CONTROLLER DESIGN FOR STABILITY AND Rollover Prevention
OF Multi-Body Ground Vehicles
With Uncertain Dynamics and Faults

Dissertation

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By

Hsun-Hsuan Huang, B.E., M.S.
Graduate Program in Aeronautical and Astronautical Engineering

The Ohio State University
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Dissertation Committee:
Rama K. Yedavalli, Advisor
Dennis Guenther
Jenping Chen
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ABSTRACT

Rollover prevention is a fundamental and significant issue for vehicle safety research. Passenger and commercial vehicles with a relatively high center of gravity are especially prone to rollover. Rollover is a threat especially for military vehicles, which operate in severe operational environments and maneuvers. However, many rollover situations cannot be prevented by driver actions alone, even when they are correctly warned. Additional assistance from active anti-rollover control systems can mitigate the deficiency in human capability. Furthermore, rollover events are subject to various perturbations in the rollover parameters (e.g., speed and road adhesion coefficient) and external disturbances (such as adverse weather and terrain conditions). Also vehicles have limited mobility under vehicle component failures resulting from fatigue or field conditions. Hence, the control system has to be fault-tolerant in order to enhance rollover prevention. Thus, with rollover prevention of military multi-body ground vehicles as the main objective of this research, in this dissertation, we first propose a novel control system analysis and design technique by extending the popular Linear Quadratic Regulator (LQR) control design method specializing it for the ‘control coupled output regulation’ problem. Specifically, in this rollover prevention problem, a ‘unified rollover index’ is proposed, which captures both the roll dynamics and lateral dynamics, explicitly
into the optimization procedure of the LQR framework, which results in a performance index with a coupled term in state and control variables. This LQR design with control coupled output regulation outperforms LQR design with state regulation only, because the cross coupling term helps to prudently allocate the weightings on states and control with the overall performance output minimization as the primary objective rather than individual state regulation. Thus, the proposed rollover prevention technique effectively incorporates the physical nature of the vehicle dynamics into the problem formulation resulting in significantly improved performance. The proposed control design technique is novel and beneficial to the ground vehicle control designers because through this technique, it is shown that the coupling in the vehicle dynamic states and control variables is taken advantage of to improve rollover prevention. In addition, the proposed technique allows us to compare different controller configurations and select the most efficient controller structure in terms of both control effort as well as cost. It is shown that because of the inherent coupling the system has, sometimes it is possible that a well designed single controller (actuator) can result in better performance than multiple controllers (actuators) with improper design. The proposed methodology is illustrated with two applications in the vehicle dynamics area. In the first application, an active steering control system is designed which clearly shows the improved rollover prevention capability of the proposed design compared to the existing designs in the literature. The second application considers a more complicated tractor semi-trailer vehicle and shows how a single active anti-roll bar system at the trailer unit gives better performance than multiple-axle actuators at tractor and trailer together with the double lane change maneuver as the external disturbance. Next the issue of robust control design
to handle uncertainties in the vehicle dynamics parameters as well as component faults. Based on the theory of ‘Linear interval parameter matrix families’, a single robust full state feedback control gain is designed by a convex combination of the control gains designed for finite points (vertices) of the uncertain parameter space. The proposed robust controller design is applied to the multi-body ground vehicle control with uncertainty in the forward speed of the vehicle and the road adhesion coefficient taken into consideration. The results clearly show the efficacy of the proposed robust controller under the assumed perturbations. Thus the proposed techniques in this dissertation help in not only preventing rollover of multi-body ground vehicles with controllers of reduced control effort (which in turn translates to considerable actuator and power savings) but also guarantee the stability and performance for vehicles with uncertain dynamics and faults.
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VITA

2000……………………………………..B. Engineering Aerospace Engineering,
Tamkang University, Tamsui, Taiwan

2004……………………………………..M.S. Aerospace Engineering, The
Pennsylvania State University

2006 to 2008…………………………..Graduate Research Assistant, Department of
Aerospace Engineering, The Ohio State
University

FIELDS OF STUDY

Major Field: Aeronautical and Astronautical Engineering
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CHAPTER 1: INTRODUCTION

1.1 Background and Motivation

Rollover prevention is a fundamental and significant issue for vehicle dynamics and has been a topic of considerable research for a long time [1-5, 14-16, 18-37]. There are two distinct types of vehicle rollover: tripped and un-tripped [20, 21]. A tripped rollover commonly occurs when a vehicle skids and digs its tires into soft soil or hits a tripping mechanism such as a curb or guardrail with a sufficiently large lateral velocity. Maneuver-induced un-tripped rollover can occur during typical driving situations and poses a real threat for the vehicles with an elevated center of gravity. Examples are excessive speed during cornering, obstacle avoidance and severe lane change maneuvers, where rollover occurs as a direct result of the lateral wheel forces induced during these maneuvers [20-22]. Furthermore, vehicle rollover can also occur during external disturbances like side-wind. Thus, passenger vehicles with a high center of gravity such as light trucks (vans, pickups, and SUVs (Sport Utility Vehicles)) are more prone to rollover accidents. Moreover, the heavy commercial vehicles with narrow track width are often involved in rollover accidents. Also, rollover prevention is of high importance for military vehicles, which operate in severe operational environments and maneuvers. Rollover prevention is a critical safety issue but there is still no safety performance standard available. In addition, there are no well-recognized rollover protection standards.
or design guidelines.

Typically, a driver does not have any indication before a rollover happens and many rollover situations cannot be prevented by driver actions alone, even when they are correctly warned. Additional assistance from active anti-rollover control can mitigate the deficiency in human capability. Hence, rollover prevention systems is classified into two stages: detection of the possibility of a rollover, and development of a mitigation control algorithm. Thus, the research on rollover prevention systems has mainly focused on two areas: rollover detection systems and anti-rollover control systems. In this dissertation, the emphasis is on developing a control algorithm for anti-rollover control systems.

1.2 Literature Survey for Rollover Detection Systems

In detection systems, the concept of a rollover index is used to determine the threshold for rollover. Based on the various signals such as vehicle roll angle, roll acceleration and lateral acceleration, a rollover detection system inputs this data into an algorithm which determines whether the rollover index exceeds the threshold or not [23]. Hence, a roll state estimator needs to be established to detect the danger of rollover [6-13]. Various rollover thresholds are derived from different factors which influence rollover such as the position of a vehicle’s center of gravity (CG), the energy of rollover, and vertical tire forces [5].

Rollover thresholds is divided to two categories: Steady-State Rollover Threshold (SSRT) and Dynamic Rollover Threshold (DRT) [5, 8, 12]. In the early study on rollover detection system, the concept of an SSRT is used. Furthermore, Static Stability Factor (SSF) equals SSRT as s first order approximation through the simplifying assumption that a vehicle behaves as a rigid body [7, 38]. SSF is the least conservative estimation among
applied quasi-static models. Since the vehicle is a dynamical system, DRT is a more appropriate index to analyze. The DRT is obtained by analyzing from different points of view. One DRT is derived from the energy point of view [5, 8, 16]. Another DRT, which is called Dynamic Stability Index (DSI), is derived based on the lateral CG offset point of view. Yet, another DRT is Lateral Transfer Ratio (LTR), which is a wheel lift threshold applied for detection of rollover [5, 6]. Furthermore, a Predictive Lateral Transfer Ratio (PLTR) is developed by incorporating the predictive influence of the driver’s steering input [13]. In general, rollover is related to lateral acceleration. Lateral acceleration needed to produce rollover is a function of the length of time it is applied. Thus, a Time-To-Rollover (TTR) Metric, which means Time-To-Wheel-Lift-Off Threshold, is defined to estimate the time until rollover occurs [9, 10]. In this dissertation, we derive a ‘unified rollover index’ using the concept of LTR by taking the torque balance for an unsprung mass about the zero-level center. Note that once wheel lift occurs, anti-rollover control is very difficult. Therefore, the concept of LTR is selected for this research.

1.3 Literature Survey for Anti-Rollover Control Systems

After the risk of rollover is detected, effective anti-rollover control systems are crucial to prevent vehicles from rollover or help vehicles to recover from rollover. In recent years with the development of advanced control technology and cost reduction of electronic and control equipment, active control has been widely used in the automotive industry in the design of anti-rollover control systems [16]. In addition to prevent the rollover, an active rollover control is used to improve both ride comfort and ride handling. There are four major active rollover control applications based on the actuation schemes: (a) Active anti-roll-bar systems; (b) Active suspension systems (c) Anti-roll
braking systems; and (d) Active steering systems [39]. Active anti-roll-bar and active suspension systems *directly control the vehicle roll motion* and the anti-roll braking systems and active steering systems *reduce vehicle oversteer and control vehicle yaw moment*. In addition, combinations of these different techniques are also considered [5, 22, 23, 29, 33, 34].

### 1.3.1 Direct Vehicle Roll Motion Control

In this section, active anti-roll-bar systems and active suspension systems, which are used to prevent vehicles’ rollover by controlling the roll motion directly, are discussed. An active anti-roll-bar hydraulically determines the variation of the equivalent stiffness of the anti-roll-bars. The vehicle load distribution is influenced by the roll-bar stiffness distribution such that the roll angle and roll moment is improved by an active anti-roll-bar. The use of an anti-roll-bar system to improve vehicle roll stability and reduce the rollover has been proposed and developed, especially for heavy road vehicles [17, 18, 25-27, 30, 31]. Active suspension systems use electrohydraulic equipment to generate controlled vertical forces to react to rollover moments and are used to gain improvements in both roll and ride performance [5, 15, 28]. In this dissertation, a 9-Degrees Of Freedom (DOF) tractor-semitrailer with active anti-roll bar is considered to represent a Multiple-Input Multiple-Output (MIMO) system. Active anti-roll-bar system is a common and achievable controller for heavy vehicles.

### 1.3.2 Vehicle Yaw Moment Control

In this section, anti-roll braking systems and active steering systems, which are used to prevent vehicles’ rollover by controlling the yaw motion, are presented. The yaw moment is generated through braking and steering and yaw moment control can
effectively reduce roll. Anti-rollover braking system is able to reduce rollover risk due to an effect. The effect is the coupling between roll, lateral, and yaw dynamics. Both anti-rollover braking system and active steering system essentially control yaw moment to reduce rollover risk indirectly due to this coupling. In addition, since roll is directly caused by the lateral acceleration of the center of mass, a decreasing lateral acceleration by anti-rollover braking control can also reduce the roll angle. An anti-rollover braking system controls the front brakes to reduce the cornering capability of the front tires, which causes the vehicle to turn less sharply and reduces its speed to prevent the rollover [1-3, 14, 16, 24, 32, 36]. Active steering systems control the steering input directly to reduce the rollover risk by reducing or reversing the steering angle to reduce or reverse the unstable roll maneuver [4, 20, 35]. In this dissertation, a 3-DOF bicycle mode with active steering control is considered to represent a Single-Input Single-Output (SISO) system. The steering input significantly influences lateral vehicle dynamics, and an excessive steering command may result in unstable vehicle motion, especially for military vehicles, which operate in severe operational environments and maneuvers. Thus, active steering is used as a control variable which is essential to prevent rollover, especially for military vehicles in emergency situations.

1.4 LQR Design in Vehicle System Dynamics

In this section, the Linear-Quadratic Regulator (LQR) method in vehicle system dynamics is discussed since an analysis and extension to LQR design with control coupled output regulation is proposed and applied to vehicle dynamics in this dissertation. A very popular method for control design of linear dynamic systems is the
LQR method [45, 46, 47]. For a considered linear dynamic system, the state space representation of the system is

\[ \dot{x} = Ax + Bu \quad x(t_0) \text{ given} \]

\[ y = Cx + Du \]

where \( A \) is \( n \) by \( n \) matrix; \( B \) is \( n \) by \( m \) matrix; \( C \) is \( k \) by \( n \) matrix; \( D \) is \( k \) by \( m \) matrix; \( x \) is \( n \) by 1 vector; \( u \) is \( m \) by 1 vector; \( y \) is \( k \) by 1 vector.

As is well-known in this method, typically the outputs (controlled variables) and the control variables are introduced into a quadratic performance index with respective weighting matrices \( Q_s \) and \( R_s \):

\[ J = \int_{t_0}^{\infty} \left[ x^T(\tau)Q_s x(\tau) + u^T(\tau)R_s u(\tau) \right] d\tau = J_x + J_u \]

where \( Q_s = C^T \bar{Q} C \) and \( R_s = D^T \bar{Q} D \); \( J_x \) is a state regulation cost, and \( J_u \) is a control effort. The weighting matrices \( Q_s \) is a symmetric positive semi-definite and \( R_s \) is a symmetric positive-definite matrix. We label this as the standard LQR problem. The standard LQR problem with standard performance index is for the state regulation problems. In general, it is a trial-and-error procedure to assign the weighting matrices \( Q \) and \( R \) in LQR design. A more generalized version of LQR is the LQR design for the control coupled output regulation problems in which there is a cross coupling term between the state and control variables with a weighting matrix \( N \) in the quadratic performance index:

\[ J = \int_{t_0}^{\infty} \left[ x^T(\tau)Qx(\tau) + 2x^T(\tau)Nu(\tau) + u^T(\tau)Ru(\tau) \right] d\tau = J_x + J_c + J_u \]
where \( Q = C^T \Phi C; N = C^T \Phi D; R = D^T \Phi D + \Phi; J_x \) is a state regulation cost, \( J_c \) is a coupling effect, and \( J_u \) is a control effort. The weighting matrix \( Q \) is a symmetric positive semi-definite and \( R \) is a symmetric positive-definite matrix. The major difference between standard LQR design and proposed LQR design for control coupled output regulation problems is that a cross coupling term appears in the proposed LQR design. Clearly, determining the weighting matrix on the cross coupling term is even more difficult without any guidelines. In this dissertation, a novel control algorithm is developed using the LQR framework with control coupled output regulation in which the rollover index is directly incorporated into the quadratic performance index. Thus, the ambiguity and labor involved in assigning the weights in LQR design is reduced.

Using rollover index in LQR framework for vehicle dynamic systems was mentioned in [15, 26-28, 30, 31]. However, the LQR design for the state regulation problem was applied to vehicle dynamics without the cross coupling term [15, 26, 30, 31]. Although, the influence of the cross coupling term between the states and control variables is considered in [27, 28], no efforts were made in them toward analyzing this influence. Moreover, no systematic methodology is given in determining the weighting matrices and designing an effective controller. In this dissertation, the extremely important role of the cross coupling term is thoroughly investigated and a very systematic methodology for designing an LQR controller for control coupled output regulation problems is provided.
1.5 Robust Control for Stability and Rollover Prevention

In this section, the issue of robust control for stability and rollover prevention is introduced. In the real world, because of variations in the rollover parameters (i.e., vehicle speed, angle, etc.), the control algorithm has to be robust to accommodate these variations. In addition, vehicles have limited mobility under vehicle component failures resulting from fatigue or field conditions, the control structure has to be fault-tolerant in order to enhance rollover prevention under the presence of faults. There is considerable literature for the design of robust controllers for stability and rollover prevention to handle uncertainties and component faults [5, 20, 21, 22, 29, 33-35, 36]. In [5], the robust active suspension is designed based on fuzzy logic. In [20, 21, 35, 36], the robust rollover prevention controllers are derived from the Linear Matrix Inequality (LMI) technique. In [29, 33, 34], the Linear Parameter Varying (LPV) method is used to design the controllers, which are robust to variations in vehicle speed. Rollover is avoided for a wide input range by integrating the active braking control but the rollover stability may be reduced by the active braking control. In this dissertation, a robust controller is designed based on the theory of linear interval parameter matrix families using the concept of convexity [37].

1.6 Contributions of this Dissertation

In this section, we summarize the contributions of this dissertation.

- We propose a novel analysis and design extension to the popular LQR method with control coupled output regulation. In the current literature, it is known that small (zero) output regulation requires large (theoretically infinite) control effort. In the proposed methodology, it is shown that the quadratic performance index $J$ is made
zero with finite control effort if coupling effect is negative. The result shows that the LQR design with control coupled output regulation outperforms LQR design with state regulation only, because the cross coupling term helps to prudently allocate the weights on states and controls with the overall performance output minimization as the primary objective rather than individual state regulation. The analysis shows that the cross coupling term makes a contribution to regulate the performance output. The criterion for designing an effective controller and a procedure for comparing the efficiency of different controller configurations are proposed.

- In order to apply the LQR design with control coupled output regulation to the rollover prevention problem, a ‘unified rollover index’ which captures both the roll dynamics and lateral dynamics explicitly into the optimization procedure. Then, a new quadratic performance index, in which we call ‘rollover performance index’, is derived.

- A 3-DOF bicycle model with active steering control as the control variable is considered to represent a Single-Input Single-Output (SISO) system. An active steering control system is designed using the proposed LQR extension method, which clearly shows the improved rollover prevention capability of the proposed design compared to the designs without considering the coupling effect. The result shows that the coupling in the vehicle dynamic states and control variables can improve roll over prevention.

- A more complicated 9-DOF tractor semi-trailer model with active anti-roll bar is considered to represent a Multiple-Input Multiple-Output (MIMO) system. The proposed technique allows us to compare different controller configurations and
select the most efficient controller structure in terms of both control effort as well as cost. The simulation, where a double lane change maneuver as external disturbance, shows a single active anti-roll-bar system at trailer unit gives better performance than multiple-axle actuators at tractor and trailer together. That is, because of the inherent coupling the system has, sometimes it is possible that a well designed single controller (actuator) can result in better performance than multiple controllers (actuators) with improper design.

- Next, the issue of uncertainty in vehicle parameters is addressed. Based on the theory of ‘Linear interval parameter matrix families’, a single robust full state feedback control gain is designed by a convex combination of the control gains designed for finite points (vertices) of the uncertain parameter space. The proposed robust controller design is applied to the multi-body ground vehicle control with uncertainty in the forward speed of the vehicle and the road adhesion coefficient taken into consideration. The results clearly show the efficacy of the proposed robust controller under the assumed perturbations.

Thus the proposed techniques in this dissertation help in not only preventing rollover of multi-body ground vehicles with controllers of reduced control effort (which in turn translates to considerable actuator and power savings) but also guarantee the stability and performance for vehicles with uncertain dynamics and faults. In addition, the proposed methodologies of this dissertation are sufficiently generic that they is applied to control problems in various other fields such as aerospace, mechanical, electrical, and other systems.
1.7 Dissertation Organization

This dissertation is organized as follows: in chapter 2, a theory for design of an effective controller using LQR design with control coupled output regulation is presented at the system level. In chapter 3, modeling of vehicle dynamics for rollover prevention control is addressed. In chapter 4, we specialize this technique to vehicle dynamic application in which we explicitly express the rollover index as a quadratic performance index in the LQR formulation. The new performance index is called ‘rollover performance index’. After the rollover performance index is introduced, the proposed LQR design with control coupled output regulation studied in chapter 2 is applied. In chapter 5, the performance of an LQR design based on a rollover performance index and a standard LQR design are compared. The comparison is based on a single-unit, multi-body ground vehicle with active steering control. In chapter 6, the proposed rollover performance index approach is applied to a multiple-unit, multi-body ground vehicle with active anti-roll-bar control. In chapter 7, a robust controller design incorporating the rollover performance index is proposed to handle the uncertain dynamics and component faults. Chapter 8 offers some concluding remarks and suggests areas for further research.
CHAPTER 2: ANALYSIS AND DESIGN EXTENSION TO LQR METHOD FOR
CONTROL COUPLED OUTPUT REGULATION
FOR A LINEAR SYSTEM

2.1 Introduction

This chapter presents a novel analysis and design extension to the popular LQR method for a linear system with control coupled output regulation as its performance objective. A very popular method for control design of linear dynamic systems is the LQR method [45, 46, 47]. In the LQR design, the stability is always satisfied because of the optimality principle.

In the state regulation problem in a linear system, the system dynamics is represented by

$$\dot{x} = Ax + Bu \quad x(t_0) \text{ given}$$

(2.1)

$$y = Cx$$

(2.2)

where $A$ is $n$ by $n$ matrix; $B$ is $n$ by $m$ matrix; $C$ is $k$ by $n$ matrix; $x$ is $n$ by 1 vector; $u$ is $m$ by 1 vector; $y$ is $k$ by 1 vector. The performance index in LQR design for the state regulation problem is

$$J = \int_{t_0}^{\infty} \left[ y^T(\tau)Qy(\tau) + u^T(\tau)Ru(\tau) \right] d\tau = \int_{t_0}^{\infty} \left[ x^T(\tau)Q_x x(\tau) + u^T(\tau)R_x u(\tau) \right] d\tau$$

(2.3)

where $Q_x = C^TQCC$ and $R_x = D^TQD$. The weighting matrix $Q_x$ is a symmetric
positive semi-definite and the weighting matrix $R_s$ is a symmetric positive-definite matrix.

In the performance index for the LQR design with state regulation, the definition of state regulation cost ($J_x$), and control effort ($J_u$) are described as

$$J_x = \int_{t_0}^{\infty} x^T(t)Q_s x(t) dt; J_u = \int_{t_0}^{\infty} u^T(t)R_s u(t) dt;$$

Hence, the performance index is represented as $J = J_x + J_u$. Note that $J_x > 0$ and $J_u > 0$.

In the control coupled output regulation problem in a linear system, the system dynamics is represented by

$$\dot{x} = Ax + Bu \quad x(t_0) \text{ given} \quad \text{(2.4)}$$

$$y = Cx + Du \quad \text{(2.5)}$$

where $A$ is $n$ by $n$ matrix; $B$ is $n$ by $m$ matrix; $C$ is $k$ by $n$ matrix; $D$ is $k$ by $m$ matrix; $x$ is $n$ by 1 vector; $u$ is $m$ by 1 vector; $y$ is $k$ by 1 vector. The performance index in LQR design for the control output regulation problem is

$$J = \int_{t_0}^{\infty} \left[y^T(\tau)\bar{Q}y(\tau) + u^T(\tau)\bar{R}u(\tau)\right] d\tau = \int_{t_0}^{\infty} \left[(Cx(\tau) + Du(\tau))^T Q(Cx(\tau) + Du(\tau)) + u^T(\tau)\bar{R}u(\tau)\right] d\tau$$

$$= \int_{t_0}^{\infty} \left[x^T(\tau)(C^T \bar{Q}C)x(\tau) + 2x^T(\tau)C^T \bar{Q}Du(\tau) + u^T(\tau)(D^T \bar{Q}D + \bar{R})u(\tau)\right] d\tau$$

$$= \int_{t_0}^{\infty} \left[x^T(\tau)Qx(\tau) + 2x^T(\tau)Nu(\tau) + u^T(\tau)Ru(\tau)\right] d\tau \quad \text{(2.6)}$$
where $Q = C^TQC; N = C^TQD; R = D^TQD + \bar{R}$. The weighting matrix $Q$ is a symmetric positive semi-definite and the weighting matrix $R$ is a symmetric positive-definite matrix.

In the performance index for the LQR design with control coupled output regulation, the definition of state regulation cost ($J_x$), control effort ($J_u$) and coupling effect ($J_c$) are described as

$$J_x = \int_{t_0}^{\infty} x^T(\tau)Qx(\tau)d\tau; J_u = \int_{t_0}^{\infty} u^T(\tau)Ru(\tau)d\tau;$$

$$J_c = \int_{t_0}^{\infty} 2x^T(\tau)Nu(\tau)d\tau = \int_{t_0}^{\infty} x^T(\tau)Nu(\tau) + u(\tau)^T N^T x(\tau) d\tau$$

Hence, the performance index is represented as $J = J_x + J_u + J_c$. Note that $J_x > 0$ and $J_u > 0$.

There is a major difference in the performance index from LQR design with control coupled output regulation in contrast to the performance index from LQR design with state regulation. The difference is that a cross coupling term between the state and control variables with a weighting matrix $N$ appears in the performance index making it non-standard. Therefore, in the performance index for control coupled output regulation problems, a cross coupling term between the state and control variables with a weighting matrix, which results in coupling effect $J_c$, is generated by relaxing some constraints on a state weighting matrix and control weighting matrix.
It is well-known that in the LQR design for state regulation problems, if an Open-Loop (OL) system is unstable, $J$ is made zero only with an infinite control effort. However, there is one major difference between the LQR design with state regulation and control coupled output regulation. That is for an unstable OL system, it is possible for $J$ to approach to zero with finite control effort if $J_c$ is negative. Furthermore, if the OL system is stable, the LQR design with the control coupled output regulation spends less control effort to make $J$ approach to zero compared to the LQR design with the state regulation because of the negative coupling effect $J_c$. If $J_c$ is negative, the coupling effect assists the controller to regulate the output such that total performance index $J$ is significantly reduced.

2.2 Controller Design for Zero Output for the Case of Square and Invertible Matrix $D$

In LQR design with control coupled output regulation, for an unstable OL system, it is possible for $J$ to approach to zero with a finite control effort if the $J_c$ is negative. Before the LQR design for control coupled output regulation problems is considered, an ideal and interesting case in which the total performance index $J$ is equal to zero is discussed in this section. In LQR problems, the purpose of the total performance index $J$ is to regulate the performance output using an optimal control. Then, for a square and invertible matrix $D$ case, it is easy to observe that when $u(\tau) = -D^{-1}Cx(\tau)$ is applied, the performance output equation is equal to a zero vector, because $y = Cx + Du = (C - DD^{-1}C)x = 0$. 

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The matrix $D$ in equation (2.2) is required to be square matrix (including a scalar) so MIMO systems with $k=m$ including SISO systems, are considered in this section. In addition, the invertible matrix $D$ is equivalent to a reasonable assumption arising from practical considerations, which the feedforward controls are linear independent. The advantage of the control $u(\tau) = D^{-1}C\dot{x}(\tau)$ is to force a performance output to become a zero vector, where $y = (C-DD^{-1}C)x = 0$, but the disadvantage is that stability of the Close-Loop (CL) system, which is $\dot{x} = (A-BD^{-1}C)x$, is not guaranteed because this controller is not obtained from optimization procedure.

2.2.1 Controller Design for the Zero Performance Output for the Case of Square and Invertible matrix $D$

In this section, the properties of the controller $u = -(D^{-1}C)x(\tau)$, which generates a zero output, are discussed. The performance index $\overline{J}_y = \int_{t_0}^{\infty} [y^T(\tau)Qy(\tau)] \, d\tau$ for the controller $u = -(D^{-1}C)x(\tau)$, which generates a zero performance output, is evaluated and is given by

$$\overline{J}_y = \int_{t_0}^{\infty} [y^T(\tau)Qy(\tau)] \, d\tau = \int_{t_0}^{\infty} \left[(C\dot{x}(\tau) + Du(\tau))^T \overline{Q}(Cx(\tau) + Du(\tau))\right] \, d\tau$$

$$= \int_{t_0}^{\infty} \left[x^T(\tau)(C^T\overline{Q}C)x(\tau) + 2x^T(\tau)C^T\overline{Q}Du(\tau) + u^T(\tau)(D^T\overline{Q}D)u(\tau)\right] \, d\tau \quad (2.7)$$

The state regulation cost ($\overline{J}_x$), the control effort ($\overline{J}_u$) and the coupling effect ($\overline{J}_c$) are
evaluated with  \( u = -\left(D^{-1}C\right)x(\tau) \) and they are given by:

\[
\overline{J}_x = \int_{t_0}^{\infty} x^T(\tau)\left(C^TQC\right)x(\tau) d\tau.
\]

\[
\overline{J}_u = \int_{t_0}^{\infty} u^T(\tau)(D^TQD)u(\tau) d\tau = \int_{t_0}^{\infty} \left[-\left(D^{-1}C\right)x(\tau)\right]^T D^TQD\left[-\left(D^{-1}C\right)x(\tau)\right] d\tau
\]

\[
= \int_{t_0}^{\infty} x^T(\tau)C^T \left(D^{-1}\right)^T D^TQDD^{-1}Cx(\tau) d\tau = \int_{t_0}^{\infty} x^T(\tau)\left(C^TQC\right)x(\tau) d\tau
\]

\[
\overline{J}_c = \int_{t_0}^{\infty} x^T(\tau)\left(2x^T(\tau)CQDu(\tau)\right)k(\tau) d\tau = -2\int_{t_0}^{\infty} x^T(\tau)\left(x^T(\tau)C^TQD(D^{-1}C)\right)k(\tau) d\tau
\]

\[
= -2\int_{t_0}^{\infty} x^T(\tau)\left(C^TQC\right)x(\tau) d\tau
\]

Note that  \( \overline{J}_x = \overline{J}_u \) and  \( \overline{J}_c = -2\overline{J}_x \). Thus, \( \overline{J}_y = \overline{J}_x + \overline{J}_u + \overline{J}_c = 0 \). That is, the special case where the performance output is equal to zero occurs when \( \overline{J}_x = \overline{J}_u \) and \( \overline{J}_c = -2\overline{J}_x \). \( \overline{J}_x = \overline{J}_u \) and \( \overline{J}_c = -2\overline{J}_x \) demonstrates that it is possible for a total performance index to approach zero with a finite control effort if \( \overline{J}_c \) is negative and the CL system  \( \dot{x} = (A - BD^{-1}C)x \) is stable.

### 2.3 Extension to the LQR Method for a Linear System with Control Coupled Output Regulation

In this section, the extension of zero output case to the LQR method for a linear system with control coupled output regulation is introduced. In the LQR design with
control coupled output regulation, the control effort is adjusted and all matrix D cases, which include square and non-square matrices, is considered. In addition, the stability is guaranteed by the procedure of the LQR design.

### 2.3.1 Solutions to Control Coupled Output Regulation Problems [45, 46, 47]

In this section, the solutions to control coupled output regulation problem is given. The Performance Index in LQR design for control coupled output regulation problem written in terms of the output equation \( y = Cx + Du \), which is denoted as \( J \), is expressed as a quadratic form in state \( x(\tau) \) and input \( u(\tau) \) shown in (2.6). That is,

\[
J = \int_{t_0}^{\infty} \left[ y^T(\tau)Qy(\tau) + u^T(\tau)Ru(\tau) \right] d\tau \\
= \int_{t_0}^{\infty} \left[ x^T(\tau)Qx(\tau) + 2x^T(\tau)Nu(\tau) + u^T(\tau)Ru(\tau) \right] d\tau \\
\text{(2.6)}
\]

where \( Q = C^TQC \); \( N = C^TQD \); and \( R = D^TQD + R \). The weighting matrix \( Q \) is a symmetric positive semi-definite and the weighting matrix \( R \) is a symmetric positive-definite matrix. Making the definition \( \bar{u}(\tau) = u(\tau) + R^{-1}N^T x(\tau) \) and the state equation (2.1) becomes equivalent to

\[
\dot{x} = \left( A - BR^{-1}N^T \right)x + B\bar{u} \\
\text{(2.8)}
\]

and the performance index in equation (2.6) is modified to
The optimal control law for $u$ is given by

$$u^* = -R^{-1}BT P x(\tau)$$

where $P$ is the solution of the following Algebraic Riccati Equation (ARE)

$$P(A - BR^{-1}N^T) + (A^T - NR^{-1}B^T)P - PB^{-1}B^T P + Q - NR^{-1}N^T = 0$$

Therefore, the optimal control law for $u$ associated with equation (2.1) and equation (2.6) is given by

$$u^* = -R^{-1}(B^T P + N^T) x(\tau) = -K x(\tau)$$

$$K = R^{-1}(B^T P + N^T)$$

In addition, the closed-loop system is asymptotically stable if

(a) The pair $({A, B})$ is stabilizable;

(b) $R = R^T > 0$ and $Q - NR^{-1}N^T \geq 0$; and

(c) The pair $\left(Q - NR^{-1}N^T, A - BR^{-1}N^T\right)$ has no unobservable mode on the imaginary axis.

Note that the weighting matrix for the cross term between the state and control variables $N$ is involved in the ARE in equation (2.10) and in the optimal controller gain $K$ in
2.3.2 Controller Design for Zero Performance Index ($P = 0$) for the case of Square and Invertible Matrix D

Before we discuss the properties of LQR design for control coupled regulation problems, which is for any matrix $D$ case, the special case, which is for a square and invertible matrix case, mentioned in section 2.2.1 which has a relationship with the solution to LQR design for control coupled output is discussed in this section. An ARE with zero forcing term is called homogeneous ARE [40]. In the LQR design for control output regulation problems, the forcing term in the ARE shown in equation (2.10) is $Q - NR^{-1}N^T$. If $Q - NR^{-1}N^T = 0$, the solution for the ARE in equation (2.10) is $P = 0$, which is given in the [40]. If matrix $D$ is invertible, $P = 0$ is a nontrivial solution such that the assumption for the LQR design for control output regulation problems, which is $Q - NR^{-1}N^T \geq 0$, has to be modified as $Q - NR^{-1}N^T \geq 0$ or $Q - NR^{-1}N^T = 0$ (i.e., $Q - NR^{-1}N^T$ is a zero for SISO systems and $Q - NR^{-1}N^T$ is a zero matrix for MIMO systems) in order to include the solution $P = 0$. The solution $P = 0$ (i.e., $Q - NR^{-1}N^T = 0$) is obtained when the user-specified matrix $\overline{R} = 0$. Furthermore, when $P = 0$, the controller is $u = -(D^{-1}C)x(\tau)$ and the performance index is equal to zero. However, this controller $u = -(D^{-1}C)x(\tau)$ exists when this controller can stabilized the original OL system. The stability of the CL system $\dot{x} = (A - BD^{-1}C)x$ has to be checked, because the assumption $Q - NR^{-1}N^T \geq 0$ which is used to guarantee the stable CL system is removed.
### 2.3.3 Coupling Effect

The coupling effect is the major difference between control coupled output regulation problems and state regulation problems. In this section, the coupling effect in the LQR design for a linear system with control coupled output regulation, in which it is derived from a cross coupling term between the state and control variables with a weighting matrix $N$, is investigated. In the LQR design with control coupled output regulation, the definition of state regulation cost ($J_x$), control regulation cost (control effort) ($J_u$) and coupling effort ($J_c$) are described as

$$
J_x = \int_{t_0}^{\infty} x^T(\tau)Qx(\tau)d\tau; \quad J_u = \int_{t_0}^{\infty} u^T(\tau)Ru(\tau)d\tau;
$$

$$
J_c = \int_{t_0}^{\infty} 2x^T(\tau)Nu(\tau)d\tau = \int_{t_0}^{\infty} x^T(\tau)Nu(\tau) + u(\tau)^T N^T x(\tau) d\tau
$$

Hence, the performance index is represented as $J = J_x + J_u + J_c$.

The controlled system using the controller in equation (2.11) to obtain the control effect is

$$
J_u = \int_{t_0}^{\infty} u^T(\tau)Ru(\tau)d\tau = \int_{t_0}^{\infty} x^T \left[ R^{-1} \left( B^T P + N^T \right) \right]^T R \left[ R^{-1} \left( B^T P + N^T \right) \right] x d\tau
$$

$$
= \int_{t_0}^{\infty} x^T \left[ (N + PB)R^{-1} \right] R \left[ R^{-1} \left( B^T P + N^T \right) \right] x d\tau
$$

$$
= \int_{t_0}^{\infty} x^T \left( NR^{-1}B^TP + NR^{-1}N^T + PBR^{-1}B^TP + PBR^{-1}N^T \right) x d\tau
$$

(2.13)
The controlled system using the controller in equation (2.11) and the coupling effect is

\[ J_c = \int_{t_0}^{\infty} \mathbf{r}^T(\tau)\mathbf{K}^T(\mathbf{r}(\tau) + \mathbf{u}(\tau) + \mathbf{x}(\tau)) \, d\tau = -\int_{t_0}^{\infty} \mathbf{r}(\tau)^T(\mathbf{K}^T\mathbf{K} + \mathbf{N}^T\mathbf{N})\mathbf{x}(\tau) \, d\tau \]

\[ = -\int_{t_0}^{\infty} \mathbf{r}(\tau)^T(\mathbf{K}^T\mathbf{R} + \mathbf{N}^T\mathbf{N})\mathbf{r}(\tau) \, d\tau = -\mathbf{r}_0^T\mathbf{K}\mathbf{r}_0 \]  

(2.14)

where \( \mathbf{P}_c \) is from the solution of the Lyapunov equation

\[ \mathbf{P}_c \mathbf{A}_{CL} + \mathbf{A}_{CL}^T \mathbf{P}_c + (\mathbf{K}^T \mathbf{K} + \mathbf{N}^T\mathbf{N}) \]

\[ = \mathbf{P}_c \mathbf{A}_{CL} + \mathbf{A}_{CL}^T \mathbf{P}_c + (\mathbf{K}^T \mathbf{K} + \mathbf{N}^T\mathbf{N}) = 0 \]

and \( \mathbf{A}_{CL} = \mathbf{A} - \mathbf{B} \mathbf{K} = \mathbf{A} - \mathbf{BR}^{-1}(\mathbf{B}^T \mathbf{P} + \mathbf{N}^T) \).

We know that \( J_x > 0 \) because \( \mathbf{Q} \succeq 0 \) and \( J_u > 0 \) because \( \mathbf{R} > 0 \). However, the definiteness of \( \mathbf{N}^{-1}\mathbf{B}^T \mathbf{P} + \mathbf{BR}^{-1}\mathbf{N}^T + 2\mathbf{NR}^{-1}\mathbf{N}^T \) is uncertain so \( J_c \) could be either positive or negative. The definiteness of the term \( \mathbf{N}^{-1}\mathbf{B}^T \mathbf{P} + \mathbf{BR}^{-1}\mathbf{N}^T + 2\mathbf{NR}^{-1}\mathbf{N}^T \) depends on the selection of the weighting matrix \( \mathbf{N} \).

### 2.3.4 A Non-Zero Performance Output Regulation Cost (\( J_y \)) in an LQR Design with Control Coupled Output Regulation

In this section, a performance output regulation cost \( J_y \) can approach zero but never be equal to zero in LQR design with the control coupled output regulation is demonstrated. As mentioned in section 2.2, the controller \( \mathbf{u} = -\mathbf{D}^{-1} \mathbf{C} \mathbf{x}(\tau) \) can force
the output to become zero. As mention in section 2.3.2, in LQR design with the control coupled output regulation, a performance index is equal to zero for a square and invertible matrix $D$ case when the user-specified matrix $R$ is equal to zero. However, $\bar{R} = 0$ yields $Q - NR^{-1}N^T = 0$ such that the assumption for applying the LQR design cannot be satisfied. That is, as long as LQR design is applied, $\bar{R} \neq 0$. Therefore, a performance output regulation cost $J_y$ can approach to zero but never be equal to zero in an LQR design for the control coupled output regulation problems.

2.4 Component-wise Cancellation

Based on the argument, in which it is possible for $J$ to approach to zero with finite control effort if $J_c$ is negative and the special case in section 2.2.1, the negative $J_c$ in the LQR design for control coupled output regulation problems possesses a significance in reducing the elements in performance output equation. In this section, a Single-Input Single Output system (SISO) system is considered to investigate the nature of negative $J_c$ and a concept of component-wise cancellation is proposed.

Note that the performance output equation shown in equation (2.5) is $y = Cx + Du$. For an SISO system, $y$, $Cx$ and $Du$ are scalars. From the performance output equation $y = Cx + Du$, it is easy to observe that a increasing (decreasing) $Cx$ has to associate with a decreasing (increasing) $Du$ in order to reduce the performance output $y$. That is, the sign of $Cx$ and $Du$ must be opposite to make $y$ approach to zero. Therefore, the coupling effect $J_c = 2 \int_{t_0}^{\infty} x^T(\tau)(C^T \bar{Q}D)u(\tau)d\tau$ with $\bar{Q} \geq 0$ is
expected to be negative for regulating the performance output. In this SISO case, it shows that negative coupling effect \( J_c \) make a favorable contribution to the LQR design for control coupled output regulation problems. Therefore, a concept of component-wise cancellation is developed and it results in a negative coupling effect \( J_c \). An effective controller for regulating the performance output is designed to have component-wise cancellation, in which \( J_c \) is negative.

2.5 Tradeoff Diagram of External Control Effort and Normalized Performance Output Regulation Cost for the LQR Design with Control Coupled Output Regulation when \( J_c < 0 \)

In the tradeoff diagram of the normalized external control effort and the state regulation cost for the LQR design with state regulation is introduced in [43]. In this section, the tradeoff diagram of the external control effort and the normalized performance output regulation cost in LQR design with control coupled output regulation is proposed.

By assuming \( \bar{Q} = \rho I \), the performance index in equation (2.3) for state regulation problems and the performance index in equation (2.6) for control coupled output regulation problems are rewritten as

\[
J = \rho \int_{t_0}^{\infty} y^T(\tau)y(\tau)d\tau + \int_{t_0}^{\infty} u^T(\tau)Ru(\tau)d\tau = \rho J_{yn} + J_{uex} \quad (2.15)
\]

where \( J_{yn} \) is normalized performance output regulation cost, \( J_{uex} \) is external control effort and \( \rho \) is a weighting scalar, which is a control variable. Increasing (decreasing)
the weighting scalar $\rho$ results in increasing (decreasing) the $J_{uex}$. Since negative $J_c$ is desired for design a controller with component-wise cancellation, it is a criterion to design an effective controller. After the range of weight scalar $\rho$ for negative $J_c$ is sought, the tradeoff diagram of external control effort and normalized performance output regulation cost for the LQR design with state regulation (without coupling effect) and the LQR design with control coupled output regulation (with coupling effect) is shown in Fig. 2.1. In Fig. 2.1, the trade-off curve for control coupled output regulation problems is always below state regulation problems if $J_c$ is negative. In addition, the ideal range of the weighting scalar $\rho$ for control coupled output regulation problems is located at an area where $J_{yn}$ approaches to a zero with a finite external control effort.

Fig. 2.1 The tradeoff diagram of the external control effort and the normalized performance output regulation cost if $J_c < 0$
2.6 Controller Selection Based on Component-wise Cancellation

In section 2.4, a concept of component-wise cancellation is introduced. In this section, a criterion and procedure to select an effective controller is proposed based on component-wise cancellation. If a system has better component-wise cancellation, the normalized performance output cost is reduced significantly. In other words, the elements of performance output equation is reduced by component-wise cancellation. This fact is illustrated in Fig. 2.2. In Fig. 2.2, when $J_{uex}$ is fixed for three systems, the $J_{yn}$ of system 1 is smaller than the $J_{yn}$ of system 2 and the $J_{yn}$ of system 2 is smaller than the $J_{yn}$ of system 3. Therefore, we can say the controller in system 1 is more efficient than the controller in system 2, and the controller in system 2 is more efficient than the controller in system 3.

![Fig. 2.2 A criterion to select an effective controller based on component-wise cancellation](image-url)
Therefore, a procedure to determine the efficiency of controllers for LQR design with control coupled output problems is

**Design Observation 2.1:** A considered state space system is given by

\[
\dot{x} = Ax + Bu, \quad y = Cx + Du.
\]

The performance index is defined as

\[
J = \rho \int_{t_0}^{\infty} y^T(\tau)y(\tau)d\tau + \int_{t_0}^{\infty} u^T(\tau)\overline{R}u(\tau)d\tau = \rho J_{yn} + J_{uex}.
\]

The normalized performance output regulation cost is defined as

\[
J_{yn} = \int_{t_0}^{\infty} x^T(\tau)(C^TC)x(\tau)d\tau + \int_{t_0}^{\infty} 2x^T(\tau)Cd(\tau)u(\tau)d\tau + \int_{t_0}^{\infty} u^T(\tau)(D^TD)u(\tau)d\tau.
\]

In LQR design for control coupled output regulation problems, the controller with better component-wise cancellation is more efficient. Therefore, a criterion and procedure to determine an effective controller is that a controller is more effective for regulating the performance output if and only if the smaller normalized performance regulation cost \(J_{yn}\) is generated compared to the other systems with a fixed \(J_{uex}\). The weighting scalar \(\rho\) and the weighting matrix \(\overline{R}\) are design variables to fix \(J_{uex}\) in LQR design with control coupled output regulation.

**2.7 Illustrative Example**

In LQR design for control coupled output regulation problems, the component-wise cancellation, which results in a negative coupling effect, plays an important role in selecting the effective controller. In this section, an example is considered to represent
that the result of the design observation proposed in section 2.6.

In order to demonstrate the LQR design for control coupled output problems, three different state space systems are considered. Matrix $A$ and matrix $B$ in equation (2.4) and matrix $C$ and matrix $D$ in equation (2.5) are given by

System 1:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -6 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, C = \begin{bmatrix} -1 & -2 \end{bmatrix}, D = 1.$$

System 2:

$$A = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \end{bmatrix}, D = 10.$$

System 3:

$$A = \begin{bmatrix} 4 & 0 \\ -1 & -10 \end{bmatrix}, B = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = 1.$$

The LQR design with control coupled output regulation is applied to these three systems. In the simulations, the initial condition $[0.1 \ 0.1]^T$ is assumed for all designs. To apply the procedure in section 2.6, the external control effort ($J_{u_{ex}}$) is fixed for three designs by varying the weighting scalar $\rho$ and the weighting matrix $\overline{R}$. The state weighting matrix $Q$, the control weighting matrix $R$, the weighting matrix for cross term between state and control variables $N$, the external control effort ($J_{u_{ex}}$), the normalized performance output regulation cost ($J_{yn}$), the coupling effort ($J_{c}$), the performance
index ($J$), controller gain ($K$), and norm of controller gain ($|K|$) for three designs are obtained:

Design 1:

\[
Q = \rho C^T C = 1 \cdot \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}; \quad N = \rho C^T D = 1 \cdot \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix};
\]

\[
R = \rho D^T D + \bar{R} = 1 \cdot 100 + 0.129 = 100.129;
\]

\[
J_{uex} = 0.0252; J_{yn} = 16.7184; J_c = -3.0714; J = 16.7436;
\]

\[
K = [27.6217 \quad -37.6428]; |K| = 46.6898.
\]

Design 2:

\[
Q = \rho C^T C = 10 \cdot \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}; \quad N = \rho C^T D = 10 \cdot \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -10 \\ -20 \end{bmatrix};
\]

\[
R = \rho D^T D + \bar{R} = 10 \cdot 1 + 0.1 = 10.1;
\]

\[
J_{uex} = 0.0252; J_{yn} = 0.0921; J_c = -2.6205; J = 0.9461;
\]

\[
K = [4.1326 \quad 1.1179]; |K| = 4.2811.
\]

Design 3:

\[
Q = \rho C^T C = 100 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 100 & 100 \\ 100 & 100 \end{bmatrix}; \quad N = \rho C^T D = 100 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix};
\]

\[
R = \rho D^T D + \bar{R} = 100 \cdot 1 + 4.44 = 104.44;
\]
\[ J_{uex} = 0.0252; J_{yn} = 6.8136 \times 10^{-5}; J_c = -1.0414; J = 0.0325; \]
\[
\mathbf{K} = [1.2062 \quad 0.8757]; |\mathbf{K}| = 1.4906.
\]

Based on the fixed \( J_{uex} \) for three designs, \( J_{yn} \) in design 3 is smaller than \( J_{yn} \) in design 2 and \( J_{yn} \) in design 2 is smaller than \( J_{yn} \) in design 1. Therefore, according to the criterion proposed in section 2.6, the controller in design 3 is better than the controller in design 2 and the controller in design 2 is better than the controller in design 1.

In the simulation results, design 3 can regulate the performance output most with less control input. Design 2 cannot regulate the output as much as design 3 and the control input in design 2 is larger than the control input in design 3. Design 1 cannot regulate the output as much as design 2 and the control input in design 1 is larger than the control input in design 2. Therefore, the controller in design 3 is better than the controller in design 2 and the controller in design 2 is better than the controller in design 1. The simulation results verify the criterion proposed in the section 2.6.

In design 3, the normalized performance output (\( J_{yn} \)) is almost zero and the normalized external control (\( J_{uex} \)) is very small. Therefore, it verifies that a performance index is able to approach zero with a small control input in the LQR design with control coupled output regulation. Also, the weighting scalar \( \rho \) in equation (2.15) is located at the ideal range for LQR design for control coupled output regulation problem when \( J_c \) is negative.
2.8 Summary for LQR Design with Control Coupled Output Regulation

This chapter presents an analysis and extension to the LQR design for the control coupled regulation problem. The LQR design for the control coupled regulation problem not only assures the stability but also satisfies the specific requirement of a performance. In LQR Design with control coupled output regulation, a cross coupling term between the state and control variables with a weighting matrix $N$, $2x^T(\tau)Nu(\tau)$, is able to be generated by relaxing some constraints on a state weighting matrix $Q$ and control weighting matrix $R$. Then, a coupling effect, which results from a cross coupling term, is generated in LQR design for control coupled output regulation problems.

We note that when the performance output is regulated by component-wise cancellation, it yields a negative coupling effect. Therefore, in LQR design for control coupled output regulation problems, a range of weighting scalar $\rho$ for negative $J_c$ is sought first. Then, determining a desired external control effort and a normalized performance output regulation cost by selecting an ideal weighting scalar $\rho$ in this range to obtain an effective controller. Based on component-wise cancellation, a criterion and procedure to determine an efficient controller in LQR design for control coupled output regulation problems are proposed. That is a controller is more effective for regulating the performance output if and only if the smaller normalized performance regulation cost $J_{yn}$ is generated compared to the other systems with a fixed $J_{uex}$. The LQR design with control coupled output regulation outperforms LQR design with state regulation when $J_c$ is negative.
3.1 Introduction

In this chapter, modeling of vehicle dynamics for rollover prevention control is addressed. For vehicle dynamics and control studies, lumped parameter models are usually employed and typically these models focus on either ride qualities or handling qualities of the vehicles. A ground vehicle with a suspension consists of two main parts: the sprung mass and the unsprung mass. The sprung mass is the mass of the body and other components supported by the suspension and the unsprung mass is the mass of suspension, wheels or tracks and other components directly connected to them, rather than supported by the suspension. The motion of a vehicle with the constraints of the road has 6 Degrees Of Freedom (DOF), classified as follows:

- Longitudinal translation (forward and backward motion)
- Lateral translation (side slip)
- Vertical translation (bounce or heave)
- Rotation about the longitudinal axis (roll)
- Rotation about the transverse axis (pitch)
- Rotation about the vertical axis (yaw)

Vehicle ride is essentially related to the vehicle vertical dynamics (bounce, pitch,
and roll) whereas handling is concerned with lateral dynamics (side slip, yaw, and roll). Note that roll is coupled both in ride and handling. Furthermore, it is well-known that there is a compromise between ride comfort and handling qualities. Ride models are typically composed of interconnected spring-mass-damper systems and are defined by a set of ordinary differential equations. The complexity of ride models is listed as

1. quarter car model (2 DOF) – two translational DOF;
2. half car model (2 DOF) – one translational DOF and one rotational DOF (e.g., bounce and pitch motions or bounce and roll motions);
3. half car model (4 DOF) – extension of the half car with additional 2 DOF by including tyre masses and elasticity;
4. full car model (multiple DOF) – bounce, roll and pitch motions are represented.

The models mentioned above are classical ride models. Higher order ride models is developed including further DOF.

Analogously to ride vehicle models, handling models also have various DOF incorporated in the models. The equivalent handling model of the quarter car is a linear single track model which describes lateral and yaw dynamic response to handling maneuvers (ignoring the effect of sprung and unsprung masses). Hence, the degrees of complexity of handling models vary based on assumptions on the rigid-body and performance objectives.

In order to develop the most complete model of vehicle roll behavior and examine roll response associated with specific maneuvering conditions, it is necessary to consider the vehicle model combining motions both in the yaw and roll planes. That is because lateral, yaw and roll dynamics are all coupled. Therefore, handling model which includes
lateral, yaw, and roll dynamics is considered as a rollover model. Pitching and bouncing motion have relatively small effects on vehicle lateral dynamics, thus they are neglected in rollover models. In the rollover models, the external inputs like road profile, steering angle from a driver, and wind gust are treated as disturbances.

A linear model is constructed for controller design. A higher order non-linear vehicle model is linearized locally around an operating point by assuming small angle approximation and by neglecting higher order dynamic terms. Note that linear models are still useful for analysis and control design purposes because the vehicle could still be unstable under small angular motions if the velocity is too high. In these models, vehicle forward velocity is taken as a constant varying parameter. Hence, in this dissertation, we focus on two vehicle models, (i) a 3 DOF bicycle model and (ii) a 9 DOF tractor semi-trailer model discussed in sections 3.2 and 3.3.

### 3.2 3-DOF Bicycle Model with Active Steering Control [4]

The considered bicycle model in this dissertation has 3 DOF, which are vehicle’s lateral, yaw, and roll dynamics, and it is shown in Fig 3.1. The equations of motion for this rollover model are shown from equation (3.1) to equation (3.3); equation (3.1) is the lateral force equation; and equation (3.2) is the yaw moment equation. Equation (3.3) is the roll moment equation. The nonlinear equations which represent the rollover (yaw-roll) model for the 3-DOF bicycle vehicle are

\[ m\ddot{y} - m_2 h \dot{\theta} \cos \phi = m_2 h r^2 \sin \phi - m v_x r + m_2 h \dot{\phi}^2 \sin \phi + F_{yf} + F_{yr} \]  

(3.1)

\[ J \ddot{\phi} = -r \left( m_2 h^2 \dot{\phi} \sin 2\phi + m_2 h \phi \sin \phi \right) + l_f F_{yf} - l_r F_{yr} \]  

(3.2)
\[ m_2 h v_y \cos \phi + (J_{2,z} + m_2 h^2) \ddot{\phi} = \]

\[- m_2 h v_x \cos \phi + \frac{m_2 h^2}{2m} r^2 \sin 2\phi - d_\phi \dot{\phi} + m_2 g h \sin \phi - c_\phi \phi = 0 \quad (3.3)\]

where \( c_f \) is front cornering stiffness, \( c_r \) rear corning stiffness, \( c_\phi \) is roll stiffness of passive suspension, \( d_\phi \) is roll damping of passive suspension, \( g \) is acceleration due to gravity, \( h_R \) is height of roll axis over ground, \( h \) is nominal height of CG2 over roll axis, \( J_{2,z} \) is roll moment of inertia of sprung mass, \( J_z \) is overall yaw moment of inertia, \( l_f \) is distance front axle to CG1, \( l \) is distance rear axle to CG1, \( m \) is overall vehicle mass, \( m_2 \) is sprung mass, \( \mu \) is road adhesion coefficient.

Fig. 3.1 Rollover model for the 3-DOF bicycle model [4]
Numerical values of the parameters of the model, shown in Appendix A, are taken from [28] and US Army Tank-Automotive Research, Development and Engineering Center. The vehicle data from [28] represents a single-unit lorry and the vehicle data from US Army Tank-Automotive Research, Development and Engineering Center represents an Army SUV.

A set of linear differential equations is obtained based on the following assumptions:

- neglect all 2nd and higher order angular terms
- neglect all angular cross terms
- apply a small angle approximation, i.e., \( \sin \phi = \phi \) and \( \cos \phi = 1 \).

Hence, the linear differential equations are

\[
\begin{align*}
    m\ddot{y} - m_2 h \dot{\phi} &= -m \nu_s r + F_{ys} + F_{yr} \\
    J_2 \ddot{r} &= l F_{ys} - l \nu_s F_{yr} \\
    m_2 \ddot{h} + (m_2 h^2) \ddot{\phi} &= -m_2 h \nu_s r - d_s \dot{\phi} + m_2 g h \phi - c_s \phi
\end{align*}
\]

Next, the lateral tire forces \( F_{ys} \) and \( F_{yr} \) are modeled. The slip angle of a tire is defined as the angle between the orientation of the tire and the orientation of the velocity vector of the wheel, which is shown in Fig. 3.2.
In Fig. 3.2, the slip angle of the front wheel is

\[ \alpha_f = \delta - \theta_{sf} \]  

(3.7)

where \( \theta_{sf} \) is the angle that the velocity vector makes with the longitudinal axis of the vehicle and \( \delta \) is the front wheel steering angle. Similarly, the rear slip angle is given by

\[ \alpha_r = -\theta_{sr} \]  

(3.8)

The lateral tire force for the front wheels of the vehicle can therefore be written as

\[ F_{sf} = c_f \mu(\delta - \theta_{sf}) \]  

(3.9)

where the proportionality constant \( c_f \) is called the cornering stiffness of front tire, \( \delta \) is the (front) steering angle, \( \theta_{sf} \) is the front tire velocity angle, and \( \mu \) is the road adhesion coefficient. Similarly, the lateral tire force for the rear wheels is written as

\[ F_{sr} = c_r \mu(-\theta_{sr}) \]  

(3.10)

where \( c_r \) is the cornering stiffness of rear tire and \( \theta_{sr} \) is the rear tire velocity angle.
The following relations is used to calculate $\theta_{sf}$ and $\theta_{yr}$:

\[
\tan(\theta_{sf}) = \frac{v_y + l_f r}{v_x} \tag{3.11}
\]

\[
\tan(\theta_{yr}) = \frac{v_y - l_r r}{v_x} \tag{3.12}
\]

Using small angle approximations,

\[
\theta_{sf} = \frac{v_y + l_f r}{v_x} \tag{3.13}
\]

\[
\theta_{yr} = \frac{v_y - l_r r}{v_x} \tag{3.14}
\]

Substituting equation (3.13) and (3.14) into equation (3.9) and (3.10),

\[
F_{sf} = c_f \mu \left( \delta - \frac{v_y + l_f r}{v_x} \right) \tag{3.15}
\]

\[
F_{yr} = c_r \mu \left( -\frac{v_y - l_r r}{v_x} \right) \tag{3.16}
\]

Substituting equation (3.15) and (3.16) into equation (3.4) and (3.5),

\[
m \ddot{v}_y - m_z h \ddot{\phi} = -m v_x r + c_f \mu \left( \delta - \frac{v_y + l_f r}{v_x} \right) + c_r \mu \left( -\frac{v_y - l_r r}{v_x} \right) \tag{3.17}
\]

\[
J_z \ddot{\phi} = l_f c_f \mu \left( \delta - \frac{v_y + l_f r}{v_x} \right) - l_r c_r \mu \left( -\frac{v_y - l_r r}{v_x} \right) \tag{3.18}
\]

Rearranging the equation (3.4) and (3.5), the linear differential equations which represent the rollover (yaw-roll) model for the 3-DOF bicycle vehicle are
\[ m\dot{v}_y - m_z h \ddot{\phi} = -mv_y r - \frac{\mu}{v_x} (c_f + c_r) v_y - \frac{\mu}{v_x} (c_f l_f - c_r l_r) r + c_f \mu \delta \] (3.19)

\[ J_z \dot{r} = -\frac{\mu}{v_x} (c_f l_f - c_r l_r) v_y - \frac{\mu}{v_x} (c_f l_f^2 - c_r l_r^2) r + c_f l_f \mu \delta \] (3.20)

\[ m_z h \dot{v}_y + (J_{2x} + m_z h^2) \ddot{\phi} = -m_z h v_y - d_\phi + m_z g h \phi - c_\phi \phi \] (3.21)

The equation (3.19), (3.20), and (3.21) is written as the state space form and the state equation is given by:

\[ E \dot{x} = Ux + Vu \quad \text{where} \quad x = [\phi \quad v_y \quad r \quad \dot{\phi}]^T ; \quad u = \delta ; \quad A = E^{-1}U; \text{ and } B = E^{-1}V. \]

Moreover,

\[
E = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & m & 0 & -hm_2 \\
0 & 0 & J_z & 0 \\
0 & -hm_2 & 0 & J_{2x} + h^2 m_2
\end{bmatrix}; \quad U = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & U_{22} & U_{23} & 0 \\
0 & U_{32} & U_{33} & 0 \\
-c_\phi + m_2 g h & 0 & hm_2 v_x & -d_\phi
\end{bmatrix}
\]

where

\[ U_{22} = -(c_f + c_r) \mu / v_x; \quad U_{23} = -(c_f l_f - c_r l_r) \mu / v_x - mv_x; \]

\[ U_{32} = -(c_f l_f - c_r l_r) \mu / v_x; \quad U_{33} = -(c_f l_f^2 + c_r l_r^2) \mu / v_x. \]

\[ V = \begin{bmatrix} 0 & c_f \mu & c_f l_f \mu & 0 \end{bmatrix}^T. \]

In addition, matrix \( E \) is assumed to be invertible. Therefore, the state equation is given by \( \dot{x} = Ax + Bu \). The four states of the system are roll angle (\( \phi \)) of sprung mass relative to unsprung mass, lateral velocity (\( v_y \)), yaw rate (\( r \)) and roll rate (\( \dot{\phi} \)) of sprung mass relative to unsprung mass. The control input of the system is steering angle (\( \delta \)).

The output equation in the state space form for the 3-DOF bicycle model, where the output is a scalar (rollover index), is derived in section 4.3.

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3.3 9-DOF Tractor Semi-trailer with Active Anti-roll-bar Control [18]

The frame flexibility is included in the tractor semi-trailer considered in this dissertation. It is to capture the influence of compliance on the distribution of roll moments between axles. The sprung mass of tractor is split, typically in proportion with the axle weights, into front and rear section, each with appropriate inertial properties. These two sections of the sprung mass of the tractor are connected with a torsional spring whose stiffness matches the torsional stiffness of the vehicle frame. A kinematic constraint between adjacent vehicle units is required at each coupling. A nonlinear tyre model is considered in this tractor semi-trailer model. The primary source of non-linearity is the variation in tyre cornering stiffness \( \frac{F_y}{\alpha} \) with vertical load \( F_z \), and this variation is described using the quadratic equation \( \frac{F_y}{\alpha} = c_1 F_z + c_2 F_z^2 \). This equation is generally suitable for lateral accelerations up to the rollover point and is widely used in heavy vehicle simulation studies [41]. The forward velocity of the vehicle is constant during any lateral maneuver.

The considered tractor semi-trailer model, which shown in Fig. 3.3, has 9 DOF: six degrees of freedom of the tractor unit (yaw, side-slip, front and rear sprung mass roll angle, steer axle roll angle and drive axle roll angle), plus the articulation angle between the tractor and trailer, the roll angle of the sprung mass of the trailer, and the roll angle of the trailer axle group. Similar to the linearization procedure mentioned in section 3.2, the nonlinear model is linearized locally around an operating point.

The equations of motion for the linear tractor semi-trailer vehicle model are
shown in equations (3.22) to (3.33). Equations (3.22) to (3.26) are five equations of motion of the tractor unit (yaw moment equation, front and rear sprung mass roll moment equations and front and rear unsprung mass roll moment equation); equation (3.27) and equation (3.28) correspond to the lateral force equation of the coupling between tractor and trailer and the shear force equation of the tractor frame, respectively; equation (3.29) is the kinematic constraint between tractor and trailer; and equations (3.30) to (3.33) are the four equations of motion of the trailer unit (yaw moment equation, sprung mass roll moment equation and unsprung mass roll moment equation and lateral force equation).

Finally, the equations of motion for the linear tractor semi-trailer vehicle model is

\[
I_{x z f 1} \ddot{\phi}_{f 1} - I_{x z f 1} \ddot{\phi}_{r 1} + I_{z z f 1} \ddot{\psi}_{1} = N_{\beta 1} \beta_{1} + N_{\psi 1} \psi_{1} + N_{\delta 1} \delta - b_{f 1} F_{y 1} 
\]  
(3.22)

\[
I_{x z f 1} \ddot{\phi}_{f 1} - I_{x z f 1} \ddot{\phi}_{r 1} = m_{s f 1} g (h_{s f 1} - r_{1}) \phi_{f 1} 
\]
\[+ m_{s f 1} (h_{s f 1} - r_{1}) (U\dot{\beta}_{1} + \dot{U} \beta_{1} + U \psi_{1}) \]
\[+ k_{f 1} (\phi_{f 1} - \phi_{r f 1}) - l_{f 1} (\dot{\phi}_{f 1} - \dot{\phi}_{r f 1}) \]
\[+ k_{b f 1} (\phi_{f 1} - \phi_{r 1}) - l_{b f 1} (\dot{\phi}_{f 1} - \dot{\phi}_{r 1}) \]
\[- F_{b f 1} b_{r f 1} + u_{f 1} \]
\]  
(3.23)

\[
I_{x z r 1} \ddot{\phi}_{r 1} - I_{x z r 1} \ddot{\phi}_{r 1} = m_{s r 1} g (h_{s r 1} - r_{1}) \phi_{r 1} 
\]
\[+ m_{s r 1} (h_{s r 1} - r_{1}) (U\dot{\beta}_{1} + \dot{U} \beta_{1} + U \psi_{1}) \]
\[+ k_{r 1} (\phi_{r 1} - \phi_{r r 1}) - l_{r 1} (\dot{\phi}_{r 1} - \dot{\phi}_{r r 1}) \]
\[+ k_{b r 1} (\phi_{r 1} - \phi_{r 1}) + l_{b r 1} (\dot{\phi}_{r 1} - \dot{\phi}_{r 1}) \]
\[- k_{r 1} (\phi_{r 1} - \phi_{r f 1}) + (r_{1} - h_{s r 1}) F_{e 1} \]
\[+ F_{b r 1} h_{b r 1} + u_{r 1} \]
\]
(3.24)

\[
- r_{1} (Y_{\beta f 1} \beta_{1} + Y_{\psi f 1} \psi_{1} + Y_{\delta f 1} \delta) = m_{u f 1} (h_{u f 1} - r_{1}) (U\dot{\beta}_{1} + \dot{U} \beta_{1} + U \psi_{1}) 
\]
\[+ m_{u f 1} (h_{u f 1} + \dot{h}_{u f 1} \phi_{f 1} - k_{f 1} (\phi_{f 1} - \phi_{r f 1}) \]
\[+ k_{r 1} (\phi_{r 1} - \phi_{f 1}) - l_{f 1} (\dot{\phi}_{f 1} - \dot{\phi}_{r f 1}) + u_{f 1} \]
\]
(3.25)
\[-r_i (Y_{\beta,i} \beta_1 + Y_{\psi,i} \psi_1) = m_{u,i} (h_{u,r,i} - r_i) (U \dot{\beta}_1 + U \beta_1 + U \psi_1) \]
\[-m_{u,i} g h_{u,r,i} \phi_{r,i} - k_{r,i} (\phi_{r,i} - \phi_{r,r,i}) \]
\[+ k_{r,i} \phi_{r,i} - l_{r,i} (\phi_{r,i} - \phi_{r,r,i}) + u_{r,i} \]
(3.26)

\[F_{e,i} = -m_{s,f,i} (h_{s,f,i} - r_i) \dot{\phi}_{f,i} - m_{s,r,i} (h_{s,r,i} - r_i) \ddot{\phi}_{f,i} \]
\[-m_i (U \dot{\beta}_1 + U \beta_1 + U \psi_1) + Y_{\psi,i} \psi_1 + Y_{\delta,i} \delta \]
(3.27)

\[F_{b,i} = (Y_{\beta,f,i} \beta + Y_{\psi,f,i} \psi_1 + Y_{\delta,f,i} \delta) \]
\[-m_{f,i} (U \dot{\beta}_1 + U \beta_1 + U \psi_1) \]
\[-m_{s,f,i} (h_{s,f,i} - r_i) \dot{\phi}_{f,i} \]
(3.28)

\[\dot{\beta}_1 - \dot{\beta}_2 + \frac{U}{U} (\beta_1 - \beta_2) + \frac{b'_1}{U} \psi_1 - \frac{b'_2}{U} \psi_2 - \frac{(r_1 - h_{u,r,i})}{U} \phi_{r,i} + \frac{(r_2 - h_{u,f,i})}{U} \phi_2 \]
\[+ \psi_1 - \psi_2 + \frac{U}{U} (\psi_1 - \psi_2) = 0 \]
(3.29)

\[I_{x'x'2} \ddot{\phi}_2 - I_{x'x'2} \ddot{\psi}_2 = N_{\beta,2} \beta_2 + N_{\psi,2} \psi_2 + b'_{f,2} F_{e,1} \]
(3.30)

\[I_{x'x'2} \ddot{\phi}_2 - I_{x'x'2} \ddot{\psi}_2 = m_{s,2} g (h_{s,i} - r_i) \phi_2 + u_{r,2} \]
\[-m_{s,2} (h_{s,i} - r_i) (U \dot{\beta}_2 + U \beta_2 + U \psi_2) \]
\[-k_{r,2} (\phi_2 - \phi_{r,r,2}) - l_{r,2} (\phi_2 - \phi_{r,r,2}) \]
\[+ k_{\phi,2} (\phi_{r,i} - \phi_2) - (r_2 - h_{u,f,2}) F_{e,1} \]
(3.31)

\[\neg (Y_{\beta,2} \beta_2 + Y_{\psi,2} \psi_2) = m_{u,r,2} (h_{u,r,2} - r_2) (U \dot{\beta}_2 + U \beta_2 + U \psi_2) \]
\[-m_{u,r,2} g h_{u,r,2} \phi_{r,i} - k_{r,2} (\phi_2 - \phi_{r,r,2}) \]
\[+ k_{r,2} \phi_{r,i} - l_{r,2} (\phi_2 - \phi_{r,r,2}) + u_{r,2} \]
(3.32)

\[m_{s,2} (h_{s,i} - r_i) \ddot{\phi}_2 = -m_2 (U \dot{\beta}_2 + U \beta_2 + U \psi_2) \]
\[+ Y_{\beta,2} \beta_2 + Y_{\psi,2} \psi_2 + F_{e,1} \]
(3.33)
where subscript 1 denotes tractor, subscript 2 denotes trailer, subscript $f$ denotes front and subscript $r$ denotes rear.

The thirteen states of the system are roll angle of front and rear sprung mass for tractor ($\phi_{f,1}$ and $\phi_{r,1}$), roll rate of front and rear sprung mass for tractor ($\dot{\phi}_{f,1}$ and $\dot{\phi}_{r,1}$), sideslip angle for tractor ($\beta_1$), yaw rate for tractor ($\psi_1$), roll angle of front and rear unsprung mass for tractor ($\phi_{u,f,1}$ and $\phi_{u,r,1}$), roll angle of sprung mass for trailer ($\phi_2$), roll rate of sprung mass for trailer ($\dot{\phi}_2$), sideslip angle for trailer ($\beta_2$), yaw rate for trailer ($\psi_2$), and roll angle of unsprung mass for trailer ($\phi_{u,r,2}$). $u_{f,1}$ is the active roll torque at the tractor steering axle, $u_{r,1}$ is the active roll torque at the tractor drive axle, and $u_{r,2}$ is the active roll torque at the trailer unit. $\delta$ is the front wheel steering angle from driver.

All parameters of interested are listed below:

- $\beta$ sideslip angle
- $\delta$ steer angle
- $\phi$ absolute roll angle of sprung mass
- $\phi_t$ absolute roll angle of unsprung mass
- $\psi$ heading angle
- $\psi'$ yaw rate
- $a'$ longitudinal distance to axle, measured forwards from center of total mass
- $b'$ longitudinal distance to articulation point, measured forwards from center of total mass
- $c_a$ tyre cornering stiffness, measured at rated vertical tyre load
- $F_b$ lateral shear force in vehicle frame
- $F_c$ lateral force in vehicle coupling
- $F_y$ lateral tyre force
- $g$ acceleration due to gravity
- $h_a$ height of articulation point, measured upwards from ground
height of frame twist axis, measured upwards from ground
height of center of sprung mass, measured upwards from ground
height of center of unsprung mass, measured upwards from ground
roll moment of inertia of sprung mass, measured about sprung center of mass
roll moment of inertia of sprung mass, measured about origin of coordinate system
yaw-roll product of inertia of sprung mass, measured about sprung center of mass
yaw-roll product of inertia of sprung mass, measured about origin of coordinate system
yaw moment of inertia of sprung mass, measured about sprung center of mass
yaw moment of inertia of total mass, measured about origin of coordinate system
suspension roll stiffness
vehicle coupling roll stiffness
vehicle frame torsional stiffness
tyre roll stiffness
suspension roll damping rate
vehicle frame torsional damping rate
total mass
sprung mass
unsprung mass
partial derivative of net tyre yaw moment with respect to sideslip angle
partial derivative of net tyre yaw moment with respect to steer angle
partial derivative of net tyre yaw moment with respect to yaw rate
height of roll axis, measured upwards from ground
forward speed
longitudinal speed
partial derivative of net tyre lateral force with respect to sideslip angle
\[ Y_\delta = \frac{\partial F_i}{\partial \delta} = -c_{a,i} \]

partial derivative of net tyre lateral force with respect to steer angle

\[ Y_\psi = \frac{\partial F_i}{\partial \psi} = \sum_j \frac{a_j c_{a,j}}{U} \]

partial derivative of net tyre lateral force with respect to yaw rate

The numerical values of the parameters of the model, shown in Appendix B, are taken from [18]. The equations of motion for the 9-DOF tractor semi-trailer model are expressed in state space form and the state equation is given in Appendix C. In the state equation for the tractor semi-trailer, which is \( x = Ax + Bu + B_\delta \delta \), \( B_\delta \delta \) is added because the front wheel steering angle (\( \delta \)) from driver is treated as a disturbance in this rollover model. Moreover, \( u \) is vector of active control torques at the active anti-roll bars. The output equations for the 9-DOF tractor semi-trailer with respect to different control configurations (\( u \)), where the output vectors are 3×1 vectors, are derived in section 6.5.1, 6.5.2 and 6.5.3, respectively. \( u \) is vector of active control torques at the active anti-roll bars. In section 6.5.1, \( u = [u_{f,1} \ u_{r,1} \ u_{r,2}]^T \). In section 6.5.2, \( u = [u_{r,1} \ u_{r,2}]^T \). In section 6.5.3, \( u = [u_{r,2}] \).

![Fig. 3.3 Tractor semi-trailer [18]](image-url)
3.4 Sensors and Actuators

In practical applications, selection of sensors and actuators is very important. The control design based on the proposed methodology is a full state feedback control. Therefore, all states in the vehicle model have to be measured or estimated. There are four major active rollover control applications based on the actuation schemes: (a) Active anti-roll-bar systems; (b) Active suspension systems (c) Anti-roll braking systems; and (d) Active steering systems [39]. The vehicle is controlled to prevent rollover by superimposing an active control input on the driver steering input. The anti-roll-bar systems at each axle consist of a pair of actuators and stiff anti-roll bar in parallel with passive springs and dampers. These anti-roll-bar systems generate additional (controlled) roll moments between the sprung and unsprung masses to prevent vehicles from rollover. An active suspension system is used to control the vertical movement of the wheels to eliminate body roll and pitch variation in many driving situations including cornering, accelerating and braking. An anti-roll braking system prevents the vehicle from rollover by controlling the braking of each tire. In this dissertation, the active steering system is applied to the 3-DOF bicycle model and the anti-roll-bar system is used by the 9-DOF tractor semi-trailer model.

3.5 Critical Parameters for Rollover

In rollover detection systems, the concept of a rollover index is used to determine the threshold for rollover. Based on the various signals such as vehicle roll angle, roll acceleration and lateral acceleration, a rollover detection system inputs this data into an algorithm which determines whether the rollover index exceeds the threshold or not [23]. Rollover thresholds is divided into two categories: Steady-State Rollover Threshold
(SSRT) and Dynamic Rollover Threshold (DRT) [5, 8, 12]. Static Stability Factor (SSF) equals SSRT as a first order approximation through the simplifying assumption that a vehicle behaves as a rigid body [7, 38]. The SSF is $\frac{T}{2h}$, where $h$ is height of center of gravity (CG) above the ground and $T$ is the track width. This simple measure of rollover threshold is often used for a first-order estimate of a vehicle’s resistance to rollover. It is especially attractive because it requires knowledge of only two vehicle parameters, which are the CG height and the track width. However, the estimates are very conservative (predicting a threshold that is greater than the actual) and are more useful for comparing vehicles rather than predicting absolute levels of performance. In other words, a vehicle with high SSF is prone to roll over and it is a useful index for building a vehicle with low rollover propensity. In addition, it shows that CG height ($h$) and track width ($T$) are two critical parameters in determining a rollover. Since the vehicle is a dynamical system, DRT is a more appropriate index to analyze. In this dissertation, a unified rollover index, in the lines of DRT and the concept of Lateral Transfer Ratio (LTR), is proposed and is discussed in detail in chapter 4.
CHAPTER 4: ROLLOVER PERFORMANCE INDEX

4.1 Introduction

A unified rollover index is derived in this chapter. Based on the ‘unified rollover index’, an approach in which the rollover index is expressed as a performance index labeled ‘rollover performance index’ is presented in this chapter. The novelty of the approach lies in expressing the traditional rollover index as a quadratic performance index in Linear-Quadratic Regulator (LQR) design with control coupled output regulation thereby integrating the roll over index directly and explicitly in the optimization procedure.

There are two major differences in the proposed rollover performance index in contrast to the standard LQR performance index. The first difference is that the state weighting matrix and control weighting matrix are determined naturally by expressing the rollover index as a quadratic performance index. The second difference is that, in the proposed methodology, a cross coupling term between the state and control variables with a weighting matrix appears in the performance index making it non-standard.

4.2 Rollover Index ($RI$)

Rollover Index is an important metric in vehicle safety assessment. A variety of rollover indices have been introduced in the literature [5]. The definition of rollover index
which applied in this dissertation is the load difference (i.e., vertical force) between the left and right wheels of the vehicle, normalized by the total load [4, 8, 9]. That is,

$$RI = \frac{\text{Load on right Tires} - \text{Load on Left Tires}}{\text{Total Load}} = \frac{F_{z,R} - F_{z,L}}{F_{z,R} + F_{z,L}} = \frac{F_{z,R} - F_{z,L}}{mg},$$

where $m$ is mass of the total vehicle and $m = m_s + m_u$ ($m_s$ is sprung mass and $m_u$ is unsprung mass). With this definition of the index, the vehicle is considered ‘rolled over’ when the rollover index ($RI$) is equal to 1 or -1. In other words, the vehicle does not roll over as long as $|RI| < 1$. For $F_{z,R} = F_{z,L}$ (i.e., $RI = 0$), the vehicle drives straight on a horizontal road. When $F_{z,R} = 0$ (i.e., $RI = -1$), the right wheels lift off the road. When $F_{z,L} = 0$ (i.e., $RI = 1$), the left wheels lift off the road.

We assume the unsprung mass is insignificant and unsprung mass rolls about a horizontal roll axis, which is along the centerline of the unsprung mass and at the ground level. Hence, it is reasonable to neglect the unsprung mass inertial contribution. Moreover, the roll angle $\phi$ is assumed to be small. The effective linear torques exerted by the suspension system about the roll center are defined as $T_{\text{spring}} = k\phi$ and $T_{\text{damper}} = c\dot{\phi}$, where $k$, $c$ denoted the total torsional spring stiffness and torsional damper coefficients of the suspension respectively. Fig. 4.1 is the vehicle rollover model. In this dissertation, the rollover index, we call unified rollover index, includes the roll and lateral dynamics. The unified rollover index is derived from the torque balance for unsprung mass about the zero-level center, which is the point S in the Fig. 4.1. The result of the torque balance is shown in equation. (4.1).
\[
\begin{align*}
\left\{ (-F_{z,R} + F_{z,L}) \frac{T}{2} + F_y h_R + k \phi + c \phi = 0 \\
F_y = m_2 a_{y,2} = m_2 (\ddot{v}_y + v_x \dot{r} - h \ddot{\phi})
\end{align*}
\] (4.1)

\[
\Rightarrow F_{z,R} - F_{z,L} = \frac{2}{T} \left( m_2 a_{y,2} h_R + k \phi + c \phi \right)
\] (4.2)

\[
= \frac{2}{T} \left[ m_2 (\ddot{v}_y + v_x \dot{r} - h \ddot{\phi}) h_R + k \phi + c \phi \right]
\]

Fig. 4.1 Vehicle rollover model
where \( a_{y,2} \) is the lateral acceleration of the CG2 and \( h_R \) is distance from the roll axis to the ground.

Note that lateral load transfer in the equation (4.2) arises from two mechanisms:

(a) \( \frac{2}{T} m_2 a_x h_R \) — lateral load transfer due to lateral acceleration.
(b) \( \frac{2}{T} (k\phi + c\dot{\phi}) \) — lateral load transfer due to vehicle roll. The suspension roll moment, \( k\phi + c\dot{\phi} \), arising from the effective linear torques exerted by the suspension system about the roll center.

Hence, the rollover index results in

\[
RI = \frac{F_{z,R} - F_{z,L}}{F_{z,R} + F_{z,L}} = \frac{2}{mg} \left( \ddot{v}_y + v_x \ddot{r} - h\ddot{\phi} \right) h_R + k\phi + c\dot{\phi}
\]

(4.3)

Since both roll and lateral dynamics can affect roll moment, the unified rollover index in the equation (4.3), including the roll and lateral dynamics is more rigorous to detect the rollover.

In [8], the author argued that the rollover estimation in [4] is not sufficient to detect the transient phase of rollover due to the fact that it is derived ignoring roll dynamics. The control input in [4] is the steering angle which significantly influences lateral dynamics of vehicle, so the rollover index without considering roll dynamics is feasibly used to judge the rollover risk for this controlled system. Nevertheless, it cannot estimate the rollover accurately if the control input is roll moment. Furthermore, the \( RI = -\frac{2(k\phi + c\dot{\phi})}{mgT} \), which is claimed in [8, 9], is also not sufficient to estimate the rollover since the lateral dynamics is ignored in the formula. The lateral acceleration is a critical factor in rollover. Hence, the rollover cannot be detected strictly if the controller
is used to control lateral dynamics such as active steering control or yaw control. It is because the variation of the lateral acceleration is not evaluated in the rollover index.

Since the controller design methodology in this paper is based on a new performance index, which explicitly incorporates the traditional rollover index into the optimization procedure of the LQR framework, the accuracy of the rollover index has an influence on the achievement of the controller. This unified rollover index captures all dynamics related to the rollover (roll and lateral dynamics) and is able to be used to estimate the rollover precisely for every system and leads the proposed methodology for a controller design more effectively.

4.3 Rollover Performance Index ($J_R$)

We now present an approach in which the rollover index is expressed as a performance index labeled ‘rollover performance index’. In order to demonstrate this approach simplistically, we consider a 3-DOF bicycle model with active steering control in [4], which is introduced in section 3.2. Notice that the states for the system are roll angle ($\phi$), lateral velocity ($v_y$), yaw rate ($r$) and roll rate ($\dot{\phi}$). Hence,

$$
\mathbf{x} = [\phi \ v_y \ r \ \dot{\phi}]^T
$$

$$
\dot{v}_y = A_{21}\phi + A_{22}v_y + A_{23}r + A_{24}\dot{\phi} + B_{21}\delta;
$$

$$
\ddot{\phi} = A_{41}\phi + A_{42}v_y + A_{43}r + A_{44}\dot{\phi} + B_{41}\delta.
$$

Therefore, rollover index is expressed as

$$
RI = c_1\phi + c_2v_y + c_3r + c_4\dot{\phi} + d_1\delta
$$

where
Thus, we can express Rollover Index as an output variable with an output equation given by

\[
y = \begin{bmatrix} \phi \\ y_r \\ \dot{\phi} \end{bmatrix} + d_1 \delta = Cx + Du = RI
\] (4.4)

For closed loop system,

\[
u = -Kx \quad \text{and} \quad y_{CL} = (C - DK)x = RI_{CL}
\]

We now incorporate the output in equation (4.4) into the performance index of Linear Quadratic Regulator framework. The object of the controller is to maintain the absolute value of \( RI \) below one. Hence, the controller is designed so that the following rollover performance index is minimized and rollover prevention is achieved by reducing the following rollover performance index.

\[
J = \int_{t_0}^{\infty} \left[ RI(\tau)^T QRI(\tau) + u^T Ru \right] d\tau = \int_{t_0}^{\infty} \left[ y^T(\tau) \bar{Q} y(\tau) + u^T \bar{R} u \right] d\tau = J_R
\]

\[
= \int_{t_0}^{\infty} \left[ (Cx(\tau) + Du(\tau))^T \bar{Q}(Cx(\tau) + Du(\tau)) + u^T \bar{R} u \right] d\tau
\]

53
\[\begin{align*}
&= \int_{t_0}^{\infty} \left[ x^T(\tau) C^T \bar{Q} C x(\tau) + 2x^T(\tau) C^T \bar{Q} D u(\tau) + u^T(\tau) D^T \bar{Q} D u(\tau) + u^T(\tau) \bar{R} u(\tau) \right] d\tau \\
&= \int_{t_0}^{\infty} \left[ x^T(\tau)(C^T \bar{Q} C) x(\tau) + 2x^T(\tau) C^T \bar{Q} D u(\tau) + u^T(\tau)(D^T \bar{Q} D + \bar{R}) u(\tau) \right] d\tau \\
&= \int_{t_0}^{\infty} \left[ x^T(\tau) Q x(\tau) + 2x^T(\tau) N u(\tau) + u^T(\tau) R u(\tau) \right] d\tau
\end{align*}\]  

(4.5)

where \( Q = C^T \bar{Q} C \); \( N = C^T \bar{Q} D \); \( R = D^T \bar{Q} D + \bar{R} \). The weighting matrix \( Q \) is a symmetric positive semi-definite and the weighting matrix \( R \) is a symmetric positive-definite matrix.

In the LQR design based on the rollover performance index, the output is the rollover index and the input depends on actuator scheme. For a small single-unit vehicle, one rollover index is sufficient for detecting the risk of rollover. For a long or articulated vehicle, every axle is possible to roll over independently due to flexibility. Hence, the multiple rollover indices for a long or articulated vehicle should be considered. Furthermore, active anti-roll-bar system is common for a long or articulated vehicle since they are able to control all axles separately.
CHAPTER 5: LQR DESIGN BASED ON ROLLOVER PERFORMANCE INDEX FOR SINGLE-INPUT SINGLE-OUTPUT (SISO) SYSTEM

5.1 Introduction

The LQR design based on rollover performance index belongs to the LQR design for the control coupled output regulation problem and the LQR design based on standard performance index belongs to the LQR design for state regulation problem. We label the LQR design for state regulation problem standard LQR design. This chapter demonstrates the properties of LQR design based on rollover performance, which make rollover prevention more effectively, by comparisons with standard LQR design based on standard performance index using single-input single-output (SISO) case. If the coupling is negative, the LQR design with control coupled output regulation problem can significantly improved the performance compared to the LQR design with the state regulation. Regarding the state space form of the rollover model used in this chapter, the state equation is introduced in section 3.2 and the output equation is introduced in section 4.3. In this chapter, the proposed methodology is compared with the standard LQR design using a 3-DOF bicycle model with active steering control mentioned in section 3.2
5.2 Comparison of LQR Design based on Rollover Performance Index and Standard LQR Design

This section illustrates that an LQR design based on rollover performance index can improve the rollover index compared with a standard LQR design by an example. As is seen from the proposed performance index, the state weighting matrix \( Q \), control weighting matrix \( R \) and the cross-coupling weighting matrix \( N \) are systematically determined with no burden on the designer to specify the weightings on the state and control independently. In fact, it turns out that there is a considerable difference in the performance of the controllers based on the proposed rollover performance index (labeled Design 1– LQR design with control coupled regulation) and the controllers based on treating state regulation and control regulation independently. In other words, simply regulating the states independently, either with a coupled state weighting matrix (labeled Design 2 – LQR design with state regulation) or with a diagonal state weighting matrix (labeled Design 3 – LQR design with state regulation) is not equivalent \( \text{per se} \) to regulating the actual rollover index.

Note that for the proposed rollover performance index the state weighting matrix \( Q \) is obtained in one-shot procedure whereas for the performance index in a control coupled output regulation problem there are too many design variables which make the selection of \( Q \) matrix in the LQR design very unwieldy and cumbersome. Thus, the proposed new rollover performance index reduces the design burden for LQR design.

5.3 Tradeoff between Rollover Index and Control Weighting Matrix \( R \)

A rollover performance index is designed to regulate the rollover index. Hence, the trade-off curves between the rollover index and the control weighting matrix \( R \) by
varying $\mathbf{R}$ is illustrated in Fig. 5.1. In Fig. 5.1, the forward velocity of 100 km/hr and an initial roll angle 10 deg are assumed. Also, the control weighting matrix $\mathbf{R}$ is varying, and displayed rollover index is the maximum rollover index during a time period. With the fixed $\overline{\mathbf{Q}} = 1$, the assigned state weighting matrix $\mathbf{Q}$ and the weighting matrix for cross term between state and control variables $\mathbf{N}$ for three designs are

**Design 1:**

$$
\mathbf{Q} = \begin{bmatrix}
13.0826 & -0.7445 & 0.0387 & 2.9030 \\
-0.7445 & 0.0424 & -0.0022 & -0.1652 \\
0.0387 & -0.0022 & 0.0001 & 0.0086 \\
2.9030 & -0.1652 & 0.0086 & 0.6441
\end{bmatrix};
\mathbf{N} = \begin{bmatrix}
8.8182 \\
-0.5019 \\
0.0261 \\
1.9567
\end{bmatrix}
$$

**Design 2:** $\mathbf{Q}$ is the same as in design 1. $\mathbf{N} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$

**Design 3:**

$$
\mathbf{Q} = \begin{bmatrix}
13.0826 & 0 & 0 & 0 \\
0 & 0.0424 & 0 & 0 \\
0 & 0 & 0.0001 & 0 \\
0 & 0 & 0 & 0.6441
\end{bmatrix};
\mathbf{N} = \begin{bmatrix} 0 \\
0 \\
0 \\
0 \end{bmatrix}.
$$

Fig. 5.1 trade-off curves between rollover index and control weighting matrix $\mathbf{R}$
The trade-off curve between the rollover index and the weighting matrix $R$ in the Fig. 4.1 demonstrates that the LQR design based on the rollover performance index (Design 1) can reduce the rollover index compared with the standard LQR design (Design 2 and 3). In other words, when the weighting matrix $R$ is varying, the LQR design based on the rollover performance index (Design 1) always yields a smaller rollover index compared with the standard LQR design (Design 2 and 3). Also, a decreasing control input leads a increasing rollover index. Hence, the control input obtained from an LQR design based on the rollover performance index can improve the rollover prevention compared to the standard LQR design. Furthermore, the increasing weighting matrix $R$ causes the weak coupling effect such that LQR design with control coupled output regulation approaches the LQR design with state regulation when the weighting matrix $R$ is sufficiently large.

5.4 Comparison of Simulation Results for LQR Design based on Rollover Performance Index and Standard LQR Design

In this section, LQR design based on rollover performance index (Design 1) and standard LQR design (Design 2 and 3) are compared by a simulation.

$\bar{Q} = \rho I$ is assumed in the rollover performance index in equation (4.6). The rollover performance index is rewritten for design 1:

$$J_R = J = \rho \int_{0}^{\infty} y(T) y(T) d\tau + \int_{0}^{\infty} u(T) \bar{R} u(T) d\tau = \rho \int_{0}^{\infty} y_m(T) d\tau + J_{uex}$$

$$= \rho \int_{0}^{\infty} x(T) C^T C x(T) + 2x(T) C^T D u(T) + u(T) D^T D u(T) d\tau + \int_{0}^{\infty} u(T) \bar{R} u(T) d\tau$$
For design 2 and 3, $\overline{Q} = \rho I$ is assumed in the standard performance index in equation (2.3).

$$J = \rho \int_{0}^{\infty} y^T(\tau) y(\tau) d\tau + \int_{0}^{\infty} u^T(\tau) \overline{R} u(\tau) d\tau = \rho J_{yn} + J_{uex}$$

$$= \rho \int_{0}^{\infty} x^T(\tau) C^T C x(\tau) d\tau + \int_{0}^{\infty} u^T(\tau) \overline{R} u(\tau) d\tau$$

Note that there is no coupling effect in design 2 and 3, which are standard LQR design.

In order to compare the efficiency of three designs mentioned in section 5.3 based on the procedure and criterion proposed in section 2.6, the external control effort ($J_{uex}$) for three designs are designed to be identical by varying the weighting scalar $\rho$ and the weighting matrix $\overline{R}$. The forward velocity of 100km/hr and an initial roll angle 10deg are assumed for all designs. The assigned state weighting matrix $Q$, the control weighting matrix $R$, the weighting matrix for cross term between state and control variables $N$, the external control effort ($J_{uex}$), the coupling effect ($J_c$) the normalized performance output regulation cost ($J_{yn}$), the rollover performance index ($J_R$) for design 1, the standard performance index ($J_s$) for design 2 and 3, controller gain ($K$), and norm of controller gain ($|K|$) for three designs are obtained:

Design 1:

$$Q = \begin{bmatrix} 13.0826 & -0.7445 & 0.0387 & 2.903 \\ -0.7445 & 0.0424 & -0.0022 & -0.1652 \\ 0.0387 & -0.0022 & 0.0001 & 0.0086 \\ 2.903 & -0.1652 & 0.0086 & 0.6441 \end{bmatrix} ; N = \begin{bmatrix} 8.8182 \\ -0.5019 \\ 0.0261 \\ 1.9567 \end{bmatrix} ; R = 6.9438$$
Design 2 and design 3 are LQR with the state regulation, because the performance out equation is $y = Cx$. Three controlled vehicle using the controller designs above are administered in the simulations. In the simulations, a yaw torque as a disturbance acting on the vehicle during the double lane change is stimulated to illustrate the objective of active steering controller design and the controller addresses the rollover prevention. An unexpected disturbance torque can lead to dangerous driving situations because of the overreactions. Then, the excessive steering from a driver can cause the vehicle to roll over. A periodic disturbance torque is not typical so a step disturbance is adopted in the simulations. The peak value of the yaw torque disturbance is $220$ K Nm and it is taken at
1-second intervals (form 1sec to 2sec). The amount of yaw torque disturbance is assigned as large to raise the rollover risk. A constant forward velocity \( v_x \) of 100km/hr and initial roll angle of 10deg are assumed in all the simulations. The simulation results of the steering angle (control input), all states, the lateral acceleration and the rollover index are shown in Fig. 5.2 through Fig. 5.8.
Fig. 5.3 Comparison of the roll angle for design 1, 2, 3 and uncontrolled vehicle

Fig. 5.4 Comparison of the lateral velocity for design 1, 2, 3 and uncontrolled vehicle
Fig. 5.5 Comparison of the yaw rate for gain 1, 2, 3 and uncontrolled vehicle

Fig. 5.6 Comparison of the roll rate for gain 1, 2, 3 and uncontrolled vehicle
Fig. 5.7 Comparison of the lateral acceleration for gain 1, 2, 3 and uncontrolled vehicle

Fig. 5.8 Comparison of the RI for gain 1, 2, 3 and uncontrolled vehicle
In the results of three designs, which have the same external control effect, indicate that performance output regulation cost ($J_{yn}$) in design 1 is larger than design 2 and design 3, because the objective of this LQR design is to minimize the rollover index instead of regulating the states and control input independently. Hence, the rollover index has been significantly reduced in design 1, but the optimal cost (rollover performance index) $J_R$ in the design 1 is not larger than the design 2 and 3 due to the contribution from the coupling regulation cost ($J_c$). The design 1 efficiently utilizes the feature of coupling between states and control to produce an optimal controller to satisfy the requirement of rollover performance. This significant effect of the coupling effect ($J_c$) in the design 1 cannot be achieved in designs 2 and 3, which emphasize individual state regulation.

It is observed that the design 1 of the proposed LQR design based on rollover performance index significantly improved the rollover performance compared to design 2 and design 3. Note that the rollover index magnitude for designs 2 and 3 is larger than the rollover index magnitude for designs 1. This is due to the fact that the goal for the rollover performance index is to reduce rollover index, not just to regulate the states independently. The trade-off is clearly apparent from the plots. The proposed rollover performance index minimization requires a little more control than the standard LQR designs but the reduction of rollover index is quite significant. Similarly the controller of design 1 transfers the effort more to the roll rate and roll angle reduction while not emphasizing reduction in yaw rate and lateral velocity, which are induced from the steer angle (control input). Thus the cross coupling term, which results from component-wise
cancellation, helps to judiciously allocate the weights on states and control with the overall rollover index minimization as the primary objective rather than individual state regulation.
6.1 Introduction

The vehicle frame is usually assumed as a rigid body for a small, single-unit vehicle. However, torsional compliance of the vehicle frame has influences on the distribution of roll moments between axle groups for articulated heavy vehicle. Winkler et al. noted that the torsional compliance of the vehicle frame is a uniquely important element in establishing the roll stability of some vehicles, particularly those with flat-bed trailers [17]. Figure 6.1 illustrates that the rear end of a flat-bed trailer rolls over independently of the front end due to torsional flexibility of the frame [17]. Hence, rollover index for every axle group is defined and a Multiple-Input (multiple controller) Multiple-Output (multiple rollover indices) system is considered for an articulated heavy vehicle.

For an MIMO system, there are too many design variables in an LQR design for a control coupled output regulation problem which makes the selection of matrix $Q$ and $R$ in an LQR design very unwieldy and cumbersome. The proposed new rollover performance index reduces the design burden for LQR design and introduces matrix $N$ integrating physical dynamics to improve the performance. Furthermore, actuators are relatively expensive and their number in engineering systems needs to be reduced to a
minimum. If the effect from component-wise cancellation can make an effort to the system, the number of the controller is reduced or the same controller(s) can spend(s) less power to reach the same performance. Regarding the state space form of the rollover model used in this chapter, the state equation is introduced in section 3.3 and the three different output equations, where the control input is different, are introduced in section 6.4.1, 6.4.2, and 6.4.3, respectively. In this chapter, three control configurations, which are designed for the 9-DOF tractor semi-trailer model with active anti-roll-bar control mentioned in section 3.3, are compared based on the proposed technique.

![Image](image.png)

Fig. 6.1 the rear end of a flat-bed trailer rolls over independently of the front end due to torsional flexibility of the frame [17]
6.2 Rollover indices (RIs)

For articulate heavy vehicles, rollover index for every axle group is properly defined. According the rollover index defined in chapter 3.3, the rollover index for every axle is shown in the equation (6.1)

\[ RI_i = \frac{F_{z, RI} - F_{z, Li}}{F_{z, Ri} + F_{z, L i}} = \frac{F_{z, Ri} - F_{z, Li}}{m_i g} = \frac{2[m_s a_y h_{R_i} + k_i \phi_{i} + c_i \dot{\phi}_{i}]}{m_i g T_i} \tag{6.1} \]

and the lateral acceleration in equation (6.2) is adopted from [4, 22]:

\[ a_y = v_x \beta + \dot{v}_x \beta + v_x \psi - h \ddot{\phi} \tag{6.2} \]

Thus,

\[ RI_i = \frac{2[m_s (v_x \beta_i + \dot{v}_x \beta_i + v_x \psi_i - h \ddot{\phi}_i) h_{R_i} + k_i \phi_{i} + c_i \dot{\phi}_{i}]}{m_i g T_i} \]

where the \( v_x \) is the forward velocity and assumed to be constant and subscript s denotes sprung mass. With this definition of the index, the right (left) wheels of an axle lift off when the rollover index (RI) with respect to the corresponding axle is equal to -1 (1). In other words, the wheels of an axle \( i \) does not lift off the road as long as \( |RI_i| < 1 \). For this tractor semi-trailer, there are three rollover indices defined. One index is for tractor steering axle, another index is for tractor drive axle, and the other index is for the group of trailer axles, where three trailer axles are lumped together as a group. From the definition of the rollover for tractor semi-trailers, this tractor semi-trailer is considered ‘rolled over’ when both the drive and trailer axles have lifted off the road [18].
6.3 Rollover performance index ($J_R$)

We now express the rollover index as a performance index, which is a rollover performance index. This tractor semi-trailer has three axles which are steering, drive and trailer axle. Notice that the states for the linear state space system are $\phi_{f,1}$, $\dot{\phi}_{f,1}$, $\phi_{r,1}$, $\dot{\phi}_{r,1}$, $\beta_1$, $\psi_1$, $\phi_{t,f,1}$, $\phi_{t,r,1}$, $\phi_2$, $\dot{\phi}_2$, $\beta_2$, $\psi_2$ and $\phi_{t,r,2}$. Three rollover indices are defined for this tractor semi-trailer model, which are the Rollover Index 1 ($RI_1$), the Rollover Index 2 ($RI_2$) and the Rollover Index 3 ($RI_3$). Expressing $RI_1$, $RI_2$ and $RI_3$ in an output equation is able to weight three rollover indices simultaneously using the proposed LQR design based on rollover performance index. Furthermore, the output vector is a $3\times1$ vector.

6.4 Controller Selection based on Component-wise Cancellation in LQR designs with Control Coupled Output Regulation

There are various possible control configurations for an MIMO system. For tractor semi-trailer, the options are (a) control all axles on vehicle; (b) control the axles on the trailer unit and the tractor drive axle; (c) control only the axles on the trailer unit. Based on the negative coupling in the LQR design with control coupled output regulation, the most efficient control configuration among these three control configurations is determined by design observation 2.1.

6.4.1 Active Roll Control at all Axles on the Vehicle

In this section, the first control configuration, where all axles on the vehicle have active roll control, is given. For this control configure, the output equation with $RI_1$, $RI_2$ and $RI_3$ as three performance output variables is given by
\[ y = RI = \begin{bmatrix} R_{1} \\ R_{2} \\ R_{3} \end{bmatrix} \begin{bmatrix} c_{1,1,1-3} \\ c_{2,1,1-3} \\ c_{3,1,1-3} \end{bmatrix} + \begin{bmatrix} d_{1,1} & d_{1,2} & d_{1,3} \\ d_{2,1} & d_{2,2} & d_{2,3} \\ d_{3,1} & d_{3,2} & d_{3,3} \end{bmatrix} \begin{bmatrix} u_{f,1} \\ u_{r,1} \\ u_{r,2} \end{bmatrix} = Cx + Du \]

where \( c_{1,1,1-3}, c_{2,1,1-3}, c_{3,1,1-3}, d_{1,1-3}, d_{2,1-3}, \) and \( d_{3,1-3} \) are shown in Appendix D.

6.4.2 Active Roll Control at the Axles on the Trailer Unit and at the Driver Axle on the Tractor Unit

In this section, the second control configuration, where the tractor driver axle and the trailer unit have active roll control, is given. For this control configuration, the output equation with \( R_{1}, R_{2} \) and \( R_{3} \) as three performance output variables is given by

\[ y = RI = \begin{bmatrix} R_{1} \\ R_{2} \\ R_{3} \end{bmatrix} \begin{bmatrix} c_{1,1,1-3} \\ c_{2,1,1-3} \\ c_{3,1,1-3} \end{bmatrix} + \begin{bmatrix} d_{1,2} & d_{1,3} \\ d_{2,2} & d_{2,3} \\ d_{3,2} & d_{3,3} \end{bmatrix} \begin{bmatrix} u_{r,1} \\ u_{r,2} \end{bmatrix} = Cx + Du \]

where \( c_{1,1,1-3}, c_{2,1,1-3}, c_{3,1,1-3}, d_{1,2-3}, d_{2,2-3}, \) and \( d_{3,2-3} \) are shown in Appendix D.

6.4.3 Active Roll Control at the Axles on the Trailer Unit

In this section, the third control configuration, where only the axles on the trailer unit have active roll control, is given. For this control configuration, the output equation with \( R_{1}, R_{2} \) and \( R_{3} \) as three performance output variables is given by

\[ y = RI = \begin{bmatrix} R_{1} \\ R_{2} \\ R_{3} \end{bmatrix} \begin{bmatrix} c_{1,1,1-3} \\ c_{2,1,1-3} \\ c_{3,1,1-3} \end{bmatrix} + \begin{bmatrix} d_{1,3} \\ d_{2,3} \\ d_{3,3} \end{bmatrix} \begin{bmatrix} u_{r,2} \end{bmatrix} = Cx + Du \]

where \( c_{1,1,1-3}, c_{2,1,1-3}, c_{3,1,1-3}, d_{1,3}, d_{2,3}, \) and \( d_{3,3} \) are shown in Appendix D.
6.5 Evaluation of the Different Control Configurations

We now illustrate the proposed methodology with an MIMO example. As is seen from the proposed performance index, the state weighting matrix \( Q \), control weighting matrix \( R \) and the cross-coupling weighting matrix \( N \) are systematically determined with no burden on the designer to specify the weightings on the state and control independently. In addition to handle the coupling between states, inputs or states and inputs systematically, expressing \( RI_1 \), \( RI_2 \) and \( RI_3 \) in an output equation is able to weight three rollover indices simultaneously. Therefore, the controller for preventing three axles from rollover simultaneously is efficiently designed according to this form of output equation without analyzing the complicated coupling for rollover character of all axles independently.

The three control configurations are simulated in this section. Configuration 1 is to control all the three axles shown in section 6.4.1, configuration 2 is to control the tractor drive axle and the trailer unit shown in section 6.4.2, and configuration 3 is control the trailer unit only shown in section 6.4.3. In order to apply design observation 2.1 for evaluating the LQR design with control coupled output regulation, the external control effort \( J_{uex} \) is fixed for three designs by varying the weight \( \rho \) and the weighting matrix \( \bar{R} \). An initial absolute roll angle of unsprung mass of the trailer unit 1deg is assumed for all simulations. The assigned state weighting matrix \( Q \), the control weighting matrix \( R \), the weighting matrix for cross term between state and control variables \( N \), and the controller gain \( K \) are listed in Appendix E. The external control effort \( J_{uex} \), the normalized performance output regulation cost \( J_{yn} \), the coupling effort \( J_c \), the
rollover performance index ($J_R$), and the norm of controller gain ($|K|$) for three designs are obtained:

Control Configuration 1:

$$J_{uex} = 2.471 \times 10^{-13}; J_{yn} = 2.471 \times 10^{-13}; J_c = -1.9276 \times 10^{-19};$$

$$J_R = 4.9419 \times 10^{-13}; |K| = 1.3076 \times 10^{19}.$$  

Control Configuration 2:

$$J_{uex} = 2.471 \times 10^{-13}; J_{yn} = 1.8165 \times 10^{-13}; J_c = -1.9093 \times 10^{-19};$$

$$J_R = 4.9419 \times 10^{-13}; |K| = 1.3095 \times 10^{19}.$$  

Control Configuration 3:

$$J_{uex} = 2.471 \times 10^{-13}; J_{yn} = 1.0822 \times 10^{-14}; J_c = -4.9421 \times 10^{-18}$$

$$J_R = 4.942 \times 10^{-13}; |K| = 5.086 \times 10^{16}.$$  

Based on the fixed $J_{uex}$ for three designs, $J_{yn}$ in design 3 is smaller than $J_{yn}$ in design 2 and $J_{yn}$ in design 2 is smaller than $J_{yn}$ in design 1. Therefore, according to design observation 2.1, the controller in design 3 is better than the controller in design 2 and the controller in design 2 is better than the controller in design 1.

6.5.1 A Simulation for Double Lane Change Maneuver

A CL driver-vehicle-controller system simulation was conducted to investigate the rollover prevention of the proposed controller. In the simulation, a double lane change maneuver is stimulated to represent human drivers in lane following situation and a
constant forward velocity is assumed, which is 60km/hr. The steering angle from a driver is shown in Fig. 6.2. The simulation results of the roll torques (control inputs) are shown in Fig 6.3, 6.4, and 6.5 for configuration 1, configuration 2, and configuration 3 respectively. The rollover index for the tractor steering axle of configuration 1, 2, and 3 are shown in Fig. 6.6. The rollover index for the tractor driver axle of configuration 1, 2, and 3 are shown in Fig. 6.7. The rollover index for the trailer unit of configuration 1, 2, and 3 are shown in Fig. 6.8.

Fig. 6.2 Steering angle from a driver
Fig. 6.3 Control input (roll torque) for the control configure 1

Fig. 6.4 Control input (roll torque) for the control configure 2
Fig. 6.5 Control input (roll torque) for the control configure 3

Fig. 6.6 $RI$ of the tractor steering axle for the control configure 1, 2, and 3
Fig. 6.7 $RI$ of the tractor drive axle for the control configure 1, 2, and 3

Fig. 6.8 $RI$ of the trailer axles for the control configure 1, 2, and 3
In the simulations, control configuration 3 can maintain that $|RI_1|$, $|RI_2|$, and $|RI_3|$ always below 1, but control configuration 1 and 2 cannot keep $|RI_1|$, $|RI_2|$, and $|RI_3|$ to be always below 1. It shows that control configuration 3 can prevent the rollover more efficiently compared to control configuration 1 and control configuration 2 under the same situation. Note that actuators are relatively expensive and their number in engineering systems is desired to be reduced to a minimum. Control configuration 3 can control the vehicle to avoid the rollover by only control the trailer unit and the magnitude of the control input in control configuration 3 is feasible. The number of control input in control configure 1 is three and the number of control input in control configuration 2 is two. Nevertheless, more control inputs cannot improve the rollover prevention more in this simulation. That is because the controller in control configuration 3 has better component-wise cancellation than the control configuration 1 and 2. Therefore, the number of the controllers in control configuration 3 is minimized and still reaches good performance compared to control configuration 1 and 2. The multiple controllers in control configuration 1 and 2 have interferences from each other, where the energy is wasted on fighting each other, such that component-wise cancellation is destroyed. Hence, the effect from component-wise cancellation is attenuated in control configuration 1 and control configuration 2. Designing a controller or a set of controller with component-wise cancellation can improve the performance and reduce the control effort to saving the expense on actuators and power.
CHAPTER 7: A ROBUST CONTROL DESIGN FOR STABILITY AND ROLLOVER PREVENTION WITH UNCERTAIN DYNAMICS AND FAULTS

7.1 Introduction

In the real world, rollover events with variations in the rollover parameters (i.e., vehicle speed, road adhesion coefficient, etc.), the control algorithm has to be robust to accommodate these variations. In addition, the control structure has to be a fault-tolerant system in order to enhance rollover prevention. In this section, a robust controller design incorporates with the proposed methodology, which is the LQR design based on a rollover performance index, is established. This robust controller is for stability and rollover prevention and is designed to handle the uncertain dynamics and component faults. That is, the performance specifications are met and the uncertain dynamics and faults are considered by the proposed robust controller design. In the proposed robust controller design, uncertain dynamics and component faults is represented as varying parameters in vehicle modeling. In the example demonstrated in this chapter, interval parameters, which are forward velocity represents \( v_x \) and road adhesion coefficient \( \mu \), are considered. The forward velocity and the road adhesion coefficient are critical parameters in determining the rollover. That is because the vehicles with high velocity are prone to roll over and the contact between the vehicle and the ground plays a key role in determining the rollover consequence. Furthermore, the robust controller design is based
on polytopic uncertain systems and the concept of convex combination is utilized. Note that the example given in this chapter only addresses the uncertain dynamics, but the component faults is considered in the way, where component faults is treat as varying parameters such that the proposed methodology is also able to handle component faults.

### 7.2 A Polytopic Uncertain System for a Vehicle Model

In order to illustrate this approach, we consider the same 3-DOF bicycle model with active steering control shown in section 3.2 The vehicle dynamic model in state space form with an active steering control and a yaw-torque disturbance is given by

\[
\dot{x} = Ax + Bu \quad x(t_0) \text{ given} \tag{7.1a}
\]

\[
y =Cx + Du \tag{7.1b}
\]

where \(x = [\phi_2 \quad v_y \quad r \quad \phi_2]^T\), and \(u\) (control input) = steering angle \((\delta)\)

Now, we take into account the parameter uncertainties resulting from bounded vehicle velocity and road adhesion coefficient variations. That is, \(\underline{v}_x \leq v_x \leq \overline{v}_x\) and \(\underline{\mu} \leq \mu \leq \overline{\mu}\), where under bar and upper bar denote the lower bound and upper bound, respectively. The matrices \(A\) and \(B\) of the vehicle dynamic model with interval parameters \(\mu\) and \(v_x\) are represented as

\[
A(\mu, v_x) = \begin{bmatrix}
0 & 0 & 0 & 1 \\
-12.073 & A_{22} & A_{23} & -3.819 \\
0 & A_{32} & A_{33} & 0 \\
-12.023 & A_{42} & A_{43} & -3.803
\end{bmatrix}; \quad B(\mu) = \begin{bmatrix}
0 \\
63.019\mu \\
32.503\mu \\
22.227\mu
\end{bmatrix}
\]

where

\[
A_{22} = -147.803 \frac{\mu}{v_x}; \quad A_{22} = 7.679 \frac{\mu}{v_x} - v_x; \quad A_{22} = 2.031 \frac{\mu}{v_x}; \quad A_{22} = -116.563 \frac{\mu}{v_x}
\]
$A_{22} = -52.13 \frac{\mu}{v_x}; \ A_{22} = 2.708 \frac{\mu}{v_x} + 8.9641 \times 10^{-17} v_x \approx 2.708 \frac{\mu}{v_x}$

Note that matrix $A$ depends in a multi-affine fashion on the parameters $q_1 = \mu/v_x$ and $q_2 = v_x$ and matrix $B$ depends in an affine fashion on the parameters $q_3 = \mu$. Moreover, all varying parameters are positive. Hence, the polytopic uncertain system for robust control design is expressed as

$$\dot{x} = A(q)x + B(q)u$$

(7.2)

$$A(q) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -12.073 & A_{22} & A_{23} & -3.819 \\ 0 & A_{32} & A_{33} & 0 \\ -12.023 & A_{42} & A_{43} & -3.803 \end{bmatrix} \quad B(q) = \begin{bmatrix} 0 \\ 63.019q_3 \\ 32.503q_3 \\ 22.227q_3 \end{bmatrix}$$

$A_{22} = -147.803q_1; \ A_{23} = 7.679q_1 - q_2; \ A_{32} = 2.031q_1; \ A_{33} = -116.563q_1; \ A_{42} = -52.13q_1; \ A_{43} = 2.708q_1$

$$\frac{\mu}{v_x} \leq q_1 \leq \frac{\bar{\mu}}{v_x}; \ v_x \leq q_2 \leq \bar{v}_x; \ \mu \leq q_3 \leq \bar{\mu}$$

Since the matrices $A$ and $B$ depend in a multi-affine fashion on the parameters $q_i$, and each component of $q_i$ is bounded and positive, it follows that the matrix $A(q)$ and $B(q)$ can always be expressed as a convex combination of a finite number of matrices. These matrices are called vertex matrices corresponding to the extreme values of components of $q_i$. For this case, matrices $A$ is expressed as a convex combination of 8 vertex matrices which are
and \( \mathbf{B} \) is expressed as a convex combination of 8 vertex matrices which are

\[
\mathbf{B}_1 = \mathbf{B}(q_1, q_2, q_3), \quad \mathbf{B}_2 = \mathbf{B}(q_1, \overline{q_2}, q_3), \quad \mathbf{B}_3 = \mathbf{B}(\overline{q_1}, q_2, q_3), \quad \mathbf{B}_4 = \mathbf{B}(q_1, q_2, \overline{q_3})
\]
\[
\mathbf{B}_5 = \mathbf{B}(q_1, q_2, q_3), \quad \mathbf{B}_6 = \mathbf{B}(\overline{q_1}, q_2, q_3), \quad \mathbf{B}_7 = \mathbf{B}(q_1, \overline{q_2}, q_3), \quad \mathbf{B}_8 = \mathbf{B}(\overline{q_1}, \overline{q_2}, q_3).
\]

Specifically, the varying parameters in matrix \( \mathbf{A} \) are \( q_1 \) and \( q_2 \) and in matrix \( \mathbf{B} \) is \( q_3 \).

That is, matrix \( \mathbf{A} \) is expressed as a convex combination of 8 vertex matrices which are

\[
\mathbf{A}_1 = \mathbf{A}(q_1, q_2, q_3), \quad \mathbf{A}_2 = \mathbf{A}(q_1, q_2, q_3), \quad \mathbf{A}_3 = \mathbf{A}(q_1, \overline{q_2}, q_3), \quad \mathbf{A}_4 = \mathbf{A}(q_1, q_2, \overline{q_3}),
\]
\[
\mathbf{A}_5 = \mathbf{A}(\overline{q_1}, q_2, q_3), \quad \mathbf{A}_6 = \mathbf{A}(\overline{q_2}, q_2, q_3), \quad \mathbf{A}_7 = \mathbf{A}(\overline{q_1}, \overline{q_2}, q_3), \quad \mathbf{A}_8 = \mathbf{A}(\overline{q_1}, \overline{q_2}, \overline{q_3}).
\]  

Hence, there are 8 vertex systems according these vertex matrices represented in equation (7.3) are

\[
\dot{x}_i = \mathbf{A}_i \mathbf{x} + \mathbf{B}_i \mathbf{u} \quad \text{for } i = 1, \ldots, 8
\]  

(7.5)

The matrices \( \mathbf{C} \) and \( \mathbf{D} \) in the performance output which is equation (7.1b) are represented according to the matrices \( \mathbf{A} \) and \( \mathbf{B} \) with weightings. Hence, the matrices \( \mathbf{C} \) and \( \mathbf{D} \) also have interval parameters \( q \). That is,

\[
y = r_1 \mathbf{A}(q) \mathbf{x} + r_2 \mathbf{B}(q) \mathbf{u} = \mathbf{C}(q) \mathbf{x} + \mathbf{D}(q) \mathbf{u}
\]  

(7.6)

where \( \mathbf{r} \) is the weighting vector acquired from the rollover index. Hence, there are 8
vertex performance outputs according the combinations of these vertex matrices. They are
\[
y_i = C_i x + D_i u = R_i
\]  (7.7)

The robust controller is designed based on this polytopic uncertain system and these vertex systems are applied in the design procedure.

7.3 A Robust Full State Feedback Control Design based on Polytopic Uncertain Systems

The robust controller is designed to guarantee stability and performance with respect to variations of forward velocity and road adhesion coefficient, where is \(v_x \leq v_x \leq v_x\) and \(\mu \leq \mu \leq \mu\). The technique for the robust state feedback controller design which satisfies stability and performance is proposed.

7.3.1 The Structure of the Matrices in the OL systems and CL Systems

The controllable plant in equation (7.1) under full state feedback \(u = -K x\) is considered. A controller \(K\) is obtained from a convex combination of individual LQR controller \(K_i = R_i^{-1} \left( B_i^T P_i + N_i^T \right)\), which are obtained from the vertex systems \(\dot{x}_i = A_i x + B_i u\ (i = 1, \ldots, 8)\) and arbitrary fixed weighting matrices \((Q_i, R_i, N_i)\) via the Algebraic Ricatti Equation (ARE), as the common controller. The convex combination of the individual LQR controllers are represented by
\[
K = \sum_{i=1}^{8} \beta_i K_i
\]  (7.8)

where the real number \(\beta_i\) satisfy \(\beta_i \geq 0\) and \(\sum_{i=1}^{P} \beta_i = 1\). Moreover, \(K_i\) are the
controller gain designed for all vertex systems in equation (7.5).

Hence, applying the controller gain in equation (7.8) to the OL system in equation (7.1) yields the CL system in equation (7.9).

\[
\begin{align*}
\dot{x} &= (A - BK)x + Bw = A_{CL}x \quad (7.9a) \\
y &= (C - DK)x + Dw = C_{CL}x = RI_{CL} \quad (7.9b)
\end{align*}
\]

The 8 vertex matrices for matrices \( A, B, C \) and \( D \) in equation (7.3) and equation (7.4) are

\[
A_i = A(q_i); B_i = B(q_i); C_i = C(q_i); D_i = D(q_i) \quad (7.10)
\]

where \( q_i \) equals its minimum or maximum value for \( i = 1, \ldots, 8 \).

Moreover, the vertex systems for the CL system are represented in the equation (7.11).

\[
\begin{align*}
\dot{x}_{CLi} &= (A_i - B_iK)x = A_{CLi}x_{CLi} \quad (7.11a) \\
y_{CLi} &= (C_i - D_iK)x = C_{CLi}x = RI_{CLi} \quad (7.11b)
\end{align*}
\]

where \( i = 1, \ldots, 8 \).

The eight vertex matrices for the CL system are in equation (7.12).

\[
A_{CLi} = A_{CL}(q_i), C_{CLi} = C_{CL}(q_i) \quad (7.12)
\]

where \( q_i \) equals its minimum or maximum value for \( i = 1, \ldots, 8 \).

### 7.4 Robust Control for Stability

**Theorem 7.1** [44]: The system (7.1) with parameters \((A_i, B_i) (i = 1 \ldots 8)\) and \((A_i, B_i)\) is simultaneously stabilisable by a state feedback \( u = -Kx \) if and only if the ARE

\[
\begin{align*}
P_i[A_i - B_i \sum_{j \neq i} \beta_j R_j^{-1}(B_j^T P_j + N_j^T)] + [A_i - B_i \sum_{j \neq i} \beta_j R_j^{-1}(B_j^T P_j + N_j^T)]^T P_i \\
- 2P_i B_i \beta_i R_i^{-1} B_i^T P_i + Q_i - 2N_i \beta_i R_i^{-1} N_i^T = 0 \quad (i = 1, \ldots, 8)
\end{align*}
\]
with arbitrary positive definite matrices \( Q_i \) and \( R_i \) have positive definite solutions \( P_i \).

If this condition is satisfied, a stabilizing controller is given by equation (7.8).

**PROOF.**

The CL system is defined in (7.9) and the vertex systems for the CL system are represented in equation (7.11). The constant controller gain \( K \) is represented as a convex combination shown in equation (7.8).

\[ \Rightarrow \]

The system (7.1) with parameters \((A_i, B_i) (i = 1 \ldots 8)\) and \((A_i, B_i)\) are vertex matrices shown in equation (7.10). The CL system is simultaneously stabilisable by a state feedback \( K = \sum \beta_i K_i \) shown in equation (7.8).

The CL vertex systems are

\[
\dot{x}_{CL_i} = (A_i - B_i K)x = \left( A_i - B_i \sum_{i=1}^{8} \beta_i K_i \right)x = \left( A_i - B_i \sum_{i \neq j}^{8} \beta_i K_j \right)x - \beta_i B_i K_i x
\]

The \( K \) is obtained using LQR design, so it is expressed as

\[
K = \sum_{i=1}^{8} \beta_i K_i = \sum_{i=1}^{8} \beta_i R_i^{-1}\left( B_i^T P_i + N_i^T \right)
\]

where \( P_i \) is solution of the ARE

\[
P_i[A_i - B_i \sum_{j \neq i} \beta_j R_j^{-1}(B_j^T P_j + N_j^T)] + [A_i - B_i \sum_{j \neq i} \beta_j R_j^{-1}(B_j^T P_j + N_j^T)]^T P_i
\]

\[ - 2P_i B_i \sum_{j} \beta_j R_j^{-1} B_i^T P_i + Q_i - 2N_i \beta_i R_i^{-1} N_i^T = 0 \quad (i = 1, \ldots, 8) \quad (7.13)
\]

Hence, the single controller which simultaneously stabilizes all vertex systems is expressed
\[ K = \sum_{i=1}^{8} \beta_i K_i. \]

The single LQR controller gain in the equation (7.8) with matrices \( P_i \) obtained from the equation (7.13) can stabilize all vertex systems. Therefore, the system (7.1) with parameters \((A_j, B_j)\) (\(i = 1...8\)) and \((A_j, B_j)\) is simultaneously stabilisable by a state feedback \( u = -Kx \) if equation (7.13) exists.

\( \Leftarrow \)

The state feedback \( K = \sum_{i=1}^{8} \beta_i K_i \) is assumed, and the CL vertex systems is

\[
\dot{x}_{CL_i} = (A_i - B_i K)x = \left( A_i - B_i \sum_{i=1}^{8} \beta_i K_i \right)x = \left( A_i - B_i \sum_{i \neq j} \beta_j K_j \right)x - \beta_i B_i K_i x
\]

The \( K \) is obtained using LQR design, and it is expressed as

\[
K = \sum_{i=1}^{8} \beta_i K_i = \sum_{i=1}^{8} \beta_i R_i^{-1} \left( B_i^T P_i + N_i^T \right)
\]

where \( P_i \) is solution of the ARE

\[
P_i [A_i - B_i \sum_{j \neq i} \beta_j R_j^{-1} (B_j^T P_j + N_j^T)] + [A_i - B_i \sum_{j \neq i} \beta_j R_j^{-1} (B_j^T P_j + N_j^T)]^T P_i - 2P_i B_i \beta_i R_i^{-1} B_i^T P_i + Q_i - 2N_i \beta_i R_i^{-1} N_i^T = 0 \quad (i = 1,...,8)
\]

It shows that the single control gain \( K \) stabilizes all vertex system, and is expressed as

\[
K = \sum_{i=1}^{8} \beta_i K_i = \sum_{i=1}^{8} \beta_i R_i^{-1} \left( B_i^T P_i + N_i^T \right)
\]

(7.14)

Therefore, if the condition shown in equation (7.13) is satisfied, the system (7.1) with
parameters \((A_i, B_i) (i = 1...8)\) and \((A_i, B_i)\) is simultaneously stabilisable by a state feedback \(u = -Kx\). In addition, the stabilizing controller is given by equation (7.14).

**Theorem 7.2** [38]: All the matrices belonging to the convex combination matrix family (with four vertex matrices \(A^{vi}\) and the ‘virtual center’ matrix \(A^v + A^v^2 + A^v^3 + A^v^4\) being Hurwitz stable) are Hurwitz stable if and only if the 8 exposed ‘Kronecker Nonsingularity Matrices’ (KN matrices) of two vertex matrices taken at a time, namely

\[
-[(L^{v1} + L^{v2})^{-1} L^{v1}] \quad \text{and} \quad -[(L^{v1} + L^{v2})^{-1} L^{v2}];
\]

\[
-[(L^{v2} + L^{v4})^{-1} L^{v2}] \quad \text{and} \quad -[(L^{v2} + L^{v4})^{-1} L^{v4}];
\]

\[
-[(L^{v4} + L^{v3})^{-1} L^{v4}] \quad \text{and} \quad -[(L^{v4} + L^{v3})^{-1} L^{v3}];
\]

\[
-[(L^{v3} + L^{v1})^{-1} L^{v3}] \quad \text{and} \quad -[(L^{v3} + L^{v1})^{-1} L^{v1}]
\]

are ‘Real Axis stable’ and the 4 ‘Kronecker Nonsingularity Matrices’ (KN matrices) of all four vertex matrices taken at a time, namely

\[
-[(L^{v1} + L^{v2} + L^{v3} + L^{v4})^{-1} L^{v1}]; \quad -[(L^{v1} + L^{v2} + L^{v3} + L^{v4})^{-1} L^{v2}];
\]

\[
-[(L^{v1} + L^{v2} + L^{v3} + L^{v4})^{-1} L^{v3}]; \quad -[(L^{v1} + L^{v2} + L^{v3} + L^{v4})^{-1} L^{v4}]
\]

are all ‘Virtually stable’.

**Corollary 7.1**: A considered CL system with interval matrices \(A\) and \(B\) is

\[
\dot{x}_{CL} = (A - BK)x = A_{CL}x.
\]

The vertex systems of the above CL system with interval matrices \(A\) and \(B\) is

\[
\dot{x}_{CLi} = (A_i - B_i K)x = A_{CLi}x.
\]
If theorem 7.2 is satisfied for the polytopic matrix family $A_{CL}$, all the matrices belonging to this polytopic matrix family are asymptotically stable.

Note that the disturbance will not affect the stability of the system. Therefore, the robust controller for stability of a given polytopic matrix family is available after the disturbance is applied.

7.5 Robust Control for Performance (Rollover Prevention)

Theorem 7.3: The robust controller for performance (rollover prevention) is obtained from one of the vertex systems. If all controlled vertex systems using the robust controller gain for stability can satisfy the requirement of performance (rollover prevention), the robust controller for stability can also guarantees desired performance ($y = |RI_{CL}| < 1$) which is robust controller for stability and performance with respect to a given polytopic matrix family.

PROOF.

In this study, the desired performance is rollover prevention, so the absolute value of the performance output in the equation (7.9b) is less than one, i.e., $|RI_{CL}| < 1$ within the range of varying parameters.

Since the absolute value function (i.e., $y(x) = |x|$) is convex, the maximum of a convex function $y$ relative to a convex set occurs at some extreme points. Extreme points in a convex set form the vertex matrices. Hence, the maximum of $y = |RI_{CL}|$ coincides with the maximum of $|RI_{CL}|$. That is, if $|RI_{CL}| < 1$, the $|RI_{CL}| < 1$ within the range of the varying parameter. Then, the theorem 7.3 is proved.
7.5.1 Level of Performance

The level of the performance which depends on the initial conditions and disturbances is defined in this section. The CL system, where the controller is the robust controller for the stability and the performance, with the disturbance is given by

\[
\begin{align*}
\dot{x} &= (A - BK)x + B_w w = A_{CL} x + Bw \\
y &= (C - DK)x + D_w w = C_{CL} x + Dw = R I_{CL}
\end{align*}
\]

where \( A_{CL} \) is stable. Assume the disturbance \( w \) is bounded so the disturbance in the equation (7.15) is described as \( |w| \leq L \) where \( L \) is a constant. In addition,

\[
0 \leq e^{\alpha(A_{CL})(t-t_0)} \leq 1 ; \quad 0 \leq 1 - e^{\alpha(A_{CL})(t-t_0)} \leq 1 ;
\]

\[
\|e^{A_{CL}(t-\tau)}\| \leq \kappa(M)e^{\alpha(A_{CL})(t-\tau)} ; \quad \|e^{A_{CL}(t-t_0)}\| \leq \kappa(M)e^{\alpha(A_{CL})(t-t_0)}
\]

where \( \alpha(A_{CL}) \) denotes the maximum real part of eigenvalues of \( A_{CL} \), \( M \) is modal matrix and \( \kappa(M) \) is the condition number of \( M \).

\[
|R_{CL}| = |C_{CL} x + D_w w| \leq |C_{CL} x| + |D_w w| \leq \|C_{CL}\| \|x\| + \|D_w w\| \leq \|C_{CL}\| \|x\| + |D_w| L
\]

where

\[
\|C_{CL}\| \|x(t)\| = \|C_{CL}\| \left\| e^{A_{CL}(t-t_0)} x_0 + \int_{t_0}^{t} e^{A_{CL}(t-\tau)} B_w w d \tau \right\|
\]

\[
\leq \left\|C_{CL}\right\| \left[ \left\|e^{A_{CL}(t-t_0)} x_0\right\| + \int_{t_0}^{t} \left\|e^{A_{CL}(t-\tau)} B_w w\right\| d \tau \right]
\]
\[
\begin{align*}
&\leq \|C_{CL}\| \left[ e^{A_{CL} (t-t_0)} \|x_0\| + \int_{t_0}^{t} e^{A_{CL} (t-\tau)} \|B_w\| \|w\| \, d\tau \right] \\
&\leq \|C_{CL}\| \left[ \kappa(M) e^{\alpha(A_{CL}) (t-t_0)} \|x_0\| + \int_{t_0}^{t} \kappa(M) e^{\alpha(A_{CL}) (t-\tau)} \|B_w\| \|w\| \, d\tau \right] \\
&\leq \|C_{CL}\| \left[ \kappa(M) e^{\alpha(A_{CL}) (t-t_0)} \|x_0\| + \frac{\kappa(M) \|B_w\| L}{-\alpha(A_{CL})} (1 - e^{\alpha(A_{CL}) (t-t_0)}) \right] \\
&\leq \|C_{CL}\| \left[ \kappa(M) \|x_0\| + \frac{\kappa(M) \|B_w\| L}{-\alpha(A_{CL})} \right]
\end{align*}
\]

Therefore,

\[
|RI_{CL}| \leq \|C_{CL}\| \|x\| + \|D\| L
\]

\[
\leq \|C_{CL}\| \left[ \kappa(M) \|x_0\| + \frac{\kappa(M) \|B_w\| L}{-\alpha(A_{CL})} \right] + \|D_w\| L
\]

\[
= \|C_{CL}\| \kappa(M) \|x_0\| + \|C_{CL}\| \left[ \frac{\kappa(M) \|B_w\| L}{-\alpha(A_{CL})} \right] + \|D_w\| L
\]

\[
= \|C_{CL}\| \kappa(M) \|x_0\| + \|C_{CL}\| \left[ \frac{\kappa(M) \|B_w\| L}{-\alpha(A_{CL})} + \|D_w\| \right] L
\]

The threshold for initial conditions and disturbance for rollover are calculated. The controlled vehicle will not roll over if the \(|RI_{CL}| < 1\).

(1) For undisturbed system (i.e., \(w = L = 0\)),

The threshold for initial conditions \((x_0)\):
\[ \|C_{CL}\|\kappa(M)\|x_0\| < 1 \Rightarrow \|x_0\| < \frac{1}{\|C_{CL}\|\kappa(M)} \]

(2) For disturbed system (i.e., \(w \neq 0\) and \(L \neq 0\)),

(2a) For given initial \((x_0)\), the threshold of the disturbance \((L)\) is

\[ \|C_{CL}\|\kappa(M)\|x_0\| + \left[\|C_{CL}\|\frac{\kappa(M)\|B_w\|}{\alpha(-A_{CL})} + \|D\|\right] L < 1 \Rightarrow L < \frac{1 - \|C_{CL}\|\kappa(M)\|x_0\|}{\|C_{CL}\|\frac{\kappa(M)\|B_w\|}{\alpha(-A_{CL})} + \|D\|} \]

(2b) For given disturbance \((L)\), the threshold of the initial condition \((x_0)\) is

\[ \|C_{CL}\|\kappa(M)\|x_0\| + \left[\|C_{CL}\|\frac{\kappa(M)\|B_w\|}{\alpha(-A_{CL})} + \|D_w\|\right] L < 1 \]

\[ \Rightarrow \|x_0\| < \frac{1 - \left[\|C_{CL}\|\frac{\kappa(M)\|B_w\|}{\alpha(-A_{CL})} + \|D_w\|\right] L}{\|C_{CL}\|\kappa(M)} \]

The thresholds of the initial condition and disturbance presented in this section are for robust performance (rollover prevention). That is, the robust performance controller guarantees the performance (rollover prevention) which is robust with respect to all varying parameters (whole uncertain system). Since these thresholds are calculated based on the worst performance case due to the many inequalities involved, these thresholds for initial condition and disturbance in this section is conservative. However, the evaluation for the level of performance proposed in this section gives the information about the quality of the designed robust controller. The robust controller is from the convex combination of the stabilizing controller of all vertex systems and an appropriate
convex combination for robust controller is still under developed. Now the selection of an appropriate convex combination is a trial-and-error procedure. Based on the level of performance represented in this section, we are able to judge the efficiency of the designed robust controller. Hence, the level of performance is useful to establish the criterion for selecting an appropriate convex combination in robust controller design.
8.1 Conclusions

In this dissertation, the main objective is to design controllers for rollover prevention of military multi-body ground vehicles. We first propose a novel control system analysis and design technique by extending the popular LQR control design method specializing it for the ‘control coupled output regulation’ problem. Specifically, in this rollover prevention problem, a ‘unified rollover index’ is proposed, which captures both the roll dynamics and lateral dynamics, explicitly into the optimization procedure of the LQR framework, which results in a performance index with a coupling term in state and control variables. Thus, the proposed rollover prevention technique effectively incorporates the physical nature of the vehicle dynamics into the problem formulation resulting in significantly improved rollover performance by the coupling in the vehicle dynamic states and control variables. In addition, the proposed technique is used to evaluate different controller configurations and select the most efficient controller structure in terms of both control effort as well as cost. It is shown that sometimes it is possible that a well designed single controller (actuator) can result in better performance than multiple controllers (actuators) with improper design because of the way the coupling effect interacts with the controller. The second issue addressed in this dissertation is a robust control design to handle uncertainty in the vehicle dynamics para-
meters as well as component faults. Based on the theory of ‘Linear interval parameter matrix families’, a single robust full state feedback control gain is designed by a convex combination of the control gains designed for finite points (vertices) of the uncertain parameter space. The proposed robust controller design is applied to the multi-body ground vehicle control with uncertainties in the forward speed of the vehicle and the road adhesion coefficient taken into consideration. The results clearly show the efficacy of the proposed robust controller under the assumed perturbations. Thus the proposed techniques in this dissertation, which focus on the controller design for vehicle dynamics, helps in not only maintaining stability and preventing rollover of multi-body ground vehicles with reduced control effort (which in turn translates to considerable actuator and power savings) but also guarantee the stability and performance for vehicles with uncertain dynamics and component faults. In addition, the proposed techniques are sufficiently generic to be applicable to other applications in fields such as aerospace, mechanical, electrical and other systems.

8.2 Future Research Directions

In practical applications, as long as the performance output which is desired to be minimized is derived and is incorporated into the performance index of LQR framework, an effective controller for regulating the performance output using LQR design with control coupled output regulation is designed. Therefore, the dynamics of different systems with different objectives will be studied in order to derive the relationship between the performance output and the states and inputs in the future. In practice, not all states is measured. In order to introduce the proposed technique to the real word, estimation systems for different applications will be developed according to the specific
performance output. Regarding the robust control issue, the robust controller is derived from the convex combination of the stabilizing controllers of all vertex systems. Present research is limited to two parameters, so further research is needed to extend it to multiple parameters.
REFERENCES


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APPENDIX A: The Numerical Vehicle Data for the 3 DOF Bicycle Model in Section 3.2

<table>
<thead>
<tr>
<th>vehicle parameters</th>
<th>numerical vehicle data from [4]</th>
<th>numerical vehicle data from TARDEC</th>
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<tbody>
<tr>
<td>front cornering stiffness</td>
<td>$c_f = 582$ kN/rad</td>
<td>$c_f = 12.002$ kN/rad</td>
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<td>rear cornering stiffness</td>
<td>$c_r = 783$ kN/rad</td>
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<td>roll damping of passive suspension</td>
<td>$d_\phi = 100$ kN/rad</td>
<td>$d_\phi = 25.499$ kN/rad</td>
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<td>acceleration due to gravity</td>
<td>$g = 9.81$ m/s$^2$</td>
<td>$g = 9.81$ m/s$^2$</td>
</tr>
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<td>height of roll axis over ground</td>
<td>$h_R = 0.68$ m</td>
<td>$h_R = 0.36$ m</td>
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<td>nominal height of CG2 over roll axis</td>
<td>$h = 1.15$m</td>
<td>$h = 0.61$ m</td>
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<td>roll moment of inertia, sprung mass</td>
<td>$J_{z,x} = 24201$ kgm$^2$</td>
<td>$J_{z,x} = 3718$ kgm$^2$</td>
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<tr>
<td>overall yaw moment of inertia</td>
<td>$J_z = 34917$ kgm$^2$</td>
<td>$J_z = 27206$ kgm$^2$</td>
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<td>distance front axle to CG1</td>
<td>$l_f = 1.95$ m</td>
<td>$l_f = 2.54$ m</td>
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<tr>
<td>distance rear axle to CG1</td>
<td>$l_r = 1.54$ m</td>
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<td>$m = 9072$ kg</td>
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<td>sprung mass</td>
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<td>$m_2 = 7258$ kg</td>
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<tr>
<td>road adhesion coefficient</td>
<td>$\mu = 1$</td>
<td>$\mu = 0.8$</td>
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<tr>
<td>track width</td>
<td>$T = 1.86$</td>
<td>$T = 0.99$</td>
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APPENDIX B: The Numerical Vehicle Data
for the 9-DOF Tractor Semi-trailer Model in Section 3.3 [18]

Vehicle parameters:

Axle and body geometry

<table>
<thead>
<tr>
<th>Vehicle unit</th>
<th>$h_s$</th>
<th>$h_y^*$</th>
<th>$b'_f$</th>
<th>$h_{u,r}$</th>
<th>$r$</th>
<th>$h_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tractor</td>
<td>1.058</td>
<td>0.530</td>
<td>1.115</td>
<td>1.250</td>
<td>0.621</td>
<td>0.776</td>
</tr>
<tr>
<td>Semi-trailer</td>
<td>1.900</td>
<td>0.530</td>
<td>5.653</td>
<td>1.100</td>
<td>0.1</td>
<td>2.050</td>
</tr>
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<td>Units</td>
<td>m</td>
<td>m</td>
<td>m</td>
<td>m</td>
<td>m</td>
<td>m</td>
</tr>
</tbody>
</table>

* Identical on both tractor axles and on all three trailer axles

Body inertia

<table>
<thead>
<tr>
<th>Vehicle unit</th>
<th>$m_s$</th>
<th>$I_{xx}$</th>
<th>$I_{zz}$</th>
<th>$I_{xz}$</th>
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</thead>
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<td>Tractor</td>
<td>4819</td>
<td>2411</td>
<td>11383</td>
<td>1390</td>
</tr>
<tr>
<td>Semi-trailer</td>
<td>30821</td>
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<td>223625</td>
<td>14577</td>
</tr>
<tr>
<td>Units</td>
<td>kg</td>
<td>Kg·m²</td>
<td>Kg·m²</td>
<td>Kg·m²</td>
</tr>
</tbody>
</table>

Axle inertia

<table>
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<tr>
<th>Vehicle unit</th>
<th>Axle</th>
<th>$m_a$</th>
<th>$I_{xx}$</th>
<th>$I_{zz}$</th>
<th>$I_{xz}$</th>
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<td>Tractor</td>
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<td>706</td>
<td>440</td>
<td>440</td>
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<td>Tractor</td>
<td>drive</td>
<td>1000</td>
<td>563</td>
<td>563</td>
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<tr>
<td>Semi-trailer</td>
<td>1,2,3</td>
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<td>564</td>
<td>564</td>
<td>0</td>
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<tr>
<td>Units</td>
<td>kg</td>
<td>Kg·m²</td>
<td>Kg·m²</td>
<td>Kg·m²</td>
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</table>

Frame and coupling properties

<table>
<thead>
<tr>
<th>Vehicle unit</th>
<th>$k_b^*$</th>
<th>$k_\phi^*$</th>
</tr>
</thead>
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<tr>
<td>Tractor</td>
<td>629</td>
<td>3000</td>
</tr>
<tr>
<td>Units</td>
<td>kNm·rad⁻¹</td>
<td>kNm·rad⁻¹</td>
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</tbody>
</table>

* For the semi-trailer, the torsional flexibilities of the coupling and vehicle frame are lumped together into $k_\phi$, and the vehicle frame is modeled as a rigid body
Suspension and tyre properties

<table>
<thead>
<tr>
<th>Vehicle unit</th>
<th>Axle</th>
<th>$k$</th>
<th>$k_t$</th>
<th>$l$</th>
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<tbody>
<tr>
<td>Tractor</td>
<td>steer</td>
<td>380</td>
<td>2060</td>
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<td>Tractor</td>
<td>drive</td>
<td>684</td>
<td>3337</td>
<td>6.68</td>
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<tr>
<td>Semi-trailer</td>
<td>1,2,3</td>
<td>800</td>
<td>1776</td>
<td>23.9</td>
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</tbody>
</table>

Units: kNm-rad$^{-1}$ kNm-rad$^{-1}$ kNm-rad$^{-1}$

Fig. B.1 Rollover model of the tractor semi-trailer [18]
APPENDIX C: The State Equation for the Tractor Semi-Trailer in Section 3.3 [18]

The state equation for the tractor-semitrailer is \( \dot{x} = Ax + B_0u + B_1\delta \).

The resulting matrix \( A, B_0, \) and \( B_1 \) are as follows:

\[
x = [\phi_{f,1} \dot{\phi}_{f,1} \phi_{r,1} \dot{\phi}_{r,1} \beta_1 \psi_1 \phi_{f,1} \phi_{r,1} \phi_2 \dot{\phi}_2 \beta_2 \psi_2 \phi_{r,2}]^T
\]

In section 6.5.1, \( u = [u_{f,1} \ u_{r,1} \ u_{r,2}]^T \) and \( B_0 = E^{-1}V \).

\[
V = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T
\]

In section 6.5.2, \( u = [u_{r,1} \ u_{r,2}]^T \) and \( B_0 = E^{-1}V \).

\[
V = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T
\]

In section 6.5.3, \( u = [u_{r,2}] \) and \( B_0 = E^{-1}V \).

\[
V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 \end{bmatrix}^T
\]

\( B_1 = E^{-1}W \). The disturbance is steering angle from a driver (\( \delta \)).

\[
W = [N_{\delta,1} - b'_{f,1} Y_{\delta,1} - h_{b,1} Y_{\delta,f,1} (r_1 - h_{u,r,1}) Y_{\delta,1} + h_{b,1} Y_{\delta,f,1} \ r_1 Y_{\delta,f,1} \ \cdots \ \cdots \ 0 \ 0 \ b'_{f,2} Y_{\delta,1} - (r_2 - h_{u,f,2}) Y_{\delta,1} 0 \ Y_{\delta,1} 0 \ 0 \ 0 ]^T
\]

\( A = E^{-1}U \)
<table>
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<tr>
<th></th>
<th>( l_{x,y} )</th>
<th>( h_{x,j} )</th>
<th>( \bar{m}_{x,j} )</th>
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APPENDIX D: The Elements for the Performance Output Equation in Section 6.4.1, Section 6.4.2, and Section 6.4.3

\[ c_{1,1} = \frac{2m_{s,f,1}h_{R,f,1}}{m_{f,1}gT} (v_x A_{5,1} - h_{f,1} A_{2,1}) + \frac{2k_{f,1}}{m_{f,1}gT}; \]

\[ c_{1,2} = \frac{2m_{s,f,1}h_{R,f,1}}{m_{f,1}gT} (v_x A_{5,2} - h_{f,1} A_{2,2}) + \frac{2c_{f,1}}{m_{f,1}gT}; \]

\[ c_{1,3} = \frac{2m_{s,f,1}h_{R,f,1}}{m_{f,1}gT} (v_x A_{5,3} - h_{f,1} A_{2,3}); \]

\[ c_{1,4} = \frac{2m_{s,f,1}h_{R,f,1}}{m_{f,1}gT} (v_x A_{5,4} - h_{f,1} A_{2,4}) \]

\[ c_{1,5} = \frac{2m_{s,f,1}h_{R,f,1}}{m_{f,1}gT} (v_x A_{5,5} - v A_{2,5}); \]

\[ c_{1,6} = \frac{2m_{s,f,1}h_{R,f,1}}{m_{f,1}gT} (v_x A_{5,6} - h_{f,1} A_{2,6} + v_x) \]

\[ c_{1,7} = \frac{2m_{s,f,1}h_{R,f,1}}{m_{f,1}gT} (v_x A_{5,7} - h_{f,1} A_{2,7}); \]

\[ c_{1,8} = \frac{2m_{s,f,1}h_{R,f,1}}{m_{f,1}gT} (v_x A_{5,8} - h_{f,1} A_{2,8}) \]

\[ c_{1,9} = \frac{2m_{s,f,1}h_{R,f,1}}{m_{f,1}gT} (v_x A_{5,9} - h_{f,1} A_{2,9}); \]

\[ c_{1,10} = \frac{2m_{s,f,1}h_{R,f,1}}{m_{f,1}gT} (v_x A_{5,10} - h_{f,1} A_{2,10}) \]

\[ c_{1,11} = \frac{2m_{s,f,1}h_{R,f,1}}{m_{f,1}gT} (v_x A_{5,11} - h_{f,1} A_{2,11}); \]

\[ c_{1,12} = \frac{2m_{s,f,1}h_{R,f,1}}{m_{f,1}gT} (v_x A_{5,12} - h_{f,1} A_{2,12}) \]

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\[ c_{1,13} = \frac{2m_{s,f,l} h_{R,f,1}}{m_{f,l} gT} (v_x A_{5,13} - h_{f,l} A_{2,13}); \quad d_{1,1} = \frac{2m_{s,f,l} h_{R,f,1}}{m_{f,l} gT} (v_x B_{5,1} - h_{f,l} B_{2,1}); \]

\[ d_{1,2} = \frac{2m_{s,f,l} h_{R,f,1}}{m_{f,l} gT} (v_x B_{5,2} - h_{f,l} B_{2,2}); \quad d_{1,3} = \frac{2m_{s,f,l} h_{R,f,1}}{m_{f,l} gT} (v_x B_{5,3} - h_{f,l} B_{2,3}); \]

\[ c_{2,1} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x A_{5,4} - h_{r,l} A_{4,4}); \quad c_{2,2} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x A_{5,2} - h_{r,l} A_{4,2}); \]

\[ c_{2,3} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x A_{5,3} - h_{r,l} A_{4,3}) + \frac{2k_{r,l}}{m_{r,l} gT}; \]

\[ c_{2,4} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x A_{5,4} - h_{r,l} A_{4,4}) + \frac{2c_{r,l}}{m_{r,l} gT}; \]

\[ c_{2,5} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x A_{5,5} - h_{r,l} A_{4,5}); \quad c_{2,6} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x A_{5,6} - h_{r,l} A_{4,6} + v_x); \]

\[ c_{2,7} = \frac{2m_{s,r,1} h_{R,f,1}}{m_{r,l} gT} (v_x A_{5,7} - h_{r,l} A_{4,7}); \quad c_{2,8} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x A_{5,8} - h_{r,l} A_{4,8}); \]

\[ c_{2,9} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x A_{5,9} - h_{r,l} A_{4,9}); \quad c_{2,10} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x A_{5,10} - h_{r,l} A_{4,10}); \]

\[ c_{2,11} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x A_{5,11} - h_{r,l} A_{4,11}); \quad c_{2,12} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x A_{5,12} - h_{r,l} A_{4,12}); \]

\[ c_{2,13} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x A_{5,13} - h_{r,l} A_{4,13}); \quad d_{2,1} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x B_{5,1} - h_{r,l} B_{4,1}); \]

\[ d_{2,2} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x B_{5,2} - h_{r,l} B_{4,2}); \quad d_{2,3} = \frac{2m_{s,r,1} h_{R,r,1}}{m_{r,l} gT} (v_x B_{5,3} - h_{r,l} B_{4,3}). \]
\[ c_{3,1} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x A_{11,1} - h_{r,2}A_{10,1}) \] ; \[ c_{3,2} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x A_{11,2} - h_{r,2}A_{10,2}) \] ;

\[ c_{3,3} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x A_{11,3} - h_{r,2}A_{10,3}) \] ; \[ c_{3,4} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x A_{11,4} - h_{r,2}A_{10,4}) \] ;

\[ c_{3,5} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x A_{11,5} - h_{r,2}A_{10,5}) \] ; \[ c_{3,6} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x A_{11,6} - h_{r,2}A_{10,6}) \] ;

\[ c_{3,7} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x A_{11,7} - h_{r,2}A_{10,7}) \] ; \[ c_{3,8} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x A_{11,8} - h_{r,2}A_{10,8}) \] ;

\[ c_{3,9} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,1}gT} (v_A x A_{11,9} - h_{r,2}A_{10,9}) + \frac{2k_{r,2}}{mgT} \] ;

\[ c_{3,10} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x A_{11,10} - h_{r,2}A_{10,10}) + \frac{2c_{r,2}}{m_{r,2}gT} \] ;

\[ c_{3,11} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x A_{11,11} - h_{r,2}A_{10,11}) \] ; \[ c_{3,12} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x A_{11,12} - h_{r,2}A_{10,12} + v_A) \] ;

\[ c_{3,13} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x A_{11,13} - h_{r,2}A_{10,13}) \] ; \[ d_{3,1} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x B_{11,1} - h_{r,2}B_{10,1}) \] ;

\[ d_{3,2} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x B_{11,2} - h_{r,2}B_{10,2}) \] ; \[ d_{3,3} = \frac{2m_{s,r,2}h_{r,r,2}}{m_{r,2}gT} (v_A x B_{11,3} - h_{r,2}B_{10,3}) \] .
APPENDIX E: The Design Variables and Control Gain for the Example in Section 6.5

In this appendix, the assigned state weighting matrix $Q$, the control weighting matrix $R$, the weighting matrix for cross term between state and control variables $N$ is calculated, and the controller gain ($K$) for the example in section 6.5 are listed.

$$Q = \rho C^T C; \quad N = \rho C^T D; \quad R = \rho D^T D + \bar{R}.$$  

Control Configuration 1:

$$\rho = 1; \quad \bar{R} = \begin{bmatrix} 2.0645 \times 10^{-31} & 0 & 0 \\ 0 & 5.4945 \times 10^{-33} & 0 \\ 0 & 0 & 1.0931 \times 10^{-28} \end{bmatrix};$$

$$C = \begin{bmatrix} 8.6787 & -119.55 & 2.8256 \\ 0.038905 & 6.4077 \times 10^{-18} & 0 \\ 27.375 & 616.63 & -17.186 \\ 0 & 0.067185 & -4.9535 \times 10^{-18} \\ -23.751 & -135.53 & 2.1597 \\ 0.85519 & 27.16 & -0.53767 \\ 40.175 & -54.428 & 1.9303 \\ 36.725 & 547.42 & -11.922 \\ -32.829 & -490.89 & 19.35 \\ -1.6927 \times 10^{-15} & -1.8292 \times 10^{-15} & 0.16845 \\ -0.094744 & -0.79574 & -3.9762 \\ -0.0042702 & -0.035865 & 0.49778 \\ 0.51198 & 4.3001 & 6.4226 \end{bmatrix}^T.$$
\[
D = \begin{bmatrix}
-4.2593 \times 10^{-21} & 0 & 7.0942 \times 10^{-20} \\
-1.5822 \times 10^{-21} & 0 & 7.6662 \times 10^{-20} \\
0 & 7.4125 \times 10^{-22} & -4.6416 \times 10^{-21}
\end{bmatrix};
\]

\[
K = \begin{bmatrix}
-2.777 \times 10^{16} & 1.622 \times 10^{18} & -3.7382 \times 10^{13} \\
1.4428 \times 10^{15} & 4.169 \times 10^{15} & -4.4725 \times 10^{12} \\
6.6883 \times 10^{16} & -8.1131 \times 10^{18} & 2.7846 \times 10^{14} \\
-4.2401 \times 10^{12} & 1.4424 \times 10^{16} & -5.0865 \times 10^{12} \\
1.4464 \times 10^{16} & 1.7249 \times 10^{18} & 8.1773 \times 10^{13} \\
4.0687 \times 10^{15} & -3.6059 \times 10^{17} & -4.4475 \times 10^{12} \\
-9.9044 \times 10^{16} & 6.8191 \times 10^{17} & -1.8095 \times 10^{11} \\
2.9942 \times 10^{16} & -7.4012 \times 10^{18} & -3.8781 \times 10^{11} \\
-2.6377 \times 10^{16} & 6.6458 \times 10^{18} & 2.4861 \times 10^{14} \\
1.8238 \times 10^{14} & 1.8479 \times 10^{15} & 7.8456 \times 10^{13} \\
7.2471 \times 10^{14} & 1.6661 \times 10^{16} & 3.1158 \times 10^{14} \\
-1.8987 \times 10^{14} & -1.5886 \times 10^{15} & -3.4126 \times 10^{13} \\
-5.2651 \times 10^{14} & -5.6544 \times 10^{16} & -6.2264 \times 10^{14}
\end{bmatrix}^T.
\]
Control Configuration 2:

\[ \rho = 1.3603; \quad \overline{R} = \begin{bmatrix} 7.4741 \times 10^{-33} & 0 \\ 0 & 8.024 \times 10^{-29} \end{bmatrix}; \]

\[ C = \begin{bmatrix} -119.55 & 2.8256 \\ 6.4077 \times 10^{-18} & 0 \\ 616.63 & -17.186 \\ 0.067185 & -4.9535 \times 10^{-18} \\ -135.53 & 2.1597 \\ 27.16 & -0.53767 \\ -54.428 & 1.9303 \\ 547.42 & -11.922 \\ -490.89 & 19.35 \\ -1.8292 \times 10^{-15} & 0.16845 \\ -0.79574 & -3.9762 \\ -0.035865 & 0.49778 \\ 4.3001 & 6.4226 \end{bmatrix}^{T}; \quad D = \begin{bmatrix} 0 & 7.6662 \times 10^{-20} \\ 7.4125 \times 10^{-22} & -4.6416 \times 10^{-21} \end{bmatrix}. \]

\[ K = \begin{bmatrix} 1.648 \times 10^{18} & -2.7688e + 013 \\ 2.6191 \times 10^{15} & -2.7962e + 011 \\ -8.1603 \times 10^{18} & 4.3003e + 014 \\ 1.2486 \times 10^{16} & -5.7382e + 012 \\ 1.716 \times 10^{18} & 9.3738e + 013 \\ -3.6249 \times 10^{17} & -1.7417e + 012 \\ 7.3641 \times 10^{17} & -9.6541e + 013 \\ -7.3869 \times 10^{18} & -7.5153e + 011 \\ 6.6314 \times 10^{18} & 3.4146e + 014 \\ 1.4574 \times 10^{15} & 1.0754e + 014 \\ 1.5269 \times 10^{16} & 4.2685e + 014 \\ -1.1341 \times 10^{15} & -4.73e + 013 \\ -5.611 \times 10^{16} & -8.4825e + 014 \end{bmatrix}^{T}. \]
Control Configuration 3:

\[ \rho = 22.832; \quad \mathbf{R} = 5.3 \times 10^{-23}; \]

\[
C = \begin{bmatrix}
2.8256 & 0 & -17.186 & -4.9535 \times 10^{-18} & 2.1597 & -0.53767 & 1.9303 & -11.922 & 19.35 & 0.16845 & -3.9762 & 0.49778 & 6.4226
\end{bmatrix}^T
\]

\[
D = -4.6416 \times 10^{-21}; \quad K = \begin{bmatrix}
-5.9307 \times 10^{15} & 9.2982 \times 10^{12} & 3.6292 \times 10^{16} & -4.3509 \times 10^{13} & -5.2534 \times 10^{15} & 1.258 \times 10^{15} & -4.1555 \times 10^{15} & 2.567 \times 10^{16} & -1.685 \times 10^{16} & 1.8064 \times 10^{15} & 7.0504 \times 10^{15} & -7.9378 \times 10^{14} & -1.3837 \times 10^{16}
\end{bmatrix}^T.
\]