DEVELOPMENT OF SURFACE WEAR AND LAPPING SIMULATION MODELS FOR HYPOID GEARS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in the Graduate School of The Ohio State University

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ABSTRACT

Surface wear is one of the failure modes observed in real-life hypoid gear sets impacting the contact patterns and the resultant noise behavior. In this study, a model is proposed to predict the surface wear of face-milled or face-hobbed hypoid gear pairs. This wear model incorporates Archard’s wear model along with loaded hypoid gear tooth contact models of different types. It combines the sliding distances computed from kinematic conditions with contact pressures predicted by the contact models to find the wear distributions along the tooth surfaces. The contact pressures are obtained along the contact zones at each rotational gear position. This model is applied to a family of hypoid gear pairs having different shaft offset values to show that a larger shaft offset results in larger sliding distances and larger wear depths. Influences of the gear and pinion position errors on the resultant wear patterns are also quantified by using the wear model.

In order to reduce the computational time required by the wear simulations, an approximate method is proposed that devises an interpolation scheme to patch instantaneous wear profiles predicted at a small number of rotational increments. This approximate model is used to perform families of wear simulations with time varying
gear and pinion position errors in an attempt to simulate a typical lapping process used for face-hobbed hypoid gears. As required by the lapping simulation methodology, the sensitivity of the contact positions to the magnitudes and directions of the mounting errors are quantified through the differential geometry. At the end, complete lapping simulations of hypoid gear pairs having theoretical surfaces as well as surfaces having manufacturing errors are carried out, and the influence of the lapping processes on the motion transmission error amplitudes are quantified.
Dedicated to my parents
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NOMENCLATURE

\( \mathbf{a}^p, \mathbf{a}^g \) \hspace{1em} Unit vectors along the rotational axes of pinion and gear

\( A \) \hspace{1em} Shaft angle error

\( A_m \) \hspace{1em} Length of a contact segment along the contact line at a contact segment \( m \)

\( b \) \hspace{1em} Hertzian half contact width

\( C_\zeta \) \hspace{1em} Number of wear cycles during \( \zeta \)-th surface geometry update cycle

\( C_{\text{tot}} \) \hspace{1em} Total number of wear cycles during the entire update cycles

\( E \) \hspace{1em} Pinion position error along the shaft offset direction

\( E_p, E_g \) \hspace{1em} Modulus of elasticity of pinion and gear material

\( E^* \) \hspace{1em} Equivalent modulus of elasticity between two materials

\( e \) \hspace{1em} Shaft offset of hypoid gears

\( \mathbf{e}_1, \mathbf{e}_2 \) \hspace{1em} Unit vectors along the principal directions

\( G \) \hspace{1em} Gear position error along the gear axial direction

\( G_{ij} \) \hspace{1em} Tooth surface geometry at a surface grid \( ij \)

\( h \) \hspace{1em} Wear depth

\( ij \) \hspace{1em} Tooth surface grid index in face width and profile directions

\( \delta h \) \hspace{1em} Wear depth accumulated between two rotational positions
Δh  Wear depth accumulated during one mesh cycle

\( h_{\text{max}} \)  Maximum wear depth on a surface

\( H \)  Total number of rows of contact grids in profile direction

\( I, J \)  Maximum value of a surface grid index in face width and profile directions

\( k \)  Wear coefficient

\( \ell \)  Index of potential contact lines

\( L \)  Intersecting line between two planes

\( m \)  Index of contact segments on a contact line

\( M \)  Total number of columns of contact grids in face width direction

\( M_{\text{co}} \)  Cross over point between two single tooth transmission motion graphs

\( M_m \)  Center point on a moving contact grid cell at a segment \( m \)

\( M_q \)  Center point of a contact pattern at position \( q \) along the lapping path

\( \bar{M} \)  Fixed contact point on a surface.

\( n \)  Index of rotational gear position

\( n_c \)  Gear rotational position when surface grid \( ij \) ends to be in contact

\( n_s \)  Gear rotational position when surface grid \( ij \) starts to be in contact

\( n \)  Surface normal vector

\( N \)  Maximum incremental rotational gear position
\( N_{cl} \) Total number of potential contact lines

\( N_{cp} \) Total number of contact segments on a contact line \( \ell \)

\( N_{cs} \) Total number of contact segments at each time step \( n \)

\( N_{CT}, N_{TC} \) Number of discrete positions along the lapping path CT, TC, respectively

\( N_{CH}, N_{HC} \) Number of discrete positions along the lapping path CH, HC, respectively

\( P \) Pinion position error along the pinion axial direction

\( P_{ij} \) Contact pressure on a surface grid \( ij \)

\( P_{\text{max}} \) Maximum contact pressure

\( q \) Index of center points of contact patterns (lapping position)

\( q_t \) Total number of positions per lapping cycle along the lapping path

\( Q(r, z) \) Surface projection plane

\( \mathbf{r} \) Position vector of a gear blank

\( R \) Equivalent radius of curvature between two contacting surfaces

\( R_p, R_g \) Radius of curvature at a contact point on a pinion and gear surfaces

\( s \) Sliding distance

\( S \) Contact pattern movement

\( S_1, S_2 \) Components of contact pattern movement \( S \) along principal directions
\( \Delta S \) Distance between two positions \( q \) and \( q+1 \) along a lapping path on a tooth surface

\( t_1 \) Unit vector along the contact line direction

\( t_2 \) Unit vector along the perpendicular direction of a contact line

\( \Delta t_q \) Duration of wear cycles at position \( q \) in the lapping simulation

\( T \) Torque

\( \hat{T}, \hat{S} \) Curvilinear parameters on a gear surface

\( \vec{T} \) Triangle

\( T_m \) Local coordinate system with frames \((t_1, t_2, n)\) at contact segment \( m \)

\( u, \theta \) Curvilinear parameters on a cutter surface

\( U_1, U_2 \) Surface coordinates along the longitudinal and profile directions

\( v_q \) Speed of a contact pattern center at position \( q \) in the lapping process

\( v \) Velocity of a contact point

\( v_r \) Velocity of a contact point relative to a tooth surface

\( v_{\omega} \) Velocity of a fixed point on a tooth surface

\( (v_{fg})_n \) Velocity of a contact point at rotational position \( n \) with respect to the gear fixed global coordinate system \( X_{fg} \)

\( W \) Total load carried by a contact segment.
\( \mathbf{X} \) Position vector of a contact point

\( \mathbf{X}_e \) Local coordinate system with frames \((\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})\)

\( \mathbf{X}_{p}, \mathbf{X}_{g} \) Pinion and gear body moving coordinate systems

\( \mathbf{X}_{fp}, \mathbf{X}_{fg} \) Pinion and gear body fixed coordinate systems

\( \mathbf{X}_\Gamma \) Local surface coordinate system with frames \((\Gamma_1, \Gamma_2, \mathbf{n})\)

\( \alpha \) Vector of cutter parameter

\( \chi \) Angle between a contact line direction and a contact line velocity direction

\( \varepsilon^\zeta \) Wear threshold for each \( \zeta \)-th update cycle

\( \varphi_c \) Cradle angle of a hypoid gear cutting machine

\( \Gamma_1, \Gamma_2 \) Unit vector in longitudinal direction and its perpendicular direction at a contact point on a tangent plane

\( \eta \) Index of contact grids in profile direction

\( \kappa \) Curvature

\( \kappa_1, \kappa_2 \) Principal curvatures

\( \lambda \) Angle between principal direction vector \( \mathbf{e}_i \) and arbitrary vector \( \mathbf{t} \) on a surface

\( \mu \) Index of contact grids in face width direction

\( \nu \) Poisson’s ratios

\( \Pi \) Tangent plane at a contact segment
\( \theta \)  Rotational position angle of pinion or gear

\( \sigma \)  Variable representing gear position errors, \( E, P \) and \( G \)

\( \Sigma \)  Tooth surface

\( \Sigma^{pg} \)  Difference surface between pinion and gear surfaces

\( \psi \)  Angle between first principal direction vectors \( e_1^p \) and \( e_1^g \) on pinion and gear surfaces

\( \zeta \)  Index of tooth surface geometry update cycles

\( Z \)  Total number of tooth geometry updates

\( Z_p, Z_g \)  Number to teeth of pinion and gear

**Superscripts:**

\( p \)  Pinion body

\( g \)  Gear body
CHAPTER 1

INTRODUCTION

1.1. Background and Motivation

Hypoid gears are used in various automotive, rotorcraft and industrial applications to transmit power between two perpendicular shafts having a certain amount of offset. They find their highest volume applications in the rear and front axles of rear-wheel or all-wheel drive passenger cars and trucks. Hypoid gears are similar to bevel gears as both have crossed-axis shafts, conical gear blanks and tapered teeth that vary both in thickness or height. However, hypoid gears are designed to accommodate shaft axes that are offset. Therefore, they can be considered as a more general form of spiral bevel gears. When the offset is removed, a hypoid gear pair reduces to a spiral bevel gear pair with additional applications in industrial gear boxes and rotorcraft drive trains.

As any other type of gearing, hypoid gears exhibit several types of failure modes including tooth breakage, scoring, pitting and surface wear [1-4]. Gear surface wear is a
phenomenon that takes place over a relatively long period of time impacting on the functionality of a gear pair in two distinct ways. One obvious impact is on the vibration and noise characteristics of the gear train [5-7]. The accumulated wear results in a motion transmission error that acts as a displacement excitation. This might have a negative influence on the dynamic behavior and the resultant noise from the gear pair [8]. With the accumulation of wear along the tooth surfaces, a gear pair that is quiet when it is new might become noisier in time during the expected life span of the product. In addition, surface wear alters load distribution and contact patterns at the gear mesh interface. Worn tooth surfaces often accelerate the occurrence of other more immediate failure modes such as pitting and tooth breakage [9].

In hypoid gears, the screwing motion due to the offset causes relative sliding in both directions of the face width and the profile, resulting in significant relative sliding formed by the sum of these two sliding vectors [10, 11]. This property makes hypoid gears more susceptible to wear problems than other types of gears. Therefore, it is desirable to have a model that can predict the wear behavior of a hypoid gear pair in the early design stages to allow the designer to take on corrective action to minimize such adverse effects.

The actual gear tooth surfaces deviate geometrically from the intended nominal shapes due to the errors caused by the manufacturing cutting process and heat treatment distortions that occur during the case carburizing process. Gears having such deviations might exhibit contact patterns that are not uniform. In case of face-hobbed hypoid gears,
another manufacturing process, called lapping, must be applied to correct some of these deviations. In the lapping process, first, a pinion and a gear are paired after heat treatment. They are operated under lightly loaded conditions with an abrasive paste (lapping compound) applied to the teeth of both gears. At the same time, the mounting positions of the pinion or the gear (or both) are varied in certain directions according to a user-defined time schedule, moving the instantaneous contact zone to different areas of the tooth surface. An accelerated surface wear process takes place, conforming tooth surfaces to each other to obtain a well-spread and uniform contact pattern. Face-hobbing process with lapping is more cost effective than face-milling process, which requires grinding or milling as the final process. They can also be made as durable and quiet as face-milled gears, provided the lapping process is applied properly. For these reasons, a great majority of hypoid gears produced globally are lapped [10]. Any attempt to model the lapping process must start with an accurate and computationally efficient wear model. Such a model is the main focus of this study.

1.2. Literature Review

While several models of varying complexity have been proposed in the literature for modeling different wear mechanisms [12], Archard’s wear model [13] has been the most widely used wear model when it comes to modeling of actual machine elements and components such as cam-follower contacts, engine piston rings, fuel injectors, and gears. Archard’s wear model assumes that wear depth is proportional to the product of normal pressure and the sliding distance at the contact interface, and lumps all other potential
parameters related to surface texture, lubricants and materials into a single dimensional wear coefficient that must be defined experimentally. Besides its reasonable accuracy in describing mild adhesive and abrasive wear [12, 14-17], this apparent simplicity of Archard’s wear model is one of the main reasons why it was used widely for more complex contact problems such as gears.

Any gear contact can be generalized as a contact of two rough surfaces under combined rolling and sliding under mixed or boundary lubrication conditions. The amount of sliding as well as normal load amplitudes varies with the position of the contact on the tooth surfaces. These corresponding difficulties associated with geometry and computation of the load distributions of gears provide additional incentives for the use of Archard’s wear model that is rather straightforward to implement.

In an early study, Wu and Cheng [18] analyzed the sliding wear of spur gears considering gear dynamics and rough-EHL condition using the sliding wear model developed by them [19]. Their model employed a pair of equivalent contacting cylinders of continuously changing radius to represent the spur gear contact. Flodin and Anderson [9] calculated the wear depth on spur gear surfaces using Archard’s wear equation in conjunction with contact pressure calculation by approximate Winkler surface or mattress model. They assumed that the maximum contact stress with the Winkler model is equal to the maximum Hertzian contact stress between the surfaces. The sliding distances during gear mating were calculated by using the involute kinematic relationship considering the Hertzian contact width. They later expanded their wear model to helical
gears by modeling a helical gear as a collection of thin spur gear slices arranged in such a way to form the desired helix angle [20]. Each slice was modeled by a combination of a stiff surface spring and a somewhat softer tooth spring, and the contact pressure was determined from Hertzian theory by assuming line-contact conditions. Afterwards, they developed a simplified model for both spur and helical gears using mean pressure values on each contact point on the gear surfaces [21].

Later models [22, 23] used the more accurate finite-element based contact model with essentially the same wear model of Archard to simulate spur and helical gear wear. These models provided improvements over earlier gear wear models, not only because their load distribution predictions were more accurate but also, they were able to include intentional tooth profile and lead corrections as well as unavoidable tooth form errors originated from manufacturing. These models were also shown to correlate well with the measured surface wear profiles. Most recently, Ding and Kahraman [24] incorporated the gear wear model of Bajpai et al [22] with a gear dynamic model to study the two-way interactions between surface wear and vibratory motions of spur and helical gears.

All of the models reported above were limited to parallel-axis spur or helical gears. As involute tooth profiles dictate the geometry of spur and helical gears, the computation of relative sliding at the contact zones is a straightforward task. In addition, the prediction of contact pressure of spur or helical gears is also a routine task as several well-established and validated load distribution models are available. The same cannot be said for hypoid gears. The geometry of hypoid surfaces is significantly more complex
than involute gears since machine tool settings and cutting tool parameters are also required to define the surfaces. As a result, the procedures to compute relative sliding and contact pressures are not readily available in the literature. Therefore, only a few studies have attempted to model wear of hypoid gears [25, 26]. One such study by Singh [25] combined a commercially available Finite Element (FE) based hypoid gear contact model CALYX [27] with Archard’s wear equation to predict the initial wear distributions of face-milled hypoid gears with or without gear misalignments. His predictions of the initial maximum pressure and wear distributions showed the direct influence of gear position errors on contact patterns and the resultant wear profiles. Singh presented his results using the ratio of wear depth to wear coefficient \( \frac{h}{k} \) as the parameter since wear coefficient \( k \) was not known. The examples in this study used spiral bevel gears (zero shaft offset) to show that certain areas on the tooth surface experience no sliding and hence no wear, which is not the case for hypoid gears.

Perhaps the only published study on hypoid wear is by Gosselin et al [26] in the context of simulation of the hypoid lapping process. This study also used Archard’s wear equation to predict wear profiles, which were then compared to actual measurements to quantify the wear coefficient. The lapping experiments were performed for different hypoid gear sets under various load and speed conditions. They suggested that the wear coefficient after lapping was always larger than the one before lapping under the same operating condition because the contact stress becomes smaller due to the larger contact area. In order to simplify their simulations, they employed a five-point parabolic interpolation scheme for computing the pressure profiles on each contact point. The
sliding distance calculation in their study was based on sliding velocities that were calculated at discrete time steps. The surface geometry was not updated during the entire lapping simulation even though the changes in the surface topography affect the pressure distribution on the gear surfaces.

1.3. Scope and Objective

The literature review presented above points out the fact that very little published work is available on simulation of wear and lapping process for hypoid gears. Accordingly, the first main objective of this study is to develop a model to predict surface wear of face-milled and face-hobbed spiral bevel and hypoid gears. Using this surface wear model, this study aims at

- quantifying the impact of the gear design and operating parameters on surface wear,

- describing the impact of the surface wear on spiral bevel and hypoid gear noise excitations and durability metrics, and,

- investigating the influence of the gear position and angle errors on surface wear.

As described earlier, the lapping process can be considered as an accelerated wear process that is combined with the predetermined kinematic motion of the gears. With this, the second main objective of this dissertation is to develop a lapping simulation model for face-hobbed and face-milled hypoid gears. This model will provide much
needed fundamental understanding of the lapping process, enabling other related objectives including

• improvement of conventional lapping process with parameters including load, speed and error profiles to obtain desired contact patterns with minimized lapping time,

• determination of the desired lapping process in early stages of design, minimizing the product development process by eliminating several iterations based on trial-and-error, and

• investigation of alternate kinematic lapping motions within the capabilities of today’s modern lapping machines.

In this study, the FE-based hypoid contact analysis model [27] and a recently developed semi-analytical hypoid gear contact model [28] will be employed under the assumption of quasi-static operating conditions. Any dynamic loading effects will not be included in this study as such effects are secondary in automotive axle applications. First, these models will be interfaced with the wear and lapping simulation models proposed above to perform detailed computational simulations. Such simulations are expected to require very significant computational effort since a large number of contact analyses must be performed at various rotational increments with varying amounts of key gear position errors. Therefore, the third objective of this study is to develop approximate wear and lapping simulation schemes that rely on contact analyses at a much smaller
number of locations. These schemes will patch these discrete contact analyses through the kinematics properties and interpolation schemes. These approximate schemes will be:

- evaluated for their accuracy through comparisons to their detailed counterparts to assess the trade-off between the computational time and accuracy,

- tested for their suitability as design tools for implementing routine tasks such as tolerance sensitivity and robustness studies, and

- used for detailed parametric studies in support of the previous two objectives.

1.4. Dissertation Outline

In Chapter 2, a surface wear formulation for hypoid gear contacts is presented. The overall computational methodology for computing contact pressures from a FE contact model [27] and determining surface sliding distances are described in this chapter. In addition, initial wear distribution predictions are presented with and without gear mounting errors.

Chapter 3 repeats the same methodology proposed in Chapter 2 by using the semi-analytical model of Kolivand and Kahraman [28] instead of the FE approach. Predictions of this computationally more efficient method will be compared with the initial wear profiles predicted by the wear model using the FE contact model. As the
semi-analytical contact model can handle local surface deviations due to wear effectively through modifications to the ease-off topography, it makes possible to perform loaded tooth contact analysis of surfaces with varying levels of wear. Accordingly, this chapter provides a methodology to update the surface geometries and corresponding pressure distribution throughout the life span of the hypoid gear set.

In Chapter 4, a new approximate approach will be introduced to allow wear predictions within a significantly reduced computational time. This scheme requires analyses at a small number of rotational increments and estimates the wear between these increments through interpolations. The accuracy of this approximate wear computation method will be assessed through comparisons of its predictions to those of the exact method introduced in Chapter 2 and 3.

Chapter 5 proposes a methodology to simulate the lapping for face-hobbed hypoid gears in combination with error sensitivity analysis to contact pattern position changes. This lapping formulation uses the approximate wear model of Chapter 4 along with a formulation that predicts the contact positions as a function of gear position errors. This chapter investigates the impact of surface wear by lapping on the anticipated vibration behavior of the hypoid gear pair. One key dynamic excitation parameter, the loaded transmission error, will be chosen in this chapter as the key parameter. Finally, a detailed summary, major conclusion, contributions of this dissertation and the recommendation for future work are presented in Chapter 6.
2.1. Archard’s Wear Model

Archard’s wear model was used widely and quite successfully in the past for predicting wear profiles of spur and helical gears [9, 20-23]. These predictions were also compared to measured gear wear distributions to demonstrate good agreement under both quasi-static [22] and dynamic conditions [24]. The same wear models will be employed here as well. Archard’s wear model describes the wear of a point on a sliding contact interface as a first-order differential equation

$$\frac{dh}{ds} = kP,$$

(2.1a)

which can be integrated to solve for the wear depth \( h \) as
\[ h = k \int Pds . \] (2.1b)

Here \( s \) is the relative sliding distance between a contact point and the mating point on the other surface, and \( P \) is the normal contact pressure. All other parameters influencing wear including lubricants, material and surface related parameters [29] are lumped in Eq. (2.1) into a single constant parameter \( k \) that is known as the wear coefficient. While this simplifies the predictions significantly, it makes the model semi-empirical as the value of \( k \) must be determined experimentally for the contact conditions in hand. For instance, Bajpai et al [22] determined that 
\( k = 9.65 \times 10^{-19} m^2/N \) experimentally for case hardened (HRC 60) and shaved helical gears lubricated with automatic transmission fluid. Later simulations in other studies for similar applications using this value resulted in accurate predictions [23, 24].

2.2. Technical Approach

According to Eq. (2.1), both contact pressure \( P \) and the sliding distance \( s \) of all of the contact points at every rotational increment of the gears must be computed in order to predict the wear distribution along the contact surfaces. The sliding distance \( s \) can be computed from the geometrical relationship between a pinion and gear, and this computation inherently involves the cutting tool geometry and the machine tool settings. While \( s \) of a given contact point at a given rotational position changes very slightly with surface wear, the changes in \( P \) with wear are significant. Therefore, contact pressure computations from a loaded tooth contact model must be updated frequently as
accumulation of wear causes the load distributions, and hence, the contact pressures to change. This necessitates an iterative computational methodology as shown in Figure 2.1 for the prediction of the progression of surface wear of hypoid gear teeth.

The methodology outlined in Figure 2.1 first requires the initial (no wear) tooth surfaces to be defined. These initial tooth surfaces can either be the nominal theoretical surfaces as defined through the simulation of cutting process or actual measured ones deviated from their theoretical counterparts due to heat treatment distortions and manufacturing errors. Initial surface geometries of pinion $p$ (driving) and gear $g$ (driven) at a given contact point $ij$ are defined as $(G_{ij}^p)^{\zeta=0}$ and $(G_{ij}^g)^{\zeta=0}$, where superscript $\zeta = 0$ indicates that these are initial surfaces with no wear. With these initial surface geometries, a loaded gear contact analysis is performed at each incremental rotational gear position $n$ ($n \in [1, N]$) to determine the initial contact pressures at each surface grid point $ij$ on both gears as $(P_{ij}^p)_n^0$ and $(P_{ij}^g)_n^0$. Here, superscripts $p$ and $g$ denote the pinion and gear, respectively.

Similarly, the sliding distances of the same grid points between two subsequent rotational increments are also computed. These pressure values and sliding distances are combined with a user-defined wear coefficient $k$ to predict wear that has occurred at each contact point over a single wear (loading) cycle. Here, a wear cycle is defined from the position where a particular gear tooth enters the meshing zone (initiates contact with a
Figure 2.1 Methodology used for computing surface wear of hypoid gears.
tooth on the mating gear for the first time) to the position where the same tooth leaves the meshing zone.

These wear cycles are repeated with the same $P$ and $s$ distributions many times to accumulate wear until the maximum wear depth becomes sufficiently large so as to warrant an update of tooth surface geometry. The worn gear tooth surfaces $(G_{ij}^p)_{\zeta}$ and $(G_{ij}^g)_{\zeta}$ are then fed into the contact model for prediction of the updated contact pressures $(P_{ij}^p)_{\zeta}$ and $(P_{ij}^g)_{\zeta}$ for all rotational positions $n \in [1, N]$ and the same process is repeated until another geometry update is needed. Here, the superscript $\zeta \in [0, Z]$ is the index for geometry (pressure) updates, where $Z$ is the maximum number of geometry updates required to reach a user-defined maximum allowable wear depth value.

2.3. Coordinate System and Surface Grids

In order to perform a wear simulation, a family of spatial Cartesian coordinate frames is defined as shown in Figure 2.2. Two coordinate frames $X_p(x_p, y_p, z_p)$ and $X_g(x_g, y_g, z_g)$ are attached to the pinion and the gear, and rotate with them, while two other coordinate frames $X_{fp}(x_{fp}, y_{fp}, z_{fp})$ and $X_{fg}(x_{fg}, y_{fg}, z_{fg})$ are established as fixed reference frames. Origins of $X_g$ and $X_{fg}$ coincide as shown in Figure 2.2. The same is true for $X_p$ and $X_{fp}$. As the rotational axes remain the same, axes $z_g$ and $z_{fg}$ (and $z_p$ and $z_{fp}$) also coincide, with the gear rotational axis pointing from the vertex of
Figure 2.2  Definition of the coordinate systems of a hypoid gear pair.
the pitch cone to the back face. The shaft offset \( e \) is defined as the (shortest) distance between \( y_{fg} \) and \( z_{fp} \) \((e \neq 0 \text{ for hypoid gears and } e = 0 \text{ for spiral bevel gears})\).

In addition, in line with the contact model that will be introduced later, a local curvilinear surface coordinate system \((\hat{T}, \hat{S})\) is used to define grids of tooth surfaces as shown in Figure 2.3. Curvilinear coordinate parameter \( \hat{T} \) divides the surfaces equally along the face width direction (from toe to heel), varying in value from \( \hat{T} = -1 \) to 1. Likewise \( \hat{S} \) along the profile direction (from root to tip) is defined from \( \hat{S} = 0 \) to 48. The location of each grid point \( ij \) is denoted by \((\hat{T}_i, \hat{S}_j)\) where \( i \in [0, I] \) and \( j \in [0, J] \). This results in \((I + 1) \times (J + 1)\) grid points at which pressure time histories and sliding distance must be computed on both the pinion and the gear surfaces.

Since the tooth surfaces of the hypoid gear are generated by a forming or generating process with complex machine settings and cutter parameters, the surface geometry cannot be described in closed-form, but must be obtained numerically. In order to simplify this problem, spatial Cartesian coordinates, \( x, y \) and \( z \) at each grid point are projected on a plane \( Q(r, z) \) as shown in Figure 2.4 where \( r = [x^2 + y^2]^{1/2} \). The extremes of the tooth surface \( A \) are projected to \( Q(r, z) \), where the rectangle \( B \) maps the tooth surface \( A \). Likewise, all the grid nodes on the tooth surface \( A \) are also projected to \( Q(r, z) \). The Cartesian coordinate values in the gear reference coordinate system \( X_g \) are obtained by using the curvilinear surface parameters of each grid node on both surfaces.
Figure 2.3 Illustration of fixed surface grids and surface parameters on the gear tooth surface

\[ \hat{T} = -1 \to 1 \]

\[ \hat{S} = 0 \to 48 \]

\[ \hat{S} = 48 \]

\[ \hat{T} = 1 \]

\[ \hat{S} = 0 \]

\[ (\hat{T}, \hat{S}) \]

TIP

HEEL

ROOT

TOE
Figure 2.4 Mapping of fixed tooth surface grids onto the $Q(r, z)$ plane.
that are provided by the contact model. Use of the coordinate system $Q(r, z)$ for wear simulation simplifies the analysis as well as presentation of the wear profiles.

2.4. Computation of Contact Pressure Distribution

The first step in the flowchart shown in Figure 2.1 is to obtain contact pressure values $(p_{ij}^{P,g})_{n}^{c}$ of each point on the tooth surfaces of both gears. For this purpose, a commercially available hypoid gear analysis package [27] will be used in this chapter. This loaded contact model has several unique features to handle the intricacies of gear contact analysis. In any typical gear contact, the width of the contact zone can be as much as two orders of magnitude smaller than the size of the gear teeth. This requires a very fine (but localized) mesh in the surface areas around the instantaneous contact zone. In addition, the contact moves along the tooth surfaces as it goes through a loading cycle. When conventional finite element (FE) methods are used, not only an extremely refined finite element mesh must be used at each contact zone, but also a complete re-meshing is required at each rotational gear contact position. In addition, the contact conditions must be handled by using nonlinear gap elements. All these pose as major difficulties in using conventional FE packages for analysis of gear contacts.

The hypoid gear FE contact model used here [27] divides each gear forming the pair into a near-field region where the contact occurs and a far-field region away from the contact. The FE method is used to compute relative deformations and stresses for points in the far-field, while a semi-analytical deformation model based on the Bousinesq and
Cerruti solutions is used in the near field within the contact zones [30, 31]. This approach does not require a highly refined mesh at the contact areas, reducing the computational and numerical difficulties, and time significantly in comparison to conventional FE models. The tooth surfaces are modeled by a large number of nodes representing the actual shape and surface modifications. This FE contact model [27] does not require either refinement of the FE mesh near the contact or re-meshing the finite elements for each contact position. Contact conditions are handled as linear inequality constraints whose solution is obtained by a revised Simplex solver.

There are several published experimental studies performed for validation of predictions of this contact model for different types of gears. In one such study, Tamminana et al [32] compared spur gear transmission error and root strain predictions of this contact model to measurements demonstrating a very good agreement under both quasi-static and dynamic conditions. Ligata et al [33] showed that planetary gear strain and planet load sharing predictions of this contact model agree well with an extensive set of measurements. More relevant to this study, Piazza and Vimercati [34] compared the hypoid gear version of this contact model to measurements of Handschuh [35] to demonstrate the accuracy of this model in predicting stresses and contact patterns of spiral bevel and hypoid gears. In view of these previous validation efforts, this FE contact model will be used in this study as well for predicting the contact pressure values 

\[
\left( P_{\mu \eta}^{P,g} \right)_{n}^{c}
\]

of each contact grid cell on the tooth surfaces as shown in Figure 2.5.
Figure 2.5 Example pressure distributions on the moving contact grids of the contact model at two different rotational positions of (a, b) the pinion, and (c, d) the gear.
At any given rotational position \( n \) for a gear pair with surface geometries \( (G_{ij}^{P,G})^{\zeta} \), the contact model brings the gear teeth in contact, and based on the magnitude of separations around the unloaded contact point, it lays a surface contact grid, as shown schematically in Figure 2.6(a). This contact grid consists of \( 2M+1 \) cells (columns) along the face width direction \( (-M \leq \mu \leq M) \) and \( 2H+1 \) cells (rows) along the profile (tooth height) direction \( (-H \leq \eta \leq H) \). The gear contact analysis is repeated for every rotational position \( n \ (n \in [1, N]) \) to obtain the distribution of instantaneous contact pressure on every moving contact grid cell \( \mu \eta \) as \( \left( P_{\mu \eta}^{P,G} \right)^{\zeta} \). Here, \( n = N \) represents the rotational gear position when a tooth leaves the gear mesh while \( n = 1 \) is the first rotational position when the tooth of interest moves into gear mesh contact. Therefore, the rotational span covered by \( n \in [1, N] \) represents a single wear cycle for that particular tooth.

At each rotational position \( n \), the contact zone is placed in a different location. As a result, the contact model uses different contact grids for each \( n \). As the coordinates of each moving contact grid cell \( \mu \eta \) on each tooth surface change with \( n \), with respect to pinion and gear reference frames, predicted pressure distributions \( \left( P_{\mu \eta}^{P,G} \right)^{\zeta} \) must be transformed to the fixed grids on each tooth surface to obtain \( \left( P_{ij}^{P,G} \right)^{\zeta}_n \). This task is referred in Figure 2.1 as the grid transformation, and is illustrated in Figure 2.6(b) schematically. Since the contact width at any rotational position \( n \) is very small
Figure 2.6 (a) A schematic illustration of the moving contact grids over the contact region, and (b) moving contact grids on the fixed surface grids.
compared to the size of the tooth, both moving and fixed grids shown in Figure 2.6(b) must be rather refined.

The contact pressure value \( (P_{\mu \eta}^{P,g})_{n}^{\zeta} \) on a moving contact grid cell \( \mu \eta \) and its coordinates \( (X_{\mu \eta}^{P,g})_{n} \) with respect to the inertial gear frame are provided by the contact model. Here, if a fixed grid node \( ij \) is contained in any of the loaded moving grid cells \( \mu \eta \) at this rotational position \( n \), then the pressure values are transferred in such a way that

\[
(P_{ij}^{P,g})_{n}^{\zeta} = (P_{\mu \eta}^{P,g})_{n}^{\zeta}.
\]

Here, the contact model predicts uniform pressure within each moving grid cell that is a reasonable assumption provided the size of each grid cell is sufficiently small.

Figure 2.7 illustrates the movement of a moving contact zone on the fixed grid, and the contact pressure variation on a fixed grid point \( ij \) with \( n \) schematically. Here, the leading edge of the contact reaches a fixed grid point \( ij \) (i.e. point \( ij \) enters the contact) at position \( n = n_s \). The same point \( ij \) travels across the contact zone until the rotation position \( n = n_e \) when \( ij \) is at the trailing edge, ready to leave the contact. This means that point \( ij \) will not bear any load \(( (P_{ij}^{P,g})_{n}^{\zeta} = 0 )\) for most of the wear cycle when \( 1 \leq n < n_s \) or \( n_e < n \leq N \). Meanwhile, \( (P_{ij}^{P,g})_{n}^{\zeta} \) climbs to a large maximum value within a very short period of time before vanishing to zero. Therefore, each rotational increment must be kept very small (\( N \) must be large) so that any grid point \( ij \) remains within the
Figure 2.7 (a) Moving contact zones at $n = n_s$ and $n_e$, (b) the resultant discrete pressure distribution.
contact zone during a sufficiently large number of consecutive positions (say 6 to 10) to capture the sudden changes in \((P_{ij}^{p,g})^n\) with \(n\).

2.5. Computation of Sliding Distance

Figure 2.8 shows a cross-section of a tooth contact zone in the direction of sliding. In Figure 2.8(a) at the position \(n = n_s\), a particular tooth surface grid point \(ij\) (one of the fixed grid nodes) on the pinion tooth surface is at the leading edge of the contact zone. At this position \(n = n_s\), it touches a particular point \(b\) on the tooth surface of gear \(g\). Both points have the same coordinates at this particular position, defined relative to \(X_{fg}\) as

\[
(X_{fg}^P)_{ij}^{n = n_s} = (X_{fg}^g)_{n = n_s}^b = \begin{bmatrix} x_{fg} \\ y_{fg} \\ z_{fg} \end{bmatrix}_{ij}.
\] (2.2)

When the gears are rotated by one increment from \(n = n_s\) to \(n = n_s + 1\) as shown in Figure 2.8(b), these two points \(ij\) and \(b\) travel by different amounts and they are no longer in contact with each other, while both are still remaining within the contact zone. The distance between the points \(ij\) and \(b\) at position \(n = n_s + 1\) is equal to the amount of relative sliding that takes place as gears are rotated from \(n = n_s\) to \(n = n_s + 1\). At this position, the coordinates of these points \(ij\) and \(b\) are given as \((X_{fg}^P)_{n = n_s + 1}^{ij}\) and \((X_{fg}^g)_{n = n_s + 1}^b\).
Figure 2.8 A schematic representation of the sliding distance between rotational position $n = n_s$ and $n = n_s + 1$. 
(X^g_{fg})_{n=n_s+1}^b$, respectively. These position vectors can be obtained from \((X^p_{fp})_{n=n_s}^{ij}\) and \((X^g_{fg})_{n=n_s}^b\) by applying certain coordinate rotations and translations. According to Figure 2.2, the position vector of point \(ij\) on the pinion surface with respect to the fixed coordinate \(X_{fp}\) at the rotational position \(n=n_s\) is obtained from the given position vector with respect to the other fixed coordinate \(X_{fg}\) as

\[
(X^p_{fp})_{n=n_s}^{ij} = R_f (X^p_{fg})_{n=n_s}^{ij} + T_f
\]

(2.3a)

where

\[
R_f = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -\cos \gamma & \sin \gamma \\
0 & \sin \gamma & \cos \gamma
\end{bmatrix}, \quad T_f = \begin{bmatrix}
-e \\
0 \\
0
\end{bmatrix}.
\]

(2.3b,c)

Here, \(\gamma\) is the shaft angle that is typically 90° and \(e\) is the offset for hypoid gears.

Next, the gears are brought to the rotational increment of \(n=n_s+1\) by rotating the pinion and gear by \(\Delta \theta_p\) and \(\Delta \theta_g = -\Delta \theta_p (Z_p / Z_g)\), respectively (\(Z_p\) and \(Z_g\) are the number of teeth of the pinion and gear). In this position

\[
(X^p_{fp})_{n=n_s+1}^{ij} = R_{\Delta \theta_p} (X^p_{fp})_{n=n_s}^{ij},
\]

(2.4a)
\begin{equation}
(X_g^{\theta})^b_{f g, n=n_s+1} = R_{\Delta \theta}^g (X_g^b)_{f g, n=n_s}, \tag{2.4b}
\end{equation}

where

\[
R_{\Delta \theta_j} = \begin{bmatrix}
\cos \Delta \theta_j & -\sin \Delta \theta_j & 0 \\
\sin \Delta \theta_j & \cos \Delta \theta_j & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad j = p, g. \tag{2.4c}
\]

Lastly, the pinion point \( i j \) with respect to the gear fixed coordinate \( X_{f g} \) is represented by

\[
(X_{fg}^p)^{ij}_{n=n_s+1} = R_f^{-1} \left[ (X_f^p)^{ij}_{n=n_s+1} - T_f \right]. \tag{2.4d}
\]

With this, the sliding distance between points \( i j \) and \( b \) that occurs as gears rotate between two consequent positions \( n = n_s \) and \( n = n_s + 1 \) is defined as

\[
(s_{ij}^p)_{n_s \rightarrow n_s+1} = \left\| (X_{fg}^p)^{ij}_{n=n_s+1} - (X_{fg}^b)_{n=n_s+1} \right\|. \tag{2.5a}
\]

Likewise, the sliding distance experienced by the fixed grid point \( i j \) between the positions \( n = n_s + 1 \) and \( n = n_s + 2 \) is defined as

\[
(s_{ij}^p)_{n_s+1 \rightarrow n_s+2} = \left\| (X_{fg}^p)^{ij}_{n=n_s+2} - (X_{fg}^b)_{n_s+2} - (s_{ij}^p)_{n_s \rightarrow n_s+1} \right\|. \tag{2.5b}
\]
This can be generalized for every contact grid point $ij$ on the pinion tooth surface between any two consecutive rotational position increments $n$ and $n+1$ as

$$
(s_{ij}^p)_{n\rightarrow n+1} = \begin{cases} 
\left\| (X^p_{fg})_{n+1}^{ij} - (X^g_{fg})_{n+1}^b \right\| - \sum_{n=n_s}^{n} (s_{ij}^p)_{(n-1)\rightarrow n}, & n_s \leq n \leq n_e, \\
0, & \text{else.}
\end{cases}
$$

These computations must be done only for the grid points that are in the contact zone (i.e. have non-zero pressures) at the rotational position $n$. Once these computations are done for a given wear update $\zeta$, they can be used for all other wear update $\zeta \in [0, Z]$ with the assumption that $(s_{ij}^p)_{n\rightarrow n+1}$ does not change with wear.

The sliding distance computations can also be done by using the sliding velocities. The sliding distance on the pinion surface at each time step can be represented by

$$
(s_{ij}^p)_{n\rightarrow n+1} \approx \left\| v^{ij}_f \right\| \Delta t
$$

where $v^{ij}_f$ is the sliding velocity vector between two mating points $ij$ and $b$ at an instant when gears are at rotational position $n$ (Figure 2.7(a)), and $\Delta t$ is the time elapsed between positions $n$ and $n+1$. The sliding velocity vector is defined as

$$
(v^{ij}_f)_n = (v^p_{fg})_n^{ij} - (v^g_{fg})_n^b
$$
where \((v_{fg}^{ij})_n\) and \((v_{fg}^{gb})_n\) are the velocities of points \(ij\) and \(b\) on the pinion and gear surfaces at position \(n\), both defined with respect to the fixed gear reference frame \(X_{fg}\) as

\[
(v_{fg}^{ij})_n = R_f^{-1} [\omega_{fp}^p \times (X_{fp}^{ij})_n],
\]

(2.8a)

\[
(v_{fg}^{gb})_n = \omega_{fg}^g \times (X_{fg}^{gb})_n,
\]

(2.8b)

where \(\omega_{fp}^p\) and \(\omega_{fg}^g\) are the angular velocities of the pinion and gear relative to their respective fixed coordinate frames as shown in Figure 2.2. With these, the sliding velocity vector is defined as

\[
(v_{ij}^j)_n = \begin{cases}
(y_g)_{gb}^b \omega^g - (y_g)_{gb}^b \omega^p \cos \gamma + (z_g)_{gb}^b \omega^p \sin \gamma \\
-(x_g)_{gb}^b \omega^g + (x_g)_{gb}^b \omega^p \cos \gamma + e \omega^p \cos \gamma \\
-[(x_g)_{gb}^b + e] \omega^p \sin \gamma
\end{cases}
\]

(2.9)

While both methods of computing sliding distances result in very close results, the first method was preferred in this study since it relates to the wear model directly.
2.6. Computation of Wear Depth

With \((P_{ij}^{PG})^\zeta_n\) and \((s_{ij}^{PG})_{n\rightarrow n+1}\) in hand \((i \in [0,I], \ j \in [0,J] \text{ and } n \in [1,N])\), the wear depth accumulated at each surface grid node \(ij\) between two rotational positions \(n\) and \(n+1\) at the \(\zeta\)-th geometry update is given as

\[
(\delta h_{ij}^{PG})_{n\rightarrow n+1} = \frac{1}{2} k \left[ (P_{ij}^{PG})^\zeta_n + (P_{ij}^{PG})^\zeta_{n+1} \right] (s_{ij}^{PG})_{n\rightarrow n+1} .
\]  

(2.10)

During one complete wear cycle, the surface grid node \(ij\) is loaded for a number of consecutive rotational positions \((n_s \leq n \leq n_e)\). Therefore, the accumulated wear depth is the sum of wear accumulated at each and every loaded rotational increment as:

\[
(\Delta h_{ij}^{PG})^\zeta = \sum_{n=1}^{N-1} (\delta h_{ij}^{PG})_{n\rightarrow n+1} .
\]

(2.11)

According to Figure 2.1, wear cycles are repeated \(C_{\zeta}\) times after each wear update \(\zeta\) until maximum wear depth of any point on either the surface of the pinion or the surface of the gear reaches a predefined threshold value \(\varepsilon^\zeta\). At this instant, the wear depth accumulated by point \(ij\) since the last geometry update \(\zeta\) is given as

\[
(h_{ij}^{PG})^\zeta = C_{\zeta} (\Delta h_{ij}^{PG})^\zeta
\]

(2.12a)
\[(h_{ij}^g)^{\zeta} = C_{\zeta} \frac{Z_p}{Z_g} (\Delta h_{ij}^g)^{\zeta}\] 

(2.12b)

where \(Z_p\) and \(Z_g\) are the number of teeth of the pinion and gear, respectively, and the total number of wear cycles \(C_{\zeta}\) are defined as

\[C_{\zeta} = \varepsilon^{\zeta} / \max \left[ (\Delta h_{ij}^g)^{\zeta} \right].\] 

(2.12c)

The wear depth accumulated after each geometry (and pressure) update are added to determine the final wear depth of point \(ij\) as

\[h_{ij}^{P,G} = \sum_{\zeta=0}^{Z} (h_{ij}^{P,G})^{\zeta}\] 

(2.13a)

where \(Z\) is the user-defined maximum number of geometry update iterations. Finally, the total number of mesh cycles required to reach \(h_{ij}^{P,G}\) is the sum of wear cycles after each geometry update as:

\[C_{\text{tot}} = \sum_{\zeta=0}^{Z} C_{\zeta} .\] 

(2.13b)
2.7. Examples of Hypoid Gear Wear Simulations

2.7.1. Variation of Surface Wear with Shaft Offset

Three face-hobbed hypoid gear pairs each having different offset values (low, medium and high) will be used here to demonstrate the proposed wear model. All three gear sets are all formed by a 12-tooth pinion and a 41-tooth gear, meshed at a 90-degree shaft angle. The ring gears of all three gear sets are identical in size, while the sizes of the pinions vary based on the shaft offset value. In general, the diameter of pinion increases with increased offset. The basic design specifications and blank dimensions of these three gear pairs are listed in Table 2.1. In addition, the parameters associated with the gear blank geometry are defined in Figure 2.9 [36]. For the wear coefficient, the numerical value established by Bajpai et al [22] for case-carburized helical gears having similar roughness and hardness parameters \( k = 9.65 \times 10^{-19} \, \text{m}^2/\text{N} \) is adapted in these simulations.

As opposed to parallel-axis gears such as spur or helical gears, hypoid gears experience sliding along both the face width and profile directions due to the screw motion. Therefore, sliding is present even along the pitch line in the facewidth direction, unlikely to the parallel-axis gears that have no sliding at the pitch point. The sliding in the face width direction increases with increased shaft offset, and the resultant sliding velocity due to the component in profile and face-width directions increases significantly. Figure 2.10 shows the sliding velocity vectors over the entire contact region of pinions of these three example hypoid gear sets, confirming this point. These figures demonstrate
<table>
<thead>
<tr>
<th>Parameters</th>
<th>Unit</th>
<th>Design-A Pinion Gear</th>
<th>Design-B Pinion Gear</th>
<th>Design-C Pinion Gear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth</td>
<td></td>
<td>12 41</td>
<td>12 41</td>
<td>12 41</td>
</tr>
<tr>
<td>Hand of spiral</td>
<td></td>
<td>Left Right</td>
<td>Left Right</td>
<td>Left Right</td>
</tr>
<tr>
<td>Shaft angle</td>
<td>[degree]</td>
<td>90.00</td>
<td>90.00</td>
<td>90.00</td>
</tr>
<tr>
<td>Shaft offset</td>
<td>[mm]</td>
<td>15.00</td>
<td>30.00</td>
<td>44.45</td>
</tr>
<tr>
<td>Outer cone distance</td>
<td>[mm]</td>
<td>106.44 116.46</td>
<td>104.21 120.86</td>
<td>103.56 127.79</td>
</tr>
<tr>
<td>Face width</td>
<td>[mm]</td>
<td>33.13 31.20</td>
<td>36.80 32.33</td>
<td>41.97 34.24</td>
</tr>
<tr>
<td>Root angle</td>
<td>[degree]</td>
<td>20.09 69.70</td>
<td>24.33 64.66</td>
<td>28.64 58.74</td>
</tr>
<tr>
<td>Spiral angle</td>
<td>[degree]</td>
<td>40.00 31.33</td>
<td>46.99 19.65</td>
<td>49.98 24.08</td>
</tr>
<tr>
<td>Pitch diameter</td>
<td>[mm]</td>
<td>73.12 218.46</td>
<td>85.86 218.46</td>
<td>99.28 218.46</td>
</tr>
</tbody>
</table>

Table 2.1 Gear blank geometry parameters of example hypoid gear pairs
Figure 2.9 Definition of the gear blank geometry parameters [36].
Figure 2.10  Comparison of the sliding velocity magnitude and direction on the gear tooth surfaces for (a) $e = 15.0$ mm, (b) $e = 30.0$ mm, and (c) $e = 44.45$ mm.
clearly that the magnitude and the direction of the sliding velocities change significantly with offset. For hypoid gears, the pitch line is located near the root on the pinion and the tip on the gear flanks. As opposed to the low offset hypoid gear set, the sliding velocity on this pitch line in the high offset gear set is quite large due to the sliding component caused by the screwing motion in face width direction. Moreover, the magnitudes of the sliding velocity vectors are much higher for the higher offset gear set. Since the magnitude of sliding velocity is directly related to the sliding distance during the gear mesh cycles according to Eq. (2.6), one would expect higher wear depths with larger shaft offsets.

Figures 2.11 and 2.12 show the initial maximum pressure distributions \((P_{ij}^{p,g}(\bar{\zeta}=0))\) and the initial wear distributions \((h_{ij}^{p,g}(\bar{\zeta}=0)/k)\) of all three example gear pairs after one complete wear cycle at two different input (pinion) torque values of \(T = 50\) and \(300\) Nm, respectively. In these figures, the numerical value of the wear coefficient was not required as the wear profiles were represented by the \(h/k\) ratio. Since these example gear sets have different blank dimensions, their contact conditions are different, resulting in different contact bearing sizes and pressure distributions for each shaft offset. It is evident from these two figures at two different torque levels that initial wear depths for third gear set having the higher shaft offset are significantly higher, illustrating the dominant influence of the relative sliding, and hence, the shaft offset on surface wear of hypoid gear pairs.
Figure 2.11 Comparison of (a) the initial pressure distributions and (b) the initial wear distributions of the pinion and gear surfaces for (a1,b1) $e = 15.0$ mm, (a2,b2) $e = 30.0$ mm, and (a3,b3) $e = 44.45$ mm at an input torque of $T = 50$ Nm.
Figure 2.12  Comparison of (a) the initial pressure distributions and (b) the initial wear distributions of the pinion and gear surfaces for (a1,b1) \( e = 15.0 \) mm, (a2,b2) \( e = 30.0 \) mm, and (a3,b3) \( e = 44.45 \) mm at an input torque of \( T = 300 \) Nm.
It is also noted in Figures 2.11 and 2.12 that the wear depth on the gear tooth surface is larger than the wear depth on the pinion tooth surface for all three designs. This is mainly due to additional sliding experienced by the gear. It must also be noted that these results represent one complete wear cycle of a pinion tooth and a gear tooth. However, when the operation of the gear pair is considered, a tooth on the pinion experiences $Z_g / Z_p = 41/12 = 3.42$ wear cycles for every wear cycle on a gear tooth. With this kinematic wear cycle rate included, the wear depth ratio of $(Z_g / Z_p)(h_{ij}^p)^{\zeta=0}/k$ for the pinion corresponds to $(h_{ij}^g)^{\zeta=0}/k$ for the gear, the former being about twice the latter.

Besides the maximum magnitude of the wear depth, the shape of the wear distribution is also different for each offset value. The gear set with $e = 44.45$ mm has no region of pure rolling within the contact zone, resulting in a relatively uniform and large wear footprint along the theoretical contact line. Since the sliding velocity changes only slightly within the contact pattern, the shape of the wear distributions is dictated mostly by the pressure distribution. As shown in Figure 2.11 and 2.12, the maximum wear observed in the middle of contact pattern for the higher offset gear set emulates the shape of the pressure distribution. This is not the case for the low offset gear set with $e = 15$ mm. Here, wear depths are increased as the contact moves farther away far from the pitch line, since the magnitude of sliding velocity near the pitch line is considerably less than those far reaching areas. In addition, some edge loading is evident in Figures
2.11(a) and 2.12(a) for this low offset gear set, resulting in excessive wear activity along the tip and root regions.

2.7.2. Variation of Surface Wear with Gear Position Errors

The contact pattern and load distribution of a hypoid gear pair are known to be very sensitive to position errors of the pinion and the gear with respect to each other. These errors, also called assembly errors or misalignments, are mainly due to manufacturing variations of the gearbox housing. Figure 2.13 defines four basic types of gear position errors that are considered in this study. They are (i) the pinion position error $E$ along the vertical (offset) direction, (ii) the pinion position error $P$ along the pinion axial (horizontal) direction, (iii) the gear position error $G$ along the gear axial direction, and (iv) the shaft angle error $A$.

Figures 2.14 and 2.15 show predicted initial wear patterns of one of the example gear pairs ($e = 44.45$ mm) under various combinations of these errors to demonstrate differences in wear distributions primarily due to the movement of the contact pattern. At the nominal, no-error position ($E = P = G = A = 0$), a properly designed gear pair should have a contact pattern located somewhere in the middle of the tooth surfaces. Likewise, any wear profile should be uniformly distributed with very little edge contact along the root and tip regions as shown on top of Figures 2.14 and 2.15. With application of the assembly errors in both directions, one at a time, the contact pattern location moves to different positions, altering the wear profiles significantly as shown in Fig. 2.14 and
Figure 2.13 Definition of gear position errors $E$, $P$, $G$ and $A$ for a left handed hypoid gear set.
Figure 2.14 Wear patterns of the pinion tooth surface for various $E$, $P$, $G$ and $A$ values at $T = 300$ Nm.
Figure 2.15 Wear patterns of the gear tooth surface for various $E$, $P$, $G$ and $A$ values at $T = 300$ Nm.
2.15 for the pinion and the gear, respectively. This indicates that any hypoid gear wear model must have the capability to include these position errors.

The movement of the contact with each assembly error varies with certain design attributes of the hypoid gear such as the hand of pinion (right or left handed), side of blank (drive or coast) and radius of cutter blade group. All three example gear sets defined in Table 2.1 have left handed pinions (right handed gear) and the contact on the drive side are considered. For this gear set with $e = 44.45$ mm, the movement of the contact bearing with position errors is well-defined as shown on the wear distributions for the nominal case.

2.8. Summary

In this chapter, a methodology was proposed for simulation of surface wear of face-milled or face-hobbed hypoid gear pairs. In this methodology, Archard’s wear model was combined with a finite element based hypoid gear contact model. Two different methods for computing sliding distance were proposed based on the geometric and kinematic conditions of the hypoid gear contact. The proposed wear methodology was demonstrated through examples from an automotive rear axle differential unit. These simulations indicate that wear depths increase with increased shaft offset. In addition, the position errors of the gears move the loaded contact pattern along the tooth surfaces, resulting in drastically different wear profiles.
Each wear simulation presented in this chapter considered $N = 100$ rotational positions with fine contact grids, requiring nearly 90 minutes of CPU time on a personal computer with a 3GHz processor. Most of the CPU time was consumed by the pressure computations of the FE-based contact model. This was one of the main reasons why the results were limited to initial wear profiles and no pressure updates were implemented. The other main reason for this was the difficulty in capturing local surface deviations due to wear through the surface fitting schemes of the contact model employed here. An ease-off based semi-analytical hypoid contact model developed recently by Kolivand and Kahraman [28] will be employed in the next chapter with the proposed wear methodology I to address these two issues.
CHAPTER 3

HYPOID GEAR WEAR ANALYSIS BY USING
A SEMI-ANALYTICAL GEAR CONTACT MODEL

3.1. Introduction

In the previous chapter, the surface wear of hypoid gears was computed by combining a wear formulation with a finite element (FE) based deformable-body hypoid gear contact model that is available commercially [27]. This contact model provided accurate predictions of contact pressures of face-milled (FM) and face-hobbed (FH) hypoid gears. It used refined contact grids that moves with the contact. An FE formulation away from the contacts and semi-analytical contact solutions Bousinesq and Cerruti [30, 31] at the contact zone were combined together to predict the contact pressure distributions accurately. While this hybrid model is superior to conventional FE packages in all aspects of gear contact analysis, its use is somewhat limited by the fact that it demands sizable CPU time. Considering that a typical wear analysis requires
analyses at a large number of rotational positions spanning a complete wear cycle, and that the same sets of analyses must be repeated at many stages of the wear accumulation, this computational time requirement becomes critical. Therefore, a more simplified contact model with significantly increased computational efficiency is very desirable even if it is not as accurate as the FE model. This is even more critical for simulation of lapping that requires a large number of contact analyses at various misalignment values as will be discussed in Chapter 5.

The FE-based model [27] of a hypoid gear pair relies on a surface curve fitting method to capture the deviations from the theoretical nominal surfaces. While this is very effective and accurate for representing global deviations due to manufacturing errors and heat treatment distortions, it might be difficult to represent local deviations such as wear in a certain area on the tooth surfaces [37, 38]. This was the main reason why the wear simulations presented in Chapter 2 were limited to initial wear profiles, and the geometry updates describe in the flowchart of Figure 2.1 were not implemented.

This chapter employs the same wear formulation described in Chapter 2 with the semi-analytical hypoid gear load distribution model of Kolivand and Kahraman [28]. As this model uses a closed-form curved shell model with a Rayleigh-Ritz formulation instead of finite elements, it is typically about an order of magnitude faster than the FE model. In addition, since the deviations (both local and global) are handled via changes to the theoretical ease-off topography, the difficulties in handling local deviations is eased to allow geometry updates during the wear process. This chapter first provides a
brief description of the load distribution model while the readers are referred to Kolivand and Kahraman [28, 38] and Kolivand [37] for further details. Next, details of implementing this contact model with the wear formulation are discussed. At the end, the complete wear analyses are performed and compared to those obtained by using the FE model.

3.2. Semi-analytical Hypoid Gear Contact Model

The family of the cutter surfaces generates the tooth flank which is enclosed by the envelope. For describing this envelope mathematically, Kolivand and Kahraman [28] defined two independent curvilinear parameters \( u \) and \( \theta \) to describe the cutter surfaces. The family of cutter surfaces corresponding to a cradle angle \( \varphi_c \) as observed from the gear blank is given as [28]

\[
\mathbf{r} = f(\mathcal{I}(\varphi_c), \mathbf{a}, u, \theta, \varphi_c)
\]  

(3.1)

where \( \mathcal{I}(\varphi_c) \) is the function of the cradle angle that represents a vector of machine setting parameters in the case of including higher order motions and in order case, a contact vector without the higher order motions. Here \( \mathbf{a} \) is a vector of cutter parameters. The envelope of \( \mathbf{r} \) is such that [28]

\[
\mathbf{n} \cdot \frac{\partial \mathbf{r}}{\partial \varphi_c} = 0
\]  

(3.2a)

where
Equation (3.2), known as the fundamental equation of meshing, results in two implicit nonlinear equations that can be solved for two unknowns $u$ and $\theta$ to define each point on the generated tooth surface.

The conventional method for performing an unloaded tooth contact analysis (UTCA) treats each of the pinion and gear tooth surfaces as arbitrary surfaces having no specific relationship to each other. With the pinion and the gear allowed to rotate about their respective axes, the contact point on a tooth surface is calculated by checking for the two contact conditions: (i) points of the pinion and the gear in contact coincide and (ii) the normal vectors on the pinion and the gear surface at this contact point are collinear [39,40]. This conventional method requires significant computation time since a set of five nonlinear equations must be solved for five unknowns. In order to increase the computational efficiency, Kolivand and Kahraman [28] proposed a detailed formulation for the UTCA based on the ease-off approach, which was originally proposed by Stadtfeld [41]. The ease-off surface is defined as the surface representing the deviation of the real gear surface from the conjugate of its mating real pinion surface [28, 37, 38]. The instantaneous unloaded potential contact lines and the contact point path (CPP) were computed using this method to determine unloaded contact patterns and the unloaded transmission error function [28].
Actual number of tooth pairs in contact is defined by several parameters such as gear ratio, roll angle of pinion and the amount of torque applied. In order to simplify the procedure to find the actual contact lines under load, $N_{cl}$ number of potential contact lines ($\ell \in [1, N_{cl}]$) at each time step $n \in [1, N]$ are extracted from the ease-off topography based on a maximum separation distance between the two surfaces under no load [28]. With all potential $N_{cl}$ contact lines identified, each contact line $\ell$ is discretized by $N_{cp}$ number of segments with the segment index $m \in [1, N_{cp}]$. Thus, a total of $N_{cs} = N_{cl}N_{cp}$ potential contact segments are defined at each time step $n$. With the knowledge of the separation value at each segment $m$ of each contact line $\ell$, Kolivand and Kahraman [28, 37] used a curved shell model of a gear tooth [43] together with a base rotation model [44, 45] and Weber equation [46] to define a compliance matrix. The corresponding compatibility and equilibrium conditions were imposed simultaneously with this compliance matrix to predict the load distributions along the contact lines [28, 37].

3.3. Construction of Contact Grids along the Contact Lines

For computation of surface wear with the load distribution predicted by the semi-analytical contact model, first, the load distributions along the contact lines (given as load intensity or force per unit length) must be spread over the actual contact width so that this instantaneous pressure distribution can be transformed to fixed surface grids on the pinion and gear tooth surfaces. On any contact line $\ell$, consider a segment $m$ of length
The half contact width \( b_m \) of this segment is found from Hertzian theory by considering a cylindrical contact

\[
b_m = \frac{4W_m R_m}{\pi E^* A_m}
\]  

(3.3a)

where \( W_m \) is the total load carried by the segment. \( R_m \) is the equivalent radius of curvature in the direction normal to the contact line segment, defined as

\[
R_m = \left( \frac{1}{(R_p)_m} + \frac{1}{(R_g)_m} \right)^{-1}
\]  

(3.3b)

where \( (R_p)_m \) and \( (R_g)_m \) are the radii of curvature of the pinion and the gear at the location of segment \( m \) along the contact line direction. In Eq. (3.3a), \( E^* \) is the equivalent modulus of elasticity that is defined as

\[
E^* = \left( \frac{1-(\nu_p)^2}{E_p} + \frac{1-(\nu_g)^2}{E_g} \right)^{-1}
\]  

(3.3c)

where \( \nu_p \) and \( \nu_g \) are Poisson’s ratios and \( E_p \) and \( E_g \) are the moduli of elasticity for the pinion and gear materials, respectively.

Figure 3.1(a) shows a schematic of a contact line \( \ell \) having \( N_{cp} \) contact segments. The center point of segment \( m \) along the contact line is denoted by point \( M_m \). Each
Figure 3.1 (a) Illustration of contact grid cells on a contact line $\ell$ and (b) definition of tangent planes at contact segments.
contact grid cell defined to correspond to each contact segment is approximated to a rectangle shape on the plane $\Pi_m$ that is tangent to the tooth surface at $M_m$. The coordinates of the corners of each grid cell are obtained here by applying the following two steps with the vectors defined with respect to the global coordinate system $X_{fg}$ unless specified.

**Step-1: Find the tangent plane $\Pi_m$:**

Tangent plane $\Pi_m$ should be first defined at each contact segment center $M_m$ in order to define the contact grid cell plane. Since any vector contained by $\Pi_m$ is perpendicular to each surface normal vector $n_m$, the plane equation representing $\Pi_m$ can be expressed by

$$n_m \cdot [X_B - X_m] = 0$$ (3.4)

where $X_B$ is the position vector of one of the corners of the plane segment $\Pi_m$ (or any other arbitrary point on $\Pi_m$) and $X_m$ is the position vector of $M_m$.

**Step-2: Define the left and right of the boundaries with adjacent cells:**

Defining the intersection of two adjacent tangent planes $\Pi_m$ and $\Pi_{m+1}$ in Figure 3.1(b) by line $L_m$ and the position vector of one of vertices (point B) on $L_m$ by $X_B$, the following line equation can be written:

$$56$$
\[ \mathbf{X}_B = \mathbf{X}_O + H (\mathbf{n}_m \times \mathbf{n}_{m+1}) \]  

(3.5a)

Here \( H \) is the line constant, and \( \mathbf{X}_O \) is the position vector of mid-point \( O \) on \( L_m \), as shown in Figure 3.1(b), such that

\[ \mathbf{X}_O = c_1 \mathbf{n}_m + c_2 \mathbf{n}_{m+1}. \]  

(3.5b)

where \( c_1 \) and \( c_2 \) are constant coefficients. Since point \( O \) is contained by both \( \Pi_m \) and \( \Pi_{m+1} \), Eq. (3.4a) can be applied to point \( O \) to obtain

\[ \mathbf{n}_m \cdot (\mathbf{X}_O - \mathbf{X}_m) = 0, \quad \mathbf{n}_m \cdot \mathbf{X}_O = \mathbf{n}_m \cdot \mathbf{X}_m, \]  

(3.6a)

and

\[ \mathbf{n}_{m+1} \cdot (\mathbf{X}_O - \mathbf{X}_{m+1}) = 0, \quad \mathbf{n}_{m+1} \cdot \mathbf{X}_O = \mathbf{n}_{m+1} \cdot \mathbf{X}_{m+1}. \]  

(3.6b)

By multiplying Eq. (3.5b) by \( \mathbf{n}_m \) and \( \mathbf{n}_{m+1} \) and combining with Eq. (3.6), the following equations are derived

\[ \mathbf{n}_m \cdot \mathbf{X}_O = c_1 \mathbf{n}_m \cdot \mathbf{n}_m + c_2 \mathbf{n}_m \cdot \mathbf{n}_{m+1} = \mathbf{n}_m \cdot \mathbf{X}_m, \]  

(3.7a)

\[ \mathbf{n}_{m+1} \cdot \mathbf{X}_O = c_1 \mathbf{n}_{m+1} \cdot \mathbf{n}_m + c_2 \mathbf{n}_{m+1} \cdot \mathbf{n}_{m+1} = \mathbf{n}_{m+1} \cdot \mathbf{X}_{m+1}. \]  

(3.7b)

\( \mathbf{X}_O \) is obtained by solving the above two equations for the coefficients \( c_1 \) and \( c_2 \). With this, \( \mathbf{X}_B \) and the coordinates of the other corner point \( C \) in Figure 3.1(b) are defined by setting \( H = \mp b_m / \sin \vartheta_m \) in Eq. (3.5a). Likewise the corners of \( \Pi_{m+1} \) along \( L_m \) are also
given by Eq. (3.5a) with \( H = \mp b_{m+1} / \sin \vartheta_{m+1} \), where \( \vartheta \) is the angle between two adjacent normal vectors. The same process is applied to other all other pairs of tangent planes to establish the coordinate of their corner points as well.

Figures 3.2(a) and (b) illustrate contact grid cells constructed on the pinion and gear tooth surfaces at \( T = 300 \) Nm for an example having shaft offset \( e = 30 \) mm. It is seen that contact points are located in the middle of contact grid cells, and all grid cells are connected to each other along the contact line \( \ell \). In Figure 3.3, these grid cells are projected onto the plane \( Q_{rz} \) of the gear at three different torque values of \( T = 50, 100 \) and \( 300 \) Nm. It is noted that the total length of the contact line and the width of each contact cell are increased with an increase in torque.

3.4. Computation of Pressure Variation

The maximum pressure \( (P_{\text{max}})_m \) at the center point \( M_m \) of the contact segment \( m \) is predicted by the hypoid gear contact model. The contact pressure distribution within the same contact cell at the same time instant is obtained by assuming a Hertzian pressure distribution of a line contact. As shown in Figure 3.4(a), the contact pressure is distributed in a semi elliptical fashion across the contact width \( 2b_m \). From Figure 3.4(b), the contact pressure \( P_{ij} \) of any fixed grid node \( ij \) that falls in \( \Pi_m \) can be approximated by
Figure 3.2 Constructed contact grid cells on the loaded (a) pinion and (b) gear surfaces at $T = 100$ Nm. Instantaneous contact grid cells at various rotational increments are superimposed here.
Figure 3.3. Projections of the contact grid cells onto $Q_{rz}$ plane of gear at (a) $T = 50$ Nm, (b) $T = 100$ Nm and (c) $T = 300$ Nm.
Figure 3.4 (a) Hertzian pressure distribution on a given contact cell, (b) variation of pressure across the contact cell.
In order to transfer the contact pressure from the contact grid cells to the fixed surface grids in a simple way, both the contact grid cells and the fixed surface grid are projected onto the plane $Q_{rz}$ as described in Section 2.2. During this projection procedure, the shape of the contact grid cell is distorted, resulting in a quadrilateral shape rather than a rectangle. The distortions of these grid cells are more severe in the case of a pinion due to the large amount of spiral angle change along the face width. Therefore, when the location of fixed surface grid $ij$ is detected within a grid cell after both contact and surface grids are projected on the plane $Q_{rz}$, the corresponding pressure values to the grid $ij$ should be calculated by using a bilinear mapping between the quadrilateral and the rectangular domains.

### 3.5. Comparison of Wear Predictions Using FE and Semi-analytical Contact Models

In this section, the initial surface wear predictions of the wear model with the semi-analytical contact model presented in this chapter and the FE model [27] of Chapter 2 will be compared. Since the pressure distributions predicted by both methods were compared by Kolivand and Kahraman [28], only the initial wear results will be compared here. These pressure comparisons showed that pressure predictions of the FE contact model were slightly higher than those of the semi-analytical contact model and they
exhibited more edge loading activity. As a result, the initial wear predictions using these two methods as contact predictor should differ to a certain extent.

Figure 3.5 shows initial wear depth distributions of a pinion tooth in their normalized form, \((h_{ij}^p)\zeta=0/k\), at three torque levels of \(T = 50, 300\) and 600 Nm. In the first column of this figure, initial wear distributions obtained by using the semi-analytical contact model are presented while the second column contains the same for the FE model. Here, the gear set used for this comparison is the Design-B specified in Table 2.1 with a shaft offset of \(e = 30\) mm. In Figure 3.5, some differences are observed between the predictions using the two contact models. While the wear patterns at the center of the tooth are rather close, the wear predictions with the FE based contact model exhibit more wear in the root and tip boundaries of the contact, especially when the torque is higher. A similar comparison is shown in Figure 3.6 for the initial wear profiles of the gear \((g_{ij})\zeta=0/k\). The same minor differences associated with edge loading at higher torque values are evident in this figure as well. Such differences might be critical to gear sets operating continuously at higher torque conditions. In various applications including automotive axles, however, most of the duty cycles are spent in lower torque ranges, making the wear simulations using the semi-analytical contact model viable. In addition, the lapping process uses very lightly loaded conditions so that the wear model using the semi-analytical contact model might be sufficient. Considering its computational advantages and its capability to handle local surface deviations accurately, the wear
Figure 3.5 Comparison of initial pinion wear \((h/k)\) distributions after one complete mesh cycle at \(T = 50, 300,\) and \(600\) Nm by using (a) the semi-analytical contact model and (b) the FE based contact model.
Figure 3.6 Comparison of initial gear wear \( (h/k) \) distributions after one complete mesh cycle at \( T = 50 \), 300, and 600 Nm by using (a) the semi-analytical contact model and (b) the FE based contact model.
model presented in this chapter using the contact model of Kolivand and Kahraman [28] can be deemed suitable for wear simulations of hypoid gears.

3.6. Example of Wear Simulations with Surface Geometry Updates

According to the computation methodology shown in Figure 2.1, tooth surface geometries must be modified periodically during the accumulation of wear in order to capture the impact of instantaneous wear profiles on the contact pressure distributions. In Chapter 2, such geometry and pressure updates were not implemented due to the difficulties in capturing the local surface deviations due to wear through the surface fitting schemes of the FE-based contact model. As the ease-off based contact scheme of the semi-analytical model can capture local surface deviations due to wear accurately [38], a set of complete wear simulations will be presented here with multiple geometry updates.

The second gear set (Design-B) in Table 2.1 was used as the example system. Here a wear threshold of $\varepsilon = 1 \, \mu m$ was used. According to Figure 2.1, as soon as any point on the tooth surface of the pinion (or gear) accumulates this amount of wear depth since the last surface update, the wear simulation is interrupted and surfaces are updated to remove material to include this additional wear, and a new contact analysis is performed. Figure 3.7(a) shows the pressure $(P_{ij}^p)_{\zeta}$ and wear depth $(h_{ij}^p)_{\zeta}$ profiles of the pinion at $T = 50$ Nm after the geometry updates of $\zeta = 0$, $4$, $8$, $12$ and $16$. Here, $\zeta = 0$
Figure 3.7 Pressure and wear depth distributions of (a) a pinion tooth and (b) a gear tooth after geometry updates of $\zeta = 0, 4, 8, 12, \text{ and } 16$ for $\varepsilon^\zeta = 1 \, \mu m$ at $T = 50 \, Nm$. 

continued
Figure 3.7 continued

(b) Gear

\[ \zeta = 0 \]

\[ \zeta = 4 \]

\[ \zeta = 8 \]

\[ \zeta = 12 \]

\[ \zeta = 16 \]

\( (P_{ij}^g)^\zeta \)

[MPa]

\( (h_{ij}^g)^\zeta \)

[\mu m]
represents the brand new tooth surfaces with no wear. Accordingly, the first row in Figure 3.7(a) has the initial maximum pressure distribution \( (P_{ij}^P)^{\zeta=0} \) and very little wear before the first geometry update.

As wear progresses, the overall maximum value of contact pressure is reduced while the contact zone spreads to a larger area. The corresponding wear profiles shown in the second column of Figure 3.7(a) indicate a gradual accumulation of wear in the loaded contact zone with the maximum wear depth at a point near the maximum contact pressure. The corresponding results for a gear tooth surface are presented in Figure 3.7(b) at the same torque value to show a very similar wear accumulation. Here the contact pressures for the pinion and gear tooth surfaces are identical while they look different since they are mapped to the fixed surface grids on the pinion and gear tooth surfaces. The qualitative shapes of the pinion and gear tooth wear are also quite similar. The difference is in the magnitudes of the wear depths accumulated. The gear pair used in this simulation has \( Z_p = 12 \) and \( Z_g = 41 \) teeth. Accordingly, a tooth on the pinion goes through \( 41/12 = 3.417 \) times more wear cycles than a tooth on the mating gear. With the differences in sliding distances, this corresponds to nearly twice more wear on pinion tooth surface compared to the wear on gear tooth surface in Figure 3.7.

The same type of simulations is performed at higher torque values next to show that the surface areas with wear expand to cover a larger portion of the tooth surfaces. Figure 3.8 shows \( (P_{ij}^P)^{\zeta} \), \( (h_{ij}^P)^{\zeta} \), \( (P_{ij}^g)^{\zeta} \) and \( (h_{ij}^g)^{\zeta} \) of the same gear pair at \( T = 300 \text{ Nm} \).
Figure 3.8  Pressure and wear depth distributions of (a) a pinion tooth and (b) a gear tooth after geometry updates of $\zeta = 0, 4, 8, 12, \text{ and } 16$ for $\varepsilon^\zeta = 1 \, \mu\text{m}$ at $T = 300 \, \text{Nm}$. 

70
Figure 3.8 continued

(b) Gear

\( \zeta = 0 \)

\( \zeta = 4 \)

\( \zeta = 8 \)

\( \zeta = 12 \)

\( \zeta = 16 \)

\( (P_{ij}^g)_{\zeta} \) [MPa]

\( (h_{ij}^g)_{\zeta} \) [\( \mu \)m]
while Figure 3.9 exhibits the predictions at $T = 600$ Nm. In both figures, the pinion wear depths are consistently larger than those of the gear due to the differences in cycle rate.

Figure 3.10 shows the variation of the maximum wear depth with number of pinion wear cycles with $k = 9.65 \times 10^{-19}$ m$^2$/N. Here the maximum wear depth is defined as the maximum of the overall wear profiles (measured from the initial geometry $(G_{ij}^{p,g})^\zeta=0$), i.e. $(h_{\text{max}}^{p})^\zeta = \max\left[(h_{ij}^{p})^\zeta\right]$ and $(h_{\text{max}}^{g})^\zeta = \max\left[(h_{ij}^{g})^\zeta\right]$. In Figure 3.10(a), $h_{\text{max}}^{p}$ and $h_{\text{max}}^{g}$ are plotted as a function of pinion wear cycles at $T = 50$ Nm, while Figure 3.10(b) and (c) show the same at $T = 300$ and 600 Nm, respectively. As evident from these figures, $h_{\text{max}}^{p}$ is typically 80 to 100% higher than $h_{\text{max}}^{g}$. It is also seen that the wear rate (slope of the wear depth versus cycles curves) reduces as the number of cycles increase. This is mainly due to the fact that the maximum pressure values are reduced with wear as the contact is spread to larger areas as seen in Figures 3.7 to 3.9.

Finally, the same data of Figure 3.10 is presented in Figure 3.11 in such a way as to quantify the influence of the transmitted torque. In Figure 3.11(a), $h_{\text{max}}^{p}$ versus the number of pinion wear cycles for $T = 50$, 300 and 600 Nm are compared, while Figure 3.11(b) does the same for $h_{\text{max}}^{g}$. As expected according to the Archard’s wear model, an increase in torque increases the contact pressures and the resultant wear profiles.
Figure 3.9 Pressure and wear depth distributions of (a) a pinion tooth and (b) a gear tooth after geometry updates of $\zeta = 0, 4, 8, 12, \text{ and } 16$ for $\epsilon = 1 \mu m$ at $T = 600$ Nm.
Figure 3.9 continued

(b) Gear

\( \zeta = 0 \)

\( \zeta = 4 \)

\( \zeta = 8 \)

\( \zeta = 12 \)

\( \zeta = 16 \)
Figure 3.10 Variation of maximum wear depth with the pinion wear cycles at (a) $T = 50$ Nm, (b) $T = 300$ Nm and (c) $T = 600$ Nm.
Figure 3.10 continued

![Graph showing Pinion and Gear wear cycles]

Pinion wear cycles (millions) vs. $h_{\text{max}}$ [μm]
Figure 3.11 Influence of torque transmitted on maximum wear depth of (a) the pinion and (b) the gear.
For instance $h_{\text{max}}^p$ at $T = 600$ Nm after 10 million cycles is about 1.5 and 4.8 times more that those for $T = 300$ and 50 Nm, respectively.

3.7. Summary

In this chapter, the hypoid gear wear simulation methodology proposed in Chapter 2 was used in conjunction with a semi-analytical contact model in place of finite elements (i) to reduce the computational time required from the simulations and (ii) to be able to perform geometry and pressure updates throughout the wear simulation. An auxiliary formulation to convert the predicted load distribution along the loaded contact lines to surface pressure distributions was presented. Comparison of initial wear distributions using the FE based and the semi-analytical contact models were provided to show that they match reasonably well, especially under lightly loaded conditions. At the end, example wear simulations with geometry update were presented to demonstrate the progression of wear as well as to show the differences between wear depths of the pinion and gear surfaces at various torque values.
CHAPTER 4

AN APPROXIMATE METHOD OF PREDICTING HYPOID GEAR WEAR

4.1. Need for an Approximate Method

Simulation of the hypoid gear lapping process is one of the main goals of this study. Lapping is an accelerated wear accumulation process that takes place across the tooth surfaces as mounting positions (called gear position errors or misalignments in the previous chapters) of gears forming the hypoid gear pair are varied continuously. When discretized, the lapping process can be viewed as a collection of a large number of wear simulations at various discrete gear position error conditions. Even with the use of the semi-analytical contact model as described in Chapter 3, each wear simulation takes considerable computational time (although it is much faster than a simulation using the FE based contact model) simply because one must perform contact analyses at a large number of rotational positions (say 100 or more) in order to obtain a smooth wear surface. If one simulates the lapping process by considering the wear behavior at several hundred
discrete position error conditions, the need for a much faster wear simulation becomes evident.

This chapter provides an approximation to the wear model of Chapter 3 aimed at achieving the goal of reduced the computational time. A method that devises an interpolation scheme to reduce the number of rotational positions (say by at least an order of magnitude) to simulate wear will be proposed in the next section. This method will be referred to as an approximate method while the methods described in Chapters 2 and 3 will be called exact throughout the rest of this dissertation.

4.2. Approximate Wear Simulation Methodology

The approximate method uses the same wear formulations as the exact method in predicting wear at certain points on the tooth surface using a larger rotational increment (time steps) and employs a interpolation (surface fitting) scheme to estimate the wear depths of other loaded tooth surface points. In Chapters 2 and 3, the exact method resulted in reasonably smooth and continuous wear profiles, provided the total number of time steps $N$ was in excess of 100 ($n \in [1, N]$). This was stated to be the reason for most of the computational demand. In the approximate method, $N$ is chosen to be an order of magnitude smaller than that of the exact method, improving the computational efficiency significantly.

The first step in the approximate method is to perform a load distribution analysis covering a complete mesh cycle through a small number of time steps $N$. Here, it is
sufficient to limit the contact analysis to one mesh cycle since the contact lines from the adjacent loaded teeth can be transferred to the surface of a loaded tooth. Figure 4.1(a) shows an example of loaded contact lines (superimposed on a tooth surface) of the example gear set Design-B specified in Table 2.1 at 10 instances (rotational increments). Here, only contact analysis was performed at only 5 rotational positions with the remaining 5 contact lines were copied from the contacts on the adjacent teeth. In other words, contacts at \( N = 10 \) instances were obtained using only 5 contact analyses.

Using the method outlined in Section 3.3, moving grid cells are defined along every contact line at every time instant \( n \), as shown in Figure 4.1(b). Focusing on one of the grid cells \( m \ (m \in [1, \ N_{cp}] ) \) along one of these lines at time \( n \), the center point \( M_m^n \) of the cell \( m \) at time \( n \) shown in Figure 4.2(a) corresponds to a point \( \bar{M}_m^n \) defined on the fixed surface grid, whose coordinates can be defined using the global coordinate frame. There are a total of \( N \times N_{cp} \) such points defined in Figure 4.1.

The approximate method aims at finding the wear depth only at these points \( \bar{M}_m^n \) instead of seeking wear along every point of a highly refined fixed tooth surface grid, for further enhancing the computational efficiency. For this, the wear depth at point \( \bar{M}_m^n \) must be computed as the corresponding moving grid cell \( m \) passes through it. At the instant when \( \bar{M}_m^n \) and the moving grid point \( M_m^n \) coincide in space, \( \bar{M}_m^n \) is on the contact line and experiences its maximum pressure value \( (P_m^n)_{\text{max}} \). \( \bar{M}_m^n \) moves across
Figure 4.1 (a) Superposition of contact lines at different rotational positions $n$ on a pinion tooth surface and (b) the contact grid cells constructed on the contact lines.
Figure 4.2 (a) Moving velocity vector of a contact grid cell $m$ at a contact point $M^n_m$ at time step $n$, and (b) the pressure time history at the fixed surface point $\tilde{M}^n_m$ having a semi-elliptical Herztian contact profile.
the moving grid cell $m$ in the direction of the surface velocity of the cell $(v_r)_m^n$ (velocity of the contact line at point $M_m^n$) as shown in Figure 4.2(a), and in the process, spends a $(\tau_d)_m^n$ period of time as a loaded tooth surface point. The time spent within the contact is given as

$$ (\tau_d)_m^n = \frac{D_m^n}{(v_r)_m^n} $$

(4.1a)

where $(v_r)_m^n = \| (v_r)_m^n \|$ and $D_m^n$ is the distance traveled within the contact cell (Figure 4.2(a)) defined as

$$ D_m^n = \frac{2h_m^n}{\sin \chi_m^n} $$

(4.1b)

With this, wear accumulated at point $M_m^n$ after one wear cycle is determined from Archard’s wear equation as

$$ \Delta h_m^n = \int_0^{(\tau_d)_m^n} kP_m^n(t)(v_s)_m^n dt $$

(4.2a)

where $(v_s)_m^n dt = (ds)_m^n$ (sliding distance) and the normal pressure time history $P_m^n(t)$ of the same fixed point can be defined by using the Hertzian theory as shown in Figure 4.2(b) as
Finally, the wear depths at all tooth surface points \( ij \ (i \in [1,I], j \in [1,J]) \) are obtained by an interpolation scheme from these wear depths at points \( \tilde{M}_m^n \) \( (m \in [1, N_{cp}], n \in [1, N]). \) Akima’s [47] surface interpolation method was used for this purpose. Algorithms and subroutines for this method are publicly available [48]. This method first computes Delaunay triangulation [49] of the given data points at \( \tilde{M}_m^n \). Then, on each triangle \( \bar{T} \) defined here, the interpolant \( h \) has the form of a bivariate quintic polynomial on \( Q_{rz}(r,z) \) plane such that

\[
h(r,z) = \sum_{m+n \leq 5} A_{mn}^T r^m z^n \quad \forall r, z \in \bar{T}. \tag{4.3}\]

Since each surface grid coordinate on \( Q_{rz}(r,z) \) is associated with surface grid index \( ij \), the results from this interpolation was plotted on each grid point on each pinion and gear surface.

The tooth surface can be locally approximated to a second polynomial surface at each contact point \( \tilde{M}_m^n \) since Hertzian contact conditions are assumed for the computation of the pressure time history. With this approximation, the contact shape becomes elliptical so that the contact line direction along the major axis of the contact ellipse can be defined. This locally approximated tooth surface is given as
\[ f(x_m, y_m, z_m) = 0 \]

where \( x_m \), \( y_m \) and \( z_m \) are the coordinates of \( \vec{M}_m^n \) on the pinion
tooth surface (or on the gear tooth surface) with respect to the global coordinate system
\( \mathbf{X}_{fg} \). For any tooth surface point near \( \vec{M}_m^n \) with coordinates \((x, y, z)\), the gear surface
expression \( f(x, y, z) = 0 \) can be written by using Taylor series [50]:

\[
\frac{\partial f}{\partial x}(x_m^n - x) + \frac{\partial f}{\partial y}(y_m^n - y) + \frac{\partial f}{\partial z}(z_m^n - z) \\
+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x_m^n - x)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(y_m^n - y)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial z^2}(z_m^n - z)^2 \\
+ \frac{\partial^2 f}{\partial x \partial y}(x_m^n - x)(y_m^n - y) + \frac{\partial^2 f}{\partial y \partial z}(y_m^n - y)(z_m^n - z) \\
+ \frac{\partial^2 f}{\partial z \partial x}(z_m^n - z)(x_m^n - x) + \ldots = 0
\]  

(4.4a)

This surface function can be simplified by retaining up to the second-order terms and by
coordinate transformation from the global system \( \mathbf{X}_{fg} \) into a local system
\( \mathbf{T}_m = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{n})_m \) attached to the tangent plane \( \Pi_m \) as shown in Figure 4.2(a) as

\[ z_t = a_1 x_t^2 + 2a_2 x_t y_t + a_3 y_t^2 \]  

(4.4b)

where \( x_t \), \( y_t \), and \( z_t \) are components of position vector on the surface with respect to the
local coordinate system \( \mathbf{T}_m \). In practice, this polynomial surface can be simplified
further by matching vectors \( \mathbf{t}_1 \) and \( \mathbf{t}_2 \) with the two principal direction vectors \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \)
at the contact point \( \vec{M}_m^n \) [50-52]. With this, the surface function (4.4b) is expressed by
principal curvatures $\kappa_1$ and $\kappa_2$ with respect to the local coordinate system 
$(X_e)_m = (e_1, e_2, n)_m$ with coordinates $(x_e, y_e, z_e)$ such that [50-52]

$$z_e = \frac{1}{2} \kappa_1 x_e^2 + \frac{1}{2} \kappa_2 y_e^2$$ (4.5)

Here $\kappa_1$ and $\kappa_2$ are normal principal curvatures along the principal directions $e_1$ and $e_2$, and are known as a second order geometric parameters at a point [52]. In this local coordinate system $(X_e)_m = (e_1, e_2, n)_m$ on the second order surfaces, moving velocity and contact ellipse direction will be expressed in the following sections.

Implementation of the above outlined formulation (4.2a) requires computation of 
$(v_r)_m^n$ and the associated parameter $D_m^n$ and $\chi_m^n$ to calculate $(\tau_d)_m^n$. Well-known differential geometry formulations based on surface normal curvatures and their directions will be used here to determine $(v_r)_m^n$.

4.2.1 Computation of the Velocity of the Contact Line

Figure 4.3(a) shows the movement of contact point on the pinion surface from a rotational position $\theta = \theta_p$ to the next consecutive position $\theta = \theta_p + \Delta \theta_p$. When the positions of the point $M$ at $\theta = \theta_p$ moves to $M'$ at the next rotational step
Figure 4.3 (a) Movement of a contact point $M$ on the surface between two consecutive pinion rotational position $\theta = \theta_p$ and $\theta = \theta_p + \Delta \theta_p$ and (b) projection of $v_r^p$ onto the principal axes on pinion surface.
\[ \theta = \theta_p + \Delta \theta_p \], the velocity vector \( \mathbf{v}^p \) of the same point on the pinion surface with respect to the global coordinate frame \( \mathbf{X}_{fg} \) is expressed as

\[
\mathbf{v}^p = \mathbf{v}_r^p + \mathbf{v}_\omega^p \tag{4.6a}
\]

where \( \mathbf{v}^p \) is given by first derivative of position vector \( \left( \mathbf{v}^p = d\mathbf{X}^p / dt \right) \), \( \mathbf{v}_r^p \) is the moving velocity of this point \( M \) on the pinion surface, and \( \mathbf{v}_\omega^p = \omega^p \times \mathbf{X}^p \). Since point \( M \) is commonly located on both pinion and gear surfaces at every instant, one can also write

\[
\mathbf{v}^g = \mathbf{v}_r^g + \mathbf{v}_\omega^g . \tag{4.6b}
\]

In order to obtain \( \mathbf{v}_r^p \) and \( \mathbf{v}_r^g \) for a contact point \( M \) on the pinion and gear surfaces, respectively, two geometric conditions are considered [52]. The first condition is the condition of contact that is given as

\[
\mathbf{X}^p = \mathbf{E} + \mathbf{X}^g \tag{4.7}
\]

where \( \mathbf{E} \) is the is the position vector representing shaft offset. The second geometric condition is in regards to the collinearity of the surface normal vectors:

\[
\mathbf{n}^p = \mathbf{n}^g . \tag{4.8}
\]
Condition (4.7) can be differentiated in time to obtain

\[
\frac{dX^p}{dt} = \frac{dX^g}{dt}
\]  

(4.9a)

stating that \( v^p = v^g \) at point \( M \). Hence, from Eq. (4.6), one obtains

\[
v^p_r + v^p_{\omega} = v^g_r + v^g_{\omega}.
\]  

(4.9b)

Here, moving velocities \( v^p_r \) and \( v^g_r \) are expressed on their respective local coordinate systems \( X_e = (e_1, e_2, n) \) on the pinion and gear surfaces as

\[
v^p_r = v^p_{r_1} e^p_1 + v^p_{r_2} e^p_2,
\]  

(4.9c)

\[
v^g_r = v^g_{r_1} e^g_1 + v^g_{r_2} e^g_2.
\]  

(4.9d)

Here, \( v^p_{r_1} \) and \( v^p_{r_2} \) are the projections of the \( v^p_r \) onto two principal direction \( e^p_1 \) and \( e^p_2 \) on the pinion surface, as shown in Figure 4.3(b). Likewise, \( v^g_{r_1} \) and \( v^g_{r_2} \) are the projections of the \( v^p_r \) onto two principal direction \( e^g_1 \) and \( e^g_2 \), respectively. With these, Eq. (4.9b) is written as

\[
v^p_{r_1} e^p_1 + v^p_{r_2} e^p_2 + \omega^p \times X^p = v^g_{r_1} e^g_1 + v^g_{r_2} e^g_2 + \omega^g \times X^g.
\]  

(4.10)

Equation (4.8) for the collinearity condition is differentiated in time to obtain
\[ \dot{n}^p = n^g \]  \hspace{1cm} (4.11a)

where

\[ \dot{n}^p = \dot{n}_r^p + \dot{n}_n^p, \]  \hspace{1cm} (4.11b)

\[ \dot{n}^g = \dot{n}_r^g + \dot{n}_n^g. \]  \hspace{1cm} (4.11c)

Rodrigues’ formula [39-40] provides the following relationship between \( v_r \) and \( \dot{n}_r \) at point \( M \)

\[ -\kappa_{1,2} v_r = \dot{n}_r \]  \hspace{1cm} (4.12)

where \( \kappa_1 \) and \( \kappa_2 \) are the two principal curvatures along the principal direction vectors \( e_1 \) and \( e_2 \), respectively. Using Rodrigues’ formula, Eq. (4.11a) is expressed by

\[
(k_1 v^p_{r1} e_1^p + k_2 v^p_{r2} e_2^p) - \omega^p \times n^p = (k_1 v^g_{r1} e_1^g + k_2 v^g_{r2} e_2^g) - \omega^g \times n^g.
\]  \hspace{1cm} (4.13)

By taking scalar product with \( e_1^g \) and \( e_2^g \) on both side of Eq. (4.10) and (4.13), a system of four coupled equations are obtained, which are solved numerically for the unknown components of moving velocity (\( v^p_{r1}, v^p_{r2}, v^g_{r1}, \) and \( v^g_{r2} \)) along the pinion and gear principal directions (\( e_1^p, e_2^p \) and \( e_1^g, e_2^g \)). Eq. (4.9c) and (4.9d) are then used to compute the moving velocity vector on the pinion and gear surfaces.
4.2.2 Calculation of the Contact Line Direction

In order to determine \((\tau_d)_m^n\) to be used in Eq. (4.2), the angle \(\chi_m^n\) between moving velocity \((v_r)_m^n\) and contact line direction vector \(t_1\) on the tangent plane \(\Pi_m^n\) between pinion and gear surfaces should be calculated. The major axis of the contact ellipse is considered to be in the same direction as the contact line direction on each contact point \(M_m^n\). In order to search for the direction of the major axis of the contact ellipse, a difference surface \(\Sigma^{pg}\) for both pinion and gear surfaces is defined [52]. The difference surface \(\Sigma^{pg}\) contacts both pinion and gear tooth surfaces at \(M_m^n\), and its normal curvature along an arbitrary surface vector \(t\) on the tangent plane \(\Pi_m^n\) is defined by

\[
\kappa_{t}^{pg} = \kappa_{t}^{p} - \kappa_{t}^{g}.
\]  

(4.14a)

The major and minor axis directions of the contact ellipse can be defined on \(\Sigma^{pg}\) by finding the principal directions, \(t_1\) and \(t_2\). These principal directions on \(\Sigma^{pg}\) represent the extreme (limiting) values of the curvatures as the direction of an arbitrary vector \(t\) is varied, as shown in Figure 4.4. By using Euler’s equation [52], each normal curvature along this vector \(t\) on each surface is expressed with its principal curvatures \(\kappa_1\) and \(\kappa_2\) by

\[
\kappa_{t}^{p} = \kappa_{1}^{p} \cos^2 \lambda + \kappa_{2}^{p} \sin^2 \lambda.
\]  

(4.14b)
Figure 4.4 Illustration of the surface local coordinate systems at an instant contact point $M$ and the instantaneous contact ellipse directions $t_1$ and $t_2$ on the tangent surface $\Pi$. 
\[
\kappa_1^g = \kappa_1^g \cos^2 (\lambda - \psi) + \kappa_2^p \sin^2 (\lambda - \psi)
\]  

(4.14c)

where \( \lambda \) is the angle between \( \mathbf{e}_1^p \) and arbitrary vector \( \mathbf{t} \), and \( \psi \) is the angle between \( \mathbf{e}_1^p \) and \( \mathbf{e}_1^g \), both defined positive in the counter-clockwise direction. In order to search the one of the principal directions, which represents contact line direction at contact point \( M_m^n \) on the difference surface \( \Sigma^{pg} \), the angles corresponding to the limiting curvature values \( \kappa_1^{pg} \) can be found by setting \( d\kappa_1^{pg} / d\lambda = 0 \). In Figure 4.4, the angle \( \lambda \) corresponding to \( \mathbf{t}_1 \) is denoted by \( \lambda^p \). Once this contact line vector \( \mathbf{t}_1 \) is found, the angle \( \chi_m^n \) is defined, given \( (\mathbf{v}_r)_m^n \).

4.3. Comparison between the Exact and Approximate Methods

Using the example gear Design-B in Table 2.1, the gear set having a shaft offset of \( e = 30 \) mm, the wear simulations are performed here by using this approximate method at different load levels of \( T = 50, 300, \) and 600 Nm. The number of rotational positions (time steps) is set to be \( N = 10 \) for this approximate simulation. Figure 4.5 shows the principal directions \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) on both the pinion and gear surfaces on each of the contact cells of a contact line at a certain instant \( n \). It is noted that the principal directions on the pinion and gear surfaces at each contact point \( M \) are different from each other. These principal directions and the corresponding curvatures are used to compute the moving velocity vectors \( (\mathbf{v}_r)_m^n \) of each contact grid cell on the pinion and gear surfaces.
Figure 4.5 Principal directions at the contact points of a contact line at position $n$ (dotted arrows are for the pinion, and the solid arrows are for the gear)
Figure 4.6 shows that these \((v_r)^n_m\) on the theoretical pinion and gear surfaces for a pinion speed of at \(\Omega_p = 2,000\) rpm. The magnitude of the velocity vector \(\|(v_r)_{ij}\|\) along the pinion and gear surfaces are shown in Figure 4.7 on the projection plane \(Q_{rz}\), obtained after the interpolation of \(\|(v_r)^n_m\|\) values within a contact zone. Both magnitudes and directions of the moving velocity vector vary from cell to cell along each contact line at each instant \(n\). The magnitudes of moving velocities shown in Figure 4.7(a) for the pinion surface are nearly two times larger than those of the gear surface shown in Figure 4.7(b) due to the geometry of the tooth surfaces of this example gear set with \(e = 30\) mm.

Figure 4.8 shows the wear distribution is constructed by using the approximate method. Here, dots represent the actual wear calculations done by using Eq. (4.3) at each of the discrete contact points \(M^n_m\) while the continuous surfaces represent the wear distribution on the pinion and gear surface obtained by using the surface fitting scheme. These fitted wear surfaces follow discrete wear values closely, with some deviations along the edges of the contact at the tip of the pinion on and the root of the gear. Such deviations are a direct result of using a small \(N\). As stated earlier, main purpose of this approximate method is to minimize the computational time for simulation of the lapping process. Since the lapping of the gears are done at very light load values (typically \(T < 50\) Nm), edge loading conditions are not typical and this approximate method can be deemed suitable.
Figure 4.6 Moving velocity vectors of the contact grid cells of a contact line along (a) the pinion surface and (b) the gear surface at $\Omega_p = 2,000$ rpm.
Figure 4.7  Distribution of the contact cell velocity magnitudes on the projection plane $Q_{rz}$ at $\Omega_p = 2,000$ rpm; (a) the pinion surface and (b) the gear surface.
Figure 4.8 (a) Pinion and (b) gear surface wear distributions. Dots are the wear depths at discrete points $M_m^n$ and the continuous surface is the wear distribution constructed using Akima’s [47] interpolation scheme.
Figures 4.9 and 4.10 compare the initial wear distributions \((h/k)\) on the pinion and gear tooth surfaces, computed by using the exact \((N = 100)\) and approximate \((N = 10)\) methods at three different torque values of \(T = 50, 300\) and \(600\) Nm. It is seen from these figures that both methods produce almost identical initial wear distributions at \(T = 50\) Nm while some slight differences are evident especially at \(T = 600\) Nm.

The exact method with the analytical contact model [28] as proposed in Chapter 3 took 60 seconds of CPU time on a 3GHz PC (10 seconds for the compliance computations and 50 seconds for wear computation with \(N = 100\)). In comparison, the approximate method with \(N = 10\) took 15 seconds of CPU time (10 seconds for the compliance calculation and 5 seconds for wear computation), indicating that it is about four times faster than the exact method with the semi-analytical contact model. This suggests that simulation of the lapping process using the approximate wear model would take only a few minutes, which is very desirable for practical engineering purposes.

4.4. Summary

An approximate method was presented in this Chapter to speed-up wear computations to make simulation of the hypoid gear lapping process feasible. It is shown that this approximate model agrees reasonably well with its exact counterpart, especially under lightly loaded conditions as it is the case for the lapping process. In the next chapter, a lapping simulation methodology will be proposed, which uses this approximate wear model extensively.
Figure 4.9 Comparison of the pinion wear distributions predicted by using (a) the exact method and (b) the approximate method at $T = 50$, 300 and 600 Nm.
Figure 4.10 Comparison of the gear wear distributions predicted by using (a) the exact method and (b) the approximate method at $T = 50$, 300 and 600 Nm.
5.1. Introduction

Lapping is an accelerated wear process taking place between two hypoid tooth surfaces, facilitated through application of a lapping compound containing abrasive media such as silicon carbide particles. This process is applied routinely to face-hobbed hypoid gears as their machined geometries are not very accurate. With the help of the lapping process, the face-hobbing method becomes a cost-effective alternative to the grinding and face-milling processes. Face-hobbed hypoid gears were reported to exhibit better noise performance after they were lapped together [53, 54].

Lapping is applied to the entire tooth flank with the pinion and/or gear position errors varied continuously in such a way that contact locations are moved to different places across the tooth surfaces while gears are rotated at a very light torque value (typically up to 10 Nm). By varying these errors in a cyclic manner, the contact zone can
be moved along a predefined path (say, from center to toe, from toe to center, from center to heel, and then from heel to center) on a tooth surface.

Any actual gear tooth surface deviates from its theoretical nominal shape to the cutting errors and the heat treatment distortions. If not lapped, teeth with such deviations experience unfavorable localized contact patterns, non-uniform pressure distributions and higher gear transmission error (TE) and other vibration excitations. With the application of lapping process, contact conditions can be improved significantly with reduced transmission error values. This reduction in unloaded TE is illustrated in Figure 5.1 schematically where the cross over point $M_{co}$ is moved to $M'_{co}$ after the lapping process [53, 54]. With the lapping process, both the discontinuity in motion transmission and the amplitudes of the TE are reduced.

While today’s lapping machines are equipped with wide ranges of CNC capabilities allowing the operator to adjust lapping parameters in many different ways, only a few of these capabilities find their places in typical lapping processes, where the most basic lapping procedures are adopted. The one main reason for this is that there is no lapping simulation tool available for fully exploiting these added capabilities. Development of the lapping process including the definition of the lapping parameters is still done based on trial-and-error method or past field experiences. The goal of this chapter is to develop a simplified lapping simulation methodology to allow the design of the lapping process effectively so that the resultant tooth surface geometries are desirable, TE amplitudes are low and the lapping time is reduced.
Figure 5.1 A schematic of a transmission error motion graph before and after lapping. [53]
5.2. Lapping Simulation Methodology

5.2.1. Simulation Parameters

Lapping processes used today vary significantly in terms of pinion rotating speed and load values as well as the ranges and sequences of position errors applied. The pinion rotational speed during lapping can be as high as 2000 rpm or as low as 200 rpm depending on the traditional methods used, and lapping input torque (pinion torque) can be between 1 to 10 Nm depending on the gear size. In addition, the duration that is required to move the contact pattern between the extreme location at the toe and heel can take from 1 minute up to 5 minutes. Also, the speed of tooth bearing movement can be controlled to obtain predefined amount of wear. A lapping model that incorporates the impact of operating conditions (speed, load and errors) would make it possible to optimize the process to minimize the time required to achieve the desired lapped surfaces. In this study, the lapping process will be discretized to a sequential collection of discrete wear events at various discrete values of $E$, $P$ and $G$ errors as illustrated in Figure 2.14. This amounts to several hundred individual wear simulations that must be performed sequentially.

During the lapping simulation, surface geometries must be updated between every wear simulation since the wear accumulated in the previous step has an influence on the next wear simulation. In a typical lapping process, the maximum wear depths can reach values between 10 to 20 μm during multiple lapping passes. Since the amount of instantaneous wear on each contact position is typically less than the geometry update
threshold (say $\zeta = 1 \mu m$), no geometry update is required during a wear simulation at a given E-P-G error combination.

5.2.2. Lapping Path

With a rather localized tooth contact at each lapping position due to a very low level of torque applied, the lapping path can be defined in its simplest form between two longitudinal extreme locations near the toe and the heel of the tooth. However, Stadtfeld [10] pointed to the fact that there are several other paths to sweep through the entire active contact region on the surface. For example, if tooth contact size is too small to cover the entire tooth active contact region through this longitudinal movement, a lapping path that covers a combination of profile and longitudinal directions can be used to reach the root and tip region better [54]. Especially, for gear surfaces on which edge loading is detected after heat treatment, the lapping path would be defined such that the tooth contact reaches to these edge regions (root or tip) [54].

Figure 5.2 illustrates the movement of the tooth contact pattern longitudinally during lapping along the paths CT (center to toe), TC (toe to center), CH (center to heel), and HC (heel to center) over the tooth surface. Each of these paths is discretized into small steps. The paths CT and TC are covered through $N_{CT}$ steps while the paths CH and HC take $N_{CH}$ steps. This corresponds to a total of $q_t = 2(N_{CT} + N_{CH})$ such increments covering a complete lapping cycle defined in Figure 5.2. With this, the task is reduced to (i) computing wear at each discrete location $q \in [1, q_t]$ of the contact through
Figure 5.2 Definition of a lapping cycle with the associated contact pattern movement along the longitudinal tooth direction.
one wear cycle using the approximate method proposed in Chapter 4, (ii) defining the number of wear cycles $C_q$ assigned to the contact pattern position $q$ so that the wear accumulation at this point can be computed, (iii) updating the surface geometry to include the wear accumulated at position $q$ before contact is moved to next position to $q+1$.

This process will be repeated for all $q \in [1,q_f]$ to determine the total wear accumulation through a complete lapping cycle. If required, the procedure can be applied several times to simulate lapping processes with multiple lapping cycles. In order to avoid any discontinuous wear distributions within two neighboring contact positions $q$ and $q+1$, the movement of the contact between these two positions must be smaller than the width of the contact pattern. In this study, an increment of 1 mm was found to be sufficient for the example cases considered.

In a typical conventional lapping process, with the previous knowledge of the $E$, $P$ and $G$ values corresponding to the toe and heel limits as shown in Figure 5.2, the $E$ and $P$ errors are varied linearly while the $G$ error kept constant. However, in reality, the tooth contact pattern moves in a nonlinear fashion with the $E$ and $P$ values. Therefore, in order to move the tooth contact pattern in such a way that a uniform wear is achieved along the lapping path, first, the sensitivity of the contact pattern movement to each position error must be determined. Since the sensitivity values and wear rate differ with contact position, a variable speed of tooth contact pattern movement must be defined in relation to the error sensitivities. The details of this sensitivity analysis will be presented in Section 5.3.
5.2.3. Speed of Tooth Contact Pattern along the Lapping Path

Once the lapping path is defined, the tooth contact pattern is moved along this path at a certain speed relative to the surface. With the amount of wear accumulated for each wear cycle on a tooth surface (of the pinion or the gear) at a lapping position \( q \) found from the approximate wear model as \((\Delta h_{ij})_q\), the total amount of wear accumulated at the same position is given as

\[
(h_{ij})_q = C_q(\Delta h_{ij})_q
\]  

(5.1a)

where \( C_q \) is the number of wear cycles (mesh cycles) of this particular tooth on each position \( q \). Here, the number of cycles for the teeth of the pinion and the gear differ as

\[
C_q^g = \left(\frac{Z_p}{Z_g}\right)C_q^p.
\]

In the lapping simulation, \( C_q \) can be expressed by

\[
C_q = \frac{\Omega_q \Delta t_q}{60}
\]  

(5.1b)

where \( \Omega_q \) is the rotational speed of the gear considered (pinion or gear, in rpm) at the lapping position \( q \) (typically kept constant at \( \Omega_q = \Omega \)). \( \Delta t_q \) is the time spent at position \( q \) written as

\[
\Delta t_q = \frac{\Delta S}{v_q}
\]  

(5.1c)
where $\Delta S$ is the distance traveled by the contact pattern between lapping positions $q$ and $q+1$ along the lapping path as seen in Figure 5.2. In other words, a constant step size $\Delta S$ is used along the entire lapping paths while time spent at each lapping position is different since the velocity of the contact pattern $v_q$ varies. With this, Eq. (5.1a) is written as

\[ (h_{ij})_q = \frac{\Omega_q \Delta S (\Delta h_{ij})_q}{60v_q} . \]  

(5.2)

This states that $(h_{ij})_q$ is linearly proportional to $\Delta t_q$, and inversely proportional to $v_q$. Any distribution of wear along the lapping path can be achieved by controlling one of these two parameters, $\Delta t_q$ or $v_q$, while the rotational speed of the gears is kept constant at $\Omega_q = \Omega$.

The discrete lapping procedure defined above requires a total of $t_l = \sum_{q=1}^{q_t} \Delta t_q$ time period to complete a lapping cycle. In the process, total wear accumulated at a tooth surface point $ij$ through a complete lapping cycle becomes

\[ h_{ij} = \sum_{q=1}^{q_t} (h_{ij})_q . \]  

(5.3)
In Eq. (5.1a), the wear amount was computed by using a wear coefficient $k$ that must be
determined experimentally. It is noted here that the value of $k$ for the lapping process is
several orders of magnitude larger than those for the long-cycle mild wear situations.

5.3. Sensitivity of the Contact Pattern Position to Gear Position Errors

In order to define the values of the gear position errors associated with each
lapping position $q$ on the lapping path, the sensitivity of the movement $S$ of the contact
pattern to each gear position error ($E, P$ and $G$, as defined in Figure 2.13) must be
determined. At a given lapping position $q$, the center point $M_q$ of the contact pattern can
be established and the components of $S$ along the principal directions $e_1$ and $e_2$ at $M_q$
can be defined as $S_1$ and $S_2$ (for the pinion or the gear). The partial derivatives of
$\partial S_1/\partial \sigma$ and $\partial S_2/\partial \sigma$ ($\sigma \equiv E, P, G$) relative to local coordinate system $X_e(e_1, e_2, n)$ on
the (pinion or gear) tooth surface define the sensitivity of $S$ to these position errors $\sigma$. In
order to derive these partial derivatives, three conditions (two conditions of meshing, and
a kinematic condition regarding the instantaneous transmission error at $M_q$ as a contact
pattern center) are employed [52]. The conditions of meshing, as defined in Chapter 4,
are as follows: $X^p = X^g + E$ and $n^p = n^g$, where superscripts $p$ and $g$ denote the pinion
and the gear, respectively. The third condition is written mathematically as

$$
\left. \frac{d\phi^g}{d\phi^p} \right|_{M_q} = -\frac{Z_p}{Z_g}.
$$

(5.4)
Sensitivities $\partial S_1^P/\partial \sigma$ and $\partial S_2^P/\partial \sigma$ on the pinion surface, and $\partial S_1^G/\partial \sigma$, and $\partial S_2^G/\partial \sigma$ on the gear surface at point $M_q$ are defined through the following steps:

**Step-1:** *Apply the meshing condition (i).* Take the partial derivative of $X^P = X^g + E$ with respect to position errors $\sigma$ to obtain [52]

\[
\frac{\partial X^P}{\partial \sigma} = \frac{\partial X^g}{\partial \sigma} + \frac{\partial E}{\partial \sigma} \tag{5.5a}
\]

where

\[
\frac{\partial X^P}{\partial \sigma} = \frac{\partial S_1^P}{\partial \sigma} \mathbf{e}_1^P + \frac{\partial S_2^P}{\partial \sigma} \mathbf{e}_2^P + \frac{\partial \phi^P}{\partial \sigma} \mathbf{a}_P \times X^P, \tag{5.5b}
\]

\[
\frac{\partial X^g}{\partial \sigma} = \frac{\partial S_1^g}{\partial \sigma} \mathbf{e}_1^g + \frac{\partial S_2^g}{\partial \sigma} \mathbf{e}_2^g + \frac{\partial \phi^g}{\partial \sigma} \mathbf{a}_g \times X^g. \tag{5.5c}
\]

Here, $\mathbf{a}_P$ and $\mathbf{a}_g$ are the unit vectors along the rotational axes of the pinion and gear, respectively, and $E$ is the offset vector that is defined between the origins of pinion and gear body frames as shown in Figure 5.3. The term $\partial E/\partial \sigma$ in Eq. (5.5a) represents the sensitivity of $E$ to the position error $\sigma$. According to the directions defined for these errors in Figure 5.3, one finds

\[
\frac{\partial E}{\partial E} = \mathbf{a}_P^g, \quad \frac{\partial E}{\partial P} = -\mathbf{a}_P, \quad \frac{\partial E}{\partial G} = \mathbf{a}_g. \tag{5.6a-c}
\]
Figure 5.3 (a) Gear position error directions on the pinion and gear body frames $X_p$ and $X_g$.
where \( a^{pg} = (a^p \times a^g) / \sin \gamma \) with \( \gamma \) being the shaft angle. With this, Eq. (5.5a) is written as

\[
\frac{\partial S^P_1}{\partial \sigma} e^p_1 + \frac{\partial S^P_2}{\partial \sigma} e^p_2 + \frac{\partial \phi^P}{\partial \sigma} a^p \times X^p = \frac{\partial S^g_1}{\partial \sigma} e^g_1 + \frac{\partial S^g_2}{\partial \sigma} e^g_2 + \frac{\partial \phi^g}{\partial \sigma} a^g \times X^g + \frac{\partial E}{\partial \sigma} .
\]

(5.7)

The following three equations are obtained by taking the scalar product of both sides of Eq. (5.7) with \( e^g_1, e^g_2, \) and \( n^g, \) respectively, as

\[
\frac{\partial S^g_1}{\partial \sigma} = \frac{\partial S^P_1}{\partial \sigma} \cos \psi + \frac{\partial S^P_2}{\partial \sigma} \sin \psi + \frac{\partial \phi^P}{\partial \sigma} (a^p \times X^p) \cdot e^g_1
\]

\[- \frac{\partial \phi^g}{\partial \sigma} (a^g \times X^g) \cdot e^g_1 - \frac{\partial E}{\partial \sigma} \cdot e^g_1 , \]

(5.8a)

\[
\frac{\partial S^g_2}{\partial \sigma} = - \frac{\partial S^P_1}{\partial \sigma} \sin \psi + \frac{\partial S^P_2}{\partial \sigma} \cos \psi + \frac{\partial \phi^P}{\partial \sigma} (a^p \times X^p) \cdot e^g_2
\]

\[- \frac{\partial \phi^g}{\partial \sigma} (a^g \times X^g) \cdot e^g_2 - \frac{\partial E}{\partial \sigma} \cdot e^g_2 , \]

(5.8b)

\[
\frac{\partial \phi^P}{\partial \sigma} (a^p \times X^p) \cdot n^g - \frac{\partial \phi^g}{\partial \sigma} (a^g \times X^g) \cdot n^g = \frac{\partial E}{\partial \sigma} \cdot n^g .
\]

(5.8c)
Step-2: Apply the meshing condition (ii). Take the partial derivative of $n^p = n^g$ to obtain [52]

$$\frac{\partial n^p}{\partial \sigma} = \frac{\partial n^g}{\partial \sigma}. \quad (5.9)$$

Using the Rodrigues’ formula [39, 40], $\partial n^p / \partial \sigma$ and $\partial n^g / \partial \sigma$ are expressed as

$$\frac{\partial n^p}{\partial \sigma} = -\left( \kappa_1^p \frac{\partial S^p}{\partial \sigma} e_1^p + \kappa_2^p \frac{\partial S^p}{\partial \sigma} e_2^p \right) + \frac{\partial \phi^p}{\partial \sigma} a^p \times n^p, \quad (5.10a)$$

$$\frac{\partial n^g}{\partial \sigma} = -\left( \kappa_1^g \frac{\partial S^g}{\partial \sigma} e_1^g + \kappa_2^g \frac{\partial S^g}{\partial \sigma} e_2^g \right) + \frac{\partial \phi^g}{\partial \sigma} a^g \times n^g \quad (5.10b)$$

such that Eq. (5.9) can be written as

$$-\kappa_1^p \frac{\partial S^p}{\partial \sigma} e_1^p - \kappa_2^p \frac{\partial S^p}{\partial \sigma} e_2^p + \frac{\partial \phi^p}{\partial \sigma} a^p \times n^p$$

$$= -\kappa_1^g \frac{\partial S^g}{\partial \sigma} e_1^g - \kappa_2^g \frac{\partial S^g}{\partial \sigma} e_2^g + \frac{\partial \phi^g}{\partial \sigma} a^g \times n^g. \quad (5.11)$$

The scalar products of both sides of Eq. (5.11) with $e_1^g$ and $e_2^g$ yield

$$\kappa_1^p \frac{\partial S^p}{\partial \sigma} \sin \psi - \kappa_2^p \frac{\partial S^p}{\partial \sigma} \cos \psi - \frac{\partial \phi^p}{\partial \sigma} a^p \cdot e_1^g = -\kappa_2^g \frac{\partial S^g}{\partial \sigma} - \frac{\partial \phi^g}{\partial \sigma} a^g \cdot e_1^g, \quad (5.12a)$$
Step-3: Apply the condition for the instantaneous transmission error at \( M_q \). At point \( M_q \), one writes from Eq. (4.10)

\[
v_{r1}^{p}e_{1}^{p} + v_{r2}^{p}e_{2}^{p} + d\phi^{p}(a^{p} \times X^{p}) = v_{r1}^{g}e_{1}^{g} + v_{r2}^{g}e_{2}^{g} + d\phi^{g}(a^{g} \times X^{g}).
\]  

(5.13)

By taking the scalar product of both sides of this relationship with \( n^{p} \) (or \( n^{g} \) since \( p = n^{g} \)), one writes

\[
\left. \frac{d\phi^{g}}{d\phi^{p}} \right|_{M_q} = \frac{(a^{p} \times X^{p}) \cdot n^{p}}{(a^{g} \times X^{g}) \cdot n^{g}}.
\]

(5.14)

With an incremental variation of error \( \sigma \), the contact pattern is moved slightly to a new center point at \( M'_q \), for which Eq. (5.4) is applied to find [52]

\[
\left. \frac{d\phi^{g}}{d\phi^{p}} \right|_{M'_q} = \frac{\left[ a^{p} \times \left( X^{p} + \frac{\partial X^{p}}{\partial \sigma} d\sigma \right) \right] \cdot \left( n^{p} + \frac{\partial n^{p}}{\partial \sigma} d\sigma \right)}{\left[ a^{g} \times \left( X^{g} + \frac{\partial X^{g}}{\partial \sigma} d\sigma \right) \right] \cdot \left( n^{g} + \frac{\partial n^{g}}{\partial \sigma} d\sigma \right)} = -\frac{Z_{p}}{Z_{g}}.
\]

(5.15a)

By neglecting the higher-order terms of \( d\sigma \), this equation is reduced to
Eq. (5.15b) (with Eq. (5.5b), (5.5c), (5.10a), and (5.10b) substituted), Eq. (5.8a,b,c) and Eq. 5.12a,b) form a system of six equations that is solved for six unknowns \( \partial S_1^p / \partial \sigma \), \( \partial S_2^p / \partial \sigma \), \( \partial S_1^g / \partial \sigma \), \( \partial S_2^g / \partial \sigma \), \( \partial \phi_1^p / \partial \sigma \) and \( \partial \phi_1^g / \partial \sigma \) for a given error parameter \( \sigma = E, P \) or \( G \).

Once these sensitivity values are obtained, the movement \( \Delta S \) of the tooth contact pattern for the pinion and gear can be obtained by chain rule. For example, on the gear tooth surface, the components of \( \Delta S \) along the principal directions are

\[
\Delta S_1^g = \frac{\partial S_1^g}{\partial E} \Delta E + \frac{\partial S_1^g}{\partial P} \Delta P + \frac{\partial S_1^g}{\partial G} \Delta G ,
\]

\[
\Delta S_2^g = \frac{\partial S_2^g}{\partial E} \Delta E + \frac{\partial S_2^g}{\partial P} \Delta P + \frac{\partial S_2^g}{\partial G} \Delta G ,
\]

where \( \Delta E = E_{q+1} - E_q \), \( \Delta P = P_{q+1} - P_q \), \( \Delta G = G_{q+1} - G_q \) with subscripts \( q \) and \( q + 1 \) representing discrete error amounts at positions \( q \) and \( q + 1 \), respectively. As stated earlier, the position error \( G \) is varied in such a way that the amount of backlash is
constant for any $P$ and $E$ values. With knowledge of $\Delta G$, Eq. (5.16) is solved for $\Delta E$ and $\Delta P$ to obtain

$$\begin{align*}
\begin{bmatrix}
\Delta E \\
\Delta P
\end{bmatrix} &= \begin{bmatrix}
\frac{\partial S_1^g}{\partial E} & \frac{\partial S_1^g}{\partial P} \\
\frac{\partial S_2^g}{\partial E} & \frac{\partial S_2^g}{\partial P}
\end{bmatrix}^{-1}
\begin{bmatrix}
\Delta S_1^g - \frac{\partial S_1^g}{\partial G} \Delta G \\
\Delta S_2^g - \frac{\partial S_2^g}{\partial G} \Delta G
\end{bmatrix}.
\end{align*}
$$

(5.17)

Next, a local surface coordinate system $X_\Gamma(X, \Gamma_1, \Gamma_2, n)$ is defined where $\Gamma_1$ is the surface vector along the longitudinal (face width) direction at each position $q$ [52]. Here, $\Gamma_1^g = n_c^g \times n^g$ where $n_c^g$ is the normal vector of the cone surface at $M_q$ as shown in Figure 5.4(a). This cone has the same angle $\varphi$ as the root angle and contains point $M_q$. The coordinate transformation between the local coordinate system $X_e(e_1, e_2, n)$ and $X_\Gamma$ is defined as

$$\begin{align*}
\begin{bmatrix}
\Delta S_1^g \\
\Delta S_2^g
\end{bmatrix} &= \begin{bmatrix}
\cos \eta^g & \sin \eta^g \\
-\sin \eta^g & \cos \eta^g
\end{bmatrix}
\begin{bmatrix}
\Delta S_1^g \\
\Delta S_2^g
\end{bmatrix}. 
\end{align*}
$$

(5.18)

where $\Delta S_1^g$ and $\Delta S_2^g$ are component of $\Delta S^g$ in $X_\Gamma(\Gamma_1, \Gamma_2, n)$ as shown in Figure 5.4(b) and $\eta^g$ is the angle between $\Gamma_1^g$ and $e_1^g$. Thus, the position error changes $\Delta E$ and $\Delta P$ are obtained as a function of $\Delta S_1^g$ and $\Delta S_2^g$ by combining Eq. (5.17) and (5.18) as
Figure 5.4 Definition of (a) the longitudinal direction vector $\Gamma_1^g$ at $\text{M}_q$ and (b) surface local coordinate system $X^g(\Gamma_1^g, \Gamma_2^g, \mathbf{n}^g)$ at $\text{M}_q$ on the gear surface along with projection of $\Delta S^g$ onto $\Gamma_1^g$ and $\Gamma_2^g$. 

120
\[
\begin{cases}
\{\Delta E\} = \\
\{\Delta P\} = \\
\begin{bmatrix}
\frac{\partial S_1^g}{\partial E} & \frac{\partial S_1^g}{\partial P} \\
\frac{\partial S_2^g}{\partial E} & \frac{\partial S_2^g}{\partial P}
\end{bmatrix}^{-1}
\begin{bmatrix}
\Delta S_1^g \cos \eta^g + \Delta S_2^g \sin \eta^p \cdot \frac{\partial S_1^g}{\partial G} \\
\Delta S_1^g \sin \eta^g + \Delta S_2^g \cos \eta^p \cdot \frac{\partial S_2^g}{\partial G}
\end{bmatrix}
\end{cases}
\]

(5.19)

Since the sensitivity values vary with position \( q \) on the tooth surface, they must be evaluated for each \( \Delta S \) increment to determine \( E \) and \( P \) values at each \( q \). It is noted during these stepwise analyses, the offset vector \( \mathbf{E} \) changes with the changes of the error values such that \( \mathbf{E} = (e + E, -P, G) \) where \( e \) is the nominal shaft offset.

This formulation allows one to determine the values of the position errors \( E, P, \) and \( G \) that must be introduced during the lapping process in order to move the contact pattern center to a certain predetermined position on the tooth surface. Figure 5.5 shows the variation of \( E \) and \( P \) errors \((G = 0)\) for gear set Design-C in Table 2.1 that are required to move the center of the contact pattern longitudinally. In this figure, when all three errors are zero, the contact pattern is at the center (zero) position. In order to move the contact pattern by 6 mm in the longitudinal direction towards the heel (from \( U_1 = 0 \) at the tooth center to \( U_1 = 6 \) mm with \( U_2 = 0 \) in both locations where \( U_1 \) and \( U_2 \) are the surface coordinates along the longitudinal and profile directions, respectively), for instance, one requires error values of \( E = -0.54 \) mm and \( P = 0.16 \) mm. Likewise, moving the contact pattern towards the toe by the same amount (from \( U_1 = 0 \) to \( U_1 = -6 \) mm) from the tooth center requires \( E = 0.44 \) mm and \( P = -0.1 \) mm.
Figure 5.5 The relationship between the $E$ and $P$ error values and the resultant contact pattern center positions when $G = 0$. 
It is noted here that the inverse problem (i.e. given $E$, $P$ and $G$, find the location of the contact pattern) can be solved by using the FE-based contact model [27] and the semi-analytical contact model [28]. For instance, for the same example of Figure 5.5, the $E$ and $P$ values computed by using the above procedure to shift the contact pattern by a certain amount were input to the semi-analytical contact model to predict the locations of the unloaded contact patterns as shown in Figure 5.6. Locations of the contact patterns predicted by the contact model match well with those displayed in Figure 5.6. The key here is to determine error values for a given desired contact pattern position to avoid any need for a trial-and-error procedure provided by the contact models.

5.4. Example Lapping Simulations

As an example gear in this simulation, the high offset hypoid gear (Design-C in Table 2.1 with $e = 44.45$ mm) was used with both theoretical and measured surfaces. In general, high-offset hypoid gears require shorter lapping time because they have higher and larger relative sliding values that remain relatively uniform along the contact zones compared to low-offset hypoid gears. The term theoretical surface is used here to represent the nominal design with intentional modifications applied to ensure a contact pattern at the center of the tooth. The measured surface is intended to represent typical deviations from the theoretical surfaces due to cutting errors and heat treatment distortions. Such deviations can be measured with the use of a coordinate measuring machine (CMM).
Figure 5.6  Longitudinal variation of the contact pattern position on the gear surface with gear position errors $E$ and $P \ (G = 0)$. 

Nominal Position

$\begin{array}{c|c}
(U_1, U_2) & (E, P) \\
\hline
(-8, 0) & (0.59, -0.14) \\
(-6, 0) & (0.44, -0.10) \\
(-4, 0) & (0.31, -0.07) \\
(-2, 0) & (0.16, -0.04) \\
(2, 0) & (-0.17, 0.04) \\
(4, 0) & (-0.35, 0.10) \\
(6, 0) & (-0.54, 0.16) \\
(8, 0) & (-0.75, 0.23) \\
(10, 0) & (-0.97, 0.30) \\
\end{array}$
A constant rotational speed of pinion $\Omega^p = 2,000$ rpm (a typical value used in industry) was used in these simulations. Total number of lapping positions was chosen to be $q_t = 32$, consisting of 8 discrete positions for the paths CT and TC, and 10 positions for the paths CH and HC as defined in Figure 5.2. This difference is due to the fact that the path CH covers a longer distance than the path CT. With this number of lapping positions per pass, lapping simulation was conducted by applying two complete lapping cycles. A pinion torque value of $T = 10$ Nm was used.

As mentioned earlier, the wear coefficient $k$ representing the wear that takes place during the lapping process must be defined experimentally. A representative value of $k = 1.95 \times 10^{-13}$ m$^2$/N was used here.

First, a lapping simulation with a constant speed $v_q$ of contact pattern movements was performed. In the simulation, two complete lapping passes (cycles) were applied with a total of 72 lapping positions with an increment of $\Delta s = 1$ mm. Figures 5.7 and 5.8 show the pressure distribution of the contact pattern at four different positions (the toe limit, the tooth center, the heel limit, and the tooth center again) during the first and the second lapping passes, respectively. The corresponding $E$ and $P$ values are also specified in these figures. It is observed that the pressure distribution at each location on the tooth surface differs within each lapping cycle. Likewise, the pressure at the same location is different at different lapping cycles. The maximum pressure values are
Figure 5.7  Contact pattern pressure distributions at different lapping positions \( q \) during the first lapping pass.
Figure 5.8 Contact pattern pressure distributions at different lapping positions $q$ during the second lapping pass.
reduced with the number of lapping passes as the tooth surfaces are made to conform to each other with a somewhat larger contact pattern.

Figures 5.9 and 5.10 show the predicted wear depths accumulated on the theoretical tooth surfaces of the pinion and the gear during the first and second passes of the lapping cycles. In Figure 5.9(a), wear is shown to accumulate between the center and toe limit locations after the completion of the path CT. The wear depths at the same worn areas are nearly doubled in Figure 5.9(b) after the completion of the path TC since the same areas are swept by the loaded contact pattern. In the following two paths of CH and HC (Figures 5.9(c, d)), wear accumulated in a similar manner, now between the tooth center and the heel limit. Application of the second lapping cycle, as shown in Figure 5.10, increased the wear depths further with the end result of a relatively uniform wear distribution along the tooth surfaces, as illustrated in Figure 5.10(d). It is also noted in Figures 5.9 and 5.10 that the wear depths on the pinion tooth surfaces is nearly twice as large as those on the gear tooth surfaces for the same reasons described in Chapters 2 and 3.

Next, a gear pair having certain surface deviations will be considered. Figure 5.11 shows example deviations on the pinion and gear tooth surfaces, as obtained from actual CMM measurements. Measurements at surface grid of 9x5 were interpolated in this figure to obtain a 25x41 deviation surface. Any positive deviation corresponds to added material while any negative variation refers to material removal as in wear to be included in the contact model. During the simulation of the lapping of these surfaces, the
Figure 5.9 Instantaneous wear distributions at the end of lapping paths CT, TC, CH and HC during the first lapping pass.
Figure 5.10 Instantaneous wear distributions at the end of lapping paths CT, TC, CH and HC during the second lapping pass.
Figure 5.11 An example of measured deviations on the pinion and the gear tooth surfaces from the theoretical surface.
speed of contact pattern movement was varied such that the same amount of maximum wear accumulation of $(h_{\text{max}})_q = 0.6 \mu m$ takes place at each position $q$.

Using the same format of Figures 9 and 10, surface changes during two complete lapping cycles of the deviated tooth surface are shown in Figures 5.12, and 5.13. While the process is quite similar to that of the theoretical profile, the resultant wear distribution shown at the end of the lapping in Figure 5.13(d) is somewhat different that its counterpart in Figure 5.10(d). The resultant wear depth distribution is not uniform in this case, exhibiting certain local peaks across the tooth surfaces. This is because the wear depth is measured from the deviated surfaces of Figure 5.10. Such nonuniformities indicate that the lapping process was indeed effective in removing additional material from the locations of the actual measured surfaces with excessive material.

Both simulation examples used the approximate version of the wear model with the semi-analytical contact model [28]. Each simulation took 18 minutes of CPU time on a 3 GHz PC, in contrast to a lapping simulation by using the exact wear model (again with the semi-analytical contact model) that required 72 minutes of CPU time.

5.4.1. Influence of Lapping on the Transmission Error

Figure 5.14 shows the impact of the lapping process on the transmission error (TE) of the gear pair. These figures compare the root-mean-square value of the loaded TE of the gear pair before and after the lapping simulation within a range of the pinion
Figure 5.12 Instantaneous wear distributions at the end of lapping paths CT, TC, CH and HC during the first lapping pass of the surfaces with deviations.
Figure 5.13 Instantaneous wear distributions at the end of lapping paths CT, TC, CH and HC during the second lapping pass of the surfaces with deviations.
Figure 5.14 Comparison of the rms TE amplitudes of a gear pair before and after the lapping process. (a) $E = P = G = 0$, (b) $E = 0.31$ mm, $P = -0.07$ mm and $G = 0$ mm, and (c) $E = -0.35$ mm, $P = 0.1$ mm and $G = 0$ mm.
Figure 5.14 continued

(c)

![Graph showing TE (rms) [μrad] vs. T [Nm].](image)

- **Before lapping**
- **After lapping**
In Figure 5.14(a), TE versus torque curves are shown for lapped and unlapped gear pairs when they are operated with no position errors $E = P = G = 0$ such that the contact is at the tooth center. In this case, the lapping process reduces the unloaded TE amplitude significantly as well as within the torque range of 200 to 600 Nm. Meanwhile, TE amplitudes increased as a result of lapping when $T > 600$ Nm. When the same gear pairs are operated with errors of $E = 0.31$ mm, $P = -0.07$ mm and $G = 0$ mm, with the moving the contact pattern towards the toe as shown in Figure 14(b), TE amplitudes for both unlapped and lapped gear pairs increased with the influence of lapping remaining the same as Figure 5.14(a). In this position, the unloaded TE is reduced from 40 to 15 $\mu$rad through lapping, indicating that vibration and noise excitations are reduced significantly under the lightly loaded operating conditions. Similar improvements of contact condition are observed in Figure 5.14(c) in terms of unloaded TE when the contact is moved towards the heel ($E = -0.35$ mm, $P = 0.1$ mm and $G = 0$ mm) while the impact of the lapping under loaded regions is still mixed. This indicates that any optimization of the lapping process must be done with the knowledge of the allowable ranges of mounting errors and the operating range of input torque. The lapping process in Figure 5.14 can be deemed successful if the gears are to be operated below 600 Nm. The model proposed is indeed capable of incorporating and quantifying the influence of such effects of lapped surfaces.
5.5. Concluding Remarks

In this chapter, a computational methodology was proposed for the lapping simulation of face-hobbed hypoid gears. The formulation to determine the sensitivity of the contact pattern movement to the gear position errors at the contact pattern center was combined with wear computation for the lapping simulation. This allowed a determination of the gear position error amounts required to move the contact pattern to any particular tooth location along the lapping path. The lapping model with a set of lapping parameters was applied to gear pairs having theoretical surfaces with no deviations and actual surfaces with deviations. Lapping simulations were presented to show that a uniform wear distribution with more conformal tooth surfaces and lower contact pressures can be achieved through lapping. The lapping process was also shown to reduce TE amplitudes of gear pairs operated with or without gear position errors unless the torque values are extreme.
6.1. Overall Summary

In this study, a methodology was proposed for simulation of surface wear of face-milled or face-hobbed hypoid gear pairs. The methodology combines Archard’s wear model with different loaded gear contact models of hypoid gears. The wear model requires the sliding distances and contact pressures to be computed along the contact zones at each rotational gear position. Formulations were proposed for computation of sliding distance along the tooth contact zones based on hypoid gear kinematics and geometry of the tooth surfaces. First, the wear model was paired with a commercially available, finite-element based hypoid gear contact model to predict the normal contact pressure distributions as well as the tooth surface geometries. This model was applied to a family of hypoid gear pairs having different shaft offset values to show that a larger shaft offset gear set results in larger sliding distances and larger wear depths. Four basic
types of pinion and gear position errors were defined as the pinion and gear axial position errors, the shaft offset error and the shaft angle error. Influences of these position errors on wear patterns were quantified by using the wear model with FE based contact model. In order to add the capability to simulate the progression of wear through interim geometry updates and to reduce the computational time required, a semi-analytical contact model was employed next along with the same hypoid gear wear model. With this contact model, the computations were shown to be at least an order of magnitude faster with very little sacrifice in terms of the accuracy of the predictions.

In order to further reduce the computational time required by the wear simulations using the semi-analytical hypoid gear contact model, an approximate method that devises an interpolation scheme to patch instantaneous wear profiles at each rotational position with a small number of rotational increments was proposed. This approximate method was shown to be nearly four times faster than the exact method using no interpolation. This approximate method was used to perform families of wear simulations with time varying pinion and gear position errors, which constitute the lapping process of face-hobbed hypoid gears. As required by the lapping simulation methodology, the sensitivity of the contact positions to the magnitudes and directions of the gear mounting errors was quantified by using differential geometry. At the end, complete lapping simulations of hypoid gear pairs having theoretical surfaces as well as surfaces having manufacturing errors were carried out, and the impact of the lapping process on the amplitudes of the motion transmission errors were quantified.
6.2. Conclusions and Contributions

The methodology proposed for simulation of long-cycle, mild wear accumulation across the pinion and gear tooth surfaces can be identified as the main contribution of this study to the field of power transmissions and gearing. Prior to this study, no such a model was available in the literature to predict progression of the hypoid gear surface wear with or without pinion and gear mounting errors.

The use of the wear model as a practical tool to account for surface wear in the hypoid gear design process was shown to be dictated by the computational demand afforded by the hypoid contact model employed. The contact model required to predict the instantaneous contact pressures must be fast and also be able to account for the local surface deviations characterizing the surface wear so that pressure distributions can be updated throughout the progression of the surface wear. For these reasons, the proposed approximate wear simulation methodology can also be viewed as another main contribution of this study.

The discretization of the hypoid gear lapping process as a number of sequential surface wear phenomena taking place under time-varying gear mounting errors allowed for a hypoid gear pair wear model to be used as the basis for the simulation of the lapping process as well. Given its computational benefits and ability to account for gear position errors as well as any local surface deviations made the approximate wear model suitable for this purpose. This lapping simulation methodology in conjunction with wear model and formulation of quantifying the direct relationship between the contact pattern
locations and the corresponding pinion and gear position errors is new and can also be viewed as original contributions to the state-of-the-art.

The example wear simulations presented in this study pointed to certain key parameters influencing wear such as shaft offset. They also demonstrated clearly that the operating conditions, especially torque and the assembly conditions defining the position errors of the pinion and the gear all have critical influences on the resultant wear behavior. Given the differences in the frequency of wear cycles experienced, the pinions surfaces were shown to accumulate significantly more wear that that on the mating pinion surfaces.

The lapping simulations presented in this study showed that the traditional way of lapping can potentially be improved further by involving more than a couple of position errors in the process and by varying the speed of the movement of the contact pattern across the tooth surfaces. It was also shown that the lapping process reduces the unloaded transmission error amplitudes as well as the loaded ones within the low to medium torque ranges.

6.3. Recommendations for Future Work

The following items can be listed as logical next steps for improving the models proposed as well as employing them to achieve actual design enhancements:

(i) One main shortcoming of the wear methodologies proposed in this study was that they both relied on an experimentally determined wear coefficient $k$ to be used in
Archard’s wear formula. Only the sliding distance and the normal contact pressure were considered to be explicit parameters impacting wear with all other relevant material, surface and lubrication related effects assumed to be accounted by $k$ implicitly. An experimental study is required to determine typical values of $k$ to represent varying gear surface conditions with different materials at typical hardness values and different surface roughness amplitudes. This study should also include the impact of lubricant parameters, temperature and surface speed on $k$ as well.

(ii) As in the simulation of wear, the proposed lapping methodology also requires an experimentally defined wear coefficient $k$. As lapping is not mild wear, but rather a more abrasive form of wear, any $k$ values established for mild wear of hypoid gears along the life cycle of the product are of very little value for the lapping simulations. An experimental study is required to determine typical values of $k$ to represent different abrasive lapping compounds, the additives and lapping speeds, in addition to the material parameters.

(iii) In addition to determining the $k$ values required by the wear and lapping models experimentally, validation of both models are required before they can be used with confidence. For this, tightly controlled wear and lapping experiments must be performed, and the surface profiles before and after wear or lapping must be compared to those from the models to assess their accuracy in predicting the wear
distributions. In addition, the predicted and measured changes in the transmission error functions can also be compared for further validation of these wear models.

(iv) This study included a limited number of example wear and lapping simulations directed towards demonstrating the capabilities of the models including several key parameters. Detailed parametric studies can be performed by using these models to define guidelines on how to design hypoid gear pairs with minimized wear as well as in defining optimized lapping procedures for improving the resultant lapped tooth surface contact conditions as well as reducing the lapping time.
BIBLIOGRAPHY


[27] CALYX (2009) (Hypoid Face-Milled and Hypoid Face-Hobbed tooth contact analysis program), Advanced Numerical Solutions Inc.


