STUDY OF IONIZATION OF QUANTUM SYSTEMS WITH DELTA POTENTIALS IN DAMPED AND UNDAMPED TIME PERIODIC FIELDS

DISSERTATION

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ABSTRACT

We first study the asymptotic behavior of the wave function in a one-dimensional model of ionization by pulses, in which the time-dependent potential is of the form $V(x, t) = -2\delta(x)(1 - e^{-\lambda t} \cos \omega t)$, where $\delta$ is the Dirac distribution.

We find the ionization probability in the limit $t \to \infty$ for all $\lambda$ and $\omega$. The long pulse limit is very singular, and, for $\omega = 0$, the survival probability is $\text{const} \lambda^{1/3}$ ($\lambda$ is a small parameter), much larger than $O(\lambda)$, the one in the abrupt transition counterpart, $V(x, t) = \delta(x)1_{\{t \leq 1/\lambda\}}$ where $1$ is the Heaviside function.

Later we study the asymptotic behavior of the wave function in a one-dimensional model on half line of ionization by pulses, where the time-dependent potential is of the form $V(x, t) = \delta(x - 1)(V + 2\Omega \sin(\omega t))$.

For $\Omega = 0$, we give a complete asymptotic decomposition to wave function in the limit $t \to \infty$ for all $V$ and $\omega$; for $\Omega \neq 0$ we give a partial decomposition, from which we give an equation that determines the condition when ionization happens.
Dedicated to my family
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CHAPTER 1
INTRODUCTION

Quantum systems subjected to external time-periodic fields which are not small have been studied in various settings.

In constant amplitude small enough oscillating fields, perturbation theory typically applies and ionization is generic (the probability of finding the particle in any bounded region vanishes as time becomes large), see [Ya], [CCL2] and references therein. However in such treatment usually only terms corresponding to slower decaying rates are extracted, and terms with faster decaying rate are omitted; a rigorous and complete analytic description of the asymptotic expansion of the wave function has not been obtained except for a few special models.

For larger time-periodic fields, a number of rigorous results have been recently obtained, see [CLT] and references therein, showing generic ionization. However, outside perturbation theory, the systems show a very complex, and often nonintuitive behavior. The ionization fraction at a given time is not always monotonic with the field, see [CCL]. There even exist exceptional potentials of the form $\delta(x)(1 + aF(t))$ with $F$ periodic and of zero average, for which ionization occurs for all small $a$, while at larger fields the particle becomes confined once again [CCLR]. Furthermore, if $\delta(x)$ is replaced with smooth potentials $f_n$ such that $f_n \to \delta$ in distributions, then
ionization occurs for all $a$ if $n$ is kept fixed. The relevance of a $\delta$-potential model (also known as zero range potential — ZRP) is discussed in detail in many publications, see e.g. [DO].

Numerical approaches are very delicate since one deals with the Schrödinger equation in $\mathbb{R}^n \times \mathbb{R}^+$, as $t \to \infty$ and artifacts such as reflections from the walls of a large box approximating the infinite domain are not easily suppressed. The mathematical study of systems in various limits is delicate and important.

In physical experiments one deals with forcing of finite effective duration, often with exponential damping. This is the setting we study in the first half of the present thesis (Chapter 3), in a simple model, a delta function in one dimension, interacting with a damped time-harmonic external forcing.

The equation is

$$i \frac{\partial \psi}{\partial t} = \left( -\frac{\partial^2}{\partial x^2} - 2\delta(x) \left(1 - A(t) \cos(\omega t)\right) \right) \psi$$

(1.0.1)

where $A(t)$ is the amplitude of the oscillation; we take

$$\psi_0 = \psi(0, x) \in C_0^\infty, \quad A(t) = \alpha e^{-\lambda t}, \quad \alpha = 1$$

(1.0.2)

(The analysis for other values of $\alpha$ is very similar.)

The quantity of interest is the large $t$ behavior of $\psi$, and in particular the survival probability

$$P_B = \lim_{t \to \infty} P(t, B) = \lim_{t \to \infty} \int_B |\psi(t, x)|^2 dx$$

(1.0.3)

where $B$ is a bounded subset of $\mathbb{R}$. 

\
There is a vast literature on ionization by pulses, see e.g. [SET] and [DDRFP]. However, there is little in the way of mathematical work (with few exceptions, see [RL] where rectangular pulses are discussed). Mathematical approaches are challenging in a number of ways. Purely time-periodic potentials can be dealt with using Floquet theory, especially in perturbative regimes. There is no known equivalent of that when $A(t)$ is not a constant and the limit when $A(t)$ goes to a constant is very singular, as the present results show. Even for the especially simple model (1.0.1), some aspects of the analysis are delicate.

If $\alpha$ is small enough, $P$ decreases exponentially (from perturbation theory and Fermi Golden Rule) on an intermediate time scale, long enough so that by the time the behavior is not exponential anymore, the survival probability is too low to be of physical interest. For all practical purposes, if $\alpha$ is small enough, the decay is exponential, following the Fermi Golden Rule, the derivation of which can be found in most quantum mechanics textbooks; the quantities of interest can be obtained by perturbation expansions in $\alpha$. This setting is well understood; we mainly focus on the case where $\alpha$ is not too small, a toy-model of an atom interacting with a field comparable to the binding potential.

The no damping case ($\lambda = 0$) is well understood for the model (1.0.1) in all ranges of $\alpha$, see [CLR]. In that case, $P(t, B) \sim t^{-3}$ as $t \to \infty$. However, since the limit $\lambda \to 0$ is singular, little information can be drawn from the $\lambda = 0$ case.

For instance, if $\omega = 0$, the limiting value of $P$ is of order $\lambda^{1/3}$, while with an abrupt
cutoff, \( A(t) = 1_{\{t: t \leq 1/\lambda\}} \), the limiting \( P \) is \( O(\lambda) \) (as usual, \( 1_S \) is the characteristic function of the set \( S \)).

Thus, at least for fields which are not very small, the shape of the pulse cut-off is important. Even the simple system (1.0.1) exhibits a highly complex behavior.

We obtain a rapidly convergent expansion of the wave function and the ionization probability for any frequency and amplitude; this can be conveniently used to calculate the wave function with rigorous bounds on errors, when the exponential decay rate is not extremely large or small, and the amplitude is not very large. For some relevant values of the parameters we plot the ionization fraction as a function of time.

We also show that for \( \omega = 0 \) the equation is solvable in closed form, and is a new integrable model. There are few nontrivial integrable examples of the time-dependent Schrödinger equation; for other exactly solvable models, see [DYO] and [DO].

If the duration of the external force is long compared to the time scale of the problem, the damping can practically be ignored, and only a time periodic potential is considered. This is the setting we study in the second part of the present thesis (Chapter 4), in a model with a delta function potential on the half-line, interacting with a time-harmonic external forcing (for related work on full line see [CRL]).

The equation is

\[
i \frac{\partial \psi}{\partial t} = \left( -\frac{\partial^2}{\partial x^2} + \delta(x - 1) (V + 2\Omega \sin(\omega t)) \right) \psi, \; x \geq 0
\]  

(1.0.4)

where \( V \) is the depth of the delta potential well and \( \Omega(t) \) is the amplitude of the oscillation. We take boundary and initial condition

\[
\psi(0, t) = 0, \; \psi_0 = \psi(0, x) \in C_0^\infty
\]  

(1.0.5)
The quantity of interest is still the large $t$ behavior of $\psi(x,t)$, and the survival probability (1.0.3).

There is extensive literature on this topic, see [Ne], [SET], and [DDRFP]. However, there are not many exact or mathematical results. For some exact results see [DYO], and for work with more mathematical approaches see [CRL] and [RL].

The single delta potential half-line problem is related to the two-delta potential full line problem through extensions. In the time independent case, a rigorous full asymptotic expansion of the wave function is derived, from which the resonance behaviors is observed, and connections with Gamow vectors and meta-stable states are made. In the more interesting time-dependent case, we obtain a partial expansion of the wave function for any frequency and amplitude; this can be conveniently used to calculate the wave function with rigorous bounds on errors. We also discuss the possibility to have localized wave function for $\Omega$ of order 1.

As a side result, we also give a theorem on calculating a continued fraction of a certain type. Complex continued fractions have been studied in great detail, but mostly in terms of convergence questions, see [JT], [LW], and [Wa]. Some theorems also give estimates on the values of the continued fraction, but a theorem concerning value of a continued fraction along a curve is, as far as this author knows, totally new. We also give suggestion on numerical calculation using this theorem.
2.1 Quantum Mechanics

2.1.1 Schrödinger equation

In quantum mechanics, the state of a particle at any time is described by a complex wave function \( \psi(x,t) \), where \( |\psi(x,t)|^2 dx \) is the probability of finding the particle between \( x \) and \( x + dx \). The wave function satisfies the Schrödinger equation:

\[
i\hbar \frac{\partial}{\partial t} \psi(x,t) = \left( -\frac{\hbar^2}{2m} \Delta + V(x,t) \right) \psi(x,t)
\] (2.1.1)

where \( \Delta = \sum_i \frac{\partial^2}{\partial x_i^2} \) is the Laplacian operator,

\[-\frac{\hbar^2}{2m} \Delta + V(x,t) := H(x,t)\]

is the Hamiltonian, and the function \( \psi(x,t) \) is called the wave function. An \( n \)-dimensional wave function corresponding to a physical solution needs to be in \( L^2(\mathbb{R}^n) \).

A normalized wave function is

\[
\int_{\mathbb{R}^n} |\psi(x,t)|^2 d^n x = 1
\] (2.1.2)

A change of variable brings

\[
i \frac{\partial}{\partial t} \psi(x,t) = (-\Delta + V(x,t)) \psi(x,t)
\] (2.1.3)
This is the normalized Schrödinger equation, in which constants that are mathematically irrelevant are eliminated. This concise equation is preferred by mathematicians and can be found in many literatures (e.g. \[CCL2\]).

2.1.2 Gamow vector

There is another type of states that are useful, especially in the study of resonances. An eigenstate of $H$ that is exponentially growing as $\|x\| \to \infty$ is called a Gamow vector or Gamow state. It is an eigenstate that corresponds to a complex energy

$$H f(x) = (E_R - i\gamma)f(x) \quad (2.1.4)$$

where $E_R$ and $-\gamma$ are the real and imaginary part of the complex energy respectively.

Note that Gamow state is very different from the physical states in the sense that no Hilbert space contains Gamow vectors; otherwise because $H$ is self-adjoint, eigenvalues of $H$ must be real.

2.1.3 Ionization problem

An ion is an atom, group of atoms, or a particle with a positive or negative charge (i.e. non-neutral). Ionization is any process that changes the electrical balance within an atom, from which an ion will be created. In an ionization process an electron is removed from the atom. More generally, any process in which a particle gets away from any given finite region can be considered as an ionization problem.

When the Schrödinger equation is specialized to a particle, the integral

$$P(D, t) = \int_D \psi^*(x, t)x\psi(x, t) d^dx = \int_D x|\psi(x, t)|^2 d^dx \quad (2.1.5)$$
gives the probability of finding this particle in the region $D$. If the particle ionizes, $P(D,t) \rightarrow 0$ as $t \rightarrow \infty$ and vice versa.

Usually it is interesting to study the stability of a given system (i.e. with given potential $V$), some related questions are:

1. Whether starting from any initial condition the particle ionizes.
2. Whether there are some special initial conditions under which the particle remains localized (ionization does not occur).
3. How does the ionization rate change when a parameter in $V$ is changing; especially when the parameter changes from one value for which the particle ionizes regardless of initial conditions to another value for which there are initial conditions that ensure localization.

\textbf{2.1.4 Resonance, metastable states}

In physics, resonance is the tendency of a system to oscillate at larger amplitude at some frequencies than at others. These are known as the system’s resonance frequencies (or resonant frequencies).

In quantum mechanics, a metastable state is a state that is not truly stationary but is almost stationary. In practice, especially in atomic and nuclear physics applications, the designation “metastable state” usually is reserved for states whose lifetimes are considerably long.

In a typical situation where potential barriers are present, a classical particle which has an energy lower than the barriers’ energy which is trapped inside the barriers
initially would oscillate inside the barriers. But a quantum particle would eventually pass through the barriers because of the tunneling effect. Each time the particle reaches one of the barriers it has a probability of tunneling. The tunneling probability depends on the initial condition. For most initial conditions, the average time that the particle is trapped inside is short; but for some special initial conditions, the particle oscillates inside the barriers for some comparably long time (in the average sense). This is the resonance behavior in the quantum mechanical sense, and the states corresponding to these special initial conditions are metastable states.

2.1.5 Zero Range Potential — ZRP

A range of a potential is basically the radius of the region where it is significantly different from zero. It is usually considered as the range where the potential interacts with the particle. For example, consider modeling a light particle in a collision process where it hits a heavy particle with shell of radius $R$. The potential chosen to represent the shell should also have range $R$ — whether it is an exponentially decaying potential or a cut-off potential depends on how “hard” one wants the shell to be.

A Zero Range Potential (ZRP) is a potential which contains only Dirac Delta functions. Such potentials serve as approximations to the Coulomb potentials, which are the exact potentials from charge interactions but which usually generate Schrödinger equations too hard to be solved. The approximation is usually accurate since Coulomb potential, on large scale, is also a short distance potential. For example, in solid physics one studies the behavior of an charged particle in a potential caused by spatially periodically positioned nuclei. One uses periodically positioned delta function
potential because the spacing between nuclei is large enough, and because a periodically positioned Coulomb potential is too complicated.

2.2 Mathematics Miscellanea

2.2.1 Asymptotic expansion, Asymptotic power series

An asymptotic expansion $\tilde{f}$ at a point $t_0$ from positive direction is a formal series of functions $f_k(t)$:

$$\tilde{f} = \sum_{k=0}^{\infty} f_k(t)$$

(2.2.1)

with

$$\lim_{t \to t_0^+} \frac{f_{k+1}(t)}{f_k(t)} = 0$$

(2.2.2)

A function $f$ is asymptotic to the formal series $\tilde{f}$ defined in (2.2.1) (denoted as $f \sim \tilde{f}$) as $t \to t_0^+$ if

$$f(t) - \sum_{k=0}^{N} f_k(t) = o(f_N(t))$$

(2.2.3)

A function $f$ is asymptotic to a power series $\sum_{k=0}^{\infty} c_k(t - t_0)^k$ as $t \to t_0^+$ if

$$f(t) - \sum_{k=0}^{N} c_k(t - t_0)^k = O((t - t_0)^{N+1})$$

(2.2.4)

For asymptotic power series one has (c.f. [Co]):

1. Addition and multiplication of asymptotic power series are defined.

2. Asymptotic power series to a given function is unique.
3. If an asymptotic power series $\tilde{f}$ is convergent and its sum is $f$, then $f \sim \tilde{f}$.

4. Assume $f$ is integrable near $z = 0$ and that

$$f(z) \sim \tilde{f}(z) = \sum_{k=0}^{\infty} c_k z^k$$

Then

$$\int_0^z f(s) ds \sim \int \tilde{f} := \sum_{k=0}^{\infty} \frac{c_k z^{k+1}}{k + 1}$$

5. Let $M > 0$ and assume $f(x)$ is analytic in the region

$$S_a = \{ x : |x| > R, |\text{Im}(x)| < a|\text{Re}(x)|^{-M} \}$$

and

$$f(x) \sim \sum_{k=0}^{\infty} c_k x^{-k}, \text{ as } |x| \to \infty$$

in any subregion $S_{a'}$ with $a' < a$, then

$$f'(x) \sim \sum_{k=0}^{\infty} (-kc_k) x^{-k-1}$$

as $|x| \to \infty$ in any subregion $S_{a'}$ with $a' < a$.

**Remark 1.** Note that the expansion (2.2.1) may not be, and usually is not, convergent. The concept of asymptotic expansion of a function is more useful compared to convergent expansion because it directly characterizes the (limiting) behavior of a function.

**Remark 2.** The asymptotic power series are inadequate for some functions. For example, the asymptotic power series of $e^{-1/z^2}$ at $z = 0$ (from both directions) is the zero series, but the function itself obviously does not behave like a zero function.
2.2.2 Laplace transform

Let $f \in L^1(\mathbb{R})$. By Fubini’s theorem and dominated convergence theorem, the Laplace transform
\[
(\mathcal{L}f)(p) := \int_0^\infty e^{-px} f(x) dx
\] (2.2.10)
is well defined and continuous in $p$ in the closed right half $p$ plane and analytic in the open right half $p$ plane.

The Laplace convolution is defined as
\[
(f * g)(p) = \int_0^p f(s)g(p-s)ds
\] (2.2.11)

One has
\[
\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)
\] (2.2.12)

and
\[
\mathcal{L}(-xf) = (\mathcal{L}f)'
\] (2.2.13)

and
\[
\mathcal{L}(f') = p(\mathcal{L}f)' \text{ if } f(0) = 0
\] (2.2.14)

The inverse Laplace transform is formally defined as the inverse transform of the Laplace transform
\[
\mathcal{L}(f(x)) = F(p) \iff \mathcal{L}^{-1}(F(p)) = f(x)
\] (2.2.15)

For functions that have nice properties, the inverse Laplace transform can be expressed using Bromwich contour integral formula:
Theorem 3. Assume \( c > 0 \), \( F(p) \) is analytic in the closed right half \( p \) plane \( \{ p : \text{Re}(p) > p_0 \} \) and assume further that \( \sup_{p \geq p_0} |F(p + it)| \leq G(t) \) with \( G(t) \in L^1(\mathbb{R}) \).

Let

\[
 f(x) = (\mathcal{L}^{-1}F)(x) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} F(p) dp \tag{2.2.16}
\]

Then for any \( p \in \{ p : \text{Re}(p) > p_0 \} \) one has

\[
 (\mathcal{L}f)(p) = \int_0^\infty e^{-xp} f(x) dx = F(p) \tag{2.2.17}
\]

Remark 4. Because of properties (2.2.12), (2.2.13), and (2.2.14), often the Laplace transform transforms a differential equation into an algebraic equation. In practice one transforms a differential equation (fully or partially) into an algebraic equation, solve it, then inverse Laplace transform it back to get the solution.

2.2.3 Watson’s lemma

An important tool that gives the asymptotic series of \( (\mathcal{L}f)(p) \) as \( p \to \infty \) is given by the Watson’s lemma (c.f. [Co]):

Lemma 5. Let \( f \in L^1(\mathbb{R}^+) \) and assume \( f(x) \sim \sum_{k=0}^\infty c_k x^{k\beta_1 + \beta_2} \) as \( x \to 0^+ \) for some constant \( \beta_i \) with \( \text{Re}(\beta_i) > 0, i = 1, 2 \). Then for \( a \leq \infty \),

\[
 F(p) = \int_0^a e^{-xp} f(x) dx \sim \sum_{k=1}^\infty c_k \Gamma(k\beta_1 + \beta_2) p^{-k\beta_1 - \beta_2} \tag{2.2.18}
\]

along any ray in the open right half \( p \) plane.
2.2.4 Euler summation formula

The Euler-Maclaurin integration formula is (see [AS])

\[
\sum_{k=1}^{n-1} f(k) \sim \int_0^n f(k) dk - \frac{1}{2} (f(0) + f(n)) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (f^{(2k-1)}(n) - f^{(2k-1)}(0))
\]

\[
\sim \int_0^n f(k) dk + C + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(n)
\]  

(2.2.19)

where the Bernoulli polynomials \( B_n(x) \) are given by the generating function

\[
\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}
\]

(2.2.20)

and Bernoulli numbers \( B_n \) are \( B_n(x) \) evaluated at 0:

\[
B_n = B_n(0)
\]

(2.2.21)

**Remark 6.** For (2.2.19) to be a valid asymptotic expansion, \( f \) should satisfy

\[
\lim_{n \to \infty} \frac{f^{(2n+1)}(n)}{f^{(2n-1)}(n)} = 0
\]

(2.2.22)

2.2.5 Lambert function

The Lambert function (or Omega function) is the inverse function of

\[
f(w) = we^w
\]

(2.2.23)

Denote the function by \( W \), then for every complex number \( z \)

\[
z = W(z)e^{W(z)}
\]

(2.2.24)

Since \( f \) is not injective, \( W \) is multivalued (except at 0). The Lambert W function cannot be expressed in terms of elementary functions.
2.2.6 Functional analysis; the analytic Fredholm Theorem

Some useful results in functional analysis are (see [RS]):

**Definition 7.** Let $X$ and $Y$ be Banach spaces. An operator $T \in \mathcal{L}(X,Y)$ is called compact (or completely continuous) if $T$ takes bounded sets in $X$ into precompact sets in $Y$. Equivalently, $T$ is compact if and only if for every bounded sequence $\{x_n\} \subset X$, $\{Tx_n\}$ has a subsequence convergent in $Y$.

**Theorem 8.** Let $X$ and $Y$ be Banach spaces, $T \in \mathcal{L}(X,Y)$, if $\{T_n\}$ are compact and $T_n \rightarrow T$ in the norm topology, then $T$ is compact.

**Definition 9.** If the range of $T$ is finite dimensional, $T$ is called a finite rank operator.

An finite rank operator is a compact operator, and by Theorem 8, an operator that is the limit of finite rank operators is also compact. The inverse of this statement is also true:

**Theorem 10.** Let $\mathcal{H}$ be a separable Hilbert space. Then every compact operator on $\mathcal{H}$ is the norm limit of a sequence of operators of finite rank.

The analytic Fredholm theorem is:

**Theorem 11.** Let $D$ be an open connected subset of $\mathbb{C}$. Let $f : D \rightarrow \mathcal{L}(\mathcal{H})$ be an analytic operator-valued function such that $f(z)$ is compact for each $z \in D$. Then, either

(a) $(I - f(z))^{-1}$ exists for no $z \in D$.

or
(b) \((I - f(z))^{-1}\) exists for all \(z \in D\backslash S\) where \(S\) is a discrete subset of \(D\) (i.e. a set which has no limit points in \(D\)). In this case, \((I - f(z))^{-1}\) is meromorphic in \(D\), analytic in \(D\backslash S\), the residues at the poles are finite rank operators, and if \(z \in S\) then \(f(z)\psi = \psi\) has a nonzero solution in \(\mathcal{H}\).

### 2.2.7 Green’s function

Let \(\mathcal{L}\) be a differential operator and assume one wants to solve the differential equation

\[
\mathcal{L} u(x) = g(x) \tag{2.2.25}
\]

with certain predefined boundary condition.

One first solves the Green’s equation

\[
\mathcal{L} G(x) = \delta(x) \tag{2.2.26}
\]

with the same boundary condition.

The solution of (2.2.26) is called the Green’s function subject to the given boundary condition. The solution to (2.2.25) for a translationally invariant (in \(x\)) \(\mathcal{L}\) can then be expressed using the Green’s function as

\[
u(x) = \int G(x - s)g(s)ds \tag{2.2.27}
\]

Note that the Green’s function is not unique, as any solution to equation \(\mathcal{L}u = 0\) can be added to the Green’s function without changing (2.2.26).
In this chapter we study

\[ i \frac{\partial \psi}{\partial t} = \left( -\frac{\partial^2}{\partial x^2} - 2\delta(x) (1 - A(t) \cos(\omega t)) \right) \psi \]  

(3.0.1)

where \( A(t) \) is the amplitude of the oscillation; we take for \( \lambda > 0 \)

\[ \psi_0 = \psi(0, x) \in C_0^\infty; \quad A(t) = e^{-\lambda t} \]  

(3.0.2)

The main purpose is to determine the large \( t \) behavior of \( \psi \) and the survival probability

\[ P_B = \lim_{t \to \infty} P(t, B) = \lim_{t \to \infty} \int_B |\psi(t, x)|^2 dx \]  

(3.0.3)

where \( B \) is a bounded subset of \( \mathbb{R} \).

### 3.1 Main results

**Theorem 12.** Let \( \psi(t, x) \) be the solution of (3.0.1) with initial condition \( \psi_0 \in C_0^\infty \).

Let

\[ g_{m,n} = g_{m,n}(\sigma) = \frac{i}{2} \int_{\mathbb{R}} e^{-\sqrt{\sigma + n\omega - m\lambda} |x'|} \psi_0(x') dx' \]  

(3.1.1)

where we suppress the dependence on \( \lambda \) and \( \omega \). Then as \( t \to \infty \) we have
\[\psi(t, x) = r(\lambda, \omega)e^{it}e^{-|x|} (1 + t^{-1/2}h(t, x)) \quad (3.1.2)\]

where \(|h(t, x)| \leq C, \forall x \in \mathbb{R}, \forall t \in \mathbb{R}^+, \) and where

\[r(\lambda, \omega) = [-A_{1,-1} - A_{1,1} + 2g_{0,0}]_{\sigma=1} \quad (3.1.3)\]

Here \(A_{m,n} = A_{m,n}(\sigma)\) solves

\[(\sqrt{\sigma + n\omega} - im\lambda - 1)A_{m,n} = \frac{1}{2}A_{m+1,n+1} - \frac{1}{2}A_{m+1,n-1} + g_{m,n} \quad (3.1.4)\]

There is a unique solution of (3.1.4) satisfying

\[\sum_{m,n}(1 + |n|)^{3/2}e^{-b\sqrt{1+|m|}}|A_{m,n}| < \infty \quad (3.1.5)\]

where \(b > 1\) is a constant. It is this solution that enters (3.1.3).

There is a rapidly convergent representation of \(r(\lambda, \omega)\), see §3.2.5.

Clearly, \(|r(\lambda, \omega)|^2\) is the probability of survival, the projection onto the limiting bound state.

For \(\omega = 0\) we have:

**Theorem 13.** (i) For \(\omega = 0\) we have

\[r(\lambda) = \int_0^\infty \frac{-e^{-q}}{1 + e^{-q}} \int_{c-i\infty}^{c+i\infty} g_k \exp \left(-\frac{2i\sqrt{-i}k}{\sqrt{\lambda}}\right)^{\lambda^{1-k}} \frac{1}{\sqrt{\Gamma(k)}} \exp \left(-\int_0^\infty e^{-kp} \frac{\sqrt{i} \lambda \left(-2 + 2e^{-p} - p^{3/2} \sqrt{\pi \lambda} \text{erf} \left(\frac{-3/2^{3/2}}{\sqrt{\lambda}}\right)\right)}{2 (-1 + e^{-p}) \sqrt{\pi} (p\lambda)^{3/2}} dp\right) dk dq \quad (3.1.6)\]

where \(g(k) = g_{k,0}\).

(ii) We look at the case when \(\psi_0 = e^{-|x|}\), the bound state of the limiting time independent system. Assuming the series of \(r(\lambda)\) is Borel summable in \(\lambda\) for \(\arg \lambda \in [0, \pi/2]\), as \(\lambda \to 0\) we have

\[r(\lambda) \sim 2^{-2/3}(-3i)^{1/6} \pi^{-1/2} \Gamma(2/3)e^{-\frac{3\lambda}{\pi}} \lambda^{1/6} \quad (3.1.7)\]

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Note: The behavior (3.1.7) is confirmed numerically with high accuracy (including the values of constants), see §3.4.3.

We also discuss results in two limiting cases: the short pulse setting (see §3.5) and the special case $\lambda = 0$ (see §3.6).

### 3.2 Proofs and further results

#### 3.2.1 The associated Laplace space equation

We study the analytic properties of the Laplace transform of $\psi$. This Laplace approach can be viewed as a mathematically rigorous way to study the Schrödinger equation in energy space, which has been used often in physics, see [DO], [GP], and [Ne].

Existence of a strongly continuous unitary propagator for (3.0.1) (see [RS] v.2, Theorem X.71) implies that for $\psi_0 \in L^2(\mathbb{R}^d)$, the Laplace transform

$$
\hat{\psi}(\cdot, p) := \int_0^\infty \psi(\cdot, t)e^{-pt}dt
$$

exists for $\text{Re}(p) > 0$ and the map $p \rightarrow \psi(\cdot, p)$ is $L^2$ valued analytic in the right half plane

$$
p \in \mathbb{H} = \{ z : \text{Re}(z) > 0 \}
$$

The Laplace transform of (3.0.1) is

$$
\left( \frac{\partial^2}{\partial x^2} + ip \right) \hat{\psi}(x, p) = i\psi_0 - 2\delta(x)\hat{\psi}(x, p) + \delta(x) \left( \hat{\psi}(x, p - i\omega + \lambda) + \hat{\psi}(x, p + i\omega + \lambda) \right)
$$

(3.2.1)
Let \( p = i\sigma + m\lambda + in\omega \) and
\[
y_{m,n}(x,\sigma) = \hat{\psi}(x, i\sigma + m\lambda + in\omega)
\] (3.2.2)
where \( i\sigma \in \{ z : 0 \leq \text{Im} z < \omega, 0 \leq \text{Re} z < \lambda \} \).

**Remark 14.** Since the \( p \) plane equation only links values of \( p \) differing by \( m\lambda + in\omega \), \( m, n \in \mathbb{Z} \), it is useful to think of functions of \( p \) as vectors with components \( m \) and \( n \), parameterized by \( \sigma \).

Thus we rewrite (3.0.1) as
\[
\left( \frac{\partial^2}{\partial x^2} - \sigma - n\omega + im\lambda \right) y_{m,n} = i\psi_0 - 2\delta(x)y_{m,n} + \delta(x) (y_{m+1,n+1} + y_{m+1,n-1})
\] (3.2.3)
When \( |n| + |m| \neq 0 \), the resolvent of the operator
\[
-\frac{\partial^2}{\partial x^2} + \sigma + n\omega - im\lambda
\]
has the integral representation
\[
\left( g_{m,n}f \right)(x) := \int_{\mathbb{R}} G(\kappa_{m,n}(x-x')) f(x')dx'
\] (3.2.4)
with
\[
\kappa_{m,n} = \sqrt{-ip} = \sqrt{\sigma + n\omega - im\lambda}
\]
where the choice of branch is so that if \( p \in \mathbb{H} \), then \( \kappa_{m,n} \) is in the fourth quadrant, and where the Green’s function is given by
\[
G(\kappa_{m,n}x) = \frac{1}{2} \kappa_{m,n}^{-1} e^{-\kappa_{m,n}|x|}
\] (3.2.5)
Define further the operator $g$ as:

$$(g f)_{m,n} = g_{m,n} f$$

**Remark 15.** If $f(x) \in C_0^\infty$, using integration by parts we have, as $p \to \infty$

$$g(f) \sim \frac{c(x)}{p} + o \left( \frac{1}{p} \right)$$

where we regard $g$ as an operator with $p$ as a parameter; see also Remark 14. Furthermore, (3.2.4) implies $c(x) \in L^2$.

Define the operator $\mathcal{C}$ by

$$(\mathcal{C} y)_{m,n} = g_{m,n} \left[ 2\delta(x)y_{m,n} - \delta(x)(y_{m+1,n+1} + y_{m+1,n-1}) \right]$$

(3.2.6)

Then Eq. (3.2.3) can be written in the equivalent integral form

$$y = ig\psi_0 + \mathcal{C} y$$

(3.2.7)

where $g$ is defined in (3.2.4).

**Remark 16.** Because of the factor $\kappa_{m,n}^{-1}$ in (3.2.5), we have, with the identification in Remark 14,

$$\mathcal{C}\phi(p) \sim \frac{c(x)}{\sqrt{p}} \phi(p)$$

as $p \to \infty$, for any function $\phi(p)$.
3.2.2 Further transformations, functional space

In this section we assume \( \psi_0 \in C_0^\infty \). As in Remark 15, we obtain

\[
ig \psi_0 = \frac{c_1(x)}{p} + O\left(\frac{1}{p^{3/2}}\right)
\]

for some \( c_1(x) \in L^2 \).

Let

\[
h_1(x, p) = c_1(x) \mathcal{L}(1_{[0,1]}(t))
\]

In the following we suppress the \( x \) dependence of \( h_1 \) for simplicity. For large \( p \) we have

\[
h_1(p) = \frac{c_1(x)}{p} + O\left(\frac{1}{p^{3/2}}\right)
\]

**Remark 17.** As a function of \( x \), \( h_1(p) \) is clearly in \( L^2 \) and

\[
\mathcal{L}^{-1}(h_1(p)) = c_1(x) 1_{[0,1]}(t)
\]

thus for \( t > 1 \) we have

\[
\mathcal{L}^{-1}(h_1(p)) = 0
\]

Substituting

\[
y = y_1 + h_1
\]

in (3.2.7) we have

\[
y_1 = ig \psi_0 - h_1 + \mathcal{E}(h_1) + \mathcal{E}y_1
\]
Let \( y_0 = ig\psi_0 - h_1 + \mathcal{C}(h_1) \). Then Remark 16 implies that for large \( p \)

\[
y_0 = O\left( \frac{1}{p^{3/2}} \right)
\]

(3.2.13)

and by construction \( y_0 \in L^2 \) as a function of \( x \).

We analyze (3.2.12) in the space \( \mathcal{H}_b = L^2(\mathbb{Z}^2 \times \mathbb{R}, \| \cdot \|_b), b > 1 \), where

\[
\| y \|_b := \left( \sum_{m,n} (1 + |n|)^{\frac{3}{2}} e^{-b \sqrt{1+|m| \| y_{m,n} \|_{L^2}}} \right)^{\frac{1}{2}}
\]

(3.2.14)

We denote by \( \hat{\psi}_1 \) the transformed wave function corresponding to \( y_1 \). Writing \( y \) instead of \( y_1 \), we obtain from (3.2.12),

\[
y = y_0 + \mathcal{C}y
\]

(3.2.15)

**Lemma 18.** \( \mathcal{C} \) is a compact operator on \( \mathcal{H}_b \), and it is analytic in \( \sqrt{-ip} \).

**Proof.** Compactness is clear since \( \mathcal{C} \) is a limit of bounded finite rank operators. Analyticity is manifest in the expression of \( \mathcal{C} \) (see (3.2.4) and (3.2.6)). \( \square \)

**Proposition 19.** Equation (3.2.15) has a unique solution iff the associated homogeneous equation

\[
y = \mathcal{C}y
\]

(3.2.16)

has no nontrivial solution. In the latter case, the solution is analytic in \( \sqrt{\sigma} \).

**Proof.** This follows from Lemma 42 and the Fredholm alternative. \( \square \)
When \( m = 0, n = 0, \) and \( \sigma = 0, \) \( C \) is singular, but the solution is not. Indeed, by adding \( 1_{[-A,A]} \), \( A > 0 \), to both sides of \( (3.2.3) \) we get the equivalent equation

\[
\left( \frac{\partial^2}{\partial x^2} - \sigma - n\omega + im\lambda + 1_{[-A,A]} \right) y_{m,n} = i\psi_0 + (1_{[-A,A]} - 2\delta(x)) y_{m,n} + \delta(x) (y_{m+1,n+1} + y_{m+1,n-1})
\]

Arguments similar to those when \( 1_{[-A,A]} \) is absent show that the operator \( C \) associated to \( (3.2.17) \) is analytic in \( \sqrt{\sigma} \), thus \( y_{m,n} \) is analytic in \( \sqrt{\sigma} \).

### 3.2.3 Equation for \( A \)

Componentwise \( (3.2.7) \) reads

\[
y_{m,n} = \int_{\mathbb{R}} \frac{1}{2} \kappa_{m,n}^{-1} e^{-\kappa_{m,n}|x-x'|} \psi_0(x') dx' + \frac{1}{2\kappa_{m,n}} e^{-\kappa_{m,n}|x|} [2y_{m,n}(0) - (y_{m+1,n+1}(0) + y_{m+1,n-1}(0))]
\]

With \( A_{m,n} = y_{m,n}(0) \), we have

\[
(\sqrt{\sigma} + n\omega - im\lambda - 1) A_{m,n} = -\frac{1}{2} A_{m+1,n+1} - \frac{1}{2} A_{m+1,n-1} + g_{m,n}
\]

where \( g_{m,n} \) is defined in \( (3.1.1) \).

**Proposition 20.** The solution to \( (3.2.18) \) is determined by the \( A_{m,n} \) through

\[
y_{m,n} = \int_{\mathbb{R}} \frac{1}{2} \kappa_{m,n}^{-1} e^{-\kappa_{m,n}|x-x'|} \psi_0(x') dx' + e^{-\kappa_{m,n}|x|} A_{m,n} - \frac{1}{\kappa_{m,n}} e^{-\kappa_{m,n}|x|} g_{m,n}
\]

It thus suffices to study \( (3.2.19) \).
Proof. Taking $x = 0$ in (3.2.18) we obtain (3.2.19); using now (3.2.19) in (3.2.18) we have

$$y_{m,n} = \int_{\mathbb{R}} \frac{1}{2} \kappa_{m,n}^{-1} e^{-\kappa_{m,n}|x-x'|} \psi_0(x') dx' + \frac{1}{2\kappa_{m,n}} e^{-\kappa_{m,n}|x'|} [2A_{m,n} - (A_{m+1,1} + A_{m+1,n-1})]$$

$$= \int_{\mathbb{R}} \frac{1}{2} \kappa_{m,n}^{-1} e^{-\kappa_{m,n}|x-x'|} \psi_0(x') dx' + e^{-\kappa_{m,n}|x|} A_{m,n} - \frac{1}{\kappa_{m,n}} e^{-\kappa_{m,n}|x|} g_{m,n} \quad (3.2.21)$$

Remark 21. If $y \in H_b$, then $A_{m,n} = y_{m,n}(0)$ satisfies (3.1.2).

Let $A^0_{m,n} = y^0_{m,n}(0)$ where $y^0_{m,n}$ is a solution to (3.2.16). The solution of (3.2.16) has the freedom of a multiplicative constant; we choose it by imposing

$$A^0_{0,0} = \lim_{\sigma \to 1} (\sigma - 1) A_{0,0} \quad (3.2.22)$$

It is clear $A^0_{m,n}$ satisfies the homogeneous equation associated to (3.2.19)

$$(\sqrt{\sigma} + n\omega - im\lambda - 1) A^0_{m,n} = -\frac{1}{2} A^0_{m+1,n+1} - \frac{1}{2} A^0_{m+1,n-1} \quad (3.2.23)$$

3.2.4 Positions and residues of the poles

Define

$$\sigma_0 = 1 - \left\lfloor \frac{1}{\omega} \right\rfloor \omega \quad (3.2.24)$$

where $\lfloor x \rfloor$ is the integer part of $x$. To simplify notation we take $\omega > 1$ in which case $\sigma_0 = 1$. The general case is very similar.

Denote

$$\mathcal{B} := \{ in\omega + m\lambda + i : m \in \mathbb{Z}, n \in \mathbb{Z}, m \leq 0, |n| \leq |m| \} \quad (3.2.25)$$
Proposition 22. The system (3.2.23) has nontrivial solutions in $\mathcal{H}$ iff $\sigma = \sigma_0 (=1$ as discussed above). If $\sigma = 1$, then the solution is a constant multiple of the vector $A^0_{m,n}$ given by

$$
\begin{align*}
A^0_{m,n} &= 0 \quad m \geq 0 \text{ and } (m,n) \neq (0,0) \\
A^0_{m,n} &= 0 \quad m \leq 0 \text{ and } m \leq n \leq -m \\
A^0_{m,n} &= 1 \quad (m,n) = (0,0)
\end{align*}
$$

(3.2.26)

and obtained inductively from (3.2.23) for all other $(m,n)$. (Note that $\sigma = 1$ is used crucially here since (3.2.23) allows for the nonzero value of $A^0_{0,0}$.)

**Proof.** Let $\sigma = 1$. By construction, $A^0$ defined in Proposition 22 satisfies the recurrence and we only need to check (3.1.5). Since

$$
A^0_{m,n} = -\frac{A^0_{m+1,n+1} + A^0_{m+1,n-1}}{2(\sqrt{\sigma + n\omega + im\lambda} - 1)}
$$

and $\sqrt{\sigma + n\omega + im\lambda} - 1 \neq 0$, we have

$$
|A^0_{m,n}| \leq C \frac{2^m}{\sqrt{(|m| + |n|)!}}
$$

proving the claim.

Now, for any $\sigma$, if there exists a nontrivial solution, then for some $n_0, m_0$ we have $A^0_{n_0,m_0} \neq 0$. By (3.2.23), we have either

$$
|A^0_{n_0-1,m_0+1}| \geq \frac{1}{2} |(\sqrt{\sigma + n_0\omega + im_0\lambda} - 1)| \cdot |A^0_{n_0,m_0}|
$$

(3.2.27)

or

$$
|A^0_{n_0+1,m_0+1}| \geq \frac{1}{2} |(\sqrt{\sigma + n_0\omega + im_0\lambda} - 1)| \cdot |A^0_{n_0,m_0}|
$$

(3.2.28)
It is easy to see that if $in_0\omega + m_0\lambda + i \in B^c$ or $\sigma \neq 1$, the above inequalities lead to

$$|A_{n,m_0+m}^0| \geq c \sqrt{m!} \quad (3.2.29)$$

for large $m > 0$ (note that in these cases $\sqrt{\sigma + n\omega + im\lambda} - 1 \neq 0$), contradicting (3.1.5).

Finally, if $\sigma = 1$, then $A^0$ is determined by $A^0_{0,0}$ via the recurrence relation (3.2.23) (note that $A^0|_{B^c} = 0$). This proves uniqueness (up to a constant multiple) of the solution. \hfill \qed

Combining Proposition 19 and Proposition 22 we obtain the following result.

**Proposition 23.** The solution $\hat{\psi}(p)$ to equation (3.2.1) is analytic with respect to $\sqrt{-ip}$, except for poles in $B$.

**Proof.** Proposition 22 shows that (3.2.23) has a solution $A^0$ for $\sigma \in B$; by Proposition 19, $A$ has singularities in $B$, and the conclusion follows from Proposition 20. \hfill \qed

So far we showed that the solution has possible singularities in $B$. To show that indeed $\hat{\psi}$ has poles for generic initial conditions, we need the following result:

**Lemma 24.** Let $H$ be a Hilbert space. Let $K(\sigma) : H \to H$ be compact, analytic in $\sigma$ and invertible in $B(0, r) \setminus \{0\}$ for some $r > 0$. Let $v_0(\sigma) \notin \text{Ran}(I - K(0))$ be analytic in $\sigma$. If $v(\sigma) \in H$ solves the equation $(I - K(\sigma))v(\sigma) = v_0(\sigma)$, then $v(\sigma)$ is analytic in $\sigma$ in $B(0, r) \setminus \{0\}$ but singular at $\sigma = 0$.

**Proof.** By the Fredholm alternative, $v(\sigma)$ is analytic when $\sigma \neq 0$. If $v(\sigma)$ is analytic at $\sigma = 0$ then $v_0$ is analytic and $v_0(\sigma) \in \text{Ran}(I - K(0))$ which is a contradiction. \hfill \qed
The operator $\mathfrak{C}$ is compact by Remark 42. The inhomogeneity $y_0$ in equation 3.2.16 is analytic in $\sqrt{\sigma}$. Furthermore, at $\sigma = 1$, $\text{Ran}(I - \mathfrak{C})$ is of codimension 1 (Proposition 22). Combined with Lemma 24 we have

**Corollary 25.** For a generic inhomogeneity $y_0$, $y(\sigma)$ is singular at $\sigma = 1$. Equivalently, $\hat{\psi}(p)$ has a pole at $p = i$.

It can be shown that $\hat{\psi}(p)$ has a pole at $p = i$ for generic $\psi_0$. We prefer to show the following result which has a shorter proof.

**Proposition 26.** The residue $R_{0,0}$ of the pole for $\hat{\psi}$ at $p = i$ is given by

$$R_{0,0} = \lim_{\sigma \to 1} (\sigma - 1)A_{0,0} = [-A_{1,-1} - A_{1,1} + 2g_{0,0}]_{\sigma = 1}$$

(3.2.30)

In particular, $R_{0,0} \neq 0$ for large $\lambda$ and generic initial condition $\psi_0$.

**Proof.** When $m = 0$ and $n = 0$, (3.2.19) gives

$$(\sqrt{\sigma} - 1)A_{0,0} = -\frac{1}{2}A_{1,-1} - \frac{1}{2}A_{1,1} + g_{0,0}$$

(3.2.31)

Clearly $A_{0,0}$ is singular as $\sigma \to 1$, which implies that $\hat{\psi}$ has a pole at $p = i$ with residue given in (3.2.30). Thus $R_{0,0}$ is not zero if the quantity $[-A_{1,-1} - A_{1,1} + 2g_{0,0}]_{\sigma = 1}$ is not zero. First, $g_{0,0}|_{\sigma = 1}$ is not zero by definition:

$$g_{0,0}|_{\sigma = 1} = i \int_{\mathbb{R}} \frac{1}{2} e^{-|x'|} \psi_0(x')dx'$$

Next, taking $m = 1, n = 1$, and $\sigma = 1$ in (3.2.19) we obtain

$$(\sqrt{1 + \omega - i\lambda} - 1)A_{1,1} = \left[-\frac{1}{2}A_{2,2} - \frac{1}{2}A_{2,0} + g_{1,1}\right]_{\sigma = 1}$$

(3.2.32)
Thus for any $c > 0$ when $\lambda$ is large enough we have

$$|A_{1,1}| \leq c^{-1} \left[ |g_{1,1}| + \max\{|A_{2,2}|, |A_{2,0}| \} \right]_{\sigma=1}$$

Estimating similarly $A_{2,2}$ and $A_{2,0}$ and so on, we see that $|A_{1,1}| = O(c^{-1})$. When $c$ is large enough we have $|A_{1,1}| < |g_{0,0}|$. Analogous bounds hold for $A_{1,-1}$, showing that $[-A_{1,-1} - A_{1,1} + 2g_{0,0}]_{\sigma=1}$ is not zero.

\[\square\]

**Corollary 27.** For generic initial condition $\hat{\psi}$ has simple poles in $B$, and their residues are given by $R_{m,n} = A_{m,n}^0$.

**Proof.** We take a small loop around $\sigma = 0$ and integrate equation (3.2.18) along it. This gives a relation among $R_{m,n}$ which is identical to (3.2.23):

$$(\sqrt{\sigma} + n\omega - im\lambda - 1)R_{m,n} = -\frac{1}{2}R_{m+1,n+1} - \frac{1}{2}R_{m+1,n-1}$$

(3.2.32)

Proposition 26 and (3.2.22) implies that $R_{0,0} = A_{0,0}^0$. The rest of the proof follows from Proposition 22.

\[\square\]

**Remark 28.** It is easy to see that there exist initial conditions for which the solution has no poles. Indeed, if the solutions $\psi_1$ and $\psi_2$ have a simple pole at $p = i$ with residue $a_1$ and $a_2$ respectively, then for the initial condition $\psi_{0,0} = a_2\psi_{1,0} - a_1\psi_{2,0}$, the corresponding solution $\psi_0$ has no pole at $p = i$.

**3.2.5 Infinite sum representation of $A_{m,n}$**

Taking $\sigma = 1$ in (3.2.19) we get

$$(\sqrt{1+n\omega - im\lambda} - 1)A_{m,n} = -\frac{1}{2}A_{m+1,n-1} - \frac{1}{2}A_{m+1,n+1} + g_{m,n}$$

(3.2.33)
For $\tau = (a_1, ..., a_N) \in \{-1, 1\}^N$, we define $\tau^0_j = (a_1, ..., a_j, 0, ..., 0)$. (Note that $\tau = \tau^0_N$). We denote $\Sigma\tau^0_j = \sum_{i=1}^j a_i$ and $\{-1, 1\}^0 = \{0\}$.

Let

$$B_{m,n} = \frac{1}{\sqrt{1 + n\omega - im\lambda - 1}}$$

and for some $\tau \in \{-1, 1\}^N$ define

$$B_{m,n,N} = B_{m,n,N}(\tau) = \prod_{j=0}^{N-1} B_{m+j, n+\Sigma \tau^0_j}$$

Equation (3.2.33) implies

$$A_{m,n} = (-1)^N \frac{1}{2N} \sum_{\tau \in \{-1,1\}^N} B_{m,n,N-1} A_{m+N,n+\Sigma \tau}$$

$$+ \sum_{j=0}^{N-1} (-1)^j \frac{1}{2^j} \sum_{\tau \in \{-1,1\}^j} B_{m,n,j} g_{m+j,n+\Sigma \tau} \tag{3.2.34}$$

As $N \to \infty$ we have

$$\prod_{j=0}^{N} B_{m+j,n} \sim \frac{1}{\sqrt{N!}}$$

and $A_{m,n}$ goes to zero as $m \to \infty$, and thus we have

$$\lim_{N \to \infty} (-1)^N \frac{1}{2N} \sum_{\tau \in \{-1,1\}^N} B_{m,n,N-1} A_{m+N,n+\Sigma \tau} = 0$$

In the limit $N \to \infty$ we obtain

$$A_{m,n} = \sum_{j=0}^{\infty} (-1)^j \frac{1}{2^j} \sum_{\tau \in \{-1,1\}^j} B_{m,n,j} g_{m+j,n+\Sigma \tau} \tag{3.2.35}$$

Remark 29. Truncating the infinite expansion to $N$, the error is bounded by

$$\left| \frac{1}{2N} \sum_{\tau \in \{-1,1\}^N} \left( \prod_{j=0}^{N} B_{m+j,n+\Sigma \tau^0_j} A_{m+N,n+\Sigma \tau}^0 \right) \right| \tag{3.2.36}$$
3.3 Proof of Theorem 12

In §3.2.4 it was shown that for a generic initial condition \( \psi_0(x) \), the solution \( \hat{\psi}(x,p) \) has simple poles in \( \mathcal{B} \), with residues \( R_{m,n} = A_{m,n} \).

Since \( y \in \mathcal{H}_b \), the inverse Laplace transform can be expressed using the Bromwich contour formula. Recall that \( y \) differs from the original vector form of \( \hat{\psi} \) by (3.2.11), we have

\[
\psi(x,t) = \mathcal{L}^{-1}\hat{\psi}(x, p) = \mathcal{L}^{-1}(h_1) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \hat{\psi}_1(x, p) dp \tag{3.3.1}
\]

The fact that \( y \in \mathcal{H}_b \) also implies that \( \hat{\psi}_1(x, p) \to 0 \) fast enough as \( p \to c \pm i\infty \). Thus the contour of integration in the inverse Laplace transform can be pushed into the left half \( p \)-plane, after collecting the residues. As a result, for some small \( c < 0 \) the contour becomes one using the Bromwich contour formula coming from \( c - i\infty \), joining \( c - i\epsilon, 0 \), and \( c + i\epsilon \) (for arbitrarily small \( \epsilon > 0 \)) in this order, then going towards \( c + i\infty \).

Thus we have

\[
\psi(t, x) = \mathcal{L}^{-1}(h_1) + \text{Res}_{p=i}(e^{pt}\hat{\psi}_1) + \frac{1}{2\pi i} \int_0^{c+i\epsilon} e^{pt} \hat{\psi}_1(x, p) dp + \frac{1}{2\pi i} \int_0^{c-i\epsilon} e^{pt} \hat{\psi}_1(x, p) dp \tag{3.3.2}
\]

By Corollary 27 we have

\[
\text{Res}_{p=i}(e^{pt}\hat{\psi}_1) = R_{0,0} = A_{0,0}^0
\]

The third term on the right hand side of (3.3.2) decays exponentially for large \( t \).
(note that $c < 0$ and the integral is bounded since $y \in \mathcal{H}_{b}$), while the last two terms yield an asymptotic power series in $1/\sqrt{t}$, as easily seen from Watson’s Lemma.

Combining these results and the fact $L^{-1}(h_1) = o(1/t)$ (Remark 17), we obtain the first part of Theorem 12 with $r(\lambda, \omega) = R_{0,0}$. The rest follows from Proposition 26.

### 3.4 Proof of Theorem 13

When $\omega = 0$, the equation

$$i \frac{\partial \psi}{\partial t} = \left( -\frac{\partial^2}{\partial x^2} - 2\delta(x) + 2\delta(x) e^{-\lambda t} \cos(\omega t) \right) \psi$$

becomes

$$i \frac{\partial \psi}{\partial t} = \left( -\frac{\partial^2}{\partial x^2} - 2\delta(x) + 2\delta(x) e^{-\lambda t} \right) \psi$$

Rewriting $A_{m,n}$ and $g_{m,n}$ as $A_n$ and $g_n$, (3.2.19) becomes

$$(\sqrt{\sigma} - im\lambda - 1)A_n = -A_{n+1} + g_n$$

Since $\omega = 0$, (3.2.35) simplifies to

$$A_n = \sum_{l=0}^{\infty} (-1)^{l-1} \prod_{j=0}^{l} \frac{1}{\sqrt{1-i(n+j)\lambda - 1}} g_{n+l}$$  \hspace{1cm} (3.4.1)

#### 3.4.1 Proof of Theorem 13, (i)

When $n = 1$ equation (3.4.1) becomes

$$A_1 = \sum_{k=1}^{\infty} (-1)^k \prod_{j=1}^{k} \frac{1}{\sqrt{1-i j\lambda - 1}} g_k$$  \hspace{1cm} (3.4.2)
With the notation

\[ h_k = \prod_{j=1}^{k} \left( \sqrt{1 - i j \lambda} - 1 \right) \]

equation (3.4.2) becomes

\[ A_1 = \sum_{k=1}^{\infty} \frac{(-1)^k g_k}{h_k} \]

Let

\[ h_k = e^{w_k} \sqrt[2]{\lambda^{k-1}(k-1)!} \]

We have

\[ w_{k+1} - w_k = \log(\sqrt{1 - i k \lambda} - 1) - \frac{1}{2} \log(\lambda k) \]

Differentiating in \( \lambda \) we obtain

\[ \frac{d}{d\lambda} (w_{k+1} - w_k) = \frac{1}{2\lambda \sqrt{1 - i k \lambda}} \]

Let \( u_k \) be so that

\[ \frac{d}{d\lambda} u_k = \frac{d}{d\lambda} w_k - \frac{i \sqrt{-i k}}{\lambda^{3/2}} \]

Then,

\[ \frac{d}{d\lambda} (u_{k+1} - u_k) = -\frac{i \sqrt{-i k}}{\lambda^{3/2}} + \frac{i \sqrt{-i k}}{\lambda^{3/2}} + \frac{1}{2\lambda \sqrt{1 - i k \lambda}} \quad (3.4.3) \]

By taking the inverse Laplace transform of (3.4.3) in \( k \) we get (we use \( p \) as the transformed variable here)

\[ (e^{-p} - 1) \mathcal{L}^{-1} \frac{d}{d\lambda} u_k = \frac{\sqrt{i} (1 - e^{-p} + e^{-ip/\lambda} p)}{2\sqrt{\pi} (p\lambda)^{3/2}} \quad (3.4.4) \]
Integrating (3.4.4) with respect to $\lambda$ gives

$$\mathcal{L}^{-1}u_k = \frac{\sqrt{i} \lambda \left( -2 + 2e^{-\rho} - i^{3/2}\sqrt{p\pi \lambda} \text{erf} \left( \frac{-i^{3/2} \sqrt{\rho}}{\sqrt{\lambda}} \right) \right)}{2 (-1 + e^{-\rho}) \sqrt{\pi} (p\lambda)^{3/2}} \cdot \frac{1}{\sqrt{\lambda^{k-1}(k-1)!}}$$

Thus

$$\frac{1}{h_k} = e^{-w_k} \sqrt{\lambda^{k-1}(k-1)!} \cdot \exp \left( - \int_0^\infty e^{-\rho p} \sqrt{i} \lambda \left( -2 + 2e^{-\rho} - i^{3/2}\sqrt{p\pi \lambda} \text{erf} \left( \frac{-i^{3/2} \sqrt{\rho}}{\sqrt{\lambda}} \right) \right) \frac{1}{2 (-1 + e^{-\rho}) \sqrt{\pi} (p\lambda)^{3/2}} dp \right) \times \exp \left( - \frac{2i \sqrt{-i \lambda}}{\sqrt{\lambda}} \frac{1}{\sqrt{\Gamma(k)}} \right)$$

Finally we obtain

$$A_1 = \sum_{k=1}^{\infty} \frac{(-1)^k g_k}{h_k} = \mathcal{L} \sum_{i=1}^{\infty} (-1)^i \mathcal{L}^{-1} \left( \frac{g_i}{h_k} \right)$$

$$= \int_0^\infty \sum_{i=1}^{\infty} (-1)^i e^{-iq} \mathcal{L}^{-1} \left( \frac{g_i}{h_k} \right) dp = \int_0^\infty \frac{-e^{-q}}{1 + e^{-q}} \mathcal{L}^{-1} \left( \frac{g_k}{h_k} \right) dp$$

$$= \int_0^\infty \frac{-e^{-q}}{1 + e^{-q}} \int_{-i\infty}^{c+i\infty} \frac{g_k \exp \left( - \frac{2i \sqrt{-i \lambda}}{\sqrt{\lambda}} \frac{1}{\sqrt{\Gamma(k)}} \right)}{\lambda^{1+k}} dp dq$$

3.4.2 Proof of Theorem 13, (ii)

Here we assume that expansion of $A_1$ as $\lambda \to 0$ is invariant under a $\frac{\pi}{2}$ rotation; that is, there are no Stokes lines in the fourth quadrant; this would be ensured by Borel summability of the expansion in $\lambda$. 
Let $\lambda = ir$ with $r < 0$, and for simplicity let $g \equiv 1$, then (3.4.2) implies

$$A_1 = \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{(-1)^k}{\sqrt{1 + kr} - 1} = \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n} (\sqrt{1 + kr} + 1)}{(-1)^k k! r^k}$$

$$= \sum_{n=1}^{\infty} \frac{\exp(\sum_{k=1}^{n} \log(\sqrt{1 + kr} + 1))}{(-1)^k k! r^k}$$

(3.4.5)

The Euler-Maclaurin summation formula gives

$$\sum_{k=1}^{n} \log(\sqrt{1 + kr} + 1) \sim \int_{0}^{n} \log(\sqrt{1 + xr} + 1) dx + C$$

$$= -\frac{1}{r} + k \log(\sqrt{1 + kr} + 1) - \frac{1}{2} k + \frac{\sqrt{1 + kr}}{r} + C$$

(3.4.6)

where

$$C \sim \sum_{k=1}^{1/r} \log(\sqrt{1 + kr} + 1) - \int_{0}^{1/r} \log(\sqrt{1 + xr} + 1) dx \sim -\frac{\log(2)}{2}$$

Therefore

$$A_1 \sim \sum_{k=1}^{\infty} \frac{\exp\left(-\frac{1}{r} + k \log(\sqrt{1 + kr} + 1) - \frac{1}{2} k + \frac{\sqrt{1 + kr}}{r}\right)}{(-1)^k k! r^k}$$

(3.4.7)

Since

$$\exp\left(-\frac{1}{r} + k \log(\sqrt{1 + kr} + 1) - \frac{1}{2} k + \frac{\sqrt{1 + kr}}{r}\right)$$

$$= \exp\left(-\frac{3}{2r} + \frac{2}{3} \sqrt{r(k + \frac{1}{r})^{(3/2)}} - \frac{\log(2)}{2} - \frac{\log(\pi)}{2} + \frac{\log(-r)}{2}\right)$$

applying the Euler-Maclaurin summation formula again we get

$$A_1 \sim \frac{2^{1/3} 3^{1/6} \Gamma\left(\frac{2}{3}\right) e^{-\frac{3}{2}} (-i\lambda)^{1/6}}{2\sqrt{\pi}}$$

(3.4.8)
3.4.3 Numerical results

Figure 3.1 shows $\log(|R_0|)$ as a function of $\log(\lambda)$ for $\omega = 0$. $R_{0,0}$ is the residue of the pole of $\hat{\psi}(p, x)$ at $p = i$, see Corollary 27.

Figure 3.1: Log-log plot of $|R_0|$ as a function of $\lambda$ for $\omega = 0$. $R_{0,0}$ is the residue of the pole of $\hat{\psi}(p, x)$ at $p = i$, see Corollary 27.

$R_{0,0}$ is the residue of the pole of $\hat{\psi}(p, x)$ at $p = i$, see Corollary 27.

3.4.3 Numerical results

Figure 3.1 shows $\log(|R_0|)$ as a function of $\log(\lambda)$, very nearly a straight line with slope $1/6$ (corresponding to the $\lambda^{1/6}$ behavior), with good accuracy even for $\lambda$ as large as 1.
3.5 Ionization rate under a short pulse

We now consider a short pulse, with fixed total energy and fixed total number of oscillations. The corresponding Schrödinger equation is

$$i \frac{\partial \psi}{\partial t} = \left( -\frac{\partial^2}{\partial x^2} - 2\delta(x) + 2\lambda \delta(x) e^{-\lambda t} \cos(\omega t) \right) \psi$$  \hspace{1cm} (3.5.1)

where $\lambda$ is now a large real parameter (note the factor $\lambda$ in front of the exponential). We are interested in the ionization rate as $\lambda \to \infty$.

By similar arguments as in §3.2.5 we have the convergent representation

$$A_{m,n} = \sum_{i=0}^{\infty} (-1)^i \left( \frac{\lambda}{2} \right)^i \prod_{\tau \in 2^i} \prod_{j=0}^{i} B_{n+j, m+|\tau|} g_{n+i, m+|\tau|}$$  \hspace{1cm} (3.5.2)

Figs. 3.2, 3.3, and 3.4 give $|R_{0,0}|$ (see Corollary 27) in terms of $\lambda$ for different values of $\omega/\lambda$.

![Figure 3.2: $|R_{0,0}|$ as a function of $\lambda$, with fixed ratio $\omega/\lambda = 5$. $R_{0,0}$ is the residue of the pole of $\hat{\psi}(p, x)$ at $p = i$, see Corollary 27](image)
3.6 Results for $\lambda = 0$ and $\omega \neq 0$

We briefly go over the case $\lambda = 0$, where ionization is complete; the full analysis is done in [CRL]. In this case $\hat{\psi}$ does not have poles on the imaginary line; we give a summary of the argument in [CRL].

The homogeneous equation now reads

$$\sqrt{\sigma + m\omega} A_m = -\frac{1}{2} A_{m+1} - \frac{1}{2} A_{m-1} + A_m$$  \hspace{1cm} (3.6.1)

Thus we have

$$\sum\limits_N \sqrt{\sigma + m\omega} A_m \overline{A_m} = -\frac{1}{2} \sum\limits_N A_{m+1} \overline{A_m} - \frac{1}{2} \sum\limits_N A_{m-1} \overline{A_m} + \sum\limits_N A_m \overline{A_m}$$

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The first sum and the second sum on the right hand side are conjugate to each other, and each term in the third sum is real. So the right hand side is real, thus the left hand side is also real.

For \( \text{Im}(\sigma) \neq 0 \), \( \text{Im} \left( \sqrt{\sigma + m\omega} A_m \overline{A_m} \right) \) has same the sign as \( \text{Im} \sigma \). Therefore the sum can not be real and the equation has no nontrivial solution. When \( \text{Im}(\sigma) = 0 \), for \( m < 0 \), all \( \text{Im} \left( \sqrt{\sigma + m\omega} A_m \overline{A_m} \right) \) have the same sign and for \( m \geq 0 \), \( \sqrt{\sigma + m\omega} A_m \overline{A_m} \) is real. Since the final sum is purely real, this means \( A_m = 0 \) for \( m < 0 \). But then, recursively, all \( A_m \) should be 0.

Zero is thus the only solution to (3.6.1). By the Fredholm alternative the solution \( A \) is analytic in \( \sqrt{\sigma} \) and thus the associated \( y \) is analytic in \( \sqrt{\sigma} \). This entails complete ionization.
3.6.1 Small $\lambda$ behavior

We expect that the behavior of the system at $\lambda = 0$ is a limit of the one for small $\lambda$. However, this limit is very singular, as the density of the poles in the left half plane goes to infinity as $\lambda \to 0$, only to become finite for $\lambda = 0$. Nonetheless, given a $\lambda$, formula (3.2.35) allows us to calculate the residue of $\hat{\psi}$.

Figure 3.5 shows the behavior of the residue versus $\omega$, for $\lambda = 0.01$.

Figure 3.5: $|R_{0,0}|$, at $\lambda = 0.01$, as a function of $\omega$. $R_{0,0}$ is the residue of the pole of $\hat{\psi}(p, x)$ at $p = i$, see Corollary 27.
CHAPTER 4

ONE DELTA POTENTIAL HALF LINE PROBLEM

In this chapter we study the Schrödinger equation on the half line

\[ i\psi_t = -\psi_{xx} + (V + 2\Omega \sin \omega t) \psi \delta(x - 1), \quad \psi(x, t) = \psi_0(x) \]  \hspace{1cm} (4.0.1)

for \( x \geq 0, \ \Omega > 0, \ \psi_0 \in C_0^\infty(\mathbb{R}) \), and with constant \( V \) and the following Dirichlet boundary condition:

\[ \psi(0, t) = 0 \]

Note that the placing of the potential at \( x = 1 \) is conventional, as it can be rescaled.

We still mainly concentrate on the large \( t \) behavior of \( \psi \) and the survival probability \((3.0.3)\).

4.1 Settings and Main results

The main result in the time independent case \((\Omega = 0)\) is the following theorem.

**Theorem 30.** For large time \( t \), the solution to equation \((4.3.1)\) has the asymptotic expansion

\[ \psi(x, t) \sim \sum_{n=1}^{\infty} \alpha_n t^{-\frac{n}{2}} + \sum_{n=1}^{\infty} r_n e^{p_n t} \]  \hspace{1cm} (4.1.1)

where

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1. The \( p_n \)'s are the poles of \( \hat{\psi} \).

2. The \( r_n \)'s are residues of \( \hat{\psi} \) at \( p_n \), and \( r_n \sim -\frac{1}{2} e^{-\sqrt{-1} p_n (x+R)} \) as \( n \to \infty \).

3. The second sum is convergent.

where \( \hat{\psi} \) is the Laplace transform of \( \psi \) and \( \alpha_n \) are constants determined by \( \hat{\psi} \), and \( R \) is upper bound of the support of \( \psi_0 \). The proof, explicit formula for \( \alpha_n \), and other details are given in section 4.3.5.

The critical exponents \( p_n \) are given by the following theorem.

**Theorem 31.** Let \( p_n \) be the poles of \( \hat{\psi} \) indexed so that \( \text{Re} p_{n-1} > \text{Re} p_n \). Then:

1. The \( p_n \)'s are given in terms of the Lambert function \( W \) by

\[
p_n = i \left( \frac{W_n(Ve^V)}{2} - V \right)^2
\]

where \( W_n(\alpha) \) gives the \( n \)-th solution to \( \beta e^\beta = \alpha \).

2. There are infinitely many \( p_n \)'s, all in left half \( p \)-plane.

3. For large \( n \),

\[
p_n \sim \frac{i}{4} \left( \log \left( \frac{(n + \frac{1}{2})\pi}{V} \right) + i \left( n + \frac{1}{2} \right)\pi \right)^2, \text{ as } n \to \infty
\]

where \( n \) is odd when \( V > 0 \) and is even when \( V < 0 \).

The proof is given in section 4.3.3.

The main results in time-dependent case are the following theorems.
Theorem 32. Ionization happens if $\hat{\psi}$ has no poles on imaginary line. A sufficient condition is that $\sqrt{\omega + i\sigma}/\pi \notin \mathbb{N}$.

The proof is given in section 4.4.7.

The condition $(\sqrt{\omega + i\sigma})/\pi \in \mathbb{N}$ is also sufficient for wave function to localize, a proof for particular choices of parameters is given in section 4.4.3.

A connection between the time-dependent case and the time independent case is given below.

Theorem 33. As $\Omega \to 0$, all poles of $\hat{\psi}(x,p)$ of the time-dependent problem approach poles of $\hat{\psi}(x,p)$ for the time independent problem ($\Omega = 0$) in $\mathbb{C}/\{i\omega\}$.

The proof is given in section 4.4.5.

A partial decomposition to the wave function is provided in the following theorem.

Theorem 34. Take any vertical line $\{p_0 + iy | y \in \mathbb{R}\}$, $p_0 < 0$ along which $\hat{\psi}$ is analytic. Let the positions of the poles of $\hat{\psi}$ for $\text{Re}p > \text{Re}p_0$ in the strip $\{p|0 < \text{Im}p < \omega\}$ be denoted as $a_m$ with residues $r_m(x)$, $m = 1, 2, ..., M$. For $t > 1$ the wave function $\psi(x,t)$ has the expansion

$$
\psi(x,t) = \sum_{m=1}^{M} \sum_{n=-\infty}^{\infty} e^{(a_m + in\omega)t} r_{m,n}(x) + e^{p_0 t} \int_{-\infty}^{\infty} e^{iyt} \hat{\psi}_1(x,p_0 + iy) dy
+ \sum_{n=-\infty}^{\infty} \left( \int_{p_0 + i(n-1)\omega + i\epsilon}^{p_0 + i(n-1)\omega + i\epsilon} e^{pt} \hat{\psi}_1(x,p) dp + \int_{p_0 + i(n-1)\omega + i\epsilon}^{p_0 + i(n-1)\omega - i\epsilon} e^{pt} \hat{\psi}_1(x,p) dp \right) \quad (4.1.3)
$$
where all the sums are convergent, $a_m$ are solutions to

\[
\frac{1}{1-\alpha_1 + \frac{1}{\alpha_2 - \alpha_3 + \ldots}} = \alpha_0 + \frac{1}{1-\alpha_{-1} + \frac{1}{\alpha_{-2} + \frac{1}{\alpha_{-3} + \ldots}}} \tag{4.1.4}
\]

and $r_{m,n}(x)$ denotes the residue of $\hat{\psi}$ at $p = a_m + i n \omega$ which satisfies the homogeneous equation (with dependence in $x$ suppressed)

\[
r_{m,n} = A_{m,n} (V r_{m,n} - i \Omega r_{m,n-1} + i \Omega r_{m,n+1}) \tag{4.1.5}
\]

Here $\psi_1$ is related of $\hat{\psi}$ by a change of variable explained in section 4.2.3. The proof is given in section 4.4.6.

Below we give a relevant theorem on calculating one type of continued fractions.

**Theorem 35.** For a curve on complex plane that does not pass through the segment $[-2i, 2i]$, consider the points $c_1, c_2, \ldots c_N$ along the curve. There exists parameters $\epsilon_0$ and $\epsilon_\Delta$ such that if $|(c_n-c_{n+1})/c_n| < \epsilon_\Delta$ for $n = 1, 2, \ldots, N-1$ and $|(c_1-\sqrt{c_1^2+4})/2| < \epsilon$ for some $\epsilon < \epsilon_0$, then the continued fraction

\[
f_c := c_N + \frac{1}{c_{N-1} + \frac{1}{c_{N-2} + \ldots + \frac{1}{c_1}}} \tag{4.1.6}
\]

satisfies

\[
|f_c - \frac{c_N + \sqrt{c_N^2 + 4}}{2} < \frac{c_N + \sqrt{c_N^2 + 4}}{2} \epsilon \tag{4.1.7}
\]
Here the square root function is defined so that

\[ |z + \sqrt{z^2 + 4}| > |z - \sqrt{z^2 + 4}| \]

And the parameters \( \epsilon_0 \) and \( \epsilon_\Delta \) can be written in explicit forms.

The proof is given in section 4.4.8.

4.2 Main strategy, some preliminary work

First we show that the half line problem can be converted to a full line problem.

4.2.1 Equivalent full line problem

Consider the odd extension of the problem

\[ \psi_{\text{ext}}(x, t) = \begin{cases} 
\psi(x, t), & x \geq 0 \\
-\psi(-x, t), & x < 0 
\end{cases} \tag{4.2.1} \]

and

\[ \psi_{\text{ext},0}(x) = \begin{cases} 
\psi_0(x), & x \geq 0 \\
-\psi_0(-x), & x < 0 
\end{cases} \tag{4.2.2} \]

The solution to (4.0.1) with Dirichlet boundary condition on a half-line is a solution to

\[ i\psi_{\text{ext},t} = -\psi_{\text{ext},xx} + (V + 2\Omega \sin \omega t) \psi_{\text{ext}}(\delta_{x-1} + \delta_{x+1}), \quad \psi_{\text{ext}}(x, t) = \psi_{\text{ext},0}(x) \tag{4.2.3} \]

don \( \mathbb{R} \) in the space \( \{ y(x) : y \in L^2(\mathbb{R}), y(-x) = -y(x) \} \). Any skew-symmetric solution to (4.2.3) is a solution to (4.0.1).

Because of this equivalence, we choose to solve the full-line problem for the general time-dependent case for simplicity, where we use the notation \( \psi \) for the extension \( \psi_{\text{ext}} \).
4.2.2 Laplace Transform, equation in \( p \) space

Existence of a strongly continuous unitary propagator for (4.0.1) (see [RS] v.2, Theorem X.71) implies that for \( \psi_0 \in L^2(\mathbb{R}^d) \), the Laplace transform

\[
\hat{\psi}(\cdot, p) := \int_0^\infty \psi(\cdot, t)e^{-pt}dt
\]  

exists for \( \text{Re}(p) > 0 \) and the map \( p \to \psi(\cdot, p) \) is \( L^2 \) valued analytic in the right half plane

\[
p \in \mathbb{H} = \{ z : \text{Re}(z) > 0 \}
\]  

(4.2.5)

The Laplace transform of (4.2.3) is

\[
\hat{\psi}_{xx}(x, p) + ip\hat{\psi}(x, p) = i\psi_0(x)
\]  

\[
+ (\delta_{x-1} + \delta_{x+1}) \left( V\hat{\psi}(x, p) + i\Omega\hat{\psi}(x, p + i\omega) - i\Omega\hat{\psi}(x, p - i\omega) \right)
\]

(4.2.6)

Let \( k = \sqrt{-ip} \) where the square root is defined as

\[
\sqrt{e^{2\alpha}} = e^{\frac{\alpha}{2}}, \text{ where } -\frac{3\pi}{2} < \alpha < \frac{\pi}{2}
\]

The resolvent to operator \( \frac{d^2}{dx^2} + ip \) is

\[
g : f(x) \to -\frac{e^{-kx}}{2k} \int_{-\infty}^x e^{kx}f(x)dx - \frac{e^{kx}}{2k} \int_x^\infty e^{-kx}f(x)dx
\]

(4.2.7)

**Remark 36.** If \( f(x) \in C_0^\infty \), using integration by parts we have

\[
g(f) \sim \frac{c(x)}{p} + O\left(\frac{1}{p^{3/2}}\right), \ p \to \infty
\]

for some \( c(x) \in L^2 \).
Inverting (4.2.6) using (4.2.7) gives

\[ \hat{\psi}(x, p) = r_1(x, p) + r_2(x, p) \]  

(4.2.8)

where

\[ r_1(x, p) = -\frac{i e^{-kx}}{2k} \int_{-\infty}^{x} e^{kx} \psi_0(x) dx - \frac{i e^{kx}}{2k} \int_{x}^{\infty} e^{-kx} \psi_0(x) dx \]  

(4.2.9)

and

\[ r_2(x, p) = \begin{cases} \frac{e^{kx}}{2k} (e^k - e^{-k}) \left( V \hat{\psi}(1, p) - i\Omega \hat{\psi}(1, p - i\omega) + i\Omega \hat{\psi}(1, p + i\omega) \right), & \text{for } x < -1 \\ \left( \frac{e^{-kx}}{2k} - \frac{e^{kx}}{2k} \right) e^{-k} \left( V \hat{\psi}(1, p) - i\Omega \hat{\psi}(1, p - i\omega) + i\Omega \hat{\psi}(1, p + i\omega) \right), & \text{for } -1 < x < 1 \\ \frac{e^{-kx}}{2k} (e^{-k} - e^k) \left( V \hat{\psi}(1, p) - i\Omega \hat{\psi}(1, p - i\omega) + i\Omega \hat{\psi}(1, p + i\omega) \right), & \text{for } x > 1 \end{cases} \]  

(4.2.10)

where the oddness of \( \hat{\psi}(x, p) \) has been used.

**Remark 37.** By writing (4.2.9) in the form

\[ r_1(p) = i \frac{e^{-kx} - e^{kx}}{2k} \int_{x}^{\infty} e^{-kx} \psi_0(s) ds + i e^{-kx} \frac{e^{kx}}{2k} \psi_0(s) ds \]  

(4.2.11)

we see that \( r_1(p) \) is analytic in \( k \), or \( \sqrt{p} \).

Let

\[ A(p) := \frac{e^{-k}}{2k} (e^{-k} - e^k) \]  

(4.2.12)
Using (4.2.10), when $x = 1$, (4.2.8) gives
\[
\hat{\psi}(1, p) = r_1(1, p) + A(p)V\hat{\psi}(1, p) - A(p)i\Omega\hat{\psi}(1, p - i\omega) + A(p)i\Omega\hat{\psi}(1, p + i\omega)
\]
(4.2.13)

Since (4.2.13) can be rewritten in the form
\[
V\hat{\psi}(1, p) - i\Omega\hat{\psi}(1, p - i\omega) + i\Omega\hat{\psi}(1, p + i\omega) = \left(\frac{\hat{\psi}(1, p) - r_1(1, p)}{A(p)}\right)
\]
equation (4.2.8) gives
\[
\hat{\psi}(x, p) = \left(\hat{\psi}(1, p) - r_1(1, p)\right) \cdot \begin{cases} 
- e^{k(x-1)}, & \text{for } x < -1 \\
\frac{e^{-kx} - e^{kx}}{e^{-k} - e^{k}}, & \text{for } -1 < x < 1 \\
e^{-k(x+1)}, & \text{for } x > 1
\end{cases}
\]
(4.2.14)

Remark 38. From (4.2.13) and (4.2.14) it is clear that solving the inhomogeneous equation (4.2.6) for $\hat{\psi}(x, p)$ is equivalent to solving the inhomogeneous equation (4.2.13) for $\hat{\psi}(1, p)$. By taking $r_1(x, p) = 0$ it is also clear that solving the associated homogeneous equation for $\hat{\psi}(x, p)$ is equivalent to solving the associated homogeneous equation for $\hat{\psi}(1, p)$. Furthermore, the solutions to the inhomogeneous equations, $\hat{\psi}(x, p)$ and $\hat{\psi}(1, p)$ share the same analyticity properties.

Remark 39. The recursive nature of equation (4.2.6) and the fact that $r_1(x, p)$ has only one branch point at $p = 0$ implies that $\hat{\psi}(x, p)$ has branch points for $p = in\omega$, $n \in \mathbb{Z}$, and by Remark 38 so does $\hat{\psi}(1, p)$.
4.2.3 Vector notation, space, and continuation of solution

In this section, all functions that are stated as analytic or meromorphic in \( p \) are understood in the sense that they are analytic or meromorphic in \( p \) except for possible branch singularities at \( p = in\omega, \ n \in \mathbb{Z} \) as pointed out in Remark 39.

Let \( D = \{ \sigma \in \mathbb{C} : 0 < \text{Im}\sigma < \omega \} \) and \( p = \sigma + in\omega \) where \( \sigma \in D \). Equation (4.2.6) relates \( \hat{\psi}(x,p) \) only with \( \hat{\psi}(x,p-i\omega) \) and \( \hat{\psi}(x,p+i\omega) \), thus we can treat \( \sigma \) and \( x \) as parameters and use \( n \) as the variable, with associated vector notations

\[
\hat{\psi} := \hat{\psi}(x,p) = \hat{\psi}(x,\sigma,n) = \hat{\psi}(\sigma,n)
\]

and

\[
r_1 := r_1(x,p) = r_1(x,\sigma,n) = r_1(\sigma,n)
\]

Equation (4.2.6) becomes

\[
\hat{\psi}(\sigma,n) = i\mathbb{C}\hat{\psi}_0 + \mathbb{C}(\hat{\psi}(\sigma,n)) \tag{4.2.15}
\]

where \( \mathbb{C} = \mathbb{C}(x,\sigma) \) is a operator defined as

\[
\mathbb{C}(\hat{\psi}(\sigma,n)) = g \left( (\delta_{x-1} + \delta_{x+1}) \cdot \left( V\hat{\psi}(\sigma,n) + i\Omega\hat{\psi}(\sigma,n+1) - i\Omega\hat{\psi}(\sigma,n-1) \right) \right) \tag{4.2.16}
\]

Remark 40. Because of the factor \( \kappa_{m,n}^{-1} \) in (4.2.7) we have

\[
\mathbb{C}\phi(p) \sim \frac{c(x)}{\sqrt{p}} \phi(p)
\]

as \( p \to \infty \), for any function \( \phi(p) \).
As stated in Remark 36, we have

\[ i \hbar \psi_0 = \frac{c_1(x)}{p} + O \left( \frac{1}{p^{3/2}} \right) \]  

(4.2.17)

for some \( c_1(x) \in L^2 \).

Let

\[ h_1(p) = h_1(x, p) = c_1(x) \mathcal{L} \left( 1_{[0,1]}(t) \right) \]  

(4.2.18)

For large \( p \) we have

\[ h_1(p) = \frac{c_1(x)}{p} + O \left( \frac{1}{p^{3/2}} \right) \]  

(4.2.19)

**Remark 41.** As a function of \( x \), \( h_1(p) \) is clearly in \( L^2 \) and

\[ \mathcal{L}^{-1}(h_1(p)) = c_1(x) 1_{[0,1]}(t) \]

thus for \( t > 1 \) we have

\[ \mathcal{L}^{-1}(h_1(p)) = 0 \]

Substituting in (4.2.15)

\[ \hat{\psi} = \hat{\psi}_1 + h_1 \]  

(4.2.20)

we have

\[ \hat{\psi}_1 = i \hbar \psi_0 - h_1 + \mathcal{C}(h_1) + \mathcal{C} \hat{\psi}_1 \]  

(4.2.21)
Let \( r_0 = i\hat{g}\psi_0 - h_1 + \mathcal{C}(h_1) \), we have

\[
\hat{\psi}_1 = r_0 + \mathcal{C}\hat{\psi}_1 \tag{4.2.22}
\]

Remark 40 implies that for large \( p \) we have

\[
r_0 = O\left(\frac{1}{p^{3/2}}\right) \tag{4.2.23}
\]

and by construction \( r_0 \in L^2 \) as a function of \( x \).

We analyze (4.2.22) in the space \( \mathcal{H} = \mathcal{H}(x,\sigma) = L^1(Z,\|\cdot\|_1) \) with norm

\[
\|y(\sigma,n)\|_1 = \sum_n |y_n|.
\]

**Lemma 42.** \( \mathcal{C} \) is a compact operator on \( \mathcal{H} \), and it is analytic in \( p \).

**Proof.** Compactness is clear since \( \mathcal{C} \) is a limit of bounded finite rank operators. Analyticity is manifest in the expression of \( \mathcal{C} \) (see (4.2.7) and (4.2.16)). \( \square \)

**Proposition 43.** Equation (4.2.22) has a unique solution if and only if the associated homogeneous equation

\[
\hat{\psi}_1 = \mathcal{C}\hat{\psi}_1 \tag{4.2.24}
\]

has no nontrivial solution in \( \mathcal{H} \). In the latter case, the solution is analytic, otherwise it is meromorphic.

**Proof.** This follows from Lemma 42 and the Fredholm alternative. \( \square \)

Let \( y_n(\sigma) := \hat{\psi}(1,\sigma + in\omega) \), \( r_n := r_1(1,\sigma + in\omega) \), and \( A_n := A(\sigma + in\omega) \). We study

\[
y_n = r_n + A_n(Vy_n - i\Omega y_{n-1} + i\Omega y_{n+1}) \tag{4.2.25}
\]

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with associated homogeneous equation

\[ y_n = A_n (Vy_n - i\Omega y_{n-1} + i\Omega y_{n+1}) \]  

(4.2.26)

The choice of space is suggested by the change of variables (4.2.20) and (4.2.14).

The Laplace integral representation (4.2.4) of \( \hat{\psi} \) is analytic, valid only in the right half \( p \) plane, and satisfies equation (4.2.8). Proposition 13 shows that the vector equation (4.2.22) has a unique meromorphic solution \( \hat{\psi}_1(n; x, \sigma) \) for all \( \sigma \in D \). The corresponding \( \hat{\psi}(x, p) \) is meromorphic on the whole \( p \) plane and it also satisfies equation (4.2.8), thus it is the analytic continuation of \( \hat{\psi} \) defined in (4.2.4).

4.2.4 A partial decomposition of wave function for large \( t \)

The wave function \( \psi(x, t) \) can be written as the inverse Laplace transform of \( \hat{\psi}(x, p) \):

\[ \psi(x, t) = \mathcal{L}^{-1}(\hat{\psi})(x, p) = c_1(x)1_{[0,1]}(t) + \mathcal{L}^{-1}(\hat{\psi}_1(x, p)) \]  

(4.2.27)

where we used (4.2.20).

For \( t > 1 \) the first term is absent and since \( \hat{\psi}_1 \in \mathcal{H} \), the inverse Laplace transform in the second term can be expressed explicitly using the Bromwich contour formula:

\[ \mathcal{L}^{-1}(\hat{\psi}_1(x, t)) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \hat{\psi}_1(x, p) dp \]

The choice of space \( \mathcal{H} \) allows us to push the contour of integration in the inverse Laplace transform into the left half \( p \)-plane. In this process the residues of the poles are collected and the contour is bent around branch singularities.

The branch singularities of \( \hat{\psi}_1 \) are at \( p = in\omega, n \in \mathbb{Z} \) as pointed out in Remark 39. In the process of pushing contour of integration, the contour has to be deformed...
in such a way that within each strip \( \{(n-1)\omega < \text{Im}p < n\omega\} \) it is vertical, and the vertical segments in different strips are connected by two horizontal segments, one just below the branch cut and one just above the branch cut. Assume \( p_0 \) is the real part of the vertical part of the contour. An integral of a function \( f(p) \) along such a contour \( C \) is
\[
\int_C f(p)dp = \sum_{n=-\infty}^{\infty} \left( \int_{i(n-1)\omega + i\epsilon}^{p_0 + i(n-1)\omega + i\epsilon} f(p)dp \right) \tag{4.2.28}
\]
\[
+ \int_{p_0 + in\omega - i\epsilon}^{p_0 + in\omega} f(p)dp + \int_{p_0 + in\omega}^{p_0 + in\omega + i\epsilon} f(p)dp
\]
where \( i\epsilon \) is understood as a small perturbation to ensure the contour to be just away from the cuts.

Furthermore, the segments in the contour can be re-grouped into one vertical line from \( p_0 - i\infty \) to \( p_0 + i\infty \) and a collection of infinite many horizontal segments, and this gives
\[
e^{p_0 t} \int_{-\infty}^{\infty} e^{iyt} \hat{\psi}_1(p_0 + iy)dy
\]
and
\[
\sum_{n=-\infty}^{\infty} \left( \int_{i(n-1)\omega + i\epsilon}^{p_0 + i(n-1)\omega + i\epsilon} e^{pt} \hat{\psi}_1(p)dp + \int_{p_0 + in\omega - i\epsilon}^{p_0 + in\omega} e^{pt} \hat{\psi}_1(p)dp \right)
\]
Thus, assuming that the poles of \( \hat{\psi}_1 \) lying to the right of the contour are positioned at \( p = a_n \) with residue \( r_n \), we have
\[
\psi(x,t) = \sum_{n=-\infty}^{\infty} e^{a_n t} r_n + e^{p_0 t} \int_{-\infty}^{\infty} e^{iyt} \hat{\psi}_1(p_0 + iy)dy
\]
\[
+ \sum_{n=-\infty}^{\infty} \left( \int_{i(n-1)\omega + i\epsilon}^{p_0 + i(n-1)\omega + i\epsilon} e^{pt} \hat{\psi}_1(p)dp + \int_{p_0 + in\omega - i\epsilon}^{p_0 + in\omega} e^{pt} \hat{\psi}_1(p)dp \right) \tag{4.2.29}
\]
Finally, all sums appearing in (4.2.29) are convergent because $\hat{\psi}_1 \in \mathcal{H}$.

**Remark 44.** Since $h_1$ has no poles, $a_n$ and $r_n$ can be equivalently defined as the positions and residues of poles of $\hat{\psi}$.

### 4.3 Study in the time independent case

#### 4.3.1 Explicit solution in $p$ plane, equation of position of singularities

Consider the limiting case $\Omega = 0$:

$$i\psi_t = -\psi_{xx} + V\psi\delta_{x-1}, \, \psi(x,t) = \psi_0(x) \quad (4.3.1)$$

The Laplace transformed equation is

$$\hat{\psi}_{xx}(x,p) + ip\hat{\psi}(x,p) = i\psi_0(x) + (\delta_{x-1} + \delta_{x+1}) \left( V\hat{\psi}(x,p) \right) \quad (4.3.2)$$

Solve first the limiting case of half-line problem

$$\hat{\psi}_{xx}(x,p) + (ip - \delta_{x-1}V) \hat{\psi}(x,p) = i\psi_0(x) \quad (4.3.3)$$

with boundary condition $\hat{\psi}(0,p) = 0$.

The associated homogeneous equation to (4.3.3) is

$$\hat{\psi}_{xx}(x,p) + (ip - \delta_{x-1}V) \hat{\psi}(x,p) = 0 \quad (4.3.4)$$

Let $\hat{\psi}_1(x,p)$ and $\hat{\psi}_2(x,p) = 0$ be solutions to (4.3.4) with boundary condition $\hat{\psi}_1(0,p) = 0$.
and

\[ \hat{\psi}_2(x, p) \sim e^{-\sqrt{-ip} x} \text{ as } x \to +\infty \]

respectively.

We have

\[
\hat{\psi}_1(x, p) = \begin{cases} 
- \frac{2\sqrt{-ip}}{2\sqrt{-ip} - V + e^{2\sqrt{-ip}} V} e^{\sqrt{-ip} x} \\
+ \frac{2\sqrt{-ip}}{2\sqrt{-ip} - V + e^{2\sqrt{-ip}} V} e^{-\sqrt{-ip} x},
\end{cases}
\text{for } 0 \leq x < 1
\]

(4.3.5)

and

\[
\hat{\psi}_2(x, p) = \begin{cases} 
- \frac{e^{-2\sqrt{-ip}} V}{2\sqrt{-ip}} e^{\sqrt{-ip} x} - \frac{2\sqrt{-ip} - V}{2\sqrt{-ip}} e^{-\sqrt{-ip} x},
\end{cases}
\text{for } 0 \leq x < 1
\]

(4.3.6)

\[ e^{-\sqrt{-ip} x}, \text{ for } 1 < x \]

The solution to (4.3.3) can thus be written as

\[
\hat{\psi}(x, p) = -\frac{\hat{\psi}_1(x, p)}{W(p)} \int_x^\infty \hat{\psi}_2(s, p) (i\psi_0(s)) \, ds
- \frac{\hat{\psi}_2(x, p)}{W(p)} \int_0^x \hat{\psi}_1(s, p) (i\psi_0(s)) \, ds
\]

(4.3.7)

where

\[
W(p) = -\frac{2\sqrt{-ip} (2\sqrt{-ip} + V - e^{2\sqrt{-ip}} V)}{2\sqrt{-ip} + (-1 + e^{2\sqrt{-ip}}) V}
\]

(4.3.8)
is the Wronskian between \( \hat{\psi}_1 \) and \( \hat{\psi}_2 \).

The solution (4.3.7) satisfies the boundary condition \( \psi(0, p) = 0 \) and \( \psi(x, p) \sim e^{-\sqrt{-ip}x} \) as \( x \to \infty \), thus it is the unique solution when \( p \) is in right half \( p \)-plane \( \{ p : \text{Re}p > 0 \} \).

For any compactly supported initial condition \( \psi_0 \), (4.3.7) remains well-defined and analytic when \( p \) extends into left half \( p \)-plane, \( \{ p : \text{Re}p < 0 \} \). Thus it is the solution to (4.3.3) on the whole \( p \) plane and its odd extension provides the solution to (4.2.6).

The function \( \hat{\psi} \) has singularities at where the Wronskian (4.3.8) is zero

\[
2 \sqrt{-ip} + V - e^{-2\sqrt{-ip}V} = 0 \tag{4.3.9}
\]

4.3.2 Connection with Gamow vectors

Consider the eigenvalue problem

\[-\phi_{xx}(x) + V(x)\phi(x) = E\phi(x) \tag{4.3.10}\]

with

\[
V(x) = \begin{cases} 
-A\delta_{x-1} & x \geq 0 \\
\infty & x < 0 
\end{cases} \tag{4.3.11}
\]

and \( A > 0 \).

In this section we look for Gamow vector solutions, i.e. solutions that grow or decay exponentially as \( x \to \infty \). Furthermore, for \( E < 0 \) we require that the solutions are bound states.

Let \( -E = k^2 \) where \( \text{arg} k \in (-\frac{3\pi}{4}, \frac{\pi}{4}) \) for \( E \) in \( (-\frac{3\pi}{2}, \frac{\pi}{2}) \).
When \( x \neq 1 \), (4.3.10) becomes

\[-\phi_{xx}(x) = -k^2 \phi(x)\]

which has solutions \( e^{kx} \) and \( e^{-kx} \).

When \( E < 0, k > 0 \). Since the solution to (4.3.10) has to be in \( L^2(0, \infty) \), it has to be of the form

\[
\phi = \begin{cases} 
Be^{kx} + Ce^{-kx} & 0 \leq x \leq 1 \\
e^{-kx} & x > 1 
\end{cases} \quad (4.3.12)
\]

where \( \phi \) satisfies

1. \( \phi \) is continuous at \( x = 1 \).

2. \( \phi'(1+) - \phi'(1-) = -A\phi(1) \).

3. \( \phi(0) = 0 \).

Using (1) and the fact that (3) requires \( B = -C \), we have

\[
B = -C = \frac{e^k}{e^k - e^{-k}} \quad (4.3.13)
\]

Thus (2) implies

\[
\frac{e^k + e^{-k}}{e^k - e^{-k}}e^{-k} = -e^{-k} + \frac{A}{k}e^{-k} \quad (4.3.14)
\]

which simplifies to

\[
2k + (-A) - (A)e^{-2k} = 0 \quad (4.3.15)
\]
For any solution to (4.3.15), (4.3.12) gives a solution to (4.3.10), thus

\[ \psi(x, t) = e^{ik^2 t} \phi(x) \]  

(4.3.16)

When \( E > 0 \), because (4.3.16) is exponentially growing in \( x \), it is a Gamow vector solution.

Every Gamow vector solution corresponds to some \( k \) that satisfies (4.3.15). Comparison between (4.3.15) and (4.3.9) shows that \( \hat{\psi} \) has a pole at \( p = i k^2 = -i E \). Thus any energy obtained from solving the Gamow-type problem gives a pole for \( p = -i E \).

### 4.3.3 Proof of Theorem 31

The poles of \( \hat{\psi} \) are given by (4.3.9),

\[ 2 \sqrt{-ip} + V - e^{-2\sqrt{-ip}} V = 0 \]

which can be written as

\[ 2 \left( \sqrt{-ip} + V \right) e^{2\sqrt{-ip} + V} = V e^V \]

(4.3.17)

Inverting it using the Lambert function \( W \) gives

\[ p = i \left( \frac{W(Ve^V)}{2} - V \right)^2 \]

(4.3.18)

This proves (1).

The fact that there are infinitely many \( p_n \)'s and \( |p_n| \to \infty \) follows from the property of the Lambert function. A direct proof is given below.
Let \(-2\sqrt{-ip} = a + ib\) where \(a\) and \(b\) are the real and imaginary part respectively.

Equation (4.3.9) becomes

\[(a - V) + bi = -Ve^{a+bi}\] (4.3.19)

Comparing the modulus of the two sides of (4.3.19) gives

\[
\sqrt{(a - V)^2 + b^2} = |V|e^a
\] (4.3.20)

Thus when \(|a|\) is large

\[
|b| \sim |V|e^a \quad \text{or} \quad a \sim \log\left(\frac{|b|}{|V|}\right)
\] (4.3.21)

It is clear from (4.3.21) that when \(|a|\) is large \(a > 0\).

Comparison between the real part of the two sides of (4.3.19) gives

\[a - V = -Ve^a \cos b\] (4.3.22)

Thus when \(|a|\) is large, \(\cos b \sim 0\). This gives

\[b \sim (k + \frac{1}{2})\pi \quad \text{for some} \quad k \in \mathbb{Z}\] (4.3.23)

Furthermore, comparison between the imaginary part of the two sides of (4.3.19) gives

\[-V \frac{|b|}{|V|} \sin b \sim b\] (4.3.24)

Since when \(b \sim (k + \frac{1}{2})\pi, b \sim \pm 1\), (4.3.24) implies that \(k\) is odd when \(V > 0\) and \(k\) is even when \(V < 0\).

Thus (4.3.21) gives

\[a \sim \log\left(\frac{(k + \frac{1}{2})\pi}{|V|}\right)\] (4.3.25)
Recalling that \(-2\sqrt{-i\bar{p}} = a + i b\), we have

\[
p \sim \frac{i}{4} \left( \log \left( \frac{(k + \frac{1}{2})\pi}{V} \right) + i \left( k + \frac{1}{2} \right) \pi \right)^2
\]  

(4.3.26)

Next we show that equation (4.3.9) has a unique solution near the points given in (4.3.26). The following calculation is under the assumption \(V < 0\). The calculation in the case \(V > 0\) is similar.

Let

\[
\sqrt{-i\bar{p}} = \frac{-1}{2} \left( \log \left( \frac{(k + \frac{1}{2})\pi}{-V} \right) + i \left( k + \frac{1}{2} \right) \pi \right) (1 + \epsilon)
\]  

(4.3.27)

Using (4.3.2) we have

\[
V - \left( -\frac{i(\pi + 2k\pi)}{2V} \right)^{1+\epsilon} V - (1+\epsilon) \left( \frac{1}{2} i(\pi + 2k\pi) + \log \left[ -\frac{\pi + 2k\pi}{2V} \right] \right) = 0
\]  

(4.3.28)

Solving for \(\epsilon\) gives

\[
\epsilon = G(\epsilon)
\]

where

\[
G(\epsilon) = -1 + \frac{\log \left( \frac{V - (1+\epsilon) \left( \frac{1}{2} i(\pi + 2k\pi) + \log \left( -\frac{\pi + 2k\pi}{2V} \right) \right)}{V} \right)}{\log \left( -\frac{i(\pi + 2k\pi)}{2V} \right)}
\]  

(4.3.29)

Since for \(k\) large we have

\[
\log \left( \frac{V - (1 + \epsilon) \left( \frac{1}{2} i(\pi + 2k\pi) + \log \left( -\frac{\pi + 2k\pi}{2V} \right) \right)}{V} \right) \sim \log \left( \frac{(1 + \epsilon)^{\frac{1}{2}} i(\pi + 2k\pi)}{V} \right)
\]  

(4.3.30)
to the leading order we obtain
\[ G(\epsilon) \sim \frac{\log(1 + \epsilon)}{\log \left( -\frac{i(\pi + 2k\pi)}{2V} \right)} \] (4.3.31)

Using this asymptotic property of \( G \), it is easy to see that the map \( G \) is contractive in the disk
\[ |\epsilon| < \frac{2}{\left| \log \left( -\frac{i(\pi + 2k\pi)}{2V} \right) \right|} \]
thus by contractive mapping theorem equation \( \epsilon = G(\epsilon) \) has a unique solution in this disk, and this proves that (4.3.9) has a solution centered at every point of the form (4.3.26).

This proves (2) and (3).

4.3.4 Order and residue of poles

For the order of poles of \( \hat{\psi} \) we have the following result.

**Lemma 45.** The poles of solution \( \hat{\psi} \) to equation (4.3.3) are all simple.

**Proof.** We write (4.3.7) as
\[ \hat{\psi} = -\frac{\hat{\psi}_1(x, p)}{W(p)} \int_x^\infty \hat{\psi}_2(s, p) (i\psi_0(s)) ds - \hat{\psi}_2(x, p) \int_0^x \frac{\hat{\psi}_1(s, p)}{W(p)} (i\psi_0(s)) ds \] (4.3.32)
where \( \hat{\psi}_1(x, p)/W(p) \) has the form
\[
\hat{\psi}_1(x, p) = \begin{cases} 
- e^{-\sqrt{-ip}x} + e^{\sqrt{-ip}x} & \text{for } 0 \leq x < 1 \\
- e^{-2\sqrt{-ip}V} + V + 2\sqrt{-ip} & \\
\frac{1}{2\sqrt{-ip}} e^{\sqrt{-ip}} - \frac{2\sqrt{-ip} + (1 + e^{2\sqrt{-ip}}) V}{2\sqrt{-ip} (2\sqrt{-ip} + V - e^{-2\sqrt{-ip}} V)} e^{-\sqrt{-ip}}, & \text{for } 1 < x
\end{cases}
\] (4.3.33)
Using (4.3.33), (4.3.6), and the fact that $\psi_0$ is compactly supported, it is evident that the expression on the right side of equation (4.3.32) is analytic except for:

1. a possible branch singularity at $p = 0$.

2. poles at those $p$ satisfying

$$-e^{-2\sqrt{-ip}}V + V + 2\sqrt{-ip} = 0$$

Since

$$\frac{d}{dp} \left(-e^{-2\sqrt{-ip}}V + V + 2\sqrt{-ip}\right) = -i \frac{(1 + V e^{-2\sqrt{-ip}})}{\sqrt{-ip}}$$

(4.3.34)

if both

$$-e^{-2\sqrt{-ip}}V + V + 2\sqrt{-ip} = 0$$

and

$$\frac{d}{dp} \left(-e^{-2\sqrt{-ip}}V + V + 2\sqrt{-ip}\right) = 0$$

are satisfied, we must have $Ve^V = -e^{-1}$, which implies $p = 0$.

Thus for those $p$ satisfying $-e^{-2\sqrt{-ip}}V + V + 2\sqrt{-ip} = 0$, we must have

$$\frac{d}{dp} \left(-e^{-2\sqrt{-ip}}V + V + 2\sqrt{-ip}\right) \neq 0$$

which means poles for $\hat{\psi}$ are simple.

For the residues of poles we have the following lemma.

---

1This is just (4.3.9).
Lemma 46. For large $p$ satisfying \(-e^{-2\sqrt{-ip}}V + V + 2\sqrt{-ip} = 0\), the residue of this simple pole is of the order \(e^{-\sqrt{-ip}(x+R)}\) where $R$ is the upper bound of the support of $\psi_0$.

Proof. In the following, the sign “∼” below means “asymptotic to” when $|p| \to \infty$ in the left half $p$ plane. For large $p$ satisfying \(-e^{-2\sqrt{-ip}}V + V + 2\sqrt{-ip} = 0\), from the fact $2\sqrt{-ip} \sim Ve^{\frac{-2}{\sqrt{-ip}}}V$ we see that (4.3.33) implies

\[
\hat{\psi}_1(x,p) \sim \begin{cases} 
-e^{-\sqrt{-ip}x}, & \text{for } 0 \leq x < 1 \\
-e^{-2\sqrt{-ip}V + V + 2\sqrt{-ip}}, & \text{for } x > 1
\end{cases}
\]

Thus (4.3.35) implies

\[
\hat{\psi}_2(x,p) \sim \begin{cases} 
e^{-\sqrt{-ip}x}, & \text{for } 0 \leq x < 1 \\
e^{-\sqrt{-ip}x}, & \text{for } x > 1
\end{cases}
\]

Thus (4.3.7) gives

\[
\hat{\psi}(x,p) \sim i \left( \frac{e^{-\sqrt{-ip}x}}{-e^{-2\sqrt{-ip}V + V + 2\sqrt{-ip}}} \int_0^\infty e^{-\sqrt{-ip}p} \psi_0(s) ds + e^{\sqrt{-ip}x} \int_0^1 e^{-\sqrt{-ip}p} \psi_0(s) ds \right)
\]

\[
\sim i \left( \frac{e^{-\sqrt{-ip}x}}{-e^{-2\sqrt{-ip}V + V + 2\sqrt{-ip}}} \int_0^\infty e^{-\sqrt{-ip}p} \psi_0(s) ds \right)
\]

Let $\psi_0$ be supported on $[0, R]$. Then,

\[
\int_0^\infty e^{-\sqrt{-ip}p} \psi_0(s) ds = \int_0^R e^{-\sqrt{-ip}p} \psi_0(s) ds = e^{-\sqrt{-ip}R} \int_0^R e^{\sqrt{-ip}p} \psi_0(R-s) ds \sim e^{-\sqrt{-ip}R} o(1)
\]

where Watson’s lemma was used in the last step.
Furthermore, since \(-e^{-2\sqrt{-ip}}V + V + 2\sqrt{-ip} = 0\) implies \(e^{-2\sqrt{-ip}}V = V + 2\sqrt{-ip}\), for large \(p\) we have
\[
\frac{d}{dp} \left( -e^{-2\sqrt{-ip}}V + V + 2\sqrt{-ip} \right) = -\frac{i}{\sqrt{-ip}} \left( 1 + V e^{-2\sqrt{-ip}} \right)
\]
\[
\sim -\frac{i}{\sqrt{-ip}} \left( 1 + V + 2\sqrt{-ip} \right) \sim -2i
\]

Thus by (4.3.38), (4.3.39), and (4.3.37) we have
\[
\text{Res}(\hat{\psi}(p)) \sim -\frac{1}{2} e^{-\sqrt{-ip}(x+R)}
\]
(4.3.40)

\[\square\]

### 4.3.5 Proof of Theorem 30

As explained in section 4.2.4, the wave function can be written in terms of contour integrals plus an explicit sum involving the poles of \(\hat{\psi}\), i.e. equation (4.2.29).

When specialized to the time independent problem, \(\hat{\psi}\) has only one branch singularity at \(p = 0\). Thus the last term in (4.2.29)
\[
\sum_{n=-\infty}^{\infty} \left( \int_{i(n-1)\omega+i\epsilon}^{p_0+i(n-1)\omega+i\epsilon} e^{pt} \hat{\psi}_1(p)dp + \int_{p_0+i\omega-i\epsilon}^{im\omega-i\epsilon} e^{pt} \hat{\psi}_1(p)dp \right)
\]

reduces to a sum of just two single integrals. Furthermore, (4.3.37) implies that
\[
|\hat{\psi}| \leq C e^{-\sqrt{-ip}}
\]
for some constant \(C\). Thus the contour can be pushed towards \(p_0 = -\infty\). In this process the second term
\[
e^{p_0 t} \int_{-\infty}^{\infty} e^{igt} \hat{\psi}_1(p_0 + iy)dy
\]

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vanishes and the last term becomes

\[- \int_{-\infty}^{-i\epsilon} e^{pt} \hat{\psi}_1(x, p) dp + \int_{0}^{-\infty+i\epsilon} e^{pt} \hat{\psi}_1(x, p) dp \]  

(4.3.41)

Since the two integrals in (4.3.41) are along the negative \(p\) direction, they are in fact usual Laplace integrals. Watson’s lemma shows that as \(t \to \infty\) we have

\[- \int_{0}^{-\infty+i\epsilon} e^{pt} \hat{\psi}_1(x, p) dp + \int_{0}^{-\infty+i\epsilon} e^{pt} \hat{\psi}_1(x, p) dp \sim \sum_{\alpha} \alpha t^{-\frac{3}{2}} \]  

(4.3.42)

The first term in (4.2.29)

\[\sum_{n=-\infty}^{\infty} e^{pt} r_n \]  

(4.3.43)

now contains all the poles of \(\hat{\psi}\) (see Remark [44]).

Lemma [31] and Lemma [46] show that

\[\text{Re}(p_n) \sim -\frac{1}{4}(1 + 2k)\pi \log \left( \frac{(\frac{1}{2} + k)\pi}{V} \right)\]

and

\[r_n = \text{Res}(\hat{\psi}(p_n)) \sim -\frac{1}{2} e^{-\sqrt{-1}p_n(x+R)} \sim \frac{(k + \frac{1}{2})\pi}{V}\]  

(4.3.44)

Thus the sum (4.3.43) converges.

**Remark 47.** By Theorem [30] the different ionization rates for different initial conditions come from the corresponding residues at \(\text{Imp}_n\). Physically, the different ionization rates originate from the resonance phenomenon of meta-stable states. Since in a normalized Schrödinger equation, \(\text{Imp}\) has physical meaning of energy, \(\text{Imp}_n\) corresponds to the energy levels of meta-stable states. In particular we see that the energy levels of the meta-stable states scale like \(n^2\).
The energy eigenvalues $p_n$ lie in the left lower complex plane (as a comparison, the physical wave functions have purely imaginary energy eigenvalues). These correspond to Gamow vector solutions (see section 4.3.2). Such correspondence is not new for other potentials, as an example, for square potentials see [MG].

Furthermore, by Theorem 30 we see that a meta-stable state which has energy level scaled as $n^2$ ionizes at a rate of order $e^{-n}$.

### 4.4 Study in the time-dependent case

#### 4.4.1 Necessary condition for $\hat{\psi}$ to have poles on imaginary line

Equation (4.2.26) is equivalent to:

$$y_n = A_nVy_n - A_ni\Omega y_{n-1} + A_ni\Omega y_{n+1}$$

(4.4.1)

If $A_n = 0$ then $y_n = 0$.

If $A_n \neq 0$ let

$$\alpha_n = \left(\frac{1}{i\Omega A_n} - \frac{V}{i\Omega}\right)$$

(4.4.2)

We have

$$\alpha_n y_n = y_{n+1} - y_{n-1}$$

(4.4.3)

Since

$$\sum_n \bar{y}_n y_{n+1} - \sum_n y_{n-1}\bar{y}_n = \sum_n \bar{y}_{n-1} y_n - \sum_n y_{n-1}\bar{y}_n \in i\mathbb{R}$$
if we multiply (4.4.3) with $\bar{\gamma}_n$ and sum it over all $n$, the right side of the result is purely imaginary:

$$\sum_n \alpha_n y_n \bar{\gamma}_n \in i\mathbb{R} \quad (4.4.4)$$

Using the definition (4.4.2) of $\alpha_n$, this implies

$$\sum_n \left(V - \frac{1}{A_n}\right) |y_n|^2 \in \mathbb{R}$$

Since $V \in \mathbb{R}$ we have

$$\text{Im} \left( \sum_n \frac{1}{A_n} |y_n|^2 \right) = 0 \quad (4.4.5)$$

Remark 48. Equation (4.4.5) always holds if the terms with $A_n = 0$ are omitted from the sum (note that $A_n = 0$ gives $y_n = 0$).

Let

$$2\sqrt{-i\sigma + n\omega} = a + bi$$

with $a$ and $b$ real.

Lemma 49. If $a \geq 0$ and $b \leq 0$ then

$$\text{Im} \left( \frac{1}{A_n} \right) \geq 0$$

If $a \geq 0$ and $b < 0$ then

$$\text{Im} \left( \frac{1}{A_n} \right) > 0$$

---

This is a summary of the related argument in [CRL].
Proof. Since
\[
\frac{1}{A_n} = \frac{2\sqrt{-i\sigma + n\omega}}{e^{-2\sqrt{-i\sigma + n\omega}} - 1}
\]
we have
\[
\text{Im} \left( \frac{1}{A_n} \right) = c \left( b(e^{-a} \cos b - 1) + a(e^{-a} \sin b) \right)
\]
with \(c > 0\) a constant.

In the case \(a \geq 0\) and \(b \leq 0\) we obviously have
\[
|be^{-a} \cos b| \leq |b|e^{-a}
\]
and
\[
|ae^{-a} \sin b| \leq ae^{-a}|b|
\]
Using the fact that \(e^{-a}(a + 1)\) is decreasing and \(e^{-a}(a + 1) \leq 1\) we have
\[
|b|e^{-a} + ae^{-a}|b| \leq |b|
\]
So
\[
b(e^{-a} \cos b - 1) + a(e^{-a} \sin b) \geq -b - |be^{-a} \cos b + ae^{-a} \sin b|
\]
\[
\geq -b - |be^{-a} \cos b| - |be^{-a} \cos b| \geq -b - |b|e^{-a} - ae^{-a}|b| \geq 0
\]
which proves the first part of the lemma.

Furthermore, if \(b \neq 0\), there cannot be equality in both (4.4.6) and (4.4.7), and the conclusion follows.

As a corollary of Lemma \[49\] we have

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**Theorem 50.** When $\Re \sigma > 0$, equation (4.4.1) has no nontrivial solution. Equivalently, $\hat{\psi}$ is analytic in the region $\Re p > 0$.

**Proof.** When $\Re \sigma > 0$ we have, by the choice of branch,

$$\Re \sqrt{-i\sigma + n\omega} \geq 0 \text{ and } \Im \sqrt{-i\sigma + n\omega} < 0$$

Thus Lemma 49 implies

$$\Im \left( \frac{1}{A_n} \right) > 0$$

Thus

$$\Im \left( \sum_n \frac{1}{A_n} |y_n|^2 \right) = \sum_n \Im \left( \frac{1}{A_n} \right) |y_n|^2 > 0$$

and equation (4.4.5) is not satisfied. \hfill $\square$

When there are poles on the imaginary line we have the following theorem.

**Theorem 51.** When $\Re \sigma = 0$, equation (4.4.1) has a nontrivial solution only if $A_{-1} = 0$. Equivalently, $\hat{\psi}$ has a pole on the imaginary line only if $(\omega + i\sigma)/\pi^2 = m^2$ for some integer $m$.

**Proof.** $\Re \sigma = 0$ implies

$$\sqrt{-i\sigma + n\omega} \in \left\{ \begin{array}{ll} \mathbb{R}^+, & n \geq 0 \\ -i\mathbb{R}^+, & n < 0 \end{array} \right.$$\hfill (4.4.1)

Therefore

$$\Im \left( \frac{1}{A_n} \right) \left\{ \begin{array}{ll} \geq 0, & n \geq 0 \\ > 0, & n < 0 \end{array} \right.$$
Thus if

\[ \text{Im} \left( \sum_{n} \frac{1}{A_n} |y_n|^2 \right) = 0 \]

then \( y_n = 0 \) for \( n < 0 \).

Furthermore, if \( A_n \neq 0 \) for all \( n \geq -1 \),

\[ y_{n+1} = \alpha_n y_n + y_{n-1} \]

is valid for \( n \geq -1 \), so the condition that \( y_n = 0 \) for \( n < 0 \) implies that \( y_n = 0 \) for all \( n \), a trivial solution.

Thus if (4.4.1) has a nontrivial solution, \( A_n \) must be 0 for some \( n \geq -1 \). Since

\[ A_n = \frac{e^{-2\sqrt{-i\sigma + n\omega}} - 1}{2\sqrt{-i\sigma + n\omega}} \]

\( A_n = 0 \) implies \( e^{-2\sqrt{-i\sigma + n\omega}} - 1 = 0 \), which implies that

\[ \sqrt{-i\sigma + n\omega} = -im\pi \]

for some integer \( n \) and \( m \). Since \( \omega > 0 \), \( n \) must be negative, thus \( n \) can only be \(-1\).

Thus

\[ \frac{\omega + i\sigma}{\pi^2} = m^2 \]

\( \square \)

**Remark 52.** From the proof of Theorem 51 we see that we must have \( y_n = 0 \) for \( n < 0 \) in order to have poles on imaginary line.
4.4.2 Sufficient condition for $\hat{\psi}$ to have poles on imaginary line

Assume that

$$\frac{\omega + i\sigma}{\pi^2} = m^2.$$ 

Define

$$\rho_n = \frac{y_{n+1}}{y_n}.$$ 

Since $y_n = 0$ for $n < 0$ (Remark 52), $A_{-1} = 0$, and $A_n \neq 0$ for other $n$. Thus (4.4.1) gives

$$\alpha_n = \rho_n - \frac{1}{\rho_{n-1}} \quad (4.4.8)$$

for $n \geq 0$.

Iterating (4.4.8) gives

$$\rho_0 = \frac{1}{-\alpha_1 + \frac{1}{-\alpha_2 + \frac{1}{-\alpha_3 + \ldots}}} \quad (4.4.9)$$

where the convergence is guaranteed by the fact that $|\alpha_n| \to \infty$ as $|n| \to \infty$.

Because $A_{-1} = 0$ implies $y_{-1} = 0$ and thus $\rho_0 = \alpha_0$, we have

$$\frac{1}{-\alpha_1 + \frac{1}{-\alpha_2 + \frac{1}{-\alpha_3 + \ldots}}} = \alpha_0 \quad (4.4.10)$$
Equation (4.4.10) is the sufficient condition to have poles on the imaginary line. In practice, one fixes $\omega$, $\sigma$, and $V$ so that the truncated equation of (4.4.10)

$$\alpha_0 = \frac{1}{-\alpha_1 + \frac{1}{1 - \alpha_2 + \frac{1}{1 - \alpha_3 + \ldots + \frac{1}{-\alpha_n}}}$$

becomes a rational equation of $\Omega$ for which the solution can simply be found by determining the roots of a polynomial.

Showing that for generic choice of parameters $\omega$, $\sigma$, and $V$ (4.4.10) has a solution $\Omega$ can be difficult. However, for given choice of $\omega$, $\sigma$ and $V$, once one finds a numerical solution using truncated equation, one can prove that there exists an exact solution to (4.4.10) near the approximate solution. See section 4.4.3 as an example.

First we need the following results.

**Lemma 53.** If $|\alpha_i| > 2$, $i = 1, 2, \ldots$, then

$$\left| \frac{1}{\alpha_1 + \frac{1}{\alpha_2 + \ldots}} \right| < 1 \quad (4.4.11)$$
Proof. For any $N \in \mathbb{N}$ we have

\[
\begin{array}{c|c|c}
1 & \alpha_1 + & < 1 \\
\hline
1 & \alpha_2 + \ldots + \frac{1}{\alpha_N} & < 1 \\
\hline
\end{array}
\]

Obviously

\[
\left| \frac{1}{\alpha_N} \right| < \frac{1}{2} < 1
\]

Assume that

\[
\begin{array}{c|c|c}
1 & \alpha_m + & < 1 \\
\hline
1 & \alpha_{m+1} + \ldots + \frac{1}{\alpha_N} & < 1 \\
\hline
\end{array}
\]

Then

\[
\begin{array}{c|c|c}
1 & \alpha_{m-1} + & < 1 \\
\hline
1 & \alpha_m + \ldots + \frac{1}{\alpha_N} & < \frac{1}{2 - 1} = 1 \\
\hline
\end{array}
\]
The rest of the proof follows by induction.

**Corollary 54.** If \(|\alpha_i| > M, i = 1, 2, \ldots\), then

\[
\frac{1}{\alpha_1 + \frac{1}{\alpha_2 + \ldots}} < \frac{1}{M - 1}
\]

**Proof.** By Lemma 53 we have

\[
\frac{1}{\alpha_2 + \frac{1}{\alpha_3 + \ldots}} < 1
\]

So

\[
\frac{1}{\alpha_1 + \frac{1}{\alpha_2 + \ldots}} < \frac{1}{\alpha_1} - \frac{1}{\alpha_2 + \frac{1}{\alpha_3 + \ldots}} < \frac{1}{M - 1}
\]

\[\square\]

**4.4.3 An example of rigorously finding the solution to (4.4.10)**

Take for instance \(\omega = \pi^2 + 0.3, \sigma = 0.3i,\) and \(V = 1\) as an example. The fifth truncated equation suggests a solution near \(\Omega = 3.8963\).
To show that there is a solution near this approximate one, first notice that \( |\alpha_n| \) decreases with respect to \( \Omega \). It is easy to see that (for \( \text{Re}\sigma = 0 \)) \( \text{Im}\alpha_n \) increases with respect to \( n \). Thus for \( n \) that \( \text{Im}\alpha_n > 0 \), \( |\alpha_n| \) increases with respect to \( n \). For \( \Omega = 4 \) we have \( \text{Im}\alpha_5 > 3.5 \), thus \( |\alpha_n| > 3.5 \) for \( \Omega < 4 \) and \( n \geq 5 \).

Rewriting (4.4.10) as

\[
\alpha_4 + \frac{1}{\alpha_3 + \frac{1}{\alpha_2 + \frac{1}{\alpha_1 + \frac{1}{\alpha_0}}}} = \frac{1}{-\alpha_5 + \frac{1}{-\alpha_6 + \frac{1}{-\alpha_7 + \frac{1}{-\alpha_8 + \ldots}}}}
\]

(4.4.12)

we see that both sides of (4.4.12) are purely imaginary. By Corollary 54, since \( |\alpha_n| > 3.5 \), the imaginary part of the right side of (4.4.12) is between \(-0.4\) and \(0.4\). The graph of the imaginary part of the left side is given in Fig 4.1. Numerical tests give that the left side at \( \Omega = 3.890 \) is \( 1.611i \) which is bigger than \( 0.4i \), and at \( \Omega = 3.899 \) is \( -2.43899i \) which is smaller than \( 0.4i \). Since the singularity of the left side is given by

\[
\alpha_3 + \frac{1}{\alpha_2 + \frac{1}{\alpha_1 + \frac{1}{\alpha_0}}} = 0
\]

which is around 3.9025, by continuity there exits a solution to (4.4.12).
4.4.4 Continued fraction representation of (4.2.26)

It is easy to see that for Re\( p < 0 \), \( A_n \neq 0 \). Thus (4.2.26) can always be written as (4.4.3) and thus (4.4.8).

Iterating (4.4.8) gives

\[
\rho_0 = \frac{1}{-\alpha_1 + \frac{1}{-\alpha_2 + \frac{1}{-\alpha_3 + \ldots}}} \tag{4.4.13}
\]

Figure 4.1: The imaginary part of the left side of (4.4.12)
and

\[ \rho_0 = \alpha_0 + \frac{1}{\alpha_{-1} + \frac{1}{\alpha_{-2} + \frac{1}{\alpha_{-3} + \ldots}}} \] \hspace{1cm} (4.4.14)

Because \(|\alpha_n| \to \infty\) as \(|n| \to \infty\) both (4.4.13) and (4.4.14) are convergent. Equating them gives

\[ \frac{1}{\alpha_{-1} + \frac{1}{\alpha_{-2} + \frac{1}{\alpha_{-3} + \ldots}}} = \alpha_0 + \frac{1}{\alpha_{-1} + \frac{1}{\alpha_{-2} + \frac{1}{\alpha_{-3} + \ldots}}} \] \hspace{1cm} (4.4.15)

This is the equation in continued fraction representation equivalent to \((4.2.26)\).

**Remark 55.** Formally, when \(p\) is on the imaginary line, the condition \(y_{-1} = 0\) and \(y_0 \neq 0\) implies that \(\rho_{-1} = \infty\) and thus \(\alpha_{-1} = \infty\). Thus the right side of (4.4.15) truncates to \(\alpha_0\) and we get (4.4.10).

### 4.4.5 Proof of Theorem 33

The poles of the time independent problem are given by (4.3.9). For a solution to (4.3.9), \(p_i = \sigma_i + in_\omega\), by (4.4.2) we have

\[ \alpha_{n_i}(\sigma_i) = 0 \] \hspace{1cm} (4.4.16)

We show that for any \(\sigma\) such that

\[ \alpha_n(\sigma) \neq 0, \forall n \] \hspace{1cm} (4.4.17)
for small enough $\Omega$, $\sigma$ is not a solution to equation (4.4.15).

For such $\sigma$ that (4.4.17) holds we have

$$\frac{1}{A_n} - V \neq 0, \forall n$$

Since $|A_n| \to 0$ for $n \to \pm\infty$, the set $\{|\frac{1}{A_n} - V|, n \in \mathbb{Z}\}$ has a minimum $m_\sigma$. For $\Omega < \frac{1}{2}m_\sigma$ we have $|\alpha_n| > 2, \forall n$. By Lemma 53 we have

$$\begin{align*}
\left| \frac{1}{A_{n-1}} + \frac{1}{A_{n-2}} + \frac{1}{A_{n-3}} + \ldots \right| &< 1 \\
\left| \frac{1}{A_0} + \frac{1}{A_{-1}} + \frac{1}{A_{-2}} + \frac{1}{A_{-3}} + \ldots \right| &> |\alpha_0| - 1 \\
\left| \frac{1}{A_{-1}} + \frac{1}{A_{-2}} + \frac{1}{A_{-3}} + \ldots \right| &> 1
\end{align*}$$

Thus the right side of (4.4.15) satisfies
But from Lemma 53, the left side of (4.4.15) satisfies

\[
\begin{vmatrix}
1 \\
-\alpha_1 + 1 \\
-\alpha_2 + 1 \\
-\alpha_3 + \ldots
\end{vmatrix} < 1
\]

Thus \( \sigma \) is not a solution.

**Corollary 56.** *From the proof of Theorem 33 we see that for small \( \Omega \), the poles of \( \hat{\psi}(x,p) \) for the time-dependent problem are positioned in regions centered around points satisfying \( \alpha_n(\sigma) = 0, \exists n \). Furthermore, the largeness of such regions can be estimated using the criteria \( m_\sigma < 2\Omega \).

### 4.4.6 Proof of Theorem 34

As discussed in section 4.2.4, the wave function can be written as (4.2.29). The position of the poles are given by (4.4.15). Since the solutions to (4.4.15) are solutions of \( \sigma \), the poles in terms of \( p \) are periodic with period \( i\omega \). Thus a solution \( \sigma = a_m \) gives poles of the type \( p = a_m + in\omega \).

Thus the first term in (4.2.29) becomes

\[
\sum_{m=1}^{M} \sum_{n=-\infty}^{\infty} e^{i(a_m + i n\omega)t} r_{m,n}
\]

Furthermore, since the inhomogeneity \( r_n \) has no poles, if we take a small loop around the singularity and integrate (4.2.25) along it, then by Cauchy theorem
(4.2.25) becomes the homogeneous equation for the residues, i.e. equation (4.1.5).

The fact that the sum appearing in the theorem is convergent follows from the fact \( \hat{\psi}_1(\sigma, n) \in \mathcal{H} \).

### 4.4.7 Proof of Theorem 32

It is clear that ionization happens if when \( t \to \infty \), \( \psi(x, p) \to 0 \) uniformly. Function \( \psi(x, p) \) has an expansion (4.1.3) given by Theorem 34. In this expansion, since \( p_0 \) has negative real part, it is clear that as \( t \to \infty \) the second term goes to zero. Furthermore, the last term (the sum of two integrals) also vanishes by Watson’s lemma. Thus only the first term, the double sum could be non-vanishing. Apparently in this double sum, those terms corresponding to \( a_m \) with negative real part vanish as \( t \to \infty \). Thus unless some \( a_m \) has zero real part, \( \psi(x, p) \to 0 \) as \( t \to \infty \) uniformly, that is, ionization happens. The rest follows from Theorem 51.

**Remark 57.** Note first that the positions of the poles of \( \hat{\psi} \) depends only on where the homogeneous has solutions, not on the choice of initial condition. Thus if a system (i.e for some parameters \( \Omega, \omega, V \)) is “essentially” ionizing (i.e (4.4.15) gives no poles on imaginary line), the wave function ionizes for any initial condition.

Furthermore, the condition that (4.4.15) has solutions on imaginary line cannot guarantee that the wave function for any initial condition localizes, because the residues corresponding to these poles could still add up to zero. In fact, there exist initial conditions for which (4.4.15) has solutions on imaginary line, but \( \hat{\psi} \) is regular on imaginary line. To see this, assume that for a fixed system there are two solutions corresponding to two initial conditions which give different residues at the same
position (positions are given by (4.4.15), thus are independent of initial conditions) on the imaginary line. There exists a linear combination of these two residues that vanishes, thus the same linear combination of the solutions has zero residue at \( a_m \), i.e. \( r_{m,0} = 0 \). Since the residues \( r_{m,n} \) satisfy homogeneous equation (4.1.5) with zero boundary condition for \( n \to \pm \infty \), it is the zero solution. This means that this combination of solutions does not have poles on imaginary line, thus \( \hat{\psi} \) is regular on imaginary line.

Another interesting observation is that by repeating the process we use to construct a solution which does not have poles on the imaginary line, we can construct a solution (with different initial condition) which has no poles in any finite region. If a rapidly ionizing solution is desired this is one way of constructing it.

### 4.4.8 Proof of Theorem 35

Define \( f_n \) to be the truncated continued fraction

\[
f_n := c_n + \frac{1}{c_{n-1} + \frac{1}{c_{n-2} + \cdots + \frac{1}{c_1}}} \tag{4.4.18}
\]

then \( f_n \) satisfies

\[
f_n = c_n + \frac{1}{f_{n-1}} \tag{4.4.19}
\]

\(^3\) A general result about difference equation with smooth coefficients is given in [CC], here, only for this special case we gave a more explicit result with a shorter and more straightforward proof.
Since the curve does not pass through the points $\pm 2i$, the functions $\pm \sqrt{z^2 + 4}$ can be analytically defined along the curve. So do the functions $f(z)^\pm = \frac{z \pm \sqrt{z^2 + 4}}{2}$. It can easily be checked that $f^+(z)f^-(z) = -1$ thus for fixed $z$ one of the two functions has a norm bigger than one and the other one has a norm smaller than one. Furthermore, if $|f^+(z)| = |f^-(z)| = 1$, since $f^+(z)f^-(z) = -1$, $f^\pm(z)$ must satisfy $\text{Re}f^+ = -\text{Re}f^-$ and $\text{Im}f^+ = \text{Im}f^-$. Since $f^+ + f^- = z$ this implies that $z$ is purely imaginary and $-2 \leq \text{Im}z \leq 2$. Since the curve does not pass through the segment $[-2i, 2i]$, the condition $|f^+(z)| = |f^-(z)| = 1$ never occurs along the curve, thus one of the two functions $f^\pm$ always has a norm bigger than 1. Denote this function by $f^+(z)$.

Let $f^\pm(c_n) = f^\pm_n$ and let $f_n = f^+_n(1 + \delta_n)$. Straightforward calculations show that $f_n = c_n + \frac{1}{f_{n-1}}$ implying that $\delta_n$ satisfy

$$\Delta_n = f^+_n\delta_n - f^-_{n-1}\frac{\delta_{n-1}}{1 + \delta_{n-1}}$$

(4.4.20)

where $\Delta_n = f^-_n - f^-_{n-1}$.

Rewriting (4.4.20) as

$$\delta_n = \frac{\Delta_n}{f^+_n + f^-_{n-1}\frac{\delta_{n-1}}{1 + \delta_{n-1}}}$$

(4.4.21)

and writing $c_{n-1} = -c_n(1 + \epsilon_\Delta)$ we have

$$\sqrt{c^2_{n-1} + 4} = \sqrt{(c^2_n + 4) \left(1 + \frac{\epsilon_\Delta c^2_n}{c^2_n + 4}\right)} \sim \sqrt{c^2_n + 4 + \frac{\epsilon_\Delta c^2_n}{\sqrt{c^2_n + 4}}}$$

(4.4.22)

Thus

$$\frac{\Delta_n}{f^+_n} = -\epsilon_\Delta \frac{1}{1 + \sqrt{1 + \frac{4}{c^2_n}}} + \epsilon_\Delta \frac{1}{1 + \frac{4}{c^2_n}}$$

(4.4.23)
Because of the choice of the square root function, the term $\sqrt{1 + \frac{4}{z^2}}$ never gets near $-1$ along the curve (otherwise we would have $\sqrt{z^2 + 4} \sim -z$). Let

$$\epsilon_{L_1} := \min \left\{ \frac{1}{2}, \left| \sqrt{1 + \frac{4}{z^2}} - 1 \right| : z \text{ along the curve} \right\}$$

(4.4.24)

Furthermore, since the curve does not pass through the points $\pm 2i$, the function $|1 + \frac{4}{z^2}|$ has a lower bound, denoted by $\epsilon_{L_2}$:

$$\epsilon_{L_2} := \min \left\{ \frac{1}{2}, \left| 1 + \frac{4}{z^2} \right| : z \text{ along the curve} \right\}$$

(4.4.25)

Finally, also because the curve does not pass through the points $\pm 2i$, the sequence $|\frac{f_{n-1}}{f_n}|$ has an upper bound, let

$$\epsilon_{L_3} := \min \left\{ \frac{1}{2}, \left| \frac{f_{n-1}}{f_n} - 1 \right| : n = 2, 3, ..., N \right\}$$

(4.4.26)

Using these notations we have

$$\left| \frac{\Delta_n}{f_n^+} \right| \leq \epsilon_\Delta \left( \frac{1}{\epsilon_{L_1}} + \frac{1}{\epsilon_{L_1} \epsilon_{L_2}} \right) < \frac{2\epsilon_\Delta}{\epsilon_{L_1} \epsilon_{L_2}}$$

(4.4.27)

Take $\epsilon_0$ and $\epsilon_\Delta \ll 1$ subject to the conditions

$$\epsilon_0 < \frac{1}{4} \epsilon_{L_3} \text{ and } \epsilon_\Delta < \frac{\epsilon_{L_1} \epsilon_{L_2} (\epsilon_{L_3} - \epsilon_0)}{2(1 - \epsilon_0)} \epsilon_0$$

(4.4.28)

and define the space $l_0 := \{|\delta_n| < \epsilon, n = 1, 2, ..., N\}$ with the norm $|\delta|_0 := \max_n |\delta_n|$.

For $\delta_n \in l_0$, the right side of (4.4.21) satisfies

$$\left| \frac{\Delta_n}{f_n^+} + \frac{f_{n-1}^-}{f_n^+} \frac{\delta_n}{1 + \delta_{n-1}} \right| \leq \left| \frac{\Delta_n}{f_n^+} \right| + \left| \frac{f_{n-1}^-}{f_n^+} \frac{\delta_n}{1 + \delta_{n-1}} \right|

< (1 - \epsilon_{L_3}) \frac{\epsilon_0}{1 - \epsilon_0} + \epsilon_{L_3} - \epsilon_0 \frac{\epsilon_0}{1 - \epsilon_0} = \epsilon_0$$

(4.4.29)

where (4.4.27) and (4.4.28) are used to obtain the second inequality.
Furthermore, $\epsilon_0 < \frac{1}{4}\epsilon L_3$ implies

$$4 - \epsilon_0 \frac{\epsilon_0}{1 + 2\epsilon_0} < \epsilon L_3 \quad (4.4.30)$$

which is equivalent to

$$\frac{2\epsilon_0 + 1}{(1 - \epsilon_0)^2} (1 - \epsilon L_3) < 1 \quad (4.4.31)$$

Take two different functions $\delta^1_n$ and $\delta^2_n$ in $l_0$. We have

$$\max_n \left| \frac{f_n^-}{f_n^+} \left( \frac{\delta^1_n}{1 + \delta^1_{n-1}} - \frac{\delta^2_n}{1 + \delta^2_{n-1}} \right) \right| < \max_n \left| \frac{(\delta^1_n - \delta^2_n) + (\delta^2_n - \delta^1_n)\delta^1_{n-1} + (\delta^1_{n-1} - \delta^2_{n-1})\delta^1_n}{(1 - \epsilon_0)^2} \right| (1 - \epsilon L_3) \quad (4.4.32)$$

Thus the contractive mapping theorem can be applied, which gives that (4.4.21) has a unique solution in $l_0$. Recall that $f_N = f_N^+(1 + \delta_n)$, this shows (4.1.7).

**Remark 58.** Theorem 35 shows that if $c_1$ is far from origin, and the curve does not pass through the segment $[-2i, 2i]$, then as we increase the number of “sample” points, the continued fraction $f_c$ converges. This process does not depend on the shape of the curve, only on the position of $c_n$.

If the curve extends to infinity, then as long as the “sample” points are dense enough so that the change in $c_n$ is smooth, Theorem 35 still holds. The reason is that the truncated curve satisfies the hypothesis of the Theorem, giving a good approximation to the resulting continued fraction. But since this result does not depend on $c_1$, the result stays the same as we extend the truncated curve.
4.4.9 Study in the far field

The far region is where $\text{Re} \, p < 0$ and $|\text{Re} \, p| \gg 1$. Define

$$\beta(p) := \frac{2\sqrt{-ip}}{e^{-2\sqrt{-ip}} - 1}$$

and $\alpha(p)$, a generalization of $\alpha_n$ defined in (4.4.2), as

$$\alpha(p) := \frac{1}{i \Omega} \left( \frac{1}{A(p)} - V \right) = \frac{1}{i \Omega} \left( \frac{2\sqrt{-ip}}{e^{-2\sqrt{-ip}} - 1} - V \right) = \frac{1}{i \Omega} (\beta(p) - V)$$

We have the following lemma.

**Lemma 59.** In the far region, $\alpha(p)$ varies slowly with respect to $p$ in the lower half $p$-plane and upper half $p$-plane respectively. Explicitly, for $\delta p \sim O(1)$ we have

$$\alpha(p + dp) \sim \alpha(p) (1 + o(1))$$

**Proof.** In the far region, in either lower or upper half $p$-plane

$$\sqrt{-i(p + \delta p)} \sim \sqrt{-ip} \left( 1 - \frac{1}{2} \frac{\delta p}{p} \right) \sim \sqrt{-ip} + \frac{i}{2} \frac{\delta p}{\sqrt{-ip}}$$

(4.4.33)

Thus

$$e^{-2\sqrt{-i(p + \delta p)}} \sim e^{-2\sqrt{-ip}} e^{-\frac{i}{2} \frac{\delta p}{\sqrt{-ip}}} \sim e^{-2\sqrt{-ip}} \left( 1 - \frac{i}{2} \frac{\delta p}{\sqrt{-ip}} \right)$$

(4.4.34)

and

$$\beta(p + \delta p) \sim \frac{2\sqrt{-ip} \left( 1 - \frac{1}{2} \frac{\delta p}{p} \right)}{e^{-2\sqrt{-ip}} \left( 1 + \frac{i}{2} \frac{\delta p}{\sqrt{-ip}} \right) - 1} \sim \frac{2\sqrt{-ip} \left( 1 - \frac{1}{2} \frac{\delta p}{p} \right)}{(e^{-2\sqrt{-ip}} - 1) - \frac{i}{2} e^{-2\sqrt{-ip}} \delta p}$$

(4.4.35)

For $p$ such that

$$\frac{i}{2} \frac{e^{-2\sqrt{-ip}} \delta p}{\sqrt{-ip}} = o \left( e^{-2\sqrt{-ip}} - 1 \right)$$

(4.4.36)
we have
\[
\beta(p + \delta p) \sim \frac{2\sqrt{-ip}}{e^{-2\sqrt{-ip}} - 1} \left(1 - \frac{1}{2} \frac{\delta p}{p} + \frac{i}{2} \frac{\delta p}{\sqrt{-ip}} \frac{e^{-2\sqrt{-ip}}}{e^{-2\sqrt{-ip}} - 1}\right) \quad (4.4.37)
\]

If \(p\) is in the upper half plane, we have
\[
e^{-2\sqrt{-ip}} = o\left(\frac{1}{p}\right) = o(1)
\]

This implies
\[
\frac{i}{2} \frac{e^{-2\sqrt{-ip}} \delta p}{\sqrt{-ip}} = o(e^{-2\sqrt{-ip}}) = o(1)
\]

thus (4.4.36) is satisfied and (4.4.37) gives
\[
\beta(p + \delta p) \sim \frac{2\sqrt{-ip}}{e^{-2\sqrt{-ip}} - 1} \left(1 - \frac{1}{2} \frac{\delta p}{p} + o\left(\frac{1}{p}\right)\right) \quad (4.4.38)
\]

Thus
\[
\alpha(p + \delta p) - \alpha(p) = \frac{\delta p}{\Omega \sqrt{-ip}} + o\left(\frac{1}{\sqrt{-ip}}\right) \quad (4.4.39)
\]

and \(\alpha(p)\) is slow-changing.

If \(p\) is in the lower half plane, the choice of branch is such that for \(|\text{Im}p|\) not very large, \(|e^{-2\sqrt{-ip}}| \gg 1\), and for \(|\text{Im}p|\) very large \(|e^{-2\sqrt{-ip}}| = O(1)\). In the former region, (4.4.37) is valid and gives
\[
\beta(p + \delta p) \sim \frac{2\sqrt{-ip}}{e^{-2\sqrt{-ip}} - 1} \left(1 + \frac{i}{2} \frac{\delta p}{\sqrt{-ip}} + o\left(\frac{1}{\sqrt{-ip}}\right)\right) \quad (4.4.40)
\]

Thus
\[
\alpha(p + \delta p) - \alpha(p) = \frac{\delta p}{\Omega e^{-2\sqrt{-ip}}} + o\left(\frac{1}{e^{-2\sqrt{-ip}}}\right) \quad (4.4.41)
\]
In the region where \(|e^{-2\sqrt{-ip}}| = O(1)|, if \(e^{-2\sqrt{-ip}} - 1 = O(1)\), \(4.4.37\) is still valid and

\[
\beta(p + \delta p) \sim \frac{2\sqrt{-ip}}{e^{-2\sqrt{-ip}} - 1} \left(1 + \frac{i \delta p}{2 \sqrt{-ip}} \frac{e^{-2\sqrt{-ip}}}{(e^{-2\sqrt{-ip}} - 1)}\right) \tag{4.4.42}
\]

This implies

\[
\alpha(p + \delta p) - \alpha(p) \sim \alpha(p) \left(\frac{1}{2\Omega} \frac{\delta p}{\sqrt{-ip}} \frac{e^{-2\sqrt{-ip}}}{(e^{-2\sqrt{-ip}} - 1)}\right) \tag{4.4.43}
\]

Finally, for \(e^{-2\sqrt{-ip}} - 1 = o(1)\), let \(p_0 = \text{Re}p\) and \(\sqrt{-ip} = -a - bi\) with \(a, b \in \mathbb{R}^+\).

First, \(e^{-2\sqrt{-ip}} - 1 = o(1)\) gives \(\sqrt{-ip} \sim n\pi i\). This implies that \(a = o(1)\) and \(b \sim n\pi\) for some \(n \in \mathbb{N}\). Next by \(2ab = -p_0\) we have

\[
a \sim \frac{-p_0}{2n\pi} \sim \frac{i}{2} \frac{-p_0}{\sqrt{-ip}} \tag{4.4.44}
\]

Then

\[
e^{-2\sqrt{-ip}} - 1 = e^{2a + 2bi} \sim (1 + 2a) - 1 \sim i \frac{-p_0}{\sqrt{-ip}} \tag{4.4.45}
\]

Thus \((4.4.36)\) is still satisfied and \((4.4.37)\) implies

\[
\beta(p + \delta p) \sim \frac{2\sqrt{-ip}}{e^{-2\sqrt{-ip}} - 1} \left(1 + \frac{i \delta p}{2 p_0}\right) \tag{4.4.46}
\]

Thus

\[
\alpha(p + \delta p) - \alpha(p) \sim \alpha(p) \left(\frac{\delta p}{2\Omega p_0}\right) \tag{4.4.47}
\]

Combining \((4.4.39)\), \((4.4.41)\), \((4.4.43)\), and \((4.4.47)\) we see that in all possible cases

\[
\alpha(p + \delta p) - \alpha(p) \sim o(\alpha(p))
\]
In practice one calculates the solution to (4.4.15) for large \( p \), which involves calculating a continued fraction along a vertical line. For \( n > 0 \), it is easy to check that \( \alpha_n \)’s are uniformly large thus either by Theorem 35 or by Corollary 54 the continued fraction is easy to calculate. For \( n < 0 \), \( \alpha_n \) is not uniformly large, thus Corollary 54 cannot be applied. But one can check that the path of \( \alpha(p) \) for large \( p \) in the lower half plane along a vertical ray is a spiral line for \( \text{Im} p \) not so large, and some irregular curve for \( \text{Im} p \) very large. Every loop of the spiral line is almost a circle with center \( \frac{V}{V_1} \). Different loops have slowly changing radius with respect to \( \text{Im} p \). The spiral line passes through the segment \([-2i, 2i]\) many times only for \( p \) in a certain region. Once the radius is large enough for some \( p_s \), the curve does not cross the segment \([-2i, 2i]\) for \(|\text{Im} p| > |\text{Im} p_s|\) any more. Lemma 59 applies \((\delta p = \omega \sim O(1))\), which makes the Theorem 35 valid in this region, and one should in practice calculate the continued fraction up to \( p_s \), and replace its tail with the estimate (4.1.7), which is very accurate in the far field.
CHAPTER 5
DIRECTION FOR FURTHER RESEARCH

In this chapter we list some interesting problems for future research.

In Chapter 3 we give a complete asymptotic expansion of the wave function $\psi$ for large $t$ in a time-dependent setting for the Schrödinger equation on $\mathbb{R}$ (Theorem 12). In Chapter 4 we give only a partial asymptotic expansion of the wave function in a half line setting (Theorem 34). One interesting problem is to obtain a complete asymptotic expansion for the half line problem.

To obtain such an asymptotic expansion one needs to show that the terms in the expansion (4.1.3) satisfy

$$
\lim_{p_0 \to -\infty} \sum_{m=1}^{M} \sum_{n=-\infty}^{\infty} e^{(a_m+i n \omega)t_{m,n}} < \infty \quad (5.0.1)
$$

where $M$ depends on $p_0$, and

$$
\lim_{p_0 \to -\infty} e^{p_0 t} \sum_{n=-\infty}^{\infty} \int_{n \omega}^{(n+1)\omega} e^{iyt} \hat{\psi}_1(p_0 + iy) dy \to 0 \quad (5.0.2)
$$

and that the integral

$$
\int_{\mathcal{C}} e^{pt} \hat{\psi}_1(p) dp \quad (5.0.3)
$$

exists, where $\mathcal{C}$ describes a contour goes from $-\infty$ around the line \{ $p | \text{Re}(p) \leq 0, \text{Im}(p) = (n - 1)\omega$ \} and back to $-\infty$. 89
The first step is to obtain the asymptotic behavior in large $|p|$ of the poles of $\hat{\psi}$ for $p$ in left half $p$ plane. Since we already showed that the poles are periodic with period $i\omega$, one needs only to show the asymptotic behavior for those poles in the strip $\{p : 0 < \text{Im}p < \omega\}$.

The key equation is (4.4.8) in section 4.4.4. One possible approach is to analyze (4.4.8) using asymptotic methods for difference equations with smoothly changing coefficients. By following this approach one could approximate the solutions to the difference equation by constant solutions first, then calculate the corrections. However, along a discrete set $\{i\omega\}$ the term $\alpha_n$ in (4.4.8) is only piecewise smoothly changing. One could in principle approximate solutions in each region by constants and then match them at the turning points. One serious difficulty to overcome is that the number of the turning points goes to $\infty$ as $p_0 \to \infty$ and the approximation error accumulates as the number of the turning points increases.

The next step is to estimate the residues of $\hat{\psi}$ in order to show (5.0.1). We already showed that the residues for those poles with period $i\omega$ decay eventually for $n \to \pm\infty$ thus the inner sum converges, and the problem boils down to the convergence of the outer sum.

One possible way to estimate the residues is to write the solution of the inhomogeneous equation (4.2.25) in terms of solutions of the homogeneous equation (4.2.26) using the Wronskian formula. Hence the residues can be estimated in terms of the homogeneous equation.

The last step is to use the estimates above to estimate $\hat{\psi}$ and thus show that (5.0.3) and (5.0.2) are both satisfied.
The method used in this thesis works well for delta potential potentials. One may apply this method to Schrödinger equation with multi-delta function potentials. Some interesting problems are:

1. Finding the long time behavior of wave function and the survival probability, at least numerically.

2. Connecting this analysis to what are known in physics as resonance expansion.

3. Study of three-dimensional problems. Special three-dimensional problems involving delta function potential shells can be interpreted as one-dimensional problems with delta function potential wells.

4. The space periodic one-dimensional delta function potential problem. Such an example of such with a time independent potential using one-dimensional periodic delta functions with equal amplitudes can be found in [Ba]. This is a very simple model of conductivity of pure materials. I want to explore the possibility whether the conductivity of an alloy can be modeled with delta function potential with uneven amplitudes.
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