KNOWLEDGE AND DEVELOPMENT OF FUNCTIONS
IN A TECHNOLOGY-ENHANCED
HIGH SCHOOL PRECALCULUS CLASS:
A CASE STUDY

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

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1993
To the Lord Jesus and to the Love He gave me.
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TABLE OF CONTENTS

ACKNOWLEDGMENTS ............................................................. iii
VITA ....................................................................................... iv
LIST OF TABLES ..................................................................... xiii
LIST OF FIGURES .................................................................. xiv

CHAPTER PAGE

I THE PROBLEM ................................................................. 1

Introduction ...................................................................... 1
Need for the Study .......................................................... 4
Statement of the Problem ............................................... 5
Research Questions ........................................................ 6
Importance of the Study .................................................... 6
Organization of this Dissertation ....................................... 6

II THEORETICAL FRAMEWORK AND REVIEW OF THE LITERATURE .............................................. 8

A Theoretical Framework for the Research ...................... 9
A Short Survey on the Historical Development of Functions................................................................. 10
Research on the Concept of Functions .......................... 15
Procedural and Conceptual Knowledge .......................... 15
Knowledge of Functions .................................................. 16
Students' Development of Functions ............................. 18
Epistemological and Cognitive Obstacles ....................... 23
Concept Image and Concept Definition of Functions ..... 24
Summary of Epistemological and Cognitive Obstacles ... 31
Overview of Computer Technology in Mathematics Education ......................................................... 31
The Use of Graphing Utilities in Mathematics Education.. 33
Computer Graphing Software .......................................... 35
Graphing Calculators ....................................................... 36
Research on Graphing Calculators and Functions .......... 39
Chapter Summary and Conclusions ............................... 39
III METHODOLOGY .......................................................... 41

Design of the Study ....................................................... 41
Why Naturalistic Inquiry? ............................................ 41
Why Case Study? ......................................................... 42
Why Concept Images ................................................ 42
Trustworthiness of the Study ....................................... 43
Credibility ................................................................. 43
Transferability ......................................................... 46
Dependability ........................................................... 47
Confirmability .......................................................... 48
Instruments ............................................................... 49
The Researcher .......................................................... 49
Observation Form and Field Notes ......................... 50
Reflective Journal ....................................................... 51
Students' Quizzes, Tests, and Exams ...................... 51
Students' Notebooks ................................................ 51
Practice Test on Functions ...................................... 52
Handout for Extra Credits ...................................... 52
Interviews ................................................................. 53
Students' Interviews ................................................ 53
Teacher's Interviews ............................................... 55
Procedures .............................................................. 55
Sampling ................................................................. 55
The Project ............................................................... 55
The School ................................................................. 56
Mr. H. and his Seventh-Period Class ...................... 56
The Students ............................................................. 56
Data Collection ........................................................ 58
Field Work ............................................................... 58
Administration of the Practice Test on Functions .... 60
Handout for Extra Credits ....................................... 60
Students' Invitation to Participate ......................... 60
Interviews ................................................................. 61
Students' Interviews ................................................ 64
Teacher's Interviews ............................................... 65
Author's Interview ................................................... 66
Gathering Data Outside the Field ......................... 66
Member Checks ......................................................... 66
Data Analysis .......................................................... 67
Interviews ................................................................. 69
Reflective Journal ....................................................... 69
Testing Materials ....................................................... 69
Students' Notebooks ............................................... 70
Writing up the Final Report .................................... 70
Summary ............................................................... 71
Some Ideas on Functions at the Beginning of the Course...... 238
Images and Definitions of Functions during the Study...... 241
Summary of Tyler's Development and Knowledge of Functions................................. 246
The Case of Sara................................................................. 247
Introduction................................................................. 247
How Sara defines a Function (Concept Definition)...... 247
Sara's mental Pictures associated with Function
(Concept Image)................................................................... 248
Equation/Formula.............................................................. 248
Graph.................................................................................. 252
Familiarity........................................................................... 253
Vertical Line Test............................................................... 253
Continuity............................................................................ 254
One Output for every Input.................................................. 257
Summary of Sara's Function Images............................... 258
Procedural and Conceptual Knowledge of Functions....... 258
Composition of Functions................................................... 258
Global Approach to Functions and Connections.......... 259
Settings: Domain, Range and Rule................................. 262
Construction of Graphs....................................................... 263
Development of Functions.................................................. 264
Some Ideas on Functions at the Beginning of the Course...
Images and Definitions of Functions during the Study...... 266
Summary of Sara's Development and Knowledge of Functions........................................ 270
The Case of Nathalie........................................................... 271
Introduction................................................................. 271
How Nathalie defines a Function (Concept Definition).... 272
Nathalie's Mental Pictures Associated with Function
(Concept Image)............................................................... 272
Equation/Formula.............................................................. 273
Graph.................................................................................. 275
A Set of Ordered Pairs........................................................ 277
Familiarity........................................................................... 279
Continuity............................................................................ 280
Vertical Line Test............................................................... 282
One Output for every Input.................................................. 283
Summary of Nathalie's Function Images....................... 284
Procedural and Conceptual Knowledge of Functions....... 284
Composition of Functions................................................... 285
Global Approach to Functions and Connections.......... 286
Settings: Domain, Range and Rule................................. 290
Construction of Graphs....................................................... 293
Development of Functions.................................................. 293
Some Ideas on Functions at the Beginning of the Course...
Images and Definitions of Functions during the Study...... 295
Summary of Nathalie's Development and Knowledge of Functions
   The Case of Griswald
   Introduction
   How Griswald defines a Function (Concept Definition)
   Griswald's Mental Pictures Associated with Function (Concept Image)
   Familiarity
   Vertical Line Test
   Continuity
   Graph
   Equation/Formula
   One Output for every Input
   Summary of Griswald's Function Images
   Procedural and Conceptual Knowledge of Functions
   Composition of Functions
   Global Approach to Functions and Connections
   Settings: Domain, Range and Rule
   Construction of Graphs
   Development of Functions
   Some Ideas on Functions at the Beginning of the Course
   Images and Definitions of Functions during the Study
   Summary of Griswald's Development and Knowledge of Functions

VI DISCUSSION AND RECOMMENDATIONS

Students' Images of Functions during the Study
   Summary of Elizabeth's Case
   Summary of Steve's Case
   Summary of Jane's Case
   Summary of Carol's Case
   Summary of Tyler's Case
   Summary of Sara's Case
   Summary of Nathalie's Case
   Summary of Griswald's Case
   Discussion
   Students' Conceptualization of Functions
   No Consideration of Domain and Range
   Tendency to Regularity
   Separation between the Algebraic and the Graphical World
   Historical Obstacles
   Piecewise Functions
   Arbitrary Functions
   Univalence
   No Consideration of Domain and Range
   Psychological Obstacles
   Pedagogical Obstacles
The Use of Technology in this Class................................. 352
Implications......................................................... 353
Recommendations.................................................. 355
Limitations of the Study.......................................... 356
Directions for further Research................................. 357

APPENDICES
A Observation Form.................................................. 359
B Items proposed to the Teacher.................................. 362
C Practice Test on Functions....................................... 364
D Handout for Extra Credits....................................... 376
E Protocols for Students' Interviews.............................. 378
F Protocols for Teacher's Interviews............................. 385
G Prospectus for Preliminary Research............................ 389
H Students' Demographic and Academic Data.................... 395
I Schedule and Place for Students' Interviews................... 399
J The Content of the Course....................................... 401

REFERENCES................................................................ 390

xii
LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>58</td>
</tr>
<tr>
<td></td>
<td>Chronology of the Study.</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Theoretical framework for the development of functions</td>
<td>10</td>
</tr>
<tr>
<td>in a technology-enhanced environment</td>
<td></td>
</tr>
<tr>
<td>2. The meeting room</td>
<td>62</td>
</tr>
<tr>
<td>3. The math office</td>
<td>62</td>
</tr>
<tr>
<td>4. The math center</td>
<td>63</td>
</tr>
<tr>
<td>5. Mr. H.'s classroom</td>
<td>76</td>
</tr>
<tr>
<td>6. Two items with the image of functions are graphs</td>
<td>87</td>
</tr>
<tr>
<td>7. An item showing that graphs represent functions</td>
<td>88</td>
</tr>
<tr>
<td>8. An item showing that graphs represent functions</td>
<td>88</td>
</tr>
<tr>
<td>9. An item with the image that functions are equations</td>
<td>89</td>
</tr>
<tr>
<td>10. Two items with the image of curve</td>
<td>91</td>
</tr>
<tr>
<td>11. Elizabeth's graph of a straight line</td>
<td>99</td>
</tr>
<tr>
<td>12. The constant function $f(x) = 5$</td>
<td>102</td>
</tr>
<tr>
<td>13. The points $(1, 0), (0, 0), (-2, 3)$</td>
<td>103</td>
</tr>
<tr>
<td>14. The graph of the function that assigns to every number different</td>
<td>108</td>
</tr>
<tr>
<td>from zero its square and to zero it assigns one</td>
<td></td>
</tr>
<tr>
<td>15. Elizabeth's graph of $x$ squared</td>
<td>111</td>
</tr>
<tr>
<td>16. Elizabeth's graph of $x$ cubed</td>
<td>111</td>
</tr>
<tr>
<td>17. Elizabeth's first graph of item $(l_2, 4)$</td>
<td>111</td>
</tr>
<tr>
<td>18. Elizabeth's rearrangement of item $(l_2, 4)$ considering one half</td>
<td>112</td>
</tr>
</tbody>
</table>
19. Elizabeth's rearrangement of item (I2, 4) considering negative one half as input................................................................. 112
20. Elizabeth's final graph in item (I2, 4)........................................ 113
21. Elizabeth's changes to item (I5, 6) to make it the graph of a function..................................................................................... 113
22. Elizabeth's graph of $x^3 + 2x + 1$............................................. 116
23. Elizabeth's graph of $2(x^3 - 2)$............................................... 117
24. The standard cube graph......................................................... 117
25. A zero point of $2(x^3 - 2)$....................................................... 119
26. A zero for the function in (I3, 5).............................................. 120
27. A vertical asymptote for the function in item (I3, 5)............... 120
28. A horizontal asymptote for the function in (I3, 5) passes through $y = \frac{1}{2}$................................................................. 120
29. Elizabeth's graph for the item (Q-02/19)................................. 122
30. Elizabeth's graph for item (I4, 5)............................................. 123
31. Steve's example of a function in definition 4......................... 130
32. Steve's nonexample of a function........................................... 130
33. Steve's example of a function in definition five..................... 131
34. Steve's examples of non-functions......................................... 137
35. Steve's straight line................................................................. 138
36. Steve's straight line with axes............................................... 138
37. Steve's use of the vertical line test in (I2, 5)......................... 139
38. Steve's first approach to (I2, 4)............................................. 142
39. A familiar graph for Steve....................................................... 143
40. Steve's transformation of a familiar parabola in (I2, 4)........... 145
<table>
<thead>
<tr>
<th></th>
<th>Steve's graph for (I2, 4)</th>
<th>145</th>
</tr>
</thead>
<tbody>
<tr>
<td>42</td>
<td>Steve's graph in (I4, 2)</td>
<td>145</td>
</tr>
<tr>
<td>43</td>
<td>Steve's explanation to figure 42</td>
<td>146</td>
</tr>
<tr>
<td>44</td>
<td>Steve's member check to (I4, 2)</td>
<td>147</td>
</tr>
<tr>
<td>45</td>
<td>Steve's first drawing in item (I3, 5)</td>
<td>153</td>
</tr>
<tr>
<td>46</td>
<td>Steve's graphs for problem (I3, 5)</td>
<td>153</td>
</tr>
<tr>
<td>47</td>
<td>Steve's graph of ( f(x) = x^2 ), only for integer numbers</td>
<td>156</td>
</tr>
<tr>
<td>48</td>
<td>Jane's graph of a function with the same outputs</td>
<td>169</td>
</tr>
<tr>
<td>49</td>
<td>Jane's use of the vertical line test to produce examples of functions and nonfunctions</td>
<td>169</td>
</tr>
<tr>
<td>50</td>
<td>Jane's use of the vertical line test</td>
<td>175</td>
</tr>
<tr>
<td>51</td>
<td>Jane's rearrangement of item (I2, 4) considering one half as an input</td>
<td>177</td>
</tr>
<tr>
<td>52</td>
<td>Jane's graph of ( f(x) = x^2 ), only for integer numbers</td>
<td>181</td>
</tr>
<tr>
<td>53</td>
<td>Carol's graph of (I2, 4)</td>
<td>204</td>
</tr>
<tr>
<td>54</td>
<td>Carol's first sketch of item (I3, 5)</td>
<td>208</td>
</tr>
<tr>
<td>55</td>
<td>Carol's graph of item (I3, 5)</td>
<td>209</td>
</tr>
<tr>
<td>56</td>
<td>Tyler's use of the vertical line test to check that a graph is a function</td>
<td>223</td>
</tr>
<tr>
<td>57</td>
<td>Tyler's graph of (I2, 4) for positive numbers</td>
<td>229</td>
</tr>
<tr>
<td>58</td>
<td>Tyler's graph of (I2, 4) for nonzero numbers</td>
<td>229</td>
</tr>
<tr>
<td>59</td>
<td>Tyler's use of the vertical line test to produce graphs of functions</td>
<td>231</td>
</tr>
<tr>
<td>60</td>
<td>Tyler's graph of item (I4, 4)</td>
<td>234</td>
</tr>
<tr>
<td>61</td>
<td>Tyler's graph of ( f(x) = x^2 ) only for integer numbers</td>
<td>235</td>
</tr>
<tr>
<td>62</td>
<td>Tyler's graph of item (I3, 6)</td>
<td>236</td>
</tr>
</tbody>
</table>
63. Tyler's graph of (I4, 2) ................................................................. 237
64. Sara's graph of (I2, 4) ................................................................. 255
65. Sara's modification to the graph in item (I2, 4) after including \( \left( \frac{1}{2}, \frac{1}{4} \right) \) ................................................................. 255
66. Sara's changes to five points to make them the graph of a function in (I5, 6) ................................................................. 256
67. Sara's resulting graph after connecting the points given in order (I5, 7) ................................................................. 256
68. Sara's asymptote of the function in (I3, 5) ................................................................. 260
69. Sara's graph of (I3, 5) including the y-intercept ................................................................. 261
70. Sara's graph of (I3, 5) including the end behavior ................................................................. 261
71. Sara's first graph of the roller coaster problem (I4, 5) ................................................................. 264
72. Sara's final graph for the roller-coaster problem ................................................................. 264
73. Nathalie's graph of a familiar function in (I1, 13) ................................................................. 279
74. Nathalie's graph of item (I2, 4) including \( \left( \frac{1}{2}, \frac{1}{4} \right) \) ................................................................. 280
75. Nathalie's first graph in (I2, 6) ................................................................. 281
76. Nathalie's last graph of item (I2, 6) ................................................................. 281
77. Nathalie's expected graph in (I3, 5) ................................................................. 287
78. Nathalie's graph of (I3, 5) ................................................................. 287
79. Nathalie's example of a function that 'goes down' ................................................................. 288
80. Nathalie's description of the effect of the absolute value on a function that 'goes down' ................................................................. 289
81. Nathalie's expected graph in (I4, 4) ................................................................. 289
82. Nathalie's final graph of (I4, 4) ................................................................. 289
83. Nathalie's graph of \( f(x) = 1 \) if \( x \geq 0 \) ................................................................. 291

xvii
84. Nathalie's graph of item (I4, 2) ................................................................. 291
85. Nathalie's figure for the univalence feature .............................................. 299
86. Griswald's graphs of two functions (I1, I3) ............................................... 303
87. Griswald's first sketch of the rational function in (I3, 5) ......................... 312
88. Griswald's second sketch of the rational function in (I3, 5) ................. 313
89. First possible graph for the rational function (I3, 5) according to Griswald................................................................................. 313
90. Second possible graph for the rational function (I3, 5) according to Griswald................................................................................. 314
91. Griswald's graph of the function that assigns one to nonnegative numbers and negative one to negative numbers ......................... 315
92. Griswald's graph of the Dirichlet's function ............................................. 316
93. Network of Elizabeth's function images .................................................. 329
94. Network of Steve's function images ........................................................ 331
95. Network of Jane's function images .......................................................... 333
96. Network of Carol's function images ......................................................... 335
97. Network of Tyler's function images ......................................................... 337
98. Network of Sara's function images ......................................................... 339
99. Network of Nathalie's function images .................................................... 341
100. Network of Griswald's function images ................................................. 343
CHAPTER I
THE PROBLEM

... functions and graphs are interesting in an instructional sense because they tend to focus on relation as well as entity, and because they are a magnificent tool for examining patterns. It is in this role that the powerful use of the computer is especially salient. (Leinhardt, Zaslavsky & Stein, 1990, p. 3)

The present research was designed to investigate the knowledge and development of the idea of functions among high-school students in a technology-enhanced precalculus class. This chapter describes the current stage of mathematics education related to functions and the use of technology in mathematics classrooms. The next sections include the need for the present study, statement of the problem, the research questions to investigate, and importance of this research.

Introduction

Knowledge is acquired gradually: learners move from the concrete to the abstract, from numbers to symbols, from the specific to the general (National Research Council, 1989). The learning of mathematics is no exception. Students in mathematics classrooms move progressively from numbers to symbols, from symbols to variables, and from variables to functions.

Functions are important inside and outside mathematics. The National Council of Teachers of Mathematics (1989, 1991) calls them a unifying concept in mathematics. Unification is an aspiration to mathematical activity,

with the consequence of a vision directed from the sublime endpoint of the activity to the cloudy sources — a vision that reveals itself mathematically and mathematically-didactically. (Freudenthal, 1983, p. 511)
Freudenthal describes the unifying power of functions.

Functions are all around in mathematics and its applications, albeit labeled in various ways: mapping, transformation, permutation, operation, process, functional, operator, sequence, morphism, functor, automaton, machine, which are used according to needs and opportunities: Function is preferred if the set of values is numerical, mappings and transformations come from geometry but serve as well, with certain attributes added in algebraic structures such as morphisms, prefixed with certain propositions or adjectives, functors, acting on morphisms; permutations is the term for a one-to-one mapping on itself, in particular, if studied in a group theory context; operation or process is the term used with certain simple standard functions (additions, root extraction). If A itself is a set (or space) of functions, then in order to avoid repetitions or ambiguities, the terms functional and operator are used for functions from A to B - the first if B is a set of numbers and the second if it is also a set of functions. The term sequence is usual, if not obligatory, for mappings from N. (p. 497)

The National Research Council (1989) adds that "the language of change and chance is conveyed by the symbolism of functions" (p. 51).

The power of functions is rooted also in their applications for mathematical modelling in several branches of human endeavor. This "mathematical modelling is the determination and use of function rules that appropriately describe the mathematical relationships among two or more relevant variables" (Zbiek & Heid, 1990, p. 32).

The importance of functions in mathematics has also had an impact on the mathematics curriculum. Several documents recognize the usefulness and importance of functions and call for curricular emphases on them (NCTM, 1989, 1991; NRC, 1989). Fey and Good (1985) propose the function concept as the center of school algebra. Some curricula have been designed having the function concept as a baseline (e.g., Usiskin & Senk, 1992). At an upper level, the College Entrance Examination Board (1985) states that

The concept of functions is central to mathematics, and students entering college not only need to understand what functions are in general but also to be familiar with examples. (p. 29)

The critical role of functions in the mathematics curriculum is unquestioned and has demanded the attention of mathematics educators. The importance of functions has generated a growing body of research around them. Researchers have explored students' learning of this concept (Orton, 1970, cited in Lovell, 1971; Sierpinska, 1988; Thomas,
students' misconceptions and difficulties with the concept (Markovits, Eylon & Bruckheimer, 1986, 1988), and the mismatch between the concept image and the concept definition of functions among students (Dreyfus & Vinner, 1989; Ferrini-Mundy & Graham, 1991; Vinner, 1983) and among teachers (Even, 1988, 1990; Even, Lappan & Fitzgerald, 1988).

Some researchers have proposed frameworks to investigate learners' knowledge of functions (Dreyfus & Eisenberg, 1982, 1983; Even, 1990), others have focused on interpretation of function graphs (Bell & Janvier, 1981; Clement, 1985; Ponte, 1985).

A graph, the geometric representation, is not the only possible way to represent a function. The issue of multiple representations is evident with functions, which also can be represented symbolically (algebraic representation), numerically (table representation), and verbally (verbal representation). With the availability of graphing technology, the use of multiple representations is particularly important from a curricular perspective. In general, it is expected that computers, and more recently graphing calculators are the kind of media that might help students to visualize appropriate representations of functions (Goldenberg, 1987). It is conjectured that the capability of graphing technology to display several representations at the same time might have a positive impact on the teaching and learning of functions.

Two issues seem to characterize the use of technology in mathematics education. First, an increasing number of recommendations to use technology in the classroom (NCTM, 1980, 1989, 1991; NRC, 1989), and second, a lack of knowledge of didactical and pedagogical effects (Fischbein, 1990; Sowder, 1989). This lack of knowledge becomes extreme if we focus on specific mathematical concepts. For example, in the case of functions, there is not much research regarding the use of graphing utilities to teach this concept (Dunham, 1990; Leinhardt, Zaslavski & Stein, 1990). This does not mean that there is no research conducted in this area. Indeed, some researchers have begun to build a
growing body of knowledge regarding the effects of using technology in the teaching of functions (Browning, 1988; Dreyfus & Eisenberg, 1987; Goldenberg, 1988; Kent, 1987; Rich, 1991; Vazquez, 1991; Vermilya, 1990; Zehavi, Gonen, Omer & Taizi, 1987; Zehavi, 1988).

Recommendations to use technology in mathematics education are abundant in the literature. However, those recommendations have often been driven by novelty rather than research. In particular, computers and calculators are being used in mathematics classrooms without definitive knowledge of the effects of doing so. Beevers and his associates (1988) point out that we do not know yet how the computer helps human learning. In the same vein, the National Research Council (1989) states "[...] we are not sure how best to teach mathematics with computers" (p. 62). Barret and Goebel (1990) mention that even in classrooms where the computer is available, often the teacher does not have a thorough understanding of the role of the computer in the teaching of mathematics. The National Research Council (1989) admits that the use of computers in teaching is an unfamiliar territory, but society has many things to gain by integrating technology in the classroom.

As a result of recommendations, Fischbein (1990) points out,

[w]e are witnessing nowadays the penetration of computers at every level of instruction without a serious research basis and without systematic attempts to evaluate their psychological and didactical effect [and] the present tendency is to include the computer in teaching programs without a careful consideration of the ensemble of its psychological and didactical effects. (p. 6)

Need for the Study

In spite of the importance of the concept of functions, relatively little research has been conducted concerning the learning of functions. One possible explanation is that research in mathematics education has focused primarily on elementary level mathematics and functions do not appear in the curriculum until upper elementary level mathematics. In addition, research that combines functions and graphing technologies is scarce. At the precalculus level, where functions are the main topic in the course, the area is almost
unexplored. We are starting to add knowledge in this area (e.g., Rich, 1991).

The present study was designed to contribute to our body of knowledge about learning functions. The contribution to mathematics education was designed at two levels. First, by focusing on the precalculus level that is almost unstudied. And second, by investigating an environment where technology is readily available: the Calculator and Computer Precalculus Project (Demana & Waits, 1988). This research investigated and documented knowledge and development of functions among high-school students who use graphing calculators as an integral part of their precalculus class.

Several documents in mathematics education encourage the use of technology for the teaching and learning of mathematics (NCTM, 1980, 1989, 1991; NRC, 1989). As Fischbein (1990) states, there is a need to investigate the effect of using technology in the mathematics classroom. Indeed, the impact of technology is still "new, unexpected, unstudied, and unpredictable" (Sowder, 1989, p. 29). On the other hand, research focusing on the development of mathematical concepts while using technology has been called for (e.g., Shumway, paper submitted for publication); in particular, research related to concepts of advanced mathematics such as functions is needed (Fischbein, 1990). However, in the words of Leinhardt, Zaslavsky, and Stein, (1990)

Little research has been done on the teaching of graphing and functions through technologies. Most of what is reported are criteria for development of software and theoretical considerations. (p. 7)

Statement of the Problem

This research investigated knowledge and development of the concept of functions among high-school students in a technology-enhanced precalculus class. Students in the Calculator and Computer Precalculus Project, C²PC, (Demana & Waits, 1988, 1990) use graphing technology, particularly graphing calculators, as an integral part of their class. A class participating in the C²PC project was the setting for this research. Eight students in the class were selected to conduct case studies of their knowledge and development of
functions. Qualitative methods were used to study the eight students included in this research.

Research Questions

This study investigated the following questions:

1. What are the concept images and the concept definition of functions that students in this technology-enhanced precalculus class have?

2. How do students in a technology-enhanced precalculus class use functions?

3. What is the knowledge of functions that students in a technology-enhanced precalculus class have? Is their knowledge procedural or conceptual?

4. What are the stages students in a technology-enhanced precalculus class go through in their attainment of the concept of functions?

Importance of the Study

Topic, level, milieu, and methodology contribute to the importance of this study. As pointed out by Leinhardt, Zaslavsky, and Stein (1990)

much of the research on mathematics learning and teaching has focused on the very earliest levels of mathematics content. Functions and graphs [...] is a topic that generally does not appear until the upper elementary grades or later. (p. 2)

In precalculus classes in particular, function, a unifying idea in mathematics (NCTM, 1989) is the central topic. Furthermore, the materials and the milieu provided by the C²PC project allow for study in a graphing-facility environment. The case study methodology used in the research will provide some insight into the general by looking at the particular (Eisner, 1990).

Organization of this Dissertation

This chapter introduced the need for the present study, statement of the problem, the research questions to investigate, and importance of this research. Chapter II reviews the literature related to this research. An overview of the chapter is given by presenting the theoretical framework developed for this study. Chapter III includes methodology and procedures used in the study. Chapter IV describes the context (physical, curricular,
pedagogic, and subject-matter) where this study was conducted. Chapter V contains the eight case studies. Chapter VI contains a discussion of the cases based on historical, psychological, and pedagogical obstacles; it concludes with implications, recommendations, limitations, and directions for further research.
CHAPTER II

THEORETICAL FRAMEWORK AND REVIEW OF LITERATURE

The development of the function concept has revolutionized mathematics in much the same way as did the nearly simultaneous rise of non-Euclidean geometry. It has transformed mathematics from a pure natural science—the queen of the sciences—into something vastly larger. It has established mathematics as the basis of all rigorous thinking—the logic of all possible relations. (Boyer, 1946, p. 13)

This research was designed to investigate the knowledge and development of the idea of functions among high-school students in a technology-enhanced precalculus class. This chapter reviews the literature relevant to the study, and as in any qualitative research, the review is "intended to establish the discourse to be addressed" (Noblit & Hare, 1988, p. 24).

The chapter includes four sections. The first section presents a theoretical framework for the research. The framework is based on the three following sections and is presented at the beginning of the chapter as an advanced organizer for the chapter. The second section describes the historical development of functions. The third section presents educational research on the concept of function. The last section is an overview of the use of computer technology in mathematics education. The section concludes with research related to the use of graphing utilities in the teaching and learning of functions. The chapter concludes with a summary in which the research questions of this study are revisited.

Other researchers have reviewed some areas related to functions and covered some topics relevant to this study. The second and third sections included in the review fall under this situation; they are based on former reviews and are not exhaustive.
A Theoretical Framework for the Research

At the outset of this research, there are at least two concurrent avenues of studying the knowledge and development of mathematical functions in a technology-enhanced precalculus class. On one side we have research on technology, and on the other side we have research on mathematical functions. Two components appear specifically related to mathematical functions: (1) historical development; and (2) concept image and concept definition. The historical development is what Sfard (1987) calls the "historical analogy" with the learning of mathematical concepts. Concept images and concept definitions are related to the kind of experiences that students have with a mathematical concept. The concept image develops with stimuli, experience, and maturation. History and learning of functions merge in research on conceptualization of functions as processes and objects (Sfard, 1991). Related to knowledge, a discussion of conceptual and procedural knowledge with special reference to functions is included in reviewing the literature on functions.

The third component—multiple representations—gives a possible intersection of technology and functions. First, mathematical functions can be represented algebraically, numerically, graphically, and verbally. Second, with the availability of graphing technologies, it is natural to think that the features of multiple representations are more salient (Goldenberg, 1987). Two mathematical representations are the main focus of the study: algebraic and graphical representations because

[they] are two very different symbol systems that articulate in such a way as to jointly construct and define the mathematical concept of function. Neither functions nor graphs can be treated as isolated concepts. (Leinhardt, Zaslavsky & Stein, 1990, p. 3)

The other two representations—verbal and numerical—are available although they were less emphasized by the teacher in this study (teacher's first interview).

Figure 1 shows the theoretical framework developed for the present study. I start by dealing with the historical development of functions. The survey is based on Boyer (1946), Kieiner (1989), Madrid de la Vega et al. (1981), Malik (1980), and Vilenkin (1968).
A Short Survey on the Historical Development of Functions

Mathematics is a cultural phenomenon and is subject to cultural forces in addition to inner forces in its own nature (Wilder, 1968). Several mathematical concepts and their use in mathematics have gone through numerous historical changes. Mathematical concepts emerged as generalizations of intuitive ideas and with gradual changes they led to formal
mathematical definitions. Sometimes the definitions included objects that were not originally intended. Mathematicians then studied the new objects and definitions were carried to a higher level of abstraction.

Implicit recognition and use of functions can be traced back to about 2000 B. C. Boyer (1946) mentions that "the use of interpolation in antiquity was an application of an elementary functional relationship, a simple proportionality" (p. 5). However, the function concept did not emerge explicitly until the 18th century. Kleiner (1989) mentions two main reasons why this did not happen earlier: insufficient symbolic notation, and the lack of a sufficient number of examples to motivate an abstract concept of function. Several developments in the period 1450-1650 such as the extension of the number concept to include real (and to some extent complex) numbers, the marriage of algebra and geometry, the introduction of motion as a central problem in science, and the invention of symbolic algebra are mentioned by Kleiner as fundamental for the emergence of functions. The invention of analytic geometry and the emergence in the 17th century of science as mathematized suggested a "dynamic, continuous view of the functional relationship as against the static, discrete, view held by the ancients" (Kleiner, 1989, p. 283). Leibniz in 1692 was the first to use the word 'function'. He and Jean Bernoulli used this word, however, to designate geometric objects that were associated with curves (tangent, ordinates, and radii of curvature), or in specific instances of functions such as power or trigonometric functions.

Four main historical stages are perceived in the development of the concept of function. Prominent mathematicians are associated with each stage. The first stage is that of an equation or a formula and it is associated with Euler (b. 1707, d. 1783), who considered functions to be graphable, with no corners. For him "a function of a variable quantity [was] an analytical expression composed in any manner whatever of this variable and constants" (Boyer, 1946, p. 12). This definition, however, did not emphasize the dependence
relationship, but the algebraic representation by an equation. For Euler, the 'analytical expression' was one of the possible manifestations of an independent abstract entity, while a curve was the other one (Sfard, 1989). Under these circumstances, an expression (Madrid de la Vega et al., 1981, p. 19) such as

\[ y = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases} \]

defined not one, but two functions. Madrid de la Vega and colleagues (1981) point out that Euler recognized as a function \( f(x) \) any curve drawn free in a \( x\)-\( y \) coordinate plane. With the theory of higher plane curves of Newton, Cramer, and Euler himself, Euler "saw that any curve drawn free hand in a plane determines a functional relationship which may not be representable, either implicitly or explicitly, in ordinary analytical form" (Boyer, 1946, p. 12). This situation implied the inadequacy of equations to express the concept of functions in the most general possible way and set the arena for the next historical stage.

It was a common characteristic among mathematicians around this time to seek regularity and an order represented by formulas and equations. This situation can be recognized in the solution to the string problem. A problem solved in 'different' ways by D. Bernoulli in 1750 and D'Alembert. The real issue, however, was the connection between function and functional dependency and the possibility to express such dependency by a formula. J. Fourier in contrast to mathematicians of this time worked with badly discontinuous functions and solved the controversy between Bernoulli and D'Alambert by showing that the sum of an infinite series of trigonometric functions can be expressed by different formulas in different intervals. Fourier (b. 1768, d. 1830) is associated with the second stage in the historical development of functions. The need to introduce a new definition for a function was based on his work on heat conduction. He introduced a definition in 1812 where the stress was on the assignment of values for the function, and pointed out that it did not matter if the assignment was given by one or more formulas.
Fourier's work led to the next stage represented by a student of his, Dirichlet, who formalized Fourier's results and gave the conditions for a function on a fixed interval to be represented by a Fourier series. By 1837, this led him to provide a modern definition of functions as "y is a function of x, for a given domain of values of x, whenever a precise law of correspondence between x and y can be stated clearly" (Boyer, 1946, p. 13). Dirichlet (b. 1805, d. 1859) stressed the importance of the unique assignment of y to every x. In this case, no formula over the whole domain of definition was required. This definition was very general and allowed for definitions of functions by words! Dirichlet introduced the well-known example of a function everywhere discontinuous:

\[ f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{if } x \text{ is not a rational number} \end{cases} \]

This was also an example of a function that does not satisfy the conditions he found for the Fourier series to converge to it. With Dirichlet's definition, mathematicians delighted with functions, and started constructing examples with unexpected properties. This led the abstraction to a higher level; with Dirichlet, functions were limited to "physical intuition and after him, the concept became more abstract, mathematically speaking" (De la Madrid et al., 1981, p. 24, translation by this researcher). Analysis emerged, and with the introduction of metric spaces and topology, mathematicians noticed that properties of functions are subject to the structures of the sets on which they are defined and where they take their values. Thus the concepts of domain and range emerged. The last stage is associated to Bourbaki\(^1\) in 1939 and is characterized by the arbitrariness of the domain and the range.

Bourbaki gave a general formulation of function as a rule of correspondence between the domain and the range, where both are allowed to be arbitrary sets. The notion of a mathematical function in this formulation merges with the concept of correspondence, mapping, and transformation (Vilenkin, 1968). Bourbaki later observed that a function "is a

\(^1\) Nicolas Bourbaki is a pseudonym for a group that has influenced formalism in mathematics.
subset of the Cartesian product of sets" (Malik, 1980, p. 491).

The historical evolution of the concept of function shows three mental ideas associated with functions: the geometric (as a curve), the algebraic (as a formula or equation), and the logical (as a correspondence). Kleiner (1989) contends that the evolution of the function concept can be seen as two "tugs of war" between pairs of these three mental ideas. The first tug of war, between the geometric and the algebraic image, is decided in favor of the algebraic representation. The second tug of war, between the algebraic - and the logical image, is decided in favor of the logical representation.

The development of mathematics has led functions to a higher abstraction. However, the learning of functions still involves "tug of wars." How and to what extent the tugs of war may be influenced with the capability of graphing technology to display several representation of functions is an important question which must be considered. In particular, Will the use of graphing technology for the teaching and learning of functions introduce a new "tug of war"?

Dreyfus (1990) mentions that

the teaching of the function concept has undergone a development analogous to the historical one; the progressive abstraction of the concept in mathematics has found its reflection in the school curriculum. (p. 120)

This thought finds support in two surveys of elementary algebra and college algebra textbooks (Kennedy & Ragan, 1969) and high-school textbooks (Cooney & Wilson, in press). Kennedy and Ragan found that before 1959, functions were defined as rules or correspondences between variables; in contrast the majority of books which were published after 1959 included definitions of functions in terms of sets of ordered pairs. Cooney and Wilson, who surveyed 16 books published between 1958 and 1986, found that the definitions of functions relied on sets, either as sets of ordered pairs or as correspondences between elements of the sets on which the function was defined. The importance of the concept function in mathematics and its role in the mathematics curriculum has attracted the
attention of mathematics educators.

**Research on the Concept of Functions**

We turn now to research on functions in mathematics education. This section is composed of three parts: (1) conceptual and procedural knowledge in mathematics with particular reference to functions, (2) students' development of functions, and (3) cognitive obstacles; the latter sets the arena for the discussion of concept images and concept definitions of functions. The last two parts explore difficulties and misconceptions that students have with the function concept.

**Procedural and Conceptual Knowledge**

Discussions in mathematics education around procedural and conceptual knowledge are not new. These discussions extend to other fields, since they relate to more general questions concerning the acquisition of knowledge. Historically the discussion has taken form with different labels. Maybe the most prominent has been the distinction between skill and understanding.

Hiebert and Lefebvre (1986) mention three differences between current discussions of procedural and conceptual knowledge and former discussions of understanding and skill. First, past discussions have been around which one (understanding or skill) or both should be stressed. Second, not so much interest was paid to investigating the relationships between them; currently the emphasis is on the links between the two. Third, current discussions include non school and informal settings, while in former discussions the scope was on school mathematics. Regardless of labels, the importance of the discussions is in their helpfulness to provide insight into mathematics learning and performance.

Hiebert and Lefebvre intend to define conceptual and procedural knowledge recognizing that the definition is not a classification scheme in which all knowledge can be included. They admit that there is not a clear demarcation between the two types of knowledge. Their distinction, however, provides an arena for interpreting students' learning
processes. They characterize conceptual knowledge as knowledge that is rich in relationships; its development is promoted by the establishment of relationships of pieces of information. This can occur in two ways, either by connecting existing pieces of knowledge or by linking new information to already existing knowledge. Procedural knowledge consists of two parts. The first one is the mathematical formal language, the form of mathematics; it involves awareness of symbols and syntax. The second part consists of rules, algorithms, or procedures used to solve mathematical tasks. Two features of procedural knowledge are distinctive from conceptual knowledge. Procedural knowledge is linearly sequential and structured; in contrast, conceptual knowledge is a complex network with multiple types of relationships.

*Knowledge of Functions*

Two studies are presented in this part. The first study described (Even, 1990) refers to procedural and conceptual knowledge, while the second study (Markovits, Eylon & Bruckheimer, 1986, 1988) refers to understanding.

Even (1990), using mathematical functions, illustrates the use of a framework to investigate teacher's subject matter knowledge about a specific mathematical topic. Her framework consists of seven aspects which are: (1) essential features, (2) different representations, (3) alternative ways of approaching, (4) the strength of the concept, (5) basic repertoire, (6) knowledge and understanding of the concept, and (7) knowledge about mathematics.

Univalence and arbitrariness are the essential features of functions. Univalence is assigning one and only one element of the codomain to each element of the domain. Arbitrariness refers to the nature of domain, codomain and rule of correspondence.

The applicability of functions is based on different representations which in turn permit establish connections in mathematics and promote rich relationships in the learner's mind. Since understanding functions in a specific representation does not mean understanding
them in a different representation, it is necessary to be familiar with multiple representations and be able to establish linkages (translations) between them. Dealing with different representations of functions might facilitate the abstraction of the concept by focusing on essential characteristics common to the different representations and ignoring specific properties of each representation (Dufour-Janvier, Bednarz & Belanger, 1987; Lesh, Post & Behr, 1987).

The third aspect, alternative ways of approaching, is closely related to different representations and translations between them. In the case of functions, two alternative approaches are: a point-wise approach and a global approach. The first approach refers to plotting points, or reading them. The second approach refers to an examination of the function over an interval or in a global way by attending to features such as extrema, points of discontinuity, asymptotes, x- and y-intercepts, domain, range among others.

The strength of a concept, the fourth aspect, refers to the new fields and opportunities that the concept leads to. The strength of functions is grounded in two new possibilities: compositions and inverses. Even's fifth aspect, a basic repertoire of functions, includes knowledge of functions included in the curriculum and structure of the sets on which these functions are defined. Knowledge and understanding of functions, the sixth aspect, refers to procedural and conceptual knowledge of functions and relationships between these two. The last aspect, knowledge about mathematics refers to knowledge of activities inherent in mathematical activity such as giving examples, conjecturing, finding non examples, refining, generalizing, and proving.

Markovits, Eylon, and Bruckheimer (1986, 1988), while investigating difficulties that junior-high school students have with the concept of function, identified four components in the understanding of functions. The components are: (1) the ability to classify relations into functions and non functions, and the ability to give examples of relations that are functions and nonfunctions, (2) the ability to identify images, preimages and pairs of these two for a
given function, and the ability to find the image of a given preimage and vice versa, (3) the ability to make translations between different representations of a function, and the ability to recognize equal functions, and (4) the ability to identify and give examples of functions satisfying given constraints. Two stages can be recognized in each component: the passive (classifying, identifying, etc.), and the active (doing something, giving examples, etc.) Their work has led them to develop items based on these components. A short version of a test they developed appears in their 1988 paper.

**Students' Development of Functions**

Thomas (1971, 1975), and Orton (1970; cited in Loveil, 1971) hypothesized that students learn the concept of function in stages. Thomas worked with above average students, eleven - through fourteen-years-old. He used a written test (N = 201) and individual interviews (N = 20). The test helped to identify the stages and the interviews to refine them. The characterization of the stages was based on three components found in the written test and the interviews: concept identification, process, and operations. Concept identification is the ability to identify examples and nonexamples of functions. Process is the ability to work with different representations of functions to find images, domain, range, preimages, and sets of images. Operations is the ability to carry out operations with functions with an indication that the result of such operation is again a function. Keeping these components in consideration, four stages were proposed. The first level is concrete-intuitive; the learner sees a function as a process. The next level is preconceptual; the learner recognizes a function in different settings and can perform translations between these settings; however, the learner does not understand the basic criteria for a relation to be a function. The third level is conceptual—a combination of stages I and II; a basic characteristic is providing adequate criteria for the discrimination of relations into functions and nonfunctions. At the highest level, the learner sees a function as an entity with properties and with which operations, such as compositions or transformations, can be
carried out.

Although the verification of the stages was only partially successful, Thomas identified students' difficulties in the learning of the function concept. These difficulties are the junctures between the stages: (1) making links between graphical and algebraic settings and (2) making the transition of functions from a process to an object.

Orton used a similar methodology based only on individual testing. He worked with seventy secondary students ranging from twelve to seventeen year-olds. His results were similar to Thomas'. Some of the students' difficulties that he identified were: an apparent need for a pattern in graphs, confusions between the meaning of many-to-one and one-to-many, and an inability to interpret a graph.

Neither Thomas nor Orton dealt with the concept of function in a broad sense (e.g., pattern discovery or relationship between phenomena); instead they focused on the learning of a specific and formal definition of function as presented by a specific curriculum. Several situations limit generalizations of these studies, for example, the populations and the sample sizes used.

A historical analogy (Sfard, 1987) between the learning of the concept function and the historical development of the concept was found by Sierpinska (1988). She studied three groups of fifteen-through seventeen-year-olds students. Her methodology was based on "didactical situations" (p. 569), which include social context, kinds of teacher interventions, and mathematical context among other features. Her frame is based on Boyer's (1946) work and has at least six stages. These stages are:

I  An implicit idea of transformation (T) of points or relationships between magnitudes.
II T described by numerical tables.
III T described by proportions.
IV T described by equations.
V T described by graphs and equations.
VI An elaborate explicit idea of relationship between variables.
Sierpinska concluded that students' conceptions of functions can be composed of several degenerate forms of historical stages II through V. She also observed that these forms can work in parallel, without a conscious link. Two categories are used to describe students' conception of functions: concrete and abstract.

The concrete category includes three conceptions: mechanical, synthetic geometrical, and algebraic. In the mechanical conception, "a function is a displacement of points" (p. 570). This conception corresponds to the historical stage I in non verbalized versions. In the synthetic-geometrical conception, a function is a geometrical object, a concrete curve. In the algebraic conception, a function is a formula; a string of symbols, variables, and constants separated by an "=" sign.

The abstract category contains four conceptions: numerical, algebraic, analytic geometrical, and physical. The numerical conception resembles the historical stage II though it might be vague or implicit in students' mind; a "function is a transformation of some things into other things" (p. 570). In the algebraic conception "a function is an equation or an algebraic expression containing variables" (p. 571). This conception is a degeneration of the historical stage IV, since the idea of the relationship between the variables given by the equation is not present (stage IV without stage I). In the analytic geometrical conception, a function is 'an abstract curve' that represents a relation. This conception is a degeneration of the historical stage V, since the curve is the function and not the relation. The physical conception, although not observed in Sierpinska's students, is close to the historical stage VI. In it, a function is a relationship between variables (some independent and others dependent). The relationship can be represented by graphs.

Sierpinska concluded that the extensive use of graphical representations in the mathematical context she used favored students' conception of functions as a curve (analytic-geometrical conception). Furthermore, the same context appeared to create obstacles to the development for conceptualizing functions as "an elaborate, explicit idea of
relationship between variables" (historical stage VI).

Mathematical knowledge is complex since some of its entities (e.g., numbers, variables, functions) can be conceptualized by the learner as object or as processes. A mathematical function can be seen as a process that associates numbers in the domain with numbers of the codomain; an algorithm to compute a number by means of other numbers. However, as the learner meets higher mathematics, say differentiation, perceiving a function as a process is insufficient to deal with mathematical tasks. It is necessary to perceive the function as an object to be operated upon. This transition from perceiving a function as a process first and then as an object is observed also in the historical evolution of functions and in Thomas' and Orton's research. Sfard's studies (1989, 1991) help to make the distinction between functions as processes and functions as objects. Conceiving a function as a process is an operational conception, while conceiving a function as an object is a structural conception.

Sfard argues that the operational conception is the first one to develop and out of it the structural conception evolves. Her work suggests that this transition is composed of three stages: interiorization, condensation, and reification. Interiorization is the stage when the learner gets familiar with processes that will lead to a new concept (e.g., algebraic manipulation in the case of functions). "In the case of functions, it is when the idea of variable is learned and the ability of using a formula to find values of the "dependent" variable is acquired" (Sfard, 1991, p. 19).

Condensation is a "squeezing" period. In this stage, the learner is able to perceive a process as a whole, without having the need to go into details. Combinations of processes, comparisons, and generalizations are some of the benefits of condensation. If the learner becomes more capable at perceiving a mapping as a whole, this person should be regarded at a more advanced status in the condensation process. "Eventually, the learner can investigate functions, draw their graphs, combine couples of functions (e.g., by composition), even to find the inverse of a given function" (Sfard, 1991. p. 19). A person
should be regarded at the condensation level as long as the concept is tied to some processes. When the concept is perceived as a "fully-fledged object," it is said that the concept has been reified.

*Reification*, therefore, is defined as an ontological shift—a sudden ability to see something familiar in a new light. Thus, whereas interiorization and condensation are gradual, quantitative rather than qualitative changes, reification is an instantaneous quantum leap: a process solidifies into object, into a static structure. (Sfard, pp. 20-21)

At this stage, processes can be applied to the object; the object is the input in the operation and other new mathematical objects can be obtained this way. The stage of reification becomes now the interiorization stage for higher-level concepts. For functions, Sfard (1991) writes, that reification

may be evidenced by proficiency in solving equations in which "unknowns" are functions (differential and functional equations, equations with parameters), by ability to talk about general properties of different processes performed on functions (such as composition or inversion), and by ultimate recognition that computability is not a necessary characteristic of the sets of ordered pairs which are to be regarded as functions. (p. 20)

Although Sfard’s framework implies a hierarchy, it is possible for the learner to take some side routes. A possibility is to identify the concept at hand with one of its representations (a graph or an equation in the case of functions) as a consequence of being unable to come to terms with the invisible objects in the reification stage. This deviation, which can be temporary or permanent, is a quasi-structural conception.

Sfard (1987) concluded two necessary conditions for reification. First, since the structural approach evolves from the operational approach, the last approach should precede the former approach in the classroom. Second, the process of reification becomes necessary when the operational approach proves to be inadequate and insufficient. In other words, there is no need to perceive a process as an object, unless we have higher-level processes to perform upon the new object. She also concluded that the operational approach promotes reification. However, reification is so intricate and difficult that for
some students the structural approach will be unreachable no matter what teaching strategy
is used.

Breidenbach, Dubinsky, Hawks, and Nichols (1992) consider three ways of thinking of
functions: prefunctions, action and processes. The action and process conceptions are
analogous to the operational and structural conceptions in Sfard's work. A prefunction
notion is "one in which it appears that the student does not have very much of a function
concept at all" (p. 252). An action conception is static, "in that the subject will tend to think
about it one step at a time" (p. 251). According to Breidenbach and colleagues, a process
conception

involves a dynamic transformation of objects according to some repeatable means that,
given the same original object, will always produce the same transformed object. The
subject is able to think about the transformation as a complete activity beginning with
objects of some kind, doing something to these objects, and obtaining new objects as a
result of what was done. (p. 251)

Their work suggests that the use of computers in an instructional program improves the
process conception of functions.

The concept of function is complex, since it entails other subconcepts such as image,
rules, and domains; it can be represented in different settings (graphs, table, rules); and the
possibility of using it to relate apparently unrelated subjects (e.g., algebra and geometry)
requires different levels of abstraction that are connected with its structure (Dreyfus &
Eisenberg, 1982). In addition, other situations such as the experience with the concept or
the learner's tendency to overgeneralize instances of the concept might add difficulties to the
learning of functions. It is not surprising that students have difficulties in learning
functions, difficulties that mathematicians also dealt with throughout the historical
development of the concept.

Epistemological and Cognitive Obstacles

This subsection deals with an epistemological perspective. The notion of obstacle is
introduced, a notion that encompasses both the ideas of concept image and concept
Bachelard (1983, cited in Herscovics, 1989; Sierpinska, 1988; and Tall, 1989b) introduced the concept of epistemological obstacle in terms of the historical development of science. Herscovics (1989, p. 61) translated the following paragraph from Bachelard (p. 13).

When one looks for the psychological conditions of scientific progress, one is soon convinced that it is in terms of obstacles that the problem of scientific knowledge must be raised. The question here is not that of considering external obstacles, such as the complexity and transience of phenomena, or to incriminate the weakness of the senses and of the human spirit; it is in the very act of knowing, intimately, that sluggishness and confusion occur by a kind of functional necessity. It is here that we will point out causes of stagnation and even regression; it is there that we will reveal causes of inertia which we will call epistemological obstacles [italics added].

The learning counterpart of an epistemological obstacle is the concept of cognitive obstacle. A cognitive obstacle is a piece of knowledge that is useful to the learner for dealing with some problems; therefore, it becomes anchored in the learner's mind. However, that piece of knowledge might be inadequate or difficult to adapt when the learner is faced with novel problems (Tall, 1989b). Epistemological obstacles were introduced in the context of scientific development and are considered natural, inherent, and scattered in the development of science. Herscovics (1989) contends that in a similar way, the acquisition of new conceptual schemata by the learner is strewn with cognitive obstacles [that should] be considered normal and inherent to the learner's construction of knowledge. (p. 61)

Mathematics educators such as Dreyfus, Tall, and Vinner have taken this idea into the theory of concept image and concept definition. The notions of obstacle and concept image "complement each other in the explanation of phenomena occurring during the learning process" (Dreyfus, 1990, p. 117).

*Concept Image and Concept Definition of Functions*

The kind and amount of experience that students have with mathematical concepts determine the ideas associated with those concepts. It is possible that students are
introduced to formal definitions, but fail to use them in deciding whether or not an object is an example of a given concept. Instead they use the ideas developed during the experience with the concept. The notions of concept image and concept definition (Dreyfus & Vinner, 1989; Tall, 1985, 1989a, 1989b; Tall, & Vinner, 1981; Vinner, 1983) provide a framework to discuss this situation. Vinner (1975) examined the idea of mental imagery in abstract mathematical thinking and introduced the idea of mental image as "the set of all pictures denoted" (p. 339) by a noun in a person's mind. A mental picture is "any kind of representation—picture, symbolic form, diagram, graphs, etc." (Dreyfus & Vinner, 1989).

Tall and Vinner (1981) write that a concept image describe[s] the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. [T]he concept definition is a form of words used to specify [a] concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole. It may also be a personal reconstruction by the student of a definition. It is then the form of words that the student uses for his own explanation of his [...] concept image. (p. 152)

The concept image of a concept develops with stimuli, experience, and maturation. It does not need to be coherent at all times. Some parts of a concept image may conflict with other parts in the concept image or with the concept definition. Mental images associated with a mathematical concept "may contain the seeds of future conflict" (Tall & Vinner, 1981, p. 152), even in the learning of a formal theory.

Up to now, the notions of concept image and concept definition have been applied mainly to concepts of calculus (e.g., Ferrini-Mundy & Graham, 1991; Tall, 1985; Tall & Vinner, 1981; Vinner, 1983), but not exclusively as in the case of learning geometry (Hershkowitz, 1990). Studies focusing on functions seem to be distinctive, either because functions are a concept of advanced mathematical thinking (Dreyfus, 1990; Tall, 1992), or because their understanding is needed to build other mathematical concepts that will appear in later courses (Buck, 1970). Both students and teachers are the populations used in these studies.
Vinner (1983) administered a questionnaire to 10th and 11th graders. Findings showed that about 60% of the students surveyed had a formal definition of functions but only 20% acted according to it. Findings of students relying on their concept images instead of their concept definition and the limited range of images associated with functions have been supported by other researchers. Tall and Vinner (1981) administered a questionnaire to investigate the concept image of continuity that students arriving at a university have. Data showed that most of the images in students responses referred to the graphical representation, and that either the graph did not have gaps or it was "all in one piece" (p. 167). Dreyfus & Vinner (1989) compared the concept image and concept definition that students entering college have before taking their calculus course. Students were split into four groups according to the number of courses in mathematics they had taken; a fifth group was composed of junior-high-school teachers. Findings showed that concept images and concept definitions of functions were 'primitive' among all students except those majoring in physics or mathematics, and among the junior-high-school teachers. In contrast, using a questionnaire Even (1988, 1989) and Even, Lappan, and Fitzgerald (1988) found that a sample of prospective secondary teachers had narrow ideas of functions. Their concept definition of functions was formal, while their concept image was mainly that of an equation. In addition, these teachers had difficulties making connections between the graphic representation of a quadratic function and the corresponding equation in standard form.

Common to the research cited above is the use of questionnaires to collect data. A different approach was used by Ferrini-Mundy and Graham (1991). They used qualitative methods to investigate the understanding of calculus concepts in college students. Six students were selected from a traditional calculus course, "taught in a large lecture format, without the use of technology" (p. 4).
Ferrini-Mundy and Graham describe the understanding of functions of one student, Sandy. They found that Sandy associated three images with the concept: algebraic formula, familiarity, and continuity. Sandy's decision for a graph to represent a function was frequently based on the existence of an equation that would represent the given graph. This image also emerged when Sandy was asked to provide examples of functions. Her response was "...like a division one, or using a quotient, or like multiplying two numbers?" (p. 7).

The image of familiarity refers to Sandy's recognition of functions based on her experience of having graphed or seen them before. The image of continuity was a tendency to connect the dots, or to perceive discrete graphs as 'separated dots'. Graphically, Sandy perceived a discontinuous graph (with one jump) as two functions. Data also suggested that Sandy had a pointwise approach to graphs.

Ferrini-Mundy and Graham concluded about knowledge and understanding of calculus concepts that:

graphical contexts and algebraic contexts may function for students as separate worlds, [...] competing, conflicting conceptions and conclusions are held quite comfortably and routinely in the development of calculus concepts, [processes to construct these concepts] seem to be influenced strongly by previous experience and knowledge. There are powerful tendencies to call upon familiar examples and frequently-used patterns...Startling inconsistencies exist between performance, particularly on procedural items, and conceptual understanding [and] students build a conception of mathematics as a result of experiencing instruction. (pp. 17-20)

Research on images and definitions of functions illuminates some of the potential conflicts that students might have in their transition to advanced mathematics, namely calculus. Students show a need for an algebraic representation of functions, a potential obstacle for the learning of differentiation and integration. Findings show that teachers constitute another side of the problem (Even, 1988, 1989; Even, Lappan, and Fitzgerald, 1988). Teachers also exhibit the need for an algebraic representation, and may create a cycle of obstacles since they
may give the formal definition and work with the general notion for a short while before spending long periods in which all examples are given by formulae. In such a case the concept image may develop into a more restricted notion, only involving formulae, whilst the concept definition is largely inactive in the cognitive structure [hence] the teaching programme itself has been responsible for this unhappy situation. (Tall & Vinner, 1981, p. 153)

The idea of cognitive obstacles is important to mathematics educators since cognitive obstacles seem to be the cause of or at least explain different difficulties students have with mathematical concepts. Dreyfus (1990), Herscovics (1989), and Leinhardt, Zaslavsky, and Stein (1990) have reviewed several contributions to the literature on functions. These reviewers present difficulties, misconceptions, and obstacles that students have with the function concept.

Two of the four areas identified as sources of difficulties in 'graphs of equations in two variables' are important to the present study: linear equations and their graphs, and interpretation of graphs. In the first case, Herscovics (1989) cites how horizontal and vertical line graphs are more difficult for students since one of the variables is missing. Lines whose y-intercept is not at the origin turn out to be more difficult to produce than lines whose y-intercept is at the origin. Interpretation of graphs has been studied in work by Janvier (1981), Bell and Janvier, (1981), and Ponte (1985). Two difficulties can be mentioned in this area. First, students have a point-by-point approach to graphs instead of a global - or interval approach. This situation may be due to the didactical situation; before calculus few students are asked to approach the graph from a global perspective, reading off intervals where the graph is increasing, decreasing, finding extrema, and so forth. The second difficulty refers to students interpreting a graph based on visual considerations or experience with the problem at hand.

Related to functions, Herscovics cites how functional notation, because it condenses so much, can create confusion for algebra students. A second source of obstacles in this area is caused by intuitions. He recognizes that "little research has been specifically oriented
towards identifying the cognitive obstacles involved at the intuitive level of the function concept" (p. 77). Wagner (1981) found that algebra students have difficulties with the arbitrary nature of variables, a concept on which the notion of function is based. Results by Dreyfus and Eisenberg (1982) suggest that students need extensive experience with tabular representations of functions before dealing with graphical representation. The last source of difficulties cited by Herscovics is the mismatch between the concept image and the concept definition of functions.

I turn now to the review of Leinhardt, Zaslavsky, and Stein (1990). Their review looks at graphs, graphing, and functions. Their work is made up of three sections: task, students learning, and teaching. I refer only to the students' learning section which is presented in "two main parts that represent two aspects of student learning: (a) intuitions, and (b) misconceptions and difficulties" (p. 24). The authors define intuitions as features of student's knowledge that arise largely from everyday experience; although in the more advanced student they may involve a mixture of everyday and deeply understood formal message. (p. 24)

Further, these authors point out that

[The theoretical discussion on intuitions emerges in debates on how best to introduce functions into the curriculum. Often, those debates center on which definition of a function (the modern or the older definition) should be used. (p. 27)]

The most widely used definition is the set definition that is contrary to students' intuitions which include "notions about dependence, causality and variation" (p. 28). In their discussion on intuitions about graphs they cite works mentioned by Herscovics and conclude that studies in this area seem to support empirically the notion that intuitions that are based on students' knowledge of real-world situations operate successfully when reasoning in the graphing domain. (p. 29)

Misconceptions "are defined as incorrect features of student knowledge that are repeatable and explicit" (p. 30). Leinhardt and colleagues discuss eight categories of misconceptions. Since three have been already presented, only five remain to be
summarized: (1) correspondence; (2) linearity; (3) continuous versus discrete graphs; (4) representations of functions; and (5) relative reading and interpretation.

"Correspondence" deals with two beliefs: first, that a function is a one-to-one correspondence and second, confusion between one-to-many and many-to-one correspondence. "Linearity" is the tendency in students to gravitate to linearity in several situations, including interpolation which is linear as taught in school. Two sources can be mentioned for this situation: the extensive work with linear graphs and the habit of connecting the dots. "Continuous versus discrete graphs" illustrates difficulties in deciding how to represent a graph. The review mentions students representing continuous graphs discretely and representing discrete graphs continuously. "Representations of functions" makes the point that moving from the graphic representation into the symbolic representation is more difficult than the other way around when students use familiar graphs. However, the degree of difficulty for translations in both directions does not decrease using less familiar graphs. Two of the three themes in "relative reading and interpretation" repeat Herscovics' discussion of interpretation of graphs. The third theme, "slope/height confusion," refers to the confusion of students between gradient and the extrema points (minimum or maximum value). Leinhardt and her colleagues add the tendency in students "to view literal symbols as representing sets, that is the letters were used merely as labels identifying sets" (p. 43).

Markovits, Eylon, and Bruckheimer (1986, 1988) add two more observations of student difficulties: first, a lack of awareness for the domain and the range of a function, which may be correlated with the difficulties experienced with functions defined on discrete domains, and second a limited repertoire of functions, more especially in the algebraic representation than in the graphic representation. Dreyfus' (1990) review is organized around three themes already discussed: concept image and concept definition, visualization, and process and object.
Summary of "Epistemological and Cognitive Obstacles"

Two complementary ideas in the learning process were introduced: cognitive obstacles and concept image and concept definition. Students' errors indicate the existence of cognitive obstacles. Misperceptions that students have are due to the natural tendency to overgeneralize their experience with initial examples used in the introduction of the function concept. Although the overgeneralizations occur naturally, they can become hurdles in the construction of the more global notion of function (Herscovics, 1989). Four broad problem areas in the learning of functions emerge: an isolation from consideration of domain and range; a desire for regularity; a pointwise approach; and a separation between the graphic and the algebraic world. Students' images of, and misconceptions and difficulties with functions may be either a reflection of their operational conception of functions, a quasi-structural conception, or their transition to the structural conception.

Overview of Computer Technology in Mathematics Education

Based on the assumption that computers and related technology can reshape the teaching and learning of mathematics, the use of these technologies in mathematics classrooms at all levels has been strongly recommended. Specific recommendations to use computers and calculators at all levels appear in several documents (NCTM, 1980, 1989, 1991; NRC, 1989). Technology entered the mathematics classroom as a response to earlier recommendations. Shumway (1989a) synthesizes the use of computers in mathematics education

Historically, during the period 1965-1988, computer use in mathematics education began with an emphasis on computations and programming, moved to drill and practice, teacher utility, information management, and tutorial uses, and today seems to focus on computations, graphics, simulations, concept learning, and problem solving. (p. 15)

Previous recommendations apparently have been based on experts' opinions instead of empirical research (Day, 1987); recommendations for the 1990's are no different. In 1989, in the preface to the Standards, the National Council of Teachers of Mathematics referred to
the document as "designed to establish a broad framework to guide reform in school mathematics in the next decade" (p. v). The Standards recommends that the power of calculators and computers needs to be used in mathematics programs. However, when the draft of the Standards was released, the Research Advisory Committee of the National Council of Teachers of Mathematics (1988) argued that

The Standards document contains many recommendations but in general it does not provide a research context for the recommendations, even when such a context is available. (p. 339)

Recommendations to use technology in mathematics education have been driven by novelty and beliefs, rather than research. In particular, computers are being used in school mathematics without knowing definitive answers to questions related to their effectiveness and utility. Beevers and his associates (1988) point out that we do not know yet how the computer helps humans in learning. In the same vein, the National Research Council (1989) agrees by saying that "we are not sure how best to teach mathematics with computers" (p. 62) and admits that the use of computers in teaching is an unfamiliar territory, but society has many things to gain by integrating technology in the classroom. Barret and Goebel (1990) mention that even in classrooms where the computer is available, the teacher does not typically have a thorough understanding of its role in the class. One situation that seems to explain why we do not know much about computers is that research on computer technology is fairly new. For example Roblyer (1988) found that almost three quarters of the studies she analyzed (38 studies and 44 dissertations) were conducted during 1985-1987.

This situation has inspired the generation of an increasing body of knowledge regarding the use of technology in the mathematics classroom at all levels, in all areas, with all kind of students and teachers. No topic has generated more attraction in the history of mathematics education. Topics of research on technology have been linked to multiple representations, redefinitions of what is fundamental in mathematics, the possibility for
students to work on real world discussions, the ways that mathematical ideas are taught, and the ways they are learned by the students (Sowder, 1989).

Some findings are: The use of calculators and the use of computers can be effective in the teaching and learning of mathematics at all levels (Suydam, 1986), except for grade 4 (Hembree & Dessart, 1986). Students can get involved in higher-order thinking processes (Farrell, 1989; Lesh, 1987), improve their understanding of the mathematical concepts involved (Ayers, 1988; Borton 1988; Loughnane, 1988), and their attitudes toward mathematics (Dick & Shaughnessy, 1988). Some studies show that the use of technology is not detrimental to students' manipulative skills (Heid, 1988; Tall & Thomas, 1991); others have shown that students who use technology in their mathematics classes achieve higher levels of conceptual understanding than their peers who do not use the technology (Heid, 1988; Palmitter, 1991). Findings on teachers are also encouraging: teachers can change their roles from dispensers of knowledge to motivators of learning (Dick & Shaughnessy, 1988; Farrell, 1989).

These research findings are creating high expectations and promising outcomes about technology. In turn, these findings promote further use. Although the impact of technology is "new, unexpected, unstudied, and unpredictable" (Sowder, 1989, p. 29), mathematics educators still recommend its use. As a result of recommendations, Fischbein (1990) points out,

[w]e are witnessing nowadays the penetration of computers at every level of instruction without a serious research basis and without systematic attempts to evaluate their psychological and didactical effect [and] the present tendency is to include the computer in teaching programs without a careful consideration of the ensemble of its psychological and didactical effects. (p. 6)

**The Use of Graphing Utilities in Mathematics Education**

I turn now, to discuss two kinds of computer technology: computer graphing facilities and graphing calculators. In the discussion, I focus on research related to functions. A brief presentation of common features between computer graphing software and graphing
calculators is given before describing some findings on them.

A graphing utility is "any device that draws a graph of a function" (Demana & Waits, 1989, p. 18). Students who use graphing utilities have access to graphing features (including three dimensions), and zoom utilities (Demana & Waits, 1987; Loughrane, 1988; Rubio Montaner, 1987; Waits & Demana, 1987). Students can graph functions given in conic, parametric or polar equations, and see on the screen, a portion of the graph in a rectangular region, called a viewing rectangle. Two features in graphing utilities are of interests to mathematics educators: zoom-out and zoom-in. Zoom-out is a process that has the effect of putting the user at greater distances from the graph, so that a bigger portion can be observed at one time. Zoom-in simulates putting the graph under a microscope. These features are powerful tools for investigating important features of the graph and therefore have implications for the teaching and learning of mathematics, particularly functions. Zoom-out allows the study of global behavior of a function, and the behavior around a given point or when $x$ increases in absolute value (end-behavior). It also provides a graphical way to find extrema. In contrast, zoom-in can be used to solve equations with a high degree of precision. These two interactive features provide students with powerful tools in problem solving (Demana & Waits, 1987; Waits & Demana, 1987), since students can avoid the use of special numerical techniques. Graphing utilities also allow students to work on realistic problems that do not appear in textbooks (Demana & Waits, 1987). The impact that the use of graphing utilities might have on the mathematics content also needs to be considered. For example, a graphical property of the derivative of a function can be found using the zoom-in feature. If using zoom-in repeatedly on a curve results in a straight line then the curve is differentiable. Zoom-in is a refinement of more sophisticated mathematical concepts and procedures: nested intervals and the bisection method (Waits & Demana, 1987).
The use of graphing utilities seems to have a positive impact on students' behavior in mathematics classes. For example, Lesh (1987) found that students who use them become questioners and are able to generate sophisticated solutions in algebra. The use of graphing utilities has also been shown to have positive effects on teacher's and students' behaviors in precalculus classes (Farrell, 1989; Rich, 1991).

The graphing capability can be an aid in specific topics such as functions. Waits and Demana (1987) suggest that by using graphing utilities students can explore enough examples of graphs to build intuition and understanding of their properties. In the words of Dreyfus and Eisenberg (1983), "a concept learned with an intuitive base is likely to be understood by the students and can then be built upon in a meaningful way" (p. 78).

Functions can be represented in different settings such as: graphical, tabular, symbolic, verbal, or iconic. As pointed out by Goldenberg (1987), a single representation is not complete, but common sense suggests that

multiple linked representations increase redundancy and thus can reduce ambiguities that might be inherent in any single representation [...] it is theoretically reasonable to suppose that appropriate visual representations help invest meaning in, and thereby promote the learning, the symbol system with which algebra students must cope; and computer technology lends itself well to this application. (p. 197)

In the same vein, Leinhard and colleagues (1990) argue that

[t]he issue of multiple representations may become more salient in the context of these technologies. Working simultaneously with at least two linked representations is more manageable, with these media. Most graphical technologies provide a graphical representation of a function and can display simultaneously at least two representations. (p. 7)

Specific recommendation to use technology for the teaching and learning of functions are abundant (NCTM, 1989, 1991; NRC, 1989).

Computer Graphing Software

Reports (e.g., Loughnane, 1988; Schwartz & Yerushalmy, 1987) and research (Ayers, 1988; Borton, 1988) on the use of computer software suggests that students who use it can improve and enhance their understanding of the mathematical concepts involved.
Zehavi, Gonen, Omer and Taizi (1987) and Zehavi (1988) have explored the use of computer software in developing intuitions of functions. Zehavi and her associates investigated the effect of graphing software in helping to develop intuitive readiness for graphical representations of linear equations. They found that seventh graders showed an "intuitive understanding of graphical solutions of linear equations and inequalities" (p. 255). In a follow-up study conducted eight months later, the same students, eighth graders then, showed a good retention of what was learned. The authors found the software to be moderately effective. Zehavi (1988), also using seventh-graders and specific software "designed to develop intuitive readiness for the encounter with graphs of linear relations" (p. 392), concluded that the software "can shape students' intuitive rule-based orientation in the coordinate system" (p. 399).

Results in computer graphing research on functions are not definite. For example, Dreyfus and Eisenberg (1987) did not find significant differences between students who used Green Globs (Dugdale, 1982) software as a microworld for the learning of transformations of functions.

**Graphing Calculators**

Compared to computers with graphing software, graphing calculators offer a cheaper alternative. Therefore, access is not as substantial an issue. Graphing calculators are a combination of computers and calculators. They have computer capabilities such as programming, memory, and screen. Furthermore, as calculators, they add portability and personalized use. Some models like the Casio's fx-7000G, fx-7500G, fx-8000G, fx-8500G, Hewlett-Packard's HP-28C, HP-28S and HP-48SX, Sharp's EL-5200, and Texas Instruments' TI-81 are graphing utilities. Foley (1987) describes the Casio fx-7000G.

This machine combines the capabilities of a scientific calculator, a programmable computer, and an interactive-graphics computer system. The display window can display a 16-character by 8-line set of textual symbols, and graphs are produced on a 95 x 63 dot matrix [...] The Casio uses true algebraic logic, which parallels standard mathematical notation. Any elementary function can be readily and rapidly
graphed. Some simple BASIC-like programming is required to graph conics, polar equations, and parametric equations, and the point-by-point plotting for polar and parametric equations is quite slow, typically taking about 60 sec. But it can be programmed to execute virtually any two-dimensional graph [...] Each unit comes with an owner's manual that describes its operation and includes a library of 14 programs (pp. 54-55)

Other features of these calculators are Range, Trace and Factor. The Range allows the user to set up the viewing rectangle and the scale. The Trace feature allows pixel-to-pixel movement along the most recently drawn graph, with the computer displaying the x- or y-coordinate associated with each pixel along the way. It is easy to stop, reverse direction, and switch from x-read-out to y or vice versa. (Foley, 1987, p. 55)

The Factor feature allows the user to zoom-in and zoom-out about a traced or plotted point. Differences between these calculators are communications with printers, memory size, screen size, notational display, implied multiplication, editing, programming, graphing several functions, statistics, matrix algebra, and algebra symbolic manipulation (Demana et al., 1990; Dion, 1990).

Demonstrations with graphing calculators are also possible. There is an overhead projector adapter for them, which allows the teacher to combine the overhead projector with the calculator and display the window on a screen. This way the teacher can maintain unity in the class.

Graphing calculators are relatively new, and therefore there is little research that considers their impact (Dunham, 1990). Most of the talks in a recent conference on technology held at The Ohio State University dealt with how a teacher can use graphing calculators in a mathematics classroom. Some of the articles that appeared in the proceedings of the conference concerned their use in calculus as a symbolic manipulator (Andrew, Morley & Neff, 1990), their use to carry out complex arithmetic and represent it graphically (Diehl, 1990), their use to calculate limits (Farris, 1990), and their use to develop limit concepts (Flores & McLeod, 1990).
Papers on graphing calculators have appeared in journals for mathematics teachers and are basically descriptive. They are on graphing calculators' capabilities (Foley, 1987), on the opportunities students have to explore problems by using them (Demana & Waits, 1987; Waits & Demana, 1987), or on their graphical applications to solve inequalities (Shumway, 1989b).

Although graphing calculators are new, there are also studies that consider their impact on students and teachers. Dick and Shaughnessy (1988) found that student attitudes toward mathematics were positively influenced by graphing calculators (HP-28S) and changes in attitudes may be different for males and females. The authors do not comment on teachers' attitudes toward the technology since the teachers volunteered; however, the teachers stated that they would increase their use of graphing calculators from 10-20% to 20-25% of total classtime. Teachers' behavior also was more exploratory. This finding is also confirmed by Farrell (1989), who studied teachers behaviors in a technology-enhanced precalculus class. Ruthven (1990) found that students who use graphing calculators develop "specific relationships between particular symbolic and graphics forms" (p. 447). He also argued that the use of graphing calculators is likely to encourage both teachers and students to make more use of graphic approaches in solving problems and developing new mathematical ideas, not only strengthening these specific relationships, but rehearsing more general relationships between graphic and symbolic forms. (p. 447)

An interesting situation related to students' behavior was found by Dunham (1990) while conducting a study on confidence and performance among students who use graphing utilities. When content is presented, Dunham found that students would appreciate attention to connections between graphic and algebraic solutions.

As research suggests, it is possible to conjecture that graphing calculators have potential in the teaching of mathematics. In particular, Dion (1990) conjectures that students will improve their intuitive understanding of functions by using graphing
Research on Graphing Calculators and Functions

A few recent studies have investigated the use of graphing calculators for learning functions at the secondary school level. Vazquez (1991) compared two eighth grade pre-algebra classes to investigate the effectiveness of using graphing calculators to acquire skills to graph linear functions. Vazquez' research, however, did not show significant differences in achievement. At the precalculus level, Rich (1991) compared two classes that used graphing calculators with three classes that did not use graphing calculators. Rich found that students in the treatment classes learn to solve problems algebraically and graphically, view graphs in a more global way: use domain, and end - and asymptotic behavior to describe graphs, conjecture and generalize more, and understand better the connection of an equation and its graph.

Chapter Summary and Conclusions

This chapter has presented different contributions to the literature on functions. A first contribution was the historical evolution of the function concept. A second contribution was a psychological account that included development of functions and cognitive obstacles. A third contribution (Sfard, 1991) emerged from merging psychological and historical accounts. Sfard's work differs from others at least in considering the "nature of mathematical entities (ontological issue) as perceived by a thinker (psychological perspective)" (p. 8) and in stressing the unity of mathematical knowledge as opposed to decomposing it into two separate components: procedural/conceptual (Hiebert, 1986); and instrumental and relational (Skemp, 1987). A fourth contribution came from research on graphing technologies for teaching and learning functions. Consideration of these contributions explicates the theoretical framework for the present study.

This chapter also examined some possible implications of the use of graphing technologies in mathematics education, with particular reference to functions. However, at
this time, their effectiveness in teaching and learning mathematics is still an open question. As Leinhard and colleagues (1990) note,

Little research has been done on the teaching of graphing and functions through technologies. Most of what is reported are criteria for development of software and theoretical considerations. (p. 7)

The literature reviewed recognizes that experiences students have with mathematical concepts contribute to their conceptions of those mathematical concepts. In the specific case of functions, the literature reported has apparently overlooked those experiences. Two kinds of possible experiences are (1) the learning environment and (2) the mathematical content to be learned. These experiences, among others, are characteristics of constructivism (Steffe, Richards & von Glasersfeld, 1979). As Noddings (1989) points out, a constructivist position leads to a constructivist methodology. The present study was designed to include the experiences students have with functions in a technology-enhanced class by using naturalistic inquiry in the research. Chapter III describes the methodology used to investigate the research questions proposed in this study.

1. What are the concept images and the concept definition of functions that students in this technology-enhanced precalculus class have?

2. How do students in a technology-enhanced precalculus class use functions?

3. What is the knowledge of functions that students in a technology-enhanced precalculus class have? Is their knowledge procedural or conceptual?

4. What are the stages students in a technology-enhanced precalculus class go through in their attainment of the concept of functions?
CHAPTER III

METHODOLOGY

If we are going to gain anything from the study of the culture of the mathematics classrooms, it will come from an understanding of the factors that contribute to their productivity in this sense. (Nickson, 1992, p. 111)

The purpose of this study was to investigate the knowledge and development of functions among high-school students in a technology-enhanced precalculus class. In this chapter, I describe the methodology used in the study. The chapter contains four sections. In the first section I state the design of the study and address three fundamental questions related to the adequacy of the methodology: the choice of the paradigm, and the use of a case-study format and concept images to investigate students' development of functions.

The second section presents four criteria related to the trustworthiness of the study and how I accomplished each of them. The third section describes the instruments I used in the study and how I developed them. The last section contains the procedures I used to collect and analyze the data, and to report the findings.

Design of the Study

This was a naturalistic study with a case-study format. Eight precalculus students from a suburban school in Central Ohio, which took part in the Calculator and Computer Precalculus Project, C²PC (Demana & Waits, 1988) were selected as participants in this study. A case study was developed for each student.

Why Naturalistic Inquiry?

Lester (1989) contends that the researchers' perception of phenomena affects the methodology used to investigate those phenomena. It was assumed that the learning of
mathematics is affected at least by teacher, learner, curriculum and tools of instruction and that these components interact. Hence, in order to understand better this interaction, naturalistic inquiry was considered as an appropriate "place to stand' from which to view the reality" (Maguire, 1987, p. 12) of the development and knowledge of functions in a technology-enhanced precalculus class. The choice of this paradigm warranted "that greater attention [could] be given to nuances, setting, interdependencies, complexities, idiosyncrasies, and context" (Patton, 1990, p. 51). This naturalistic approach is grounded in an interpretative tradition because it seeks "an explanation for social or cultural events based upon the perspectives and experiences of the people being studied" (Noblit & Hare, 1988, p. 12).

**Why Case Study?**

Case studies are the most common method used in naturalistic inquiry. However, the choice of this format for the current study was based on several reasons. First, case studies "become particularly useful where one needs to understand some special people, particular problem, or unique situation in great depth" (Patton, 1990, p. 54). Second, case studies "may be epistemologically in harmony with the reader's experience and thus to that person a natural basis for generalization" (Stake, 1978, p. 5). Finally, a case study might be a fruitful method to investigate concept formation (Johnson, 1966).

**Why Concept Images?**

The literature on concept definitions and concept images of mathematical concepts suggests that although students are often able to state a formal definition of a concept, they do not use the definition to decide if a given representation of the concept (algebraic, numeric, graphic, or verbal) is an instance or non-instance of the concept. Instead, they use their concept image, which "is a result of [their] experience with examples and non examples of the concept" (Dreyfus & Vinner, 1989). This study was designed to investigate the development of students' concept images of functions.
Trustworthiness of the Study

Guba (1981), Guba and Lincoln (1982), and Lincoln and Guba (1985) present four criteria for judging the trustworthiness of findings in a naturalistic study: credibility, transferability, dependability, and confirmability. These criteria and their relations with the design of the study are discussed below.

Credibility

This criterion is related to the question "Do the data sources (most often humans) find the inquirer's analysis, formulation, and interpretations to be credible (believable)?" (Guba & Lincoln, 1982, p. 246). Guba (1981), Guba and Lincoln (1982), and Lincoln and Guba (1985) suggest use of the following to add credibility to the findings: prolonged engagement, persistent observation, peer debriefing, triangulation, referential adequacy materials, and member checks.

One objective of spending a long time in a culture—prolonged engagement—is to avoid distortions that the presence of the researcher may cause among the locals. A second objective is for locals to become familiar with the researchers. Persistent observation during an extended period aims to identify and obtain a better understanding of salient qualities, atypical characteristics, and essential features in the culture. Simultaneously, researchers can identify and ignore irrelevant aspects and determine, pursue, and narrow on critical characteristics (Eisner, 1985).

I followed the recommendation of Wolcott (1988) to observe the culture for one cycle (12 months), modifying them to suit the context. First, I interpreted 'the cycle' as one scholastic year. Second, I limited my observations to the part of the year when functions were presented. I started my observations the second day of classes, August 27, 1991, and concluded them on April 9, 1992. The content of the course—trigonometry—led me to decide to stop the observations at this time. As treated in this curriculum, the introduction of trigonometry was unrelated to functions, a topic that would not be revisited until the end of
the month. My observations were conducted during eight months; however, my
engagement with the class lasted for the whole scholastic year. After stopping my
observations, I kept coming back to the school and visited the class for a few minutes before
it ended. This allowed me to keep in touch with the students and to schedule the last two
interviews and the member checks. I kept in touch with the teacher even after the school
year concluded (December).

I observed the class on a daily basis every week on a consistent schedule. However,
there were times when I could not come to the school, either for personal reasons or because
of confusion due to a shift in school activities. This confusion occurred only in the first
weeks of the research. My observations included testing situations when they were
scheduled during the class period.

I arrived at the school before the period started and left after the period ended. Before
the class started, I waited in the attendance office. I used this time to become familiar with
the staff and the school facilities. During my field work I had the opportunity to talk
informally with the students. Sometimes I participated in talks not related to the class.
Students accepted me as part of the class, though they recognized that I was not a teacher,
nor a student. I was in the middle. At the beginning of the year, the teacher introduced me
to the class as Mr. Martinez. The students addressed me this way, when he was around;
otherwise they called me Armando. By the end of the year, everybody addressed me as
Armando, whether or not the teacher was around. Students were inclined to ask questions
about my background, reasons for me being there, and, a few times, about the content of the
course—What was going on, or why they had to study a certain topic. After the class, I
stayed often to talk to the teacher. Our conversations covered different topics; sometimes
they included findings of the research, other times there was no relation with the research
nor the class. We did not limit our talks to the school setting; we had lunch together
several times in the school neighborhood.
*Peer debriefing* keeps the inquirers 'honest' and provides them "the opportunity to test their growing insights and to expose themselves to searching questions" (Guba, 1981, p. 85). In order to be aware of my own biases and to work toward minimizing or avoiding them, I discussed the research and the emergent design with professors and colleagues. Sometimes my conversations aimed to seek help (Wolcott, 1990). In my conversations with the teacher, I also exposed to him my insights in the research; however, he was not a peer debriefer, even when I encouraged him to give me some feedback, he said "this is your research, and I do not want to impose anything on you."

*Triangulation* is the use of multiple methods to study the same object. Denzin (1989) describes four types of triangulations: (1) data triangulation; (2) researcher triangulation; (3) theory triangulation; and (4) methodological triangulation. Data triangulation refers to the use of multiple data sources. In this study, this was achieved by focusing on eight students and the teacher, by having the same set of protocols for students' interviews; and by being an observer in the class. Researcher triangulation involves multiple rather than a single observer. Since this study emanated from my dissertation research, I was the only researcher and could not include this type of triangulation. Theory triangulation is the use of several perspectives instead of one in studying the same object. There were several perspectives in this study: constructivism; cognitive obstacles with particular reference to concept images and concept definitions; literature on the historical development of functions; procedural and structural conception of functions; and literature on multiple representations. Methodological triangulation is the use of as many research techniques as possible to study an object. Different methods used in the research were students' quizzes, tests, exams, interviews, observations in class, demographic data, and informal conversations.

*Referential adequacy materials* are documents, videotapes, pictures, or slice-of-life materials collected during the study and kept without analysis. I kept several magazines.
that the teacher gave me, describing the activities in the school. I also took and kept pictures of the students while working in small groups and pictures of the teacher's writing and drawing on the board. I videotaped the class once during the year, in April. I also made and kept copies of examination papers of the whole class I observed. There were times when I also kept copies of tests from the other two precalculus classes the teacher taught.

*Member checks* refers to the opportunity for participants, data sources, to react to data categories, interpretations, and conclusions. I conducted member checks at the formal and informal level (Patton, 1990) both throughout and at the end of the study (Guba, 1981). The study also included a comprehensive member check (Lincoln & Guba, 1985) with the teacher and one of the directors of the C²PC project. The teacher read drafts of chapters IV, V, and VI, and one of the directors of the C²PC project is a member of my dissertation committee. The section on Procedures describes the member checks in more detail.

**Transferability**

The naturalistic inquirer does not look for generalizations, though he believes some degree of transferability is possible, due to similarities between the transferring and the receiving context. This possibility is enhanced through thick description and theoretical purposive sampling.

*Thick description* is the detailed reporting of the cultural and social context, "to impart a vicarious experience of it" and "to facilitate judgments about the extent" to which the findings from that context can be transferred to another similar context (Guba & Lincoln, 1982, p. 248). Geertz (1973) writes

> culture is not a power, something to which social events, behaviors, institutions, or processes can be causally attributed; it is a context, something within which they can be intelligibly—that is thickly—described. (p. 14)

Chapter IV contains a detailed report of the context: curricular, physical, pedagogical, and content-specific subject matter (functions).
Theoretical purposive sampling is aimed to maximize the range of information collected and "is governed by the emergent insights about what is important and relevant" (Guba, 1981, p. 87). I selected the students using this kind of sampling and describe the details in the section on Procedures.

**Dependability**

An exact replication of a naturalistic study is not possible because designs are emergent and include changes with conscious intent. The concept of dependability is intended to convey stability after discounting such changes (conscious and unpredictable). Guba (1981), Guba and Lincoln (1982), and Lincoln and Guba (1985) suggest the use of overlap methods, stepwise replication and the dependability audit to add dependability to the study.

The use of overlap methods is a kind of triangulation to produce complementary results. Stepwise replication means to split data and inquirers in halves to investigate data independently. I argued already in researcher triangulation, why this study did not use this method.

The dependability audit is similar to the process that an auditor goes through to check that the accounts were kept in an acceptable professional practice. This detailed record is called an audit trail. A code is given to each source of data and when conclusions are made, the codes of contributing data are recorded. Lincoln and Guba (1985) describe in detail this technique based on the work of Edward S. Halpern, which includes six categories. The following list indicates the data collected and coded within Halpern's categories.

1. Raw Data
   a). Interview notes.
   b). Audio tapes and transcripts of interviews.
   c). Field notes.
   d). Teacher's testing materials.

2. Data reduction and analysis.
   a). Member checks.
   b). Reflective journal.
3. Data reconstruction and synthesis.
   a). Member checks.
   b). Reflective journal.
   c). Final report.

   a). Reflective journal.

5. Intentions and dispositions.
   a). Prospectus for research to the school district.
   b). Research proposal.
   c). Reflective journal.

6. Instruments development information.
   a). Interview protocols.
   b). Proposed test items.

The audit trail permits an auditor to follow the development of ideas and theory. For this dissertation, my advisor is the auditor of record.

**Confirmability**

In naturalistic inquiry the responsibility for objectivity is put on the data and not on the inquirer; we are interested in the confirmability of the data and not on the certifiability of the researcher. Guba (1981), Guba and Lincoln (1982), and Lincoln and Guba (1985) suggest triangulation, practicing reflexivity, and the confirmability audit to add confirmability to the study.

*Practicing reflexivity* takes the form of a reflective journal that is kept in the field and on which the researchers include introspections, and "document shifts and changes in [their] orientations" (Guba, 1981, p. 87). The *confirmability audit* is similar to the dependability audit. In this case the auditor certifies that there exist data supporting the interpretations, and that every interpretation is meaningful and consistent with the data available. A second objective of the confirmability audit is that findings and data clusters can be traced back to the original data, i.e. other researchers can derive the same conclusions, if starting with the same data. The dependability audit described above was used as the confirmability audit.
Instruments

I used different instruments to provide a holistic perspective of the context under study: myself, an observation form, field notes, a reflective journal, students’ tests, quizzes, exams, and notebooks, a Practice Test on Functions, a handout students completed for extra credit, and students’ and teacher’s interviews. Additional data were collected via informal talks with the teacher and the students, and students academic data.

The Researcher

In naturalistic inquiry, the researcher is the main instrument. In the following paragraphs, I write about myself and my training. I also comment on my expectations about the class before I started the research.

I was born in Mexico City. I got a B.S. in mathematics in the National University of Mexico and became interested in the teaching of mathematics at the end of my undergraduate studies. I taught in Mexico for six years at the university level as a teaching assistant. The next two years, I became involved in mathematics education at a formal level. I taught a different population of students: middle-school students and teachers, elementary-school teachers, and engineers in four-years schools; I also participated in several projects related to the use of technology at different levels. One of the objectives of these projects was to develop computer activities for teachers’ use. This is how my interest in the use of technology appeared.

I came to the doctoral program at Ohio State intending to pursue my doctoral thesis on the use of technology for teaching mathematics. Following this interest, I chose Instructional Design and Technology as my minor area of studies. My interest in how students learn mathematics originated in and developed throughout the doctoral program. What Dubinsky (Jackson, 1988) says about himself happened to me: "I looked everywhere but at the students" (p. 1129).
Identifying the dissertation topic emerged from my interest in functions, an interest based on their importance, first, as mathematical objects and second, for the role they play in the mathematics curriculum. The dissertation topic promised a reflection of my personal interests: a topic of advanced mathematical thinking, the students, and the use of technology. It was the nature of the problem I chose to investigate which made me increase my awareness and knowledge of naturalistic inquiry. This awareness and knowledge developed formally in classes and also in independent studies.

I started my doctoral studies with a set of beliefs about the use of technology in the mathematics classroom. Some beliefs changed, other disappeared, and new ones emerged. I believe that the use of technology can have an enormous impact in the teaching and learning of mathematics. I think that activities that mathematicians do such as exploring, conjecturing, providing counterexamples, refining the conjectures, and proving are accessible to the students when technology is available. I also believe that teachers with an adequate training can shift their behavior in class from a teacher-centered presentation to a student-centered class. I had this same set of beliefs at the beginning of the study. These beliefs created a set of expectations for the classroom where this research took place. However, during my daily field work, I learned that changes are difficult to make even when the resources are available. As the time passed, I acquired a more complete view of what schooling is about and discovered that the potential of technology is limited to context. This more complete view of the classroom helped me to add other variables to the study that I did not even imagine initially: teacher's limitations. The limitations that come from the system, the traditional roles of students and the teacher, and the internal contradiction of the teacher who wants to change but cannot (Skemp, 1987).

Observation Form and Field Notes

Appendix A contains the observation form I used in the study. I developed this form during the two years I supervised student teachers at Ohio State. Observations in this
research focused on teacher, students, and curriculum. In order to register some identified
areas in teacher's and students' activities in class, I modified the form accordingly. With the
emergence of the design, I had to modify it in the course of the study. For example, the last
section, Topic (Functions) was added during the first three weeks of fieldwork. I used a
notebook for fieldnotes that on a regular basis I transcribed into a reflective journal.

Reflective Journal

The journal was the depository of observations, reflections, insights, and emergence of
the design. I also included in it methodological steps and decisions made during the study
to facilitate the dependability audit. Keeping this journal provided the opportunity to reflect
throughout the study on my own subjectivity (Peshkin, 1988). I typed the journal in the
computer.

Students' Quizzes, Tests, and Exams

I collected testing materials from students as additional sources of information. Sometimes
I repeated test items in the interviews. I also proposed that the teacher include
some test items to test working hypothesis generated during my field work, in the
interviews, or during the peer-debriefing activity. Appendix B contains the items I proposed
to the teacher.

Students' Notebooks

To compensate for the days I could not observe the class, I used participants'
notebooks to have, at least, the students sense of the content covered during those days. I
hoped that this activity would point to some directions to investigate the symbolism and
syntax that students applied to functions (and mathematics in general). I did not inform the
students of this method to gather data for I wanted them to keep their notes in the most
natural possible way.
Practice Test on Functions

Markovits, Eylon, and Bruckheimer (1986, 1988) developed a test to investigate students' difficulties and misconceptions about functions. I used a short version of the test that appears in their 1988 paper, intending to gather information on students' entry skills on functions. Appropriateness of the test was based on the four components of functions considered in its content. The components are: (1) the ability to classify relations into functions and non-functions, and the ability to give examples of relations that are functions and non-functions, (2) the ability to identify images, preimages and pairs of these two for a given function, and the ability to find the image of a given preimage and vice versa, (3) the ability to make translations between different representations of a function, and the ability to recognize equal functions, and (4) the ability to identify and give examples of functions satisfying given constraints. In this study, the test was called Practice Test on Functions (Appendix C). I added the last item which asks students to state a definition of function. On the cover page, students were asked personal questions such as name, grade, birth date, gender, and whether or not they have used computers or graphing calculators in their mathematics classes. They were also asked to state their choice of career and/or major in college, if they had already made a decision.

Handout for Extra Credits

In the middle of the second semester, both students' interest in the class and performance on tests declined. By the end of the year, this situation was compounded by scarce time to collect data from participants. In order to gather information from the participants, but still be fair to the class, I proposed to the teacher to administer to the whole class three questions for extra credits. The teacher accepted and I designed a 'handout for extra credit' (Appendix D). The two last problems were included to gather data on images of functions defined on non-numerical or discrete domains. Problem 2 is a modified version of a problem appearing in the Standards (NCTM, 1989, p. 55). Problem 3 comes
from Vilenkin (1968).

**Interviews**

The purpose of an interview is to have access to what is in and on another's mind (Patton, 1990). I interviewed the students and the teacher. All interviews were structured. Students' interviews were flexible enough to address specific situations with each student. Two reasons led me to develop structured interviews. First, to gather comparable data, and second to accommodate my schedule into students' schedules, for I could meet them only during their free time at school. Below I discuss the selection and development of the protocols I used.

*Students' Interviews*

Five protocols for students interviews were selected or developed in the course of the study (Appendix E). The first interview was intended to gather data on students' perception, goals, expectations, and evaluation of the class. Some items are modified versions of items appearing in Clarke and colleagues' paper (1990). Items 13 and 14 were targeted on students' images of functions.

The second interview was taken from Dreyfus and Vinner (1989), though the second and third graphs are not the same. For example, a reason to change the third graph was to test the image of regularity with a function that is constant from a point on. Item 7 was included to examine the image of equation. This item appears in Even (1988, 1989), and Even, Lappan, and Fitzgerald (1988).

The next three interviews were developed to investigate working hypotheses that arose during my observations or from previous interviews. For example, for the third interview, I noted in my field work that the class emphasized procedures and did not make explicit the importance of function: either conceptually or for applications. Items 1 and 3 were intended to triangulate this idea. Items 2 and 4 addressed specifically images and definitions of functions. Item 5 had two objectives. First, it targeted procedural and conceptual
knowledge of functions, by contriving the use of the calculator. I observed in class that students could find the zeros of a function (either by graphing or by factoring) and that they could graph a rational function using the graphing calculator. I noticed that the teacher did not mention explicitly the connections between the zeros of the function in the numerator and the zeros of the rational function. The teacher did not mention either the relationships between the zeros of the function in the denominator and the graph of the rational functions. Students were asked to describe the graph, using the calculator to graph any piece except the whole expression. The problem was a novel situation, since most of the examples in class were easily factorable. The second objective of this item was to investigate the global approach to functions. Item 6 also intended to allow observation of students conceptualization of the problem. In addition, it aimed to gather additional data on the image of equation, and investigate their reactions to a piecewise function defined on a bounded interval.

The protocol for the fourth interview focused on functions and the use of technology in class. Items 1, 2, and 3 investigated images of functions. Item 1 aimed to investigate the components, and item 2 the difficulties that students have with piecewise functions. Item 2 is a function version of a sequence item that appears in Tall and Vinner (1981). The functions in item 3 used a discrete domain. The graph in item 3b) comes from Ferrini-Mundy and Graham (1991). I added the graph in part a) since I wanted to explore also students reactions when they had the graph of a function. Items 4 and 5 were intended to gather information on the global understanding of functions. Item 4 differs from item 5 in the third interview in the use of the calculator. Item 5 also addressed the conceptual and procedural knowledge of functions, by dealing specifically with construction of graphs. This item comes from the Standards (NCTM, 1989). Item 6 aimed to investigate students use of the graphing calculator. Item 7 intended to triangulate observations on small-group activities in class.
Items in the fifth interview explored functions as objects. Item 1 dealt with the
definition of a function, while items 2, 3, and 4 addressed images of the concept. Item 4 is a
modification of item 7 in the second interview. Items 2 and 3 also addressed the
conceptualization that students had made of functions. Properties and representations of
functions were never explicitly mentioned in class. Item 5 was a 'grand finale' question.
This item was suggested by my advisor. Item 6 investigated the image of continuity, while
item 7 investigated the image of equation. Both items are slight modification of items

Teacher's Interviews

Two protocols were developed for teacher's interviews. The first one was designed to
gather data on students' entry skills and knowledge about functions, and second, to
determine teacher's expectations regarding what students should know about functions at
the end of the course. The content of the protocol was based on topics related to functions.
The second protocol aimed to find out about his philosophy of teaching and the use of the
technology in class. My advisor suggested to me some of the items in the second interview.
Both protocols appear in Appendix F.

Procedures

Sampling

The Project

During the first two years at The Ohio State University I supervised student teachers.
This experience allowed me to observe first hand the American educational system and
several ongoing projects in the schools. In particular, I had the opportunity to observe a
student teacher in a high school that used the C²PC materials.

One of the goals and some of the benefits claimed by the authors of the C²PC project
provided a rationale to conduct this research in a class in this project. The aforementioned
goal is "[t]o improve student understanding of functions and related graphical concepts"
(Demana & Waits, 1988, p. 47). Related benefits claimed are:

[i]t provides a focus on functions and relations and their graphs, important fundamental concepts [...] It provides students with a thorough understanding of the behavior of a significant expanded class of functions and relations through a graphical approach [...] It gives students the opportunity to establish connections between problems, equations describing the problems, and graphs of the equations describing the problem. (Demana & Waits, 1988, p. 54)

The School

This research took place in the same school where I had the opportunity to supervise the student teacher that used the C²PC materials. There were practical situations that led me to choose this school: I knew several teachers in the school and it was a familiar setting.

Mr. H. was the teacher I worked with.

Mr. H. and his Seventh-Period Class

It is interesting the way I contacted Mr. H. I had in mind another teacher to work with, Mr. B. However, Mr. B. was not going to teach precalculus during the year 1991-1992, and recommended Mr. H. to me. I explained to Mr. H. over the telephone that I had submitted to the district a prospectus for research (Appendix G) and would like to invite him to participate in my study. He was kind and willing to cooperate. Mr. H. taught three of the seven precalculus classes offered in 1991-1992. I selected his seventh-period class to conduct this research after I observed each class once. Among the reasons I considered were that all the students in this period were seniors, this was the schedule that best fit my other obligations, and other practical matters (Bogdan & Biklen, 1980). Another consideration was having mornings available for scheduling students' interviews.

The Students

Selection of the eight students from this class took about 6 weeks. The possibility of any of the students dropping the course made me consider starting with eight students. If two dropped the course, at least I would end up with six. I planned to have four males and four females. I never intended to have volunteers in the study; therefore, during my
observations I made notes about students' participation in class and their conversations with me. Other criteria that I used were achievement, goals of and interests in the course, and ethnic identity. I selected 16 students including eight who were considered in case any of the first choice did not want to participate. When I had made a list of eight first choices, I asked the teacher to rank the students according to their achievement in order to triangulate my observations. There was agreement in our ranking (hence the study included students ranging from low to high achievers). There was no need to use the second options for the first eight students invited agreed to participate. These eight students were: Jane, Carol, Tyler, Griswald, Tom, Steve, Sara, and Nathalie. These names are the ones that students selected for the report. Specific situations led me to select some of the students as my first choices. For example, Jane was the only minority student in class, Tyler was the only student who was taking the class pass/fail, Griswald did not seem to do his homework or be attentive in class, but still got the best grades in the course, and Steve seemed to be the student that played around the most with the graphing calculator. Two more students—Elizabeth and Jose—were invited to take part later, the reason being that in the first interview Griswald did not seem to take things seriously. Although I thought of replacing him, my advisor suggested keeping him. This made ten students. The original plans for eight students came back again in the final report. During the second semester two students were dropped, Jose and Tom. Jose did not complete the second interview and for the third interview we only had 15 minutes to talk. In the case of Tom, it was difficult to schedule the interviews with him. For example, it was not possible to schedule the second interview before the first week of February. The report therefore includes eight cases. Appendix H contains demographic and academic students data, and teacher's ranking on the eight final participants. Table 1 on the next page outlines the chronology of the study.
Table 1. Chronology of the Study.

<table>
<thead>
<tr>
<th>Date</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>August, 1991</td>
<td>Prospectus of Preliminary Research to the District</td>
</tr>
<tr>
<td>23</td>
<td>Meeting with Mr. H.</td>
</tr>
<tr>
<td>27</td>
<td>Field work started</td>
</tr>
<tr>
<td>September, 1991</td>
<td>Administration of the Practice Test on Functions</td>
</tr>
<tr>
<td>6</td>
<td>Teacher’s first interview</td>
</tr>
<tr>
<td>9</td>
<td>Data collection from quizzes and tests started</td>
</tr>
<tr>
<td>October</td>
<td>Selection and invitation of students</td>
</tr>
<tr>
<td>31</td>
<td></td>
</tr>
<tr>
<td>November</td>
<td>Interviewing students for first time started</td>
</tr>
<tr>
<td>First week</td>
<td>Interviewing students for the second time started</td>
</tr>
<tr>
<td>Second week</td>
<td></td>
</tr>
<tr>
<td>December</td>
<td>Collection of students’ notebooks started</td>
</tr>
<tr>
<td>20</td>
<td></td>
</tr>
<tr>
<td>January</td>
<td>Collection of data from exams started</td>
</tr>
<tr>
<td></td>
<td>Inclusion of items in tests to test working hypothesis started.</td>
</tr>
<tr>
<td>February</td>
<td>Interviewing students for the third time started</td>
</tr>
<tr>
<td>19</td>
<td>Transcription of the interviews started</td>
</tr>
<tr>
<td>March</td>
<td>Field work</td>
</tr>
<tr>
<td>April</td>
<td>Interviewing students for the fourth time started</td>
</tr>
<tr>
<td>14</td>
<td>Videotape of the class</td>
</tr>
<tr>
<td>16</td>
<td>Handout for extra credits</td>
</tr>
<tr>
<td>24</td>
<td>Member checks by students of partial interpretation started.</td>
</tr>
<tr>
<td>May</td>
<td>Interviewing students for the last time started</td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>June</td>
<td>Mr. H. member check of final interpretations started</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Data Collection

*Field Work*

I met Mr. H. personally on Tuesday, the first week of classes. We talked about the research, the methodology, and the instruments to be used. In return, he told me about his activities in the school. During the first week of classes I had the opportunity to observe his three precalculus classes and select one of them to conduct the study. Mr. H. introduced
me to the class and explained the reason I was there. He also mentioned that I was going to be around for a long period of time. I used the last seat in the right corner, next to the window for the first semester and the last seat in the left corner for the second semester (see Figure 5 in Ch. IV). I used the observation form at the beginning and at the end of the period. I came into the classroom, sat down and recorded attendance. Then I started my daily field notes with the date. I registered the time the bell rang, and the time the class started. Then I observed the class. I kept track of every shift, writing the time it lasted: attendance, going over the homework, the presentation of the topic, the use of technology, testing situation, assignment, working with individuals, or small group activity. I wrote down notes about students and the teacher when the topic was related to functions. In the case of the teacher I wrote down his examples, exercises, metaphors, and drawings. In the case of the students, I wrote their comments, questions, answers, and vocabulary. This method of taking field notes was the same throughout the study. At the end of the class I added up the time spent on the activities and marked some observations about the presentation of the class on my observation form.

In testing situations, I received the test sheet at the same time that students did. I took field notes and wrote comments in the testing materials. Observations at these times were just seeing the students working on their tests. I used the time to solve the test, and to make remarks about the items.

Two situations allowed me to be a participant in the class. First, when the teacher put the students to work in small groups or as he used to say "work with the smart person next to you", and second, the time that students had at the end of the period to get started with their homework. My participation consisted in providing help when students raised their hands. Other times, I sat with a group, listened to their talk, and served as consultant when needed. In both situations, I took notes when the discussion was related to functions.
In either testing situations or in small group activities, Mr. H. provided a handout to work on. I kept them all. In the case of testing situations, the teacher graded the test or quizzes and handed them back usually one or two days after all the class had taken the test. Students had the opportunity to go over the test at the same time Mr. H. provided the answers. After this, he asked the students to hand the tests back so that I could look at them. His request was always accompanied by a phrase such as "if you don't mind that Mr. Martínez look at your tests". No student refused for me to see the tests. I made xerox copies of all the tests, quizzes, and exams from the whole class and returned them usually the next day. I kept a copy of the tests from the other two precalculus classes Mr. H. taught when the tests included an item that I had proposed. Sometimes, he made a copy for me only of the page that included the proposed item.

*Administration of the Practice Test on Functions*

Mr. H. administered the test at the beginning of the period on September 6. He explained to the students that there were items that students might not know, but encouraged them to do their best. Students had thirty five minutes to work on it. At the end of the period I collected them all. Mr. H. made few comments about the content in the next two days, but did not go over the test.

*Handout for Extra Credits*

Mr. H. administered the handout for extra credits on April 19, 1992, at the end of the period. Students had about 20 minutes to work on it. The decision of how many extra credits to give came from him. This was the only time that I graded students materials. I made xerox copies and returned them to Mr. H. the next day.

*Students' Invitation to Participate*

I asked Mr. H. to invite the students to participate in my study. I decided to do so because I wanted to respect his position in the classroom hierarchy and make the invitation more formal. He accepted and explained about my research in more detail at the beginning
of the period on October 31. He also commented that I would like to ask eight students to participate in the study more closely. He announced the names of the selected students and asked for their cooperation. He requested them to come to his desk at the end of the period. Before the period ended, he explained to them that I would like to interview them several times during the year. He also told them that I would be flexible about the schedule of the interviews. Students showed interest and willingness to participate. They gave me the times we could schedule the first interview. Later, after the first interview with Griswald, I asked Mr. H. to invite Jose and Elizabeth to participate in the study. He accepted and invited them on an individual basis. I was not present at the time he invited them, but during the interviews, Jose and Elizabeth appeared interested in cooperating.

*Interviews*

When I requested a place to interview the students, Mr. H. mentioned the possibility of using two places: the meeting room behind the attendance office, and the math office. The math office, in contrast with the meeting room was an open area at one of the corners in the hall behind the attendance office. The office was surrounded by the teachers' lounge, the social sciences office, and the foreign languages office. Appendix I contains the schedule for the interviews for each student and the place where they took place, since only the meeting room offered privacy. There were times, although few, when I was interviewing a student in the math office and a teacher was making a phone call, or several teachers were talking around us. Figure 2 shows the meeting room and figure 3 the math office (the rooms are not in scale). Interviews with the teacher were conducted in one of the four rooms in the math center (Fig. 4). The speckled shading indicates where the interviewee sat and the solid shading indicates where I sat.
Figure 2. The meeting room.

Figure 3. The math office.
Before I interviewed the students, I showed Mr. H. the protocols. He rated the items as appropriate. I also asked professors in the Department of Mathematics Education to react to the protocols. They offered suggestions and changes were made.

I scheduled the interviews at the convenience of the students and the teacher, and on an individual basis. I usually asked the students to schedule the following interview either at the beginning or at the end of the period, in order not to interfere with the class. I asked the teacher to schedule the first interview with him in our first meeting. We scheduled the second interview in our informal talks.

I audiotaped all the interviews. Sometimes, I had the opportunity to set up the tape recorder before the interview. Other times, I set it up at the same time I was talking to the interviewee. In any circumstance, I never turned it on until I asked the interviewee's permission to do so. All the interviews started with a friendly talk to establish a rapport and concluded with a "thank-you" remark for their cooperation.

Spradley (1979) comments on four characteristics of an interview that I considered while interviewing the participants. First, turn taking in the interview was less balanced than in a friendly conversation--I asked almost all the questions and the participants answered most of them. Second, I restated what the participant said sometimes and repeated the questions. Third, I expressed my interest to hear what the interviewee had to say in verbal and nonverbal manners; at the beginning of the meeting encouraging the students to talk,
and at the end of the interview to facilitate the scheduling of the next meeting. Finally, I encouraged the participants to expand on what they had said in any item.

_Students' Interviews_

I had a notebook for each student, in order to keep their scratch work together. At the beginning of the interviews I made the notebook available so that they had paper to make any drawing they wanted. I asked the students the questions and heard their answers. Sometimes, I wrote notes about what the student was saying. When I felt that he or she had finished I asked questions about the notes I wrote to clarify my thinking. Other times I had the students read the questions, particularly when there was a graph involved. Sometimes, I used two print outs of the protocol, one for the student and one for me. At times and depending on students reactions to an item or the answer they provided, I had to be more inquisitive and provided additional help to gather more information. I always had control in the interviews and tried to follow the protocols, however, at some times we followed a different order. In the first interview, for example, some students' comments altered the planned order.

In the first interview, I explained to the students the purpose of the research, and the kind of data I was going to collect. I explained to them that the data collected would be confidential and the reporting of the findings would be anonymous. However, they were also aware that sometimes, I would discuss with the teacher some of the data collected in the interviews. To maintain the data as confidential, I never shared with the teacher the personal comments that students made on the class. At the end of the interview, I gave the students the set of questions for the second interview. I told them that those would be the questions for the next meeting, saying also that I thought it would be convenient for them to try to solve the items in advance. I also told them that as soon as they gave me the handout back I was going to schedule the second interview. On the average, students took about a week to hand the handouts back. Scheduling for the second and forthcoming interviews was more
disperse than for the first meeting (Appendix 1).

Starting on the third interview, I included test items related to functions for some students in addition to the protocols. My decision to do so was based on the observation that students only wrote a final answer without the process to find that answer. At the end of the third interview, I asked them to choose a name they would like me to use in the reporting of the cases. Where applicable, the fifth interview started by addressing students' reactions to their cases as I had written them up so far. The protocols also included students answers to the handout for extra credit.

*Teacher's Interviews*

I interviewed the teacher two times. At the beginning of the first interview I mentioned to him the purpose of the study and how his answers would help me in the research. The interview lasted for about 35 minutes. As in the students' interviews I wrote down some notes for further clarification when I considered he had provided a complete answer. During the interview, one student came for help and the session was interrupted once.

The need for a second interview appeared with the unfolding of the research. I had a structured protocol but the interview developed into an exchange of ideas. It extended for about one hour and a half, and there were no interruptions. This was the time when the issue of reactivity—how he changed because of me and how I changed because of him was discussed openly (Patton, 1990). We talked about my expectations about the class, the use of technology, a more formal statement of my findings, and how my beliefs about implementing technology in the mathematics classroom had changed during the year. I mentioned to him ethical issues and how I was going to address them in the final report (Spradley, 1979). Mr. H. talked basically of how my conversations with him had made him reflect on his teaching.
Author's Interview

I planned originally to interview one of the authors of the C²PC materials. However, this interview could not be conducted. The interview was planned to gather data on the intended curriculum in the project. I thought of using data from this interview to triangulate teacher's expectations regarding what students should know about functions at the end of the course. I planned to use a similar protocol as that of the teacher's first interview. Since the interview did not take place, I used written materials that the authors have written on the project to accomplish my triangulation.

Gathering Data Outside the Field

Throughout the study I had informal conversations with Mr. H. and with the students. In the case of Mr. H. I got to know him better. I learned about his schooling, his teaching experience and style, his perceptions of the project, his knowledge of the students, and his philosophy about mathematics and the use of technology. In the case of the students, sometimes I asked about the events that occurred in the school, their feelings in class, or their activities outside the school.

Member Checks

This study included informal, formal, and comprehensive member checks. The informal member checking occurred mainly during the interviews. This checking took the form mainly of a concluding remark such as "So, you said..." or "Let me see, if I understand you..." or "Is [...] what you are saying?".

The formal member checks with the students were conducted in April and May. All the students had the opportunity to read a preliminary interpretation of their cases, which included data collected from observations, all testing materials, the first three interviews, and the handout for extra credits. Students were given their cases and asked to react to them. Every case was handed in an envelope with a note explaining the content, encouraging them to make remarks in my work, remarking on the importance of their reactions, and
appreciating their cooperation in this procedure. The eight participants had at least one week to read their cases. Six students (Tyler, Griswald, Sara, Nathalie, Steve, and Carol) received their cases before the fifth interview. All but Carol read them prior to the meeting, therefore for the other five participants, I had the opportunity to hear their reactions in the last meeting. In the case of Carol, Jane, and Elizabeth, I talked to them and listened to their reactions at the end of the period when I had arranged to pick up the cases. Either in an interview or in class, the procedure to member check the cases was the same: I asked them if they had read the case, what they thought about what I had written, and whether or not they had suggestions or changes to make. In general, all students agreed with the content. Minor changes were made. Independently of the written work, all the students heard a summary of what they had said in the fourth interview at the beginning of the fifth interview.

In addition, to the students, Mr. H. also read some of the preliminary interpretations of the cases. I met with him in the math center, where we discussed the content of the cases. His cooperation as a member checker was also at the comprehensive level. In this case, he read drafts of chapters IV, V, and VI. I handed to him the chapters in an envelope and met with him on campus to talk about the content of each chapter. He had at least one week to read each chapter. As with students' reactions, he agreed with the content reported for the most part. Minor changes were made. Teacher member checking for students preliminary interpretations were conducted in April and May. The comprehensive member check occurred in September and October.

Data Analysis

There is no strict time distinction between data gathering and data analysis for they were simultaneous during the study. This situation allowed me to accommodate the design according to the presentation of functions in class and develop the protocols for the interviews. Discussion on how I analyzed the data follows.
The study aimed to build a grounded theory explaining the facts observed. I used an inductive analysis of the data in the building of such a theory. Every student was treated as a case study. Emergent categories from interviews, testing materials, and fieldnotes were used to answer the research questions on page 6.

I used a domain analysis (Spradley, 1979) and a coding paradigm (Lincoln & Guba, 1985; Miles and Huberman, 1984b) to analyze the interviews, reflective journal, and testing materials.

Spradley (1979) describe eight universal semantic relationship to conduct an analysis of semantic domains. These relationships are:

1. Strict inclusion          X is a kind of Y
2. Spatial                 X is a place in Y, X is part of Y
3. Cause-effect            X is a result of Y, X is a cause of Y
4. Rationale               X is a reason for doing Y
5. Location for action     X is a place for doing Y
6. Function                X is used for Y
7. Means-end               X is a way to do Y
8. Sequence                X is a step (stage) in Y
9. Attribution             X is an attribute (characteristic) of Y

Three semantic relationships appeared to be associated to functions in students' minds: strict inclusion, function, and attribution. Examples of each relationship follow. The image of equation in some of the participants—"Functions are equations"—is an example of strict inclusion. "Functions can be represented by equations" is an example of a function relationship. "Functions are always continuous" is an example of an attribute relationship. These same relationships emerged on the testing materials from Mr. H. (see Chapter IV, Section 4).
Interviews

Interviews were transcribed to have the data readily available. Since sometimes the students made personal comments about the class, only the pieces needed in the research were transcribed as a way to handle the promise of confidentiality. In the case of the teacher, both interviews were transcribed completely.

Categories in the analysis of students' interviews emerged from the data, although pertinent literature was considered. For example, Ferrini-Mundy and Graham (1991) describe the case of Sandy, a student with the image of familiarity. Tyler used the word "experience" meaning the same image as Ferrini and Graham's student. I decided to use the category "familiarity" to make the discourse uniform.

I followed Miles and Huberman's (1984a) suggestion to build matrices for the interviews. Each student's answer to an item in a particular interview corresponded to an entry in the matrix. Five matrices were built. This allowed me to have access to the data for comparisons across the students.

Reflective Journal

I had a print out of the journal to have readily access to it. The journal was of particular importance in writing up Chapter IV.

Testing Materials

I separated each student's testing materials including the Practice Test on Functions, and the handout for extra credits. In the case of Mr. H.'s testing materials, I focused only on the items dealing with functions. I went over the items trying to justify students' answers based on their scratch work, when there was some. For each student, I made memos and a brief summary of each test. This analysis provided another source of data for students procedural and conceptual knowledge of functions. I constantly looked for cluster' or links between these two kinds of knowledge. In the case of the Practice Test on Functions and the Handout for Extra Credit, I analyzed them completely. I built another matrix for the
items that I proposed to the teacher and the Handout for Extra Credit.

*Students' Notebooks*

Collection of data from students' notebooks was not as productive as was hoped. Most of the participants did not write notes on a regular basis or they wrote down exactly what the teacher had written on the board. Their homework only contained answers to exercises without any process explaining how they found their answers.

*Writing up the Final Report*

Chapter IV describes the context where this study took place. The report is based on literature on the project, magazines from the school, interviews with the teacher, and the reflective journal. Chapter V includes the case studies. Data from the first interview with Mr. H. sets the arena for the presentation of the cases.

Each student case contains the following sections: introduction, definitions and images of functions, discussions of procedural and conceptual knowledge of and development of the concept. Each case concludes with a discussion of the content included. Data for the introduction were observations, informal talks with the students, and the first interview. The sections on definitions and images of functions come from interviews and the Handout for Extra Credit. In writing the procedural and conceptual knowledge section, I reported the categories in an effort to make evident interrelationships between the two kinds of knowledge.

In the report of the development of function I started the discussion with the data collected in the Practice on Functions. For the data collected in the study, I used a combination of writing the things as I found them (Wolcott, 1990) and the historical development of the concept of function. In building categories (stages) for the development of functions, the concept definition of function was at the top of the list. Concept images of functions held by students were used for intermediate categories. I looked for relationships or discrepancies between definition and images of functions. I also looked for students'
conceptualization of functions as objects and as processes.

There were times when two or more students solved an item in the interviews using a similar approach. In this case, to avoid repetition in the chapter, I reduced the report by making a reference to a previous student. Chapter VI contains the discussion of the cases based on historical, psychological, and pedagogical obstacles. In writing it I used the matrices I built from the interviews and testing materials. In addition, the chapter contains a summary of each student. In each case, I included a network representing function images and their relationships. The network emerged from each student's data.

**Summary**

This chapter presented the methodology used to investigate the knowledge and development of functions among eight high-school students who used graphing calculators in their precalculus class. Four research questions were investigated:

1. What are the concept images and the concept definition of functions that students in this technology-enhanced precalculus class have?

2. How do students in a technology-enhanced precalculus class use functions?

3. What is the knowledge of functions that students in a technology-enhanced precalculus class have? Is their knowledge procedural or conceptual?

4. What are the stages students in a technology-enhanced precalculus class go through in their attainment of the concept of functions?

The first section of this chapter discussed the adequacy of naturalistic inquiry, case study format, and concept images to go about answering these research questions. The second section described the methods used to achieve criteria dealing with the trustworthiness of the study. The third section reported the development of the instruments used. The fourth section, contains the procedures to collect and analyze data, and the reporting of findings.
CHAPTER IV
THE CONTEXT

Mathematics education research should be sensitive to context, and particularly to didactical intention, as a significant contextual element. (Wheeler, 1989, p. 286)

In this chapter, I describe the specific context where this study took place. The description is made up of four pieces. The first piece refers to the particular project and its curricular materials. Then I move to describe the school setting. This section also describes the classroom, and provides additional information related to the students in the class I studied. In the third section, I describe the pedagogical context, meaning the methods and style of the teacher, and the structure of the class. Finally, I describe the way Mr. H. taught functions. The intention of this section is twofold, first to set up an arena for the presentation of the case studies in the next chapter and second, to provide the reader with a vicarious experience of the course. Appendix I contains a list of topics of the course that were related to functions.

The Calculator and Computer Precalculus Project, C²PC

This study took place in a school that used the C²PC materials (see Chapter III, pp. 55-56). In this section, I describe the project. The content is based on the description given by the directors of the project (Demana & Waits, 1988).

The Calculator and Computer Precalculus Project, C²PC, developed at The Ohio State University is

[...] a nontraditional approach to teaching precalculus mathematics that makes use of calculators and computers to accurately graph functions and relations and to solve a wide variety of "difficult" but "interesting and realistic"
problems...[A]ccurate computer graphs can be used to determine the behavior of functions and relations and to develop geometric intuition important in the study of calculus and advanced mathematics. (p. 47)

The course "includes most of the traditional topics covered in a post Algebra II functions, trigonometry and analytic geometry course" (p. 47). The use of technology permits the teacher to emphasize problem solving and real world applications, without forgetting important algebraic skills and techniques. In addition to providing a new approach, the use of graphing utilities gives students the opportunity to check and interpret geometrically the solutions found by algebraic methods.


Training of participant teachers occurred in two phases. The first phase was an initial selection of teachers to receive training to use the C2PC materials. The directors of the project conducted workshops for the teachers at Ohio State. The second phase was the involvement of trained teachers in the training of other teachers at their sites.

Students in precalculus classes in the project are students "who have successfully completed Algebra I, Geometry, [and] Algebra II" (p. 47). Access to a device that has an effective graphing utility is the only requirement for these students. At the beginning of the project, teachers used the computer software Master Grapher (Waits & Demana, 1989) to teach the course. Currently, the use of graphing calculators is common practice in all schools and colleges that have joined the project.

The school where this study took place has witnessed the development of the project since its beginning. Curriculum materials were pilot tested in the school in 1987. One of the directors in the project knew Mr. B. and invited the school to participate in piloting the
materials. Mr. H. remembers how the project started officially in the school:

After about the first couple years of piloting, it just, it became, to us it became obvious that it was going to be a good course...Better than what we were doing and it was going to incorporate technology. And the decision was made as a department that this was going to be the course we were going to teach. We were going to teach pre-calculus from this book...Cause we felt that it was headed in the right direction. Even if the book wasn't as good as it was going to be, it was better than anything we had up to now, especially for technology. (II)²

The School Setting

The School

Data included in this section refer to the year 1991-1992. The school is a four-year high school located in an upper-middle class suburban area in Central Ohio. It occupies a two-story building and, in addition to classrooms, offers to the students several facilities to develop their computer literacy as well as athletic skills. Three computer labs, the library, and a two-story computerized learning center compose the computer facility. Athletics facilities include a stadium for football, soccer and lacrosse, two gymnasiums, a natatorium, baseball and softball diamonds, weight-training room, running track, and tennis courts.

School is in session 180 days a year. Each school day is from 8:00 a.m. until 3:30 p.m. with eight 50-minutes class periods except Thursdays, when periods are 45 minutes long. Teachers in the school average at least fifteen years of classroom experience. Approximately 72% of the faculty have Master's degrees and 3% doctorates; 55.6% are males, 44.4% females; and only 0.9% are from minority groups.

Student enrollment is about 1500, and the graduating class 350. Student population is overwhelmingly white, with 3% minorities (mainly Asian), 52% males, and 48% females. Student competitiveness about getting into college is reflected in their plans: four-year

² From this point in, I quote materials from the interviews. The code (I#, #) stands for (interview, number of the interview, and item number in the protocol). When the name of the interviewee is not clear from the context, I indicate it before the code. I adopt the same code to report students cases in Chapter V.
colleges (92%), two-year colleges (3%), vocational/occupational (1%), and other (4%). An estimated 0.5% of the students drop out during the school year.

The curriculum is comprehensive and includes Advanced Placement, honors, occupational, vocational, and adjustment programs. Twenty units of credit are required for graduation. In addition, seniors must complete a senior thesis, "an original and analytical work, linking concepts encountered within the student's senior seminar courses" (School magazine).

In mathematics, there are three sequencing options for students in grades 11 and 12: Regular, College Prep, and Honors. Regular refers to students not intending to go college. College Prep and Honors refer to students intending to go college. The difference between these two is that Honors students study mathematics one more year (calculus). Students in this study were in the College Prep option. In addition, the school has a Student Math Center to help students who are experiencing difficulties in mathematics. At least one mathematics teacher is available in the math center every period.

The Classroom

Mr. H. taught three of the seven precalculus classes offered in this year. His three precalculus classes met in room 241, on the second floor of the building. The windows in the room faced one of the school parking lots. Figure 5 in the next page depicts the physical facilities that Mr. H.'s classroom offered. The boxes in column indicate students' seats.

The Students in the Class

There were few peculiarities about the seventh-period class. It started out with 26 students, fifteen females and eleven males. In Autumn 1991, two females dropped the course. All the students but Jane were white. One student (Pam) had taken the course before; according to her, she failed it because the class was difficult.
I perceived that students were friendly and got along among them. There was however, a natural tendency to form small groups among close friends. This friendship was reflected particularly in the small-group activity in class.

![Diagram of classroom layout](image)

Figure 5. Mr. H. classroom.

The Pedagogical Context

Mr. H.

Mr. H. is a 40-year-old native of Ohio, white, tall, and with a sturdy appearance; he has a pleasant personality and a strong character. Educationally, he holds an M.A. in mathematics education from The Ohio State University. His fifteen years of teaching experience include 9 years in high school and 6 years in middle school. He has taught most of the courses at both levels: 7th and 8th grade math, general/business math, algebra I and II, geometry, trigonometry, and precalculus.

Mr. H. started participating in the C²PC project the last year of the pilot test (1987-1988), when the school decided to expand the number of teachers that were teaching the course.

[S]o they just started picking people and saying, Would you like to do this? and I said 'Yeah, I'd love to.' (I2)
Mr. H. taught the course for a quarter, without having participated in the project training workshop. He participated in the workshop the following quarter around Christmas. The formal training was a two-day workshop where they bring you in. They take you through the materials, not through the book so much as through the technology. They show you what you can do with the technology...And how they designed their questions to be used with the technology and helped you with graphing, helped you with programming...So they had a two day workshop that helped a lot. (12)

However, as many teachers in the project, most of what he learned probably came from other teachers who had a more intense experience. In his case, Mr. B. helped him and provided him with ideas to teach the course.

Most of [Mr. B.'s ideas] were just on, since he had been through the course, three years when I started or two years when I started. Most of [his ideas] were just ideas on how to work in the computer and the graphs into the problems. How to use them to, if you were doing a factoring question, if you were doing problem solving, or linear equations. It was mostly just helping out with the technology. How to get the graph to look the way you wanted it to, the idea of zooming in...Most of his help came in the area of the technology. Not the problems themselves but he helped me to get the technology involved in the course. (12)

Mr. H. considered that he perceived the goals in the project as not much different from what the authors had in mind. According to him the objectives were

[t]o use technology to make what we're doing more visually obvious. I really think that's the idea. To stop doing everything by paper and pencil and making it so hard, just so hard to understand what's going on let alone to actually answer the question...But to actually to be able to put a picture in front of you...Take the difficulty out of, out of graphing the equation and make it easier to understand the graph and read the information, use the information, rather than have to spend so much time just getting the graph that you don't even care what the graph's all about anymore...But now that's just the opposite, the graph is easy to come up with. It gives you much better understanding of what's going on, what that graph represents and you can start using the information to solve problems, word problems. (12)

The authors insist that using graphing utilities, students have access to the power of visualization (e.g., Demana & Waits, 1988). Mr. H. also understood this as a primary goal in the project. In his opinion, visualization would be the idea of, if I start talking about a quadratic equation, I want [the students] to automatically envision a parabola...And to understand that since we've
been through so many of them to automatically understand that if it's a parabola, it's either increasing and then decreasing or decreasing and then increasing, but they have that shape come to their mind immediately. And to know there's a vertex, to know that it's possible not to have any zero points...Because of the way you can move it around on the screen. Or a cubic equation, I mean anything with an odd degree, to have them understand immediately what it looks like or what it should look like. And that it has to have a zero point somewhere...Things like that...To have the picture in their mind of what that function represents and then you can spend more time discussing what the zero point means or what the y-intercept means. What it means to be decreasing. Because they have the pictures in their minds they don't have to sit there and spend two hours getting the picture. They can start interpreting what they see. That's what I think of it. (12)

Mr. H. stated that he liked the course and overall was happy with it. The incorporation of technology, in particular, allowed students and teachers to get rid of an incredible amount of work manipulating algebraic expressions or the tedious point-by-point graphing process of 'complicated' functions that took, just a lot of time. The use of technology simply provided the opportunity to approach a graph more globally.

I really enjoyed that. I could, you could get right to the graph and if you wanted to make a little change, like a stretch factor or a, what we call stretch factor now, we didn't call them that back then. A stretch factor or a shift or a reflection, things like that...It's so much easier to talk about now...You don't have to regraph any points, you don't have to sit there and analyze what happens to every single point, point by point. You can do it as a function. It's much easier. I really enjoy what they do with it that way. It just makes much more sense now. I realize that the technology wasn't available, you couldn't, you just literally couldn't do that, it wasn't that we weren't doing it, you just couldn't do it...But now that we have it, yeah, I like that a lot. (12)

Although Mr. H. considered that the use of technology benefited the learning of mathematics, he also discovered that its use was somewhat detrimental for students' algebraic skills. This feeling about technology was one of the aspects he did not like in the project.

One of them is being educated...Having been educated one way all your life. And then all of a sudden, you change... And the things that you spent most of your life learning and teaching are no longer important. Or at least they're, you're told that they're not as important any more...That's a very difficult thing to get used to. And this, one thing this course does is it drastically reduces the amount of mechanical skills that the kids learn...They just don't learn to factor or to solve, uh, solve by factoring, solve linear equations, they don't learn to manipulate radicals. The stuff that we used to be told are important...They don't do that as much anymore. And for
a while I thought that was okay. I really, I bought into the idea of technology and it's okay to skip all that. But now that I've been teaching it for three or four years that would be the one drawback, I wish there was a little bit more mechanical stuff. Cause I'm concerned that we send too many kids out without those skills...They're going on to college and they don't have the manipulative skills I think they should have. That's one drawback. (I2)

The Structure of the Class

Daily Routine

Mr. H.'s class followed a traditional format (Romberg & Carpenter, 1986). There was consensus among the participants in the study about the daily routine in Mr. H. class.

Nathalie described it in detail.

We go in, we sit down, we talk for a little bit and then Mr. H. gets up, and checks our homework from the night before, and then he'll answer any question that we have and then he'll teach a new lesson and he'll go through. He'll explain what we'll be doing for the day, he'll title it like sign charts. We all start taking notes and he'll walk us through to get us the basic idea of what we are doing and then he will walk us through a couple of problems, and then he'll ask us to do a couple on our own and if we have any questions, he'll answer any questions that we have and then he will do this probably until there is only about 10 or 15 minutes left in the period and then he'll give us our homework and we'll work on our homework while he walks around answering questions that anybody might have. (I1, 6)

Mr. H. added his intentions and how time was allocated.

I check homework a lot. So daily I would come in and check and see if they've got homework done...I probably check on the average of two out of every three assignments I check...the class you're in is not a very strong class academically, so I spend the first part trying to check homework and see if they've attempted it, done the best they can or not. Whatever...Then we simply try to go over the homework. I've tried, again I don't know if this is what you'd notice or not, but I try not to answer anything that they don't ask...I'm not going to stand up there and go over every question. They have to be assertive enough to ask me what they want. And if they don't ask they don't get it...But I don't want to sit up there and go every single question...And then we try to move into whatever has to be done that day. Whether it's a new section, an old section, a review section. I try to spend the rest of the time that I want to spend on the next set of problems and I always try to leave ten to fifteen minutes at least at the end of class and they'll work. That doesn't mean they all work. But I always [leave time], some teachers want to go bell to bell, I don't do that, I try to always leave time at the end where they can at least get started. (I2)
Class Activities

Teaching Practices

Mr. H.'s teaching style was a lecture-type. The arrangement of the seats in Figure 5 was usual in his class. He said that his reasons for lecturing, as opposed to having the students participate actively were partially based on the passive role that students have played in the mathematics classroom, and also a matter of convenience.

[Students] weren't used to [other method of instruction] and they're still not...They still want to be taught, so to speak. They don't wanna necessarily do it [themselves], they don't want to learn it on their own, they want to be taught. You tell me, I'll do it. And I still find myself, and I'm sure you've noticed in class, I'm sure I still do too much. I should not do what I do. I should make it far more independent...It's much easier to give them what they want and not have to deal with the problems of making them do it on their own. (I2)

He explained that another reason was the size of the classes he taught.

So, it's just easier for me to have control of all those classes, because I have so many kids. The other teachers that are teaching pre-calculus, their classes are [much smaller]...Very irritating. But that's the way it is....Mine are very large and it's easier for me to have control than it is to try to do it the other way. It would be better if I made the students more in control of the class but it's hard cause I have so many of them. It's easier to do it this way. I've gotta be honest. (I2)

The use of technology impacted his style of teaching. Before he started using graphing calculators, Mr. H. spent much less time at the end, reviewed more, and included more mechanical content. His class was

[a] lot of problem solving, or equation solving, I should say...A lot of manipulations of polynomials, fractions...We would have spent a lot more time on the homework, it would have been put on the board more often...That sort of stuff. And I would have spent most of the rest of the time presenting the new material because it's mechanical in nature...A lot of examples I'd present that they'd do. Um, so there wouldn't have been much time at the end...That's changed a lot. I spend less time on the homework and less time presenting the material. (I2)

Mr. H. admitted that using technology his teaching style would be in continuing change.

I changed a lot, but I think it's going to change even more. I think the [second edition of the textbook in the project] will allow you go even faster using these calculators. [We used to take] a lot of time to teach the calculator, we don't have to do that anymore. That has changed the way I taught a lot. I don't lecture as much, I try to get more time for them to do work. Yeah, it has changed a lot, a great
Procedural versus Conceptual Knowledge

Mr. H. perceived mathematics as applications, problem solving, and modeling, as the ability to analyze a problem, to solve it...What [mathematics] used to be is manipulations. Can you manipulate algebra? I don't think that is the case. [I'm thinking of] analyzing and can you solve? I hope that's where we are heading because it wasn't that way some years ago. (11)

His teaching reflected partially his thinking, and emphasized mostly the solving part. However, this solving part was implemented procedurally: apply the algorithm or the formula, recognize the method, memorize the process. Procedures to solve problems were modeled making his instruction to appear as algorithmic.

Mr. H. introduced new materials by lecturing, interacting with students, but making most of the talk. A great majority of his questions were closed questions. He used to introduce a new concept, definition, or theorem, and then provide examples of the concept. Sometimes, he mentioned nonexamples and counterexamples. Next, he gave the students exercises to practice the new material. After this there was an extension of the new material, followed by the assignment for the next day.

The following episode comes from one day in January, and illustrates his teaching style. The topic was composition of functions. After the usual activities that occurred before introducing new material, he announced the topic and wrote it on the board "Composition of Functions." He continued writing "A process of putting 2 or more simple fcns, together to form a more complex one" (sic). As usual in his lecturing, he explained that the composition of functions meant to "take two functions and put them together." He gave the functions $f(x) = \sqrt{x - 4}$ and $g(x) = x^2$ and evaluated both compositions at 2. Then he asked the students to evaluate the compositions at 10 and 6. After students solved them, he explained only how the first result was found. Then he moved to the next material, that was to find a "rule for the composition $f \circ g(x)$." He said that "what $[f \circ g(x)]$ means is
literally to stick the \( g(x) \) instead of the \( x \)." As usual, his comments were accompanied by his writing on the board. He wrote "\( f \circ g(x) = f(g(x)) = \sqrt{x^2 - 4} \)" and said to substitute \( x^2 \) for \( x \) to get \( \sqrt{x^2 - 4} \) [sic]. The composition \( g(f(x)) \) was handled as "\( g \circ f(x) = g(f(x) = ( \quad )^2 \)" while saying "instead of \( x \), you stick the entire function." After this he wrote on the board"\((\sqrt{x - 4})^2 = x - 4\)." When he wrote the equal sign, he said "which is." After this he assigned homework for the students to get started.

Mr. H. only addressed the procedural content of composition of functions. The conceptual content of a mathematical lesson was rather hidden. For example, in his discussion of composition of functions in the rest of the year, he did not discuss that it was possible for the composition not to exist or that the resulting rule was again a function. Subsequent mention of the topic was handled in the same way—algorithmically.

The presentation that Mr. H. gave for composition of functions was typical of his teaching style. Similarly, he modeled procedures to factor, find zeros of functions, asymptotes of rational functions, and the algebraic expression of a given graph. Procedures were emphasized also in solving problems, as with the box problem or in annuity and mortgage problems.

*The Use of Technology*

Mr. H. used several instructional media to deliver instruction. A combined use of the overhead, the board, overhead projector adapter, and handouts characterize his teaching. The overhead, the graphing calculator, the board, and the overhead projector adapter aided mainly to model processes, "to walk the students through" to use Nathalie's words. He considered the technology as an aid in instruction. For example, he said that the main idea when he used the graphing calculator in the overhead projector was to check whether or not all the students were together or as a means to check himself, making the students to participate in class. (12)
Mr. H. explained that, as time passed, there was an inverse relation in the use of the graphing calculator in class. He used it less, while the students used it more.

First of all, after the first month or so I try not to use [mine during class]. I try to make them do it. I quit using the overhead for example...Because now that I'm pretty convinced that they all know how to graph, I try not to, I try to teach without it in my hands...And try to make them use it on their own. Um, I still try to do it, whether I'm effective or not, I don't know, as an aid...I still try to use it as an aid. But that's what I try, whenever we're doing a question, you know, I'll say 'you take your calculators out and you can check this', or somebody take it out and they usually do. They'll check and see, is this what it's supposed to look like? To see if I made a mistake. Um, I still use it as an aid. I try not to be dependent upon it. I don't know if that rubs off on the kids or not. But I rarely get mine out anymore and have it in my hand. I expect them to do that. But not me...Cause I want to teach as though it's an aid to me, not as a crutch that I have to use...So I don't know if I'm effective that way or not but that's what I try to do. (I2)

The use of graphing calculators was on a daily basis and a part of each activity, unless he stated otherwise explicitly.

I guess I go into my class with the expectation that every day I'm going to use it...Every day I never tell the kids not to unless I specifically [have something else in mind]. In other words, I don't say today is a calculator day or today's not. Um, every day is a calculator day...If it's a situation where I don't want them to use it, I'll tell them. Otherwise we use it every day. I expect the homework questions to be set up for calculators every day...If they can get away without them fine, but...I also, I try to point out when the questions in the homework or in the book are not intended to be done with the calculator. They're there to check. I'll tell them this is not a calculator question. Use it to check yourself. But it's not a question. You're supposed to be able to do that without one...So I try to point that out as often as I can. We don't do that, this is an algebra question. (I2)

Mr. H. considered that students should learn to decide when to use the calculator to solve a problem and when not to, but still they had not reached that point. Therefore, his wording of test questions and specific instructions in class suggested when to use the calculator and when not to.

[T]he truth is most of the kids are finding the calculators to be the easiest way all the time and I'm trying to safeguard against that. Cause I think that's unfair. I think they're going to walk out of here and be totally dependent on that and I don't want that to happen. So when I tell them that I want that exact value, that's their indication that I don't want [them to] trace. I want them to do it algebraically...The truth is, yeah, it would be a whole lot better for them to make that decision but I don't think they can do that. So I try to tell them what I want them to do. (I2)
Small Groups

Work in small groups was also frequent in Mr. H.'s class, though at the end of the period. He provided a handout and told the students to join the 'smart person' behind them to work on the sheet. He usually walked around the groups to provide additional help. Sometimes the students moved their seats to join some friends, other times they joined the person behind or next to them. At the beginning of the course some students, like Griswald, worked by themselves. By the end of the year, this situation still happened but was more unusual. Mr. H. describes his intentions for this activity.

Um, my intention is that they just, that they [don't] get up and go work with their friend. That's my intention. You stay where you are but please compare your answer to the person behind you...Don't try to do this on your own...I know how to work on my own...I know how to learn math on my own. But that's, that's very intense and it can be very boring if you got to spend hours at it by yourself. I would, my personal style of teaching is I would much rather have those kids with somebody working together. I think it's more fun...I guess I should be honest, my only intention is that they'll work on the task at hand...And if one of them can, is assured that they have the right answer will help the other...Or if neither one's sure, they will talk it out...That's my intention. I know they're never going to explore...They're going to do what's, needs to be done and then move to the next question. (L2)

Usually, Mr. H. had an answer sheet on his desk so that students could check their work.

Two situations were common in class. Either the students worked in a small group or they did not. In the first case, students worked with their peers, checked their answers and then copied the remaining solutions from Mr. H.'s desk. Usually in this case students raised their hands to get help from the teacher. In the second case, students copied down the answer sheets as early as they could.

The Subject Matter Context: Functions

Concept Definition of Functions

During the first week of classes, several classroom activities allowed the students to become familiar with calculator usage. Through the activities students could deal with
functions graphically without a formal introduction of the concept. Such introduction occurred in the second week of classes.

Mr. H. introduced functions formally as special cases of relations. He wrote on the board the definition of a relation as "a set of ordered pairs" and the definition of a function as "a relation in which each input (called x) is paired with exactly one output (y)" [sic]. The introductory notation was framed around several examples, that he had presented earlier: "f(x), P(n) (profit), g(x), D(R) (distance is a function of rate)." He referred to x, n and R as independent variables.

Mr. H. gave two examples to illustrate the idea: \(x^2 + y = 25\) is a function, but \(x^2 + y^2 = 25\) is not. Then he prompted "how do you know? Use your common sense." If you can find that the same number is sent to two outputs, then we do not have a function. He also made the comment that it is possible to have the same output for different inputs, like in \((4, 9)\) and \((-4, 9)\) for the parabola (that students recognized as such). In the first example, he drew the corresponding graph, and checked that it was a function, noting that there each x had only one y.

The second example was found not to be a function. In this case, Mr. H. used the vertical line test without a formal presentation. He mentioned that \((0, 5)\) and \((0, -5)\) satisfy the equation. Then he drew the graph of the function and a vertical line saying "the vertical line test fails."

Mr. H. added two more examples, but this time he did not provide a graphical illustration: \(y^2 = x\) is not a function, but \(x^2 = y\) is. He also mentioned that a "vertical line" is not a function. This was the day when he introduced function notation.

What does \(f(0)\) mean? This is function notation. \(f(0)\) means what is \(y\) if \(x = 0\)? \(f(x) = 0\) means. What is \(x\) if \(y = 0\)?

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3 The discussion refers to a one-day presentation.
In finding $x$ such that $f(x) = 1$, he said "I can literally draw a line one" and drew it.

The domain of a function was referred as the set of $x$'s and the range as the set of $y$'s. A more formal definition of the domain was given two weeks later. Then, his writing on the board read as "Understood domain is the largest set of $x$'s for which $f(x)$ makes sense." The formal definition of the range was given.

The domain and the range appeared in subsequent discussions of functions. The discussions presented them as isolated, not as an integrated whole. It was common that Mr. H. asked for the domain and the range of a function in class or in a test, but the question sounded more like another item that students had to answer.

Although Mr. H. defined functions in a modern way, he also associated with the concept several images. There were subtle messages in his teaching and examination materials conveying different images of a function. At times he equated functions with equations or graphs. Other times he simply said 'I'm giving you a function' or implied that functions could be represented by graphs or equations. Other times he referred to functions as 'rules'. An analysis of the problems that dealt with functions in tests ($N = 52$) parallel the images associated with functions in class. What follows is a description of Mr. H.'s images of functions that emerged from his class and examination materials.

**Concept Images of Functions**

Mr. H.'s images of functions are clustered in the following categories: graph, the vertical line test, equation, rule, curve, functions, and others. Examples of each category follow.

**Graph**

Two main images were conveyed associated with graphs: functions are graphs or graphs are representations of functions. It was common practice for Mr. H. to draw a graph and say "this is the graph of a function" or "that's a function."
A similar situation occurred in examinations. The two ideas are contained in 12 items from the first semester. In seven problems, functions and graphs are equated. Two examples of this category appear in figure 6.

This graph is not a function, but using function notation, answer;

\[ f(0) = \]
\[ f(-3) = \]
\[ f(2) = \]

(Q-09/11, 6b).

Use the graph at right to answer [this question]. In the interval \((-2, 1)\) is this function increasing or decreasing?

(Q-12/10, 9).

Figure 6. Two items with the image of functions are graphs.

Five of the problems convey the idea that functions can be represented by graphs. Figure 7 and figure 8 show two examples.
The graph at right represents a complete graph of a function \( f(x) \). Find all solutions to \( f(x) = 0 \).

Figure 7. An item showing that graphs represent functions.

Use these graphs to answer, which of the graphs above represent "functions"? (E-01/14, 1).

Figure 8. An item showing that graphs represent functions.

The Vertical Line Test

The vertical line test was mainly associated with graphical representations—as a method to check whether a given graph is a function or not. Students met the vertical line test
explicitly in only one problem in their first quiz. It seems that Mr. H. was looking for
differences between relations and graphs that did not depend on graphical representations.

Explain the difference between relation and function (no need to mention the vertical
line test) (Q-09/11, 3)4.

Equation

The image of equation appeared in Mr. H.'s presentations throughout the whole year.
As with graphs, two images emerged: functions are equations, and functions can be
represented by equations. Sometimes, Mr. H. wrote an equation saying "this is the
function" or "it is a function." Thirty items in tests paralleled this situation. However, there
was not a clear cut off in the two categories. Some problems, though few, belonged to both.

Seventeen problems belong to the first category. An example appeared in (Q-01/09, 8).

A fifth degree equation touches the x-axis three times and is known to have no
complex zeros. Sketch such a situation and label those "touches" as A, B, C. Then
write an equation for this graph in factored form.

Eqn:

Figure 9 shows another example of this situation.

Write a possible equation for the
graph at right. The answer should
be a rational function in expanded
form. (No. abs. value).
(Q-02/07, bonus).

Figure 9. An item with the image that functions are equations.

4 The coding used in the testing materials is the following. T stands for test, Q for quiz, and E for exam.
The date (month and day) and item number follow the type of examination. The same code is used in
Chapter V. In this instance, the notation indicates quiz on September 11, item 3.
Seventeen problems belong to the second category. Some examples are:

A rectangular area is being fenced in next to a concrete wall. There are 300 feet of fence available for the three sides remaining. Let x be the length of the fence perpendicular to the wall. Write an equation of the area of this region as a function of x (T-09/20, 18a).

Let \( f(x) = \sqrt{x} \). Write an equation which represents \( f(x) \) under these transformations performed in order:
- A reflection over the y-axis followed by a horizontal shift two units left. Then the graph is moved vertically up 1 unit, vertically stretched by a factor of 3 and finally reflected over the x-axis (Q-01/09, 2)

\[ f(x) = \_\_\_\_\_\_\_\_\_\_\_\_\]

An example of a problem that belonged to both categories was:

Let \( f(x) = \frac{1}{x} \), \( h(x) = \frac{2}{x} \), \( g(x) = 4x^2 \), \( p(x) = \sqrt{x + 4} \). Answer these questions:

(A) Find an equation for \( \text{gofoh}(x) \):

(C) Find and equation for \( \text{pof}(x) \):

(D) Find an equation for \( \text{gof}(x) \):

(E) Find the range for the equations in........

Rule

The rule image was mentioned in class only when referring to compositions (See Procedural versus Conceptual Knowledge, p. 81). In testing items appeared also associated with compositions in one item.

Use [the functions] \( f(x) = \frac{1}{x + 4} \), \( g(x) = x^2 - 3 \), \( h(x) = \sqrt{x + 3} \) [to] find a "rule" for [the] compositions \([f \circ g(x)\text{ and } g \circ h(x)\)] and also to find the domain & range. A picture is not mandatory, but you may want to sketch one to help you find them. Simplify your "rules"! (Q-01/29,7 and 8).

Functions

Only a few times were functions referred to as such in class. Instead the images presented above were used. Six test items also reflected this situation. Two examples of this category are

A fifth degree function (polynomial) has, at most, ____ extrema (E-01/14, 50).
Let \( f(x) = 2x; \ g(x) = x + 3; \ h(x) = \sqrt{x}; \ j(x) = x^2; \ p(x) = \frac{1}{x}. \) Find the function represented by these compositions:

(A) \( g \circ f \circ j(x) = \)__________  
(B) \( g \circ f \circ p(x) = \)___________.  

(Q-01/29, 9).

**Curve**

This image was not mentioned in class. It only emerged in testing items. Three items contained this image. Figure 10 shows two examples of this situation.

Sketch a complete graph of these curves. (T-04/08, 1 and 2).

\[ y = \log_3 (4 - x) \quad \quad \quad y = -\log_{\frac{1}{2}} (4x + 8). \]

![Figure 10. Two items with the image of curve.](image)

**Others**

Under this heading, there are reported three situations: no reference to function is given, and functions are referred as pictures or expressions. Only the first appeared in class. Mr. H. sometimes said "this is \( f \), find the domain and the range." This situation also appeared in three test items. An example is

Let \( f(x) = x^4 - 2x^2 + 3x - 2. \) Which of these integers represents zeros of \( f(x); x = -2, -1, 0, 1? \) (Q-12/10, 13)

The other two situations appeared only in examinations. Only one problem contained the idea of picture and one problem by the end of the year used the word expression
The Use of Technology for Teaching and Learning Functions

Technology in Mr. H.'s class had merely demonstrative purposes. Four media were used: the overhead projector, the graphing calculator, the overhead projector adapter, and the board. It was common that Mr. H. used them separately. For example with the overhead projector he introduced exponential functions, gave three examples of them and asked the students to graph them on their graphing calculator. A similar situation occurred with the transformations $f(x + b)$, $f(x) + b$, and $af(x)$. In this case, he illustrated the effect of a constant in the function $f(x)$, by drawing the resulting graph on the transparency.

Other topics were presented only using the board. Definitions, theorems such as the remainder theorem and the factor theorem, and algebraic procedures like long - and synthetic division were presented or illustrated in this way.

Other times, Mr. H. combined the use of the graphing calculator with the overhead. In this case, he used the overhead projector adapter. He sat on a stool and modeled for the students the procedures at the same time he taught which calculator keys to use. Activities which make repeated use of the trace and the zooms such as solving inequalities or systems of equations, graphing, reading off the range, finding extrema, zeros, or intervals where a function increased or decreased were presented in this way. Combination of the three media was the common norm at the beginning of the year. By the end of the year, it was more frequent to see Mr. H. using only the board and the overhead without the overhead projector adapter.

Summary and Overview of Forthcoming Chapters

This chapter described the particular context where this study took place. The description includes the curricular, physical, didactical, and subject-matter context. While as Patton (1990) says, we can never completely understand how a person thinks, it is true that different sources of data and in particular the didactical context in mathematics education
(Wheeler, 1989) can help us to understand better how people think of and learn mathematics. This has been the intention of this chapter.

In Chapter V, I present the case studies. Two brief descriptions about students knowledge of functions precede the cases. One includes their experience with functions prior to the course and from Mr. H.'s point of view. The other includes what he expected them to know after going through the precalculus course. The order of presentation of the cases is a matter of convenience. Chapter VI is the last chapter in this dissertation and includes the discussion of the cases from three different perspectives based on cognitive obstacles: historical, psychological, and pedagogical. The chapter ends with conclusions, implications, recommendations, limitations, and directions for further research.
CHAPTER V
THE CASES

All students engage in a great deal of invention as they learn mathematics; they impose their own interpretation on what is presented to create a theory that makes sense to them. Students do not learn simply a subset of what they have been shown. Instead they use new information to modify their prior beliefs. As a consequence, each student’s knowledge of mathematics is uniquely personal. (NCTM, 1989, p. 2)

This chapter contains the cases. Two brief sections precede them. The first one describes students’ experience with functions from the point of view of the teacher, Mr. H. The second section describes what he expected the students to know about functions at the end of the course. Mr. H.’s quotes in each section come from the first interview.

Students’ Experience with Functions at the Beginning of the Course

Mr. H. considered that students’ experience with functions was at the procedural level: algebraic manipulation without any link to functions.

I don’t think they have a very good understanding [of function] at all...What they normally have had is algebra skills, manipulative skills. We always stress the ability to solve equations, but we never, well I shouldn’t say never. We have not sufficiently stressed the understanding of the function itself. It was more the ‘how you solve the equation, we do the functions later’. That’s wrong, but that’s what we normally did.

However, he said that students knew how to add, subtract, and multiply functions. The experience with graphs was also at the procedural level and from a pointwise approach. Mr. H. considered that students did not experience the link between a graph and a real-world situation.

During their 7th, 8th grade year, they would have been in a typical algebra course and they would have done a lot of graphing. That graph would never really mean anything. Well, it would mean, ‘here’s the graph of a straight line. Here’s the graph of a parabola.’ But I don’t think it would mean too much to them. We
didn't attach any meaning to it. Like I'm trying to do...in class. The graph
represents area versus length. You know, trying to give more meaning to it. We
didn't do that in the past...They should know intercept, slopes, how to read an
ordered pair. They should know the basics about a graph....They should know the
vocabulary, except in my mind, the broad concept, this is a graph of what. They
know how to graph, but they don't know how to connect to a particular problem.

Although students studied families of functions such as straight lines, parabolas,
hyperbolas, logarithms and not very much of trigonometric functions, Mr. H. admitted
that he felt

comfortable with hyperbolas, parabolas, and straight lines. That's about
it...They've seen logarithms, but I still wouldn't assume they know what is going
on.

Mr. H.'s Expectations about what Students should know about Functions

at the End of the Course

Mr. H.'s expectations were based on his understanding of what the textbook was
designed for—a global approach to functions. He also expected students to discriminate
functions from non functions and recognize that discontinuous graphs can also represent
functions.

What I hope about functions, they get out of there is that functions have behaviors
and we can learn to predict their behaviors. I want them to learn that if you have a
function, or a problem I should say, if you have a problem, then you can interpret
it, I should say if you can interpret it as a function, then you can do some
predicting later on, you can analyze the function graphically. I don't want them to
be afraid of analyzing a picture. I want them to be able to look at the graph, that's
my hope...and I can ask them anything about that graph. I can ask them end
behavior...behaviors around a point...what's on the x-axis, what's on the y-axis. I
expect them to be able to identify what a function is and isn't and a big one is I
think, continuity. They've never seen discontinuous functions and when they
leave here I hope they can identify discontinuous functions but still understand
that is a function. Because they have never dealt with that before. These are
pretty much the goals, because that's what I think the book is designed, to try to
teach the behavior of functions, and we spend a lot of time on rational functions,
too. I guess...we get more specific with functions than they have in the past. We
don't just do parabolas anymore. We don't do hyperbolas, we do some ellipses,
which are not functions...and conics, rational functions...logarithms and then we
do trig...So we'll do all the trig functions sine, cosine, tangent. I want them to be
able to look at the graph and read it and know what it means. That's my hope. [...] they'll learn about maximums, minimums, increasing, decreasing on certain
intervals, end behaviors. They'll do that for every function that we study...So
those kind of specific questions about end behavior, or increasing, decreasing, yeah, they should know those at the end of this year.

Mr. H. recognized that the course emphasized little on compositions and inverses.

We don't do a lot with [compositions]. They know what compositions means but we will not do a lot with them. There are two sections but we don't teach it a lot. It comes back periodically and it comes back very specifically at the end of the book, but my class won't get there, we don't do a lot.

In the case of inverses, it was clear that the emphases was at the procedural level.

They did it last year, but they will not understand what or why we have inverses. We'll do it again this year. This will not be the first introduction to it, but we'll do it every, almost every section. We'll do inverse linear functions, inverse trig functions. We do inverse for a lot of stuff...I don't think they'll get a strong understanding. They'll know how to find one but that's about it. I don't think they will get a real understanding, not mathematicalsematicsementical understanding, why there are inverse functions. That doesn't bother me. We want them to understand how to find one for right now.

Mr. H. expected students to be aware of the domain and the range while working with functions, maybe as a consequence of having worked with the algebraic representation and the problem representation. However, he said "I don't have any fantasies that all are going to do that."

The Case of Elizabeth

Introduction

Elizabeth is dynamic, enthusiastic, and a cheerleader for the basketball team. She says that mathematics is one of her favorite subjects and understands it pretty well. She finds it interesting, "not precisely as a breeze," but she likes the challenge that mathematics puts on her. She likes science too and is also taking physics this year.

Elizabeth is taking precalculus because she likes mathematics. A second reason is college preparation. In college, she plans to study to be a nurse. She thinks that precalculus will be useful in her classes, particularly in calculus.
Elizabeth likes using a calculator in precalculus. She considers that mathematics is easier with its help, because it does simple formulas or simple functions. It also helps her to concentrate on more important and advanced content, and go into more details, rather than just learning how to graph. However, she considers that in the long run the use of the calculator is hurting, since people rely on it even for the simplest work. She considers that its use makes her lose some algebraic skills. She thought the algebraic skills of her sister, who did not use a graphing calculator, were stronger.

Elizabeth likes the graphing on the calculator. At the beginning of the course she explored all that the calculator could do and how different functions are graphed, but when she got used to it, her interest declined. By the end of the year, the calculator was not appealing anymore.

Elizabeth likes seeing how the material that she learns in precalculus is connected to her physics class. However, neither of her teachers makes references to the content of the other class. The use of the calculator in the physics class is occasional and at times limited to numerical calculations.

How Elizabeth defines a Function (Concept Definition)

Elizabeth provided six definitions of a function during the study. The first definition contains two ideas: graphic representation and the relation between $x$- and $y$-values. In (PTF16)\(^5\) she defined a function as "[a] graph of a relation."

The second definition is Mr. H.'s definition stated in class and contains the idea of univalence.

A relation is the set of ordered pairs and a function states for every $x$ there is one and only one $y$. (Q-09/11)

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\(^5\) PTF refers to the Practice Test on Functions (Appendix C). The number indicates the item in the test. In this case, PTF16 indicates item 16 in the test.
Her third definition includes the algebraic representation which establishes a relationship between the x- and the y-values, but missed the univalence property.

An equation that relates the x, y-values, x to the y-values. (I2, 8)

The fourth definition contains pieces of the first three definitions: an equation with the univalence property, and a suggestion of the graphic representation (ordered pairs).

An equation that when you put x in you get a different y-value, for each one, or you'll never get, like, you'll never have two points like two comma four and then two comma six. For each x-value, you get a different y-value. (I3, 2)

Elizabeth's fifth definition is independent of the graphic representation and refers to an expression and the univalence property. Her images suggest that she was thinking of equations when she referred to 'an expression that relates the x- and the y-values'; not to an arbitrary correspondence. She wrote the following

An expression that x & y values are related. For every x value one & only one y value. (XC1)

Close to the end of the year, her definition is a blend of the first four definitions and adds the idea of the vertical line test,

it's an equation that when graphed that you get one, for one x-value every y, one y-value, it can either be an equation that can be graphed and then when it's graphed you can use the vertical line test or the horizontal line test [...] you use the vertical line test to decide whether it's a function and then inverse, when you have the graph of the inverse you use the horizontal line test. (I5, 1)

Elizabeth's Mental Pictures Associated with Function (Concept Image)

Several images associated with functions emerge from the data collected from Elizabeth. Images have been separated but repetitions in several categories imply that there is not a clear separation between them. The images associated with functions are the following: graph, vertical line test, one y for every x, equation, familiarity, and continuity. Indications of each cluster follow.
Graph

This image was probably one of the most common images associated with functions (see first definition, p. 97) at the beginning of the course. In the first interview, she said

Any function?...It's on a graph, it has to do basically with the x-values, like if it were x squared, like if it's just going to be a single x, it's going to be a straight line [drawing figure 11].

Figure 11. Elizabeth's graph of a straight line.

Three ideas appear in this image:

1) functions can be represented by graphs,
2) graphs can be functions, and
3) an intermediate step to decide whether or not an equation is a function

The first idea in the graph image is that functions can be represented by graphs. Her answer to 'what makes a function' (I4, 1) relates to this point.

When I think of a function, I think of a graph, but, it doesn't necessarily have to be a graph, but it can be a set of points where that's, there's only one x-value for every y or only one value for every x...it doesn't have to be a graph, it can be an equation...or a set of equations. I know that functions just aren't graphs.

The second idea in the graph image is graphs are functions if they pass the vertical line test. An instance of this situation appears on Elizabeth's last definition (p. 98). The third idea in the graph image is to use the graphical representation as an intermediate step to decide whether an equation is a function or not. For example, an equation is not a function "when the equation doesn't graph the way functions should" (I2, 7).
**Vertical Line Test**

The vertical line test was a means to decide whether or not a graph \((I2, 2, 5; I4, 3b)\) or an equation \((I2, 7; I5, 1)\) is a function. For example in the graph of a function with four points \((I4, 3b)\) she said that the graph wasn't a function "cause there's two points, there's the same, um, x value, I mean there's two points on the same x line, or y line." In the discontinuous graph given in \((I2, 2)\), she answered correctly and "just used the vertical line test."

In the case of equations, \((I2, 7)\), she said that to check if an equation is a function, she would graph the equation and after that she would do "[t]he vertical line test, to see if it works."

**One Output for every Input**

There are two ideas in this image of functions:

1) a property of functions, and
2) an implicit equivalence of the univalence criterion and the vertical line test.

The first idea--a property of functions--appears in Elizabeth's definitions explicitly (2nd, 4th, 5th, 6th definition; I3, 2; I4, 1). For example in 'what a function is' (I3, 2) she said

An equation that when you put x in you get a different y-value, for each one, or you'll never get, like, ... you'll never have two points like two comma four and then two comma six. For each x-value, you get a different y-value.

See 2nd, p. 97; 4th, 5th, 6th definition, p. 98; I4, 1, p. 99.

The second idea is an implicit equivalence of this image and the vertical line test (I2, 1, 3). For the graph of the non function in (I2, 1) she said "no, because it hits here and here" and in the continuous graph of a function, (I2, 3), she said 'yes' because "it only hits in one place."
Equation

This image appeared in all the interviews and it seems to be the strongest image anchored on her mind. Three ideas appear in this image:

(1) a relationship between $x$ and $y$;
(2) functions come from equations; and
(3) functions can be represented by equations.

Elizabeth had a need for a relationship between the $x$- and $y$-values (I1, 13, 14c; I2, 5, 7, 8; I3, 6; I4, 1; I5, 6, 7; XC2). This image appeared sometimes as a pattern (I2, 6; I3, 6; I4, 3a) and sometimes as an expression (see 4th definition, p. 98; I2, 4; I4, 2).

Describing a function with her eyes closed (I1, 13), she mentioned, "$x$, $x$ squared, $x$ cube, $x$ fourth, $x$ fifth," as ideas for functions. The following dialogue took place

**Me**  Is your idea of function, $x$, $x$ squared, $x$ cube, $x$ fourth, $x$ fifth, and so on?

**Eliz**  No, I think all functions, they relate like or if this is another way of saying $y$ and then how your answer to $y$ is what $x$ pertains, by substituting $x$ in the equation.

**Me**  So, in order to have a function you need an equation?

**Eliz**  Mhh, I think so...I don't know if that's right.

**Me**  Is there anything else that you need for having a function, just an equation? Or do you need something else?

**Eliz**  An equation, with an $x$ and y-values, numbers, I think that's all.

**Me**  Do you need numbers?

**Eliz**  Yeah, it can't just be like, it can be just like $x$ squared but usually it's plus like a number.

**Me**  So you need an $x$, something, and numbers. That's basically an equation, right?

**Eliz**  Mha.

When asked about her ideas about $f(x)$, (I1, 14c), she said that $f(x)$ "usually stands for $y$ and the change of $x$, like the $x$ usually stands for a number, and then [if you change the $x$], that's your new number." See (I2, 8), p. 98.
Although the most important property of functions is that of a relationship, the image of equation imposes obstacles to dealing with the arbitrariness of functions where the relationship is not obvious as in a constant function given verbally (I2, 5); a set of points either graphically (I2, 7; I4, 3a; I5, 6), or algebraically (I5, 7), or in a non numerical set (XC2); a piecewise function (I3, 6); and one defined arbitrarily (I2, 6). For the constant function in (I2, 5), she could not think of any drawing for this situation. I asked her to propose a particular value for y. She picked up five. I suggested two points, (3, 5) and (2, 5). The following dialogue took place.

Eliz  Two, five, OK. So [drawing figure 12] it'd be

![Figure 12. The constant function f(x) = 5.](image)

It would be a straight line. So the answer is yes...that will work.

Me   Why is that you think it is a function?

Eliz  Because the x and y are related, as substituting a number for x you can see the y value ... I guess it wouldn't be because if it is a straight line, it wouldn't work. [changing] It passes the vertical line test.

Later, when I asked her if she could write an equation for this function, she said

Hmm, I can't think of any, because it doesn't matter what x you put in as long as you, well you can put like a zero and just put any x value in and that would be y equals and the, anything else would be zero. I don't know.

When asked if equations and functions were related, (I2, 7), she said they were

because for every equation that x and y are related it's a function ... could be equations, well function is like f of x and that they are related like in an equation. Things like y equals mx, like I don't know, just f of x and then from that you can write an equation which is a function.
Also in this item, we were talking about whether a set of three points might be a function. She said

Eliz  No, I mean if you are just plotting points, they have to be related, somehow they have to be related in order for there to be an equation.

Me   Suppose that I give you three points, one, zero, zero, zero and negative two, three and those are the only points that I have. Could you please plot them?

Eliz  [plotting the points as shown in figure 13.]

Figure 13. The points (1, 0), (0, 0), (-2, 3).

Me   The question is, is that a function? Only those three points?

Eliz  I say no, because I don't. I can't figure out the way like all those points are related to all the y-values.

Me   So you would need something like an equation?

Eliz  Like this is like the square or y or something, in order to get y you just square, square then I say, it could be an equation...a function behind, maybe these points related to each other.  

In the graph of a function with four points, (I4, 3a), she said it is not a function "cause it doesn't, they're just plotted points. It doesn't really have a pattern or an equation that follows that. I think there needs to be more information given."

When asked about the existence of a function that passes through four points given algebraically, (I5, 7), she rejected the four points as being a function "because there's not, no, I don't see any direct relationship" and in the non numerical set in (XC2) she circled no "because they are items with prices and no certain relationship."

When finding a rule of a piecewise function given graphically, (I3, 6), she found the domain and range. After this she asked "what else do you want me [to do]" and said
"see, I don’t know how these points relate to each other, like in a pattern they follow some times." I asked her if she was trying to find one single rule for the whole graph, and she answered "yes." I asked her "why?" and she replied

I'm just trying to find like looking at all the numbers that you gave us, I don't know, the first thing I thought like looking if something is triple or double, but I don't see anything.

it was clear in (I4, 1, 3a; I5, 6) why she rejected a discrete set of points as a function. The following dialogue took place in what makes up a function (I4, 1) when we retook the question.

Me    Um, tell me, what is a function?
Eliz  Um, an equation that such that for every x-value there is a y-value.
Me    Uh huh.
Eliz  It's not real, I mean, I don't know, it's not really, I don't think, I don't consider it just a set of points, like if you give like these pictures [pointing to I4, 3a, 3b]
Me    Uh huh.
Eliz  I don't consider them to be functions, but I don't know, I might be wrong.
Me    Okay. So those are not functions, because those are not equations?
Eliz  They have no relation.
Me    No relation. Okay. No relation, do you mean like, uh, something like, you expect that some way they are related?
Eliz  Mmm hmm.
Me    The x and the y's?
Eliz  Yeah.
Me    By that equation.
Eliz  Mmm hmm.
Me    Is that?
Eliz And these just being, these are just plotted points, so I don't see how they're related to one another that they could be a function.

A similar explanation appeared in making changes to five graphed points to make them the graph of a function, (15, 6), where she said "[i]t's, see it's hard for me to tell without having the equation to see how these points are related."

Difficulties with the function defined arbitrarily (12, 6) appeared also because she was not sure what integers were. The following dialogue took place after giving her some examples of integers and non integers. It illustrates how she is trying to find an equation using the operations she is familiar with, which in turn shows the obstacle that the image of equations generates.

Eliz It couldn't be square, cause that just would ... if you squared a non integer, you still get a non integer.

Me OK, to begin with, could you please write just two numbers. I mean, two points that will be like, say I give you zero, that will be an integer and something an output whatever. What would you use for y if it has to be non integer?

Eliz Zero plus one.

Me But plus one would be an integer.

Eliz Oh, you just want it like the non integers?

Me What would you write? Anything.

Eliz x minus five.

Me But you would get

Eliz OK, oh, it would be like.

Me Just, just for zero, what would be your?

Eliz Zero divided, well, zero is a non integer right?

Me Zero is an integer.

Eliz And you want a non integer...zero divided by something.

Me But if we divide we will still get zero.
Eliz: Yeah, if you multiply you still get a zero.

Me: Aha.

Eliz: Should we use radical?

Me: But radical zero will be zero.

Eliz: Anything you do is zero.

Me: What about one half?

Eliz: Am I trying to get an integer number?

Me: No, no if you use the point zero, one half. I'm just trying to make one example.

Eliz: [writing (0, 1/2)]

Me: This is the kind of points that you will get, give me an input that is integer like zero and I'll give you something back. The idea is that you can make it up. That's the point. Now, can you give now another? One that is non integer say, one third and let's get an integer. Whatever it is.

Eliz: Divided by like, three over one to get one, you multiply. One, that will be an integer, [1/3 3/1]

Me: So we have two points, so at least we know we can go point by point. Now, the question is, can we do it for all the points? Just remember this has to be to have domain all reals.

Eliz: See you have to be able to do it for every radical.

Me: Yeah, something like that.

Eliz: If you get something like that, we can always multiply by its reciprocal.

Me: When we get a fraction. The problem is that the non integer might not be a fraction.

Eliz: Oh, I know, if it's like radical or something, they are non integers. We can multiply by one over radical 2. That'll give you one.

Me: Probably you just want to divide by radical two.

Eliz: Would it still be one?

Me: If you just divide by radical two.

Eliz: Ah, OK [correcting.]
Me  Now, the question is you are saying that some [non integers] are fractions, some might be radicals, but how do we know that we get all the numbers by using radicals?

Eliz  Isn't it all decimals are going to be a fraction, or radicals?

Me  There are other numbers that are not radicals, that are not fractions, like pi.

Eliz  Sorry, I'm totally lost.

Me  OK. Let me ask you a question. Why did you try to multiply or divide? what is that you are trying to get?

Eliz  Find a pattern ... for the equation, I mean the x-value.

Me  Are you trying to find an equation for the inputs?

Eliz  Mha

Me  Is that what you are trying to do?

Eliz  Yeah.

In Elizabeth's relationship image, some times there was an explicit need for an equation to decide if a graph was a function (I2, 4; I4, 2). In this case there seems to be a gap between the graphical representation and the algebraic representation: it does not suffice to show a graph with the desired properties (i.e., passing the vertical line test), but there is the need to give an equation representing the graph and hence to have a function. For example, Elizabeth had difficulties understanding the task in (I2, 4), the existence of a function that sent every nonzero number to its square and zero to one. After rewording the problem using inputs and outputs, she said "I'm sure there is. I don't know the equation of that." She could not draw the correct graph (see protocol in Continuity, pp. 111-113). After she asked if her guess was correct, I showed her the correct graph (figure 14). When she saw it, she asked "What is the equation for that?"
Figure 14. The graph of the function that assigns to every number different from zero, its square and to zero it assigns one.

A more detailed account of this situation appeared during the protocol for the piecewise function in (I4, 2).

Me    Yeah, how many functions are involved in this whole expression, right,
Eliz  Oh, two, cause you graph, don't you graph both these?
Me    Uh huh.
Eliz  And then where they, where they, um, intersect is where, is that what f of x is?
Me    Uh huh.
Eliz  Is that right, or no?
Me    Um, tell me about your ideas, and,
Eliz  No. Oh, no.
Me    Forget if it is right.
Eliz  Um, well, it doesn't really have an equation, it's just saying, well one if x is greater than or equal to zero and then negative one if x is less than zero.
Me    Uh huh. So, okay,
Eliz  So, go ahead.
Me    Yeah, I was going to ask you when you mentioned it doesn't have an equation,
Eliz  Mmm hmm.
Me    It seems like you need to have the equation for being a function.
Eliz     Mmm hmm.
Me      Yeah. So in that situation do you think that this is not a function?
Eliz    Yeah. Right. Cause you don't have an equation.
Me      Okay. Okay, because you don't have an equation.
Eliz    Mmm hmm.
Me      Okay. Um, now how many functions, or, yeah, I should say how many expressions, yeah, how many functions are involved here?
Eliz    No, well, it doesn't seem like these are functions.
Me      Oh, they don't seem like functions?
Eliz    No, cause you don't really have anything, well maybe, I don't think you know enough about them to be able to tell if it's a function.
Me      Okay. Okay. Okay.
Eliz    Does that make sense?
Me      What you are telling me, yes, what you are telling me, it does make sense. Now let me explore another thing. If I asked you to graph it,
Eliz    Mmm hmm.
Me      Uh, can you make a drawing of it in, just right here.
Eliz    Well this seems like more like the time, you about, like they give you an equation and they say
Me      Uh huh.
Eliz    These are like restrictions or something.
Me      Uh huh.
Eliz    So it doesn't even seem like you could even graph it.

A second idea in the equation image is that functions come from equations (l2, 7; l3, 2, 6). In (l2, 7), relationship of equations and functions, I asked her if given the function, there was an equation that describes it. The following dialogue took place.

Me      Now, for every function that you have, do you always have to have an equation that describes it?
Eliz: Hmm, unless when you graph it [...] or you can just have a set of points and you graph it.

Me: So, you don't necessarily need

Eliz: an equation, but in order to get these points you need to have something, equations, to get the points.

Elizabeth started finding a rule for the graph of a piecewise function, (I3, 6), by asking "do you want me to get an equation for this graph?" I corrected and said "an expression." She could not come up with the mathematical expression for this item. I asked her if it was difficult to find it and she said

Yeah, I mean, you just think how they get the graph, it is from an equation you have to get the graph from an equation wouldn't you? That's why I always think that all functions are equations.

This idea of functions coming from equations might be the cause for the obstacles encountered in the need for an equation (p. 102-107).

The last idea in the equation image is that functions can be represented by equations (I3, 4a, 6; I4, 1; I5, 1). For example, in (I3, 4a), components of a function, she said that a function is "an equation that has both y and x-values. An equation that is set equals to y, I think." See I4, 1, p. 99; I3, 6, above; and I5, 1, p. 98.

**Familiarity**

The image of familiarity refers to recognizing some functions based on previous experience with them. We use some functions so frequently that they become part of our schemas to the point that we do not bother checking whether or not they are functions. This is the case of straight lines (except vertical ones).

This image appeared only in the first interview (I1, I3), describing a function with eyes closed. There is only one idea associated with this image: the nature of examples.

The following protocol illustrates the familiar functions that came to Elizabeth's mind.

Eliz: Any function?...It's on a graph, it has to do basically with the x-values, like if it were x squared, like if it's just going to be a single x it's going to be a
straight line [see figure 11, p. 95] and then could be an $x$ squared, it would be a parabola [drawing figure 15]

Figure 15. Elizabeth's graph of $x$ squared.

and then I don't know how the $x$ cube, I think it's like [drawing figure 16]

Figure 16. Elizabeth's graph of $x$ cube.

put on the graph, and just like alternated even powers are the same and odd powers are like this, like $x$ to the fourth would be the same as $x$ squared.

**Continuity**

Elizabeth said that functions are not always continuous when I asked her directly (I5, 2). In that case, the question referred explicitly to continuity. However, the image of continuity during the interviews was more a habit of connecting dots (I2, 4; I5, 6). This was the only idea attached to the continuity image. In the existence of a function discontinuous at one point, (I2, 4), she had difficulties understanding the problem, so I reworded the question in terms of inputs and outputs. She was still not sure of the existence of such a function. So I asked her to plot some points for that situation. She made the drawing as shown in figure 17.

Figure 17. Elizabeth's first graph of item (I2, 4)
Later, I asked her "What is the output for one half?" and "How would the drawing be now?" The following dialogue took place.

**Eliz** I don't know. Does this still graph like a parabola?

**Me** Yeah, I mean, because we are squaring.

**Eliz** [drawing figure 18]

![Figure 18. Elizabeth's rearrangement of item (12, 4) considering one half as input.](image)

**Me** Would it be something like that? It would be like a parabola and then has like a little wiggle?

**Eliz** Yeah, it would be [outside] like a regular parabola.

**Me** We didn't have exactly an equation. Is it a function?

**Eliz** Yes.

**Me** Could you please plot negative one half [as an input]? Where does it go?

**Eliz** [plotting a dot as shown in figure 19].

![Figure 19. Elizabeth's rearrangement of item (12, 4) considering negative one half as input.](image)

**Me** Could you please rearrange the wiggle there?

**Eliz** Still I don't know how to graph.

**Me** What would your educated guess be?

**Eliz** It could be like [drawing figure 20].
Is it right?

I answered 'no' and showed her the correct graph (see discussion of the end of the protocol in p. 107).

The habit of connecting dots appeared also in (I5, 6), making changes to five plotted points to make them the graph of a function. Her first reaction to this item was to connect the dots resulting in the graph shown in figure 21.

Summary of Elizabeth's Function Images

Seven function images emerged in the data collected from Elizabeth during the study: equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, continuity, and relationship. Several links between these images suggest the existence of a network. Elizabeth's network of function images is initially dominated by the relationship image, which supports a strong equation image. There is a very strong association between functions and equations throughout the study. Elizabeth's function images reflect an operational and quasi-structural conceptualization of functions.
Perceiving a function as an equation puts cognitive obstacles in dealing with the arbitrary nature of functions.

**Procedural and Conceptual Knowledge of Functions**

Elizabeth was an average student in the class. Her participation in class always took the form of a "how do I do it" question. In this section I present her approaches to several tasks to illustrate her knowledge of functions.

On her tests and quizzes, errors were common on finding the domain or range of functions. This situation appeared throughout the year either in her notation, reading the range from the calculator (Q-02/07, 2b; T-02/19, 4), or composition of functions. An example of her notation was with the function \( g(x) = x^2 - 3 \) in (Q-01/29). She wrote the domain as "\( x \geq 2 \cup x \leq 2 \)" and the range as "\( R \)." Reading the range from the calculator appeared to be another difficulty. In (T-02/19, 4), students were asked, among other things, to find the range for the function \( g(x) = \frac{2x^3 + 3x^2 + 2x + 3}{x^2 - 1} \) and sketch the graph. Elizabeth found the range to be "\((\infty, -3.75] \cup [2.8, \infty)\)," although her graph suggests that the range is all reals. It might be the case that the trust in the calculator is more appealing than the graphical representation. The discussion of composition deserves a special heading.

**Composition of Functions**

Elizabeth was able to carry out composition of functions in the usual testing; however, she remains at the procedural level. Her difficulties seem to be rooted in the familiarity with the resulting composition (an equation whose domain and range are well known). For example, in (Q-01/29) students were asked to find a rule for \( g \circ h(x) \) when \( g(x) = x^2 - 3 \) and \( h(x) = \sqrt{x + 3} \). Elizabeth found \( g \circ h(x) \) to be \( x \) and wrote the domain correctly as "\( x \geq -3 \)," however for the range she wrote "\( R \)." Similar situations occurred later too. In (T-02/19, 3), students were asked to find an equation for \( g \circ f(x) \) and \( g \circ p(x) \) where \( f(x) = \frac{1}{x} \), \( h(x) = \frac{2}{x^2} \), \( g(x) = 4x^2 \), and \( p(x) = \sqrt{x} + 4 \). Elizabeth found \( g \circ f_0 h(x) = x^2 \).
and wrote the domain as "R." She also found \( p^g(x) = 2\sqrt{x^2 + 1} \), the domain as "R" and wrote the range as "y = 4."

The usual testing suggests that she has a procedural knowledge of compositions. Later in the composition of \( f(x) = 5 \) and \( h(x) = x \), (Q-02/07, see Appendix B) her lack of conceptual knowledge of compositions became more evident. On that quiz, she wrote "5x" for \( f(h(x)) \) and "5x^2" for \( h(f(x)) \). In (13) she explained to me how she carried out the first composition. Her procedure also suggests what she might have done on the second one.

**Eliz** I remember I plugged in, took x for this and substituted x in for five and then I just got 5x, multiplied.

**Me** So what did you substitute?

**Eliz** I took this x, I put it into the f equation, I just multiplied the two.

**Me** So you had, like, you had a five and then you substituted, since you had f of h, you substituted. See this was 5 and then you just put the x.

**Eliz** Yeah. Well, I took this value and you put it into the equation h of x, and then I just multiplied the f in...kind h of x. Do you know what the right answer is?

**Me** Five.

**Eliz** You just put x = 5?

**Me** What happens is that this parenthesis...doesn't mean multiplication, means composition this is f will be applied to h.

**Eliz** Is \([h(f(x))]\) five also?

**Me** Yeah.

**Eliz** In this one I probably just took 5.

**Me** Yeah, what you're saying makes sense. What I would like to see is how you go to do it. That's an easier way. If you have x, you substitute for h and you put here x, and then you know what f of x is.

**Eliz** That's what really...the parenthesis means equals instead of multiplying.
Me  Yeah. Can you recall why is that you came up with $x$ squared in this one?

Eliz  I just probably figured that out...I think, I don't know. I probably put it as five $x$, cause $f$ is five, I don't know. I was stupid and then, multiplied 5$x$, multiplied times $x$.

Global Approach to Functions and Connections

The next protocol (I3, 5), graphing a rational function with limited use of the graphing calculator, illustrates lack of connections to older material; little conceptualization of the problem; the weakness of her conceptual knowledge (rich relationships); and the procedures emphasized in the course.

Eliz  I can factor out the two in the bottom. I don't know if that helps.

[writing \( \frac{x^3 + 2x + 1}{2(x^3 - 2)} \)]

And then, do you want me to find out how the graph looks like?

Me  Yeah. You can do anything you want with the calculator.

Eliz  So you can graph like these two separately?

Me  Yes, you can and you can tell me why is that you decided to graph one piece.

Eliz  OK, for this one [graphing \( x^3 + 2x + 1 \)] the graph is [drawing figure 22]

Figure 22. Elizabeth's graph of \( x^3 + 2x + 1 \).

and this one [graphing \( 2(x^3 - 2) \), see figure 23]
Figure 23. Elizabeth’s graph of \(2(x^3 - 2)\)

... you can put like the domain, I mean knowing the domain, the asymptotes, so domain would be all reals... because both graphs are continuous, they go on, so the same with the range, would it be where these two connect? Or where they intersect?

Me I don’t understand the question.

Eliz Would the graph start like where the two intercept?

Me Do you mean \(\frac{x^3 + 2x + 1}{2x^3 - 4}\)?

Eliz Yeah, this whole graph.

Me No, because you are dividing... It is good that you did those graphs, but now the question is, how will that information, or the information that each of this provides help you to do this one? For example, what information can you get from the top and what information do you get from the bottom part?

Eliz How do, they both cube, they’re both cube, ah, which the standard cube graph looks like [drawing figure 24]

Figure 24. The standard cube graph.

... and now it is basically this just shifted, like it’s a, I don’t know after that.

Me Suppose, this graph, which is the bottom part. What information would it give you about the whole thing?
Eliz: It's end behavior, they're both cubic, so the end behavior would be $y$ equals, well, I take the $x$ cube over two $x$ cube, so $y$ equals two, but that doesn't fit. Isn't it right?

Me: No. The end behavior analysis that you did is correct, but remember that this one is not going to give you by separate the end behavior because you are dividing.

Eliz: Oh, yeah, so you can think that the end behavior would go to two? [writing $\frac{x^3}{2x^3} y = 2$].

Me: The problem is that [what you wrote,] is right but, $[y = 2]$ is wrong.

Eliz: Oh, it's one half [correcting to $y = 1/2$]. OK. So you can get that.

Me: But you don't get the end behavior from the graph.

Eliz: Right, not from [the graph of any of these drawings] because they are separate.

Me: What I would like you to tell me is what kind of information can you get from these graphs by separate, they will tell you something about this [rational expression].

Eliz: Maybe asymptotes if there are any asymptotes.

Me: Which one?

Eliz: Both, I would say both.

Me: Both of them?

Eliz: Oh, no, I say [the top part]...No, the bottom part would give you the asymptotes.

Me: Where is the asymptote?

Eliz: I don't think, there isn't any one, [looking at figure 23, p. 117].

Me: Yeah, but remember this.

Eliz: Oh, that is not the picture of the whole graph.

Me: Yeah...where is the asymptote?

Eliz: Two, or this cube root of, [looking at figure 23]

Me: and graphically?
Eliz  ...in this picture?

Me    Yeah, where would you trace to find the asymptote?

Eliz  The zero point, [circling in figure 25]

Figure 25. A zero point of 2(x^3 -2).

Me    The zero point, so there is an asymptote in that point...Are there more asymptotes?

Eliz  Maybe one on the y-intercept, that would be horizontal...maybe, I don't know.

Me    Now, what about the top?

Eliz  That will give you the zero points.

Me    How many are there?

Eliz  Three.

Me    Three zero points, which ones?

Eliz  I don't know where they are. I just took that from the power ...you have three x values but some of them might not be reals ...they might be imaginaries ... [I corrected to complex].

Me    So how many? From the graph you can say.

Eliz  I say well, you really don't know. Oh, I say there is one ...because it is right over here [marking in figure 22 the zero].

Me    So we know the zeros, whatever that zero is, can you mark it [on new axes]? Because now we are putting the things together.

Eliz  It's going to be negative, [marking a dot, see figure 26] close to negative one or something.
Figure 26. A zero for the function in (13, 5).

Me    Can you mark where...the vertical asymptote is?

Eliz  It's over here [drawing a vertical line, see figure 27].

Figure 27. A vertical asymptote for the function in item (13, 5).

Me    What other things do we know? The end behavior?

[bell rang]

Eliz  $y$ equals a half [marking, see figure 28], it's over here.

Figure 28. A horizontal asymptote for the function in (13, 5) passes through $y = \frac{1}{2}$.

We ran out of time and Elizabeth needed to go to class; the interview finished here.

Some relationships (conceptual knowledge) appeared in the protocol that need to be mentioned. First, her comment after drawing the standard cube (p. 117) suggests an attempt to carry out a transformation. However, she could not continue with the task. Two more situations are the relationship between the degree of the polynomial and the number of zeros (p. 119), and the geometric connection between real zeros and x-intercepts.
Elizabeth's description of another rational function, (I4, 4), was poor even using the calculator. She found domain, range, horizontal asymptote, end behavior, and quadrants where the graph was located. In the case of the domain, she said

[okay, um, well, I think, well, there looks like, sort of like the domain there's not a...It doesn't look like it's all reals, but it's in between, um, let me trace, I mean, if you just ask me, I think it's between like negative infinity to negative a half to um, positive a half to infinity. I mean, that's just a guess...I didn't get into real values.

In the case of the range, the difficulty with order in the real numbers appeared again. The following protocol illustrates the point.

Eliz for the range, it's negative infinity to one, or to two, I mean.
Me But the range is negative infinity to two? Is that what you told me?
Eliz Oh I'm sorry, positive infinity to two.
Me Okay. You mean two to positive infinity
Eliz Yeah, two to a positive, I'm sorry
Me Okay. That's okay.
Eliz Two to positive infinity.

Elizabeth also found a horizontal asymptote at y = 2 and the end behavior "both, both positive infinity, no matter which way you go," and located the graph: "it's like both in quadrant, just in one and two." Her analysis concluded by saying "I don't know. That's about it, I'd say."

**Settings: Domain, Range, and Rule**

The course dealt mainly with examples where the domain was either the set of real numbers or the set of real numbers minus at most five points. The range appeared to be the union of intervals at most. This might explain Elizabeth's difficulties in problems where the domain or the range were arbitrary sets; functions with a discrete domain (XC2; Q-02/19, see appendix B); or word problems as in the items in (I2). For example, in the function that assigns to every triangle its area, (XC3), she wrote the domain as "R"
and the range as "y = 10." In (I5) she said, she used the function y = 10 in this problem. This implies the use of the equation image to solve the problem.

The function that assigns price to pieces of clothing, (XC2), was defined on a nonnumerical discrete domain. She chose the given table not to be a function, "because they are items with prices and so no certain relationship." Again the obstacle that the relationship image puts is evident.

Elizabeth's definitions of functions do not mention domain nor range as components. They seem to be isolated. However, this case is not totally clear, since in the parabola defined only for integer numbers, (Q-02/19), she drew the graph in figure 29.

She wrote "R" for the domain and "y ≥ 0" for the range. But in (I4) she said that she "didn't know what [Mr. H.] meant for integer numbers." So it seems that the wrong answer came from being unsure about the domain given, not from ignoring it. In this case she proceeded by familiarity.

Construction of Graphs

The construction of the graph for the roller-coaster problem in (I4, 5) caused an initial difficulty. However, Elizabeth completed the task successfully (see figure 30). She said

it would be the opposite of the track...Cause as it goes down the speed goes up...And then as it goes, as it goes up, the speed goes down.
Development of Functions

This section is composed of two parts. The first part describes some of the ideas on functions that students brought to the class. Data are drawn from the Practice Test on Functions (Appendix C). The second part is a discussion of the development of the concept of functions based on the concept images and definitions provided by students throughout the study.

Some Ideas on Functions at the Beginning of the Course

Some of Elizabeth's ideas on functions were captured on the Practice Test on Functions. Her definition of a function was "a graph of a relation" without distinguishing functions from relations. She was familiar with simple functions. In item 3 when asked for examples and non-examples of functions she drew a constant line as an example of a function and one branch of an hyperbola-like graph in quadrants I and IV as a nonexample of a function.

Images associated with functions were graph (see definition, p. 97), vertical line test, equation/formula, continuity, and familiarity. Some of her answers suggest that she was familiar with the vertical line test to decide whether a graph was a function or not. However, it was not clear that she could use it to produce graphs of functions. For example, when six points were given and students had to draw the graph of a function
whose domain included the given points (PTF14), she connected the dots but did not check whether her resulting graph was or was not a function.

Other answers where equality of functions was asked (PTF8), suggest that she perceived functions as a formula/equation and ignored the domain and the range. More evidence supporting this emerges from her answer to (PTF10), where she drew a continuous straight line as the graph of a function with domain and range equal the set of natural numbers. Furthermore, the equation image might also explain her use of the function \( g(x) = -7x \) instead of \( f(x) = -7 \) to evaluate the constant function \( f(x) \) at different numbers.

The next image associated with function is that of continuity—connecting dots (8d, 13, 14). For example, in 8d she wrote "no not complete function" for the set of discrete points. Also, when two points were given graphically and students had to draw the graph of a function whose domain included the given points 9PTF13), she drew a smooth continuous graph and chose the option "more than 2 but fewer than 10" without explanation.

As expected, at times the distinction between images is difficult, as in the case of equation/formula and familiarity. For example, her answer concerning whether a piecewise relation was a function (PTF2) suggests two situations: either the decision was based only on one piece "because of the \( x^2 \) in the equation" or the familiarity with the parabola.

Some of the items that she answered incorrectly might have been caused by lack of familiarity either with the notation, with the vocabulary, or with functions used in the items. Regarding vocabulary, it seems that she did not use the terms images and preimages as terms for functions. The use of piecewise functions on the test exhibited, maybe, the emphasis of former courses. Regarding notation, she did not seem to be
aware of the use of open circles and dots in piecewise functions. Other mistakes with piecewise functions were in finding the range or recognizing graphs with given domain and range.

Difficulties with the syntax of mathematics became evident with the notation she used for sets. For example, for a piecewise function (PTF9) she wrote the domain and the range as "D = \{x/ -3 < 5 \} R = \{y/ 4 > 2 \}.

The Practice Test on Functions exhibited mastery of content of former courses and images associated with functions. In the next section, I discuss the changes that occurred in Elizabeth's images and definition of functions during the study.

Images and Definitions of Functions during the Study

There are some changes in the six definitions that Elizabeth provided during the study. Three images seem to be central in the definitions: equation, graph, and univalence. The vertical line test image joined these images to form a network at the end of the course. Particularly the last definition suggests such a network (p. 98). The image of equation is the most consistent. It manifests itself as a description of a relationship between x and y.

Elizabeth's definition of function in (I2) incorporated the image of equation explicitly (see p. 98). However, the definition does not discriminate between functions and relations. In spite of this situation, it is clear that she can decide whether a graph is or is not a function. She answered correctly the first three items on the interview. Her use of the image of one y for every x on (I2, 1, 3) and the vertical line test on (I2, 2) also suggests that she can use these two images as equivalent. There is evidence of a network of images in this interview. In particular, during (I2, 7), relationship of equations and functions, the images of equation, graph, and the vertical line test appeared in the protocol. However, the weakness of this network became apparent in other items.
The idea of relationship in the equation image puts obstacles in dealing with functions where no such relationship is evident (see pp. 102-107). For example, all her attempts to solve (I2, 6), existence of a piecewise function discontinuous everywhere, were unsuccessful because she tried to find a formula for the given function.

Most of the problems in (I2) were word problems. It was clear in the interview that Elizabeth did not attempt to graph any item. This allows one to conjecture that she experiences difficulty moving from the verbal representation into the graphic representation. A second explanation might be the lack of heuristics to solve a problem.

During (I3) there is also evidence of a network of two images: equation and univalence (see fourth definition, p. 98). Again, it became evident the weakness of the equation image. For example in (I3, 6), finding a rule for the graph of a piecewise function, she wrote the domain and the range for the piecewise graph, but could not find a mathematical expression.

Relative progress in this regard appeared with the constant function. The difficulties that she experienced in I2, 5 (p. 102) did not appear in finding an equation for a constant function given graphically, (Q-01/29, see Appendix B). In this last case, she got all the items correct.

A relative contradiction and relative progress appeared in the components of a function, (I3, 4a). There she said that "all equations, yeah, ... are [functions] because if it is just a set of points, if you are just giving points, I don't think that's really, well that could be a function. I don't know." The relative contradiction is related to all equations being functions, without any extra requirement on the equation. The relative progress is in the insecurity of a set of points being a function, as opposed to (I2, 7), relationship of equations and functions, where she said that three points could not be a function. However, a deeper analysis of her protocols showed that she was thinking that a discrete
set of points belonged to an interval and therefore she proceeded to connect the dots
given.

During (I4) there is also evidence of a network of three images. The image of graph
has joined the images of equation and univalence. However, the graph is dependent on
equations, because "in order to graph [a function] you have to have an equation" (I4, 1).

Although the image of univalence appears in the network, it is the equation image
that is used to work with the concept of function. This image was an obstacle to deal
with the piecewise function in (I4, 2) and with a set of points in (I4, 3a). The two
functions were rejected as functions; the first because she did not see an equation (see pp.
108-109); and the second one because there was not a pattern (see p. 103). However, in
(I4, 3b) where there was a pattern, she applied the vertical line test (p. 100).

In the last interview the vertical line test is mentioned in the definition, joining the
images of equation, graph, and univalence (p. 98). The equation needs to satisfy the
univalence property. A way to know that is to graph the equation and use the vertical line
test.

The question on properties of functions (I5, 2) provided insight into the graph image.
During this item the image of graph seems to have its own identity (a difference from the
two proceeding interviews). She said that not all functions were equations, "sometimes
they are just graphs, I mean, like if you have a graph of a function you can't necessarily
always get an equation from it." She said that all functions can be graphed. This is in
contrast, with her saying "no" to the question "Are functions defined only for numbers?"
Later, in (I5, 4), relationship of graphs and functions, the graph is a way to "get more
information out of the function...like the domain and range...and that, the end behavior
and, there isn't any, holes in the graph." Although not recognized by Elizabeth, this is a
benefit of the graphing calculator.
Elizabeth's idea on representation of functions at the end of the year reduces to graphs, equations, and a "plot of points" (15, 3). This plot of points resembles the idea of a table. It is in this interview, "how to explain functions to precalculus students, (15, 5), when the domain and the range appeared in a definition. I asked her, "How would you tell Mr. H. to define functions?" and she answered "an equation that has a set of x and y values for, that no two points are alike." This seems to be a big conceptual jump. Since the "one y for every x" has shaped into two sets.

Summary of Elizabeth's Development and Knowledge of Functions

Elizabeth started with and maintained a view of functions as relationships throughout the study. This is a basic but fundamental idea about functions (Sfard, 1989). However, Elizabeth experiences a strong tendency to equations (a quasi-structural conceptualization), which impedes her dealing with functions where the relationship is not evident; in particular with discrete domains, simple correspondences, or piecewise functions.

Elizabeth experienced two kinds of development: procedural and conceptual. The course emphasized the procedural one and she benefited with strong algorithms. However, at times there was not much understanding of them. Conceptually, she seems to have attained a network of function images. The network, however, seems to be dominated by the equation image. This causes her to miss one of the essential feature of functions--arbitrariness. The other essential feature of functions--univalence--emerged few times. Chapter VI contains a comprehensive summary of Elizabeth's case and a diagram representing her network of function images.
The Case of Steve

Introduction

Steve is a competitive student, who likes mathematics. He has always liked it because it involves thinking, and is taking precalculus to do a higher level of mathematics. He thinks that the course will help him on his major in college—architecture; he also thinks that by taking the course he will expand on subjects that he has already learned. Steve thinks that mathematics is pretty tough and challenging.

Steve used a graphing calculator in algebra and thinks that using it in precalculus is helpful. He is enthusiastic about this. In his opinion, the calculator helps him to understand mathematics when he sees the graph, to reason why things work, and to understand how he gets an answer. However, in his view, even with the graphing calculator, mathematics is still tough.

Steve's enthusiasm for the calculator has made him the class expert using it. He likes playing around with the calculator, uses it in physics, and has some programs on it. He has helped several students in the class to take advantage of it. He uses the calculator as a tool to double check himself, to check that his algebraic manipulations are correct for example, or sometimes on the tests, when he is not sure that his answer is right.

Steve has been working hard in the class. At the beginning of the year he started as a B student and by the end of the year his grade moved up to A's. He enjoys the small-group activities because different people see things differently and peers can explain things in different ways, and at the level of anyone in the group.

How Steve defines a Function (Concept Definition)

Steve provided the following definitions of function during the study. The first definition contains the image of graph and an implicit image of equation. In (PTF16) he defined a function as "[a]nswer to graph".
Steve reversed the univalence criterion in his second definition:

A relation is a set of ordered pairs. A function is when you have exactly one input for one output (Q-09/11).

His third definition contains the image of univalence: "...the definition of a function, it's for every x there is one corresponding y" (I2, 8).

The fourth definition (I3, 2) is a blend of four images: equation, graph, vertical line test, and univalence (although with some hesitation).

I guess, it'd be like an equation or a graph and where there's only one, I think it's one x-value for y or for every input there is one output so there is only one, you can't cross like the equation like twice, like it has to be something either like that [drawing Figure 31]

![Figure 31](image-url)

Figure 31. Steve's example of a function in definition 4.

So it doesn't cross, like the vertical line test. If it's like a line straight down like something like a circle or something, [drawing Figure 32]

![Figure 32](image-url)

Figure 32. Steve's nonexample of a function.

It wouldn't work cause there is two values that solve it...like that one would have two numbers [marking two dots in figure 32].

The fifth definition is similar, except for the equation image. The univalence image is used in terms of the graphing calculator (trace).
FOR EVERY ONE X value there can only be ONE y value to match it. NO
matter where on a graph every x is matched up with a different y value. For
example if you trace down the line you will get a different y value for every x
different. Always be different. [figure 33 shows his scratchwork] (Q-03/04; see
Appendix B).

Figure 33. Steve's example of a function in
definition five.

In the sixth definition, only the univalence image is mentioned and with hesitation again
(see 4th definition, p. 130).

See we had that on one of, that one thing. I think. Uh, I'm not sure whether it's
for every input you get one output or for every out you get one in. I'm not
sure...You know, for every input you get exactly one output. (I4, 1)

The seventh definition is similar to the fifth one (p. 128). Here he did not refer to the use
of the graphing calculator and hesitated stating the univalence criterion.

(A function) In a function for every y-value there is a different x-value. For
every input there is exactly one output. Must pass the vertical line test. (XC1)

The last definition is the realization of the univalence property but with more flexibility
stated explicitly: the possibility of 'many to one'.

Um, let's see, let's see, for a, exactly every input you'd get one output, for every x
you'd get a y, no two, you can never have the same x values, you can have the
same y values but no, not the same x values. (I5, 1)

Steve's Mental Pictures Associated with Function (Concept Image)

Several images associated with functions emerge from the data collected from Steve.

Images have been separated, but overlapping images suggest a network instead of a
partitioned set. The images associated with functions are the following:
equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, continuity, and regularity. Indications of each cluster follow.

**Equation/Formula**

This image appeared in all the interviews but the fourth one. It seems to be the most anchored image on Steve's mind. This image manifests itself in several ways:

1. functions can be represented by equations (I1, 13, 14c; I4, 1; I5, 3),
2. functions are equations (I2, 6; I5, 1, 7), and
3. not all functions are equations ((I5, 2; I2, 7; I5, 1; Q-03/04 in I4).

The first idea in the equation/formula image is that functions can be represented by equations. Describing a function with his eyes closed, (I1, 13), he said

> I think like an equation, like a graph or something like y equals like three x cube plus two x or whatever I think like a little, I visualize it, I see it on the graphics calculator like graph functions, things like that.

And later, when I asked his ideas about f(x), he said: "I think like an equation, on a graph" (I1, 14c). In what makes up a function (I4, 1) he hesitated on the question. So I asked the same question in (I5) after going over the member check of the fourth interview. The following dialogue took place:

**Stev** Let's see. This other, uh, well, then, it would probably have to be, like, I don't know, variables that are like that are raised to certain powers and put into like an equation,

**Me** Mmm hmm.

**Stev** So when it comes out it gives you possibilities, when you trace it will give you possibilities of what your variables could be, so if your variable is one your other variable has to be another one, cause it's kind of set up that way, on this one you like graph it, the answer's on the graph, if you plug that in to your calculator.

**Me** Mmm hmm.

**Stev** You'll get the, like if you plug in the x answer, you'll get the, you'll get a y answer out for that

**Me** Mmm hmm.

**Stev** So it's just for every possible variable, there's a, for x, there's a different y one that works.
The issue of representation appeared more explicitly in (I5, 3), when I asked him "how can you represent functions?", he said "in the equation form, a graph."

The second manifestation of the equation/formula image is that functions are equations. Hence the need to have or look for an equation in order to decide if a representation either verbal (I2, 6), geometric (I3, 6), or algebraic (I5, 7) is a function. This conception of functions imposes an obstacle in any case. In (I2, 6), existence of a piecewise function discontinuous everywhere, he answered positively and gave the following explanation:

I said like, I think this is more like, I took the squared root. The squared root, of uh, like x or [\sqrt{x}] whatever and I got like non-integers and then, when I took, when I square you get integers but when you take the squared root you get non-integers...Say like take, like if you have something if you take the squared root of like three that will be a non integer, and then if you take, if you square three that will be an integer...that's how, that's what I thought.

Steve could not find a mathematical expression for the piecewise function in (I3, 6). He could find the domain and the range, however. The following dialogue illustrates his difficulties.

Me When it comes about the expression, you said that
Stev It's going to be a linear equation, I think.
Me Why is that it's going to be a linear equation?
Stev Because, I don't know, cause those are lines, I guess.
Me Can you think of any expression that might give you that [graph]?
Stev Ah, I'm not sure.
Me Have you seen a familiar problem?
Stev Yeah, some things, but I'm not sure how, with this one...That's the only thing I can think of.
Me Any guess? If you were to guess, what would your guess be?
Stev Of what the equation would be?
Me    uh uh

Stev  I'm not sure, I have no idea.

The following dialogue (I5, 7) illustrates his strategy to solve a problem to decide if there exists a function passing through four points given algebraically:

Me    The question is, is there a function such that f(2) = 0, f(-3) = 7, f(-4) = 1, and f(\frac{1}{2}) = -1? [handing him paper for scratch work], is there a function like that?

Stev  Let's see, can I use my calculator?

Me    You can use, yeah.

Stev  Okay. Two and five, [writing these numbers], you plug two and will be zero, let's see, um, my first instinct without either, with just looking at it

Me    Mmm hmm.

Stev  I'd probably have to say, I would think, it would be, you plug two in, let's see, it would be zero there, hmm, I would probably have to say no.

Me    You would probably say no, and your reason is?

Stev  Um, if you plug in two and you get out zero, if you multiply something by two to get zero, it's gotta be zero, therefore, if you multiply by negative three times zero, four and one half,

Me    Okay.

Stev  Then to make this zero if you did negative or two minus two, it would be zero, then if you have two minus three it would be five,

Me    Okay.

Stev  And it would just be, it just doesn't work.

Me    Am I correct if I understand that you are trying to find an equation for those four points?

Stev  Yeah.

Me    And uh

Stev  And see, plug those numbers in and see if they work, substitution,

Me    And they didn't work.
Stev  No. I don't think it works, it might work, but
Me   Okay. So you think that there is not such a function.
Stev  No.
Me    No. Okay. Um, and it is because you were trying to find an equation and
      you couldn't find it.
Stev  Yeah. Is there a function?
Me    Yeah.
Stev  It's probably something really long.

The third idea in the equation/formula images is that not all equations are functions. This idea appeared as a property of functions (I5, 2):

    Me    Let me ask you about other ones, for example, equations. Are all
          functions equations?
    Stev  Not all of them.
    Me    Okay. So a property is that not all functions are equations.
    Stev  Yeah.

However, equations can be functions if they pass the univalence property (I2, 7; I5, 1, p. 131); or when graphed they pass the vertical line test (Q-03/04 in I4). In the case of the univalence property (I2, 7), concerning the relationship of equations and functions, he said

    Equations is like, that gives you the line and that tells you it's a function of... if
    you take your equation and you have an x, y value plugged, your actual y, you
    should have one for every x, should have one like one correspondent y or
    whatever. You just plug in like, the number to see if there is one for one changed
    of.

The idea that equations are functions if they pass the vertical line test appeared explicitly in (I4), when he was explaining to me his definition of a function (Q-03/04, p. 131).

    I guess, it'd be like an equation or a graph and where there's only one x value for
    y or for every input there is one output, so there is only one, you can't cross like
    the equation, like twice.
Graph

Two ideas emerged associated with the graph image in the data collected from Steve:

(1) functions can be represented by graphs (PTF16; I1, 13; I5, 2, 3, 4), and
(2) not all graphs are functions (I5, 4; Q-03/04).

The first idea was the initial image in Steve's first definition (p. 129; I1, 13, p. 132).

At the end of the year this image became a property. In the last interview, I asked him
directly "can functions be graphed?" in the discussion of properties functions have. He
said "yeah, functions can be graphed." I pursued this idea by asking him "so all functions
can be graphed?" and he answered "yeah, all functions can be graphed." The idea of
functions being graphable as a property of graphs continued in the relationship of graphs
and functions in the same interview (I5, 4):

Me  Okay. Let me ask you about question number four, are functions and
    graphs related?
Stev  Um yeah
Me  Yeah?
Stev  Because you can always graph functions.
Me  You can always graph functions.
Stev  Yeah. If it's a function, you can graph it.
Me  So if it is a function?
Stev  Yeah. Then you will be able to graph it.
Me  You will be able to graph it.
Stev  Uh huh.
Me  So that's something that you said [in (I5, 2), p. 133], right?
Stev  Yeah.
Me  All functions can be graphed.
Stev  Properties.
Me  Uh huh, you'll be able to graph it, so that's the way they are related.
Stev Yeah.

Me Uh huh, you'll be able to graph it, so that's the way they are related.

Stev Yeah.

See (15, 3) in p. 133.

The second idea associated with the graph image is 'not all graphs are functions'.

This idea emerged in the continuation of the dialogue presented above (15, 4):

Me However, going the other way around, are all graphs functions?

Stev No.

Me No?

Stev Let's see, you can have a graph that won't be a function.

Me Okay, can you give me an example?

Stev Um, let's see, you won't be able to do it on your calculator because your calculator, it doesn't really graph non-function things, but like say, there's like a drawing, can I do it up here?

Me Yeah.

Stev Like there's one [drawing figure 34] that's like that, the sideways parabolas,

Figure 34. Steve's examples of non-functions.

Me Okay.

Stev Um, let's see, there will be, like if you have, you can't graph like a circle [overlapping on figure 34] like that,

Me Okay.

Stev Just things like that.

Me Okay, so those are examples of graphs that are not functions, right?
However, graphs can be functions if they have the univalence property (Q-03/04, pp. 130-131); or if they pass the vertical line test (Q-03/04, p. 135).

Vertical Line Test

The vertical-line-test image manifested itself in different ways too:

(1) a means to decide if a graph is a function (I2, 1, 3, 5; Q-03/04; I4, 3; XCl),
(2) a property of functions (I5, 2), and
(3) an equivalent statement to the univalence feature (I2, 1, 8).

Steve used the vertical line test to decide if a graph was a function or not. For the graph of a non function, (I2, 1), he said

I just put it wasn't like a function. I know I've seen things with curves like that, like they don't cross one another in the same like quadrant or whatever and I guess I looked at the, like if you drew a line through, there are two points for like x right there, there and there, and then I said it wasn't a function because from the definition of function like for every like x there is a y, exactly one y, stuff like that.

In (I2, 5), existence of a function all of whose values are the same, the following dialogue took place:

Stev yes, because it'd be like, I said for example if you had like a straight line that has values, that are equal to one another. Always has the same, like it's going to be the same y but different x's [drawing figure 35]

Figure 35. Steve's straight line.

Me Could you please draw your axes there?

Stev [drawing figure 36]

Figure 36. Steve's straight line with axes.

Then, I just said that, it is a line like that, it is going to have the same y but different x's along it [drawing a vertical line, figure 37].
Steve also applied the vertical line test to graphs with a discrete domain. The following dialogue (I4, 3), graphs of relations with four dots, illustrates this situation:

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td><strong>Steve</strong></td>
<td>The first one, yeah, I'd have to say yeah, just from what I've like learned, I don't know, I'd just probably go and draw a line through and make sure that only on one line could go through the point.</td>
</tr>
<tr>
<td><strong>Me</strong></td>
<td>Uh huh. So, uh, you'd use the vertical line test.</td>
</tr>
<tr>
<td><strong>Steve</strong></td>
<td>Yeah. I'd probably just use that.</td>
</tr>
<tr>
<td><strong>Me</strong></td>
<td>Okay. So in part b, what would be the case?</td>
</tr>
<tr>
<td><strong>Steve</strong></td>
<td>I'd do the same thing</td>
</tr>
<tr>
<td><strong>Me</strong></td>
<td>Mmm hmm.</td>
</tr>
<tr>
<td><strong>Steve</strong></td>
<td>If this one looks like it kind of, it looks like they kind of go through,</td>
</tr>
<tr>
<td><strong>Me</strong></td>
<td>Mmm hmm.</td>
</tr>
<tr>
<td><strong>Steve</strong></td>
<td>So I would probably say it wasn't a function.</td>
</tr>
<tr>
<td><strong>Me</strong></td>
<td>This is not because of the vertical line test.</td>
</tr>
<tr>
<td><strong>Steve</strong></td>
<td>Yeah.</td>
</tr>
</tbody>
</table>

Other instances of Steve's use of the vertical line test to decide whether or not a graph is a function appear in the following questions: I2, 3, p. 143; XC1, p. 131; Q-03/04, p. 135; I5, 6, pp. 147-148.

The second idea associated with the vertical line test is being a property of functions (Q-03/04; I5, 2). In (I5, 2), properties functions have, he said

Say, I don't know, they have be like function, they have to have only one x thing, you can pass the vertical line test, that's kind of like a property to be a function.

The idea of the vertical line test as a property of functions also appeared in Steve's fifth definition (Q-03/04, pp. 130-131).
The last idea associated with the vertical line test is that of being equivalent to the univalence feature. In (I2, 8), definition of a function, I asked him if he saw any relationship between the definition of functions as for every \( x \) there is only one \( y \) and the vertical line test. He said

Yeah, I say, they are the same thing because this way if it were reverse, if it was rather for every \( y \) there is one different \( x \), you have to have like a horizontal test but it's just like just like the same, I think. If you have a line that, they are basically the same thing, I think... one is words and one is something that you do. One is the definition and one is like things to do.

One Output for every Input.

There are several ideas associated with this image of functions:

1. definition (Q-9/11; I2, 8; XC1; I5, 1),
2. equivalence to vertical line test (I2, 1, 2; I4, 2),
3. property (I2, 7; Q-03/04; I4, 1; I5, 2); and
4. a means to decide whether or not equations or graphs are functions (Q-03/04).

The definition idea appeared in four specific items that asked for a definition of function (2nd and 3rd definition, p. 130; 7th and 8th definition, p. 131).

The second idea associated with the one-output-for-every-input image is that of being equivalent to the vertical line test (see above). Steve's approaches to several problems convey this idea too (I2, 1, 2; I4, 2; Q-01/29). In (I2, 2), discontinuous graph of a function, he said

I wasn't sure, I kinda guessed because I remembered an open circle means like greater than or equal to or something like that. I thought the open was greater or equal to and that goes to that point and this one is there, so they don't actually have the same point there, they shift but there is no two at the same point.

A second instance of the equivalence between the vertical line test and the univalence criterion for functions appeared in the fourth interview. There he explained that he decided the constant function \( f(x) = 7 \) given graphically (Q-01/29, see Appendix B) to be a function,

[c]ause, it's got a different like \( x \) value for every \( y \), so right here, it has, it'll have the same \( y \) value but different \( x \) values.
Other instances of the vertical line test and the univalence criterion as equivalent ideas appeared in deciding whether or not a graph is a function (I2, p. 138) and in his discussion of how many functions there are in a piecewise function (I4, 2, pp. 145-147).

The third idea is that one output for every input is a property of functions (I2, 7; Q-03/04; I4, 1; I5, 2). In "relationship of equations and functions", he said

...if you take your equation and you have an x, y value plugged, your actual y, you should have one for every x, should have one corresponding y or whatever. You just plug in like, the number to see if there is one for one changed of. (I2, 7)

In what makes up a function, he was unsure about it, and said:

See we had that on one of, that one thing. I think. Uh, I'm not sure whether it's for every input you get one output or for every out you get one in. I'm not sure...You know, for every input you get exactly one output. (I4, 1)

The idea of property also appeared in Steve's fifth definition (Q-03/04, p. 131) and in his discussion of the properties functions have (I5, 2, p. 139).

The last idea associated with the one-output-for-every-input image is to use it as a means to decide whether or not equations or graphs are functions (Q-03/04). Steve's discussion appears in p. 135.

Familiarity

Steve used this image in interviews (I1, 13; I2, 3, 4) and in tests (Q-01/29; Q-02/07). There are two ideas in the familiarity image. The first idea is familiarity with examples of two types: familiar examples of functions (I1, 13), and familiar shapes (I2, 4). The second idea in the familiarity-image is to use this knowledge to decide if a given representation is a function or not.

An instance of familiar examples of functions appeared when I asked him to describe a function with his eyes closed, (I1, 13), he said "[the first drawing that comes to my mind is] kind like the cube function."
Familiarity with shapes of functions appeared in the existence of a function discontinuous at one point, (I2, 4). Steve's first approach to this problem was to say "it's probably stretched out but, it'd be like a parabola" [drawing figure 38].

Figure 38. Steve's first approach to (I2, 4).

The second idea in the familiarity image is to use this knowledge to decide if a given representation is a function or not. This idea appeared in the fourth interview (Q-01/29; Q-02/07, Appendix B). In the first case (Q-01/29), finding an equation for the constant function, he explained why the given graph was a function.

Stev That this is a function?
Me Yeah.
Stev Cause, it's got a different like x value for every y, so right here, it has, it'll have the same y value but different x values.
Me Okay.
Stev And it's like a straight, it's a straight line.
Me Mmm hmm. Does it make it a function? Being a straight line?
Stev Well, most of the time it's a function, if it's a straight line unless it goes like straight up and down.
Me Say that again.
Stev Unless it goes like straight up and down.

In the second case (Q-02/07), composition of f(x) = 5 and h(x) = x, there was no need to graph the algebraic representation.

Me Did you graph for any of those?
Stev No, because, cause I figured if the graph right here was five,
Me Mmm hmm.
If y equals five, I know it's a straight just like that one of seven

Mmm hmm.

So I knew that the range would only be on line with five

Okay.

I knew that it would go to negative infinity that way. But on this one I knew it would be a diagonal line, because it would be the slope of whatever

Uh huh. And so how is it that you decided for both being functions?

How?

How?

Um, let's see, well, both of them are straight lines again, and not, and they aren't vertical

However, this image produces obstacles when a given representation does not look familiar. For example, Steve decided correctly that the continuous graph in (12, 3) was a function, by using the vertical line test. However, he told me in the interview that he was not really sure at the beginning. I asked him what happened and he said "[it's straight, like this line confused me...because it doesn't like follow the normal behavior, like graphs I've seen." So when I asked him what are the graphs that he had seen, he answered if the graph looks like, if you draw like a sine [drawing figure 39]

Figure 39. A familiar graph for Steve.

or something how it goes up and down, stuff like that. It's just I haven't seen something that goes from a curve until like probably a straight line.
Continuity

Two ideas associated with continuity emerge in the data collected from Steve:

(1) as a property of functions (I5, 2; I5, 6), and
(2) as connecting dots (I2, 4; I4, 2; I5, 6).

The first idea, continuity is a property of functions, appeared in Steve's discussion of properties functions have (I5, 2). The following dialogue took place.

Me  Let me ask you another property. What about continuity? Are all functions continuous?

Stev  Uh, functions are always continuous.

Me  Are all continuous?

Stev  Yeah. I'd say yes. From what I know.

The idea of continuity as a property of functions also appeared in the changes needed to five dots to make them the graph of a function (I5, 6, pp. 147-148).

The second idea in the continuity image, connecting dots, appeared in (I2, 4; I4, 2; I5, 6). Steve's approach to a discontinuous function at one point, (I2, 4), was different from Elizabeth's (pp. 111-113). First, he seemed to be driven by the familiarity image (p. 142) and started with a familiar global graph (a parabola). A second difference with Elizabeth's approach is that I did not insist on asking for him to plot additional points such as one half and negative half (and their respective images). I only mentioned that the image of zero was one. The following dialogue took place, when he realized that (0, 1) belonged to the graph.

Stev  This moves it up one.

Me  Will it move the whole thing?

Stev  No, it just changes the bottom.

Me  How will it be? Can you make a drawing?

Stev  How would it change the bottom of it...I think that like this part [referring to both branches of the parabola] will be the same but instead of going down to zero [drawing figure 40]
Figure 40. Steve's transformation of a familiar parabola in (I2, 4).

it'd be changing curve short, like that [drawing figure 41]

Figure 41. Steve's graph for (I2, 4).

But everything else would be the same except for that one, except that one number...when is getting closer to the vertex, or whatever...it's getting closer and closer.

A similar situation appeared with a piecewise function defined algebraically, (I4, 2).

Steve recognized it as one function. I asked him to draw the graph of it. The following dialogue took place.

Me Can you make a drawing, if possible?
Stev I'll try to.
Me Yeah, I mean
Stev I'll try, alright, let's see, see, so if it's greater than or equal to zero, it equals one. Correct?
Me Mmm hmm.
Stev So this would be one right here [drawing figure 42].

Figure 42. Steve's graph in (I4, 2).
Me  Mmm hmm.

Stev  It could go there, and then, it could go down and go to negative one, I think, sort of like that, I'm not sure [drawing figure 43].

Figure 43. Steve's explanation to figure 42.

Me  What [did you say?], is that what it's doing in this middle?

Stev  Possibly, I'm not sure.

Me  Okay.

Stev  It just goes like that, I was just making this darker [pointing to figure 42].

Me  Okay. Now is that a function?

Stev  Uh, yeah. It would probably be, it could have holes, couldn't it?

Me  Umm.

Stev  Because this one's, this one covers everything equal to zero,

Me  Mmm hmm.

Stev  And this one's, the rest I guess.

Me  Okay. So what is happening in the middle?

Stev  Well, it's coming out like, meeting up so that it covers all the things like this, I'm not sure.

Me  But it is still a function?

Stev  Yeah.

Me  Why?

Stev  Mmm hmm. Um, let's see, cause it, uh, cause none of the x's overlap, they have like their own x points.

During the member check in the fifth interview he modified the graph. The following dialogue took place.

Stev  I think it's, or I just drew it like that, but
Me    Can you make it, yeah, down there

Stev I just drew it like that but it was supposed to be like [drawing figure 44], you know, how it goes, it's curved but it goes along.

Figure 44. Steve's member check to (I4, 2).

Me    Oh, okay, so is that what you are saying?

Stev Yeah.

Me    That it could be like that one?

Stev Yeah.

A third example is making changes to five plotted points to make them the graph of a function, (I5, 6), where Steve connected the given points. The following dialogue shows his hesitation based on a didactical obstacle compounded with the continuity obstacle.

Stev Does a function have to be like connected line, can it be just points like these?

Me    Why don't you answer yourself and tell me what you think?

Stev I'm not really sure, I'd say [functions] can be points, but I'm not sure if you want

Me    Why would you say that they can be points?

Stev Well, actually I probably think they can't be.

Me    They cannot?

Stev Yeah, because we've never really had like just points, I mean we were working functions and

Me    Okay, so

Stev I mean we haven't been doing points or whatever.
Me  If you think that it is not yet a function, why don't you make any changes to make it the graph of a function? What changes could you make?

Stev  Oh, you could do this, cause, well, let's see, that'd be a function.

Me  Okay, so your changes could be connected dots and that could make it the graph of, and the graph that you made is the graph of a function.

Stev  Yeah and it goes through all these points.

Me  Okay. How do you know that it is the graph of a function?

Stev  How do I know?

Me  Yeah.

Stev  I mean because it doesn't, it passed the vertical line test and

Me  It passes.

Stev  And it's continuous, and it looks kind of like one of those sine waves that we've been doing the last couple of weeks

**Regularity**

The last image associated with functions is regularity. This image only appeared with the continuous graph of a function, (I2, 3) and seems to be related to familiarity. At the end of the dialogue in p. 143, I asked him if the problem was that the graph changes suddenly, and he replied, "yeah,...from the curve to the straight."

**Summary of Steve's Function Images**

Seven function images emerged in the data collected from Steve during this study: equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, continuity, and regularity. Several links between these images suggest the existence of a network. Steve's network of function images is dominated by the equation image. There is a strong association between functions, equations, and univalence throughout the study. Steve's function images reflect an operational and quasi-structural conception of functions. Several cognitive obstacles in dealing with the arbitrary nature of functions were associated with his conception of functions.
Procedural and Conceptual Knowledge of Functions

Steve's grades in precalculus reflect an improvement in his procedural knowledge. Difficulties at the beginning of the year were associated with algebraic manipulation, syntax, and finding the domain and range of functions. As the year progressed these errors were less frequent and his grades reflected the improvement by moving from B to A. Some of his mistakes related to algebraic manipulation appeared in solving systems of equations or factoring. An example of a syntax mistake appeared in (Q-01/29, 6), where students were asked to find the domain and the range of $h(x) = \sqrt{x} - 3$. He answered correctly the domain as "$x \geq 3$," but wrote the range as "all reals $\geq 0$". Some of his mistakes in finding the domain and the range are discussed below in "Composition of Functions."

Steve was the only student in class who openly discussed the lack of conceptual understanding in the class. In the following protocol we were talking about how to explain functions to precalculus students (I5, 5).

Stev  Yeah. [Mr. H.] kind of just gives the like, this is how you do it and he doesn't really say, goes in and say these are the properties,

Me    Mmm hmm.

Stev  He does a little, but not into great detail.

Me    Yeah. That's what I'm saying, so that's what you could suggest to him.

Stev  Yeah. So you understand why you're doing what you're doing not rather than just doing it.

Me    Okay. What other advice could you tell him? What about the kind of examples you would like him to use?

Stev  Probably on the examples instead of just skipping through, make sure he like shows each step like the first couple times.

Me    Yeah. Do you think that you got into the, a lot of how functions are being used? Would you tell him, you know, was it clear for you what functions are used for?

Stev  Not really, but I just knew how to like solve and whatever.
Composition of Functions

Steve was able to carry out composition of functions in the usual testing. However, they remained at the procedural level. His difficulties occurred in finding domain and range of the compositions. For example, in (T-02/19), students were asked to find an equation for \( g \circ h(x) \) and \( g \circ p(x) \) where \( f(x) = \frac{1}{x} \), \( h(x) = \frac{2}{x} \), \( g(x) = 4x^2 \), and \( p(x) = \sqrt{x + 4} \). Steve found that \( g \circ h(x) = x^2 \) and wrote the domain as "all Reals." He also found \( g \circ p(x) = 4x + 16 \) and again wrote the domain as "all reals." It seems, that both answers were driven by the familiarity image.

Steve, as other students, used the substitution procedure emphasized in class to find composites. His difficulties were interpreting the outcomes (see above). For example, in (Q-02/07), where students were asked to find the composition of \( f(x) = 5 \) and \( h(x) = x \), he wrote "5x" for \( f(h(x)) \) and "5" for \( h(f(x)) \). When I asked him to explain to me what he understood by composing two functions with this particular example, he said:

Um, let's see, you just have to, I don't know, follow this, I guess, you just plug h of x, you have x there, then you put that into the, you just work outside the parentheses and plug the inner number into the next one out.

This idea of "substituting" seems to be an obstacle when it is not obvious where to substitute. As in the case with the Elizabeth (pp. 115-116), Steve ended up multiplying "5 times x" in the first composition.

Oh, okay, see, well, that's just the way I. See you put x into h, so, or h of x, you put h into x...So you put h into f, I figured, cause I know that you put f, if you put this in place of that it will just be five but I wasn't sure what would happen if you put the x in place of the five...So, well I'm not sure, so I just put five x.

Global Approach to Functions and Connections

Steve was the student who provided the most accurate construction of the graph of the rational function, \( f(x) = \frac{x^2 + 2x + 1}{2x^3 - 4} \) with limited use of the graphing calculator, (I3, 5). The following dialogue, however, illustrates that there was no conceptualization of
the problem, but rather an immediate need to apply a familiar procedure, in this case
dfactorization. The calculator was used as a tool at several stages of the problem, a notable
difference from the other students.

Stev From this, just from what I'm looking at it, for like, it is pretty, we usually
do it like division ones like in class and just from looking at it, like he
usually told us the end behavior is from the front and those cancel out so
just from looking at it I say that the end behavior gets towards one half.
See what else, cause usually...factor out an x from [the numerator] and
then I have a quadratic formula on my calculator and I plug that in to
figure out some possible like zero, yeah, zero points [manipulating top and
bottom]

Me What are you trying to find out?

Stev I'm just factoring like an x out of [the top, writing \(x(x^2+2)+1\)]

Me and you said you wanted to apply the quadratic formula, what would give
you that?

Stev On the top? I think it gives you the zero points of the function. Where the
function equals zero and the bottom would give you the asymptotes of the
graph, where it can't be equal. I'm not really sure in this one. It seems
kind of, we usually have like, I don't know. It seems like we always have
[something that is factorable]. It's probably not factorable, it seems the
ones that we have are usually easy to factor. This is kind of confusing to
me... I'm not really sure on this because it is not like the ones we have
done in class.

Since Steve became confused I asked him about other features in the graph, without
mentioning them, in addition to the end behavior. The following dialogue shows his use
of the calculator and some connections to the rational root theorem.

Stev I'm not sure. It has zero points somewhere but I'm not sure...Let me think,
I try, I think and then [writing

\[
\frac{x^3 + 2x + 1}{2x^3 - 4}
\]

and [substituting numbers]. I think, yeah, I think that will work. Wouldn't
negative one be a factor?

Me What are you trying to do?

Stev OK, let see. I guess it isn't negative one [using the calculator to evaluate].
Me  So, what you are trying to do is substituting numbers?

Stev  Yeah, finding a number that will make the top zero...I just thought of [negative one], but it doesn't work, so I was thinking one squared, it will work.

Me  Why did you choose negative one and one?

Stev  Cause I saw there is a plus one there and I thought since that was cube, that will give me positive one, so that'll be two and then I was thinking, negative one and two, but then it didn't, I thought, when I punched it up, I was doing it the wrong way... another thing that I can figure out is like with the y, the y-intercept...plugging in zero for x, so it will be negative one fourth

Steve showed a strong ability to use the calculator, but it was diminished by his lack of rich mental relationships. In the following piece, he used the calculator frequently.

Me  What are you doing now?

Stev  Seeing if I can figure out the zeros for the bottom.

Me  Are you trying to graph something?

Stev  No. I'm trying to figure out the zeros for the bottom.

Me  Are you trying to do trial and error again?

Stev  Yeah, I'm just trying some numbers in the bottom.

Me  What are the numbers you are trying?

Stev  Hmm, I'm just trying to figure out, so yeah that will work, [keeps working evaluating].

Me  Are you finding something?

Stev  Not really.

At this moment, I referred to the work in class and asked him what would be the usual questions that Mr. H. would ask. He was so engaged using the calculator to substitute numbers that I had to repeat the question. I also mentioned that to work out his problem by substitution was difficult.

Stev  Let's see [keeps working], I'm not sure like the exact value, I think, I came up with a number that might work for there.
Me  Which one?

Stev  Let's see if it works. Yeah, it's pretty close.

Me  What is the number?

Stev  I think that the bottom, zero would be when the asymptotes like one point two six, around there.

Me  Can you explain me what you did in order to find [it]?

Stev  I just rounded the number, I knew that I had to multiply by two, so I found a number that's when is cubed would equal two. So that number cubed that equals two which would [multiply by 2] so that you get the four, to cancel it out.

Me  With that information, what would you do?

Stev  With the half? Well I'm not sure how it looks like, I would say it goes toward like one half or whatever so, and the intercept one fourth, it's gonna be, something, let me see, it's gonna be I think, I don't know whether it goes on which side of the line, whether it comes from here and goes up, and here and goes up, I'm not sure [drawing figure 44].

Me  Where are you using the one point two six?

Stev  That'll be, I think, that would be one half, it goes towards that and then one point two six would be a vertical somewhere like right there so probably it has to be like this [changing figure 44 to figure 45]. Then, it's gonna go like this. See it probably goes like this.

Figure 44. Steve's first drawing in item (I3, 5).

Figure 45. Steve's graph for problem (I3, 5).
Me  When you say 'probably', why?

Stev  Because from characteristics is usually in opposite like quadrants like I and III. I don't know from what I've noticing it kinda always comes in opposite of the things, but I'm not sure.

Steve admitted that he needed to provide the information where the graph "crosses zero," and that that information came "from the top," but he did not how to figure it out. He was so confident on his usage of the calculator, that he said "I'm sure if I can stay here for a while I can probably figure it out." His final remark on this problem was related to the end behavior: he needed to say "whether the end behavior would go to a slant or to a straight line."

Steve's description of a different rational function, \( f(x) = \frac{x^2 + 1}{|x|} \), using the calculator (14, 4) exhibited his facility to use the calculator and some procedures where he made the connections by himself. He gave asymptotes, domain, range, and end behavior and a feeling of symmetry.

Steve found an asymptote at zero and a slant asymptote at \( y = x \). I asked him how he knew that \( y = x \) was an asymptote, he answered that he "just looked at it and graphed \( y = x \) and it went right along that line." When I asked him where he got the feeling that it was \( y = x \), he said

I don't know, I just, I don't know, from other problems... I just kind of like saw the first thing, cause I saw that they were both coming in like that.

He mentioned that the end behavior was \( y = x \).

Steve was aware of mentioning domain and range. The domain was \((-\infty, 0) \) and \((0, \infty) \). When I asked him how he found the domain, he said

by looking at the graph, cause [I know the graph] is going continuously that way... And it comes, and it's got an asymptote at zero so it's not going to go at zero.
His answer for the range was \([2, \infty)\). He mentioned that the graphed looked "like it's reflected", but he did not mention the word symmetry, instead he just plugged two values: 2 and -2 and said "absolute value does that thing."

**Settings: Domain, Range, and Rule**

The course included examples of functions where the domain was either the set of real numbers or the set of real numbers minus five points at most. The range was the union of intervals at most. This might explain some of the difficulties that Steve experienced in problems where the domain and the range were arbitrary sets or where functions had domain other than the set of real numbers (XC2, Q-02/19, Appendix B). For example with the function that assigns to every triangle its area, (XC3), he wrote the domain and the range as "\((-\infty, \infty)\)." In (I5) he explained to me that he was working with two numbers (height and base) and the domain "would be the same for both because you could just switch them around." When I asked him to tell me what the domain would be, he said:

> It would probably have to be, um, just from like, well I guess you could have a negative, if you have both negative, then they cancel out.

When I asked him if the relation given was a function, he said "yes" and graphed it.

> It's a straight line along ten because no matter what numbers you plug in it's always going to equal ten.

Steve's answers suggests that he was working with the constant function, \(f(x) = 10\); an answer that he obtained from the problem statement \(\frac{5 \times 4}{2} = 10\). Steve realized here that this was in discrepancy with the answer he gave for the range and said:

> Well, then that would make it that the range would have to be at ten then. But I was thinking that as a picture on the graph as a triangle.

Steve's explanation implies the use of the equation image to solve the problem. Another situation is the interpretation of each side of the triangle as a function.
The function that assigns price to pieces of clothing, (XC2), was defined on a non-numerical discrete domain. He chose the given table to be a function because if you graph for example 25.00 (x) along that line, the x value will tell you how many blouses you have bought and how much it will cost. It will work with all the other items as well.

During his discussion of this problem in the fifth interview, it was clear that, similar to (XC3, above), he used several functions. In this case, he used four functions, each one of the form "price times number of items." Again the obstacle that the equation image puts is evident.

Familiarity was another image associated with settings not used in class. In the parabola defined only for integer numbers, (Q-02/19), he drew a correct graph (figure 46). He wrote "all real integers" for the domain and "[0, ∞)" for the range. In (14) he explained to me how he found them. In the case of the domain, "because it's only, it only said for only integer numbers" and in the case of the range.

Let's see, well, it starts at zero...And it can go, it's a parabola so it keeps on squaring for the numbers...So it would go from zero to infinity.

![Graph of f(x) = x^2](image)

Figure 46. Steve's graph of f(x) = x^2, only for integer numbers.

Steve's decision for this relation to be a function was based on familiarity: "Well, I just learned that parabolas were functions, I guess." However, it might have been also the case that he mentally connected the dots he drew. After he said that parabolas were
functions, I asked him if all parabolas were functions he said "No, except for the ones that go like sideways or whatever."

Steve was the only student who perceived that functions could be represented in more ways than equations and graphs (I5, 3). He said:

Um, they can probably be represented in maybe like a story problem where it tells you certain things and you have to, there are certain like facts and you have to put them in a formula [and] maybe you can represent them on like charts like on sign charts.

Construction of Graphs

The construction of the graph for the roller-coaster problem (I4, 5) caused an initial difficulty deciding which variable goes in which axes.

Trying to think which would go on which axis, position, speed, velocity, so it goes like, it goes, trying to think how to do this, I'm not sure at all on this one, cause I know if you put position here...It would have to go down, but then the speed's going up, I guess, so I'm not really sure how to do this, you can't put position on here.

Steve made the correct graph after I gave him the hint of putting position in the x-axis and speed in the y-axis. However, when he finished, he said "maybe that's what it looks like, I'm not sure."

Development of Functions

This section is composed of two parts. The first part describes some of the ideas on functions that Steve had at the beginning of the course. Data is drawn from the Practice Test on Functions (Appendix C). The second part is a discussion of the development of the concept of functions based on the concept images and definitions provided by him throughout the study.

Some Ideas on Functions at the Beginning of the Course

Some of Steve's ideas on functions were captured on the Practice Test on Functions. His definition of a function was "answer to graph" which suggested an implicit idea of equation. However, this definition lacked the univalence property. Images associated
with functions were graph (see definition), vertical line test, equation/formula, continuity, and familiarity.

Steve's answers showed that he knew that the vertical line test was equivalent to the univalence property. For example in the piecewise discontinuous function (PTF1), he marked the graph as a function because the "vertical line test works since there are open circles." However, it is not clear if he used the vertical line test to produce graphs of functions. For example, when six points were given graphically and students had to draw the graph of a function whose domain included the given points, (PTF14), he connected the dots without checking that the resulting graph was not a function.

Other answers suggested that he perceived functions as a formula/equation and ignored the domain and the range. For example, for the piecewise function given algebraically in (PTF2), he drew two functions and wrote "it does not pass the line test, therefore it's not a function." So each piece was understood as a function with domain all reals. His decisions for functions being equal (PTF8) were based only on the algebraic expression, and his graph of a function with domain and range equal the set of natural numbers (PTF10) was a continuous straight line. The equation/formula image might explain his use of the function \( h(x) = -7x \) instead of \( g(x) = -7 \) to evaluate the constant function \( g \) at different numbers.

The next image associated with functions was that of continuity--connecting dots. For example, when two points were given graphically and students had to draw the graph of a function whose domain included the given points (PTF13), he drew a straight line (see also his answer to PTF14, above).

The last image associated with functions was familiarity. Steve was familiar with simple functions. In item 3, he drew a straight line as an example of a function and a horizontal parabola as a nonexample of a function (see PTF13, above).
In the pretest, Steve could read the domain of continuous (PTF6) and piecewise continuous functions (PTF9); however he had difficulties identifying graphs with given domain and range (PTF11).

Some of the items that he answered incorrectly might have been caused by lack of familiarity either with the vocabulary or with the functions used in the items. Regarding vocabulary, it seemed that he had not used the terms "image and preimage" before (PTF4 and 5). The piecewise function was difficult when given algebraically (PTF2). Difficulties with the syntax of mathematics appeared in his notation for sets. For example for the piecewise function with constant pieces in (PTF9), he wrote the domain as "Dom) all reals > 3 all reals < 3" and the range as "Rang all reals = 2 & = 4."

Images and Definitions of Functions during the Study

During the course of the study, Steve showed some change in his definitions of functions. Three images are frequently cited in his definitions: graph, vertical line test, and univalence. Of these images, univalence is the central image (mentioned seven times). This image appeared in terms of the graphing calculator (trace) in the fifth definition (p. 131). The equation image appeared once in the definitions (see fourth definition, p. 130).

Steve's definition of a function in (I2) was given solely in terms of the univalence property (p. 130). However there are other images that emerge and compete in his decisions. The images of the vertical line test, familiarity, regularity, continuity, and equation/formula appeared in the dialogues showing the existence of a network of function images. It is clear that he can use the definition and the vertical line test as equivalent ideas to test that a graph is a function (see I2, 8, p. 140). For example, for the graphs in (I2, 1 and 3) he used the vertical line test, but in the graph (I2, 2) he used the definition. The familiarity image was used to provide examples. The regularity image emerged as a hesitation, but was dominated by the vertical line test image (see p. 143).
The continuity image was a matter of connecting dots and seem to be related to, and perhaps dominated by the familiarity image (see I2, 4, pp. 144-145). The equation/formula image appeared in items where the existence of a function was asked (I2, 6) and explicitly in the relationship between functions and equations (I2, 7). In the first case, this image was an obstacle to finding a function, since he tried to write down a formula for a set of four points. In the second case, equations that satisfy the univalence property are functions (see I2, 7, p. 135).

Most of the problems in (I2) were word problems. Steve was the only student who attempted to make drawings to solve some items in this interview. This shows, both, more awareness of translations between different representations and more understanding that when having the graph, the vertical line test can be applied.

Steve's definition of a function in (I3) is a blend of four images: equation, graph, vertical line test, and univalence (see I3, 2, p. 130). When asked to find a rule for a graph of a piecewise function given algebraically (I3, 6) he exhibited the weakness of the equation/formula image at the translation level (see I3, 6, pp. 133-134). However, the difficulty might be rooted in the given function being defined piecewise, since he recognized the pieces as linear and thought that the equation would be "a linear equation". He did not have difficulties with a constant function (I2, 5, pp. 138-139).

In the second interview, Steve shows an awareness of a relationship between graphs and equations: "from looking at the equation you can know what the graph looks like and vice versa". However, he recognizes that there are difficulties in moving from one representation to another. In the case of word problems, "the hardest part is coming up with the equation [and] once you do that, and get the graph, you just need like trace or whatever" (I3, 1).
Even though the course emphasized applications, he had difficulties seeing connections between the real world and school mathematics. He thought functions were important but did not know why: "because...you learn a lot of things in school but you don't have to use them outside the school...It's probably just important to learn them" (I3, 3).

Steve proved to be skillful with the calculator in this interview (I3, 4, pp. 149-150). In addition he was the only student who used the calculator as a tool to factor. When we were talking about the helpfulness of the calculator he said that he used it to double check my anwers, and I also sometimes if I come up with a graph and I simplify it, I graph the first thing and then graph the second thing and see if they overlap over each other to make sure they're the same thing, but one simplified and other one is not. Just to make sure they're exactly the same, make sure I simplified it right. (I3, 3)

This idea of overlapping graphs appeared again in finding the end behavior of a rational function (I4, 4, p. 154).

Steve's sixth definition (I4) is given in terms the univalence property (p. 131). Several images of functions appeared in the interview: familiarity, continuity, equation/formula, univalence property, and vertical line test. He used the familiarity image to recognize functions (Q-01/29; Q-02/07, pp. 142-143), and to find the range of the parabola defined only for integer numbers (Q-02/19, pp. 156-157).

As in the third interview (I3, 6, p. 157), a piecewise function caused him difficulties. In this interview, the continuity image appeared when I asked him to graph a piecewise function (I4, 2, pp. 145-147). It might be the case that the difficulty was caused partially by the function being defined piecewise, but also by the pieces given as constant functions. He struggled understanding the rule. However, there is evidence of progress. He did not see the pieces as two different functions, but as one. So the continuity image
might be a reflection of him seeing the pieces as one function (that should be represented by one graph and not two!!) His graph in this interview (p. 145) suggests a conceptual problem with the univalence property. However, his modification during the member check in (I5, pp. 146-147) eliminates the doubts about him not understanding this property graphically.

In the case of moving from the graph to the equation, Steve did not experience difficulties finding a rule for a constant function (Q-01/29, p. 142). However, he answered incorrectly the question on finding "x such that f(x) = 2." This suggests that the equation/formula image is more anchored than the graph image. He said in the interview that he had misunderstood the problem and he was solving the equation \[ 2 = \frac{7}{x} \], though it was not clear why this equation. He did not use the graph given to solve the problem.

The use of the univalence property to decide if a given graph is a function shows conceptual progress in the understanding of the property. In deciding that the graph given in (Q-01/29), a constant, was a function he said: "it's got a different, like x-value for every y, so right here, it has, it'll have the same y-value but different x-values." Similarly, the decision for the graph (I4, 2, pp. 145-146) was "cause none of the x's overlap, they have like their own x points." Finally, he used the vertical line test to decide if a finite set of points given graphically (I4, 3, p. 139) was a function.

In the last interview, he defined a function in terms of the univalence property (p. 131). The question on properties of functions (I5, 2) left the door open for other images to emerge. He mentioned by himself the univalence property and the vertical line test. The next properties were mentioned when I asked the questions "Are all functions equations? Can functions be graphed? Are all functions continuous?" He said that not all functions are equations; all functions can be graphed; and functions are always
continuous. In the last situation, he acted by familiarity, saying "from what I know."

In this interview (I5, 3), Steve showed a network of representations, which included not only equations and graphs (see p. 157). However, the graph image creates a potential obstacle by admitting that all functions can be graphed (see above).

The continuity image appeared in 'making changes to five plotted points to make them the graph of a function' (I5, 6, pp. 147-148). This situation might be explained by a pedagogical obstacle since the class "[hadn't] been doing points or whatever." Three ideas emerged when I asked him how he knew that the graph was a function. First, "it passed the vertical line test." Second, "it's continuous." Third "it looks kind of like one of those sine waves that we've been doing the last couple weeks." The latter is related to familiarity, since students were studying trigonometric functions. Steve's connecting dots seems to contradict his ability to recognize functions defined on a discrete domain (see I4, 3, p. 139). Related to this situation is the dominant role of the equation/formula image over the graph image (see I5, 7, pp. 134-135) and over the univalence property (XC2, 3, pp. 155-156).

It is in this interview, how to explain functions to precalculus students (I5, 5), that he recognized the need to make more connections in class. He said that it was necessary to talk specifically about the properties that functions have, because "I don't think we really like learned that, we just learned it by, you know, by using them so much." Secondly, he recognized that Mr. H. emphasized procedures instead of concepts and Steve would suggest to Mr. H. to emphasize concepts "so you understand why you're doing what you're doing not just rather than just doing it." Steve also admitted that he did not have a clear idea of what for functions were used. He just "knew how to, like, solve, and whatever."
By the end of the year, Steve's ideas about functions were concrete. During the member check in the last interview, he answered "yes" when I asked him if he thought a function was different from what the definition was. He said about functions

I've just been trying to put it together, the uh, the equation, what the graphs look like, I was trying to put it together for the, what the function is and the definitions kind of, what makes up a function's kind of different from what a function is...What makes up [a function] is all the things that go into that. Like the equation and the graphing and all those things. (I5, 1)

More data supporting his conceptualization of functions at a lower level than structural appeared in what makes up a function (I4, 1).

I can only tell by, like, looking at them. I can't really tell by, I can tell some by, like, what the equation is but I can't really tell, tell if they are or not without looking at them.

**Summary of Steve's Development and Knowledge of Functions**

Steve experienced the development of functions at the procedural and conceptual level. The course provided him with strong algorithms. Particularly, there is a link between the use of the graphing calculator as a tool and the conceptual knowledge of functions. Although Steve's definition for a function was abstract, he sees function as their representations, a quasi-structural conceptualization in terms of Sfard's (1991) work. He ended the course with a quasi-structural conceptualization: functions are equations. This conceptualization puts obstacles in dealing with the arbitrary nature of functions. The emerging image--univalence-- might help him to deal with arbitrariness, however, a tug of war will occur until he realizes that equations are insufficient to represent functions. Chapter VI includes a comprehensive summary of Steve's case and a network of function images.
The Case of Jane

Introduction

Jane is a quiet girl in precalculus. She likes mathematics and, knowing that it has not been easy for her, she tries to work hard at it. She considers the subject to be important for her college preparation: "I mean, I could stop taking math and if I stop where would I be? I mean if I go to college I wouldn't know anything." She finds mathematics sometimes interesting and sometimes boring, depending on the interaction between the teacher and the student in the classroom. She thinks that mathematics helps people to identify their strengths and weaknesses, and enables them to recognize the fields they should go into. Jane is taking precalculus to be prepared for college courses, to know more about the kinds of mathematics that there exists, and to gain more understanding of it.

Jane did not have any experience with graphing calculators before getting into this class. She thinks that using a calculator in precalculus makes math easier—it saves a lot of time particularly on the graphing. She likes using the calculator in physics as well. However, she does not consider that learning mathematics with the calculator is better because it is possible to become attached to and dependent on the calculator. She thinks that in tests where they are not allowed to use the calculator, it is easy to make mistakes because they use the calculator for homework and everything. In the same vein, she does not like the fact that there were processes, like mental arithmetic, that she used to do in her head and now she uses the calculator to do them.

Jane thinks that all activities done in class are positive. She likes the small-group work because she can obtain answers to more questions, and "it is usually easier to understand [when] we help each other." Overall, she feels confident in class but shaky sometimes.

During the interviews, Jane repeatedly admitted that she did not understand me. It seemed to her sometimes that I was looking for a specific answer to the questions instead of trying to understand how she thought.
How Jane defines a Function (Concept Definition)

Jane provided seven definitions of function during the study. The univalence property appears in all of them. In six definitions the vertical line test is mentioned, and hence the graph image is evoked. The first definition contains three images: equation, vertical line test, and the univalence property. These images form a network – the equation image as a representation of functions, and an implicit equivalence between the vertical line test and the univalence property:

   equation or formula which you substitute #'s in for variable that pass vertical line test where coordinates not repeat [sic]. (PTF16)

The second definition is Mr. H.'s definition stated in class,

   a relation is a set of order pairs and a function is for each input there is only 1 output. In a relation you can have more than 1 output per input [sic]. (Q-09/11)

Jane's third definition is based on the univalence property and included an implicit equivalence with the vertical line test.

   ...for every input there is one output and then I mean, I think about the vertical line test. (I2, 8)

Her fourth definition is similar to the third definition but has an implicit reference to the equation representation.

   It's just when you put in for an x-value, you only get one y-value for, I guess it's how you did the vertical line test and all of that, I mean. Uh, you can say just by the same it is a relation and bla bla bla [sic]. (I3, 2)

In her fifth definition the graph image is added to the images appearing in the fourth definition. Jane did not mention the vertical line test in this definition.

   [A function] can be a pt, a line...relation for every input only one output f(x) = equation which is basically the same thing as y. has domain ranges, end behaviors & models...[sic]. Q-03/04)

Jane's sixth and seventh definitions are similar to the third definition (see above).

   For every input there is only 1 output. It is a function if it passes the vertical line test. (6th definition, XC1)
For every input there's one output, and the vertical line test. (7th definition, I5, 1)

Jane's Mental Pictures Associated with Function (Concept Image)

Several images associated with functions emerge from the data collected from Jane. Images have been separated, but overlapping images suggest a network instead of a partition. The images associated with functions in Jane's work are the following: one output for every input, vertical line test, graph, equation/formula, familiarity, and continuity.

Indications of each cluster follow.

One Output for every Input

In the data gathered from Jane, there are several ideas associated with the image of functions as generating a unique output for each input:

(1) definition (PTF16, Q-09/11; I1, 13; I2, 8; I3, 2; Q-03/04; XC1; I5, 1),
(2) equivalent to the vertical line test (I1, 13; I3, 2; XC1; I4, 3; I5, 1),
(3) property (I5, 2),
(4) a means to decide whether or not equations are functions (I3, 2), and
(5) a means to decide whether or not graphs are functions (I4, 3).

The definition idea appeared in specific questions that asked for a definition of functions (see seven definitions, pp. 166-167) and when I asked her to describe a function with her eyes closed (I1, 13).

...a function is where like for every input there is only like one output and before the way I used to remember it was the vertical line, where it can only like cross one point in the parabola or whatever it was and if it crossed more than one then it wasn't a function.

This remark led us to the idea of the vertical line test being equivalent to the one-output-for-every-input image. I asked her "can you combine [these two ideas] or are they separated?" and she answered,

I think it can be one or the other. I guess it's the same thing, because if it crosses twice then for that input you have more than certain outputs.

The unique output criterion also appeared in Jane's 4th and 6th definitions (p. 166), and in her discussion of univalence as a criterion to check if a graph is a function (p. 168).
When asked what "properties functions have" (I5, 2). Jane started answering the question by mentioning transformations of graphs. In her answer, although not explicitly stated, the univalence feature is a property of functions.

Like leading coefficients...I guess vertical stretch and stuff like that, end behaviors, behavior models. I guess anything that applies to the definition....Mmm hmm. I guess there are so many different properties, you can say, I guess, anything.

The use of the univalence as a criterion to decide if equations are functions appeared when she gave the fourth definition (p. 166). I asked her to elaborate on the "bla, bla, bla." She said:

OK it’s where what I was saying a second ago for like each number you get one number and I mean out, I mean if it repeats like if you, in the vertical line test if you have two, six and five, six, like that, it’s not a function.

The use of univalence as a criterion to check if a graph is a function appeared in the graphs with four dots (I4, 3). In the case of the function (part a), she said

Yeah, I guess...Because, um, I guess for each input there’s like only one output like at zero, zero, or whatever D is, one, ten or I don’t know what the numbers you’re using.

In the case of a non function (part b), she said

No. Um, because if you say this is like negative three or something, it could be B or D. So if you put that into some equation, I don’t know what the equation would be but it has more than one output for out, or one output, each input has two outputs.

Vertical Line Test

The vertical line test image also manifested itself in different ways in Jane’s work:

(1) a means to decide whether or not a graph is a function (I2, 1, 2, 3, 5; XC1),
(2) to produce graphs of functions and non functions (I2, 8), and
(3) equivalent to the univalence property.

Jane used the vertical line test to decide if a given graph was a function. For example for the graph of a nonfunction (I2, 1), she said "I thought it wasn’t because of the vertical line test, because it crosses more than once in this area." A similar argument appeared with graphs of functions (I2, 2, 3). In the case of a piecewise function (I2, 2), she said ".since it has an open, open dot. It doesn’t cross twice. It’s a function."
In (12, 5), existence of a constant function, she produced a graph with only one point as an example of the situation given (see figure 47).

![Graph](image)

Figure 47. Jane's graph of a function with the same outputs.

I asked her if that was a function and she said "I guess so, if it passes the vertical line test... Yeah [it is a function]."

The second idea Jane associated with the vertical line test is using it to produce graphs of functions and non functions. After she gave her 6th definition (p. 166), I asked her about the two drawings that she had made on the handout (figure 48). She said "yes, I used the vertical line test" referring to them.

![Graph](image)

Figure 48. Jane's use of the vertical line test to produce examples of functions and non functions.

**Graph**

There are three ideas in the graph image:

1. Functions can be represented by graphs (1st and 5th definition; I3, 4a; I5, 3, 4),
2. Not all graphs are functions (12, 8; I5, 4), and
3. All functions can be graphed (I5, 2).
The first idea, functions can be represented by graphs appeared in definitions and tasks. For example in (I3, 4a), she said

What makes up a function? Uh, I guess it would be line, points, I mean, it can go on I guess infinitely. You know, uh, I don't know what you are asking.

Since she pointed out that she did not understand what I was asking, I paraphrased the question in terms of a house and the things that made up a house. However, the question still was not clear to her, since she said:

it could be a point, lines, a variable maybe like x, and maybe a constant or something. I don't know what you really want to know.

The most explicit indication of functions being represented by graphs appeared twice. First, when I asked Jane how functions can be represented (I5, 3), and second, during her discussing the relationship of functions and graphs (I5, 4). In the first case, her immediate answer was "by graphs, um by equations. That's all I can think of." In the second case, she said "yes [they are related, a graph] is a visual, it's a visual representation of a function." These were consistent with her first and fifth definitions of functions (p. 166).

The second idea, that not all graphs are functions, was introduced in the use of the vertical line test to produce graphs of functions and non functions (fig. 48, p. 169). A second situation where this idea emerged was after her comment on graphs being a visual representation of functions (see above). She continued "but not always [a graph] is a function."

The last idea appeared when I asked her "can you graph all functions?" (I5, 2) and she said "yes."

Equation/Formula

This image appeared in all the interviews. It seems to be one of the two most anchored images on Jane's mind. This image emerges in several presentations:

(1) functions can be represented by equations (I1, 14c; I5, 4);
(2) functions are equations (I2, 4, 5, 6, 7; I3, 2, 4a, 6; I4, 1; Q-03/04; I5, 2, 7; XC2, XC3); and
(3) not all equations are functions (I3, 2; 5th definition).

The first idea in the equation/formula image is that functions can be represented by equations. When I asked what she thought when she heard or saw \( f(x) \) (I1, 14c), she said "\( f(x) \) is just like a function, uh,...if I know what you're plugging in for the problem or something." This idea appeared explicitly when she was asked how functions can be represented (I5, 4); her answer to this question was "by graphs, um by equations. That's all I can think of."

The second manifestation of the equation/formula image is that functions are equations. Hence the need to have or look for an equation in order to decide if a representation either verbal (I2, 4, 5, 6; XC3); tabular (XC2); or algebraic (I5, 7) presents a function. This idea of functions imposes an obstacle in any case as can be seen in the discussion to follow.

Jane was unable to decide the existence of a discontinuous function at one point (I2, 4). She said "See, I just wasn't sure what the question really, I mean is it like, y to the, y squared and then does it assign one, zero or something like that? So I wasn't." Her scratch work on the handout suggests that she was looking for an algebraic expression. I suggested that she make a graph by plotting points. The discussion of this situation is presented on pp. 176-177.

The existence of a constant function (I2, 5) provided the opportunity to get a better picture of Jane's ideas on functions. Jane presented several alternatives when asked about the existence of a function all of whose values were equal. Initially she said that the answer was 'no'. This answer did not reveal a lack of understanding of the question but a lack of clarity about the 'many-to-one' situation. She said

I'm thinking that, uh, the way that I'm thinking of it is if you are given a certain equation or something and what comes out is all the same answer. It's all I'm perceiving in the questioning. I mean if you look it that way, if you keep coming down with the same y's it is not a function.
I asked her to make a drawing of such a situation. After she drew figure 47 (p. 169) as a graph of the situation given, she said "it would be just like a point" and recognized that she was confusing the same input with different outputs. So she changed her mind and said "so the inputs could be different. It could be a function, then, I mean." She provided several equations as explanations. For example, the square function gives the same output: "I'm thinking of square because if you have negative three squared and if you had three, do you see what I'm saying? Squared." I said "yes" and asked "what if you had two?" and she said "they all will be the same thing again." I thought she was referring to a function with a discrete domain until I asked her to show me where the two would go. She said "I guess it depends what the equation is, I mean." So I pursued the idea and said "make up the value since all the values are the same." Her answer made me realize that she was confusing inputs with outputs again:

\[ x \text{ plus one [writing]} \text{ and then } x \text{ plus one will be one and then if you put } x \text{ as one, and everything is always like that...I guess it would be the } x \text{ plus one and if you put in } x \text{ equals one in and you put two for } x. \]

So she was working (though unconsciously) with the family of functions, \( x + 1 \), where for each value of \( x \), a constant function is obtained.

The need for an equation also appeared in Jane's work on the existence of a piecewise function discontinuous everywhere (12, 6). Jane had difficulties understanding the task. After I explained the question to her in terms of inputs and outputs, she said "if you had something like \( x \) to the one fourth or if you had \( x \) to the one third, if you use three." Even with directions, the task turned out to be difficult for her, so I decided to stop the task.

Maybe the most clear evidence of Jane's image that functions are equations appeared when she was asked about the relationship of functions and equations. She said

I put the equation represents the problem situation, because I was thinking it has restrictions and stuff and like when we graph on the calculator, it will show totally everything, so when you look at the equation, it will make restrictions so I mean I think basically they are the same thing. (12, 7).
For Jane, the distinction between functions and equations was based on algebraic notation, as illustrated in the following protocol (I2, 7 continued):

Me  Suppose that you are given a function and your teacher tells you that there are three zeros. Suppose that the zeros are one, zero, and two for this polynomial function. And he asks you to write down an equation. What would you do?

Jane  I go x minus two, x minus one, x [writing].

Me  Is this a function or is this the equation?

Jane  It, it can be both.

Me  If you want it to be a function, what would you do? Do we have to write something here so that it's a function?

Jane  f of x.

Me  If we want it to be an equation?

Jane  We just put y.

The need for an equation also appeared in the function that assigns to every triangle its area, (XC3). She wrote that H was a function because "you can plug in different #'s for $T_1$ causing variables in the outcomes."

When presented with a different representation (tabular), Jane expressed an explicit need for an equation. In the absence of one, the rule that assigns to each cloth item its price, (XC2), was rejected as a function. She wrote:

there are no variables if you tried to graph them they would be different points. More of a price chart. They are different items, so not comparing the price. No one equation for it to be 1 function [sic].

The equation image was also an obstacle dealing with an algebraic representation. When asked about the existence of a function that passes through four points given algebraically (I5, 7), she said

No, not unless they were separate functions. Meaning if you leave f of two equals zero one function, f of negative three equals seven another function, and so on.
I asked her if there was a function that satisfied those four conditions and she said "well, not really, I guess that means that you can just seem, I don't think so." I asked her if she was looking for an equation for the function and she said "yes."

Jane linked the graphical representation to the equation image, but in this case the image did not cause any obstacle. It seems that the wording of the problem suggested to Jane that she could write one equation (see p. 182).

The idea that functions are equations also emerges in Jane's description (I3, 4a, p. 170; I4,1) and defining (4th and 5th definition, p. 166) of functions. For example when asked what makes up a function (I4, 1), she said:

Um, let's see, a variable or something like that, maybe a leading coefficient and stuff like that... Um like y equals and I mean, just like that... whatever the equation is.

At the beginning of the fourth interview, I asked her to read her fifth definition (Q-03/04, p. 166). After she did it, I asked her "do functions have to be equations?" and she said "I think so."

The third idea in the equation/formula image is 'not all equations are functions'. The occurrence of this idea in Jane's data appeared in 'what is a function?' (4th and 5th definition, p. 166). In both cases, an equation is a function if it satisfies the univalence feature.

**Familiarity**

Jane used the familiarity image in interviews and in tests. There are two ideas in the familiarity image. The first idea is familiarity with examples of two types: familiar examples of functions (I1, 13; I2, 7, 8), and familiar shapes (I2, 4, I3, 4). The second idea in the familiarity image is using this knowledge to decide if a given representation is a function or not (I2, 5, Q-02/7; T-02/19).

Jane was familiar with easy examples of functions and as she recognized "obsessed with parabolas." For example, when I asked her to explain to me how the vertical line test
worked (11, 13), she drew a parabola and a 'zig-zag' graph (figure 49).

![Fig. 49. Jane's use of the vertical line test.]

During the second interview, when I asked her to give me an example of a function and an equation (I2, 7), she said "just f of x equals x squared [writing]." Other examples of parabolas appeared in (I2, 8, p. 169).

Familiarity with shapes of functions appeared in (I2, 4). Although she was unable to decide about the existence of a discontinuous function at a point, the wording of the problem suggested a familiar shape—a parabola. After I asked her to solve the problem graphically, she said "I guess, it would be like a parabola." I told her that that would be for the numbers different from zero, but zero would be sent to one and I asked her again, what the graph would look like and she repeated "it's like a parabola." See the discussion on continuity (p. 176-177)

The use of the familiarity image to decide if a given representation is a function appeared in Jane's discussion of a constant function, (I2, 5). I gave her an example where the inputs were different, but the outputs were all the same. Next, I asked her if there was a function that satisfied that. She said "Yeah, it would be a total horizontal line."

Familiarity with a constant function also appeared in (Q-02/7):

Me  Tell me about things that you know about the constant function? For example, here you have one, f of x equals 5." She said

Jane  it's just a line, I mean a straight line, continuous. I mean if you had, if you put, plug an x equals one half, it would be, y would be still equals 5...so it's a function.
Continuity

There are two ideas associated with the continuity image:

(1) not all functions are continuous (I5, 2), and
(2) connecting dots (I2, 4; I5, 6).

Jane was one of the few students who answered "no" to the question "Are all functions continuous?" (see Elizabeth, p. 111). However, she seemed to attach a different idea to continuity than other students. For example, in the piecewise function in (I3, 6) she asked "this isn't continuous, right?" so I asked her "what do you mean by continuous?" She said "meaning it goes on." So it seems that the idea of continuity refers to defined over the set of real numbers and not the intuitive idea introduced in class as "being able to draw the graph without lifting the pencil."

The second idea associated in continuity was "connecting dots." The two questions when this idea appeared were drawing a graph and making changes to five points to make them the graph of a function. Both answers reflected a pedagogical obstacle: the habit of connecting dots. For example, in the first case, existence of a function discontinuous at zero (I2, 4), I asked her to make a graph for the item, since she could not answer the problem. Her initial image was "I guess, it would be like a parabola." So she expected to have a parabola. Her first drawing (by plotting points) resulted in a similar graph to Elizabeth's (figure 17, p. 111). Next, I asked her to plot one half and its image. After she did so, I asked her "does the parabola still apply." She said "I guess no." The following dialogue took place

Me    Now, that point is like outside of this little parabola, you drew. Now, could we say that that is not a function or?
Jane  Well, maybe just could be more like that and maybe then go back up. I don't know.
Me    Could you please make a drawing of the new possibility?
Jane  [drawing figure 50].
Maybe it would be like that and maybe go back up or something.

However, she said "it can't have" when I said "suppose that this number were like three [pointing to the negative branch]. So, she crossed out this graph. However, the idea of keeping the graph continuous made her say:

I'm still wondering if it goes down a little bit. I guess it wouldn't work because the numbers will keep going down, so it's probably not a parabola.

The teaching practice of connecting the dots appeared when I asked her "why did you try to connect the points?" and she said

Why? It's just a habit. I mean, you learn to do that, I mean. I was always taught to connect the dots, so...Yeah, I remember 'connect the points', so...that's what we were taught in Algebra 2. They're connected! [laughing].

The second item when the idea of connecting dots appeared was in making changes to five plotted points to make them the graph of a function (I5, 6). In this case, she connected the points with a continuous smooth curve and said "you could just go like that. It won't have any straight lines." Connecting the dots seems to be a reaction to seeing a set discrete of points.

I asked her, referring to the original graph, "before you connected [the points] uh, is that a function?" and she said "yeah...I guess, yes." So I asked her "why did you connect them if it was already a function?" and she admitted "I don't know."
Summary of Jane’s Function Images

Six function images emerged in the data collected from Jane during this study: equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, and continuity. Several relationships between these images indicate the existence of a network. Jane’s work suggest a strong tendency to equations and univalence. The equation image, in particular, created several cognitive obstacles to deal the arbitrary nature of functions. Jane’s function images reflect an operational and quasi-structural conceptualization of functions.

Procedural and Conceptual Knowledge of Functions

Jane was an average student in class and had difficulties working against time. It was common for her to be one of the last students handing back testing materials at the end of the period or to come to see Mr. H. in the math center to finish a test that she had not completed. Jane was a quiet student in class, who never asked questions. However, she interacted a lot during the small-group activity. In this section I present her approaches to several tasks to illustrate her knowledge of functions.

Jane had fewer difficulties associated with algebraic manipulations than other students did. Her tests show a consistent knowledge of algorithms or with the applications of concepts. Her difficulties might be instead associated with the pressure of time. Similar to other students (e.g., Steve, p. 149), she had difficulties finding the range of functions. For example in (T-10/21-22), students were asked to find the range of a graph. The correct answer was [-8, ∞), but she wrote instead "all reals." Other mistakes associated with domain and range are discussed below.

Composition of Functions

Jane was able to carry out compositions of functions in the usual testing. However, it seems that she worked at the procedural level. She had difficulties finding the range of composites. For example, in (Q-01/29), students were asked to find a "rule", domain, and
range for the composition \( f \circ g(x) \) where \( f(x) = \frac{1}{x + 4} \), and \( g(x) = x^2 - 3 \). Jane found the correct rule and domain for \( f \circ g(x) = \frac{1}{x^2 + 1} \). However, for the range she wrote "y ≤ 0."

Mistakes suggesting difficulties with the range also appeared in the compositions in (T-02/19). There, students were asked to find an equation, domain, and range for \( g \circ p(x) \) and \( p \circ g(x) \) where \( g(x) = 4x^2 \), and \( p(x) = \sqrt{x + 4} \). Jane found the correct equation and domain for both, \( p \circ g(x) = \sqrt{4x^2 + 4} \) and \( g \circ p(x) = 4x + 16 \). However, she wrote the range of both composites as "all reals." It seems, then, that the familiarity image influenced the answer in the case of the composition \( g \circ p(x) = 4x + 16 \).

Another example of Jane's procedural knowledge of compositions appeared in (Q-02/07). There students were asked to find the compositions \( f(g(x)) \) and \( g(f(x)) \) when \( f(x) = 5 \) and \( h(x) = x \). Jane as other students (e.g., Elizabeth, pp. 114-115) used the "substitute procedure" emphasized in class to find the composition of functions. However, the limitations of the procedure were blocked by the notation, which suggests familiar multiplication instead of a composition.

Jane: I think you were just supposed to substitute the answers...I put it, uh, the outside had f and f was f of x equals five so I had I guess a five like that. The way I was doing it...I put this [writing 5(x)] and h of x equals x, that's what I was doing.

Me: Oh, you substituted that way so you got the product.

Jane: Right, and I guess that's what happened in that too...See I always get confused with that but someone told me when you just have like f of x and h of x you just substitute each in other [writing x(5)].

Global Approach to Functions and Connections

Before I present two tasks where I asked Jane to describe a function, I present her discussion of a new problem she could solve. Jane chose "sign functions and charts."

However, when I asked her to explain to me what sign charts were, she said

It's just where the zero points and if they give (x - 6), the zero point is just like 6, and you put that, and just the equations you put in through, I mean it's kind of hard to explain. (II, 5)
Next, I present the tasks where I asked Jane to describe a function. In both cases she was unsuccessful. The first task, the description of the graph of the rational function $f(x) = \frac{x^3 + 2x + 1}{2x^3 - 4}$ with limited use of the calculator (I3, 5), made evident the weakness of her conceptual knowledge of functions (rich relationships); no conceptualization of the problem; the procedures emphasized in class; and her struggle with algebraic manipulations.

Jane spent considerable time before she admitted that she was stuck and thought she was able to solve the problem. However, her only attempts during the task were to factor the numerator and the denominator. In the denominator, she factored out the two. In the numerator, she wrote "(x - 2)(x - 1)." She never tried to use the calculator. I decided to move to the next exercise. First, because when I asked her "what are you trying to do?" after she factored the numerator, she said "I don't know, I'm just staring at this, trying to see if something comes to me." Second, after having some time to think, I asked her if she was stuck and she said "yes."

The second task was the description of the rational function $f(x) = \frac{x^2 + 1}{|x|}$ using the calculator (I4, 4). Jane's description, not only lacked the appropriate language but complete information. Her description included range (partially), symmetry, continuity (though incorrectly), end behavior, and domain.

Um, well, um it's on positive y's...Um, I guess you could say they look symmetric...it looks continuous...I guess you could say the end behavior model looks like, you know, just like a line...it doesn't pass through the origin...That's all I can think of right now.

**Settings: Domain, Range, and Rule**

The course included examples of functions where the domain was either the set of real numbers or the set of real numbers minus five points at most. The range was no more complicated than the union of intervals. This might explain some of the difficulties that Jane experienced in problems were the domain and range were arbitrary sets or functions with a discrete domain (XC2, XC3, Q-02/19). For example, for the function that assigns to
every clothing item its price, she wrote that it was not a function because

there are no variables if you tried to graph them they would be different points.
More of a price chart. They are different items so not comparing the price. No one
equation for it to be 1 function. (XC2)

The crucial role of the equation image is evident (see p. 173). During the fifth interview she
said she "thought they were separate functions....There wasn't a constant variable...that's
what I thought." To rely in the equation image also appeared in the function that assigns to
every triangle its area (XC3). For the domain, she wrote "all real positive #',s," and for the
range "greater than zero since can't have negative Δ." She wrote "yes" for H being a
function since "you can plug in different #',s for T1 causing variables in the outcome." In
the fifth interview, she said that she
got confused with [the] question...and [she] was trying to relate it with points...and it
was given for all triangles..[She] didn't really know what [I was] asking.

Familiarity was another image associated with settings not used in class. For example,
in the case of the parabola defined only for integer numbers (Q-02/19), she drew a partially
correct graph (figure 51).

\[ \cdot \hspace{1cm} \cdot \hspace{1cm} \text{integer} \neq 0 \]

Figure 51. Jane's graph of \( f(x) = x^2 \),
only for integer numbers.

She wrote the domain as "all reals," and the range as "positive integers." During the fourth
interview she told me that she came up with that graph because "I think I just chose whole
numbers like two or one, and I just...plug that in, plug x squared." In the case of the domain
she changed her answer while she was reading "it could be all reals..and integers, yeah...I
guess if it's only for integers it would be all integers then." However, she could not
remember how she found the range. To decide then that the graph was a function was
based on the familiarity image: "I think I just made it into a parabola or something like that."

Jane was one of the few students who was able to find a mathematical expression for a piecewise function defined on a bounded interval (I3, 6), which she wrote as

\[
y = 4 \text{ at } 3 \text{ through } 10 \\
y = -2 \text{ at } -2 . . . . . 3.
\]

She kept thinking after she wrote it. Then the following dialogue took place.

**Me**  Can I Interrupt your thinking? What are you doing now?

**Jane**  I was separating the two and I was just saying where y equals four, it's between three and ten, and the same thing the other way, y equals negative two to three...I was just thinking that combining maybe an equation together. Maybe make two separate ones, and maybe combine or something equals.

**Me**  So, do you want to combine them?

**Jane**  If it can be done, not sure. I don't know, I had them separately, I thought maybe easier.

**Me**  Why do you want to combine them?

**Jane**  Well you said 'one expression'. If it's fine, it's OK. I'm sure you can do it in two separate things.

Jane answered correctly all the questions related to the constant function in (Q-29/01) and as indicated above could find an algebraic expression for a piecewise function where each piece was a constant function on a bounded interval. However, she had difficulties recognizing how many functions were in a piecewise function (I4, 2), where each piece was a constant function. The following dialogue illustrates her difficulties to graph the function.

**Jane**  Well, is that just, this one whole equation, or, I mean is that like the domain and all, or they're separated?

**Me**  Yeah, that, that says everything. That says everything for this relation.

**Jane**  Well, I guess it's one, but if you

**Me**  You guess that it's one.

**Jane**  Want to separate them or something, it could be infinite.

**Me**  OK. May I ask you to make a drawing of it?
Jane: Well, it depends on the equation, too... It's one if x is greater than or equal to zero.

Me: Right.

Jane: I don’t know, do you want us to make our own equation or

Me: No. I want you to make a graph.

Jane: One if x is greater than or equal to zero? I'm not sure I understand this problem.

Jane's difficulties were moving from the algebraic representation into the graphic representation and not from the graphic representation into the algebraic representation as it is documented in the literature. Her difficulties might be due to the use of constant functions.

Construction of Graphs

Jane was one of the four students who could not draw correctly the graph in the roller-coaster problem (14, 5). Her graph of the position versus the speed had the same shape as the roller coaster track.

I think it'd be probably, it would change as the graph, as the roller coaster does too, so it'd probably be like sort of the same thing.

So I asked her "based on what?" and she said:

I don't know, a good guess... I don't know, it may not be a good guess... Well, let me see, for starting in A since it's not a steady thing, it's not really a line or anything... So that's why I think it would be changing as the position would change, so would the speed.

Development of Functions

This section is composed of two parts. The first part describes some of the ideas on functions that Jane had at the beginning of the course. Data is drawn from the Practice Test on Functions (PTF, Appendix C). The second part is a discussion of her development of the concept of functions based on the concept images and definitions provided by her throughout the study.
Some Ideas on Functions at the Beginning of the Course

Some of Jane's initial ideas on functions were captured on the Practice Test on Functions. Her definition of a function was "equation or formula which you substitute #s in for variables that pass vertical line test where coordinates not repeat" [sic]. Three ideas appear in this definition: equation, graph, and an implicit equivalence between the vertical line test and the univalence property. Jane displayed two more images in the test: continuity and familiarity.

Jane's pretest answers show that she perceived the vertical line test as equivalent to the univalence property. For example in the piecewise discontinuous function (PTF1), she marked the graph as not a function because "the vertical line test shows it passes more than 1 pt" [sic] (see her definition, above). Her answer is incorrect, but she erred based on the notation used. She was not aware of the use of open dots in graphs. It is not clear if Jane used the vertical line test to produce graphs of functions. She produced a continuous graph in (PTF14) as Steve did (p. 158).

Other items suggest that she was aware of domain and range when a function was given. For example, when asked about the equality of functions to a given function defined on the set of natural numbers (PTF8), she wrote "when both are natural and are = then is a = function." So it seems that she knew that two functions are equal if they have the same rule over the same domain and codomain. This way she ruled out the first function which was defined for the set of real numbers, although the rule of correspondence was the same. She wrote "no not same as counting numbers include 0..." In part b, with the same domain and codomain, the original function was divided by two; she accepted this as an equal function by writing "yes when reduce." So, here her problem appeared to be an algebra problem. For the graph in (PTF8c), a continuous straight line, she wrote "No not match with equation" while for the dots in (PTF8c) she wrote "yes match."
Jane was inconsistent in her awareness of domain and range. Her graph of a function with domain and range as the set of natural numbers (PTF10) was a continuous straight line.

The equation/formula image emerged in her use of the function \( h(x) = -7x \) instead of \( g(x) = -7 \) to evaluate the constant function \( g \) at different numbers (PTF7). Mistakes associated with the equation/formula appeared in her attempt to find an algebraic form for functions. Given the graph of a piecewise function (PTF9), she answered correctly for the domain, but had difficulties reading the range; she did not write any algebraic form. A second problem was to find an algebraic form of a function that passes through three points (PTF15). There she wrote that the number of different such examples was "0" and explained "not possible."

The image of continuity appeared in the same items as in the case of Steve (see p. 158), but two differences appeared. First, she chose the "infinite" option as the number of different functions passing through two points (PTF13b) because "many functions that impossible to count or do all", and second she connected the dots with a smooth curve.

The last image associated with functions is familiarity. There are two ideas here. The first one refers to recognizing familiar functions and the second idea is being familiar with examples. Jane decided that the piecewise function given algebraically in (PTF2) was "a function because it is a parabola." There she drew a parabola; which opened down. Her decision seemed to come from looking at one piece of the algebraic form (and perhaps her tendency to use parabolas, pp. 174-175).

Examples of functions that Jane (PTF3) gave were similar to Steve's (p. 158). In finding an example in algebraic form of a function from the real numbers to the natural numbers (PTF12a) she tried the familiar functions "\( y = mx + b, -2x \)" and wrote that the number of different such functions was "infinite, depending what you make as #s and variables."
Jane could read the domain of the graph of a piecewise function (PTF9), but had difficulty with the range. For the three continuous graphs in (PTF6) she wrote "range" next to the y-axis in the three graphs. However, she identified correctly two (out of three) functions with given domain and range (PTF11).

Some of the items that she answered incorrectly might have been caused by the lack of familiarity with the vocabulary or with the functions used in the items. Regarding vocabulary, it seems that she did not use the terms "image and preimage" before (PTF4, 5). The piecewise function was difficult when given algebraically (PTF2).

In the Practice Test on Functions, Jane exhibited knowledge of the content of former courses and her images associated with functions. Before I discuss the changes that occurred in Jane's images and definitions of functions during the study, I present her perception of the importance of the topic of functions as she described it in the first interview.

Jane was one of the few students who realized at the beginning of the course that functions were important. She said "how functions are used" was the most important thing she had learned in precalculus (II, 3): "I mean, now I understand more about it. Last year we just talked about it but we didn't go into any depth, as we are doing in this class." When I asked her why she thought functions were the most important thing, she said

It's just, I mean, it's something that we have been doing through over and over the last few years, I mean. We have been building up to it. So I think, I mean, there must be a reason why we are building up to it.

Images and Definitions during the Study

There are several changes observed in the definitions of function that Jane provided in the study. All her definitions contain two central images: univalence and vertical line test. Maybe her perception of them as being equivalent made her cite them simultaneously.
Jane's first definition (p. 166) contained the equation image, maybe because former courses emphasized it. The equation image was less cited in later definitions and seems to have been dominated by the graph image—an image that appeared implicitly in all definitions where the vertical line test is mentioned.

The definition that Jane provided in March (5th definition, p. 166) is a landmark in her thinking. Her definition included univalence, vertical line test, graph, equation, and a big jump in her concept of functions: domain and range. Few students mentioned them as parts of functions in the study.

Some changes in Jane's work can also be traced to the images that she associated with functions during the interviews. In the second interview, she defined a function in terms of the univalence feature only (p. 166). However, other images emerged and competed in her decisions in the tasks presented in the interview. The images of vertical line test, equation, familiarity, and continuity appeared in the protocol; Jane seemed to exhibit a network of images instead of a partitioned set.

Jane used the vertical line test to decide if a graph was a function and to produce graphs of functions and non functions (pp. 168-169). By contrast, she used the equation/formula image in items where the existence of a function was asked (I2, 4, 5, 6) and in discussing the "relationship between functions and equations" (I2, 7). In the first case, the equation/formula image was an obstacle finding the functions, since she tried to write down a formula for them (pp. 171-174). In the second case, "basically [functions and equations] are the same thing" (pp. 172-173). The familiarity image was used by Jane to identify functions (I2, 4, 5) and to provide examples (I2, 7, 8). In the first case, the familiarity image helped her to recognize that a straight horizontal line was a function, but it turned out to be an obstacle when she was faced with the wording of problem 4; the function that assigns the square to every number different from zero. In this situation, Jane tried to work with a
parabola. The use of familiarity to provide examples of functions appeared graphically (p. 169) and algebraically (p. 175). For Jane, the continuity image was a matter of connecting dots; she was the student who observed that this was a teaching practice (obstacle?). While she was drawing the graph of (12, 4) by plotting points, I asked her "why did you try to connect the points?" she answered me

Why? It's just a habit. I mean, you learn to do that, I mean. I was always taught to connect the dots, so...that's what we were taught in Algebra 2. They're connected! [laughing].

Most of the problems in the second interview were word problems. Jane did not attempt to graph any item. This allows one to conjecture that she experiences difficulty moving from verbal representations to graphic representations. A second alternative explanation might be a lack of heuristics to solve a problem; a third alternative could be the dominance of other function images such as the equation or familiarity over the graph image.

Jane's definition of a function in the third interview is a blend of three images: univalence, vertical line test, and equation (I3, 2, p. 166). In addition to these images, the graph image appeared in the interview.

The univalence feature appeared as a means to decide if equations are functions (p. 168) and as equivalent to the vertical line test (4th definition, p. 166). The equation image appeared related to the graph image. After she mentioned the vertical line test I said "I assume you are supposing or you're assuming, I mean, that you can graph your function." She answered "yeah," so I went ahead and asked "Is it always possible to graph it?" and she answered "Mmm I guess if you have enough information, I mean, if you have too many variables maybe you can't." This hesitation from the algebraic representation into the graphic representation did not appear in going the other way around. I asked her "Now, suppose that I give you a graph. Can you come up with an expression that tells you the graph, how it should be?" and she said "yes, that's what we have been doing, I mean, like the
one last in the quiz."

Jane mentioned graphs, forms of graphs, equations and inequalities and concepts related to a global approach to functions ("zero points, maximum, minimum, absolute value, asymptote, just domain, range") as the topics they had learned in the class up to that point (I3, 1). However, she did not have an understanding of the importance of functions (I3, 3). Her answer to this question reflects the authority of the teacher: "I'm not sure what you mean by that. I mean, we wouldn't be studying it if it wasn't important." So I asked her "do you think it's going to be important later" and she said:

never know, maybe. I may want to go somewhere, into another field. I don't know what I want to do right at this minute. Maybe one day, if I say I am really interested, maybe in medicine or something, so I can go into that.

The interrelationship of functions and graphs appeared in the representation of functions (p. 170). For Jane, the equation image caused an obstacle to a more general perception of functions. After she elaborated on the "bla bla bla" (p. 168), I asked her if functions were only for numbers and she said

I think that's the only thing you work with, I don't know why you are asking me numbers, I mean, if you are talking about variables or something, I don't know.

Jane's solution to finding a rule for a piecewise function given graphically (I3, 6, p. 182) suggests an accommodation in her network of ideas associated with the equation image. Students who relied on the equation image were unsuccessful in this task (e.g., Steve's case, pp. 133-134).

The process of describing the graph of a rational function with limited use of the calculator (I3, 5) made evident the weaknesses of her conceptual knowledge of functions; no conceptualization of the task; the procedures emphasized in class; and her struggle with novel algebraic expressions (p. 180).
Jane provided a definition of a function previously to the fourth interview (5th definition, p. 166). Here, her definition is a blend of graph, equation, and univalence images. This definition contains domain and range as part of a function. During the interview, I asked her, referring to that definition, 'do functions have to be equations?' and she said "um I think so." This perception of functions as equations appeared as an obstacle in what makes up a function (I4, 1, p. 174) and in how many functions there are in a piecewise function (I4, 2, pp. 182-183).

The description of the graph of a rational function using the calculator (I4, 4) showed an inconsistency with her definition and with her answer to (I3, 1, above). She described the graph as follows.

Um, well, um it's on positive y's...Um, I guess you could say they look symmetric...it looks continuous...I guess you could say the end behavior model looks like, you know, just like a line...it doesn't pass through the origin...That's all I can think of right now.

The two problems related to the description of a graph using the calculator (I3, 5; I4, 4, p. 180) exhibited how poor her use of the calculator is. However, she recognized that the calculator was very useful in precalculus. She admitted that she used it in the homework, "sometimes just to check [herself], and also when you have to do graphs and that." The question on the usefulness of the calculator in tests made her prompt: "yeah, I mean, I don't know where I'd be without it." She admitted that she "fooled around with it" at the beginning of the course and recognized that the class did not "really use it for all the uses of the calculator either." In this case, she was referring to all the keys that the calculator has.

The last task related to graph construction, the roller coaster problem (I4, 5, p. 183), exhibited Jane's reliance on the iconic features of the problem to draw the graph.

The univalence criterion for functions, in addition to the definition, emerged as a means to decide whether or not a finite set of points given graphically is a function (I4, 3, p. 168).
In the last interview, Jane defined a function in terms of the univalence feature and the vertical line test (7th definition, p. 167). The univalence feature was the only one characteristic that she mentioned in how to explain functions to precalculus students (I5, 5).

The question on properties of functions (I5, 2) allowed other images and ideas to emerge. For example, the equation image emerged as all functions are equations. The dominance of the equation image over the graph image appeared also in the existence of a function that passes through four points (I5, 7). She said that there was not such a function, "unless they were separate functions, meaning if you leave f of two equals zero one function, f of negative three equals seven another function, and so on." This separation of functions appeared several times (e.g., XC2, p. 181).

The graph image also appeared with the question "Can you graph all functions?" where she "yes." The continuity image appeared in making changes to five plotted points to make them the graph of a function (I5, 6, p. 177). She connected the dots. However, this is again a reflection of the "habit" of connecting them since when I asked the question "Is this a function?" referring to the original points, she said "yes." So I asked her "Why did you connect them?" and she said "I don't know."

Two ways to represent functions appeared: equations and graphs (I5, 3). These two ideas might create a potential obstacle. For example, by admitting that all functions can be graphed and using this as a criterion to test if a given representation is a function. A similar situation might hold when saying all functions are equations. On the other hand, she admitted that a graph "is a visual representation of a function...but not always is a function" (I5, 4, p. 171).

Jane showed a consistent use of these two images in the interview. Other ideas emerged during the protocol, for example, not all functions are continuous, and all functions have domain and range.
Summary of Jane's Development and Knowledge of Functions

Jane experienced the development of functions at the procedural and conceptual level. The course provided her with strong algorithms, however, at times it is difficult for her to make connections between procedures and recognize where to apply them.

It seems that Jane had a less concrete notion of functions (as compared to the other participants). Her definition of function contains the idea of domain and range. For example, in properties of functions, she said "anything that applies to the definition." However, her conceptualization of functions is a quasi-structural conceptualization; as equations mainly. This is a cognitive obstacle to deal with the arbitrary nature of functions.

Univalence was also a strong image anchored also in Jane's mind. It allowed her to deal with some unfamiliar situations. Jane had no difficulty stating the univalence criterion for functions. This might be a reflection of the strength of this image and the tug of war that exists between the equation and univalence images. The war might be decided in favor of the univalence, when she realizes that equations are insufficient to represent functions.

Chapter VI includes a comprehensive summary of Jane's case and a network describing her function images and relationships among them.
The Case of Carol

Introduction

Carol is a quiet girl in precalculus and barely asks questions. She does not like mathematics, a subject that has been always difficult for her. She finds it more challenging than her other courses. In addition, she does not consider herself as talented for it, hence she works really hard on it. She did well on her tests and as a consequence, her grades were always above B+.

Carol is taking precalculus because it is a requirement and because it looks good on her transcript. She likes the idea of using the graphing calculator in the class. In particular, she likes using the calculator for graphing instead of plotting points. However, she thinks that she is dependent on the calculator, which she perceives as bad, because there are a lot of tests, like the SAT, where students are supposed to be adept at algebraic manipulations. She also thinks that mathematics is easier with the calculator because it works things out; that way, she thinks, students can come up with better answers. She had some experience using computers in her mathematics classes before coming into this class.

Carol enjoyed the use of the calculator in precalculus. At the beginning of the course, she "graphed all different graphs...just tried to graph the graphs to see what they would look like and programming and, but not very much." However, by the end of the year, she did not do that anymore, the calculator was not "interesting anymore." The calculator, however, gave her a sense of security in mathematics. Unfortunately, she used the calculator in her physics class only as a regular calculator (I4, 7).

Carol was very nervous during the interviews, especially in the third one. She said that the reason she was nervous was probably because she "did not know the answers [and] that makes [her] feel stupid." However, she said her answers were not affected by her feelings. During the member check at the end of the year, she stated that she did not perform well under pressure.
How Carol defines a Function (Concept Definition)

Carol provided seven definitions of functions during the study. All of them contain the univalence feature. Her first definition contains an implicit reference to the graph image as a set of ordered pairs.

A set of ordered pairs where no two x's are repeated. (PTF16)

Carol's second definition is Mr. H.'s definition given in class.

A relation is any set of ordered pairs. But for a function for each input there is only one possible output. (Q-09/11)

The third definition is based solely in the univalence feature but leaves the door open to different representations of functions:

Anything that doesn't go through two of the same x points. (12, 8)

The fourth definition contains a contradictory statement (functions are not relations) and four images (univalence, graph, vertical line test, and equation).

It's a, not a relation, but a set of points...for the, for every x there can't be like, there has to be different y's and like for what you put in for x you'll get the y out, you know, like f of x you put something in for x and you get the y point out...And then you can graph it...and can pass the vertical line test. (13, 2)

The contradictory statement was probably due to a set-inclusion difficulty: functions are relations, but not all relations are functions.

Carol's fifth definition is based on the univalence feature and the equation image.

It means that whatever x equals can be put into an equation of f(x) to find the y value for the x value of a function there can only be one y value. When x is put into an equation f(x) the solution is the y part of the ordered pair [sic]. (Q-03/04)

Her sixth definition adds the graph image to the images in the fifth definition.

For every x value there is only one y value. Whatever value is put into an equation f(x) the solution which is created is a y value and the value that is put in the equation is a x value. These two values created and ordered pair that can be graphed. (XC1)

The last definition that Carol gave is the recognition of a relationship between x and y, and a blend of three images (graph, univalence, and vertical line test).
It's a relationship between \( x \) and \( y \), um, two points on a graph, for a value you put in for \( x \) you'll get the \( y \) out and you can graph it and it will pass the vertical line test. (I5, 1)

**Carol's Mental Pictures Associated with Function (Concept Image)**

Several images associated with functions emerge from the data collected from Carol. Images have been separated for presentation, but overlapping images suggest a network instead of a partition. The images associated with functions are the following: relationship between \( x \) and \( y \); equation/formula, graph, vertical line test, one output for every input, familiarity, and continuity. Indications of each cluster follow.

**Relationship**

This image appeared in the middle of the study (I3, 4a, pp. 197-198); explicitly stated as a definition (I5, 1, pp. 194-195). Mr. H. never used the word "relationship" in his explanation of functions in class.

At times, the idea that Carol attached to her relationship image was different than the one that Elizabeth associated with functions (pp. 101-102). Carol did not always need a pattern as Elizabeth did. For example, in the graphs with four points (I4, 3), after she recognized the first graph of a function, the following dialogue took place.

Carol: Yeah [it is a function], I mean I would assume if it is in the same equation, I mean

Me: Which equation?

Carol: Well, I mean if there's just, yeah, I mean it would be, yeah it's a function.

Me: Yeah, it is a function, why did you mention the equation?

Carol: Well, I mean, I mean it's possible that they're not related at all.

Me: Mmm hmm.

Carol: I assume they were, their points were derived from the same function...or the same equation anyway.

In contrast, the need of a pattern appeared in the existence of a function that satisfies four algebraic conditions (I5, 7, p. 197). The need of the pattern seems to be a reflection of the
equation image, which is discussed below.

*Equation/Formula*

This image appeared in all the interviews; it seems to be one of the two most anchored images in Carol's mind (the other one is univalence). This equation/formula image emerged in several ways:

1. functions can be represented by equations \((11, 13; 13, 2; 15, 3)\),
2. functions are equations \((11, 13, 14c; 12, 7; 13, 4a; 14, 1; 15, 2)\), and
3. not all equations are functions \((12, 7)\).

The first idea in the equation/formula image is that functions can be represented by equations. When she was describing a function with her eyes closed \((11, 13)\), she said:

A function is, uh, something \(f\) of \(x\) or \(g\) of \(x\) or like, I guess, any letter you want and then \(x\) in parenthesis, I mean equals an equation containing the \(x\) like \(x\) squared minus two or any equation, it can be an absolute value and then if you want to solve it, well if they give you a number like \(f\) of four, you put the four in like for the \(x\), usually you put that in. I don't know...that's about it.

This idea continued to the end of the year, I asked her "How can you represent functions?" \((15, 3)\) and she said: "in an equation form, or you can graph them. That's all I know."

See also Carol's fourth definition, \((13, 2)\), p. 194.

The second manifestation of the equation/formula image is that functions are equations. Hence the need to have or look for an equation in order to decide if a representation either verbal \((XC3)\); tabular \((XC2)\); or algebraic \((15, 7)\) is a function. This conception of function imposes an obstacle in any case.

In the case of the relation \(H\) that assigns to every triangle its area \((XC3)\), she decided that \(H\) was a function, "because for each \(x\) value that you put in the equation there is only one \(y\) value." In this case, she wrote the domain and the range of \(H\) as "all reals." During the last interview she clarified more about her answer. She remembered that she found the domain to be all reals "because you can put any \(x\) value in and it doesn't depend on anything. You can put any number in and nothing restricts you so it can be all, any number." However, she did not know "where to put the \(x\)."
The tabular representation was used in the relation that assigns to every clothing item its price (XC2). In this case, the relation was ruled out as a function "because the items are not related in any way by an equation." In the last interview, I asked her how she had solved the problem. She said

Well, for that question, when I was reading it I didn't know what, I was, I was like, I thought it could be a function, because you could graph it... You know? I mean, like, but also, there was no equation, so I was, you know. I didn't answer, that, when I answered that I wasn't completely sure, either way... Either way, I thought it could go either way so I just kind of, cause I didn't really know. There was no equation... I mean, I didn't know if you could graph it but I mean this. The way it's set up it looks like something you could graph like a table with x's and y's, you know what I mean?... You could pair up. I wasn't sure and that's why I put no.

The algebraic representation appeared in the existence of a function that passes through four points (I5, 7). Carol did not know if there was such a function. When I asked her.

"Why is it that you don't know?", she answered "Well, I don't know. What I'm thinking was that you would have to find an equation that would like relate all those points together."

I asked her, what if you do not find an equation and she said "it's not a function, the points are not related at all."

The idea that functions are equations also appeared at the beginning of the study in describing a function with her eyes closed (I1, 13, p. 196) and when she answered what she thought of when she saw or heard f of x (I1, 14c).

That's a function. I don't, you set it up equals to something and then a number can be put in for x and solve different equations, but I don't know. That's it.

One of the most explicit examples of her view that functions are equations appeared in her answer to the relationship of equations and functions (I2, 7). She wrote on her handout "A function has to be [an] equation." After she read her answer she said "but an equation doesn't necessarily have to be a function."

Another explicit example appeared in discussing the components of a function (I3, 4a). She said:
Well, I mean you need like a representation of the two points, either by like f of x, like a number or like f of x and like a relation. You need something, you need both parts, so you'll come out with like a point somehow...I mean, I guess you need like an x and a y somehow.

A third explicit example of functions being equations appeared when I asked her "What makes up a function?" (I4, 7). Her answer conveys the need of an algebraic representation that expresses a relationship between x and y.

I mean, you, you need a formula...Like an equation...Um, you need, I mean, some x value and the y value represented...I mean you need whatever is the equation. Somehow you need it to convey that you can get an ordered pair out of it somehow.

I asked her, "somehow you need that, because you want to graph it?" and she continued: "Well, yeah, plus, I mean that's what a function is, a relationship...for that ordered pair."

A characteristic of the equation/formula image is the omission of the univalence feature or the omission of the domain and range. Carol did not mention the univalence feature until I asked for conditions that the equation needed to satisfy. The following dialogue, a continuation of her discussion of "what makes up a function," illustrates the point.

Me  Is there anything else you need besides the formula?

Caro  Well, you just need f of x equals something, anything...And then that's a function...Well, no that's not true.

Me  Why is it that it is not true?

Caro  I don't know, I'm just confused. You just need f of x equals an equation and then you have a function.

Me  You need f of x equals [something] and you have a function.

Caro  Yeah.

Me  OK. Anything else?

Caro  No.

Me  Any condition that that equation has to satisfy?

Caro  Yeah, there can't be two of the same y values for one x.
At the end of the year, the equation/formula image is attached to functions and emerges as an obstacle. I asked her if all functions were equations (I5, 2) and she said

I don't know...I always see functions with equations, but that, or, a function, I always see equations and functions or functions with equations, but I mean I don't know if they can exist without them.

The third idea in the equation/formula image is that not all equations are functions. This idea appeared in the relationship of equations and functions (I2, 7, p. 197) and as a property of functions (I5, 2, above).

However, equations can be functions if when you graph them they pass the vertical line test (I3, 2, p. 194; I5, 1, pp. 194-195) or if they have the univalence feature (5th definition, p. 194). The idea that equations are functions if they pass the vertical line test appeared explicitly in the 4th interview, when she was explaining to me her 5th definition. I asked her "how do you decide that an equation is a function?" and she said "you could graph it and see...A vertical line test...Or if it's an equation that I know, like x squared."

Graph

Four ideas associated with the graph image emerge in the data collected from Carol:

1. functions can be represented by graphs (I3, 2; I3, 4a; I5, 1, 3),
2. not all graphs are functions (Q-03/04; I1, 13; I3, 2; I5, 1, 4),
3. a means to test that an equation is a function (I4), and
4. all functions can be graphed (I5, 2, 4).

The first idea in this image is that functions can be represented by graphs. This idea appeared in Carol's 4th and last definitions (pp. 194-195), when I asked her to mention the components of a function (I3, 4a, pp. 197-198), and when I asked her "How can you represent functions?" (I5, 3). In the last case, she said: "in an equation form, or you can graph them. That's all I know."

The second idea associated with the graph image is "not all graphs are functions." An extension of this situation is "graphs can be functions if they have the univalence property" (5th definition, p. 194) or if they pass the vertical line test (I1, 13, p. 196; I3, 2, p. 194; I4,
An explicit expression of graphs being functions if they pass the vertical line test appeared when Carol was describing a function with the eyes closed (I1, 13). I asked her if she had seen a graph when she closed her eyes and she said yes. So I asked her to make a drawing of the graph or describe it in case she could not graph it. She said

It can well, it can be, a function can be anything, except for, it can't like can't the vertical line, they can't, they can't like, it can be a parabola, but it can't be a sided parabola because that line goes through. I don't know, that's what I always remember. Cause if you try over the vertical line test, it can't be, if you have the vertical line, it can't go through a function twice. It can be a line or it can be anything.

The third idea Carol associated with the graph image is using it as a means to check that an equation is a function. For example, during the 4th interview, after she read her 5th definition I asked her "how do you decide that an equation is a function?" and she said "you could graph it and see...A vertical line test...Or if it's an equation that I know, like x squared."

The last idea associated with the graph image is that all functions can be graphed. This idea appeared at the end of the year. I asked her "Can you graph all functions?" in properties that functions have (I5, 2) and she answered "yes." Also in the question "Are functions and graphs related?" (I5, 4), she said "yes, um, cause all functions can be graphed...But a graph doesn't necessarily have to be a function."

**Vertical Line Test**

The vertical line test image also emerged in different ways in Carol's data:

1. (1) as a means to decide that a graph is a function (I2, 1; I4, 3; I5, 1, p. 195; I5, 6);
2. (2) as a property of functions (I5, 2);
3. (3) as an equivalent statement to the univalence feature (I2, 1, 2, 8; I3; I4, 3); and
4. (4) as a means to decide that an equation is a function (I5, 1, p. 195).

Carol used the vertical line test to decide if a graph was a function or not. For the graph of a non function (I2, 1), she said "no because if you do the vertical line test to [it], it crosses it twice like right there."
Carol also applied the vertical line test to graphs with a discrete domain (I4, 3; I5, 6). In the case of the graph of a function with four points (I4, 3a), she said that the graph was a function "because each x value has its own y-value. They don't cross, I mean if you did a vertical line test it wouldn't on the lines cross." In the case of a non function (I4, 3b), she said "it is not a function...because of the vertical line test and like for these x, for this x value, well, yeah, for those x values there's more than one y."

Carol was one of the few students who did not make changes to five plotted points to make them the graph of a function (I5, 6). She simply said "I wouldn't make any changes...Because of the vertical line test."

The second idea associated with the vertical line test is being a property of functions. Carol did not explicitly say that a function had to pass the vertical line test. However, when I asked her "What properties do functions have?" (I5, 2) she said:

they would be the same properties that I just said in the definition, just that it, for each x there can only be one y value...That's the only property I can think of.

Carol's definition in the last interview ends saying "and you can graph it and it will pass the vertical line test" (pp. 194-195).

The third idea associated with the vertical line test is that of being equivalent to the univalence feature. This idea allows her to use either of these images even in the same representation. For example, she used the vertical line test to decide that the graph in (I2, 1) was not a function (p. 200), but she used the univalence feature (though incorrectly stated) to decide that a discontinuous graph (I2, 2) was a function. She said "it's a function because it doesn't have the same y point twice." An explicit recognition of this equivalence appeared in "What is a function in your opinion?" (I2, 8). She wrote on her handout "anything that doesn't go through two of the same x points" so I asked her "why is that?" She replied "I was just referring to the vertical line test." Other recognition of the equivalence between the vertical line test and the univalence feature appeared when she told me how she solved the
compositions \( f(g(x)) \) and \( g(f(x)) \) (pp. 211-212). The last idea associated with the vertical line is to use it to test that an equation is a function. In this case, the equation is graphed and then the vertical line test is applied to the graph (p. 200).

One Output for every Input

The univalence image is the other image most anchored in Carol's mind. She associated several ideas with it:

(1) definition,
(2) equivalence to the vertical line test,
(3) property (I5, 2), and
(4) as a means to decide if a given graphic (continuous (I2, 2, 5) or discrete (I2, 4; I4, 3)) representation is a function.

The definition idea appeared in all the definitions that she provided for functions (pp. 194-195). As discussed above, Carol's second idea associated with the univalence image is that of being equivalent to the vertical line test. The third idea is that the univalence feature is a property of functions (I5, 2). Carol's answer to the question "What properties do functions have?" was:

they would be the same properties that I just said in the definition, just that it, for each \( x \) there can only be one \( y \) value...That's the only property I can think of.

The fourth idea associated with the univalence feature is that it can be used to test whether or not a given representation is a function. Carol used the univalence feature to decide if an equation, a graph, or a set of points is a function. However, she had difficulties explaining the reasoning. For example for the graph of a discontinuous function (I2, 2) she said "yes, it's a function because it doesn't have the same \( y \) point twice."

The use of the univalence feature to test if a set of points is a function (though with hesitation) appeared when I asked 'when you don't have a graph, how do you decide [that the problem is a function]?’ (I2, 4). There she said

Well, if you just have a set of points and two \( y \)-values are the same, just for the same \( x \) like it can't be, two of the set of points can't be the same, the \( y \)-values can't be the same. Maybe I'm wrong, I don't know.
Carol used also the univalence feature with graphs with four points (I4, 3, p. 200).

A final comment to this image is that Carol had difficulty stating it as illustrated above. For example, in the case of a function all of whose y values are the same, (II, 5), she said first: "two y values can't be the same." In the hesitation, she changed her mind to "they are the same, they can't be the same for the same x values." Next to "two points can't be the same, I mean, I don't know, in a function the x's and the y's can't be the same. I'm sorry." And finally to "two x's can't be the same." At this point, I asked her "Is there a function such that all the y values are the same?" and she answered "yeah, it would be a straight line." When I asked her to draw it, she drew a similar figure to Elizabeth's (figure 12, p. 102).

Familiarity

There are two ideas in the familiarity image. The first idea is familiarity with examples of functions. Carol was familiar with easy examples of functions and mentioned them when asked about functions (II, 13, p. 195). However, she knew more examples, since she admitted that "a function can be anything" that passes the vertical line test (II, 13, p. 200). The second idea in the familiarity image is to use this knowledge to decide if a given representation is a function or not (5th definition, pp. 194-195).

Continuity

There is one idea in the continuity image—connecting the dots. This idea appeared in the existence of a function discontinuous at one point, (II, 4). Carol's protocol for this problem was similar to Elizabeth's (pp. 111-113). Carol drew an intermediate graph (figure 53) between figures 17 and 20, after she plotted several points. This picture suggests the dominance of the familiarity image over the idea of connecting dots in the continuity image (since she did not include the point (0, 1)), 
Figure 53. Carol's graph of (I2, 4).

When I asked Carol "what happens with zero, one?", she said "I don't know. Oh, I forgot about that little one" and drew figure 20 (p. 113) identical with Elizabeth's; a continuous graph that passed through (0, 1). A dramatic difference with Elizabeth's protocol was that Carol did not rely on the equation image. The protocol ended when she said "yes" to the question "Is [the graph you drew] a function?" I did not show the correct graph (fig. 14, p. 108) to Carol as I did with Elizabeth.

Summary of Carol's Function Images

Seven function images emerged in the data collected from Carol during this study: equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, continuity, and relationship. Several links between these images suggest the existence of a network. Carol's work suggest a strong tendency to equations and univalence. A third image--relationship--was strong also in Carol, and signifies a healthy operational conceptualization of functions. Carol's network of function images reflect a quasi-structural conception of functions. The equation image created cognitive obstacles to deal with the arbitrary nature of functions. However, the strong tendency to univalence allowed Carol to deal with some unfamiliar tasks.

Procedural and Conceptual Knowledge of Functions

Carol was an above average student in class. Part of her success in the class was based on her ability to use the calculator. She was a quiet student and asked few questions during the year, though she interacted a lot during the small-group activity, usually with Jane. In this section I present her approaches to several tasks to illustrate her knowledge of
functions.

Carol's grades reflect her ability to carry out algorithms. She had difficulties with algebraic manipulations, and reading points from the calculator. An example of her difficulties with algebraic manipulations appears in the dialogue of (13, 5), pp. 206-207.

An example of her difficulties reading answers from the calculator was related to the syntax of mathematics. In (Q-03/04) students were asked to solve the inequality "\( \sqrt{4 - x^2} \geq x - 1 \)." Her answer was "\((-1.46, 1.33) \cup (1.46, 1.33)\)." During the third interview she explained to me that she graphed both sides of the inequality and continued.

This one is greater than that...Which is just in that little spot in the middle...I mean I understood that. I just didn't understand how he wanted me to write it. I think at that form. I put like both points.

The correct answer to this question was the interval \((-1.46, 1.46)\), but Carol wrote her answer as the union of two intervals. When she traced on the calculator, she read the \(x\) and the \(y\) coordinates for the two intersections of the graphs. Similar mistakes appeared in other testing materials.

**Composition of Functions**

Carol was successful finding the compositions of functions in the usual testing. However, she remained at the procedural level. She was one of the few students who found \( f(h(x)) = 5 \) when \( f(x) = 5 \) and \( h(x) = x \) (Q-02/07). During the third interview, she explained to me how she found this answer:

Um, by putting the, putting \( h \), of \( x \). See I'm still not even sure. But, \( h \) of \( x \) is \( x \)...and then you put it into \( f \) which is five...So, and it is just five.

However, for the composition \( h(f(x)) \), she wrote "5x." In this case, her answer was based on a belief about testing:

I don't know, I just thought, I don't know. Since I got five for one it wouldn't be five for both...So I put five \( x \).
Global Approach of Functions and Connections

In this section I present three tasks where Carol described a function. In the first case, (13, 4b), I gave her the following task: "Imagine that you have a graph that you need to describe to a friend over the telephone. What are the things that you would tell to your friend to describe the graph?"

Carol's first reactions to this task were two questions: "about the shape of the graph?" and "by using an equation?" I answered "yes" to the first question, but "no" to the second. The following dialogue illustrates her description. This description reflects either the need of being asked questions, the need of a specific case, or the lack of connections of the material studied (global approach to graphs).

Carol: I mean you could just describe it physically, like it's a parabola or...a parabola or something like that...Hyperbola or I don't know, a curving line, I don't know, those are physical features.

Me: OK. What other things would you tell [your friend]?

Carol: Like if there are any asymptotes or something like that.

Me: OK.

Carol: Like...like breaks in the line or, um...That's all I can't think of.

Me: What else? I mean, nothing more?

Carol: Like what the end behavior is?

Me: For example, behavior.

Carol: x goes to infinity, f of x goes to infinity or whatever it is.

Me: OK. Anything else?

Carol: Oh, the zero points...x and y intercepts...Whether it's one to one.

Me: Whether it is one to one. OK. Anything else?

Carol: See if it passes the vertical line test.

Me: And anything else?

Carol: No. I don't think so.
The process of constructing the graph of the rational function, \( f(x) = \frac{x^3 + 2x + 1}{2x^3 - 4} \) with limited use of the calculator was less successful (I3, 5). The task made evident the weakness of her conceptual knowledge (rich relationships); little conceptualization of the problem; the procedures emphasized in the class; and her struggle with algebraic manipulations.

She started as other students by factoring the top and the bottom (e.g., Elizabeth, p. 116). Her first factorization of the numerator was "\( x(x^2 + 2) + 1 \)" followed by "\( (x + 1)(x^2 + 2) \)." In the denominator she wrote "\( 2(x^3 - 2) \)." There was a long delay before she asked if I wanted a graph. She admitted befuddlement:

I don't know how to do it, like now, without graphing the whole thing...I don't know how to get the asymptotes, get it...well, I guess I do...Then again, I don't know if I'm right. I don't remember this stuff.

Carol became nervous in this task. Her answers at each step were a hesitation. For example, she continued,

Would there be an asymptote at zero?...and then plus or minus cube root of two? I don't know what I'm doing. I'm mixing this up. I don't know what I'm doing.

I decided to interact with her hoping she would feel comfortable and confident. This is the dialogue we had.

**Me**  
What expression are you using for the [asymptotes]?

**Caro**  
I was using the bottom.

**Me**  
OK. For the asymptotes you are using the bottom. Now, how is it that you come up with the values where the asymptotes are?

**Caro**  
I don't know, I just, I don't know if you just, I don't know if it. I don't know...I just thought you would have that equal to zero. Like you do when you just solve for zero point. You know what I mean?

**Me**  
Sort of.

Carol knew that the zeros of the denominator would give the vertical asymptotes of the curve. However, her lack of ability manipulating algebraic expressions made her to deduce
wrong statements.

    Carol I don't know then, I would, I mean, I know it's not right but I would put
    asymptotes at zero and then plus or minus cube root of two.

    Me Where did you come up with zero?

    Carol From the two, I don't know...you know if you have, you're like, if, I was
    trying to find the zero points.

In other words, she had the equation $2(x^3 - 2) = 0$ and obtained as roots $x = 0$ and
$\pm \sqrt{2}$. Similar mistakes appeared in the factorization of the numerator, which she wrote
as "$x(x^2 + 2) + 1 = (x + 1)(x^2 + 2)$." 

Carol knew that the top part would give the zero points. So she found the zeros to be at
"negative one and squared root of negative two." However, she changed her mind about the
zeros, when I asked her to solve the equation $x^2 + 2 = 0$. She solved the equation and said
"there aren't any there, for that...Cause it's not a real number."

Carol started putting this information into a graphic form. She drew figure 54 for the
asymptotes.

\[ x = 0 \quad x = \pm \sqrt{2} \]

Figure 54. Carol's first sketch of item (I3, 5).

The end behavior model also was obscured by the wrong algebraic manipulation. She said
that the graph looked "diagonal line...that's what I think it would be." Carol knew that the
terms with the highest degree would give the end behavior and she proceeded to simplify
them. In this case, she had "like it would be x cubed over two x cubed." She came up with
two possible answers: "2 or 2x", but she did not know "which one" to choose. I told her
that the correct answer was one half and asked her 'how that information could be used to
draw the graph' and she answered:

I don't know because usually there's like something there, usually you end up with an x or an x squared. I mean, one half, should just be like a straight line across, like if you graphed it, but I don't know.

Carol was not sure of what we were doing. She was unsure where the one half fit in the graph. When I asked her to graph the line she said "but that's not right...I don't know, because that's not going to be a straight line across." So I told her "remember that this is not, it is the general behavior of the graph. That doesn't belong to the graph." With this information she attempted to make a better approximation for the graph (figure 55).

![Graph of y = 1/2 with x = 0 and x = ±√2](image)

Figure 55. Carol's graph of item (I3, 5).

Carol did not know how the graph behaved in the middle, she just said that it might be a parabola. When I asked her what else she could say about the graph, she answered "um, that as x goes to infinity, or as f of x goes to infinity x goes to one half?" The interview finished here.

The third task refers to the description of a graph of the rational function, \( f(x) = \frac{x^2 + 1}{|x|} \), using the calculator (I4, 4). In this case, her answer was limited to finding domain, range, and positiveness of the graph.

Carol's first descriptor was that the graph was a function. A distinctive statement from other participants.

It's a function...[why?] because it looks like one...Well, I mean, like that, yeah, it's a function, cause it passes the vertical line test.
Carol did not rely only on the window she was using but used other windows to check the graph: "I was just checking...I was making sure it didn't cross over in there somewhere."

Carol also found the domain and the range.

Its domain looks like it's all reals. Well, well no, wait, I take that back, it's like negative, it looks like an asymptote. I take that back...I'm sorry...Looks like there's an asymptote at, for, so the domain would be negative infinity to zero...zero to infinity.

However, she did not indicate whether the domain included the endpoints or if the domain was the union of the intervals until I asked her to do so. Her replies were correct. In the case of the range, she hesitated "um, two, two to infinity?" Again she did not indicate if the interval included the two until I asked her. In this case, she was not sure—she guessed.

Carol also described the positiveness of the graph: "not negative, because it's absolute value...um, that's all I can think of."

**Settings: Domain, Range and Rule**

The course included examples of functions where the domain was either the set of real numbers or the set of real numbers minus five points at most. The range was the union of intervals at most. This might explain some of the difficulties that Carol experienced in problems where the domain and range were arbitrary sets or with functions defined on a discrete domain (XC2, XC3, Q-02/19). For example, for the function that assigns to every clothing item its price, she wrote that it was not a function "because the items are not related in any way by an equation" (XC2).

In the case of the relation H that assigns to every triangle its area (XC3), she decided that H was a function, "because for each x value that you put in the equation there is only one y value." In this case, she wrote the domain and the range of H as "all reals."

Familiarity was another image associated with settings not used in class. For example, in the case of the parabola defined only for integer numbers (Q-02/19), she drew a similar graph to Jane's (figure 51, p. 181), just with more dots, but still without the point (0, 0).
Carol wrote the domain as "all integers," but the range as "all integers greater than or equal to 0." However, when I asked her in the 4th interview if she had 'three in the range" she said "no. Wait. Hmm. I don't know. I don't think so." It seems natural to extend the information we are familiar with. For example, if for a parabola defined over the real numbers the range is the set of non negative real numbers, then for a parabola defined over the integers the range is the set of non negative integers. Carol used the familiarity image to decide that this was a function.

Because I mean, just from the graph...Like from the equation you can tell it's a function...I mean x squared is a parabola, you kind of know that...You kind of know that graph well, you kind of know it's a function...From what I've, you know, known before, that x squared is generally a function.

Carol was another student (e.g., Jane, p. 182) who could write the rule over each interval for the piecewise function defined on a bounded interval (I3, 6). She wrote the rules using the notation f(x) (a difference from Jane who used y's). However, she was confused by the familiarity image with a straight line. After writing the rules, she said "but I don't know what to do...it stops and I don't know why it does that." When I asked her if she was surprised because it stopped, she admitted:

yeah, cause I don't, I don't think I've ever learned anything like, mmm, I don't know...cause usually if a y is straight, it just goes on. I mean this would be something like, it's strange something like.

Carol also found the domain and wrote it as two intervals (without a union sign). After doing it, she said "I mean, well just, but I don't know how to like get that across to your equation." So she was trying to find a single equation for the expression. I tried to make her talk about piecewise functions to see if she could perceive that the task was complete, however, she said "I vaguely remember that, but...I don't know how to do it." Here the task ended.

Her knowledge of a constant function appeared after she explained to me how she had solved the compositions f(g(x)) and g(f(x)) when f(x) = 5 and g(x) = x (Q-02/07). I asked
her to tell me what she knew about a constant function. She said that she had not heard about them. So I told her that the answer she came up with for one of the compositions was a constant function, \( f(h(x)) = 5 \). We continued:

Caro: So it's not like an equation. It's just equal to a number... What do I know about it?... It's a graph... it's either a vertical line or a horizontal line, I don't know, I'm not sure... Hmm. Would it be, I'm not sure, would it be \( y \) equals five? [drawing a constant line].

Me: OK. And what can you tell? First, you drew the graph and then?

Caro: I drew the line cause, maybe that's alright. I don't know.

Me: OK.

Caro: That's what I think it was because, I don't know well, yeah because all the \( y \) values are different. So that should be right because it doesn't change.

Me: OK.

Caro: No matter what you put in for \( x \), it will always be five.

Me: OK. Now, is it a function?

Caro: Yeah.

Me: Why?

Caro: Cause, none of the \( y \) values are the same.

Me: Because of what?

Caro: Well, you can do the vertical line test.

Carol did not have difficulties finding the domain as "all reals" and the range as "five." She also answered correctly all the questions related to the constant function \( f(x) = 7 \) given graphically (Q-29/01).

Carol was one of the few students (e.g., Griswald), who recognized that a piecewise relation (I4, 2) was one function. However, her way of interpreting the relation suggests her difficulties with piecewise functions, with constant pieces. The following dialogue illustrates her understanding of the relation.

Me: How many functions do you have there?
Caro: Just one.
Me: Just one. OK. Now is it a function?
Caro: Yeah.
Me: Yeah? How do you know that it is a function?
Caro: Because it is, each different point and they're not, I mean they're just points and they're different points and they not going to...they're just, two separate points. I mean they're not lines so they're not going to cross or anything, I mean.
Me: How do you know that they are not going to cross?
Caro: I'm just kind of guessing, I'm not guessing from like, I don't know, I'm like thinking that they won't but I don't know that for sure.

Carol could not draw the graph for this item. She did not know where to put the two points she was associating with the graph:

I mean, it just seemed like it would be two different points and not be lines...It would just be the points at negative one and one and it wouldn't be a line, but maybe I'm wrong...I don't even know where [the points] would be, I mean, x is right there. Cause, there's no, no real defined.

Carol could not draw the points, but still thought the relation given was a function. She said the domain was "all reals", and the range "one and negative one."

Construction of Graphs

Carol's solution of the construction of the graph for the roller-coaster (I4, 5) was similar to Steve's (p. 157). She also had troubles deciding which variable goes in which axes and after some direction drew the correct graph.

Development of Functions

This section is composed of two parts. The first part describes some of the ideas on functions that Carol had at the beginning of the course. Data is drawn from the Practice Test on Functions. The second part is a discussion of the development of the concept of functions based on the concept images and definitions provided by her throughout the study.
Some Ideas on Functions at the Beginning of the Course

Some of Carol's ideas on functions were captured on the Practice Test on Functions. Her definition of function was: "A set of ordered pairs where no two x's are repeated," which reflects the use of the univalence property. This definition suggests the image of graph (as ordered pairs). Images associated with functions in the test were univalence (definition), equation/formula, continuity, familiarity, and vertical line test.

Carol used the univalence feature as definition of a function and to check if an algebraic expression was a function. For example, in the piecewise function given algebraically (PTF2), she answered (though incorrectly) "B) It is not a function because their can be the same value for x twice" [sic]. Although the procedure was correct, the incorrect conclusion might have come out of perceiving the given representation as two rules (i.e., she ignored the domain for each piece) and concluded that each x had two images.

Ignoring the domain of a function and looking only at the algebraic expression might have been a reflection of the equation/formula image. She used this equation/formula image to evaluate the constant function at different points (PTF7) and to decide if two functions were equal (PTF8). In the first case, she used \( h(x) = -7x \) instead of \( g(x) = -7 \) to evaluate the constant function \( g \) at different values. In the second case, she ignored the domain and codomain of a given function to decide if it was equal to another function. For example, given the line \( f(x) = 4x + 6 \) from the natural numbers to the natural numbers, she wrote

a) for the line \( f(x) = 4x + 6 \) from the real numbers to the real numbers "the functions are equal and make the same graph", and

b) for the line \( f(x) = 2x + 3 \) from the natural number to the natural numbers "the function is the same but has been reduced." In this case, an algebraic mistake was the origin of the error. When the graphical representation was given, the equation/formula image might have been followed by the familiarity image. For example for the two graphs given in part c and d, she wrote "No, this isn't the graph of the function." So it might be the case that
she had an equation, mentally drew a continuous line (by familiarity), and expected to have a line defined for all the reals.

Carol did not totally omit consideration of domain and range. For example, in item 10, students were asked to draw the graph of the function \( g(x) = x - 2 \) from the natural numbers to the natural numbers. She plotted points only in the first quadrant \( (2, 0), (3, 1), (4, 2), (5, 3) \) and proceeded to connect them. Her line is also in the first quadrant and has an arrow pointing to the upper right. So she considered the domain, since she did not draw the line going on to the other direction. Her connecting of the points might have been caused by the continuity image (or perhaps, better stated, by the habit of connecting dots).

Having an equation might also be a requirement for a function. For example Carol's scratch work shows that she was using the function \( f(x) = x \) when students were asked to find an algebraic form for a piecewise function given algebraically (PTF9).

The familiarity image also appeared in giving examples of functions. She drew a vertical parabola as an example of a function and a horizontal parabola as an example of a non-function (PTF3).

Carol's answers suggest that she perceived the vertical line test as equivalent to the univalence property. For example in the piecewise discontinuous function (PTF1), she wrote "A) the relation is a function because the vertical line test only passes through one point."

Carol had difficulties recognizing continuous and discontinuous graphs with a given domain and range (PTF11). She did not attempt the remaining items in the test. However, her test exhibited content of former courses and images associated with functions.

Images and Definitions of Functions during the Study

There are several changes observed in the seven definitions that Carol provided during the study. All of them contain the univalence feature. Although at the beginning of the course, her definition is abstract, by the end of the year, the definition stabilized as a
relationship between $x$ and $y$ that can be represented graphically and passes the vertical line test.

Some changes can also be traced on the images that Carol associated with functions during the interviews. Her definition of a function in the second interview states explicitly the univalence feature but other images are suggested: a graphical representation that passes the vertical line test. Three more images emerged in the dialogues as well: equation/formula, familiarity and continuity.

Carol used the vertical line test to decide if a graph was a function (I2, 1, p. 200) and perceived this image as equivalent to the univalence feature (I2, 8, p. 201). This generates a rich network of relationships, since it provides two alternatives for testing that a graph is a function. The univalence feature appeared, in addition to definition of functions (p. 194), as a means to decide that a continuous (I2, 2, 5, pp. 202-203) or discrete (I2, 4, p. 202) graph was a function. However, Carol showed difficulties with the univalence feature. For example, she was confused with the wording of the existence of a function all of whose values were the same (I2, 4). I asked her to make a graph of the given situation and then check if the graph passed the vertical line test at the beginning of the task. She said

Well, if you just have points and two $y$ values are the same, just for the same $x$ like, it can't be, two of the set of points can't be the same, the $y$ values can't be the same. Maybe, I'm wrong. I don't know.

The equation/formula image emerged as 'functions are equations', but not all functions are equations (I2, 7, p. 197). The familiarity image appeared in the existence of a function discontinuous at one point, (I2, 4, p. 203). In this case, familiarity became an obstacle initially since she tried to work with a parabola as other students did (e.g., Elizabeth, pp. 110-111). When she realized that the graph was not a parabola, the continuity image emerged as a matter of connecting dots (I2, 4, pp. 203-204).

Most of the tasks included on the second interview were word problems. Carol, as other students (e.g., Jane, pp. 187-188) did not attempt to graph any item. This allows one
to conjecture that she experiences difficulty moving from verbal representation into graphic representation. A second alternative explanation might be the lack of heuristics to solve a problem. A third alternative is the dominance of other function images such as equation/formula or familiarity over the graph image.

Carol's definition of function in the third interview is a blend of four images: univalence, vertical line test, graph, and equation (p. 194). Her statement contains a confusion—functions are not relations. One more image emerged in the dialogue—relationship.

The univalence feature appeared as a definition. The idea of equivalence between univalence and vertical line test continued consistently. She referred to both when she showed me how she decided that \( f(x) = 5 \) was a function (pp. 211-212). The graph image appeared as a way to represent functions, but she acknowledged that not all graphs are functions (I3, 2, p. 194; I3, 4a, pp. 197-198). The equation/formula emerged as a way to represent functions (I3, 2, p. 194) and as a conception of functions—functions are equations (I3, 4a, pp. 197-198).

It was in the third interview when the relationship image emerged (pp. 195-196). Elizabeth was the other student who attached this image to functions. However, unlike Elizabeth, Carol did not manifest any need for a pattern in the the interview.

Carol's definition did not include the idea of equation and this allowed her to make translations from graphic representations to algebraic representations. However, the image of equation still causes some obstacles to make those translations. For example, finding the mathematical expression of a piecewise function, (I3, 6), she could find the rule and the domain for each piece, but could not complete the task because she "did not know how like to get it across of [the] equation"

This difficulty might be due partially to the function being piecewise. Carol did not have difficulty answering items related to a constant function given graphically (Q-01/29).
Carol provided a definition previous to the 4th interview (5th definition, p. 194). This
definition is based only on the equation image and the univalence feature. During the 4th
interview other images appeared: graph, vertical line test, and familiarity. The
equation/formula image emerged as an algebraic representation of functions (I4, 1, p. 198)
and made her conclude that functions were equations. This is the second appearance of the
relationship image. In this case, a function is "a relationship, a formula that tells you a
relationship between x and y." This conceptualization is the most basic idea of function.

The univalence feature appeared in what makes up a function (I4, 1), though incorrectly
stated. After she said that she needed an equation, I asked her, "any condition that that
equation has to satisfy" and she said "yeah, there can't be two of the same y values for one
x." The graph image was connected to the equation image. In this case, she knew that a
way to decide that an equation is a function is to graph it and verify that it passes the vertical
line test (pp. 199-200). Carol used the vertical line test to decide that a graph is a function
and as an equivalent statement to the univalence feature in the same item (I4, 3, p. 201). Her
idea of not needing a pattern in the relationship image appeared in this item. She said that
the points in the first graphs were a function and continued "it's possible that they're not
related at all." The familiarity image appeared as a way to decide that a known equation is a
function (p. 199).

Carol was one of the two students (the other one was Griswald), who answered
correctly that there was only one function in a piecewise relation (I4, 2). However, she gave
the wrong reason: "they're just, two separate points, I mean they're not lines so they're not
going to cross or anything, I mean." She was unable to graph the function.

I'm not really sure how to graph it...I mean, it just seemed like it would be two
different points and not be lines...Yeah. It would just be the points at negative one
and one and it wouldn't be a line., but maybe I'm wrong.

It might be the case that the confusion was rooted in the notation used in piecewise
functions. The course did little to draw piecewise functions with constant functions.
Carol's description of the graph of a rational function using the calculator (I4, 4) was limited to finding the domain, range, and its positiveness. However, she knew she was dealing with a function and stated so. She used several windows to check the graph in other intervals. I asked how she knew it was a function and she answered:

"It looks like one...I mean, it looks like that, yeah it's a function, cause it passes the vertical line test...[I zoomed in] I was just checking...I was making sure it didn't cross over in there somewhere."

Carol drew the correct graph in the roller coaster problem, but as other students (e.g., Steve) she had difficulties labeling the axis.

Carol was able to read the domain and the range of functions (piecewise (I4, 2) and with discrete domain (I4, 3), from the calculator (I4, 4)). However, she had difficulties with the end points using the calculator (pp. 209-210).

Carol provided a definition previous to the fifth interview. Her definition is based on the equation, graph, and univalence images (6th definition, p. 194). The dominance of the equation image over the univalence feature emerged when she faced a function with a discrete domain and range (XC2, p. 210) or a function with arbitrary sets (XC3, p. 210).

Carol's last definition of a function includes four images: relationship, graph, univalence, and vertical line test (pp. 194-195). During the 5th interview other images emerged: continuity, and equation.

The relationship image appeared this time as the need for a pattern (I5, 7, p. 197). This seems to be a regression, since she did not attach this idea before to this image (e.g., I4, 3, p. 195). Three ideas appeared related to the graph image: (1) as a representation for functions (I5, 1, pp. 194-195; I5, 3, p. 199); (2) not all graphs are functions (I5, 1, pp. 194-195; I5, 4, pp. 199-200); and all functions can be graphed (I5, 2, 4, p. 200).

The univalence feature appeared as a property of functions (I5, 2, p. 202), in addition to the definition. Three ideas appeared related to the vertical line test image: (1) a means to decide that a graph is a function (I5, 1, pp. 194-195; I5, 6, p. 201); (2) a property that
functions satisfy (15, 2, p. 201); and (3) as a means to decide that an equation is a function (15, 1, pp. 194-195). This last idea is related to the graph image as an intermediate representation.

Carol's continuity image appeared to be related to the familiarity image. I asked her if all functions were continuous (15, 2), and she said "I don't know...I haven't seen enough examples." The equation image also emerged when I asked her if all functions were equations (15, 2). Her answer suggests how the equation image represents a pedagogical obstacle.

I don't know...I've never learned that I guess...I always see functions with equations but that or, a function. I always see equations with functions and or functions with equations, but I mean I don't know if they can exist without them.

Carol perceived only two possible ways to represent functions: equation and graph (15, 3). This led her to state the relationship between graphs and functions (15, 4): "all functions can be graphed...but a graph doesn't necessarily have to be a function."

Carol was one of the few students who did not make changes to five points to make them the graph of a function. She recognized them as a function at the beginning by using the vertical line test. However, she could not decide if there was a function that passed through four points given algebraically (15, 7), even though she plotted the points correctly (p. 197).

Carol was another student who called for more conceptual understanding of functions instead of emphasizing how to use them. She did not have specific suggestions to make for Mr. H., however.

Maybe go over what it, a function really means, in more depth. I mean, I don't know how you would go about it, but...I don't think we spent much time like on the concept of what a function really is. We just used them more than like learned to, like, stuff about a function is an equation, stuff that I don't know. (15, 5)

Maybe the questions in the protocols made Carol realize that there was a broad range of functions. She suggested to have "more uncommon [examples]...not ones that are just
parabolas or something, you know, exposing us to different functions."

Although Carol enjoyed the use of the calculator, she felt that she was dependent on it. She suggested to use it "as something to check whether something is a function, but not like depend on it" (IS, 5).

**Summary of Carol's Development and Knowledge of Functions**

Carol experienced the development of functions at the procedural and conceptual level. The course provided her with strong algorithms, however she had difficulties with algebraic manipulations and perhaps her potential was unrealized.

Conceptually, Carol developed a network of relationships. However, the richness of the network is blocked at times for the equation image. Carol's conceptualization of the idea of functions seems to be a quasi-structural conception—as equations—but not in a strict sense. The relationship image that Carol associated with functions seems to reflect a pure operational conception. Carol started becoming aware of her conception of functions and started to question that maybe not all functions are equations or graphs. So her weakness, may become, paradoxically, her strength.

Carol perceived the univalence feature as an essential feature. This allowed her to deal with the other essential feature of functions—arbitrariness. She, however, still has contradictory reactions in items dealing with arbitrary rules for functions or arbitrary domains and ranges. This is some of the effects of the equation image, which is winning the tug of war.

Finally, Carol's difficulty expressing her mathematical ideas, in particular the univalence criterion of functions might have been compounded with her nervousness. How much these difficulties got in the way of her development of mathematical ideas are beyond the scope of this study.
The Case of Tyler

Introduction

Tyler is a quiet and attentive student in precalculus. He barely asks questions in class, and seems to write a lot of what Mr. H. does in class. Tyler likes mathematics although he finds it tough and challenging for at least two reasons. First, it does not come easy to him and he needs to work really hard to get decent grades. Second, he feels he has not had very good mathematics teachers. He admits that he "learned the materials just to take the test and do well on it."

Tyler is having difficulties in precalculus and anticipates having them in college mathematics. He thinks he needs to go slower and "should be in an adjusted precalculus class." He feels "confused sometimes, or most of the times actually." For example, he always does his homework, makes an attempt to answer the questions; however, he often does not know for sure if his answers are right.

After doing poorly on the first two tests Tyler thought of dropping the course; however he decided to take it as pass/fail, since it is important to have it on his college transcript, and because he is planning to major in business in college. He thinks that a general knowledge of precalculus will be useful then. However, taking the course as pass/fail has its negative side; it allows him "kinda coast through the course" and still pass the class.

Tyler used a graphing calculator in Algebra 2 and loves learning mathematics with it. He thinks this way of learning mathematics is a lot easier and that using a calculator helps him to understand things better because he can visualize them. The use of calculators in precalculus gives him some satisfaction and hopes, because the more he learns to use the calculator, the better he is going to be. He knows that much of the figuring with pencil and paper is what he has trouble with. But, with the calculator he can graph whatever they ask him. He did not find it appealing to do explorations with it outside school "because his interests were somewhere else" (14, 6).
Two personal notes about Tyler. First, he had the highest GPA at the beginning of the class among all the participants in this study. Second, he insisted during the first two interviews that he did not know if he was the right subject for me to study because he was taking the class pass/fail. He thought that if I was supposed to be looking at the improvement students made "probably [he was] going to stay where [he was] at" (I2).

How Tyler defines a Function (Concept Definition)

Tyler provided seven definitions of function during the study. The absence of the most common image—the vertical line test—is notable in his first three definitions. All the definitions contain the idea of graph either explicitly or implicitly. The first definition does not distinguish between functions and non functions and suggests an implicit image of the graph representation: "A set of ordered pairs" (PTF16).

The second and third definitions are similar to the first one, but the graph image is explicitly stated.

A relation is a set of ordered pairs, plotting points in a graph, a function is the line connecting these points. (2nd definition, Q-09/11)

A function is some numbers that are in ordered pairs and can be graphed. (3rd definition, I2, 8)

It is not until the fourth definition that the univalence feature appears in terms of the vertical line test. The graph image is evoked with examples.

It is a line...I'm going simply, say, it can be a parabola, or it can be a hyperbola, or just a plane that crosses the graph once, so you can tell it's a function [drawing figure 56]

Figure 56. Tyler's use of the vertical line to check that a graph is a function.
You can make any graph as long as the line, the vertical line test works.

Tyler used the term "plane" to refer to a line. Asking him to clarify what a plane meant, he said "or a line, like if you ever draw a vertical line to it, that's how you can tell it's a function" (I3, 2).

The fifth definition is a regression to the 2nd definition, but adds the equation image (in terms of variables).

A function is a set of order pairs or numbers that can be put on grid and graphed. Using functions you are able to use variables in place of numbers and graph them. Functions can go from -infinity to infinity or -.001, .001 they do not have a defined size [sic]. Q-03/4)

The sixth definition is based only on the univalence feature. A suggestion of equivalence between the vertical line test and the univalence feature appears. Again the graph image is implicit: "For every input there is one output. The vertical line test is successful" [sic]. (XC1)

The seventh definition is given in terms of two images: graph and vertical line test.

It's a graph with an x and y axis, that fits into the theory of the vertical line test, and it can be a line or it can be a series of dots, or single points on the graph. (I5, 1)

Tyler's Mental Pictures Associated with Functions (Concept Image).

Several images associated with functions emerge from the data collected from Tyler. Images have been separated for presentation, but overlapping images suggests a conceptual network instead of a partition. The images associated with functions are the following: equation/formula, graph, familiarity, continuity, vertical line test, and one output for every input. Indications of each cluster follow.

Equation/formula

Two ideas emerge associated with the equation/formula image in the data collected from Tyler:

(1) functions can be represented by equations (I1, 5; I5, 3 ), and
(2) functions are equations (I1, 13; I2, 7; I4, 1; I5, 2).
The idea that functions can be represented by equations appeared in solving problems and explicitly, in the question about representations of functions. For example, when I asked him for a new problem he could solve, he chose the "box problem." His explanation conveys the idea of representing functions by equations (and graphs).

OK. It's where, it is a problem that gives you the, it gives you the length and you have to figure out the height, the height of the box so you can figure out the volume and use your calculator to find the graph, the equation, and when you graph the equation, the top of the point on the graph is where the, let see, you can figure out the maximum volume of the box, the size deck of its dimensions.

Another example of this idea appeared explicitly in how functions can be represented (I5, 3). In this case, his answer was "graph and equation, I don't know how else, that's the only thing that we usually do when using functions."

The second idea in the equation image is that functions are equations. This idea emerged in the relationship between functions and equations (I2, 7). Tyler showed some hesitation in this question. However, he decided on the idea that functions and equations are the same thing.

I said they were [related]. I don't know what exactly the definition of an equation is, cause, I just thought that equations were the same thing as, like a function would be the same thing as an equation...Well not exactly the same thing...I said they were related, I just wasn't sure, I kinda thought they were pretty much the same, like a function would be an equation. I guess I think they were kind of the same thing.

A second example of this idea appeared in "What makes up a function?" (I4, 1). He answered to this question: "numbers, variables, exponents. I guess, graphs. That's about it."

Tyler, as other students, associated the idea of equation/formula to mathematical expressions. This view of mathematical expressions is limited and was an obstacle for finding the mathematical expression of a piecewise function (I3, 6, pp. 236-237) and deciding whether or not the relation that assigns its area to every triangle was a function (p. 235).
The graph image emerged in several presentations in Tyler's work:

(1) functions can be represented by graphs (II, 5, 13, 14c; 2nd, 3rd definition; 15, 3),
(2) graphs are functions if they pass the vertical line test (I3, 2; I5, 1),
(3) functions can be graphed (Q-03/04; I5, 2, 4), and
(4) functions are graphs (I4, 1).

The first idea, functions can be represented by graphs, appeared describing a function with his eyes closed (I1, 13). When I asked him 'what do you see?' he answered "I see what's in all functions, I mean I see the graph, I see lines." However, this idea of graph is concrete and needs another representation to exist: the equation. I asked him "can you describe your graph?" and he said

I guess it depends on what kind of equation I'm graphing...I know that the x and y-axis are supposed to be there but I don't know what the graph is supposed to look like, I mean.

Another example of the idea of graphs representing functions appeared in what he thought when he saw or heard f(x) (II, 14c). In this case, his answer was "f of x. Isn't it supposed to be the vertical axis of the graph which would be y?" He did not add anything else, when I asked if he had more ideas on that.

The second idea associated with the graph image, graphs are functions if they pass the vertical line test appeared in two definitions (4th definition, pp. 223-224; 7th definition, p. 224).

The third idea associated with graphs is that functions can be graphed (5th definition, p. 224). In the 4th interview, I asked him "is a function something that you can graph?" and he said "yeah." Another example of this idea appeared in properties of functions (I5, 2). I asked several questions "Are all functions equations? Can you graph all functions? Are all functions continuous?" and he said "Well, functions, all of them can be graphed...and they have a varying domain and range." So I asked him "do all functions have domain and range?" and he said "yeah." When I asked him again "Are all functions equations?" He
said "all functions are equations, all functions should be able to be represented by equations."

Another example of this idea appeared in the relationship of graphs and functions (I5, 4). In this case, he said that functions and graphs were related because

functions are always visible on the graph. So you can, graphs make it easier to understand what the function is about.

The last idea associated with graphs is that functions are graphs and appeared in what makes up a function (I4, 1, p. 225).

_Familiarity_

There are two ideas in the familiarity image. The first idea is familiarity with examples of two types: familiar examples of functions (I3, 2; Q-02/19), and familiar shapes (I2, 4; Q-02/19). The second idea in this image is to use this knowledge to decide if a given representation is a function or not (Q-02/19; I4, 3).

Tyler used the familiarity image to illustrate what functions are. His 4th definition includes curves such as hyperbolas, parabolas, and curves that pass the vertical line test (I3, 2, pp. 223-224). His final comment in this question was "It's just the first example that came into my head" (p. 230).

An example of familiar shapes appeared in his response concerning the existence of a function discontinuous at one point (I2, 4). This problem was difficult for him to understand, and he asked me to explain it in another way, so I asked him "Can you think of a function that will assign to every number its square?" He answered "a parabola." When I asked him, "How does the parabola look like?" he drew a parabola as Steve did (figure 38, p. 142). See (Q-02/19, pp. 235-236).

The second idea in the familiarity image is to use this knowledge to decide if a given representation is a function or not (I2, 1, 2, 3, 4). This situation made apparent his lack of conceptual knowledge (rich relationships) and the weakness of this image. He was the only
student who did not apply the vertical line test in the graphs of the second interview. He relied on the "experience of have seen graphs like these." For the graph of a non function (I2, 1), he said

I don't remember ever graphing an item like this before...I just looked at the picture and I don't remember ever seeing a function that looked like that before, I don't know how to interpret it and see if it's possible...I said no...from my experience with functions, I didn't, I've never seen a graph graphed like that before.

Similarly, the graph of a piecewise function was ruled out as a function (I2, 2). In this case, the familiarity image did not include straight lines.

I couldn't remember if functions could be straight lines or not, I don't think, I don't think they can be straight lines. For some reason, I think they are parabolas or hyperbolas or whatever...I didn't think [functions] were [straight lines], I couldn't remember, so I said no again.

The graph of a continuous function (I2, 3) was also difficult for Tyler. Again, the exclusion of straight lines from the familiarity image was the source of error. He ruled out the graph to be a function because

I don't know, see, I thought like this part [pointing to the curvy part] could be like a function but I wasn't sure what. This [pointing to the straight part] confused me a little bit so I just, because this is going to be like a straight line and I, I just, I went again with the answer I had [in the second graph, I2, 2, above], and said no.

Tyler also ruled out graphs with four dots (I4, 3) to be functions because had not seen graphs like that. However, he could read the domain and range of the graph.

I say no. I just, just from past experience, I don't ever remember seeing a function that is just dots on the graph, could be a function (I4, 3a) [and the second graph] pretty much looks the same.

Another example of Tyler's experience with graphs to recognize functions (I2, 4) is discussed below.

*Continuity*

There is only one idea associated with the continuity image: connecting the dots. It appeared in the existence of a discontinuous function at one point (I2, 4). Tyler's approach to this task was similar to Steve's (pp. 144-145). A difference with Steve's approach was
that Tyler required the question to be paraphrased and it was more guided. First, I asked him to think of a function that assigned to every number its square. He answered "a parabola" and graphed it. The next step was to think of a function that assigned to every number different from zero its square but to 0 it assigned one. It seems that he wanted to come up with an algebraic expression, because when I told him "you don't have to come up with an algebraic formula, now," he said "you just want a picture?" After this clarification, I asked him to draw the curve for the positive numbers and he said "it would be just a line on this side" and drew figure 57.

![Figure 57. Tyler's graph of (I2, 4) for positive numbers.](image)

The next step was to draw the graph for the negative numbers. He drew figure 58, saying "just this side of."

![Figure 58. Tyler's graph of (I2, 4) for nonzero numbers.](image)

The idea of connecting dots and hence the sense of continuity, emerged when I asked "And for zero?" He said "to zero it assigns one [and will be] over here zero, one," plotted the point, drew a graph similar to Steve's (figure 41, p. 145), and said "so it will be up here, so it will be just moved up. I don't understand." Another difference with Steve's protocol was that I corrected this drawing and showed to him the correct graph (figure 14, p. 108). The
limitations of the familiarity image appeared again when I asked him if there was a function whose graph would be like this' and he said "I haven't seen graphs like this one." The protocol ended here.

Another example of the idea of connecting the dots appeared in the changes needed to make six dots the graph of a function (15, 6). In this case, he connected the dots with a smooth curve as other students did (e.g., Elizabeth, p. 113). I asked him if the dots were a function and he said "no" because he had said so with other graphs (14, 3, p. 228).

Tyler said that all functions are continuous (15, 2). However, this idea meant defined on an infinite interval. When we clarified that he was referring to this idea he changed his mind and said "some functions are continuous."

Vertical Line Test

There are two ideas associated with the vertical line test in the data collected from Tyler:

(1) as a means to test that a graph is a function (Q-02/19), and
(2) as a means to produce graphs of functions (13, 2).

The first idea associated with the vertical line test is to use it as a tool to test that a graph is a function. Tyler developed this idea late in the course. For example, he used the vertical line test in the parabola defined only for integer numbers (Q-02/19, pp. 235-236).

The second idea associated with the vertical line test is that it can be used to produce graphs of functions. After he gave his 4th definition (pp. 223-224), I asked him "Can you think of something else or other examples?" and he said

Well, just, I guess you can make any graph as long as the line, the vertical line test works can be a function. You can do all kinds of weird stuff I guess [drawing figure 59]. I mean essentially that's going to be a parabola but I'm not sure...not sure of many more examples than the ones I just gave you. I'm sure of those but no more. It's just the first example that came into my head.
Figure 59. Tyler's use of the vertical line test to produce graphs of functions.

Tyler used the term "essentially" to indicate the end behavior of the function he drew, something that he "just gathered from what [they had] been talking about in class." It might have been the case that his comment on not being sure about more examples referred to not knowing the names of other functions.

One Output for every Input

The univalence image appeared only once in Tyler's work. His sixth definition (p. 224) was stated using this image. No other mention of this image emerged in the protocols.

Summary of Tyler's Function Images

Six function images emerged in the data collected from Tyler during this study: equation/formula, graph, vertical line test, one output for every input, familiarity, and continuity. Several relationships among these images suggest the existence of a network. Two images are dominant in Tyler's network: graph and familiarity. The familiarity image created obstacles to deal with unfamiliar tasks and was understood as "experience."

Equation and univalence appeared isolated in the network. Tyler's network of function images suggests an operational and quasi-structural conceptualization of functions.

Procedural and Conceptual Knowledge of Functions

Tyler was not successful in the usual testing. Examination of his tests and quizzes reveals difficulties with algebraic manipulations, lack of understanding of the material included in the course, difficulties interpreting outcomes in the calculator, and difficulties
with the function concept. Some of these difficulties might be explained by the images
Tyler associated with functions.

Composition of Functions

Tyler was successful composing functions in the usual testing. He seemed to have
mastered the "substitution procedure" stressed in class. His difficulties were interpreting
the result of the composition. For example, in (Q-01/29), students were asked to "find the
'rule' for [the] compositions and also find the domain & range" of $g^0h(x)$ and $f^0g(x)$ when
$f(x) = \frac{1}{x + 4}$, $g(x) = x^2 - 3$ and $h(x) = \sqrt{x + 3}$. Tyler found the correct rule for $g^0h(x)$ and
he wrote it as "$(\sqrt{x + 3})^2 - 3$," which he obtained by substituting $h(x)$ in $g(x)$. However, for
the domain he wrote "$x \geq 0$" and range "$[6, \infty)$." He found the domain and the range using
the calculator but with the wrong rule. He entered "$(\sqrt{x + 3})^2 + 3$" instead. In the case of
$f^0g(x)$ he found the correct rule again, $f^0g(x) = \frac{1}{x^2 - 3 + 4}$. However for the domain he
wrote "all reals except -2 and 2" and for the range "$[2, 9]$." Again, he used the calculator
with the wrong rule. In this case, he used "$\frac{1}{(x - 3)^2 + 4}$."

Tyler was one of the few students who found $f(h(x))$ and $h(f(x))$ to be 5 when $f(x) = 5$
and $h(x) = x$. He found the domain correctly and wrote they both were functions.
However, he wrote the range in both cases as "$(-\infty, \infty)$." In the 3rd interview he said that he
was rushing for time. I did not have an opportunity to ask him how he solved the
compositions.

Global Approach to Functions and Connections

In this section, I present two tasks where Tyler described a function. The first problem
was to construct the graph of a rational function with limited use of the calculator (13, 5).
The task made evident other weaknesses of his conceptual knowledge (rich relationships);
little conceptualization of the problem; the procedures emphasized in class; and a sense of
being lost without the calculator.
As other students, he tried to factor the top and the bottom. However, a difference with other students is that he knew "like if you put a number in here and it makes this, makes this zero, then you know what the, you know, where the vertical asymptote of the graph is and something like that." He also thought that the zeros of the denominator were "where the graph crosses the x axis" but he was not sure since he said "I don't remember, I have it in my notes."

Tyler did not know how to use the calculator to factor, and hence was limited to his procedural algebraic knowledge: "I don't know, I can't factor this by using my calculator...to get this you have to do a lot of trial and error" and wrote for the top "(x(x + ))( )".

I stopped this strategy and focused on the denominator. However, the change was not successful either. Tyler admitted that the expression in the denominator was 'easier' (by which he meant shorter), but he could not factor it. He simply wrote it as "(x - 2)( )."

The next strategy was another procedure used in class: division. This attempt was not successful either: "We have done that before, but I don't know if I. I know the stuff we have done before, I don't know the circumstances when it works, so I don't know. I'm stuck."

The limitations of his algebraic knowledge did not impede his knowing what he was aiming for. I asked him "what are other things that you have been doing in class?" and he said

End behavior, domain and range. I can find the end behavior for this...to find the end behavior you take the denominator and you take the numerator and subtract it from the denominator or the denominator from the numerator, so I mean it's two.

Tyler was using the term "subtract" to indicate reduce. Like in Carol's case (pp. 206-209) there were two possibilities either zero or two.

x cubed, it'd still be zero. I'm not sure what to do with the two if it becomes two...I don't remember because I have a number down here with my x, if the end behavior becomes two takes on, takes on that number.

A difference with Carol's case is that Tyler knew how the graph looked like if it had an end behavior of two.
At this point the interview became a guided interview. I allowed Tyler to use the calculator because he said that he knew how to solve the problem only with the calculator. He found the domain as "all reals except one point three," which he wrote as "Dom. ≠ 1.3." He also found the range as "everything but the positive five," which he wrote as "Ran ≠ .5." His description of the graph included a zero "something around negative point six," the y-intercept "about negative point three," and the vertical asymptote "would be at one point three." The calculator was helpful to solve this task and he proved to be skillful to use it.

The end behavior, however, still caused him some difficulties, mainly because he took 'x approaches infinity' but looked at the vertical asymptote. The end behavior model was also difficult. He ended up saying: "I can guess, I just say x, well, I change to x cube."

The second task in this section is the description of the graph of the rational function, \( f(x) = \frac{x^2 + 1}{|x|} \), with the calculator as an aid (I4, 4). Tyler's description included domain, reflection, end behavior ("form of parabola"), positiveness, vertical asymptote, and horizontal asymptote (though incorrectly).

In this quadrant, it got, it goes from negative infinity to zero and in the other quadrant goes from zero to infinity...the graph is, at least it appears to be a reflection of each other across the y axis...would be a form, a form of a parabola...it's a, never crosses the x axis, so it'd be positive and it never crosses the y axis because of the asymptote at zero...it must have, must have a horizontal, horizontal asymptote...cause they both don't break, they don't go, they don't break. They're both the same distance down, so there will be a horizontal asymptote at that line, cause it comes to that line (figure 60). That's about it.

![Diagram](image)

Figure 60. Tyler's graph of item (I4, 4).
Tyler's understanding of horizontal asymptote resembles the idea of bound instead.

**Settings: Domain, Range, and Rule**

The course included examples of functions where the domain was either the set of real numbers or the set of real numbers minus at most five points. The range was the union of intervals at most. This might explain some of the difficulties that Tyler experienced in problems where the domain and range were arbitrary sets or functions with a discrete domain (XC2, XC3, Q-02/19). For example, for the function that assigns to every clothing item its price, he wrote that it was not a function because each item has a set price there is no way to draw a graph of this situation and have it fit into the definition of a function.

The equation image appeared in the function that assigns to every triangle its area (XC3). In this case he said that the relation was a function because whatever the input for the base & height is it will be proportional to the sq. inches. Therefore a direct relationship can be implied.

Familiarity was another image associated with settings not used in class. For example, in the case of the parabola defined only for integer numbers (Q-02/19), he drew figure 61.

![Figure 61](image)

Figure 61. Tyler's graph of \( f(x) = x^2 \)

only for integer numbers

He wrote the domain as "all real integers," and range as "\( y \geq 0 \) all integers." His answers are similar to Carol's (pp. 210-211). During the 4th interview I asked him how he built his graph. He said
Well, I know how the graph of $x$ squared looks like and it is just a parabola, and it said it was only integer numbers, which would be the whole numbers and not the stuff in between, so I put the dots on the whole numbers, going on forever.

Tyler told me how he found the domain and the range, and when I asked him if he "could get three for $y$?" he said, "I'm not sure I guess," but did not attempt to use the graph. There is another possible explanation for the range he found. By looking at his graph, it seems that the dots are at the points with integer coordinates for the $x$'s and positive integer for the $y$'s. So he might have read the range from his graph and then conclude that three was indeed a possible output.

Tyler decided that this was a function because you can use the vertical line test and it's a parabola...[I used] both...I just know from past experience that $x$ squared is. Cause it confused me at first because it'd have just been dots and I wasn't sure if the vertical line test would work and then I think, it goes right through the dots, so it worked.

Another example of the difficulties that Tyler had was finding the mathematical expression for a the piecewise function on a bounded interval (13, 6). I asked him this question in the 4th interview since we ran out of time on the 3rd interview.

Tyler started asking me if I wanted "an equation to get this graph?" I said "no" and explained that I wanted a mathematical expression, not necessarily an equation. He was able to find the domain of the graph and said

this tell you where the lines are on the graph...I don't know how the, I think I have to do another one and say that this is, negative is below the line...I don't know how to do this, I think if I would just put [the domain], it could be, the line could be from here to here (figure 62)

![Figure 62. Tyler's graph of item (13, 6)](image-url)
and there and there...cause like this is negative two to five and this is still five to fifteen, but it is not the same graph because this is, you can pretty much put this line anywhere and five and fifteen in the vertical plane. I don't know how, how to write it, I guess I just have to put another.

Tyler was partially successful. I modified the task to include only one piece. In this case, he wrote the domain and could read the rule of correspondence: "it'd be equals negative three, uh, and you have to specify like that, negative two and five," which he wrote as "y = 3, [-2, 5] -2 < x or 5 > x" [sic]. He also was successful doing the other part, but he did not remember what piecewise functions were and could not finish the task, so I moved on to another task.

Tyler's explanation conveys that he was focusing on the domain and the range. However, he did not realize that the range was the rule of correspondence in this case. Probably the difficulty was based on the fact that the course did not include for students to write the mathematical expressions of piecewise graphs, nor constant functions.

Another difficulty with piecewise functions was knowing how many functions appear in a piecewise relation (I4, 2). Tyler had difficulties interpreting the question, "How many functions are there in the relationship

\[ f(x) = \begin{cases} 
  1 & \text{if } x \geq 0, \\
  -1 & \text{if } x < 0.
\end{cases} \]

He said: "I don't understand the question. Can I draw?" I answered "yes" and he drew figure 63.

![Figure 63. Tyler's graph of (I4, 2).](image_url)
I asked him about the shading and he said

I was trying to get a picture if I could. And it didn't make sense, but x's if the number is greater than or equal to zero, I just can. I screw up right there, because that's all the numbers greater than or equal to zero [in the 1st and 2nd quadrants] and then I was trying to figure out where the y came in. I don't know. I guess if the y would be across the graph right there [referring to the line] and I just did the same thing for the other one, negative one in the quadrants three and four, but I don't understand.

He did not know what the domain nor the rule were. This might be a consequence of the his perceiving functions as equations (ignoring domain and range). He did not know how many functions were in the item and could not answer if the relation given was a function.

Construction of Graphs

The roller-coaster problem was difficult for Tyler. He, as other students (e.g., Steve, p. 157), had difficulties labeling the axes, but finally got them right. However, as other students (e.g., Jane, p. 183), he used the iconic context of the problem and drew an incorrect graph.

Development of Functions

This section contains two sections. The first one describes some of Tyler's ideas on functions at the beginning of the course. Data is drawn from the Practice Test on Functions. The second section is a discussion of the development of the concept of function based on the concept images and definitions that he provided throughout the study.

Some Ideas on Functions at the Beginning of the Course

Some of the ideas that Tyler had about functions were captured on the Practice Test on Functions. His definition of a function was "a set of ordered pairs." This definition does not distinguish functions from relations that are not functions. Other images associated with functions were equation/formula, continuity, and familiarity.

Tyler, as other students (e.g., Steve, pp. 157-158) who also perceived functions as an equation/formula, ignored the domain and range. For example, students were asked to decide if four functions were equal to the line \( f(x) = 4x + 6 \) defined from the set of natural
numbers to the set of natural numbers (PTF8). Tyler decided in part a, that \( f(x) \) was equal to the function \( g(x) = 4x + 6 \) defined from the set of real numbers to the set of real numbers because "same thing G & F and irrelevant." In part b) having the same domain and codomain but with a different rule, \( g(x) = 2x + 3 \), he decided that \( f(x) \) and \( g(x) \) were not equal because "not same line." It is possible to conjecture that he graphed the lines on the calculator and obtained different graphs. This procedure is correct when the domain and the codomain are the same. However, his answer suggests his image of equation/formula. In part c, students were given the graph of the line \( g(x) = 4x + 6 \) on the first quadrant. Tyler decided that it was equal to \( f(x) \) because "same graph." For the correct graph, in part d, he decided that it was not equal to \( f(x) \) because "not the same graph as \( 4x + 6 \).

The equation image also emerged in graphing functions. Students were asked to graph the line \( g(x) = x - 2 \) from the set of natural numbers to the set of natural numbers (PTF10). In this case, Tyler drew a continuous line with negative slope. Again, he only focused on the rule. In this case, either he did not use the graphing calculator or when he used it, he entered a different rule.

Deciding whether or not numbers are in the domain also allowed the equation/formula to emerge. Students were asked which of the numbers 2, -1, 0, 11.5, 1267 was a preimage of \( f(x) = 4x + 6 \), from the set of natural numbers to the set of natural numbers (PTF5). In this case he wrote "2, -1, 11.5 are all on the line." We might think that he was not familiar with the terminology used, however, it may not be this case. In part c), students were given three points and asked which one of them was a (preimage, image) pair of \( f(x) \). Tyler chose the only one correct and wrote "5, 26 spot on the graph."

The equation/formula image also appeared with piecewise functions. In this case, there was an overlapping with the familiarity image. Students were asked if a piecewise relation given algebraically was a function (PTF2). Tyler looked only at the first piece and wrote "yes it is a function because it is a parabola." The familiarity image also appeared in giving
examples of functions and non functions (PTF3). Tyler drew the graphs of a wavy function and a vertical line.

The continuity image emerged in two different situations. First, functions are continuous. In this case, Tyler wrote that the piecewise discontinuous graph of a function was not a function "because functions are continuous" (PTF1). The second situation was a matter of connecting dots. For example, given two points graphically, students were asked to draw the graph of a function that passed through the given points. Tyler connected the dots with a smooth continuous curve (not a straight line) and wrote that there were two different such functions because there are "two ways to do it" (PTF13). It seems that the second way was to travel the trajectory he had starting at his endpoint this time. The idea of connecting points also appeared in a similar task but with six points (PTF14). Tyler connected the dots with a smooth continuous curve (BFCEAD) that it was not a function. Again he wrote there were two such functions because "done 2 ways."

Tyler specified correctly the domain and range of a piecewise function (PTF9) and of a continuous function on a bounded interval (PTF11c). However, he had difficulties identifying graphs with given domain and range (PTF11).

Some of the items that he answered incorrectly or that he did not try might have been caused by lack of familiarity with the vocabulary or with the functions used in the items. Regarding vocabulary, it seems that the terms image and preimage are not well understood (PTF4). He did not attempt the item on the constant function (PTF7).

The answers to the Practice Test on Functions that other students (e.g., Steve pp. 157-159) gave exhibited the content of former courses. However, this is not the case for Tyler. A dramatic difference from the other participants was the total omission of the vertical line test (and hence the univalence feature).
Images and Definitions during the Study

There are several changes observed in the definitions of a function that Tyler provided in the study. In all the definitions the graph image appears. During the first quarter, his definitions do not distinguish between functions and relations. Later the univalence feature appeared but only in terms of the vertical line test. This seems to be related to the graph image. His fifth definition however misses the univalence feature and is based on two images: graph and equation. The last two definitions state that a graph of a function passes the vertical line test. Only one of his definitions uses the one output for every input image.

Tyler’s definitions seem to have a schematic progression: from graph to a graph that passes the vertical line test. An intermediate step occurred as a regression: equation and graph without the vertical line test.

Some changes can also be traced in the images that Tyler associated with functions during the interviews. He defined a function in the second interview only in terms of the graph image (p. 223). Other images emerged and competed in his decisions in the tasks presented in the interview. The images of familiarity, equation, and continuity appeared in the protocols exhibiting a network.

The most salient image in Tyler’s work is familiarity, a process he uses to decide whether or not a graph is a function. This idea, however, is poor, since it does not depend on the definition of a function, but on having the “experience” of having seen an identical or similar graph. In the case of Tyler it is completely isolated from the definition. Tyler used the familiarity image, incorrectly, to decide if a given representation of a relation was a function or not (pp. 227-228).

The equation/formula image appeared in items where the existence of a function was asked (I2, 4) and in relationship between functions and equations (I2, 7). In the first case, the equation image emerged and disappeared. However, I believe the equation image might have been an obstacle if I would not have made the comment “you don’t have to come up
with an algebraic formula now."
He got involved in drawing the picture and ignored the equation requirement. In the second case, the equation image dominated other images by admitting "functions and equations are the same thing."
Finally, the continuity image emerged as the habit of connecting dots.

Most of the problems on (I2) were verbal and it was evident that he did not attempt to make a graph of any situation until I asked him to do so, even when three of the definitions that he provided for functions previous to the interview relied on graphs heavily and he said in the first interview he was a "visual person."

Tyler's definition of a function in the third interview is the first appearance of the univalence feature and it is stated in terms of the vertical line test. The graph image is implicit as in the 1st definition. A third image that appeared in his definition was familiarity.

In addition to his appearance in the definition, the vertical line test is used to produce graphs. However, these two ideas are at odds with the construction of the piecewise function, \( f(x) = \frac{3|x| - 2}{x + 1} \), that students were asked to do in a quiz (Q-02/07). Tyler graphed, on the calculator, each function that resulted from examining the cases \( x \geq 0 \) and \( x < 0 \). Next he drew both graphs overlapped as the answer. During the interview he explained he did not know he was dealing with a function.

Although the use of the vertical line to construct graphs of functions suggests a movement away from familiarity, it does not imply that Tyler is independent from the familiarity image in the sense that emerged during the second interview—making decisions in terms of experience (I2).

The familiarity image also appeared with the examples that he provided in his definitions. The graph image appeared as two ideas: (1) functions can be represented by graphs; and (2) graphs are functions if they pass the vertical line test.
Tyler mentioned as the things that they had been learning in the class (I3, 1):

functions, how to graph them, what variations of, functions using absolute value, just recently learning about asymptotes, and how they relate to functions, and really right now is just about asymptotes and how they affect the graph, the graphs that we come up with.

However, this global knowledge of functions remained isolated from the importance of functions. His answer recognized that there he made no connections to see their importance.

Honestly, they have to be important somehow or somewhere along the line, but that part of the learning hasn't been implemented. They haven't showed us, I don't feel [Mr. H.] has shown us how we are going to use this in actual life. You know they have to be important, we wouldn't waste the whole year doing something.

With this lack of connections students like Tyler tend to underestimate the scope of functions. In the third interview, I asked him "What do you think it's an important aspect of a function?" and he answered

I do not know. I do not know what I'm going to be using a function for, when I'm done with school. I don't know how it's going to relate to my life outside of education after I graduate and all that. I'm more likely to forget about them.

The course stressed applications, but still Tyler perceived functions with a narrow scope. I asked him "Can you think of some examples of situations or applications of functions?" and he said

Not really. Probably if you're going to go into engineering or something like that, because that's a lot of the word problems we have. They try to relate things that use functions for, but at the top of my head I can't think of any examples.

The process of constructing the graph of a rational function with limited use of the calculator (I3, 5) made evident the procedures emphasized in class; little conceptualization of the problem; and his difficulties with algebraic skills. However, he had a more conceptual understanding of what the zeros of the numerator and denominator meant in the graphic representation (pp. 232-233) as compared with other students (e.g., Carol pp. 207-209).

Tyler provided a definition of function previous to the 4th interview (5th definition, p. 224). His definition is a regression since it lacks the univalence feature. In the 4th
interview he explained that the 'defined size' referred to the stretching factor: "graphs can be infinitely small or fairly large." Other images that appeared in the interview were equation/formula, graphs, and familiarity. The equation/formula image emerged as the idea that functions are equations (I4, 1) and as an obstacle to deciding how many functions are in a piecewise function (I4, 2). In this case he was confused between interpreting the rule of correspondence and the domain (pp. 237-238). The graph image emerged as the idea that functions are graphs (I4, 1). Tyler did not make any comment about the univalence criterion in any case.

The familiarity image appeared again as "experience" (I4, 3. p. 228). Here, Tyler rejected graphs with four dots as functions because he had not seen such graphs. In the case of Tyler, the weakness of the familiarity image is based on the separation from constructing one's own knowledge as opposed to receiving or experiencing it.

Tyler's use of the calculator to describe a rational function (I4, 4, pp. 234-235) showed his ability to use the calculator but a weak description of the graph, which only included vertical asymptote, domain (without explicit mention), symmetry, and an understanding of horizontal asymptote as lower bound. The roller-coaster problem (I4, 5, p. 238) showed a tendency to use the context to build the graph.

Tyler's remarks concerning the activities in small group (I4, 7) revealed that he thought other students in class did not "understand what's going on." He said that when he worked with a friend "it's not really he is helping me out or I'm helping him out because when I work with him, we're pretty much, we don't know what's going on." He also said that he and his friend did not try to understand how to solve a problem; they were interested in getting the work done.

We don't ask a lot of [what if] questions, we ask what is the answer?' How to get it, not what did I do wrong?
Tyler's last definition of a function is based on three images: graph, univalence, and vertical line test. The vertical line test was the only image he mentioned in "how to explain functions to precalculus students" (I5, 5). Other images that emerged in the protocol were equation/formula, and continuity.

The equation/formula image emerged in properties that functions have (I5, 2, pp. 226-227). Tyler admitted that all functions should be able to be represented by equations. Similarly with graphs, he said that all functions could be graphed. Only two possible ways to represent functions appeared: equation and graph (I5, 3, p. 225).

Two ideas were attached to the continuity image. First, Tyler, as other students (e.g., Jane, pp. 176-177) meant defined over an infinite interval when he referred to continuous functions. After we clarified this meaning, he said that not all functions were continuous (I5, 2, p. 230). The second idea in the continuity image was the 'habit of connecting the dots'. In the case of making changes to six dots to make them the graph of a function (I5, 6, p. 230), he connected them with a continuous curve. However, he was not sure about this procedure.

Umm, I don't know. I guess it just depends on the way you connect the dots. Or are you supposed to connect the dots to make it a function?...I don't know how to change it to make it a function.

As with other students, after he connected the dots, I asked him if he thought the dots were a function and he said "no."

The fifth interview was the first time that Tyler mentioned domain and range as components of functions. However, the equation and familiarity image might have led him to say that the domain and range had to be a set of numbers (I5, 2).

Tyler did not perceive the vertical line test and the univalence feature as equivalent. Other students showed a grasp of this equivalence in the first two interviews. I raised the question if they perceived the equivalence when I thought the question was appropriate. I never felt this way with Tyler. I decided to ask him in the last interview if he saw "any
connection between the vertical line test and the definition" as one output for every input. He answered "no, not really. Maybe, I'm sure there is one and not just thinking right."
However, he did not find any relation (I5, 2).

Tyler recognized functions as "an essential part of precalculus" but did not offer specific suggestion to Mr. H. about teaching functions. He simply suggested "to go a little slower, [and] spend more time on it instead of trying immediately jump and get going" (I5, 5).

**Summary of Tyler's Development and Knowledge of Functions**

Tyler experienced the development of his procedural knowledge about functions. However, few relationships were built in his network of conceptual knowledge. The strength of the procedures is based on when to apply them and not just knowing how to apply them. This became a weakness in Tyler.

Tyler's conception of functions seems to be in transition between two quasi-structural conceptions: graphs and equations. His difficulties seeing connections between one output for every input and the vertical line test suggests that he is in the interiorization stage of the univalence feature of functions. An obstacle to move to a higher stage is his familiarity image. The other essential feature of functions—arbitrariness—is far from Tyler. His ideas on functions resemble an Eulerian conception of functions: a function is any curve drawn free in the x-y coordinate plane.
The Case of Sara

Introduction

Sara is a quiet girl, who barely asks questions in class. She is observant and intends to major in psychology in college. She likes mathematics, but perceives it as challenging and more difficult than other subjects, because according to her, mathematics is either right or wrong.

Sara is taking precalculus for several reasons. She wants to have four years of mathematics, precalculus is the next level of mathematics she has to take, and it looks good on her transcript. She likes precalculus, but it is not clear to her what precalculus is about. She just wants to do well in anything that they teach her.

Sara had some computer experience in her former mathematics classes, but she did not see any difference in learning to graph functions with the computer and without it. In contrast, she thinks that it is very useful to have the graphing calculator in precalculus. She likes it very much and does not want to do precalculus without the calculator for two reasons: it helps to visualize things, and saves a lot of time.

Sara likes when Mr. H. uses the overhead in class. According to her, Mr. H. goes into great detail and she can see the screen. However, she does not like the small group activity, because it takes Mr. H. forever to get around to each group. If her group has questions, they do not really get that much done.

How Sara defines a Function (Concept Definition)

Sara provided six definitions of functions during the study. All definitions, but one, mention the univalence feature. The first definition is stated solely in terms of the univalence feature but has an implicit idea of the graph image (coordinates): "When one x coordinate is paired with only one y coordinate" (PTF16).

The second definition is Mr. H.'s definition stated in class: "A relation is a set of ordered pairs and a function is one input can have only one output" (Q-09/11).
Sara's third definition is based solely on the graph image without making any distinction between functions and relations: "A graph" (I2, 8).

Her fourth definition is a blend of several images: univalence, equation, graph, vertical line test, and familiarity.

A function is an equation or a graph where I think, it's for every, let's see, x value there can be only one y value. I don't know if it's the other way around or not, you know, it has to pass the vertical line test. A function, I mean, I don't know. We do them all the time. [f(x) = 5] that's a function, it's just the way it's written. That's what I think a function is. (I3, 2)

The fifth definition is based on three images: univalence, equation, and graph (implicitly mentioned). She wrote, a function is a

set of ordered pairs where an x value can only have one y-value. A function can be a composition equations, and can represent real life situations. (XC1)

Sara's sixth definition is based on three images: equation, graph, and vertical line test. This time she sees the existence of a 'real definition' based on the univalence feature.

Um, well, a function is an equation, that has numbers and variables and um, you can graph it and it would have to pass, for it to be a real function, it would have to pass the vertical line test. And let's see, I know, the real definition that I learned was the part about x values having only one y value...That was the basic, when we were first introduced to functions, that was the first thing that we were ever told...And then later when we got into more difficult things...like Mr. H. did a lot of real world problems. (I5, 1)

Sara's Mental Pictures Associated with Functions (Concept Image)

Several images associated with functions emerge from the data collected from Sara. Although, these images have been separated for presentation, their overlapping suggests a network instead of a partition. The images associated with functions are the following:

Equation/formula vertical line test, familiarity, continuity, one input for every output, and graph. Indications of each cluster follow.

Equation/formula

This image appeared in all the interviews and emerges in several presentations in Sara's data:
(1) functions are equations (I1, 14c; I2, 7; I3, 4a; I4, 1; I5, 4),
(2) not all equations are functions (I2, 7; I5, 1),
(3) not all functions are equations (I2, 6, 7),
(4) a function is an equation that satisfies the univalence property (I3, 2), and
(5) functions can be represented by equations (I5, 4).

The first idea in the equation image is that functions are equations. For example, when I asked her what she thought when she saw or heard $f(x)$ (I1, 14c), she said

function of $x$, whatever you plug in for $x$ would be the value for $x$, like if you had an equation that had $x$ on it and $f$ of $x$ equals that, plug in a number for $x$, and plug it in into that equation.

A second example of this idea appeared in what makes up a function (I3, 4a). Her answer was concrete and referred to the equation image solely: "$x$ and $y$, numbers, and, I don't know. That's about it."

I asked this question again in the 4th interview (I4, 1). Although, equations appeared in her answers, there is a sense of arbitrariness in the domain. She said then,

anything, you know, like the $y$ equals two plus three or something. Variables and numbers...Well, you can like involve, well, like your variables can stand for objects, but I mean, you know...Like, you know, you have twelve dresses and five shirts at like different prices and you know, you can set up an equation...Functions are equations.

A third example of this idea appeared in the relationship of graphs and functions (I5, 4). In this case she said: "A function is usually some sort of equation, but if you want to see what the function looks like then you'd put it on a graph. It's just a visual function."

Sara's last definition of a functions (I5, 1, p. 248) is also an example of this idea. A last example of functions being equations appeared in 'relationship of functions and equations' (I2, 7). In this case, she said that functions and equations were related because

like if you had an equation $y$ equals $x$ squared, that would be a function, cause it would be a parabola and I guess not all equations are functions. But they are related. You can get a function out of an equation.

This statement led us to the second idea associated with functions: not all equations are functions. I asked her to give me "an example of an equation that is not a function" and she
said "like one of those parabolas, like that [drawing a parabola opening to the right]. I can't remember what the equation would be, like negative or I can't remember."

This led us to the third idea associated with functions: not all functions are equations. I insisted and asked her if she needed "always an equation for a function" and she said "no." I asked her if she could give me "an example of a function where you don't have an equation" and she said

No, but like we were doing before, plotting points (12, 6, below). I didn't have an equation. I just plotted points.

The fourth idea, functions are equations that satisfy the univalence property, appeared in her 4th definition (p. 248). The last idea, functions can be represented by equations emerged in her answer to the question "How can functions be represented?" (15, 3). She said "by graphs. Equations. That's all."

The equation image can cause several obstacles, since it can become dominant and impede a student to work with different representations of functions. Below I illustrate several obstacles generated by the equation image that appeared in Sara's work. For example, when asked about the existence of a function that assigns its square to every nonzero number and one to zero, she wanted to make changes to an equation (12, 4, pp. 255-256).

An obstacle also appeared in the thinking about the existence of a function that sends integers to non integers and non integers to integers (12, 6). In this case, there was the need for an equation.

Yeah, mmmh, if, could I have numbers and then take the square root of it and that would be a non integer, right?

When I mentioned "that doesn't work always," she said "true, square root of four is two. I guess that wouldn't be a function." Maybe her conclusion was made because the formula proposed did not work out. However, she insisted on having a formula different from the square root. "I'm trying to think. There has to be a formula where you put in a number and
get out a non integer."

Sara said "no," when I asked her if she always needed a formula, and continued "I could plot points, but if I just plot points, then yes, probably it would be." Apparently, it was possible to have a graph that passed the vertical line test to solve the problem. She plotted points like one and point five, and two and one point five, and three and two point five, and so on. That's a line. That will work.

So the continuity image appeared (connecting the dots). I pointed out, that the dots were only for integer numbers, but we did not know what happened between the dots. So I asked her to work out with non integers. Again she plotted points, and connected them. Her final comment about the formula suggests that having a graph that passed the vertical line test was indeed apparent.

Negative point five and one, negative one point five and two, negative five and three. That would be a straight line. But we don't know the formula.

I realized that Sara did not know what integers were. She was confusing negative numbers with non integer numbers. However, she said she knew what integers and non integers were when I asked her at the beginning of the task.

A last example of the need for an equation to have a function, appeared when Sara rejected the graph of a function with four dots (I4, 3a) as a function "because there's no way you could, you'd need four different equations...this whole graph wouldn't be a function."

The equation image was also an obstacle moving from the graphical representation into the algebraic representation. For example, Sara tried to find an equation for a piecewise function (I3, 6, p. 263).

Another obstacle that the equation image generates is perceiving a function only as the rule and ignoring the domain and range. For example in the case of the function \( f(x) = x^2 \) only for integer numbers, she solved the problem by focusing only on the rule (Q-02/19, pp. 263-264).
There are five ideas in the graph image in Sara's work:

1. Functions are graphs (I1, 13),
2. Graphs are functions (I2, 6),
3. All functions can be graphed (I2, 8),
4. A function is a graph that passes the vertical line test (I3, 2; I5, 1), and
5. Functions can be represented by graphs (I5, 2; I5, 4).

The first idea in the graph image is functions are graphs. For example when I asked her to describe a function with her eyes closed (I1, 13), the following dialogue took place.

Sara: I can picture one, like a cube root function.
Me: Can I ask you to make a drawing?
Sara: uhuh [drawing the square root function]. It goes like that and then, is that the square root function?
Me: That's the square root.
Sara: OK. This is the cube root [drawing x cube].
Me: That's x to the three.
Sara: x cube?
Me: Yeah.
Sara: Is not the root?
Me: No.
Sara: OK. I don't know, do you want me to think of others?
Me: What are the ideas that you have when you think of a function?
Sara: A graph.

The second idea associated with the graph image is graphs are functions. This idea emerged in the existence of a function that assigns integer numbers to non integer numbers and non integer numbers to integer numbers (I2, 6, pp. 250-251). This conception of functions is independent from the equation image and gives to the graph representation its own identity. She "didn't have an equation. [She] just plotted points."
The third idea associated with the graph image is 'all functions can be graphed'. This idea emerged in 'what is a function' (I2, 8). Sara said "it is a graph." So, I asked her if she always had a graph for a function and she answered "uhu."

The fourth idea in the graph image, functions are graphs that pass the vertical line test, can be observed in her 4th and last definitions (p. 248).

The fifth idea associated with graphs is related to representations and appeared in 'how can functions be represented' (I5, 3): "by graphs. Equations. That's all" and in the relationship of graphs and functions (I5, 4, p. 249).

*Familiarity*

There are two ideas in the familiarity image. The first idea is familiarity with examples of three types: familiar examples (I1, 13; I2, 7), familiar non examples (I2, 7), and familiar shapes (I2, 4, 6).

Sara attached the graph image to functions and thought of familiar examples when asked about functions. However, at times with the wrong name (see I1, 13, p. 252). She was also familiar with the shape of some functions and recognized them in several tasks (I2, 4, pp. 255-256; I2, 6, pp. 250-251). Her familiarity with examples and nonexamples of functions appeared in the relationship of functions and equations (I2, 7, pp. 249-250).

The second idea associated with the familiarity image is the use of this knowledge to decide if a given representation is a function or not (Q-02/19, pp. 263-264).

*Vertical Line Test*

The vertical line test image manifested itself in different ways too:

1. a means to decide whether or not a graph is a function (I2, 1, 2, 3, 5; Q-02/19; I4, 3b);
2. a condition for graphs and equations for being a function (I3, 2), and
3. an equivalent statement to the univalence feature (I3, 2).

Sara used the vertical line test to decide if a given graph was a function. For example for the three graphs in the second interview she used the vertical line test to find out which
ones were functions. However, in the interview it was clear that she was confusing the horizontal line test and the vertical line test. After some clarification she realized which test to use and said "I knew that you just need to use one of the tests to see if it is [a function] or not." She also applied the vertical line test with a graph with four points (I4, 3b). In this case, she rejected the graph as a function "because it wouldn't pass the vertical line test." See (I2, 5, pp. 257-258; Q-02/19, p. 263-264).

The second idea associated with the vertical line test is a condition for graphs and equations for being a function. This appeared in her 4th definition (p. 248).

Sara perceived the vertical line test as equivalent to the univalence feature. This idea appeared in her 4th definition, too. However, the equivalence between them was not always clear. Particularly at the end of the year when she kept confusing the vertical line test and the horizontal line test. For example, I asked her in the last interview if she saw any relationship between the vertical line test and the univalence feature. As others students (e.g., Nathalie), Sara had difficulties stating the univalence feature and concluded that she did not know whether or not they were related.

Cause if you, you only had one y value for every x value then you wouldn't get a vertical line...Oops, if I only had, I can have as many x values as I want, but only, wait, x and y, maybe that wouldn't make sense, cause I get a vertical line test...If I had only one y value. Then, okay, now I can get a horizontal line...I guess, maybe, well in some way [the univalence feature and the vertical line test] must be opposite otherwise it wouldn't pass the vertical line test...Well, I could never remember if you're supposed to use a horizontal line test or the vertical line test...So, I don't know.

Although, Sara associates several ideas with the vertical line test, she was the only student in this study who had difficulties applying it. For example, she thought that a vertical line was a function "because it passes the vertical line test" (pp. 259-260).

Continuity

Sara had one idea associated with the continuity image—connecting dots (I2, 4, 6; I5, 6, 7). For example, in discussing the existence of a function discontinuous at zero (I2, 4, 1
asked her to make a graph for the item, since she could not answer the problem. Her first drawing was the parabola \( x^2 \), that she got by plotting points that she found in a table. However, when I pointed out that \((0, 1)\) was not in her drawing, she rearranged her graph, drew figure 63 and said "it comes like a box!"

![Figure 63](image)

Figure 63. Sara's graph of the function that assigns its square to every number different from zero and one to zero including \((0, 1)\).

I told her that it was not the case. As with other students (e.g., Elizabeth, pp. 111-113), I asked her to plot the image of one half and negative one half. She plotted the points in her graph, but she did not know what to do to include them. In other words, she did not try to connect the graph in such a way that included those two points as Elizabeth did (figure 20, p. 113). However, her reasoning was based on the equation image and maybe on a confusion interpreting the stretching factor.

You are not supposed to do something differently, equation or?...to do something different with the equation, that it will make it not smaller than that. Like when you square a fraction it makes it smaller.

So I asked her to add more points to the graph: the image of one fourth and negative one fourth. This time, she came up with a graph that "looks like it's going down, maybe...It's just, I mean, it's a bigger parabola" [drawing figure 64].

![Figure 64](image)

Figure 64. Sara's modification to the graph in item (I2, 4) after including \(\frac{1}{2}, \frac{1}{4}\).
She recognized that this was not the correct graph, since it did not go through the point 
(0, 1). However, she did not know what to do.

A second example of the habit of connecting dots appeared in the changes needed to 
make five points the graph of a function (I5, 6). Sara was the only student who moved the 
points and connected them. She made two possible graphs [figure 65] and said "I usually 
think of graphs being continuous." This idea seems to be different from the "habit of 
connecting the dots."

![Graph](image)

**Figure 65.** Sara's changes to five points to make them the graph of a function (I5, 6).

A third example of connecting dots appeared in the existence of a function that passes 
through four points given algebraically (I5, 7). Sara plotted the points, connected them, and 
said that such a function existed. However, when I asked her why she said so, she said 
"cause, it just depends on which way you connect them, though...If I, if you connect them in 
order, see you go to that and to that [figure 66], that wouldn't be a function."

![Graph](image)

**Figure 66.** Sara's resulting graph after connecting the points given in order (I5, 7).
Sara said "you are supposed to [connect the dots in order]" when I asked her "How do we know that we don't have to connect them in order?" She concluded that in the given order, that would not be a function.

Another example of connecting dots appeared in the function that assigns non integers to integers and integers to non integers (I2, 6, pp. 250-251).

One Output for every Input

There are two ideas associated with the univalence feature:

(1) definition, and
(2) equivalent to the vertical line test.

The first idea associated with this image is conceiving it as the definition of a function. It appeared in all but one definition (pp. 247-248). The second idea, equivalent to the vertical line test, appeared in her 4th definition (p. 248). The discussion has been elaborated in the vertical line test section (pp. 253-254)

Sara, as other students (e.g., Carol, p. 203) also was confused with the wording of the univalence feature. For example, in the case of a function all of whose values are the same (I2, 5), she said first that there was not a function because "if every value equals, you mean, you have four output four, six output six." I corrected by saying "all the outputs are the same." And then she said: "isn't it that the rule for functions that for every input has to have only one output? or is it the reverse?" I told her that the first one she mentioned was the correct one. She referred to the existence of a discontinuous function at one point (I2, 4, pp. 254-256): "that's why this is wrong, because there are two ones." I mentioned again that there was only one output for every input. When she realized that the y's were the outputs, she said "OK, so it'd be four, four, seven, four", drew a horizontal line after she plotted the points and said "yeah, that works, because it passes the vertical line test." After this I asked her if she could "write an equation for this function" and she said "it has to be y equals four" and wrote "y = 4."
Summary of Sara's Function Images

Seven function images emerged in the data collected from Sara during this study: equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, regularity, and continuity. Several relationships between these images suggest the existence of a network. Two images are dominant in the network: univalence and equation. The equation manifested at times as regularity which in turn created cognitive obstacles to deal with the arbitrary nature of functions. Sara's network of function images suggests an operational and quasi-structural conception of functions.

Procedural and Conceptual Knowledge of Functions

Sara was not very successful in the usual testing. She felt that she did "stupid mistakes" in the tests, and sometimes she interpreted the problems differently. Her performance of the procedures is at times surprising. She was the only student in the class who got a perfect score on the second part of the exam in January. The interviews provided more information on her knowledge of functions.

Composition of Functions

Sara as other students (e.g., Elizabeth, pp. 114-116), used the "substitution" procedure emphasized in class to find the composition of functions. The extreme case of a constant function made evident her lack of conceptual knowledge of composition of functions. She found \( f(h(x)) \) and \( h(f(x)) \) to be both equal to \( 5x \) when \( f(x) = 5 \) and \( h(x) = x \) (Q-02/19). In the third interview, she said that she tried to substitute but she was confused.

That's why I just put the \( x \) there, because I didn't know what to do. If there would have been like an \( x \) already, or maybe say \( [h] \) was \( x \) squared and \( [f] \) like squared root of \( x \) then it'd be five \( x \).

The second composition was also confusing to her.

I just combined both terms. This one should be \( x \), isn't it? I don't know, it just confused me when you don't have an \( x \) value in one of the functions.
Global Approach to Functions and Connections

In this section, I present Sara's tasks related to the constant function and tasks where she described a function. I had the opportunity to ask Sara several times about the constant function. The first time was during the second interview (I2, 4, pp. 254-256). In that case, she did not have any difficulty finding the equation of a horizontal line, when I asked her to do so. The second time was during a quiz (Q-01/29), when students where asked to find the equation of a constant function given graphically and to evaluate the function at two points. In this case, she had difficulties reading the range, which she wrote as "[7, ∞)," but answered correctly the other items.

The third occasion was during the third interview. After she explained to me how she had found the compositions of the constant function and the identity function (p. 258), I asked her to tell me what she knew about a constant function. Her answer shows her confusion with straight lines: "Goes on, it is continuous. Like x, like a constant function would be y equals mx plus b." She drew a line with positive slope. I told her that that was not a constant function and she asked "is it a parabola?" I said 'no' and told her that the example used in the composition, f(x) = 5, was a constant function. When I asked her to make a graph of it, she said "f of x equals five would be y equals 5, right?" and drew a vertical line.

Sara first said that her graph was a function "because it passes the vertical line test," but changed her mind: "well, I mean it's a vertical line...I guess it doesn't. No, it is not a function." I told her that the drawing she had made was x = 5 and asked her to draw y = 5. This time she made a horizontal line. However she had difficulties reading the range: "y greater than five, equals to, greater to equals to five." She explained that she did not have any method to find the range: "I usually look at the graph and figure it out. I don't really have any method." She knew that the range was given by the y-values, but got confused. She said next the range would "be all reals," next "there is no range" and finally "there must
be a range. It's just five, I mean."

There are three tasks where I asked Sara to describe a function. The first task was in the 3rd interview. I asked Sara to describe the graph of a function to a friend over the telephone (I3, 4b). Her description focused mainly in the shape of the graph.

Any function, I'm trying to describe? Probably I'd say in which quadrants it was in, and describe like certain shapes maybe, uhm, like if it is a parabola, I mean, I'd say it was like a smile or something and quadrants one or two or something, but, I don't know. I can tell him to plot points, but I don't know.

The second task was the description of the rational function, \( f(x) = \frac{x^3 + 2x + 1}{2x^3 - 4} \), with limited use of the calculator (I3, 5). In the task, Sara exhibited difficulties making connections between the material studied in class; little conceptualization of the problem; the procedures emphasized in the class; and her difficulties with algebraic manipulations.

Sara started by identifying the end behavior with some difficulties: "OK, let's see, if I take x cube over two x squared, the end behavior would be 1 over 2 x maybe." She was aware that the "top gives the zero points and bottom is for domain," and proceeded to apply the method emphasized in class: factorization. But she could not do it and moved to the bottom. There she was successful, found the "asymptote [at] cube root of two" and drew figure 67.

![Figure 67. Sara's asymptote of the function in (I3, 5).](image-url)
Next, she substituted $x = 0$ in the function to find the y-intercept at negative one fourth and marked the new information to get figure 68 saying "so it goes through there or something."

![Figure 68. Sara's graph of (13, 5) including the y-intercept.](image)

Sara found the correct end behavior, $\frac{1}{2}$, but had difficulties interpreting it. She added a horizontal line at $y = \frac{1}{2}$, and said "so it's going towards that, then it could be going towards infinity", while she was drawing figure 69.

![Figure 69. Sara's graph of (13, 5) including the end behavior.](image)

She changed her mind when she realized that the graph had a vertical asymptote and was not possible to have a graph like this. She said she needed the zeros, but everything (factoring or completing the square) she tried did not work. She did not know another approach to find the zeros. The domain turned out to be "probably everything except cube root of two," but she could not continue with the task.
The third task is the description of the graph of \( f(x) = \frac{x^2 + 1}{|x|} \) using the calculator (I4, 4). Her description included shape, domain, range, and symmetry.

Well, everything on the bottom part of the equation would have to be positive because of its absolute value...And then anything on top would be positive because it's squared, so, and the graph takes on like, like sort of a parabola shape...And, I'd say, domain would probably be all reals...And the range would be. It looks like it'd be from one to infinity, but I don't know. I'd have to assume it was to infinity...It's symmetrical...And it probably, if the absolute value wasn't there, I think one part of these stems would be down...That's about it.

\textbf{Settings: Domain, Range, and Rule}

The course included examples of functions where the domain was either the set of real numbers or the set of real numbers minus five points at most. The range was the union of intervals at most. This might explain some of the difficulties that Sara experienced in problems where the domain and range were arbitrary sets or with functions defined on a discrete domain (XC3, Q-02/19). For example for the relation H that assigns to every triangle its area, she decided H was a function "because it passes the vertical line test." She wrote that the domain was "all R's except zero" and the range "all R's." During the last interview she said she did not understand the question. Her explanation to find the domain was "probably because you can't have zero in the bottom of the equation." In the case of the range, she said she guessed. So it seems that her answer was driven by the equation image.

The equation image also appeared in finding the mathematical expression for a piecewise function on a bounded interval (I3, 6). She confused the information given and the procedures emphasized in class.

When you have holes at x equal three, you have to have, may come up even at the top [writing \( \frac{x - 3}{x - 3} \)] like if it has to cancel...so it cannot be equals three. So that would have to be an asymptote. I don't know, it's a strange problem.

Sara found the rule for one of the pieces but did not know how to write it as an equation. She believed there was a way to find an equation. The bell rang and she just said "it looks piecewise."
Identifying how many functions there are in a piecewise function (14, 2) was also difficult for Sara. She did not understand the notation used in the problem. Finally she understood the notation and I asked her to make a drawing. Sara was one of the few students who could come up with a graph of this item, although incorrectly. She drew two horizontal lines—one at \( y = 1 \) and other at \( y = -1 \). She added that the graph was not a function, actually there were two functions.

Familiarity was another image associated with settings not used in class. For example, in the case of the parabola defined only for integer numbers (Q-02/19), she drew a continuous graph as Elizabeth did (figure 29, p. 122). She wrote the domain as "all R's," and the range as "(0, \( \infty \)) integers." During the 4th interview, she said that she "did not understand what [Mr. H.] wanted." She just knew that "f of x equals x squared...is a parabola" and based all her answers on that. To decide that the f(x) was a function, she said it passes the vertical line test...I just kind of know that's a function, you know what I mean?...I mean x squared is basic, so...I wouldn't even look at that and wonder if it was a function.

Sara was one of the few students (cf. Nathalie), who chose the relation that assigns to every item of clothing its price to be a function (XC2). However, her argument was partially incorrect. She wrote that the relation was a function "because there are no two values alike. if it was set up + graphed it would pass the vertical line test" [sic]. During the last interview she said what she "was trying to say was the part about the x value with only one y output...cause none of these values are the same." When I asked about the graph of the function, she admitted she did not know, she had guessed in that part.

Construction of Graphs

Sara did not draw the correct graph for the roller coaster problem. The other three students who drew an incorrect graph (Jane, Tyler, and Nathalie) relied on the iconic feature of the problem to make the graph. Sara relied instead on her familiarity with roller-coaster rides. First, as other students she had difficulties labeling the axes. After some guidance
she got them right. She made two graphs (figures 70 and 71).

Well, let's say position A is going at five...and then B will go at ten. C will go, I think it will go faster and faster and faster and then it will slow down right at the end [figure 70].

![Figure 70. Sara's first graph of the roller coaster problem (I4, 5)](image)

Actually, well it would probably slant off for a while because a ride for along time would be going the same speed and also on real quick. This is from knowing what roller-coaster rides are like...And then it would just stop really fast...Well, let's say that D through L will be on here and then M and N will just be...the stopping time...It starts off and then it will maintain its normal speed throughout most of the ride and then when you're going to come into the gate it stops [figure 71].

![Figure 71. Sara's final graph for the roller-coaster problem.](image)

**Development of Functions**

This section contains two sections. The first section describes some of the ideas of functions that Sara had at the beginning of the course. Data are drawn from the Practice Test on Functions. The second section is a discussion of her development of the concept of functions based on the concept images and definitions provided by her throughout the study.
Some Ideas on Functions at the Beginning of the Course

Although Sara attempted few questions in the Practice Test on Functions, some of her function images appeared on the test. Her definition of function was "when one x coordinate is paired with only one y coordinate." This definition is based on the univalence property and suggests the image of a graph (coordinates). This was the only time when the univalence feature was mentioned. Other images associated with functions were vertical line test, familiarity, equation/formula, and continuity.

Sara used the vertical line test to check that a given piecewise graph was a function (PTF1). However her answer does not suggest a connection with the univalence feature. She simply wrote "it passes the vertical line test." No further use of the test appeared.

The familiarity image appeared when students were asked to draw the graph of a function and a non function (PTF3). Sara drew a vertical parabola as an example of a function and a sideways parabola as an example of a non function.

It is possible to conjecture that she used the familiarity image also to decide whether a piecewise relation was a function or not (PTF2). In this case she chose the algebraic relation given to be a function, but did not write any explanation. It is also possible that the equation/formula image might have been evoked first, since the first piece in the relation given is a parabola.

The equation/formula image might also explain her use of the function $h(x) = -7x$ instead of the function $g(x) = -7$ to evaluate the constant function $g(x)$ at different numbers (PTF7). Although without completing the answer, she wrote "$y = 2/3x + 2??" for one of the constant pieces of the piecewise graph given in (PTF9).

Sara's answer to item 2 (above) allows speculation that her perception of functions is as a formula (ignoring the domain and range). This situation appeared when students were asked to graph the line $g(x) = x - 2$ from the set of natural numbers to the set of natural numbers (PTF10). Her scratch work is a table of $x$-values with the integers from 1 to 5 and
the corresponding y-values. Sara plotted the points and connected them with a line. This tendency to connect the dots might have been a reflection of the continuity image (as the habit of connecting the dots).

Sara only attempted to answer one more item (PTF11a) though incorrectly. This suggests that she had difficulties recognizing a graph with a given domain and range.

In the Practice Test on Functions, Sara exhibited knowledge of the content of former courses and her images associated with functions. In the next section, I discuss the changes that occurred in Sara's images and definitions of functions.

**Images and Definitions during the Study**

There are several changes observed in the six definitions of function that Sara provided in the study. All her definitions except the third one contain the univalence criterion for functions. The third definition seems to be a regression in Sara's thinking, since it is based only in the graphic representation: "a graph." The last three definitions are based on the equation image and the graphic representation, that either satisfy the univalence criterion or the vertical line test.

Some changes in Sara's work can also be traced to the images that she associated with functions during the interviews. In the second interview, she defined a function in terms of the graphic representation only: "a graph." However, other images emerged and competed in her decisions in the tasks presented in the interview. The images of vertical line test, equation, familiarity, continuity, and one output for every input appeared in the protocol. Sara exhibited a network, and not a partitioned set of images.

Sara used the vertical line test to decide if a graph was a function (p. 253-254). In contrast she used the equation image in items where the existence of a function was asked (I2, 6) and in discussing the relationship between functions and equations (I2, 7). In the first case, the equation image created an obstacle to finding functions, since she tried to write down a formula for them (p. 250-251). In the second case three ideas emerged: functions
are equations, not all equations are functions, and not all functions are equations (p. 249). Sara used the familiarity image to provide examples and nonexamples of functions (p. 253). She also exhibited familiarity with the shape of some functions and recognized particular functions in several tasks (pp. 250-251 and pp. 254-255). For Sara, the continuity image was a matter of connecting dots. This image emerged after she plotted points for the tasks at hand (p. 255). The univalence criterion flashed in her decision about the existence of a function all of whose values were the same (I2, 5), but was not used in the protocol. She mentioned it, but the dialogue suggested her confusion understanding it.

Most of the problems in the second interview were word problems. Sara did not attempt to graph any item in order to solve it until I suggested that she do so. This suggests that she experiences difficulty moving from verbal representations to graphic representations. A second alternative explanation might be a lack of heuristics to solve a problem; a third alternative could be the dominance of other function images such as equation or familiarity over the graph image.

Sara's definition of a function in the third interview is a blend of five images: univalence, equation, graph, vertical line test, and familiarity (p. 248). The univalence criterion appeared as a definition and an equivalent statement to the vertical line test. The vertical line test appeared then as a means to decide whether a graph or an equation is a function (I3, 2, p. 248). The equation image also appeared as a component of a function (I3, 4a) and manifested as functions are equations (p. 249). An obstacle created by the equation image emerged in finding the mathematical expression for a piecewise graph (I3, 6). Sara, as other students (e.g., Steve pp. 133-134) tried to write down an equation. During the task she confused the information given (open dots in the graph) with the procedures emphasized in class (if it is a hole, then the denominator of a rational expression equals zero at that point) (p. 259-260).
Sara mentioned sign charts, factoring quadratic equations, factoring by grouping, absolute value, functions, graphs, equations, and solution of problems as the topics that the class had learned up to that point (I3, 1). Her understanding of the importance of functions reduced to applications "to figure out problems in different ways" as the ones that "Mr. H. will ask [them] to find" (I3, 3). In her discussion, an emphasis on the procedures and an absence of concepts are clear.

I think of a function as a pretty generic term though, I mean. It's always, OK, we do function of, and [Mr. H.] writes the problem. I don't really ever think about what a function really is, I guess.

The graph image, in addition to the definition, appeared in her description of a graph to a friend (p. 260). Her description focused mainly on the shape of the graph. Sara's description of a rational function with limited use of the calculator (I3, 5) made evident the weaknesses of her conceptual knowledge of functions; little conceptualization of the problem; the procedures emphasized in class, and her difficulties with algebraic manipulations (pp. 260-261).

Sara provided a definition of a function prior to the fourth interview (5th definition, p. 248). Here, her definition is based on three images: univalence, equation, and graph. The vertical line test and the familiarity image joined these images in the protocol. The equation image appeared in what makes a function (I4, 1) and manifested itself as functions are equations (p. 249). Her answer contains a taste of arbitrariness for the domain.

The equation image turned out to be an obstacle in Sara's dealing with a piecewise function (I4, 2, p. 263). Sara said that there were two functions in the relation given. This presentation of the equation image suggests that Sara perceived functions as the rule only (and hence ignored the domain and range). Another obstacle created by the equation image emerged in deciding whether or not the graph with four points was a function (I4, 3a). In this case, there was the need for an equation (p. 251). By contrast, Sara applied correctly the vertical line test to decide whether or not the graph with four points in (I4, 3b) was a
function. In this case, it seems that the sense of regularity in the second graph was more appealing for the vertical line test while the sense of irregularity evoked the equation image.

The familiarity image appeared in the roller-coaster problem (I4, 6). Sara was the only student who relied on her experience with roller-coaster rides to draw the graph asked (pp. 263-264).

Sara exhibited her skills using the calculator in the two tasks where the calculator was needed (I3, 5, and I4, 4). However, her description of the rational function when students were allowed to use the calculator (I4, 4) only included shape, domain, range, and symmetry.

In the last interview, Sara defined a function in terms of three images: equation, graph, and vertical line test. This time she states the existence of a 'real definition' based on the univalence feature (6th definition, p. 248). The continuity image joined these images in the protocol.

The equation and the graph images appeared intertwined. For example, in properties that functions have, she mentioned either algebraic properties and operations or descriptors of graphs such as asymptotes, open up or down, piecewise, continuous (I5, 2).

The relation of graphs and equations also appeared in the relationship of graphs and functions (I5, 4). Sara said that a function was "some sort of equation, but if you want to see what the function looks like then you'd put it on a graph. It's just a visual function." This interrelationship also emerged in the question about representation of functions. She mentioned only two possible ways to represent functions: equations and graphs (I5, 3).

The continuity image appeared in two items: (1) the changes needed to make five points the graph of a function (I5, 6) and (2) the existence of a function that passes through four points given algebraically (I5, 7). Sara was the only student who moved the points in the first task before she connected them. She proposed two possible changes: a parabola and a graph that resembles an increasing function (figure 65, p. 256). In the second task, she plotted the points given and connected them. Both answers have a taste of regularity.
However, a regression in her thinking is the order in which the points should be connected. She said that the points should be connected in the given order and under that circumstance, the resulting graph was not a function (pp. 256-257).

Sara suggested the use of the definition she stated in this interview in her discussion of how to teach functions to precalculus students (15, 5). She mentioned the domain and range and called for more examples of functions, since it was difficult to find them in some complicated graphs. She also called for an emphasis on conceptual understanding instead of procedural understanding. For example, she suggested "maybe explain why if a graph was a certain shape, why it wouldn't be a function because of the vertical line test...Instead of just saying you use the vertical line test to tell whether or not it's a function."

Summary of Sara's Development and Knowledge of Functions

Sara started with a graphical image of functions and moved at the end of the year to a conceptualization of functions as equations. The graph image reduced to a visual representation that resulted from graphing the equation. She experienced two kinds of development: procedural and conceptual. The course emphasized the procedural one and she benefited with powerful algorithms. However, at times there was not much understanding of them. Conceptually there is a network of images. The richness of the network seems to be blocked sometimes by the equation image. The equation image, unfortunately, contains a taste of regularity, which impedes her to deal with one of the essential features of functions—arbitrariness (either with the rule or with the domain and range). The second essential feature—univalence—creates conflicts sometimes. Sara forgets it or confuses the vertical line test with the horizontal line test. However, this situation seems to be at the verbal level, since she can apply both statements (univalence and vertical line test) accordingly.
The Case of Nathalie

Introduction

Nathalie is a friendly girl who likes to talk; her willingness to socialize is evident. She has a profound respect for education and has taken four years of every subject in high school in order to have as much knowledge as she can. Although she is not sure yet about what she wants to do in college, she is thinking about majoring in law, literature, languages, or veterinary medicine.

Nathalie likes mathematics but the subject is challenging and difficult for her. She also gets frustrated when she does not understand it. She feels that her algebra skills are good, and in contrast her geometry skills are not.

Nathalie thinks precalculus will prepare her for more advanced subjects in college. Her expectations about the course are beyond the content. She expects the class will give her the mathematical skills she needs to think logically instead of just going in emotions—a problem-solving approach to real life situations.

Nathalie had mixed feelings about the use of the calculator in precalculus: love and hate. She thought that when she used the calculator she was not learning to do things, but learning how to use the calculator. This situation was compounded by the fact that the college she applied to does not allow students to use their calculators on any mathematics test or final. That made her nervous. She was worried specifically about her graphing skills that were not very good. She thought the calculator was a crutch, but helped a lot to learn mathematics: it made it easier to learn the subject. At the beginning of the school year, she used the calculator to explore how graphs appeared on the screen, just for the fun of it. That enthusiasm declined by the end of the year. She suggested "that if less use of the calculators was stressed, it would [have been] better."
How Nathalie defines a Function (Concept Definition)

Nathalie provided seven definitions of functions in the study. All of them mention explicitly the univalence feature. The first and second definitions are similar: "one x with only one y" (1st definition, PTF16) and "Relation is a set of ordered pairs. Function has for each input exactly one output" (2nd definition, Q-09/11). Her second definition was Mr. H.'s definition stated in class.

Nathalie's third definition adds the graph image to the univalence feature: "A function is a graph where a number for an x value can be used once and only once" (I2, 8).

Her fourth definition adds the equation image to the third definition: "A function is an equation that produces a graph where for every x there is only one y or yeah, that's what I think" (I3, 2).

The fifth definition is similar to the third definition. Her writing shows a confusion stating the univalence feature.

A mathematical relationship identified as a function means that you will have only one x value for every y value on the graph. This is important in identifying functions because if there is more than one x for every y you would end up with simply a relation. For example a circle. [Made the drawing of a circle]. (Q-03/04)

Nathalie's sixth definition is similar to the fourth definition: "a function is an equation that shows a graph where for every x there is only one y" (XC1).

Her seventh definition is based solely in the univalence feature. Again, there was confusion stating it:

one input for every output...or is it one output for every input?...oh, why do I keep getting this wrong? I hate that. I mean I know what I want to say, I just can't, so in other words it's one y value for every x value. (I5, 1)

Nathalie's Mental Pictures Associated with Function (Concept Image)

Several images associated with functions emerge from the data collected from Nathalie. The images have been separated for presentation, but overlapping suggests a network instead of a partition. The images associated with functions are the following:
equation/formula, graph, a set of ordered pairs, familiarity, continuity, vertical line test, and one output for every input. Indications of each cluster follow.

**Equation/Formula**

Nathalie associated three ideas with the equation/formula image of functions:

1. functions are equations (I1, 14c; I2, 7),
2. functions are equations with the univalence feature (I3, 2; I4, 1), and
3. functions can be represented in equation form (I5, 2, 3).

The first idea, functions are equations appeared in discussing her ideas when she heard or saw $f(x)$ (I1,14c). Her answer is based on the equation image.

\[ f(x) \text{. Usually I think of a function. It's } f \text{ of } x \text{. You just take whatever the number would be, like here it's } f \text{ of } x \text{ but if you have } f \text{ of three you'll put a three in for whatever } x \text{ is in the problem that you have. It's going to be } f \text{ of } x \text{ will equal, isn't it } y? \text{ It's like the } y \text{ value, kind of, you're solving for } y, \text{ like if you had } y \text{ equals three } x \text{ plus two, you take } y \text{ equals three times three plus two. So it will be nine plus two, so } y \text{ is going to be equals eleven. Is that right? I don't know. That's what I think. } f \text{ of } x \text{ is } y. \]

An explicit statement about functions being equations appeared in the relationship of functions and equations (I2, 7). In her discussion there was some confusion of what an equation was. She could not identify which one of $x^2 + 2$ and $y = x^2 + 2$ was an equation. However, she said that functions and equations were related.

\[ \text{Because a function comes from an equation. You have to have an equation to get a function. You can just draw it but there is a function, I mean an equation that comes from behind it. There is an equation that makes it like that.} \]

This idea of functions being equations also appeared in the third interview. The definition of a function in this case requires for an equation to satisfy the univalence feature (I3, 2, p. 272). This is the second idea that Nathalie associated with equations. During her explanation of this definition, she said that even when you had a graph, there was an equation behind it.

\[ \text{I mean if you like, just draw an } x-y \text{ axis and you put something on there, I mean just draw a function, I mean, there is going to be an equation you can get from that, isn't there? I mean, you can figure something out. Maybe, I don't know.} \]
A second example of functions being equations with the univalence feature appeared in 'what makes up a function' (I4, 1). Nathalie’s answer is based in three images initially: graph, set of points, and equation. Later the univalence criterion appears.

Um, what makes up a function. You need an equation set equal to y, um, with, that gives you a set of points for a graph like when you solve it, it will give you points on a graph, you know. You can get your y, x and y intercepts, and you need, you know numbers to make up a function...But basically [a function] is an equation set equals to y that can give a picture, that will help you to have a picture of a graph...an you have, like what makes up function? You need one x for every y and...you need domain, range...and y intercepts and x intercepts and that sort of things.

The last idea that Nathalie associated with equations is related to representations—functions can be represented in an equation form. This idea appeared in her discussion of the properties functions have (I5, 2) and how functions can be represented (I5, 3). In the first case, Nathalie said

[functions] usually form an equation, but not necessarily, I guess if you could call x equals two an equation...then, that's also an equation for a function...[functions] can be written, they're written in equation form.

In the case of how functions can be represented (I5, 3), she said: "Um, in a graph...um, in an equation...A set of ordered pairs...I suppose, um by name."

Perceiving a function as an equation can be an obstacle to deal with tasks where an equation is not evident (I3, 6; XC3; p. 288) or can create the image that functions are only defined for numbers, restricting the arbitrary nature of functions. For example, in properties that functions have, she mentioned "all functions have domain and range." I asked her "What type of domain and range can you have?" Her answer shows a numerical conceptualization of domain and range.

Um, that's pretty much endless, I think...You can have all reals or you can have it not equaling, like x can't equal two or x can't equal negative five.

I asked her then about the nature of the set: "Do you always have to have numbers for the domain?" Nathalie's answer suggests how her set of functions is limited to algebraic rules: "Well what else would they be for?...I can't think of anything else you'd define a function
Graph

Seven ideas associated with graphs emerge in the data collected from Nathalie:

1. graphs are functions if they pass the vertical line test (I2, 1, 2, 3),
2. functions can always be graphed (I2; I5, 2, 4),
3. graphs are functions (I2, 4),
4. functions are graphs with the univalence feature (I2, 8),
5. graphs come from equations (I2, 7; I3, 2)
6. functions can be represented by graphs (I5, 3), and
7. not all graphs are functions (I5, 4).

The first idea in the graph image is an interrelationship between graphs and the vertical line test: graphs are functions if they pass the vertical line test. This idea appeared after Nathalie explained to me how she used the vertical line test to decide whether or not a graph is a function (p. 280). I asked her if the shapes of graphs of functions could be "crazy" and she replied:

Sure, they can do everything they want as long as they keep the basic rule of a function. I mean, I don't care, because you can always check the vertical line test and see, no matter how they look like. They can be all over the place, I don't really have any problem.

The second idea that Nathalie associated with graphs is functions can always be graphed. This idea appeared in the dialogue of the shapes of graphs being crazy. I asked her, "What is the basic rule that you are talking about?" She answered "the basic rule is that for every x value there can only be one y value." I paraphrased her answer, "you said you can always apply the vertical line test. Does it mean that given a function, you can always graph it and then you can try the vertical line test?" and she said "yeah, I think so."

Another instance of functions can always be graphed appeared in her discussion of the properties functions have (I5, 2). I asked her, "Can you always graph [functions]? I mean, is that a property? Functions can always be graphed?" Her answer specifically states this possibility. "Yeah, you can always graph them, you have, yeah, you have to be able to graph them."
The third idea associated with graphs that emerged in Nathalie's work is graphs are functions. This idea gives its own identity to a graph; it can exist without the equation image. This conceptualization of graphs contains a taste of arbitrariness. For example, Nathalie tried to provide a graph to show the existence of a function that sends integers to non integers and non integers to integers (I2, 6, pp. 279-280). Although her attempt was unsuccessful, it was remarkable as compared to the students who tried to attach an equation to the function (e.g., Elizabeth, pp. 105-107). This idea was only in formation in Nathalie's mind, since in the next item, relationship of functions and equations (p. 273), she said "you can just draw it but there is...an equation that comes from behind it. There is an equation that makes it like that."

The fourth idea in the graph image is that functions are graphs with the univalence feature. An example of this situation appeared in her answer to what a function is (I2, 8). Nathalie said "a function is a graph where a number for an x value can be used once and only once."

The fifth idea associated with graphs is 'graphs come from equations'. This idea appeared in the relationship of functions and equations (I2, 7, p. 273) and in her fourth definition (I3, 2, p. 272).

Nathalie also associated a representation idea with graphs. This is the sixth idea that emerges from her data: functions can be represented by graphs, and appeared in how functions can be represented (I5, 2, p. 274).

The last idea that emerges from Nathalie's data associated with graphs is "not all graphs are functions." This idea appeared in her discussion of the relationship of graphs and functions (I5, 4). Nathalie said that graphs and functions were related because "you can graph a function." And continued "I mean, you can have, yeah, you have a graph of a function. Cause graphs are either functions or relations. And that is basically judged by a vertical line test."
A Set of Ordered Pairs

Three ideas associated with the set of ordered pairs image emerge in the data collected from Nathalie:

1) an intermediate step between equations and graphs (I4, 1),
2) a way to represent functions (I5, 3), and
3) functions can be a set of ordered pairs if they have the univalence criterion (I2, 8; I4,3; I5, 7).

The first idea, an intermediate step between equations and graphs appeared in her discussing what makes up a function (I4, 1).

You need an equation set equal to y, um, with, that gives you a set of points for a graph like when you solve it, it will give you points on a graph, you know.

The second idea associated with the set of ordered pairs image is related to representations—functions can be represented as a set of ordered pairs. When I asked her how functions can be represented (I5, 3), she said: "Um, in a graph...um, in an equation...A set of ordered pairs...I suppose, um by name."

The third idea associated with the set of ordered pairs image is that sets of ordered pairs are functions if they meet the univalence criterion. This idea emerged in the her discussing the equivalence between the vertical line test and the univalence criterion (I2, 8). Her recognition of the equivalence between them includes how a set of ordered pairs can be a function,

because if you have two y values for an x then the vertical line test is not going to work. It's not going to be a function. See, what you could do, if you had like a set of ordered pairs and you see a one more than once then that's like doing the vertical line test, just by looking at the set of ordered pairs and so actually looking at the graph. [They are the same], they just apply differently...Well if you have a graph in front of you, then it's easier just to look at the vertical line test, but if you don't, then you need to do like points, like if you have a set of ordered pairs in front of you and they don't say like 'graph that' or anything. It's going to be, if you had the set of ordered pairs in front of you, it's going to be faster to look and see, but if you do have the graph sitting in front of you, I think it's going to be faster if you just look at the graph.
Perceiving a function as a set of ordered pairs allowed Nathalie to deal partially with the arbitrary nature of functions—functions defined on a discrete domain, even when the domain was a nonnumerical set. For example, she answered correctly why the tabular relation that assigns price to every clothing item was a function (XC2): "There are completely different items with different prices so they won't overlap. 1 y for every x." During the fourth interview she told me how she had solved the problem. Her strategy although correct suggests a difficulty understanding the univalence feature.

Well, what I did is basically just put numbers in, like the blouse would be the number one, the shirt would be number two, the pants, three and four...and here you have a set of ordered pairs where you've got one, twenty five, two, twenty six, you know, cause see none of these had the same price, and none of them were the same type of item...So that means that none of these are alike and none of these are alike so you can have a function.

Other instances of Nathalie's flexibility to deal with discrete domains appeared in the changes needed to make five dots the graph of a function (I5, 6), the existence of a function that satisfies five algebraic conditions (I5, 7), and deciding whether or not a graph with four dots was a function (I4, 3, pp. 283-284). In the first case, she did not make any changes. She simply said "well, it is the graph of a function."

In the case of the existence of a function that satisfies five algebraic conditions, \( f(2) = 0, f(-3) = 7, f(-4) = 1, \) and \( f(\frac{1}{2}) = -1, \) she wrote them as ordered pairs first "\((0, 2) (7, -3) (1, -4) (1, \frac{1}{2})\)" and concluded "those would be a set of points...And as you can see there is only, for every x value there's only one y." Although the strategy is remarkable, since she was not blocked by the equation image or the continuity image, she did not interpret the information correctly. She read and wrote the x's as y's and vice versa.

The set of ordered pairs image also allowed Nathalie to remember the univalence feature more easily than other students. Actually when she was stating her last definition (p. 272) she drew the graph of a relation that did not have the univalence property to remember it. Her discussion included the ordered pairs \((3, 4)\) and \((3, 5)\) (fig. 85, p. 299).
Familiarity

There are two ideas in Nathalie's familiarity image. The first idea is familiarity with examples of two types: familiar examples of functions (I1, 13), and familiar shapes (I1, 13; I2, 4; I3, 5). The second idea is using this knowledge to recognize if a given representation is a function or not (I2, 5; Q-02/07).

Nathalie was the only student who did not provide a parabola or a straight line as examples of functions. For instance when I asked her to describe a function with her eyes closed (I1, 13), she said

the most common function that pops into my head would probably be like sine, you know how goes like a squiggling weird thing [drawing figure 73] I forgot if this is the sine or cosine, but, that one. That's what I think of.

Figure 73. Nathalie's graph of a familiar function in (I1, 13).

Examples of familiar shapes appeared in the existence of a function discontinuous at one point (I2, 4) and with the shape of a rational function (I3, 5). The first example is presented in the discussion of her continuity image (p. 275). In the second case, Nathalie had expectations about the graph for the item. She was the only student who had a rough idea how the graph of the rational function at hand should look (figure 77, p. 287).

The second idea associated with familiarity is using this knowledge to recognize a function. For example, in the existence of a constant function (I2, 5), she was a little confused at the beginning with the meaning of "value." As other students (e.g., Sara, pp. 257-258), she thought that "same value" meant "same x and y." However, when I told her that the only requirement in the problem was that the y-values were the same, she said "it's a
straight line, oh, then yes you can, so the answer is yes."

Nathalie's discussion of the composition of the identity function with a constant function also illustrates the second idea associated with familiarity (Q-02/07, p. 285)

*Continuity*

Nathalie was one of the few students (e.g., Elizabeth, pp. 111-113) who attached only one meaning to continuity: connecting the dots. For example, in the existence of a function that assigns its square to every number different from zero and one to zero (I2, 4) she was not sure about the function asked for, so I asked her to make a graph. She computed several values for the function, plotted them (including (0, 1)), connected them, and obtained a parabola similar to Elizabeth's (figure 17, p. 111). The tendency to connecting dots (but ignoring (0, 1)) emerged again, though temporarily, when I asked her where one half went to. She said

*One half? It would go to the square root of one half, I mean the square, right? I don't know, I always forget to do this. One fourth. So it will go right there. It is just getting smaller, smaller, smaller, but it won't pass zero [drawing figure 74]. It goes like right, wait! I guess not, it goes right here, like a quarter right? I'm confused now, because this doesn't seem like it will work, if for zero it will be zero, one. That's up here, right, but for one it would be up here, but one quarter will be down here. That doesn't seem like it will work.*

![Figure 74. Nathalie's graph of item (I2, 4) including \(\frac{1}{2}, \frac{1}{4}\).](image)

Nathalie's next guess was similar to Elizabeth's final graph (figure 20, p. 113), but she did not know how to continue. She concluded that maybe there was not such a function.

A second instance of the habit of connecting dots emerged in the existence of a function that sends integer numbers to non integer numbers and non integer numbers to
integer numbers (12, 6). Nathalie was not sure initially about what integer numbers were. After I explained to her what they were, she thought that sometimes a function like that might exist. I asked her to give an example or make a drawing of a situation like that. The following dialogue took place.

Nath: I don't know. The only drawing I can make would be like that one and then it would be point five, and then say two like, one point five right there, or three and point two would be down here and four and five point three, [laughing]. [drawing figure 75] It's a function, yeah!

![Figure 75. Nathalie's first graph in (12, 6).](image)

Me: OK, but remember that for the non integers has to be an integer.

Nath: So point five would be two, and for one point five will be four, yeah, still [drawing figure 76].

![Figure 76. Nathalie's last graph of item (12, 6).](image)

Oh, not if you connect them, or why would you connect them? Yeah, it'll work.

Me: Is that a function?

Nath: Yeah, I think, I think so, I don't know if I'm right or wrong. I'm not sure.
Nathalie was one of the few students who recognized that not all functions were continuous (I5, 2). This is in contrast with other students who thought all functions were continuous (e.g., Steve, pp. 144-148) or did not know the answer since they had not seen enough examples (e.g. Carol, p. 220).

**Vertical Line Test**

Nathalie associated three ideas with the vertical line test:

1. a method to decide whether or not a graph is a function (I2, 1, 2, 3; Q-02/07; I5, 4),
2. graphs are functions if they pass the vertical line test (I2), and
3. an equivalent statement to the univalence feature (I2, 1, 2, 3, 8).

The first idea that Nathalie associated with the vertical line test was to use it to decide whether a graph was function or not. For example, for the graph of a non function (I2, 1) she said

*This is not a function because of the vertical line test, it will go through two points. There will be more than one y for one x. So here [referring to the graph] it will be like say one and there will be three different y values for that, you can't have it so it's not a function.*

Nathalie also used the vertical line test with continuous (I2, 3) and discontinuous functions (I2, 2). For example, in the case of a discontinuous graph (I2, 2), she said

*This one, it's a function because that circle right there means it doesn't go through that point and so I did the vertical line test and there is only one x, one y for every x.*

Two more examples of this idea appeared in Nathalie's discussion of the composition of the identity function and a constant function (Q-02/07, p. 285) and in her idea that not all graphs are functions (I5, 4, p. 276).

The second idea associated with the vertical line test is that graphs are functions if they pass the vertical line test. This idea was discussed in the graph image (p. 275).

The third idea associated with the vertical line test is the equivalence with the univalence feature (I2, 8). This situation can be observed in the use of the vertical line test to decide if a graph is a function or not (see above). This idea explicitly emerged after she told me what a
function was in the second interview (I2, 8). I asked her if she saw any connection between the definition of a function in terms of the univalence feature and the vertical line test. She admitted that she had always seen them as different but that they were the same,

because if you have two y values for an x then the vertical line test is not going to work. It's not going to be a function. See, what you could do, if you had like a set of ordered pairs and you see a one more than once then that's like doing the vertical line test, just by looking at the set of ordered pairs and so actually looking at the graph. [They are the same], they just apply differently... Well if you have a graph in front of you, then it's easier just to look at the vertical line test, but if you don't, then you need to do like points, like if you have a set of ordered pairs in front of you and they don't say like 'graph that' or anything. It's going to be, if you had the set of ordered pairs in front of you, it's going to be faster to look and see, but if you do have the graph sitting in front of you, I think it's going to be faster if you just look at the graph.

One Output for every Input

This image also emerged in different presentations in Nathalie's data.

(1) definition,
(2) basic rule of functions,
(3) equivalent to the vertical line test, and
(4) a means to decide if a set of ordered pairs is a function (I2; I4, 3).

Nathalie mentioned the vertical line test in all the items where a definition was asked (p. 272), this is the first idea associated with functions. The second idea associated with the univalence feature is describing it as the basic rule of functions (I2). This idea appeared after she answered the question about shapes of graphs being crazy (p. 275). Then I asked her to tell me what the basic rule was that she was talking about. She answered "the basic rule is that for every x value there can only be one y value." The third idea, equivalent to the vertical line test, is discussed as the last image associated with the vertical line test above.

The last idea associated with the univalence feature is to use it to decide that a set of ordered pairs is a function. This idea emerged in the equivalence between the vertical line test and the univalence feature (I2, 8, pp. 282-283) and in graphs with four points (I4, 3). Nathalie used the univalence feature to decide whether or not a graph with four points was a function. In the case of a function (I4, 3a) she said
[Yes, it is a function] because, well because you only have one x for every y, so it's not like, it's a function because of that and I don't think you need to have like a ray or a parabola or anything to define a function...You know, it's just, here's the four points and they, you know there's only one x for every y, so you have a function.

In the case of a non function (I4, 3b) the argument was similar.

Because this will go, A and C are on the same, they have the same x. A and C, and D and B have the same x and you can't have that for a function.

Nathalie was one of the students who had the most difficulties verbalizing the univalence criterion for functions. One of her difficulties was reversing the x's and the y's (or the inputs and outputs) verbally (I4, 3a; above), written (see definitions, p. 272), and reading graphic information (Q-01/29, pp. 291-292; I5, 7, p. 292; I5, 2, p. 274).

Summary of Nathalie's Function Images

Seven function images emerged in the data collected from Nathalie during the study: equation/formula, graph, a set of ordered pairs, familiarity, continuity, vertical line test, and one output for every input. Several links between these images suggest the existence of a network. Nathalie's function images reflect a tug of war between two images: equation and univalence. Her function images reflect an operational and quasi-structural conceptualization of functions. However, the strong tendency to univalence and her perception of functions as ordered pairs allow Nathalie to deal partially with the arbitrary nature of functions.

Procedural and Conceptual Knowledge of Functions

Nathalie was an average student in class. At the beginning of the year, she started getting A's. Later her grades decreased and stabilized around B's. She asked a lot of questions in class and interacted in the small groups a lot. However, at times she tended to socialize as opposed to doing class work.

Nathalie's grades reflect her ability to carry out algorithms. Her tests reveal strong algebraic skills and confidence on them. There were items where students were allowed to use the calculator and instead she solved the problems algebraically. She used to visit the
math center in school very often. This seems to have helped her to work out some of the mistakes that she did in tests.

Nathalie's understanding of some problems included in the interviews are at the procedural level sometimes. In this section, I present her approaches to several tasks to illustrate her knowledge of functions.

**Composition of Functions**

Initially, Nathalie had difficulties finding the composition of functions in the tests. For example, in (Q-01/29), students were asked to find the compositions \( g \circ p(x) \) and \( g \circ h \circ j(x) \) when \( g(x) = x + 3 \), \( f(x) = 2x \), \( h(x) = \sqrt{x} \), and \( p(x) = \frac{1}{x} \). She found \( g \circ p(x) = \frac{1}{2(x + 3)} \) which is the answer for \( p \circ g(x) \), and \( g \circ h \circ j(x) = "2|x| + 3" \) which should be \( 2|x| + 3 \). In subsequent tests, her mistakes decreased. However, her knowledge of compositions is procedural. She was one of the few students who found \( h(f(x)) = 5 \) when \( f(x) = 5 \) and \( h(x) = x \) (Q-02/07). During the third interview, she explained to me how she found this answer and how she decided it was a function.

Well, \( f \) of \( x \) equals five, so you put five in here and then you have \( x \) and then, I guess I just figured it was just, it was five...It was a function. Um, well that's the way, I mean it was like the form of the function. It was like \( f \) of \( x \), or \( f \) of \( h \) of \( x \), that's like how you write a function...And, um, I don't know. Five is a function cause it's a straight line and you do a vertical line test, that sort of thing.

However, for the composition \( f(h(x)) \) she wrote \( 5x \)." In this case, she did not know where to apply the 'substitute' procedure and gravitated to multiplication as other students did (e.g., Sara, p. 258)

OK, I was a little confused with this one...but you just take \( h \) of \( x \) and put it into \( f \) of \( x \) or \( f \) of \( h \) of \( x \), you put \( x \) in the five, so there is no \( x \), I just, I didn't, I thought that may be should be five since there isn't any, \( x \) kind of disappear but I wasn't sure so I just put five \( x \) down and went from there.

Nathalie wrote the final answer as the product, \( 5x \). Her explanation was: "[I multiplied] because there are no any, like plus or minus signs in there. I don't know."
Global Approach of Functions and Connections

Before I present three tasks where Nathalie described a function, I introduce her discussion of the box problem, a new problem she could solve (I1, 5). In her explanation she recognizes that she did not understand the problem conceptually.

The box problem is when you have, let's see, a cardboard, a piece of paper and you need to take so many, oh, inches from one side, like take out corners of each and then fold each side up so that you can have a box instead of just a flat piece of cardboard. [To solve it], you take the length of the box and then you need, you have to take out a certain amount of the box out of the cardboard so that you can have a place to fold it up and so there's an x, cause you don't know how much to take out, an x on either side, so you take the length minus two x's and say you are trying to figure out what area of this box is gonna be. So you're going to have the length minus two x and then the width minus two x and then times the number. I never really understood that, why you take times the number you took out but I just do it. So take x and you multiply that through and then you put certain numbers and that way you can figure out how much you need to take out of each side to make the box as deep as you need it.

The first task where Nathalie described a function was in the third interview. I asked her to tell me the important features a function has (I3, 4b). Her answer is related to two images: graphs and equations.

You need the variable. No you don't, I don't know. You need the different parts like the vertical shift and the horizontal shift, the stretch, shrink factors. Those are important. You need those to decide the side and shape and place of your graph. You need like the end behavior model, you need to know how it is supposed to look. That's about it, I mean, I don't really know. You need to know that there is like a negative in front of it, because then it's reflected over there.

The second task also took place in the third interview. In this case, I asked Nathalie to describe the graph of the rational function \( f(x) = \frac{x^3 + 2x + 1}{2x^3 - 4} \) with limited use of the calculator (I3, 5). The task made evident her mastery of the procedures emphasized in class; her strength in algebraic manipulations (as compared to other students); and familiarity with graphs of rational functions.

OK. Well to find the end behavior model, I take the highest exponent from the top and divide it by the highest exponent in the bottom, since you have x cube and x cube so those are going to cancel out and you are left with one half. That's going to be the end behavior model and then the end behavior is going to be as x approaches infinity, f of x is going to approach one half, right and then, you can't factor [the numerator] but you can factor the bottom a little bit...you get a two, you get x cube
minus two and your domain is gonna be all reals, basically isn't it? Because you can't, yeah, domain is going to be all reals, range probably will be too. I don't know yet. The zero points, put zero in. It's gonna be one [evaluating the top at zero] and your y-intercept is gonna be one, I mean negative one fourth. Hmm, let's see you are going to have a horizontal asymptote at one half. So, oops, equals no negative one half, equals one half. Can I draw it out?...Yeah, probably it's going to be something like this. I think, I don't know [drawing figure 77].

![Figure 77. Nathalie's expected graph in (13, 5).](image)

It is not clear if Nathalie was guided by the familiarity image with rational functions to expect a graph like this. A confusion with her algebraic outcomes and her predicted graph emerged when I asked her why she expected the graph to be that way.

Oh, no, it won't be like that because, the domain is all reals, so you won't have, you have a horizontal asymptote at y equals one half, and, I don't know, you can't. There are any, the range, the domain is all reals. There are any vertical asymptotes, so I guess it can be just about everything. I don't know. It's just confusing.

I encouraged her to use the algebraic outcomes she found to come up with a graph. The task was partially successful.

Well there is going to be a y intercept at negative one fourth and there is going to be zero point at one. So it could be...something like that I guess [drawing figure 78].

![Figure 78. Nathalie's graph of (13, 5).](image)
However, she was still not sure that the graph would look like this.

You don't [know that it looks this way], I mean it could go up, no it has to go down because it's going to cross the x axis, and this is going to keep going down and down and this is going to approach one half but it'll never touch it and I guess you can have something up top but no, you can't, no, that's it. I think that's it. I don't know. I think. It makes sense. You can't have anything up top because this is going to be, if you did something up here it's gonna make it, it's not going to be a function.

I asked her to explain to me where she used every piece she had found. Her answer shows an incomplete understanding of the end behavior model: only when x goes to infinity.

Yeah, for y, and then as x goes to infinity, which is over here, f of x is going to approach one half, so we got that. Domain is all reals, which we have. Range, y can't be equal one half which it doesn't. Zero point at one, [y-intercept at] negative one fourth, horizontal asymptote. It makes sense.

Nathalie's description ended here.

The third task is the description of the rational function $f(x) = \frac{x^2 + 1}{|x|}$ using the calculator (14, 4). Nathalie entered the graph in the calculator and changed the range several times. Her description starts with the effect of the absolute value.

Um, well, since it's an absolute value, there can't be any negatives, so if this was like go down [figure 79],

Figure 79. Nathalie's example of a function that "goes down."

it would bounce back because of the absolute value signs [figure 80]. It's gonna, it'll reflect it off the x axis, like if normally it was like that. It would be, it would come down and it would go up like that...It would just flip it over...So it can't have, it can't be less than zero.
Figure 80. Nathalie's description of the effect of the absolute value on a function that "goes down."

Her next thought, although with doubts, was driven by the familiarity image: "Um, since it's x squared I think it's going to be a parabola...I'm not sure though. Um, I don't know." This insecurity led her to change again the viewing rectangle. The graph in this case was driven by the continuity image. But she was not sure yet: "Or maybe you just, it might just go like this and go up like that [drawing figure 81].

Figure 81. Nathalie's expected graph in (I4, 4).

I don't know. Let me zoom in on it...I'm just trying to see if it goes straight up or if it connects...If it like goes up and I'm trying to see whether it does this, just up like that [figure 81] or if it goes like this [drawing figure 82]. And I think it just goes like this [figure 82]...I don't think it connects, I think it just goes up and where it's zero is, uh, zero is an asymptote...Well the domain is x can't equal zero...Because if x equals zero you would be dividing by zero, and you can't divide anything by zero.

Figure 82. Nathalie's final graph of (I4, 4).
Her concluding remarks is in contradiction with the domain she found: "And there should be an asymptote at y at positive one...If you put zero in for x you get one...That's about it."

**Settings: Domain, Range, Rule**

The course included examples of functions where the domain was either the set of real numbers or the set of real numbers minus five points at most. The range was the union of intervals at most. This might explain some of the difficulties that Nathalie had with problems where the domain and range were arbitrary sets (XC3, Q-02/19). For example, for the relation H that assigns to every triangle its area (XC3), she decided that H was a function, "because it is explaining height as a form of T and therefore you should get only one x for every y." Although this answer is correct, she found the domain to be "R", and the range "y ≥ 0." During the last interview she said she had difficulties understanding the problem and mentioned how she associated an equation with the problem:

I, okay, there weren't any, there wasn't any way that the bottom of this equation could equal zero so there weren't any restrictions to the domain.

But she could not remember how she found the domain or the range.

Familiarity was another image associated with settings not used in class. For example, in the case of the parabola defined only for integer numbers (Q-02/19), she drew a similar graph to Elizabeth's (figure 29, p. 122). However, Elizabeth was aware of the domain, while Nathalie was not. During the fourth interview she told me what she did: "Well, x squared is a parabola...I think...And so I drew a parabola, but I guess I was wrong."

Nathalie also had difficulties finding a mathematical expression of a piecewise function defined on a bounded interval (13, 6). I asked her this question in the fourth interview. As other students (e.g., Sara, p. 262), Nathalie tried to find an equation. In addition, she had difficulties interpreting the information given and she could not solve the task.

Oh, that's weird looking. This is a hole. Well, x can, it does equal five because it equals five down here. x is five. Um, so I guess you'd have something like x minus five times x plus two, um, hmm, I don't know. Let's see, but x cannot equal, it can't...I don't know. That's about it.
Difficulties with piecewise functions also appeared moving from the algebraic representation into the graphic representation. Nathalie said that there were two functions in a piecewise function (I4, 2), but she was one of the few students who made a graph.

Um, one if x is greater than or equal to zero, so you, there's one and it'd be, like that [drawing figure 83], hmmmmm, I don't know.

![Figure 83. Nathalie's graph of f(x) = 1 if x ≥ 0.](image)

I think that's right. If x is greater than or equal to, yeah, and then for negative one, if x is less than zero you'd have, oops, like that [figure 84]. So you have two functions right there. Don't you? Yeah. Two functions.

![Figure 84. Nathalie's graph of item (I4, 2).](image)

Nathalie did not realize that she was dealing with constant pieces. However, it is not clear which functions she graphed. Her answer to what the arrows meant suggests that she interpreted the domain (x ≥ 0) as the range (y ≥ 0).

Um, well this one means that it's, it'd be going up, so it would be greater than or equal to zero, a hole at zero...It's greater than or equal to, so it's going to be, you can put a hole there because it does equal zero, but you put an arrow going up because it's greater than zero...And this one, it's not equal to zero, so you put a hole and you go down because it's less than zero.

Other difficulties with the constant functions emerged in her graphical representations. Nathalie could not find f(2), f(\(\frac{1}{2}\)) or an x such that f(x) = 2, when the constant function, f(x) = 7 was given graphically (Q-01/29). She did not use the graph to read the information asked. She wrote "0" as answers in each case. In the fourth interview she said that the
reason was because the given point on the graph was (0, 7) and she thought that all the x's were zero. This situation of reversing x's and y's appeared very often in Nathalie's reasoning (see also (15, 7) below).

Nathalie was one of the few students who dealt correctly with discrete domains. She answered correctly why the tabular relation that assigns to every clothing item its price was a function: "There are completely different items with different prices so they won't overlap, 1 y for every x." During the fourth interview she told me how she had solved the problem. Her strategy although correct suggests a difficulty understanding the univalence feature.

Well, what I did is basically just put numbers in, like the blouse would be the number one, the shirt would be number two, the pants, three and four...and here you have a set of ordered pairs where you've got one, twenty five, two, twenty six, you know, cause see none of these had the same price, and none of them were the same type of item...So that means that none of these are alike and none of these are alike so you can have a function.

The confusion with the univalence feature seems to be at the verbal level in this case. I asked her what if she had two twenty-fives for the y's and she said

if you had like two, I think that's still okay. You just couldn't have two blouses for...Yeah, you can have two items for twenty five dollars but you can't have two blouses, one for twenty five and one for twenty six.

Nathalie was also one of the few students (e.g., Carol, p. 197), who did not connect five dots to make them the graph of a function (15, 7). She simply said "well, it is the graph of a function" and did not make any changes.

Nathalie was one of the two students (e.g., Griswald) who recognized the existence of a function that satisfies five algebraic conditions (15, 7). Given the conditions \( f(2) = 0, f(-3) = 7, f(-4) = 1, \) and \( f\left(\frac{1}{2}\right) = -1, \) she wrote them as ordered pairs first "'(0, 2) (7, -3) (1, -4) (1, \frac{1}{2})" and concluded "those would be a set of points...And as you can see there is only, for every x value there's only one y." Although the strategy is remarkable, since she was not blocked by the equation image or the continuity image, she did not interpret the information correctly. She read and wrote the x's as y's and vice versa.
Construction of Graphs

Nathalie had difficulties drawing the graph of the roller-coaster problem (14, 5). Like other students, (e.g., Steve, p. 157) she could not label the axes initially, but finally got them right. She did not draw the correct graph either. As with other students (e.g., Jane, p. 183), her graph was driven by the iconic features of the problem.

Development of Functions

This section is composed of two parts. The first part describes some of the ideas on functions that Nathalie had at the beginning of the course. Data are drawn from the Practice Test on Functions. The second part is a discussion of the development of the concept of functions based on the concept images and definitions provided by her throughout the study.

Some Ideas at the Beginning of the Course

Some of Nathalie's ideas on functions emerged on her answers on the Practice Test on Functions. Her definition of function is based only on the univalence feature: "one x with only one y." Other images associated with functions in the test were equation/formula, continuity, and familiarity.

Nathalie used the univalence feature also as a way to check that a piecewise graph (PTF1) and a piecewise expression (PTF2). In the first case she wrote, though incorrectly, that the graph was not a function "because there is more than 1 y for each x." The confusion arose from the use of open dots. In the second case, she wrote that the relation given was a function "because there is only one y with every value." It is likely that she answered this item driven by the equation/formula image-by focusing on only one piece; the quadratic expression.

Nathalie also used the equation/formula image to draw the graph of the function $g(x) = x - 2$ defined from the set of natural numbers to the set of natural numbers (PTF10). In this case, she drew a continuous line on the first quadrant. She ignored the domain and
Nathalie also used the equation/formula image to evaluate a constant function at
different values (PTF7) and to decide if two functions were equal (PTF8). Nathalie was the
only student who used the identity function $h(x) = x$ to evaluate the constant function
$g(x) = -7$ at different points. Other students used $h(x) = -7x$. Nathalie's answers suggests
that she confused reals numbers with positive numbers. When students were asked if there
exists a real number $x$ such that $g(x) = -7$, she wrote "no" because "-7 isn't real" (PTF7c).
Using the identity function, she found "$g(-7) = -7$" (PTF7a).

A complete omission of the domain and range in the equation/formula image did not
occur. Students were asked to decide if some functions were equal to $f(x) = 4x + 6$ from
the natural numbers to the natural numbers (PTF8). Nathalie wrote for each case the
following.

(a) for the line $g(x) = 4x + 6$ from the real numbers to the real numbers "yes, real &
natural are the same."

(b) for the line $g(x) = 2x + 3$ from the natural numbers to the natural numbers "no it's
been divided in 1/2."

When the graphical representation was given, the equation formula might have been
influenced by the familiarity image. For example for the continuous line given in part c, she
wrote "yes it is the graph" and for the discrete graph in part d, "no." The familiarity image
also appeared in giving examples of functions. She drew a vertical parabola as an example
of a function and a horizontal parabola as an example of a non function (PTF3).

The continuity image also appeared in items where she was asked to draw a graph. For
example, when two points were given (PTF13), she connected them with a straight line. She
said that there was an infinite number of such graphs because "there aren't any other points
to confine the line." When six points were given (PTF14), she connected the following
points in the same order D-B-A-E-C (which is not a function) and concluded that there was
not a function like that since "the lines are restricting."
Nathalie had difficulties reading the domain and range of a piecewise graph (PTF9), and recognizing graphs with a given domain and range (PTF11). Other items that she answered incorrectly or did not attempt might have been caused by the lack of familiarity with the vocabulary used in the items (PTF4, 5, 6).

Images and Definitions of Functions during the Study

There are several changes observed in the definitions of function that Nathalie provided in the course. Those changes seem to resemble partially images in the historical development of functions: from graphs to equations to univalence. The distinction is based on the use of the univalence feature every time the concept of function is evoked.

Some changes can also be observed on the images that Nathalie associated with functions during the interviews. She defined a function in the second interview in terms of the graph and univalence images: "a function is a graph where a number for an x value can be used once and only once." Four more images appeared in the protocols: vertical line test, familiarity, continuity, and equation/formula. The graph image in addition to the definition, emerged as functions can always be graphed (p. 275). This image was related to the vertical line test.

Nathalie used the vertical line test to decide if a graph is a function (I2, 1, 2, 3, p. 282) and perceived this image as equivalent to the univalence feature (I2, 1, 2, 8, pp. 282-283). This establishes a rich network of relationships, since it provides two equivalent methods to decide whether or not graph is a function. The univalence feature also emerged, in addition to the definition as a "basic rule for functions" (p. 283).

The familiarity image emerged as familiar shapes (I2, 4) and as familiar functions (I2, 5). In the case of familiar shapes, Nathalie expected the function that sends each number different from zero to its square and zero to one, to be a parabola (p. 280). In the case of familiar functions, she decided that a function all of whose values were the same (I2, 5) existed because it was "a straight line" (pp. 279-280).
The continuity image emerged as the habit of connecting dots. All the graphs that she built in the problem of existence of a function discontinuous at one point (I2, 4) were continuous functions (p. 280). However, the strategy she followed was to plot points and then connect them. A similar situation occurred with the existence of a function that sends integer to non integers and non integers to integers (I2, 6). Nathalie said that such a function might exist sometimes. She provided first several points with integer coordinates and non integer coordinates, plotted them and connected them. When she realized that the non integers had to be paired with integer numbers, she provided two more points, rearranged the graph and still connected the points. She was not sure though that her answer was correct (pp. 281-282).

Nathalie's strategy in this problem suggests an independence from the equation/formula image, which gives her a sense of arbitrariness. However the habit of connecting dots might be also a reflect of a sense of regularity. The equation image also appeared as 'functions come from equations' and as 'functions are equations' (I2, 7, p. 273).

Most of the problems in the second interview were word problems. Nathalie was one of the few students who attempted to make some drawings in some of the tasks. This might be a result of the graph getting its own identity. This is also a remarkable difference with others students (e.g., Jane, p. 188) who were dominated and blocked by the equation image. This has several implications for problem solving: Drawing a graph becomes a heuristic strategy, and the path from verbal representation into graphic representation strengthens her network of relationships.

Nathalie's definition of a function in the third interview, "a function is an equation that produces a graph where for every x there is only one y," is based in three images: equation, graph, and univalence feature.

The equation image appeared associated with the graph image, in addition to the definition, as graphs come from equations and important features of functions. In the first
case, she said that given a graph there was always possible to find an equation that gave the graph (I3, 2, p. 272). This situation takes away the sense of identity from the graph image that appeared in the second interview. It might be that the identity of the graph image was in formation, and hence still weak.

The second link between graphs and equations appeared in important features of a function (I3, 4b). In this case, she referred to graphical and algebraic features (p. 286).

Nathalie identified several topics related to functions (e.g., compositions, rational functions, graphing) as the topics studied in the course (I3, 1). She also recognized that functions were important (I3, 3), however in a future sense. She did not know where she was going to use them in real life.

Nathalie's description of a rational function with limited use of the calculator (I3, 5) showed her mastery of the procedures emphasized in class; her strengths to do algebraic manipulations (as compared to other students); familiarity with graphs of rational functions, but a lack of rich relationships (pp. 286-288).

Nathalie provided a definition previous to the fourth interview (6th definition, p. 272). This definition although based on the univalence feature and related to the graph image is stated incorrectly: "only one x value for every y value." During the fourth interview, other images appeared: familiarity, the vertical line test, equation/formula, and continuity. The univalence feature was cited at different moments in the interview and at times it was stated incorrectly. However, it seems a confusion at the verbal level, since she discriminated correctly—applying the univalence feature, although stated incorrectly—the graph of a function from the graph of a non function, both with discrete domain (I4, 3, pp. 283-284).

Nathalie used the familiarity image to find the domain and range of the resulting composition of the identity function and a constant function and to recognize it as a function. The vertical line test was mentioned as a method to test that a graph is a function (p. 282). Another appearance of the familiarity image was related to the parabola defined
only for integer numbers (Q–02/19). In this case, she knew that \( x^2 \) was a parabola and used this knowledge to find the domain and range (p. 290). This situation might also be a consequence of the equation/formula image, since she only focused on the rule and ignored the domain and the range.

The equation image emerged as an obstacle in finding the mathematical expression for a piecewise function defined on a bounded interval (I3, 6). I asked her this question in the fourth interview. Nathalie tried to find an equation and had difficulties interpreting the information given. She could not solve the task (p. 290). The equation image appeared related to the graph image again. In this case, her answer to what makes up a function (I4, 1) was given in terms of algebraic and graphic features (pp. 273-274). A conceptual shift appeared: "functions have domain and range."

Nathalie was one of the few students (e.g., Sara) who drew a graph for the piecewise function with constant pieces (I4, 2). However, she had difficulties understanding the information given and made an incorrect graph. This might have been also a result of reversing x's and y's (pp. 291-292). With this reversal situation she answered that there were two functions. However, Nathalie's functions were defined on intervals while Sara drew two constants, both with domain equal the set of real numbers (p. 263). She also made an incorrect graph for the roller–coaster problem (I4, 5). As other students (e.g., Jane), she was driven by the iconic situation of the problem.

The familiarity and continuity images flashed in the description of a rational function using the graphing calculator (I4, 4, pp. 288-289). Some difficulties appeared with her algebraic manipulations. In this question she showed her abilities to use the calculator.

Nathalie provided a definition previous to the fifth interview. Her definition is based on the equation, graph, and univalence feature (6th definition, p. 272). The dominance of the equation image over the univalence feature emerged when she was faced with a function
defined for an arbitrary set (XC3, p. 290). In contrast, the univalence feature dominated other images in the case of a function with discrete domain and range (XC2, p. 278).

Nathalie's last definition of a function is based only on the univalence feature (p. 272). However, she spent a considerable time stating it correctly. She kept reversing the x's and the y's, even when she drew a correct figure for a function in her reflections (figure 85).

(3,4) (3,5)

Figure 85. Nathalie’s figure for the univalence feature.

She also used the univalence feature in how to explain functions to precalculus students, and as a means to decide if a set of ordered pairs was a function (I5, 7). Two more images appeared during the protocol: graph and equation. The equation image appeared as a property, functions are equations (15, 2). Other properties of functions where related to the algebraic or graphic representation: Functions have end behaviors, "all have domain and range," and they can be graphed. The graph image emerged in the relationship of functions and graphs (15, 4) as all functions can be graphed. The equation image emerged blocking the arbitrariness of possible sets for the domain and range of functions. Nathalie said that functions were only for numbers. She just could not "think of anything else you'd define a function with" (pp. 274-275).

Nathalie and one other student (Steve, p. 157) perceived functions could be represented not only by equations or graphs. She added "a set of ordered pairs." In the case of Nathalie, this perception helped her to recognize a set of points with the univalence feature as a function. She was one of the few students who did not make any changes to five points given graphically (I5, 6) to make them the graph of a function. She said that she would make changes "unless you wanted to change the overall look." This idea also benefited her
when the points were given algebraically (I5, 7). In this case, she first wrote the points as pairs (though in the reverse order), and concluded "as you can see there is only, for every x value there's only one y." Her suggestions for teaching functions targeted at properties of functions, means to recognize functions, and the ability to move between the graphic and the algebraic representation, the univalence feature (which this time she stated correctly) as a rule for functions, recognizing a function on a graph using the vertical line test or checking the ordered points, "being able to determine what a function or what an equation is going to look like, just from the algebraic equation, and also being able to come up with an algebraic equation from a graph" (I5, 5).

**Summary of Nathalie's Development and Knowledge of Functions**

Nathalie started the course having the univalence image for functions. During the study she narrowed her conception of functions to equations and graphs with the univalence criterion. By the end of the year her function concept started becoming independent of the graph and equation images. However, this situation is not completely consistent and several regressions in her thinking were observed. Her strategies to solve the tasks illustrates a tug of war between two function images: equation and univalence. Nathalie experienced two kinds of development of functions: procedural and conceptual. The course emphasized the procedural one and she benefited with strong algorithms. At times, there is not much understanding of them. Conceptually there is a network of images, whose richness is enhanced by the univalence criterion. Hence, she can deal with several situations involving the arbitrary nature of functions (either with the rule or with the domain and range). This seems distinguish her among all the participants. The other essential feature of functions—univalence—creates conflicts at the verbal level. Nathalie had difficulties stating this criterion. She was one of the students (the other is Griswald) who do not seem to have a total quasi-structural conception of functions. She seems to be close to the process of reification instead.
The Case of Griswald

Introduction

Griswald is an athletic 17-years-old male. He likes only what he perceives as the challenging part of mathematics. He is a straight-A student in precalculus and is taking the course for two reasons at least: to get some preparation for college calculus, and as a logical progression—the next course to take. His expectations about the course are honest: "a good grade, and to be able to get through college calculus." The precalculus class is not very challenging for Griswald, and consequently he feels pretty bored in the class. However, he is afraid of moving into a calculus course because he considers himself lazy.

Griswald finds the use of the graphing calculator in precalculus kind of boring because mathematics is more challenging, in his opinion, without the calculator. He thinks that the calculator is a way to check himself, but it also takes all the fun away. Learning mathematics is easier with the calculator but bad: the calculator gives the right answer when the right equation is input; but, according to him, there is no way to check if someone puts in the wrong equation.

By the end of the year, Griswald thought it was "a bad idea to use the calculators." In his opinion, he "would have learned a lot more without the calculator." For example in the case of graphs, he couldn't name very many graphs, like what they are going to look like...[He could] tell what x squared looks like and like x cubed but...if they get too tough, [he was] not going to be able to [describe them].

Griswald thought that without the calculator he would have learned to describe them. Instead, he learned "how to put something into a calculator." Furthermore he thought using the calculator was cheating: "you can cheat even like on algebraic stuff if you don't know how to do it right." For example if he did not know how to solve an item, he will "figure out how to graph it somehow and trace, and then figure out how [he] should have done the
work." In summary, he though he was "not learning anything." This might explain why he never felt attracted to play around with the calculator.

A personal note on Griswald. He was the student with the lowest overall GPA among all eight participants at the beginning of this study.

**How Griswald defines a Function (Concept Definition)**

Griswald provided five definitions of functions during the study. The first definition is based on the graphical representation and the univalence criterion (stated as the vertical line test): "A graphed answer that would pass the vertical line test" [sic] (I2, 8).

Griswald's second definition is based on the same images included in the first definition.

I just think of, sometimes it's, like is never straight up and down, it doesn't cross any place like vertically. I always think of the vertical line test, like kind of visualize. (I3, 2)

His last three definitions are phrased in terms of the univalence criterion. The third definition discriminates between functions and non functions. He wrote on the quiz,

To me a function is a relation that for every x value there is one and only one y value. It is important because a function is a special relation. (Q-03/04)

The fourth definition states that a function is a relation: "A relation in which for each x value there is one and only one y value" (XC1).

Griswald's last definition is based only on the univalence criterion: "a function's for every x there's one y" (I5, 1).

**Griswald's Mental Pictures Associated with Function (Concept Image)**

Several images associated with functions emerge from the data collected from Griswald. Images have been separated for discussion, but repetitions in several categories suggest a network instead of a partition. The images associated with functions are the following: familiarity, vertical line test, continuity, graph, equation/formula, and one input for every output. Indications of each cluster follow.
Grigswald associated several ideas with the familiarity image of functions. The first idea is familiarity with examples of three types: familiar examples of functions (I1, 13, 14c), familiar shapes of functions (I4, 6), and familiar nonexamples of functions (I5, 4). The second idea in the familiarity image is using this knowledge to decide whether or not a given representation is a function (I2, 4; I3, 2).

An instance of familiar examples of functions appeared when I asked him to describe a function with his eyes closed. He said:

I see a big little parabola, crosses the y axis about negative two, there is also another line coming across, like a straight line, it cuts across two points of the parabola.

When I asked him to make a drawing of it, he drew figure 86 and said "I thought of two functions. The second one grabbed upon me."

![Figure 86. Griswold's graphs of two functions (I1, 13).](image)

Another instance of familiarity with examples of functions emerged when I asked him about his ideas about f(x) (I1, 14c). He replied, "I think of some kind of a function. The first thing that would hit me would be a straight line."

Familiarity with shapes appeared in his discussion of the use of graphing calculators in the class (I4, 6). Griswold did not like the use of the graphing calculator because he could not name many graphs "like what they're going to look like." However, he was familiar with some shapes, since he could "tell...what x squared looks like and like x cubed, but...if they get too tough, [he was] not going to be able to [describe them]."
Familiarity with non examples of functions emerged in Griswald's discussion of the relationship between functions and graphs (I5, 4). He said functions and graphs were related because by using the vertical line test he could decide if a relation was a function or not. So I asked him if he knew other procedures different from the vertical line test. His answer included non examples of functions.

Yeah...if you get a y squared equal x...just from that I know that that's not a function...that's just one example, like if you, if you see something, you can see that there's two posible answers for y's.

The use of the familiarity image to decide if a given representation is a function appeared in Griswald's discussion of the function that sends every number different from zero to its square and zero to one (I2, 4). At the beginning, Griswald said he did not understand the question and asked for some help. So I suggested that he make a graph. He calculated several points, plotted them, connected them, and said "it's a parabola, so it's a function."

A second example of using the familiarity image to decide whether or not a given expression is a function appeared in his discussion of the components of a function (I3, 2). I asked him, "what would you do to decide that an expression is a function? He answered "just that it sounds to me, like if it is like y equals x squared then you know. I know it is a function."

**Vertical Line Test**

The vertical line test also emerged in different ways in Griswald's work:

1. a means to decide whether or not a graph is a function (I2, 1, 2, 3, 4, 5, 8; I4, 1, 3; I5, 4),
2. an equivalent statement to the univalence criterion (I2, 8),
3. as a property of functions (I5, 2).

Griswald used the vertical line test to decide whether or not a graph was a function. For example, for the three graphical items in the second interview he said "I just did the vertical line test for all these." He also applied the test for the graph of a function
discontinuous at one point (I2, 4, figure 14, p. 108, see dialogue pp. 305-306).

Another application of the vertical line test to decide whether or not a graph is a
function appeared in the existence of a function all of whose values are equal (I2, 5). In this
case, Griswald interpreted the equal values condition as same x's. So, he said initially that
the answer was "no, because that will be a vertical line at some point. [He] used a visual.
[He] was thinking of a function like that and then [he] did the vertical line test mentally."
However, when I asked "when you talk about same values, are you looking at the first
coordinate?", he drew a horizontal line and said "it is a function, it passes the vertical line
test." Other examples of the use of the vertical line test appeared in his discussion of how
many functions are in a piecewise function (I4, 2, p. 317) and the relationship of functions
and graphs (I5, 4 p. 304).

The second idea that Griswald associated with the vertical line test is equivalence to the
univalence criterion for functions. In the second interview I asked him if there was a
difference between them (I2, 8). He answered "no, the vertical line test is just doing that. I
mean is the same thing. Yeah." Griswald said that given a graph he would use the vertical
line test to decide whether or not it was a function. But he would use the definition "just if
someone asks what a function was."

The third idea associated with the vertical line test is that of being a property of
functions. Griswald’s answer to what properties functions have was "for each x there's one
y, that's pretty, uh, that pass the vertical line test" (I5, 2).

**Continuity**

Only one idea associated with continuity emerges from the data collected from
Griswald—connecting dots (I2, 4; I5, 6). This idea appeared in Griswald’s work when he
was asked to make graphs. For example, in the function that assigns to every number
different from zero its square and one to zero, he calculated several points, plotted them, and
connected them (I2, 4, p. 304). When I told him "zero doesn't go to zero", he modified his graph and drew a similar graph to Elizabeth's (figure 20, p. 113). The difference was that Griswald's graph intersected the x-axis at two points, while Elizabeth's graph did not. However, he was not sure that this was the graph we were talking about. He reflected for a second and suddenly said:

Oh, no, it wouldn't go like that actually, I don't think. OK, how about this [drawing the correct graph, figure 14, p. 108]?...And there is a dot right there and there are two open nuts. That's my new guess. I want to stick with that one.

When I asked him, "Why did you decide for this one?" He said "as far as I can tell it fits everything there." The last question I asked him about this problem was "Is that a function?" He said "yeah...it passes the vertical line test and for every x there's only one y."

Another instance of connecting the dots appeared in making changes to five plotted points to make them the graph of a function (I5, 6). Griswald proposed several changes. One of them being connecting the dots: "I could connect all the dots like this...if you do it like left to right. Hitting every one...You could put them all in a straight line. You could move them all around, up and down." Griswald however, recognized the dots as a function. After he proposed the changes, I asked him "What about leaving them the way they are?" He answered "yeah, I suppose it's a function."

Graph

Griswald associated five ideas with the graph image:

1. graphs are functions if they pass the vertical line test (I2, 4, 8; I5, 4),
2. graphs are functions if they satisfy the univalence criterion (I2, 4),
3. all functions can be graphed (I2, 8; I3, 4a),
4. functions can be represented by graphs (I3, 2; I5, 3), and
5. a means to decide whether a relation is a function or not (I5, 4)

The first idea, graphs are functions if they pass the vertical line test, appeared in Griswald's definition of function in the second interview. He wrote a function is "a graphed answer that would pass the vertical line test." This idea also appeared in his discussion of the relationship between functions and graphs (I5, 4, p. 307).
The second idea, graphs are functions if they satisfy the univalence criterion appears at the end of the dialogue of a function discontinuous at one point. He said "yeah [it is a function]. It passes the vertical line test and for every x there's only one y" (I2, 4).

The third idea, all functions can be graphed, appeared after he told me what a function was in his opinion in the second interview (I2, 8). However, there is some hesitation in whether or not all functions can be graphed. I asked him, "Can you graph any function?" He answered "I think, but probably you are coming with an example [that cannot be graphed]." Another appearance of this idea emerged when I asked him directly if all the functions or expressions could be put into a graph. His answer suggests that the answer is yes: "Everything? I guess. I can't, it would be pointless to graph something that there is no answer to it" (I3, 2). I did not ask Griswald directly what he meant by "answer." However, it seems that he was referring to 'evaluate' as can also be deduced from his second definition of function: "a graphed answer that would pass the vertical line test" (I3, 2).

The fourth idea associated with the graph image is that functions can be represented by graphs. For example, in his definition of a function in the third interview, he indicated:

I just think of [a function] sometimes it's, like, never straight up and down, it doesn't cross any place like vertically. I always think of the vertical line test, like kind of visualize. (I3, 2)

Representing functions by graphs emerged specifically in Griswald's discussion of how functions can be represented (I5, 3). He simply said "it can be represented in graphing or equation."

The fifth idea in the graph image is that graphs are a means to decide whether a relation is a function or not. This idea emerged in Griswald's discussion of the relationship between graphs and functions (I5, 4). He said that functions and graphs were related because "you can tell if something's a function by looking at the graph [by using] the vertical line test."
The equation/formula image also emerged in several presentations in the data collected from Griswald:

(1) functions are equations (I2, 6, 7; I3, 4a),
(2) functions are equations with the univalence criterion (I4, 1),
(3) functions can be represented by equations (I5, 2), and
(4) not all equations are functions (I5, 4).

The first idea in the equation/formula is that functions are equations. Hence the need to provide an equation or formula to show the existence of a function. For example, in the case of the function that assigns integers to nonintegers and nonintegers to integers (I2, 6), Griswald's attempts were unsuccessful. He started by saying that there were "many, many, many answers to this." So I asked him to give me an example of one. He said "OK one coordinate would be one, one half", made the graph of a straight line, wrote \( m = 2 \) next to it and said "the slope is two." So I asked him "What about zero? it's going to be zero, zero. Both. But zero is supposed to go to a non integer" while I was pointing to his graph. He said "allright we move it, we go to a parallel line next to it," pointing with his finger to the graph. Then he realized that his example did not work and said "that example wouldn't change integers into nonintegers, but if you give me like a couple weeks, I can come up with an example of nonintegers into integers." The dialogue ended here.

A second instance of functions being equations appeared in the relation between equations and graphs (I2, 7). Griswald said that functions and equations were related because "a function is an equation." He also said students in class were using functions and equations in the same way "sometimes...because you know, you make like, like for a story problem, you make an equation into a function." I pursued this idea in a specific situation: the box problem. I asked him how he would solve it. His answer implied that functions and equations are not the same. He said "an equation first, I think. Because you have to get rid of one variable and make like the two equations or whatever and when you
finish that, turn the last into a function."

This idea also appeared in Griswold’s discussion of the components of a function. His answers also convey his conceptualization of functions as concrete entities: "What do you need? I’m not really sure. If I can see it, I can tell you whether or not [it is a function]...I need to see it. I can’t think of the actual components you need. It’s like y squared and x squared or something like that, then you can get that. I need to see the actual problem...See the function, the written out function and numbers" (13, 4a).

The second idea associated with the equation image is functions are equations with the univalence criterion. This idea emerged in Griswold’s discussion of what makes up a function (14, 1). In this case, he answered "a formula that for each x will only produce one y...that would fit that qualification."

Griswold’s third idea associated with the equation image is functions can be represented by equations. This idea emerged specifically in Griswold’s discussion of how functions can be represented (15, 2). He simply said "it can be represented in graphing or equation."

The last idea that Griswold associated with the equation image is not all equations are functions. This idea emerged in Griswold’s discussion of the relationship between functions and graphs (15, 4). After he mentioned the vertical line test as a procedure to decide whether or not a graph is a function (p. 299), I asked him if he knew other procedures. He answered:

yeah...if you get a y squared equal x...just from that I know that that's not a function...that's just one example, like if you, if you see something, you can see that there's two possible answers for y's.

One Output for every Input

In the data gathered from Griswold, there are several ideas associated with the univalence criterion for functions:

(1) an equivalent statement to the vertical line test (12, 8),
(2) equations are functions if they have the univalence criterion (14, 1),
(3) definition,
(4) property of functions, 
(5) a means to decide if an equation is a function (I5, 4), and 
(6) a means to decide if a discrete set of points is a function (I5, 6).

The discussion of equivalence between the vertical line test and the univalence criterion for functions appears in the vertical line test section (p. 304). The second idea associated with the univalence criterion, equations are functions if they have the univalence criterion, was presented in the discussion of Griswald's equation/formula image (p. 308).

The third idea associated with the univalence criterion is that of definition. This idea emerged in Griswald's three last definitions (p. 302). The fourth idea associated with the univalence criterion is being a property of functions. This idea emerged in Griswald's discussion of properties that functions have. His answer was "for each x there's one y, that's pretty, uh, that pass the vertical line test" (I5, 2).

The fifth idea in the univalence criterion image is using it to decide whether or not an equation is a function. This idea appeared in Griswald's discussion of the relationships of graphs and functions (I5, 4). After he mentioned that he would use the vertical line test to decide whether or not a relation was a function, I asked him if he knew other procedures. He said:

yeah...if you get a y squared equal x...just from that I know that that's not a function...that's just one example, like if you, if you see something, you can see that there's two possible answers for y's.

The last idea associated with the univalence criterion is using it to decide whether or not a discrete set of points is a function. This idea appeared in the existence of a function that satisfies four conditions given algebraically (I5, 6). Griswald plotted the points given and decided that they were a function "cause for each x there's one y."

Summary of Griswald's Function Images

Six function images emerged in the data collected from Griswald in this study: equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, and continuity. Several relationships between these images suggest the existence of a
network. Two images appear strongly anchored in Griswald's work: univalence and equation. The strong tendency to univalence allowed him to deal successfully with most of the tasks presented. However, he still relied on the equation image when the tasks were unfamiliar and involved numerical sets.

**Procedural and Conceptual Knowledge of Functions**

Griswald was a high achiever in precalculus. His grades were always in the range A and A-, except at the end of the year, when all students' interest in the class decreased.

Griswald participated little in class. He did not work hard either. For example, he did his homework a few times and when he did it, he was working on it while Mr. H. was checking other students'. At times, it was common to see Griswald reading a book while Mr. H. was teaching a topic. In this section, I present his approaches to several tasks to illustrate his knowledge of functions.

**Composition of Functions**

Griswald as other students used the substitution procedure emphasized in class to find the composition of functions. However, he was the only student who found correctly the composition of a constant and the identity function. Griswald did not rely on the equation image to find the composition. He made the substitution on each case and used the function at hand.

**Global Approach of Functions and Connections**

In this section I present three tasks where Griswald described a function. The first task was in the third interview. I asked Griswald the important features of the graph of a function (I3, 4b). His description at the beginning was partial, but with more encouragement the description was more global:

OK maybe all zeros, asymptotes, extremes, that would be enough for me...on what intervals is going up and down, and if it was a straight line or a curve, parabola, or whatever. Shapes in between points, I guess.
The second task is the description of the rational function, \( f(x) = \frac{x^3 + 2x + 1}{2x^3 - 4} \), with limited use of the calculator (I3, 5). The task made evident the weaknesses of his conceptual knowledge of functions (rich relationships) and little conceptualization of the problem.

Griswold tried to manipulate algebraically the expression, but was unsuccessful and said "I don't know what to do with that." After some encouragement he said "I can do like the zero points [writing \( x^3 + 2x + 1 = 0 \)]." He found \(-.4509695\) for the \( x \)-intercept with the calculator and substituting \( x = 0 \) in the expression he found the \( y \)-intercept at negative one fourth. His next strategy was to evaluate the expression at one point, \( x = 5 \), mentally evaluate the end behavior, and with all this information to sketch a possible graph.

like try a point or whatever, this is like one half. That would mean to me, that it goes like this [drawing figure 87]...like this is some kind of asymptote.

![Figure 87. Griswold's first sketch of the rational function in (I3, 5).](image)

I asked him how he had found the asymptote, but he was not sure. "How? I'm guessing it is one half. But I don't know. So it crosses the \( x \) axis here in some place, but I figured that out I think, and I think. Am I way off?" I answered no. His next strategy then was to evaluate the expression at \( x = -10 \). This situation highlighted his poor conceptualization of the task:

figuring out where negative ten would be. Try another point. OK it comes up like that [drawing figure 88]. Do you think it's right? It's kind, like, I don't know how to deal with this problem actually. I mean the graph doesn't show me where...
Griswald moved next to examine the denominator since "it has to be undefined some place." He input the denominator in the calculator, but misinterpreted the information: "about 1.25 and that would be a hole right there." He drew a hole in figure 87 in the $x$-intercept with positive $x$. The next information that Griswald found was the end behavior. But again he was not sure and could not remember how Mr. H. found it:

$$\text{well all these numbers are about half of where this is and that as close I can come because I couldn't really factor...[Mr. H.] factors them I think, maybe.}$$

With this information, Griswald drew two possible graphs for the function. The end behavior gave the two possibilities. In the case of the first graph, he said: "I think it looks like. I'm not sure if it comes up or over [figure 89]."

In the case of the second graph, he said: "Yeah, there is an intercept there, either it goes like or is going to be like this, I'm not sure and then again probably goes like that [figure 90]."
I asked directly about the domain and range. Griswald read both from his graph:

domain is all numbers, all real numbers except for the negative one point two five and the range is like, uh, see, I'm not sure if it is below point four five. I'm not sure if that's extreme or not, but it would be something around and then up to one half or maybe a little bit higher than one half if it goes over.

He also decided that the graph was a function as far as "[he] had drawn it." The description ended here.

The third task is the description of the graph of \( f(x) = \frac{x^2 + 1}{|x|} \) using the calculator (I4, 4). Griswald started by saying "it looks like, it's a function" based on the vertical line test. His description follows.

its range is everything above zero...the domain is all x's except zero, [deduced] from the x on the denominator...it looks like a reflection...cause of the, no, it's one of these deals, bent up, because of the x.

Griswald used different windows to get a better picture of the graph. However he was "running out of interesting things to tell [me] about the [graph]." Trying out different windows and the trace feature he found a different range: "it starts at one actually," but he was not sure about it. So he decided to evaluate the function at an specific value, \( x = .25 \), just to check something below one...That's just, generally something [he is] gonna try to see how low it's going to go...cause zero won't work in it so [he] was thinking point two five would just run on there.

Griswald did not make any further description of the function. His strategy at the end of his discussion resembles the idea of limit of a function around a point; he was trying to investigate the behavior of the function in a neighborhood that contained zero. 1 only
observed this kind of analysis of a graph in Griswald's data.

**Settings: Domain, Range, Rule**

The course included examples of functions where the domain was either the set of real numbers or the set of real numbers minus five points at most. The range was the union of intervals at most. However this did not represent an obstacle for Griswald in dealing with the majority of the arbitrary functions included in tests and interviews. This makes Griswald especially distinctive from the other seven participants in the study. For example in the third interview, we were talking about the components of a function (p. 309). In his discussion he mentioned that he needed to "see the function, the written out function and numbers" to decide whether or not it was a function. So I went ahead and asked him if the relation given by \( f(x) \) as one for the nonnegative numbers and negative one for negative numbers was a function. He said "that appears to be, yeah. Do you want me to draw the graph?" I said yes and he drew the correct graph [figure 91].

![Figure 91. Griswald's graph of the function that assigns one to nonnegative numbers and negative one to negative numbers.](image)

I remarked that he had drawn a graph and from it made a decision. I pursued the idea further and asked him, "Do you think that all the functions or expressions can be put into a graph?" His answer suggests that the answer is 'yes': "Everything? I guess. I can't, it would be pointless to graph something that there is no answer to it." So I gave him the Dirichlet's function, \( f(x) \), one for rational numbers and negative one for irrational numbers. Although Griswald hesitated in the drawing he said
Jesus, I'm going to say it's a function too. I can't draw that one, honestly, but...because it looks like it just would be dots, up and down, the whole way across.

However with a little encouragement he drew a correct graph [figure 92] while he said "it wouldn't be like straight up, I mean aligned [drawing vertical lines]." And insisted referring to the possibility of always being able to draw a graph, "no, you can't really visualize the graph. You can't really draw it realistically." So it seems that Griswald's understanding of putting a function into a graphic form refers to "an exact drawing."

Figure 92. Griswald's graph of the Dirichlet's function.

Griswald was one of the three students (see Jane, p. 182, Carol, p. 211) in the study who could find a mathematical expression for the piecewise function defined over a bounded interval (I3, 6). Griswald did not try to write down an equation as Jane did. He started by saying "I guess I go like [writing]

\[
\begin{align*}
  f(x) & \rightarrow -3 \\
  [-2, 3) & \\
  (3, 10) & \rightarrow 4
\end{align*}
\]

I thought initially that the arrows referred to an assignment notation. He confirmed my thought. When I asked him what the arrows indicated, he answered "like for the numbers, when x is through this set of numbers...sort of equal I guess [changing] I think that would be better." His final expression then read as follows

\[
\begin{align*}
  f(x) & \rightarrow -3 \\
  [-2, 3) & = 3 \\
  (3, 10) & = 4
\end{align*}
\]
Only two students in the class solved correctly all the items related to the function $x^2$ on the set of integer numbers (Q-02/19). Griswald was one of them. The other student was not a participant in the study. During the fourth interview, Griswald said that he plotted the points because he "knew it wasn't a line...so [he] just plotted, well, [he] knew what it was, [he] knew what it looked like." He "picked one and then two and then [he] knew where they'd be because of the square deal." Griswald did not rely on the familiarity image; he found the domain "from the problem", the range "from the y values," and decided it was a function by using "the vertical line test."

Griswald was also one of the few participants who recognized that there was only one function in the piecewise relation given as

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Furthermore, he was the only participant who drew a correct graph for this item and recognized it as a function by doing "a mental vertical line test" (I4, 2).

Griswald was one of the participants (see Nathalie, pp. 283-284) in the study who identified correctly which graph was a function and which graph was not a function in the graphs with four points given in (I4, 3). In both cases he used the vertical line test to make the decision.

Griswald also dealt successfully when the function was given as a table and with a discrete domain. In the relation that assigns its price to every clothing item (XC2), he chose the relation to be a function because "all of the answers are different x values." During the last interview I asked him about his answer. He elaborated and said "for each x there's one y. Like [he] had the items as the x's and the prices as the y's."

None of the students in the class answered correctly the items related to the relation that assigns its area to each triangle (XC3). There were two possibilities related to the equation image. Either students tended to work with an equation or with the specific case of a
constant function. During the last interview, Griswald said he had used the constant function, $y = 10$, but did not remember what he did with it. However, there is a mismatch with his written answers. He wrote the domain as "[0, infinity)," the range as "[0, infinity)" and decided the relation was a function because "there will only be one ordered pair that will fit this equation."

**Construction of Graphs**

Griswald also drew the correct graph for the roller-coaster problem. He simply said "it's going to be like the...it's like the inverse...cause when it's going down hill, it's going to be going faster." He was the only participant who did not draw any axes in order to draw the graph. When I asked him about them, he said "I didn't need the axes...cause I'm a great mind."

**Development of Functions**

This section is composed of two parts. The first part describes some of the ideas on functions that Griswald had at the beginning of the course. Data are drawn from the Practice Test on Functions. The second part is a discussion of the development of the concept of functions based on the concept images and definitions that he provided during the study.

**Some Ideas on Functions at the Beginning of the Course**

Griswald said in the first interview that the Practice Test on Functions was difficult; he got bored and felt he was not going to answer anything. He only answered the first four items. In the first item, students were asked to decide if a given piecewise graph was a function or not (PTF1). He decided correctly on the graph being a function, but without any explanation. The second item he answered, though incorrectly asked whether a piecewise relation given algebraically was a function or not (PTF2). He did not provide any explanation.
The third item he answered asked students to draw an example and a non example of a function (PTF3). He drew a straight line in the first case and a sideways parabola in the second case. The last item he attempted asked students to decide which points on a graph represented an image, preimage, a (preimage, image) pair, or a point that does not represent a (preimage, image) pair (PTF4). He simply wrote "does not represent a preimage image pair." In the next section, I discuss some of the changes that occurred in Griswald's images and definitions of functions.

Images and Definitions of Functions during the Study

One specific change can be observed in the five definitions that Griswald provided during the study: from the graphical representation that passes the vertical line test to stating simply the univalence criterion. This seems however a conceptual leap: from a concrete representation to an abstract concept of function.

Other changes in Griswald's work can also be traced to the images that he associated with functions during the interviews. In the second interview, he defined a function in terms of the graphical representation and the univalence criterion (p. 302). Other images emerged and competed in his decisions in the tasks presented in the interview. The images of vertical line test, equation, familiarity, continuity, and one output for every input appeared in the protocol. Griswald also exhibited a network of images instead of a partitioned set.

Griswald used the vertical line test to decide if a given graph was a function (pp. 304-305). In contrast he used the equation image in items where the existence of a function was asked (I2, 6) and in discussing the relationship between functions and equations (I2, 7). In the first case, the equation image created an obstacle to finding functions, since he tried to write down a formula (pp. 307-308). This might be explained by his idea of functions being equations that appeared in his discussion of the relationship between functions and equations.
Griswald was the only student who solved correctly the existence of a function that assigns its square to every number different from zero and one to zero (I2, 4). He, as the other seven participants was driven initially by the familiarity and continuity images: familiarity with parabolas and connecting the dots. However, it seems that the arbitrary nature of functions was stronger for him. He drew the correct graph and concluded that it was a function because it "passes the vertical line test and for every x there's only one y."

Most of the problems in the second interview were word problems. Griswald did not attempt to graph any item in order to solve it before the interview. However, in the interview it was clear that his problem solving heuristics, as related to the graphical representation, were strong.

Griswald's definition of a function in the third interview is based in three images: graph, vertical line test, and univalence (implicitly stated as the vertical line test). Two more images appeared in the protocol: equation and familiarity.

The equation image emerged associated with the components of a function (I3, 4a) and presented itself as functions are equations. In his discussion, he mentioned that he needed to see "the written out function and numbers." This implies a concrete conceptualization of functions. However, this conceptualization is not an obstacle to deal with arbitrary functions (Dirichlet's type, p. 315-316) or moving from the graphical representation into the algebraic representation (I3, 6, p. 316). The familiarity image emerged as knowledge of familiar functions (p. 303).

Griswald mentioned "problems on functions and how they relate different things," intercepts, asymptotes, behaviors, and extremes as the topics that the class had learned up to that point (I3, 1). His understanding of the importance of functions reduced to "really hard to find one, few story problems." Griswald did not "think [he] will ever run into one of those in [his] life." However, he said that spending so much time on functions was because "it helps to think...You can have more structured thought type thinking...you learn how to
think, how to go about those sort of things."

Griswald's description of a rational function with limited use of the calculator (I3, 5) made evident the weaknesses of his conceptual knowledge of functions (rich relationships) and little conceptualization of the problem (pp. 312-314).

Griswald provided a definition of a function previous to the fourth interview (3rd definition, p. 302). His definition is based only on the univalence criterion. He consistently used this definition in deciding that the relation that assigns its price to every clothing item was a function. In contrast, he used the equation image in the relation that assigns its area to each triangle (p. 317). During the fourth interview the equation and vertical line test images emerged.

The equation image appeared in what makes up a function (I4, 1) and manifested itself as functions are equations. Griswald's answer to this question was "a formula that for each x will produce one y." He was the only participant in the study who could deal with the arbitrary nature of functions and recognized a piecewise function as composed of one function and drew the correct graph (I4, 2, p. 317). He used the vertical line test to deduce that the graph was a function. The vertical line test appeared in other instances where Griswald dealt successfully with the arbitrary nature of functions. He also used the vertical line test with graphs having a discrete domain. For example with graphs having four points (I4, 3) or with the domain being the integer numbers (Q–02/19). In the first case, he applied the vertical line test to deduce that the first graph was a function, while the second one was not. In the second case, he just "plotted points...cause [he] knew it wasn't a line" (p. 317).

Two tasks in the interview were related to constructing graphs. In the first one, Griswald's description of the graph of a rational function using the calculator was poor (I4, 5). His description included domain, range (though incorrectly), and symmetry (which he referred as reflection). He also recognized the graph as a function by using the vertical line test. In the second task, Griswald was successful constructing the graph for the roller-
coaster problem. He was the only participant who did not draw any axes (I4, 6).

Griswald's fifth definition (p. 302) is more or less the same definition that he stated one month before (4th definition, p. 302). They both are stated in terms of the univalence feature. The equation, graph, vertical line test, familiarity, and continuity images emerged during the protocol of the last interview. The univalence feature emerged in Griswald's answer to the existence of a function that satisfies four algebraic conditions (I5, 6). He plotted the points and decided the relation was a function "cause for each x there's one y."

The equation image appeared as functions can be represented by equations and not all equations are functions. In the first case, he said that functions could be represented by graphs and equations (I5, 2, p. 309). In the second case, there was an implicit suggestion of not all functions being equations, when he provided nonexamples of functions in algebraic form (I5, 4 p. 309).

The graph image emerged as the second possible way to represent functions (I5, 4) and related to the vertical line test in two ways: graphs are functions if they pass the vertical line test, and as a means to decide whether a relation is a function or not (I5, 4). The vertical line test presented itself with two ideas: a means to decide whether or not a graph is a function (I5, 4, p. 304) and as a property of functions (I5, 2 p. 305). The familiarity image emerged in Griswald's knowledge of familiar nonexamples of functions (I5, 4). The continuity image appeared in the changes needed to make five points the graph of a function (I5, 6). Griswald proposed several changes join the points or move them before joining them. However, there is a taste of arbitrariness in moving the points. The other student who moved the points was Sara (p. 256). However, she arranged the points with a sense of regularity.

Griswald suggested introducing functions to precalculus students first in a graphical way (using the vertical line test) and then in the equation representation. He did not think the class was "taught the equation. [He thought] either you picked it up or you
didn't...There wasn't much time spent on it." He also suggested "to get rid of [the
calculators] unless it's going to become like a nationwide thing...if in colleges...you could
use it everywhere for all your classes...because they're not like readily accepted."

Griswald was the only student who said he did not learn much in the course and part of
it he blamed on the use of the calculator. However, I think that he learned many things but
he was not aware of them. He started with a concrete conceptualization of functions as
graphs that pass the vertical line test and by the end of the year his concept of function was
more independent of the equation image and tended to rely more on the univalence criterion.
Certainly, the tasks included in the interview allowed him to exhibit other images. Griswald
relied on the equation image when the task was unfamiliar and included numerical sets.

Summary of Griswald's Development and Knowledge of Functions

Griswald experienced two kinds of development of functions: procedural and
conceptual. The course emphasized the procedural ones and he benefited with powerful
algorithms. His grades exhibit his mastery of the procedures. A poor conceptualization of
the problem of describing the graph of a rational function with limited use of the calculator
(I3, 4), appeared once.

There is a conceptual network of images associated with functions. The strength of
this network is enhanced by the univalence criterion. Under this circumstance, Griswald can
deal with several situations where the arbitrary nature of functions is presented. Few
participants in the study could solve problems that involved arbitrary rules, domains, or
ranges. Nathalie was another student who dealt successfully with some of those items.
However, a distinction between Griswald and Nathalie is the time when they started dealing
with arbitrariness. Griswald started in the second interview (December), while Nathalie
started close to the end of the scholastic year. Griswald indeed seems to be close to the
reification stage of the concept of functions.
CHAPTER VI

DISCUSSION AND RECOMMENDATIONS

Go forth now. Go forth and question. Ask and listen. The world is just beginning to open up to you. Each person you question can take you into a new part of the world. For the person who is willing to ask and listen the world will always be new. The skilled questioner and attentive listener knows how to enter into another's experience. (From Halcolm's Epistemological Parables, cited in Patton, 1980, p. 278)

This research was designed to investigate the knowledge and development of the idea of functions among high-school students in a technology-enhanced precalculus class. A class participating in the C²PC project was the setting of this research. Eight students were selected from a class to conduct case studies of their knowledge and development of functions. Four research questions were investigated in this study:

1. What are the concept images and the concept definition of functions that students in this technology-enhanced precalculus class have?

2. How do students in a technology-enhanced precalculus class use functions?

3. What is the knowledge of functions that students in a technology-enhanced precalculus class have? Is their knowledge procedural or conceptual?

4. What are the stages students in a technology-enhanced precalculus class go through in their attainment of the concept of function?

The purpose of this chapter is discuss tentative answers to these questions. First, a brief description of the images students associated with functions is given. Next, a summary of each case is presented to establish the arena for the discussion of each research question. Each summary includes a directed graph to illustrate the network of function images and links between them of each student. Implications, recommendations, limitations of the study, and directions for future research conclude the chapter and the dissertation.
Students' Images of Functions during the Study

Nine function images were identified among the participants in this study: graph, vertical line test, univalence, equation/formula, familiarity, continuity, relationship, set of ordered pairs, and regularity.

The first image is the graph image, which refers to the graphical representation of a function. The second image is the vertical line test. Students used it as a means to recognize graphs of functions. The procedure was as follows. Given a graph, students would check that no vertical line intersects the graph more than once. In this case, the graph is a function.

The third image is the univalence image, which was the formal definition to introduce functions in class: One $y$ for every $x$. Students stated the univalence criterion sometimes as "one output for every input."

The fourth image is the equation/formula image, which refers to the algebraic representation of a function. Students' data suggest that they had in mind a 'chain' of variables and numbers (a formula), when referring to this image.

The fifth image is the familiarity image. In this case, all of the students attached a similar meaning to it: familiarity provides examples of functions and nonfunctions, and using this knowledge enables one to recognize functions. One of the students, Tyler, attached an extra meaning to this image: his experience. He recognized a function when he had seen or graphed a similar or identical graph. Otherwise, he would reject a function based on his experience.

The sixth image is the continuity image. Three meanings were associated with it. The first meaning, and the one used by the teacher, refers to the intuitive idea of drawing a graph
without lifting the pencil. The second meaning was a habit of connecting dots. The third meaning was observed in Jane. Sometimes she said that a continuous function was the graph of a function that "goes on" (e.g., it continues.)

The seventh image is the relationship image. This image emerged in two students only: Elizabeth and Carol. For Elizabeth, this image was the need of a pattern. She rejected graphs with a discrete set of points if she could not find a relationship among the points. By contrast, Carol used the relationship image as an operational conception of functions: the way that \( x \) and \( y \) are related.

The eighth image is the set of ordered pairs image. This image emerged in Nathalie only. For her, this image was a way to represent functions and an intermediate step to make translations from the algebraic representation to the graphical representation. A benefit of this image was the possibility to deal with discrete domains and ranges.

The last image is the regularity image. Regularity is the word that I use to describe the difficulties that students had dealing with the arbitrary nature of functions. In the discussion of the research questions, this is the meaning intended. In Steve and Sara's network this image was included since they showed a more explicit need for regularity. In Steve's case, it only emerged briefly in the second interview. In the case of Sara, it was more persistent; for example, by the end of the study she would connect a set of points in the given order and hence produce a graph of a non function, when the graph was a function.

These images are used in describing a student's network of function images. Each network is a dynamic and evolving model that contains the images students associated with functions throughout the study. Each network is a first approximation to a student's set of function images. Inclusion of each image was decided on its appearance in a student's approach to a task. The thickness of each boundary indicates from a very weak (very thin
line) to a very strong (very thick line) anchored image in each student. For example, in Elizabeth's network (figure 93) the univalence image is denoted by a very thin boundary while the relationship image is denoted by a very thick boundary, indicating that the relationship image was very strongly anchored in Elizabeth's work. By contrast the univalence image was very weakly anchored on her approaches to the tasks.

The use of arrows indicates a relationship between the images connected by the arrow. For example, an arrow starting at graph and ending in vertical line test is usually connected by "functions if." In this case, the relationship indicated in the student's network reads as "graphs are functions if they pass the vertical line test." An edge without arrow indicates also a relationship; it represents the nature of the task when the image emerged; except in the case of the edges joining the univalence and the vertical line test image. In this case, the edges indicate the student's recognition of such equivalence between these two statements.

The arrows starting at the familiarity image are labeled as "examples" and end usually in graphs, equations, and continuity. The arrows in this case indicate that students familiar examples were given as continuous graphs, and equations. An exception, to this situation is Tyler's network (figure 97) which has an arrow labeled "function if" from graph to familiarity. As stated above, this relationship, in the case of Tyler, reads as "the graph is a function if it is a familiar graph." Students' summaries follow.
Summary of Elizabeth's Case

Seven function images emerged in the data collected from Elizabeth during this study: equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, continuity, and relationship. Several links between these images suggest the existence of a network (figure 93). Elizabeth's network of function images is initially dominated by the relationship image, which supports a strong equation image. There is a very strong association between functions and equations throughout the study. By the end of the year, the association became Elizabeth's conceptualization of functions: functions are equations. Her conceptualization of a function usually lacks attention to domain, range, and the univalence criterion, and it specifically calls for a pattern (relationship). Hence it is not just incomplete but a cognitive obstacle in dealing with the arbitrary nature of functions.

The equation image appeared to be related to two images: graph and one output for every input. The graph seemed to be an intermediate step to recognize when an equation is a function either by passing the vertical line test or by having the univalence criterion. A graph was also a means to represent a function. The images of familiarity and continuity were related to the graph image; they appeared mainly in tasks involving graphical representations.

The link between the univalence image and the equation image is stated as not all equations are functions (since equations are functions if they have the univalence criterion). This link was weak and may be dominated by the image: functions are equations. Elizabeth had little difficulty stating the univalence criterion for functions, when needed. She recognized the vertical line test and the univalence criterion as equivalent. However, the strong tendency toward equations inhibited the use of either image even in graphical tasks.

The relationship (pattern) image was related to equations and graphs. If Elizabeth recognized a pattern in the task given, she would use the graph image (and the vertical line
test if this was the case). By contrast, if she did not recognize a pattern, she would rely on her equation image, which usually was an obstacle to deal with the absence of patterns. The relationship image also emerged in Elizabeth's work as a need for regularity. The equation image might be a facet of the relationship image instead of another image. Continuity might also reflect regularity.

Elizabeth's establishment of relationships in her network of functions images was constrained by the dominance of the equation relationship. Her conceptualization of functions as equations reflects a quasi-structural conceptualization and resembles the historical stage associated with Euler: the equation image has not emerged as insufficient to represent functions.

![Diagram](image)

Figure 93. Network of Elizabeth's function images.
Summary of Steve’s Case

Seven function images emerged in the data collected from Steve during this study: equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, continuity, and regularity. Several links between these images suggest the existence of a network (figure 94). Steve's network of function images is dominated by the equation image. There is a strong association between functions and equations throughout the study. By the end of the year, the association became Steve's conceptualization of functions: functions are equations. This conceptualization of a function at times lacks domain and range, or the univalence criterion, and hence is incomplete. Several cognitive obstacles in Steve's work are associated with this image: difficulties in dealing with different representations and with unfamiliar settings (domain, range, rule).

The equation image appeared related to two images: graph and one output for every input. The graph seemed to be an intermediate step to recognize when an equation is a function, either by passing the vertical line test or by having the univalence criterion. A graph was also a means to represent functions. The images of familiarity, regularity, and continuity were related to the graph image; they emerged mainly in tasks involving graphical representations. The regularity image emerged only once (recognizing the graph of a function). Both, continuity and regularity might be reflections of the equation image. The continuity image also became a cognitive obstacle when Steve was dealing with unfamiliar settings (discrete domains).

The link between the univalence and equation images is stated as not all equations are functions (since equations are functions if they have the univalence criterion). This link was weak and maybe dominated by the equation image. Steve exhibited difficulties stating the univalence criterion of functions, but after some reflection he could state it correctly. This might have either contributed to the dominance of the equation image or been a reflection of
the tug of war between the univalence, an emerging image, and the equation image. He recognized the vertical line test and the univalence criterion for functions as equivalent. However, the strong tendency to equations inhibited the use of either image even in graphical tasks.

Steve's establishment of links between his function images was constrained by the dominance of the equation image in his network of images. His conceptualization of functions as equations reflect a quasi structural conceptualization. However, he seems to be close to a tug of war between the univalence and the equation images. The war will take place not until the equation image emerges as insufficient to represent functions.

![Network of Steve's function images]

Figure 94. Network of Steve's function images.
Summary of Jane's Case

Six function images emerged in the data collected from Jane during this study: equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, and continuity. Several links between these images suggest the existence of a network (figure 95). Jane's network of function images reflects a tug of war between two images: equation and univalence. There is a strong tendency toward each image throughout the study. By the end of the year, however, the war favors the equation image which became Jane's conceptualization of functions: functions are equations. Such conceptualization of a function, in Jane's case lacks domain and range and hence is incomplete. Several cognitive obstacles in Jane's work were associated with this image: difficulties in dealing with different representations and with unfamiliar settings (domain, range, rule). However, the strong tendency to univalence permits Jane to deal with some unfamiliar tasks.

The equation image appeared related to two images: graph and univalence. The graph seemed to be an intermediate step used to recognize when an equation is a function, either by passing the vertical line test or by meeting the univalence criterion. Jane was one of the few students who mentioned the use of the vertical line test to produce graphs. This expanded her network of relationships. A graph was also a means to represent functions. The familiarity and continuity images where related to the graph image; they appeared mainly in tasks involving graphical representations. A distinction in the continuity image was a habit of connecting dots and not a tendency to regularity as other students showed.

The link between univalence and equations, stated as not all equations are functions, was strong. Jane had no difficulty stating the univalence criterion of functions; this might reflect the strength of this image. She also recognized the vertical line test and the univalence criterion as equivalent. She showed a stronger reliance on the vertical line test, as her network of images indicates.
Jane's establishment of links between her function images was constrained by the dominance of the equation image in algebraic tasks and in arbitrary non-numerical functions. Her conceptualization of functions as equations reflects a quasi-structural conceptualization. However, a tug of war is taking place between an emerging and each time stronger image–univalence–and the equation image. The tug of war favors the equation image until Jane realizes it is insufficient to represent functions.

Figure 95. Network of Jane's function images.
Summary of Carol's Case

Seven function images emerged in the data collected from Carol during this study: equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, continuity, and relationship. Several links between these images suggest the existence of a network (figure 96). Carol's network of function images reflects a tug of war between two images: equation and univalence. There is a strong tendency toward each of these images throughout the study. By the end of the year, the war was decided in favor of the equation image which became Carol's conceptualization of functions: functions are equations, although with a distinction based in the influence of the relationship image. This is a less contaminated quasi-structural conceptualization, since Carol's relationship image is purely operational. For her, the relationship image is a relationship between $x$ and $y$. Such conceptualization of a function at times lacks domain and range and hence is incomplete.

Several cognitive obstacles in Carol's work are associated with this image: difficulties with different representations and with unfamiliar settings (domain, range, rule). However, the strong tendency toward univalence permits Carol to deal with unfamiliar tasks.

The equation image appeared related to three images: graph, univalence, and relationship. The graph seemed to be an intermediate step to recognize when an equation is a function, either by passing the vertical line test or by having the univalence criterion. A graph was also a means to represent functions. The graph was also related to familiarity and continuity. Both emerged mainly in tasks involving graphical representations.

The link between univalence and equations—not all equations are functions—was strong. Carol had difficulties stating the univalence criterion for functions. This might have reflected a tug of war between these two images or contributed to the dominance of the equation image. Both, univalence and equations were related to the relationship image. If Carol perceived a pattern, she looked at the equation image; however, when she did not see a pattern, she turned to the univalence image. She recognized the vertical line test and the
univalence criterion for functions as equivalent.

A weakness in Carol's network is the dominance of the equation image in algebraic tasks or in arbitrary nonnumerical functions, hence conceptual knowledge of functions was constrained. Carol's conceptualization of functions as equations reflects a quasi-structural conceptualization. However, a tug of war is taking place between univalence and equations. The relationship image has cooperated in the emergence of the univalence image and will help it to win the war, when she realizes the limitations of equations to represent functions.

Figure 96. Network of Carol's function images.
Summary of Tyler's Case

Six function images emerged in the data collected from Tyler during this study: equation/formula, graph, vertical line test, one output for every input, familiarity, and continuity. Several links between these images suggest the existence of a network (figure 97). Tyler's network of function images is dominated by two images: graph and familiarity. There is a very strong association between functions and graphs throughout the study. By the end of the year, the association became Tyler's conceptualization of functions: functions are graphs. Such conceptualization of a function, in Tyler's case, lacks domain, range, or the criterion of one output for every input in terms of the vertical line test. Hence Tyler's conceptualization of functions was not just incomplete but a cognitive obstacle to deal with the arbitrary nature of functions.

The familiarity image was the other function image most anchored in Tyler's mind. He was the only student who attached a special meaning to this image. He rejected graphs of functions if he did not have the experience of having seen or graphed identical or similar graphs. The limitations of this image were notable at the beginning of the study when he even rejected graphs of straight lines because he did not remember if lines could be functions. This situation was compounded by the minimal work on lines that the class included. This image became an obstacle with unfamiliar graphs. By the end of the year, he still relied on this image to recognize graphs of functions defined on a discrete set of points.

The continuity image was related to graphs; it emerged in tasks involving graphical representations and presented itself as a matter of connecting dots. The vertical line test was connected to the graph image in two ways: as a means to recognize graphs and as a means to produce graphs. Only Tyler and Jane mentioned explicitly the second link in their function images.

Tyler was the only participant in this study who had function images isolated in his network of function images: the equation and univalence criterion were not connected to any
other function image. In particular, he did not recognize the vertical line test and the univalence criterion as equivalent at the end of the year.

During the study, Tyler was introduced to new procedures and new content, which encouraged a rearrangement of his network of function images. However, the establishment of linkages was slow and definitely incomplete as compared to other students. In particular, the dominance of the graph and familiarity images in the network was a limitation for conceptual development.

Tyler's conceptualization of functions as graphs reflects a quasi-structural conceptualization, which resembles an Eulerian conception of functions: a function is any curve drawn free in the x-y coordinate plane. Maybe the next stage for Tyler in his development of functions is equations. The univalence criterion and hence the arbitrary nature of functions seem to be far from him.

Figure 97. Network of Tyler's function images.
Summary of Sara's Case

Seven function images emerged in the data collected from Sara during this study: equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, regularity, and continuity. Several links between these images suggest the existence of a network (figure 98). Sara's network of function images is dominated by the equation image; she shows evidence of a strong association between functions and equations throughout the study. By the end of the year, the association became Sara's conceptualization of functions: functions are equations. Such conceptualization of functions at times lacks reference to domain and range, the univalence criterion, and at times calls for regularity. Hence, this conceptualization is not just incomplete, but a cognitive obstacle to deal with the arbitrary nature of functions.

The equation image was related to three images: graph, regularity, and univalence. The graph seemed to be an intermediate step to recognize when an equation was a function (by passing the vertical line test). A graph was also a means to represent functions. The images of familiarity and continuity were related to the graph image; they emerged mainly in tasks involving graphical representations. The continuity image emerged as a matter of connecting dots but also with some sense of regularity. Both continuity and regularity might be a reflection of the equation image.

The link between the univalence image and the equation images was stated as not all equations are functions (only if they meet the univalence criterion.) This link was weak and dominated by the equation image (functions are equations.) Sara experienced difficulties stating the univalence criterion for functions, recognizing the equivalence between the vertical line test and the univalence criterion, and often confused the vertical line test with the horizontal line test. This seems to suggest that univalence was an emerging image.

Several situations contrived Sara's conceptual development of functions: the dominance of the equation image, the inconsistency recognizing the vertical line test and the univalence
criterion as equivalent, and the confusion between the vertical line test and the horizontal line test.

Sara's conceptualization of functions as equations reflects a quasi-structural conceptualization. Her difficulties in stating the univalence criterion might be a sign of a forthcoming tug of war.

Figure 98. Network of Sara's function images.
Summary of Nathalie's Case

Seven function images emerged in the data collected from Nathalie during this study: equation/formula, graph, a set of ordered pairs, familiarity, continuity, vertical line test, and one output for every input. Several links between these images suggest the existence of a network (figure 99). Nathalie's network of function images reflects a tug of war between two images: equation and univalence. By the end of the year, the war was decided in favor of the equation image and became Nathalie's conceptualization of functions: functions are equations. Her conceptualization of a function, however lacks domain and range at times and hence is incomplete. Several cognitive obstacles in Nathalie's work are associated with this image: difficulties in dealing with different representations and with unfamiliar settings (domain, range, and rule). However, the strong tendency to univalence allowed Nathalie to deal with some unfamiliar tasks.

The equation image appeared related to three images: graph, set of ordered pairs, and univalence. Nathalie was the only student who has a link going from graph to equations (graphs come from equations.) All the students except Tyler had a link going from equations to graphs (an intermediate step to decide whether or not an equation is a function). A graph was a means to represent functions and appeared related to univalence (dealing with discrete graphs), familiarity, and continuity. These three images emerged mainly in tasks involving graphical representations.

The link between equations and set of ordered pairs was an intermediate step to move to the graphical representation. Nathalie was the only student who made strong emphasis on this translation process: given an equation, a set of ordered pairs is obtained, which in turn will provide a graph. Nathalie's perception of functions as set of ordered pairs also allowed her to deal with discrete domains. This situation is indicated with an arrow joining the set of ordered pairs image with the univalence feature--a set of ordered pairs is a function if there is one y for every x.
The link between univalence and equations—not all functions are equations—was strong; it was dominated by the equation image in unfamiliar tasks, but not in tasks involving a discrete set of points. Nathalie had difficulties expressing the univalence criterion. This might have reflected a tug of war between these two images or contributed to the dominance of the equation image. Although, Nathalie’s reversing of x’s and y’s is scattered in her work, she solved the tasks correctly. She recognized the vertical line test and the univalence criterion as equivalent.

Nathalie's network of function images was dominated by the equation image, imposing a weakness in the network. However, her strong tendency to univalence allowed her to deal with some unfamiliar tasks.

Nathalie's conceptualization of functions as equations reflects a quasi-structural conceptualization. However, a tug of war is taking place between univalence and equations. Maybe her meeting new functions will help her to decide the war in favor of the univalence criterion and reach the reification stage.

Fig. 99. Network of Nathalie's function images
Summary of Griswald's Case

Six function images emerged in the data collected from Griswald in this study: equation/formula, graph, vertical line test, one output for every input (univalence), familiarity, and continuity. Several links between these images suggest the existence of a network (figure 100). Griswald's network of function images reflects a tug of war between two images: equation and univalence. There is a strong tendency toward both images throughout the study. By the end of the year, this war was decided in favor of the equation image and became Griswald's conceptualization of functions: functions are equations. At times his conceptualization of functions lacked domain and range and hence it was incomplete. Griswald exhibited few cognitive obstacles (all related to the equation image) in the tasks presented. He relied on the equation image when unfamiliar tasks involved numerical sets. However, the strong tendency to univalence allowed him to deal with most of the tasks included in the study.

The equation image appeared related to two images: graph and univalence. The graph seemed to be an intermediate step toward recognizing when an equation was a function (by passing the vertical line test). A graph was also a means to represent functions and was related to familiarity and continuity. Both of these emerged mainly in tasks involving graphical representations.

The link between univalence and equations—not all equations are functions—was weak. Griswald was the only student who seemed to start making a link from univalence to equations. He knew "by seeing it" that an equation which gave two y's for one x would not be a function. He and Tyler were the only two students who did not have difficulties stating the univalence criterion. Tyler was far from a tug of war between univalence and equation. By contrast, Griswald seemed to be moving away from a quasi-structural conception (functions are equations), due to the strength of the univalence crierion. He recognized the vertical line test and the univalence criterion as equivalent.
Griswald exhibited a very strong tendency toward univalence during the study. This allowed him to deal with unfamiliar tasks that the rest of the students could not handle. Hence, his conceptual development of functions was stronger.

Griswald's conceptualization of functions is partially quasi-structural. His tendency to rely on the univalence criterion, puts him close to a reification stage. The leap, however, will not occur until he meets new arbitrary functions.

Fig. 100. Network of Griswald's function images.
Discussion

Discussion of tentative answers to the research questions investigated in this study follow.

1. What are the concept images and the concept definition of functions that students in this technology-enhanced precalculus class have?

Nine images associated with functions were observed in the participants in this study: graph, vertical line test, univalence, equation, familiarity, continuity, set of ordered pairs, relationship and regularity. The first six images were observed in all the students, although with different characteristics.

Collectively these images point to four main ideas: a concrete and quasi-structural conception of function, no consideration of domain and range, a tendency to regularity, and a separation between the algebraic and the graphical world.

Students' Conceptualization of Functions

Functions were introduced in this class using the univalence criterion: one y for every x. However, the implemented curriculum used this image little and provided extensive use of graphical and algebraic representations. This situation and the object-oriented language used by the teacher built the students' images of functions. Furthermore, the most tangible objects for functions were those that represented them: equations and graphs. This is a quasi-structural conceptualization since the representatives became the object that they were intended to represent (Sfard, 1989). Two quasi-structural conceptions were observed: graphs and equations. Both conceptualizations are concrete conceptions of functions. All the participants in the study showed a strong need "to see the equation or the graph" to describe a function. Furthermore, their concept of domain was limited to numerical sets. Even in tasks dealing with composition of functions, there was little evidence that students realized that the resulting composition, when it existed, was another function. However, they were familiarized with the "substitute" process of finding the composite function. These
situations lead to the conclusion that none of the students in this study exhibited an explicit conceptualization of a function as an object.

No Consideration of Domain and Range

As discussed above, students in this study had a concrete and quasi-structural conceptualization of functions. In particular, all of the students showed a strong tendency to equations. This situation was particularly evident in unfamiliar situations (discrete domain or rule). At the end of the year, all the students associated equations with functions to some degree.

For many students, the equation image of a function was characterized by focusing on the formula part without consideration of domain and range. As a consequence, several students ignored these in constructing graphs or in their definitions.

Tendency to Regularity

Students' tendency to regularity (connecting dots with a smooth continuous curve, seeking for a pattern, looking for an equation) can be explained in terms of their conceptualizations of functions. As discussed above, students in this study had a concrete conceptualization of functions, as equations mainly. Equations express some regularity and students might have abstracted this property from the extensive work with equations—a learner's natural tendency to overgeneralize from examples. Regularity, as opposed to the arbitrary nature of functions, created cognitive obstacles in unfamiliar settings (rule, domain, and range). For these students, regularity might be a facet of the equation image.

Separation between the Algebraic and the Graphical World

There are several instances that suggest a separation between the algebraic and the graphical world. Specifically, students recognized the univalence criterion and the vertical line test as equivalent, but had difficulties applying them. For example, when students were asked to decide if a set of points was a function, some students would try to find an equation, instead of applying the vertical line test to the points. Although some students
would apply the vertical line test successfully in this case, these same students had difficulties when the points were given algebraically. In this case, they tried to find an equation for the set of points instead of plotting the points and apply the vertical line test. This situation was apparent due to the fact that students' answers were driven by the most anchored image of the function concept. They relied on their function images, due to the absence of links between different function representations and images. Even when the links existed, students lacked recognition of such links. Discussion of students' function images in the light of cognitive obstacles follows.

**Historical Obstacles**

Four cognitive obstacles related to the historical development of functions were observed in students' work: piecewise and arbitrary functions, univalence, and no consideration of domain and range. All of them are related to some extent to the equation image.

**Piecewise Functions**

Difficulties students experienced with discontinuous functions resemble ideas of mathematicians of the 18th century, who did not recognize expressions such as

\[
f(x) = \begin{cases} 
  x^2 & \text{if } x \geq 0 \\
  3x & \text{if } x < 0.
\end{cases}
\]

as single functions, but as two functions. One possible explanation for students' difficulties with piecewise functions is the cognitive obstacle that the equation image generates: conceiving functions only as the algebraic expression without consideration of the domain and range. A student whose conceptualization of function is that of an equation, effectively sees two rules (i.e., two formulas and hence two functions). Difficulties with piecewise functions were observed moving from the algebraic representation to the graphic representation and vice versa.
Arbitrary Functions

Difficulties students experienced with arbitrary functions resemble also ideas of mathematicians of the 18th century, who did not recognize as functions, relations that were not given by an "analytical expression." Rejection of arbitrary functions also brings to mind the historical dispute over Dirichlet's definition of a function. As in the case of piecewise functions, a possible explanation for students' difficulties with arbitrary functions might be rooted in the equation image: Equations express, to some extent, regularity. A second explanation is the extensive work with 'standard and reasonable' examples of functions in class. The implemented curriculum did not include examples of functions defined with arbitrary domains, ranges, or rules; all the examples were limited to equations.

Univalence

Historically, univalence was an aid in dealing with the arbitrary nature of functions and allowed mathematicians to extend the concept of functions. Similarly some of the students who overcame the obstacles that the equation image created dealt successfully with some tasks involving arbitrary functions (mainly involving discrete domains and arbitrary rules). By contrast, students who could not overcome the obstacles that their equation image created could not deal with unfamiliar situations that involved arbitrary rules, domains or ranges.

No Consideration of Domain and Range

Historically, the concepts of domain and range emerged in the last stage of the development of functions and after mathematicians noticed that properties of functions were subject to the structures of the sets on which they are defined and where they take their values.

Although students were introduced to domain and range and used them to describe functions, some students were inconsistent in recognizing that the graph of a function on the set of real numbers, for example, would change when it is defined on the set of integer
numbers. Similar to the historical development of functions, students' recognition of domain and range might be also the last concept to appear in the development of the function concept.

**Psychological Obstacles**

Three function images emerge in some of the participants' work as psychological obstacles: relationship, familiarity, and continuity. Two students, Elizabeth and Carol, had a strong tendency to relationship, although with different meanings. For Elizabeth, the relationship image was expressed as the need for a pattern, by contrast for Carol the relationship image was a relationship between $x$ and $y$.

In the case of Elizabeth this image turned out to be an obstacle to deal with the arbitrary nature of functions, or where the relationship was not evident; she hesitated even in the case of a constant function. Elizabeth's work showed a strong link between equation and relationship images. Here, the relationship might be a different facet of the equation image.

In the case of Carol, the relationship image was an operational conception of function, and hence a healthy conception. However, as in the case of Elizabeth, it was strongly related to the equation image, which in turn emerged as an obstacle to deal with the arbitrary nature of function.

Familiarity was the second function image that emerged as a psychological obstacle, although only in one student. Tyler was the only student who attached an extra, and stronger, meaning to the familiarity image. He did not recognize the graph of a function if he had not seen a similar or identical graph. He rejected even straight lines as functions at the beginning of the course. At the end of the course, he used the vertical line test as a means to decide whether or not a graph was a function. However, he still regressed in unfamiliar graphs (e.g., set of dots) and relied on his experience to obtain an answer. Needless to say, there are limitations to his familiarity image.
The third psychological obstacle was continuity. Some of the students at the end of the year believed functions should be continuous. However, not all the students attached the same meaning to the notion of continuity. For example, Jane meant by a continuous graph "a graph that goes on." This notion of continuity seems to refer to the graph being defined over the set of all reals as opposed to the intuitive notion introduced in class of being able to draw the graph on a paper without lifting the pencil. To some extent the continuity image in all the participants was a matter of connecting dots. However, this image became an obstacle for some of the students to recognize functions defined over a discrete set of points or in making changes to a discrete set of dots to make them the graph of a function.

Pedagogical Obstacles

At the end of the study, six students conceptualized functions as equations, and one as graphs. These two conceptualizations are quasi-structural since the representative became the object that it intended to represent. These conceptualizations seem to be rooted in the nature of the examples presented in class. Excessive work with equations and graphs and the teacher's equation-graph oriented language might have contributed to shaping students' image of the function concept.

The nature of examples might have also contributed to two images: regularity and continuity. Although students were introduced to discontinuous functions, the examples were few, as compared to continuous functions, and presented in simplified contexts, hence when students abstracted properties of functions, they probably overgeneralized the idea of continuity. Moreover, the lack of consideration of arbitrary rules, domains, and ranges in the implemented curriculum created obstacles to deal with the arbitrary nature of functions.

2. How do students in a technology-enhanced precalculus class use functions?

Observations of the class showed emphases in applications. Students used graphing technology to explore topics that are usually taught in a calculus course: maxima, minima, zeros, intervals where a function is increasing or decreasing, end behavior, end behavior
model, and continuity. The graphing calculator facilitated a global exploration of graphs. It also amplified the scope of the problems (amount included, applications, and difficulty) and the content of a traditional precalculus curriculum. For example, students could solve inequalities with functions that are not usually included in a typical precalculus class. They also started dealing with advanced mathematical concepts, such as limits and continuity. Indeed the potential of the graphing calculator allowed them to deal with higher mathematical concepts without being introduced to the procedures that usually precede them.

However, the presentation of the content emphasized procedures instead of concepts and hence students' use of functions was mainly at the algorithmic level. This approach to functions was harmful to students' recognition of the importance of functions and to their establishment of rich conceptual relationships. Students recognized that functions were important, but the reason most of the participants gave was the time spent on the topic, rather than their broad scope and nature of the applications. It was not clear for the participants in this study where and when they were going to apply functions in "real life." Some of the students thought that functions might be useful in their college classes, but there was a sense of insecurity in them.

Limitations of the algorithmic approach to functions appeared in novel situations. For example, none of the students drew accurately the graph of the rational function \( f(x) = \frac{x^3 + 2x + 1}{2x^3 - 4} \), with limited use of the calculator (13, 5). In this case, the emphasis on algorithms in class was reflected in each of the students' approach to this task. In addition, some of the students' applications of algorithms were compounded with difficulties carrying out algebraic procedures.

3. What is the knowledge of functions that students in a technology-enhanced precalculus class have? Is their knowledge procedural or conceptual?
As was discussed above, the course emphasized algorithms. In this respect, students' knowledge of functions was procedural. Limitations of procedures without rich relationships appeared in unfamiliar tasks or where the procedure was not immediately applied. If we represent conceptual knowledge of functions (rich relationships) as a directed complete graph (multiedges are possible) where every vertex is a function image and each edge joining two images is a relationship, then every student missed important links in the network. This might be explained by the learner's resistance to conceptual change; knowledge became compartmentalized in a way such as not to interfere with existing knowledge (Hiebert, & Lefebvre, 1986). This situation was compounded by little work done to help students to make those connections. The lack of connections between procedures related to functions and the concept itself may have caused the students to apply the procedures in inappropriate ways.

4. What are the stages students in a technology-enhanced precalculus class go through in their attainment of the concept of function?

Students' answers to the pretest (Practice Test on Functions) exhibited function images similar to the ones associated at the end of the year. However, during the study a more clear and complete picture of each image was obtained. The equation image was the most dominant in each students' mind and became a quasi-structural conception in six students. Inevitably, all the students passed through the quasi-structural conception of function: a current and potential cognitive obstacle.

All students' conceptualization of functions were concrete: as graphs (Tyler) or as equations (the other seven participants). However, there were slight differences between students. Only Nathalie and Griswald appeared to be close to the reification stage at the end of the study.

The students appeared distributed along a continuum of function conceptualizations: from Tyler (as a graph) to Griswald (moving away from equations). This is in parallel with
the historical development of functions. Furthermore, the operational conceptualization was the first to develop. All the students identified at some point a function with its representation (quasi-structural conception). A structural conceptualization of functions was not explicitly observed in any of the students. It is necessary to remark, however, that there was no such intention by the teacher.

The Use of Technology in this Class

This study assumed a continuous use of technology; indeed, the class had ready access to graphing technology. Paradoxically, its use was continuous but not to its full potential. The teacher did not model the use of technology or provide an environment and tasks where the students could explore and recognize the power of the calculator. Furthermore, the teacher's use of the technology without explicit comments to its benefits might have shaped students' attitudes toward it. No work was done to help students realize the access to a broader and richer (conceptually) approach to precalculus. Unfortunately the best students in the study (e.g., Nathalie and Griswold) perceived the calculator as detrimental in their learning, work, and future. Griswold thought that he had not learned anything from the use of calculator. In addition, he felt that he was "cheating" when he used the calculator. Nathalie expressed mixed feelings (love and hate) and a fear toward tasks that involved graphing skills, since the college she was admitted to does not allow students the use of calculators in any tests. These feelings toward the technology were not exclusive to these students. Jane also perceived that the calculator had negative effects on her mathematical skills. In contrast, Tyler "loved" the use of the calculator in class. He felt more confident in his procedural skills. Carol admitted that she would have an advantage in college over students who had not used a calculator.

Although the students attitudes toward the technology vary, little recognition of its benefits appeared in the students. Furthermore, the limited range of function images observed in the study and students' lack of conceptual knowledge of functions should make
us rethink the implementation of technology. For example, Steve was the technology-oriented student in the study, but he did not solve the tasks any better than the other participants.

Although the study focused on students' development of functions, teacher's interviews and observations of the class permit statement of several conclusions regarding the use of technology in Mr. H.'s class. First, his perception of technology included also the overhead projector. Hence each time he used it, he perceived he was using the technology. Second, he recognized that the use of technology had affected his teaching style. He also recognized that further use of technology was going to continue affecting his teaching style.

Mr. H.'s attitudes toward the use of technology are an open question. Little use of technology in the class as described in the Standards (NCTM, 1989, 1991) might be a reflection of little training and the overcrowded curriculum to teach.

Implications

A theoretical contribution of this study is the theoretical framework developed and used in the research. The appropriateness of concept images and concept definitions to investigate the development of functions proved to be successful. Moreover, the use of qualitative methods helped the researcher to build and interpret students' network of relationships between function images.

A contribution of this research is a descriptive network of students' function images. A second contribution is a more detailed description of each function image. A third contribution is the addition of different meanings that students associated with functions (e.g., continuity). A fourth contribution is the finding of two more function images not described in the literature: set of ordered pairs and relationship. All these contributions allow an explanation for some of the difficulties and misconceptions that students have with the concept of function.
Research findings from other studies receive support from the findings in this study which are similar to those studies involving concept images and concept definition of functions (Even, 1989; Tall, & Vinner, 1981; Vinner, 1983; Vinner, & Dreyfus, 1989). Students relied primarily on their function images instead of their function definition and exhibited a limited range of images associated with functions.

Findings in this study about Tyler are partially similar to those reported by Ferrini-Mundi and Graham (1991) about their student named Sandy. Specifically the image of familiarity was the same in both students. The difference between these students is the context. Sandy is a college student from a traditional calculus course taught in a large lecture format, without the use of technology.

Even (1989) found that prospective secondary mathematics teachers have a narrow image of functions. This study found that an experienced teacher used an object-oriented language in teaching functions. Hence, even when teachers have a rich network of relationships, the way they implement the curriculum might be rooted in narrow function images and create a cycle.

Sierpinska (1988) found that extensive use of graphical representations promoted a quasi-structural conceptualization of functions (the graph is the function and not the functional relationship) and that this conceptualization created an obstacle to conceptualizing functions as explicit relationships between variables. Findings in this study are similar for other function images. The extensive use of graphical and algebraic representations promoted quasi-structural conceptualization of functions (the equation is the function). This conceptualization might be a potential obstacle to conceptualizing functions as objects. Although it has to be recognized, that in the context of the present study there was no intention for the students to conceptualize functions structurally.

The language and approaches to functions that participants in this study used support partially Rich's (1991) findings. Rich found that students in the C²PC project learned to
solve problems algebraically and graphically and developed a global approach to graphs. The same conclusions were observed in the participants in this study.

**Recommendations**

The *Standards* (NCTM, 1989) describe what the mathematics curriculum should include in terms of content priority and emphases. The reform proposed in the document is summarized as students gaining mathematical power. In the document, mathematical power "denotes an individual's abilities to explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems" (p. 5). In the document also, the function concept is recognized as a unifying concept in mathematics that should be emphasized in high school mathematics.

Several recommendations arise from findings in the present research in the spirit of the *Standards* (NCTM, 1989). First, conceptual understanding of functions might be promoted by helping students to make connections between their function images, since these connections do not occur spontaneously (Hiebert, & Lefebvre 1986). The networks of function images described in this research might be a good point to start to help students to recognize their function images, the links between them, and the advantages and limitations of each image and link.

This calls for the second recommendation. The *Standards* (NCTM, 1989, 1991) recognizes the teacher as the key factor in the reform proposed. Therefore, teachers need to be prepared to help students to establish relationships in students' networks of function images. Furthermore, it is the case that with the current style of teaching a quasi-structural conceptualization of functions is unavoidable. Both need to be considered in teacher education programs.

Third, curriculum developers of nontraditional curricula need to consider the object-oriented language that teachers use to implement an innovative program. In addition, there is a need to conduct formative and summative evaluations of teachers' implementation of
such programs. In particular, curriculum developers need to devise programs to constantly help teachers to recognize the potential of technology in the classroom. How can we expect teachers to implement a nontraditional program if they still regress to traditional ways of teaching even with the availability of technology?

**Limitations of the Study**

Limitations of this study include methods of collecting data, specific curriculum, and number of participants and tasks. Two methods of collecting data were specially limited: interviews and observations. In the first case, at times there was a need to have a guided interview in order to get more information on students' ideas about functions. It can be argued that this guiding might have suggested a specific way to approach the task and here students exhibited more clearly a particular function image (the one suggested by the guiding question). Second, students' interviews were constrained by time. They ranged from 30 to 45 minutes, since they were scheduled during students' free periods in the school. In particular, students were less willing to participate actively during the last interview. This situation may be due to the proximity of the end of the year; students' interest in class was also little at this time. In the case of the observations, they were limited to the structure of the class and by the location of the researcher in the classroom.

The curriculum is another limitation to the present study. This study focused on the learning of a specific definition of function in a specific curriculum, and not in a broad sense (e.g., pattern discovery or relationship between phenomena). In particular, the definition used by the teacher in this study was formal (one $y$ for every $x$). Related to this situation is also the implementation of technology in Mr. H.'s class. Certainly, the class was a technology-enhanced precalculus class. However, contrived use of technology as a tool impedes making or examining claims about the benefits of using technology in mathematics classrooms.
The number of participants impeded making a more detailed account of function images. For example, examination of testing materials suggested that students used several approaches (not necessarily correct) to solve a problem. Specifically, Tyler showed a tendency to linear interpolation in dealing with a quadratic function. No consideration of linearity or other images was made in the study due to the number of participants.

This study included few tasks to identify a structural conceptualization of functions. The overwhelming amount of questions related to an operational conceptualization of functions might be a reason that such a conceptualization was observed most commonly.

Finally, although the study provided a better picture of students' function images, it is not a complete picture of the network, but a first approximation.

Directions for Further Research

Several questions are proposed below as directions for further research. First, similar studies to this one need to be conducted with different teachers: Teachers that emphasize other representations of functions (verbal and numerical), teachers for whom using technology in the classroom is a first priority, and teachers with different years of experience. In any case, what kind of function images and obstacles emerge in students and which ones tend to decline? Do students in a technology-intensive class build different function images than students in a technology-enhanced class?

Second, a more comprehensive picture of students' network of function images needs to be devised. Studies attempting to improve the description of students' network of function images are needed. Are there more function images? How are they related? When do they emerge?

Third, studies like the present need to be conducted with students in elementary and middle schools. Specifically, which cognitive obstacles and function images are the first to arise? Why? When? In the same vein, similar studies need to be conducted with college students. Specifically, do new function images emerge? Do old function images tend to
disappear? Why? When?

At the college level, which function images become a cognitive obstacle, say dealing with functions of many variables (e.g., univalence always was stated as one y for every x, what happens with $f(x, y) = z$?)

Fourth, the interviews provided the students with an opportunity to think about different kinds of functions for the first time. Some students could not cope with those tasks, however, other students made evident a rearrangement of their network of function images and hence a development in their conception of functions. Further investigation of this idea needs to be developed. How can examples of functions that are not handled by the graphing calculator be incorporated in the implementation of nontraditional curricula?

To what extent students' images of functions reflect their teacher's function images is an important question that needs to be addressed. This study focused on students. Other students focussing on teachers and students are needed. Do students' network of function images parallel their teacher's network of function images? A similar question can be stated for students and curriculum. Do students' networks of function images parallel the implemented curriculum?

Finally, Kulm (1990) contends that students value what is tested; they consider important what they are being asked. This suggests that we should explore the kind of concept images that students build due to the wording of test questions. Specifically, what are the function images that teachers suggest in their testing materials? Are those images consistent with the images used in class?
APPENDIX A

OBSERVATION FORM
OBSERVATION FORM
High School
Class and topic: 2nd period, Precalculus
Length of class (minutes): ____________
Mr. H.
Date: ____________
# Females ____________ # Males ____________

PROGRAM SETTING/PHYSICAL, SEATING ARRANGEMENTS
Board ____________ Desk ____________

----- ----- ----- ----- ----- ____________
----- ----- ----- ----- ----- ____________
----- ----- ----- ----- ----- ____________
----- ----- ----- ----- ----- ____________
----- ----- ----- ----- ----- ____________

CLASSROOM ENVIRONMENT
Human, social environment, the way people organize themselves:

Attitudes

LESSON PRESENTATION

Beginning:
Activities and strict interactions
Tchr/stds__________ Studs/stds__________ Stud/technology__________

Informal interactions and unplanned activities

Native language of Program/Participants

Nonverbal communications:

Closure, signs
What is said?

How is completion related to the program?

EXAMPLES
How many? Tally them: ____________ Variety? Specify them: ____________
Utility? Clarify: ____________ Concreteness? ____________
Realistic? ____________
CONCEPTS VS PROCEDURES
Problem solving? Calculation?
Comprehension? Algorithmic thinking?
Mathematical thinking? Connections?
Reasoning: Estimation:

EMPHASIS IN CONNECTIONS (between & among)
Graphic Symbolic Numeric Verbal

EMPHASIS IN COMMUNICATION
Graphic Symbolic Numeric Verbal

USE OF MEDIA (Specify)

TEACHER
Body language?
Work on individuals?
Lecturer: Inquirer: Model: Coach:
Metaphors:

STUDENTS
Use of mathematical language
Problem Solvers: Questioners: Observers:
Body language?
Metaphors (write them down):

HOMEWORK
ACTIVITIES (time spent)
Lecture Small groups Tchr-led discussion of content Tchr-led discussion of studs reactions and concerns Tchr using audiovisual equipment Non-instructional time (roll, transitions) Testing:

What is it said?
How is it said?
When is it said?
Who said it?

Comments
APPENDIX B

ITEMS PROPOSED TO THE TEACHER
Proposed Items

1. (Q-01/29, 11). Find an equation for the relation graphed below:

![Graph with point (0, 7) and axes labeled x and y.]

Eqn: __________

Domain: __________

Range: __________

Find: \(f(2) = \) __________

\(f\left(\frac{1}{2}\right) = \) __________

Find \(x\) such that \(f(x) = 2\) __________

Is \(f(x)\) a function? __________

2. (Q-02/07, 5). Let \(f(x) = 5\) and \(h(x) = x\).

(A) Find \(f(h(x))\) __________

\(h(f(x))\) __________

Domain: __________

Domain: __________

Range: __________

Range: __________

Is this a fcn.? __________

Is this a fcn.? __________

3. (T-02/19, 7). Let \(f(x)\) be a relation defined only for integer numbers as:

\(f(x) = x^2\).

(A) Sketch a graph of \(f(x)\) at right:

(B) Domain: __________

(C) Range: __________

(D) Is \(f(x)\) a function? __________

4. (Q-03/04, 4). In your opinion: When a mathematical relationship is identified as being a "function", exactly what does that mean? Why is it important? (you may discuss the graph, if you feel that's important, but do not mention the vertical line test.)
APPENDIX C

PRACTICE TEST ON FUNCTIONS
PRACTICE TEST ON FUNCTIONS

This is a practice test. It will not affect your grade.

Please complete the following.

NAME:__________________________________________________________.
GRADE:__________________________
BIRTH DATE:______________
GENDER (circle one):

Male
Female

Please answer the following questions:

1. Have you used computers in any of your mathematics classes?__________.
   In which?______________________________________________________.
2. Have you used graphing calculators in any of your mathematics classes?______.
   In which?______________________________________________________.
3. If you have made a career choice in college, state it:______________________________.

Directions

You are permitted to use a calculator or a computer on this test. This test consists of 16 items. You will have at most 45 minutes to work on the test. Please give this test your best effort.

In each item, you are asked to answer a question and explain your answer, or draw a graph.

Please try to answer all items. There is no penalty for guessing. If you happen to get stuck on a particular item, move on and come back to it later. If you need to do scratch work, do so on the test paper itself. Please do your own work.

Return the test paper to your teacher when you are finished. Thank you for your cooperation.

PLEASE DO NOT TURN THIS PAGE UNTIL YOU ARE ASKED TO DO SO
Practice Test on Functions

1. Mark the correct statement and explain.
   a) The relation is a function
   b) The relation is not a function.

\[
\begin{array}{c}
\text{Explanation} \\
\end{array}
\]

2. Mark the correct statement and explain.
   a) The relation \( f \) is a function.
   b) The relation \( f \) is not a function.

\[
\begin{array}{c}
f : \{\text{real numbers}\} \rightarrow \{\text{real numbers}\} \\
\begin{array}{c}
x \rightarrow \\
5 & x < 0 \\
\end{array} \\
\end{array}
\]

\[
\begin{array}{c}
\text{Explanation} \\
\end{array}
\]
3. Sketch the graph of a relation that is a function and the graph of a relation that is not a function.

a) Function

b) Not a function

4. For each of the given points and the function represented by the graph, decide if it represents an image, preimage, a (preimage, image) pair, or a point that does not represent a (preimage, image) pair.
5. Given the function $f$:

$$f : \{\text{natural numbers}\} \rightarrow \{\text{natural numbers}\}$$

$$f(x) = 4x + 6$$

a) Which of the numbers 2, -1, 0, 11.5, 1267 is a preimage of $f$?

Explain your answers: ________________________________.

b) Which of the numbers -2, 10, 8, 46, 23 is an image under $f$?

Explain your answers: ________________________________.

c) Which of the following ordered pairs $(5, 26), (0.5, (2, 10))$ is a (preimage, image) pair of $f$?

Explain your answers: ________________________________.

6. For each of the graphs of the functions given, mark the elements of the range that are the images of the point $A$ in the domain.
7. For the function $g$

\[ g : \{ \text{real numbers} \} \rightarrow \{ \text{real numbers} \} \]

\[ g(x) = -7 \]

a) Complete the following

\[ g(4) = \quad g(-7) = \quad \]

\[ g(0) = \quad g(3.5) = \quad \]

b) Is there a real number $x$, such that $g(x) = 3$?

How many such $x$'s exist?

Explain

\[ \]

b) Is there a real number $x$ such that $g(x) = -7$?

How many such $x$'s exist?

Explain

\[ \]

8. Given the function $f$

\[ f : \{ \text{natural numbers} \} \rightarrow \{ \text{natural numbers} \} \]

\[ f(x) = 4x + 6 \]

For each of the following, decide whether it describes a function equal to $f$ and explain:

a) $g : \{ \text{real numbers} \} \rightarrow \{ \text{real numbers} \}$

\[ g(x) = 4x + 6 \]

Explanation:

\[ \]

\[ \]
b) \( g : (\text{natural numbers}) \rightarrow (\text{natural numbers}) \)

\[ g(x) = 2x + 3 \]

Explanation: ____________________________

c) 

![Graph](image1)

Explain: ____________________________

d) 

![Graph](image2)

Explain: ____________________________

9. Find the algebraic form of the function shown in the graph, specifying its domain and range.
10. Draw the graph of the function:

\[ g : \{\text{natural numbers}\} \rightarrow \{\text{natural numbers}\} \]

\[ g(x) = x - 2 \]

11. Indicate those graphs that represent a function with domain \( x \in (2, 6) \) and range \( y \in (-1, 4) \).
12. a) Give an example in algebraic form of a function from the real numbers to the natural numbers: ________________________.

b) The number of such functions is

. 0
. 1
. 2
. more than 2 but fewer than 10
. more than ten but not infinite
. infinite

Explain your answer: ________________________________.
13. a) In the given coordinate system draw the graph of a function such that the coordinates of each of the points A, B represent a preimage and the corresponding image of the function.

b) The number of different such function that can be drawn is

- 0
- 1
- 2
- more than 2 but fewer than 10
- more than ten but not infinite
- infinite

Explain your answer:
14. a) In the given coordinate system draw the graph of a function such that the coordinate of each of the points A, B, C, D, E, F represent a preimage and the corresponding image of the function.

b) The number of different such functions that can be drawn is

. 0
. 1
. 2
. more than 2 but fewer than 10
. more than ten but not infinite
. infinite

Explain your answer: ________________________________.
15. a) Given an example in algebraic form of a function $f$ for which

\[ f(3) = 4, f(6) = 7, \text{ and } f(8) = 13 \]

b) The number of different such examples is

. 0
. 1
. 2
. more than 2 but fewer than 10
. more than ten but not infinite
. infinite

Explain your answer:

16. State a definition of a function:

END OF THE TEST

You may go back and check your work if you wish
APPENDIX D

HANDOUT FOR EXTRA CREDITS
Handout for Extra Credits

NAME: ___________________________ DATE: ___________________________

The following questions are related to functions. Here is the deal. Try your best for every question. For credits give the correct answer AND a correct reason. The credits will be added to your next quiz. On the contrary, if your responses are incorrect or incomplete, your grade in the next quiz will not be affected.

1. Write a definition of function.

2. This item has two questions.

   a) Does the following table represent a function? Circle one choice.

      YES               NO

      Item            Price (in dollars)
      Blouse          $25.00
      Shirt           $26.00
      Pants           $23.00
      Shoes           $45.00

   b) Explain here your choice in part a.

3. Consider the following relation H that is given for all triangles. If T is a triangle then H(T) is equal to the area of T. For example, suppose that T₁ is a triangle with height equals 5 inches and basis equals 4 inches. Then $H(T₁) = \frac{5 \times 4}{2} = 10$ square inches.

   a) What is the domain of H?

   b) What is the range of H?

   c) Is H a function?

   d) Why? (explain here you answer to part c)
APPENDIX E

PROTOCOLS FOR STUDENTS' INTERVIEWS
Protocol Students: First Interview

2. Why are you taking this course?
3. What is the most important thing you have learned in this course?
4. What do you expect to learn in this course?
5. Tell me one new problem you can do now.
6. Suppose that I do not know anything about this course. If I ask you to describe the daily routine in the class, what would you say?
7. How do you feel in this class?
8. What activity (activities) do you like in this course the most?
9. Tell me new words you have learned in this course.
10. How do you like learning mathematics with a calculator?
11. Do you think that learning mathematics becomes 'easier' (better?) with a calculator?
12. What are the things you enjoy doing with the calculator?
14a. What ideas come to your mind, when you see (or hear) \( x \)?
14b. What ideas come to your mind, when you see (or hear) \( y \)?
14c. What ideas come to your mind, when you see (or hear) \( f(\text{of} \\ x) \)?
Students Second Interview

Does there exist a function whose graph is:

1. 

2. 

3. 

4. Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

5. Does there exist a function all of whose values are equal to each other?

6. Does there exist a function whose values for integer numbers are non integers numbers and whose values for non integer numbers are integer numbers?

7. Are equations and functions related?

8. What is a function, in your opinion?
Protocol Students: Third Interview

1. Tell me what you have been learning in this class.

2. Please tell me what is a function.

3. If I ask you "What is the importance of functions?" what would you say?

4. What are the components of a function?

   What are things (or features) you consider important to look for in a function?

5. Let's talk about the following relation:

   \[ f(x) = \frac{x^3 + 2x + 1}{2x^3 - 4} \]

Please tell me everything you can about its graph. Feel free to make any sketch. You can use your calculator at any moment, except for punching in the whole expression.

6. Find a rule (mathematical expression) for the following relation:
Students Fourth Interview

1. What makes up a function?

2. How many functions are involved in the relation \( f(x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
-1 & \text{if } x < 0 
\end{cases} \)

(dependring on what the student say, try the following question) Is it a function? why?

3. Is the set of four points displayed on the plane, a function?

   a)

   b)

Why?

4. Describe the graph of \( f(x) = \frac{x^2 + 1}{|x|} \). Say about it as much as you can.

(you may use your calculator).
5. This is a picture of a roller-coaster track.

6. I would like to know how you use your calculator, but not for precalculus-related activities. Tell me if you use it for other classes, how and what you use it for, and if you play around with it, exploration.

7. Tell me what you do when you are in a small group in precalculus: Check answers, explain problems, explore, discuss, discover things, nothing? You don't get together with anyone.
Students Fifth Interview

1. Tell me what a function is.

2. What properties do functions have?

3. How can functions be represented?

4. Are functions and graphs related? How?

5. How do you think functions should be explained to precalculus students next year (a class like the one you had and in the same school)?

6. What changes would you make in this graph, to make it the graph of a function?

7. Is there a function, such that \( f(2) = 0, f(-3) = 7, f(-4) = 1, \) and \( f(\frac{1}{2}) = -1 \)?

   Explain your answer.

8. (Items 2 and 3 from the handout for extra credits—see attached handout).
APPENDIX F

PROTOCOLS FOR TEACHER'S INTERVIEWS
Teacher's Protocol. First Interview.

Teacher's opinions on students' knowledge of functions.

1. (For this question probably what we need is to mention the emphases that the Chicago material puts on the concept of function) What is the idea of function that the student have at this point? What are the operations they are able to perform with functions? Which functions did they use? Did the students use a graphing calculator last year?

   The next set of questions refer to the use of the C²PC material. They all refer to function concepts and I tried to separate them just for convenience.

2. What is the idea of function that you expect the students to have acquired at the end of this course?

3. What are the concepts related to functions (image, range, domain, pair, rule of correspondence, relation) that you expect the students to have learned?

4. What are the particular connections between representations of functions (graphic, symbolic, numeric, verbal) that his course will emphasize? Will the students be able to move from one representation into another?

5. What are the activities related to function that the students will be able to perform with graphing calculators (solve problems, scales, find intersections)?

6. Given the graph of a function will students be able to come up with a symbolic expression for the function? Or at least recognize it in terms of reflections, shifts, translations, or maybe more precisely as compositions?

   Given the symbolic representation of a function will students be able to predict how the graph of the function might look like?

7. Will the students at the end of the course be able to recognize functions as objects that they can perform operations with (addition, subtraction, multiplication, quotient, compositions, inverses)?

8. Which properties of functions will the students be able to manipulate, recognize, express (constant, continuity, increasing, decreasing, extrema points—maxima, minima)?

9. What kind of functions will the students be able to manipulate, work with (polynomials, exponentials, logarithmic, rational, trigonometric)? Please, feel free to mention others you can think of.

10. What will be the technical vocabulary the students will use to describe behaviors of functions?

11. What other students' abilities do you think might be influenced by the use of graphing calculators (attitudes toward mathematics, for example)?
12. Are there any difficulties that the students will face by the use of graphing calculators (for example, the appearance of asymptotes that are not supposed to be on the screen)?

13. How general the concept of function will be? Say, what about discontinuous or piecewise functions?

14. What is mathematics?
Teacher's Protocol. Second Interview.

1. How did the school get involved in the C²PC project? How did you get involved in this project? When was that?

2. What kind of training did you get for this project?

3. Tell me what are the goals of this project?

4. Compare and contrast your teaching style before you used the graphing calculator with your current style (having the graphing calculator).

5. How do you like teaching with the graphing calculator?

6. In your opinion, what are the benefits that the students are getting by using graphing calculators?

   Are there areas, where you think that the graphing calculator is detrimental?

7. How would you feel if someone (with the authority) decided not to use the graphing calculators anymore in your precalculus class next year? How would you teach the course?

8. I have been telling you that most of the students I have been interviewing have the idea that functions are equations. Why do you think that is happening?
APPENDIX G

PROSPECTUS FOR PRELIMINARY RESEARCH
Prospectus for Preliminary Research

Title: Use, Understanding, and Development of the Concept of a Mathematical Function among Selected High School Students in a Graphing-facility Environment.

Submitted: Summer Quarter 1991

Date: August 23, 1991

Investigator: Armando Moisés Martínez Cruz

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Faculty Advisors: Dr. Suzanne K. Damarin Telephone: 292-4872

Dr. Patricia A. Brosnan Telephone: 292-1045

I. Statement of the Problem

This research is aimed to investigate the use, understanding, and development of the concept of a mathematical function among students in a graphing-facility environment. Students in the Computer and Calculator Precalculus Project, C2PC, (Demana & Waits, 1990) use graphing technology intensively, in particular graphing calculators, in their precalculus classes. Students in a graphing facility
II. Methodology

A. Population

The population is high school students enrolled in a precalculus class in the C²PC project. A sample of six students will be selected to participate in the study.

B. Design

This is a naturalistic study that will use a case study format. Qualitative methods such as prolonged engagement, persistent observations, triangulation, and referential adequacy materials will be used (Guba & Lincoln, 1982). A class currently participating in the C²PC project will be selected on the basis of availability. During the two first weeks of Autumn quarter, 1991, six students will be selected to participate in the study.

C. Instrumentation and Data

The researcher will be the “instrument” in this research. The researcher will be an observer in a class and will take notes on students use, will keep notes on what the teacher does so that students get certain ideas on mathematical functions. A reflexive journal with thick descriptions of the observations performed along the quarter will be kept.

A Practice Test on Functions (Markovits, et al., 1988; see Appendix) will be administered to the whole class to gather information on students entry skills. The selection of the six students will be based on their performance on the Practice Test on Functions, researcher observations, and teacher suggestions. Information on the
six students' use, understanding, and development of the concept of a mathematical function will be gathered through interviews. Each selected student will be interviewed two times during the quarter. In the first interview students will be asked to go over their answers to the Practice Test on Functions. In the second interview students will be asked to describe, use, apply, and manipulate mathematical functions in verbal, graphic, symbolic, and numeric representations. Both interviews will be audio-taped, and some of them video-taped.

The teacher will also be interviewed at the beginning of the quarter. In this case, the data collected will target two objectives: first, to triangulate data on students' entry skills and knowledge on functions and second, to determine teachers expectations regarding what students should know about functions at the end of the quarter. The interview will be audio-taped.

Referential adequacy materials such as students homework, quizzes, exams and scratchwork will be used to collect additional data on students' ideas of a mathematical function.

D. Analysis

A grounded theory that explains the facts observed is the goal of this study (Guba & Lincoln, 1982). In particular, relationships between conceptual and procedural knowledge about functions will be sought.

Strict anonymity and confidentiality will be maintained during the data analysis and reporting of the results.
III. Time Schedule

This research is proposed for Autumn quarter, 1991. The administration of the Practice Test on Functions will take a class period (45 minutes). Students' interviews will last approximately 45 to 50 minutes for each student. They will be interviewed during their free time, not during a scheduled class. The first interview will be scheduled several days after the administration of the test. The second interview will be scheduled four to five weeks after the first interview. The teacher's interview will be scheduled during the three first weeks of the quarter and will last approximately 45 minutes. The interview will be scheduled according to teacher's preferences.

It is expected that this research will be conducted during Autumn quarter, 1991. The district will receive a report of the findings of this research in Winter quarter, 1992.

References


Benefits to School Involved

1. The school will have the opportunity to examine the advantages of using computer technology in precalculus classes.

2. The school will have the opportunity to identify students' entry skills and knowledge of mathematical functions.

2. Students will have the opportunity to take a Practice Test on Functions.

3. Selected students will have the opportunity to develop their skills to communicate mathematically.

School District

XXXXXXXXXXXXXXXXXX (High School)
APPENDIX H

STUDENTS' DEMOGRAPHIC AND ACADEMIC DATA
Students' Demographic and Academic Data

Elizabeth

Birth Date: December 20, 1973

GPA 3.35 (1991-1992)
Rank 147 of 355

ACT Scores
December 1991
Mathematics 19
Elem/Alg. 10
Alg./Geom. 10
Geom./Trig. 9

Steve

Birth Date: July 29, 1973

GPA 3.29 (1991-1992)
Rank 162 of 355

ACT Scores
October 1991
Mathematics 27
Elem/Alg. 13
Alg./Geom. 13
Geom./Trig. 16

Jane

Birth Date: June 11, 1973

GPA 3.35 (1991-1992)
Rank 195 of 355

ACT Scores
October 1991
Mathematics 23
Elem/Alg. 10
Alg./Geom. 13
Geom./Trig. 12
Carol

Birth Date:  October 3, 1973

GPA 3.50 (1991-1992)
Rank 116 of 355
ACT Scores
December 1991
Mathematics 19
Elem/Alg. 10
Alg./Geom. 10
Geom./Trig. 9

Tyler

Birth Date:  September 14, 1973

GPA 3.54 (1991-1992)
Rank 116 of 355
ACT Scores
April 1991
Mathematics 21
Elem/Alg.
Alg./Geom.
Geom./Trig.

Sara

Birth Date:  January 9, 1974

Rank 188 of 355
ACT Scores
April 1991
Mathematics 21
Elem/Alg. 11
Alg./Geom. 11
Geom./Trig. 11
Nathalie

Birth Date: March 3, 1974

GPA 3.28 (1991-1992)
Rank 166 of 355
ACT Scores
June 1991
Mathematics 21
Elem/Alg. 12
Alg./Geom. 11
Geom./Trig. 10

Griswold

Birth Date: July 15, 1974

GPA 3.54 (1991-1992)
Rank 225 of 355
ACT Scores
February 1991
Mathematics 27
Elem/Alg. 15
Alg./Geom. 14
Geom./Trig. 13
APPENDIX I

SCHEDULE AND PLACE FOR STUDENTS' INTERVIEWS
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Note: The date is indicated by the initial of the day (M, T, W, T., etc.) and is followed by month and day.

Legend:
- NR stands for meeting room, and MO for multi-office.

Interview Settings:
- Steve
- Elizabeth
- Giswald
- Jane
- Carol
- Same
APPENDIX J

THE CONTENT OF THE COURSE
The Content of the Course. The Implemented Curriculum.

The following topics, related to functions, were covered in the course during the observations conducted for this study.

Translations between different representations of functions

- From algebra to graph (graphing calculator use) and vice versa.
- From sign charts to algebra and vice versa (polynomial functions only).
- From sign chart to graph and vice versa (polynomial functions only).

Applications: From story problems to algebraic representation to graphic representation.

- Solving equations and graphically (graphing calculator use).

Finding information of functions given algebraically or graphically

- Zeros, x- and y-intercepts, domain, range, asymptotes, location and values of extremes (local and absolute), end behavior and end behavior model, intervals where the functions are increasing/decreasing, evaluation at a given point, finding x such that f(x) equals a given y.

Connections between algebraic and graphic representations

- Zeros and x-intercepts; zeros and asymptotes; zeros and factors.

Composition of functions

- Evaluation at a given point, finding rules of compositions, few decompositions. Few applications.

Transformations

- Only pointwise, did not include rational functions here.

Types of functions

All these tasks were associated with functions in one or more of the following categories: Any polynomial functions up to degree five, rational functions up to degree 3, square root function.
References


Shumway, R. J. (submitted for publication). Computer programming as a context for the developmental study of the concept of quantified variable.


