Mechanical Behavior and Heat Transfer in Polymer Fiber Melt-Spinning and Drawing Processes

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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1997
This dissertation addresses several open problems in the modeling of industrial polymeric fiber manufacturing processes. In melt spinning, the molten polymer is extruded through a spinneret, from which it emerges as many molten fibers. Then the molten fibers are then exposed to quench air, which cools and solidifies the polymer. After spinning, the solidified polymer fiber is called an "undrawn" or "as-spun" fiber. Undrawn fibers exhibit unsatisfactory mechanical characteristics for use in textiles and as structural reinforcement. To remove this unsatisfactory mechanical characteristic, the as-spun fiber undergoes an additional process in which it is drawn.

With the help of empirical studies, polymer fibers have been manufactured for many years without a deep investigation of the underlying physical nature of the spinning and drawing processes. As the requirements placed on polymer fibers have increased and quality control has become more important, the necessity of studying the theoretical fundamentals of polymer fiber spinning and drawing has grown. My Ph.D. research has sought to improve the modeling capability in both the spinning and drawing processes of polymer fiber manufacture.

This dissertation creates a fiber-air interaction model incorporating the influence of temperature change in the density of quench air, a feature which has not been modeled in previous research. Neglecting the temperature change of quench air and the resulting density change in the quench air results in an underestimation the variation
of fiber behavior through the fiber bundle. In addition, the fiber-air interaction model of this dissertation for the first time incorporates momentum conservation, which enables the model to deduce the boundary layer volume flow rate entrained by each fiber from first principles rather than an empirical formula. This removal of empiricism means fewer experiments are needed to model the fiber-air interaction. A single fiber spinning model incorporating the temperature-dependent density is presented in this dissertation. In addition, the model incorporates the radial variation of temperature and pressure throughout the entire model. The new melt-spinning model predicts delayed cooling and stretching of the fiber, and a lower maximum tensile stress than does the conventional theory. As a preliminary modeling of the drawing process, this dissertation developed the fundamental governing equations for a fiber moving and wrapped on a pulley. The conservation of momentum gives a generalized capstan formula, which takes the inertia of the fiber into account. The model is applied to solve the torque transmission problem of a belt connection two pulleys.
This is dedicated to my parents
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CHAPTER 1

INTRODUCTION

This dissertation addresses several open problems in the modeling of industrial polymeric fiber manufacturing processes.

Melt spinning is a commonly-used method to form polymer fibers, because of its simplicity and economy. Figure 1.1 shows a schematical configuration of the melt spinning process. The molten polymer is extruded through a spinneret, from which it emerges as many molten fibers; the number of fibers from a single spinneret can range from less than one hundred for reinforcing fibers to many hundreds for textile fibers. The molten fibers are then exposed to quench air, which cools and solidifies the polymer.

After spinning, the solidified polymer fiber is called an “undrawn” or “as-spun” fiber. Undrawn fibers exhibit unsatisfactory mechanical characteristics for use in textiles and as structural reinforcement. Note from Figure 1.2 that after an initial elastic response, the fiber undergoes extreme permanent elongation with very little increment of force. To remove this mechanical characteristic, the as-spun fiber undergoes an additional process in which it is drawn. Drawing consists of an irreversible elongation of the solidified spun fiber to 20-2000% of its original length, through the entire plateau of Figure 1.2, thereby extending and parallelizing the macromolecules.
Figure 1.1: Schematical configuration of melt spinning process
Figure 1.2: Typical load-extension curve for the drawing of an as-spun polymer, from [27] and [29].

and crystallites along the fiber axis and causing a change in phase structure [29]. Figure 1.3 shows a schematical configuration of a drawing process. The undrawn fiber is stretched by a factor \( \frac{v_c}{v_f} \), called the draw ratio, where the surface speed \( v_f \) of the feeding godets is less than the surface speed \( v_c \) of the take-up godets.

With the help of empirical studies, polymer fibers have been manufactured for many years without a deep investigation of the underlying physical nature of the spinning and drawing processes. As the requirements placed on polymer fibers have increased and quality control has become more important, the necessity of studying the theoretical fundamentals of polymer fiber spinning and drawing has grown.
Figure 1.3: Scheme of a continuous drawing process. 1: Feeding godets; 2: oven; 3: take-up godets; $v_f$: feed velocity; $v_r$: drawing velocity; $L$: drawing path; from [29].

My Ph.D. research has sought to improve the modeling capability in both the spinning and drawing processes of polymer fiber manufacture. In modeling the spinning process, it is convenient to consider the spinning model as divided into a model for the behavior of a single fiber in a prescribed quench air environment, which we call the *single fiber model*, and a model for the interaction of the row of fibers with the quench air stream, which we call the *fiber-air interaction model*. The complete fiber spinning model intricately couples these two models into one, global, *multi-fiber spinning model*.

In Chapter II the emphasis is on the interaction between quench air and spinning fibers. A fiber-air interaction model is developed, which is combined with a
commonly-referenced single fiber model to generate a multi-fiber spinning model that describes the entire process.

In Chapter III the focus is shifted to improving the single fiber model. The single fiber model presented in Chapter III incorporates several advances over the conventional single fiber models, such as the model used in Chapter II. Each of these improvements were developed somewhat separately. I was primarily involved in the modeling of temperature-dependent density: thermally-induced shrinkage is modeled in the conventional treatment with the *a posteriori* substitution of a temperature-dependent density function into the slender-fiber equations for an incompressible material. I present a single fiber spinning model based on a thermodynamically-consistent 3-D constrained theory for a material with temperature-dependent density. Compared to the conventional single fiber spinning model, the thermodynamically-consistent model predicts slower cooling, delayed stretching and lower maximum tensile stress in the spinline. I have also incorporated improvements in the modeling of temperature-dependence of specific heat and radial temperature gradients with the fiber, developed primarily by other graduate students in the group, to produce a state-of-the-art single fiber model. In this dissertation I also examine the effect of incorporating all of the improvements in the single fiber model.

Chapter IV presents the preliminary modeling of the drawing process from the point of view of isothermal processes. The field equations, which hold for any constitutive equations in the steady state drawing process are presented. Several constitutive equations are chosen to apply to the drawing model in torque transmission problem and necking problem. In addition, the capstan formula is generalized to include the
inertia force and the deformation of the material according the field equation derived in this chapter. Chapter V is a conclusion.
CHAPTER 2

MODEL FOR THE INTERACTION AND COOLING OF MULTIPLE FIBERS

The focus of this chapter is the development of a fiber-air interaction model. This model is combined with an existing, commonly referenced single fiber model to produce the complete multi-fiber spinning model. For the single fiber model, I adopt a single fiber model which is due to Kase and Matsuo [16], with features from Yasuda et al [28].

2.1 A conventional single fiber model

A single fiber model is a mathematical model that predicts the velocity, temperature, cross-sectional area, and tensile force along the fiber in the fiber spinning process given the upstream and downstream process conditions, material characterization, and quench air environment of the fiber. Figure 2.1 shows the spinning of a single fiber: molten polymer is extruded through the spinneret and then passes through a quench air zone to form a solidified fiber which is collected by the take-up roll. In Figure 2.1 the thickness of the fiber and the amount of transverse deflection have been exaggerated. With the assumptions that

- the process is in steady state,
Figure 2.1: Schematical representation of single fiber spinning.
- temperature and axial velocity in the fiber are uniform over each cross section,

and

- radiative and axial heat transfer are negligible.

Kase and Matsuo [16] deduced one-dimensional fiber spinning equations for the molten polymer from conservation of mass, conservation of momentum in the axial direction, and the energy equation:

\[
\frac{df_f}{dz} = - \frac{G g}{v_f(z)} + \frac{dv_f(z)}{dz} G + \pi c_q(z) r_f(z) \rho_a(z) v_f^2(z), \tag{2.1}
\]

\[
\frac{dv_f}{dz} = \frac{f_f(z) \rho_f(z) v_f(z)}{\beta(z) G}, \tag{2.2}
\]

\[
\frac{d\theta_f}{dz} = -\frac{2 \pi r_f(z)}{G c_p(z)} h(z) (\theta_f(z) - \theta_a(z)). \tag{2.3}
\]

(In Chapter III, the assumption that temperature is uniform over the cross section will be shown to be inconsistent with the second law of thermodynamics, and removed.)

The primitive unknowns in (2.1)–(2.3) are

\[
f_f(z) = \text{axial force in the fiber at location } z,
\]

\[
v_f(z) = \text{axial velocity in the fiber at location } z,
\]

\[
\theta_f(z) = \text{absolute temperature of the fiber at location } z.
\]

In (2.1)–(2.3) \( G \) is the constant mass flow rate in the steadily spinning fiber, \( \theta_a(z) \) is the ambient air absolute temperature, \( g = 981 \text{ cm/s}^2 \) is the acceleration of gravity, and \( \rho_a(z) \) is the density of the ambient air,

\[
\rho_a(z) = \frac{0.353 \text{ g K}}{\theta_a(z) \text{ cm}^3}. \tag{2.4}
\]

The polymer density \( \rho_f(z) \) and specific heat \( c_p(z) \) are assumed to be linear functions of fiber temperature \( \theta_f \),

\[
\rho_f(z) = 1.493 \frac{g}{\text{cm}^3} - 5 \times 10^{-4} \frac{g}{\text{K cm}^3} \theta_f(z), \tag{2.5}
\]
\[ c_p(z) = 0.1362 \frac{\text{cal}}{\text{g K}} + 6 \times 10^{-4} \frac{\text{cal}}{\text{g K}^2} \theta_f(z). \]  

(Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut elit tellus, luctus nec ullamcorper nec, velit nec pretium rhoncus. Nulla vivamus. 

(Looking ahead, in Chapter III the modeling of temperature-dependent density in this theory will be shown to be the thermodynamically incorrect, and a new, thermodynamically-consistent theory is given.) \( r_f(z) \) is the fiber radius,

\[ r_f(z) = \sqrt{\frac{G}{\pi \rho_f(z) v_f(z)}}. \]  

\( c_q(z) \) is a dimensionless air drag coefficient [20],

\[ c_q(z) = 0.375 \left[ \frac{2 r_f(z) v_f(z)}{v_a(\theta_a)} \right]^{-0.61}. \]  

where \( v_a(\theta_a) \) is the kinematic viscosity of air as a function of absolute ambient temperature \( \theta_a \), and \( h(z) \) is the coefficient for heat loss per area per degree per time, given by the empirical formula [20],

\[ h(z) = 4.73 \times 10^{-5} \frac{\text{cal}}{\text{K s}^{\frac{3}{2}} \text{cm}^3} \left( \frac{v_f(z)}{\pi r_f(z)^2} \right)^{0.334} \left[ 1 + \left( \frac{8 v_a(z)}{v_f(z)} \right)^{-2.167} \right], \]  

in which \( v_a(z) \) is the quench air transverse velocity. The quantity \( \beta(z) \) is the elongational viscosity of melt polymer, assumed to be temperature-dependent:

\[ \beta(z) = 3 \frac{\text{g-force}}{\text{cm}^2 \text{s}} \left( \frac{\text{g}}{\text{dl}} \right)^{5.1} \eta^{5.1} \exp \left[ 2.303 \left( \frac{3280 \text{ K}}{\theta_f(z)} - 1.54 \right) \right], \]  

where \( \eta = 0.58 \frac{\text{dl}}{\text{g}} \) is the intrinsic viscosity of the polymer melt. The upstream and downstream boundary conditions that must be specified in order to integrate the equations (2.1)–(2.3) are the upstream field velocity and temperature, and downstream velocity,

\[ v_f(0) = v_0, \]  

\[ \theta_f(0) = \theta_0. \]  

\[ v_f(l) = v_{\text{take-up}}, \]
Importantly, the quench air temperature $\theta_a(z)$ and cross-flow velocity $v_a(z)$, are specified functions of axial location in the single fiber model.

We employ the equation for the deflection $x_f(z)$ of the fiber derived from the momentum conservation in the quench air cross flow direction given by Yasuda et al [28]

$$\frac{d^2 x_f(z)}{d^2 z} = \frac{-\rho_a(z) v_a^2(z) r_f(z) c_q(z) + \pi \rho_f(z) r_f^2(z) g x_f(z)}{f_f(z) - \pi \rho r_f^2(z) v_f^2(z)}.$$  \hspace{1cm} (2.14)

This second order ordinary differential equation, driven by the solution of (2.1)–(2.3), demands the upstream and downstream boundary conditions on the fiber deflection,

$$x_f(0) = x_{\text{spinneret}},$$  \hspace{1cm} (2.15)

$$x_f(l) = x_{\text{take-up}},$$  \hspace{1cm} (2.16)

2.2 The fundamentals of fiber-air interaction, and notation

A single fiber model like the one given in the previous section predicts the spinning behavior given a known quench air temperature $\theta_a(z)$ and known cross flow velocity $v_a(z)$. In melt spinning, a row of molten fibers is cooled and deflected by the quench air stream. In return, however, each row of molten fibers heats the quench air and deflects some of the quench air stream down the fibers in an entrained boundary layer. Hence the quench air stream striking the second row of fibers in a multi-row process is in general hotter and slower, with less flow, than the stream that struck the first row of fibers, and so forth. Thus, the quench conditions that the fibers experience differ from row to row.

In the modeling of a multi-row process, the $z$ axis is chosen to be in the spinning direction, the $x$ axis in the direction of the quench air flow, and the $y$ axis in the
transverse direction connecting the spinneret holes in a fiber row. The number of fiber rows is \( n \); a particular row is denoted by the index \( j = 1, 2, \ldots, n \). For example, the axial length of row \( j \), from spinneret to take-up location, is \( l(j) \).

With this labeling, Figure 2.2 illustrates the fundamentals of fiber-air interaction in the multi-row spinning process, and the notation employed in this dissertation. The quench air blows on the \( j \)-th row of fibers with temperature \( \theta_a(j, z) \) and cross-flow velocity \( v_a(j, z) \); the \( j \) and \( z \) indicate that these quantities pertain to the quench air experienced by \( j \)-th row at the axial position \( z \). After passing through the \( j \)-th row of fibers, the quench air has different temperature \( \theta_a(j+1, z) \) and cross-flow velocity \( v_a(j+1, z) \) and becomes the quench air for the \( j+1 \)-row of fibers, if present. Some of the air flow does not penetrate to the next, \( j+1 \)-th, row, but instead is drawn down in the boundary layer with volume flow rate \( q(j, z) \) entrained by each fiber in the \( j \)-th row; this boundary layer grows with \( z \). Note that in multi-fiber spinning model the same symbols are used for the variables that appear in the single fiber model, except that an integer index is added to indicate the row dependence of the quantity.

## 2.3 Previous models for fiber-air interaction

The first work modeling quench air conditions in multiple fiber spinning was by Yasuda et al [28], who used an empirical formula to determine the quantity of air entrained by each row of fibers, and conservation of energy and mass to calculate the quench temperature and cross-flow velocity. However, the empirical formula for entrainment used in [28] is not specified, and hence the analysis cannot be applied to other situations.
Figure 2.2: Schematic representation of the fiber-air interaction.
Dutta [7] proposed a model obtained by applying conservation of mass to the control volume in Figure 2.3 and assuming the quench air is isothermal. Hence in his model the quench air is not heated as it flows through the fiber bundle, the density of the quench air remains constant, and conservation of mass becomes conservation of quench air volume:

\[ v_a(j+1, z) = v_a(j, z) - \frac{q(j, z) - q(j, z - dz)}{\Delta y dz}. \] (2.17)

This relation (2.17) is combined with an empirical formula due to Matsui [20] for the quantity of air entrained by the fibers,

\[ q(j, z) = -4 \pi \rho_0 r_f^2(j, z) v_a(j, z) \int_{\psi_{\text{eff}}}^{1} \left[ 1 - \int_{\psi}^{1} \frac{c_q(z) Re_a d\psi}{\psi + (\psi^2 + \lambda(1-\psi^2)^2)^{\frac{1}{2}}} \right] d\psi^*. \] (2.18)
In (2.17), $\Delta y$ is the distance between two neighboring holes in the same row. In (2.18), $\rho_0$ is the constant quench air density, $Re_a$ is the Reynolds number,

$$Re_a = \frac{2 r_f(j, z) v_f(j, z)}{\nu_a(\theta_a(j, z))}, \quad (2.19)$$

and $\lambda$ and $\psi$ are given by

$$\lambda = 0.5 K^2 Re_a^2 c_q(j, z), \quad (2.20)$$
$$\psi = \left( \frac{r_f(j, z)}{r_b(j, z)} \right)^{\frac{1}{2}}, \quad (2.21)$$

where $K$ is a constant related to Prantl's mixing length, determined by experiment, and $r_b(j, z)$ is the effective radius of the boundary layer, so that $r_b(j, z) - r_f(i, z)$ is the thickness of boundary layer. From (2.17) and (2.18), Dutta [7] solves for the cross-flow velocity profile $v_a(j+1, z)$ of the next-row quench air from a given profile $v_a(j, z)$, but $\psi_{eff}$ and $K$ are not specified. This model neglects the density change of air due to temperature change; later in the dissertation the consequences of this neglect are discussed. Dutta's model also relies on the empirical relation (2.18). In my fiber-air interaction model, the density change of the quench air due to temperature change is included, and the empirical formula (2.18) is replaced by using of conservation of momentum and energy. I am unable to quantitatively assess the complete Dutta's model and compare its predictions to mine, since Dutta does not publish values for $K$ and $\psi_{eff}$.

### 2.4 The fiber-air interaction model

To develop the improved fiber-air interaction model of this dissertation for an $n$-row spinning process, it is assumed that:
Figure 2.4: Example of spinneret geometry modeled by the fiber-air interaction model.

- each fiber in the same row experiences the same quench conditions, \( \theta_a(j, z) \) and \( v_a(j, z) \), i.e. quench conditions have no \( y \) dependence,

- each fiber in the same row has the same behavior,

- the quench air flow is unidirectional, and perpendicular to each row of fibers.

As shown in Figure 2.4, the rows of holes in the spinneret do not necessarily have the same width. The only constraint on the geometry of the spinneret is that every row of spinneret holes is either longer than or of the same length as the previous row until the row of maximum width, after which each row is not wider than the previous one.
In addition to the flexibility of row width, this model does not restrict each row to have the same number of holes, nor does the spacing between rows have to be uniform. It is assumed, however, that the holes are equally spaced within each row so that the fibers in a given row experience uniform quench conditions.

Air is entrained by the spinning fibers along the fiber axial direction and forms a boundary layer, which grows down the spinline as $z$ increases. The radii of the boundary layers surrounding the fibers will grow until at some axial location $z_m(j)$ the boundary layers in the converging fiber bundle meet. Beyond $z_m(j)$ to the end of the fiber at $z = l(j)$, the boundary layers have merged into a single boundary layer surrounding the entire bundle of fibers, and there is no quench air flow through the fibers bundle. Therefore the modeling of fiber-air interaction is divided into two domains,

- $0 \leq z \leq z_m(j)$: the domain with distinct boundary layers surrounding each fiber, and quench air cross flow through the bundle;

- $z_m(j) < z \leq l(j)$: the domain with a single boundary layer surrounding the entire bundle, and hence no quench air cross flow through the bundle.

### 2.4.1 Model for the domain $0 \leq z \leq z_m(j)$ with distinct boundary layers surrounding each fiber

Two control volumes are employed to derive the equations of the fiber-air interaction, shown in Figure 2.6 and 2.7. Both control volumes encompass the row $j$ of spinning fibers covering the axial domain from $z-\Delta z$ to $z$; $w(j, z)$ is the width of the control volume in the $y$ direction, surrounding row $j$ of fibers; see Figure 2.5. The
two control volumes differ only in that the control volume of Figure 2.6 excludes the fibers and the control volume of Figure 2.7 includes the fibers.

The cross-flow velocity and absolute temperature of the quench air entering either control volume are \( v_a(j,z) \) and \( \theta_a(j,z) \), respectively. To ensure continuity, \( v_a(j+1,z) \) and \( \theta_a(j+1,z) \) are the velocity and absolute temperature of quench air both exiting the control volume centered on row \( j \) and entering the control volume centered on row \( j+1 \). A volume flow rate \( q(j,z-dz) \) of air entrained in the boundary layer surrounding each fiber enters through the top of the control volume with a mean velocity \( \bar{v}_b(j,z-dz) \); \( q(j,z) \) and \( \bar{v}_b(j,z) \) are the volume flow rate and mean velocity of boundary layer surrounding each fiber exiting the bottom of control volume and, to ensure continuity, entering the top of the next control volume, i.e. the control volume covering the axial domain from \( z \) to \( z+dz \).

When there is quench air cross flow through the bundle, the quench air conditions, \( v_a(j+1,z) \) and \( \theta_a(j+1,z) \), for the next row and the volume flow rate, \( q(j,z) \), and mean velocity, \( \bar{v}_b(j,z) \), of boundary layer exiting the control volume, are calculated.
Figure 2.6: Control volume for fiber-air interaction model excluding the fibers.

from the quench air conditions, $v_a(j, z)$ and $\theta_a(j, z)$, of the current row, the volume flow rate $q(j, z-dz)$ and mean velocity $\bar{v}_b(j, z-dz)$ entering the top of the control volume, and the fiber response, through conservation of mass, conservation of energy, and conservation of linear momentum applied to the control volume.

**Conservation of mass**

For conservation of mass the control volume of Figure 2.6, which excludes the fibers, is employed. The mass per time of quench air flowing into the volume, both transversely through the side of the volume from the previous rows of fibers and axially through the top of the volume in the boundary layer, must equal the air mass per time of the air flowing out, both to the next row of fibers and out the bottom in
the boundary layer:

\[
\rho_a(\theta_a(j, z)) v_a(j, z) w(j, z) \, dz \\
+ \zeta(j) \left[ \rho_a(\theta_a(1, z)) v_a(1, z) - \rho_a(\theta_a(j, z)) v_a(j, z) \right] \left[ w(j, z) - w(j-1, z) \right] \, dz \\
+ m(j) \rho_a(\theta_{avg}(j, j+1, z - dz)) q(j, z - dz) \\
= \rho_a(\theta_a(j+1, z)) v_a(j+1, z) w(j, z) \, dz \\
+ m(j) \rho_a(\theta_{avg}(j, j+1, z)) q(j, z),
\]

(2.22)

where \( m(j) \) is the number of fibers in row \( j \), \( \rho_a(\theta_a) \) is the density of the quench air, given in (2.4), and we assume

\[
\theta_{avg}(j, j+1, z) = \alpha_1 \theta_a(j, z) + \alpha_2 \theta_a(j+1, z) + \alpha_3 \theta_f(j, z),
\]

(2.23)

with

\[
\alpha_1 + \alpha_2 + \alpha_3 = 1.
\]

(2.24)

If the previous row is shorter than the current row, some of the quench air entering the control volume is virgin quench air, with velocity \( v_a(1, z) \) and density \( \rho_a(\theta_a(1, z)) \), rather than velocity \( v_a(j, z) \) and density \( \rho_a(\theta_a(j, z)) \) of the air from the immediately preceding row. The second term on the left hand side is used to include this phenomenon, in which \( \zeta(j) \) is defined as

\[
\zeta(j) = \begin{cases} 
0 & \text{if } w(j, z) \leq w(j-1, z) \\
1 & \text{if } w(j, z) > w(j-1, z).
\end{cases}
\]

(2.25)

**Conservation of energy**

A given volume flow rate \( q_a \) of air with temperature \( \theta_a \) and moving laminarly with average velocity \( v_a \) has kinetic energy per time of

\[
\frac{\text{kinetic energy}}{\text{time}} = \frac{1}{2} q_a \rho_a(\theta_a) v_a^2,
\]

(2.26)

20
Figure 2.7: Control volume for fiber-air interaction model including the fibers.

and, we assume, an internal energy per time of

\[
\frac{\text{internal energy}}{\text{time}} = q_a \rho_a(\theta_a) C_a \theta_a,
\]  

(2.27)

where [18]

\[
C_a = 0.24 \frac{\text{cal}}{g \ K} = 1.0046 \times 10^7 \ \frac{\text{cm}^2}{\text{K sec}^2}.
\]

(2.28)

For conservation of energy we consider the control volume in Figure 2.6 that excludes the fibers. The volume flow rate, temperature, and velocity of the air entering the left side of the control volume are

\[
q_a = v_a(j, z) w(j, z) \, dz,
\]

\[
\theta_a = \theta_a(j, z),
\]

\[
v_a = v_a(j, z);
\]

(2.29)
the volume flow rate, temperature, and velocity of the air exiting the right side of the control volume are

\[ q_a = v_a(j+1, z) w(j, z) \, dz, \]
\[ \theta_a = \theta_a(j+1, z), \]
\[ v_a = v_a(j+1, z); \]  \hspace{1cm} (2.30)

the volume flow rate, temperature, and velocity of the air entering the top of the control volume are

\[ q_a = q(j, z-dz), \]
\[ \theta_a = \theta_{avg}(j, j+1, z-dz), \]
\[ v_a = \bar{v}_b(j, z-dz); \]  \hspace{1cm} (2.31)

and the volume flow rate, temperature, and velocity of the air exiting the bottom of the control volume are

\[ q_a = q(j, z), \]
\[ \theta_a = \theta_{avg}(j, j+1, x), \]
\[ v_a = \bar{v}_b(j, z); \]  \hspace{1cm} (2.32)

Conservation of energy applied to the control volume is

\[
\left[ \frac{\text{kinetic energy}}{\text{time}} + \frac{\text{internal energy}}{\text{time}} \right]_{\text{air flowing into control volume}} + \left[ \frac{\text{heat}}{\text{time}} \right]_{\text{from the fibers into control volume}} = \left[ \frac{\text{kinetic energy}}{\text{time}} + \frac{\text{internal energy}}{\text{time}} \right]_{\text{air flowing out of control volume}}
\]  \hspace{1cm} (2.33)

Heat leaves the fibers and enters the control volume through both convection and radiation. We assume that the heat convection per time is

\[ \frac{\text{heat convection from the fiber}}{\text{time}} = H(j, z) \, dz, \]  \hspace{1cm} (2.34)
where
\[ H(j, z) = h(j, z) \left[ \theta_f(j, z) - \theta_a(j, z) \right] \]  (2.35)

is the heat loss per length per time from the fiber and \( h(j, z) \) as heat transfer coefficient given in (2.9), and heat radiation per time is
\[ \frac{\text{heat radiation from the fiber}}{\text{time}} = \gamma \varepsilon \left[ \theta_f^4(j, z) - \theta_a^4(j, z) \right] s(j, z) \, dz, \]  (2.36)

where \( \gamma \) is the Stefan-Boltzmann constant for blackbody radiation, \( \varepsilon \) is the emittance of non-black surface, \( s(j, z) \) is the circumference of spinning fiber, and \( \theta_r \) is room temperature. Therefore, conservation of energy applied to the control volume is
\[
\begin{align*}
\rho_a(\theta_a(j, z)) v_a(j, z) w(j, z) \, dz & \left[ C_a \theta_a(j, z) + \frac{1}{2} v_a^2(j, z) \right] \\
+ \zeta(j) \rho_a(\theta_a(1, z)) v_a(1, z) \left[ w(j, z) - w(j-1, z) \right] \, dz & \left[ C_a \theta_a(1, z) + \frac{1}{2} v_a^2(1, z) \right] \\
- \zeta(j) \rho_a(\theta_a(j, z)) v_a(j, z) \left[ w(j, z) - w(j-1, z) \right] \, dz & \left[ C_a \theta_a(j, z) + \frac{1}{2} v_a^2(j, z) \right] \\
+ m(j) \rho_a(\theta_{avg}(j, j+1, z-dz)) q(j, z-dz) \left[ C_a \theta_{avg}(j, j+1, z-dz) + \frac{1}{2} v_a^2(j, z-dz) \right] \\
+ m(j) H(j, z) \, dz \\
+ m(j) \gamma \varepsilon \left[ \theta_f^4(j, z) - \theta_a^4 \right] s(j, z) \, dz \\
= \rho_a(\theta_a(j+1, z)) v_a(j+1, z) w(j, z) \, dz \left[ C_a \theta_a(j+1, z) + \frac{1}{2} v_a^2(j, z) \right] \\
+ m(j) \rho_a(\theta_{avg}(j, j+1, z)) q(j, z) \left[ C_a \theta_{avg}(j, j+1, z) + \frac{1}{2} v_a^2(j, z) \right].
\end{align*}
\]  (2.37)

When nondimensionalizing equation (2.37) for conservation of energy, one can find that in all practical fiber-spinning regimes the terms corresponding to kinetic energy and heat radiation are much smaller than the other terms in the equation. For example, when the velocity \( v_a(j, z) \) and temperature \( \theta_a(j, z) \) of cross-flow quench air are 50 cm/sec and 30 °C, respectively, the ratio of internal energy to kinetic energy...
in the quench air system is

\[
\frac{\text{internal energy}}{\text{kinetic energy}} = \frac{C_a \theta_a(j, z)}{\frac{1}{2} \nu_a^2(j, z)} = \frac{3.04 \times 10^9}{1250} = 2.43 \times 10^6
\]  \hspace{1cm} (2.38)

Thus, for such quench condition the kinetic energy in the quench air is negligible compared to its internal energy. Similarly, when there are 20 fibers in a row experience the above mentioned quench air, the ratio of internal energy flow rate term for the cross-flow quench air to the radiation term at the spinneret is

\[
\frac{\text{internal energy flow rate}}{\text{radiation energy}} = \frac{\rho_a(\theta_a(j, z))v_a(j, z) w(j, z) C_a \theta_a(j, z)}{m(j)\gamma \varepsilon \left[\theta_f^4(j, z) - \theta_a^4\right] s(j, z)} = \frac{1.06 \times 10^9}{7.53 \times 10^6 \varepsilon}
\]  \hspace{1cm} (2.39)

where \(\varepsilon\) is the emittance of non-black surface and is usually much smaller than 1, and we choose \(s(j,s)\) as 0.035 cm and \(\theta_f(j, z)\) as 270 °C for the spinneret to have a conservative estimate. Thus, for simplicity we can delete these terms from equation (2.37), leaving

\[
\begin{align*}
\rho_a(\theta_a(j, z))v_a(j, z) w(j, z) d\xi \, C_a \, \theta_a(j, z) \\
+ \zeta(j) \, \rho_a(\theta_a(1, z))v_a(1, z) \left[w(j, z) - w(j-1, z)\right] d\xi \, C_a \, \theta_a(1, z) \\
- \zeta(j) \, \rho_a(\theta_a(j, z))v_a(j, z) \left[w(j, z) - w(j-1, z)\right] d\xi \, C_a \, \theta_a(j, z) \\
+ m(j) \, \rho_a(\theta_{\text{avg}}(j, j+1, z-d\xi)) q(j, z-d\xi) \\
+ m(j) \, H(j, z) \ d\xi \\
= \rho_a(\theta_a(j+1, z))v_a(j+1, z) w(j, z) d\xi \, C_a \, \theta_a(j+1, z) \\
+ m(j) \, \rho_a(\theta_{\text{avg}}(j, j+1, z)) q(j, z) \, C_a \, \theta_{\text{avg}}(j, j+1, z). 
\end{align*}
\]  \hspace{1cm} (2.40)
Conservation of linear momentum

For conservation of linear momentum, we consider the control volume in Figure 2.7 that includes the fibers. Conservation of linear momentum along the axial direction of fiber demands that the total force exerting on the control volume in axial direction is equal to the change of linear momentum per time in the same direction. This gives

\[
m(j) \left[ f_f(j, z) - f_f(j, z - dz) \right] \\
+ m(j) \pi \rho_f(j, z) g r_f^2(j, z) dz \cdot \sec \varphi(j, z) \cos \varphi(j, z)
- m(j) q(j, z) \rho_a(\theta_{avg}(j, j + 1, z)) \bar{v}_b(j, z)
- m(j) q(j, z - dz) \rho_a(\theta_{avg}(j, j + 1, z - dz)) \bar{v}_b(j, z - dz)
+ w(j, z) dz v_a^2(j + 1, z) \rho_a(\theta_a(j + 1, z)) \cdot \sin \varphi(j, z)
- w(j, z) dz v_a^2(j, z) \rho_a(\theta_a(j, z)) \cdot \sin \varphi(j, z)
+ m(j) G(j) \left[ v_f(j, z) - v_f(j, z - dz) \right] \tag{2.41}
\]

where \( \varphi(j, z) \) is the angle between the vertical and the deflected fiber. Similarly, in the direction normal to fibers we have

\[
-m(j) \pi \rho_f(j, z) g r_f^2(j, z) dz \cdot \cos \varphi(j, z) \sin \varphi(j, z)
= w(j, z) dz v_a^2(j + 1, z) \rho_a(\theta_a(j + 1, z)) \cos \varphi(j, z)
- w(j, z) dz v_a^2(j, z) \rho_a(\theta_a(j, z)) \cos \varphi(j, z) \tag{2.42}
\]

Summary

When a multi-row fiber spinning process with the same row width is investigated, the field equations (2.22), (2.40), (2.41), and (2.42) for the domain before the merging of the boundary layers can be simplified to:
For $0 \leq z \leq z_m(j)$:

$$\theta_{avg}(j, j+1, z) \frac{d}{dz} q(j, z) - q(j, z) \frac{d}{dz} \theta_{avg}(j, j+1, z)$$

$$- \frac{w(j, z)}{m(j)} \left[ \frac{v_a(j, z)}{\theta_a(j, z)} - \frac{v_a(j+1, z)}{\theta_a(j+1, z)} \right] \theta_{avg}^2(j, j+1, z) = 0,$$  \hspace{1cm} (2.43)

$$v_a(j+1, z) - v_a(j, z) + \frac{m(j)}{w(j, z)} \frac{d}{dz} q(j, z)$$

$$- \frac{m(j) \left( \theta_f(j, z) - \theta_a(j, z) \right)}{0.353 C_a w(j, z)} h(j, z) = 0,$$ \hspace{1cm} (2.44)

$$\frac{d}{dz} \left( \frac{q(j, z) \rho_a(j, z)}{\theta_a(j, z)} \Delta \theta_a(j, z) \right) - \frac{d}{dz} f_f(j, z) + G(j) \frac{d}{dz} v_f(j, z) - \pi \rho_f(j, z) \sigma r_f^2(j, z)$$

$$- \frac{w(j, z)}{m(j)} \left[ v_a^2(j, z) \rho_a(j, z) - v_a^2(j+1, z) \rho_a(j+1, z) \right] \sin(\varphi) = 0,$$ \hspace{1cm} (2.45)

$$\theta_a(j+1, z) v_a^2(j, z) - \theta_a(j, z) v_a^2(j+1, z)$$

$$- \pi \frac{m(j) \theta_a^2(j+1, z) \rho_f(j, z) \sigma r_f^2(j, z) \sec(\varphi) \tan(\varphi)}{0.353 \frac{g}{cm^3} w(j, z)} = 0.$$ \hspace{1cm} (2.46)

These are four ordinary differential equations for the average boundary layer velocity $\Delta \theta_a(j, z)$, the boundary layer volume flow rate $q(j, z)$, and the velocity $v_a(j+1, z)$ and temperature $\theta(j+1, z)$ of the quench air exiting row $j$ and passing on to row $j+1$. Note that equations (2.43), (2.44) and (2.46) are decoupled from $\Delta \theta_a(j, z)$ and can be used to solve for $v_a(j+1, z)$, $\theta_a(j+1, z)$ and $q(j, z)$ before using (2.45) to solve for $\Delta \theta_a(j, z)$. These equations are solved by shooting method with Runge-Kutta integration.
2.4.2 Model for the domain $z_m(j) \leq z \leq l(j)$ with no cross-flow quench air through the bundle

As the thicknesses of the boundary layers grow with increasing $z$, the boundary layers surrounding the fibers in some processes merge into a single boundary layer at some downstream location $z = z_m(j)$. Specifically, $z_m(j)$ is the axial location where the individual boundary layer thickness of a fiber in row $j$ reaches half of the distance $w(j, z) / m(j)$ between two neighboring fibers in row $j$. Thus, with the effective radius $r_b(j, z)$ of the boundary layer determined from $q(j, z)$ and $\bar{v}_b(j, z)$ by

$$
r_b(j, z) = \sqrt{\frac{q(j, z)}{\pi \bar{v}_b(j, z)}}, \tag{2.47}
$$
$z_m(j)$ is the solution of

$$r_b(j, z_m(j)) = \frac{w(j, z_m(j))}{2 \, m(j)}.$$  \hfill (2.48)

For $z > z_m(j)$ there is no cross flow of quench air through the fiber bundle, i.e. $v_a(j, z) = 0$, so that the entrained air is the only quench air in this region. We use $\theta_q(j, z)$ as the temperature of the entrained air at this region.

**Conservation of mass**

Since there is no cross-flow quench air, mass conservation for the control volume in Figure 2.8, excluding the fibers, is

$$m(j) \, \rho_a(\theta_q(j, z - dz)) \, q(j, z - dz) + 2 \, v_{in}(j, z) \, w(j, z) \, dz \, \rho_a(\theta_r)$$

$$= m(j) \, \rho_a(\theta_q(j, z))q(j, z),$$  \hfill (2.49)

where $v_{in}(j, z)$ is the velocity of incremental air flowing into the entrained boundary layer within the domain from $z-dz$ to $z$, i.e. the air entrained from the lateral of the control volume.

**Conservation of energy**

Applying the law of thermodynamics on the control volume in Figure 2.8 gives

$$m(j) \, \rho_a(\theta_q(j, z - dz))q(j, z - dz) \left[ C_a \, \theta_q(j, z - dz) + \frac{1}{2} \frac{v^2_b(j, z - dz)}{} \right]$$

$$+ m(j) \, h(j, z) \, (\theta_f(j, z) - \theta_q(j, z)) \, dz$$

$$+ m(j) \, c(\theta_f(j, z) - \theta_r) \, s(j, z) \, dz$$

$$+ 2 \, v_{in}(j, z) \, w(j, z) \, dz \, \rho_a(\theta_r) \left[ C_a \, \theta_r + \frac{1}{2} \frac{v^2_{in}(j, z)}{} \right]$$

$$= m(j) \, \rho_a(\theta_q(j, z))q(j, z) \left[ C_a \, \theta_q(j, z) + \frac{1}{2} \frac{v^2_b(j, z)}{} \right],$$  \hfill (2.50)
Figure 2.9: Control volume for fiber-air interaction model including the fibers at $z > z_m(j)$. 

**Conservation of linear momentum**

From the control volume in Figure 2.9, including the fibers, conservation of linear momentum in vertical direction demands

\[
\begin{align*}
    & m(j) f_f(j, z) \cdot \cos\varphi(j, z) - m(j) f_f(j, z-dz) \cdot \cos\varphi(j, z-dz) \\
    + & m(j) \pi \rho_f(j, z) g \tau_f^2(j, z) \, dz \cdot \sec\varphi(j, z) \\
    = & m(j) q(j, z) \rho_a(\theta_q(j, z)) \, \bar{v}_b(j, z) \cdot \cos\varphi(j, z) \\
    - & m(j) q(j, z-dz) \rho_a(\theta_q(j, z-dz)) \, \bar{v}_b(j, z-dz) \cdot \cos\varphi(j, z-dz) \\
    + & m(j) G(j) \left[ v_f(j, z) \cdot \cos\varphi(j, z) - v_f(j, z-dz) \cdot \cos\varphi(j, z-dz) \right], \quad (2.51)
\end{align*}
\]

For the domain beyond the merging of the boundary layers into an single boundary layer surrounding the entire fiber bundle, the four unknowns $\theta_q(j, z)$, $\bar{v}_b(j, z)$, $q(j, z)$, and $v_{in}(j, z)$ are related by only three fields equation (2.49)–(2.51). This is because,
with no cross flow through the fiber bundle, there is no transverse momentum equation.

To regain closure I assume the radial velocity profile within the boundary layer proposed by Sakiadis [24], from which follows a relation between average boundary layer velocity \( \bar{v}_b(j, z) \) and the boundary layer volume flow rate \( q(j, z) \)

Sakiadis’s assumed profile is

\[
v_b(j, z, r) = v_f(j, z) \left( 1 - \frac{1}{\vartheta(j, z)} \ln \left( \frac{r}{r_f(j, z)} \right) \right)
\]

(2.52)

where \( v_f(j, z) \) and \( r_f(j, z) \) are the velocity and radius of the fibers, and \( \vartheta(j, z) \) is related to the boundary layer radius thickness \( \delta(j, z) \) through

\[
\vartheta(j, z) = \ln \left( \frac{\delta(j, z)}{r_f(j, z)} \right).
\]

(2.53)

Hence the boundary layer volume flow rate is

\[
q(j, z) = \int_{r_f}^{\delta+r_f} 2\pi r v_q(j, z, r) \, dr
\]

\[
= \int_{r_f(j, z)}^{r_f(j, z)(1+\exp(\vartheta(j, z)))} 2\pi r v_f(j, z) \left( 1 - \frac{1}{\vartheta(j, z)} \ln \left( \frac{r}{r_f(j, z)} \right) \right) \, dr.
\]

(2.54)

and the average boundary layer velocity is

\[
\bar{v}_b(j, z) = \frac{q(j, z)}{\pi \left[ (\delta + r_f)^2 - r_f^2 \right]} = \frac{q(j, z)}{\pi \left[ (r_f(j, z)(1+\exp(\vartheta(j, z)))^2 - r_f^2 \right].
\]

(2.55)

In differential form, (2.49)–(2.51) are

\[
\frac{d}{dz} q(j, z) \theta_q(j, z) - q(j, z) \frac{d}{dz} \theta_q(j, z) - \frac{v_{in}(j, z) w(j, z)}{m(j)} \theta_q^2(j, z) = 0,
\]

(2.56)

\[
\frac{d}{dz} q(j, z) - \frac{w(j, z)}{m(j)} v_{in}(j, z) - \frac{h(j, z)(\theta_f(j, z) - \theta_q(j, z))}{0.353 \frac{Kg}{cm^2} \cdot C_a} = 0,
\]

(2.57)

\[
\frac{d}{dz} \left( q(j, z) \rho_a(j, z) \bar{v}_b(j, z) \right) - \frac{d}{dz} f_f(j, z) + G(j) \frac{d}{dz} v_f(j, z)
\]

30
\[-\pi \rho_f(j, z) g r_j^2(j, z) = 0. \tag{2.58}\]

These three equations and (2.54), (2.55) are five equations for five unknowns \(\theta_q(j, z), \bar{v}_b(j, z), q(j, z), v_{in}(j, z), \) and \(\theta(j, z).\)

### 2.5 Application of the fiber-air interaction model, and comparison with isothermal quench air model

The fiber-air interaction model of section 2.4 is now combined with the single fiber model of section 2.1 to produce a multi-fiber spinning model. This model simulates the complete melt spinning process: The behavior of the first row of fibers is determined by the single fiber model with known spinning conditions, material properties, and the supplied quench air conditions. Then the quench air conditions for the second row of fibers are generated by the fiber-air interaction model from spinning conditions, material properties, and the fiber behavior of row one, which is now known from the quench air conditions for row one. The behavior of the second row of fibers is determined by the single fiber model with known spinning conditions, material properties, and the quench air conditions for the second row of fibers generated by the fiber-air interaction model. Then the quench air conditions for the third row of fibers are generated by the fiber-air interaction model from spinning conditions, material properties, and the fiber behavior of row two, which is now known from the quench air conditions for row two, and so forth.

A nine-row melt spinning process is modeled, with spinning conditions given in Table 2.1. The material properties are specified in section 2.1, and the parameters for \(\theta_{avg}(j, j+1, z)\) in the fiber-air interaction model are chosen as \(\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}.\) Several quench conditions are investigated: \(\theta_a(1, z) = 25\) or \(30\) °C and \(v_a(1, z) = 30\)
<table>
<thead>
<tr>
<th>Spinning conditions</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spinneret hole radius</td>
<td>$1.78 \times 10^{-2}$ cm</td>
</tr>
<tr>
<td>Spinline length</td>
<td>120 cm</td>
</tr>
<tr>
<td>Throughput $G$</td>
<td>$8.3 \times 10^{-3}$ g/sec</td>
</tr>
<tr>
<td>Extrudate temperature $\theta_0$</td>
<td>270 °C</td>
</tr>
<tr>
<td>Take-up velocity $v_t$</td>
<td>1200 cm/sec</td>
</tr>
<tr>
<td>Number of rows</td>
<td>9</td>
</tr>
<tr>
<td>Number of fibers per row</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 2.1: Spinning conditions used in the simulations of multi-fiber spinning process

or 50 cm/s, and the corresponding simulation are presented in Figure 2.10–2.16. In all of these simulations the quench air temperature and velocity profiles striking the first row of fibers are uniform down the spinline, although the fiber-air interaction creates spatial nonuniformity of quench conditions for the subsequent eight rows.

As shown in Figure 2.10, 2.12, 2.14, and 2.16, the quench air conditions are different for each row. Being heated by the spinning fibers, the quench air temperature $\theta_a(j, z)$ increases monotonically in $j$ at fixed $z$, and decreases in $z$ at fixed $j$; the quench temperature at a given axial distance from the spinneret increases as one travels into the fiber bundle in the direction of the quench flow, whereas the quench temperature for each fiber cools down the spinline. As $j$ increases, $\theta_a(j, 0^+)$ marches toward the spinneret temperature. For a given $\theta_a(1, z)$, the quench air is heated more in the case with $v_a(1, z) = 30$ m/s than the case with $v_a(1, z) = 50$ m/s, since in the $v_a(1, z) = 50$ m/s case more air is used in a given time to cool the same amount of fibers, and there is less time for the air to be heated while passing through the fibers. Quantitatively, the temperature of quench air at the axial location 0.25 m from the spinneret ranges from 25 °C entering the first row to 26.71 °C entering the
second row to 42.78 °C leaving the ninth row for the case with \( v_a(1, z) = 30 \text{ cm/s} \) and \( \theta_a(1, z) = 25 \text{ °C} \), and ranges from 25 °C to 26.14 °C to 36.66 °C at the same position and rows for the case with \( v_a(1, z) = 50 \text{ cm/s} \) and \( \theta_a(1, z) = 25 \text{ °C} \).

As the quench air passes through the spinning fibers, part of it is diverted to the boundary layer surrounding the spinning fibers. This causes the mass of air passing through the spinning fibers to decrease as the row number \( j \) increases. If the temperature change of air is ignored, the quench air cross flow velocity would be expected to decrease correspondingly. However, except near the spinneret the cross-flow velocity of quench air increases as \( j \) increases. This is because the density of the quench air decreases sufficiently, due to the heating of the quench air, to increase the volume flow rate of the cross flow air, even though the mass flow rate is decreasing. See Figures 2.10, 2.12, 2.14, and 2.16. Near the spinneret there is a region in which the cross-flow velocity decreases, because of the quantity of air needed for the initiation of the boundary layers. The cross-flow velocity of quench air ranges from 30 cm/s striking the first row of fibers to 30.07 cm/s striking the second row to 30.78 cm/s striking the ninth row at the axial location 0.25 m from the spinneret for the \( v_a(1, z) = 30 \text{ cm/s} \) and \( \theta_a(1, z) = 25 \text{ °C} \) case, and it ranges from 50 to 50.09 to 50.91 cm/s at the same position for the \( v_a(1, z) = 50 \text{ cm/s} \) and \( \theta_a(1, z) = 25 \text{ °C} \) case. In addition to the difference of the range in quench velocity at a given axial location, the shape of the quench velocity axial profiles from row to row are quite different in the \( v_a(1, z) = 30 \text{ cm/s} \) and \( v_a(1, z) = 50 \text{ cm/s} \) cases.

The boundary layer volume flow rate per fiber down a given fiber increases monotonically. It is also noted that the boundary layer volume flow rate per fiber at any axial location increases monotonically as \( j \) increases in the two \( v_a(j, z) = 30 \text{ m/s} \)
cases but not in the two \( v_a(j, z) = 50 \text{ m/s} \) cases. The axial gradient of boundary layer volume flow rate per fiber, \( \frac{dq(j, z)}{dz} \), can be used to determine whether the cross-flow velocity \( v_a(j+1, z) \) experienced by the next row of fibers is larger than the cross-flow velocity \( v_a(j, z) \) experienced by the current row of fibers. If we rewrite (2.44) as

\[
v_a(j+1, z) = v_a(j, z) - \frac{m(j)}{w(j, z)} \frac{d}{dz} q(j, z) + \frac{m(j)(\theta_f(j, z) - \theta_a(j, z)) h(j, z)}{0.353 C_a w(j, z)}, \tag{2.59}
\]

it is seen that if the term \( \frac{m(j)}{w(j, z)} \frac{d}{dz} q(j, z) \) is less than the term \( \frac{m(j)(\theta_f(j, z) - \theta_a(j, z)) h(j, z)}{0.353 C_a w(j, z)} \) accounting for the volume increase of cross-flow air due to temperature rise, the cross-flow velocity \( v_a(j+1, z) \) experienced by the next row is larger than the velocity \( v_a(j, z) \) experienced by the current row \( j \). If \( \frac{m(j)}{w(j, z)} \frac{d}{dz} q(j, z) \) is larger than \( \frac{m(j)(\theta_f(j, z) - \theta_a(j, z)) h(j, z)}{0.353 C_a w(j, z)} \), the cross-flow velocity \( v_a(j+1, z) \) experienced by the next row is smaller than that of the current row, \( v_a(j, z) \). This latter case happens in the vicinity of the spinneret and far from the spinneret in Figure 2.10, 2.12, 2.14, and 2.16, in whereas the cross-flow velocity \( v_a(j+1, z) \) experienced by the next row is larger than that of the current row, \( v_a(j, z) \), in an intermediate interval.

Since the quench air temperature increases from row one to row nine \( (\theta_a(j, z) \) increases as \( j \) increases), the fiber temperature profiles show that there is a delaying of cooling of the fibers as \( j \) increases. This is more pronounced in the \( v_a(1, z) = 30 \text{ cm/s} \) cases than in the \( v_a(1, z) = 50 \text{ m/s} \) cases. The fiber temperature ranges from 64.26 °C in row one to 89.17 °C in row nine at the axial location 0.25 m from the spinneret for the \( v_a(1, z) = 30 \text{ cm/s} \) and \( \theta_a(1, z) = 25 \text{ °C} \) case, and from 61.32 for row one °C to 79.39 °C for row nine at that location for the \( v_a(1, z) = 50 \text{ cm/s} \) and \( \theta_a(1, z) = 25 \text{ °C} \) case. For the \( v_a(1, z) = 30 \text{ cm/s} \) and \( \theta_a(1, z) = 30 \text{ °C} \) case, the
range of fiber temperature is from 68.89 °C for row one to 100.21 °C for row nine at 0.25 m from the spinneret, more than the corresponding 65.93 °C to 83.94 °C for the \( v_a(1, z) = 50 \text{ cm/s} \) and \( \theta_a(1, z) = 30 \text{ °C} \) case.

Fiber behavior is dominated by the fiber temperature profile. As the temperature of a fiber reduces, the intrinsic viscosity \( \beta(j, z) \) increases and the fiber is more rigid, moving with a velocity closer the take-up velocity \( v_{\text{take-up}} \). Thus the first row of fibers, with the most rapid cooling, approaches to the take-up velocity quicker than the second row of fibers, the second row of fibers approaches the take-up velocity quicker than the third row of fibers, and so forth. Indeed, as \( j \) increases, the fibers march to the take-up velocity more reluctantly due to the slower cooling. To reflect how the velocities of fibers are influenced by the fiber temperatures, the velocities of fibers are isolated at the axial location 0.125 m from the spinneret (it is not good to sample at 0.25 m from the spinneret since all of the fiber velocities have already reached 99% of the take-up velocity at 0.25 m from the spinneret). At 0.125 m from the spinneret, the fiber velocity varies from 10.98 m/s to 9.09 m/s for the \( v_a(1, z) = 30 \text{ cm/s} \) and \( \theta_a(1, z) = 30 \text{ °C} \) case, and from 11.10 m/s to 10.22 m/s for \( v_a(1, z) = 50 \text{ cm/s} \) and \( \theta_a(1, z) = 30 \text{ °C} \) case.

The change of density \( \rho_f(j, z) \) with the assumed linear function (2.5) of temperature is about 0.03% for every 1 °C of temperature change. The change of temperature during the spinning process is 220 °C, which produces a density change of 6.6%. This change in \( \rho_f(j, z) \) is negligible when compared to the change of \( \rho_f \) due to the change in the velocity profile. Consequently, mass conservation, \( G = \pi r_f^2(j, z) \rho_f(j, z) v_f(j, z) = \) constant, suggests that the square of the fiber radius \( r_f(j, z) \) is nearly proportional to \( v_f^{-1}(j, z) \) in the process. The radius \( r_f(j, z) \) of the fiber drops faster when \( j \) is
smaller, since \( v_f(j,z) \) grows with increasing \( j \). Specifically, the fiber radius varies from 0.1366 mm in row one to 0.1511 mm in row nine at 0.125 m from the spinneret for \( v_a(1,z) = 30 \text{ cm/s} \) and \( \theta_a(1,z) = 30 \text{ °C} \) case and varies from 0.1357 mm to 0.1420 mm at 0.125 m from the spinneret for \( v_a(1,z) = 50 \text{ cm/s} \) and \( \theta_a(1,z) = 30 \text{ °C} \) case.

The variation of tensile force in fibers also reflects the influence of fiber temperature, since the intrinsic viscosity \( \beta(j,z) \) is a function of fiber temperature. At 0.25 m from the spinneret, the tensile force in the fibers varies from 101.0 dyne in row one to 72.00 dyne in row nine for the \( v_a(1,z) = 30 \text{ cm/s} \) and \( \theta_a(1,z) = 30 \text{ °C} \) case and varies from 112.5 dyne to 90.77 dyne for the \( v_a(1,z) = 50 \text{ cm/s} \) and \( \theta_a(1,z) = 30 \text{ °C} \) case.

Recall that the analysis of Dutta [7] ignores both the heating of the quench air stream as it cools the fiber bundle, and the density change of the quench air due to this temperature change. To investigate the error due to ignoring the temperature change of the quench air in the fiber-air interaction model, Dutta’s formula (2.17) is used to determine the cross-flow air velocity \( v_a(j,z) \) using the boundary layer volume flow rate predicted by the air in fiber-air interaction model presented in this dissertation. Comparing equation (2.17) with (2.59), one finds that the difference is

\[
\frac{m(j)(\theta_f(j,z) - \theta_a(j,z)) h(j,z)}{0.353 C_a w(j,z)}
\]

the term that accounts for the volume increase of cross-flow air due to its temperature change. The comparison of computed cross-flow quench air velocities through the fiber bundle is shown in Figures 2.18 for the \( v_a(1,z) = 30 \text{ cm/s} \) and \( \theta_a(1,z) = 25 \text{ °C} \) case. When the temperature change of quench air is ignored, the deflection of quench air into the fiber boundary layers makes the cross-flow air slow down much more than what is predicted by the air-fiber interaction model presented in this dissertation, as shown in Figure 2.18. Figure 2.19 shows that the fiber behavior
from the multi-fiber model with isothermal quench air and cross-flow air velocity
produced by Dutta’s formula (2.17); note the row to row variation in Figure 2.19 is
much less than that predicted by my model, shown in Figure 2.11.
Figure 2.10: Quench air temperature and quench air velocity profiles in the multi-row fiber spinning process of Table 1 with the quench conditions $\theta_q(1, z) = 25 \degree C$ and $v_q(1, z) = 30 \text{ m/s}$. The horizontal axis is the distance from the spinneret in meters.
Figure 2.11: Fiber temperature, velocity, and tensile force, and boundary layer volume flow rate per fiber in the multi-row fiber spinning process of Table 1 with the quench conditions $\theta_{q}(1, z) = 25 ^\circ C$ and $v_{q}(1, z) = 30 \text{ m/s}$. The horizontal axis is the distance from the spinneret in meters.
Figure 2.12: Quench air temperature and quench air velocity profiles in the multi-row fiber spinning process of Table 1 with the quench conditions $\theta_a(1, z) = 30^\circ C$ and $v_a(1, z) = 30$ m/s. The horizontal axis is the distance from the spinneret in meters.
Figure 2.13: Fiber temperature, velocity, and tensile force, and boundary layer volume flow rate per fiber in the multi-row fiber spinning process of Table 1 with the quench conditions $\theta_a(1,z) = 30^\circ C$ and $v_a(1,z) = 30$ m/s. The horizontal axis is the distance from the spinneret in meters.
Figure 2.14: Quench air temperature and quench air velocity profiles in the multi-row fiber spinning process of Table 1 with the quench conditions $\theta_a(1,z) = 25^\circ C$ and $v_a(1,z) = 50$ m/s. The horizontal axis is the distance from the spinneret in meters.
Figure 2.15: Fiber temperature, velocity, and tensile force, and boundary layer volume flow rate per fiber in the multi-row fiber spinning process of Table 1 with the quench conditions $\theta_a(1, z) = 25 \, ^\circ C$ and $v_a(1, z) = 50 \, m/s$. The horizontal axis is the distance from the spinneret in meters.
Figure 2.16: Quench air temperature and quench air velocity profiles in the multirow fiber spinning process of Table 1 with the quench conditions $\theta_a(1, z) = 30^\circ C$ and $v_a(1, z) = 50$ m/s. The horizontal axis is the distance from the spinneret in meters.
Figure 2.17: Fiber temperature, velocity, and tensile force, and boundary layer volume flow rate per fiber in the multi-row fiber spinning process of Table 1 with the quench conditions $\theta_a(1, z) = 30$ °C and $v_a(1, z) = 50$ m/s. The horizontal axis is the distance from the spinneret in meters.
Figure 2.18: The variation of quench air cross-flow velocity profile determined by Dutta's air-fiber interaction model with isothermal quench air (the curves with dots) and the air-fiber interaction model presented in this dissertation (the curves without dots) for the process of Table 1 with $v_a(1,z) = 30$ cm/s, $\theta_a(1,z) = 25^\circ C$. 
Figure 2.19: Fiber temperature, velocity and tensile force profiles produced by the single fiber model given the quench conditions predicted by Dutta's isothermal quench air model (see Figure 21), for $\theta_d(1, z) = 25 ^\circ C$ and $v_d(1, z) = 30$ cm/s. Compare with Figure 14.
CHAPTER 3

NONISOTHERMAL FIBER SPINNING WITH TEMPERATURE-DEPENDENT DENSITY

3.1 Introduction

Existing thin-fiber models for nonisothermal melt spinning with thermally-induced shrinkage are based on the \textit{a posteriori} substitution of a temperature-dependent density function into the thin-fiber equations for an incompressible material. An example is the single fiber model used in the previous chapter. This \textit{a posteriori} substitution is in violation of the second law of thermodynamics. Here a fiber-spinning model is presented, based on the thermomechanically consistent theory for a material with prescribed temperature-dependent density developed in [5] to account for the effects of shrinkage. In addition to temperature-dependent density, the thin-fiber model incorporates the leading-order radial resolution of temperature and stress recently presented in [14]. To examine the effects of correctly modeling radial gradients and temperature dependence of density, solutions of our theory with and without radial temperature variation are given, and compared both with solutions from the conventional theory based on the incompressible theory with \textit{a posteriori} substitution of a temperature-dependent density and the assumption of radially-independent temperature. It is found that the consistent modeling of temperature-dependent density
predicts slower cooling, delayed stretching, and lower maximum tensile stress in the
spinline than does the conventional theory based on the incompressible theory, and
also a stress reduction near the fiber’s solidification point.

3.2 The thermomechanically constrained theory for a material with prescribed temperature-dependent density

Polymer fibers are manufactured in nonisothermal spinning processes in which
molten polymer is extruded from a circular die into cooling air and drawn continu-
ously into a solidified thin fiber. In melt spinning the polymer experiences a large
temperature change, so that it is necessary in modeling the process to account for
the shrinkage experienced by the polymer as it cools. The standard practice in ex-
isting models is to *a posteriori* substitute a temperature-dependent expression for
density into the thin-fiber equations for an incompressible material, either directly
(Papanastasiou *et al.* [22], Hayashi *et al.* [13], Dutta [8]), or in a form in which the
incompressibility constraint is modified to account for temperature-induced shrink-
age (Kase & Matsuo [16]). Neither of these approaches is thermomechanically self-
consistent. Specification of material density as a function of temperature amounts to
the imposition of a thermomechanical constraint, i.e. a condition on the temperature
and deformation state of the material that must be satisfied *a priori* by any motion.

The first general mechanical theory of internal constraints was developed by Noll
[21]. The generalization to thermomechanical constraints was made by Green, Naghdi
was first recognized as a material constraint by Green *et al.* [10] and the resulting
balance equations governing a material subject to this constraint have been studied
by Cao, Bechtel, and Forest [5], and Bechtel, Cao, and Hsiao [1].

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For a material with prescribed temperature-dependent density, conservation of mass becomes the thermomechanical constraint

\[ \frac{\rho'(\theta)}{\rho(\theta)} \dot{\theta} + \text{div} \, \mathbf{v} = 0, \quad (3.1) \]

In equation (3.1), \( \theta \) absolute temperature, \( \mathbf{v} \) the velocity, \( \text{div} \) the Eulerian divergence operator, \( \rho(\theta) \) the prescribed temperature density, and \( \rho'(\theta) \) the derivative of this function with respect to temperature,

\[ \rho'(\theta) = \frac{d}{d\theta} \rho(\theta). \quad (3.2) \]

To support the constraint (3.1) there must be a constraint response \( p \) which appears not only in the Cauchy stress as an indeterminate pressure added to the constitutive part \( \mathbf{T} \) of the stress tensor,

\[ \mathbf{T} = \hat{\mathbf{T}} - p \mathbf{I}, \quad (3.3) \]

but also as an additive term in the internal energy,

\[ \varepsilon = \dot{\varepsilon} + p\theta \frac{\rho'}{\rho^2}, \quad (3.4) \]

where \( \dot{\varepsilon} \) is given by a constitutive assumption. (3.4) shows that for a material with prescribed temperature-dependent density the term \( p\theta \frac{\rho'}{\rho^2} \) is needed in the internal energy to offset the entropy created by the constraint pressure in the stress, so that the net entropy generated by the entire response maintaining the constraint (3.1) is zero. For an incompressible material, the constraint response appears only in stress, not in the internal energy.

With the constraint (3.1) and the results (3.3) and (3.4), the field equations for a material with temperature-dependent density are [5]

\[ \text{div} \, \mathbf{v} = -\frac{\rho'(\theta)}{\rho(\theta)} \dot{\theta}, \quad (3.5) \]
\begin{align}
\rho(\theta)\dot{\mathbf{v}} &= \text{div} \, \mathbf{T} - \text{grad} \, p + \rho(\theta)\mathbf{g}, \quad \text{(3.6)}
\rho(\theta)c(\theta)\dot{\theta} + \frac{\rho'(\theta)}{\rho(\theta)} \theta \ddot{\mathbf{p}} + \frac{\rho \theta}{\rho(\theta)} \dot{\theta} (\rho''(\theta) - 2 \frac{\rho'^2(\theta)}{\rho(\theta)}) = \mathbf{T} \cdot \mathbf{D} + \text{div} \, (k \text{ grad } \theta), \quad \text{(3.7)}
\end{align}

where in the energy equation (3.7) neglected radiation is neglected, specific heat \(c(\theta)\) is assumed to be temperature-dependent, and Fourier heat conduction is also assumed. For simplicity, here a constant thermal conductivity \(k\) is assumed. In equations (3.5)–(3.7) \(\mathbf{g}\) is the acceleration of gravity and \(\mathbf{D}\) the symmetric part of the velocity gradient. To explicitly describe the standard practice for accounting for cooling-induced shrinkage mentioned in the abstract and earlier in this section, the thin-fiber models in Papanastasiou et al. [22], Hayashi et al. [13], Dutta [8] follow from the equations for incompressible material with \textit{a posteriori} substitution of temperature-dependent density,

\begin{align}
\text{div} \, \mathbf{v} &= 0, \quad \text{(3.8)}
\rho(\theta)\dot{\mathbf{v}} &= \text{div} \, \mathbf{T} - \text{grad} \, p + \rho(\theta)\mathbf{g}, \quad \text{(3.9)}
\rho(\theta)c(\theta)\dot{\theta} &= \mathbf{T} \cdot \mathbf{D} + \text{div} \, (k \text{ grad } \theta), \quad \text{(3.10)}
\end{align}

and the thin-fiber model in Kase & Matsuo [16] follows from equations (3.9) and (3.10) with constraint (3.5) rather than (3.8). Both of these sets of equations ((3.8)–(3.10) and (3.5), (3.9), (3.10)) miss the constraint response in the energy equation (compare (3.10) with (3.7)). The constraint of temperature-dependent density is fundamentally different from incompressibility: one cannot merely insert a temperature-dependent function for density \textit{a posteriori} in the conventional incompressible balance equations. Constraint response terms must be included in the energy equation as well as the momentum equation, and the incompressibility constraint must be modified.
Cao [5] and Bechtel et al. [1] show that modeling nonisothermal plane, capillary, and annular Poiseuille flows with equations (3.5)–(3.7) yields solutions quantitatively and qualitatively much different from what is obtained from a posteriori substitution of $\rho(\theta)$ into the equations for an incompressible material; the constraint response is found to have a significant effect on the velocity and temperature profiles and the mass flow vs. pressure gradient relation. The constraint response term in the energy equation also leads to differences in the modeling of fiber spinning, as is demonstrated here.

### 3.3 The 3-dimensional boundary value problem for melt spinning

The melt spinning processing ignoring transverse deflection is shown in Figure 3.1. It is assumed that the melt prior to solidification is a Newtonian fluid, which is generally accepted for poly(ethylene terephthalate) (PET) fiber spinning, and the flow is assumed to be axisymmetric along the direction of gravity.

The boundary value problem for the melt spinning of a fluid with prescribed temperature-dependent density consists of field equations (3.5)–(3.7), together with the kinematic free surface boundary condition,

$$\left\{ \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) F \right\}_s = 0,$$

the kinetic boundary condition,

$$t^a - T_s n = \sigma \kappa n,$$

and the convective heat loss boundary condition,

$$- (k \text{ grad } \theta) \cdot n = h(\theta - \theta_s)|_s,$$
Figure 3.1: The melt spinning process
at the axisymmetric free surface

\[ F = \phi(z,t) - r = 0. \]  

(3.14)

In (3.11)–(3.13) "|\(\vartheta\)|" denotes the value of the function at the surface (3.14), \(\mathbf{n}\) the outward normal unit vector at the surface, \(\kappa\) the mean curvature of the surface, \(\sigma\) the surface tension (assumed constant), and \(h\) the heat loss coefficient (assumed to be a known function of the fiber velocity and free surface radius). The symbols with superscript \(a\), namely \(\mathbf{t}^a\) and \(\theta^a\), are the traction and temperature of the ambient air, respectively, at the fiber surface. The ambient temperature \(\theta^a\) is a known function of axial location \(z\) and time \(t\). The boundary ambient traction \(\mathbf{t}^a\) is expressed in the form

\[ \mathbf{t}^a = -(\tilde{p} + \rho^a)\mathbf{n} - t^a\mathbf{m}, \]  

(3.15)

where \(\mathbf{m}\) is the unit vector tangent to the surface (3.14) in the \(r, z\) plane (see Figure 23), the normal and shear traction components \(p^a\) and \(t^a\) due to air drag are known functions of \(z\) and \(t\), and \(\tilde{p}\) is a known constant barometric pressure.

For the axisymmetric surface

\[ \mathbf{n} = \frac{\mathbf{e}_r - \phi_{,z}\mathbf{e}_z}{(1 + \phi^2_{,z})^{1/2}}, \quad \mathbf{m} = \frac{\phi_{,z}\mathbf{e}_r + \mathbf{e}_z}{(1 + \phi^2_{,z})^{1/2}}, \quad \kappa = \frac{1}{\phi(1 + \phi^2_{,z})^{1/2}} - \frac{\phi_{,zz}}{(1 + \phi^2_{,z})^{3/2}}. \]  

(3.16)

For the melt a Newtonian constitutive model for stress with an Arrhenius temperature dependence of viscosity is adopted,

\[ \dot{\mathbf{T}} = 2\mu(\theta)\mathbf{D}, \quad \mu(\theta) = \mu_0\exp\left(\frac{E}{R\theta}\right), \]  

(3.17)

where \(R\) is the gas constant, and \(\mu_0\) and \(E\) are material constants. Solidification in the spinline is modeled by assuming that when fiber temperature decreases to a specified
temperature \( \theta_g \) the polymer becomes rigid. For definiteness, it is assumed that the density and specific heat are linear functions of temperature,

\[
\rho(\theta) = \rho_0 - \rho_1 \theta, \quad c(\theta) = c_0 + c_1 \theta.
\]  

(3.18)

The boundary value problem is nondimensionalized with characteristic values \( r_c, z_c, t_c, v_c, \tau_c, p_c, \theta_c, \rho_c, c_c, \mu_c, h_c \) for transverse length, axial length, time, axial velocity, stress, pressure, absolute temperature, density, specific heat, viscosity, and heat loss, respectively. The slenderness parameter \( \epsilon \) is the ratio of the characteristic transverse length to the characteristic axial length,

\[
\epsilon = r_c / z_c.
\]  

(3.19)

The dimensionless independent variables \( \tilde{r}, \tilde{z}, \tilde{t} \) are defined through

\[
\tilde{r} = r_c \tilde{r}, \quad \tilde{z} = z_c \tilde{z}, \quad \tilde{t} = t_c \tilde{t}.
\]  

(3.20)

The dimensionless dependent variables, namely the nonzero polar cylindrical components \( \tilde{v}_r, \tilde{v}_z, \tilde{T}_{rr}, \tilde{T}_{\theta\theta}, \tilde{T}_{zz}, \tilde{T}_{rz} \) of velocity vector and determinate stress tensor, gage pressure \( \tilde{p} \), temperature \( \tilde{\theta} \) (functions of \( \tilde{r}, \tilde{z} \), and \( \tilde{t} \) only, due to axisymmetry) and free surface radius \( \tilde{\phi} \) (a function of \( \tilde{z} \) and \( \tilde{t} \)) are defined through

\[
\begin{align*}
\tilde{v}_r(r, z, t) &= \epsilon v_c \tilde{v}_r(\tilde{r}, \tilde{z}, \tilde{t}), & \tilde{v}_z(r, z, t) &= \epsilon v_c \tilde{v}_z(\tilde{r}, \tilde{z}, \tilde{t}), & \phi(z, t) &= r_c \tilde{\phi}(\tilde{z}, \tilde{t}), \\
\tilde{T}_{rr}(r, z, t) &= \tau_c \tilde{T}_{rr}(\tilde{r}, \tilde{z}, \tilde{t}), & \tilde{T}_{\theta\theta}(r, z, t) &= \tau_c \tilde{T}_{\theta\theta}(\tilde{r}, \tilde{z}, \tilde{t}), \\
\tilde{T}_{zz}(r, z, t) &= \tau_c \tilde{T}_{zz}(\tilde{r}, \tilde{z}, \tilde{t}), & \tilde{T}_{rz}(r, z, t) &= \epsilon \tau_c \tilde{T}_{rz}(\tilde{r}, \tilde{z}, \tilde{t}), \\
p(r, z, t) - \tilde{p} &= p_c \tilde{p}(\tilde{r}, \tilde{z}, \tilde{t}), & \theta(r, z, t) &= \theta_c \tilde{\theta}(\tilde{r}, \tilde{z}, \tilde{t}).
\end{align*}
\]  

(3.21)

The known functions for ambient air drag, temperature, and heat convection are scaled as

\[
\begin{align*}
p^a(z, t) &= p_c \tilde{p}(\tilde{z}, \tilde{t}), & t^a(z, t) &= \epsilon d_c \tilde{t}^a(\tilde{z}, \tilde{t}), & \theta^a(z, t) &= \theta_c \tilde{\theta}(\tilde{z}, \tilde{t}),
\end{align*}
\]  

55
\[ h(z, t) = h_c \hat{h}(\tilde{z}, \tilde{t}). \]  

(3.22)

The characteristic density \( \rho_c \), specific heat \( c_c \), and viscosity \( \mu_c \) are selected as

\[ \rho_c = \rho(\theta_c) = \rho_0 - \rho_1 \theta_c, \quad c_c = c(\theta_c) = c_0 + c_1 \theta_c, \quad \mu_c = \mu(\theta_c) = \mu_0 \exp \left( \frac{E}{R \theta_c} \right). \]  

(3.23)

The dimensionless boundary value problem in the melt consists of the dimensionless field equations:

\[ \tilde{v}_{r, \tilde{t}} + \frac{\tilde{v}_r}{\tilde{r}} + \tilde{v}_{z, \tilde{z}} = \frac{P}{1 + P - P\tilde{\theta}} \left( \beta \tilde{\theta}_{, \tilde{t}} + \tilde{v}_r \tilde{\theta}_{, \tilde{r}} + \tilde{v}_z \tilde{\theta}_{, \tilde{z}} \right), \]  

(3.24)

\[ \epsilon^2 (1 + P - P\tilde{\theta}) (\beta \tilde{v}_{r, \tilde{t}} + \tilde{v}_r \tilde{v}_{r, \tilde{r}} + \tilde{v}_z \tilde{v}_{z, \tilde{z}}) = B \left[ \tilde{T}_{rr, \tilde{t}} + \frac{\tilde{T}_{rr}}{\tilde{r}} + \frac{\tilde{T}_{r\theta, \tilde{z}}}{\tilde{r}} - \epsilon^2 \tilde{T}_{rz, \tilde{z}} - \alpha \tilde{p}_{, \tilde{r}} \right], \]  

(3.25)

\[ (1 + P - P\tilde{\theta}) (\beta \tilde{v}_{z, \tilde{t}} + \tilde{v}_r \tilde{v}_{z, \tilde{r}} + \tilde{v}_z \tilde{v}_{z, \tilde{z}}) = B \left[ \tilde{T}_{rz, \tilde{r}} + \tilde{T}_{zz, \tilde{z}} + \frac{\tilde{T}_{rz}}{\tilde{r}} - \alpha \tilde{p}_{, \tilde{z}} \right] + \frac{1 + P - P\tilde{\theta}}{Fr}, \]  

(3.26)

\[ (1 + P - P\tilde{\theta}) \left[ 1 - C + C\tilde{\theta} - 2\epsilon^2 U \alpha \frac{(\tilde{p} + A)\tilde{\theta}}{(1 + P - P\tilde{\theta})^3} \right] (\beta \tilde{\theta}_{, \tilde{t}} + \tilde{v}_r \tilde{\theta}_{, \tilde{r}} + \tilde{v}_z \tilde{\theta}_{, \tilde{z}}) \]  

\[ - \frac{PU\alpha\tilde{\theta}}{1 + P - P\tilde{\theta}} (\beta \tilde{p}_{, \tilde{t}} + \tilde{v}_r \tilde{p}_{, \tilde{r}} + \tilde{v}_z \tilde{p}_{, \tilde{z}}) \]  

\[ = U \left[ \tilde{T}_{rr, \tilde{t}} \tilde{v}_r, \tilde{r} + \tilde{T}_{r\theta, \tilde{z}} \tilde{v}_r, \tilde{z} + \frac{1}{2} \tilde{T}_{rz} (\epsilon^2 \tilde{v}_r, \tilde{z} + \tilde{v}_z, \tilde{r}) \right] \]  

\[ + \frac{1}{Gz} \left( \tilde{\theta}_{, \tilde{r}} + \frac{1}{\tilde{r}} \tilde{\theta}_{, \tilde{z}} + \epsilon^2 \tilde{\theta}_{, \tilde{z}, \tilde{z}} \right), \]  

(3.27)

\[ \tilde{T}_{rr} = Es\tilde{v}_{r, \tilde{r}} \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right], \]  

(3.28)

\[ \tilde{T}_{r\theta} = Es\tilde{v}_r \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right], \]  

(3.29)

\[ \tilde{T}_{zz} = Es\tilde{v}_z \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right], \]  

(3.30)

\[ \tilde{T}_{rz} = \frac{Es}{2} (\tilde{v}_{r, \tilde{z}} + \epsilon^{-2} \tilde{v}_{z, \tilde{r}}) \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right], \]  

(3.31)
and dimensionless boundary conditions:

\begin{align}
(\beta \tilde{\phi}_{\hat{z}} - \tilde{v}_r + \tilde{v}_s \tilde{\phi}_{\hat{r}})|_{\hat{r}=\hat{\phi}} &= 0, \quad (3.32) \\
(\alpha \tilde{\rho} - \tilde{T}_{rr} + \epsilon^2 \tilde{T}_{rz} \tilde{\phi}_{\hat{z}})|_{\hat{r}=\hat{\phi}} &= Y \left( \frac{1}{\tilde{\phi}(1 + \epsilon^2 \tilde{\phi}_{\hat{r}}^2)^{\frac{1}{2}}} - \frac{\epsilon^2 \tilde{\phi}_{\hat{r}}^2}{(1 + \epsilon^2 \tilde{\phi}_{\hat{r}}^2)^{\frac{3}{2}}} \right) + \epsilon^2 \frac{\tilde{d}_c \tilde{v}_a \tilde{\phi}_{\hat{z}}}{\tau_c}, \quad (3.33) \\
[\tilde{T}_{rz} + (\alpha \tilde{\rho} - \tilde{T}_{zz}) \tilde{\phi}_{\hat{r}}]|_{\hat{r}=\hat{\phi}} &= Y \left( \frac{1}{\tilde{\phi}(1 + \epsilon^2 \tilde{\phi}_{\hat{r}}^2)^{\frac{1}{2}}} - \frac{\epsilon^2 \tilde{\phi}_{\hat{r}}^2}{(1 + \epsilon^2 \tilde{\phi}_{\hat{r}}^2)^{\frac{3}{2}}} \right) \tilde{\phi}_{\hat{z}} + \frac{\tilde{d}_c \tilde{v}_a}{\tau_c}, \quad (3.34) \\
\frac{1}{(1 + \epsilon^2 \tilde{\phi}_{\hat{r}}^2)^{\frac{1}{2}}} (\tilde{\phi}_{\hat{r}} - \epsilon^2 \tilde{\phi}_{\hat{r}} \tilde{\theta}_{\hat{z}})|_{\hat{r}=\hat{\phi}} &= -\text{Bi} \tilde{h} (\tilde{\theta}|_{\hat{r}=\hat{\phi}} - \tilde{\theta}^a). \quad (3.35)
\end{align}

The dimensionless boundary value problem (3.24)–(3.35) involves the following dimensionless combinations of material properties and characteristic scales:

\begin{align}
\epsilon &= \frac{r_c}{z_c}, \quad B = \frac{\tau_c}{\rho_c v_c^2}, \quad Fr = \frac{v_c^2}{g z_c}, \quad Y = \frac{\sigma}{\tau_c r_c}, \quad \alpha = \frac{p_c}{\tau_c} \\
\beta &= \frac{z_c}{t_c v_c}, \quad Es = \frac{2 \mu_c v_c}{\tau_c z_c}, \quad Gz = \frac{\rho_c c_v r_c^2}{k z_c}, \quad Bi = \frac{h_c r_c}{k}, \\
U &= \frac{\tau_c}{\rho_c c_v \theta_c}, \quad P = \frac{\rho_l \theta_c}{\rho_c}, \quad C = \frac{c_l \theta_c}{c_c}, \quad \frac{\mathcal{E}}{R \theta_c}, \quad A = \frac{\tilde{p}}{\rho_c}. \quad (3.36)
\end{align}

Products of these numbers give dimensionless groups

\begin{align}
\text{Re} &= \frac{1}{B Es} = \frac{\rho_c v_c z_c}{2 \mu_c}, \quad W = \frac{1}{BY} = \frac{\rho_c v_c^2 \gamma_c}{\sigma}, \quad X = \frac{UEs}{2} = \frac{mu_c v_c}{\rho_c c_v \theta_c z_c}, \\
Z &= \frac{\text{Bi}}{Gz} = \frac{h_c z_c}{\rho_c c_v \gamma_c}, \quad (3.37)
\end{align}

which will appear later in reduced systems of equations. In a particular melt-spinning process, each of the dimensionless numbers in (3.36) and (3.37) can be expressed through its order in slenderness parameter \( \epsilon \). For example, \( K = \tilde{K} \epsilon^n \) where \( \tilde{K} \) is a scalar of \( \mathcal{O}(\epsilon^0) \), i.e. \( \epsilon < \tilde{K} < \epsilon^{-1} \). The relative magnitudes in \( \epsilon \) of the numbers (3.36) and (3.37) indicate the relative importance of the competing physical effects.
in a specific process, and hence provide an explicit description of the regime of fiber behavior prior to solidification.

3.4 The thin-fiber model incorporating temperature-dependent density and radial variation of temperature and pressure

The 3-dimensional boundary value problem (3.24)–(3.35) is now reduced to a 1-dimensional leading order thin-fiber model through integration over the fiber cross section.

In the perturbation theory for the isothermal problem (Bechtel et al. [2]) to produce closure to all orders in a slender jet perturbation theory it is sufficient to postulate the following radial dependence in the velocity components \( v_r, v_z \) and free surface radius \( \phi \),

\[
\hat{v}_r(\hat{r}, \hat{z}, \hat{t}) = \sum_{n,m \geq 0} \epsilon^{2n+m} \hat{r}^{2n} v_z^{n,m}(\hat{z}, \hat{t})
\]

\[
= v_z^{0,0}(\hat{z}, \hat{t}) + \epsilon v_z^{0,1}(\hat{z}, \hat{t}) + \epsilon^2 \left[ v_z^{0,2}(\hat{z}, \hat{t}) + \hat{r}^2 v_z^{1,0}(\hat{z}, \hat{t}) \right] + \mathcal{O}(\epsilon^3), \quad (3.38)
\]

\[
\hat{v}_r(\hat{r}, \hat{z}, \hat{t}) = \sum_{n,m \geq 0} \epsilon^{2n+m} \hat{r}^{2n+1} v_r^{n,m}(\hat{z}, \hat{t})
\]

\[
= \hat{r} v_r^{0,0}(\hat{z}, \hat{t}) + \epsilon \hat{r} v_r^{0,1}(\hat{z}, \hat{t}) + \epsilon^2 \left[ v_r^{0,2}(\hat{z}, \hat{t}) + \hat{r}^2 v_r^{1,0}(\hat{z}, \hat{t}) \right] + \mathcal{O}(\epsilon^3) \quad (3.39)
\]

\[
\hat{\phi}(\hat{z}, \hat{t}) = \sum_{m \geq 0} \epsilon^m \phi^{(m)}(\hat{z}, \hat{t})
\]

\[
= \phi^{(0)}(\hat{z}, \hat{t}) + \epsilon \phi^{(1)}(\hat{z}, \hat{t}) + \epsilon^2 \phi^{(2)}(\hat{z}, \hat{t}) + \mathcal{O}(\epsilon^3). \quad (3.40)
\]

In Schultz & Davis [25] it is shown that, for some leading order balances of physical effects in the isothermal problem, slender geometry implies these expansions for selected regimes ([25] deduces the steady reduction of (3.38)–(3.40), with time dependence suppressed, but the results extend to the dynamic case as shown). The
leading order terms of each of these expansions corresponds to the standard thin-
fiber approximation for the modeling of isothermal slender fiber spinning as seen in
Petric [23] and Matovich & Pearson [19]. The isothermal leading order problem in
its most primitive form consists of seven one-space, one-time differential equations;
the 1-dimensional unknowns in this leading order problem are the first terms in the
velocity expansions,

\[ v_z^0 (\tilde{z}, \tilde{t}), \quad v_r^0 (\tilde{z}, \tilde{t}), \]  \hspace{1cm} (3.41)

the averages of the diagonal components of the determinate part \( \hat{T} \) of the Cauchy
stress tensor,

\[ \int \hat{T}_{rr} da, \quad \int \hat{T}_{\theta\theta} da, \quad \int \hat{T}_{zz} da, \]  \hspace{1cm} (3.42)

the average constraint pressure,

\[ \int \hat{p} da, \]  \hspace{1cm} (3.43)

and the leading order term in the free surface radius,

\[ \phi^{(0)} (\tilde{z}, \tilde{t}). \]  \hspace{1cm} (3.44)

Shear stress does not appear in the leading order problem of the integrated form and
only the leading order problem are concerned here.

If one does not explicitly assume a form for the radial dependence of the velocity
components, as done in (3.38) and (3.39), one still obtains seven leading order equa-
tions, but for ten unknowns, since the two velocity quantities (3.41) are replaced by
the five integrations

\[ \int v_r da, \quad \int v_z da, \quad \int v_r^2 da, \quad \int v_z^2 da, \quad \int v_r v_z da. \]  \hspace{1cm} (3.45)

This classic loss-of-closure situation is avoided by the perturbation expansions (3.38)–
(3.40), which is also sufficient to obtain the governing equations to any order.
To generalize from the isothermal problem to the nonisothermal problem with temperature-dependent density (3.24)–(3.35), there are two obstacles. The first obstacle, also discussed in [14], is due to the radial variation of temperature. In melt spinning the convective cooling of the fiber by the ambient quench air stream, embedded in the right-hand side of the thermal boundary condition (3.35), is necessarily a dominant effect. In this regime, characterized by the parameter Bi being leading order, there must be a leading-order radial temperature variation, no matter how slender the fiber is (Chung & Iyer [6], Vassilatos et al. [26]). Because of the temperature variation across fiber, closure difficulties arise. Looking ahead to equations (3.70)–(3.72), integration of the energy equation over the fiber cross section produces one additional one-space, one-time differential equation, but many additional unknowns, namely

$$\tilde{\theta}(\tilde{z}, \tilde{t}) = \frac{\int_{\text{cross section}} \tilde{\theta} \, da}{\int_{\text{cross section}} \, da} = \text{dimensionless average temperature}, \quad (3.46)$$

$$\tilde{\theta}|_{\partial}(\tilde{z}, \tilde{t}) = \tilde{\theta}(\tilde{r} = \tilde{\phi}, \tilde{z}, \tilde{t}) = \text{dimensionless surface temperature}, \quad (3.47)$$

$$\int_{\text{cross section}} \tilde{\theta}^n \exp \left( \frac{E}{\tilde{\theta}} \right) \, da, \quad n = 0, 1, 2, \quad (3.48)$$

$$\int_{\text{cross section}} \tilde{\theta}^n \tilde{\theta}_{\tilde{r}}^2 \, da, \quad n = 0, 1 \quad (3.49)$$

$$\int_{\text{cross section}} \tilde{\theta}^n \tilde{\theta}_{\tilde{z}}^2 \, da, \quad n = 0, 1 \quad (3.50)$$

$$\int_{\text{cross section}} \tilde{\theta}^n \tilde{\theta}_{\tilde{\phi}} \, da, \quad n = 1, 2 \quad (3.51)$$

$$\int_{\text{cross section}} \tilde{\theta}^n \tilde{\theta}_{\tilde{r}} \, d\tilde{r}, \quad n = 0, 1, 2 \quad (3.52)$$

$$\int_{\text{cross section}} \tilde{\theta}^n \, da, \quad n = 1, 2, 3, 4, 5, \quad (3.53)$$

where $da = 2 \pi \tilde{r} \, d\tilde{r}$ is the dimensionless area element. The many integrations of higher powers of $\tilde{\theta}$ are present due to the temperature dependence of specific heat.
and density, and the nonlinear products in the constraint response within the energy equation. Closure is recovered by assuming that the radial temperature profile is parabolic, based on a study of the exact solution for the temperature distribution in a rigid cylinder [14],

$$
\hat{\theta}(\tilde{r}, \tilde{z}, \tilde{t}) = \hat{\theta}^a(\tilde{z}, \tilde{t}) + \left[1 - S(\tilde{z}, \tilde{t}) \tilde{r}^2\right] D(\tilde{z}, \tilde{t}). \tag{3.54}
$$

From the definition of average temperature $\tilde{\theta}$,

$$
\tilde{\theta} = \frac{2}{\phi^2} \int_0^\phi [\hat{\theta}^a + (1 - S\tilde{r}^2)D(\tilde{z}, \tilde{t})] \tilde{r} d\tilde{r}, \tag{3.55}
$$

we deduce to leading order

$$
\tilde{\theta} = \tilde{\theta}^a + D - \frac{S}{2} \frac{D}{\phi^{(0)}^2}, \tag{3.56}
$$

We also require the leading order equation from the pointwise heat loss boundary condition (3.35) is satisfied, which implies

$$
-2SD\phi^{(0)} = -\frac{B_i}{\phi^{(0)}} (1 - S\phi^{(0)}^2)D, \tag{3.57}
$$

where the evolving local Biot number is defined as

$$
B_i = B_i(\tilde{z}, \tilde{t}) = \frac{h(z, t)}{k} \phi(z, t) = B_i \tilde{h}(\tilde{z}, \tilde{t}) \phi^{(0)}(\tilde{z}, \tilde{t}). \tag{3.58}
$$

Solving (3.56) and (3.57), $S$ and $D$ are expressed as functions of average temperature $\tilde{\theta}$, free surface radius $\phi^{(0)}$, and known functions $\tilde{h}(\tilde{z}, \tilde{t})$ and $\tilde{\theta}^a(\tilde{z}, \tilde{t})$:

$$
S(\tilde{z}, \tilde{t}) = \frac{B_i}{(2 + B_i) \phi^{(0)}^2}, \tag{3.59}
$$

$$
D(\tilde{z}, \tilde{t}) = \frac{(4 + 2 B_i)(\tilde{\theta} - \tilde{\theta}^a)}{4 + B_i}. \tag{3.60}
$$
and inserting (3.59) and (3.60) into (3.54) gives

\[
\tilde{\theta}(\tilde{r}, \tilde{z}) = \tilde{\theta}^a(\tilde{z}) + \left[ 1 - \frac{Bi_l \tilde{r}^2}{(2 + Bi_l)\phi^{(0)}_2} \right] \left( \frac{\tilde{\theta} - \tilde{\theta}^a}{1 - \frac{Bi_l}{4 + 2Bi_l}} \right). \tag{3.61}
\]

With the radial distribution (3.54), all of the thermal quantities (3.46)–(3.53) can be explicitly related to the single thermal quantity of average temperature \( \tilde{\theta} \) and the free surface radius \( \phi^{(0)} \), specified ambient temperature \( \tilde{\theta}^a \), functions \( S(\tilde{z}, \tilde{r}) \), \( D(\tilde{z}, \tilde{r}) \) and their derivatives:

\[
\tilde{\theta}|_{\theta} = \tilde{\theta}^a + (1 - S \phi^{(0)}_2) D = \tilde{\theta}^a + \frac{4(\tilde{\theta} - \tilde{\theta}^a)}{4 + Bi_l},
\]

\[
\int \tilde{\theta}^n \exp \left( \frac{E}{\tilde{\theta}} \right) da = 2\pi \int_0^{\phi^{(0)}} \left( \tilde{\theta}^a + D(1 - S \tilde{r}^2) \right)^n \exp \left[ \frac{E}{\tilde{\theta}^a + D(1 - S \tilde{r}^2)} \right] \tilde{r} d\tilde{r}, \quad n = 0, 1, 2,
\]

\[
\int \tilde{\theta}^2 da = 2\pi S^2 D^2 \phi^{(0)}^4
\]

\[
\int \tilde{\theta} \tilde{\theta}^2 da = 2\pi S^2 D^2 \phi^{(0)}^4 (\tilde{\theta} - \frac{1}{6} \phi^{(0)}_2 S D)
\]

\[
\int \tilde{\theta}^2 \tilde{z} da = 2\pi \int \left( \tilde{\theta}^a + Dz - 2(Sz D + SDz) \tilde{r} \right)^2 d\tilde{r}, \quad 62
\]

\[
\int \tilde{\theta} \tilde{\theta}^2 \tilde{z} da = 2\pi \int \left( \tilde{\theta}^a + D \left( 1 - S \tilde{r}^2 \right) \right) \left( \tilde{\theta}^a + D_z - 2(S_z D + S D_z) \tilde{r}^2 \right)^2 d\tilde{r},
\]

\[
\int \tilde{\theta} \tilde{\theta} \tilde{z} \tilde{z} da = 2\pi \int \left( \tilde{\theta}^a + D \left( 1 - S \tilde{r}^2 \right) \right) \left( \tilde{\theta}^a + D_z - 2(S_z D + S D_z) \tilde{r}^2 \right) d\tilde{r},
\]

\[
\int \tilde{\theta}^2 \tilde{\theta} \tilde{z} \tilde{z} da = 2\pi \int \left( \tilde{\theta}^a + D \left( 1 - S \tilde{r}^2 \right) \right)^2 \left( \tilde{\theta}^a + D_z - 2(S_z D + S D_z) \tilde{r}^2 \right) d\tilde{r},
\]
\[ \int \tilde{\theta} \tilde{\tau} d\tilde{r} = -SD \phi^{(0)^2}, \]

\[ \int \tilde{\theta} \tilde{\tau} d\tilde{r} = -\frac{2}{3} SD \phi^{(0)^3} \left( \tilde{\theta} - \frac{1}{10} SD \phi^{(0)^2} \right), \]

\[ \int \tilde{\theta}^2 \tilde{\tau} d\tilde{r} = -\frac{2}{3} SD \phi^{(0)^3} \left( \tilde{\theta} - \frac{1}{3} (\tilde{\theta}^a + D) SD \phi^{(0)^2} + \frac{7}{20} (\tilde{\theta}^a + D) S^2 D^2 \phi^{(0)^4} \right), \]

\[ \int \tilde{\theta}^2 da = \pi \phi^{(0)^2} \left( \tilde{\theta}^2 + \frac{1}{12} \phi^{(0)^4} S^2 D^2 \right), \]

\[ \int \tilde{\theta}^3 da = \pi \phi^{(0)^2} \left( \tilde{\theta}^3 + \frac{1}{4} \phi^{(0)^4} S^2 D^2 (\tilde{\theta}^a + D) - \frac{1}{8} \phi^{(0)^6} S^3 D^3 \right), \]

\[ \int \tilde{\theta}^4 da = \pi \phi^{(0)^2} \left( \tilde{\theta}^4 + \frac{1}{2} \phi^{(0)^4} S^2 D^2 (\tilde{\theta}^a + D)^2 - \frac{1}{2} \phi^{(0)^6} S^3 D^3 (\tilde{\theta}^a + D) \right. \]
\[ + \frac{11}{80} \phi^{(0)^8} S^4 D^4, \]

\[ \int \tilde{\theta}^5 da = \pi \phi^{(0)^2} \left( \tilde{\theta}^5 + \frac{5}{6} \phi^{(0)^4} S^2 D^2 (\tilde{\theta}^a + D)^3 - \frac{5}{4} \phi^{(0)^6} S^3 D^3 (\tilde{\theta}^a + D)^2, \right. \]
\[ + \frac{11}{16} \phi^{(0)^8} S^4 D^4 (\tilde{\theta}^a + D) - \frac{13}{96} \phi^{(0)^10} S^5 D^5, \] \hspace{1cm} (3.62)

where \( B_{ir}, S \) and \( D \) are the functions of \( \tilde{\theta}, \tilde{\theta}^a, \phi^{(0)}, \) and \( \tilde{h} \) given in (3.58)–(3.60).

The second obstacle to closure is special to our constrained theory for materials with temperature-dependent density. The difficulty arises because, like temperature and stress, the constraint pressure in melt spinning has a leading-order radial gradient [14]. This causes no problem in the incompressible theory for melt spinning. Since in the field equation of that theory \( p \) appears only linearly and the leading-order problem only involves the cross-sectionally averaged pressure. In the constrained theory, however, nonlinear combinations of radially dependent pressure and temperature appear in the constraint response in the energy equation (3.27). To recover closure
pressure is treated like temperature: a parabolic radial distribution is posited and the
definition of cross-sectional average and the pointwise kinetic boundary condition are
used to relate coefficients in this distribution to the average pressure. Specifically,

\[ \bar{p}(\bar{r}, \bar{z}, \bar{t}) = \left[ 1 + T(\bar{z}, \bar{t}) \bar{r}^2 \right] V(\bar{z}, \bar{t}) \]  

(3.63)
is assumed. From the definition of average pressure \( \bar{p} \),

\[ \bar{p} = \frac{\int_{\text{cross section}} \bar{p} \, da}{\int_{\text{cross section}} da} = \frac{2}{\phi^2} \int_0^\phi \left( 1 + T \bar{r}^2 \right) \bar{V} \bar{r} \, d\bar{r} \]  

(3.64)

It is deduced to leading order

\[ \bar{p} = \left( 1 + \frac{T}{2} \phi^{(0)}^2 \right) V. \]  

(3.65)

It is also required that the leading order equation from the pointwise radial component
(3.33) of the stress boundary condition is satisfied, which implies (assuming \( E_s \) and
\( Y \) are \( \mathcal{O}(1) \))

\[ \alpha \left( 1 + T \phi^{(0)}^2 \right) V - E_s v_r^{0,0} \exp \left[ E \left( \frac{1}{\vartheta_{|\vartheta}} - 1 \right) \right] = \frac{Y}{\phi^{(0)}} \]  

(3.66)

Solving (3.65) and (3.66), \( T \) and \( V \) are expressed as functions of average pressure
\( \bar{p} \), average temperature \( \bar{\theta} \), radial velocity \( v_r^{0,0} \), free surface radius \( \phi^{(0)} \), and known
functions \( \bar{h} \) and \( \bar{\theta}^a \):

\[ T(\bar{z}, \bar{t}) = \frac{2 \frac{Y}{\phi^{(0)}} - 2 \alpha \bar{p} + 2 E_s v_r^{0,0} \exp \left[ E \left( \frac{1}{\vartheta_{|\vartheta}} - 1 \right) \right]}{2 \alpha \phi^{(0)}^2 \bar{p} - \phi^{(0)}^2 E_s v_r^{0,0} \exp \left[ E \left( \frac{1}{\vartheta_{|\vartheta}} - 1 \right) \right] - Y \phi^{(0)}} \]  

\[ V(\bar{z}, \bar{t}) = \frac{2 \bar{p} - \frac{Y}{\alpha \phi^{(0)}} - \frac{E_s}{\alpha} v_r^{0,0} \exp \left[ E \left( \frac{1}{\vartheta_{|\vartheta}} - 1 \right) \right]}{\bar{\theta}_{|\vartheta}} \]  

(3.67)

where \( \bar{\theta}_{|\vartheta} \) is the function of \( \bar{\theta}, \bar{\theta}_a, \phi^{(0)}, \) and \( \text{Bi}_l \) given in (3.62).
With the radial distribution (3.63), one can explicitly relate all the thermal quantities arising from the radial dependence of pressure in the integration of energy equation over the fiber cross section (look ahead to (3.72)) to the single thermal quantity of average temperature $\tilde{\theta}$ and the free surface radius $\phi^{(0)}$, specified ambient temperature $\tilde{\theta}^a$, and functions $T, V, S$, and $D$:

$$\int \tilde{p} \tilde{\theta} \tilde{\theta}_{,\xi} da = \pi \phi^{(0)}^2 V (\tilde{\theta}^a + D) (\tilde{\theta}_{,\xi}^a + D_{,\xi})$$

$$\quad + \left[ TV (\tilde{\theta}^a + D) (\tilde{\theta}_{,\xi}^a + D_{,\xi}) - V S D (\tilde{\theta}_{,\xi}^a + D_{,\xi}) - V (\tilde{\theta}^a + D)(S_{,\xi} D + S D_{,\xi}) \right]$$

$$\quad + \left[ \frac{\pi}{2} \phi^{(0)}^4 \right]$$

$$\quad + \left[ V S D (S_{,\xi} D + S D_{,\xi}) - TV (S_{,\xi} D + S D_{,\xi})(\tilde{\theta}^a + D) - TV S D (\tilde{\theta}_{,\xi}^a + D_{,\xi}) \right]$$

$$\quad + \left[ \frac{\pi}{3} \phi^{(0)}^6 \right]$$

$$\quad + \left[ \frac{\pi}{4} \phi^{(0)}^8 TV S D (S_{,\xi} D + S D_{,\xi}) \right]$$

$$\int \tilde{p} \tilde{\theta} \tilde{\theta}_{,\zeta} da = -\left[ \frac{\phi^{(0)}^4}{2} V (\tilde{\theta}^a + D) + \frac{\phi^{(0)}^6}{3} (TV (\tilde{\theta}^a + D) - S D V) - \frac{\phi^{(0)}^8}{4} TV S D \right]$$

$$\quad \cdot 2 \pi S D$$

$$\int \tilde{p} \tilde{\theta} \tilde{\theta}_{,\zeta} da = \pi \phi^{(0)}^2 V (\tilde{\theta}^a + D) (\tilde{\theta}_{,\zeta}^a + D_{,\zeta})$$

$$\quad + \left[ TV (\tilde{\theta}^a + D) (\tilde{\theta}_{,\zeta}^a + D_{,\zeta}) - V S D (\tilde{\theta}_{,\zeta}^a + D_{,\zeta}) - V (\tilde{\theta}^a + D)(S_{,\zeta} D + S D_{,\zeta}) \right]$$

$$\quad + \left[ \frac{\pi}{2} \phi^{(0)}^4 \right]$$

$$\quad + \left[ V S D (S_{,\zeta} D + S D_{,\zeta}) - TV (S_{,\zeta} D + S D_{,\zeta})(\tilde{\theta}^a + D) - TV S D (\tilde{\theta}_{,\zeta}^a + D_{,\zeta}) \right]$$

$$\quad + \left[ \frac{\pi}{3} \phi^{(0)}^6 \right]$$

$$\quad + \left[ \frac{\pi}{4} \phi^{(0)}^8 TV S D (S_{,\zeta} D + S D_{,\zeta}) \right]$$

$$\int \tilde{p} \tilde{\theta} \tilde{\theta}_{,\xi} da = 2 \pi TV \left[ \frac{\phi^{(0)}^4}{2} (\tilde{\theta}^a + D) - \frac{\phi^{(0)}^6}{3} S D \right]$$

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\[
\int \tilde{p}_{i} \tilde{\theta}^2 \tilde{r} \, da = 2\pi TV \left[ \frac{\phi^{(0)}_4}{2} (\tilde{\theta}^a + D)^2 - \frac{2\phi^{(0)}_6}{3} SD (\tilde{\theta}^a + D) + \frac{\phi^{(0)}_8}{4} S^2 D^2 \right]
\]

\[
\int \tilde{p}_{i} \tilde{\theta} \, da = \pi \phi^{(0)}_2 V_{i} (\tilde{\theta}^a + D) + \frac{\pi \phi^{(0)}_4}{2} \left( (T_{i} \tilde{V} + TV_{i}) (\tilde{\theta}^a + D) - V_{i} SD \right)
- \frac{\pi \phi^{(0)}_6}{3} SD (T_{i} \tilde{V} + TV_{i})
\]

\[
\int \tilde{p}_{i} \tilde{\theta}^2 \, da = \pi \phi^{(0)}_2 V_{i} (\tilde{\theta}^a + D)^2
+ \frac{\pi \phi^{(0)}_4}{2} \left( (T_{i} \tilde{V} + TV_{i}) (\tilde{\theta}^a + D)^2 - 2V_{i} SD (\tilde{\theta}^a + D) \right)
+ \frac{\pi \phi^{(0)}_6}{3} \left[ V_{i} S^2 D^2 - 2(T_{i} \tilde{V} + TV_{i}) (\tilde{\theta}^a + D) SD \right]
+ \frac{\pi \phi^{(0)}_8}{4} (T_{i} \tilde{V} + TV_{i}) S^2 D^2
\]

\[
\int \tilde{p}_{i} \tilde{\theta} \, da = \pi \phi^{(0)}_2 V_{i} (\tilde{\theta}^a + D) + \frac{\pi \phi^{(0)}_4}{2} \left( (T_{i} \tilde{V} + TV_{i}) (\tilde{\theta}^a + D) - V_{i} SD \right)
- \frac{\pi \phi^{(0)}_6}{3} SD (T_{i} \tilde{V} + TV_{i})
\]

\[
\int \tilde{p}_{i} \tilde{\theta}^2 \, da = \pi \phi^{(0)}_2 V_{i} (\tilde{\theta}^a + D)^2
+ \frac{\pi \phi^{(0)}_4}{2} \left( (T_{i} \tilde{V} + TV_{i}) (\tilde{\theta}^a + D)^2 - 2V_{i} SD (\tilde{\theta}^a + D) \right)
+ \frac{\pi \phi^{(0)}_6}{3} \left[ V_{i} S^2 D^2 - 2(T_{i} \tilde{V} + TV_{i}) (\tilde{\theta}^a + D) SD \right]
+ \frac{\pi \phi^{(0)}_8}{4} (T_{i} \tilde{V} + TV_{i}) S^2 D^2
\]  

\[ (3.68) \]

The thin-fiber equations produced in this dissertation are in the regime in which all physical effects with all dimensionless numbers, i.e. \( B, Fr, Y, Es, Gz, Bi, U, E, C, \) and \( P \) are all assumed of order \( \epsilon^0 \). To obtain the 1-dimensional leading order problem,
first the expansions (3.38)–(3.40) are inserted into the boundary value problem (3.24)–
(3.35). Then, integration of the constraint (3.24) over the cross section gives

\[ v_r^{0,0} = -\frac{1}{2(1 + P - P\bar{\theta})} \left( \beta \bar{\theta}_t + v_{z,\bar{z}}^{0,0} \right) + \frac{P}{2} v_{z,\bar{\theta}}^{0,0}. \]  

(3.69)

Hereafter (3.69) is used to eliminate \( v_r^{0,0} \). With (3.40) and (3.69), the kinematic boundary condition (3.32) becomes

\[ \beta \phi_t^{(0)} + v_z^{0,0} \phi_z^{(0)} + \frac{\phi_z^{(0)}}{2} v_{z,\bar{z}}^{0,0} = \frac{P}{2(1 + P - P\bar{\theta})} (\beta \bar{\theta}_t + v_{z,\bar{\theta}}^{0,0}). \]  

(3.70)

Integrating \( \bar{r} \) times (3.25) over the cross section creates the resultant \( \int \tilde{p} \tilde{r} d\tilde{r} \). Similarly, the integration of (3.26) over the fiber cross section produces \( [\int \tilde{p} \tilde{r} d\tilde{r}]_t \). Hence to eliminate the pressure resultant, the former integrated equation is differentiated with respect to \( \bar{z} \) and subtracted from the latter integrated equation. Use of the stress boundary condition (3.33) and incorporation of the constitutive equations (3.28)–(3.30) then results in the momentum equation,

\[ (1 + P - P\bar{\theta})(v_z^{0,0} + v_z^{0,0} v_{z,\bar{z}}^{0,0}) = \frac{1}{W} \frac{\phi_z^{(0)}}{\phi^{(0)}_z} \phi_t^{(0)} + \frac{2Bd}{\alpha} \bar{\theta} + \frac{1 + P - P\bar{\theta}}{Fr} \]

\[ + \frac{BE_s \exp(-E)}{2 \pi \phi^{(0)}_z} \left[ (3v_z^{0,0} - P v_z^{0,0} \bar{\theta}) \int \exp \left( \frac{E}{\bar{\theta}} \right) da \right]_t. \]  

(3.71)

The one-dimensional energy equation is obtained by integrating \( (1 + P - P\bar{\theta})^2 \) times (3.27) over the cross section of the fiber, and using the constitutive assumptions (3.28)–(3.30) and (3.69):

\[ l_0 + l_1 + l_2 \]

\[ = \frac{U E_s}{2 \pi} \exp(-E) \left( \frac{1}{2} \left( \frac{P}{1 + P - P\bar{\theta}} \right)^{v_z^{0,0} \bar{\theta} - v_{z,\bar{z}}^{0,0} \bar{\theta}} + \left( v_{z,\bar{\theta}}^{0,0} \right)^2 \right) \]

\[ \cdot \left[ (1 + P)^2 \int \exp \left( \frac{E}{\bar{\theta}} \right) da - 2 (1 + P) P \int \bar{\theta} \exp \left( \frac{E}{\bar{\theta}} \right) da + P^2 \int \bar{\theta}^2 \exp \left( \frac{E}{\bar{\theta}} \right) da \right] \]

\[ - \frac{(1 + P)^2}{Gz} \frac{\tilde{h} \left( \bar{\theta} \right)|_{\bar{z} = \bar{\theta}^a} - \bar{\theta}^a}{\phi^{(0)}} \]
\[
- \frac{c^2 (1 + P)^2}{G_z} \left( 2 (\phi^{(0)}_{,zz} \tilde{\theta} + 2 \phi^{(0)}_{,x} \phi^{(0)}_{,zz} \tilde{\theta}) + \phi^{(0)}_{,zz} \tilde{\theta} + 4 \phi^{(0)} \phi^{(0)}_{,iz} \tilde{\theta}_{,i} \right) \\
- \frac{c^2 (1 + P)^2}{G_z} \left( (\phi^{(0)}_{,zz} \tilde{\theta})_{|z=\hat{z}} + 2 \phi^{(0)} \phi^{(0)}_{,zz} \tilde{\theta} + \phi^{(0)}_{,zz} \tilde{\theta}_{|z=\hat{z}} + 4 \phi^{(0)} \phi^{(0)}_{,iz} \tilde{\theta}_{,i} \right) \\
+ 2P (1 + P) \left( \tilde{\theta}_{|z=\hat{z}} \right) \left( B \tilde{h} (\tilde{\theta}_{|z=\hat{z}} - \tilde{\theta}^a) \phi^{(0)} + \frac{1}{2\pi} \int \tilde{\theta}^2 da \right) \\
- \frac{c^2 P (1 + P)}{\pi G_z} \left( \frac{d}{dz} \int \tilde{\theta} \tilde{\theta}_{,z} da - \frac{d}{dz} \int \tilde{\theta}^2_{,z} da \right) \\
- \frac{P^2}{G_z} \left( \tilde{\theta}^2_{|z=\hat{z}} \right) B \tilde{h} (\tilde{\theta}_{|z=\hat{z}} - \tilde{\theta}^a) \phi^{(0)} + \frac{1}{\pi} \int \tilde{\theta} \tilde{\theta}^2_{,z} da \right) \\
+ \frac{c^2 P^2}{2\pi G_z} \left( \frac{d}{dz} \int \tilde{\theta}^2 \tilde{\theta}_{,z} da - 2 \frac{d}{dz} \int \tilde{\theta} \tilde{\theta}^2_{,z} da \right), \quad (3.72)
\]

where

\[
\begin{align*}
\lambda_0 &= b_0 \beta \left( \frac{\phi^{(0)}_{,zz} \tilde{\theta}}{2} + (\tilde{\theta} - \tilde{\theta}_{|z=\hat{z}}) \phi^{(0)} \phi^{(0)}_{,t} \right) \\
&+ \frac{b_1}{2} \beta \left( \frac{1}{2\pi} \frac{d}{dt} \int \tilde{\theta}^2 da - \tilde{\theta}_{|z=\hat{z}}^2 \phi^{(0)} \phi^{(0)}_{,t} \right) \\
&+ \frac{b_2}{3} \beta \left( \frac{1}{2\pi} \frac{d}{dt} \int \tilde{\theta}^3 da - \tilde{\theta}_{|z=\hat{z}}^3 \phi^{(0)} \phi^{(0)}_{,t} \right) \\
&+ \frac{b_3}{4} \beta \left( \frac{1}{2\pi} \frac{d}{dt} \int \tilde{\theta}^4 da - \tilde{\theta}_{|z=\hat{z}}^4 \phi^{(0)} \phi^{(0)}_{,t} \right) \\
&+ \frac{b_4}{5} \beta \left( \frac{1}{2\pi} \frac{d}{dt} \int \tilde{\theta}^5 da - \tilde{\theta}_{|z=\hat{z}}^5 \phi^{(0)} \phi^{(0)}_{,t} \right) \\
&+ \frac{b_5}{2} A \beta \left( \frac{1}{2\pi} \frac{d}{dt} \int \tilde{\theta}^2 da - \tilde{\theta}_{|z=\hat{z}}^2 \phi^{(0)} \phi^{(0)}_{,t} \right) \\
&+ \frac{b_5}{2\pi} \int \tilde{\theta} \tilde{\theta}_{,z} da + \frac{b_6}{2\pi} \int \tilde{\theta}_{,z} \tilde{\theta} da + \frac{b_7}{2\pi} \int \tilde{\theta}_{,t} \tilde{\theta}^2 da,
\end{align*}
\]

\[
\begin{align*}
\lambda_1 &= \frac{b_0 P}{2} \left( \frac{P}{1 + P - \hat{\theta}} \right) \left( \phi^{(0)}_{,zz} \tilde{\theta}_{,z} - \phi^{(0)}_{,zz} \tilde{\theta} \right) \phi^{(0)}_{,t} (\tilde{\theta}_{|z=\hat{z}} - \tilde{\theta}) \\
&+ \frac{b_1}{4} \left( \frac{P}{1 + P - \hat{\theta}} \right) \left( \phi^{(0)}_{,zz} \tilde{\theta}_{,z} - \phi^{(0)}_{,zz} \tilde{\theta} \right) \phi^{(0)}_{,t} (\tilde{\theta}_{|z=\hat{z}} - \tilde{\theta}) \\
&+ \frac{b_2}{6} \left( \frac{P}{1 + P - \hat{\theta}} \right) \left( \phi^{(0)}_{,zz} \tilde{\theta}_{,z} - \phi^{(0)}_{,zz} \tilde{\theta} \right) \phi^{(0)}_{,t} (\tilde{\theta}_{|z=\hat{z}} - \tilde{\theta}) \\
&+ \frac{b_3}{8} \left( \frac{P}{1 + P - \hat{\theta}} \right) \left( \phi^{(0)}_{,zz} \tilde{\theta}_{,z} - \phi^{(0)}_{,zz} \tilde{\theta} \right) \phi^{(0)}_{,t} (\tilde{\theta}_{|z=\hat{z}} - \tilde{\theta}) \\
&+ \frac{b_4}{10} \left( \frac{P}{1 + P - \hat{\theta}} \right) \left( \phi^{(0)}_{,zz} \tilde{\theta}_{,z} - \phi^{(0)}_{,zz} \tilde{\theta} \right) \phi^{(0)}_{,t} (\tilde{\theta}_{|z=\hat{z}} - \tilde{\theta}) \\
&+ \frac{b_5}{4} \left( \frac{P}{1 + P - \hat{\theta}} \right) \left( \phi^{(0)}_{,zz} \tilde{\theta}_{,z} - \phi^{(0)}_{,zz} \tilde{\theta} \right) \phi^{(0)}_{,t} (\tilde{\theta}_{|z=\hat{z}} - \tilde{\theta}) \\
&+ \frac{b_5}{2\pi} \frac{d}{dt} \int \tilde{\theta} \tilde{\theta}_{,z} da + \frac{b_6}{2\pi} \frac{d}{dt} \int \tilde{\theta}_{,z} \tilde{\theta} da + \frac{b_7}{2\pi} \frac{d}{dt} \int \tilde{\theta}_{,t} \tilde{\theta}^2 da + \frac{1}{2\pi} \frac{d}{dt} \int \tilde{\theta} \tilde{\theta}^2 da + \frac{1}{2\pi} \frac{d}{dt} \int \tilde{\theta} \tilde{\theta}^2 da + \frac{1}{2\pi} \frac{d}{dt} \int \tilde{\theta} \tilde{\theta}^2 da,
\end{align*}
\]

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\begin{align*}
    l_2 &= b_0 v_z^{0,0} \left( \frac{\phi^{(0)} \bar{\phi}^{(0)}}{2} + (\tilde{\theta} - \bar{\theta})|_{\tilde{r} = \phi} \phi^{(0)} \phi^{(0)} \right) \\
    &+ \frac{b_1}{2} v_z^{0,0} \left( \frac{1}{2\pi} \frac{d}{d\tilde{z}} \int \tilde{\theta}^2 \text{d}a - \tilde{\theta} |_{\tilde{r} = \phi} \phi^{(0)} \phi^{(0)} \right) \\
    &+ \frac{b_2}{3} v_z^{0,0} \left( \frac{1}{2\pi} \frac{d}{d\tilde{z}} \int \tilde{\theta}^3 \text{d}a - \tilde{\theta} |_{\tilde{r} = \phi} \phi^{(0)} \phi^{(0)} \right) \\
    &+ \frac{b_3}{4} v_z^{0,0} \left( \frac{1}{2\pi} \frac{d}{d\tilde{z}} \int \tilde{\theta}^4 \text{d}a - \tilde{\theta} |_{\tilde{r} = \phi} \phi^{(0)} \phi^{(0)} \right) \\
    &+ \frac{b_4}{5} v_z^{0,0} \left( \frac{1}{2\pi} \frac{d}{d\tilde{z}} \int \tilde{\theta}^5 \text{d}a - \tilde{\theta} |_{\tilde{r} = \phi} \phi^{(0)} \phi^{(0)} \right) \\
    &+ \frac{b_5}{2} v_z^{0,0} \left( \frac{1}{2\pi} \frac{d}{d\tilde{z}} \int \tilde{\theta}^2 \text{d}a + \frac{b_6}{2\pi} v_z^{0,0} \int \tilde{\theta} \text{d}a + \frac{b_7}{2\pi} v_z^{0,0} \int \tilde{\theta}^2 \text{d}a, \right) \tag{3.73}
\end{align*}

with

\begin{align*}
    b_0 &= (1 + P)^3 (1 - C), \\
    b_1 &= (1 + P)^2 (4PC - 3P + C), \\
    b_2 &= 3P(1 + P)(P - C - 2PC), \\
    b_3 &= P^2(4PC - P + 3C), \\
    b_4 &= -CP^3, \\
    b_5 &= -2P^2 U \alpha, \\
    b_6 &= -(1 + P)PU \alpha, \\
    b_7 &= P^2 U \alpha. \tag{3.74}
\end{align*}

Note how each of the integrals in (3.62) and (3.68) occur in (3.70)–(3.72). When combined with the identifications (3.62) and (3.68), equations (3.70)–(3.72) are three equations for the three unknowns average temperature \( \bar{\theta} \), axial velocity \( v_z^{0,0} \), and
free surface radius $\phi^{(0)}$. Armed with a solution $\tilde{\theta}, v_{z}^{0,0}, \phi^{(0)}$, the radially dependent temperature is determined from (3.54). Continuing, with temperature $\theta$ known as a function of $r$, stress and pressure are deduced as functions of $r$ through (3.63) and the leading order forms of (3.28)–(3.30), namely,

\[
\tilde{T}_{rr} = \left( \frac{E s P}{1 + P - P \tilde{\theta} \tilde{v}_{z}^{0,0} \tilde{\theta}_{,z} - v_{z,0}^{0,0}} \right) \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right], \tag{3.75}
\]

\[
\tilde{T}_{\theta\theta} = \left( \frac{E s P}{1 + P - P \tilde{\theta} \tilde{v}_{z}^{0,0} \tilde{\theta}_{,z} - v_{z,0}^{0,0}} \right) \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right], \tag{3.76}
\]

\[
\tilde{T}_{zz} = E s v_{z,0}^{0,0} \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right], \tag{3.77}
\]

3.5 Other theories

As part of this dissertation, the predictions of the theory of the previous subsection, which consistently models the effects of both temperature-dependent density and radial variation of temperature and pressure, are compared with the predictions of three other theories, namely

(i) one which includes both the radial dependence of temperature and pressure and the necessary constraint response in the energy equation for a material with temperature-dependent density, but assumes viscosity, density, and specific heat depend on temperature only through average temperature,

(ii) one which ignores the radial temperature variation (and hence is incompatible with the second law of thermodynamics) but includes the constraint response for a material with temperature-dependent density, and
(iii) the conventional theory which ignores both the radial variation of temperature and the constraint response in the energy equation that should accompany temperature dependence of density.

The consistent theory of subsection 3.4 is labelled Constrained Theory 1; (i) is labelled Constrained Theory 2; (ii) is labelled Constrained Theory 3, and (iii) is labelled the Conventional Theory.

### 3.5.1 Conventional theory without radial variation of temperature and pressure and ignoring the constraint response

All existing nonisothermal thin-fiber models tacitly assume

\[
\tilde{\theta}(\tilde{r}, \tilde{z}, \tilde{t}) = \tilde{\theta}(\tilde{z}, \tilde{t}), \quad \tilde{p}(\tilde{r}, \tilde{z}, \tilde{t}) = \tilde{p}(\tilde{z}, \tilde{t}).
\]  

(3.78)

When cross-sectionally uniform temperature and cross-sectionally uniform pressure are assumed, the thermal quantities (3.46)–(3.53), and (3.68) collapse trivially to the single one-dimensional thermal variable \( \tilde{\theta}(\tilde{z}, \tilde{t}) \),

\[
\tilde{\theta} = \tilde{\theta},
\]

\[
\tilde{\theta}|_{\theta} = \tilde{\theta},
\]

\[
\int \tilde{\theta}^n \exp \left( \frac{E}{\tilde{\theta}} \right) da = \pi \phi^{(0)2} \tilde{\theta}^n \exp \left( \frac{E}{\tilde{\theta}} \right), \quad n = 0, 1, 2,
\]

\[
\int \tilde{\theta}^n da = \pi \phi^{(0)2} \tilde{\theta}^n,
\]

\[
\int \tilde{\theta}^{\tilde{z}} da = 0,
\]

\[
\int \tilde{\theta} \tilde{\theta}^{\tilde{z}} da = 0,
\]

\[
\int \tilde{\theta}^2 \tilde{z} da = \pi \phi^{(0)2} \tilde{\theta}^2 \tilde{z},
\]

\[
\int \tilde{\theta} \tilde{\theta} \tilde{z} da = \pi \phi^{(0)2} \tilde{\theta} \tilde{\theta} \tilde{z},
\]

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\[
\int \tilde{\theta} \tilde{\partial}^2_{zz} \, da = \pi \phi^{(0)} \tilde{\theta} \tilde{\partial}^2_{zz},
\]
\[
\int \tilde{\theta}^2 \tilde{\partial}^2_{zz} \, da = \pi \phi^{(0)} \tilde{\theta}^2 \tilde{\partial}^2_{zz},
\]
\[
\int \tilde{p} \tilde{\theta} \, da = \pi \phi^{(0)} \tilde{\theta} \tilde{p},
\]
\[
\int \tilde{p} \tilde{\theta} \tilde{\partial}^2_{zz} \, da = \pi \phi^{(0)} \tilde{\theta} \tilde{\partial}^2_{zz} \tilde{p},
\]
\[
\int \tilde{p} \tilde{\theta} \tilde{\partial}_{z} \, da = 0,
\]
\[
\int \tilde{p} \tilde{\theta} \tilde{\partial}_{\bar{z}} \, da = \pi \phi^{(0)} \tilde{\theta} \tilde{\partial}_{\bar{z}} \tilde{p},
\]
\[
\int \tilde{p} \tilde{\theta} \tilde{\partial}_{\bar{z}} \, da = 0,
\]
\[
\int \tilde{p} \tilde{\theta} \tilde{\partial}^2 \, da = 0,
\]
\[
\int \tilde{p} \tilde{\theta} \tilde{\partial}^2 \, da = \pi \phi^{(0)} \tilde{\theta} \tilde{\partial}^2 \tilde{p},
\]
\[
\int \tilde{p} \tilde{\partial}^2 \tilde{\partial} \, da = \pi \phi^{(0)} \tilde{\partial}^2 \tilde{p},
\]
\[
\int \tilde{p} \tilde{\partial}^2 \tilde{\partial} \, da = 0,
\]
\[
\int \tilde{p} \tilde{\partial}^2 \tilde{\partial} \, da = \pi \phi^{(0)} \tilde{\partial}^2 \tilde{p} \tilde{\partial} \tilde{\partial},
\]
\[
\int \tilde{p} \tilde{\partial}^2 \tilde{\partial} \, da = \pi \phi^{(0)} \tilde{\partial}^2 \tilde{p} \tilde{\partial} \tilde{\partial},
\]
\[
(3.79)
\]

The conventional thin-fiber equations use these identifications (3.79) instead of (3.62) and (3.68) in the integrated equations (3.70)–(3.72), ignore the thermal expansion term on the right hand side of (3.7), and ignore the constraint response in (3.7), resulting in the primitive equations

\[
\beta \phi^{(0)} + v_{\bar{z}}^2 \phi^{(0)} + \frac{1}{2} \phi^{(0)} v_{\bar{z}}^2 = 0,
\]
\[
(1 + P - P \tilde{\theta})(v_{\bar{z}}^0 + v_{\bar{z}}^0 v_{\bar{z}}^0)
\]
\[
= \frac{1}{W} \frac{\phi^{(0)}}{\phi^{(0)}} + \frac{2B}{\phi^{(0)}} \alpha \tilde{p} + B \alpha \tilde{p} + \frac{1 + P - P \tilde{\theta}}{\text{Fr}}
\]
\[
+ \frac{3}{\text{Re}} \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right] \left[ v_{\bar{z}}^0 + v_{\bar{z}}^0 \left( 2 \frac{\phi^{(0)}}{\phi^{(0)}} - E \frac{\tilde{\partial}^2}{\tilde{\theta}^2} \right) \right],
\]
\[
(3.81)
\]
\[(1 + P - P \tilde{\theta})(1 - C + C \tilde{\theta})(\tilde{v}_z + v_z^{0,0} \tilde{\theta}_z)\]

\[= -2Z \frac{\bar{h}}{\phi}(\tilde{\theta} - \tilde{\theta}^0) + 3X (v_z^{0,0})^2 \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right], \tag{3.82}\]

for axial velocity \(v_z^{0,0}\), free surface radius \(\phi^{(0)}\), and fiber temperature \(\tilde{\theta}\), all functions of \(\tilde{z}\) and \(\tilde{t}\). The remaining unknowns are determined from the solution \(v_z^{0,0}, \phi^{(0)}, \tilde{\theta}\) through

\[v_r^{0,0} = -\frac{1}{2} v_z^{0,0}, \tag{3.83}\]

\[p^{0,0} = \frac{A_{rr}}{\phi^{(0)2}} + \frac{1}{BW \phi^{(0)}}, \tag{3.84}\]

\[A_{rr} = A_{\theta\theta} = -\frac{1}{2} A_{zz} = -\frac{1}{BRe} \exp[E(\frac{1}{\tilde{\theta}} - 1)]\phi^{(0)2} v_z^{0,0}, \tag{3.85}\]

where \(A_{rr}, A_{\theta\theta}, A_{zz}\) are the stress resultant

\[A_{rr} = 2 \int_0^{\phi^{(0)}} T_{rr} \tilde{r} d\tilde{r}, \quad A_{\theta\theta} = 2 \int_0^{\phi^{(0)}} T_{\theta\theta} \tilde{r} d\tilde{r}, \quad A_{zz} = 2 \int_0^{\phi^{(0)}} T_{zz} \tilde{r} d\tilde{r}. \tag{3.86}\]

Note that the conventional theory requires no value of the thermal conductivity \(k\) in the calculation.

### 3.5.2 Constrained Theory 2 in which viscosity, density, and specific heat are assumed to depend on temperature only through average temperature

In this theory the viscosity, density, and specific heat are assumed to depend on temperature only through average temperature,

\[\mu = \mu_c \exp \left[ E \left( \frac{1}{\tilde{\theta}} - \frac{1}{\theta_c} \right) \right], \tag{3.87}\]

\[\rho = \rho_0 - \rho_1 \tilde{\theta}, \tag{3.88}\]

\[c_{eff} = C_0 + C_1 \tilde{\theta} + \frac{2\rho_1}{(\rho_0 - \rho_1 \tilde{\theta})^3} p \tilde{\theta}, \tag{3.89}\]
rather than pointwise temperature. This has the effect of inserting identifications \((3.79)_{3}-(3.79)_{20}\) into equations \((3.70)-(3.72)\), while still retaining \((3.62)_{1}\). The resulting equations are

\[
\beta \phi_{,t}^{(0)} + v_{z}^{0,0} \phi_{,z}^{(0)} + \frac{\phi^{(0)}}{2} = \frac{P}{\left(1 + P - P \tilde{\theta} \right)} \phi^{(0)} \tilde{\theta}^{,t} + v_{z}^{0,0} \tilde{\theta}^{,z}, \tag{3.90}
\]

\[
(1 + P - P \tilde{\theta})(v_{z}^{0,0} + v_{z}^{0,0} v_{z}^{0,0}) = \frac{1}{W \phi^{(0)2}} + \frac{2B}{\phi^{(0)}} \dot{\tilde{\theta}}^{a} - B \alpha \tilde{\theta}^{a} + \frac{1 + P - P \tilde{\theta}}{Pr} \\
\frac{3}{Re} v_{z,zz}^{0,0} \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right] + \frac{3}{Re} v_{z,zz}^{0,0} \left( 2 \frac{\dot{\phi}^{(0)}}{\phi^{(0)}} - E \frac{\tilde{\theta}}{\tilde{\theta}^{2}} \right) \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right] \\
\frac{P}{Re} \left( 2 \frac{\dot{\phi}^{(0)}}{\phi^{(0)}} - E \frac{\tilde{\theta}}{\tilde{\theta}^{2}} \right) \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right] \tilde{\theta}^{,t} \tilde{\theta}^{,z} \\
- \frac{P}{D_{4} Re} L(\tilde{\theta}, \tilde{t}) \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right], \tag{3.91}
\]

\[
D_{4} \tilde{\theta}^{,t} + v_{z}^{0,0} \tilde{\theta}^{,z} - PX \tilde{\theta} \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right] \left( v_{z,zz}^{0,0} + v_{z}^{0,0} v_{z,zz}^{0,0} \right) = -2Z \tilde{\theta}^{a} \left( f(\tilde{\theta}, Bi_{t}) - \tilde{\theta}^{a} \right) \\
+ 3X (v_{z,zz}^{0,0})^{2} \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right] + PU \tilde{\theta} \left[ \alpha \left( \tilde{\theta}^{a} + v_{z}^{0,0} \tilde{\theta}^{a} \right) + \frac{Y v_{z}^{0,0}}{2 \phi^{(0)}} \right], \tag{3.92}
\]

where

\[
L(\tilde{\theta}, \tilde{t}) = -C \tilde{\theta} \tilde{\theta} + v_{z}^{0,0} \tilde{\theta}^{,z} - 3XE \left( \frac{v_{z,zz}^{0,0}}{\tilde{\theta}} \right) \tilde{\theta}^{,z} \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right] \\
+ 6X v_{z,zz}^{0,0} \tilde{\theta}^{,z} \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right] - 2Z \tilde{\theta}^{a} \left( f(\tilde{\theta}, Bi_{t}) - \tilde{\theta}^{a} \right) + f(\tilde{\theta}, Bi_{t}) \tilde{\theta}^{,z} + f_{Bi_{t}}(\tilde{\theta}, Bi_{t}) \tilde{\theta}^{a} \\
- \frac{2Z}{\phi^{(0)2}} \left( \tilde{\theta}^{a} \phi^{(0)} + \tilde{\theta}^{a} \right) \left( f(\tilde{\theta}, Bi_{t}) - \tilde{\theta}^{a} \right) - \frac{\tilde{h}_{\phi^{(0)}}}{\phi^{(0)}} \phi^{(0)} \left( f(\tilde{\theta}, Bi_{t}) - \tilde{\theta}^{a} \right), \tag{3.93}
\]

\[
D_{4} = (1 + P - P \tilde{\theta})(1 - C + C \tilde{\theta}) + PX v_{z,zz}^{0,0} \left( 2 \frac{E}{\tilde{\theta}} \right) \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right], \tag{3.94}
\]

with

\[
f(\tilde{\theta}, Bi_{t}) = \tilde{\theta}^{a} + \frac{4}{4 + Bi_{t}} (\tilde{\theta} - \tilde{\theta}^{a}). \tag{3.95}
\]
The remaining unknowns are determined from the solution \( \tilde{\theta}, v^0_z, \phi^0 \) through the relations

\[
v^0_r = -\frac{1}{2(1 + P - P \tilde{\theta})} \left( \beta \tilde{\theta}_r + v^0_z \tilde{\theta}_z \right) + \frac{P}{2} v^0_z \tilde{\theta}_z. \tag{3.96}
\]

\[
p^0 = \frac{A_r}{\phi(0)^2} + \frac{1}{BW \phi(0)}, \tag{3.97}
\]

\[
A_r = A \tilde{\theta} = -\frac{1}{2} A_{zz} = -\frac{1}{B \text{Re}} \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right] \phi(0)^2 v^0_z \tilde{\theta}_z. \tag{3.98}
\]

### 3.5.3 Constrained Theory 3 for the spinning of fibers with temperature-dependent density but ignoring the radial temperature variation

When the thermodynamically-consistent constrained theory (3.5)–(3.7) is used in combination with the assumption that temperature is uniform over the fiber cross section, the kinematic boundary condition (3.32) and constraint equation (3.24) give

\[
\beta \phi^{(0)} + v^0_z \phi^{(0)} + \frac{\phi^{(0)}}{2} v^0_z = \frac{P}{2(1 + P - P \tilde{\theta})} \phi^{(0)} (\tilde{\theta}_r + v^0_z \tilde{\theta}_z), \tag{3.99}
\]

the momentum equations (3.25)–(3.26) reduce to

\[
(1 + P - P \tilde{\theta}) (v^0_z \tilde{\theta}_z + v^0_r v^0_z) = \frac{1}{W \phi^{(0)^2}} + \frac{2Bd}{\phi^{(0)}} + \frac{1 + P - P \tilde{\theta}}{Fr} \left( \frac{3}{v^0_z \tilde{\theta}_z - P v^0_z \tilde{\theta}_z} \right) \int \exp \left( \frac{E}{\tilde{\theta}} \right) da \tag{3.100}
\]

and, using identifications (3.79), the energy equation (3.72) simplifies to

\[
(1 + P - P \tilde{\theta}) \left[ 1 - C + C \tilde{\theta} - 2P^2U \alpha \tilde{\theta} \right] \left( \frac{z_c}{c} \tilde{\theta}_r + \tilde{\theta}_z \tilde{\theta}_z \right) = \frac{PU \alpha \tilde{\theta}}{1 + P - P \tilde{\theta}} \left( \frac{z_c}{c} \tilde{\theta}_r + \tilde{\theta}_z \tilde{\theta}_z \right) \tag{3.101}
\]

\[
= U \text{Es} \left[ 2(v^0_r)^2 + (v^0_z)^2 \right] \exp \left[ E \left( \frac{1}{\tilde{\theta}} - 1 \right) \right].
\]
The remaining unknowns are deduced from the solution \( \tilde{\theta}, v_z^{(0,0)}, \phi^{(0)} \) through the relations

\[
v_r^{0,0} = -\frac{1}{2(1 + P - P \tilde{\theta})} \left( \beta \tilde{\theta} \tilde{z} + v_z^{0,0} \right) + \frac{P}{2} v_z^{0,0} \tilde{z}. \tag{3.102}
\]

\[
p^{0,0} = \frac{A_{rr}}{\phi^{(0)}}, \tag{3.103}
\]

\[
A_{rr} = A_{\theta \theta} = -\frac{1}{2} A_{zz} = -\frac{1}{B \text{Re} \exp[E(\frac{\tilde{z}}{\tilde{\theta}} - 1)]} \phi^{(0)} v_z^{0,0}. \tag{3.104}
\]

### 3.6 Comparison of the Constrained Theories and Conventional Theory

We now simulate a particular process for the spinning of poly(ethylene terephthalate) (PET) fibers using all four theories. The material properties of PET and processing conditions used in the numerical simulation are listed in Table 3.1. For the process described in Table 3.1, the characteristic values for nondimensionalization are listed in Table 3.2. The dimensionless numbers and boundary values are computed from these characteristic scales, as shown in Table 3.3. Note that for this typical process all of the dimensionless parameters in Table 3.3 are \( \mathcal{O}(\epsilon^0) \), i.e. they are between \( \epsilon = 5.664 \times 10^{-5} \) and \( \epsilon^{-1} = 1.766 \times 10^4 \). Hence the leading order problem (3.69)–(3.72) is valid. (The process is in the regime in which all effects contribute to the leading order fiber behavior.) The dimensionless number \( P \) in Table 3.3 reflects the temperature dependence of density of PET.

Amorphous PET melt experiences glass transition at 67°C. The polymer melt is modeled as a Newtonian fluid above glass transition and the solidified polymer as rigid. The empirical formulae for the dimensional heat loss coefficient \( h \) and the
<table>
<thead>
<tr>
<th>PET properties</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density coefficient $\rho_0$</td>
<td>1,493 kg · m$^{-3}$ †</td>
</tr>
<tr>
<td>Density coefficient $\rho_1$</td>
<td>0.5 kg · m$^{-3}$ · K$^{-1}$ †</td>
</tr>
<tr>
<td>Specific heat coefficient $c_0$</td>
<td>711.0 J · kg$^{-1}$ · K$^{-1}$ †</td>
</tr>
<tr>
<td>Specific heat coefficient $c_1$</td>
<td>2.364 J · kg$^{-1}$ · K$^{-2}$ †</td>
</tr>
<tr>
<td>Glass transition temperature $\theta_g$</td>
<td>340.2 K ‡</td>
</tr>
<tr>
<td>Thermal conductivity $k$</td>
<td>0.147 W · m$^{-1}$ · K$^{-1}$ ‡</td>
</tr>
<tr>
<td>Surface tension $\sigma$</td>
<td>$25 \times 10^{-3}$ N · m$^{-1}$ ‡</td>
</tr>
<tr>
<td>Intrinsic viscosity $[\eta]$</td>
<td>0.6450 dl · g$^{-1}$ ‡</td>
</tr>
<tr>
<td>Viscosity $\mu_0$ at 558.2K</td>
<td>204.8 Pa · s</td>
</tr>
<tr>
<td>Activation energy $\mathcal{E}$</td>
<td>$56.54 \times 10^3$ J · mole$^{-1}$ †</td>
</tr>
<tr>
<td>Gas constant $R$</td>
<td>8.314 J · mole$^{-1}$ · K$^{-1}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Spinning conditions</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spinneret hole radius $r_0$</td>
<td>$0.2 \times 10^{-3}$ m</td>
</tr>
<tr>
<td>Spinline length $z_0$</td>
<td>1.2 m</td>
</tr>
<tr>
<td>Throughput $G$</td>
<td>$1.5 \times 10^{-5}$ kg · s$^{-1}$</td>
</tr>
<tr>
<td>Extrudate temperature $\theta_0$</td>
<td>558.2 K</td>
</tr>
<tr>
<td>Take-up velocity $v_t$</td>
<td>50 m · s$^{-1}$</td>
</tr>
<tr>
<td>Cooling air temperature $\theta^a$</td>
<td>303.2 K</td>
</tr>
</tbody>
</table>

† Hayashi et al. [13].
‡ Polymer Handbook [3].
¶ Calculated by $\mu(\theta) = [\eta]^{5.15}\exp\{\frac{\mathcal{E}}{R\theta} - 2.3\}$.

Table 3.1: PET properties and spinning conditions used in the simulations of melt spinning of PET with temperature-dependent density
Table 3.2: The characteristic values of physical properties and conditions for the PET melt spinning process of Table 1

dimensional air drag $t^a$ from Matsui [20] are used:

$$h(z, t) = 1.352 \times \left( \frac{v_{0,0}^{0.333}}{(\phi(0))^{0.667}} \right) \frac{J/(m^2 sK)}{(m^{0.334 s^{0.333}})},$$  \hspace{1cm} (3.105)

$$t^a(z, t) = 4.643 \times 10^{-6} \left( \frac{(v_{0,0})^{1.39}}{(\phi(0))^{0.61}} \right) \text{Pa} \cdot m^{0.61}/s^{1.39}.$$ \hspace{1cm} (3.106)

(Note that for $c_e, r_e$, $t^a = 1.323 \text{Pa}$, which justifies the scaling for the shear stress in (3.22).)

For the steady state problem, the boundary conditions relevant to the fiber spinning process are specifying the fiber radius, axial velocity, and temperature at $\bar{z} = 0$, and take-up velocity at $\bar{z} = 1$,

$$\phi(0)(0) = \phi(0), \quad v_{z,0}(0) = \bar{v}_0, \quad \bar{\theta}(0) = \bar{\theta}_0, \quad v_{z,0}(1) = 1.$$ \hspace{1cm} (3.107)

where the characteristic radial length is selected to be the downstream fiber radius; the characteristic axial length to be the fiber length from end of the extrudate swell zone to the takeup position; and the characteristic velocity to be the takeup velocity.
<table>
<thead>
<tr>
<th>Dimensionless numbers</th>
<th>Formula</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slenderness parameter $\epsilon$</td>
<td>$r_c/z_c$</td>
<td>$5.664 \times 10^{-5}$</td>
</tr>
<tr>
<td>Arrhenius number $E$</td>
<td>$E/(R\theta_c)$</td>
<td>19.99</td>
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<tr>
<td>Degree of temperature dependence of density $P$</td>
<td>$\rho_c \theta_c/\rho_c$</td>
<td>0.1286</td>
</tr>
<tr>
<td>Degree of temperature dependence of specific $C$</td>
<td>$C_1 \theta_c/c_c$</td>
<td>0.5308</td>
</tr>
<tr>
<td>Reynolds number $Re$</td>
<td>$\rho_c v_c z_c/\mu_c$</td>
<td>0.01971</td>
</tr>
<tr>
<td>Ellis number $Es$</td>
<td>$2 \mu_c v_c/(\tau_c z_c)$</td>
<td>256.2</td>
</tr>
<tr>
<td>Froude number $Fr$</td>
<td>$v_c^2/(gz_c)$</td>
<td>1698.9</td>
</tr>
<tr>
<td>Weber number $W$</td>
<td>$\rho_c v_c^2 r_c/\sigma$</td>
<td>1124.0</td>
</tr>
<tr>
<td>$B$</td>
<td>$\tau_c/(\rho_c v_c^2)$</td>
<td>0.3961</td>
</tr>
<tr>
<td>$Y$</td>
<td>$\sigma/(\tau_c r_c)$</td>
<td>$2.246 \times 10^{-3}$</td>
</tr>
<tr>
<td>Graetz number $Gz$</td>
<td>$\rho_c c_v r_c/(k z_c)$</td>
<td>0.3281</td>
</tr>
<tr>
<td>Biot number, $Bi$</td>
<td>$h_c r_c/k$</td>
<td>0.6932</td>
</tr>
<tr>
<td>$U$</td>
<td>$\tau_c/(\rho_c c_v \theta_c)$</td>
<td>$1.921 \times 10^{-3}$</td>
</tr>
<tr>
<td>$X$</td>
<td>$\mu_c v_c/(\rho_c c_v \theta_c z_c)$</td>
<td>0.2460</td>
</tr>
<tr>
<td>$Z$</td>
<td>$h_c z_c/(\rho_c c_v r_c \theta_c)$</td>
<td>2.113</td>
</tr>
<tr>
<td>Dimensionless boundary values</td>
<td>Definition</td>
<td>Value</td>
</tr>
<tr>
<td>Upstream radius $\bar{r}_0$</td>
<td>$r_0/r_c$</td>
<td>23.54</td>
</tr>
<tr>
<td>Upstream velocity $v_0$</td>
<td>$v_0/v_c$</td>
<td>$1.966 \times 10^{-3}$</td>
</tr>
<tr>
<td>Upstream temperature $\bar{\theta}_0$</td>
<td>$\theta_0/\theta_c$</td>
<td>1.641</td>
</tr>
<tr>
<td>Downstream velocity $\bar{v}_t$</td>
<td>$v_t/v_c$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3.3: The dimensionless numbers and boundary values for the PET melt spinning process of Table 1
Figure 3.2: Comparison of the axial profiles of fiber radius, velocity, average temperature, and tensile stress obtained from the four models in Table 5.
<table>
<thead>
<tr>
<th>model</th>
<th>radial dependence of temperature</th>
<th>constraint response in energy equation</th>
<th>pointwise description of viscosity and specific heat</th>
</tr>
</thead>
<tbody>
<tr>
<td>CT1</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>CT2</td>
<td>✓</td>
<td>✓</td>
<td>-</td>
</tr>
<tr>
<td>CT3</td>
<td>-</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>ConvT</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

CT1: constrained theory 1  
CT2: constrained theory 2  
CT3: constrained theory 3  
ConvT: conventional theory

Table 3.4: Description of the models used in the comparison

Recall the state-of-the-art theory, Constrained Theory 1, incorporates two improvements for the modeling of fiber spinning processes: 1) the effects of shrinkage of the fiber due to cooling are modeled in a thermodynamically consistent manner; and 2) the radial temperature variation within the fiber is considered. Figure 3.2 compares the predictions of this theory to the predictions of the intermediate model, Constrained Theory 3, which takes into account the effects of shrinkage but retains the assumption of surface temperature equal to average temperature, the intermediate model, Constrained Theory 2, which assumes only that the average viscosity and specific heat are the viscosity and specific heat at the average temperature, and the Conventional Theory which contains neither of the advances of the Constrained Theory 1. Figure 3.2 reveals the new theory CT1 predicts a phenomenon of stress reduction which has not been observed before. The tensile stress in the fiber peaks near the glass transition position and then drops almost 50 percent. In contrast,
the tensile stress calculated from the commonly used model ConvT does not have this peak value and is ever increasing. Without taking into account the effects of shrinkage, the relation \( \psi_{r,0}^0 = -\frac{1}{2} \psi_{z,0}^0 \) is commonly borrowed from the classical result in isothermal elongational flows, and applied to nonisothermal fiber spinning modeling in every elsewhere in the literature. This relation approximately holds when the correction term \( \frac{1}{2} P \psi_{r,0}^0 \tilde{\theta}_{,z} \) is small in (3.69). That is the case for the initial stage of fiber spinning where the velocity is low and the axial gradient of temperature is moderate but the axial gradient of velocity is very high. When the velocity achieves certain level, the correction is no longer negligible. The stretching process of the fiber is almost complete when the tensile stress reaches its peak value, as can be seen from the fiber radius profile in Figure 3.2. Thus, at this location \( \psi_{r,0}^0 \) is small. In steady state (3.69) can be rewritten as

\[
\psi_{r,0}^0 = -(1 + P - P \tilde{\theta}) \left( 2\psi_{r,0}^0 + P \psi_{z,0}^0 \tilde{\theta}_{,z} \right).
\]  

(3.108)

It follows that when \( \psi_{r,0}^0 \) is small, the correction \( P \psi_{z,0}^0 \tilde{\theta}_{,z} \) is significant even though the negative temperature gradient is still moderate, since axial velocity has increased substantially. Neglecting of effects of shrinkage results in a significant overestimate of velocity gradient and therefore results in a qualitative error in prediction of fiber tensioning behavior.

Taking into account the radial temperature variation shifts the tensile stress profile down, so that the peak value of tensile stress is significantly lower than what is predicted if the radial variation of temperature is ignored. Also, the cooling and stretching of the fiber predicted by the thin-fiber model with radial temperature variation are slower than predicted by the conventional uniform temperature assumption.
Figures 3.3–3.7 show the influence of ambient air temperature, spinneret temperature, take-up velocity, spinneret hole size, and polymer mass flow rate on the fiber spinning behavior. Note that both the spinneret temperature and the take-up velocity have large influences on the axial stress. Figure 3.4 indicates that the axial stress increases greatly as the spinneret temperature decreases.

With the use of the constrained theory, the thin-fiber model incorporating the effects of fiber’s shrinkage has been derived. To account for the radial temperature variation, the parabolic form of temperature distribution was proposed to replace the conventional assumption of radially independent temperature. The local Biot number was employed to evaluate the evolution of radial temperature distribution along the spinline.

The new thin-fiber model predicted slower cooling and stretching of the fiber, and much lower maximum tensile stress than the conventional theory did. It was discovered that there exists a stress reduction zone near the fiber’s glass transition and this phenomenon is attributed to the effects of fiber’s shrinkage.
Figure 3.3: The effect of ambient temperature on the axial profiles of fiber’s radius, velocity, average temperature, and tensile stress obtained from Constrained Theory 1, i.e. the 1-dimensional thin-fiber model derived under the constrained theory with radial temperature variation. All process conditions other than ambient temperature are as given in Table 2. The horizontal axis is the distance from the spinneret in meters.
Figure 3.4: The effect of spinneret temperature on the axial profiles of fiber’s radius, velocity, average temperature, and tensile stress obtained from Constrained Theory 1, i.e. the 1-dimensional thin-fiber model derived under the constrained theory with radial temperature variation. All process conditions other than spinneret temperature are as given in Table 2. The horizontal axis is the distance from the spinneret in meters.
Figure 3.5: The effect of take-up velocity on the axial profiles of fiber’s radius, velocity, average temperature, and tensile stress obtained from Constrained Theory 1, i.e. the 1-dimensional thin-fiber model derived under the constrained theory with radial temperature variation. All process conditions other than take-up velocity are as given in Table 2. The horizontal axis is the distance from the spinneret in meters.
Figure 3.6: The effect of spinneret radius on the axial profiles of fiber's radius, velocity, average temperature, and tensile stress obtained from Constrained Theory 1, i.e. the 1-dimensional thin-fiber model derived under the constrained theory with radial temperature variation. All process conditions other than spinneret radius are as given in Table 2. The horizontal axis is the distance from the spinneret in meters.
Figure 3.7: The effect of mass flow rate on the axial profiles of fiber’s radius, velocity, average temperature, and tensile stress obtained from Constrained Theory 1, i.e. the 1-dimensional thin-fiber model derived under the constrained theory with radial temperature variation. All process conditions other than mass flow rate are as given in Table 2. The horizontal axis is the distance from the spinneret in meters.
CHAPTER 4

MECHANICAL BEHAVIOR OF POLYMERIC FIBERS IN DRAWING

In this chapter a preliminary modeling of the drawing process from the point of view of isothermal processes is presented. The leading order field equations, which hold for any constitutive equation in a steady state drawing process, are derived in Section 4.1. A linearly elastic constitutive equation and two nonlinearly elastic constitutive equations are presented in Section 4.2, and the drawing model is applied to solve the torque transmission problem in Section 4.3. In Section 4.4 an elastic constitutive equation with two stress-free states is introduced and the drawing model is applied to the fiber necking problem.

4.1 Mathematical Model of Fiber Drawing

Consider a fiber wrapped around two pulleys, one the driven, or feed, pulley with radius \( r_1 \) and angular velocity \( \omega_1 \), and another the driving, or take-up, pulley with radius \( r_2 \) and angular velocity \( \omega_2 \), as shown in Figure 4.1. The peripheral speed of the driving pulley, \( v_2 = r_2 \omega_2 \), is faster than the peripheral speed \( v_1 = r_1 \omega_1 \) of driven pulley, and thus the fiber is stretched during this process, called the drawing process. To develop the governing equations, the problem is parameterized with the arclength
Figure 4.1: Schematical configuration of drawing process.

$s$ of the fiber, where $s$ is taken to be increasing in the direction of pulley rotation. An Eulerian description is adopted, with a typical point $s$ fixed in space. The process is assumed to be in steady state so that the conditions at location $s$ are independent of time. Gravity is ignored in this chapter.

### 4.1.1 Conservation of mass

The mass per volume of the fiber is $\rho$; the cross-sectional area and speed of the fiber are $A(s)$ and $v(s)$, respectively. In this steady state process conservation of mass requires that the mass flow rate is constant along the draw line,

$$
\text{mass flow rate } G = \rho \ A(s) \ v(s) = \rho \ A_0 \ v_0 = \text{constant}
$$

for all $s$ either on or between the pulleys. In (4.1), $A_0$ and $v_0$ are the reference cross-sectional area and the reference velocity of the fiber, typically the cross-sectional area and velocity of the undrawn fiber. For an incompressible material ($\rho=$constant), (4.1) has the alternative form

$$
v(s) = v_0(1 + \epsilon(s)), \quad \quad A(s) = \frac{A_0}{1 + \epsilon(s)}
$$

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where $\epsilon$ is the strain from the reference state.

Conservation of momentum has different forms for the fibers on or off the pulleys, and thus it is necessary to discuss the two cases separately:

### 4.1.2 Conservation of momentum for a fiber on a rotating pulley

Figure 4.2 shows a control volume fixed in space containing the segment of filament from $s$ to $s + ds$, subtending an angle $d\theta$. The speed of the fiber entering this domain is $v(s)$ and the speed exiting it is $v(s + ds)$; the tension in the fiber where it enters is $T(s)$ and where it exits is $T(s + ds)$. The friction force between the fiber and the pulley is $f(s) ds$, with the force per length $f(s)$ assigned to be positive if it tends to accelerate the fiber. The normal compressive force between the fiber and the pulley is $n(s) ds$. In this dissertation, aerodynamic and adhesive force are ignored, so that the force per length $n(s)$ must be non-negative,

$$n(s) \geq 0. \quad (4.3)$$

Conservation of linear momentum in the tangential direction requires

$$T(s + ds) \cos\left(\frac{d\theta}{2}\right) - T(s) \cos\left(\frac{d\theta}{2}\right) + f(s) ds$$

$$= G v(s + ds) \cos\left(\frac{d\theta}{2}\right) - G v(s) \cos\left(\frac{d\theta}{2}\right), \quad (4.4)$$

Conservation of momentum in the normal direction requires

$$n(s) ds - T(s) \sin\left(\frac{d\theta}{2}\right) - T(s + ds) \sin\left(\frac{d\theta}{2}\right)$$

$$= -G v(s) \sin\left(\frac{d\theta}{2}\right) - G v(s + ds) \sin\left(\frac{d\theta}{2}\right). \quad (4.5)$$
Figure 4.2: Control volume for momentum conservation on the pulley.

The subtended angle $d\theta$ is considered to be small, $|d\theta| \ll 1$, so that $\cos\left(\frac{d\theta}{2}\right) \approx 1$ and $\sin\left(\frac{d\theta}{2}\right) \approx \frac{d\theta}{2}$. Expressing $T(s+ds) = T(s) + dT(s)$ and $v(s+ds) = v(s) + dv(s)$, assuming $dT(s)$ and $dv(s)$ are small, and neglecting the product of small quantities, (4.4) and (4.5) reduce to

\[ dT(s) + f(s) \, ds = G \, dv(s), \quad (4.6) \]

and

\[ n(s) = \frac{T(s) - G \, v(s)}{r}, \quad (4.7) \]

respectively, where I have used $ds = r \, d\theta$, with $r$ the radius of the pulley.

As an alternate derivation of (4.6), by considering the control volume in Figure 4.2, conservation of angular momentum about the center of the pulley requires

\[ T(s+ds) \, r + f(s) \, ds \, r - T(s) \, r = G \, v(s+ds) \, r - G \, v(s) \, r, \quad (4.8) \]

which reduces to (4.6) with the assumption that $dT(s)$ and $dv(s)$ are small quantities.
The requirement (4.3) that $n(s)$ is non-negative everywhere together with (4.7) demands that

$$T(s) - G v(s) \geq 0. \quad (4.9)$$

Hence at any point $s$ on the fiber there must be a tensile force in the fiber greater than or equal to the mass flow rate of the fiber times its speed. If not, the pulley would have to pull on the fiber to keep it in contact and accelerating in a circular path. This the pulley cannot do, and the insufficiently-tensioned fiber will fly off the pulley.

If the fiber is slipping on the pulley, or if the slipping is impending, the magnitude of the frictional force per unit length $f(s)$ is assumed to be proportional to the normal force per unit length $n(s)$,

$$|f(s)| = \mu n(s), \quad (4.10)$$

where the proportionality constant $\mu$ is the coefficient of either static or kinetic friction, depending on if the pulley peripheral speed is the same as the fiber speed (static, impending slip) or not (kinetic). The direction of the force (i.e. the sign of $f(s)$) is dictated by whether the pulley peripheral speed is larger or smaller than the fiber speed for the kinetic friction case, and the direction necessary for the friction to accelerate the fiber with the moving pulley surface for the static friction, impending slip case. In kinetic friction case, if $r \omega > v(s)$, where $r$ and $\omega$ are the radius and angular velocity of the pulley, then $f(s)$ is positive (friction tends to accelerate the fiber in the direction of the pulley rotation); if $r \omega < v(s)$ then $f(s)$ is negative (friction tends to decelerate the fiber). Note that (4.10) can also be rewritten as

$$f(s) = \text{sign}(f(s)) \mu n(s), \quad (4.11)$$
where \( \text{sign}(f(s)) = \frac{f(s)}{|f(s)|} \) is the sign of the friction force \( f(s) \).

If the fiber is not slipping on the pulley and slipping is not impending, then

\[
|f(s)| < \mu n(s), \tag{4.12}
\]

with both the sign and magnitude of \( f(s) \) determined by conservation of momentum.

**Special case: The capstan formula**

When the fiber is fixed in the space and wrapped on a pulley which is either rotating or fixed, then \( G, v, \) and \( dv \) are zero and (4.6) and (4.7) become

\[
dT(s) + f(s) \, ds = 0, \tag{4.13}
\]

\[
n(s) = \frac{T(s)}{r}. \tag{4.14}
\]

Differentiating (4.14) results in

\[
dT(s) = r \, dn(s), \tag{4.15}
\]

which when inserted in (4.13) gives

\[
f(s) = -r \frac{dn(s)}{ds}. \tag{4.16}
\]

If the pulley is rotating then the friction is kinetic (recall that the fiber is not moving). We assume that if both the pulley and fiber are stationary, that slipping is impending. In either case, equation (4.12) holds.

Equation (4.16) together with (4.11) gives

\[
-r \frac{dn(s)}{ds} = \text{sign}(f(s)) \mu n(s), \tag{4.17}
\]

which is integrated to yield

\[
n(s) = n(s_0) \exp \left( -\text{sign}(f(s)) \frac{\mu}{\mu} \frac{(s - s_0)}{r} \right), \tag{4.18}
\]

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with

\[ n(s_0) = \frac{T(s_0)}{r} \]  \hspace{1cm} (4.19)

as the normal force per unit length at a given point \( s_0 \) on the pulley and \( T(s_0) \) the tensile force at \( s_0 \). (4.14) and (4.18) then give what is commonly called the capstan formula [9]

\[ T(s) = T(s_0) \exp\left(-\text{sign}(f(s)) \frac{\mu (s - s_0)}{r}\right). \]  \hspace{1cm} (4.20)

The capstan formula is a valid special case of the general theory strictly only if the fiber is fixed in space, with either a rotating pulley or a nonrotating pulley with impending slip. In particular we must have \( G, v, \) and \( dv \) (the mass flow rate, speed, and change of speed of the fiber) equal to zero. The formula also holds if the motion of the pulley and fiber is assumed quasistatic. Historically, the capstan formula has been applied to moving fibers on pulleys [12]; as can be seen from the previous analysis this is correct only if the speed of the fiber is small.

The advance of (4.6) and (4.7) over the capstan formula (4.20) is that (4.6) and (4.7) model the inertia of a moving fiber. Since the capstan formula (4.20) ignores inertia, it is the same for the fiber slipping on the pulley or fixed on the pulley. When inertia is included, there are three different situations, each of which demands a separate set of equations: the fiber slipping on the pulley and moving faster than the pulley’s peripheral speed, the fiber slipping on the pulley and moving slower than the pulley’s peripheral speed, and the fiber rotating with the pulley:
Fiber slipping on the pulley

When the fiber speed is different from the pulley peripheral speed \((v(s) \neq r\omega)\), slip occurs. By differentiating (4.7) one has

\[
r\, dn(s) = dT(s) - C\, dv(s), \tag{4.21}
\]

which together with (4.6) gives

\[
r\, dn(s) = -f(s)\, ds. \tag{4.22}
\]

Combining (4.11) and (4.22) yields

\[
\frac{dn(s)}{ds} = -\frac{\text{sign}(f(s))\, \mu\, n(s)}{r}. \tag{4.23}
\]

Equation (4.23) in turn can be integrated to give

\[
n(s) = n(s_0) \exp \left( -\text{sign}(f(s)) \frac{\mu\, (s - s_0)}{r} \right). \tag{4.24}
\]

Combining (4.11) and (4.24) yields

\[
f(s) = \text{sign}(f(s)) \, \mu \, n(s_0) \exp \left( -\text{sign}(f(s)) \frac{\mu\, (s - s_0)}{r} \right). \tag{4.25}
\]

When the belt speed is less than the pulley peripheral speed, \(f(s)\) is positive, and (4.24) and (4.25) become

\[
n(s) = n(s_0) \exp \left( -\frac{\mu\, (s - s_0)}{r} \right), \quad \text{when } v < r\omega, \tag{4.26}
\]

\[
f(s) = \mu \, n(s_0) \exp \left( -\frac{\mu\, (s - s_0)}{r} \right), \quad \text{when } v < r\omega. \tag{4.27}
\]

From (4.26) and (4.27), the normal force per unit length and the friction force per unit length are monotonically decreasing with arclength along the direction of pulley rotation when the belt is moving slower than the pulley surface,

\[
\frac{dn}{ds} < 0 \quad \text{and} \quad \frac{df}{ds} < 0 \quad \text{when } v < r\omega. \tag{4.28}
\]
When the belt speed exceeds the pulley peripheral speed, \( f(s) \) is negative, (4.24) and (4.25) become

\[
n(s) = n(s_0) \exp \left( \frac{\mu (s - s_0)}{r} \right), \quad \text{when } v > r\omega, \quad (4.29)
\]

\[
f(s) = -\mu n(s_0) \exp \left( \frac{\mu (s - s_0)}{r} \right), \quad \text{when } v > r\omega. \quad (4.30)
\]

From equations (4.29) and (4.30), it is noted that the normal force per unit length and the magnitude of the friction force per unit length on the belt are monotonically increasing with arclength along the direction of pulley rotation when the belt is moving faster than the pulley surface,

\[
\frac{dn}{ds} > 0 \quad \text{and} \quad \frac{df}{ds} > 0 \quad \text{when } v > r\omega. \quad (4.31)
\]

Combining (4.7), (4.19), (4.26) and (4.29) gives explicit relations between the tensile force and the velocity of the fiber on a pulley for the cases of the fiber moving faster than the surface of the pulley,

\[
T(s) - Gv(s) = \left( T(s_0) - Gv(s_0) \right) \exp \left( \frac{\mu (s - s_0)}{r} \right),
\]

\[\text{for } v(s) > r\omega, \quad (4.32)\]

and the fiber moving slower than the surface of the pulley,

\[
T(s) - Gv(s) = \left( T(s_0) - Gv(s_0) \right) \exp \left( -\frac{\mu (s - s_0)}{r} \right),
\]

\[\text{for } v(s) < r\omega. \quad (4.33)\]

Equations (4.32) and (4.33) will also be used in Section 4.3 to determine the arc over which the fiber slip over the pulley as long as the constitutive equation of the fiber is known.
Fiber not slipping

When the fiber moves with the pulley surface, it does not slip on the pulley and

\[ v(s) = r \omega. \]  \hfill (4.34)

This implies

\[ \frac{dv}{ds} = 0, \]  \hfill (4.35)

and conservation of momentum in the tangential direction \((4.6)\) reduced to

\[ \frac{dT}{ds} - f(s) = 0, \]  \hfill (4.36)

i.e. the gradient of tensile force is equal to the frictional force for the fiber moving with the pulley surface. Since the fiber moves with the pulley surface, \((4.2)_1\) and \((4.35)\) require that the strain gradient of fiber on the pulley is zero when there is no slip, i.e.

\[ \frac{de}{ds} = 0, \quad \text{for } v(s) = r \omega. \]  \hfill (4.37)

For the fiber in which the tensile force is a function of strain, \((4.37)\) means that the tensile force is a constant along the fiber over any interval in which the fiber is not slipping on the pulley.

4.1.3 Conservation of momentum for a fiber in the free span

To derive the form of conservation of momentum for the fiber in the free span between the pulleys, consider the control volume in Figure 4.3, in which the fiber segment spans \(s\) to \(s + ds\), with tensile force \(T(s)\) and velocity \(v(s)\) at location \(s\) and
tensile force \( T(s+ds) \) and velocity \( v(s+ds) \) at location \( s+ds \). Ignoring aerodynamic forces, momentum conservation demands

\[
T(s+ds) - T(s) = G\left(v(s+ds) - v(s)\right).
\] (4.38)

Equations (4.38) gives to leading order,

\[
dT = G \, dv.
\] (4.39)

Comparing with the form of linear momentum for the fiber on the pulley, we see that (4.39) is the special cases of (4.6) with no frictional force, but there is no analog of (4.7). In the free span, since the motion is rectilinear conservation of linear momentum in the transverse direction reduces to zero equals zero. The number of scalar equations demanded by conservation of momentum collapses from two for the case of the fiber on the pulley to one in the free span.

### 4.2 Constitutive equations

To complete the mechanical theory for the drawing of fibers, a constitutive equation must be provided relating the tensile force in the fiber to its deformation. Qualitative discussion of material behavior in drawing are available. For instance, Ziabicki
[29] notes there are several fundamental types of deformation characteristics for undrawn polymer fibers, such as plastic fracture, strain hardening, and neck-type deformation. However there are no quantitative treatments, proposing explicit constitutive equations. The governing equations in the previous section hold for any type of material. To solve the torque transmission problem of Section 4.3, one linearly elastic and two nonlinear elastic constitutive equations are proposed in this section. Later, in Section 4.4, a linearly elastic constitutive equation with two stress-free reference states is proposed to investigate the necking problem.

### 4.2.1 A linearly elastic material

As a first constitutive model, it is proposed that the tensile force $T(s)$ in the fiber at location $s$ is linearly proportional to the strain $\epsilon(s)$ from the reference state at $s$:

$$T(s) = k_1 \epsilon(s) \quad k_1 > 0 \quad (4.40)$$

For a fiber of this material moving on a pulley, the requirement (4.9) that the normal force per length $n(s)$ is non-negative can be rewritten as

$$\left( k_1 - G v_0 \right) \epsilon(s) - G v_0 \geq 0. \quad (4.41)$$

In a drawing process the force in the fiber is always tensile ($T(s) > 0$), and thus $\epsilon(s) = \frac{T(s)}{k_1}$ is always positive. In addition, $G$ and $v_0$, and thus $G v_0$, are always positive. Consequently (4.41) implies that

$$\left( k_1 - G v_0 \right) \epsilon(s) \geq G v_0 \geq 0, \quad (4.42)$$

and

$$k_1 - G v_0 \geq 0. \quad (4.43)$$
Hence

\[ k_1 \geq G \, v_0 \geq 0, \]  

which is a necessary condition for the fiber with material response \( T(s) = k_1 \epsilon(s) \) to move on the surface of a pulley. Condition (4.44) means that not all linearly elastic fibers can have steady state motion on a velocity-prescribed pulley. This can be seen if (4.44) is rewritten in terms of Young’s modulus \( E = \frac{k_1}{A_0} \), noting that \( G = \rho \, A_0 \, v_0 \):

\[ A_0 \, E \geq \rho \, A_0 \, v_0^2, \]  

or

\[ E \geq \rho \, v_0^2. \]  

Therefore the reference velocity \( v_0 \), density, and Young’s modulus \( E \) determine if the fiber can reach a steady state motion on a pulley.

To find the strain of the fiber at \( s \), (4.32) and (4.33) are used to obtain

\[
\epsilon(s) = \frac{G \, v_0 + [(k_1 - G \, v_0) \, \epsilon(s_0) - G \, v_0] \, \exp \left( \frac{\mathbf{u}(s - s_0)}{r} \right)}{k - G \, v_0},
\]

for \( v(s) > r \omega \)  

(4.47)

and

\[
\epsilon(s) = \frac{G \, v_0 + [(k_1 - G \, v_0) \, \epsilon(s_0) - G \, v_0] \, \exp \left( -\frac{\mathbf{u}(s - s_0)}{r} \right)}{k_1 - G \, v_0},
\]

for \( v(s) < r \omega \)  

(4.48)

When (4.44), or equivalently (4.46), is true, (4.47) reveals that if any section of the fiber on the surface of the pulley moves faster than the peripheral speed, then \( v(s) \) increases monotonically on the pulley in that section. Conversely, from (4.48) it is
seen that if any section of the fiber on the surface of the pulley moves slower than the peripheral speed, than \( v(s) \) decreases monotonically on the pulley in that section. Importantly, \( v(s) - r \omega \) never changes sign on the same pulley for a fiber of a linearly elastic material.

4.2.2 Nonlinearly elastic material I

Since only a tensile force is transmitted through the fiber during the drawing process, negative strain is not a concern in this chapter. A constitutive equation that proposes the tensile force \( T \) is a quadratic function of the strain \( \epsilon \) from the reference state is used,

\[
T(s) = k_1 \epsilon(s) + k_2 \epsilon^2(s), \quad k_1, \epsilon(s) > 0. \tag{4.49}
\]

For a fiber of this material moving on a pulley, the requirement (4.9) that the normal force per length \( n(s) \) is non-negative can be rewritten as

\[
k_2 \epsilon^2(s) + \left( k_1 - G v_0 \right) \epsilon(s) - G v_0 \geq 0, \tag{4.50}
\]

or

\[
(\epsilon(s) - \epsilon_1^*) (\epsilon(s) - \epsilon_2^*) \geq 0, \tag{4.51}
\]

where

\[
\epsilon_1^* = \frac{G v_0 - k_1 + \sqrt{(k_1 - G v_0)^2 + 4 k_2 G v_0}}{2 k_2}, \tag{4.52}
\]

\[
\epsilon_2^* = \frac{G v_0 - k_1 - \sqrt{(k_1 - G v_0)^2 + 4 k_2 G v_0}}{2 k_2}, \tag{4.53}
\]

are two characteristic strains for the drawing process. Note that

\[
\epsilon_1^* > \epsilon_2^*, \quad \text{if } k_2 > 0,
\]
\[ \epsilon_2^* > \epsilon_1^*, \quad \text{if} \ k_2 < 0. \quad (4.54) \]

To have a fiber of this material move steadily on the pulley,

\[ \epsilon(s) \geq \max\{\epsilon_1^*, \epsilon_2^*\} \quad \text{or} \quad \epsilon(s) \leq \min\{\epsilon_1^*, \epsilon_2^*\}. \quad (4.55) \]

In addition to (4.55), it is also required that \( \epsilon(s) > 0 \), since there is no negative strain in the process. To find the strain of the fiber at any point on the pulley, (4.32) and (4.33) can be rewritten to give

\[ \epsilon(s) = \frac{G v_0 - k_1 + \sqrt{\mathcal{G}(s, \epsilon(s))}}{2 k_2}, \quad \text{for} \ v(s) > r \omega, \quad (4.56) \]

\[ \epsilon(s) = \frac{G v_0 - k_1 - \sqrt{\mathcal{G}(s, \epsilon(s))}}{2 k_2}, \quad \text{for} \ v(s) < r \omega, \quad (4.57) \]

with \( \epsilon(s_0) \) the known strain at a point \( s_0 \) and

\[ \mathcal{G}(s, \epsilon(s_0)) = (k_1 - G v_0)^2 + 4 k_2 (G v_0 + k_2 \epsilon^2(s_0)) + (k_1 - G v_0) \epsilon(s_0) \exp \left( \frac{\mu}{r} (s - s_0) \right) \quad (4.58) \]

In equation (4.56) \( \epsilon(s) \) is an increasing function of \( s \); thus if \( v(s_0) > r \omega \) at a point \( s_0 \) on the pulley, \( v(s) = v_0(1 + \epsilon(s)) \) increases monotonically on the pulley after that point \( s_0 \) for the fiber of this material. Similarly, if \( v(s_0) < r \omega \) at a point \( s_0 \) on the pulley, \( v(s) = v_0(1 + \epsilon(s)) \) decreases monotonically on the pulley after that point \( s_0 \) for the fiber of this material.

### 4.2.3 Nonlinear elastic material II

The constitutive equation that is often used in modeling strain hardening and strain softening is proposed here. Specifically, it is proposed that the tensile force \( T \)
in the fiber is a cubic function of the strain $\epsilon$ from the reference state,

$$T(s) = k_1\epsilon(s) + k_3\epsilon^3(s), \quad k_1, \epsilon(s) > 0. \quad (4.59)$$

When $k_3 > 0$, the material is strain hardening. The force $T(s)$ corresponding to a certain strain $\epsilon(s)$ for strain hardening material is greater than that of a linearly elastic material with spring constant $k_1$, and the difference grows cubically. When $k_3 < 0$, the material is strain softening. The force $T(s)$ corresponding to a certain strain $\epsilon(s)$ for strain softening material grows is less than that of a linearly elastic material.

For a fiber of this material moving on a pulley, the requirement (4.9) that the normal force per length $n(s)$ is non-negative can be rewritten as

$$k_3 \epsilon^3(s) + (k_1 - G v_0) \epsilon(s) - G v_0 \geq 0. \quad (4.60)$$

This is a necessary condition for the fiber of this material to move steadily on the pulley. To find the strain of the fiber at a point $s$, equation (4.32) and (4.33) are rewritten as

$$\epsilon^3(s) + p \epsilon(s) + q_1 = 0, \quad \text{for } v(s) > r \omega, \quad (4.61)$$

$$\epsilon^3(s) + p \epsilon(s) + q_2 = 0, \quad \text{for } v(s) > r \omega, \quad (4.62)$$

with

$$ p = \frac{k_1 - G v_0}{k_3}, $$

$$ q_1 = -\frac{k_3}{G v_0} \left( \frac{k_3 \epsilon^3(s) + (k_1 - G v_0) \epsilon(s)}{k_3} \right) \exp \left( \frac{-\mu (s - s_0)}{r} \right), $$

$$ q_2 = -\frac{G v_0}{k_3} \left( \frac{k_3 \epsilon^3(s) + (k_1 - G v_0) \epsilon(s)}{k_3} \right) \exp \left( \frac{-\mu (s - s_0)}{r} \right). \quad (4.63)$$

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4.3 The torque transmission problem

Equating a belt with a fiber, the steady torque transmission problem of Figure 4.4 is a drawing process. Here I apply the mathematical model derived in this chapter with those material properties described in previous section to solve the torque transmission problem.

When a torque $M$ is transmitted between two pulleys of the same radius $r$ as shown in Figure 4.4, the tensions $T_1$ on the tight free span and $T_2$ on the slack free span are

\[
T_1 = T_0 + \frac{M}{2r}, \quad (4.64)
\]
\[
T_2 = T_0 - \frac{M}{2r}, \quad (4.65)
\]

obtained by summing the moments on either pulley to zero; $T_0$ is the initial tensile force in the belt, and we have assumed that the bearings are frictionless.

In this section, the moment transmitted $M$, the angular velocity $\omega_1$ of the driving pulley, the initial tension tensile force $T_0$ in the belt, and the coefficient of friction
μ between the belt and the pulleys are assumed to be specified. The product of \( M \) and \( \omega_1 \) is the power input of the system. The unknowns to be determined by a solution of the torque transmission problem are the angular velocity \( \omega_2 \) of the driven pulley and the subtended angles \( \phi_1 \) and \( \phi_2 \) in which the belt is slipping on the driving and driven pulley, respectively (see Figure 4.5). From this solution one can obtain maximum moment, \( M_{\text{max}} \), that the system can transmit. \( M_{\text{max}} \) is the value of \( M \) for which one or both \( \phi_1 \) and \( \phi_2 \) are \( \pi \), i.e. the value of the moment for which the belt is slipping over its entire contact within the driving and/or driven pulley.

Equation (4.2) requires that the velocity is proportion to \( 1 + \epsilon(s) \), thus

\[
\omega_2 = \frac{(1 + \epsilon_s) \omega_1}{1 + \epsilon_t},
\]

with \( \epsilon_t \) the strain in the tight free span and \( \epsilon_s \) the strain in the slack free span. The tensile force in the belt in this problem is not a function of material property, but the strain in the belt is. Hence, the angular velocity of driven pulley is function of material property. For the linearly elastic material,

\[
\begin{align*}
\epsilon_t &= \frac{T_0 + \frac{M}{2r}}{k_1}, \\
\epsilon_s &= \frac{T_0 - \frac{M}{2r}}{k_1},
\end{align*}
\]

and (4.66) gives

\[
\omega_2 = \frac{(T_0 + \frac{M}{2r} + k_1) \omega_1}{T_0 - \frac{M}{2r} + k_1}.
\]

When the belt is made of nonlinearly elastic material I, \( \epsilon_t \) and \( \epsilon_s \) are determined by substituting (4.64) and (4.65) into (4.49),

\[
\epsilon_t = \frac{-k_1 + \sqrt{k_1^2 + 4k_2(T_0 + \frac{M}{2r})}}{2k_2},
\]
\[ \epsilon_s = -k_1 + \frac{\sqrt{k_1^2 + 4 k_2 (T_0 - \frac{M}{2r})}}{2 k_2}. \quad (4.69) \]

Thus

\[ \omega_2 = \frac{\left( 2 k_2 - k_1 + \sqrt{k_1^2 + 4 k_2 (T_0 - \frac{M}{2r})} \right) \omega_1}{2 k_2 - k_1 + \sqrt{k_1^2 + 4 k_2 (T_0 + \frac{M}{2r})}}. \quad (4.70) \]

When the belt is made of nonlinearly elastic material II, substituting (4.64) and (4.65) into (4.59) gives

\[ \epsilon_t^3 + \frac{k_1}{k_3} \epsilon_t - \frac{T_1}{k_3} = 0, \quad (4.71) \]

\[ \epsilon_s^3 + \frac{k_1}{k_3} \epsilon_s - \frac{T_2}{k_3} = 0, \quad (4.72) \]

for \( \epsilon_t \) and \( \epsilon_s \). And then \( \omega_2 \) is determined by (4.66).

### 4.3.1 Solution from the capstan formula

The capstan formula (4.20), which ignore the inertia of the belt, demands

\[ \exp(\mu \phi_1) = \exp(\mu \phi_2) = \frac{T_0 + \frac{M}{2r}}{T_0 - \frac{M}{2r}}. \quad (4.73) \]

Result (4.73) is independent of material properties, and from which the subtended angles by the slip zone are

\[ \phi_1 = \phi_2 = \phi = \frac{1}{\mu} \ln \left( \frac{T_0 + \frac{M}{2r}}{T_0 - \frac{M}{2r}} \right). \quad (4.74) \]

The maximum moment \( M_{\text{max}} \) can be transmitted by this system is obtained by substituting \( \phi \) and \( M \) with \( \pi \) and \( M_{\text{max}} \) into (4.73),

\[ M_{\text{max}} = \frac{2r T_0 \left[ \exp(\mu \pi) - 1 \right]}{1 + \exp(\mu \pi)}. \quad (4.75) \]

For the specific torque transmission problem: A belt with \( k =25000 \text{ kN} \) is used to transmit a torque \( M = 4 \text{ N-m} \) between two pulley of the same radius \( r =0.2 \text{ m} \). The
friction coefficient between the belt and the pulley is \( \mu = 0.15 \), and the initial tension of the belt is \( T_0 = 100 \text{ N} \). Then the tension in the tight side \( T_1 \) and the tension in the slack side \( T_2 \) are

\[
T_1 = T_0 + \frac{M}{2r} = 110 \text{ N},
\]
\[
T_2 = T_0 - \frac{M}{2r} = 90 \text{ N},
\]

and the strains on the belt are \( \epsilon_t = 4.4 \times 10^{-3} \) in the tight side and \( \epsilon_s = 3.6 \times 10^{-3} \) in the slack side. If the velocity of the driving pulley is given as, \( v_1 = r \omega_1 = 25.11 \text{ m/s} \), then the reference velocity can be found from mass conservation \((4.2)_1\) as

\[
v_0 = \frac{v_1}{1 + \epsilon_t} = 25 \text{ m/s},
\]

and the angular velocity of the driven pulley is determined from \((4.2)_1\),

\[
\begin{align*}
\omega_2 &= \frac{v_2}{r} = 125.45, \\
v_2 &= v_0(1 + \epsilon_s) = 25.09 \text{ m/s},
\end{align*}
\]

The belt slips over an arc \( r \phi \) on each pulley and \( \mu \phi \) is given by the capstan formula

\[
\mu \phi = \ln \left( \frac{T_1}{T_2} \right) = 0.2007,
\]

so \( \phi = 1.3378 \left(76.650^\circ\right) \). And the maximum torque \( M_{\text{max}} \) is 9.2542 N-m.

### 4.3.2 Torque transmission by the belt of linearly elastic material

When the torque is transmitted by a linearly elastic belt (Section 4.2.1), the strain \( \epsilon_t \) in the tight free span and \( \epsilon_s \) in the slack free span are given in \((4.67)\). The subtended angle over which the belt slips on the pulley can be determine by \((4.33)\) as

\[
\phi_1 = -\frac{1}{\mu} \ln \left( \frac{T(\epsilon_s) - G v(\epsilon_s)}{T(\epsilon_t) - G v(\epsilon_t)} \right), \tag{4.76}
\]
Figure 4.5: Belt speed in the torque transmission problem. The specified speed $r \omega_1$ of the driving pulley is greater than unknown speed $r \omega_2$ of the driven pulley.

and by (4.32) as

$$\phi_2 = \frac{1}{\mu} \ln \left( \frac{T(\epsilon_t) - G v(\epsilon_t)}{T(\epsilon_s) - G v(\epsilon_s)} \right).$$

Thus,

$$\phi_1 = \phi_2 = \phi = \frac{1}{\mu} \ln \left( \frac{(T_0 + \frac{M}{2 r}) (k_1 - G v_0 k_1)}{(T_0 - \frac{M}{2 r}) (k_1 - G v_0 k_1)} \right).$$

I now simulate the specific torque transmission problem: A belt with $k_1 = 25000$ kN is used to transmit a torque $M = 4$ N-m between two pulley of the same radius $r = 0.2$ m. The friction coefficient between the belt and the pulley is $\mu = 0.15$, and the initial tension of the belt is $T_0 = 100$ N. Then the tension in the tight side $T_1$ and the tension in the slack side $T_2$ are

$$T_1 = T_0 + \frac{M}{2 r} = 110 \text{ N},$$
$$T_2 = T_0 - \frac{M}{2 r} = 90 \text{ N},$$

and the strains on the belt are $\epsilon_t = 4.4 \times 10^{-3}$ in the tight side and $\epsilon_s = 3.6 \times 10^{-3}$ in the slack side. If the velocity of the driving pulley is given as, $v_1 = r \omega_1 = 25.11$
m/s, then the reference velocity can be found as

\[ v_0 = \frac{v_1}{1 + \epsilon_t} = 25 \text{ m/s}, \]

and the velocity of the driven pulley is determined from

\[ v_2 = v_0(1 + \epsilon_s) = 25.09 \text{ m/s}, \]
\[ \omega_2 = \frac{v_2}{r} = 125.45, \]

If the mass flow rate of the belt is \( G = 0.5 \text{ kg/s} \), the belt slips over an arc \( r \phi \) on each pulley and \( \mu \phi \) is given by the generalized capstan formula

\[ \mu \phi = \ln \left( \frac{T_1}{T_2} \right) = 0.2296, \]

which reduce to \( \phi = 1.5306 \) (87.697°). \( M_{\text{max}} \) is 8.0973, N-m which is different from what predicted by the capstan formula.

On the segment of the belt that stick on the pulley, the strain remains on a constant value. For the belt of this linearly elastic material, this means there is no friction force between the belt and pulley on this segment. (4.78) determines the length of the arc over which the belt slip on the pulley, but the position of the slip arc is not known yet. The segment of belt entering the driving pulley is the tight one with tension \( T_1 \) and velocity \( v_1 \), and the other side is the slack segment with tension \( T_2 \) and velocity \( v_2 \). Since the velocity of the belt will increase or decrease monotonically on the pulley, the peripherial speed of the driving pulley should be \( v_1 \) and the belt slips and decrease its speed only on the segment immediate before the belt leaves the pulley. Similarly, the driven pulley rotates with peripheral speed \( v_2 \) and the belt increases its speed only on the segment immediate before it leaves the pulley.
4.3.3 Torque transmission by the belt of nonlinearly elastic material I

Here the torque transmission problem is investigated with the nonlinearly elastic material of the type \( T = k_1 \epsilon + k_2 \epsilon^2 \). The strains \( \epsilon_t \) is determined by (4.69).

The subtended angle over which the belt slips on each pulley can be determined by (4.32) and (4.33) as

\[
\phi_1 = \phi_2 = \phi = \frac{1}{\mu} \ln \left( \frac{2 \, k_2 (T_1 - G \, v_0) + G \, v_0 (k_1 - \sqrt{k_1^2 + 4 \, k_2 \, T_1})}{2 \, k_2 (T_2 - G \, v_0) + G \, v_0 (k_1 - \sqrt{k_1^2 + 4 \, k_2 \, T_2})} \right). \tag{4.79}
\]

For a belt with \( k_1 = 2.5 \times 10^4 \) kN and \( k_2 = 1000 \) N, in the case 4 N-m torque is transmitted by two pulleys of 0.2 m radius with \( G = 0.5 \) kg/s, \( v_0 = 25 \) m/s, \( T_1 = 110 \) N, \( T_2 = 90 \) N, and \( \mu = 0.15 \), (4.79) predicts that \( \phi = 1.5271 \) (87.496°) and the capstan formula estimates that \( \phi = 1.3378 \) (76.650°). The positions of the slip segments are the same as the linearly elastic material, i.e. immediately before the belt leaves the pulley.

4.3.4 Torque transmission by the belt of nonlinearly elastic material II

When investigating the torque transmission problem with the linearly elastic belt of the type \( T = k_1 \epsilon + k_3 \epsilon^3 \), \( k_1, \epsilon > 0 \) material, the strains \( \epsilon_t \) and \( \epsilon_s \) in the tight and slack side of belt are determined from (4.72). With the information of \( \epsilon_t \) and \( \epsilon_s \), the subtended angle over which the belt slips on each pulley can be determined by (4.32) and (4.33) as

\[
\phi_1 = \phi_2 = \phi = \frac{1}{\mu} \ln \left( \frac{T_1 - G \, v_1}{T_2 - G \, v_2} \right). \tag{4.80}
\]

For a belt with \( k_1 = 2.5 \times 10^4 \) kN and \( k_3 = 1000 \) N, in the case 4 N-m torque is transmitted by two pulleys of 0.2 m radius with \( G = 0.5 \) kg/s, \( v_0 = 25 \) m/s, \( T_1 = 110 \)
N, $T_2 = 90$ N, and $\mu = 0.15$, (4.72) gives $\epsilon_t = 4.3999 \times 10^{-3}$ and $\epsilon_s = 3.5999 \times 10^{-3}$. (4.80) gives $\phi = 1.5314$ (87.743°). With different material properties, $k_1 = 2.5 \times 10^4$ kN and $k_3 = -1000$ N, one have $\epsilon_t = 4.5525 \times 10^{-3}$ and $\epsilon_s = 3.7550 \times 10^{-3}$, and $\phi = 1.5306$ (87.691°), but the capstan formula estimate that $\phi = 1.3378$ (76.650°).

### 4.4 Modeling of necking

A phenomenon observed in fiber drawing is necking. Essentially, the fiber has two stress-free states, one called the “undrawn” fiber, with the larger cross-sectional area, and the other called the “drawn” fiber with a smaller cross-sectional area (see Chapter I). If a fiber in its undrawn state is loaded sufficiently, it will transform through an irreversible process called “necking”, after which upon unloading the fiber reverts to its drawn state. The necking transformation occurs at a location of an abrupt reduction of cross-sectional area, called a “neck”, or, “neck shoulder”.

When necking takes place during the drawing process, the field equations derived in Section 4.1 still hold for a region of the fiber away from the neighborhood of the neck shoulder. To model necking, in this section a constitutive equation is used in the drawing process which has two stress-free states; state one refers to the fiber before the abrupt cross-sectional area change (i.e. the undrawn fiber) and state two refers to the fiber after the abrupt cross-sectional area change (i.e. the undrawn fiber). It is emphasized that in both state one and state two the fiber is under no load. The cross-sectional area of state one (the undrawn fiber) is $A_{10}$, and the cross-sectional area of state two (the drawn fiber) is $A_{20}$, with $A_{20} \ll A_{10}$. Strain $\epsilon(s)$ is defined from the undrawn state one; draw ratio $\lambda$ is defined as the current length divided by
the length of the fiber in stress-free state one. Hence

\[ \lambda(s) = 1 + \epsilon(s). \] (4.81)

It is assumed that, as long as the force is not too large, the behavior from the undrawn state one is elastic. Said differently, it is assumed that if a fiber in state one is loaded by a force that is not too large, upon unloading the fiber returns to state one. Assuming a linear dependence of force to strain, this implies that in state one

\[ T(s) = k_1 \epsilon(s) \quad \text{for} \quad T(s) < \hat{T}, \] (4.82)

where \( k_1 \) is a positive constant and \( \hat{T} \) is the critical tension at which necking occurs. After necking, the unloaded fiber is longer than it was in its unloaded undrawn state and is longer by a factor of \( (1 + \epsilon^*) \), where \( \epsilon^* \) is the characteristic strain of state two from state one. It is assumed that deformation from this drawn state two is also linearly elastic,

\[ T(s) = k_2 (\epsilon(s) - \epsilon^*) \] (4.83)

where \( k_2 \) is a positive constant. It is assumed that the material in both state is incompressible, and that volume is preserved in the necking. Hence there is a single density \( \rho \) throughout the drawing process.

The necking initiates and begins to propagate at critical strain \( \hat{\epsilon} \) under the critical tensile force \( \hat{T} \). Consider a fiber of infinite length that is being drawn steadily between two pulleys. The feeding velocity \( v_f = r \omega_1 \) and the take-up velocity \( v_r = r \omega_2 \) (see Figure 3) are specified and can be tuned so that necking can take place either between pulleys or on a pulley. The particle in the fiber moves with a speed less than \( v_r \) when it enters the second pulley and is pulled by the friction force between the fiber and the
Figure 4.6: Constitutive responses for the polymer fiber with necking used in the drawing process.

surface of pulley so that it has been accelerated on the pulley and leaves the pulley with the take-up velocity \( v_r \). The fiber is transferred from state one into state two at the position of abrupt change of cross sectional area.

To analyze the drawing process, the processes are sorted into three types:

- necking takes place in free span,
- necking takes place on feeding pulley, or
- necking takes place on take-up pulley.

### 4.4.1 The drawing process with necking in the free span

Consider the fiber leaving the feeding pulley with velocity \( v_1 \geq v_f \) and entering the take-up pulley with velocity \( v_2 \leq v_r \). In this section it is assumed that the neck occurs between the two pulleys. With (4.82) and (4.83), conservation of momentum (4.39) for the free span gives that away from the neck,

\[
(G v_0 - k_i) d\varepsilon = 0, \quad \text{for } i = 1, 2, \quad (4.84)
\]
where $v_0$ is the speed of the unloaded state-one fiber. Equation (4.84) implies that $\epsilon$ is not a function of position $s$ on either side of the neck in free span if $Gv_0 \neq k_1$ and $Gv_0 \neq k_2$. Thus, unless the process is tuned precisely, the particles in free span upstream of the neck all have the same velocity $v_1$, and all those downstream of the neck have velocity $v_2$. In the following it is assumed $Gv_0 \neq k_1$ and $Gv_0 \neq k_2$. 

Figure 4.7: Free body diagram for momentum conservation at the neck shoulder in the free span.
Now consider the particles in the hatched area in Figure 4.7. Momentum conservation for this material volume requires

\[(T_2 - T_1) \, dt = \left( \frac{A_{10}}{1 + \varepsilon_1} \, v_1 \, dt \, \rho \right) (v_2 - v_1), \tag{4.85} \]

where \(T_1\) and \(T_2\) are the tensile forces in the fiber in the free span upstream (state one or undrawn) and downstream (state two or drawn), respectively; (4.85) leads to

\[T_2 - T_1 = G v_0 (\varepsilon_2 - \varepsilon_1). \tag{4.86} \]

with \(\varepsilon_1\) and \(\varepsilon_2\) the strains in the section of fiber in state one and two in the free span, respectively. Hence the difference of the tensile forces on both sides of the abrupt cross-sectional area change is proportional to the mass flow rate \(G\), reference velocity \(v_0\), and the difference of strains \(\varepsilon_2 - \varepsilon_1\) between the two sides. Substituting (4.82) and (4.83) into (4.86) gives

\[\varepsilon_2 = \frac{(k_1 - G v_0) \varepsilon_1 + k_2 \varepsilon^*}{k_2 - G v_0}. \tag{4.87} \]

In this case, \(\varepsilon_1\) is nothing more than \(\dot{\varepsilon}\), since necking takes place when the draw between the two pulleys.

On the pulley, (4.32) and (4.33) imply that \(v(s) - r_i \omega_i, i = 1, 2\) never change sign for the fiber in the same state. Also, with the help of (4.32) and (4.33), one can determine the velocity profile of the fiber and the distance for the fiber to accelerate to expected velocities on the pulleys.

Considering the material with the properties (4.82) and (4.83), (4.32) and (4.33) imply that

- When the sign of \(k_i - G v_0\) is the same as the sign of \(v(s) - r_j \omega_j, i, j = 1, 2\) in a certain interval of \(s\), \(\varepsilon(s)\) is an increasing function of \(s\) in this interval i.e.

  \[\varepsilon(s) \text{ is an increasing function if } v(s) > r_j \omega_j \text{ and } k_i > G v_0\]
\[ i, j = 1, 2 \] (4.88)

- When the sign of \( k_i - G v_0 \) is different from the sign of \( v(s) - r_j \omega_j, \) \( i, j = 1, 2 \)
in a certain interval of \( s, \) \( \epsilon(s) \) is a decreasing function of \( s \) in this interval i.e.

\[
\epsilon(s) \text{ is an decreasing function if } \quad v(s) < r_j \omega_j \text{ and } k_i > G v_0
\]
\[ i, j = 1, 2 \] (4.89)

It can be seen from (4.88) and (4.89) that the same fiber cannot decrease/increase its velocity and then increase/decrease its velocity on the same pulley in the same state. With the neck in the free span, the fiber is deformed from reference state one on the feed pulley and from state two on the take-up pulley.

### 4.4.2 The drawing process with necking on a pulley

As discussed previously, (4.84) requires that \( \epsilon(s) \) is constant between two pulleys if necking does not take place in free span. This also demands that the velocity and tensile force region are constants in free span.

In this subsection, the neck shoulder is on the pulley. The fiber is fed onto the pulley with a strain lower than \( \dot{\epsilon}, \) and then accelerated and stretched on the pulley. To describe the problem, let the position where necking takes place be \( \dot{s}, \) which is between \( s_1 \) and \( s_0. \) The velocity \( v(s), \) strain \( \epsilon, \) and tensile force \( T \) are discontinuous at \( \dot{s}, \) but the momentum and mass flow rate are continuous at \( \dot{s}. \) Hence (4.32) or (4.33) holds on this pulley in the domains \( s_0 < s < \dot{s} \) and \( \dot{s} < s < s_1. \) To find the strain \( \epsilon(\dot{s}^+)\), we consider the control volume in Figure 4.8, the conservation of angular momentum about the center of pulley gives

\[
T_2(\dot{s} + \frac{\Delta s}{2}) - T_1(\dot{s} - \frac{\Delta s}{2}) + f(\dot{s} - \frac{\Delta s}{2}) \frac{r \, d\theta}{2} - f(\dot{s} + \frac{\Delta s}{2}) \frac{r \, d\theta}{2}
\]
Figure 4.8: Control volume for momentum conservation at the neck shoulder on the pulley.

\[
= G v(\hat{s} + \frac{\Delta s}{2}) - G v(\hat{s} - \frac{\Delta s}{2}).
\] (4.90)

As \(\Delta s\) approaches an infinitesimal quantities for abrupt change of cross-sectional area, the friction force terms are so small that (4.90) is equivalent to (4.86). Thus, \(\epsilon(\hat{s}^+)\) is given by the alternative form of (4.87), which is

\[
\epsilon(\hat{s}^+) = \frac{(k_1 - G v_0) \epsilon(\hat{s}^-) + k_2 \epsilon^*}{k_2 - G v_0},
\] (4.91)

with \(\epsilon(\hat{s}^-) = \hat{\epsilon}\) for necking propagation.
CHAPTER 5

CONCLUSION

This dissertation presents models and simulations of multi-fiber melt spinning process, and drawing process. Because of their complexity, little research has been done in modeling these process. In modeling the spinning process, I divide the model into a model for the behavior of a single fiber in a prescribed quench air environment, called the single fiber model, and a model for the interaction of the a row of fibers with the quench air stream, called the fiber-air interaction model. These two models are then coupled into one, global, multi-fiber spinning model.

This dissertation creates a fiber-air interaction model incorporating the influence of temperature change in the density of quench air, a feature which has not been modeled in previous research. Neglecting the temperature change of quench air and the resulting density change in the quench air results in an underestimation the variation of fiber behavior through the fiber bundle (compare Figures 22 and 14).

In addition, the fiber-air interaction model of this dissertation for the first time incorporates momentum conservation, which enables the model to deduce the boundary layer volume flow rate entrained by each fiber from first principles rather than an empirical formula. This removal of empiricism means fewer experiments are needed to model the fiber-air interaction.
A single fiber spinning model incorporating the temperature-dependent density constraint of Cao [4] is presented in this dissertation. In addition, the model incorporates the radial variation of temperature throughout the entire model, rather than assuming that the cross-sectionally averaged viscosity and specific heat can be approximated by the viscosity and the specific heat at the average temperature. The radial variation of fiber temperature forces the pressure to also have a radial variation. Two intermediate constrained theories, which each include some of the improvements of the new theory but ignore others, as well as the conventional theory, without any of the improvements, are compared with the new theory, to investigate the separate effects of the constraint from density-dependent density, temperature variation, and pressure variation. Then the influence from the variation of several spinning conditions within the new theory are investigated, namely, ambient air temperature, spinneret temperature, take-up velocity, spinneret radius, and mass flow rates.

The new melt-spinning model predicts delayed cooling and stretching of the fiber, and a lower maximum tensile stress than does the conventional theory. It was discovered that there exists a stress relaxation zone near the fiber's glass transition; this phenomenon is attributed to the effects of fiber shrinkage.

As a preliminary modeling of the drawing process, Chapter IV developed the fundamental governing equations for a fiber moving and wrapped on a pulley. The conservation of momentum gives a generalized capstan formula (4.32), (4.33), which takes the inertia of the fiber into account. This formula reduces to the original capstan formula (4.26) when the fiber is fixed in the space and slip between the fiber and pulley is assumed to be impending.
The model is applied to solve the torque transmission problem of a belt connection two pulleys. A comparison of the predictions of this model, which includes the inertia of the belt, and a published treatment [15] which ignores this inertia, reveals that the neglect of inertia results in an overprediction of the moment that can be transmitted before the belt slips on the drive pulley. I also present an elastic constitutive model for the fiber which has two stress-free states, and combine this with the momentum equations to derive an isothermal solution of the steady necking phenomenon in fiber drawing.


