Likelihood as a Method of Multi Sensor Data Fusion for Target Tracking

A Thesis

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ABSTRACT

This thesis addresses the problem of detecting and tracking objects in a scene, using a distributed set of sensing devices in different locations, and in general use a mix of different sensing modalities. The goal is to combine data in an efficient but statistically principled way to realize optimal or near-optimal detection and tracking performance. Using the Bayesian framework of measurement likelihood, sensor data can be combined in a rigorous manner to produce a concise summary of knowledge of a target’s location in the state-space. This framework allows sensor data to be fused across time, space and sensor modality. When target motion and sensor measurements are modeled correctly, these “likelihood maps” are optimal combinations of sensor data. By combining all data without thresholding for detections, targets with low signal to noise ratio (SNR) can be detected where standard detection algorithms may fail. For estimating the location of multiple targets, the likelihood ratio is used to provide a sub-optimal but useful representation of knowledge of the state space. As the calculation cost of computing likelihood or likelihood ratio maps over the entire state space is prohibitively high for most practical applications, an approximation computed in a distributed fashion is proposed and analyzed. This distributed method is tested in simulation for multiple sensor modalities, displaying cases where it is and is not a good approximation of central calculation. Detection and tracking
examples using measured data from multi-modal sensors (Radar, EO, Seismic) are also presented.
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CHAPTER 1

Introduction

As computing power and sensing technology improve, distributed sensor networks (DSNs) are rapidly becoming a reality. With DSNs come the promise of robust, inexpensive, widespread sensing, as well as challenges of data management, processing, transmission, and fusion. Especially for a wireless sensor network, constraints on power, computation speed, bandwidth and time complicate this process [4]. While many applications of DSNs have been suggested, target detection and tracking is perhaps the most common.

Many approaches to data fusion for target tracking in DSNs have been proposed in literature. Most traditional methods involve processing the data to make “hard” decisions, whether detections or classifications, and associating this decision with a current or new track [3,6–9,16,17]. While computationally much simpler, this pre-processing of sensor measurements can result in misclassification, missed detection of targets in noise, or false alarms resulting in creation of false tracks. For multiple target tracking, the inherent data association problem reduces certainty in the accuracy of tracks.

The Kalman Filter in its many forms has been successfully employed for target tracking for many years. However, it is sub-optimal for non-linear, non-Gaussian
sensor or motion models. While the extended Kalman filter (EKF) greatly reduces errors caused by non-linear measurement, it requires a global mean to be estimated at each iteration from which to linearize the model [5]. This requires transmission of sensor measurements to a fusion center at each iteration. The EKF is also not well matched for problems in which the noise which cannot be approximated as additive and Gaussian. Additionally, the Kalman filter is inherently a single target estimator, so for the multiple target case sensor measurements must still be associated with a particular target, with multiple Kalman filters run in parallel.

Bayesian approaches in which the likelihood function or posterior probability are calculated numerically on a grid have been explored, as in [15]. However, large state spaces or multiple targets make the solution computationally intractable for real time estimation. The use of particle filters for state estimation have been explored as in [13] and [14], though problems remain with using particle filters for multiple target tracking, as well as accurately representing areas in the state space of low probability density.

Other work related to target detection and tracking has been done in distributed or decentralized Bayesian calculation state estimation. Rao and Durrant-Whyte have considered decentralized Kalman filters [12], however this method requires full communication across the sensor network of each local state estimate at each time step. Additionally, the local state space is identical to the global one, so there is no computational savings for decentralization. Direct extension of this work to non-linear, non-Gaussian measurements would require communication and processing ability beyond the capacity of most sensor networks. More recently, Olfati-Saber has proposed
a distributed Kalman filter [10]. While this method reduces computation by calculating a local state estimation in a lower dimension than the global state space, it requires linear measurements in parallel or orthogonal dimensions.

We approach the problem of combining multi-modal sensor measurements from the Bayesian perspective of likelihood. We are motivated by the likelihood principle, which allows the statement that all information on the state space available from the sensors is included in the likelihood function, assuming proper modeling of sensors, noise, and target motion. As calculation of the likelihood function is computationally intense for practical problems, we explore the use of distributed calculation to approximate the likelihood function.

The likelihood function is inherently single target, as there can be only one “true” state. While it can be directly extended to $n$ targets if the number of targets is known to be exactly $n$, this requires the dimension of the full state space to be $nM$, where $M$ is the dimension of the state space for a single target. As this is impractical for most applications, we view the likelihood function on an $M$ dimensional state space as a suboptimal representation of the multi-target state space. As scaling of the likelihood function becomes arbitrary in the multi-target case, we evaluate the use of the likelihood ratio as a method to normalize the likelihood function to produce a representation that is approximately the sum of the likelihood ratios for the single target case. This approach intends to achieve near-optimal performance in target detection and tracking while keeping computation and communication costs to a tractable amount suitable for implementation in a wireless sensor network.

The rest of this thesis is organized as follows: In Chapter 2 we discuss the properties of the likelihood function, and present and analyze a distributed calculation
to approximate it. In Chapter 3 we discuss the likelihood ratio, and its usefulness to both single and multiple target state estimation. In Chapter 4 we present several sensor models used in simulation and on measured data, and discuss two motion models and the resulting state prediction equations. Chapter 5 presents results from simulations that give insight into cases where methods discussed in Chapters 2 and 3 may be useful. Chapter 6 shows results from an experiment combining measured radar, video, and seismic data to detect walking and running persons.
CHAPTER 2

Likelihood as a Method of Sensor Fusion

Combining information from multiple sensors is a challenging problem, particularly for sensors of different modalities and arbitrary measurement geometries. This challenge is increased for sensors that do not take measurements in spatial coordinates. Fortunately, Bayesian statistics provides a framework to link sensor measurements to the underlying “state” of nature. This then allows information to be combined in a rigorous manner, properly fusing sensor measurements across sensor modalities and time.

For our formulation of the problem we assume a standard state-space model of the form:

\[ x_k = h(x_{k-1}, n_p) \quad y_k = g(x_k, n_m) \]  \quad (2.1)

where \( x_k \in X_k \subset \mathbb{R}^N \) is the \( N \)-dimensional state vector at time instant ‘\( k \)’, \( y_k \in Y_k \subset \mathbb{R}^L \) is the \( L \)-dimensional vector all sensor measurements taken at time ‘\( k \)’, \( h() \) is a possibly non-linear state transition model, \( g() \) is a possibly non-linear measurement model, \( n_p \) is process noise, and \( n_m \) is measurement noise. Thus current sensor measurements are only based on the current state of the system and measurement noise, and the current state of the system is only dependent on the previous state and process (or state transition) noise. It is assumed that the state transition and measurement
models are known, and that the distributions of the process and measurement noises are known.

2.1 Likelihood Maps

Bayesian statistics provides us with an optimal way to combine sensor data using the likelihood function $f(y|x)$. This estimation is optimal in the sense that it contains all relevant information on the state $x$ available from the sensor measurements $y$, assuming proper probabilistic modeling of sensor measurements and target motion. This statement is based on the likelihood principle, which is stated by James Berger as, “In making inferences or decisions about $x$ after $y$ is observed, all relevant experimental information is contained in the likelihood function for the observed $y$. Furthermore, two likelihood functions contain the same information about $x$ if they are proportional to each other (as functions of $x$)” [2]. This principle allows us to combine sensor measurements in a cohesive, optimal framework.

We define the instantaneous likelihood to be $f(y_k|x_k)$. The total likelihood is defined as the likelihood of all sensor measurements up to time ‘$k$’ given the current state $x_k$. Note that for independent noise across time and sensor measurements,

$$f(y_{1:k}|x_k) = \prod_{i=1}^{k} f(y_i|x_k)$$  \hspace{1cm} (2.2)

where $y_{1:k}$ is the vector of all sensor measurements for all time up to ‘$k$’. We are primarily interested in the total likelihood, as it contains all available information from both current and past measurements. For the single target case, note that the total likelihood is proportional to the posterior probability density function (pdf) $f(x_k|y_{1:k})$ assuming the prior distribution $f(x_k)$ is uniformly distributed.
It is possible to update the total likelihood sequentially as new measurements arrive. To see this, note that under the assumption of conditional independence of the sensor measurements and laws of total probability:

\[
f(y_1, y_2|x_2) = f(y_1|x_2)f(y_2|x_2) = f(y_2|x_2) \int_{-\infty}^{\infty} f(y_1|x_1)f(x_1|x_2)dx_1
\]

The same holds for the third time step:

\[
f(y_1, y_2, y_3|x_3) = f(y_1, y_2|x_3)f(y_3|x_3) = f(y_3|x_3) \int_{-\infty}^{\infty} f(y_1, y_2|x_2)f(x_2|x_3)dx_2
\]

By induction, we can conclude that the total likelihood is given by

\[
f(y_{1:k}|x_k) = f(y_k|x_k) \int_{-\infty}^{\infty} f(y_{1:k-1}|x_{k-1})f(x_{k-1}|x_k)dx_{k-1}
\]

This is significant, in that the current total likelihood is dependent only on the current instantaneous likelihood \(f(y_{1:k}|x_k)\), the previous total likelihood \(f(y_{1:k-1}|x_{k-1})\), and a motion, or state transition model \(f(x_{k-1}|x_k)\). In this sense it is quite similar to the structure of the Kalman filter, in which the the next state is predicted and then updated as new measurements arrive.

Figure 2.1 shows the structure of the likelihood update described in (2.7). As both likelihood maps and Kalman filters are optimal estimates given available data, for a single target with linear motion and measurement models and additive Gaussian noise, the estimates are identical.

While sequentially updating the total likelihood reduces computation versus complete recalculation, the computational load is still significant, particularly for large dimensional state spaces. For non-linear processes or non-Gaussian noise, there is
generally no closed form expression to evaluate the integral in (2.7). It must then be evaluated numerically on a grid in the state space. Computing this integral numerically on an $n$-point grid requires on the order of $n^2$ multiply-accumulates. Consider the a simple example of a 4-dimensional state space with 2-dimensional position and velocity, $[x_1 \ x_2 \ v_{x_1} \ v_{x_2}]^T$ with 100 points on the grid in each dimension. There are then $10^8$ points in this state space, and $10^{16}$ multiply-accumulate computations are needed to update the state estimate via grid approximation of (2.7). A 1-petaflops supercomputer would take at least 10 seconds to complete this computation. This number can be greatly reduced for most applications, because in general the state transition model assumes a very low probability of large jumps in the state space. This allows the numerical integration to be calculated only in a local region around the expected location of each point, $E(x_k|x_{k-1})$. However, even with this approximation the computation is still generally too large to be calculated in real time for most applications.

Calculating the total likelihood using a particle filter is a promising method, however problems remain in keeping sufficient particle diversity with multiple targets in
the state space as well as reliably locating targets entering the scene. In addition to computation costs, any method of centrally calculating the total likelihood requires sensor measurements or instantaneous likelihoods to be transmitted to the fusion center at each iteration. This imposes a heavy load on the communication resources (bandwidth and energy) of the nodes in the sensor network.

2.2 Distributed Calculation of Likelihood Maps

The high computational and communication costs of centrally calculating the total likelihood motivates the use of a distributed approximation of central calculation. Here we look to exploit the fact that frequently the dimension in which a sensor’s measurements are taken is lower than that of the global state space of interest. An example would be an omni-directional sensor whose measurement informs only in the single range dimension from the sensor to the object being measured. Define this lower \((M < N)\) dimensional state space as \(Z_{i,k} \subset \mathbb{R}^M\) with points in the state space defined by the \(M\)-dimensional vector \(z_{i,k}\). There exists some known transform between the two spaces:

\[
z_{i,k} = A_i(x_k)
\]  

(2.8)

The information provided by the sensor is limited to a subspace of the global state space. At each sensor or local group of sensors, a local likelihood map, \(f(y_{i,1:k}|z_{i,k})\), can be calculated on this lower-dimensional state space. The global likelihood map is then approximated by projecting the local likelihood map onto the higher dimensional global likelihood map and multiplying:

\[
f(y_{1:k}|x_k) \approx \prod_{i=1}^{S} [f(y_{i,1:k}|A_i(x_k))]
\]  

(2.9)
where $S$ is the total number of sensors, and $\mathbf{y}_{i,1:k}$ is the vector of all sensor measurements from sensor ‘$i$’ for all time up to ‘$k$’. The computation is not identical to the centrally calculated likelihood map except under certain conditions which are discussed in section 2.3. However, in many cases it is a good approximation that may be useful for target tracking, as will be shown in Chapters 5 and 6.

### 2.3 Analysis of Likelihood Approximation

As local calculation of likelihood maps in (2.9) is an approximation of central calculation in (2.7), it is desirable to understand the sources of error and give conditions for which the approximation is exact. Toward this goal, we consider a simple case of a non-linear measurement model with additive Gaussian noise. Using one-dimensional range measurements from multiple sensors, we wish to estimate the two-dimensional position of a target. We define the state space $X_k \subset \mathbb{R}^2$ with points in the space defined by vector $\mathbf{x}_k = [x_{1;k} \ x_{2;k}]^T$. Additionally we define the local state space for sensor ‘$i$’ as $Z_{i;k} \subset \mathbb{R}$ with points defined by $z_{i;k}$. The measurement vector at each time step is the set of sensor measurements, $\mathbf{y}_k = [y_{1;k} \ y_{2;k}]^T$.

Here we analyze the two sensor, two time-step case. For simplicity we define our axes based on one of the intersection points of the first measurements $y_{1;1}$ and $y_{2;1}$ as shown in Figure 2.2. Using the far field approximation, we assume the curves around the intersection point are linear. The measurement made at sensor ‘$i$’ and time step ‘$k$’ is modeled as shown below,

$$y_{i;k} = z_{i;k} + n_{i;k} \quad n_{i;l} \sim N(0, \sigma_n) \quad (2.10)$$

where $z_{i;k}$ is the distance from the target to the sensor. Knowledge of sensor positions $s_1$ and $s_2$ gives $z_{i;k} = A_i(\mathbf{x}_k)$. As we define our axes based on the first measurements,
Figure 2.2: Sensor Location and Axes Definition for Distributed Likelihood Analysis

$A_i(x_k)$ is given by

$$A_1(x_k) = x_{1;k} + y_{1;1} \hspace{1cm} (2.11)$$

$$A_2(x_k) = x_{1;k} \cos(\theta) + x_{2;k} \sin(\theta) + y_{2;2} \hspace{1cm} (2.12)$$

This allows a translation from the local likelihood to the global likelihood:

$$f(y_{i;k}|x_k) = f(y_{i;k}|A_i(x_k)) \hspace{1cm} (2.13)$$

For this example we further assume Brownian motion, giving the motion update equation:

$$x_{k+1} = x_k + n_p \hspace{0.5cm} n_p \sim N \left( 0, \begin{bmatrix} \sigma^2_x & 0 \\ 0 & \sigma^2_x \end{bmatrix} \right) \hspace{1cm} (2.14)$$

For the first time step the centrally calculated (equation 2.7) and locally calculated (equation 2.9) likelihood maps are identical,

$$f(y_{1}|x_1) = f(y_{1;1}|x_1)f(y_{2;1}|x_1) = f(y_{1;1}|A_1(x_1))f(y_{2;1}|A_2(x_1)) \hspace{1cm} (2.15)$$
For the second time step it can be shown that the centrally calculated likelihood is given by,

\[
f(y_{1:2}|x_2) = Ce^{-\frac{(\sigma_n^2 \sin^2 \theta (y_{1:2}^2 + x_{2:2}^2) + \sigma_m^2 (y_{1:2}^2 + (y_{1:2} \cos \theta + x_{2:2} \sin \theta)^2))}{2(\sigma_n^2 \sin^2 \theta + 2\sigma_m^2 \sigma_n^2 + \sigma_m^4)}}
\]

and the locally calculated likelihood can be shown to be,

\[
f(y_{1:2}|x_2) f(y_{2:1:2}|x_2) = Ce^{-\frac{-(x_{1:2}^2 + (x_{1:2} \cos \theta + x_{2:2} \sin \theta)^2)}{2(\sigma_m^2 + \sigma_n^2)}}
\]

Note that the second two terms in (2.16) and (2.17) are identical. While the first terms are not the same, there are several interesting similarities. For the asymptotic case where \(\sigma_n^2 \gg \sigma_m^2\), the two are asymptotically identical, and both expressions reduce to:

\[
f(y_{1:2}|x_2) = f(y_{1:2}|x_2) f(y_{2:1:2}|x_2) = Ce^{-\frac{-(y_{1:2}^2 + (y_{1:2} \cos \theta + x_{2:2} \sin \theta)^2)}{2\sigma_n^2}}
\]

This shows that in the case where the update frequency is high (\(\sigma_m^2\) small), the locally calculated likelihood will be very close to the centrally calculated one. For the opposite asymptotic case in which \(\sigma_m^2 \gg \sigma_n^2\), previous measurements provide essentially no information for calculating the current likelihood. As the instantaneous likelihood is identical for local and central calculation, the result is also asymptotically the same:

\[
f(y_{1:2}|x_2) = f(y_{1:2}|x_2) f(y_{2:1:2}|x_2) = Ce^{-\frac{(y_{1:2}^2 + (y_{1:2} \cos \theta + x_{2:2} \sin \theta)^2)}{2\sigma_n^2}}
\]
Additionally, two special cases exist when $\theta = 0$ and $\theta = \frac{\pi}{2}$. For $\theta = \frac{\pi}{2}$, the two sensors are essentially measuring two orthogonal spaces. Since the motion model is independent for $x_{1,k}$ and $x_{2,k}$, local calculation is identical to central calculation of the likelihood map:

$$f(y_{1:2}|x_2) = f(y_{1:1:2}|x_2) f(y_{2:1:2}|x_2) = Ce^{-\frac{(x_{1,2}^2 + x_{2,2}^2)}{2\sigma_n^2}} \times e^{-\frac{(y_{1,2}-(x_{1,2}+y_{1,1}))^2}{2\sigma_n^2}} \times e^{-\frac{(y_{2,2}-(x_{1,2}\cos\theta + x_{2,2}\sin\theta + y_{2,1}))^2}{2\sigma_n^2}} \times e^{-\frac{(y_{1,2}-(x_{1,2}+y_{1,1}))^2}{2\sigma_n^2}}$$

(2.20)

For $\theta = 0$, where the sensors are co-linear with the estimate of the target’s position, we find the two likelihoods to be very similar in form although not identical.

$$f(y_{1:2}|x_2) = Ce^{\frac{(x_{1,2}^2)}{\sigma_n^2}} \times e^{-\frac{(y_{1,2}-(x_{1,2}+y_{1,1}))^2}{2\sigma_n^2}} \times e^{-\frac{(y_{2,2}-(x_{1,2}\cos\theta + x_{2,2}\sin\theta + y_{2,1}))^2}{2\sigma_n^2}}$$

(2.21)

$$f(y_{1:1:2}|x_2)f(y_{2:1:2}|x_2) = Ce^{\frac{(x_{1,2}^2)}{\sigma_n^2}} \times e^{-\frac{(y_{1,2}-(x_{1,2}+y_{1,1}))^2}{2\sigma_n^2}} \times e^{-\frac{(y_{2,2}-(x_{1,2}\cos\theta + x_{2,2}\sin\theta + y_{2,1}))^2}{2\sigma_n^2}}$$

(2.22)

These expressions are identical except the centrally calculated expression has twice the motion variance of the locally calculated. This means that the locally calculated likelihood reports a falsely low motion variance (by a factor of 2). Replacing the local motion variance $\sigma_m^2$ with $2\sigma_m^2$ makes the two identical. This approach generalizes to the case of $n$ co-linear sensors. Replacing the local motion variance with $n\sigma_m^2$ results in perfect approximation. This result is similar to that found by Olfati-Saber in his paper on distributed Kalman filtering [10] in which $M_\mu = nM$ where $M$ is the central Kalman gain, and $M_\mu$ is the distributed (micro) Kalman gain.

For the case where $0 < \theta < \pi/2$, the difference between local and central calculation cannot be completely corrected, since the covariance matrix in the first term of
(2.16) cannot be matched by (2.17) for any chosen $\sigma_m^2$. To see this, recall that the second two terms in (2.16) and (2.17) are the same, and note that the first term in (2.16) can be written in matrix form as:

$$\exp\left(-\frac{1}{2} x_2^T \Sigma^{-1} x_2\right)$$

(2.23)

where $\Sigma^{-1}$ is the inverse of the covariance matrix of the centrally calculated likelihood given by,

$$\Sigma^{-1}_C = \begin{bmatrix}
\sigma_m^2 \sin^2(\theta) + \sigma_n^2 \cos(\theta) & \sigma_m^2 \sin(\theta) \cos(\theta) \\
(\sigma_m^2 + \sigma_n^2) \sin(\theta) & \sigma_m^2 \sin^2(\theta) + \sigma_n^2 \\
(\sigma_m^2 + \sigma_n^2) & 1
\end{bmatrix}$$

(2.24)

and the first term in (2.17) can be written as:

$$\exp\left(-\frac{1}{2} x_2^T \Sigma^{-1} x_2\right)$$

(2.25)

where $\Sigma^{-1}_D$ is the inverse of the covariance matrix of the distributed calculation of the likelihood given by,

$$\Sigma^{-1}_D = \begin{bmatrix}
1 + \cos^2(\theta) & \sin(\theta) \cos(\theta) \\
(\sigma_m^2 + \sigma_n^2) & \sin^2(\theta)
\end{bmatrix}$$

(2.26)

It can be easily seen that no choice of a local $\sigma_m$ can result in perfect approximation of the centrally-calculated likelihood in the general case. There are three equations to be equalized (Two position variances and a covariance) and only a single variable. Only in the special cases where the three equations are linearly dependent is this possible, as discussed above.

We do however, wish to give some bounds on the maximum error for this special case. For this we use the Rao distance, a measurement based on the Fisher-Rao metric, which is calculated from the Fisher information matrix. For p-variate normal distributions with identical mean vector $\mu$ and covariance matrices $\Sigma_C$ and $\Sigma_L$, [14]
Atkinson and Mitchell show in [1] that the Rao distance is given by

$$s^2(\mu, \Sigma_C, \Sigma_D) = \frac{1}{2} \sum_{i=1}^{p} (\log \lambda_i)^2$$  \hspace{1cm} (2.27)

where \(\lambda_i\) are the generalized eigenvectors of \(\Sigma_C\) and \(\Sigma_D\) given by the solutions to the equation

$$|\Sigma_C - \lambda \Sigma_D| = 0$$  \hspace{1cm} (2.28)

While the solution to this equation is not tractable for general \(\theta\), simulations to compute the Rao distance numerically showed that for all \(\sigma_n\) and \(\sigma_m\), the Rao distance is greatest when \(\theta = \epsilon \approx 0\) (Note that the Rao distance is undefined for \(\theta = 0\)). For this case, it is possible to show that the Rao distance is given by

$$s^2_{\theta=\epsilon}(\mu, \Sigma_C, \Sigma_D) = \frac{1}{2} \log^2 \left( \frac{\sigma_n^2}{\sigma_{mD}^2 + \sigma_n^2} \right) + \frac{1}{2} \log^2 \left( \frac{2\sigma_m^2 + \sigma_n^2}{\sigma_{mD}^2 + \sigma_n^2} \right)$$  \hspace{1cm} (2.29)

where \(\sigma_{mD}^2\) is the chosen local motion variance. It can be further shown that the value of \(\sigma_{mD}^2\) that minimizes this distance is given by

$$\sigma_{mD}^2 = \sigma_n^2 \left( \sqrt{1 + \frac{2\sigma_m^2}{\sigma_n^2}} - 1 \right)$$  \hspace{1cm} (2.30)

Note that this minimizes the maximum Rao distance, but does not minimize the error for the general case where \(\theta > \epsilon\).
CHAPTER 3

Likelihood Methods for Multi-Target Tracking

For cases where it is known that there is exactly one target in the state space, the likelihood map is an optimal estimation of the state space. Using Bayes theorem, at any point the likelihood function can be scaled to a probability density function by multiplying by a prior density on $x$ and scaling such that the total probability is 1:

$$f(x|y) = \frac{f(y|x)f(x)}{\int_{-\infty}^{\infty} f(y|\alpha)f(\alpha)d\alpha}$$  \hspace{1cm} (3.1)

However, for the case where the number of targets is unknown, Bayes theorem does not directly apply. The likelihood function alone is an arbitrary scaling, and essentially gives no information about the actual probability of a target being at any location in the state space. Rather, it gives a measure of the relative probability of the location of objects in the space.

Our goal is a scaling of the likelihood function such that the result is informative as to the number of targets in the scene, and their location and distribution. Thus the representation for multiple targets should be approximately the sum of the individual representations for a single target. Additionally, when there are no targets in the scene, the representation should tend toward zero. To properly scale the likelihood function, we investigate both the use of the likelihood ratio, and calculation of a posterior probability density for multiple targets.
3.1 Likelihood Ratio for the Single or No Target Case

We first examine the likelihood ratio for the case where it is known that there will be only zero or one targets in the state space. The likelihood ratio is a classic tool for examining two hypotheses with the purpose of deciding between them [11]. For each unit area in the state space, we define the two hypotheses as $H_{1x_k}$, a target is at a point in the state space $x_k$, and $H_{0x_k}$, a target is not at that point. For each of these hypotheses, there is a corresponding probability distribution function on the sensor signals received. Using the ratio of these two functions and including any prior information, we can calculate which hypothesis is more likely. The likelihood ratio then effectively scales the total likelihood to give a measure of the probability that a target is at a particular location in the space. The total likelihood ratio $L_{x_k}(y_{1:k})$, calculated at point $x_k$ on sensor measurements $y_{1:k}$ is defined as:

$$L_{x_k}(y_{1:k}) = \frac{f(y_{1:k}|H_{1x_k})}{f(y_{1:k}|H_{0x_k})}$$ (3.2)

Note that the numerator in this equation is a change in notation from the previous chapter. Here we use $H_{1x_k}$ to designate that a target is at the point $x_k$. The expressions $f(y_{1:k}|H_{1x_k})$ and $f(y_{1:k}|x_k)$ are the same. The $H_i$ terminology is added to distinguish a location from target existence at the location. The calculation of the numerator in this equation has thus been discussed previously in Chapter 2.

The denominator can be broken down as follows:

$$f(y_{1:k}|H_{0x_k}) = f(y_k|H_{0x_k})f(y_{1:k-1}|H_{0x_k})$$ (3.3)

To calculate the first term we implicitly assume, similarly to the instantaneous likelihood in the numerator, that $f(y_k|H_{0x_k})$ is the likelihood of the sensor measurements
given no target at point \( x_k \) and \( \text{no target anywhere else in the state space} \), essentially assuming that the signal is composed only of noise. While this assumption may seem unreasonable, in the context of the likelihood ratio, this assumption in both the numerator and denominator exactly cancel one another out. Consider the following generic likelihood ratio with time indices dropped, where \( H_{0_{x \cap}} \) denotes that no targets exist in the complement of \( x \):

\[
\frac{f(y|H_{1_{x \cup}}, H_{0_{x \cap}})}{f(y|H_{0_{x \cup}}, H_{0_{x \cap}})} = \frac{f(H_{1_{x \cup}}, H_{0_{x \cap}} | y) P(H_{0_{x \cup}}, H_{0_{x \cap}})}{P(H_{0_{x \cup}} | y) f(H_{1_{x \cup}}, H_{0_{x \cap}})} \frac{f(y)}{f(y|H_{0_{x \cup}})}
\]

(3.4)

The second term in (3.3) can be separated by total probability into the case where a target was present somewhere in the state space at the previous time step, and the case where no target was present anywhere:

\[
f(y_{1:k-1}|H_{0_{x_k}}) = P(H_{1_{k-1}} | H_{0_{x_k}}) f(y_{1:k-1}|H_{1_{k-1}}, H_{0_{x_k}}) + (1-P(H_{1_{k-1}} | H_{0_{x_k}})) f(y_{1:k-1}|H_{0_{k-1}}, H_{0_{x_k}})
\]

(3.5)

Here \( P(H_{1_{k-1}} | H_{0_{x_k}}) \approx P(H_{1_{k-1}}) \) where \( P(H_{1_{k}}) \) is the probability that a target existed somewhere in the state space at time ‘\( k \)’. This is a prior probability that must be chosen. Terms in equation (3.5) can be further separated as follows:

\[
f(y_{1:k-1}|H_{1_{k-1}}, H_{0_{x_k}}) = \int_{-\infty}^{\infty} f(y_{1:k-1}|x_{k-1}) f(x_{k-1}|H_{0_{x_k}}) dx_{k-1}
\]

(3.6)

and

\[
f(y_{1:k-1}|H_{0_{k-1}}, H_{0_{x_k}}) \approx f(y_{1:k-1}|H_{0_{k-1}})
\]

(3.7)

as discussed above. The remaining term to calculate is then \( f(x_{k-1}|H_{0_{x_k}}) \) the likelihood that a target was at some point in the state space in the previous time step.
given that there is nothing at another point in the current time step. This term can be accurately approximated by \( f(x_{k-1}) \), a chosen prior density on \( x_{k-1} \):

\[
f(x_{k-1}|H_{0k}) \approx f(x_{k-1}) \tag{3.8}
\]

Equation 3.6 can now be written as,

\[
f(y_{1:k-1}|H_{1k-1}, H_{0k}) = f(y_{1:k-1}|H_{1k-1}) = \int_{-\infty}^{\infty} f(y_{1:k-1}|x_{k-1}) f(x_{k-1}) dx_{k-1} \tag{3.9}
\]

Equation 3.5 can then be viewed as representing the pdf of \( y_{1:k-1} \):

\[
f(y_{1:k-1}|H_{0k}) = f(y_{1:k-1}) = P(H_{1k-1}) f(y_{1:k-1}|H_{1k-1}) + (1-P(H_{1k-1})) f(y_{1:k-1}|H_{0k-1})
\]

Thus the total likelihood ratio for zero or one targets is given by,

\[
L_{x_k}(y_{1:k}) = \frac{f(y_{k}|x_k) f(y_{1:k-1}|x_k)}{f(y_{k}|H_{0k}) f(y_{1:k-1})} \tag{3.10}
\]

### 3.2 Likelihood Ratio for an Unknown Number of Targets

While knowledge that there is exactly zero or one target in the state space is sometimes available, for most tracking scenarios the total number of targets is unknown. Here we investigate the use of the likelihood ratio for this case.

As there may be more than one target in the state space, the likelihood function is no longer an optimal combination of sensor measurements. Rather, it is approximately the product of the likelihood functions for individual targets. The likelihood ratio provides a way to normalize this multi-target likelihood function to produce a representation that is approximately a weighted sum of what their single target likelihood ratios would be, as well as a mechanism for declaring target detections.

In Section 3.1, it was assumed that there was zero or one target in the state space. As here this assumption is not possible, the calculation of the likelihood ratio in (3.2)
is slightly different. Instead we assume that there is a region in the state space $R(x)$ around each point where the likelihood is associated primarily with no more than a single target. In other words, we assume that within a specified region around a point, there is exactly zero or one target and the likelihood in the region is not significantly increased by the presence of targets outside this region. This region is chosen such the probability that a target existed in the region in the previous time step, given the presence of a target at point $x_k$, is approximately 1,

$$P(H_1R(E[x_{k-1}|x_k])|x_k) \approx 1$$ (3.11)

where $E[x_{k-1}|x_k]$ is the expected value of $x_{k-1}$ given $x_k$, which is given by the motion model, and $P(H_1R(E[x_{k-1}|x_k])|x_k)$ is the probability that a target existed in the specified region at time ‘$k - 1$’, given a target at point $x_k$.

This assumption does not affect the calculation of the likelihood function in the numerator in (3.2), whose calculation remains the same as discussed previously in Chapter 2. The calculation of the denominator $f(y_{1:k}|H_0x_k)$ can be split as before in (3.3) into an instantaneous likelihood given nothing at $x_k$ and a “predicted likelihood” of the past given nothing at $x_k$. The second term in (3.3) can now be written as:

$$f(y_{1:k-1}|H_0x_k) = P(H_1R(E[x_{k-1}|x_k])|H_0x_k)f(y_{1:k-1}|H_1R(E[x_{k-1}|x_k]), H_0x_k)$$

$$+ (1 - P(H_1R(E[x_{k-1}|x_k])|H_0x_k))f(y_{1:k-1}|H_0R(E[x_{k-1}|x_k]), H_0x_k)$$ (3.12)

where $P(H_1R(E[x_{k-1}|x_k])|H_0x_k)$ is a prior probability that an object exists within the region $R(E[x_{k-1}|x_k])$ given no target exists at $x_k$. This value can accurately be approximated as,

$$P(H_1R(E[x_{k-1}|x_k])|H_0x_k) \approx P(H_1R(E[x_{k-1}|x_k]))$$ (3.13)
As in Section 3.1, terms in (3.12) can be further separated as follows:

\[
f(y_{1:k-1}|H_1R(E[x_{k-1}|x_k]), H_{0x_k}) = \int_{R(E[x_{k-1}|x_k])} f(y_{1:k-1}|x_{k-1})f(x_{k-1}|H_{0x_k})dx_{k-1}
\]  

(3.14)

and

\[
f(y_{1:k-1}|H_0R(E[x_{k-1}|x_k]), H_{0x_k}) = f(y_{1:k-1}|H_{0x_k-1})
\]  

(3.15)

Again, the term \( f(x_{k-1}|H_{0x_k}) \) in (3.14) can be approximated as,

\[
f(x_{k-1}|H_{0x_k}) \approx f(x_{k-1})
\]  

(3.16)

which is a chosen prior density on \( x_{k-1} \). As before, equation 3.12 can be now viewed as being the pdf of \( y_{1:k-1} \):

\[
f(y_{1:k-1}|H_{0x_k}) = f(y_{1:k-1}) = P(H_1R(E[x_{k-1}|x_k]))f(y_{1:k-1}|H_{1R}(E[x_{k-1}|x_k]))
\]

\[+(1 - P(H_1R(E[x_{k-1}|x_k])))f(y_{1:k-1}|H_{0R}(E[x_{k-1}|x_k]))
\]  

(3.17)

The total likelihood ratio for multiple targets is then the same as for the case where there are zero or one target, except the calculation of \( f(y_{1:k-1}) \) requires the additional regional assumptions discussed above.

\[
L_{x_k}(y_{1:k}) = \frac{f(y_k|x_k)f(y_{1:k-1}|x_k)}{f(y_k|H_0)f(y_{1:k-1})}
\]  

(3.18)

While this method works well for separated targets with likelihoods of approximately the same order, there are some cases where objects of extremely high likelihood can spread into a neighboring region, artificially causing the likelihood ratio at the smaller target to be significantly reduced. This is particularly apparent in sensors where targets can have small differences in distance, but large disparities in signal power. For example, in radar systems, target signal power drops off at a rate of \( r^4 \). A
target at 10 meters would have 16 times the return power of a similarly sized target at 20 meters. Depending on the measurement noise level, the likelihood for the first target could be many orders of magnitude higher than the more distant target, even though the separation may not be that great. To reduce this problem, we place a “soft” threshold on the likelihood ratio using an arctan() function. Thus at each iteration \( f(y_{1:k}|x_k) \) is transformed into \( \hat{f}(y_{1:k}|x_k) \) as follows:

\[
\hat{f}(y_{1:k}|x_k) = f(y_{1:k}|H_{0_k}) \frac{2\pi L_{\text{max}}}{\pi f(y_{1:k}|x_k)} \arctan \left( \frac{2L_{\text{max}} f(y_{1:k}|H_{0_k})}{\pi f(y_{1:k}|x_k)} \right)
\]

(3.19)

where \( L_{\text{max}} \) is the maximum likelihood ratio allowed. The new term \( \hat{f}(y_{1:k}|x_k) \) is then used to replace \( f(y_{1:k}|x_k) \) in all calculations of the likelihood ratio for the next time step. This equation can also be viewed in terms of the likelihood ratio:

\[
\hat{L}_{x_k}(y_{1:k}) = \frac{2\pi L_{\text{max}}}{\pi} \arctan \left( \frac{\pi L_{x_k}(y_{1:k})}{2L_{\text{max}}} \right)
\]

(3.20)

Figure 3.1 shows the properties of this function. For

\[
L_{x_k}(y_{1:k}) < \frac{L_{\text{max}}}{10}
\]

the function can be viewed as 1 : 1 linear. For

\[
L_{x_k}(y_{1:k}) > \frac{L_{\text{max}}}{10}
\]

the slope of the function begins to reduce asymptotically as \( L_{x_k}(y_{1:k}) \to \infty \) and \( \hat{L}_{x_k}(y_{1:k}) \to L_{\text{max}} \). This “soft” threshold is preferable to a “hard” limiting threshold in that it still allows finding the most likely point of a target’s location, rather than a region of clipped likelihood ratios. The maximum likelihood ratio \( L_{\text{max}} \) can be viewed as a method to reduce “domination” of the state space by a single (or multiple) heavily weighted single target likelihood functions. It can also be viewed as a method
to reduce the “certainty” of the sensor and motion models. It prevents the likelihood ratio of being too high, or too “sure” about its estimate.

3.3 Bayesian Formulation for a Single Target

In this section we consider the problem of sequentially estimating the probability of existence of at most one target over the state space from a Bayesian point of view. We assume a binary hypothesis model: The null hypothesis corresponds to the target being outside the scene of interest, the composite alternative is one target exists in the scene parametrized at state $x$. We assume prior to any sensor measurements the
probability of existence of the target outside the state space is given by \( P(H_0|y_0) \) and the distribution of the target probability over the state space is given by the density \( f(x_0|y_0) \). We note that the density function \( f(x_0|y_0) \) is non-negative and integrates over the state space due to complement hypothesis \( H_0 \), i.e.:

\[
\int f(x_0|y_0)dx_0 = 1 - P(H_1|y_0). \tag{3.21}
\]

The discrete time probability model is completed by specifying the transition probabilities between the two hypothesis. We assume at each discrete time step the target can transition from \( H_0 \) into the state space with probability \( \delta_1 \) and can vanish from the state space into \( H_0 \) with probability \( \delta_0 \). Given a set of measurements \( y_{1:k} \) we can calculate the posterior probability \( P(H_0_k|y_{1:k}) \) and the posterior density \( f(x_k|y_{1:k}) \) over the state recursively using a combination and prediction and update steps:

- **Prediction:**

\[
f(x_k|y_{1:k-1}) = (1 - \delta_0) \int f(x_{k-1}|y_{1:k-1})f(x_k|x_{k-1})dx_{k-1}
+ \delta_1 P(H_{0k-1}|y_{1:k-1})g(x) \tag{3.22}
\]

\[
P(H_{0k}|y_{1:k-1}) = P(H_{0k-1}|y_{1:k-1})(1 - \delta_1) + (1 - P(H_{0k-1}|y_{1:k-1}))\delta_0 \tag{3.23}
\]

where \( g(x) \) is a distribution on \( x \) for a target entering the scene from \( H_0 \)

- **Update:**

\[
f(x_k|y_{1:k}) = \frac{f(y_k|x_k)f(x_k|y_{1:k-1})}{P(H_{0k}|y_{1:k-1})f(y_k|H_{0k}) + \int f(y_k|x)f(x|y_{1:k-1})dx} \tag{3.24}
\]

\[
P(H_{0k}|y_{1:k}) = \frac{P(H_{0k}|y_{1:k-1})f(y_k|H_{0k})}{P(H_{0k}|y_{1:k-1})f(y_k|H_{0k}) + \int f(y_k|x)f(x|y_{1:k-1})dx} \tag{3.25}
\]
We note that Equation (3.24) can be written as:

\[
    f(x_k|y_{1:k}) = \frac{f(y_k|x_k)f(x_k|y_{1:k-1})}{f(y_{1:k-1})} \tag{3.26}
\]

\[
    = \frac{f(y_k|x_k)f(y_{1:k-1}|x_k)}{f(y_{1:k-1})f(y_{1:k-1})} \tag{3.27}
\]

Which has the form of Equation (3.18), however the calculation of various terms in the two expressions are slightly different. Equations in Sections 3.1 and 3.2 assume a constant $P(H_{0k})$, which may not be a valid assumption in some cases. Extending this formulation to multiple targets has not been fully studied for this thesis, and is a topic for future research.
CHAPTER 4

Sensor and Motion Models

The basis for calculating total likelihood functions and total likelihood ratios involves calculating instantaneous likelihoods from sensor measurement models, and likelihood predictions from state transition and motion models. Here we give several sensor models used in both simulation and collection of real data, as well as their resulting likelihood functions. Additionally, we look at potential choices of motion models and derive likelihood prediction equations.

4.1 Sensor Measurement Models

Five sensor modalities are presented here, each of which is used in either simulations in Chapter 5 or the real data experiment in Chapter 6. For each modality, a model of the sensor signal is given, as well as the resulting likelihood function \( f(y|x) \) and the likelihood given no target in the state space \( f(y|H_0) \).

4.1.1 Radar

The noise in a range-doppler radar is not additive Gaussian, and thus presents a good example of the non-linear, non-Gaussian capability of likelihood maps. Each “bin” of a range-doppler map is a complex number. The phase of this complex number
is random and uniformly distributed. Additionally, the real and imaginary components each have additive, uncorrelated Gaussian noise. The measurement model is shown below for each bin ‘i’ in the range-doppler map.

\[ y_i = \sqrt{y_{R_i}^2 + y_{I_i}^2} = |y_{R_i} + jy_{I_i}| \]

\[ y_{R_i} = A_i(x) \cos(\alpha_i) + n_{R_i} \]

\[ y_{I_i} = A_i(x) \sin(\alpha_i) + n_{I_i} \]  \hspace{1cm} (4.1)

\[ n_{R_i}, n_{I_i} \sim N(0, \sigma_n^2) \quad \alpha_i \sim U(−\pi, \pi) \]

Here \( A_i(x) \) is a known function that maps position to amplitude in the range doppler map and \( U(−\pi, \pi) \) is the uniform distribution between \(-\pi\) and \(\pi\). This function is dependent on the target, as well as on the transmitted waveform and bandwidth. As the noise on the real and complex channels can be modeled as additive Gaussian noise, the resulting distribution of the amplitude is Rician. It can be shown that the resulting distribution on \( y_i | x \) is of the form,

\[ f(y_i | x) = \frac{y_i}{2\pi\sigma_n^2} \exp\left(-\frac{y_i^2 + A_i^2(x)}{2\sigma_n^2}\right) \int_{-\pi}^{\pi} \exp\left(\frac{A_i(x)y_i \sin(\theta)}{\sigma_n^2}\right) d\theta \]  \hspace{1cm} (4.2)

which is the Rice distribution. This derivation is shown in Appendix A. For the no target case, \( A_i = 0 \) and the expression reduces to the Rayleigh distribution:

\[ f(y_i | H_0) = \frac{y_i}{2\pi\sigma_n^2} \exp\left(-\frac{y_i^2}{2\sigma_n^2}\right) \]  \hspace{1cm} (4.3)

4.1.2 Passive Energy Sensors

Passive energy sensors are sensors such as acoustic, seismic, or magnetic for which the signal decays with distance to the target, and can be modeled as having additive Gaussian noise. We model the received signal at sensor ‘i’ as follows.

\[ y_i = \sqrt{\frac{A}{\epsilon + r_i(x)^\alpha}} + n_i \quad n_i \sim N(0, \sigma_n^2) \]  \hspace{1cm} (4.4)
The distribution on $y_i|x$ is then

$$f(y_i|x) = \frac{1}{\sqrt{2\pi\sigma^2_n}} \exp\left(\frac{-\left(y_i - \mu_i\right)^2}{2\sigma^2_n}\right) \quad \mu_i = \sqrt{\frac{A}{\epsilon + r_i(x)^\alpha}}$$  \hspace{1cm} (4.5)

Here $A$ is the target amplitude, $r_i(x)$ is the distance from the target to the sensor, $\alpha$ is the signal decay rate, and $\epsilon$ is a term to prevent asymptotic behavior around $r_i = 0$. When no target is present, the signal is entirely noise and the likelihood is given by:

$$f(y_i|H_0) = \frac{1}{\sqrt{2\pi\sigma^2_n}} \exp\left(\frac{-y_i^2}{2\sigma^2_n}\right)$$  \hspace{1cm} (4.6)

### 4.1.3 Detection Sensors

Here we define detection sensors as those which pre-process their sensor measurements to form detections and give some estimate of a target’s location in its local frame of reference $z_i$. While more accurate models could be constructed for each modality of this type of sensor, here we give a simple model that can apply to several sensor modalities. The sensors are modeled as having some probability of detection $P_D$, which may vary with location. When a target is detected, its position estimate is calculated as having zero mean additive Gaussian noise with covariance matrix $\Sigma$.

To allow for the possibility of multiple detections, the sensor is also assumed to have a constant probability of false alarm, $P_{FA}$ which is uniformly distributed in the local state space. Examples of this type of sensor would be a video sensor that detects targets within its field of view and gives an estimate of their position, or a range-doppler radar which pre-processes its measurements to give estimates of range and radial velocity of a target. We treat each “detection” as a separate sensor measurement. The
model of a detection sensor is as follows:

\[
y_i = \begin{cases} 
    z_i(x) + n & \text{w.p. } P_D \\
    \text{‘No Detection’} & \text{w.p. } 1 - (P_D + P_{FA}) \\
    u & \text{w.p. } P_{FA}
\end{cases}
\]  

(4.7)

\[n \sim N(0, \Sigma) \quad u \sim U(Z_i)\]

The likelihood is then given by

\[
f(y_i|\mathbf{x}) = \begin{cases} 
    \frac{P_D}{(2\pi^{M/2} |\Sigma|^2)^{1/2}} \exp \left( -\frac{1}{2} (y_i - z_i(x))^\top \Sigma^{-1} (y_i - z_i(x)) \right) + \frac{P_{FA}}{Z_i} & y \in \mathbb{R}^M \\
    1 - (P_D + P_{FA}) & y = \text{‘No Detection’}
\end{cases}
\]  

(4.8)

The likelihood with no target present is dependent only on the rate of false alarm, and is given by:

\[
f(y_i|H_0) = \begin{cases} 
    \frac{P_{FA}}{Z_i} & y \in \mathbb{R}^M \\
    1 - P_{FA} & y = \text{‘No Detection’}
\end{cases}
\]  

(4.9)

### 4.1.4 Binary Detection Sensors

Similarly to detection sensors, binary sensors pre-process sensor measurements to produce either a 1 (detection) or 0 (no detection). These sensors are distinct in that they make no estimate of a target’s location, and generally have very short range. There is some known probability of detection \(P_D\) which may be dependent on distance to target, and a fixed, known probability of false alarm \(P_{FA}\). We model \(P_D\) as decaying with distance. The signal at sensor ‘\(i\)’ is given by:

\[
y_i|x = \begin{cases} 
    1 & \text{w.p. } P_D + P_{FA} - P_DP_{FA} \\
    0 & \text{w.p. } 1 - (P_D + P_{FA} - P_DP_{FA})
\end{cases}
\]  

(4.10)

with \(P_D = 1 - \Phi(\frac{r_i(x) - \mu_d}{\sigma_d})\) where \(\Phi()\) is the normal cumulative distribution function, \(r_i(x)\) is the range to the target, \(\mu_d\) is the mean detection distance, and \(\sigma_d\) is the detection range uncertainty. This gives a likelihood function in the form:

\[
f(y_i|x) = \begin{cases} 
    1 - \Phi(\frac{r_i(x) - \mu_d}{\sigma_d}) & y = 1 \\
    \Phi(\frac{r_i(x) - \mu_d}{\sigma_d}) & y = 0
\end{cases}
\]  

(4.11)
The likelihood function given no targets is:

\[ f(y_i|H_0) = \begin{cases} P_{FA} & y = 1 \\ 1 - P_{FA} & y = 0 \end{cases} \]  
(4.12)

Although the sensors do not directly measure range or velocity, the measurement model in combination with a motion model can be used to find global or local total likelihood functions for both range and velocity.

### 4.1.5 Bearing Only Sensors

Bearing only sensors are similar to binary detection sensors, except that the sensor also makes a bearing estimate. An example of this type of sensor would be an acoustic or seismic array where beam forming is used to estimate signal angle of arrival. It is assumed that there is some known \( P_{FA} \) which is omni-directional, and some known \( P_D \). Error in angular measurement is assumed to be Gaussian and independent of distance. The sensor measurement model is given below for the presence of a target at \( x \):

\[ y_i|x = \begin{cases} \theta(x) + n_\theta & \text{w.p. } P_D \\ \text{‘No Detection’} & \text{w.p. } 1 - (P_D + P_{FA}) \\ n_{FA} & \text{w.p. } P_{FA} \end{cases} \]

\( n_\theta \sim N(0, \sigma_\theta^2) \)

\( n_{FA} \sim U(-\pi, \pi) \)

This results in a likelihood function of the form:

\[ f(y_i|x) = \begin{cases} \frac{P_D}{\sqrt{2\pi}\sigma_\theta^2} \exp\left(\frac{-(\theta(x)-y)^2}{2\sigma_\theta^2}\right) + \frac{P_{FA}Z_i}{y \in [-\pi, \pi]} \\ 1 - (P_D + P_{FA}) \end{cases} \]

(4.13)

For no target present, the probability of false alarm is uniformly in \([-\pi, \pi]\), and the likelihood is given by:

\[ f(y_i|H_0) = \begin{cases} \frac{P_{FA}Z_i}{y \in [-\pi, \pi]} \\ 1 - (P_{FA}) \end{cases} \]

(4.15)
4.2 Motion Models

To properly combine sensor data over time, some target motion model must be assumed. From these models, predictions of target location and state can be formed given a current location and state. Here we look at two simple but useful motion models for tracking of ground targets.

4.2.1 Gaussian Motion Model

While not applicable to all situations, the Gaussian motion model is nonetheless commonly used for target tracking. It is used in some form for state prediction in all Kalman filters, and is used in all simulations in chapter 5, and in the likelihood calculations on real data in chapter 6.

Most state space examples in this thesis are 4-dimensional, with $x_k = [x_k^1 \ x_k^2 \ v_k^1 \ v_k^2]^T$

Here we assume that motion and position in the two directions $x_1$ and $x_2$ is uncorrelated. Thus for simplicity and without loss of generality, we will look at the two dimensional case with $x_k = [x^k \ v^k]^T$. We model the current velocity as being the previous velocity plus some random acceleration multiplied by time.

$$v^k = v^{k-1} + N_a^k \Delta t \quad (4.16)$$

where $N_a^k$ is a random acceleration. Here we model it as being zero mean white Gaussian noise:

$$N_a^k \sim N(0, \sigma_a^2)$$

Current position is by definition equal to the previous position added to the mean velocity multiplied by time:

$$x^k = x^{k-1} + \mu_v^k \Delta t \quad (4.17)$$
where $\mu_k^v$ is the mean velocity of the target between time step $k-1$ and $k$. We model
$\mu_k^v$ as the average of the $v^k$ and $v^{k-1}$ plus some random velocity deviation that results
from non-constant acceleration.

$$\mu_k^v = \frac{v^k + v^{k-1}}{2} + N_v^k$$  (4.18)

with $N_v^k$ modeled as being zero mean white Gaussian noise.

$$N_v^k \sim N(0, \sigma_v^2)$$

Replacing $v^k$ in equation (4.18) with equation (4.16), and $\mu_v$ in equation (4.17) with
equation (4.18) leads to a position model of the form:

$$x^k = x^{k-1} + (v^{k-1} + N_v^k)\Delta t + \frac{N_a^k\Delta t^2}{2}$$  (4.19)

The motion model is thus linear with additive Gaussian noise, and can be written in
the form:

$$\begin{bmatrix} x^k \\ v^k \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^{k-1} \\ v^{k-1} \end{bmatrix} + \begin{bmatrix} N_a^k\Delta t^2 \\ N_v^k\Delta t \end{bmatrix}$$  (4.20)

This can be straightforwardly extended to 4 dimensions by assuming that motion in
perpendicular directions is independent, which leads to the following linear relationship:

$$\begin{bmatrix} x_1^k \\ x_2^k \\ v_1^k \\ v_2^k \end{bmatrix} = \begin{bmatrix} 1 & \Delta t & 0 \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^{k-1} \\ x_2^{k-1} \\ v_1^{k-1} \\ v_2^{k-1} \end{bmatrix} + \begin{bmatrix} N_{a_x}^k\Delta t^2 + N_{v_x}^k\Delta t \\ N_{a_y}^k\Delta t^2 + N_{v_y}^k\Delta t \\ N_{a_x}^k\Delta t \\ N_{a_y}^k\Delta t \end{bmatrix}$$  (4.21)

As the state space update equations are modeled as having only additive Gaussian
process noise, the probability density function on $x^k$ given $x^{k-1}$ is a multivariate
Gaussian random variable of the form:

$$f(x^k|x^{k-1}) \sim N(\mu_x^k, \Sigma_x)$$  (4.22)
where $\mu^k_x$ is the expected value of $x^k$ given $x^{k-1}$, $E[x^k|x^{k-1}]$, and $\Sigma_x$ is the covariance matrix. It is straightforward to show that, for the 2-dimensional case,

$$E[x^k|x^{k-1}] = \begin{bmatrix} x^{k-1} + v^{k-1}\Delta t \\ v^{k-1} \end{bmatrix}$$

and,

$$\Sigma_x = \begin{bmatrix} \sigma_v^2\Delta t^2 + \sigma_a^2\Delta t^4/4 & \sigma_a^2\Delta t^3 \\
\sigma_a^2\Delta t^3/2 & \sigma_a^2\Delta t^2 \end{bmatrix}$$

And for the 4-dimensional case,

$$E[x^k|x^{k-1}] = \begin{bmatrix} x_x^{k-1} + v_x^{k-1}\Delta t \\
x_2^{k-1} + v_2^{k-1}\Delta t \\
v_x^{k-1} \\
v_2^{k-1} \end{bmatrix}$$

and,

$$\Sigma_x = \begin{bmatrix} \sigma_v^2\Delta t^2 + \sigma_a^2\Delta t^4/4 & 0 & \sigma_a^2\Delta t^3 & 0 \\
0 & \sigma_v^2\Delta t^2 + \sigma_a^2\Delta t^4/4 & 0 & \sigma_a^2\Delta t^3 \\
\sigma_a^2\Delta t^3/2 & 0 & \sigma_a^2\Delta t^2 & 0 \\
0 & \sigma_a^2\Delta t^3/2 & 0 & \sigma_a^2\Delta t^2 \end{bmatrix}$$

4.2.2 Uniform Motion Model

The uniform motion model defines a region around the expected value of a target’s location in which the probability density of target given its previous location is uniform. The probability that the target does not fall within this region is assumed to be zero. As with the Gaussian motion model, we will derive equations for the 2-dimensional case ($x_k = [x^k \ v^k]^T$) and expand to the 4-dimensional assuming independence of motion in the two directions $x_1$ and $x_2$.

Here we model the velocity as being equal to the previous velocity plus uniform process noise:

$$v^k = v^{k-1} + U_v^k$$

(4.27)
where $U^k_v$ is a uniform random variable on the interval $[-\frac{1}{2}, \frac{1}{2}]$. $x^k$ is modeled as being the sum of the current previous location, the average velocity multiplied by the length of the time step, and uniformly distributed position noise:

$$x^k = x^{k-1} + \mu^k_v \Delta t + U^k_x \tag{4.28}$$

with $\mu^k_v$ given by,

$$\mu^k_v = \frac{v^k + v^{k-1}}{2} \tag{4.29}$$

This leads to,

$$x^k = x^{k-1} + v^{k-1} \Delta t + U^k_v \frac{\Delta t}{2} + U^k_x \tag{4.30}$$

where $U^k_x$ is a uniform random variable on the interval $[-\frac{1}{2}, \frac{1}{2}]$. The motion model is thus linear with additive uniform noise and can be written as:

$$\begin{bmatrix} x^k \\ v^k \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^{k-1} \\ v^{k-1} \end{bmatrix} + \begin{bmatrix} U^k_v \frac{\Delta t}{2} + U^k_x \\ U^k_v \end{bmatrix} \tag{4.31}$$

An alternate useful form is:

$$\begin{bmatrix} x^k \\ v^k \end{bmatrix} = \begin{bmatrix} E[x^k|x^{k-1}] \\ E[v^k|x^{k-1}] \end{bmatrix} + \begin{bmatrix} U^k_v \frac{\Delta t}{2} + U^k_x \\ U^k_v \end{bmatrix} \tag{4.32}$$

Finding the joint probability distribution function on $x^k$ and $v^k$ first involves solving for $U^k_x$ and $U^k_v$ in terms of $x^k$ and $v^k$. These are given by:

$$U^k_x = x^k - \frac{\Delta t}{2} v^k - E[x^k|x^{k-1}] + \frac{\Delta t}{2} E[v^k|x^{k-1}] \tag{4.33}$$

and

$$U^k_v = v^k - E[v^k|x^{k-1}] \tag{4.34}$$

with

$$E[x^k|x^{k-1}] = x^{k-1} + v^{k-1} \Delta t \tag{4.35}$$
and

\[ E[v^k|x^{k-1}] = v^{k-1} \]  \hfill (4.36)

The joint pdf for \( x^k \) and \( v^k \) is then given by:

\[
f_{X^k,V^k}(x^k, v^k) = \frac{1}{J(x^k, v^k)} \times f_{U^k_x, U^k_v}(x^k - \frac{\Delta t}{2} v^k - E[x^k|x^{k-1}] + \frac{\Delta t}{2} E[v^k|x^{k-1}], v^k - E[v^k|x^{k-1}]) \]  \hfill (4.37)

where \( J(x^k, v^k) \) is the Jacobian of \( x^k \) and \( v^k \). As \( U^k_x \) and \( U^k_v \) are independent random variables, their joint pdf is the product of the two individual pdfs, and is given by:

\[
f_{U^k_x, U^k_v}(u^k_x, u^k_v) = \begin{cases} \frac{1}{l_{x,u} l_{v,u}} & |u^k_x| \leq \frac{l_x}{2}, |u^k_v| \leq \frac{l_v}{2} \\ 0 & \text{otherwise} \end{cases} \]  \hfill (4.38)

Combining equations (4.37) and (4.38) and calculating the Jacobian \( J(x^k, v^k) = 1 \) gives an expression for the joint pdf of \( x^k \) and \( v^k \):

\[
f_{X^k,V^k}(x^k, v^k) = \begin{cases} \frac{1}{l_{x,u} l_{v,u}} |x^k - \frac{\Delta t}{2} v^k - E[x^k|x^{k-1}] + \frac{\Delta t}{2} E[v^k|x^{k-1}]| \leq \frac{l_x}{2} \\ |v^k - E[v^k|x^{k-1}]| \leq \frac{l_v}{2} \\ 0 & \text{otherwise} \end{cases} \]  \hfill (4.39)

This defines a parallelogram in \( x^k \) and \( v^k \), with larger values of \( v^k \) corresponding to larger values of \( x^k \), with the converse also true.

The uniform motion model is generally less robust than the Gaussian motion model, as motion outside of expected bounds is not accounted for. However, it can still be useful if target motion is heavily constrained. The main advantage over the Gaussian motion model is that it contains only multiplication and addition within the non-zero region, which allows for quicker computation.
CHAPTER 5

Simulation Results

Computer simulation of sensor measurements, noise, and target motion were useful for understanding and testing the algorithms discussed in Chapters 2 and 3 of this thesis. We first look at the comparisons between central and local calculation of likelihood maps for several sensor types. We then look at using the likelihood ratio for tracking of a single target, and finally the likelihood ratio for tracking of multiple targets. Comparisons between central and local calculation are given.

5.1 Distributed vs. Central Likelihood Calculation

Here we evaluate distributed calculation of likelihood maps as discussed in Chapter 2 by comparison to centrally calculation. We evaluate several very different sensor modalities to judge appropriateness for central calculation. Here Doppler radar, passive energy sensors, binary detection sensors and bearing-only sensors are evaluated. For each modality, the expected value for position and velocity is calculated from the likelihood map for both central and local calculation and plotted over time with error bars showing two standard deviations. For all simulations, the state space is chosen to be 4-dimensional, with $\mathbf{x} = [x_1 \ x_2 \ v_{x_1} \ v_{x_2}]^T$. Calculations were done on a grid of $40 \times 40m^2$ with velocity calculated between $-6$ and $6m/s$ for both $v_{x_1}$.
\(v_x\) and \(v_y\). Grid spacing is 1 meter in position and 1 m/s in velocity. All simulations are of a single target in circular motion around the center of the scene with radius 10 meters. For likelihood calculations, the motion model is chosen as being linear with Gaussian process noise, as given by (4.21). It should be noted that the motion model is mismatched to the actual motion of the target.

The important result to note from the simulations is not the accuracy of the estimate of target position in the state space, as this is arbitrary to the noise level chosen. Rather, the relevant comparison is the accuracy of the locally calculated likelihood to the centrally calculated one, since the centrally calculated likelihood map is the optimal estimation given the sensor measurements and models. For each sensor, the expected value of the target state given the measurements taken, \(E(x|y)\) is shown for both locally and centrally calculated likelihood maps. Tables 5.1 and 5.2 gives a quantitative comparison showing the improvement in computation time for each of the given examples. Comparison is given for both the “worst case” where the local likelihood maps are transmitted and combined at each iteration, as well as for transmission and combination every 5th iteration. Note that the values shown in Table 5.2 are total computation time, hence total local computation for 3 radars is significantly less than that for 36 passive energy sensors, although the actual local computation time is comparable.

5.1.1 Radar

Here we consider an example in which three range-doppler radars are spaced evenly at 120\(^{\circ}\), pointing inward to the scene center. Each radar produces a 60 \(\times\) 64 bin range-doppler map with noise, and each bin is regarded as a separate \(y_i\). Bin spacing is
Table 5.1: Computation time components for local and central likelihood map calculation

<table>
<thead>
<tr>
<th>Sensor Modality</th>
<th>Mean Likelihood Calculation Time</th>
<th>Mean Integration Calculation Time</th>
<th>Mean Projection Calculation Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central Radar</td>
<td>492 s</td>
<td>10,929 s</td>
<td>N/A</td>
</tr>
<tr>
<td>Distributed Radar</td>
<td>8 s (3 × 2.7 s)</td>
<td>5 s (3 × 1.7 s)</td>
<td>15 s (3 × 5 s)</td>
</tr>
<tr>
<td>Central Energy</td>
<td>1 s</td>
<td>7,684 s</td>
<td>N/A</td>
</tr>
<tr>
<td>Distributed Energy</td>
<td>&lt; 1 s</td>
<td>52 s (25 × 2.1 s)</td>
<td>122 s (25 × 4.9 s)</td>
</tr>
<tr>
<td>Central Binary</td>
<td>2 s</td>
<td>7,623 s</td>
<td>N/A</td>
</tr>
<tr>
<td>Distributed Binary</td>
<td>&lt; 1 s</td>
<td>53 s (25 × 2.1 s)</td>
<td>126 s (25 × 5 s)</td>
</tr>
<tr>
<td>Central Bearing</td>
<td>3 s</td>
<td>7,712 s</td>
<td>N/A</td>
</tr>
<tr>
<td>Distributed Bearing</td>
<td>&lt; 1 s</td>
<td>130 s (25 × 5.2 s)</td>
<td>134 s (25 × 5.4 s)</td>
</tr>
</tbody>
</table>

Table 5.2: Total computation time comparison for local and central likelihood map calculation

<table>
<thead>
<tr>
<th>Sensor Modality</th>
<th>Transmission and Combination Every Iteration</th>
<th>Transmission and Combination Every 5th Iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central Radar</td>
<td>Total Time (41 Iterations) 457,332 s</td>
<td>Computation Reduction</td>
</tr>
<tr>
<td>Distributed Radar</td>
<td>1,143 s</td>
<td>400:1</td>
</tr>
<tr>
<td>Central Energy</td>
<td>307,401 s</td>
<td>663 s</td>
</tr>
<tr>
<td>Distributed Energy</td>
<td>7,082 s</td>
<td>43:1</td>
</tr>
<tr>
<td>Central Binary</td>
<td>305,002 s</td>
<td>3,178 s</td>
</tr>
<tr>
<td>Distributed Binary</td>
<td>7,286 s</td>
<td>42:1</td>
</tr>
<tr>
<td>Central Bearing</td>
<td>308,603 s</td>
<td>3,254 s</td>
</tr>
<tr>
<td>Distributed Bearing</td>
<td>10,694 s</td>
<td>29:1</td>
</tr>
</tbody>
</table>

38
2 meters in range and \( \approx 0.8m/s \) in radial velocity. The model for this sensor is given in (4.1) with instantaneous likelihood given in (4.2). Figure 5.1 give expected position for both central \( E_C(x|y) \) and local \( E_L(x|y) \) calculation of the likelihood maps for perpendicular directions \( x_1 \) and \( x_2 \). Figure 5.2 give expected velocities for both central, \( E_C(v|y) \) and local, \( E_L(v|y) \) calculation of the likelihood maps for perpendicular velocities \( v_1 \) and \( v_2 \). A slice of the 4-D log-scale likelihood map for time step \( k = 41 \) is shown in Figure (5.3).

In this example, the local estimate of target position and velocity is very close to that of the central estimate. One large advantage that doppler radar has over other sensor modalities discussed here is that radial velocity can be estimated with a single measurement. This greatly reduces the variance of the estimate in (thus increasing the effect of) the instantaneous likelihood map, which is identical for central and distributed estimation.

### 5.1.2 Passive Energy Sensors

For passive energy sensors we consider an example of a \( 5 \times 5 \) grid of sensors with 10 meter spacing, covering the \( 40 \times 40m^2 \) scene. These sensors are modeled as in (4.4). Unlike range-doppler radar, passive energy sensors do not directly measure velocity. However, successive measurements and a motion model allow for both central and local velocity estimations. Figure 5.4 gives expected position for both central \( E_C(x|y) \) and local \( E_L(x|y) \) calculation of the likelihood maps for perpendicular directions \( x_1 \) and \( x_2 \). Figure 5.5 give expected velocities for both central, \( E_C(v|y) \) and local, \( E_L(v|y) \) calculation of the likelihood maps for perpendicular velocities \( v_1 \) and \( v_2 \). A slice of the 4-D log-scale likelihood map for time step \( k = 41 \) is shown in Figure (5.6).
Figure 5.1: Comparison of expected position for central and distributed calculation for radar example

Figure 5.2: Comparison of expected velocity for central and distributed calculation for radar example
From these figures it is clear that for energy sensors, local calculation is a better approximator of position than velocity. This exposes a weakness in the approximation of likelihood maps by local calculation. An energy sensor may only have a few measurements with which to compute a radial velocity, increasing the error of its estimation. Central computation has the advantage of multiple measurements from several sensors available to estimate velocity. Thus indirect estimation of state space variables that benefit from spatially diverse measurements (such as velocity) tend to be higher in error for local calculation than for central calculation.

5.1.3 Binary Detection Sensors

As with passive energy sensors, we consider an example of a $5 \times 5$ grid of sensors with 10 meter spacing, covering the $40 \times 40 m^2$ scene. The sensors are modeled as in (4.10). For this simulation, the sensors have a mean detection radius of 10 meters.
Figure 5.4: Comparison of expected position for central and distributed calculation for energy sensor example

Figure 5.5: Comparison of expected velocity for central and distributed calculation for energy sensor example
Figure 5.6: Comparison of 2-D slice of a 4-D likelihood map for central and distributed calculation for energy sensor example. Time step $k = 41$.

with a standard deviation of 5 meters. Figures 5.7 and 5.8 plot expected positions and velocities respectively over a 40 second period. A slice of the 4-D log-scale likelihood map for time step $k = 41$ is shown in Figure (5.9).

While not perfect, local position estimation is still quite good considering that no global information is available to the local sensors. As with energy sensors, the local estimation of the velocity is worse than that of the central calculation. This is true for the same reasons as above, and the additional constraint that binary detection sensors make no range measurement. Rather, all radial velocity measurements are based on binary detection measurements and a motion model. For example, no detections followed by repeated detections allow for inference that the target is moving toward the sensor. While the local velocity estimation is not good, considering the limitations it is remarkable that it is nearly always within two standard deviations of the central estimate.
Figure 5.7: Comparison of expected position for central and distributed calculation for binary sensor example

Figure 5.8: Comparison of expected velocity for central and distributed calculation for binary sensor example
5.1.4 Bearing Only Sensors

The case of bearing only sensors is interesting in that there is not a direct transform from the global motion model to the local one, as the local dimensions are bearing angle $\theta$ and angular velocity $\frac{d\theta}{dt}$. As with passive energy and binary sensors, no measurement of target velocity is made. Rather, angular velocity is estimated from multiple measurements of target bearing. However, no distance measurements are made, except using knowledge of detection range. Thus there is no direct translation from the local state space to the global one. It should be noted that it is possible to make distance and radial velocity estimates based on detection ranges as is done with binary sensors, however the local state space is then 4-dimensional, and there is no computation savings for local calculation. For this simulation, local velocity and acceleration variances were chosen to be high, due to the large uncertainty in target distance. Sensor measurements are modeled as in (4.13). We consider
again an example with a $5 \times 5$ grid of sensors at 10 meter spacing, totaling a scene
of size $40 \times 40m^2$. Each sensor has a mean detection radius of $\mu_d = 20$ meters,
with a standard deviation of $\sigma_d = 5$ meters. If a detection is made, a position
estimate is also made with a standard deviation of $5^\circ$. As shown in Figures 5.10 and
5.11, the position estimate is very good, but the locally calculated velocity estimate
significantly worse, as angular velocity estimation without range estimation makes
it impossible to accurately calculate tangential velocity. A slice of the 4-D log-scale
likelihood map for time step $k = 41$ is shown in Figure (5.12).

5.2 Single Target Likelihood Ratio

Here simulation results are given to demonstrate the usefulness likelihood ratio
for the case where there is at most one target in the scene. The likelihood ratio is
calculated here using the equations given in Section 3.1. Examples are given for both
radar and energy sensors. Both central and distributed calculations are shown.

5.2.1 Radar

This radar example is identical in design to the experiment in Section 5.1.1. How-
ever, in this example the likelihood ratio is calculated, which gives a measure of the
certainty that a target is in the scene. Figure 5.13 gives a comparison of the likelihood
ratio maps for a single target in circular motion around the scene center. While not
perfect, the maps are very similar, and give information on both target existence and
it’s probability distribution in the state space.
Figure 5.10: Comparison of expected position for central and distributed calculation for bearing-only sensor example

Figure 5.11: Comparison of expected velocity for central and distributed calculation for bearing-only sensor example
5.2.2 Passive Energy Sensors

This example for passive energy sensors is also identical in design to that given in Section 5.1.2. Again, the difference is that here the likelihood ratio map is calculated, rather than only the likelihood map. Figure 5.14 gives a comparison of the likelihood ratio maps for a single target in circular motion around the scene center. While the likelihood ratio maps are similar, although not exact. One notable point is that the distributed likelihood ratio map over-estimates the likelihood ratio. As the local state spaces are smaller that the global state space, the likelihood that a target is in a given unit region in the state space is higher, given a target is in the state space. In the likelihood ratio calculations, this artificially increases the locally calculated likelihood ratio, which when compounded over many sensors results in noticeable error.
Figure 5.13: Time series comparison of central and distributed single target likelihood ratio for radar example. Time spacing is 4 s. 2-D slice shown contains maximum likelihood ratio.
Figure 5.14: Time series comparison of central and distributed single target likelihood ratio for radar example. Time spacing is 4 s. 2-D slice shown contains maximum likelihood ratio.
5.3 Multiple Target Likelihood Ratio

Simulations were also designed and run to test the utility of likelihood ratio maps for multi target tracking when the number of targets is unknown. A three target scenario is demonstrated for range-doppler radar, and a two target scenario is demonstrated for passive energy sensors. Results for both central and distributed calculation of the likelihood ratio are given.

5.3.1 Radar

For this example twelve radars are spaced around a state space measuring 70 meters $\times$ 30 meters. Calculations in the 4-D state space take place on a grid with 1 meter spacing in position and 1 meter/second spacing in velocity. The radars are range-doppler as modeled in (4.1). Three targets move in circular motion as shown in Figure 5.15 with a period of 20 seconds. The motion is such that at 5 seconds, two targets are at the same location but moving in different directions. At 15 seconds, two targets are at the same location traveling in the same direction and subsequently split.

To represent the 4-D likelihood ratio map concisely, a simple tracker was designed that finds points with a likelihood ratio of 1000 or greater, and which also have a higher likelihood ratio than points immediately surrounding them. These points can then be plotted giving an idea of the motion of the targets in the state space. Figures 5.16, 5.18, 5.20 and 5.22 give the $x$, $y$, $v_x$, and $v_y$ values of the targets in the state space, as measured by the centrally calculated likelihood ratio map. Figures 5.17, 5.19, 5.21 and 5.23 give the $x$, $y$, $v_x$, and $v_y$ values of the targets in the state space, as measured by the distributed likelihood ratio map.
Both the central and distributed likelihood maps provide accurate target tracking, and give a level of confidence in the track, as well as the actual distribution in the state space. The central likelihood map has difficulty with the intersection of the two targets moving in the same direction at the same location which occurs at time step 15. Even though the likelihood ratio is soft limited at $10^5$ by an arctan function, in this region it is so large that a wide region is assigned a likelihood ratio of $10^5$. This causes the position uncertainty seen in the tracks using the central calculation of the likelihood ratio maps. However, is should be noted that the assumption of the likelihood ratio calculation was broken here, that in a specified region around a point there are exactly zero or one targets.
Figure 5.16: 'X' position of targets in the state space over time from central likelihood ratio map, radar example

Figure 5.17: 'X' position of targets in the state space over time from distributed likelihood ratio map, radar example
Figure 5.18: 'Y' position of targets in the state space over time from central likelihood ratio map, radar example

Figure 5.19: 'Y' position of targets in the state space over time from distributed likelihood ratio map, radar example
Figure 5.20: ‘Vx’ velocity of targets in the state space over time from central likelihood ratio map, radar example

Figure 5.21: ‘Vx’ velocity of targets in the state space over time from distributed likelihood ratio map, radar example
Figure 5.22: 'Vy' velocity of targets in the state space over time from central likelihood ratio map, radar example

Figure 5.23: 'Vy' velocity of targets in the state space over time from distributed likelihood ratio map, radar example
5.3.2 Passive Energy Sensors

For this multi-target example, energy sensors are spaced 10 meters apart on a grid measuring 70 meters × 30 meters, for a total of 32 sensors. As with the radar example, likelihood ratio calculations take place on a grid with 1 meter spacing in position and 1 meter/second spacing in velocity. The energy sensors are modeled as in (4.4). Two targets move in circular motion as shown in Figure 5.24 with a period of 20 seconds. Because of the additive properties of the energy sensors, in that energy from multiple targets can be summed in a single measurement, position estimates are more reliable for targets that have greater spacing.

As with the radar example, a simple tracker was employed to find peaks with high likelihood ratio. Figures 5.25, 5.27, 5.29 and 5.31 give the $x$, $y$, $v_x$, and $v_y$ values of the targets in the state space, as measured by the centrally calculated likelihood ratio.
map. Figures 5.26, 5.28, 5.30 and 5.32 give the $x$, $y$, $v_x$, and $v_y$ values of the targets in the state space, as measured by the distributed likelihood ratio map.

As is to be expected, the estimates provided by the energy sensors are significantly worse than those from the radar sensors, particularly for the velocity, and hence the resulting tracks are of lower quality. Several factors cause this, including the poorer range estimation available from the energy sensors, and the fact that the sensors themselves are located in the scene of interest, resulting in irregularly shaped distributions which may have multiple peaks. Additionally, although the two targets were well spaced, several sensors still received additive signals from both targets, resulting in an estimate of range that was artificially lowered, and introducing another source of error. Still, with all these problems, position estimates from both central and distributed calculation methods were quite good.
Figure 5.25: 'X' position of targets in the state space over time from central likelihood ratio map, energy sensor example

Figure 5.26: 'X' position of targets in the state space over time from distributed likelihood ratio map, energy sensor example
Figure 5.27: 'Y' position of targets in the state space over time from central likelihood ratio map, energy sensor example

Figure 5.28: 'Y' position of targets in the state space over time from distributed likelihood ratio map, energy sensor example
Figure 5.29: 'Vx' velocity of targets in the state space over time from central likelihood ratio map, energy sensor example

Figure 5.30: 'Vx' velocity of targets in the state space over time from distributed likelihood ratio map, energy sensor example
Figure 5.31: 'Vy' velocity of targets in the state space over time from central likelihood ratio map, energy sensor example

Figure 5.32: 'Vy' velocity of targets in the state space over time from distributed likelihood ratio map, energy sensor example
In order to validate likelihood ratio maps as a method of multi-modal sensor fusion, we collected data from multiple sensors of participating human subjects walking and running through a scene on the Ohio State University campus. Data was collected using acoustic, seismic, radar, and overhead video sensors. While we found the acoustic sensors were not useful for localizing footsteps due to the large amount of background noise, the seismic, radar, and video were informative to target position and velocity. Ground level sensors (seismic and radar) were arrayed as shown in Figure 6.1.

The radars used in this experiment were ground level micro radars with 75 MHz of bandwidth, a range of 120 meters with 2 meter resolution in range and 0.8 m/s resolution in doppler. Three radars were used in this experiment. The radar output is a 2D array of object detections indexed by range and Doppler. The tracks are converted into detections at specific ranges and velocities. An example of this data is shown below in Figure 6.2, where the x-axis is time, y-axis is range, and color is radial velocity. The radars were modeled as a detection sensor as in (4.7) with local frame $z_i(x) = [r_i, v_{r_i}]'$ where $r_i$ is range and $v_{r_i}$ is radial velocity. There was assumed to be
zero mean uncorrelated Gaussian error in the detection ranges and radial velocities with variances $\sigma_r^2$ and $\sigma_{v_r}^2$.

The video sensors were standard 320 $\times$ 240 r.g.b. cameras mounted on a building overlooking the scene. Two cameras were used with different vantage points. Example frames from the two cameras are shown in Figure 6.3. Sequential video frames were subtracted for simple change detection. The images were then de-noised and areas of high change were clustered. Cluster centroids were projected from the pixel space onto the scene of interest and output as detections. The sensor was modeled as in (4.7). Total measurement, processing, and projection noise was modeled as uncorrelated Gaussian error in the detection location with variance $\sigma_x^2$.

The seismic sensors used in this experiment were Geospace GS-14-L3 single axis geophones. These sensors measure vertical vibrations in the ground such as those caused by human footsteps. Seismic was of limited use in our experiment for walking targets due to the limited vibration caused by the footsteps, however detection was possible when the person was running. The seismic sensors were modeled as energy
sensors with received signal of the form given in (4.4) with instantaneous likelihood given in (4.5).

The sensor data were combined using the likelihood ratio equations described in Chapter 3. Results showed effective sensor fusion in the tracking of human targets transiting the scene. Figure 6.4 shows the log likelihood ratio for a sequence of a person walking at constant speed from east to west. Frames are spaced at 5 second intervals. In this example only radar and video sensors are used in the formation of the likelihood map. Approximate radar fields of view are shown by lines drawn over
Figure 6.3: Building mounted video cameras' field of view

the image. The vertical line in the center marks the point where the subject becomes obscured by trees to the overhead video cameras. The first frame shows the likelihood ratio after the first measurements are taken. The radars have no detections, resulting in a lower likelihood within their field of view. A false alarm from the video camera results in a slightly higher likelihood around the $-30$ meter mark which quickly fades as other measurements come in. In subsequent frames, at approximately the $-17$ meter mark trees begin to obscure the target from the video camera, resulting in greater uncertainty in the north-south direction as only the radars are available for position estimation. The subject is tracked throughout the sequence, with uncertainty in position accurately conveyed by the likelihood ratio map. Interestingly, in this example a non-participant enters the southeast corner in the last few frames and is clearly visible in the map with a positive log likelihood ratio, demonstrating the multi-target capability of the likelihood ratio.
Figure 6.4: Sequence of log likelihood ratio maps produced using radar and video data. Human subject is walking east to west at a constant rate. Frames are spaced at 5 second intervals.
In a second example, two human subjects are running from east to west, one in front of the other, with approximately 8 meter spacing. As a running person produces much stronger ground vibrations than a walking person, the seismic sensors were useful for estimating target position in this example. Figure 6.5 shows the log likelihood ratio for this experiment. Here frames are spaced at 2 second intervals. There are several interesting observations to note in this example. As there were two persons in the experiment, the radar systems had trouble deciding how many targets are in the scene. Some frames individual radars tracked both targets, and other frames only the closer target was tracked. The result is significantly greater uncertainty in the position estimate. The video sensor generally tracked the rear target, as did the two east radars, resulting in a higher likelihood ratio for the rear target. As the variance of the position estimation provided by the video and radar sensors is much less than that of the seismic sensors, these sensors are significantly more useful for position estimation. The contribution of the seismic sensors to the state estimation was primarily providing information where the targets were not. In most of the frames, darker “dots” are visible. These are seismic sensors that register a very low likelihood of receiving the current signal given a target right on top of them. While making some contribution, the small number of seismic sensors in this experiment limited their effectiveness to significantly contribute to the multi-sensor state estimation.
Figure 6.5: Sequence of log likelihood ratio maps produced using radar video and seismic data. Human subject is running east to west at a constant rate. Frames are spaced at 2 second intervals.
CHAPTER 7

Conclusion

In this thesis we have discussed using both the likelihood function for single target and the likelihood ratio for multi-target tracking, for non-linear or non-Gaussian sensor or motion models. As these methods are computationally intractable for real or near-real time systems, we have proposed a distributed computation for sensor networks who’s individual sensors inform in a lower dimension than the global state space. As the majority of the computation takes place in these lower dimensional state spaces, total computation time is greatly reduced. Additionally, in a distributed implementation, sensors do not have to transmit measurements to a fusion center at each iteration. We have analyzed circumstances in which this approximation is very close to, or identical to, central calculation, and provided insight into the sources of error. A full discussion was also given for all sensor and motion models used in both simulations and on real data in this thesis.

In order to provide a more complete picture of the utility of likelihood and likelihood ratio maps, simulations were designed and run testing these algorithms on four commonly used sensor modalities. These simulations demonstrated the usefulness of likelihood and likelihood ratio maps for target tracking, as well as the computational
savings of distributed calculation and it’s similarity to central calculation. Scenarios for both single target and multiple targets were presented.

Finally, the concept of likelihood ratio maps for multi modal sensor fusion was tested on two tracking scenarios using real seismic, radar, and video data collected on participating walking and running human subjects. Results showed effective sensor fusion, with the data shown in an easy to interpret format.

Future work in this area could include investigation into methods to further reduce computation times. Promising areas include particle filters and using Gaussian mixture models to approximate the likelihood functions. Additionally, a fully distributed implementation on a real sensor network would test the feasibility of these algorithms and provide insight for any future deployment.
APPENDIX A

Range-Doppler Radar Rician Distribution Derivation

Define the complex random variable $Y_C = Y_R + jY_I$ to be the received signal of a single pixel in a range-doppler image where random variables $Y_R$ and $Y_I$ are the real and imaginary parts of $Y_C$ respectively, given by:

\begin{align*}
Y_R &= A \cos(\alpha) + N_R \\
Y_I &= A \sin(\alpha) + N_I
\end{align*} \tag{A.1} \tag{A.2}

where $N_R$ and $N_I$ are independent identically distributed normal random variables with zero mean and variance $\sigma_n^2$. $A$ is a known signal amplitude and $\alpha$ is a random phase uniformly distributed on $[-\pi, \pi]$. As $Y_R$ and $Y_I$ are independent given $A$ and $\alpha$, their joint pdf is the product of the individual pdfs, and is given by:

\begin{align*}
f_{Y_R,Y_I}(y_R, y_I|A, \alpha) &= \frac{1}{2\pi\sigma_n^2} \exp\left(-\frac{(y_R - A \cos(\alpha))^2}{2\sigma_n^2}\right) \exp\left(-\frac{(y_I - A \sin(\alpha))^2}{2\sigma_n^2}\right) \\
&= \frac{1}{2\pi\sigma_n^2} \exp\left(-\frac{1}{2\sigma_n^2}\left(y_R^2 - 2y_RA \cos(\alpha) + A^2 \cos^2(\alpha) + y_I^2 - 2y_I A \sin(\alpha) + A^2 \sin^2(\alpha)\right)\right) \\
&= \frac{1}{2\pi\sigma_n^2} \exp\left(-\frac{\|Y_C\|^2 + A^2}{2\sigma_n^2}\right) \exp\left(\frac{y_RA \cos(\alpha) + y_I A \sin(\alpha)}{\sigma_n^2}\right) \tag{A.3}
\end{align*}

The $\alpha$ terms are then removed using total probability:

\begin{align*}
f_{Y_R,Y_I}(y_R, y_I|A) &= \int_{-\infty}^{\infty} f_{Y_R,Y_I}(y_R, y_I|A, \alpha) f(\alpha) d\alpha \tag{A.4}
\end{align*}
\[ f_{Y_R,Y_I}(y_R, y_I|A) = \frac{1}{2\pi \sigma_n^2} \exp \left( -\frac{(||y_C||^2 + A^2)}{2\sigma_n^2} \right) \int_{-\pi}^{\pi} \frac{1}{2\pi} \exp \left( \frac{A(y_R \cos(\alpha) + y_I \sin(\alpha))}{\sigma_n^2} \right) d\alpha \]  
(A.5)

By trigonometric identity,

\[ f_{Y_R,Y_I}(y_R, y_I|A) = \frac{1}{2\pi \sigma_n^2} \exp \left( -\frac{(||y_C||^2 + A^2)}{2\sigma_n^2} \right) \int_{-\pi}^{\pi} \frac{1}{2\pi} \exp \left( \frac{A(||y_C|| \sin(\alpha + \gamma))}{\sigma_n^2} \right) d\alpha \]  
(A.6)

where

\[ \gamma = \arcsin \left( \frac{y_R}{||y_C||} \right) + \begin{cases} 0 & y_I \geq 0 \\ \pi & y_I < 0 \end{cases} \]  
(A.7)

Let \( \alpha' = \alpha + \gamma \), which gives

\[ f_{Y_R,Y_I}(y_R, y_I|A) = \frac{1}{2\pi \sigma_n^2} \exp \left( -\frac{(||y_C||^2 + A^2)}{2\sigma_n^2} \right) \int_{-\pi}^{\pi} \frac{1}{2\pi} \exp \left( \frac{A(||y_C|| \sin(\alpha'))}{\sigma_n^2} \right) d\alpha' \]  
(A.8)

Define \( y = ||y_C|| \)

\[ y = \sqrt{y_I^2 + y_R^2} \]  
(A.9)

Note that:

\[ F_Y(y) = \int_{y_I = -y}^{y} \int_{y_R = -\sqrt{y^2 - y_I^2}}^{y} f_{Y_R,Y_I}(y_R, y_I|A) dy_R dy_I \]  
(A.10)

by differentiation:

\[ f_Y(y) = \int_{y_I = -y}^{y} \frac{y}{\sqrt{y^2 - y_I^2}} \left( f_{Y_R,Y_I}(\sqrt{y^2 - y_I^2}, y_I|A) + f_{Y_R,Y_I}(\sqrt{y^2 - y_I^2}, y_I|A) \right) dy_I \]  
(A.11)

Combining (A.11) and (A.8) gives:

\[ f_Y(y) = \int_{y_I = -y}^{y} \frac{2}{\sqrt{y^2 - y_I^2} 2\pi \sigma_n^2} \exp \left( -\frac{y^2 - y_I^2 + A^2}{2\sigma_n^2} \right) \times \int_{-\pi}^{\pi} \frac{1}{2\pi} \exp \left( \frac{A(sqrty^2 - y_I^2 + y_I^2) \sin(\alpha')}{\sigma_n^2} \right) d\alpha' dy_I \]  
(A.12)
\[ f_Y(y) = \frac{y}{\pi \sigma_n^2} \exp \left( -\frac{y^2 + A^2}{2\sigma_n^2} \right) \int_{-\pi}^{\pi} \frac{1}{2\pi} \exp \left( \frac{A y \sin(\alpha')}{\sigma_n^2} \right) d\alpha' \int_{y_I = -y}^{y} \frac{1}{\sqrt{y^2 - y_I^2}} dy_I \]

\[ (A.13) \]

\[ = \frac{y}{\pi \sigma_n^2} \exp \left( -\frac{y^2 + A^2}{2\sigma_n^2} \right) \int_{-\pi}^{\pi} \frac{1}{2\pi} \exp \left( \frac{A y \sin(\alpha')}{\sigma_n^2} \right) d\alpha' \arcsin \left( \frac{y_I}{y} \right) \bigg|_{y_I = -y} \]

\[ (A.14) \]

\[ = \frac{y}{\pi \sigma_n^2} \exp \left( -\frac{y^2 + A^2}{2\sigma_n^2} \right) \int_{-\pi}^{\pi} \frac{1}{2\pi} \exp \left( \frac{A y \sin(\alpha')}{\sigma_n^2} \right) d\alpha' \]

\[ (A.15) \]

\[ = \frac{y}{\sigma_n^2} \exp \left( -\frac{y^2 + A^2}{2\sigma_n^2} \right) \int_{-\pi}^{\pi} \frac{1}{2\pi} \exp \left( \frac{A y \sin(\alpha')}{\sigma_n^2} \right) d\alpha' \]

\[ (A.16) \]

\[ = \frac{y}{\sigma_n^2} \exp \left( -\frac{y^2 + A^2}{2\sigma_n^2} \right) I_0 \left( \frac{A y}{\sigma_n^2} \right) \]

\[ (A.17) \]

where \( I_0() \) is the zero order modified Bessel function of the first kind. The final result in equation (A.17) is the Rician distribution on the amplitude of the received signal \( y \).
BIBLIOGRAPHY


