the above injective homomorphism, and $0 \to K_4 \xrightarrow{\phi} \Gamma_{1,3} \xrightarrow{\lambda} \sum_3 \to 0$ be the above short exact sequence. Then the image $\phi(K_4)$ is contained in $I(N(\mathbb{Z}/p) / \mathbb{Z}/p)$.

**Proof.** For an arbitrary $[w] \in \phi(K_4)$, we have $\lambda([w]) = id$. It satisfies therefore the relation in Lemma 6.1.3, hence $w$ lifts, and by Lemma 6.1.2(a), $[w] \in I(N(\mathbb{Z}/p) / \mathbb{Z}/p)$. \[ \text{Q.E.D.} \]

**Lemma 6.1.5.** Let $\mathbb{Z}/p = \langle \alpha \rangle$, where $\mathbb{Z}/p < \Gamma_{p-1}^{1}$. Write $\delta_1(\alpha) = (1|\beta_2, \beta_3, \beta_4)$ for the fixed point data. Let $I : N(\mathbb{Z}/p) / \mathbb{Z}/p \to \Gamma_{1,3}$, and $0 \to K_4 \xrightarrow{\phi} \Gamma_{1,3} \xrightarrow{\lambda} \sum_3 \to 0$ be the above short exact sequence. Then we have

- $\lambda I(N(\mathbb{Z}/p) / \mathbb{Z}/p) = \sum_3 = \langle 23 \rangle, (34) >$, if $\beta_2 = \beta_3 = \beta_4$;
- $\lambda I(N(\mathbb{Z}/p) / \mathbb{Z}/p) = \sum_2 = \langle 23 \rangle >$, if $\beta_2 = \beta_3 \neq \beta_4$;
- $\lambda I(N(\mathbb{Z}/p) / \mathbb{Z}/p) = \sum_2 = \langle 24 \rangle >$, if $\beta_2 = \beta_4 \neq \beta_3$;
- $\lambda I(N(\mathbb{Z}/p) / \mathbb{Z}/p) = \sum_2 = \langle 34 \rangle >$, if $\beta_3 = \beta_4 \neq \beta_2$;
- $\lambda I(N(\mathbb{Z}/p) / \mathbb{Z}/p) = \text{trivial}$, if $\beta_2 \neq \beta_3 \neq \beta_4$.

**Proof.** If $\beta_2 = \beta_3 = \beta_4$, the result follows from Lemma 6.1.3 and Lemma 6.1.2(a).
If $\beta_2 = \beta_3 \neq \beta_4$, by Lemma 6.1.3 and Lemma 6.1.2(a), we know that $\lambda I(N(\mathbb{Z}/p)/\mathbb{Z}/p)$ contains $\sum_2 =< (23) >$. Since $\Gamma^{1}_{p-1}$ has period 2, we know that $N(\mathbb{Z}/p) = C(\mathbb{Z}/p)$.

Suppose $\theta \in N(\mathbb{Z}/p) = C(\mathbb{Z}/p)$, $< \alpha > = \mathbb{Z}/p$, and $\theta \alpha \theta^{-1} = \alpha$. We can represent $\theta$ by $h$, $\alpha$ by $y$. Then $h$ must map any fixed point of $y$ to another fixed point of $y$ such that the $y$ has the same rotation angles at each of these two fixed points.

Then the numbers $\beta_i, \beta_j$ associated to these two fixed points of $y$ are equal. Hence

$$\lambda I(N(\mathbb{Z}/p)/\mathbb{Z}/p) = \sum_2 =< (23) >.$$ 

The other cases are similar. \textbf{Q.E.D.}

All previous lemmas are for the purpose of the following theorem.

**Theorem 6.1.1.** Let $\mathbb{Z}/p =< \alpha >$, where $\mathbb{Z}/p < \Gamma^{1}_{p-1}$. Write $\delta_1(\alpha) = (1|\beta_2, \beta_3, \beta_4)$ for the fixed point data. Let $I : N(\mathbb{Z}/p)/\mathbb{Z}/p \rightarrow \Gamma^{1,3}$, and $0 \rightarrow K_4 \xrightarrow{\phi} \Gamma^{1,3} \xrightarrow{\lambda} \sum_3 \rightarrow 0$ be the above short exact sequence. Then we have

1. If $\beta_2 = \beta_3 = \beta_4$, there is a group extension:

$$0 \rightarrow \phi(K_4) \rightarrow I(N(\mathbb{Z}/p)/\mathbb{Z}/p) \rightarrow \sum_3 =< (23), (34) > \rightarrow 0,$$

and since $K_4 \cong \phi(K_4)$, therefore also

$$0 \rightarrow K_4 \rightarrow N(\mathbb{Z}/p)/\mathbb{Z}/p \rightarrow \sum_3 =< (23), (34) > \rightarrow 0;$$

2. If $\beta_2 = \beta_3 \neq \beta_4$, there is a group extension:

$$0 \rightarrow K_4 \rightarrow N(\mathbb{Z}/p)/\mathbb{Z}/p \rightarrow \sum_2 =< (23) > \rightarrow 0;$$

If $\beta_2 = \beta_4 \neq \beta_3$, there is a group extension:

$$0 \rightarrow K_4 \rightarrow N(\mathbb{Z}/p)/\mathbb{Z}/p \rightarrow \sum_2 =< (24) > \rightarrow 0;$$
If \( \beta_3 = \beta_4 \neq \beta_2 \), there is a group extension:

\[
0 \to K_4 \to N(\mathbb{Z}/p)/\mathbb{Z}/p \to \Sigma_2 = \langle (34) \rangle \to 0;
\]

(3) If \( \beta_2 \neq \beta_3 \neq \beta_4 \), then

\[
K_4 \approx N(\mathbb{Z}/p)/\mathbb{Z}/p.
\]

**Proof.** We just use Lemma 6.1.4 and Lemma 6.1.5.

Knowing the above group extension, we can use spectral sequence to calculate \( H^*(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) \). We need determine the action of \( \Sigma_3 \) on \( H^*(K_4, \mathbb{Z}) \), which is the following lemma.

**Lemma 6.1.6.** [Co4] \( H^*(K_4, \mathbb{Z}) \) is generated as an algebra by the one dimensional cohomology classes \( B_{42} \) and \( B_{43} \). The only relation in \( H^*(K_4, \mathbb{Z}) \) is \( B_{42} B_{43} = 0 \).

Moreover, the action of \( \Sigma_4 \) on \( H^1(K_4, \mathbb{Z}) \) is as follows:

\[
\begin{align*}
B_{42} & : -B_{42} & B_{42} + B_{43} & - B_{42} \\
B_{43} & : B_{42} + B_{43} & - B_{43} & B_{42} + B_{43}
\end{align*}
\]

**Remark.** The above Lemma is still true if we pass to the cohomology with \( F_p \) coefficients. In fact, \( H^i(K_4, F_p) = Hom(H_i(K_4, \mathbb{Z}), F_p) \oplus Ext(H_{i-1}(K_4, \mathbb{Z}), F_p) = Hom(H_i(K_4, \mathbb{Z}), F_p) \). This is because \( H_0(K_4, \mathbb{Z}) = H^0(K_4, \mathbb{Z}) = \mathbb{Z}, \; H_1(K_4, \mathbb{Z}) = H^1(K_4, \mathbb{Z}) = \oplus_2 \mathbb{Z}, \; H_i(K_4, \mathbb{Z}) = H^i(K_4, \mathbb{Z}) = 0 \) for \( i > 1 \). It is easy to check that the generators are still linearly independent and the corresponding relations are still true, if we pass to cohomology with \( F_p \) coefficients.

Now we can calculate \( H^*(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) \).

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Theorem 6.1.2. Let $\mathbb{Z}/p = \langle \alpha \rangle$, where $\mathbb{Z}/p < \Gamma^1_{p-1} \text{ and } p > 3$. Write $\delta_t(\alpha) = (1|\beta_2, \beta_3, \beta_4)$ for the fixed point data. Then we have

(1) If $\beta_3 = \beta_4 = \beta_2$, then $H^0(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = F_p$, $H^1(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = 0$, $H^*(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = 0$, if $* > 1$.

(2) If $\beta_3 \neq \beta_4 \neq \beta_2$, or $\beta_3 = \beta_2 \neq \beta_4$, or $\beta_4 = \beta_2 \neq \beta_3$, then $H^0(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = F_p$, $H^1(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = F_p$, $H^*(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = 0$, if $* > 1$.

(3) If $\beta_3 \neq \beta_4 \neq \beta_2$, then $H^0(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = F_p$, $H^1(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = F_p$, $H^*(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = 0$, if $* > 1$.

Proof.

We prove the first case.

By Theorem 6.1.1, we have a group extension

$$0 \to K_4 \to N(\mathbb{Z}/p / \mathbb{Z}/p) \to \Sigma_3 = \langle 23, 34 \rangle \to 0.$$ 

The Hochschild-Serre spectral sequence takes the form

$$E_2^{ij} = H^i(\Sigma_3, H^j(K_4, F_p)) \Rightarrow H^{i+j}(\mathbb{Z}/p / \mathbb{Z}/p, F_p).$$

Since $p > 3$, $E_2^{ij} = H^i(\Sigma_3, H^j(K_4, F_p)) = 0$ if $i > 0$. Hence $H^j(N(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = H^j(K_4, F_p)^{\Sigma_3}$.

By Lemma 6.1.6, we have $H^0(N(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = F_p$, $H^1(N(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = 0$, $H^*(N(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = 0$, if $* > 1$.

For the second case, we get $H^j(N(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = H^j(K_4, F_p)^{\Sigma_2}$. By lemma 6.1.6, we can get $H^0(N(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = F_p$, $H^1(N(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = F_p$, $H^*(N(\mathbb{Z}/p / \mathbb{Z}/p, F_p) = F_p$. 

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\[ H^*(N(Z/p)/(Z/p), F_p) = 0, \text{ if } * > 1. \]

The third case is trivial. \textbf{Q.E.D.}

A similar theorem as above can also be obtained:

**Theorem 6.1.3.** Let \( Z/p \equiv < \alpha > \), where \( Z/p < \Gamma^1_{p-1} \). Write \( \delta_1(\alpha) = (1|\beta_2, \beta_3, \beta_4) \) for the fixed point data. Then we have

1. If \( \beta_3 = \beta_4 = \beta_2 \), then \( H^0(N(Z/p)/Z/p, Z)_{(p)} = Z_{(p)} \),
   \[ H^1(N(Z/p)/Z/p, Z)_{(p)} = 0, \quad H^*(N(Z/p)/Z/p, Z)_{(p)} = 0, \text{ if } * > 1. \]

2. If \( \beta_3 = \beta_4 \neq \beta_2 \), or \( \beta_3 = \beta_2 \neq \beta_4 \), or \( \beta_4 = \beta_2 \neq \beta_3 \),
   then \( H^0(N(Z/p)/Z/p, Z)_{(p)} = Z_{(p)}, \quad H^1(N(Z/p)/Z/p, Z)_{(p)} = Z_{(p)}, \)
   \[ H^*(N(Z/p)/Z/p, Z)_{(p)} = 0, \text{ if } * > 1. \]

3. If \( \beta_3 \neq \beta_4 \neq \beta_2 \), then \( H^0(N(Z/p)/Z/p, Z)_{(p)} = Z_{(p)}, \quad H^1(N(Z/p)/Z/p, Z)_{(p)} = Z_{(p)}, \)
   \[ \oplus_2 Z_{(p)}, \quad H^*(N(Z/p)/Z/p, Z)_{(p)} = 0, \text{ if } * > 1. \]

Here \( A_{(p)} \) stands for the “\( p \)-localization” of the abelian group \( A \), i.e., \( A_{(p)} = A \otimes Z_{(p)} \).

As usual, \( Z_{(p)} \) stands for the ring of rationals with no \( p \) in their denominators.

**PROOF.** The proof is similar to the one of Theorem 6.1.2. Since the tensor product with \( Z_{(p)} \) preserves exactness, one has still all the spectral sequences after \( p \)-localization. \textbf{Q.E.D.}

Using short exact sequence \( 0 \to Z/p \to N(Z/p) \to N(Z/p)/Z/p \to 0 \), we can find \( H^*(N(Z/p), Z)_{(p)} \) in the following two ways. One is by Serre spectral sequence; the other is by Künneth theorem.
Theorem 6.1.4. Let $\mathbb{Z}/p = \langle \alpha \rangle$, where $\mathbb{Z}/p < \Gamma_p^1$. Write $\delta_1(\alpha) = (1|\beta_2, \beta_3, \beta_4)$ for the fixed point data. Then we have the following:

If $\beta_2 = \beta_3 = \beta_4$, then

$$\hat{H}^i(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} = \begin{cases} \mathbb{Z}/p & \text{i = 0 mod}(2) \\ 0 & \text{i = 1 mod}(2) \end{cases}.$$ 

If $\beta_2 = \beta_3 \neq \beta_4$, or $\beta_2 = \beta_4 \neq \beta_3$, or $\beta_3 = \beta_4 \neq \beta_2$, then

$$\hat{H}^i(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} = \begin{cases} \mathbb{Z}/p & \text{i = 0 mod}(2) \\ \mathbb{Z}/p & \text{i = 1 mod}(2) \end{cases}.$$ 

If $\beta_2 \neq \beta_3 \neq \beta_4$, then

$$\hat{H}^i(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} = \begin{cases} \mathbb{Z}/p & \text{i = 0 mod}(2) \\ \mathbb{Z}/p \oplus \mathbb{Z}/p & \text{i = 1 mod}(2) \end{cases}.$$ 

Proof. We only prove the first case. (The other cases are similar, but easier.)

We apply the spectral sequence again to the central extension

$$0 \to \mathbb{Z}/p \to N(\mathbb{Z}/p) \to N(\mathbb{Z}/p)/\mathbb{Z}/p \to 0$$

in the form

$$E_2^{ij} = H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z})) \Longrightarrow H^{i+j}(N(\mathbb{Z}/p), \mathbb{Z}).$$

Since the extension is central ($N(\mathbb{Z}/p) = C(\mathbb{Z}/p)$), $N(\mathbb{Z}/p)/\mathbb{Z}/p$ has trivial action on $H^i(\mathbb{Z}/p, \mathbb{Z})$. We claim that $H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = 0$ for $i > 1$. In fact,

$$H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} \quad \text{for} \quad j = 0,$$
\[ H^i(\mathbb{Z}/p, \mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = 0, \text{ for } j > 0 \text{ odd}, \]
\[ H^i(\mathbb{Z}/p, \mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^i(\mathbb{Z}/p, \mathbb{Z}/p, \mathbb{Z}), \text{ for } j > 0 \text{ even}. \]

By Theorem 6.1.2 and Theorem 6.1.3, all these groups are 0 for \( i > 1 \). So, we only consider the cases when \( i = 0 \) or \( i = 1 \). We then have the following:

\[ H^0(\mathbb{Z}/p, \mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^0(\mathbb{Z}/p, \mathbb{Z}/p, \mathbb{Z}), \text{ for } j = 0, \]
\[ H^0(\mathbb{Z}/p, \mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = 0, \text{ for } j > 0 \text{ odd}, \]
\[ H^0(\mathbb{Z}/p, \mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^0(\mathbb{Z}/p, \mathbb{Z}/p, \mathbb{Z}) = \mathbb{Z}/p, \text{ for } j > 0 \text{ even}. \]
\[ H^1(\mathbb{Z}/p, \mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^1(\mathbb{Z}/p, \mathbb{Z}/p, \mathbb{Z}) = 0, \text{ for } j = 0, \]
\[ H^1(\mathbb{Z}/p, \mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = 0, \text{ for } j > 0 \text{ odd}, \]
\[ H^1(\mathbb{Z}/p, \mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^1(\mathbb{Z}/p, \mathbb{Z}/p, \mathbb{Z}) = 0, \text{ for } j > 0 \text{ even}. \]

The spectral sequence collapses again. We therefore get the ordinary cohomology \( H^*(\mathbb{Z}/p, \mathbb{Z})_{(p)} \) in high dimension. So we obtain the the Farrell-Tate cohomology as above.

Q.E.D.

ALTERNATIVE PROOF.

Consider the central extension \( 0 \to \mathbb{Z}/p \to N(\mathbb{Z}/p) \to \mathbb{Z}/p \to 0 \). By Theorem 6.1.2, \( H^2(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z}/p) = 0 \), thus the extension splits. Since it is also central, \( N(\mathbb{Z}/p) = \mathbb{Z}/p \times N(\mathbb{Z}/p)/\mathbb{Z}/p \). Using the Künneth Theorem,

\[ H^n(\mathbb{Z}/p, \mathbb{Z}) = \oplus_{i+j=n} H^i(\mathbb{Z}/p, \mathbb{Z}) \otimes H^j(\mathbb{Z}/p, \mathbb{Z}) \oplus (\oplus_{i+j=n-1} Tor^\mathbb{Z}_1(H^i(\mathbb{Z}/p, \mathbb{Z}), H^j(\mathbb{Z}/p, \mathbb{Z})). \]

For \( n = 2k \), a large even integer, by Theorem 6.1.3, we have
\[ H^{2k}(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} = \]
\[ H^{2k}(\mathbb{Z}/p, \mathbb{Z}) \otimes H^0(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} \oplus \text{Tor}_1^\mathbb{Z}(H^{2k-2}(\mathbb{Z}/p, \mathbb{Z}), H^{1}(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)}) \]
\[ = \mathbb{Z}/p. \]

For \( n = 2k - 1 \), a large odd integer, we get similarly

\[ H^{2k-1}(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} = 0 \text{ if } \beta_2 = \beta_3 = \beta_4, \]
\[ H^{2k-1}(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} = \mathbb{Z}/p \text{ if } \beta_2 = \beta_3 \neq \beta_4, \text{ or } \beta_2 = \beta_4 \neq \beta_3, \text{ or } \beta_3 = \beta_4 \neq \beta_2, \]
and

\[ H^{2k-1}(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} = \mathbb{Z}/p \oplus \mathbb{Z}/p \text{ if } \beta_2 \neq \beta_3 \neq \beta_4. \]

Q.E.D.
6.2 Counting conjugacy classes of subgroups of order $p$ in
\[ \Gamma_{p-1}^1 \]

In this section, we will count the conjugacy classes of subgroups of order $p$ in $\Gamma_{p-1}^1$.

By Proposition 4.2.1, it is a number-theoretical problem of counting the number of different unordered integer tuples $(1|\beta_2, \beta_3, \beta_4)$ such that $1 + \sum_2^4(\beta_i) = 0 \pmod{p}$, where $0 < \beta_i < p$ for all $i$.

**Lemma 6.2.1.** Consider the solutions of the following equation:

$1 + \sum_2^4(\beta_i) = 0 \pmod{p}$, where $(1|\beta_2, \beta_3, \beta_4)$ is regarded as unordered tuple, $0 < \beta_i < p$, and $p > 3$. We have $1$ solution if $\beta_2 = \beta_3 = \beta_4$; $(p - 3)$ solutions if $\beta_2 = \beta_3 \neq \beta_4$, or $\beta_2 \neq \beta_3$, or $\beta_3 = \beta_4 \neq \beta_2$; $(p^2 - 6p + 11)/6$ solutions if $\beta_2 \neq \beta_3 \neq \beta_4$.

**Proof.** Case 1: $\beta_2 = \beta_3 = \beta_4$

If $p = 3k + 2, k \geq 1$, we obtain $\beta_2 = \beta_3 = \beta_4 = (2p - 1)/3$. So, we only have one choice. If $p = 3k + 1, k \geq 1$, we obtain $\beta_2 = \beta_3 = \beta_4 = (p - 1)/3$. So, we again have one choice. Since $p > 3, p \neq 3k$, in Case 1, we therefore have one choice.

Case 2: $\beta_2 = \beta_3 \neq \beta_4$

If $\beta_2 + \beta_3 + \beta_4 = p - 1$, then the solutions are $\beta_4 = 2, \beta_2 = \beta_3 = (p - 1 - 2)/2$; $\beta_4 = 4, \beta_2 = \beta_3 = (p - 1 - 4)/2$; ... $\beta_4 = p - 3, \beta_2 = \beta_3 = 1$, totally we have $(p - 3)/2$ choices. If $\beta_2 + \beta_3 + \beta_4 = 2p - 1$, then we get solutions $\beta_4 = 1, \beta_2 = \beta_3 = (2p - 1 - 1)/2$; $\beta_4 = 3, \beta_2 = \beta_3 = (2p - 1 - 3)/2$; ... $\beta_4 = p - 2, \beta_2 = \beta_3 = (p + 1)/2$, totally we have $(p - 1)/2$ choices. Altogether, we have $(p - 3)/2 + (p - 1)/2 = p - 2$ choices. However, in the counting we did not exclude the choice $\beta_2 = \beta_3 = \beta_4$, so the final answer will
be $p - 3$. Similarly, when $\beta_3 = \beta_4 \neq \beta_2$, we have $p - 3$ choices, and if $\beta_2 = \beta_4 \neq \beta_3$, we have $p - 3$ choices.

Case 3: $\beta_2 \neq \beta_3 \neq \beta_4$

First, we notice that $\beta_2$ is uniquely determined by $\beta_3$ and $\beta_4$. In fact, if $0 < \beta_3 + \beta_4 < p - 1$, then $\beta_2 = p - 1 - (\beta_3 + \beta_4)$. If $p - 1 < \beta_3 + \beta_4 < 2p - 1$, then $\beta_2 = 2p - 1 - (\beta_3 + \beta_4)$. So, we only need to consider the possible choices of $\beta_3$ and $\beta_4$. We have $p - 1$ choices of $\beta_3(\beta_4)$, hence we have $(p - 1)(p - 1)$ choices for $\beta_3$ and $\beta_4$. However, in all these choices, $\beta_3 + \beta_4 = p - 1$ never works. So we need to exclude $p - 2$ choices. We have $(p - 1)(p - 1) - (p - 2) = p^2 - 3p + 3$ choices. In all these choices, the cases $\beta_2 = \beta_3 = \beta_4$, $\beta_2 = \beta_3 \neq \beta_4$, $\beta_3 = \beta_4 \neq \beta_2$, $\beta_2 = \beta_4 \neq \beta_3$ are included. So we need to subtract $1 + 3(p - 3)$ choices. Totally, we are left with $p^2 - 3p + 3 - (1 + 3(p - 3)) = p^2 - 6p + 11$ choices. As unordered triples, we thus find $(p^2 - 6p + 11)/6$ choices.

Q.E.D.

Now we can prove the main theorem.

Proof of Theorem 6.0.1.

We only prove it for the case of $\Gamma_{p-1}^1$. By Proposition 4.2.1 and Lemma 6.2.1, we have the following results: If $\beta_2 = \beta_3 = \beta_4$, we only have one conjugacy class. If $\beta_2 = \beta_3 \neq \beta_4$, or $\beta_2 = \beta_4 \neq \beta_3$, or $\beta_3 = \beta_4 \neq \beta_2$, we have $(p - 3)$ conjugacy classes. If $\beta_2 \neq \beta_3 \neq \beta_4$, we have $(p^2 - 6p + 11)/6$ conjugacy classes. By Theorem 6.1.4, the main theorem follows.

Q.E.D.
CHAPTER 7

THE $p$-TORSION OF THE FARRELL COHOMOLOGY

OF $\Gamma^N_{3(p-1)/2}$

In this chapter, the following main theorem is proved:

**Theorem 7.0.1.** Let $p$ denote an odd prime number. Then for $p > 5$, we have the following:

1. If $i$ is an odd integer, $\tilde{H}^i(\Gamma^2_{3(p-1)/2}, \mathbb{Z})_{(p)} = \prod_{l=1}^{5p^3-17p^2+19p-7}/6 \mathbb{Z}/p$; if $i$ is an even integer, then the order of $\tilde{H}^i(\Gamma^2_{3(p-1)/2}, \mathbb{Z})_{(p)}$ is $(7p^3 - 25p^2 + 35p - 23)/6 \times p$, and there is no $\mathbb{Z}/p^3$ in the cohomology.

2. If $i$ is an odd integer, $\tilde{H}^i(\Gamma^3_{3(p-1)/2}, \mathbb{Z})_{(p)} = \prod_{l=1}^{5p^3-19p^2+27p-17}/2 \mathbb{Z}/p$; if $i$ is an even integer, then the order of $\tilde{H}^i(\Gamma^3_{3(p-1)/2}, \mathbb{Z})_{(p)}$ is $(7p^3 - 27p^2 + 39p - 25)/2 \times p$, and there is no $\mathbb{Z}/p^3$ in the cohomology.

3. If $i$ is an odd integer, $\tilde{H}^i(\Gamma^4_{3(p-1)/2}, \mathbb{Z})_{(p)} = \prod_{l=1}^{5(p^3-4p^2+6p-4)} \mathbb{Z}/p$; if $i$ is an even integer, then the order of $\tilde{H}^i(\Gamma^4_{3(p-1)/2}, \mathbb{Z})_{(p)}$ is $7(p^3 - 4p^2 + 6p - 4) \times p$, and there is no $\mathbb{Z}/p^3$ in the cohomology.

4. If $i$ is an odd integer, $\tilde{H}^i(\Gamma^5_{3(p-1)/2}, \mathbb{Z})_{(p)} = \prod_{l=1}^{5(p^3-4p^2+6p-4)} \mathbb{Z}/p$; if $i$ is an even integer, then the order of $\tilde{H}^i(\Gamma^5_{3(p-1)/2}, \mathbb{Z})_{(p)}$ is $7(p^3 - 4p^2 + 6p - 4) \times p$, and there is no $\mathbb{Z}/p^3$ in the cohomology.
(5) \( \tilde{H}^i(\Gamma_{3(p-1)/2}^n, \mathbb{Z})_{(p)} = 0 \) for \( n > 5 \).

In the first section, we calculate \( \tilde{H}^i(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} \) for \( \mathbb{Z}/p < \Gamma_{3(p-1)/2}^n \) with normalizer \( N(\mathbb{Z}/p) \). In fact, as before, we get that \( N(\mathbb{Z}/p)/\mathbb{Z}/p \) is an extension of \( K_5 \), the pure mapping class group of the sphere with 5 punctures, by a subgroup of \( \sum_3 \). Then, using Serre spectral sequence on this group extension, we get the \( p \)-torsion of the cohomology of \( N(\mathbb{Z}/p)/\mathbb{Z}/p \). Note that the actions of \( \sum_3 \) on \( H^*(K_5, \mathbb{Z}) \) which can be found in [Co4] are not as simple as those mentioned in Section 6.1. This makes the computation of \( H^*(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} \) more complicated. Another difficulty is that, although the short exact sequence \( 0 \to \mathbb{Z}/p \to N(\mathbb{Z}/p) \to N(\mathbb{Z}/p)/\mathbb{Z}/p \to 0 \) is central, we do not know whether it splits or not. As a result, there are two possible forms of \( \tilde{H}^i(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} \) if \( i \) is an even integer. In the second section, we count the conjugacy classes of subgroups of order \( p \) in \( \Gamma_{3(p-1)/2}^n \) for \( n = 2, 3, 4, 5 \). (In fact, we actually count the conjugacy classes of subgroups of order \( p \) in \( \Gamma_{k(p-1)/2}^n \) for \( n = k - 1, k, k + 1, k + 2 \), where \( k \) is a positive integer.)

In this chapter, we will focus on the case \( \Gamma_{3(p-1)/2}^2 \). In fact, the other three cases are easier than this one, and they follow along the same lines. Hence, we will only discuss the propositions concerning \( N(\mathbb{Z}/p) \) in \( \Gamma_{3(p-1)/2}^2 \). Recall that Proposition 3.2.2 says that \( \Gamma_{3(p-1)/2}^n \) contains a subgroup of order \( p \) if \( n \leq 5 \); furthermore, every element of order \( p \) in \( \Gamma_{3(p-1)/2}^n \) acting on \( S_{3(p-1)/2} \) has 5 fixed points when \( p > 5 \). To avoid overcomplexity, we will assume \( p > 5 \) throughout this chapter.
7.1 Calculation of the $p$-torsion of the Farrell cohomology of the normalizer of $\mathbb{Z}/p$ in $\Gamma^2_{3(p-1)/2}$

Lemmas 7.1.1 to 7.1.5 are similar to that of Section 6.1, we will state them without proofs.

**Lemma 7.1.1.** Let $\mathbb{Z}/p < \Gamma^2_{3(p-1)/2}$ with $N(\mathbb{Z}/p)$ the normalizer in $\Gamma^2_{3(p-1)/2}$. There is an injective homomorphism $I : N(\mathbb{Z}/p)/\mathbb{Z}/p \to \Gamma^{2,3}$, where $\Gamma^{2,3} = \pi_0 \text{Diff}_{\text{eot}}(S^2, \text{fixing } P_1, P_2, \text{permuting } \{P_3, P_4, P_5\})$.

**Lemma 7.1.2.** Let $\mathbb{Z}/p = < \alpha > < \Gamma^2_{3(p-1)/2}$, and let $y$ be an orientation preserving diffeomorphism of $S_{3(p-1)/2}$ of order $p$ with the fixed points $P_1, P_2$, representing $\alpha$. Now consider the $< y >$ action on $S_{3(p-1)/2}$, with five fixed points, $P_1, P_2, P_3, P_4, P_5$, and orbit space $S^2$. Consider the following diagram:

$$
\begin{array}{ccc}
S_{3(p-1)/2} - \{P_1, P_2, P_3, P_4, P_5\} & \longrightarrow & S_{3(p-1)/2} - \{P_1, P_2, P_3, P_4, P_5\} \\
\downarrow \pi & & \downarrow \pi \\
S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4, \hat{P}_5\} & \overset{w}{\longrightarrow} & S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4, \hat{P}_5\},
\end{array}
$$

where $\pi$ is the projection induced by the $< y >$ action, and $S^2$ is the orbit space.

We then have:

(a) $\text{Im}(I : N(\mathbb{Z}/p)/\mathbb{Z}/p \to \Gamma^{2,3}) = \{[w] \in \Gamma^{2,3} | w \text{ lifts}\}$.

(b) Let $[w] \in \Gamma^{2,3}$. Then $w$ lifts to a diffeomorphism $h : S_{3(p-1)/2} - \{P_1, P_2, P_3, P_4, P_5\} \to S_{3(p-1)/2} - \{P_1, P_2, P_3, P_4, P_5\}$ if and only if every closed curve of $S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4, \hat{P}_5\}$ which lifts to a closed curve of $S_{3(p-1)/2} - \{P_1, P_2, P_3, P_4, P_5\}$ maps (via $w$) to a closed curve of $S^2 - \{\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4, \hat{P}_5\}$ which lifts to a closed curve of $S_{3(p-1)/2} - \{P_1, P_2, P_3, P_4, P_5\}$.
(c) Let \( \gamma \) be a closed curve in \( S^2 - \{ \hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4, \hat{P}_5 \} \), and \( [\gamma] = \prod x_i^{n_i} \prod x_i^{m_i} \ldots \prod x_i^{k_i} \in \pi_1(S^2 - \{ \hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4, \hat{P}_5 \}) \), where \( x_i \) is represented by a simple closed curve around \( P_i \). Then, \( \gamma \) lifts to a closed curve if and only if
\[
\sum n_i \beta_i + m_i \beta_i + \ldots + k_i \beta_i = 0 \pmod{p},
\]
where \( (\beta_1, \beta_2|\beta_3, \beta_4, \beta_5) \) is the fixed point data of \( y \).

**Lemma 7.1.3.** Let \([w] \in \Gamma^{2,3}\) where \( \lambda([w]) = \delta \) is an element of \( \Sigma_3 = \langle (34), (45) \rangle \). Consider the same commutative diagram as in Lemma 7.1.2, where \( \pi \) is the projection induced by the \( < y > \) action, \( y \in \text{Diff} \rightarrow \text{eo}^+ (S_3(p-1)/2, P_1, P_2) \) and \( y^p = \text{id} \). Suppose the fixed point data of \( y \) is \( \delta_2(y) = (\beta_1, \beta_2|\beta_3, \beta_4, \beta_5) \), where \( 0 < \beta_i < p \), \( \sum_1^5 (\beta_i) = 0 \pmod{p} \). Then for \( w \in \text{Diff} \rightarrow \text{eo}^+ (S^2 - \{ \hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4, \hat{P}_5 \}) \), \( w \) lifts if and only if the following condition is true: if \( \sum m_i \beta_i = 0 \pmod{p} \), the \( m_i \)'s are all integers, then \( \sum m_i \beta_{5(i)} = 0 \pmod{p} \).

**Lemma 7.1.4.** Let \( \mathbb{Z}/p < \Gamma^{2,3}_{3(p-1)/2} \) with normalizer \( N(\mathbb{Z}/p) \). Let \( 0 \to K_5 \overset{\phi}{\to} \Gamma^{2,3} \overset{\lambda}{\to} \Sigma_3 \to 0 \) be the short exact sequence similar to that of Section 6.1, and \( I : N(\mathbb{Z}/p)/\mathbb{Z}/p \to \Gamma^{2,3} \) be the above injective homomorphism. Then the image \( \phi(K_5) \) is contained in \( I(N(\mathbb{Z}/p)/\mathbb{Z}/p) \).

**Lemma 7.1.5.** Let \( \mathbb{Z}/p = \langle \alpha \rangle \), where \( \mathbb{Z}/p < \Gamma^{2,3}_{3(p-1)/2} \), with fixed point data \( \delta_2(\alpha) = (1, \beta_2|\beta_3, \beta_4, \beta_5) \). Consider the short exact sequence \( 0 \to K_5 \overset{\phi}{\to} \Gamma^{2,3} \overset{\lambda}{\to} \Sigma_3 \to 0 \) as before, and \( I : N(\mathbb{Z}/p)/\mathbb{Z}/p \to \Gamma^{2,3} \). Then
\[
\lambda I(N(\mathbb{Z}/p)/\mathbb{Z}/p) = \Sigma_3 = \langle (34), (45) \rangle \text{ if } \beta_3 = \beta_4 = \beta_5;
\]
\[
\lambda I(N(\mathbb{Z}/p)/\mathbb{Z}/p) = \Sigma_2 = \langle (34) \rangle \text{ if } \beta_3 = \beta_4 \neq \beta_5;
\]
\[
\lambda I(N(\mathbb{Z}/p)/\mathbb{Z}/p) = \Sigma_2 = \langle (35) \rangle \text{ if } \beta_3 = \beta_5 \neq \beta_4;
\]

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\[ \lambda I(N(Z/p)/Z/p) = \sum_2 = (45) \text{ if } \beta_4 = \beta_5 \neq \beta_3; \]
\[ \lambda I(N(Z/p)/Z/p) = \text{trivial if } \beta_3 \neq \beta_4 \neq \beta_5. \]

As Section 6.1, we have the following theorem on extensions of groups.

**Theorem 7.1.1.** Let \( Z/p = < \alpha > \), where \( Z/p \subset \Gamma^2_{3(p-1)/2} \), and let \( \delta_2(\alpha) = (1, \beta_2|\beta_3, \beta_4, \beta_5) \) be the fixed point data. As before, we have \( 0 \to K_5 \xrightarrow{\phi} \Gamma^2 \xrightarrow{\lambda} \sum_3 \to 0 \), a short exact sequence, and \( I : N(Z/p)/Z/p \to \Gamma^2 \). Then

1. If \( \beta_3 = \beta_4 = \beta_5 \), there is a group extension:
   \[ 0 \to \phi(K_5) \to I(N(Z/p)/Z/p) \to \Sigma_3 = (34), (45) \to 0, \text{ and therefore also} \]
   \[ 0 \to K_5 \to N(Z/p)/Z/p \to \Sigma_3 = (34), (45) \to 0; \]

2. If \( \beta_3 = \beta_4 \neq \beta_5 \), there is a group extension:
   \[ 0 \to K_5 \to N(Z/p)/Z/p \to \Sigma_2 = (34) \to 0; \]
   
   If \( \beta_3 = \beta_5 \neq \beta_4 \), there is a group extension:
   \[ 0 \to K_5 \to N(Z/p)/Z/p \to \Sigma_2 = (35) \to 0; \]
   
   If \( \beta_4 = \beta_5 \neq \beta_3 \), there is a group extension:
   \[ 0 \to K_5 \to N(Z/p)/Z/p \to \Sigma_2 = (45) \to 0; \]

3. If \( \beta_3 \neq \beta_4 \neq \beta_5 \), \( K_5 \cong N(Z/p)/Z/p \).

In order to apply a spectral sequence to obtain \( H^*(N(Z/p)/Z/p) \), we need to consider the action of \( \sum_3 \) on \( H^*(K_5, Z) \).

**Lemma 7.1.6.** [Co4] The integral cohomology of \( K_5 \) is torsion free, with Poincaré series \( (1 + 2t)(1 + 3t) \). Furthermore, \( H^*(K_5, Z) \) is generated as an algebra by the
one dimensional cohomology classes $B_{42}$, $B_{43}$, $B_{52}$, $B_{53}$, $B_{54}$, which form a $\mathbb{Z}$-basis of $H^1(K_5, \mathbb{Z})$. A basis for $H^2(K_5, \mathbb{Z})$ is given by $B_{42}B_{52}$, $B_{42}B_{53}$, $B_{42}B_{54}$, $B_{43}B_{52}$, $B_{43}B_{53}$, $B_{43}B_{54}$.

The relations in $H^*(K_5, \mathbb{Z})$ can be found in [Co4]. They are

1. $B_{42}B_{43} = B_{52}B_{53} = 0$,
2. $B_{52}B_{54} = B_{42}(B_{53} + B_{54})$,
3. $B_{53}B_{54} = -B_{42}B_{53} + B_{43}B_{54}$,
4. $B_{ij}B_{ij} = 0, 2 \leq j < i \leq 5, i \geq 4$,
5. $B_{ij}B_{st} = -B_{st}B_{ij}, 2 \leq j < i \leq 5, i \geq 4, 2 \leq t < s \leq 5, s \geq 4$.

The action of $\Sigma_5$ on $H^1(K_5, \mathbb{Z})$ is as follows:

<table>
<thead>
<tr>
<th>(34)</th>
<th>(45)</th>
<th>(35)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{42}$ : $-B_{42}$</td>
<td>$B_{52}$</td>
<td>$B_{42} - B_{52}$</td>
</tr>
<tr>
<td>$B_{43}$ : $B_{42} + B_{43}$</td>
<td>$B_{53}$</td>
<td>$B_{52} + B_{53} + B_{54} - B_{42}$</td>
</tr>
<tr>
<td>$B_{52}$ : $B_{52} - B_{42}$</td>
<td>$B_{42}$</td>
<td>$-B_{52}$</td>
</tr>
<tr>
<td>$B_{53}$ : $B_{53} + B_{54}$</td>
<td>$B_{43}$</td>
<td>$B_{52} + B_{53}$</td>
</tr>
<tr>
<td>$B_{54}$ : $-B_{54}$</td>
<td>$B_{52} + B_{53} + B_{54} - B_{42} - B_{43}$</td>
<td>$-B_{52} - B_{53} + B_{42} + B_{43}$</td>
</tr>
</tbody>
</table>
Remark. The above lemma is still true if we pass to the cohomology with $F_p$ coefficients. In fact, $H^i(K_5, F_p) = Hom(H_i(K_5, \mathbb{Z}), F_p) \oplus Ext(H_{i-1}(K_5, \mathbb{Z}), F_p) = Hom(H_i(K_5, \mathbb{Z}), F_p)$. This is because $H_0(K_5, \mathbb{Z}) = H^0(K_5, \mathbb{Z}) = \mathbb{Z}$, $H_1(K_5, \mathbb{Z}) = H^1(K_5, \mathbb{Z}) = \mathbb{Z}$, $H_2(K_5, \mathbb{Z}) = H^2(K_5, \mathbb{Z}) = \mathbb{Z}$, $H_3(K_5, \mathbb{Z}) = H^3(K_5, \mathbb{Z}) = 0$ for $i \geq 3$. It is easy to check that the generators are still linearly independent and the corresponding relations are still true if we pass to cohomology with $F_p$ coefficients.

Having Theorem 7.1.1 and Lemma 7.1.6, we can now calculate $H^*(\mathbb{Z}/p/\mathbb{Z}/p, F_p)$. Due to the complexity of $H^*(K_5, \mathbb{Z})$ and action of $\sum_5$ on $H^*(K_5, \mathbb{Z})$, it is harder to obtain the following theorem than to get the corresponding theorem in previous chapter.

**Theorem 7.1.2.** Let $\mathbb{Z}/p = < \alpha >$, where $\mathbb{Z}/p < \Gamma_3^{2(p-1)/2}$, with fixed point data $\delta_2(\alpha) = (1, \beta_2 | \beta_3, \beta_4, \beta_5)$.

1. If $\beta_3 = \beta_4 = \beta_5$, then $H^0(\mathbb{Z}/p/\mathbb{Z}/p, F_p) = F_p,$


2. If $\beta_3 = \beta_4 \neq \beta_5$, or $\beta_3 = \beta_5 \neq \beta_4$, or $\beta_4 = \beta_5 \neq \beta_3,$

   then $H^0(\mathbb{Z}/p/\mathbb{Z}/p, F_p) = F_p, H^1(\mathbb{Z}/p/\mathbb{Z}/p, F_p) = F_p \oplus F_p \oplus F_p,$

   $H^2(\mathbb{Z}/p/\mathbb{Z}/p, F_p) = F_p \oplus F_p \oplus F_p, H^*(\mathbb{Z}/p/\mathbb{Z}/p, F_p) = 0$, if $* > 2$.

3. If $\beta_3 \neq \beta_4 \neq \beta_5$, then $H^0(\mathbb{Z}/p/\mathbb{Z}/p, F_p) = F_p, H^1(\mathbb{Z}/p/\mathbb{Z}/p, F_p) = \oplus_5 F_p,$

   $H^2(\mathbb{Z}/p/\mathbb{Z}/p, F_p) = \oplus_6 F_p, H^*(\mathbb{Z}/p/\mathbb{Z}/p, F_p) = 0$, if $* > 2$. 

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PROOF. We only prove the first case. By Theorem 7.1.1, we have the group extension

$$0 \rightarrow K_5 \rightarrow N(\mathbb{Z}/p)/\mathbb{Z}/p \rightarrow \Sigma_3 = \langle (34), (45) \rangle \rightarrow 0.$$ 

The Hochschild-Serre Spectral sequence takes the form

$$E_2^{ij} = H^i(\Sigma_3, H^j(K_5, F_p)) \Rightarrow H^{i+j}(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p).$$ 

Since $p > 3$, $E_2^{ij} = H^i(\Sigma_3, H^j(K_5, F_p)) = 0$ if $i > 0$. Hence $H^j(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) = H^j(K_5, F_p)^{\Sigma_3}$. First, we calculate $H^1(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) = H^1(K_5, F_p)^{\Sigma_3}$.

If

$$m_1 B_{42} + m_2 B_{43} + m_3 B_{52} + m_4 B_{53} + m_5 B_{54} \in H^1(K_5, F_p)^{(34)}, \text{ where } m_i \in F_p,$$

then

$$(34)(m_1 B_{42} + m_2 B_{43} + m_3 B_{52} + m_4 B_{53} + m_5 B_{54})$$

$$= -m_1 B_{42} + m_2 (B_{42} + B_{43}) + m_3 (B_{52} - B_{42}) + m_4 (B_{53} + B_{54}) + m_5 (-B_{54}),$$

$$= (-m_1 + m_2 - m_3)B_{42} + m_2 B_{43} + m_3 B_{52} + m_4 B_{53} + (m_4 - m_5)B_{54}.$$

By comparing coefficients, we have $m_1 = -m_1 + m_2 - m_3$, and $m_5 = m_4 - m_5$. Similarly, if $m_1 B_{42} + m_2 B_{43} + m_3 B_{52} + m_4 B_{53} + m_5 B_{54} \in H^1(K_5, F_p)^{(45)}$, then we have $m_1 + m_5 = m_3, m_2 + m_5 = m_4$. Hence, if $m_1 B_{42} + m_2 B_{43} + m_3 B_{52} + m_4 B_{53} + m_5 B_{54} \in H^1(K_5, F_p)^{(34),(45)}$, then $m_1 = -m_1 + m_2 - m_3, m_5 = m_4 - m_5, m_1 + m_5 = m_3, m_2 + m_5 = m_4$, i.e., $m_1 = 0, m_2 = m_5, m_3 = m_5, m_4 = 2m_5$. Therefore, $H^1(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) = H^1(K_5, F_p)^{\Sigma_3} = F_p$.

Second, we calculate $H^2(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) = H^2(K_5, F_p)^{\Sigma_3}$. Notice that a basis for $H^2(K_5, F_p)$ is given by $\alpha_1 = B_{42}B_{52}, \alpha_2 = B_{42}B_{53}, \alpha_3 = B_{42}B_{54}, \alpha_4 = \ldots$.
$B_{43}B_{52}$, $\alpha_5 = B_{43}B_{53}$, and $\alpha_6 = B_{43}B_{54}$. Consider the action (34) on these basis elements:

$$(34)(\alpha_1) = (34)(B_{42}B_{52})$$
$$= (-B_{42})(B_{52} - B_{42})$$
$$= -B_{42}B_{52}$$
$$= -\alpha_1. $$

(Since $B_{42}B_{42} = 0$). Similarly, we can get $(34)(\alpha_2) = -\alpha_3 - \alpha_2$, $(34)(\alpha_3) = \alpha_3$, $(34)(\alpha_4) = \alpha_1 + \alpha_4$, $(34)(\alpha_5) = \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6$, $(34)(\alpha_6) = -\alpha_3 - \alpha_6$.

Also, we have the action of (45) on these basis.

$(45)(\alpha_1) = -\alpha_1,$
$(45)(\alpha_2) = -\alpha_4,$
$(45)(\alpha_3) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$
$(45)(\alpha_4) = -\alpha_2,$
$(45)(\alpha_5) = -\alpha_5,$
$(45)(\alpha_6) = \alpha_5 + \alpha_6.$

Similar as before, we get $H^2(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) = H^2(K_5, F_p)^{\Sigma_3} = F_p$. Moreover, since $H^*(K_5, F_p) = 0$ for $* > 2$, $H^*(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) = 0$, if $* > 2$.

For the second case, we get $H^1(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) = H^1(K_5, F_p)^{\Sigma_2}$. In the same
way we can get $H^0(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) = F_p$, $H^1(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) = F_p \oplus F_p \oplus F_p$, $H^2(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) = F_p \oplus F_p \oplus F_p$, $H^*(N(\mathbb{Z}/p)/\mathbb{Z}/p, F_p) = 0$, if $* > 2$.

The third case is trivial. \[Q.E.D.\]

A similar theorem as above can also be obtained.

**Theorem 7.1.3.** Let $\mathbb{Z}/p = \langle \alpha \rangle$, where $\mathbb{Z}/p < \Gamma_{3(p-1)/2}^2$, with fixed point data $\delta_2(\alpha) = (1, \beta_2|\beta_3, \beta_4, \beta_5)$.

1. If $\beta_3 = \beta_4 = \beta_5$, then $H^0(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} = \mathbb{Z}_{(p)}$, $H^1(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})(p) = \mathbb{Z}(p)$, $H^2(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} = \mathbb{Z}_{(p)}$, $H^*(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} = 0$, if $* > 2$.

2. If $\beta_3 = \beta_4 \neq \beta_5$, or $\beta_3 = \beta_5 \neq \beta_4$, or $\beta_4 = \beta_5 \neq \beta_3$,

then $H^0(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} = \mathbb{Z}_{(p)}$, $H^1(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})(p) = \mathbb{Z}(p) \oplus \mathbb{Z}(p) \oplus \mathbb{Z}(p)$, $H^2(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} = \mathbb{Z}(p) \oplus \mathbb{Z}(p) \oplus \mathbb{Z}(p)$, $H^*(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} = 0$, if $* > 2$.

3. If $\beta_3 \neq \beta_4 \neq \beta_5$, then $H^0(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} = \mathbb{Z}_{(p)}$, $H^1(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} = \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}$, $H^2(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} = \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}$, $H^*(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} = 0$, if $* > 2$.

Again $A_{(p)}$ stands for the “$p$-localization” of the abelian group $A$, i.e., $A_{(p)} = A \otimes \mathbb{Z}_{(p)}$.

**Proof.** It is similar to the proof of Theorem 7.1.2. Since the tensor product with $\mathbb{Z}_{(p)}$ preserves exactness, one has still all the spectral sequences after $p$-localization. \[Q.E.D.\]
We now consider $\tilde{H}^i(N(\mathbb{Z}/p),\mathbb{Z})_\langle p \rangle$. Unlike in Section 6.1, we cannot use Küneth Theorem to get the result because we do not know whether the short exact sequence $0 \to \mathbb{Z}/p \to N(\mathbb{Z}/p) \to N(\mathbb{Z}/p)/\mathbb{Z}/p \to 0$ splits or not. Hence, we apply a spectral sequence to obtain the following theorem.

**Theorem 7.1.4.** Let $\mathbb{Z}/p = \langle \alpha \rangle$, where $\mathbb{Z}/p < \Gamma_{3(p-1)/2}^2$ with fixed point data $\delta_2(\alpha) = (1, \beta_2|\beta_3, \beta_4, \beta_5)$.

If $\beta_3 = \beta_4 = \beta_5$, then

$$\tilde{H}^i(N(\mathbb{Z}/p),\mathbb{Z})_\langle p \rangle = \begin{cases} \mathbb{Z}/p \oplus \mathbb{Z}/p \text{ or } \mathbb{Z}/p^2 & i = 0 \mod(2) \\ \mathbb{Z}/p & i = 1 \mod(2) \end{cases}.$$  

If $\beta_3 = \beta_4 \neq \beta_5$, or $\beta_3 = \beta_5 \neq \beta_4$, or $\beta_4 = \beta_5 \neq \beta_3$, then

$$\tilde{H}^i(N(\mathbb{Z}/p),\mathbb{Z})_\langle p \rangle = \begin{cases} \oplus_1 \mathbb{Z}/p \text{ or } \oplus_2 \mathbb{Z}/p \oplus \mathbb{Z}/p^2 & i = 0 \mod(2) \\ \oplus_3 \mathbb{Z}/p & i = 1 \mod(2) \end{cases}.$$  

If $\beta_3 \neq \beta_4 \neq \beta_5$, then

$$\tilde{H}^i(N(\mathbb{Z}/p),\mathbb{Z})_\langle p \rangle = \begin{cases} \oplus_1 \mathbb{Z}/p \text{ or } \oplus_2 \mathbb{Z}/p \oplus \mathbb{Z}/p^2 & i = 0 \mod(2) \\ \oplus_3 \mathbb{Z}/p & i = 1 \mod(2) \end{cases}.$$  

**Proof.** We only prove the first case. Apply the spectral sequence to the central extension

$$0 \to \mathbb{Z}/p \to N(\mathbb{Z}/p) \to N(\mathbb{Z}/p)/\mathbb{Z}/p \to 0$$
in the form

\[ E_2^{ij} = H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} \implies H^{i+j}(N(\mathbb{Z}/p), \mathbb{Z})_{(p)}. \]

We first compute \( H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} \). Notice that the extension is central because \( N(\mathbb{Z}/p) = C(\mathbb{Z}/p) \), so \( N(\mathbb{Z}/p)/\mathbb{Z}/p \) has trivial action on \( H^j(\mathbb{Z}/p, \mathbb{Z}) \). We now claim that \( H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = 0 \) for \( i > 2 \). In fact,

\[ H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)}, \quad \text{for } j = 0, \]

\[ H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = 0, \quad \text{for } j > 0 \text{ odd}, \]

and

\[ H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)}, \quad \text{for } j > 0 \text{ even}. \]

By Theorem 7.1.2 and Theorem 7.1.3, all these groups are 0 for \( i > 2 \). Hence, we only need to consider the case \( i = 0, 1 \). We have the following:

\[ H^0(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^0(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} = \mathbb{Z}_{(p)}, \quad \text{for } j = 0, \]

\[ H^0(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = 0, \quad \text{for } j > 0 \text{ odd}, \]

\[ H^0(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^0(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p, \quad \text{for } j > 0 \text{ even}. \]

\[ H^1(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^1(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)}, \quad \text{for } j = 0, \]

\[ H^1(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = 0, \quad \text{for } j > 0 \text{ odd}, \]

\[ H^1(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^1(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p, \quad \text{for } j > 0 \text{ even}. \]
\[ H^2(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^2(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)}, \quad \text{for} \quad j = 0, \]

\[ H^2(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = 0, \quad \text{for} \quad j > 0 \quad \text{odd}, \]

\[ H^2(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} = H^2(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p, \quad \text{for} \quad j > 0 \quad \text{even}. \]

The spectral sequence collapses, because each differential originates or terminates at 0. So, both the \(E_2\) and the \(E_\infty\) page looks as follows:

\[
\begin{array}{cccccc}
| & \mathbb{Z}/p & \mathbb{Z}/p & \mathbb{Z}/p & 0 & 0...
| & 0 & 0 & 0 & 0 & 0...
| & \mathbb{Z}/p & \mathbb{Z}/p & \mathbb{Z}/p & 0 & 0...
| & 0 & 0 & 0 & 0 & 0...
| & \mathbb{Z}/p & \mathbb{Z}/p & \mathbb{Z}/p & 0 & 0...
| & 0 & 0 & 0 & 0 & 0...
| & \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} & 0 & 0...
\end{array}
\]

Hence, \(\hat{H}^1(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} = \mathbb{Z}/p\), and \(\hat{H}^2(N(\mathbb{Z}/p), \mathbb{Z})_{(p)} = \mathbb{Z}/p \oplus \mathbb{Z}/p\) or \(\mathbb{Z}/p^2\).

The other two cases are similar. Q.E.D.
7.2 Counting conjugacy classes of subgroups of order $p$

Now our task is to count the conjugacy classes of $\mathbb{Z}/p$. By Proposition 4.3.1., it is a number theoretical problem of counting the number of different unordered integer tuples. We will use the following notation.

**Definition 7.2.1.** Denote by $A_n := \text{Cardinality of } \{(1, \beta_2, \beta_3, ..., \beta_{n+2}) : (\beta_2, \beta_3, ..., \beta_{n+2}) \text{ as ordered tuple, } 1 + \beta_2 + ... + \beta_{n+2} = 0( \mod p), 0 < \beta_i < p\}$

$B_n := \text{Cardinality of } \{(1, \beta_2, \beta_3, ..., \beta_{n+1}|\beta_n, \beta_{n+1}, \beta_{n+2}) : (\beta_2, \beta_3, ..., \beta_{n-1}) \text{ as ordered tuple, } \beta_n = \beta_{n+1} = \beta_{n+2}, 1 + \beta_2 + ... + \beta_{n+2} = 0( \mod p), 0 < \beta_i < p\}$

$C_n := \text{Cardinality of } \{(1, \beta_2, \beta_3, ..., \beta_{n-1}|\beta_n, \beta_{n+1}, \beta_{n+2}) : (\beta_2, \beta_3, ..., \beta_{n-1}) \text{ as ordered tuple, } (\beta_n, \beta_{n+1}, \beta_{n+2}) \text{ as unordered tuple, } \beta_n \neq \beta_{n+1} = \beta_{n+2}, 1 + \beta_2 + ... + \beta_{n+2} = 0( \mod p), 0 < \beta_i < p\}$

$D_n := \text{Cardinality of } \{(1, \beta_2, \beta_3, ..., \beta_{n-1}|\beta_n, \beta_{n+1}, \beta_{n+2}) : (\beta_2, \beta_3, ..., \beta_{n-1}) \text{ as ordered tuple, } (\beta_n, \beta_{n+1}, \beta_{n+2}) \text{ as unordered tuple, } \beta_n \neq \beta_{n+1} \neq \beta_{n+2}, 1 + \beta_2 + ... + \beta_{n+2} = 0( \mod p), 0 < \beta_i < p\}$

**Lemma 7.2.1.** We have the following results:

1. $A_n = \frac{(-1)^n+(p-1)^{n+1}}{p}$
2. $B_n = \frac{(-1)^{n-2}+(p-1)^{n-1}}{p}$
3. $C_n = \frac{2(-1)^{n-1}+(p-1)^n-(p-1)^{n-1}}{p}$
4. $D_n = \frac{A_n-B_n-3C_n}{6}$

**Proof.** (1) If $1 + \beta_2 + ... + \beta_{n+2} = 0( \mod p), 0 < \beta_i < p$, then

$1 + \beta_2 + ... + \beta_{n+2} = p,$
or

\[ 1 + \beta_2 + \ldots + \beta_{n+2} = 2p, \]

or

\[ \ldots, \]

\[ 1 + \beta_2 + \ldots + \beta_{n+2} = np. \]

We can conclude that \( \beta_{n+2} \) is uniquely determined by \( \beta_2, \ldots, \beta_{n+1} \). In fact,

if \( 0 < \beta_2 + \ldots + \beta_{n+1} < p - 1 \), then \( \beta_{n+2} = p - 1 - (\beta_2 + \ldots + \beta_{n+1}) \);

if \( p - 1 < \beta_2 + \ldots + \beta_{n+1} < 2p - 1 \), then \( \beta_{n+2} = 2p - 1 - (\beta_2 + \ldots + \beta_{n+1}) \);

if \( 2p - 1 < \beta_2 + \ldots + \beta_{n+1} < 3p - 1 \), then \( \beta_{n+2} = 3p - 1 - (\beta_2 + \ldots + \beta_{n+1}) \);

\[ \ldots \]

if \( (n - 1)p - 1 < \beta_2 + \ldots + \beta_{n+1} < np - 1 \), then \( \beta_{n+2} = np - 1 - (\beta_2 + \ldots + \beta_{n+1}) \).

Now we consider the choices of \( \beta_2, \ldots, \beta_{n+1} \). We have \((p - 1)\) choices for \( \beta_2, \beta_3, \ldots, \beta_{n+1} \) respectively, so totally we have \((p - 1)^n\) choices. In all these choices, \( \beta_2 + \ldots + \beta_{n+1} = p - 1 \); \( \beta_2 + \ldots + \beta_{n+1} = 2p - 1 \); \ldots; \( \beta_2 + \ldots + \beta_{n+1} = (n - 1)p - 1 \) do not work. The cardinality of cases which do not work is \( A_{n-1} \). Therefore, we have \( A_n = (p-1)^n - A_{n-1} \). Also, it is not hard to see that \( A_0 = 1 \). By a simple calculation, we obtain \( A_n = \frac{(-1)^n + (p-1)^{n+1}}{p} \).

(2) If \( 1 + \beta_2 + \ldots + \beta_{n+2} \equiv 0 \pmod{p} \), \( 0 < \beta_i < p \), then

\[ 1 + \beta_2 + \ldots + \beta_{n+2} = p, \]

or

\[ 1 + \beta_2 + \ldots + \beta_{n+2} = 2p, \]

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or

\[ 1 + \beta_2 + \ldots + \beta_{n+2} = np. \]

We can conclude that \( \beta_2 \) is uniquely determined by \( \beta_3, \ldots, \beta_{n+2} \). In fact,

if \( 0 < \beta_3 + \ldots + \beta_{n+2} < p - 1 \), then \( \beta_2 = p - 1 - (\beta_3 + \ldots + \beta_{n+2}) \);

if \( p - 1 < \beta_3 + \ldots + \beta_{n+2} < 2p - 1 \), then \( \beta_2 = 2p - 1 - (\beta_3 + \ldots + \beta_{n+2}) \);

if \( 2p - 1 < \beta_3 + \ldots + \beta_{n+2} < 3p - 1 \), then \( \beta_2 = 3p - 1 - (\beta_3 + \ldots + \beta_{n+2}) \);

...

if \( (n - 1)p - 1 < \beta_3 + \ldots + \beta_{n+2} < np - 1 \), then \( \beta_2 = np - 1 - (\beta_3 + \ldots + \beta_{n+2}) \).

Now we consider the choices of \( \beta_3, \ldots, \beta_{n+2} \). We have \((p-1)\) choices for \( \beta_3, \beta_4, \ldots, \beta_{n-1} \) respectively, so we have \((p-1)^{n-3}\) choices. Also, we have \((p-1)\) choices for \( \beta_n, \beta_{n+1} \) and \( \beta_{n+2} \). Hence, totally we have \((p-1)^{n-2}\) choices. In all these choices, \( \beta_3 + \ldots + \beta_{n+2} = p - 1; \beta_3 + \ldots + \beta_{n+2} = 2p - 1; \ldots; \beta_3 + \ldots + \beta_{n+2} = (n - 1)p - 1 \) do not work. The cardinality of cases which do not work is \( B_{n-1} \). Therefore, we have

\[ B_n = (p - 1)^{n-2} - B_{n-1}. \]

Also, it is not hard to see that \( B_2 = 1 \). By a simple calculation, we obtain

\[ B_n = \frac{(-1)^{n-2} + (p - 1)^{n-1}}{p}. \]

(3) Similarly, we can get equation:

\[ C_n + B_n = (p - 1)^{n-1} - (C_{n-1} + B_{n-1}). \]

Hence

\[ C_n = \frac{2(-1)^{n-1} + (p - 1)^{n-1} - (p - 1)^{n-2}}{p}. \]

(4) It is not hard to obtain \( A_n = B_n + 3C_n + 6D_n \). Hence, \( D_n = \frac{A_n - B_n - 3C_n}{6} \).

Q.E.D.

**Theorem 7.2.1.** We have the following results:

(1) For \( \mathbb{Z}/p = < \alpha > = \Gamma_{n(p-1)/2}^{n-1}, \delta_{n-1}(\alpha) = (1, \beta_2, \beta_3, \ldots, \beta_{n-1}, \beta_n, \beta_{n+1}, \beta_{n+2}) \),
there are three categories of conjugacy classes of subgroups of order \( p \):

(a) if \( \beta_n = \beta_{n+1} = \beta_{n+2} \), we have \( B_n \) conjugacy classes of subgroups of order \( p \).

(b) if \( \beta_n \neq \beta_{n+1} \neq \beta_{n+2} \), we have \( C_n \) conjugacy classes of subgroups of order \( p \).

(c) if \( \beta_n = \beta_{n+1} \neq \beta_{n+2} \), we have \( D_n \) conjugacy classes of subgroups of order \( p \).

(2) For \( \mathbb{Z}/p = \langle \alpha \rangle \cong \Gamma^{n+1}_{n(p-1)/2} \), \( \delta_n(\alpha) = (1, \beta_2, \beta_3, \ldots, \beta_{n-1}, \beta_n|\beta_{n+1}, \beta_{n+2}) \),

there are two categories of conjugacy classes of subgroups of order \( p \):

(a) if \( \beta_{n+1} = \beta_{n+2} \), we have \( B_n + C_n \) conjugacy classes of subgroups of order \( p \).

(b) if \( \beta_{n+1} \neq \beta_{n+2} \), we have \( (6D_n + 2C_n)/2 = 3D_n + C_n \) conjugacy classes of subgroups of order \( p \).

(3) For \( \mathbb{Z}/p = \langle \alpha \rangle \cong \Gamma^{n+1}_{n(p-1)/2} \) or \( \mathbb{Z}/p = \langle \alpha \rangle \cong \Gamma^{n+2}_{n(p-1)/2} \),

\( \delta_{n+2}(\alpha) = (1, \beta_2, \beta_3, \ldots, \beta_{n-1}, \beta_n, \beta_{n+1}, \beta_{n+2}) \), we have \( B_n + 3C_n + 6D_n = A_n \) conjugacy classes of subgroups of order \( p \).

**Proof.** It is by Proposition 4.3.1. \( \text{Q.E.D.} \)
Proof of Theorem 7.0.1.

We first prove the case for $\Gamma_3^{2(p-1)/2}$. By Lemma 7.2.1 and Theorem 7.2.1, for $\mathbb{Z}/p = \langle \alpha \rangle \leq \Gamma_3^{2(p-1)/2}$, $\delta_2(\alpha) = (1, \beta_2, \beta_3, \beta_4, \beta_5)$,

(a) if $\beta_3 = \beta_4 = \beta_5$, we have $B_3 = p - 2$ conjugacy classes of subgroups of order $p$.

(b) if $\beta_3 \neq \beta_4 = \beta_5$, we have $C_3 = p^2 - 4p + 5$ conjugacy classes of subgroups of order $p$.

(c) if $\beta_3 \neq \beta_4 \neq \beta_5$, we have $D_3 = (p^3 - 7p^2 + 17p - 17)/6$ conjugacy classes of subgroups of order $p$.

By Theorem 7.1.4., the main theorem follows.

We second prove the case for $\Gamma_3^{3(p-1)/2}$. By Lemma 7.2.1 and Theorem 7.2.1, for $\mathbb{Z}/p = \langle \alpha \rangle \leq \Gamma_3^{3(p-1)/2}$, $\delta_3(\alpha) = (1, \beta_2, \beta_3, \beta_4, \beta_5)$,

(a) if $\beta_4 \neq \beta_5$, we have $B_3 + C_3 = p^2 - 3p + 3$ conjugacy classes of subgroups of order $p$.

(b) if $\beta_4 = \beta_5$, we have $3D_3 + C_3 = 3(p^3 - 7p^2 + 17p - 17)/6 + p^2 - 4p + 5 = (p^3 - 5p^2 - 9p - 7)/2$ conjugacy classes of subgroups of order $p$.

By Theorem 7.1.4., the main theorem follows.

We third prove the case for $\Gamma_3^{4(p-1)/2}$ or $\Gamma_3^{5(p-1)/2}$. By Lemma 7.2.1 and Theorem 7.2.1, for $\mathbb{Z}/p = \langle \alpha \rangle \leq \Gamma_4^{4(p-1)/2}$, $\delta_4(\alpha) = (1, \beta_2, \beta_3, \beta_4, \beta_5)$, we have $B_3 + 3C_3 + 6D_3 = p - 2 + 3(p^2 - 4p + 5) + 6(p^3 - 7p^2 + 17p - 17)/6 = p^3 - 4p^2 + 6p - 4$ conjugacy classes of subgroups of order $p$.

By Theorem 7.1.4., the main theorem follows.

Q.E.D.
CHAPTER 8

SOME DISCUSSIONS ABOUT THE $P$-TORSION OF THE FARRELL COHOMOLOGY OF $\Gamma^I_{N(P-1)/2}$

This chapter generalizes the results of Chapters 5, 6, and 7. Not all of the results here are complete; because the $p$-torsion of the Farrell cohomology of $\Gamma^i_{n(p-1)/2}$ is so complex, some results are only partially obtained. However, they do give us a general idea and they are useful in dealing with other cohomological properties of the mapping class group.

This chapter consists of two sections. In the first section, we present an algorithm to calculate $\hat{H}^*(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_p$ for $\mathbb{Z}/p < \Gamma^i_{n(p-1)/2}$ with normalizer $N(\mathbb{Z}/p)$. In fact, we get that $N(\mathbb{Z}/p)/\mathbb{Z}/p$ is an extension of $K_{n+2}$, the pure mapping class group of the sphere with $n + 2$ punctures, by a subgroup of $\sum_{n+2-i}$. The cohomology of $K_{n+2}$ with integer coefficients and the action of $\sum_{n+2}$ on $\hat{H}^*(K_{n+2}, \mathbb{Z})$ are well known ([Co4]). Hence it is not hard to get the algorithm to calculate $\hat{H}^*(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_p$.

Also, we have that the short exact sequence $0 \to \mathbb{Z}/p \to N(\mathbb{Z}/p) \to N(\mathbb{Z}/p)/\mathbb{Z}/p \to 0$ is central, and that $N(\mathbb{Z}/p)$ has $p$-period 2. Therefore, we know that if the Serre’s spectral sequence is applied to the above short exact sequence, it collapses at $E_2$ page. We did not get $\hat{H}^*(N(\mathbb{Z}/p), \mathbb{Z})_p$ completely—we only got the $E_\infty$ page of the spectral sequence—but this still gives us a lot of important properties of $\hat{H}^*(N(\mathbb{Z}/p), \mathbb{Z})_p$. For
example, we can calculate the order of $\hat{H}^*(N(\mathbb{Z}/p), \mathbb{Z}_{(p)})$ in both even and odd dimensions. Also, from the $E_\infty$ page, we can find the upper bound of $t$ which satisfies the condition that the $p$-torsion $\mathbb{Z}/p^t$ is contained in $\hat{H}^*(N(\mathbb{Z}/p), \mathbb{Z}_{(p)})$ for fixed $\Gamma_{n(p-1)/2}^i$. In the second section, we generalize the idea introduced in Chapter 7, Section 2, this time counting the conjugacy classes of subgroups of order $p$ in $\Gamma_{n(p-1)/2}^i$ for $i \leq n+2$. Recall that Proposition 3.2.2 says that $\Gamma_{n(p-1)/2}^i$ contains a subgroup of order $p$ if $i \leq n + 2$; furthermore, every element of order $p$ in $\Gamma_{n(p-1)/2}^i$ acting on $S_{n(p-1)/2}$ has $n + 2$ fixed points when $p > n + 2$. To avoid overcomplexity, we will assume $p > n + 2$ throughout this chapter.
8.1 Discussions about the $p$-torsion of the Farrell cohomology of the normalizer of $\mathbb{Z}/p$ in $\Gamma_{n(p-1)/2}^i$

Lemma 8.1.1. There are no subgroup of order $p$ in $\Gamma_{n(p-1)/2}^i$ for $i > n + 2$, i.e.,
$\hat{H}^i(\Gamma_{n(p-1)/2}^i, \mathbb{Z}/(p)) = 0$ for $i > n + 2$.

From now on, we only need consider the case $i \leq n + 2$. Lemmas 8.1.2 to 8.1.6 are similar to those of previous chapters. We will omit all proofs.

Lemma 8.1.2. Let $\Gamma_{n(p-1)/2}^i$ be the punctured mapping class group, where $n$ is a positive integer, $1 \leq i \leq n + 2$, an integer. Let $\mathbb{Z}/p < \Gamma_{n(p-1)/2}^i$ with $N(\mathbb{Z}/p)$ the normalizer in $\Gamma_{n(p-1)/2}^i$, then there is an injective homomorphism $I : N(\mathbb{Z}/p)/\mathbb{Z}/p \rightarrow \Gamma_{n+2-i}^i$, where $\Gamma_{n+2-i}^i = \pi_0 \text{Diff}^+(S^2, \text{fixing } P_1, P_2, ..., P_i, \text{permuting } \{P_{i+1}, P_{i+2}, ..., P_{n+2}\})$.

Lemma 8.1.3. Let $\mathbb{Z}/p = < \alpha > \subset \Gamma_{n(p-1)/2}^i$, and let $y$ be an orientation preserving diffeomorphism of $S_{n(p-1)/2}$ of order $p$ with the fixed points $P_1, P_2, ..., P_i$, representing $\alpha$. Now consider the $< y >$ action on $S_{n(p-1)/2}$, with $n + 2$ fixed points, $P_1, P_2, ..., P_{n+2}$, and the orbit space $S^2$.

Consider the following diagram:

$$
\begin{array}{ccc}
S_{n(p-1)/2} - \{P_1, P_2, ..., P_{n+2}\} & \longrightarrow & S_{n(p-1)/2} - \{P_1, P_2, ..., P_{n+2}\} \\
\downarrow \pi & & \downarrow \pi \\
S^2 - \{\hat{P}_1, \hat{P}_2, ..., \hat{P}_{n+2}\} & \longrightarrow & S^2 - \{\hat{P}_1, \hat{P}_2, ..., \hat{P}_{n+2}\},
\end{array}
$$

where $\pi$ is the projection induced by the $< y >$ action, and $S^2$ is the orbit space.

We then have:
(a) \( \text{Im}(I : N(\mathbb{Z}/p)/\mathbb{Z}/p \to \Gamma_{i,n+2-i}) = \{ [w] \in \Gamma_{i,n+2-i} | w \text{ lifts} \} \).

(b) Let \([w] \in \Gamma_{i,n+2-i} \). Then \(w\) lifts to a diffeomorphism

\[
h : S_{n(p-1)/2} - \{ P_1, P_2, ..., P_{n+2} \} \to S_{n(p-1)/2} - \{ P_1, P_2, ..., P_{n+2} \}
\]

if and only if every closed curve of \( S^2 - \{ \hat{P}_1, \hat{P}_2, ..., \hat{P}_{n+2} \} \) which lifts to a closed curve of \( S_{n(p-1)/2} - \{ P_1, P_2, ..., P_{n+2} \} \), maps (via \(w\)) to a closed curve which lifts to a closed curve.

(c) Let \( \gamma \) be a closed curve in \( S^2 - \{ \hat{P}_1, \hat{P}_2, ..., \hat{P}_{n+2} \} \),

\[
[\gamma] = \prod x_i^{n_i} \prod x_i^{m_i} ... \prod x_i^{k_i} \in \pi_1(S^2 - \{ \hat{P}_1, \hat{P}_2, ..., \hat{P}_{n+2} \}),
\]

where \( x_i \) is represented by a simple closed curve around \( \hat{P}_i \). Then, \( \gamma \) lifts to a closed curve if and only if \( \sum n_i \beta_i + m_i \beta_i + ... + k_i \beta_i \equiv 0 \pmod{p} \), where \((\beta_1, \beta_2, ..., \beta_{n+1}, ..., \beta_{n+2})\) is the fixed point data of \( y \).

**Lemma 8.1.4.** Let \([w] \in \Gamma_{i,n+2-i} \), where \(\lambda([w]) = \delta\) is an element of \(\sum_{n+2-i} \). Consider the same commutative diagram as in Lemma 8.1.3, where \(\pi\) is the projection induced by the \(<y>\) action, \(y \in Diff^+(S_{n(p-1)/2}, P_1, ..., P_i)\) and \(y^p = \text{id} \). Suppose the fixed point data \(\delta(y) = (\beta_1, ..., \beta_{i+1}, ..., \beta_{n+2})\), where \(0 < \beta_j < p\), \(\sum_{i=1}^{n+2} (\beta_j) = 0\pmod{p} \). Then for \(w \in Diff^+(S^2 - \{ \hat{P}_1, \hat{P}_2, ..., \hat{P}_{n+2} \})\), \(w\) lifts if and only if the following condition is true: if \(\sum m_j \beta_j = 0 \pmod{p}\), the \(m_j\)'s are all integers, then \(\sum m_j \beta_j = 0 \pmod{p}\).

**Lemma 8.1.5.** Let \(\mathbb{Z}/p < \Gamma_{n(p-1)/2} \) with normalizer \(N(\mathbb{Z}/p)\). Let \(0 \to K_{n+2} \to \Gamma_{i,n+2-i} \to \sum_{n+2-i} \to 0\) be similar short exact sequence as in previous chapters, and
\[ I : N(\mathbb{Z}/p)/\mathbb{Z}/p \to \Gamma^{i,n+2-i}. \] be the above injective homomorphism. Then the image
\[ \phi(K_{n+2}) \] is contained in \[ I(N(\mathbb{Z}/p)/\mathbb{Z}/p). \]

**Lemma 8.1.6.** Let \[ \mathbb{Z}/p = \langle \alpha \rangle, \] where \[ \mathbb{Z}/p < \Gamma_{n(p-1)/2}^i, \] and let
\[ \delta_i(\alpha) = (1, \beta_2, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_{n+2}) \] be the fixed point data. As before, we have
\[ 0 \to K_{n+2} \xrightarrow{\phi} \Gamma_{i,n+2-i}^{i,n+2-i} \xrightarrow{\lambda} \sum_{n+2-i} \to 0, \] a short exact sequence,
and \[ I : N(\mathbb{Z}/p)/\mathbb{Z}/p \to \Gamma^{i,n+2-i}. \] Then
\[ \lambda(I(N(\mathbb{Z}/p)/\mathbb{Z}/p)) = \sum_{n+2-i}, \text{ if } \beta_{i+1} = \beta_{i+2} = \ldots = \beta_{n+2}; \]
\[ \lambda(I(N(\mathbb{Z}/p)/\mathbb{Z}/p)) = \sum_{n+1-i}, \text{ if } \beta_{i+1} = \beta_{i+2} = \ldots = \beta_{n+1} \neq \beta_{n+2}; \]
\[ \ldots \]
\[ \lambda(I(N(\mathbb{Z}/p)/\mathbb{Z}/p)) = \sum_2, \text{ if two of the } \beta_j \text{'s are equal, where } \beta_j \in \{ \beta_{i+1}, \beta_{i+2}, \ldots, \beta_{n+2} \}; \]
\[ \lambda(I(N(\mathbb{Z}/p)/\mathbb{Z}/p)) = \text{trivial, if none of the } \beta_j \text{'s are equal, where } \beta_j \in \{ \beta_{i+1}, \beta_{i+2}, \ldots, \beta_{n+2} \}. \]

Therefore, as in previous chapters, we obtain the following group extensions.

**Theorem 8.1.1.** Let \[ \mathbb{Z}/p = \langle \alpha \rangle, \] where \[ \mathbb{Z}/p < \Gamma_{n(p-1)/2}^i, \] and let
\[ \delta_i(\alpha) = (1, \beta_2, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_{n+2}) \] be the fixed point data. As before, we have
\[ 0 \to K_{n+2} \xrightarrow{\phi} \Gamma_{i,n+2-i}^{i,n+2-i} \xrightarrow{\lambda} \sum_{n+2-i} \to 0, \] a short exact sequence, and
\[ I : N(\mathbb{Z}/p)/\mathbb{Z}/p \to \Gamma^{i,n+2-i}. \] Then

1. If all of the \( \beta_j \)'s are equal, where \( \beta_j \in \{ \beta_{i+1}, \ldots, \beta_{n+2} \} \), then there is a group extension:
\[ 0 \to \phi(K_{n+2}) \to I(N(\mathbb{Z}/p)/\mathbb{Z}/p) \to \sum_{n+2-i} \to 0 \] and therefore also
\[ 0 \to K_{n+2} \to N(\mathbb{Z}/p)/\mathbb{Z}/p \to \sum_{n+2-i} \to 0; \]

2. If all of the \( \beta_j \)'s except one of them are equal, where \( \beta_j \in \{ \beta_{i+1}, \ldots, \beta_{n+2} \} \), then there is a group extension:
\[ 0 \to K_{n+2} \to N(\mathbb{Z}/p)/\mathbb{Z}/p \to \sum_{n+2-i-1} \to 0; \]
(k) If $k$ of all the $\beta_j$'s are equal, where $\beta_j \in \{\beta_{i+1}, \ldots, \beta_{n+2}\}$, then there is a group extension:

$$0 \rightarrow K_{n+2} \rightarrow \mathbb{N}(\mathbb{Z}/p)/\mathbb{Z}/p \rightarrow \sum_k \rightarrow 0;$$

...(n+2-i) If none of the $\beta_j$'s are equal, where $\beta_j \in \{\beta_{i+1}, \ldots, \beta_{n+2}\}$, then

$$K_{n+2} \cong \mathbb{N}(\mathbb{Z}/p)/\mathbb{Z}/p.$$ 

We need to deal with the action of $\sum_{n+2-i}$ on $H^*(K_n, \mathbb{Z})$ which is the following.

**Lemma 8.1.7. [Co4]** The integral cohomology of $K_n$ is torsion free with Poincaré series

$$[1 + 2t][1 + 3t] \ldots [1 + (n - 2)t].$$

Furthermore, $H^*(K_n, \mathbb{Z})$ is generated as an algebra by one-dimensional cohomology classes

$$B_{ij}, \; n \geq i > j \geq 2, \; i \geq 4.$$ 

A basis for $H^0(K_n, \mathbb{Z})$ is given by

$$B_{i_1j_1} \ldots B_{i_qj_q}, \; 4 \leq i_1 < i_2 < \ldots < i_q \leq n.$$ 

The relations in $H^*(K_n, \mathbb{Z})$ together with the action of $\sum_n$ are given in section 3 of [Co4]; they are all consequences of $B_{42}B_{43} = 0$ together with equivariance.

Based on the group extensions given in Theorem 8.1.1, theoretically, the computation of $H^0(\mathbb{N}(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z}/p)$ is done by the following theorem. However, in practice, to obtain the answer, a lot of computation is involved for $i$ and $n.$
Theorem 8.1.2. Let $\mathbb{Z}/p = \langle \alpha \rangle$, where $\mathbb{Z}/p < \Gamma_{n(p-1)/2}^i$; with fixed point data 
$\delta_i(\alpha) = (1, \beta_2, ..., \beta_i|\beta_{i+1}, ..., \beta_{n+2})$. If $k$ of all the $\beta_j$'s are equal, 
where $\beta_j \in \{\beta_{i+1}, ..., \beta_{n+2}\}$, then by Theorem 8.1.1, there is a group extension:

$$0 \rightarrow K_{n+2} \rightarrow N(\mathbb{Z}/p)/\mathbb{Z}/p \rightarrow \sum_k \rightarrow 0.$$ 
Therefore, we have the following cohomology:

$H^q(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} \cong H^q(K_{n+2}, \mathbb{Z})_{(p)}$; $H^q(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z}/p) \cong H^q(K_{n+2}, \mathbb{Z}/p)_{\Sigma_k}$.

In particular, if $i = n + 1$ or $i = n + 2$, then $H^q(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} \cong H^q(K_{n+2}, \mathbb{Z})_{(p)}$;

$H^q(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z}/p) \cong H^q(K_{n+2}, \mathbb{Z}/p)$ for all $\mathbb{Z}/p < \Gamma_{n(p-1)/2}^i$.

Remark. The proofs of the above theorems are similar to the proofs of the corresponding theorems in the previous chapters, so we just omit them.

Using the central extension $0 \rightarrow \mathbb{Z}/p \rightarrow N(\mathbb{Z}/p) \rightarrow N(\mathbb{Z}/p)/\mathbb{Z}/p \rightarrow 0$, we now discuss the properties of $H^q(N(\mathbb{Z}/p), \mathbb{Z})_{(p)}$.

Lemma 8.1.8. Let $\mathbb{Z}/p = \langle \alpha \rangle$, where $\mathbb{Z}/p < \Gamma_{n(p-1)/2}^i$. Let $0 \rightarrow \mathbb{Z}/p \rightarrow N(\mathbb{Z}/p) \rightarrow N(\mathbb{Z}/p)/\mathbb{Z}/p \rightarrow 0$ be a central extension. Then the Serre spectral sequence related to the above central extension collapses at the $E_2$ page, i.e.,

$E_{\infty}^{i,j} \cong H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} \otimes H^j(\mathbb{Z}/p, \mathbb{Z})$.

PROOF. First, we claim that $(E_2^{i,j}) \cong H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)}$

$\cong H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)} \otimes H^j(\mathbb{Z}/p, \mathbb{Z})$. We need consider the following three cases:

Case (1): If $j$ is an odd positive integer, then

$H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, H^j(\mathbb{Z}/p, \mathbb{Z}))_{(p)} \cong H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, 0)_{(p)} \cong 0 \cong H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, 0)_{(p)} \otimes 0$

Case (2): If $j$ is an even positive integer, then

$LHS \cong H^i(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z}/p)$
\[ \cong \text{Hom}(H_i(N(Z/p)/Z/p, Z), Z/p) \otimes \text{Ext}(H_{i-1}(N(Z/p)/Z/p, Z), Z/p). \] By Theorem 8.1.2 and Lemma 8.1.6, we know that \( H_{i-1}(N(Z/p)/Z/p, Z) \) is torsion free. Hence,

\[ LHS \cong H_i(N(Z/p)/Z/p, Z) \otimes Z/p. \]

\[ RHS \cong H^i(N(Z/p)/Z/p, Z) \otimes Z/p \cong H_i(N(Z/p)/Z/p, Z) \otimes Z/p. \]

Case (3): If \( j = 0 \), then it is trivial.

Second, we need prove that the restriction map \( i^* : H^2(N(Z/p), Z) \rightarrow H^2(Z/p, Z) \) is surjective. By chapter 1, we know that \( Y(N(Z/p), p) = 2 \). That implies \( i^* \) is surjective.

Now we prove that the spectral sequence collapses. Since \( H^*(Z/p, Z) \) is generated by \( H^2(Z/p, Z) \), all differentials are 0. The spectral sequence collapses therefore at the \( E_2 \) page.

**Remark.** By Lemma 8.1.7, \( H^i(K_{n+2}, Z) \) is 0 if \( i > n - 1 \), hence, by Theorem 8.1.2, \( H^i(N(Z/p)/Z/p, Z)_{(p)} = 0 \) if \( i > n - 1 \). Then by the \( E_\infty \) page in Lemma 8.1.8., the order of \( \hat{H}^*(N(Z/p), Z)_{(p)} \) is determined. Also, the upper bound \( t \) for which the \( p \)-torsion \( Z/p^t \) is contained in \( \hat{H}^*(N(Z/p), Z)_{(p)} \) is determined; even though we don’t know \( \hat{H}^*(N(Z/p), Z)_{(p)} \) exactly. However, if we know that \( 0 \rightarrow Z/p \rightarrow N(Z/p) \rightarrow N(Z/p)/Z/p \rightarrow 0 \) splits, then \( \hat{H}^*(N(Z/p), Z)_{(p)} \) is completely determined. So far it is an open question.
8.2 Counting conjugacy classes of subgroups of order $p$

Now our task is to count the conjugacy classes of $\mathbb{Z}/p$. By Proposition 4.3.1., it is a number theoretical problem of counting the number of different unordered integer tuples. We have the following notation.

**Definition 8.2.1.** Denote by $A(i, n+2, k) := \text{Cardinality of } \{ (1, \beta_2, ..., \beta_i | \beta_{i+1}, ..., \beta_{n+2}) : (\beta_2, \beta_3, ..., \beta_i) \text{ as ordered tuple, } (\beta_{i+1}, \beta_{i+2}, ..., \beta_{n+2}) \text{ as unordered tuple, } k \text{ of all the } \beta_j \text{'s are equal, where } \beta_j \in \{ \beta_{i+1}, ..., \beta_{n+2} \}, \ 1 + \beta_2 + ... + \beta_{n+2} = 0(\mod p), 0 < \beta_i < p \} \}

**Lemma 8.2.1.** For $\mathbb{Z}/p = \langle \alpha \rangle \subset \Gamma_{n(p-1)/2}$, $\delta_i(\alpha) = (1, \beta_2, \beta_3, ..., \beta_i | \beta_{i+1}, \beta_{i+2}, ..., \beta_{n+2})$, there are $(n + 2 - i)$ types of conjugacy classes of subgroups of order $p$. If $k$ of all the $\beta_j$'s are equal, where $\beta_j \in \{ \beta_{i+1}, ..., \beta_{n+2} \}$. Then we have $A(i, n+2, k)$ different conjugacy classes of subgroups of order $p$ with the fixed point data of above type.

**Remark.** In applications, it is not easy to get $A(i, n+2, k)$ immediately. Recall from section 7.2, we used a recursive process. This ideal can be generalized. We thus set up a relation between $A(i, n+2, k)$ and $A(i - 1, n+1, k)$. Then we can solve the recursion. For example, exactly as in the proof of Lemma 7.2.1, we get $A(i, n+2, n+2 - i) = (p-1)^{i-1} - A(i-1, n+1, n+2 - i)$. Hence, $A(i, n+2, n+2 - i) = (-1)^{i-2} + \frac{(-1)^{i-1} + (p-1)^{i}}{p} + (-1)^{i-2}A(1, n+3 - i, n+2 - i)$. It also involves a lot of calculation in practice to get the complete answer.
Appendix A

MORE DISCUSSION ABOUT MAPPING CLASS GROUPS

A.1 The difficulty in calculating the $p$-torsion of the Farrell cohomology of $\Gamma_g^i$

Calculating the $p$-torsion of the Farrell cohomology of a mapping class group seems to be a never-ending exercise. In this section, we address the difficulty in resolving this problem and check how far the theorems in this thesis will go. Before we start, we need to give the following definition.

**Definition 1.** Let $S_g$ be a Riemann surface,
\[ \Gamma_g^{i,j} = \pi_0 Diff^e(S_g, P_1, P_2, ..., P_i; \{P_{i+1}, ..., P_{i+j}\}), \]
where $Diff^e(S_g, P_1, P_2, ..., P_i; \{P_{i+1}, ..., P_{i+j}\}) = \{\text{the group of orientation preserving diffeomorphisms of } S_g \text{ which fix the points } P_1, ..., P_i \text{ individually and permute the points } P_{i+1}, ..., P_{i+j}\}$.
Now let $\Gamma^i_g$ be the punctured mapping class group. The computation again relies on the result that $\Gamma^i_g$ is $p$-periodic for $i \geq 1$, so that by Theorem 0.1.4,

$$\hat{H}^q(\Gamma^i_g, \mathbb{Z})_{(p)} \to \prod_{\mathbb{Z}/p \in \Omega} \hat{H}^q(N(\mathbb{Z}/p), \mathbb{Z})_{(p)}$$

is an isomorphism, where $\Omega$ is a set of representatives for the conjugacy classes of subgroups of order $p$ in $\Gamma^i_g$ and $N(\mathbb{Z}/p)$ stands for the normalizer of $\mathbb{Z}/p$ in $\Gamma^i_g$.

First, we need to consider if $\mathbb{Z}/p$ is in $\Gamma^i_g$. By Propositions 3.2.1 and 3.2.2, we can determine this by considering non-negative integer solutions $(h, t)$ of the Riemann Hurwitz equation with $t \neq 1$ and $t \geq i$. As we know from Chapter 4, the value of $t$ corresponds to the number of fixed points of the $\mathbb{Z}/p$ action on $S_g$, and the number of fixed points depends only upon the conjugacy class of $\mathbb{Z}/p$. Therefore, different values of $t$ correspond to different conjugacy classes. We need to consider each of the solutions $(h, t)$ individually. (Note that in Chapter 8, we assumed $p > n + 2$. In that case there is only one solution: $(h, t) = (0, n + 2)$.)

Second, for fixed $(h, t)$, we can get results similar to Lemmas 8.1.2 through 8.1.6; therefore, we can obtain an analogous version of Theorem 8.1.1, namely that there is a group extension $0 \to \Gamma^t_h \to N(\mathbb{Z}/p)/\mathbb{Z}/p \to A \to 0$, where $A$ is a subgroup of $\sum_{t-i}$ which is determined by the fixed point data of a generator $\alpha$ of $\mathbb{Z}/p$. ($\delta_1(\alpha) = \langle 1, \beta_2, ..., \beta_i| \beta_{i+1}, ..., \beta_t >.$) In order to find $H^q(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z})_{(p)}$ or $H^q(N(\mathbb{Z}/p)/\mathbb{Z}/p, \mathbb{Z}/p)$, we need to know $H^q(\Gamma^t_h, \mathbb{Z})_{(p)}$ and the action of $A$ on it. Note that in Chapter 8 what we got was the group extension $0 \to \Gamma_0^{n+2} \to N(\mathbb{Z}/p)/\mathbb{Z}/p \to A \to 0$, where $A$ is a subgroup of $\sum_{n+2-i}$. By Birman's notation, $\Gamma_0^{n+2} \cong K_{n+2}$. $H^q(K_{n+2}, \mathbb{Z})_{(p)}$ and the action of $\sum_{n+2-i}$ on $H^q(K_{n+2}, \mathbb{Z})_{(p)}$ are well known. Here
the situation is more complicated. Let us consider the simplest case, in which \( t = i \).
By analogy with Theorem 8.1.1, \( H^q(N(Z/p)/Z/p, Z)(p) \cong H^q(\Gamma^t_h, Z)(p) \). Even though
\( H^q(\Gamma^t_h, Z)(p) \) should be easier to calculate than \( H^q(\Gamma^g_h, Z)(p) \) (because \( h < g \)), in gen-
eral, we still cannot get \( H^q(\Gamma^t_h, Z)(p) \). We have calculated \( \hat{H}^q(\Gamma^t_h, Z)(p) \) for some special
values of \( t \) and \( h \). However, \( H^* \) and \( \hat{H}^* \) coincide only in high dimension; we don’t
know much about \( H^* \) itself.

Third, the short exact sequence \( 0 \rightarrow Z/p \rightarrow N(Z/p) \rightarrow N(Z/p)/Z/p \rightarrow 0 \) may
not split in general. This leads to another difficulty in calculating the \( p \)-torsion of the
Farrell cohomology of \( \Gamma^i_g \). (We have already noticed it in Chapter 8.) For general \( \Gamma^i_g \),
it seems impossible to calculate the \( p \)-torsion of the Farrell cohomology; therefore,
knowing that the period is 2 is very useful.

A.2 The \( p \)-torsion of the Farrell cohomology of \( \Gamma^i_{n(p-1)/2} \) for

\[ i = n + 1 \text{ or } i = n + 2 \]

By Theorem 8.1.2, we know that \( H^q(N(Z/p)/Z/p, Z)(p) \cong H^q(K_{n+2}, Z)(p) \) for
\( Z/p < \Gamma^i_{n(p-1)/2} \) and \( i = n + 1 \) or \( i = n + 2 \). Cohen calculated \( H^q(K_{n+2}, Z)(p) \) com-
pletely; this was cited in Lemma 8.1.7. Now if the short exact sequence \( 0 \rightarrow Z/p \rightarrow N(Z/p) \rightarrow N(Z/p)/Z/p \rightarrow 0 \) splits, then the \( p \)-torsion of the Farrell cohomology of
\( \Gamma^i_{n(p-1)/2} \) can be obtained by applying Künneth’s Theorem for \( i = n + 1 \) or \( i = n + 2 \).
However, the short exact sequence generally does not split. In fact, Adem gives an
example of this: there is a short exact sequence \( 0 \rightarrow N \rightarrow G_n \rightarrow U_{2n} \rightarrow 0 \), where
\( N \cong Z/p, U_{2n} \) is a torsion free group and \( G_n \) has period 2, such that \( \hat{H}^m(G_n, Z)(p) \)
contains $\mathbb{Z}/p^2$, that is, the short exact sequence does not split. Mapping class groups are special, but we do not know if their special properties cause the short exact sequence to split.

A.3 The period of $\Gamma_{g}^{0,i}$

We know from Chapter 6 that there is a short exact sequence $0 \to \Gamma_{g}^{i} \to \Gamma_{g}^{0,i} \to \sum_{i} \to 0$. Since $\Gamma_{g}^{i}$ has period 2, the natural question is to ask: Is $\Gamma_{g}^{0,i}$ $(p)$-periodic, if it is so, what is the $(p)$-period? It is well known that $\sum_n$ is not periodic in general. Hence, there is reason to believe $\Gamma_{g}^{0,i}$ is not periodic for general values of $g$ and $i$. Then, is it possible to give sufficient (or necessary) conditions for the periodicity? We are interested in making further investigation.
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