2-ARC TRANSITIVE POLYGONAL GRAPHS OF LARGE GIRTH AND VALENcy

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * *

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2009

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A near-polygonal graph is a graph $\Gamma$ which has a set $\mathcal{C}$ of $m$-cycles for some positive integer $m$ such that each 2-path of $\Gamma$ is contained in exactly one cycle in $\mathcal{C}$. If $m$ is the girth of $\Gamma$ then the graph is called polygonal. Up until now, the only examples of 2-arc transitive polygonal graphs with arbitrarily large valency had girth no larger than seven, and the 2-arc transitive polygonal graph with largest girth had valency five and girth twenty-three (in fact, even with no restrictions on the automorphism group, there were no examples of polygonal graphs with odd girth greater than twenty-three). This thesis provides a construction of an infinite family of polygonal graphs of arbitrary girth $m$ with 2-arc transitive automorphism groups, showing that there are 2-arc transitive polygonal graphs of arbitrarily large valency for each girth $m$. Furthermore, this thesis also provides a construction that, given a polygonal graph of valency $r$ and girth $m$, produces a polygonal graph of valency $r$ and girth $3m$, and that the graphs constructed via this method will be 2-arc transitive if the original graph was 2-arc transitive. Finally, this thesis provides a construction of a new infinite family of near-polygonal graphs of valency 10 and a method for determining which graphs can have a given girth, which yields a few new examples of polygonal graphs.
To my parents
ACKNOWLEDGMENTS

I would first and foremost like to thank my advisor, Dr. Ákos Seress. His direction, patience, and encouragement, as well as countless hours of his time, are what made this dissertation possible.

I would like to thank Dr. Boris Pittel and Dr. Neil Robertson for giving up valuable hours of their time to serve on my dissertation committee.

I am indebted to my all friends and family for the constant support throughout the years. Their positivity has made everything much easier, and they have always been able to bring me comfortably back to the real world no matter how far my work drags me away.

I would especially like to thank Jenn, whose support has allowed me to remain sane through this process. She was there to help at every bump in the road, offering encouragement, and her computer skills made writing this dissertation infinitely easier.

Finally, I must thank my parents. They have always been there for me and supported me in every possible way. I have been given every opportunity in life to succeed, and I owe everything to them.
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CHAPTER 1

INTRODUCTION

1.1 Background

Let \( \Gamma \) be a graph. For a positive integer \( l \), an \( l \)-walk of \( \Gamma \) is a sequence of vertices \((\alpha_0, \alpha_1, ..., \alpha_l)\) such that \( \alpha_i \) is adjacent to \( \alpha_{i+1} \) for \( 0 \leq i \leq l - 1 \). If in addition \( \alpha_{i-1} \neq \alpha_{i+1} \) for \( 1 \leq i \leq l - 1 \), then an \( l \)-walk is called an \( l \)-arc; while if further all the \( \alpha_i \) are distinct then the \( l \)-arc is called an \( l \)-dipath (directed path). The identification of an \( l \)-dipath \((\alpha_0, \alpha_1, ..., \alpha_l)\) and its reverse \((\alpha_l, ..., \alpha_1, \alpha_0)\) is called an \( l \)-path, and denoted by \([\alpha_0, \alpha_1, ..., \alpha_l]\). An \( m \)-cycle is an \((m-1)\)-path \([\alpha_1, ..., \alpha_m]\) such that \( \alpha_m \) is adjacent to \( \alpha_1 \) [6].

A near-polygonal graph is a graph \( \Gamma \) with a set of distinguished cycles \( \mathcal{C} \), all of length \( m \) for some \( m \in \mathbb{N} \), such that every 2-arc of \( \Gamma \) is contained in a unique member of \( \mathcal{C} \). Furthermore, if \( m \) is also the girth, or length of the shortest cycle, of \( \Gamma \), then \( \Gamma \) is called a polygonal graph. If \( \Gamma \) is a polygonal graph such that the set \( \mathcal{C} \) of distinguished \( m \)-cycles is in fact the set of all \( m \)-cycles of \( \Gamma \), then \( \Gamma \) is called a strict polygonal graph. Finally, if a graph \( \Gamma \) has an automorphism that maps a given 2-arc to any other 2-arc, then \( \Gamma \) is said to be a 2-arc transitive graph. If the element of the automorphism group mapping one 2-arc to another is always unique, the automorphism group is said to
be *sharply 2-arc transitive*. A graph that is both polygonal and 2-arc transitive is a *2-arc transitive polygonal graph*.

Polygonal graphs are a natural generalization of the edge- and vertex-set of polygons and Platonic solids, and one immediately notes that these are themselves strict polygonal graphs, with the special set of cycles being the polygon itself or the faces of the solid, respectively. $K_n$, the complete graph on $n$ points, is a strict polygonal graph of girth 3, and the Petersen graph is a polygonal graph of girth 5 that is not a strict polygonal graph [14].

Manley Perkel invented the notion of a polygonal graph in [8] and a near-polygonal graph in [12]. Even though polygonal graphs have been around for a few decades and occur both in topological and algebraic graph theory and in geometries related to finite groups, the theory of the subject is still in its infancy. Polygonal graphs are far from being classified; indeed, the amount of symmetry they must possess in unclear, the exact structure of their automorphism groups is unknown, and, in fact, up until now, only a scarce supply of these graphs could even be constructed. [14]

### 1.2 Preliminary Results

In this section, we detail some of the known results involving polygonal graphs. First, a graph is said to be *regular* if all its vertices have the same valency, or number of adjacent edges. We have the following lemma:

**Lemma 1.2.1.** [8, Lemma 2.1] A near-polygonal graph is regular.

Hence it makes sense to discuss the valency of a near-polygonal graph, since it is the same at every vertex.
1.2.1 Trivalent Polygonal Graphs

As is noted in [14], many examples of trivalent polygonal graphs come from topological constructions. The reason for this is made clear by the following observation:

Observation 1.2.2. [14, p. 180] If a trivalent graph $\Gamma$ of girth $m$ can be embedded on a (not necessarily orientable) surface such that each face is an $m$-gon, then the set of faces form the distinguished cycle set $C$ of a polygonal graph. Conversely, if $\Gamma$ is a trivalent polygonal graph with a distinguished cycle set $C = \{C_1, C_2, ..., C_k\}$, then attaching a 2-cell to each $C_i$ we get a 2-dimensional topological space which can be embedded on some surface.

Proceeding in this fashion has yielded the following results:

Theorem 1.2.3. [10, p. 46] The only trivalent polygonal graphs of girth 3, 4, and 5 are, respectively, the edge- and vertex-sets of the tetrahedron, cube, and dodecahedron.

Theorem 1.2.4. [7] All trivalent strict polygonal graphs of girth 6 are duals of triangulations of the torus or the Klein bottle (of which there are infinitely many).

Theorem 1.2.5. [11, Theorem 5] There exists a trivalent polygonal graph of girth 7 with $14n$ vertices for all $n \in \mathbb{N}$.

Theorem 1.2.6. [1] For all $n$ divisible by 4, there exist trivalent polygonal graphs of girth $n$.

1.2.2 Polygonal Graphs with Valency at Least 4

Obviously, considering the complete graphs $K_n$ for $n \geq 3$ shows that there are infinitely many polygonal graphs of girth 3, and each $n \geq 5$ yields a polygonal graph
of valency at least 4. We obtain the following lemma by considering the points and edges of the regular cube in $k$-dimensional space, $k \geq 2$:

**Lemma 1.2.7.** [9, p. 1311] *For all $k \geq 2$ there is a polygonal graph with valency $k$ and girth 4.*

There are three main techniques for constructing polygonal graphs of valency greater than 3. The first is the “girth-doubling” construction outlined in [2]:

**Theorem 1.2.8.** [2, Theorem 1.1] *If there exists a polygonal graph $\Gamma$ of valency $r$ and girth $m$, then there exists a polygonal graph $\Gamma_2$ of valency $r$ and girth $2m$. Moreover, if $\Gamma$ is a strict polygonal graph, then so is $\Gamma_2$.*

By starting with the complete graph $K_n$, which has valency $n - 1$, and repeatedly applying this theorem, we obtain:

**Corollary 1.2.9.** [2, Corollary 1.2] *For all $r \geq 2$, there exist polygonal graphs of valency $r$ and arbitrarily large girth.*

This is a remarkable theorem that produces a large number of graphs. However, we are forced to start with a specific polygonal graph, and, since there are very few known graphs with odd girth, we are limited in the graphs we construct. It is also not immediately clear what the automorphism of the newly constructed graph will be; for example, if we start with a 2-arc transitive polygonal graph $\Gamma$ of girth $m$, does the graph $\Gamma_2$ of girth $2m$ that we construct necessarily have a 2-arc transitive automorphism group?

The other two primary techniques for constructing polygonal graphs of valency greater than 4 will be discussed briefly here, but, because they play a large role in
the results of this thesis, will be discussed in much greater depth in the following chapters. Both techniques are algebraic in nature, and rely on the following group-theoretic construction of the coset graph. For a group $G$ and a subgroup $H < G$, denote by $[G : H]$ the set of right cosets of $H$ in $G$. For an element $g \in G \setminus H$ with $g^2 \in H$, the \textit{coset graph} $\Gamma := \text{Cos}(G, H, HgH)$ is defined as the graph with vertex set $[G : H]$ such that two vertices $Hx$ and $Hy$ are adjacent if and only if $yx^{-1} \in HgH$; alternatively we can view the edge set as $\{(Ht, Hgt) : t \in G\}$. Observe that from the condition $g^2 \in H$ it follows that $HgH = Hg^{-1}H$ and $Hx$ and $Hy$ are adjacent if and only if $Hy$ and $Hx$ are adjacent, implying that $\Gamma$ is undirected. Denote by $\alpha$ the vertex $H$ of $\Gamma$ and by $\beta$ the vertex $\alpha g = Hg$, and let $G_{\alpha_1\alpha_2...\alpha_n}$ be the stabilizer in $G$ of the vertices $\alpha_1, \alpha_2, ..., \alpha_n$. Note that $\beta^g = \alpha g^2 = \alpha$, $G_{\alpha} = H$, $G_{\beta} = H^g$, and $G_{\alpha\beta} = H \cap H^g$. The neighbor set $N_\Gamma(\alpha)$ of $\alpha$ consists of the cosets in $HgH$, and the valency of $\alpha$ is the index $|G_{\alpha} : G_{\alpha\beta}| ([6])$.

One technique relies on the following theorem:

\textbf{Theorem 1.2.10.} [12, Theorem 2.3] Let $G$ be a group, and $H$ a subgroup of $G$. Let $g \in G \setminus H$ with $g^2 \in H$, and suppose that $G = \langle H, g \rangle$. Let $h \in H \setminus H^g$, $K = H \cap H^g$, and $L = H \cap H^g \cap H^{gh}$. Suppose that

(i) $H = K \cup KhK$,

(ii) if $L \leq K^k$ for some $k \in H$, then $k \in K \cup Kh$, and

(iii) if $t \in G$ with $L^t \leq K$, then $L^t = L^s$ for some $s \in H$.

Then either $G = H \cup HgH$ (in which case the graph we have constructed is a complete graph) or there is a near-polygonal graph $\Gamma$ with $G \leq \text{Aut}(\Gamma)$. 

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Observation 1.2.11. [14, p.183-184] The near-polygonal graphs constructed by Theorem 1.2.10 are 2-arc transitive.

Carefully chosen families of groups will thus allow us to construct an infinite family of 2-arc transitive near-polygonal graphs. The obvious limitation here is that in no way does this theorem guarantee that the graph we have constructed is polygonal, i.e. we have no idea what the girth of our constructed graph is without specifically checking. Some of the polygonal graphs obtained using this construction have been detailed in [12] and [13], and the results can be summarized in Table 1.1 from [14].

As is noted in [14], up until now, the graphs with girth 23 listed in Table 1.1 actually had the largest girth of any known 2-arc transitive polygonal graph.

The final technique, known as “spinning the cycle,” is based upon the following:

Theorem 1.2.12. [6, Corollary 1.2] Let $\Gamma$ be a graph and $G \leq \text{Aut}(\Gamma)$ such that $G$ acts sharply 2-transitively on the 2-arcs of $\Gamma$. Assume that for an arc $(\alpha, \beta)$ of $\Gamma$ there exists an involution $g \in G$ such that $(\alpha, \beta)^g = (\beta, \alpha)$. Then $\Gamma$ is near-polygonal.

This method of construction yields the following:

Theorem 1.2.13. [6, Theorems 1.3, 1.4, 1.5, 1.6] There exist 2-arc transitive polygonal graphs of arbitrarily large valency of girth 5, 6, and 7, respectively.

As is noted in [14], up until now, the only examples of 2-arc transitive polygonal graphs with arbitrarily large valency had girth no larger than 7, and those graphs are the ones constructed above.
<table>
<thead>
<tr>
<th>$G$</th>
<th>valency</th>
<th>girth</th>
<th>strict polygonal?</th>
</tr>
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<tbody>
<tr>
<td>$\text{PSL}(2, 3^2)$</td>
<td>$5$</td>
<td>$3$</td>
<td>strict</td>
</tr>
<tr>
<td>$\text{PSL}(2, 31)$</td>
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</tr>
<tr>
<td>$\text{PSL}(2, 431)$</td>
<td>$5$</td>
<td>$9$</td>
<td>strict</td>
</tr>
<tr>
<td>$\text{PSL}(2, 199)$</td>
<td>$5$</td>
<td>$11$</td>
<td>strict</td>
</tr>
<tr>
<td>$\text{PSL}(2, 43^2)$</td>
<td>$5$</td>
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</tr>
<tr>
<td>$\text{PSL}(2, 1951)$</td>
<td>$5$</td>
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<td>$\text{PSL}(2, 36209)$</td>
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<td>$\text{PSL}(2, 522919)$</td>
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<td>$\text{PGL}(2, 13^2)$</td>
<td>$6$</td>
<td>$12$</td>
<td>non-strict</td>
</tr>
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</table>

Table 1.1: 2-arc transitive polygonal graphs constructed using Theorem 1.2.10

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1.3 Outline of Thesis

The rest of the thesis is organized as follows.

Chapter 2 provides a construction for an infinite family of near-polygonal graphs of valency 10 via the method presented in Theorem 2.1.3. Section 2.2 proves the following theorem:

**Theorem 1.3.1.** Let $q$ be a prime power such that $q \equiv 1, 19 \pmod{30}$. Then there exist $H \leq PSL(3, q)$ and $g \in PSL(3, q)$ such that, if $G := \langle H, g \rangle$, $\Gamma := \text{Cos}(G, H, HgH)$ is a near-polygonal graph.

In Section 2.3 we determine which prime powers $q$ yield near-polygonal graphs with cycles of length $m$ for a given $m$. We first prove the following lemma:

**Lemma 1.3.2.** If $\Gamma$ is a near-polygonal graph constructed as in Theorem 1.3.1, then the length of the special cycles in $\Gamma$ must be even.

The next two lemmas construct sequences of numbers $a'_n, b'_n$ that are crucial to determining which prime powers will yield near-polygonal graphs with cycles of a given length. First, given that

\[ c_2 := \frac{308052 + 239699\sqrt{-3} + 82209\sqrt{5} - 321908\sqrt{-15}}{71907 - 82209\sqrt{-15}} , \quad (1.1) \]

we have:

**Lemma 1.3.3.** If $x$ is a root of $x^2 - (1 + c_2)x - 1 = 0$, then for any positive integer $n$ we have $x^n = a_nx + b_n$, where

(i) $a_2 = (1 + c_2)$,
Lemma 1.3.4. If $a'_n, b'_n$ are coefficients such that $x^{4n} = a'_n x + b'_n$, then

(i) $a'_n = a_n^4 (1 + c_2) ((1 + c_2)^2 + 2) + 4a_n^3 b_n ((1 + c_2)^2 + 1) + 6a_n^2 b_n^2 (1 + c_2) + 4a_n b_n^3$,

(ii) $b'_n = a_n^4 ((1 + c_2)^2 + 1) + 4a_n^3 b_n (1 + c_2) + 6a_n^2 b_n^2 + b_n^4$.

Using these sequences, we prove the following:

Theorem 1.3.5. If $q = p, p^2$ for some prime $p \neq 2, 3, 5$, the near-polygonal graph $\Gamma$ constructed from $\text{PSL}(3, q)$ as in Theorem 1.3.1 has special cycles of length $m$, and $a'_m, b'_m$ are defined as above, then the equation

$$(1 - b'_m)^2 - a'_m (1 + c_2) (1 - b'_m) - a_m^2 = 0 \quad (1.2)$$

must hold in $GF(q)$.

This provides us with a number that must be 0 in $GF(q)$. The complex norm of the numerator of this number is an integer that must be 0 in $GF(q)$, and hence the prime divisors of this integer (or their squares) are the only possibilities for $q$.

In Section 2.4, using the above method, we are able to provide some new examples of 2-arc transitive polygonal graphs, as summarized by Table 1.2.

In Chapter 3, we construct new 2-arc transitive polygonal graphs via the method presented in Theorem 1.2.12, culminating in the following theorem proved in Section 3.2:
Table 1.2: New 2-arc transitive polygonal graphs of valency 10

<table>
<thead>
<tr>
<th>Group</th>
<th>Cycle Length</th>
<th>Polygonal?</th>
<th>Strict Polygonal?</th>
</tr>
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<tbody>
<tr>
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<td>6</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>PSL(3, 121)</td>
<td>6</td>
<td>Yes</td>
<td>Yes</td>
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<tr>
<td>PSL(3, 79)</td>
<td>8</td>
<td>Yes</td>
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<tr>
<td>PSL(3, 19)</td>
<td>10</td>
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<td>No</td>
</tr>
<tr>
<td>PSL(3, 59^2)</td>
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<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>PSL(3, 239^2)</td>
<td>12</td>
<td>Yes</td>
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<tr>
<td>PSL(3, 29^2)</td>
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<td>Yes</td>
</tr>
</tbody>
</table>

Theorem 1.3.6. There are 2-arc transitive polygonal graphs of arbitrarily large valency and girth m for all odd m ≥ 3.

In Chapter 4, we construct even more 2-arc transitive polygonal graphs via the method presented in Theorem 1.2.12, this time culminating in the proof of the following:

Theorem 1.3.7. There are 2-arc transitive polygonal graphs of arbitrarily large valency and girth m for all even m ≥ 4.

Corollary 1.3.8. There are infinitely many 2-arc transitive polygonal graphs of girth m for all m ≥ 3, and there 2-arc transitive polygonal graphs of arbitrarily large valency for each m.

Chapter 5 shows that the same construction used to prove Theorem 1.2.8 can be extended slightly, leading to the proof of the following in Section 5.3:
**Theorem 1.3.9.** If there exists a polygonal graph $\Gamma$ of valency $r$ and girth $m$, then there exists a polygonal graph $\Gamma_3$ of valency $r$ and girth $3m$. Moreover, if $\Gamma$ is a strict polygonal graph, then so is $\Gamma_3$.

Furthermore, we are able to show that many properties of the automorphism group are preserved by this construction, namely:

**Theorem 1.3.10.** If $\Gamma$ is a vertex transitive, edge transitive, arc transitive, or 2-arc transitive polygonal graph, then so is $\Gamma_k$, the graph constructed via Theorem 1.2.8, for $k = 2,3$. 
CHAPTER 2

A FAMILY OF NEAR-POLYGONAL GRAPHS OF VALENCE 10

2.1 Preliminary Results

Following Perkel ([12]), we will let \( \Omega \) be the vertex set of a graph \( \Gamma \). For a vertex \( \delta \in \Omega \), let \( W^{(r)}(\delta) \) denote the set of vertices reachable by an \( r \)-long walk from \( \delta \). We will also denote \( W^{(1)}(\delta) \) by \( N_\Gamma(\delta) \) and refer to it as the neighbor set or simply as the neighbors of \( \delta \). We denote by \( \text{Aut}(\Gamma) \) the set of automorphisms of the graph \( \Gamma \). Finally, for \( G \leq \text{Aut}(\Gamma) \) and \( \alpha, \beta, \gamma, \ldots \) elements of \( \Omega \), \( G_{\alpha \beta \gamma \ldots} \) denotes the subgroup of \( G \) fixing each vertex \( \alpha, \beta, \gamma, \ldots \), and for a subgroup \( H \leq \text{Aut}(\Gamma) \), \( \Omega(H) \) is the set of vertices of \( \Gamma \) fixed by every element of \( H \).

Our construction uses the following results of Perkel ([12]).

Lemma 2.1.1. [12, Lemma 2.1] Let \( \Gamma \) be a connected graph on a set \( \Omega \), and let \( G \leq \text{Aut}(\Gamma) \), with \( G \) transitive on \( \Omega \). Let \( \alpha, \beta, \gamma \in \Omega \) with \( \beta, \gamma \in N_\Gamma(\alpha) \), and \( \beta \neq \gamma \). Suppose that

(i) \( G_\alpha \) is 2-transitive on \( N_\Gamma(\alpha) \),

(ii) \( G_{\alpha \beta \gamma} \) fixes no vertices of \( N_\Gamma(\alpha) \setminus \{\beta, \gamma\} \), and
(iii) \( G_{\alpha\beta\gamma} \) is conjugate in \( G_\alpha \) to every \( G \)-conjugate of \( G_{\alpha\beta\gamma} \) which lies in \( G_{\alpha\beta} \), i.e. for \( g \in G, (G_{\alpha\beta\gamma})^g < G_{\alpha\beta} \Rightarrow (G_{\alpha\beta\gamma})^g = (G_{\alpha\beta\gamma})^h \) for some \( h \in G_\alpha \). (Note in particular that this condition holds if \( G_\alpha \) contains just one conjugacy class of subgroups isomorphic to \( G_{\alpha\beta\gamma} \).)

Then connected components of the subgraph of \( \Gamma \) induced by \( \Omega(G_{\alpha\beta\gamma}) \) are either isolated vertices of cycles of the graph.

**Theorem 2.1.2.** [12, Theorem 2.2] Assume the hypothesis of Lemma 2.1.1 and assume \( \Gamma \) is not the complete graph on \( \Omega \). Then we can find a set \( \mathcal{C} \) of \( m \)-cycles of \( \Gamma \) such that \( \Gamma \) is a near-polygonal graph, and \( G \leq \text{Aut}(\Gamma) \).

Using the coset graph as a basis, Perkel proves the following theorem which gives some explicit conditions when a group will satisfy Lemma 2.1.1.

**Theorem 2.1.3.** [12, Theorem 2.3] Let \( G \) be a group, and \( H \) a subgroup of \( G \). Let \( g \in G \setminus H \) with \( g^2 \in H \), and suppose that \( G = \langle H, g \rangle \). Let \( h \in H \setminus H^g, K = H \cap H^g, \) and \( L = H \cap H^g \cap H^{gh} \). Suppose that

(i) \( H = K \cup KhK \),

(ii) if \( L \leq K^k \) for some \( k \in H \), then \( k \in K \cup Kh \), and

(iii) if \( t \in G \) with \( L^t \leq K \), then \( L^t = L^s \) for some \( s \in H \).

Then either \( G = H \cup HgH \) (in which case \( \Gamma \) is the complete graph on \( \Omega \)) or there is a near-polygonal graph \( \Gamma \) with \( G \leq \text{Aut}(\Gamma) \).

Using Theorem 2.1.3 as a basis, we can now prove the following Corollary:
Corollary 2.1.4. Let $G$ be a group and $H \leq G$ isomorphic to $PSL(2, r)$. Let $K$ be a maximal subgroup of $H$ isomorphic to $F_{r(r-1)}$, i.e. the Frobenius group $r : \frac{r-1}{2}$, and let $L$ be a cyclic subgroup of $K$ of order $\frac{r-1}{2}$. Suppose that:

(i) There exists $g \in G$ such that $g \notin H, g^2 \in H, K = H \cap H^g$, and $G = \langle H, g \rangle$.

(ii) There exists $h \in H \setminus H^g$ such that $L = H \cap H^g \cap H^{gh}$.

Then the coset graph $\Gamma := \text{Cos}(G, H, H^g H)$ is a near-polygonal graph.

Proof. We will show that our group $G$ satisfies the conditions of Theorem 2.1.3. We first determine the size of $KhK$. Suppose $k_1hk_2 = k_3hk_4$. Then $(k_1^{-1}k_3)^h = k_2k_4^{-1} \in K$, i.e. $(k_1^{-1}k_3)^h \in K^h \cap K = L$. Given $k_1$, there are only $\frac{r-1}{2}$ possible choices for $k_3$ (since $|L| = \frac{r-1}{2}$). If we are also given $k_2$, then, once $k_3$ is chosen, $k_4$ is uniquely determined. Thus for any given $k_1, k_2$, there are precisely $\frac{r-1}{2}$ pairs $k_3, k_4$ such that $k_1hk_2 = k_3hk_4$.

Hence

$$|KhK| = \frac{|K| \cdot |K|}{\frac{r-1}{2}} = \frac{r(r-1)}{2} \cdot \frac{r(r-1)}{2} \cdot \frac{2}{r-1} = \frac{r^2(r-1)}{2}. \quad (2.1)$$

Now, since $h \notin K$, $K \cap KhK = \emptyset$,

$$|K \cup KhK| = \frac{r(r-1)}{2} + \frac{r^2(r-1)}{2} = \frac{r(r^2-1)}{2} = |H|, \quad (2.2)$$

and so $H = KhK$ as desired.

Next, by [4] we know that $H$ contains precisely $(r+1)$ conjugates of $K$. We also know that $H$ contains $(r+1)$ distinct commutative groups of order $r$, one in each subgroup conjugate to $K$. Furthermore, each group conjugate to $K$ is the normalizer in $H$ of its respective commutative $r$-group. Each group conjugate to $K$ contains $r$ conjugate subgroups of order $\frac{r-1}{2}$. Being that there are $(r+1)$ conjugates of $K$ in $H$, this counts $r(r+1)$ cyclic subgroups of order $\frac{r-1}{2}$. Since there actually are a total of
conjugate cyclic groups of order \( \frac{r-1}{2} \) in \( H \), this means each cyclic \( \frac{r-1}{2} \)-subgroup normalizes exactly two commutative \( r \)-subgroups, i.e. \( L \leq K, L \leq K^h \) and that’s all. Thus if \( L \leq K^k \) for some \( k \in H \), then either \( K^k = K \) or \( K^k = K^h \), and since \( N_H(K) = K \), we have \( k \in K \cup K^h \), as desired.

Finally, if \( L^t \leq K \) for some \( t \in G \), then \( L^t \) is another cyclic subgroup of order \( \frac{r-1}{2} \) in \( K \). However, again by [4], all the cyclic subgroups of order \( \frac{r-1}{2} \) in \( H \) are conjugate, so there is an \( s \in H \) such that \( L^t = L^s \). We can now apply Theorem 2.1.3 to see that \( \Gamma = \text{Cos}(G, H, HgH) \) is either near-polygonal or a complete graph.

Suppose now that \( \Gamma \) is a complete graph. Since \( Hg \) and \( Hgh \) are distinct vertices of \( \Gamma \) that are both adjacent to \( H \), there must exist \( t \in G \) such that \( (Ht, Hgt) = (Hg, Hgh) \). So, comparing \( Hgt \) and \( Hgh \), there must be an element \( k \in K \) such that \( t = kh \), but then \( Hg = Ht = H \), a contradiction. Hence \( \Gamma \) is near-polygonal.

\[ \square \]

### 2.2 A New Family of Polygonal Graphs

In order to apply Corollary 2.1.4, we note that for \( q \equiv 1, 19 \pmod{30} \), the group \( \text{PSL}(3, q) \) contains a subgroup isomorphic to \( \text{PSL}(2, 9) \cong A_6 \) ([3]). Note also that for all primes \( p \neq 2, 3, 5 \), we have \( p^2 \equiv 1, 19 \pmod{30} \). For calculation purposes, we view \( \text{PSL}(3, q) \) as the image of \( \text{SL}(3, q) \) modulo scalar matrices.

Consider \( \text{SL}(3, q) \) where \( q \) is a prime power and \( q \equiv 1, 19 \pmod{30} \). Then \( \text{SL}(3, q) \) has a subgroup \( H' \cong 3.A_6 \) (see for example [3]). From [3], we see furthermore that \( H' = \langle T', M', A' \rangle \), where

\[ T' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (2.3) \]
\[ M' = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} - t & -t \\ t - \frac{1}{2} & t & -\frac{1}{2} \\ t & -\frac{1}{2} & 1 - t \end{bmatrix}, \quad \text{and} \]

\[ A' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega^2 & 0 \end{bmatrix}, \quad (2.5) \]

where \( \omega^3 = 1 \neq \omega \) and \( t \) satisfies \( 4t^2 - 2t - 1 = 0 \) (e.g., \( \omega = \frac{-1+\sqrt{-3}}{2}, t = \frac{1+\sqrt{5}}{4} \)).

Let \( B' = A'M'T' \)

\[ \begin{bmatrix} -\frac{1}{2} & \omega t & -\frac{1}{2} \omega \omega(t - \frac{1}{2}) \\ \omega t & t & \frac{1}{2} \omega^2 \\ -\omega^2(\frac{1}{2} - t) & \omega^2 t & -\frac{1}{2} \end{bmatrix} \]

Then note that \( 3.A_6 = \langle A', B' \rangle \) with relations \( A'^2 = B'^{12} = (A'B')^5 = 1 \). If \( \phi : \text{SL}(3, q) \to \text{PSL}(3, q) \) is the natural map with \( \phi(A') = A \) and \( \phi(B') = B \), we have

\[ H = A_6 \cong \phi(3.A_6) = \langle A, B | A^2 = B^4 = (AB)^5 = 1 \rangle. \]

Furthermore, we have \( K' = 3.F_{36} \leq 3.A_6 \) with \( 3.F_{36} = \langle A', C' \rangle \), where \( C' = B'A'B'A'B'B' \). Naturally, if \( \phi(C') = C \), then \( K = F_{36} \cong \langle A, C \rangle \).

Define \( S'_1 := C'A'C'C'A' \) and \( S'_2 := C'C'A' \). Note that \( S'_1 \) and \( S'_2 \) generate the normal subgroup of order 27 in \( 3.F_{36} \) (and hence \( \phi(S'_1) = S_1 \) and \( \phi(S'_2) = S_2 \) generate the normal subgroup of order 9 in \( K = F_{36} \)).

**Lemma 2.2.1.** There is an order 4 element \( g \) in \( \text{PSL}(3, q) \) such that \( g \notin A_6, g^2 \in A_6, \) and \( A_6 \cap A_6^g = F_{36} \).

**Proof.** We will explicitly construct \( g \). First note that

\[ A'^{B'-1}A'^{B'-2} = A'^{C'-1}A' = \begin{bmatrix} -\frac{1}{2} & t - \frac{1}{2} & -t \\ t - \frac{1}{2} & -t & -\frac{1}{2} \\ -t & -\frac{1}{2} & t - \frac{1}{2} \end{bmatrix} \quad (2.6) \]

is an element of \( 3.F_{36} \) and \( 3.A_6 \). This element has order 2. We introduce the following change of basis matrix:

\[ P' := \begin{bmatrix} 1 & c \omega^2 & c \\ 1 & \frac{1}{c} \omega & c \\ 1 & \omega^2 & \frac{1}{c} \end{bmatrix}, \quad (2.7) \]
where \( c = -\frac{1+2\omega t}{2(1+2\omega^2 t)} \) (and naturally \( P := \phi(P') \)). Note that

\[
P^{-1} = \frac{1}{(1 + 2c\omega)(1 - c\omega)} \begin{bmatrix} 1 + c\omega & -c^2\omega^2 & -c^2\omega \\ -c & c(1 + c\omega) & -c^2 \omega \\ -c\omega^2 & -c^2 & c\omega^2(1 + c\omega) \end{bmatrix}.
\] (2.8)

We now, unbelievably, have the following nice matrices:

\[
A^{P^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},
\] (2.9)

\[
S_1^{P^{-1}} = \begin{bmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{bmatrix},
\] (2.10)

\[
A^{C_{v^{-1}}P^{-1}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -\omega^2 \\ 0 & -\omega & 0 \end{bmatrix}, \text{ and}
\] (2.11)

\[
S_2^{P^{-1}} = \begin{bmatrix} 0 & 0 & d \\ \frac{\omega^2}{d} & 0 & 0 \\ 0 & \omega & 0 \end{bmatrix},
\] (2.12)

where \( d = \frac{c}{2(1+2\omega)(1-c\omega)}(c^2(\omega - 4\omega t - 2t) + c(-4t - 2\omega + 4\omega t) + \omega) \).

Now look at

\[
x' = \begin{bmatrix} \frac{1}{\sqrt{-3}} & -\frac{1}{3a} & -\omega^2 \\ \frac{a}{\sqrt{-3}} & \frac{3a}{\sqrt{-3}} & \frac{\omega^2}{\sqrt{-3}} \\ \omega a & \frac{\omega^2}{\sqrt{-3}} & \frac{1}{\sqrt{-3}} \end{bmatrix},
\] (2.13)

where \( a = \frac{\omega^2}{\sqrt{-3}} \). Note that \( x'^2 = A^{C_{v^{-1}}P^{-1}} \), so \( x' \) has order 4 and is an element of \((3.F_{36})^{P^{-1}}\). Also, \( A^{P^{-1}}x' = A^{P^{-1}} \) and \( S_1^{P^{-1}}x' = S_2^{P^{-1}} \cdot \omega^2I \), i.e. \( \phi(x') = x \) normalizes \( K = F_{36} \). Since \( x \notin K \), \( A_6 \) is simple, and \( F_{36} \) is a maximal subgroup of \( A_6 \), \( x \notin H \). Thus our desired \( g \) is simply \( x^P \).

Now let \( h := \phi(B'^{-1}T'B') \). Note that \( h \in H \setminus H^g \) and \( h \) has order 2. We can now prove the main theorem.
Proof of Theorem 1.3.1. We let $H$ be a copy of $A_6 \cong \text{PSL}(2,9)$ in $\text{PSL}(3,q)$ and $g$ the element provided by Lemma 2.2.1 above. Furthermore, we note that $L := H \cap H^g \cap H^{g^h}$ is a cyclic subgroup of order 4 (generated by the element $\ell := CCAC$). Therefore, by Corollary 2.1.4, $\Gamma := \text{Cos}(G,H,HgH)$ is a near-polygonal graph. 

2.3 Determining the length of the special cycles

We continue with the notation from the previous section. Explicit calculation shows that $gh\ell(\ell gh)^{-1} = I$, i.e.

$$gh \in C_G(L). \quad (2.14)$$

Since, by construction, $L$ is the stabilizer of the 2-arc $(Hgh, H, Hg)$, $L$ must also stabilize the special cycle containing the 2-arc $(Hgh, H, Hg)$. Also by construction, we note that $L$ fixes precisely the special cycle containing the 2-arc $(Hgh, H, Hg)$ (see the proof of [12, Lemma 2.1]). On the other hand, (2.14) implies that $L$ fixes $H(gh)^j$ for any integer $j > 0$. Thus the circuit $(H, Hgh, Hghgh, Hghghgh, ...)$ must be the special cycle containing the 2-arc $(Hgh, H, Hg)$, and finding the length of the special cycles is equivalent to finding the smallest $m \in \mathbb{Z}$ such that $(gh)^m \in H$. Note further that if $(gh)^m \in H$, then $(gh)^m$ stabilizes our circuit and hence $(gh)^m \in L$.

In order to determine when a group $\text{PSL}(3,q)$ will yield a near-polygonal graph with cycles of length $m$ for a given $m$, we will start working in the group $\text{SL}(3,\mathbb{C})$ with matrices defined as in the above section (note that this reintroduces the scalar matrices). The following computations were done with the help of Mathematica [16]. Now we have
The numbers $g_1, \ldots, g_9$ are listed in Appendix C. Note that $g$ and $h$ still have orders 4 and 2, respectively, but $\ell$ now has order 12. Further calculation shows that

$$\chi_{gh}(x) = x^3 - c_2 x^2 + c_1 x - 1,$$

(2.19)

where

$$c_2 = \text{Tr}(gh) = \frac{308052 + 239699i\sqrt{3} + 82209\sqrt{5} - 321908i\sqrt{15}}{719097 - 82209i\sqrt{15}},$$

(2.20)

and the constant coefficient is -1 since $\det(gh) = 1$. The numbers $c_{11}, \ldots, c_{18}$ are listed in Appendix C.

Calculation also shows that $\chi_{gh}(-1) = -1 - c_2 - c_1 - 1 = 0$, meaning $(x+1)|\chi_{gh}(x)$.

We conclude that the remaining roots of the characteristic polynomial (and hence the remaining eigenvalues) must be roots of
Lemma 2.3.1. If $\Gamma$ is a near-polygonal graph constructed as in Theorem 1.3.1, then the length of the special cycles in $\Gamma$ must be even.

Proof. First, since $\text{Det}(\ell) = 1$, $\text{Tr}(\ell) = -\frac{1+i\sqrt{3}}{2}$, and $\text{Tr}(\ell^{-1}) = \frac{1-i\sqrt{3}}{2}$, we see that

$$\chi_\ell(x) = x^3 - \frac{-1 + i\sqrt{3}}{2}x^2 + \frac{1 - i\sqrt{3}}{2}x - 1,$$  

and the roots of $\chi_\ell(x)$ (and hence the eigenvalues of $\ell$) are $-\frac{1+i\sqrt{3}}{2}$, $\frac{i+\sqrt{3}}{2}$, and $-\frac{i-\sqrt{3}}{2}$.

Now suppose that $\Gamma$ has special cycles of length $m$, $m$ odd. Thus $(gh)^m \in L = \langle \ell \rangle$. Since $\ell$ and $gh$ commute (see (2.14)), as matrices $\ell$ and $gh$ are simultaneously diagonalizable. Mathematica [16] gives us the following eigenvectors of $\ell$:

$$v_1 = (v_{11}, v_{12}, 1),$$  

$$v_2 = (v_{21}, v_{22}, 1),$$  

$$v_3 = (v_{31}, v_{32}, 1).$$

The numbers $v_{11}, \ldots, v_{32}$ are listed in Appendix C. We now define a change of basis matrix $Q$ by $Q = [v_1 \ v_2 \ v_3]$ (vectors written in column form), and from this we see that

$$Q^{-1}ghQ = \begin{bmatrix} -1 & 0 & 0 \\ 0 & r_1 & 0 \\ 0 & 0 & r_2 \end{bmatrix},$$  

where $r_1$ and $r_2$ are the other two roots of $\chi_{gh}(x)$ (their exact values are unimportant for this proof), and so the change of basis matrix $Q$ puts the eigenvalue -1 in the first
row, first column. Applying the change of basis to \( \ell \), we see that

\[
Q^{-1}\ell Q = \begin{bmatrix}
\frac{i-\sqrt{3}+3\sqrt{5}+\sqrt{15}}{i+\sqrt{3}-3i\sqrt{5}+\sqrt{15}} & 0 & 0 \\
0 & \frac{i(i-\sqrt{3}+3i\sqrt{5}+\sqrt{15})}{i+\sqrt{3}-3i\sqrt{5}+\sqrt{15}} & 0 \\
0 & 0 & \frac{1+i\sqrt{3}+3\sqrt{5}+i\sqrt{15}}{i+\sqrt{3}-3i\sqrt{5}+\sqrt{15}}
\end{bmatrix}.
\] (2.28)

Hence we have

\[
\frac{i-\sqrt{3}+3\sqrt{5}+\sqrt{15}}{i+\sqrt{3}-3i\sqrt{5}+\sqrt{15}} = \frac{-1+i\sqrt{3}}{2} = \omega
\]
in the first row, first column.

Now \( m \) odd implies that the first row, first column of \( Q^{-1}(gh)^mQ \) is -1. On the other hand, for any integer \( j \), the first row, first column of \( Q^{-1}\ell^jQ \) must be one of 1, \( \omega \), \( \omega^2 \), none of which are -1, a contradiction. Hence the cycles of \( \Gamma \) must have even length.

Lemma 2.3.2. If we consider \( g, h, \ell \) as elements of \( SL(\mathbb{C}) \) and \( (gh)^m = s \in \langle \ell \rangle \), then \( s^4 = I \).

Proof. As in the above proof, we note that the first row, first column of a diagonalized \( (gh)^m \) must be 1. If the first row, first column entry of \( \ell^j \) is 1, then \( j = 3, 6, 9, 12 \). Thus \( s = \ell^3, \ell^6, \ell^9, \ell^{12} \) and, in any case, \( s^4 = I \).

Lemma 2.3.3. If \( x \) is a root of \( x^2 - (1 + c_2)x - 1 = 0 \), then for any positive integer \( n \) we have \( x^n = a_nx + b_n \), where

(i) \( a_2 = (1 + c_2) \),

(ii) \( b_2 = 1 \),

(iii) \( a_{n+2} = (1 + c_2)(a_n(1 + c_2) + b_n) + a_n \),

(iv) \( b_{n+2} = a_n(1 + c_2) + b_n \).

Proof. The first two parts are clear from \( x^2 = (1 + c_2)x + 1 \). For the other two parts, we proceed by induction. We have our base case, so assume the conclusion is true for \( k = n \). Then
\[ x^{n+2} = x^2 \cdot x^n \] (2.29)

\[ = ((1 + c_2)x + 1)(a_n x + b_n) \] (2.30)

\[ = (1 + c_2)a_n x^2 + b_n x + a_n x + b_n \] (2.31)

\[ = (1 + c_2)a_n((1 + c_2)x + 1) + b_n x + a_n x + b_n \] (2.32)

\[ = ((1 + c_2)(a_n(1 + c_2) + b_n) + a_n)x + (a_n(1 + c_2) + b_n), \] (2.33)

as desired.

**Lemma 2.3.4.** If \( a'_n, b'_n \) are coefficients such that \( x^{4n} = a'_n x + b'_n \), then

(i) \( a'_n = a_n^4(1 + c_2)((1 + c_2)^2 + 2) + 4a_n^3b_n((1 + c_2)^2 + 1) + 6a_n^2b_n^2(1 + c_2) + 4a_nb_n^3, \)

(ii) \( b'_n = a_n^4((1 + c_2)^2 + 1) + 4a_n^3b_n(1 + c_2) + 6a_n^2b_n^2 + b_n^4. \)

*Proof.* We note that \( x^{4n} = (x^n)^4 = (a_n x + b_n)^4 \). Multiplying this out and repeatedly substituting \((1 + c_2)x + 1\) for \( x^2 \) as in the proof of Lemma 2.3.3 yields the desired result.

**Theorem 2.3.5.** If \( q = p, p^2 \) for some prime \( p \neq 2, 3, 5 \), the near-polygonal graph \( \Gamma \) constructed from \( PSL(3, q) \) as in Theorem 1.3.1 has special cycles of length \( m \), and \( a'_m, b'_m \) are defined as above, then the equation

\[ (1 - b'_m)^2 - a'_m(1 + c_2)(1 - b'_m) - a_m^2 = 0 \] (2.34)

must hold in \( GF(q) \).

*Proof.* Since \( \Gamma \) has cycles of length \( m \), we know that \((gh)^m \in L\). By Lemma 2.3.1, we know that \( m \) is even, and by Lemma 2.3.2 we know that \((gh)^{4m} = I\) when we consider
Table 2.1: Results obtained from Theorem 2.3.5

\[
\begin{array}{|c|c|c|}
\hline
m & z \cdot \overline{z} & n \\
\hline
4 & 2^{1022} & 2^{37} \cdot 5 \\
6 & 2^{1566} & 2^{51} \cdot 5 \cdot 7^2 \cdot 11^2 \\
8 & 2^{2110} & 2^{71} \cdot 5 \cdot 79^2 \\
10 & 2^{2654} & 2^{83} \cdot 5^3 \cdot 19^2 \cdot 59^2 \\
12 & 2^{3198} & 2^{101} \cdot 5 \cdot 7^2 \cdot 11^2 \cdot 239^2 \\
14 & 2^{3742} & 2^{115} \cdot 5 \cdot 13^2 \cdot 29^2 \cdot 1009^2 \\
\hline
\end{array}
\]

\(gh\) as a matrix in SL(3, \(\mathbb{C}\)). This means that the roots of \(x^2 - (1 + c_2)x - 1 = 0\), the eigenvalues of \(gh\) that are not -1, must satisfy \(x^{4m} = a'_m x + b'_m = 1\). Solving for \(x\), we get

\[
x = \frac{1 - b'_m}{a'_m}.
\]  

(2.35)

Substituting (2.35) into \(x^2 - (1 + c_2)x - 1 = 0\) and clearing the denominator, we get (2.34), as desired. \(\square\)

### 2.4 Calculations

Table 2.1 contains the results obtained from the above recursion using Mathematica ([16]), determining the primes \(p\) for which PSL(3, \(p\)) or PSL(3, \(p^2\)) admits a near-polygonal graph of a given girth \(m\). In each case, the numerator of the number obtained from (2.34) was of the form \(n \cdot z\), where \(n\) was an integer and \(z \cdot \overline{z}\) was a power of 2.

Note that we were able to calculate and factor the numerator through \(m = 38\), and that we probably could factor many numbers beyond that. However, the limits of modern computing makes it impossible to determine the girth of these graphs, and, since we are searching for polygonal graphs, the results were omitted.
<table>
<thead>
<tr>
<th>Group</th>
<th>Cycle Length</th>
<th>Polygonal?</th>
<th>Strict Polygonal?</th>
</tr>
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<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
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<td>PSL(3, 79)</td>
<td>8</td>
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</tr>
<tr>
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<td>10</td>
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<td>PSL(3, 29^2)</td>
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<tr>
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<td>PSL(3, 1009)</td>
<td>14</td>
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Table 2.2: New 2-arc transitive polygonal graphs of valency 10

In Table 2.2, we include the results obtained from analyzing some of these near-polygonal graphs using GAP ([15], see Appendices A and B).

Hence our construction has explicitly produced seven new examples of polygonal graphs, and, once we have access to computers with better memory, possibly more.
CHAPTER 3

A CONSTRUCTION OF AN INFINITE FAMILY OF 2-ARC TRANSITIVE Polygonal GRAPHS OF ARBITRARY ODD GIRTH

3.1 Preliminary results

As in [6], an \((l,m)\)-path-cycle cover of a graph \(\Gamma\) is a set \(C\) of \(m\)-cycles such that each \(l\)-path of \(\Gamma\) is covered by at least one cycle in \(C\); sometimes an \((l,m)\)-path-cycle cover is simply called an \((l,m)\)-cover. If in addition every \(l\)-path of \(\Gamma\) lies in a constant number \(\lambda\) of cycles of \(C\), then \(C\) is called a regular \(\lambda\)-(\(l,m\))-cover, or simply called a \(\lambda\)-(\(l,m\))-cover. Hence near-polygonal graphs are the graphs that have a \(1\)-(2\(m\))-cover for some \(m\).

For a graph \(\Gamma\) and a group \(G \leq \text{Aut}(\Gamma)\), an \((l,m)\)-cover \(C\) is called \(G\)-symmetrical if \(C = \{C_1, C_2, ..., C_n\}\) is such that

(i) the restriction \(G|_{C_i}\) of \(G\) to each \(C_i\) contains all rotations of \(C_i\);

(ii) \(G\) induces a transitive action on \(C\).
There are two possibilities for $G|_{C_i}$ to contain all rotations of $C_i$, namely $C_i \cong \mathbb{Z}_m$ or $C_i \cong D_{2m}$. The corresponding symmetrical covers will be called $G$-rotary or $G$-dihedral, respectively. For a positive integer $l$, a graph $\Gamma$ is called $(G,l)$-arc transitive, $(G,l)$-dipath transitive, or $(G,l)$-path transitive if $G$ acts transitively on $l$-arcs, $l$-dipaths, or $l$-paths of $\Gamma$, respectively. In the case of dipath and path transitivity, we also require that $l$-dipaths or $l$-paths exist in $\Gamma$, respectively.

We now present the basis for the construction of near-polygonal graphs in [6].

**Lemma 3.1.1.** [6, Lemma 1.1] Let $\Gamma$ be a regular graph of valency at least three, let $G \leq \text{Aut}(\Gamma)$, and let $l \geq 1$ be an integer. Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) holds for the following four statements (a)-(d).

(a) $\Gamma$ has a $G$-dihedral $(l,m)$-cover for some $m \geq 3$.

(b) $\Gamma$ is $(G,l)$-dipath transitive.

(c) $\Gamma$ has a $G$-rotary $(l,m)$-cover for some $m \geq 3$.

(d) $\Gamma$ is $(G,l)$-path transitive.

Moreover, if $\Gamma$ has a $G$-dihedral $(l,m)$-cover $C$ and $G$ acts sharply transitively on the $l$-dipaths in $\Gamma$ then $C$ is a 1-$(l,m)$-cover.

We will use Lemma 3.1.1 in conjunction with the following method for construction.

**Construction 3.1.2.** [6, Construction 2.1] Let $(\alpha_0, ..., \alpha_l)$ and $(\alpha_1, ..., \alpha_l, \alpha_{l+1})$ be $l$-dipaths in a graph $\Gamma$ (allowing that $\alpha_0 = \alpha_{l+1}$), let $G \leq \text{Aut}(\Gamma)$, and suppose that there exists $g \in G$ such that $\alpha_i^g = \alpha_{i+1}$ holds for $0 \leq i \leq l$. Let $C$ be the cycle generated by the vertices $\alpha_0^{(g)}$, and let $C = C^G$. 26
We shall refer to the method described in Construction 3.1.2 as \textit{spinning an \( l \)-dipath}. In order to apply Lemma 3.1.1, we need to ensure that a target group \( G \) occurs as a group of automorphisms of some graph \( \Gamma \). That goal can be achieved by defining \( \Gamma \) as a coset graph. [6]

For constructing near-polygonal graphs as in [6], we want to spin a 2-dipath \( (\gamma, \alpha, \beta) \). The spinning element can be described easily in terms of the coset graph.

**Lemma 3.1.3.** [6, Lemma 2.3] For a coset graph \( \Gamma = \text{Cos}(G, H, HgH) \) with \( g^2 \in H \), let \( \alpha = H \) and \( \beta = \alpha^g \). Then an element \( f \in G \) maps \( \alpha \) to \( \beta \) if and only if \( f \in G_{\alpha g} \). Furthermore, for \( f \in G \) such that \( \alpha^f = \beta \), we have that \( \beta \neq \alpha^{f^{-1}} \) if and only if \( f \in G_{\alpha g \setminus G_{\alpha \beta g}} \).

Finally, we will use the following well-known lemma as stated in [6].

**Lemma 3.1.4.** [6, Lemma 2.4] Let \( \Gamma \) be a graph, and let \( G \leq \text{Aut}(\Gamma) \) be transitive on the vertex set of \( \Gamma \). Then \( \Gamma \) is \((G, 2)\)-dipath transitive if and only if \( G_\alpha \) acts \( 2 \)-transitively on \( N_\Gamma(\alpha) \); furthermore, \( \Gamma \) is sharply \((G, 2)\)-dipath transitive if and only if \( G_\alpha \) acts sharply \( 2 \)-transitively on \( N_\Gamma(\alpha) \).

### 3.2 A family of polygonal graphs with cycles of a fixed odd length

This is a generalization of the construction in [6, Section 3]. Let \( m = 2k + 1, k \in \mathbb{N} \).

We define \( G \) to be the direct product of \( k \) copies of \( \text{PGL}(2, q) \), where \( q \) is a prime power, i.e. \( G := \prod_{i=1}^{k} \text{PGL}(2, q) \). The elements of \( \text{PGL}(2, q) \) can be identified with equivalence classes of \( 2 \times 2 \) invertible matrices over the field \( \text{GF}(q) \), with two matrices equivalent if and only if they are scalar multiples of each other. With a slight abuse of notation, we shall write that the matrices themselves are
elements of $\text{PGL}(2, q)$, and that the elements of $G$ are $k$-tuples of matrices. We identify $H \cong \text{AGL}(1, q)$ with the set of (equivalence classes of) lower triangular matrices $\left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in \text{GF}(q), ac \neq 0 \right\}$ and define $\overline{H} := \text{Diag}(\prod_{i=1}^{k} \text{AGL}(1, q)) = \{(h, h, \ldots, h) : h \in H\} \leq G$.

For $a \in \text{GF}(q)$, let $p(a) := \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \in H$ and $\overline{p}(a) := (p(a), p(a), \ldots, p(a)) \in \overline{H}$. Moreover, if $a \neq 0$ then let $d(a) := \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \in H$ and $\overline{d}(a) := (d(a), d(a), \ldots, d(a)) \in \overline{H}$.

Let $P = \{p(a) : a \in \text{GF}(q)\}$, $\overline{P} = \{\overline{p}(a) : a \in \text{GF}(q)\}$, $D = \{d(a) : a \in \text{GF}(q)^*\}$, and $\overline{D} = \{\overline{d}(a) : a \in \text{GF}(q)^*\}$. Then $H = PD$, $\overline{H} = \overline{P}D$, $P \triangleleft H$, and $\overline{P} \triangleleft \overline{H}$.

For $y \in \text{GF}(q)^*$, let $g(y) = \begin{bmatrix} 0 & y \\ -1 & 0 \end{bmatrix}$. Then, for

$$g = g(y_1, y_2, \ldots, y_k) := (g(y_1), g(y_2), \ldots, g(y_k)) \in G$$

we have $g^2 = 1 \in \overline{H}$, so we can define the coset graph

$$\Gamma = \Gamma(y_1, y_2, \ldots, y_k) := \text{Cos}(G, \overline{H}, \overline{H}g\overline{H}).$$

Let $\alpha$ denote the vertex $\overline{H}$ and let $\beta$ denote the vertex $\overline{H}g$. First we determine the number of vertices, the valency, and bound the number of components of $\Gamma$. The number of vertices is

$$|G : \overline{H}| = |G|/|\overline{H}| = q^{k-1}(q - 1)^{k-1}(q + 1)^k.$$

For any $\overline{d} \in \overline{D}$ we have $\overline{d}^q = \overline{d}^{-1}$, so $G_{\alpha\beta} = \overline{H} \cap \overline{H}^q \geq \overline{D}$. Since $\overline{D}$ is a maximal subgroup of $\overline{H}$, we must have equality here, and so the valency of $\Gamma$ is

$$|G_{\alpha} : G_{\alpha\beta}| = |\overline{H} : \overline{D}| = q.$$

For a vertex $\delta$ of $\Gamma$, let $W^{(r)}(\delta)$ denote the set of vertices reachable by an $r$-long walk from $\delta$. Then we have the following, which has the same proof as [6, Lemma 3.1]:

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Lemma 3.2.1. \( W^{(r)}(\alpha) = \overline{H}g\overline{P}g\overline{P}...g\overline{P} \) (\( r \) iterations of \( g\overline{P} \)).

Now we will bound the number of components of \( \Gamma \).

Lemma 3.2.2. Let \( y_1, y_2, ..., y_k \) be distinct elements of \( \text{GF}(q)^* \). Then \( \Gamma \) has at most \( 2^{k-1} \) components.

Proof. The component containing \( \alpha \) consists of the cosets reachable by some walk in \( \Gamma \), that is, the cosets in \( \bigcup_{r \geq 0} \overline{H}(g\overline{H})^r = \langle \overline{H}, g \rangle \). If we define \( G^* := \langle \overline{H}, g \rangle \), then the number of components of \( \Gamma \) will be \( |G : G^*| \). Since the \( y_i \) are all distinct, \( G^* \) contains a nondiagonal subgroup \( G^{**} \) that projects bijectively on each coordinate of \( \prod_{i=1}^k \text{PSL}(2,q) \) with a different map for each coordinate, and this can only happen when \( G^{**} = \prod_{i=1}^k \text{PSL}(2,q) \). If \( q \) is even then \( \text{PGL}(2,q) = \text{PSL}(2,q) \) and so \( G = G^* = G^{**} \), implying that \( \Gamma \) is connected. If \( q \) is odd then \( |\text{PGL}(2,q) : \text{PSL}(2,q)| = 2 \) and \( G \geq G^* > G^{**} \) (since each coordinate of \( G^* \) surjects onto \( \text{PGL}(2,q) \), this last containment is strict). \( |G : G^{**}| = 2^k \), so this implies that the number of components is

\[
|G : G^*| \leq 2^{k-1}, \tag{3.5}
\]

and its exact value depends on how many pairwise \( y_iy_j \) are squares in \( \text{GF}(q) \). \( \square \)

Lemma 3.2.3. For all \( y_1, y_2, ..., y_k \in \text{GF}(q)^* \), the graph \( \Gamma \) is near-polygonal.

Proof. The group \( \overline{H} = G_\alpha \) acts sharply 2-transitively on the cosets of \( \overline{D} = G_{\alpha\beta} \) so, by Lemma 3.1.4, the graph \( \Gamma = \Gamma(y_1, ..., y_k) \) is sharply \( (G, 2) \)-dipath transitive. We will show that \( \Gamma \) has a \( G \)-dihedral \( (2, n) \)-cover for some \( n \), which, by Lemma 3.1.1, will show that \( \Gamma \) is near-polygonal. To this end, we define

\[
f = f(y_1, y_2, ..., y_k) := p(1) \cdot (g(y_1), ..., g(y_k)) = \left( \begin{array}{c} 0 & y_1 \\ -1 & y_1 \end{array} \right), ..., \left( \begin{array}{c} 0 & y_k \\ -1 & y_k \end{array} \right). \tag{3.6}
\]
We have \( f \in \bar{H}g \backslash Dg \) and so, by Lemma 3.1.3, the vertex
\[
\gamma := \alpha^f = \bar{H} \left( \begin{bmatrix} y_1 & -y_1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} y_k & -y_k \\ 1 & 0 \end{bmatrix} \right)
\] (3.7)
is different from \( \beta \) and \((\gamma, \alpha, \beta)\) is a 2-dipath. Spinning the dipath as in Construction 3.1.2, we obtain a cycle \( C = [\delta_0 = \alpha, \delta_1 = \beta, \delta_2, \ldots, \delta_{n-1} = \gamma] \) for some \( n \), and the \((2, n)\)-cover \( C = C^G \). We claim the group element
\[
z := \left( \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \right)
\] (3.8)
is an involution satisfying \( \delta_i z = \delta_{n-i} \) for \( 1 \leq i \leq n-1 \). Direct computation shows that \( z^2 = 1 \) and \( f^z = f^{-1} \). So \( \beta^z f = \beta f^{-1} z = \alpha z = \alpha \), implying \( \delta_i^z = \beta^z = \alpha f^{-1} = \gamma = \delta_{n-1} \).

Then, by induction on \( i = 1, 2, \ldots, n-1 \), it follows that \( \delta_{i+1}^z = \delta_i^z f = \delta_i^{z f^{-1}} = \delta_{n-i}^z = \delta_{n-i-1} \). Hence \( \langle f, z \rangle \) acts as the dihedral group on \( C \) and so \( C \) is a \( G \)-dihedral \((2, m)\)-cover. Consequently, Lemma 3.1.1 implies that \( \Gamma \) is a near-polygonal graph.

Next, we describe how to choose the values \( y_1, \ldots, y_k \) such that \( \Gamma = \Gamma(y_1, \ldots, y_k) \) has a \( 1-(2, m) \) cover. We define a sequence of polynomials \( u_n(y) \) by \( u_0(y) := 0 \),
\[
u_{2n+1}(y) := \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} y^{n-j}, \tag{3.9}
\]
\[
u_{2n+2}(y) := \sum_{j=0}^{n} (-1)^j \binom{2n+1-j}{j} y^{n-j} \text{ for } n \geq 0. \tag{3.10}
\]

We now have the following, which as far as we know was first stated in [17]:

**Lemma 3.2.4.** Let \( j \geq 1 \) be an integer.

(a) The roots of \( u_j(y) \) in \( \mathbb{C} \) are the numbers \( \xi + \xi^{-1} + 2 \), where \( \xi \) is a \( j \)th root of unity different from \( \pm 1 \).
(b) For all $j \geq 1$, we have $u_{2j}(y) = u_{2j-1}(y) - u_{2j-2}(y)$ and $u_{2j+1}(y) = yu_{2j}(y) - u_{2j-1}(y)$.

(c) For all $j \geq 0$, we have

$$u_{2j+1}(x + x^{-1} + 2) = \sum_{i=0}^{2j} x^{-j+i}, \quad (3.11)$$

$$u_{2j+2}(x + x^{-1} + 2) = j \sum_{i=0}^{j} x^{-j+2i}. \quad (3.12)$$

(d) Let $f(y) := \begin{bmatrix} 0 & y \\ -1 & y \end{bmatrix}$. Then for all $j \geq 1$, we have $f(y)^j = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix}$, where $a_j = -y^{\lfloor \frac{j}{2} \rfloor}u_{j-1}(y), b_j = y^{\lfloor \frac{j+1}{2} \rfloor}u_j(y), c_j = -y^{\lfloor \frac{j}{2} \rfloor}u_j(y)$, and $d_j = y^{\lfloor \frac{j}{2} \rfloor}u_{j+1}(y)$. We can use Lemma 3.2.4 to prove the following:

**Lemma 3.2.5.** Let $q \equiv \pm 1 \pmod{m}$, and let $\zeta = \zeta_m$ be a primitive $m^{th}$ root of unity in $\text{GF}(q^2)$. Then, if $y_i := \zeta^i + \zeta^{m-i} + 2 \in \text{GF}(q)^*$ for $1 \leq i \leq k$, all the $y_i$ are distinct and the graph $\Gamma = \Gamma(y_1, ..., y_k)$ has a 1-(2,m)-cover.

**Proof.** First we prove that $y_i \in \text{GF}(q)$. Indeed, computing in $\text{GF}(q^2)$ we have

$$y_i^q = (\zeta^i + \zeta^{m-i} + 2)^q = \zeta^{qi} + \zeta^{q(m-i)} + 2 = \zeta^i + \zeta^{m-i} + 2 = y_i \quad (3.13)$$

since $q \equiv \pm 1 \pmod{m}$, which implies that $y_i \in \text{GF}(q)$. Note further that $y_i = 0$ if and only if $\zeta^i = -1$, and since $m$ is odd, $\zeta^i \neq -1$ and so for all $1 \leq i \leq k$, $y_i \in \text{GF}(q)^*$. We also have $y_i \neq y_j$ for $i \neq j$ because

$$y_i - y_j = \frac{(\zeta^i - \zeta^j)(\zeta^{i+j} - 1)}{\zeta^{i+j}} \neq 0 \quad (3.14)$$

by our assumptions on $i$ and $j$. By Lemma 3.2.4(c), $u_m(y_i) = 0$ and then Lemma 3.2.4(b) implies $u_{m+1}(y_i) = -u_{m-1}(y_i)$ for $1 \leq i \leq k$. Hence, by Lemma 3.2.4(d), for
the spinning element \( f = f(y_1, ..., y_k) \) as defined in Lemma 3.2.3 we have \( f^m = 1 \). Moreover, since \( \zeta \) is a primitive \( m \)th root of unity, we have \( u_n(y_1) \neq 0 \) for \( 1 \leq n < m \) because \( \zeta \) is not an \( n \)th root of unity. Therefore \( f^n \notin \overline{H} \). This means that the cycle spun by \( f \) has length \( m \) and so \( \Gamma \) has a 1-(2,\( m \))-cover.

We now proceed to show that the girth of the graph \( \Gamma \) we have just constructed is at least \( m \). We define
\[
  r^{(l)}(y) := \prod_{i=1}^{l} g(y)p(a_i),
\]
where \( y \in \text{GF}(q)^* \) and for all \( i, a_i \in \text{GF}(q)^* \). Note that we view \( r^{(l)}(y) \) as a function of \( y \) given arbitrary fixed units \( a_1, ..., a_l \in \text{GF}(q)^* \).

Lemma 3.2.6. If \( r^{(l)}(y) = \begin{bmatrix} r^{(l)}_{11}(y) & r^{(l)}_{12}(y) \\ r^{(l)}_{21}(y) & r^{(l)}_{22}(y) \end{bmatrix} \), then for all \( l \geq 2 \),

(a) \( r^{(l)}_{12}(y) = yr^{(l-1)}_{11}(y) \).

(b) \( \deg(r^{(l)}_{11}(y)) = l \).

(c) \( r^{(l)}_{11}(y) = y^{\left\lfloor \frac{l}{2} \right\rfloor} s_l(y) \), where \( s_l(y) \) is a degree \( \left\lfloor \frac{l}{2} \right\rfloor \) polynomial in \( y \) with leading coefficient \( a_1a_2...a_l \).

(d) \( s_l(y) = \begin{cases} ya_{l}s_{l-1}(y) - s_{l-2}(y), & l \text{ even;} \\ a_{l}s_{l-1}(y) - s_{l-2}(y), & l \text{ odd.} \end{cases} \)

(e) For all \( l \), the coefficient of \( y^{\left\lfloor \frac{l}{2} \right\rfloor - 1} \) in \( s_l(y) \), denoted by \( T_l \), is the opposite of the sum of \( l - 1 \) terms, each of which is the product of \( l - 2 \) different \( a_i \), \( \left\lfloor \frac{l-2}{2} \right\rfloor \) of which are odd \( i \).

(f) The constant term of \( s_l(y) \), denoted by \( K_l \), is \( (-1)^\frac{l}{2} \) for \( l \) even and \( (-1)^{\frac{l-1}{2}}(a_1 + a_3 + a_5 + \cdots a_l) \) for \( l \) odd.
(g) The linear term of $s_l(y)$ for $l$ even has coefficient $S_l$, where $S_l$ is the product of $(-1)^{\frac{l}{2}+1}$ and the sum of $\frac{(-1)^{\frac{l}{2}+1}}{2}$ terms of the form $a_ia_j$, where $i$ is odd and $j$ is even.

(h) $r_{22}^{(l)} = yr_{21}^{(l-1)}(y)$.

(i) $\deg(r_{21}^{(l)}(y)) = l - 1$.

(j) $r_{21}^{(l)}(y) = y^{\lceil \frac{l-1}{2} \rceil} t_l(y)$, where $t_l(y)$ is a degree $\lfloor \frac{l-1}{2} \rfloor$ polynomial in $y$ with leading coefficient $-a_2a_3...a_l$.

(k) $t_l(y) = \begin{cases} y a_l t_{l-1}(y) - t_{l-2}(y), & l \text{ odd;} \\ a_l t_{l-1}(y) - t_{l-2}(y), & l \text{ even.} \end{cases}$

(l) The constant term of $t_l(y)$, denoted by $K'_l$, is $(-1)^{\frac{l+1}{2}}$ for $l$ odd and $(-1)^{\frac{l}{2}}(a_2 + a_4 + a_6 + \cdots + a_l)$ for $l$ even.

Proof. Note first that $g(y)p(a) = \begin{bmatrix} ya & y \\ -1 & 0 \end{bmatrix}$. Thus

$$r^{(1)}(y) = \begin{bmatrix} ya_1 & y \\ -1 & 0 \end{bmatrix} \quad (3.16)$$

and

$$r^{(2)}(y) = \begin{bmatrix} y(ya_1a_2 - 1) & y(ya_1) \\ -ya_2 & -y \end{bmatrix}. \quad (3.17)$$

This will prove our inductive step in (b) and (c). Furthermore, we note that

$$r^{(l)}(y) = r^{(l-1)}(y) \cdot \begin{bmatrix} ya_l & y \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11}^{(l-1)} & r_{12}^{(l-1)} & ya_l & y \\ r_{21}^{(l-1)} & r_{22}^{(l-1)} & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} ya_l & y \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} ya_l r_{11}^{(l-1)}(y) - r_{12}^{(l-1)}(y) & yr_{11}^{(l-1)}(y) \\ ya_l r_{21}^{(l-1)}(y) - r_{22}^{(l-1)}(y) & yr_{21}^{(l-1)}(y) \end{bmatrix}. \quad (3.18)$$
This proves (a), and this also gives us

\[
\begin{align*}
    r_{11}^{(l)}(y) &= y a_l r_{11}^{(l-1)}(y) - r_{12}^{(l-1)}(y) \\
    &= y a_l r_{11}^{(l-1)}(y) - y r_{11}^{(l-2)}(y),
\end{align*}
\]

immediately using the result from (a).

To prove (b), we proceed by induction and assume it is true for all \( n < l \). By (3.19), \( r_{11}^{(l)}(y) \) is by inductive hypothesis the difference of a degree \( l \) polynomial in \( y \) and a degree \( l - 1 \) polynomial in \( y \) and hence is itself a degree \( l \) polynomial in \( y \).

To prove (c), we again proceed by induction on \( l \) and assume it holds for all \( n < l \). Applying our inductive hypothesis,

\[
\begin{align*}
    r_{11}^{(l)}(y) &= y a_l r_{11}^{(l-1)}(y) - y r_{11}^{(l-2)}(y) \\
    &= y a_l \cdot y^{\lceil \frac{l-1}{2} \rceil} s_{l-1}(y) - y \cdot y^{\lceil \frac{l-2}{2} \rceil} s_{l-2}(y) \\
    &= y^{\lceil \frac{l}{2} \rceil} \left( y^{\lceil \frac{l-1}{2} \rceil - \lceil \frac{l}{2} \rceil} a_l s_{l-1}(y) - s_{l-2}(y) \right).
\end{align*}
\]

(3.20)

Since \( s_{l-1}(y) \) has leading coefficient \( a_1 a_2 ... a_{l-1} \), we now see that \( s_l(y) \) has leading coefficient \( a_1 a_2 ... a_{l-1} a_l \), and

\[
\begin{align*}
    \text{deg}(s_l(y)) &= \text{deg}(r_{11}^{(l)}(y)) - \left\lceil \frac{l}{2} \right\rceil \\
    &= l - \left\lceil \frac{l}{2} \right\rceil = \left\lfloor \frac{l}{2} \right\rfloor,
\end{align*}
\]

(3.21)
as desired.

For (d), we note that \( s_1(y) = a_1, \ s_2(y) = ya_2 - 1 \) and proceed by induction. We
assume the result holds for \( n = l - 2, l - 1 \). Then

\[
\begin{align*}
r^{(l)}(y) &= r^{(l-1)}(y) \cdot \begin{bmatrix} ya_l & y \\ -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} r_{11}^{(l-1)} & r_{12}^{(l-1)} \\ r_{21}^{(l-1)} & r_{22}^{(l-1)} \end{bmatrix} \cdot \begin{bmatrix} ya_l & y \\ -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} ya_l r_{11}^{(l-1)}(y) - r_{12}^{(l-1)}(y) & yr_{11}^{(l-1)}(y) \\ ya_l r_{21}^{(l-1)}(y) - r_{22}^{(l-1)}(y) & yr_{21}^{(l-1)}(y) \end{bmatrix},
\end{align*}
\] (3.22)

and so using (a),(b) we find that

\[
\begin{align*}
r_{11}^{(l)}(y) &= y^{\left\lceil \frac{l}{2} \right\rceil} s_l(y) \\
&= ya_l r_{11}^{(l-1)}(y) - r_{12}^{(l-1)}(y) \\
&= ya_l(y^{\left\lceil \frac{l}{2} \right\rceil} s_{l-1}(y)) - y(y^{\left\lceil \frac{l-2}{2} \right\rceil} s_{l-2}(y)).
\end{align*}
\] (3.23) (3.24) (3.25) (3.26)

Checking the cases for \( l \) odd and \( l \) even distinctly yields the desired result.

For (e), we proceed by induction. Given that \( s_2(y) = ya_1a_2 - 1 \), \( s_3(y) = ya_1a_2a_3 - (a_1 + a_3) \), we have our base case. Assume the result holds for \( n = l - 1 \). By (d) and (c),

\[
\mathcal{L}_l = a_1(\mathcal{L}_{l-1}) - a_1a_2...a_{l-2},
\] (3.27)

so we have \((l - 2) + 1 = (l - 1)\) terms, each of which is the product of \((l - 2)\) different \(a_i, \left\lceil \frac{l-2}{2} \right\rceil\) are odd \(i\).

For (f), we proceed by induction with base cases \( s_1(y) = a_1 \), \( s_2(y) = ya_1a_2 - 1 \), \( s_3(y) = ya_1a_2a_3 - (a_1 + a_3) \), and assume the result is true for \( n = l - 2, l - 1 \). We note by (d) that for \( l \) even,

\[
\mathcal{K}_l = -\mathcal{K}_{l-2},
\] (3.28)
proving the even case, and for $l$ odd,

$$K_l = -K_{l-2} + a_l K_{l-1}$$  \hfill (3.29)

$$= -(K_{l-2} + a_l),$$  \hfill (3.30)

which proves the result.

For (g), we again proceed by induction. We have base case $s_2(y) = ya_1a_2 - 1$. We now assume that the result is true for $n = l - 2$ even. By (d),

$$S_l = -S_{l-2} + a_l K_{l-1}.$$  \hfill (3.31)

Using (f), this gives the desired result.

For (h), we again use induction and refer simply to (3.22).

For (i) and (j), we use induction and refer to (3.16) and (3.17) as our base cases. (3.22) gives us

$$r_{21}^{(l)}(y) = ya_l r_{21}^{(l-1)}(y) - r_{22}^{(l-1)}(y)$$

$$= ya_l r_{21}^{(l-1)}(y) - y r_{21}^{(l-2)}(y),$$  \hfill (3.32)

immediately using the result from (h).

To prove (i), we proceed by induction and assume it is true for all $n < l$. By (3.32), $r_{21}^{(l)}(y)$ is by inductive hypothesis the difference of a degree $l - 1$ polynomial in $y$ and a degree $l - 2$ polynomial in $y$ and hence is itself a degree $l$ polynomial in $y$.

To prove (j), we again proceed by induction on $l$ and assume it holds for all $n < l$.  

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Applying our inductive hypothesis,
\[ r_{21}^{(l)}(y) = ya_l r_{21}^{(l-1)}(y) - yr_{21}^{(l-2)}(y) \]
\[ = ya_l \cdot y^{\left\lfloor \frac{l-2}{2} \right\rfloor} t_{l-1}(y) - y \cdot y^{\left\lfloor \frac{l-3}{2} \right\rfloor} t_{l-2}(y) \]
\[ = y^{\left\lfloor \frac{l+1}{2} \right\rfloor} \left( y^{\left\lfloor \frac{l+1}{2} \right\rfloor} a_l t_{l-1}(y) - t_{l-2}(y) \right). \tag{3.33} \]

Since \( t_{l-1}(y) \) has leading coefficient \(-a_2 a_3 \ldots a_{l-1}\), we now see that \( t_l(y) \) has leading coefficient \(-a_2 a_3 \ldots a_{l-1} a_l\), and
\[ \deg(t_l(y)) = \deg(r_{21}^{(l)}(y)) - \left\lfloor \frac{l-1}{2} \right\rfloor \]
\[ = (l - 1) - \left\lfloor \frac{l-1}{2} \right\rfloor = \left\lfloor \frac{l-1}{2} \right\rfloor, \tag{3.34} \]
as desired.

For (k), we again use induction with base cases \( t_1(y) = -1, t_2(y) = -a_2, t_3(y) = y(-a_2 a_3) + 1 \). We assume the result is true for all \( n < l \), and using (h),(i), and (3.22) we find that
\[ r_{21}^{(l)}(y) = y^{\left\lfloor \frac{l+1}{2} \right\rfloor} t_l(y) \tag{3.35} \]
\[ = ya_l r_{21}^{(l-1)}(y) - r_{22}^{(l-1)}(y) \tag{3.36} \]
\[ = ya_l (y^{\left\lfloor \frac{l+2}{2} \right\rfloor} t_{l-1}(y)) - y(y^{\left\lfloor \frac{l+3}{2} \right\rfloor} t_{l-2}(y)). \tag{3.37} \]

Checking the cases for \( l \) odd and \( l \) even distinctly yields the desired result.

For (l), we proceed by induction with base cases \( t_1(y) = -1, t_2(y) = -a_2, t_3(y) = y(-a_2 a_3) + 1, t_4(y) = y^2(-a_2 a_3 a_4) + (a_2 + a_4) \) and assume the result is true for \( n < l \).

We note by (k) that for \( l \) odd,
\[ K'_l = -K'_{l-2}, \tag{3.39} \]
proving the odd case, and for \( l \) even, 

\[
\mathcal{K}_i' = -\mathcal{K}_{i-2}' + a_i \mathcal{K}_{i-1}'
\]

\[
= -(\mathcal{K}_{i-2}' + a_i),
\]

which proves the result and the lemma.

It should be noted that only parts (a)-(c) of the above lemma are applied in this chapter. Parts (d)-(l) are needed for Chapter 4. We now can prove our main result.

**Proof of Theorem 1.3.6.** If \( m = 2k + 1 \) for some \( k \in \mathbb{N} \), \( q \equiv \pm 1 \pmod{m} \), \( \zeta \) is a primitive \( m \)-th root of unity in \( \text{GF}(q^2) \), and \( y_i := \zeta^i + \zeta^{m-i} + 2 \), then, by Lemma 3.2.5, \( \Gamma = \Gamma(y_1, \ldots, y_k) \) is a near-polygonal graph with cycles of length \( m \). Assume \( \Gamma \) has girth \( n < m \). This means that \( \alpha \in W^{(n)}(\alpha) = \overline{H} \prod_{i=1}^{n} gP \) by Lemma 3.2.1 and that there were no returns along our \( n \)-walk, i.e. we have an \((n-1)\)-dipath \((\delta_0 = \alpha, \delta_1, \ldots, \delta_{n-1})\) with \( \delta_{n-1} \) adjacent to \( \alpha \) and no shorter dipaths from \( \alpha \) to itself in \( \Gamma \). Thus \( \overline{H} \prod_{i=1}^{n} gP = \overline{H} \), which means that there exist \( a_1, a_2, \ldots, a_n \in \text{GF}(q) \) such that \( \prod_{i=1}^{n} g\overline{P}(a_i) \in \overline{H} \), which in turn implies that \( \prod_{i=1}^{n} g(y_j)\overline{P}(a_i) \) is a lower diagonal matrix for each \( 1 \leq j \leq k \). Note that for \( 1 \leq t \leq (n-1) \), \( a_t \neq 0 \), since otherwise

\[
\prod_{i=1}^{n} g\overline{P}(a_i) = g\overline{P}(a_1) \cdots g\overline{P}(a_{t-1}) g1 g\overline{P}(a_{t+1}) \cdots g\overline{P}(a_n)
= g\overline{P}(a_1) \cdots g\overline{P}(a_{t-1} + a_{t+1}) \cdots g\overline{P}(a_n),
\]
which is at most a nontrivial \((n-1)\)-walk from \(\alpha\), contradicting our assumption of girth \(n\). Note that

\[
\prod_{i=1}^{n} g(y)p(a_i) = r^{(n-1)}(y) \cdot g(y)p(a_n)
\]

\[
= \begin{bmatrix}
    r_{11}^{(n-1)} & r_{12}^{(n-1)} \\
    r_{21}^{(n-1)} & r_{22}^{(n-1)}
\end{bmatrix} \cdot \begin{bmatrix}
    ya_n & y \\
    -1 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    ya_n r_{11}^{(n-1)}(y) - r_{12}^{(n-1)}(y) & y r_{11}^{(n-1)}(y) \\
    ya_n r_{21}^{(n-1)}(y) - r_{22}^{(n-1)}(y) & y r_{21}^{(n-1)}(y)
\end{bmatrix},
\]

(3.43)

and so

\[
\prod_{i=1}^{n} g\overline{p}(a_i) = \left( \begin{array}{c}
    y_1 a_n r_{11}^{(n-1)}(y_1) - r_{12}^{(n-1)}(y_1) & y_1 r_{11}^{(n-1)}(y_1) \\
    y_1 a_n r_{21}^{(n-1)}(y_1) - r_{22}^{(n-1)}(y_1) & y_1 r_{21}^{(n-1)}(y_1)
\end{array} \right), \ldots,
\]

\[
\left( \begin{array}{c}
    y_k a_n r_{11}^{(n-1)}(y_k) - r_{12}^{(n-1)}(y_k) & y_k r_{11}^{(n-1)}(y_k) \\
    y_k a_n r_{21}^{(n-1)}(y_k) - r_{22}^{(n-1)}(y_k) & y_k r_{21}^{(n-1)}(y_k)
\end{array} \right),
\]

(3.44)

and by Lemma 3.2.6(c)

\[
y r_{11}^{(n-1)}(y) = y \cdot y^{\left\lceil \frac{n-1}{2} \right\rceil} s_{n-1}(y).
\]

Hence each \(y_i\) is a root of the following equation in GF\((q)\) since \(\prod_{i=1}^{n} g\overline{p}(a_i)\) is lower diagonal:

\[
y \cdot y^{\left\lceil \frac{n-1}{2} \right\rceil} s_{n-1}(y) = 0.
\]

(3.46)

We know from Lemma 3.2.5 that for all \(i\), \(y_i \neq 0\). Hence each \(y_i\) for \(1 \leq i \leq k\) must be a root of \(s_{n-1}(y)\) in GF\((q)\) if the girth of \(\Gamma\) is \(n\). The \(y_i\) are all distinct, and so this implies that \(\text{deg}(s_{n-1}(y)) \geq k\). But, by Lemma 3.2.6(c),

\[
\text{deg}(s_{n-1}(y)) = \left\lfloor \frac{n-1}{2} \right\rfloor \leq \left\lfloor \frac{2k - 1}{2} \right\rfloor = k - 1,
\]

(3.47)

since \(n < m\), a contradiction. Therefore the girth of \(\Gamma\) is at least \(m\), and \(\Gamma\) is polygonal of girth \(m\). 

\[\square\]
CHAPTER 4

A CONSTRUCTION OF AN INFINITE FAMILY OF 2-ARC TRANSITIVE POLYGONAL GRAPHS OF ARBITRARY EVEN GIRTH

We proceed as in Section 3.2, except we let $m = 2k + 2$, $k \in \mathbb{N}$, as opposed to letting $m = 2k + 1$. The results concerning the number of vertices, the valency, and bound for the number of components of $\Gamma$ is exactly the same as in Section 3.2, and Lemmas 3.2.3 and 3.2.5 hold with the same proof verbatim.

We now proceed to show that the girth of the graph $\Gamma$ we have just constructed is at least $m$.

**Lemma 4.0.7.** Let $m = 2k + 2$ for some $k \in \mathbb{N}$, $q$ a prime power such that $q \equiv \pm 1 \pmod{m}$ and let $\Gamma$ be a near-polygonal graph with special cycles of length $m$ as constructed in Lemma 3.2.5. Then $\Gamma$ does not have girth less than $m - 1$.

**Proof.** The proof is the same as the proof of Theorem 1.3.6. \hfill \square

**Theorem 4.0.8.** Let $m = 2k + 2$ for some $k \in \mathbb{N}$, $p$ a prime, $q = p^n$ for some $n \in \mathbb{N}$, $q \equiv \pm 1 \pmod{m}$ and let $\Gamma$ be a near-polygonal graph with special cycles of length $m$ as constructed in Lemma 3.2.5. Then, if $\Gamma$ has girth $(m - 1)$, either

(i) $p | (k - 1)$ and $p | (2^{k-1} - 1)$, or

(ii) $p | (2^{k-1} - 1)$ and

(iii) $p | (2^{k-1} - 1)$ and $p | (2^{k-1} + 1)$.
(ii) $p | (4k + 3)$ and $p | (5k - 4 \cdot 3^k)$.

**Proof.** We must look precisely at the case when we have a cycle of length $m - 1$. Since we have a 2-arc transitive automorphism group and $\Gamma$ is $(G, 2)$-dipath transitive, we may assume that our cycle of length $m - 1$ includes $[\gamma = \overline{H}f^{-1}, \alpha = \overline{H}, \beta = \overline{H}g]$, i.e. we may assume that $\gamma \in W^{(2k-1)}(\beta)$. Now, $W^{(2k-1)}(\beta) = \overline{H}g \cdot \overline{P}g \cdots \overline{P}g$ (2$k - 1$ copies of $\overline{P}g$). Thus $\overline{H}g \cdot \overline{P}g \cdots \overline{P}g = \overline{H}f^{-1}$, or

$$g\overline{P} \cdots g\overline{P}gf \in \overline{H},$$  \hspace{1cm} (4.1)

or for some $a_1, a_2, \ldots, a_{2k-1}$,

$$g\overline{p}(a_1)g\overline{p}(a_2) \cdots g\overline{p}(a_{2k-1})gf \in \overline{H}.$$  \hspace{1cm} (4.2)

Note that the $a_i$ are nonzero since there can be no returns. So let’s look at

$$g(y)p(a_1) \cdots g(y)p(a_{2k-1})g(y)f(y) = \gamma^{(2k-1)}(y) \cdot \begin{bmatrix} -y & y^2 \\ 0 & -y \end{bmatrix}$$

$$= \begin{bmatrix} y^{k+1}(-s_{2k-1}(y)) & y^{k+1}(ys_{2k-1}(y) - s_{2k-2}(y)) \\ y^k(-t_{2k-1}(y)) & y^{k+1}(t_{2k-1}(y) - t_{2k-2}(y)) \end{bmatrix}$$  \hspace{1cm} (4.3)

based on Lemma 3.2.6(c),(j). We will use repeatedly throughout this proof the fact that, since $q > k$, two polynomials of degree $k$ or smaller differ by a constant if they have the same roots. Noting that plugging in each value of $y_i$ in for $y$ makes this, coordinate-wise, an element of $\overline{H}$, we can conclude the following:

1. The row 1, column 2 entry of (4.3) must be zero for each value of $y_i$. Since $deg(ys_{2k-1}(y) - s_{2k-2}(y)) = k$, this means that for some constant $c_1$ we have

$$c_1(ys_{2k-1}(y) - s_{2k-2}(y)) = u_{2k+2}(y).$$  \hspace{1cm} (4.4)
Note that, by Lemma 3.2.6(f), the constant term on the left is $-c_1(-1)^{k-1} = c_1(-1)^k$ and on the right the constant term is $(-1)^k \binom{2k+1-k}{k} = (-1)^k(k+1)$, and so $c_1 = (k+1)$, giving us:

$$(k+1)(ys_{2k-1}(y) - s_{2k-2}(y)) = u_{2k+2}(y). \quad (4.5)$$

Note that comparing leading coefficients gives us

$$(k + 1)a_1 \ldots a_{2k-1} = 1. \quad (4.6)$$

2. The ratio of the row 1, column 1 entry over the row 2, column 2 entry in (4.3) must be constant for all $y_i$. Thus for some constant $c_2$ we have

$$\frac{y^{k+1}(-s_{2k-1}(y))}{y^{k+1}(t_{2k-1}(y) - t_{2k-2}(y))} = c_2. \quad (4.7)$$

Hence $-s_{2k-1}(y) = c_2(t_{2k-1}(y) - t_{2k-2}(y))$. Note that this is at most a degree $k - 1$ polynomial in $y$ ($s_{2k-1}(y), t_{2k-1}(y)$ are degree $k - 1$, $t_{2k-2}(y)$ is degree $k - 2$) with at least $k$ distinct roots, and so it must be identically 0. Furthermore, the leading coefficient of $-s_{2k-1}(y)$ is $-a_1a_2 \ldots a_{2k-1}$ and the leading coefficient of $c_2t_{2k-1}(y)$ is $-c_2a_2a_3 \ldots a_{2k-1}$. Since all the $a_i$ are nonzero (again, no returns and also (4.6)), we have $c_2 = a_1$, and so

$$s_{2k-1}(y) + a_1t_{2k-1}(y) - a_1t_{2k-2}(y) = 0. \quad (4.8)$$

3. Finally, the ratio of the row 2, column 1 entry over the row 1, column 1 entry in (4.3) must be constant for all $y_i$. Thus for some constant $c_3$ we have
\[
\frac{y^k(-t_{2k-1}(y))}{y^k(-ys_{2k-1}(y))} = c_3. \tag{4.9}
\]

So \(-c_3ys_{2k-1}(y) + t_{2k-1}(y) = 0\) must hold for \(y = y_1, \ldots, y_k\). Noting that, by Lemma 3.2.6(c),(j), \(-c_3ys_{2k-1}(y) + t_{2k-1}(y)\) is a degree \(k\) polynomial in \(y\), for some constant \(c_4\) we get that
\[
c_4(-c_3ys_{2k-1}(y) + t_{2k-1}(y)) = u_{2k+2}(y). \tag{4.10}
\]

Since, by Lemma 3.2.6(l), the constant term of \(c_4t_{2k-1}(y)\) is \(c_4(-1)^k\) and the constant term of \(u_{2k+2}(y)\) is \((k+1)(-1)^k\), we get \(c_4 = k+1\). Now, the leading coefficient of \(- (k+1)c_3ys_{2k-1}(y)\) is \(- (k+1)c_3 a_1 \ldots a_{2k-1}\), and the leading coefficient of \(u_{2k+2}(y)\) is \(1 = (k+1)a_1 \ldots a_{2k-1}\) (from (4.6)), so \(c_3 = -1\), finally giving us
\[
(k + 1)(ys_{2k-1}(y) + t_{2k-1}(y)) = u_{2k+2}(y). \tag{4.11}
\]

Comparing (4.5) and (4.10), we get that
\[
s_{2k-2}(y) = -t_{2k-1}(y). \tag{4.12}
\]

Using Lemma 3.2.6(c),(j), the leading coefficient of \(t_{2k-1}(y)\) is \(-a_2 \ldots a_{2k-1}\) and the leading coefficient of \(-s_{2k-2}(y)\) is \(-a_1 a_2 \ldots a_{2k-2}\), which means that
\[
a_{2k-1} = a_1 \tag{4.13}
\]

Now, from (4.8), since \(t_{2k-1}(y) = -s_{2k-2}(y)\), we get
\[
s_{2k-1}(y) - a_1 s_{2k-2}(y) = a_1 t_{2k-2}. \tag{4.14}
\]

Noting (4.13), from part (d) of Lemma 3.2.6 we get:
\[
s_{2k-1}(y) - a_1 s_{2k-2}(y) = -s_{2k-3}(y). \tag{4.15}
\]
Combining these together, we get

\[ s_{2k-3}(y) = -a_1 t_{2k-2}(y). \] \hspace{1cm} (4.16)

The leading coefficient of \(-s_{2k-3}(y)\) is \(-a_1 a_2 \ldots a_{2k-3}\) and the leading coefficient of \(a_1 t_{2k-2}\) is \(-a_1 a_2 \ldots a_{2k-3} a_{2k-2}\), and so we conclude that

\[ a_{2k-2} = 1. \] \hspace{1cm} (4.17)

We now claim that

\[ a_1 = a_3 = a_5 = \ldots = a_{2k-3} = a_{2k-1}, \] \hspace{1cm} (4.18)

\[ a_2 = a_4 = a_6 = \ldots = a_{2k-4} = a_{2k-2} = 1. \] \hspace{1cm} (4.19)

We will proceed inductively. Assume that for some \(i > 0\), we have that

\[ a_{2k-1} = a_{2k-3} = \ldots = a_{2k+1-2i} = a_1, \] \hspace{1cm} (4.20)

\[ a_{2k-2} = a_{2k-4} = \ldots = a_{2k-2i} = 1, \] \hspace{1cm} (4.21)

and that for all integers \(1 \leq j \leq i\) we have

\[ s_{2k-2j}(y) = -t_{2k-2j+1}(y), \] \hspace{1cm} (4.22)

\[ s_{2k-2j-1}(y) = -a_1 t_{2k-2j}. \] \hspace{1cm} (4.23)

Again using part (d) of Lemma 3.2.6 and (4.21), we have

\[
\begin{align*}
  s_{2k-2i}(y) &= ya_{2k-2i} s_{2k-2i-1}(y) - s_{2k-2i-2}(y) \\
  &= ys_{2k-2i-1}(y) - s_{2k-2i-2}(y).
\end{align*}
\] \hspace{1cm} (4.24)
From part (k) of Lemma 3.2.6, (4.22), (4.20), and (4.23), we have:

\[
s_{2k-2i}(y) = -t_{2k-2i+1}(y)
= ya_{2k-2i+1}(-t_{2k-2i}(y)) + t_{2k-2i-1}(y)
= ya_{1}(-t_{2k-2i}(y)) + t_{2k-2i-1}(y)
= ys_{2k-2i-1}(y) + t_{2k-2i-1}(y).
\] (4.25)

Comparing (4.24) and (4.25), we get

\[
s_{2k-2i-2}(y) = -t_{2k-2i-1}(y).
\] (4.26)

The leading coefficient of \( s_{2k-2i-2}(y) \) is \( a_{1}a_{2}\ldots a_{2k-2i-2} \) and the leading coefficient of \( -t_{2k-2i-1} \) is \( a_{2}\ldots a_{2k-2i-2}a_{2k-2i-1} \), and so we conclude that

\[
a_{2k-2i-1} = a_{1}.
\] (4.27)

Again using part (d) of Lemma 3.2.6 and (4.27), we have

\[
s_{2k-2i-1}(y) = a_{2k-2i-1}s_{2k-2i-2}(y) - s_{2k-2i-3}(y)
= a_{1}s_{2k-2i-2}(y) - s_{2k-2i-3}(y).
\] (4.28)

From part (k) of Lemma 3.2.6, (4.23), (4.21), and (4.26), we have:

\[
s_{2k-2i-1}(y) = a_{1}(-t_{2k-2i}(y))
= a_{1}(a_{2k-2i}(-t_{2k-2i-1}(y)) + t_{2k-2i-2}(y))
= a_{1}(-t_{2k-2i-1}(y)) + a_{1}t_{2k-2i-2}(y)
= a_{1}s_{2k-2i-2}(y) + a_{1}t_{2k-2i-2}(y).
\] (4.29)

Comparing (4.28) and (4.29), we get

\[
s_{2k-2i-3}(y) = -a_{1}t_{2k-2i-2}(y).
\] (4.30)
The leading coefficient of \( s_{2k-2i-3}(y) \) is \( a_1a_2...a_{2k-2i-3} \) and the leading coefficient of 
\(-a_1t_{2k-2i-2} \) is \( a_1a_2...a_{2k-2i-3}a_{2k-2i-2} \), and so we conclude that 
\[ a_{2k-2i-2} = 1. \] (4.31)

Therefore, by induction, we can continue this process indefinitely, and thus (4.18) and (4.19) must hold. Going back to (4.6), we have \((k + 1)a_1...a_{2k-1} = 1\), and so 
\((k + 1)a_1^k = 1\), or
\[ a_1^k = \frac{1}{k + 1}. \] (4.32)

Now, we go back to (4.5), \((k + 1)ys_{2k-1}(y) - (k + 1)s_{2k-2}(y) = u_{2k+2}(y)\), and look 
at the coefficient of \( y^{k-1} \). From part (e) of Lemma 3.2.6, the coefficient of \( y^{k-1} \) in 
\((k + 1)ys_{2k-1}(y)\) is
\[ -(k + 1)(2k - 2)a_1^{k-1}. \]

The coefficient of \( y^{k-1} \) in \((k + 1)s_{2k-2}(y)\) is just the leading coefficient, which is
\[ -(k + 1)a_1...a_{2k-2} = -(k + 1)a_1^{k-1}. \]

Finally, the coefficient of \( y^{k-1} \) in \( u_{2k+2}(y) \) is
\[ (-1)^1 \binom{2k + 1 - 1}{1} = 2k. \]

Noting that
\[ a_1^{k-1} = \frac{1}{a_1(k + 1)}, \]
from (4.5) we have
\[ \frac{2k - 2}{a_1} + \frac{1}{a_1} = 2k, \]

or
\[ a_1 = \frac{2k - 1}{2k}. \] (4.33)
Again, we go back to (4.5), but this time we look at the coefficient of $y$. From part (f) of Lemma 3.2.6, the linear term of $(k + 1)y_{s_{2k-1}}(y)$ is

$$(k + 1)(-1)^{k-1}(a_1 + a_3 + ... + a_{2k-1}) = (k + 1)(-1)^{k-1}(ka_1)$$

$$= (-1)^{k-1}(k + 1)\frac{2k - 1}{2}$$

From part (g) of Lemma 3.2.6, the linear term of $(k + 1)s_{2k-2}$ is

$$(-1)^k(k + 1)\frac{(k - 1)k - 2k - 1}{2} = (-1)^k\frac{(k + 1)(k - 1)(2k - 1)}{4}.$$ 

Finally, the linear term in $u_{2k+2}(y)$ is

$$(-1)^{k-1}\left(\frac{2k + 1 - k + 1}{k - 1}\right) = (-1)^{k-1}\frac{(k + 2)(k + 1)k}{6}.$$ 

Putting these together, from (4.5) we get

$$\frac{(k + 1)(2k - 1)}{2} + \frac{(k + 1)(k - 1)(2k - 1)}{4} = \frac{(k + 2)(k + 1)k}{6},$$

which simplifies to

$$(4k + 3)(k - 1) = 0. \quad (4.34)$$

CASE 1. If $(k - 1) = 0$, then $k = 1$ in GF$(q)$. We know that

$$a_1 = \frac{2k - 1}{2k} = \frac{1}{2}$$

in GF$(q)$. Going back to (4.6), we get that

$$(k + 1)a_1^k = 1$$

$$(2)\left(\frac{1}{2}\right)^k = 1$$

$$2^{k-1} - 1 = 0 \quad (4.35)$$
Thus we must have \( p \mid (k - 1), p \mid (2^{k-1} - 1) \).

CASE 2. If \( (4k + 3) = 0 \), then \( k = -\frac{3}{4} \) in \( \text{GF}(q) \). We know that

\[
a_1 = \frac{2k - 1}{2k} = \frac{5}{3}
\]

in \( \text{GF}(q) \). Going back to (4.6), we get that

\[
(k + 1)a_1^k = 1
\]

\[
\left(\frac{1}{4}\right)\left(\frac{5}{3}\right)^k = 1
\]

\[
5^k - 4 \cdot 3^k = 0 \tag{4.36}
\]

Thus we must have \( p \mid (4k + 3), p \mid (5^k - 4 \cdot 3^k) \).

\[\square\]

We now can easily prove the main results.

**Proof of Theorem 1.3.7.** For a given even \( m = 2k+2 \), we construct our infinite family of near-polygonal graphs of girth \( m \) as in Lemma 3.2.5. By Lemma 4.0.7 these graphs all have girth at least \( (m - 1) \), and by Theorem 4.0.8 for only finitely many primes \( p \) can the constructed graph have girth \( (m - 1) \). Hence there are infinitely many 2-arc transitive polygonal graphs of girth \( m \) for all even \( m \geq 4 \).

\[\square\]

**Proof of Corollary 1.3.8.** Combining the results of Theorem 1.3.6 and Theorem 1.3.7 proves this result.

\[\square\]
CHAPTER 5

AN ANALYSIS OF THE GIRTH-DOUBLING
CONSTRUCTION FOR POLYGONAL GRAPHS

5.1 The construction

The following section is a (slight) generalization of [2, Section 2] which was cer-
tainly known to the authors of [2] but has not been published. We begin with a
polygonal graph $\Gamma$ of valency $r$ and set of special cycles $C$ of girth $m$, and we let $V$
and $E$ be the vertex-set and edge-set of $\Gamma$, respectively. Let $T$ be the edge set of a
spanning tree in $G$. For a fixed integer $k \in \mathbb{N}$, we define a group

$$H := \prod_{e \in E \setminus T} \mathbb{Z}_k.$$  \hspace{2cm} (5.1)

Group elements are thus $n$-tuples for $n = |E \setminus T|$, where each coordinate is generated
by an element of order $k$. The coordinates are indexed by the edges in $E \setminus T$, and
$|H| = k^n$.

Next, to each edge $e \in E$ we assign a group element of $H$, which is referred to as
the \textit{voltage} of $e$ [5]. We will denote this group element $g_e$. If $e \in T$, then $g_e$ is the
identity of $H$, i.e. an $n$-tuple with 0 in every coordinate. If $e \in E \setminus T$, then $g_e$ is the
generator of the cyclic group of order $k$ in the coordinate indexed by $e$ and 0 in every other coordinate.

We now use this assignment to define $\Gamma_k$. Its vertex set is $V_k = V \times H$, and we define the edge set $E_k$ in the following manner. For each edge $e \in E$, we fix an ordering $(\alpha, \beta)$ of the two vertices incident to $e$. Then, for all $g \in H$, we add $\{(\alpha, g), (\beta, g + g_e)\}$ to the edge set $E_k$. We say that the directed edge $(\alpha, \beta)$ has voltage $g_e = g_{\alpha \beta}$ and the directed edge $(\beta, \alpha)$ has voltage $-g_e = g_{\beta \alpha}$. Note that we have constructed $k^n$ edges for each $e \in E$, one for every choice of $g$.

There is a natural covering map $\phi : \Gamma_k \to \Gamma$ defined by supressing the group element. Note for each $\alpha \in V$ and $e \in E$ we have

$$|\phi^{-1}(\alpha)| = |\phi^{-1}(e)| = k^n. \quad (5.2)$$

Furthermore, if $\alpha$ has neighbors $\beta_1, \beta_2, \ldots, \beta_r$ in $\Gamma$ whose connecting directed edges are assigned voltages $g_1, \ldots, g_r$, going from $\alpha$ to each respective $\beta_i$, then $(\alpha, g) \in V_k$ has exactly $r$ neighbors. Since $\Gamma$ is $r$-regular, so is $\Gamma_k$.

Now consider an $m$-cycle $C \in \mathcal{C}$. Since $C$ has $m$ edges and is 2-regular, $\phi^{-1}(C)$ must contain $k^nm$ edges and must be 2-regular. Thus $\phi^{-1}(C)$ is the disjoint union of cycles $C_1, \ldots, C_l$ for some $l$. Start at a vertex $\alpha$ in $C$ and walk along the cycle. In $\Gamma_k$ we start at $(\alpha, g)$ for some $g \in H$. Upon returning to $\alpha$ in $C$, the path in $\Gamma_k$ is at $(\alpha, g + h)$, where $h$ is the sum of the voltages along $C$. A second walk leads us to $(\alpha, g + 2h)$, a third to $(\alpha, g + 3h)$, etc., finally returning to $(\alpha, g)$ after $k$ trips around $C$ in $\Gamma$. Hence each cycle $\tilde{C} \in \phi^{-1}(C)$ is $k$ times as long as $C$, and there are $k^{n-1}$ disjoint cycles in $\phi^{-1}(C)$.
Note that if \( P \) is a path of length 2 contained in \( C \), any \( \tilde{P} \in \phi^{-1}(P) \) is covered by a unique \( \tilde{C} \in \phi^{-1}(C) \). It follows that

\[
\tilde{C} = \{ \phi^{-1}(C) \mid C \in \mathcal{C} \}
\]  

(5.3)

covers each path of length two in \( \Gamma_k \) exactly once. Thus \( \Gamma_k \) is a near-polygonal graph with special cycles of length \( km \).

**Lemma 5.1.1.** If \( \Gamma \) is connected, then so is \( \Gamma_k \).

**Proof.** Let \( \alpha \in V \), the vertex set of \( \Gamma \). By construction, every vertex in \( \Gamma_k \) is connected to a vertex of the form \((\alpha, g)\) for some \( g \in H \). It thus suffices to show that \((\alpha, 0)\) is connected to \((\alpha, g)\) for all \( g \in H \). Let \( g = \sum_{i=1}^{t} g_{\alpha_i \beta_i} \), where \( g_{\alpha_i \beta_i} \) is the voltage for the edge going from vertex \( \alpha_i \) to vertex \( \beta_i \) (note that the \( g_{\alpha_i \beta_i} \) are not necessarily distinct). Begin at \((\alpha, 0)\) in \( \Gamma_k \) and start a walk. In \( \Gamma \) we can walk from \( \alpha \) to \( \alpha_1 \) using only edges in \( T \), which do not contribute voltage. Walk on the edge from \( \alpha_1 \) to \( \beta_1 \), then walk from \( \beta_1 \) back to \( \alpha \) using only edges in \( T \). In \( \Gamma_k \) we have walked from \((\alpha, 0)\) to \((\alpha, g_{\alpha_1 \beta_1})\). We now repeat this process for each \( i \). This is altogether a closed walk in \( \Gamma \) with voltage \( g \) that begins and ends at \( \alpha \). Hence \((\alpha, 0)\) and \((\alpha, g)\) are connected in \( \Gamma_k \), and so \( \Gamma_k \) is connected.

**Lemma 5.1.2.** If \( \Gamma \) is a polygonal graph of girth \( m \), \( k > 3 \) is an integer, and \( \Gamma_k \) is the near-polygonal graph constructed as above, then \( \Gamma_k \) is not polygonal.

**Proof.** Take two special cycles \( C_1 \) (with total voltage \( h_1 \)) and \( C_2 \) (with total voltage \( h_2 \)) in \( \Gamma \) containing an edge \( e \) with vertices \( \alpha \) and \( \beta \) with voltage \( g \) going from \( \alpha \) to \( \beta \). Traverse \( C_1 \) starting with \( e \), and then traverse \( C_2 \) ending with \( e \). Now traverse \( C_1 \) in the opposite direction to \( \beta \) and then \( C_2 \) in the opposite direction to \( \alpha \). From
this, we have a closed walk \( W \) of length \( 4m - 2 \) in \( \Gamma \) with no returns. We lift \( W \) to a walk in \( \Gamma_k \), starting at \((\alpha, 0)\): we go to \((\beta, g)\), then to \((\alpha, h_1)\), to \((\beta, h_1 + h_2 + g)\) to \((\alpha, h_1 + h_2)\), to \((\beta, h_2 + g)\) and finally back to \((\alpha, 0)\). This is a closed walk in \( \Gamma_k \), and it has length \( 4m - 2 \). Hence the special cycles in \( \Gamma_k \), which have length \( km \geq 4m \), are longer than the girth, and \( \Gamma_k \) is not polygonal.

5.2 Another construction

In order to fully analyze the case \( k = 3 \) and the automorphism groups of our graphs, we need the following construction, which is a slight modification of the one in the previous section. We begin with a polygonal graph \( \Gamma \) of valency \( r \) and set of special cycles \( \mathcal{C} \) of girth \( m \), and we let \( V \) and \( E \) be the vertex-set and edge-set of \( \Gamma \), respectively, as above. For a fixed integer \( k \in \mathbb{N} \), we define a group

\[
G := \prod_{e \in E} \mathbb{Z}_k.
\]

Group elements are thus \( N \)-tuples for \( N = |E| \), where each coordinate is generated by an element of order \( k \). The coordinates are indexed by the edges in \( E \), and \( |H| = k^N \).

Next, to each edge \( e \in E \) we assign a group element of \( H \), which is referred to as the voltage of \( e \) [5]. We will refer to this group element as \( g_e \). For all \( e \in E \), \( g_e \) is the generator of the cyclic group of order \( k \) in the coordinate indexed by \( e \) and 0 in every other coordinate. If \( e \) is incident with vertices \( \alpha \) and \( \beta \) and directed from \( \alpha \) to \( \beta \), we will keep the terminology of the previous section and let \( g_{\alpha \beta} = g_e \) and \( g_{\beta \alpha} = -g_e \).

We define a graph \( \Gamma_k^* \) with vertex set \( V \times G \) and edge set analagously as above. Note that \( \Gamma_k^* \) has valency \( r \) and is near-polygonal with cycles of length \( km \). Also, there is still a natural covering projection \( \phi^* : \Gamma_k^* \to \Gamma \) defined by suppressing the
group element. However, as we will show later, $\Gamma_k^*$ is not connected. We will now prove the following:

**Theorem 5.2.1.** If $\Gamma$ is a polygonal graph of valency $r$ and girth $m$, then $\Gamma_3^*$ is a polygonal graph of valency $r$ and girth $3m$. Moreover, if $\Gamma$ is a strict polygonal graph, then so is $\Gamma_3^*$.

**Proof.** First, given a polygonal graph $\Gamma$ of valency $r$ and girth $m$, we may use the construction above to produce a near-polygonal graph $\Gamma_3^*$ of valency $r$ and special cycles of length $3m$. We must now show that $\Gamma_3^*$ has girth at least $3m$.

Let $C^*$ be a cycle of shortest possible length in $\Gamma_3^*$. As above, we have a natural map $\phi^*: \Gamma_3^* \to \Gamma$ from suppressing the group elements (i.e. voltages). Consider $\phi^*(C^*)$ in $\Gamma$. We note first that $\phi^*(C^*)$ is a closed walk in $\Gamma$ of the same length as $C^*$ where no edge is traversed consecutively in opposite directions, i.e. no returns (Theorem 2.1.1 of [5]). Since $\phi^*(C^*)$ is a closed walk in $\Gamma$, $\phi^*(C^*)$ must contain at least one cycle in $\Gamma$. Since $\Gamma$ has girth $m$, this walk must have length at least $m$.

We walk along $\phi^*(C^*)$ as we walk along $C^*$. If $\phi^*(C^*) = C$ for some cycle $C$ in $\Gamma$, then once we start to walk along $C^*$ in one direction, we must walk along $C$ in that direction. Since there are no returns, there is exactly one direction along $C$ in $\Gamma$ we are able to walk. Thus we walk in this one direction the entire way, never leaving the cycle $C$. Walking the cycle this way requires us to traverse the cycle three times in order to cancel the voltage, so we have walked at least $3m$ edges. Hence if $C^*$ has length less than $3m$, then $\phi^*(C^*)$ must contain a cycle $C$ and an edge $e$ not contained in $C$.

We now make a couple of observations about our closed walk in $\Gamma$. First, if we “leave” a vertex $\alpha$ at least three times along our closed walk $\phi^*(C^*)$, then we must come back
to this vertex $\alpha$ as many times as we left. Each time we leave and come back, since there are no returns, we must have traversed a cycle in $\Gamma$ before getting back to $\alpha$. This cycle has length at least $m$ by assumption, so if we leave a vertex three times in our closed walk, our walk is at least $3m$ edges long.

Next, we note that since every edge in $\Gamma$ has a unique nonzero voltage, every edge we traverse in $\phi^*(C^*)$ must be used at least twice, i.e. once forwards and once backwards, three times forward, or three times backwards, in order to cancel out the voltage and return to our starting point in $C^*$ in $\Gamma^*_3$.

Without a loss of generality, we may assume that we begin our walk at a vertex $\alpha$, traverse a cycle $C$, and then leave $\alpha$ again along an edge $e$ not in $C$. Note that since this is a closed walk, we must eventually come to the vertex $\alpha$ again along our walk.

CASE 1. We come back to $\alpha$ along a walk that is edge-disjoint with $C$.

Note that $C$ is of length at least $m$ and this new walk beginning with edge $e$ also must have length at least $m$ (no returns). However, since we must cancel out the voltage of each edge we traversed, we now must walk each of these again, which is more than $3m$ edges.

CASE 2. We leave $\alpha$ along an edge $e$ and come back to $C$ at a vertex $\beta$, where we then traverse an edge in $C$.

Suppose the path along $C$ from $\alpha$ to $\beta$ is $i$ edges long. Then, since the girth of $\Gamma$ is $m$, the other path along $C$ from $\alpha$ to $\beta$ and the path from $\alpha$ to $\beta$ that is edge-disjoint from $C$ both have length at least $m - i$. Note that these two paths themselves form a closed walk in $\Gamma$, and so we have:

$$2(m - i) \geq m.$$  \hspace{1cm} (5.5)
All of these edges must be traversed at least twice, and:

\[ 2(2(m - i) + i) = 2m + 2(m - i) \geq 3m, \quad (5.6) \]

so \( |\phi^*(C^*)| \geq 3m \). Hence any cycle in \( \Gamma^*_3 \) has length at least \( 3m \) and \( \Gamma^*_3 \) is polygonal.

Now suppose that \( \Gamma \) is strict polygonal. In order to show that \( \Gamma^*_3 \) is also strict polygonal, we must show that all \( 3m \)-cycles in \( \Gamma^*_3 \) are lifts of \( m \)-cycles of \( \Gamma \). Assume that \( C^* \) is a cycle in \( \Gamma^*_3 \) of length \( 3m \) with image \( \phi^*(C^*) \) such that \( \phi^*(C^*) \) contains branching vertices (i.e. vertices incident to at least three different edges of \( \phi^*(C^*) \)).

Let \( \alpha \) be an arbitrary branching vertex. We may assume that the closed walk \( \phi^*(C^*) \) starts at \( \alpha \), and so \( \phi^*(C^*) \) can be written as the concatenation of \( k \) closed walks \( W_1, \ldots, W_k \), where each \( W_i \) is from \( \alpha \) to \( \alpha \), and \( \alpha \) does not occur as an inner vertex in any \( W_i \). Since the girth of \( \Gamma \) is \( m \), each \( W_i \) has length at least \( m \). Hence, if \( k \geq 4 \) then the length of \( \phi^*(C^*) \) is at least \( 4m \), a contradiction. Moreover, since \( \alpha \) is branching, \( \alpha \) is incident to at least three edges, and each of these edges must be traversed at least twice to cancel their voltage; hence \( k \geq 3 \). Summarizing, we obtain that \( k = 3 \), each \( W_i \) has length \( m \), \( \alpha \) is incident to exactly three edges in \( \phi^*(C^*) \), and each of these three edges are traversed exactly twice, in opposite directions.

Let \( (\alpha, \beta) \) be the first edge of \( W_1 \). We have seen that \( (\beta, \alpha) \) must be the last edge of \( W_i \) for some \( i \leq 3 \). If \( i = 1 \) then \( W_1 \) contains a proper subwalk from \( \beta \) to \( \beta \) of length \( m - 2 \) that contains no returns, a contradiction to the fact that \( \Gamma \) has girth \( m \). The case \( i = 3 \) is also impossible, since \( \phi^*(C^*) \) cannot contain consecutive edges \( (\beta, \alpha) \) and \( (\alpha, \beta) \). Hence \( i = 2 \).

We claim now that \( \beta \) is branching. Indeed, if \( \beta \) is not branching, then let \( \{\beta, \gamma\} \) be the unique (undirected) edge different from \( \{\alpha, \beta\} \) and incident to \( \beta \) in \( \phi^*(C^*) \).
The second edge of $W_1$ must be $(\beta, \gamma)$ and the $(m - 1)^{st}$ edge of $W_2$ must be $(\gamma, \beta)$. Hence the 2-arc $(\alpha, \beta, \gamma)$ occurs in $W_1$ and the 2-arc $(\gamma, \beta, \alpha)$ occurs in $W_2$. Since $W_1$ and $W_2$ are closed walks of length $m$, in fact they must be cycles. Moreover, since $\Gamma$ is strict polygonal and both $W_1$ and $W_2$ contain the 2-path $(\alpha, \beta, \gamma)$, we must have that $W_1 = W_2$, only they are traversed in opposite directions. This is a contradiction, because the last edge of $W_1$ and the first edge of $W_2$ are a return in $\phi^*(C^*)$. Hence $\beta$ is branching.

So far, we have seen that each branching vertex must occur exactly three times in $\phi^*(C^*)$ and must partition $\phi^*(C^*)$ into the concatenation of three closed walks of length $m$. This is a final contradiction, because we have also shown that the branching vertex $\beta$ has two occurrences of distance $2m - 2$ from each other. Therefore, $\Gamma^*_3$ must be strict polygonal, as desired.

We now begin our analysis of the automorphism group of $\Gamma^*_k$.

**Theorem 5.2.2.** Each $\sigma \in \text{Aut}(\Gamma)$ lifts to $\sigma^* \in \text{Aut}(\Gamma^*_k)$, i.e. there exists $K \leq \text{Aut}(\Gamma^*_k)$ such that $K \cong \text{Aut}(\Gamma)$.

*Proof.* For $\sigma \in \text{Aut}(\Gamma)$, we define a permutation $\tilde{\sigma}$ of $G$, the voltage group of $\Gamma^*_k$, the following way. Let $\{\alpha, \beta\} \in E$ (recall that $E$ is the edge set of $\Gamma$), and let $\delta = \alpha^\sigma, \epsilon = \beta^\sigma$. Then $\{\delta, \epsilon\} \in E$ because $\sigma \in \text{Aut}(\Gamma)$. For $g_{\alpha\beta} \in G$, define $g_{\alpha\beta}^{\tilde{\sigma}} = g_{\delta\epsilon}$. Since every edge in $E$ has a voltage, and the edges are being permuted by $\sigma$, $\tilde{\sigma}$ is a permutation of the edge voltages. Every element $g \in G$ can be written uniquely as a sum of the edge voltages, i.e. if $g = \sum_{i=1}^{l} g_i$, where each $g_i$ is an edge voltage (not necessarily all distinct), then this expression is unique up to permuting the $g_i$. Thus $\tilde{\sigma}$ extends naturally to the group $G$ itself defined by $g^{\tilde{\sigma}} = \sum_{i=1}^{l} g_i^{\tilde{\sigma}}$.

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We may now define a map $\sigma^*$ of $\Gamma_k^*$ as follows: for each vertex $(\alpha, g)$ of $\Gamma_k^*$, we define $(\alpha, g)^{\sigma^*} = (\alpha^\sigma, g^\tilde{\sigma})$. This map clearly permutes the vertices of $\Gamma_k^*$. If we have two adjacent vertices $(\alpha, g)$ and $(\beta, g + g_{\alpha\beta})$ in $\Gamma_k^*$, then $\sigma^*$ takes these to $(\alpha^\sigma, g^\tilde{\sigma})$ and $(\beta^\sigma, g^\tilde{\sigma} + g_{\alpha\beta^\sigma})$, preserving their adjacency. Furthermore, if $(\alpha, g)^{\sigma^*}$ and $(\beta, h)^{\sigma^*}$ are adjacent, we simply apply $(\sigma^{-1})^*$ to them to note that $(\alpha, g)$ and $(\beta, h)$ must have been adjacent also. Therefore $\sigma^*$ is an automorphism of $\Gamma_k^*$.

Finally, we note that two different automorphisms $\sigma_1$ and $\sigma_2$ of $\Gamma$ must have different actions on $V$. Hence $\sigma_1^*$ and $\sigma_2^*$ have different actions on the vertices of $\Gamma_k^*$, and we see that there exists $K \leq \text{Aut}(\Gamma_k^*)$ such that $K \cong \text{Aut}(\Gamma)$. \hfill \Box

**Corollary 5.2.3.** If $\Gamma$ is vertex transitive, edge transitive, arc transitive, or 2-arc transitive, then so is $\Gamma_k^*$.

**Proof.** The proof of each statement is analogous to the proofs of the others, so we will only show that 2-arc transitivity is maintained under the construction. Suppose that $\Gamma$ is 2-arc transitive, and look at two 2-arcs of $\Gamma_k^*$, namely $((\alpha, g - g_{\alpha\beta}), (\beta, g), (\gamma, g + g_{\beta\gamma}))$ and $((\delta, h - g_{\delta\epsilon}), (\epsilon, h), (\rho, h + g_{\epsilon\rho}))$. Since $\Gamma$ is 2-arc transitive, there exists $\sigma \in \text{Aut}(\Gamma)$ such that $(\alpha, \beta, \gamma)^\sigma = (\delta, \epsilon, \rho)$. By Lemma 5.2.2, there exists $\sigma^* \in \text{Aut}(\Gamma_k^*)$ such that

$$((\alpha, g - g_{\alpha\beta}), (\beta, g), (\gamma, g + g_{\beta\gamma}))^{\sigma^*} = (\delta, g^\tilde{\sigma} - g_{\delta\epsilon}, (\epsilon, g^\tilde{\sigma}), (\rho, g^\tilde{\sigma} + g_{\epsilon\rho})).$$

(5.7)

Since $\Gamma_k^*$ is a voltage graph, for all $t \in G$ there exists an automorphism $\Phi_t$ of $\Gamma_k^*$ defined by taking each vertex $(\alpha', g')$ of $\Gamma_k^*$ to $(\alpha', g' + t)$ (Section 2.2.1 of [5]). Therefore,

$$((\alpha, g - g_{\alpha\beta}), (\beta, g), (\gamma, g + g_{\beta\gamma}))^{\sigma^* \Phi_{t-g^\tilde{\sigma}}} = ((\delta, h - g_{\delta\epsilon}), (\epsilon, h), (\rho, h + g_{\epsilon\rho})), $$

(5.8)

and $\Gamma_k^*$ is 2-arc transitive, as desired. \hfill \Box
5.3 Relating the two constructions

Since the above constructions are so similar, it is natural to ask how \( \Gamma_k \) and \( \Gamma_k^* \) are related when constructed from the same graph \( \Gamma \). Throughout this section we will assume that the initial graph \( \Gamma \) is connected (and hence, by Lemma 5.1.1, so is \( \Gamma_k \)).

**Lemma 5.3.1.** There exists a covering projection \( \phi' : \Gamma_k^* \to \Gamma_k \).

*Proof.* These is seen by suppressing the group elements corresponding to the voltages along the spanning tree \( T \), giving us a natural covering projection like the ones that map \( \Gamma_k^* \) and \( \Gamma_k \) onto \( \Gamma \). \( \square \)

This leads to the following immediate corollary:

**Corollary 5.3.2.** Each component of \( \Gamma_k^* \) projects onto \( \Gamma_k \).

*Proof.* This is obvious since we are assuming that \( \Gamma \) (and hence \( \Gamma_k \)) is connected. \( \square \)

We now note that there is an interesting embedding of the voltage group \( H \) of \( \Gamma_k \) into the the voltage group \( G \) of \( \Gamma_k^* \). Let \( f = \{\beta, \gamma\} \in E \setminus T \), and suppose that we gave the orientation \( (\beta, \gamma) \) to \( f \) at the definition of the voltage \( g_f \) in \( H \) and \( G \). Let \( W_f = (\gamma = \gamma_0, \gamma_1, \ldots, \gamma_l = \beta) \) be the unique path in \( \Gamma \) using only edges of \( T \), and define

\[
g_f^* := g_f + \sum_{i=0}^{l-1} g_{\gamma_i \gamma_{i+1}}.
\] (5.9)

(The voltages \( g_{\gamma_i \gamma_{i+1}} \) are defined only in \( G \).) The map taking \( g_f \) to \( g_f^* \) on the generating set of \( H \) extends naturally to a homomorphism \( \varphi : H \to G \). Clearly, \( \varphi \) is injective, since \( \sum_{f \in E \setminus T} a_f g_f^* = 0 \) implies \( \sum_{f \in E \setminus T} a_f g_f = 0 \), and the latter equation holds if and only if all \( a_f = 0 \). We shall refer to \( \varphi(H) \) as \( H^* \).
Let $\alpha$ be a distinguished vertex of $\Gamma$. For any edge $f = \{\beta, \gamma\} \in E \setminus T$ there are two unique paths $W_{\alpha\beta}$ and $W_{\alpha\gamma}$ using only edges of $T$ from $\alpha$ to $\beta$ and $\gamma$, respectively. Note that if we traverse $W_{\alpha\beta}$, then walk across $f$, and then walk back to $f$ using $W_{\alpha\gamma}$ backwards, the resulting walk $W_{\alpha,f}$ has voltage $g_f^*$; indeed, we only added edges from a path $P$ from $\alpha$ to a vertex in $W_f$ (if we added any at all) to $W_f$. Each edge in $P$ is traversed exactly twice in $W_{\alpha,f}$, giving us a total voltage of exactly $g_f^*$ for $W_{\alpha,f}$. The following lemma makes the relationship between $\Gamma^*_k$ and $\Gamma_k$ more explicit by analyzing the embedding $H^* < G$:

**Lemma 5.3.3.** Vertices $(\alpha, g_1)$ and $(\alpha, g_2)$ are in the same component of $\Gamma^*_k$ if and only if $g_1 - g_2 \in H^*$.

**Proof.** First, suppose that $g_1 - g_2 \in H^*$. Then we have $g_1 - g_2 = \sum_{i=1}^{k} g_{f_i}^*$, where each $f_i \in E \setminus T$ (the $f_i$ are not necessarily distinct). We start at the vertex $(\alpha, g_1)$ in $\Gamma^*_k$ and begin a walk. Since $\Gamma^*_k$ is a covering space of $\Gamma$, a closed walk in $\Gamma$ will lift to a unique walk in $\Gamma^*_k$ starting at the vertex $(\alpha, g_1)$. Let $W$ be the concatenation of the walks $W_{\alpha,f_i}$ as defined in the previous paragraph. Then $W$ is a closed walk, and the sum of the voltages along $W$ is $\sum_{i=1}^{k} g_{f_i}^* = g_1 - g_2$. Since $\Gamma^*_k$ is a covering space of $\Gamma$, the walk $W$ will lift to a walk in $\Gamma^*_k$ starting at $(\alpha, g_1)$ and ending at $(\alpha, g_1)$.

Conversely, suppose that $(\alpha, g_1)$ and $(\alpha, g_2)$ are in the same component of $\Gamma^*_k$. Let $W$ be a walk in $\Gamma^*_k$ from $(\alpha, g_1)$ to $(\alpha, g_2)$ with no returns and let $((\beta_1, h_1), (\gamma_1, h_1 + g_{e_1})),...,((\beta_j, h_j), (\gamma_j, h_j + g_{e_j}))$ be the edges in $W$ that project down to edges $e_1, ..., e_j$ in $\Gamma$ that are in $E \setminus T$, and suppose that we traverse them along $W$ in precisely this order. Hence every other edge in $W$ projects down to an element of $T$. Note that since $T$ is a tree and there are no returns in $W$, there is a unique path $P_i$ along edges of $T$ that goes from each $\gamma_i$ to $\beta_{i+1}$ without returns. We will now modify our walk as follows:
for \(i = 1, \ldots, j - 1\), we replace \(P_i\) by \(P_i^*\), where \(P_i^*\) is the unique walk in \(\Gamma\) that is the concatenation of two paths in \(T\): one from \(\gamma_i\) to \(\alpha\) and the other from \(\alpha\) to \(\beta_{i+1}\), i.e. the concatenation of \(W_{\alpha\gamma_i}\) (as defined above) traversed in the opposite direction and \(W_{\alpha\beta_{i+1}}\). Note that there may be some returns along \(P_i^*\), but it is unique. Note also that the voltage of \(P_i^*\) is the same as \(P_i\) since we only (possibly) modified \(P_i\) by going to \(\alpha\) along some edges in \(T\) and then returning along those same edges in \(T\). Thus \(W^*\) also realizes that \((\alpha, g_1)\) and \((\alpha, g_2)\) are in the same component and has the same voltage as \(W\). However, the voltage of \(W^*\) is clearly

\[
\sum_{i=1}^{j} g^*_e \in H^*,
\]

and hence \(g_1 - g_2 \in H^*\).

Given this lemma, it is now very easy to prove the following:

**Theorem 5.3.4.** \(\Gamma^*_k\) is isomorphic to \(k^{|T|}\) disjoint copies of \(\Gamma_k\).

**Proof.** From Lemma 5.3.3, \((\alpha, g_1)\) and \((\alpha, g_2)\) are in the same component of \(\Gamma^*_k\) if and only if \(g_1 - g_2 \in H^*\). Thus each coset of \(H^*\) in \(G\) determines a component of \(\Gamma^*_k\). Since

\[
|G : H^*| = |G : H| = k^{|T|},
\]

\(\Gamma^*_k\) has \(k^{|T|}\) connected components. Each of these connected components must project down onto \(\Gamma_k\), so each component has at least \(k^{|E|-|T|}\) vertices. However, \(\Gamma^*_k\) has exactly \(k^{|E|}\) vertices, and therefore each of the \(k^{|T|}\) components of \(\Gamma^*_k\) must be isomorphic to \(\Gamma_k\).
Proof of Theorems 1.3.9, 1.3.10. We have already shown the corresponding theorems for $\Gamma_k^*$ (Theorem 5.2.1, Corollary 5.2.3), and Theorem 5.3.4 tells us that $\Gamma_k^*$ is isomorphic to $k^{|T|}$ disjoint copies of $\Gamma_k$. The results follow. \qed
The following is a program for GAP that determines whether a near-polygonal graph constructed using Theorem 1.3.1 is a polygonal graph and also whether it is a strict polygonal graph.

\begin{verbatim}
girth10:= function(p, bound)
local t, half, unity, r3, changebase, inv, d, c, niceg, invniceg, i1, x, y, cycle, A, B, C, g, h, H, K, rcosets, rprst, sign, baseset, buildup, oldbuildup, built1, built2, built, ii, count, w, z, totalcount;

if not (IsPrimePowerInt(p) and (p mod 30) in \{1,19\}) then
  return "not good prime power";
fi;

/t:= (Z(p)^0 + SquareRoots(GF(p), 5*Z(p)^0)[1])/(4*Z(p)^0);

half:= 1/(2*Z(p)^0);
\end{verbatim}
unity := (-Z(p)^0 + \text{SquareRoots}(GF(p), (-3)*Z(p)^0)[1])/(2*Z(p)^0);

r3 := \text{SquareRoots}(GF(p), Z(p)^0*(-3))[1];

A := Z(p)^0*[[[-1,0,0],[0,0, unity],[0, -unity^2,0]];

B := Z(p)^0*A*[[[-half, half - t, -t],[t - half, t, -half],
[t, -half, half - t]]*\text{DiagonalMat}([-1,1,1]);

H := \text{Group}(Z(p)^0*A, Z(p)^0*B);

if (not (Order(H) = 1080)) then
    return "failed to make A_6";
fi;

C := Z(p)^0*B*A*B*A*B*B;

K := \text{Group}(Z(p)^0*A, Z(p)^0*C);

if (not (Order(K) = 108)) then
    return "failed to make F_{36}";
fi;

c := -half*(Z(p)^0 + Z(p)^0*2*unity*t)/(Z(p)^0 + Z(p)^0*2*unity^2*t);
changebase:= Z(p)^0*[[1, c*unity^2, c], [1, 1/c, unity^2],
[1, unity, unity/c] ];

inv:= Inverse(changebase);

d:= Z(p)^0*c*(c^2*(unity - 4*unity*t - 2*t)
+ c*(-4*t - 2*unity + 4*unity*t) + unity)/
(2*(1+2*c*unity)*(1 - c*unity));

x:= unity^2/(d*r3);
niceg:= Z(p)^0*[[1/r3, -1/(3*x), -unity^2/(3*x)],
[x, unity/r3, unity/r3], [unity*x, 1/r3, unity/r3]];

invniceg:= Inverse(niceg);

g:= inv*niceg*changebase;

h:= Inverse(B)*DiagonalMat([-Z(p)^0,-Z(p)^0,Z(p)^0])*B;

rcosets:= RightCosets(H,K);

rprst:= List(rcosets, z->Representative(z));

for i1 in [1..10] do
if rprst[i1] in K then
    sign:=i1;
    fi;
od;
Remove(rprst,sign);

baseset:= [g*rprst[1], g*rprst[2], g*rprst[3], g*rprst[4],
g*rprst[5], g*rprst[6], g*rprst[7], g*rprst[8], g*rprst[9]];  

H:= AsSet(Elements(H));

buildup:=ShallowCopy(baseset);
   ii:=3;
repeat
    oldbuildup:=buildup;
    buildup:=[ ];
    for z in oldbuildup do
       for y in baseset do
          Add(buildup, z*y);
       od;
    od;
    ii:=ii+1;
until ii=bound;
built1:= [];
built1:= List(buildup, w->w*g);
built1:= AsSet(built1);

built2:= [];
built2:=List(oldbuildup, w->w*g*h);
built2:=AsSet(built2);

built:=[];
for z in built2 do
  for w in H do
    Add(built, w*z);
  od;
  od;
built:= AsSet(built);

count:= Length(Intersection(built1, built));
if (not (count = 0)) then
  return ["Girth at most ", 2*bound-1];
fi;

built2:= [];
built2:=List(buildup, w->w*g*h);
built2:=AsSet(built2);
built:=[];
for z in built2 do
for w in H do
Add(built, w*z);
od;
od;
built:= AsSet(built);

totalcount:= Length(Intersection(built1, built));

if (totalcount = 0) then
return ["Girth at least ", 2*bound+1];
fi;

if (totalcount = 1) then
return ["Strict polygonal of girth ", 2*bound];
fi;

if (not (totalcount in [0,1])) then
return ["Non-strict polygonal of girth ", 2*bound];
fi;

end;
APPENDIX B

ANOTHER GIRTH-FINDING PROGRAM

The following is a program for GAP that determines whether a near-polygonal graph constructed using Theorem 1.3.1 is a polygonal graph also whether or not it is a strict polygonal graph. It is less time-efficient than the program contained in Appendix A but requires less memory, so it should be used when the program contained in Appendix A fails to run due to memory constraints.

\texttt{newgirth10:= function(p, bound)}
\texttt{local t, half, unity, r3, changebase, inv, d, c, niceg,}
\texttt{invniceg, i1, x, y, cycle, A, B, C, g, h, H, K, rcosets,}
\texttt{rprst, sign, baseset, leveltwo, buildup, oldbuildup, built1,}
\texttt{built2, built, ii, count, jj, w, z, totalcount;}

\texttt{if not (IsPrimePowerInt(p) and (p mod 30) in [1,19]) then}
\texttt{return "not good prime power";}
\texttt{fi;}

\texttt{t:= (Z(p)^0 + SquareRoots(GF(p), 5*Z(p)^0)[1])/(4*Z(p)^0);}
half := 1/(2*Z(p)^0);

unity := (-Z(p)^0 + SquareRoots(GF(p), (-3)*Z(p)^0)[1])/(2*Z(p)^0);

r3 := SquareRoots(GF(p), Z(p)^0*(-3))[1];

A := Z(p)^0*[[[-1,0,0],[0,0,-unity],[0,-unity^2,0]];

B := Z(p)^0*A*[[[-half, half - t, -t],[t - half, t, -half],
[t, -half, half - t]]*DiagonalMat([-1,-1,1]);

H := Group(Z(p)^0*A, Z(p)^0*B);
if (not (Order(H) = 1080)) then
  return "failed to make A_6";
fi;

C := Z(p)^0*B*A*B*A*B*B;

K := Group(Z(p)^0*A, Z(p)^0*C);
if (not (Order(K) = 108)) then
  return "failed to make F_36";
fi;
c := -half*(Z(p)^0 + Z(p)^0*2*unity*t)/(Z(p)^0 + Z(p)^0*2*unity^2*t);

changebase := Z(p)^0*[[1, c*unity^2, c], [1, 1/c, unity^2],
[1, unity, unity/c]];

inv := Inverse(changebase);

d := Z(p)^0*c*(c^2*(unity - 4*unity*t - 2*t)
+ c*(-4*t - 2*unity + 4*unity*t) + unity)/
(2*(1+2*c*unity)*(1 - c*unity));

x := unity^2/(d*r3);
niceg := Z(p)^0*[[1/r3, -1/(3*x), -unity^2/(3*x)],
[x, unity/r3, unity/r3], [unity*x, 1/r3, unity/r3]];

invcnegg := Inverse(niceg);

g := inv*niceg*changebase;

h := Inverse(B)*DiagonalMat([-Z(p)^0,-Z(p)^0,Z(p)^0])*B;

rcosets := RightCosets(H,K);

rprst := List(rcosets, z->Representative(z));
for $i_1$ in [1..10] do
    if $r_{prst}[i_1]$ in $K$ then
        $\text{sign}:=i_1$;
    fi;
od;
Remove($r_{prst}$,$\text{sign}$);

baseset:= [$g*r_{prst}[1]$, $g*r_{prst}[2]$, $g*r_{prst}[3]$, $g*r_{prst}[4]$, $g*r_{prst}[5]$, $g*r_{prst}[6]$, $g*r_{prst}[7]$, $g*r_{prst}[8]$, $g*r_{prst}[9]$];

$H:= \text{AsSet(Elements(H))}$;

buildup:=ShallowCopy(baseset);
    $ii:=3$;
repeat
    oldbuildup:=buildup;
    buildup:=[];
    for $z$ in oldbuildup do
        for $y$ in baseset do
            Add(buildup, $z*y$);
        od;
    od;
    $ii:=ii+1;$

until ii=bound;

built2:= [];
built2:=List(buildup, w->w*g*h);
built2:=AsSet(built2);

for y in [1..Length(buildup)] do
    built1:=[];
built1:= List(baseset, z -> buildup[y]*z*g);
built:= [];
for z in built1 do
    for w in H do
        Add(built, w*z);
    od;
    od;
built:= AsSet(built);
count:= Length(Intersection(built, built2));
if (not (count = 0)) then
    return ["Girth at most ", 2*bound-1];
fi;
leveltwo := List([0..80], w→ baseset[QuoInt(w,9)+1] *baseset[(w - QuoInt(w,9)*9)+1]);
leveltwo := AsSet(leveltwo);

totalcount := 0;
count := 0;

for y in [1..Length(buildup)] do
    built1 := [];
    built1 := List(leveltwo, z→ buildup[y]*z*g);
    built := [];
    for z in built1 do
        for w in H do
            Add(built, w*z);
        od;
    od;
    built := AsSet(built);
    count := Length(Intersection(built, built2));
    totalcount := totalcount + count;
od;
if (totalcount = 0) then
return ["Girth at least ", 2*bound+1];
fi;

if (totalcount = 1) then
return ["Strict polygonal of girth ", 2*bound];
fi;

if (not (totalcount in [0,1])) then
return ["Non-strict polygonal of girth ", 2*bound];
fi;

end;
APPENDIX C

LARGE NUMBERS

The following is a list of large numbers that would not fit neatly onto the page in Section 2.3:

\[ g_1 := \frac{-39+11i\sqrt{3}+9\sqrt{5}-i\sqrt{15}}{-66+18i\sqrt{15}} \]

\[ g_2 := \frac{21-11i\sqrt{3}-9\sqrt{5}-29i\sqrt{15}}{-132+36i\sqrt{15}} \]

\[ g_3 := \frac{111+11i\sqrt{3}+9\sqrt{5}-7i\sqrt{15}}{-66i\sqrt{3}-54\sqrt{5}+18i\sqrt{15}} \]

\[ g_4 := \frac{-78-44i\sqrt{3}-36\sqrt{5}-2i\sqrt{15}}{-51+51i\sqrt{3}-63\sqrt{5}-21i\sqrt{15}} \]

\[ g_5 := \frac{-27-95i\sqrt{3}+27\sqrt{5}+19i\sqrt{15}}{6(17-17i\sqrt{3}+21\sqrt{5}+7i\sqrt{15})} \]

\[ g_6 := \frac{-129+61i\sqrt{3}+15\sqrt{5}-23i\sqrt{15}}{6(17+17i\sqrt{3}-21\sqrt{5}+7i\sqrt{15})} \]

\[ g_7 := \frac{-36-14i\sqrt{3}+6\sqrt{5}+4i\sqrt{15}}{-33+33i\sqrt{3}+27\sqrt{5}+9i\sqrt{15}} \]
\[ g_8 := \frac{-39 - 61i\sqrt{3} - 15\sqrt{5} - i\sqrt{15}}{-66 + 66i\sqrt{3} + 54\sqrt{5} + 18i\sqrt{15}} \]

\[ g_9 := \frac{-3 - 8i\sqrt{7} - 24i\sqrt{5} - 5i\sqrt{15}}{-33 - 33i\sqrt{3} - 27\sqrt{5} + 9i\sqrt{15}} \]

\[ gh_1 := \frac{86i\sqrt{3} - 46\sqrt{5}}{-7 - 7i\sqrt{3} + 99\sqrt{5} - 33i\sqrt{15}} \]

\[ gh_2 := \frac{495 - 495i\sqrt{3} - 21\sqrt{5} - 7i\sqrt{15}}{6(7 + 7i\sqrt{3} - 99\sqrt{5} + 33i\sqrt{15})} \]

\[ gh_3 := \frac{-21 + 7i\sqrt{3} - 99\sqrt{5} - 99i\sqrt{15}}{6(7 + 7i\sqrt{3} - 99\sqrt{5} + 33i\sqrt{15})} \]

\[ gh_4 := \frac{-93i\sqrt{3} + 145\sqrt{5}}{-223 + 119i\sqrt{15}} \]

\[ gh_5 := \frac{1785 + 1785i\sqrt{3} - 669\sqrt{5} - 223i\sqrt{15}}{-2676 + 1428i\sqrt{15}} \]

\[ gh_6 := \frac{-669 + 223i\sqrt{3} + 357\sqrt{5} + 357i\sqrt{15}}{-2676 + 1428i\sqrt{15}} \]

\[ gh_8 := \frac{-237 + 53i\sqrt{15}}{6(7 + 33i\sqrt{15})} \]

\[ gh_9 := \frac{-795 - 265i\sqrt{3} - 237\sqrt{5} - 237i\sqrt{15}}{84 + 396i\sqrt{15}} \]

\[ c_{11} := -5371383613575 \]

\[ c_{12} := -1454745577717 \]
\[c_{13} := 9416703784329\]

\[c_{14} := 1706532297823\]

\[c_{15} := -511956893469\]

\[c_{16} := 511956893469\]

\[c_{17} := -3222830168145\]

\[c_{18} := -1074276722715\]

\[v_{11} := -\frac{2(\sqrt{3}+i\sqrt{5})}{5i+\sqrt{3}+i\sqrt{5}}\]

\[v_{12} := \frac{1-\sqrt{5}}{1+\sqrt{5}}\]

\[v_{21} := -\frac{(1+i)(-3i+3i\sqrt{3}-3\sqrt{3}+\sqrt{15})}{(2+3i)+(1-2i)\sqrt{3}+i\sqrt{5}-(1+2i)\sqrt{15}}\]

\[v_{22} := -\frac{(3+4i)-i\sqrt{3}+\sqrt{5}+i\sqrt{15}}{(2+3i)+(1-2i)\sqrt{3}+i\sqrt{5}-(1+2i)\sqrt{15}}\]

\[v_{31} := -\frac{(1+i)(-3i-3i\sqrt{3}+3\sqrt{3}+\sqrt{15})}{(3+2i)-i\sqrt{3}+\sqrt{5}+(2+i)\sqrt{15}}\]

\[v_{32} := -\frac{(3-4i)-i\sqrt{3}+\sqrt{5}+i\sqrt{15}}{(2-3i)-(1+2i)\sqrt{3}-i\sqrt{5}+(1+2i)\sqrt{15}}\]
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