A LOOK AT THE ANTENNA RADIATION PROBLEM
IN THE TIME DOMAIN

DISSERTATION

Presented in Partial Fulfillment of the Requirement for
the Degree Doctor of Philosophy in the Graduate
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By

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** *** **

The Ohio State University
1972

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1972
To:

"Time, Timidity and Tenacity"
ACKNOWLEDGMENTS

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pp. 278-279, 1 June 1966.

"Variable Focal Length Lenses using Materials with Intensity
Dependent Refractive Indices," Nature, Vol. 211, No. 5053,
pp. 1081-1082, 3 September 1966.

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Length → ℓ → dℓ
∫_ℓ closed line integral
∫_ℓ open line integral

Surface or Area → S → dS
∫_S closed surface integral
∫_S open surface integral

Volume → V → dV
∫_V Volume integral

\( \vec{A} \) = Vector Notation
\( \vec{r}^o \) = Unit vector notation
\( \vec{E} \) = Electric Field
\( \vec{H} \) = Magnetic Field
\( \vec{D} \) = Electric flux density
\( \vec{B} \) = Magnetic flux density
\( \varepsilon \) = Permittivity of medium = \( \varepsilon_0 \) for vacuum
\( \mu \) = Permeability of medium = \( \mu_0 \) for vacuum
\( v = (\mu \varepsilon)^{-\frac{1}{2}} \) velocity of electromagnetic wave in medium
\( c = (\mu \varepsilon_0)^{-\frac{1}{2}} \) velocity in vacuum
\( \xi = (\mu / \varepsilon)^{\frac{1}{2}} \) impedance of medium
LIST OF NOTATIONS USED (cont.)

\( \vec{\pi} \) = Hertzian vector potential
\( q \) = Total electric charge
\( \rho \) = Volume electric charge density
\( \rho_s \) = Surface electric charge density
\( \vec{I} \) = Total electric current
\( \vec{J} \) = Volume electric current density
\( \vec{J}_s \) = Surface electric current density
\( \vec{A} \) = Magnetic vector potential
\( \phi \) = Electric scalar potential
\( \sigma \) = Conductivity
\( p \) = \( \frac{v}{c} \)
\( \Delta \) = Length of one period
\( \omega \) = Angular frequency radians/sec
\([t]\) = \((t - R/c)\) Retarded time
CHAPTER I
INTRODUCTION

A. Background and Objectives

The antenna problem and the more general radiation problem, including scattering, historically developed in the frequency domain. This is quite natural since most radiation of interest is essentially harmonic in nature, of relatively narrow bandwidth and of long duration. One shortcoming of the concentration of effort in the frequency domain, however, has been an obscuring of some of the underlying principles contributing to the radiation processes as well as an undue complication of some problems which might more easily have been handled or, at least, more easily understood if explored in the time domain.

Until quite recently, for instance, little or no consideration had been given to the transient effects on the field patterns of radiating structures excited by aperiodic or nonharmonic signals. This was due, quite possibly, to the limited need for such detailed analysis, but more probably to the prodigious amount of work required by the Fourier methods employed to exact even an approximate solution to these problems. The Fast Fourier Transform method of computer
solution has and most certainly will aid here, but it may not aid in the understanding of the underlying principles involved.

It is the purpose here, therefore, to present the radiation problem from the time domain point of view and to develop the radiation equations and associated relationships purely in the time domain, resorting to Fourier techniques only for purposes of analogy and example. Solutions to specific problems are then outlined in order to illustrate this approach. It should be emphasized that this method is not intended to be a substitute for the traditional harmonic approach, but rather a supplementary method yielding useful engineering solutions for pattern response, pulse shapes, pulse duration, etc., particularly from radiating structures excited by aperiodic signals previously studied by no other means. The calculations can often be carried out by hand or by simple computer programs. Of even more importance than this, however, are the pedagogic possibilities this approach gives to the understanding of the radiation phenomenon.

The presentation, it is hoped, will be quite logical. Following a brief review of historic work in this area, the study will proceed quite naturally from a review of Maxwell's equations and the boundary conditions to a development of the radiation equations. Utilizing the Kirchhoff-Huygens principle the radiation equation in integral form is solved in the time domain. It is this relationship which is the basis of investigation for the remainder of the study. Single and multiple
aperture arrays as well as monopole and dipole elements are studied as to their pattern and signal responses to periodic and aperiodic excitations. At selected points throughout, the harmonic case is examined as a specific example to illustrate the corresponding relationships as they appear in the frequency domain. Where experimental results are available these will be presented and compared with theory.

B. Brief History of Transient Radiation Analysis

Probably the earliest look into the nature of transient radiation from linear conductors was made by Mannbeck, (1923), in his Ph.D. thesis[1] taking to task some earlier theories by Steinmetz. Mannbeck was concerned with two basic problems: the pulse distortion associated with the transients caused by large, rapid voltage and current variations, and the amount of power radiated by two wire transmission lines as compared to the ohmic loss on long distance high voltage power lines. Although Mannbeck's work neglected terms of $1/r^2$ and less, Schelkunoff[2] later showed it to be an exact solution to Maxwell's equations and to match the boundary conditions everywhere, including the surface of the conductor. It was shown that the radiation pattern is caused by discontinuities in the conductor, e.g., bends or changes in impedance. This is difficult to picture as being a correct explanation when only the steady state harmonic case is examined since the
radiating currents are zero at the tips of a cut dipole radiator, where the radiation is supposed to occur.

Little interest in the investigation of radiation from a time domain point of view was shown until 1960 when Polk[3] considered the problem as the Fourier transform of a wide band signal. The details of his work were not handled very well, as was pointed out by Mayo[3], but his fundamental approach and results were correct. This was essentially the first look at the aperture problem in the time domain and showed that here, for the uniform illumination case, the radiation pattern can again be considered as being an interference pattern caused by signals radiated from the edges or discontinuities of the aperture. Also considered was the transient time to initiate the main and subsequent lobes of the field pattern. Tseng and Cheng[4] and, independently, Chernovsov[5] considered this problem in more detail. Tseng and Cheng's work was two dimensional and, therefore, basically scaler, considering instantaneous uniform and non-uniform illumination cases and several different excitation signals. Chernovsov solved the three dimensional problem using the Kirchoff-Huygens principle to develop a quite general aperture integral in the time domain. In a subsequent paper[6] he used this relationship to determine the scattered pulse shape and pattern from a circular disk excited at an arbitrary angle by a unit step pulse.
All of the above references were concerned with the current on the radiating structure, and no attention was given to the exciting device, or any mutual effects in the structure-feed system, or any mutual effects between elements. During this same time, but independently, work was being conducted on the short pulse response of long single wires or monopoles. This work and the results of experiments conducted are summarized in a recent book by King and Harrison[7] in which they devote a chapter to this topic. Here a long thin conducting monopole is excited by a capacitor discharge pulse and the resulting current pulse moves down the radiating structure, reflecting alternately from the tip and the feed point. Radiation occurs each time the signal is reflected, and since the signal is so short that its length is only a fraction of the length of the monopole, the radiation phenomenon is quite easily seen. The theoretical analysis of the pattern response is quite straightforward, being a Fourier sum of the harmonics contained in the exciting voltage wave shape multiplied by the corresponding admittance for each harmonic component. Little time is devoted to the explanation of the computer techniques used, as this apparently was not felt by the authors to be their main contribution. The theory and experiment match quite well, showing very little pulse distortion in the first few reflections, except for attenuation, and the voltage and current pulse shapes are nearly identical.
This, of course, does not by any means represent all the work done in this area, but is intended only to summarize the highlights of what is felt to be the significant work pertaining to this study. Other pertinent references will be cited along the way.

One final comment is in order. The physical process of radiation by accelerating charged particles is fairly well understood, and its explanation can be found in almost any medium or advanced text on electricity and magnetism, cf., Panofsky and Phillips[8].

The analysis is made in the time domain and shows the radiated field to be due to charge acceleration or for circuit analysis the time derivative of the current (dI/dt). This implies then that the far field signal from a radiating structure should be expected to be in some way related to dI/dt and not the input signal I(t), and for an extended radiating structure a surface integral is, of course, involved.

In the harmonic case there is no problem. The time derivative is accomplished by multiplying the signal by jω, basically a scaling factor; the surface integration performed; and the field pattern obtained. The far field signal will have the same form as the input but with a scale and possible phase change which usually can be ignored. So for the harmonic case this matter of differentiation is really immaterial due to the unique nature of the harmonic function. All well and good, but what about the general time varying signal, what is its waveform in
the far field? What are the effects of differentiating the signal, of the surface integration, etc? These are pertinent questions when radar pulse durations of nano-seconds are being transmitted and the scattered returns are to be analyzed, and when these signals may no longer be single frequency harmonic signals but broadband capacitor discharge pulses. It is this question of the general nature of radiation from extended radiating structures which has intrigued the author and provided the main impetus for this study. If it is possible to stimulate some interest, provoke some thought, and afford some further insight, the work will have fulfilled its purpose. It is hoped also that the presentation is clear enough so that the reader is able to grasp a physical feel for the radiation process.
CHAPTER II

REVIEW OF THE BASIC CONCEPTS

Before beginning our study of the antenna radiation problem for current and field distributions that are arbitrary functions of time it is best to first consider the form and nature of some of the fundamental relationships which have been developed for the time harmonic case showing how they would appear in the time domain.

It is quite important that we understand how such concepts as the radiation integral, the pattern function, reciprocity, etc. are to be interpreted when applied to the general time case. Thus our development has been undertaken completely in the time domain so as to clearly illustrate which properties are fundamental and which are unique to the time harmonic function. We have attempted to do this as concisely and thoroughly as possible but without boring the reader. Thus of necessity many of the details and proofs have been placed in the Appendix. We begin our study then by looking at Maxwell's equations and their implications.
A. Maxwell's Equations and the Associated Vector Potentials

It is from Maxwell's equations in a source free region that we are able to obtain a clue as to how we might approach the antenna radiation problem. Here we have

(1)  
\begin{align*}
\text{a)} \quad \nabla \cdot \overrightarrow{D} &= 0 \\
\text{b)} \quad \nabla \cdot \overrightarrow{B} &= 0 \\
\text{c)} \quad \nabla \times \overrightarrow{E} + \frac{\partial \overrightarrow{B}}{\partial t} &= 0 \\
\text{d)} \quad \nabla \times \overrightarrow{H} - \frac{\partial \overrightarrow{D}}{\partial t} &= 0
\end{align*}

with

(2)  
\begin{align*}
\text{a)} \quad \overrightarrow{D}(t) &= \epsilon \overrightarrow{E}(t) \\
\text{b)} \quad \overrightarrow{B}(t) &= \mu \overrightarrow{H}(t)
\end{align*}

where \( \overrightarrow{B}, \overrightarrow{D}, \overrightarrow{E} \) and \( \overrightarrow{H} \) are arbitrary functions of time and \( \epsilon \) and \( \mu \) are the time independent permittivity and permeability, respectively, within this region.

Looking now at Eqs. (1a) and (1b) the usual procedure is to note that the divergence of the curl of a vector \( \overrightarrow{A} \) is zero \( (\nabla \cdot (\nabla \times \overrightarrow{A}) = 0) \) and postulate the existence of two vector potentials of the form

\[ \overrightarrow{E}^m = -\frac{1}{\epsilon} \nabla \times \overrightarrow{F} \quad \text{and} \quad \overrightarrow{H}^e = \frac{1}{\mu} \nabla \times \overrightarrow{A} \]

where we have employed Eqs. (2a) and (2b). The quantities \( \overrightarrow{F} \) and \( \overrightarrow{A} \) are the electric and magnetic vector potentials, respectively, and \( \overrightarrow{E}^m \) is referred to as the electric field of the magnetic type.
that is results from \( \overline{F} \), while \( \overline{H^e} \) is referred to as the magnetic field of the electric type, i.e., resulting from \( \overline{A} \).

Two companion equations can now readily be obtained, one for \( \overline{E^e} \) in terms of \( \overline{A} \) and a second for \( \overline{H^m} \) in terms of \( \overline{F} \). This is most easily done by substituting \( \overline{E^m} \) for \( \overline{E} \) in Eq. (1c) and \( \overline{H^e} \) for \( \overline{H} \) in Eq. (1d) and then integrating from \( t_0 \), a time early enough so that the system has not yet been turned on, to \( t \), the present time, yielding

\[
\overline{E^m} = -\frac{1}{\mu \epsilon} \int_{t_0}^{t} \nabla \times \nabla \times \overline{A} \, dt
\]

and

\[
\overline{H^e} = -\frac{1}{\mu \epsilon} \int_{t_0}^{t} \nabla \times \nabla \times \overline{F} \, dt
\]

where again we have used Eqs. (2a) and (2b). It is then obvious that a complete expression for \( \overline{E} \) and \( \overline{H} \) in terms of vector potentials would be of the form

\begin{align*}
(3) & \quad \overline{E} = \overline{E^e} + \overline{E^m} = -\frac{1}{\epsilon} \nabla \times \overline{F} - \frac{1}{\mu \epsilon} \int_{t_0}^{t} \nabla \times \nabla \times \overline{A} \, dt \\
& \quad \text{and} \\
(4) & \quad \overline{H} = \overline{H^m} + \overline{H^e} = \frac{1}{\mu} \nabla \times \overline{A} - \frac{1}{\mu \epsilon} \int_{t_0}^{t} \nabla \times \nabla \times \overline{F} \, dt
\end{align*}

These equations can be written in a slightly different but often more useful form by employing the vector identity
\[ \nabla \cdot (\nabla \vec{A}) = \nabla^2 \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A}) \]

in conjunction with the fact that the vector potentials can be shown to satisfy the homogeneous wave equation

\[ \nabla^2 \vec{A} = -\mu_\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} \]

Substituting the vector identity into the left side of the wave equation and again integrating from \( t_0 \) to \( t \) yields the identity

\[ \frac{1}{\mu_\epsilon} \nabla \int_{t_0}^{t} \nabla \cdot \vec{A} \, dt - \frac{\partial \vec{A}}{\partial t} = -\frac{1}{\mu_\epsilon} \int_{t_0}^{t} \nabla \times \nabla \times \vec{A} \, dt. \]

From this identity we see that Eqs. (3) and (4) can be expressed alternately as:

\[ (3') \quad \vec{E} = -\frac{1}{\epsilon} \nabla \times \vec{F} + \frac{1}{\mu_\epsilon} \nabla \int_{t_0}^{t} \nabla \cdot \vec{A} \, dt - \frac{\partial \vec{A}}{\partial t} \]

and

\[ (4') \quad \vec{H} = \frac{1}{\mu} \nabla \times \vec{A} + \frac{1}{\mu_\epsilon} \nabla \int_{t_0}^{t} \nabla \cdot \vec{F} \, dt - \frac{\partial \vec{F}}{\partial t} \]

It is these forms of Eqs. (3) and (4) which are usually referred to as the radiation or aperture integrals.

In order to use Eqs. (3') and (4'), however, it is necessary to know something of the nature of \( \vec{A} \) and \( \vec{F} \), that is, the manner in which they are related to the radiating source. We will state those
relationships here without proof (for details see Appendix, Section D).

Here we find

\begin{equation}
\overline{A}(\overline{r}, t) = \frac{\mu}{4\pi} \int_V \frac{\overline{J}(\overline{r}_s, t - R/c)}{R} \, dV
\end{equation}

and

\begin{equation}
\overline{F}(\overline{r}, t) = \frac{\varepsilon}{4\pi} \int_V \frac{\overline{M}(\overline{r}_s, t - R/c)}{R} \, dV
\end{equation}

where \(\overline{J}\) is volume electric current density on the radiating structure, \(\overline{M}\) is an equivalent volume current density of fictitious magnetic charges and \(R = |\overline{r} - \overline{r}_s|\). The quantity \(\overline{r}_s\) is the distance from the origin of coordinates to a point in \(dV\) in the volume enclosing the radiating source and \(\overline{r}\) is the distance from the origin of coordinates to the point of observation \(O(r)\) at which the antenna radiation field \(E, H\) is to be determined (see Fig. 1). The quantity \(t - R/c\) is defined as the "retarded time" which shifts the time reference from the radiation point to the point of observation \(O(r)\) and \(c\) is the velocity of light.

When Eqs. (5) and (6) are substituted into Eqs. (3') and (4') we have relationships which give the radiated \(E, H\) fields in terms of volume currents on the radiating structure. To illustrate that these \(E, H\) fields are the only ones which the currents \(\overline{J}\) and \(\overline{M}\) can radiate,
as well as showing forms for $\vec{A}$ and $\vec{F}$ in terms of more easily measurable quantities, we look next at the application of Huygens Principle to antenna theory.

B. $\textbf{The Huygens' Principle and the Radiation Integral}$

It was Huygens who in 1690 first proposed as an explanation for the bending of light at the edge of an obstacle the rule that each point on a wave front may be regarded as a new source of waves.

The adaptation of this principle to electric and magnetic fields is usually referred to as the equivalence principle.

Consider for instance the source or sources contained in a finite volume enclosed by the surface $S$ in a homogeneous isotropic
medium, usually free space. According to Huygens' principle we may remove the source and replace it by a current sheet coinciding with surface $S$. These currents must be such as to produce the same field distribution outside $S$ as existed with the original source. This current distribution is called an equivalent current distribution, and the fields outside $S$ may be evaluated in terms of these equivalent currents on $S$.

The equivalence principle follows from the uniqueness theorem and boundary condition relationships for electromagnetic fields (see Appendix, Section C). The uniqueness theorem states that the field in a region is uniquely determined by the sources within the region plus the tangential component of electric field $\mathbf{E}_S$ over a surface $S$ enclosing the region or the tangential component of magnetic field $\mathbf{H}_S$ over $S$ or tangential $\mathbf{E}_S$ over part of $S$ and tangential $\mathbf{H}_S$ over the remainder of $S$.

To illustrate let $\mathbf{E}, \mathbf{H}$ be the field outside the surface $S$ enclosing the source region in Fig. 2. Furthermore, let the field $\mathbf{E}, \mathbf{H}$ be the original field due to the sources within $S$, but with the field inside $S$ zero, that is, with the sources removed. In order to support this total field that is zero within $S$ and $\mathbf{E}, \mathbf{H}$ outside $S$, there must exist surface currents $\mathbf{J}_S$ and $\mathbf{M}_S$ on $S$ to account for the discontinuity in $\mathbf{E}$ and $\mathbf{H}$ at $S$. 
Fig. 2.--Illustration of Huygens' principle.

The relationships between the currents $\mathbf{J}_S$ and $\mathbf{M}_S$ and the tangential components of the $E$ and $H$ fields on the surface are found by applying the boundary condition criterion for electromagnetic fields at the surface $S$ (See Appendix, Section C) giving

\begin{align}
(7) \quad \mathbf{J}_S(r_S, t) &= \mathbf{n}^o \times \mathbf{H}(r_S, t) \\
(8) \quad \mathbf{M}_S(r_S, t) &= \mathbf{E}(r_S, t) \times \mathbf{n}^o
\end{align}

where $\mathbf{n}^o$ is the outward unit vector normal to the surface $S$.

With these relationships between surface fields and equivalent surface currents it is possible to write Eqs. (5) and (6) in the form
\begin{align}
(9) \quad \overline{A}(r, t) &= \frac{\mu}{4\pi} \oint_S \frac{\overline{n} \times \overline{H}(\overline{r}_S, t - R/c)}{R} \, dS \\
(10) \quad \overline{F}(r, t) &= -\frac{\epsilon}{4\pi} \oint_S \frac{\overline{n} \times \overline{E}(\overline{r}_S, t - R/c)}{R} \, dS
\end{align}

It is this form for $\overline{A}$ and $\overline{F}$ which we shall use next in our solution of the aperture integral.

Let us consider then a solution of Eqs. (3') and (4') in spherical coordinates and in the time domain. We will include here only those terms in $r$ which are of the order of $1/r$, i.e., excluding those terms of $1/r^2$ and higher. This then puts us outside of what is defined as the "Near Field Region" (see Appendix, Section E). When Eqs. (9) and (10) are substituted into Eqs. (3') and (4') and the vector operations performed in spherical coordinates the following expressions are obtained for the $E, H$ fields in terms of the surface fields $\overline{E}_S$ and $\overline{H}_S$ of Fig. 2:

\begin{align}
(11) \quad \overline{E}(\overline{r}, t) &= -\frac{1}{4\pi\epsilon r} \left[ \overline{r} \times \oint_S \left[ \overline{n} \times \frac{\partial \overline{E}(\overline{r}_S, [t]) + \overline{r} \cdot \overline{r}_S/c - \psi(\overline{r}_S)}{\partial [t]} \right] \right] \\
&\quad -\xi \left[ \overline{r} \times \left[ \overline{n} \times \frac{\partial \overline{H}(\overline{r}_S, [t]) + \overline{r} \cdot \overline{r}_S/c - \psi(\overline{r}_S)}{\partial [t]} \right] \right] \, dS
\end{align}
\[ (12) \quad \mathbf{H}(\mathbf{r}, t) = -\frac{1}{4\pi \varepsilon_0 r} \left[ \overrightarrow{\mathbf{r}} \times \oint_S \left\{ \frac{1}{\varsigma} \overrightarrow{\mathbf{r}} \times \left[ n_0 \times \frac{\partial \mathbf{E}(\mathbf{r}_S, [t]) + \mathbf{r}_S \cdot \mathbf{r}_S / c - \psi(\mathbf{r}_S)}{\partial [t]} \right] \right\} 
\right. \\
\left. + \left[ -\frac{n_0}{\partial [t]} \right] \right\} dS \]

where

\[ (13) \quad \psi(\mathbf{r}_S) = \left\{ -\frac{(\mathbf{r}_S \times \overrightarrow{\mathbf{r}})^2}{2rc} + \phi(\mathbf{r}_S) \right\} \]

The quantity \( \phi(\mathbf{r}_S) \) is an arbitrary phase function and retarded time \([t] = t - r/c\) is a parameter which does not depend on the coordinates in which the integration is performed. \( \varsigma = \sqrt{\mu / \varepsilon} \) is the impedance of the medium, usually free-space.

There is, of course, a distance beyond which the \( (\mathbf{r}_S \times \overrightarrow{\mathbf{r}})^2 / 2rc \) term can be neglected. This region is referred to as the far-field or Fraunhofer region, and it is this region which we shall consider from here on. The details of how Eqs. (11), (12) and (13) were obtained as well as a discussion of how the "Near Field", Fresnel zone and far field regions are defined in the time domain may be found in Appendix, Section E.
C. The Radiation Intensity and Energy Distribution Curves

One of the more useful concepts in antenna analysis is the "power density pattern" or just "power pattern". It is a graphical plot of the average radiated power at a particular radius \( r \) versus look angle. It is obtained from the far-field relationships for the \( \mathbf{E} \) and \( \mathbf{H} \) fields through the Poynting vector theorem

\[
S(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)
\]

where \( S(\mathbf{r}, t) \) is the instantaneous power per unit area. The power pattern for the time harmonic case is then simply a three dimensional plot of \( \frac{1}{2} S(\mathbf{r}, \theta, \phi) \).

Since we are in the far-field for these plots we see from Eqs. (11) and (12) that \( \mathbf{E} \) and \( \mathbf{H} \) are related quite simply by

\[
\mathbf{H}(\mathbf{r}, t) = \frac{1}{z_0} \left[ \mathbf{r} \times \mathbf{E}(\mathbf{r}, t) \right]
\]

where \( z_0 = \sqrt{\mu_0/\varepsilon_0} \) is the impedance of free-space. Thus we see that the power pattern can be obtained from either \( \mathbf{E} \) or \( \mathbf{H} \) alone since they are related through a constant. A graph of the single frequency rms values of \( \mathbf{E} \) or \( \mathbf{H} \) is called a "field plot". We shall see in the next chapter that in the time domain, however, the field plot must be presented in both space and time to be completely described.
When the power density is multiplied by the square of the radius $r$ at which it is measured, we obtain the power per unit solid angle or the radiation intensity, $P(\theta, \phi)$. For the single frequency case this would be $P(\theta, \phi) = r^2/2 \ S(\theta, \phi)$. It is this pattern which we will consider in some detail in Section III-C. To determine an equivalent pattern for the general time case is somewhat more involved. In the case of the aperiodic function we must compute the total energy radiated per unit solid angle which can be done by evaluating the integral

$$W(\theta, \phi) = \int_{T_1}^{T_2} r^2 S(r, \theta, \phi, t) \, dt$$  \hspace{1cm} (16a)$$

where $T = T_2 - T_1$, the duration of the radiation, and $W(\theta, \phi)$ is the "energy distribution function" for the particular aperiodic excitation on the radiating structure. For a periodic excitation on the other hand we need only perform the integration of Eq. (16a) over one period of the function $T = 1/f$ and then divide by that period thus obtaining

$$P(\theta, \phi) = \frac{1}{T} \ W(\theta, \phi)$$  \hspace{1cm} (16b)$$

the power pattern for any periodic excitation.

D. Reciprocity

According to the time harmonic reciprocity theorem for a linear, passive, bilateral system, the voltage produced in a detector
divided by the current applied at the source remains constant when source and detector are interchanged, as long as the frequency and all impedances are left unchanged. This theorem when applied to the electromagnetic antenna system has proven to be one of the most useful concepts in antenna theory in that it implies that the antenna pattern in the receiving mode is the same as in the transmitting mode.

Cheo[9] and others have shown that this theorem can be extended to the general time case and his proof as well as some illustrative examples have been included in Appendix, Section F. In this section, however, we wish to investigate the physical implications of the reciprocity theorem as it applies to the time domain case. We will do this by first looking at the time harmonic case and then logically proceed to the general time form.

Let us look then at the form of Cheo's reciprocity relationship as it appears in the frequency domain

\begin{equation}
(17a) \quad \int_{V_a} \overline{E}_a \cdot \overline{J}_b \, dv = \int_{V_b} \overline{E}_b \cdot \overline{J}_a \, dv.
\end{equation}

Equation (17a) is the familiar Rayleigh-Carson form of the reciprocity theorem the excitation \( e^{j\omega t} \) being understood. The quantities \( \overline{J}_a \) and \( \overline{J}_b \) are the electric current densities of the source (c.f. Fig. 3) when it is on the radiating structure (a) and when it is on the structure (b), respectively. The resulting electric field intensities at any point in
space is given by \( \mathbf{E}_a \) when the source is in (a) and similarly for \( \mathbf{E}_b \).

The integration would in general extend to all space, but it can of course be limited to the sources since \( J_a \) is only non-zero within volume \( V_a \) enclosing Ant. (a) and similarly for \( J_b \).

If it is possible to isolate the source terminals of the two radiating structures as in Fig. 3 then the integrals of Eq. (17a) can in principle be performed (see Appendix, Section F) yielding

\[ V_{a \text{ in } b} I_b = V_{b \text{ in } a} I_a \]  

the Rayleigh-Carson theorem in circuit form. \( I_a \) in Eq. (17b) is the current supplied by the source in (a) and \( V_b \text{ in } a \) is the open circuit voltage at the same terminals (a) when (b) is energized similarly for the left hand side. Equation (17b) can alternatively be written in the form...
\[ (17c) \quad \frac{V_a \text{ in } b}{I_a} = \frac{V_b \text{ in } a}{I_b} = Z_m(r, \phi, \phi') \]

where \( Z_m \) is a mutual impedance between Antenna a and b.

Equation (17c) expresses the reciprocity theorem in algebraic form, i.e., that the ratio of the source current on the transmitting antenna to the induced open circuit voltage in the receiving antenna remains the same when the roles of the antennas are reversed, as long as the frequency and the impedance remain unchanged.

To see how the reciprocity theorem would apply in the general time case let us look at the possible form of the mutual impedance \( Z_m \). It is obvious from Eq. (17c) that \( Z_m \) need not be real but may be complex. Indeed we find that it will in general be inductive in nature of the form \( Z_m = j\omega M_m \) where \( M_m \) is a mutual inductance which is a function of the coupling between the two antennas. This being the case we see that \( I_a \) must then be related to the induced voltage \( V_a \) through its derivative. This fact is of little consequence in the time harmonic case since the derivative of the harmonic \( e^{j\omega t} \) is \( j\omega e^{j\omega t} \), another harmonic weighted by \( \omega \) and shifted in phase by 90°. In the general time case we would, however, obtain an output waveform \( V_a \) which would in general not be the same as the source waveform \( I_a \).
For example, consider a typical circuit element which satisfies the reciprocity criterion of Eq. (17c), the ideal transformer. From physics we know that the voltage $V_a$ on one side is related to the current $I_a$ on the other as $V_a = M_m (dI_a/dt)$ which reduces to Eq. (17c) for harmonic excitation. Clearly the two waveforms will in general be different since we know a d.c. voltage on the transformer input would produce no current output. Thus we see that the reciprocity theorem does not say that the input and output waveforms will be the same. To see what it does say let us now look at Cheo's reciprocity relationship (the interested reader is directed to Appendix, Section F for details)

\begin{equation}
\int_{-\infty}^{\infty} dt \int_{V_a} dv \, \overline{J_a}(\overline{r}, \tau - t) \cdot \overline{E_b}(\overline{r}, t)
= \int_{-\infty}^{\infty} dt \int_{V_a} dv \, \overline{J_b}(\overline{r}, t) \cdot \overline{E_a}(\overline{r}, \tau - t)
\end{equation}

where $\overline{E_a}$ and $\overline{E_b}$ are the induced electric fields and $\overline{J_a}$ and $\overline{J_b}$ are the source current densities as in the time harmonic case but with the following conditions on $J_a$ and $J_b$

(A) $\overline{J_a}(\overline{r}, t) = \overline{J_b}(\overline{r}, t) = 0$, for $t < t_o (t_o > \infty)$ and all $\overline{r}$

(B) $\overline{J_a}(\overline{r}, t) = \overline{J_b}(\overline{r}, t) = 0$, for $|\overline{r}| > r_o (r_o < \infty)$ and all $t < \infty$.

In other words, we assume that our sources were turned on at some finite time ($t_o$) in the past and are of finite size ($\overline{r}$).
If we again assume that the input and output terminals of the radiating system can be isolated and the volume integration of Eq. (17) performed we obtain

\[(17d) \quad \int_{-\infty}^{\infty} V_a \text{ in } b(\tau - t) I_b(t) \, dt = \int_{-\infty}^{\infty} V_b \text{ in } a(t) I_a(\tau - t) \, dt\]

where \(V_a\) and \(V_b\) are the induced voltages and \(I_b\) and \(I_a\) are the input currents.

It can then be shown (see Appendix, Section F) that Eq. (17d) implies that there is a transfer function of the form \(Z_{b \text{ to } a}(\tau - t)\) for the left hand side, for instance, which relates the output voltage \(V_b \text{ in } a\) on \(a\) to the input \(I_a\) on \(a\) through the convolution integral

\[V_b \text{ in } a(\tau) = \int_{-\infty}^{\infty} I_a(t) Z_{b \text{ to } a}(\tau - t) \, dt\]

and likewise for the right hand side

\[V_a \text{ in } b(\tau) = \int_{-\infty}^{\infty} I_a(\tau - t) Z_{a \text{ to } b}(t) \, dt\]

Further, these transfer functions are the same, that is,

\[Z_{a \text{ to } b(\tau, \tau)} = Z_{b \text{ to } a(\tau, \tau)}\]

To illustrate what this means physically let us assume that Ant. (a) of Fig. 3 is excited by a current waveform \(I_a\). Then from above, the induced voltage on \(b\) will be \(V_a \text{ in } b\). Let us then assume that \(b\) is now excited by a current waveform \(I_b\) and that the induced
voltage on a is $V_{bina}$. Now if the current waveforms $I_a$ and $I_b$ are identical, the induced voltage waveforms $V_{ainb}$ and $V_{bina}$ will be identical but they will not in general be the same form as $I_a$ and $I_b$. Thus reciprocity in the general time case does not imply that the input and output waveshapes are identical only that the transfer function between a and b is the same as between b and a.

The transfer-function $Z(r,t)$ is in general a rather complicated function depending on the distance between and the relative orientations of the two antennas as well as the intervening media, etc. In the next chapter we shall look at the nature of the transfer-function for some simple radiating structures. We will in particular be looking at that portion of $Z(r,t)$ which involves the radiating antenna. Thus we will be concerned with the relationship between the input current waveform $I_a$ and the far-field waveform $E_a$. 

CHAPTER III

SOME PROPERTIES OF TIME-DEPENDENT RADIATION IN THE FAR-FIELD REGION

Now that we have developed the fundamental principles of antenna theory for the general time case we are ready to look at some specific radiating structures to determine their far-field response to arbitrary time signals. To do this we shall look at three familiar examples from the frequency domain: the rectangular aperture, the line source, and the phased array. Our primary interest will be to determine how and in what form such concepts and definitions as the field and power patterns, radiation resistance, etc. would carry over to the general time case. We will as well be alert to any other physical phenomena which might appear.

A. The Rectangular Source

Let us begin by considering the surface of a rectangular aperture antenna, such as that shown in Fig. 4, having linear dimensions a and b and coinciding with the yz plane so that the normal to the antenna coincides with the direction of the x axis. The far-field intensities E and H will then be found from Eq. (11) and Eq. (12) respectively.
Rewriting Eq. (11) in terms of surface currents (Eqs. (7) and (8)) we have

\[
\overline{E}(\overrightarrow{r}, t) = \frac{1}{4\pi c r} \oint_{S_a} \left\{ \overrightarrow{r} \times \left[ \frac{\partial \overline{J}_s(\overrightarrow{r}_s, [t])}{\partial [t]} + \frac{1}{c} \frac{\overrightarrow{r}_s \cdot \overrightarrow{r}_s - \phi(\overrightarrow{r}_s)}{\partial [t]} \right] \right\} + \\
+ \overrightarrow{r} \times \left[ \frac{\partial \overline{M}_s(\overrightarrow{r}_s, [t])}{\partial [t]} + \frac{1}{c} \frac{\overrightarrow{r}_s \cdot \overrightarrow{r}_s - \phi(\overrightarrow{r}_s)}{\partial [t]} \right] \right\} dS
\]

where \( \overline{\psi}(\overrightarrow{r}_s) = \frac{\phi(\overrightarrow{r}_s)}{c} \) and \([t] = t - \frac{\overrightarrow{r}_s}{c} \) in the far-field region. The surface currents \( \overline{J}_s \) and \( \overline{M}_s \) should now be separated into \( x \) and \( y \) components of the form

\[
\overline{J}_s(\overrightarrow{r}_s, t - \frac{\overrightarrow{R}}{c}) = \overline{J}_y(\overrightarrow{r}_s, t - \frac{\overrightarrow{R}}{c}) + \overline{J}_z(\overrightarrow{r}_s, t - \frac{\overrightarrow{R}}{c})
\]

thus enabling us to express the following elements of Eq. (18) in spherical coordinates

\[
\overrightarrow{r} \times \overline{J}_s = \overrightarrow{r} \times \left[ \overrightarrow{r} \times \overrightarrow{y} \right] J_y + \left[ \overrightarrow{r} \times \overrightarrow{z} \right] J_z
\]

\[
= \overrightarrow{r} \times \left\{ \overrightarrow{r} \sin \theta \sin \phi + \overrightarrow{r} \cos \theta \sin \phi + \overrightarrow{\phi} \cos \phi \right\} J_y + \\
+ \left[ \overrightarrow{r} \cos \theta - \overrightarrow{\phi} \sin \theta \right] J_z
\]

\[
= \overrightarrow{\phi} \cos \phi \sin \phi \overrightarrow{y} + \overrightarrow{\phi} \cos \phi \overrightarrow{z} J_y - \left[ \overrightarrow{\phi} \sin \theta \right] J_z
\]

\[
\overrightarrow{r} \times \overrightarrow{J}_s = \overrightarrow{r} \times \left[ \overrightarrow{r} \times \overrightarrow{y} \right] J_y + \left[ \overrightarrow{r} \times \overrightarrow{z} \right] J_z
\]

\[
= -\left[ \overrightarrow{\phi} \cos \phi \sin \phi + \overrightarrow{\phi} \cos \phi \right] J_y + \overrightarrow{\phi} \sin \theta J_z
\]

\[
\overrightarrow{r} \cdot \overrightarrow{s} = (\overrightarrow{x} \sin \theta \cos \phi + \overrightarrow{y} \sin \theta \sin \phi + \overrightarrow{z} \cos \theta) \cdot (\overrightarrow{y} \overrightarrow{y} + \overrightarrow{z} \overrightarrow{z})
\]

\[
= y \sin \theta \sin \phi + z \cos \theta.
\]
Substituting these equations back into Eq. (18) and noting the definite limits on the integral we have

\begin{equation}
\bar{E}(r, t) = \frac{1}{4\pi cr} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \{ \xi \left[ -[\bar{\theta}^0 \cos \theta \sin \phi + \bar{\phi}^0 \cos \phi] \right. \\
\left. + \bar{J}_y(y, z, [t]) + \bar{\theta}^0 \sin \theta \bar{J}_z(y, z, [t]) \right] \\
+ [\bar{\phi}^0 \cos \theta \sin \phi - \bar{\theta}^0 \cos \phi] \bar{M}_y(y, z, [t]) \\
- \bar{\phi}^0 \sin \theta \bar{M}_z(y, z, [t]) \} \}
\end{equation}

where

\[
\dot{J}_y(y, z, [t]) = \frac{\partial}{\partial [t]} J_y(y, z, [t]) + \frac{1}{c} (y \sin \theta \sin \phi \\
+ z \cos \phi - \phi(y, z)).
\]

In Eq. (23) we now have a quite general expression for the rectangular aperture valid for any type of excitation and current distribution. It does not give us much more physical insight than Eq. (11) except that we note there is no radial component as we suspected and we see that both \( M \) and \( J \) generate \( \theta \) and \( \phi \) field components. It is well at this point to consider a specific current distribution on our aperture and using Eq. (23) determine the resulting fields. The easiest and most illustrative to begin with is the uniform illumination case.

1. The Uniformly Illuminated Rectangular Aperture

Let us assume then that our rectangular aperture is illuminated by a TEM plane wave propagating in the x direction. Thus the field at
any point on the aperture surface at a given instant of time has the same value. We will further assume that the electric intensity vector \( \mathbf{E} \) of the antenna field coincides with the \( y \) axis, and the magnetic intensity vector \( \mathbf{H} \) of the field coincides with the direction of the \( z \) axis as shown in Fig. 4. Thus Eq. (23) becomes

\[
\overline{\mathbf{E}}(r, t) = \frac{1}{4\pi cr} \int_{-b/2}^{b/2} dz \int_{-a/2}^{a/2} dy \left\{ -\xi \left[ \Omega \cos \theta \sin \phi 
\right.
\right.
\]
\[
+ \phi^o \cos \phi \right] \rho_y(y, z, [t]) - \phi^o \sin \theta \dot{M}_z(y, z, [t]) \right\} .
\]

Since we assumed our antenna surface to be illuminated by a TEM wave the \( E \) and \( H \) fields on the surface are related quite simply through Eq. (15) as

\[
E_y = \xi H_z .
\]

Fig. 4.--Rectangular aperture.
From the surface equivalency equations we have

\[ (26) \quad \overline{m}_s = - [\overline{n}^0 \times \overline{E}_s] = - [\overline{x}^0 \times \overline{y}^0] E_y = - \overline{z}^0 E_y = \overline{z}^0 M_z \]

and

\[ (27) \quad \overline{j}_s = [\overline{n}^0 \times \overline{H}_s] = [\overline{x}^0 \times \overline{z}_0] H_z = - \overline{y}^0 H_z = \overline{y}^0 J_y \]

and we see that the magnitudes \( E_y, H_z, J_y, M_z \) are very simply related for this problem. Equation (24) can then be simplified by expressing the currents \( J_y \) and \( M_z \) in terms of a single variable.

Quite arbitrarily we shall choose to express the currents in terms of the surface electric current \( J_y \), giving for \( M_z \)

\[ (28) \quad M_z = \xi J_y \]

Substituting for \( M_z \) back into Eq. (24) and combining terms we have

\[ (29) \quad \overline{E}(r, t) = \frac{-\mu}{4\pi r} \int_{-b/2}^{b/2} dz \int_{-a/2}^{a/2} dy \left[ \overline{\phi}^0 \cos \theta \sin \phi + \overline{\phi}^0 \cos \phi \sin \theta \right] \frac{\partial}{\partial [t]} J_y(y, z, [t]) \]

Again from our assumption of a plane wave it is obvious that

the form of \( J_y \) must be

\[ (30) \quad J_y(y, z, [t]) = J_\circ f([t] + \alpha y + \beta z) = J_\circ f(t) \]
where $\alpha = \frac{1}{c} \sin \Theta \sin \phi$, $\beta = -\frac{1}{c} \cos \Theta$ and since the propagation is assumed to be along the x axis, $\phi(y,z) = 0$. $J_y$ is thus seen to be a function of $x$ and $y$ through the phase term only. All that remains then is to evaluate the integral of Eq. (29).

From the fact that $J_y(y,z,[t])$ is a traveling wave we have

$$
\frac{\partial J_y}{\partial [t]} = J_0 \frac{\partial f}{\partial [t]} = \frac{1}{\alpha} \frac{\partial J_y}{\partial y} = \frac{1}{\beta} \frac{\partial J_y}{\partial z}
$$

since

$$
\frac{\partial \{t\}}{\partial [t]} = 1, \quad \frac{\partial \{t\}}{\partial y} = \alpha, \quad \frac{\partial \{t\}}{\partial z} = \beta \quad \text{and} \quad \{t\} = \{ [t] + \alpha y + \beta z \}.
$$

Replacing the time derivative in the integral by a spatial derivative and at the same time changing the limits of integration we obtain

$$
(32) \quad \int_{-b/2}^{b/2} dz \int_{-a/2}^{a/2} dy \frac{\partial}{\partial [t]} J_y(y,z,[t])
$$

$$
= \int_{-b/2}^{b/2} dz \int_{-a/2}^{a/2} dy \frac{\partial}{\partial [t]} J_y(\alpha \frac{a}{z}, z, [t])
$$

$$
= \frac{1}{\alpha \beta} \int d\{t\} \int dJ_y(t) J_y(\{t\} + \alpha \frac{a}{z} + \beta z) J_y(\{t\} - \alpha \frac{a}{z} + \beta z)
$$
where we have made a second substitution for \( dz \) and a change of variables, transforming our spatial integral into a temporal integral. This simple transformation back and forth between space and time is a fundamental property of the traveling wave. The equation for \( \overline{E}(\tau, t) \) may now be expressed in the form

\[
(33) \quad \overline{E}(\tau, t) = \frac{-\mu_{ab}}{4\pi r} \frac{1}{4\tau_a \tau_b} \left[ \overline{\phi}^\circ \cos \theta \sin \phi + \overline{\phi}^\circ (\cos \phi + \sin \theta) \right] \\
\left\{ \begin{array}{l}
\left[ \frac{\partial}{\partial t} \right] \left[ t + \tau_a + \tau_b \right] \\
\left\langle J_y(t) \right\rangle \delta(t) \\
\left[ \frac{\partial}{\partial t} \right] \left[ t + \tau_a - \tau_b \right] \\
\left\langle J_y(t) \right\rangle \delta(t)
\end{array} \right\} - \int \left\langle J_y(t) \right\rangle \delta(t) \\
\left\{ \begin{array}{l}
\left[ \frac{\partial}{\partial t} \right] \left[ t + \tau_a + \tau_b \right] \\
\left\langle J_y(t) \right\rangle \delta(t) \\
\left[ \frac{\partial}{\partial t} \right] \left[ t + \tau_a - \tau_b \right] \\
\left\langle J_y(t) \right\rangle \delta(t)
\end{array} \right\}
\]

where \( \tau_a = \frac{a}{2c} \sin \theta \sin \phi \) and \( \tau_b = \frac{b}{2c} \cos \theta \).

If we express \( J_y(t) \) in terms of the surface charge density \( \rho_s(t) \) we have

\[
(34) \quad J_y(t) = \frac{\partial \rho_s}{\partial t}
\]

and Eq. (33) can be written

\[
(35) \quad \overline{E}(\tau, t) = \frac{-\mu_{ab}}{4\pi r} \frac{1}{4\tau_a \tau_b} \left[ \overline{\phi}^\circ \cos \theta \sin \phi + \overline{\phi}^\circ (\cos \phi + \sin \theta) \right] \\
\left\{ \rho_s([t] + \tau_a + \tau_b) - \rho_s([t] + \tau_a - \tau_b) \\
\right. \\
\left. - \rho_s([t] - \tau_a + \tau_b) + \rho_s([t] - \tau_a - \tau_b) \right\}.
\]
Equation (35) is valid for any temporal function \( \rho_S(t) \) no matter how complicated and it shows the far-field \( \overline{E}(r, t) \) to result from four inhomogeneous spherical waves originating from the four corners of the rectangular aperture. This is rather interesting since the far-field of a short dipole results from the second derivative with respect to time of the source charge function. (See Appendix). It might be beneficial at this point to test Eq. (33) with the harmonic function and see if it renders the proper field-pattern.

For the harmonic case let

\[
(36) \quad J(t) = J_0 \cos \omega t = J_0 \Re \{ e^{j\omega t} \} \quad \text{or} \quad \rho_S(t) = J_0 \Re \frac{1}{j\omega} e^{j\omega t}
\]

Substituting into either Eq. (33) or Eq. (35), the real part being understood, we have

\[
(37) \quad \overline{E}(r, t) = \frac{-\mu_{ab} J_0}{4\pi r} \frac{e^{j\omega t}}{4 \tau_a \tau_b} \left[ \overline{\theta} \cos \theta \sin \phi + \overline{\phi} \left( \cos \phi + \sin \theta \right) \right]
\]

\[
\frac{1}{j\omega} \left\{ e^{j\omega(\tau_a + \tau_b)} - e^{j\omega(\tau_a - \tau_b)} - e^{-j\omega(\tau_a - \tau_b)} + e^{-j\omega(\tau_a + \tau_b)} \right\}
\]

\[
= \frac{-\mu_{ab} J_0}{4\pi r} \frac{e^{j\omega t}}{j\omega \tau_a \tau_b} \left[ \overline{\theta} \cos \theta \sin \phi + \overline{\phi} \left( \cos \phi + \sin \theta \right) \right]
\]

\[
\left\{ e^{j\omega \tau_a} - e^{-j\omega \tau_a} \right\} \left\{ e^{j\omega \tau_b} - e^{-j\omega \tau_b} \right\}
\]
$$\frac{-j \omega \mu}{4 \pi r} ab J_0 e^{j \omega t} \left[ \overline{\theta^o \cos \theta \sin \phi + \overline{\varphi^o} \cos \phi + \sin \theta} \right] \left[ \begin{array}{c} \sin \left( \frac{ka}{2} \sin \theta \sin \phi \right) \\ \frac{ka}{2} (\sin \theta \sin \phi) \end{array} \right] \left[ \begin{array}{c} \sin \left( \frac{ka}{2} \cos \theta \right) \\ \frac{ka}{2} \cos \theta \end{array} \right]$$

where \( k = \frac{2\pi}{\lambda} \), \( \omega[t] = \omega t - kr \), \( \omega \tau_a = \frac{ka}{2} \sin \theta \sin \phi \) and \( \omega \tau_b = \frac{kb}{2} \cos \theta \), which is of course the familiar far-field pattern of a uniformly illuminated harmonically excited planar rectangular aperture (c.f. Wolff[10]). Thus Eq. (35) does indeed reduce properly for the harmonic case.

Let us now look at the radiation field at points of space which lie in the symmetry planes (the xy and xz planes). All of the points of the surface xy, for instance, have the spherical coordinate \( \theta = 90^\circ \) and therefore the integral of Eq. (33) becomes

$$\lim_{\tau_b \to 0} \frac{1}{4 \tau_a \tau_b} \int \frac{[t] + \tau_a + \tau_b}{[t] + \tau_a - \tau_b} J_Y(t) d(t)$$

$$= \lim_{\tau_b \to 0} \frac{1}{2 \tau_a} J_Y([t] + \tau_a) \left\{ \frac{\tau_b - (-\tau_b)}{2\tau_b} \right\}$$

$$= \frac{1}{2 \tau_a} J_Y([t] + \tau_a) ,$$
where we have used the "mean-value theorem for integrals" (c.f., Taylor[11]). We could just have easily used Eq. (35) to obtain Eq. (38) but here we would have used the definition of the derivative (c.f., Taylor[12]).

In the symmetry plane xy the form of Eq. (33) thus becomes

\[
\overline{E}_{xy}(r, t) = - \phi^0 \frac{\mu ab(1 + \cos \phi)}{4\pi r} \frac{1}{2\tau_a} \left\{ J_y([t] + \tau_a) - J_y([t] - \tau_a) \right\}
\]

where \( \tau_a = \frac{a}{2c} \sin \phi \). Similarly for the radiation field in the xz plane we obtain by setting \( \phi = 0 \).

\[
\overline{E}_{xz}(r, t) = - \phi^0 \frac{\mu ab(1 + \sin \theta)}{4\pi r} \frac{1}{2\tau_b} \left\{ J_y([t] + \tau_b) - J_y([t] - \tau_b) \right\}
\]

where \( \tau_b = \frac{b}{2c} \cos \theta \). This special problem was first investigated by Chernousov[5] by solving Eq. (29) in only the xy and xz planes and thus obtaining Eqs. (39) and (40) directly.

We see from Eqs. (39) and (40) that the response of our aperture in the symmetry planes is again expressed in a very simple manner directly in terms of the temporal function \( J_y(t) \) with again no integration or differentiation required. Here the far-field radiation results from the superposition of two inhomogeneous spherical waves outgoing from the antenna edges, and as in Eq. (35) these
waves coincide in form with the signal waveform exciting the antenna. The character of this dependence is determined, in Eq. (39) for example, by the product between the factor \((1 + \cos \phi)/(1/2 \tau_a)\) and the quantity contained within the braces.

Let us look again at the harmonic case. Here Eq. (39), for example, takes the form

\[
\overline{E}_{xy}(\overline{r}, t) = - \phi_0 \frac{\mu \alpha J_0 (1 + \cos \phi)}{4\pi r} \frac{1}{2\tau_a} e^{j\omega t} \left\{ e^{j\omega \tau_a} - e^{-j\omega \tau_a} \right\}
\]

\[
= - \phi_0 \frac{j \omega \mu \alpha J_0 \alpha e^{-jkr}}{4\pi r} \left( 1 + \cos \phi \right) \left\{ \frac{\sin \omega \tau_a}{\omega \tau_a} \right\}
\]

\[
= - \phi_0 j e^{j\omega t} \frac{\mu \alpha J_0 e^{-jkr}}{2\lambda r} \left( 1 + \cos \phi \right) \left\{ \frac{\sin \left( \frac{ka}{2} \sin \phi \right)}{\left( \frac{ka}{2} \sin \phi \right)} \right\}.
\]

which as we see from Eq. (37) is indeed the far-field pattern in the equatorial plane as we expected. And we see that Eqs. (39) and (40) do correctly predict the field \(\overline{E}(\overline{r}, t)\) for the harmonic case.

We should by now be able to guess at the form of \(\overline{E}(\overline{r}, t)\) for points along the axis normal to the aperture (the x-axis). We might well expect the expression to be similar in form to that for the short-dipole as indeed we shall see it is.
Looking at Eq. (39) for instance, we see that in the limit as

\[ \phi \to 0 \] we have

\[
E_x(r, t) = -\frac{\phi^0}{2\pi r} \lim_{\tau_a \to 0} \frac{J_y(t + \tau_a) - J_y(t - \tau_a)}{2\tau_a}
\]

\[ = -\frac{\phi^0}{2\pi r} \frac{d}{d[t]} J_y(t) \]

where we have here used the definition of the derivative to arrive at our result. In Eq. (42) the temporal derivative appears explicitly and we note that the dimensions and thus the geometry of the aperture do not affect the form of \( E(r, t) \) along the normal axis, only its magnitude.

In summary we see that only along the normal axis to the aperture (the axis for which \( \tau_a = 0 \) in this case) do we receive the time derivative of the exciting signal \( J(t) \). At all other points the received waveform is some combination of the signal or its time integral. We shall later investigate in more detail some of the properties of these functions.

2. The line source

We continue our investigation of the radiating properties of specific geometric structures by looking next at the dual of the narrow rectangular aperture, the line source shown in Fig. 5. Since we
are still primarily interested in the far-field region we will again use Eq. (23) which for this structure takes the form

\begin{equation}
\overline{E}(r, t) = \overline{E}_\theta(r, t) = \theta^0 \frac{\xi \sin \theta}{4\pi cr} \int_{-b/2}^{b/2} \frac{\partial}{\partial [t]} I_z(z,[t]) \, dz
\end{equation}

where

\[ I_z(z,[t]) = \int_{-a/2}^{a/2} J_z(y, z,[t]) \, dy \]

and \( J_y = M_y = M_z = 0 \). We note from the derivation of Eq. (23) that we are here approximating our wire by a thin strip. This is, however, a justifiable assumption since for this example we are assuming a wire of negligible diameter (\( a \ll b \)). Let us now assume that our line, instead of being uniformly illuminated, is fed from the \(-b/2\) end
by a signal which travels in the +z direction with a velocity \( v \). The relationship for the current thus has the form

\[
I_Z(z, [t]) = I_0 f([t] + \frac{1}{c} [z \cos \theta - \phi(z)]) = I_Z([t])
\]

where \( \phi(z) = z/p, \ v = pc[13] \). Again from the wave nature of Eq. (44) we have

\[
\frac{\partial}{\partial [t]} I_Z([t]) = \frac{c}{\cos \theta - \frac{1}{p}} \frac{\partial}{\partial z} I_Z([t])
\]

and Eq. (43) becomes

\[
\bar{E}_d(x, t) = \bar{\theta}^0 \frac{\zeta \sin \theta}{4\pi r \left(\cos \theta - \frac{1}{p}\right)} \left( \int \frac{I_Z([t] + \tau_p)}{I_Z([t] - \tau_p)} \right)
\]

\[
= \bar{\theta}^0 \frac{\mu b \sin \theta}{8\pi r \tau_p} \left( I_Z([t] + \tau_p) - I_Z([t] - \tau_p) \right)
\]

where \( \tau_p = b/2c (\cos \theta - 1/p) \) and \( \mu = \zeta/c \). Thus Eq. (46) for the end-fed line source is identical to Eqs. (39) and (40) for the equatorial planes of the uniformly or instantaneously illuminated aperture case except for the \( \sin \theta \) and \( 1/p \) phase terms.

It should be apparent that the uniformly illuminated situation is only a special case where the phase velocity \( v = \infty \) and therefore \( 1/p = 0 \). The \( 1 + \cos \phi \) term appears in place of \( \sin \theta \) in Eqs. (39) and (40) because in these two equations we have included
both the electric and the magnetic dipoles which are oriented at 
right angles to each other. And thus for our example one is iso-
tropic in the xy plane while the other is isotropic in the xz plane.

Looking again at Fig. 5 for the line source it should be apparent 
that the symmetry of the bracketed portion of Eq. (46) results from 
our selection of the coordinate center and that by moving the center 
to -b/2 the bracketed portion would take the form \( I_z(\lfloor t \rfloor + 2\tau_p) \) 
- \( I_z(\lfloor t \rfloor) \). This is the usual coordinate reference for end-fed 
radiators. The far-field \( \mathbf{E}(r, t) \) for the harmonic case thus becomes

\[
(47) \quad \mathbf{E}(r, t) = \mathbf{E}_0 \frac{\mu b \sin \theta J_0}{8\pi r \tau_p} \cdot e^{j\omega \lfloor t \rfloor} \cdot (e^{j\omega 2\tau_p} - 1) \\
= \mathbf{E}_0 \frac{\mu b \sin \theta J_0}{4\pi r} \cdot j\omega \left( \frac{\sin \omega \tau_p}{\omega \tau_p} \cdot (e^{j\omega \lfloor t \rfloor + \tau_p}) \right),
\]

where we note the similarity to Eq. (41) with the phase term \( \omega \tau_p \) 
in the exponent resulting from our change of coordinates.

We leave our study of the line source for now, returning to it 
later in Section E. There we will study the formation of radiating 
spherical waves on infinitely thin wires and will at that time con-
sider the case of reflected current from the source and tip thus 
introducing the concept of resonant modes. We will as well look 
into the physical principles governing the radiation resistance.
The line source was introduced here for two specific reasons. One, to illustrate the applicability of the Huygens principle to the line source radiator, and, two, to introduce the concept of non-infinite velocity \((p < \infty)\) across the surface of the radiating structure. This will be used in the next section to illustrate the concept of the phased array.

3. An alternate method (the Fourier integral)

Before leaving this section we might comment briefly on an alternative approach which could well have been used to handle this problem, that of applying the Fourier integral directly to the pattern function for harmonic excitation (the bracketed portion of Eq. (47)). Here we sum over the harmonic spectrum \(S(\omega)\) of the time function weighting each harmonic component (Eq. (47)) accordingly and thus obtaining the expression

\[
\overline{E}(r, t) = \frac{\mu b \sin \theta I_0}{4\pi r} \int_{-\infty}^{\infty} j\omega S(\omega)e^{j\omega ([t] + \tau_p)} \frac{\sin \omega \tau_p}{\omega \tau_p} d\omega
\]

\[
= \frac{\mu b \sin \theta I_0}{4\pi \tau_p} \int_{-\infty}^{\infty} S(\omega) \left\{ \frac{e^{j\omega \tau_p} - e^{-j\omega \tau_p}}{j 2} \right\} e^{j\omega ([t] + \tau_p)} d\omega
\]
\[ = \vartheta \frac{\mu_b \sin \theta I_o}{8\pi \tau_p} \left\{ \int_{-\infty}^{\infty} S(\omega) e^{i\omega[t] + 2\tau_p} \, d\omega \right\} \]

\[ - \int_{-\infty}^{\infty} S(\omega) e^{i\omega[t]} \, d\omega \]

It should be obvious that the third form of Eq. (48) reduces directly to an expression similar to Eq. (46), namely,

\[ E(r, t) = \vartheta \frac{\mu_b \sin \theta I_o}{8\pi \tau_o} \{ \mathcal{F}[t] + 2\tau_p \} - \mathcal{F}[t] \]

when we realize that the Fourier transform of our spectral distribution \( S(\omega) \) is

\[ \int_{-\infty}^{\infty} S(\omega) e^{i\omega[t] + 2\tau_p} \, d\omega = \mathcal{F}[t + 2\tau_p] \]

and \( I_o \mathcal{F}(t) = I_s(t) \).

Both methods thus yield similar results which we note are not time independent nor is the temporal function \([t]\) in general separable from the spatial function \((\tau_p)\) except in the special case of monochromatic excitation, i.e., \( S(\omega) = \delta(\omega - \omega_0) \).

The Fourier approach is fundamentally the one taken by Polk [3] and Tseng and Cheng[4] for the one dimensional or line source radiator. They obtained results similar to those above (Eq. (46)). Skolnik[14] in his book also discusses this problem from a similar point of view. His main interest is the band-limited harmonic case,
in particular the special problem of constant spectral distribution over a finite frequency interval. Unfortunately in his integration of the harmonic spectrum he neglects the weighting factor $j\omega$ and the kernel $e^{j\omega t}$. The resulting expression for $\overline{E}_\theta$ is time independent and since we saw that the time and spatial function were inseparable the time dependence cannot be later added. His result is thus in conflict with our results and with physical observation.

B. **The Time Dependent Far-Field Response of a Rectangular Aperture**

From the previous section it was seen that the introduction of an aperture into the path of a plane wave generates, in the equatorial plane, two waves which appear to radiate from the two aperture edges normal to this plane (see Eq. (39) and Fig. 6a). These two waves replace the original plane wave phase front. For finite apertures these two wave sources will be separated by the aperture width and since electromagnetic radiation travels at finite velocity the two signals will arrive at a point in the far-field not simultaneously but separated in time by an interval determined by the difference in transit time $\Delta t_\phi = 2\tau(\phi)$ (see Eq. (52)). A plot of $\Delta t_\phi$ as a function of angle for $p = \infty$ is shown in Fig. 6b (here labeled Region 2). The effects and limitations imposed by $\Delta t_\phi$ will be the subject of this and the remaining sections of this chapter.
Fig. 6a.--Ray diagram of planar aperture.

Fig. 6b.--Space-time schematic of the evolution of the far-field electric field in the equatorial plane resulting from a sudden excitation on the aperture. In Region (1) the observer at angle \( \phi \) has received no signal, in Region (2) the observer has received signal from the near edge of the aperture but not the far edge, in Region (3) the observer is now receiving signal from both edges of the aperture.
We will be particularly interested here in the criteria for lobes and nulls in the E and H fields and in the energy distribution, duration and signal distortion as a function of angle. No consideration will be given to the effects of self and mutual impedance or scattering.

For illustrative purposes then we shall limit our examples to the equatorial plane of the aperture, i.e., \( \theta = \pi/2 \) of Fig. 4 (see Fig. 6a). This will simplify the equations but will not detract from the general understanding of the problem. Equation (39) will then be the one used for our study with \( \tau \) defined as in Section A.1.

That is,

\[
E(\phi, t) = \frac{\mu ab(1 + \cos \phi)}{4\pi r} \frac{1}{2\tau} \{J(t + \tau) + J(t - \tau)\}
\]

where

\[
\tau = \frac{a}{2c} (\sin \phi - \frac{1}{p}) \quad p = \frac{v}{c}
\]

and we have for simplicity eliminated the retarded time notation, i.e., \([t] = t\).

We thus include the case of illumination by obliquely incident wave fronts as well as phase velocities less than \( c \).
1. **Aperiodic signal excitation**

Let us begin by considering the field resulting from a sudden excitation.

\[(53) \quad J(t) = u(t)F(t)\]

uniformly applied to an aperture (Figs. 6a) where \(u(t)\) is the unit step function applied at \(t = 0\) and \(F(t)\) is the signal function. Substituting Eq. (53) into Eq. (51) yields

\[(54) \quad E(\phi, t) = \frac{\mu ab(1 + \cos \phi)}{4\pi r} \frac{1}{2\tau} \left\{ F(t + \tau)u(t + \tau) - F(t - \tau)u(t - \tau) \right\} \]

Adding and subtracting \(F(t + \tau)u(t - \tau)\) to Eq. (54) and rearranging terms gives

\[(54') \quad E(\phi, t) = \frac{\mu ab(1 + \cos \phi)}{4\pi r} \left\{ F(t + \tau) \left[ \frac{u(t + \tau) - u(t - \tau)}{2\tau} \right] + u(t - \tau) \left[ \frac{F(t + \tau) - F(t - \tau)}{2\tau} \right] \right\} \]

The two step functions of Eq. (54) combine to give three distinct regions shown in Fig. 6b for \(p = \infty\). Writing the expressions for \(E(\phi, t)\) in these regions separately, we have

\[(55a) \quad E = 0 \quad \text{for} \quad t < -\tau \quad \text{Region 1}\]

\[(55b) \quad E = \frac{\mu ab(1 + \cos \phi)}{8\pi r} \frac{F(t + \tau)}{\tau} \quad \text{for} \quad -\tau \leq t < \tau \quad \text{Region 2}\]
\( (55c) \quad E = \frac{\mu ab(1 + \cos \phi)}{4\pi r} \left( F(t + \tau) - F(t - \tau) \right) \quad t > \tau \quad \text{Region 3} \)

Note that the point \( t = \tau = 0 \) is not included. For this we must look at Eq. \((54')\) and consider the limit of \( E(\phi, t) \) as \( t \to \tau \to 0 \) which gives

\( (55d) \quad E(0, 0) = \lim_{t \to \tau \to 0} E(\tau(\phi), t) \)

\[ = \frac{\mu ab(1 + \cos \phi)}{4\pi r} \left\{ F(t) \delta(t) + u(t)F'(t) \right\} . \]

Equation \((55d)\) shows that a spike occurs at \( t = 0 \) in the \( \tau = 0 \) direction unless the current is switched as it passes through zero. That is, unless \( F(0) = 0 \). This is as we would expect since from Eq. \((42)\) we saw that along the \( \tau = 0 \) axis we get a field waveform that is the derivative of the input waveform \( J(t) \). A current discontinuity would, therefore, be expected to generate a delta function. That this is not an abrupt discontinuity with \( \phi \) is seen by looking at the limit of Eq. \((55d)\) as \( \tau \to 0 \) for \( F(0) \neq 0 \) which approaches \( 1/\tau \) as \( t \to 0 \). This will be illustrated in our example.

Looking now at the three regions of Fig. 6b it is seen that the two curves A and B, which delineate Region 2, indicate the arrival of the signal from the points A and B of the aperture (Fig. 6a). They cross at the angle for which \( \Delta t_{\phi} = 0 \). Here \( \phi = 0 \) since \( p = \infty \). It is along this axis that the signals arrive simultaneously
producing the derivative. From Eq. (52) we see that as \( p \to +0 \) this axis would move toward \( \phi = \pi/2 \) thus increasing \( \Delta t_\phi \) in the \(-\phi\) region and decreasing it in the \(+\phi\) region. For \( p < 1 \), however, there will be no \( \phi \) for which \( \tau = 0 \) and thus no direction in which the signal derivative is received. A discontinuity in the exciting signal waveform would not then produce a spike. Phase velocities less than \( c \) can occur, particularly on linear wire radiators and will be discussed in Section E.

a. Excitation by a raised cosine function

Let us look now at the electric field resulting from a typical aperiodic signal excitation. We have chosen for our example one cycle of the raised cosine function (see Fig. 7)

\[
J(t) = \frac{\omega}{2\pi} \left( 1 - \cos \omega t \right) u(t) - u \left( t - \frac{2\pi}{\omega} \right)
\]

(56)

![Graph](image)

Fig. 7. -- Sketch of one cycle of raised cosine function Eq. (56).
This function, while being simple, continuous, and of finite
duration, has many interesting and illustrative properties. Its
shape is similar to the Gaussian and thus so are many of its pro-
properties. It is a trigonometric function so its values are easily
computable. Since we have normalized to unit area with \( \frac{\omega}{\pi} \) we
have in the limit as \( \omega \to \infty \)

\[
(56') \quad \lim_{\omega \to \infty} J(t) = \lim_{\omega \to \infty} (1 - \cos \omega t) \left( \frac{u(t) - u \left( t - \frac{2\pi}{\omega} \right)}{\frac{2\pi}{\omega}} \right) = \delta(t).
\]

Thus we should be able to infer from our results the electric field
which would result from a delta function excitation.

If we now select an aperture width, a value of \( p \), and normal-
ize with respect to \( \mu b/4\pi r \), our specific \( J(t) \) function can be sub-
stituted back into Eq. (55) and programmed for the digital computer.
Then with the aid of a three dimensional plotting routine the result-
ing \( E(\phi, t) \) can be displayed in both space and time for any range of
\( \phi \) and \( t \). This has been done for our raised cosine function and the
resulting \( S-T \) (space-time) plot is shown in Fig. 8. Here we have
shown \( E(\phi, t) \) in two sketches over the range \(-4\pi/\omega \leq t \leq 6\pi/\omega \) and
\(-\pi/2 \leq \phi \leq \pi/2 \) in Fig. 8a and \(-\pi/2 \leq \phi \leq 0 \) in Fig. 8b for \( p = \infty \)
and \( a = 3\lambda \), which makes the transit time across the aperture 3 times
the signal duration.
Fig. 8a. - S-T plot of the wave shape of the far-field electric field $E_\phi$ resulting from a single cycle raised cosine function excitation (Eq. (56)) suddenly impressed on the uniformly illuminated aperture shown at the left. The exciting waveform is shown along the lower edge between A and A'. Note the three regions.
Fig. 8b.--Rear half of S-T plot shown in Fig. 8a showing signal along $\phi = 0$ direction. Note waveshape for $E_\phi=0$ is precisely the derivative of the raised cosine function (i.e., the derivative of the waveshape shown between B and B').
In Figs. 8a and b a number of things are apparent, the most obvious being that along the \( \tau = \phi = 0 \) axis \( E_\phi(0, t) \) has its largest value, shortest duration and a wave shape which is the exact derivative of the exciting waveform \( J(t) \) (see Fig. 8b). We note also that the d.c. component has disappeared and that in no direction is the exciting waveform \( J(t) \) received. Two distinct, overlapping transit time spreading curves similar to those of Fig. 6b are quite evident in Fig. 8a and have been delineated with unprimed letters for the phase front associated with the leading edge of the signal and primed letters for that associated with the trailing edge.

At angles where \( \Delta t_\phi \) exceeds the pulse duration \( PW(\Delta t_\phi \geq PW = \lambda/c) \) two distinct signal pulses appear, both having the same wave shape, that of \( J(t) \), but opposing polarities. They will be referred to as the signal wave \( W_s \) and the signal nulling wave \( W_n \) and are so labeled. It is the interaction of the \( W_n \) of one pulse with the \( W_s \) and \( W_n \) of a subsequent pulse or pulses which generates the nodes and lobes in the field plot as will be discussed in the next section.

As those angles where \( W_s \) and \( W_n \) do appear as distinct pulses it should be possible, by employing proper gating techniques in the receiver, to obtain one or the other of the signals separately. This is, of course, only possible when the pulse duration is less than \( \Delta t_\phi \), which can be quite long in the back fire direction of end fed wire radiators.
The electric field $E_\phi(\phi, t)$ resulting from a delta function excitation $J(t)$ should by now be obvious. It will be a doublet function along the $\tau = \phi = 0$ direction and at all other angles two distinct delta functions of opposite polarity separated in time by $\Delta t_\phi$ the relative magnitudes of which vary with $\phi$ as $(1 + \cos \phi)/\tau$.

2. **Lobes and nulls in the S-T plot**

(periodic signal excitation)

One of the dominate characteristics of the electric field resulting from harmonic excitation is the appearance, in the field plot, of lobes and nulls. As an example an S-T plot of the electric field resulting from a suddenly initiated harmonic function, $J(t) = u(t) \sin \omega t$, is illustrated in Fig. 9 for the time interval $-3\pi/\omega \leq t \leq 7\pi/\omega$, aperture width $a = 3\lambda$ and $1/p = 0$.

Here Region 2 is again delineated by the letters A and B, but it now appears as a "transient" region. Any change in the waveform, such as modulating or switching, of the signal would spread out in a similar transient pattern. To the right of the transient region is Region 3, the "steady state" region. We note here the appearance of the characteristic $\sin \omega t/\omega t$ pattern which begins to emerge at $t = 0$ and spreads out from $\tau = \phi = 0$. This pattern, as we remember, results from the constructive and destructive interference of the two wave functions.
Fig. 9.—S-T plot of suddenly excited time-harmonic function. Note the formation of lobes and nodes out from the $\phi=0$ direction for $t>0$. 
To determine the criterion for and the location of lobes and nulls resulting from a general function we look at the portion of Eq. (51) (here, for convenience we use Eq. (49)) which governs this, that is,

\[(57) \quad E_\Phi(\phi, t) \sim \frac{J(t+2\tau) - J(t)}{2\tau} \]

From Eq. (57) it should be apparent that the electric field will have nulls if the numerator can be made zero at some value or values of \(\phi\) for all time \(t\) \((\tau(\phi') \neq 0)\). This will occur if \(J(t)\) is of the form

\[(58) \quad J(t+2\tau(\phi')) = J(t) = J(t+\Delta)\]

for all \(t\). Such a function is, of course, a periodic function of period \(\Delta \leq a/c (1+1/p)\).

To determine where and how soon after signal initiation these nulls occur we look at our relationship for \(\Delta t_\phi\) given by

\[(52') \quad \Delta t_\phi = n\Delta = a/c (\sin \phi_n - 1/p)\]

where \(n\) is a non-zero positive or negative integer. The angular location of the \(n\)th null \(\phi_n'\) is thus given by the relationship

\[(59) \quad \phi_n' = \sin^{-1} (n\lambda/a + 1/p)\]

where \(\lambda =\) wavelength of our function \(J(t)\).
The nulls for a suddenly excited periodic function will appear in succession out from the main beam as illustrated in Fig. 9, the time needed to generate the nth lobe being \( n/f \).

Consider next the location of lobe maxima/minima in the field plot. Here we are looking for extrema, that is, those values of \( \tau(\phi') \) which make \( \partial E/\partial \tau = 0 \). From Eq. (57) then we have

\[
\frac{\partial E}{\partial \tau} \sim \frac{1}{\tau} \left\{ \frac{\partial J(t+2\tau)}{\partial t} - \frac{J(t+2\tau) - J(t)}{2\tau} \right\}
\]

where \( \frac{\partial}{\partial \tau} J(t+2\tau) = 2 \frac{\partial}{\partial t} J(t+2\tau) \). Equation (60) is defined everywhere except at \( \tau = 0 \). Here employing L'Hospital's rule we have

\[
(60') \quad \lim_{\tau \to 0} \frac{\partial E}{\partial \tau} = \lim_{\tau \to 0} \frac{f(\tau)}{g(\tau)} = \lim_{\tau \to 0} \frac{\partial f/\partial \tau}{\partial g/\partial \tau}
\]

\[
\sim \lim_{\tau \to 0} 2 \frac{\partial}{\partial t} \left[ \frac{\partial J(t+2\tau)}{\partial t} \frac{J(t+2\tau) - J(t)}{2\tau} \right] = 0
\]

which is valid for all forms of \( J(t) \) since the limit of the right-hand term in the brackets is the left-hand term. A lobe extremum will thus appear along the \( \tau = 0 \) axis regardless of the waveform of \( J(t) \) as we would expect.

From both forms of Eq. (60) we see that extrema appear in the S-T plot at those angles where the relationship

\[
(61) \quad \frac{\partial J(t+2\tau)}{\partial t} = \frac{J(t+2\tau) - J(t)}{2\tau}
\]

is satisfied.
To illustrate this let us consider three examples starting with the harmonic case \( J(t) = \sin \omega t \). Substituting into Eq. (61) gives

\[
(62a) \quad 2\omega \tau \cos \omega (t + 2\tau) = \sin \omega (t + 2\tau) - \sin \omega t
\]

\[
(62b) \quad \omega \tau [\cos \omega (t + \tau) \cos \omega \tau - \sin \omega (t + \tau) \sin \omega \tau] = \cos \omega (t + \tau) \sin \omega \tau
\]

\[
(62c) \quad \omega \tau \cos \omega (t + \tau) - \omega \tau \sin \omega (t + \tau) \tan \omega \tau = \cos \omega (t + \tau) \tan \omega \tau
\]

We see that the lobes occur at those points in time and space where \( \sin \omega (t + \tau) = 0 \) and

\[
(63) \quad \omega \tau_L = \tan \omega \tau_L
\]

Thus the lobes occur very nearly at \( \omega \tau_{LN} = N \frac{\pi}{2} \), where \( N \) is an odd integer \( \geq 3 \) (see Fig. 9 and the Power Pattern Fig. 14). They actually occur slightly before; however the approximation improves as \( N \) increases.

In order to illustrate more clearly the generation of individual lobes and nulls in the S-T plot and to show some interesting scanning and pulse shaping possibilities we have selected for our second example a pulse train consisting of only two pulses. Here we shall use two half-wave rectified pulses separated in time by twice the pulse-width. Thus we have
\[ J(t) = \{ \pm [u(t) - u(t-PW)] - [u(t-PW-PS) - u(t-2PW-PS)] \} \sin \omega t \]

where the two bracketed terms are the first and second pulses, respectively, as labeled. The pulse width is \( PW = \frac{\pi}{\omega} \), the pulse spacing is \( PS = \frac{2\pi}{\omega} \), and for this illustration we shall let \( \frac{a}{c} = 6PW \) and \( \frac{1}{p} = \frac{2}{3} \). Note that because \( PS + PW = \Delta = \frac{3\pi}{\omega} \), the \((-)\) sign in front of the pulse (2) makes it a positive pulse.

From the illustration of the raised cosine function (Fig. 8) we should be able to intuitively infer which of the two signs on pulse (1) would give a null and which a lobe. From Fig. 8 it is obvious that any interaction between the two pulses must be between the \( W_n \) of pulse (1) and \( W_s \) of pulse (2). If they are of the same sign we have a lobe, and of opposite sign a null. Since \( W_n \) has the opposite polarity from \( W_s \), the two pulses (1) and (2) must be of opposite sign for a lobe and of similar sign for a null as is confirmed in Figs. 10 and 11.

To show this analytically we substitute Eq. (64) back into the equation for extrema, Eq. (61), giving
Fig. 10.—S–T plot of the electric field resulting from excitation by two pulses of similar shape but opposite sign (shown along the lower edge and marked \( W_s(1) \) and \( W_s(2) \)). Note the two main beam pulses in the \( \phi_m \) direction and one side lobe pulse along the \( \phi_L \) direction.
Fig. 11.--S-T plot resulting from two pulses similar to Fig. 10 except now the pulses have the same sign. Note now we have a null in the $\phi_n$ direction.
(1)  
$2\omega \tau \{ \pm [u(t+2\tau) - u(t-\pi/\omega+2\tau)] - [u(t-3\pi/\omega+2\tau) - u(t-4\pi/\omega+2\tau)] \}$

$W_n(1)$
$- u(t-4\pi/\omega+2\tau) \} \cos \omega(t+2\tau) = \{ \pm [u(t+2\tau) - u(t-\pi/\omega+2\tau)] - W_n(1)$
$- [u(t-3\pi/\omega+2\tau) + u(t-4\pi/\omega+2\tau)] \} \sin \omega(t+2\tau) -$
$W_s(2)$
$W_n(2)$
$- \{ \pm [u(t) - u(t-\pi/\omega)] - [u(t-3\pi/\omega) - u(t-4\pi/\omega)] \} \sin \omega t -$

From Eq. (65a) we see that only waves $W_n(1)$ and $W_s(2)$ can be
made to overlap in time and space and this would occur for $2\tau = 3\pi/\omega$
and somewhere in the time range $-2\pi/\omega \leq t \leq 3\pi/\omega$, that is, after
$W_n(1)$ has passed but before $W_n(2)$ has arrived. Putting this
restraint on Eq. (65a) we have.

(65b)  
$2\omega \tau \{ \pm (1-1) - [u(t) - u(t-\pi/\omega)] \} \cos(\omega t + 3\pi) =$
$= \{ \pm [1-1] - [u(t) - u(t-\pi/\omega)] \} \sin(\omega t + 3\pi)$
$- \{ \pm [u(t) - u(t-\pi/\omega)] - [0-0] \} \sin \omega t$

(65c)  
$2\omega \tau [u(t) - u(t-\pi/\omega)] (-\cos \omega t) = (-1 \pm 1) [u(t) - u(t-\pi/\omega)] \sin \omega t$

(65d)  
$3\pi = (-1 \pm 1) \tan \omega t$

valid for $-2\pi/\omega \leq t \leq 3\pi/\omega$ and $2\omega \tau = 3\pi$. It is obvious that a solution
to Eq. (65) is only possible if we select the negative sign, since for
the positive sign the right side is zero (the criterion for a null, see
Eq. (57)) and we would have no extremum. Thus we see that a train
of pulses of alternate sign would produce a lobe next to the main beam while a train of pulses of similar sign would produce a null. Both the null and the lobe would occur at the same point on the S-T plot (here \( \phi_L = \phi_n \sim 90^\circ \)) as is seen from the analysis and from Figs. 10 and 11. As the pulse spacing is reduced, additional nulls and lobes will begin to appear alternately out from the main beam as is seen in Fig. 12 for the full-wave rectified pulse train which is our next example.

From Figs. 10 and 11 it is also obvious that as the pulse spacing is varied the direction in which the lobe or null appears is changed. Thus a scanning effect is produced by varying the pulse spacing in the pulse train. The main beam which here occurs at \( \phi_m \sim 47^\circ \) does not, however, change with variations in pulse spacing.

For end-fed wire radiators where \( 1/p \geq 1 \) and \( \cos \phi \) replaces \( 1 + \cos \phi \) in Eq. (51) it is possible to eliminate the main beam all together. This will be discussed in the section on the monopole. We again see, particularly along the \( \phi = \pm \pi/2 \) axis, that the pulse shape changes with \( \phi \), and thus the possibility of pulse shaping exists.

We look finally at our third example, the full-wave rectified pulse train

\[
J(t) = \{ U(t) - 2u(t - \pi/\omega) + 2u(t - 2\pi/\omega) - 2u(t - 3\pi/\omega) + \ldots \} \sin \omega t
\]
Fig. 12. -- $S-T$ plot for a suddenly excited full-wave rectified pulse train a portion of which is shown between A and B. Note in no direction is that wave shape received.
shown in Fig. 12. The pulse duration, aperture width and phasing
are the same as for the harmonic case of Fig. 9 (i.e., $PW = \pi/\omega,$
a/c $= 3PW$, $PS = 1/p = 0$). The signal has been initiated at $t = 0$ and
we note the familiar shape of the transient region. Three cycles of
the waveform can be seen along the $\phi = \pm \pi/2$ axis between A and B.
We again note that after steady state has been reached the waveform
of the received signal changes with angle and in no direction is it
that of the exciting waveform $J(t)$.

The nulls and lobes of Fig. 12 appear to be located along the
same angles as in Fig. 9. This is as we might expect for the nulls
but not necessarily for the lobe tips. To verify that the lobe tips
do indeed lie along the same angles as the harmonic case of the
same period we again employ Eq. (61) which gives upon substitution
of Eq. (66) for $J(t)$

$$
(67a) \quad 2\omega \tau \{u(t + 2\tau) - 2u(t - \pi/\omega + 2\tau) + \ldots\} \cos \omega(t + 2\tau) = \\
= \{u(t + 2\tau) - 2u(t - \pi/\omega + 2\tau) + \ldots\} \sin \omega(t + 2\tau) - \{u(t) \ldots\} \sin \omega t .
$$

Combining terms and making trigonometric substitutions we have

$$
(67b) \quad \{u(t + 2\tau) \ldots\} \left[ (2\omega \tau \cos 2\omega \tau - \sin 2\omega \tau) \cos \omega t - \right. \\
\left. - (\cos 2\omega \tau + 2\omega \tau \sin 2\omega \tau) \sin \omega t \right] \\
= - \{u(t) \ldots\} \sin \omega t
$$
which will be valid for those points in time and space where $\sin \omega t = 0$ and

$$2\omega \tau_L = \tan 2\omega \tau_L$$  \hspace{1cm} (68)

Note that $2\omega = 2\pi /PW = \omega_f$, the frequency of the rectified wave.

Equation (68) is then identical to the lobe-tip relationship obtained for a time harmonic signal of the same frequency, Eq. (63). A comparison of the power patterns for these two waveforms is shown in Fig. 17.

a. The equivalent Fourier spectrum solution

It will be quite useful at this point to now examine the null and lobe location criteria using the Fourier transform representation.

The general form of the exciting function $J(t)$ then becomes

$$J(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \sin\{n\omega t + \phi_n\}$$  \hspace{1cm} (69)

where $\omega = 2\pi /\Delta$. The transient portion will not be considered since the discrete Fourier spectrum represents a periodic function of infinite extent.

Substituting Eq. (69) into Eq. (57) gives

$$E(\phi, t) \sim \sum_{n=1}^{\infty} A_n n\omega \cos\{n\omega(t + \tau) + \phi_n\} \sin n\omega t/n\omega \tau$$  \hspace{1cm} (70)
From Eq. (70) we see that the d.c. component has vanished and that $E(\phi, t) \sim \partial J(t)/\partial t$. Each of the $n$ components in Eq. (70) has been weighted by the function $n\omega (\sin n\omega \tau /n\omega \tau)$. The $\sin n\omega \tau /n\omega \tau$ portion has a maximum value of 1 and a value of zero for all values of $n$ when $\omega \tau = N\pi (N=1, 2, \ldots)$. Thus there are directions in which all of the harmonic components have nulls simultaneously, therefore, our S-T plot will have nulls in those directions. The relationship for the null location is Eq. (59) as before. The $n\omega$ component of the weighting function on the other hand weights the higher frequency components of $\partial J(t)/\partial t$ more, thus increasing their effect on $E(\phi, t)$.

To illustrate the difficulty in locating the lobe tips using the Fourier spectrum representation let us consider again the full wave rectified pulse train whose spectrum is

\begin{equation}
J(t) \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2n\omega \tau t}{(2n-1)(2n+1)}
\end{equation}

\begin{equation}
= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\omega \tau t + \pi/2)}{(2n-1)(2n+1)}
\end{equation}

Substituting this back into Eq. (61) we see that this would appear to require that there be one $\omega \tau$ such that

\begin{equation}
n\omega \tau \tau = \tan n\omega \tau
\end{equation}
for all \( n \). But we saw from Eq. (63) that this is not possible. It appears then that the determination of the exact location of the lobe tips using the Fourier spectral components will be a rather complicated process necessitating the examination of all of the harmonic components. This problem has not been explored.

b. Amplitude modulation

From the investigation of the full-wave rectified pulse train we saw that it is possible to obtain from some periodic excitations field and power patterns which have narrower beam widths and correspondingly lower side lobe levels than those obtained from harmonic excitations of the same period (see Fig. 17). And although there was considerable variation in wave shape with look angle it is of interest to investigate the possibilities of amplitude modulating arbitrary periodic signals. To do this let us consider the fields resulting from excitation of the form

\[
J(t) = G(t)F(t)
\]

Substituting Eq. (73) back into Eq. (51) yields

\[
E(\phi, t) = \frac{\mu ab(1 + \cos \phi)}{4\pi} \frac{1}{2\pi} \left\{ G(t + \tau)F(t + \tau) - G(t - \tau)F(t - \tau) \right\}
\]

which upon adding and subtracting \( G(t + \tau)F(t - \tau) \) and rearranging becomes
(74') \[ E(\phi, t) = \frac{\mu ab(1 + \cos \phi)}{4\pi r} \left\{ G(t + \tau) \frac{F(t + \tau) - F(t - \tau)}{2\tau} + F(t - \tau) \frac{G(t + \tau) - G(t - \tau)}{2\tau} \right\} \]

If we further assume that \( F(t) \) is a much more rapidly varying function than \( G(t) \) so that

(75) \[ \frac{F(t + \tau) - F(t - \tau)}{2\tau} > > \frac{G(t + \tau) - G(t - \tau)}{2\tau} \sim \frac{\partial G(t)}{\partial t} \]

for all \( t \) and \( \tau \), then \( F(t) \) can be considered to be the carrier and \( G(t) \) the modulating wave. We then note that \( G(t) \) will be radiated undistorted in all directions if the second term in the brackets can be neglected.

To illustrate this let us consider the amplitude modulated harmonic signal. Here let

(76) \[ G(t) = 1 + A \sin \omega_m t \quad (0 \leq A \leq 1) \text{ and } F(t) = \sin \omega t \]

Then Eq. (74') becomes

(77) \[ E(\phi, t) = \frac{\mu ab(1 + \cos \phi)}{4\pi r} \left\{ \omega(1 + A \sin \omega_m (t + \tau)) \cos \omega t \frac{\sin \omega \tau}{\omega \tau} \right. \]

\[ + A \omega_m \sin \omega(t - \tau) \cos \omega_m t \frac{\sin \omega_m t}{\omega_m t} \left\} = \right. \]

\[ = \frac{\omega \mu ab(1 + \cos \phi)}{4\pi r} \left\{ \left[ \cos \omega t + \frac{A}{2} \left[ \sin[(\omega + \omega_m) t + \omega_m \tau] - \sin[(\omega - \omega_m) t - \omega_m \tau] \right] \right] \frac{\sin \omega \tau}{\omega \tau} \right. \]

\[ + \frac{A \omega_m}{2\omega} \left[ \sin[(\omega + \omega_m) t - \omega \tau] \right] + \sin[(\omega - \omega_m) t - \omega \tau] \right\} \frac{\sin \omega_m \tau}{\omega_m \tau} \]
From Eq. (77) several things are apparent. First we see two distinct field patterns each having two identical side-bands. These side-bands, however, have different phase constants. Thus the upper side-bands of the two patterns will destructively interfere when 
\[(\omega - \omega_m)t = N\pi \text{ for } N \text{-odd while the lower side-bands interfere destructively when } (\omega + \omega_m)t = N\pi \text{ for } N \text{-even.} \]
Since the secondary pattern can be considered nearly constant \((\sin \omega_m t / \omega_m t \sim 1 \text{ for all } \phi)\), this interference effect will be predominate near the nulls of the dominate pattern. In any event the magnitudes of the two side-bands will in general be less than that of the main beam. It should be noted that the ratio of the magnitudes of the two patterns is the ratio of the two frequencies \(\omega_m / \omega\). For the A.M. broadcast band where \(\omega_m \leq 10 \text{ Kc and } \omega \text{ lies between } 500 \text{ KH and } 1600 \text{ KH, this ratio would vary between } 1/50 \text{ and } 1/160, \text{ implying effectively distortionless reception.} \)

C. The Far Field Distribution of Energy (Power) With Look Angle (\(\phi\)) "The Pattern Function"

Up to now we have been concerned primarily with the distribution of the E and H fields in space and time. Of equal importance however is how the energy is radiated in space and time. For the harmonic case this is the familiar "Power Pattern". Its relative shape is quite readily obtainable by squaring either the E or H patterns. In the general time case, however, the evaluation will be
somewhat more involved and will require an integration of the Poynting vector over the time duration of interest. From Chapter II we saw that the energy radiated by our antenna as a function of direction was given by

\begin{equation}
W(\theta, \phi) = \int_{T_1}^{T_2} r^2 S(r, \theta, \phi, t) dt = \frac{r^2}{\epsilon_0} \int_{T_1}^{T_2} E^2 (r, \theta, \phi, t) dt
\end{equation}

and for a periodic function this would become the average power

\begin{equation}
P(\theta, \phi) = \frac{1}{\Delta} W(\theta, \phi)
\end{equation}

In Eq. (16b) \( W(\theta, \phi) \) is integrated over only one cycle (\( \Delta \)) of the periodic function, i.e., from \( T_1 \) to \( T_1 + \Delta \).

If the relationship for the radiated electric field from the aperture Eq. (51) is substituted into Eq. (16a) we then have

\begin{equation}
W(\phi) = \frac{(\mu_0 \epsilon_0)^2}{4\pi^2} \frac{(1 + \cos \phi)^2}{4\pi^2} \int_{T_1}^{T_2} \left( J(t + \tau) - J(t - \tau) \right)^2 dt
\end{equation}

the energy distribution function in the equitorial plane for the planar aperture.

To illustrate the versatility of this equation we shall evaluate it for four typical examples representative of some of the types of exciting functions we might expect. Each of the pattern functions will then be plotted and compared with the steady state \((\sin \omega_0 \tau / \omega_0 \tau)^2\) pattern. In all cases the same aperture width will be used, i.e.,
a/c = 3PW for aperiodic functions and a/c = 3Δ for periodic functions.

Let us begin by looking at the pattern for a raised cosine excitation

\[ J(t) = \frac{\omega_0}{2\pi} (1 - \cos \omega_0 t)(u(t) - u(t - \frac{2\pi}{\omega_0})) \]

This function is of finite duration and thus as was seen from Fig. 8a is of finite extent in space. There may be therefore look angles for which the signals from the two aperture edges do not overlap in space and time. In these regions we shall see that the same energy distribution pattern is obtained regardless of the excitation waveform J(t).

It will be convenient for this discussion to define the aperture width a in terms of λ or the width of the pulse PW/c as a = mλ. Then τ may be written

\[ \tau = \frac{a}{2c} \sin \phi = \frac{m\lambda}{2c} \sin \phi = \frac{m\pi}{\omega_0} \sin \phi \]

Substituting Eq. (56) for J(t) into Eq. (78) and evaluating we obtain

\[ W(\phi) = \frac{(1 + \cos \phi)^2}{4\pi^2 \omega_0 \tau^2} \left\{ (\omega_0 \tau - \pi)(2\sin^2 \omega_0 \tau) + 2\pi + \omega_0 \tau \right\} - \frac{3}{2} \sin \omega_0 \tau \]

\[ \frac{(\mu \alpha \beta)^2}{\xi^2 (4\pi)^2} \]

for τ ≤ π/ω_0, the region of signal overlap, and
\[(80b) \quad W(\phi) = \frac{(\mu ab)^2}{\zeta (4\pi)^2} \frac{3(1 + \cos \phi)^2}{4\pi \omega_0 \tau^2} \frac{\omega_0}{\zeta (4\pi)^2} \frac{3(1 + \cos \phi)^2}{4\pi^3 m^2 \sin^2 \phi} \]

for \( \tau \geq \pi/\omega_0 \), the region of no signal overlap. Equations (80a and b) have been plotted in Fig. 13 for \( m = 3 \).

It is interesting to note that the energy distribution curve in Fig. 13 is wider than the power pattern for the time harmonic case. That this should be so becomes apparent when we look at the frequency spectrum \( F(\omega) \) of the exciting function (Eq (56)) which is

\[(81) \quad F(\omega) \sim \frac{\pi}{\omega_0} \left\{ \frac{\sin \pi \omega/\omega_0}{\pi \omega/\omega_0} + \frac{\sin(\omega - \omega_0)\pi/\omega_0}{2(\omega - \omega_0)\pi/\omega_0} + \frac{\sin(\omega + \omega_0)\pi/\omega_0}{2(\omega + \omega_0)\pi/\omega_0} \right\} \]

where we have suppressed the phase terms. Since the energy distribution curve, from the frequency point of view, can be thought of as resulting from the superposition of weighted power patterns of the form \( \omega^2 (\sin \omega t/\omega t)^2 \) resulting from the frequencies contained in the spectrum and since from Eq. (81) it is seen that the preponderance of the energy in this frequency spectrum is concentrated at the low end of the spectrum, the first term of Eq. (81), a wide central beam should be expected.

Looking next at the cross-over where we switch from Eq. (80a) to (80b) we note that as either \( m \) or \( \omega_0 \) is increased Eq. (80b) applies to a larger and larger portion of the \( \phi \) region until in the limit as \( \omega_0 \to \infty \) Eq. (80b) applies for all \( \phi \). From Eq. (56') we remember that the
\[ m = \frac{a}{\lambda} = 3 \]

1. \[ J(t) = \frac{1}{2} (1 - \cos 2\pi t) \{ u(t) - u(t-1) \} \]
2. \[ J(t) = \sin 2\pi t \]

Fig. 13.--Normalized energy (power) pattern for a raised cosine signal excitation (a plot of Eq. (80)).
exciting function in this limit becomes the delta function $J(t) = \delta(t)$, so that Eq. (80b) can be thought of as being the energy distribution curve for a delta function excitation.

We consider next the case of excitation by a sine wave of finite duration. That is

$$J(t) = \sin \omega_o t \left( u(t) - u \left( t - \frac{2N\pi}{\omega_o} \right) \right) \quad N = 1, 2, 3$$

For this case we should expect that the shape of the energy distribution curve will tend toward $(\sin \omega_o \tau/\omega_o \tau)^2$ in the limit as $N \to \infty$ while its magnitude increases without limit.

Substituting Eq. (82) into Eq. (78) yields again two cases for $W(\phi)$

$$W(\phi) = \frac{(\mu ab)^2}{\xi(4\pi)^2} \left( 1 + \cos \phi \right)^2 \{ 2N\pi \sin^2 \omega_o \tau + \omega_o \tau \cos 2\omega_o \tau - 1/2 \sin 2\omega_o \tau \}$$

for $\tau \leq N\pi/\omega_o$, and

$$W(\phi) = \frac{(\mu ab)^2}{\xi(4\pi)^2} \frac{N\pi (1 + \cos \phi)^2}{2\omega_o \tau^2}$$

for $\tau \geq N\pi/\omega_o$. Equations (83a and b) have been plotted in Fig. 14 for $N = 1, 2, 3$ and 50 and all curves have been normalized. Figure 14 can be compared with the S-T plot of Fig. 9. We note the emergence of the first null and lobe with only two cycles present ($N = 2$). Note
Fig. 14.---Normalized energy (power) pattern for a harmonic excitation of N cycles duration (a plot of Eq. (83)).

\[
m = \frac{a}{\lambda} = 3
\]

\[
J(t) = \sin 2\pi t \left\{ u(t) - u(t - \frac{2N\pi}{\omega_0}) \right\}
\]

\[
N = \text{No. Cycles}
\]

1
2
3
50

\(-\pi/2\)

\(\phi\)

\(\pi/2\)
also that the width of the main beam is this time narrower than the \((\sin \omega_0 \tau / \omega_o \tau)^2\) pattern but gets wider with increasing N. That this is as it should be can again be seen by looking at the frequency spectrum of Eq. (82) which is

\[
F(\omega) = \frac{\sin(\omega - \omega_0) T}{(\omega - \omega_0)} + \frac{\sin(\omega + \omega_0) T}{(\omega + \omega_0)}, \quad T = \frac{2N \pi}{\omega_0}
\]

the last two elements of Eq. (81). It is seen to be symmetrically centered about \(\omega_0\). Now since the energy radiated by a harmonic wave is proportional to \(\omega\), proportionally more energy will be radiated by those harmonics above \(\omega_0\) than those below \(\omega_0\) thus we expect the narrower central beam. We also note that \(W(\phi)\) does indeed tend in the limit of increasing N toward \((\sin \omega_0 \tau / \omega_0 \tau)^2\). Finally we see that for a sine wave of finite duration there will always be some energy radiated in all directions. This was particularly evident from Fig. 9.

The final aperiodic excitation which we wish to discuss here is of the class which is easily integrated since it is assumed to have existed for all time. Thus there will be no discontinuity in the integration. We consider then the case of the Gaussian modulated harmonic

\[
J(t) = e^{ct^2} \cos \omega_0 t
\]
which has been plotted in Fig. 15 from \(-3 \leq t \leq 3\) for \(\alpha = 1\) and \(\omega_0 = 2\pi\). Equation (85) is roughly the shape of the output pulse of a Q-switched laser. The energy distribution curve for this excitation is

\[
W(\phi) = \frac{(\mu_0 b)^2}{8(4\pi)^2} \frac{(1 + \cos \phi)^2}{2\pi^2} \sqrt{\frac{\pi}{2\alpha}} \left\{ 1 - e^{-\alpha \tau^2} \cos 2\omega_0 \right\}
\]

where Eq. (78) is here integrated from \(-\infty \leq t \leq \infty\). It has been plotted in Fig. 16 for \(\alpha = 1\). We again see a slight narrowing of the main beam as well as the beginning of nulls and lobes. Equation (86) is seen to tend toward the single frequency pattern as \(\alpha \to \infty\). A look at the frequency spectrum of the Gaussian modulated harmonic,

![Graph of f(t)](image)

**Fig. 15.** Sketch of the modulated Gaussian excitation used to generate Fig. 16 (a plot of Eq. (85)).
\[ m = \frac{a}{\lambda} = 3 \]

1. \( J(t) = e^{t^2} \cos 2\pi t \)
2. \( J(t) = \sin 2\pi t \)

Fig. 16. -- Normalized energy (power) pattern for the modulated Gaussian excitation shown in Fig. 15 (a plot of Eq. (86)).
\[ F(\omega) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{(\omega - \omega_0)^2}{4\alpha}} \]

again shows a spectrum symmetric around \( \omega_0 \). Thus we see that it is quite a straightforward and rather simple process to compute the energy distribution pattern resulting from any arbitrary signal excitation on a planar aperture and, as we shall see next, the periodic case is equally straightforward.

Let us look then at our final example, the periodic exciting function. Only one example of this type of excitation will be illustrated here since all others can be handled in a similar manner. Let us consider the full-wave rectified pulse train

\[ J(t) = \{ u(t - 2\pi/\omega_0) + 2u(t - 2\pi/\omega_0) - \cdots \} \sin \omega_0 t \]

whose electric field S-T plot is shown in Fig. 12. Substituting into Eq. (78) and integrating from \( 2\tau \leq t \leq 2\tau + \pi/\omega_0 \) and dividing by \( 2\pi\omega_0 \) we obtain

\[ P(\phi) = \frac{(uab)^2}{\xi(4\pi)^2} \frac{(1 + \cos \phi)^2}{4\omega_0^2 \tau^2} \left\{ (1 - 4f/\pi) \sin^2 f + 2/\pi (f - 1/2 \sin 2f) \right\} \]

where we have replaced \( \omega_0 \tau = |m\pi \sin \phi| \) by

\[ f = |m\pi \sin \phi| \cdot \frac{(N-1)/2}{\pi} \text{ for } (N-2)/\pi \leq \omega_0 \tau \leq (N/2) \pi \]

\[ N = 1, 2, \cdots 2m \]

What we have done here is evaluated the integral Eq. (78) over the main beam region and, realizing that this integral is also valid for the
several side lobes but with a magnitude decrease due to the 

\[(1 + \cos \phi)^2 / \tau^2\], we just shift the region of integration by replacing 

\(\omega_0 \tau\) by \(f\). Equation (87) has been plotted in Fig. 17. We note here 

that the main beam is significantly narrower than that of the harmonic 
of the same periodicity and that the side lobes are reduced. A look 
at the Fourier spectrum of Eq. (66)

\[
J(t) \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2n \omega_0 t}{(2n-1)(2n+1)}
\]

shows that all of the radiated energy is contained in harmonics higher 
than the fundamental frequency \(\omega_0\) thus we would expect a narrower 
main beam. Noting also that since the Fourier spectrum of all 
periodic functions is made up of harmonics with frequencies greater 
than the fundamental \(\omega_0\) we should expect that the \((\sin \omega_0 \tau / \omega_0 \tau)^2\) 
pattern is the upper limit of any power pattern which results from a 
periodic function of period \(\Delta = 2\pi / \omega_0\).

D. **Radiation From Spatial Discontinuities**

(The Array)

With some of the characteristics of aperture radiation in the 
time domain in mind we shall now turn our attention to an array of 
elements. Here we will be looking for those radiation character-
istics unique to an array of isotropic sources such as the criterion 
for pattern nulls, transient duration etc. as well as a look at the
\[ m = \frac{a}{\lambda} = 3 \]

1. \[ J(t) = \sin \pi t \sum \{u(t) - 2u(t-1) + 2u(t-2) + \ldots \} \]
   (Fullwave Rectified Pulse Train)
2. \[ J(t) = \sin 2\pi t \]

---

**Fig. 17.** -- Normalized power pattern for the full-wave rectified pulse train (a plot of Eq. (87)).
techniques associated with arrays of finite sources such as pattern multiplication and phased and scanned arrays.

1. **General considerations**

We begin by looking at the general relationship for the electric field from a phased array

\[
E(t,\phi) = \frac{ab(1 + \cos \phi_x)}{4\pi cr}
\]

\[
\sum_{I=1}^{N} B_I \left\{ \frac{F(t+[N-1-2(I-1)]T_1 + \tau_2) - F(t+[N-1-2(I-1)]T_1 - \tau_2)}{2\tau_2} \right\}
\]

where \( B_I \) is the amplitude of the \( I \)th element, \( a \) is the element width, \( d \) is the spacing between elements and \( N \) is the number of elements and \( T_1 = d/2c \left( \sin \phi_1 - 1/p_1 \right) \) and \( T_2 = a/2c \left( \sin \phi_2 - 1/p_2 \right) \).

Equation (89) is merely a superposition of the electric fields of the individual elements as determined from Eq. (51) with a phasing term \([N-1-2(I-1)]T_1\) added to each element so as to center the array around the x-axis as shown in Fig. 18.

Rearranging the terms of Eq. (89) slightly so that we have
Fig. 18. --(a) Ray diagram for an array of 3 elements with dimensions as noted. (b) Ray diagram for an array of 3 slots with the same dimensions as (a).
\[(89') \quad E(\phi, t) = \frac{N a b(1 + \cos \phi_2)}{4 \pi c r} \frac{1}{N \tau_2} \left\{ \begin{array}{l}
B_1 F(t + (N - 1)\tau_1 + \tau_2) - \\
- B_N F(t - (N - 1)\tau_1 - \tau_2) \end{array} \right\} + \\
\sum_{I=1}^{N-1} B_{I+1} F(t + (N - 2I)\tau_1 \\
- \tau_1 + \tau_2) - B_1 F(t + (N - 2I)\tau_1 + \tau_1 - \tau_2) \right\} \]

and taking the limit as \(B_{I+1} \rightarrow B_I\), and setting \(a = d\), \(\tau_1 = \tau_2\) we see that
the terms in the sum cancel each other so that the summation is
equal to zero. Thus Eq. (89') reduces to the relationship for the
single aperture of width \(N a\) with \(2\tau = 2N\tau_1\). Thus Eq. (89) is self
consistent and merely a quantized form of the aperture integral in
Eq. (39).

Let us look next at the form of Eq. (89) in the limit as \(a \rightarrow 0\)
for the array of point sources. Now Eq. (89) reduces to the form
\[(89'') \quad E(\phi, t) = \frac{b(1 + \cos \phi_2)}{4 \pi c r} \sum_{I=1}^{N} B_I' \frac{\partial}{\partial t} F(t + [N + 1 - 2I] \tau_1) \]

where \(B_I' = \lim_{a \rightarrow 0} aB_I\). It is this form of Eq. (89) which gives the
array pattern characteristics.

To illustrate this let us again consider the uniformly illumi-
nated harmonically excited case. Here \(B_I = B_{I+1} = 1\) and \(F(t) = \sin \omega t =
\text{Im}\{e^{j\omega t}\}\) and we have
\[(90) \quad E(t, \phi) = \frac{b(1 + \cos \phi_2)}{4 \pi c r} \text{Im}\left\{ \sum_{I=1}^{N} j\omega \exp[j\omega(t + [N + 1 - 2I] \tau_1)] \right\} \]
which becomes

\[(90') \quad E(t, \phi) = \frac{Nbc(1 + \cos \phi_2)}{4 \pi cr} \cos \omega t \left\{ \frac{\sin N \omega \tau_1}{N \sin \omega \tau_1} \right\} \]

We recognize Eq. \text{(90')} as the familiar array pattern for harmonic time variation.

2. \textbf{Nulls in the array pattern}

To determine the criterion which gives nulls in the array pattern let us begin by looking at the point source array and signal function shown in Fig. 19. For this illustration we consider only arrays with an even number of elements. For nulls in the array pattern, it is obvious that signals from adjacent elements must cancel each other for all time. Thus our signal function must be periodic and symmetric about the time axis and further this symmetry property must be unaffected by differentiation with respect to time. That is,

\[(91) \quad F(t) = -F(t + \Delta/2)\]

and

\[(91') \quad \frac{\partial}{\partial t} F(t) = -\frac{\partial}{\partial t} F\left(t + \frac{\Delta}{2}\right)\]

where \(\Delta\) is the period of the function. Such a function would produce nulls in the array pattern in those directions where \(2\tau_1 = M\Delta/2, \ M\) an odd integer. From Fourier Analysis we remember that the
Fig. 19a. --Ray diagram of an N element point source array N-even.

Fig. 19b. --Sketch of a typical periodic function with the symmetry properties of Eq. (91).
Note for $d/c \sin \phi = 1/2$ (the period) adjacent waves cancel each other.
symmetry property of Eq. (91) is characteristic of periodic functions containing only odd harmonics. Thus it is the harmonic content of our function and not the phasing etc. which gives us our criterion for nulls. We shall then look at the Fourier spectrum to determine the null criterion for an array of N elements.

Substituting the general Fourier spectrum relationship for $F(t)$, Eq. (69), into Eq. (89′) yields

$$E(t, \phi) = \frac{N\omega(1 + \cos \phi_2)}{4\pi cr} \sum_{n=1}^{\infty} n A_n \cos(n \omega t + \phi_n) \left\{ \frac{\sin Nn\omega \tau_1}{N \sin n \omega \tau_1} \right\} \times \left\{ \frac{\sin n \omega \tau_2}{n \omega \tau_2} \right\}$$

Nulls will occur due to the array in those directions where the first bracketed term is zero. That is, where

$$\sin Nn\omega \tau_1 = \sin Nn\pi \frac{2\tau_1}{\Delta} = 0 \text{ or } Nn \frac{2\tau_1}{\Delta} = \text{integer}$$

while

$$\sin n\pi \frac{2\tau_1}{\Delta} \neq 0 \text{ or } n \frac{2\tau_1}{\Delta} = \text{fraction}$$

This then requires that $2\tau_1/\Delta$ be a fraction such that $N$ is a multiple of the denominator but $n$ is not. For example, let $N$ be any even number and

$$\frac{2\tau_1}{\Delta} = \frac{M}{2} \quad M \text{ any odd integer}$$
then \(N_n(M/2)\) = an integer for any \(n\). However, \(n(M/2)\) is non-integer only for \(n\) odd. Thus as was shown before, for an even element array nulls are possible when the exciting function contains only odd harmonics. Using the above argument an array with any number of elements can be analyzed to determine the requirements on the harmonic content of the exciting signal \(J(t)\) which assures nulls in the array pattern. It can then easily be shown that pattern multiplication applies. General criteria for lobe location in the array pattern have not been developed.

3. The phased array

The phased array is becoming increasingly important in radar and communication systems. Figure 18 illustrated two types of phased arrays. The element array, a matrix of individually excited apertures, and the slotted array of Fig. 18b which is fundamentally a transmission diffraction grating. In the slotted array the phasing of the array and the individual elements are in the same direction, that is, \(\phi_1 = \phi_2\) and \(1/p_1 = 1/p_2\). The direction of zero transient for the array and the element is thus the same. For the element array, on the other hand, \(1/p_1\) may or may not equal \(1/p_2\) and \(\phi_1\) may or may not equal \(\phi_2\). Here the direction of zero transient for the array generally is not the same as for the element. The field S-T plot for the two arrays with identical time delays between element excitations
will not then be the same. This is illustrated in Fig. 20 for a two element array excited by a raised cosine function. Here \( a = 2/3d = 2PW \) and \( 1/p_1 = 1/\lambda \). Figure 20a shows the S-T plot for the element array and Fig. 20b the S-T plot for the slotted array. The difference in the two field patterns is obvious.

### 4. Scanning

A detailed analysis of the effect of high speed scanning on the far-field pattern is also possible with the computer program written to represent Eq. (89). Here we can consider three types of scanning. The first is electronic scanning where in \( \tau_1 \) we let

\[
(93) \quad \frac{1}{p_1} = f(t)
\]

with \( f(t) \) being the scanning function the maximum and minimum values of which determine the limits on the angle of scan.

The other two types are both mechanical scanning where here we have two cases: rotation of the entire array structure

\[
(94) \quad \phi_1 = \phi_2 = \phi - \phi'(t)
\]

and pivoting of the individual elements about their respective axis. In this latter case \( \phi_2 \) is fixed and \( \phi_1 = \phi - \phi'(t) \).

If we make the above substitutions into the computer program we can then observe the effects on the S-T plot. Due to lack of
Note
\frac{1}{P_1} = 1/\sqrt{2}
\frac{1}{P_2} = 0
d = 3\text{ PW}
a = 2\text{ PW}
d - a = 1\text{ PW}

Fig. 20a. \(S-T\) plot of a raised cosine excited array of 2 elements phased for a maximum radiation at \(\phi=45^\circ\).

Note the main energy lobe is still in \(\phi=0^\circ\) direction with a smaller lobe in the \(\phi=45^\circ\) direction.
Note

$\frac{1}{P_1} = \frac{1}{P_2} = \frac{1}{\sqrt{2}}$

$d = 3\text{PW}$

$a = 2\text{PW}$

$d - a = 1\text{PW}$

Fig. 20b. --S-T plot of a raised cosine excited array of 2 slots phased for a maximum radiation at $\phi = 45^\circ$.

Note here in contrasts to Fig. 20a the main energy lobe is in the $\phi = 45^\circ$ direction.
time this has not been done here. However, some discussions of
the effects on high-speed scanning are found in References 15, 16
and 17.

E. The Linear Antenna

The linear antenna under single frequency steady-state
excitation has been extensively investigated by King [18]. The
transient response on the other hand has been given relatively little
study until this past decade and then most of the investigation was
done in the frequency domain [19-31]. Of utmost interest to
recent investigators has been the input impedance at the antenna
terminals. It was apparent to all at the onset that this relationship
should be independent of the length of the radiator and thus relation-
ships for the input impedance similar to that of Schelkunoff [32]
(c.f. Eq. (95))

\[
Z_0 \approx 60 \ln \left( \frac{b}{a} \right)
\]

(95) (where \(b\) = length of the wire and \(a\) = wire diameter) were inadequate
for the transient and short pulse problem (i.e., duration of \(f(t) < 2bc\)).

Papas [33] in 1949 was probably the first to investigate this
problem and arrive at a viable solution for the input impedance of
an end-fed radiating conductor (c.f. Eq. (96))
(96) \[ Z_0(\omega) \approx 60 \ln \left( \frac{c}{\omega a} \right) - j30\pi \]

Wu[21] in 1961, 12 years later, arrived at a similar but more complete relationship which we shall use in our discussion. Wu, during his investigation, also considered the highly theoretical case of a step function voltage generator connected directly to the dipole antenna terminals investigating the impedance seen by the generator [19] and the current waveform on the conductor as a function of time [22]. This was a highly theoretical investigation as the author readily admits.

During this same decade a limited amount of experimental work was done. Probably the first of any consequence was Schmitt's work[20] in 1960. Here we have an attempt to explain and illustrate the step function response of a dipole radiator. Later more experimental work was performed by Schmitt et al.[23, 26, 29] with considerably more accuracy. Most recently Beam et al.[30, 31] performed some experiments on the end fed radiator in both the near and far fields. It will therefore be of some interest to compare these experimental results with those obtained by our present model.

We shall begin this section with a study of the isolated monopole or end-fed wire radiator. This is one which is located in free space far from a ground plane. We shall develop our relationship
for the far-field region only and will use as our fundamental relationship Eq. (49).

1. The free-space monopole radiator

The end-fed wire radiation problem (Fig. 21) was first considered by Mannbeck[1] in 1922. He, however, assumed his work to be accurate only in the far field. Later Alford[34] considered the problem using Hertzian potentials. His relationships are valid for all $r$ and include the static term; they reduce to Manbeck's

Fig. 21.--The wavefront of a single traveling wave starting at the end of a monopole radiator (p=1).
results for the far-field. Finally, Schelkunoff[2] again looked at
the problem of radiation from a single wire radiator. His results
were identical to Alford's and contained the static term. Schelkunoff
was able to show that Mannbeck's relationships were valid for all r
and matched the boundary conditions at the wire, something Mannbeck
had been unable to do. In all of these analyses three things were
assumed: (1) the radius a of the wire was infinitesimally small,
(2) the impedance of the wire was independent of time and space
and (3) the velocity of the current wave was the speed of light. Our
relationships assume the first two but allow the current velocity
to have any value. It is of interest to note that Eq. (49) is exact
(that is, not an approximation) for all r until the current wave reaches
the end of the wire. Thus there is no distinction between near and
far-field regions until the current reaches the termination of the
wire conductor (see Fig. 21).

Let us look now at the implications of Eq. (49) in order to see
how the far field patterns are formed. It may be written

\[ E(t, \theta) = \frac{\mu b \sin \theta}{4\pi r} \frac{1}{2 T_I} \{ J(t) - J(t - 2 T_I) \} \]

where \( t = t' - r/c, \ T_I = -b/2c(\cos \theta - 1/p) \) and \( p = v/c \). Until this
point no concern was given as to how the current got on the
radiating structure or where it went after it had traversed it. For
the end-fed wire radiator, however, we see that what we actually have is a radiating transmission line so now we must address ourselves to the problem of impedance matching at each junction point. Implicit in Eq. (97), therefore, is the fact that when the current reaches the far end of the wire it feeds into a load impedance \( Z_L \) equal to the line impedance \( Z_0 \) \( (Z_L = Z_0) \), thus there was no reflected current. In actual practice, however, this is seldom the case. In fact, generally \( Z_L \geq \infty \) and the generator impedance \( Z_s \neq Z_0 \). Thus the transmission line is in general unmatched at both ends and we find that the current will be reflected back and forth. Thus we must modify Eq. (97) to include this phenomenon as Bulgakov et al. [35] have suggested. We have then

\[
E(\theta, t) = \frac{ub \sin \theta \theta_1}{4 \pi r} \left[ \frac{1}{T_I} \{ J(t) - J(t - 2T_I) \} + \right.
\]

\[
+ \frac{\Gamma_e}{T_R} \{ J(t - 2T_I) - J(t - 2T_I - 2T_R) \} +
\]

\[
+ \frac{\Gamma_f s}{T_I} \{ J(t - 2T_I - 2T_R) - J(t - 4T_I - 2T_R) \} + ---
\]

where \( \Gamma_e \) = reflection coefficient from the tip, \( \Gamma_s \) = reflection coefficient \( (Z_0 - Z_s)/(Z_0 + Z_s) \) from the generator feed line (c.f., Refs. 23 and 36) and \( T_R = b/2c(\cos \theta + 1/p) \). Equation (98) is quite general but is of course only valid in the far-field region for times \( t - 2T_I > 0 \).
Let us now look at some specific examples. We consider first the most common case where \(|v| = c\), \(Z_L = \infty\) and \(Z_S = Z_0\) then
\(|p| = -\Gamma_e = 1\) and \(\Gamma_s = 0\) and Eq. (98) becomes

\[
E_d(\theta, t) = \frac{\mu b \sin \theta}{4\pi r} \left[ \frac{J(t) - J(t - \frac{b}{c}(1 - \cos \theta))}{\frac{b}{c}(1 - \cos \theta)} - \frac{J(t - \frac{b}{c}(1 - \cos \theta)) - J(t - 2(\frac{b}{c}))}{\frac{b}{c}(1 + \cos \theta)} \right].
\]

Note that the directions in which the signal derivative is radiated, \(\theta = \pm \pi/2\), are the very directions for which \(\sin \theta = E_\theta = 0\). Thus the wire radiator does not radiate the signal derivative. If we note that \(\sin \theta/(1 - \cos \theta) = (1 + \cos \theta)/\sin \theta\) and \(\sin \theta/(1 + \cos \theta) = (1 - \cos \theta)/\sin \theta\) then Eq. (99) becomes

\[
E_d(\theta, t) = \frac{\mu c}{4\pi r \sin \theta} \left[ (1 + \cos \theta)J(t) - 2J(t - b/c(1 - \cos \theta)) + (1 - \cos \theta)J(t - 2(b/c)) \right].
\]

Equation (99') shows that there are three distinct waves radiated from the ends of the wire. One each time the wave arrives at a spatial discontinuity. These three wave fronts are graphically illustrated in Fig. 22 for the time ranges noted. In Fig. 23 we see the S-T plot of the raised cosine function excitation on a linear radiator of length \(b = 3\lambda\) or 3PW. Note the directions of maximum radiation first at \(\theta \sim 30^\circ\) and later at \(\theta \sim 150^\circ\).
Fig. 22.--The wavefronts of waves originating from the ends of a finite length wire monopole (c.f. Fig. 21) with $Z_S = Z_0 \neq Z_L$ and $p=1$. 
Fig. 23. -- The $S$-$T$ plot of the radiation from the monopole of Fig. 21 excited by a raised cosine function with $Z_L = \infty$. 
a. **Harmonic excitation**

Let us look now at the conventional case of harmonic excitation

\[ J(t) = I_0 \sin \omega t \quad . \]

Here we shall consider the specific resonant case where \( b = n\lambda /2 \).

This problem has been considered previously (Stratton[37]) therefore we will be able to compare results. Substituting Eq. (100) into Eq. (99') with the above restriction on \( b \) gives

\[ E_\theta(\theta, t) = \frac{\mu_c I_0}{4\pi r \sin \theta} \left[ 2 \sin \omega t - 2 \sin(\omega t - n\pi(1 - \cos \theta)) \right] \quad . \]

Making use of the trigonometry relationship

\[ \sin \omega t - \sin(\omega t - n\pi(1 - \cos \theta)) = 2 \sin(n\pi/2 \cos \theta) \times \]

\[ \times \cos(\omega t + n\pi/2 \cos \theta) \quad n \text{ - even} \]

or

\[ = 2 \cos(n\pi/2 \cos \theta) \cos(\omega t + n\pi/2 \cos \theta) \quad n \text{ - odd} \]

and substituting Eq. (102) back into Eq. (101) gives

\[ E_\theta(\theta, t) = 120 I_0 \left[ \frac{\sin(n\pi/2 \cos \theta)}{\sin \theta} \right] \cos(\omega t + n\pi/2 \cos \theta) \quad n \text{ - even} \]

\[ E_\theta(\theta, t) = 120 I_0 \left[ \frac{\cos(n\pi/2 \cos \theta)}{\sin \theta} \right] \sin(\omega t + n\pi/2 \cos \theta) \quad n \text{ - odd} \]
where we note $\mu c/\pi \sim 120$. Our result differs from that of Stratton in the phase term $n\pi/2 \cos \theta$, since Stratton uses the center as his reference and we use the end, and also in that Eq. (103) is twice as large as Stratton's result. This is because he assumes a current distribution of the form

\[ J(x, t) = I_0 \sin(2\pi/\lambda(b - |x|) \sin \omega t \]

which already assumes a steady state condition with a current maximum of $I_0$. For our transient analysis we do not assume any current distribution but only an input waveform and have it satisfy the boundary conditions. We discover as in all transmission analysis that after steady state is reached the current maximum of our standing wave is $2I_0$. We note that our input impedance must therefore have changed and is now determined by the boundary condition. We shall consider this again when we look at the center-fed dipole.

2. **Radiation from end-fed linear radiators where $p < 1$**

We consider here the case where the current velocity $v$ is less than $c$ thus making $p < 1$. This was most recently discussed by Bolle and Jacobs[38] who were concerned about the direction of maximum radiation from short pulse excitations on end-fed radiators matched to the characteristic impedance, Eq. (97) (see Figs. 21 and
Looking at Eq. (97) which we have written again for convenience

\[ E(\theta, t) = \frac{\xi_0}{4\pi r} \frac{\sin \theta}{1/p - \cos \theta} \{ J(t) - J(t-b/c(1/p - \cos \theta)) \} \]

it is obvious that the variation of intensity with direction is determined solely by the function \( \sin \theta / (1/p - \cos \theta) \) at least for the short pulse and transient condition. To determine the angle of maximum intensity we must set

\[ \frac{d}{d\theta} \frac{\sin \theta}{1/p - \cos \theta} = 0 \]

When this is solved we discover that this requires that \( \cos \theta = p \) for maximum radiation. This is rather interesting since we saw that for \( p > 1 \) maximum radiation occurred in the direction for which \( \cos \theta = 1/p \) (see Fig. 24a). If, however, instead of Eq. (97) we use Eq. (99) then the direction of maximum radiation for both cases will shift slightly. This is illustrated in Figs. 25a and b for a raised cosine excitation with \( b = 2 \text{PW} \). Here we see in Fig. 25a and b for \( p = \sqrt{2} \) that the maximum occurs at \( \theta \sim 50^\circ \) and for \( p = 1/\sqrt{2} \) the maximum occurs at \( \theta \sim 42^\circ \). Both of these are very near the predicted \( 45^\circ \). The larger reflected wave is seen therefore to only slightly affect the direction of maximum radiation. We note that the current wave and the radiated wave proceeding in the same direction do so at differing velocities. This in no way affects the radiating process, only the radiation pattern and waveforms.
Fig. 24. -- The wavefronts of waves originating from the ends of a finite length wire monopole resulting from currents with phase velocities as noted.
Fig. 25a. -- The S-T plot of the radiation from the monopole of Fig. 21 excited by a raised cosine function with $Z_L = \infty$ and $p = \sqrt{2}$. Compare with Fig. 24a.
Fig. 25b. --Same S-T plot as Fig. 25a but with $p = 1/\sqrt{2}$.
Compare with Fig. 24b.
3. **The center-fed dipole**

The center-fed dipole is essentially a two wire open-ended transmission line as shown in Fig. 26. Instead of the two wires being parallel to each other, thus minimizing the radiated power, the two wires lie 180° to each other so as to maximize radiated power. In order to analyze the radiation from such a configuration we can consider it as two end-fed linear radiators back to back. Here again we shall assume no dispersion on the line and that the voltage and current waveforms are identical. The relationship for the radiated electric field thus takes the form

\[
E_\theta(\theta, t) = \frac{\mu b \sin \theta}{8\pi r} \left\{ \begin{aligned}
&\frac{1}{T_I} \{ J(t) - J(t - 2T_I) \} \\
&\frac{\gamma_e}{T_R} \{ J(t - 2T_I) - J(t - 2T_I - 2T_R) \} \\
&\frac{\gamma_e \gamma_s}{T_I} \{ J(t - 2T_I - 2T_R) - J(t - 4T_I - 2T_R) + \cdots \} \\
&\frac{1}{T_R} \{ J(t) - J(t - 2T_R) \} + \frac{\gamma_e}{T_I} \{ J(t - 2T_R) - J(t - 2T_R - 2T_I) \} \\
&\frac{\gamma_e \gamma_s}{T_R} \{ J(t - 2T_R - 2T_I) - J(t - 4T_R - 2T_I) \} + \cdots \end{aligned} \right\}
\]

where

\[ T_I = \frac{b}{2c} \left( \frac{1}{p} - \cos \theta \right) \]

and
Fig. 26. -- Wavefront schematic of waves originating from the center and tips of a center-fed dipole for $Z_o = Z_s$. The $\times$ and $\cdot$ symbols show the polarity of the H-field.
\[ T_R = \frac{b}{2c} \left( \frac{1}{\lambda_p} + \cos \theta \right) . \]

Combining terms we have

\begin{equation}
E_0(\theta, t) = \frac{\mu b \sin \theta}{8\pi r} \left[ \frac{1}{T_1} \left\{ J(t) - J(t - 2T_I) + \Gamma_e J(t - 2T_R) \right\} \right. \\
\left. - \Gamma_e (t - 2T_R - 2T_I) + \Gamma_e \Gamma_s J(t - 2T_I - 2T_R) \right. \\
\left. - \Gamma_e \Gamma_s J(t - 4T_I - 2T_R) + \ldots \ldots \right\} + \\
\left. + \frac{i}{T_R} \left\{ J(t) - J(t - 2T_R) + \Gamma_e J(t - 2T_I) \right\} \\
\left. - \Gamma_e (t - 2T_R - 2T_I) + \ldots \ldots \right\} \right].
\end{equation}

If we now assume \( p = 1 \), i.e., \( v = c \), and again noting that

\[ \sin \theta / (1 - \cos \theta) = (1 + \cos \theta) / \sin \theta \]
and \( \sin \theta / (1 + \cos \theta) = (1 - \cos \theta) / \sin \theta \)

Eq. (106) may be written

\begin{align}
\text{(1) } E_0(\theta, t) &= \frac{\xi_0}{4\pi r \sin \theta} \left[ 2J(t) - (1 - \Gamma_e) \{ J(t - b/c(1 - \cos \theta)) \right. \\
\text{(2) } &\left. - J(t - b/c(1 + \cos \theta)) \} - 2\Gamma_e (1 - \Gamma_s) J(t - 2b/c) - \\
\text{(3) } &\left. - \Gamma_e \Gamma_s (1 - \Gamma_e) \{ J(t - 2b/c - b/c(1 - \cos \theta) \right. \\
\text{(4) } &\left. + J(t - 2b/c - b/c(1 + \cos \theta)) \} \\
\text{(5) } &\left. - 2\Gamma_e^2 \Gamma_s (1 - \Gamma_s) J(t - 4b/c) + \ldots \ldots \right\} \right].
\end{align}

Usually it is assumed that \( \Gamma_e = -1 \) since the two radiators are terminated in an open circuit. We shall as well make this assumption.
for the time being. Later, however, it will be expedient to let \(|\Gamma_e| < 1\) in order to include the effects of radiation loss.

To illustrate Eq. (107) let us consider the usual harmonic excitation with the antenna matched to the source, \(Z_0 = Z_s\), then \(\Gamma_s = 0\) and \(J(t) = I_o \sin \omega t\), \(\omega/c = 2\pi/\lambda\). Substituting into Eq. (107) yields

\[
E_\theta(0, t) = \frac{E_0}{2\pi r \sin \theta} \left[ \sin \omega t + \sin(\omega t - 2b(2\pi/\lambda)) - \{\sin(\omega t - b(2\pi/\lambda))
\right.

\]

\[
+ b(2\pi/\lambda)\cos \theta + \sin(\omega t - b(2\pi/\lambda) - b(2\pi/\lambda)\cos \theta) \right]\.
\]

Making a trigonometric rearrangement yields

\[
E_\theta(t, \theta) = -\frac{120I_o}{r} \left[ \frac{\cos(b(2\pi/\lambda)\cos \theta) - \cos(-b(2\pi/\lambda))}{\sin \theta} \right]
\]

\[
\sin(\omega t - b(2\pi/\lambda))
\]

which compares to Kraus [13], Eq. (5-81) p. 141, except for the expected factor of 2 and the phase term \(b(2\pi/\lambda)\). This has been plotted in Fig. 27 for \(b = .75\lambda\) and the step excitation \(I_o = u(t)\sin \omega t\). Note that the lobes are alternately out of phase as Kraus has suggested (see Ref. 13, p. 142).

An S-T plot of the raised cosine excitation is shown in Figs. 28a and b for \(b = 2\)PW and it is of tutorial interest to compare Fig. 28 with the wave front sketch of Fig. 26c. Again we have assumed \(\Gamma_e = -1\) and \(\Gamma_s = 0\).
Fig. 27. --S-T plot for a center-fed dipole step excited by a harmonic function.
Fig. 28a. -- S-T plot for a center-fed dipole excited with a raised cosine function.
Fig. 28b.--Rear half of Fig. 28a showing waveform for $\theta = \pi/2$. 
It is of some interest to next look at the radiated wave from
the infinitesimal dipole, i.e., one where \( \frac{b}{c} \ll PW \). Here we again
assume the same reflection coefficients as above. Thus we have

\[
E(\theta, t) = \frac{60}{r \sin \theta} \left[ J(t) + J(t - 2b/c) - J(t - b/c(1 - \cos \theta))
- J(t - b/c(1 + \cos \theta)) \right].
\]

Expanding each of the terms in the bracket in a Taylor series about
\( t \) we find that the largest term is the second derivative and Eq. (109)
becomes

\[
E(\theta, t) \approx \frac{60 \sin \theta}{r} J''(t)(b/c)^2 \quad \frac{b}{c} \ll PW
\]

This is illustrated in Fig. 29 for the raised cosine function with
\( \frac{b}{c} \approx 0.01 \ PW \). Equation (109) was used for the computation and we
note that the waveform is very nearly the second derivative of
the input as indicated (see Ref. 29, p. 125) by Eq. (110). The
above equations and figures are also applicable to the end-fed
dipole over a ground plane.

4. Evaluating the reflection coefficients

In order to proceed with the analysis of the linear radiator in
the time domain it is necessary to determine the two reflection
coefficients. To do this it will be necessary to have values for the
input or surge impedance. By surge impedance we mean the
Fig. 29. -- S-T plot for an electrically short center-fed dipole excited with a raised cosine function as in Fig. 28. Note the output waveshape is the second derivative of the exciting waveform (compare with Fig. 28a).
relationship between the magnitude and waveform of the input voltage and input current to the radiator, before the effects of the end reflections are felt at the feed point (c.f. Ref. 36). In other words it is the impedance seen by the generator at the antenna feed for short pulse excitation which, we find, is essentially resistive making the voltage and current waveforms nearly identical.

As was mentioned at the beginning of this section, Wu[21] has considered this problem. He analyzed it from the cylindrical waveguide point of view where the wave travels along the outside of the guide. He developed a relationship for the input impedance of an infinitely long base-driven antenna over a ground screen. The result of this work is that if the voltage pulse has finite rise so that its upper frequency limit is $\omega_C/2\pi$ and if $\omega_C$ and the antenna radius (a) have such values that for all frequencies in the pulse the following inequality is applicable

\begin{equation}
\pi^2 \ll \left| \ln\left(\frac{c}{\omega a}\right) \right|^2, \quad \frac{c}{\omega a} = \frac{\lambda}{2\pi a}
\end{equation}

i.e., the cut-off frequency for the waveguide is much less than $\omega_C$, then the characteristic or input impedance at the antenna feed is given approximately by the average impedance $Z_0(\omega)$ for the range of frequencies in the pulse where

\begin{equation}
Z_0(\omega) \approx \frac{Z_0}{2\pi} \left[ \ln \frac{c}{\omega a} - 0.5772 - j \frac{\pi}{2} \right].
\end{equation}
If $\omega_c$ is the upper angular frequency limit in the pulse, then

$$Z_0 = \overline{Z_0(\omega)} = \frac{1}{\omega_c} \int_0^{\omega_c} F(\omega) Z_0(\omega) d\omega \tag{113}$$

where $F(\omega)$ is the spectrum function for the pulse. As an approximation let it be assumed that $F(\omega) \approx 1$ over the range $0 \leq \omega \leq \omega_c$.

In this case

$$\frac{\xi_0}{2\pi \omega_c} \int_0^{\omega_c} \left[ ln \frac{c}{\omega a} - 0.5772 - j \frac{\pi}{2} \right] d\omega \tag{114}$$

$$= \frac{\xi_0}{2\pi} \left( 1 + ln \frac{c}{\omega_c a} - 0.5772 - j \frac{\pi}{2} \right)$$

Thus, the average impedance is approximated by an expression that involves only the impedance at the upper frequency limit. That is,

$$\overline{Z_0(\omega)} = Z_0(\omega_c) + \frac{\xi_0}{2\pi} \tag{115}$$

As might be expected, the characteristic impedance is a function of the rise time of the pulse or, as is shown, the highest frequency contained in the pulse spectrum. Equation (114) is somewhat of a hybrid since it is frequency independent but contains a capacitive quadrature component. For the range of $\omega_c$'s where Eq. (111) applies the imaginary part can be ignored and $\overline{Z_0(\omega)}$ can be considered purely resistive which appears to match experiment.
The magnitude of the reflection coefficient is given by the usual expression

\[
\Gamma = \left| \frac{Z_o(\omega) - Z}{Z_o(\omega) + Z} \right|
\]

(116)

An alternate formulation is to define the average reflection coefficient directly in the form

\[
\bar{\Gamma} = \frac{1}{2\omega_c} \int_{-\omega_c}^{\omega_c} \frac{Z_o(\omega) - Z}{Z_o(\omega) + Z} \, d\omega
\]

(117)

While this latter form is more accurate, it is much more difficult to evaluate and the difference in accuracy is quite minimal over the range of validity of Eq. (111) (see Ref. 23 where Eqs. (116) and (117) have been plotted over 5 decades of \(1/2\pi a\)).

5. **Radiation loss**

Before we can compare our theory with the experimental results available, it is necessary to give some consideration to the effect on the antenna current of the radiation loss. If this loss is small enough per pass it can be lumped into the reflection coefficient \(\Gamma_e\) making \(\Gamma_e\) something less than unity. Krasilinikow[39] has suggested a possible means of determining a value for non-harmonic radiation resistance for the center fed dipole which we will outline here.
Let us consider what would initially happen when a constant voltage source such as a battery were connected to a center fed dipole (i.e. Fig. 26). We have assumed then a current excitation of the form \( I_0 u(t) \), i.e., a step function. Clearly in the steady state there is no current and thus no radiation from this excitation so that any radiation can occur only during the transient period. This radiation occurs during the time the voltage and current pulses bounce back and forth between the source and the dipole tips as the voltage on the dipole builds up to the source voltage. To determine the effective radiation resistance we compute the average power radiated during the time it takes the leading edge of the wave to travel to the antenna tips and back to the feed point once (see Fig. 26c). Since the current is constant during that time we can quite easily determine the radiation resistance from \( I_0^2 R \) which is equal to the average power radiated. From Eq. (16a) we have the average power radiated as

\[
P(\theta, \phi) = \frac{1}{\tau} \int_0^\tau r^2 S(\theta, \phi, r, t) dt = \frac{1}{\tau} \int_0^\tau \frac{r^2}{\zeta_0} E^2(\theta, \phi, r, t) dt
\]

where \( \tau = 2b/c \) and \( E \) is given by Eq. (109). Letting

\[
J(t) = I_0 u(t)
\]
and noting from Fig. 26c the symmetry about $\theta = \pm \pi/2$ we shall consider only the region $0 < \theta < \pi/2$ and double the result. Only the shaded region contributes to the integral and thus Eq. (16a) becomes

$$
(16a) \quad P(\beta, \phi) = \frac{c}{2b} \int_0^{2b/c} \frac{b/c(1-\cos \theta)}{4\pi^2 \sin^2 \theta} \, dt + \frac{b/c(1-\cos \theta)}{4\pi^2 \sin^2 \theta} \int_0^{2b/c} \frac{\xi_0 I_0^2}{4\pi^2 \sin^2 \theta} \, dt \\
= \frac{\xi_0 I_0^2}{4\pi^2} \left\{ \frac{1-\cos \theta}{\sin^2 \theta} \right\}.
$$

The total average radiated power is then found by integrating over $\theta$ and $\phi$ giving

$$
(119) \quad \overline{P} = \int_0^{2\pi} d\phi \int_0^{\pi} P(\theta, \phi) \sin \theta \, d\theta \\
= \frac{\xi_0 I_0^2}{2\pi^2} \int_0^{2\pi} d\phi \int_0^{\pi/2} \frac{1-\cos \theta}{\sin \theta} \, d\theta \\
= 120 I_0^2 \int_0^{\pi/2} \tan \theta/2 \, d\theta \\
= 120 I_0^2 \ln^2.
$$

It is interesting to note that Eq. (119) is independent of the dipole length $b$ and thus so is the effective radiation resistance which we see must have the value

$$
(120) \quad R_{rad} \approx 83 \Omega.
$$
For the end fed dipole the radiation resistance would of course be just one-half this value.

The substitution of Eq. (119) into Eq. (116) would give for the effective radiation reflection coefficient

\[ \Gamma_e = \left| \frac{Z_0(\omega) - 41.5}{Z_0(\omega) + 41.5} \right| \]  

(121)

A comment on the physical significance of Eq. (121) might be in order at this point. We have lumped the effect of radiation loss into a reflection coefficient \( \Gamma_e < 1 \) whereby implying that all the radiation and thus all the current reflection occurs at the dipole tips and is thus not a distributive effect along the dipole length. The current pulse is thus assumed to travel undistorted down the dipole to the tips where it is reflected with a change in sign and a decrease in magnitude but is otherwise unchanged (see Fig. 26).

The exactness of this assumption will be seen in the next section.

Looking at the case where the loss is due solely to the radiation, that is, \( |\Gamma_S| = 1 \), the number of round trip passes \( n \) necessary to reduce the amplitude of \( E_0 \) to \( 1/e \) of its initial value is given by

\[ n = -\frac{1}{\ln \Gamma_e} \]  

(122)

Thus the smaller \( \Gamma_e \) the smaller \( n \). This brings up the interesting point that if Eq. (114) is used for \( Z_0(\omega) \), which we note is a function
of $\omega_c^{-1}$, then the reflection coefficient is seen to decrease with increased $\omega$. Thus it would seem that pulses with steep slopes, i.e., larger $\omega_c$, would damp out more rapidly than pulses with more gentle changes in current (voltage), all things being equal. We also note that the damping/pass is independent of length but dependent on wire radius $a$. This would appear to make sense when we remember that radiated energy is $\sim \omega$. The decrease with $\omega$ of the reflection coefficient from a resistive load $\Gamma_\text{g}$ is shown graphically by King et al. [23]. It is not known, however, whether or not the reduction of $\Gamma_\text{e}$ with $\omega_c$ has been shown before.

6. The effect of $\Gamma_\text{g}$ and $\Gamma_\text{e}$ on the S-T plot
(a comparison with experimental data)

Finally let us examine how Fig. 28 would be modified for various $\Gamma_\text{g}$'s and compare our results with the results obtained using Fourier methods as well as some of the limited experimental data available. [20, 23, 26, 29, 30, 31, 35] Specifically, we shall compare our results with what Schmitt et al. [29] and Palciauskas et al. [30] have obtained for the vertical end-fed line over a ground plane (Fig. 30a). Thus we will now look at how the S-T plot changes with source impedance $Z_\text{g}$. We compare from the reflection coefficients and Eq. (107) the relative heights, waveshapes and spacing of the radiated pulses which result from the single short pulse excitation of the end-fed dipole shown in Fig. 30a.
Fig. 30. -- (a) Geometry of the transmission experiment (Schmitt[29]). (b) Calculated, (c) Measured time history of the radiation field $E_\theta(t)$ along the ground plane ($\theta=\pi/2$) for the excitation $V_s$ shown. (time scale = 1.25 µs/cm).
To compute the reflection coefficients we need, of course, a value for the surge or input impedance $Z_o$ of the dipole and as was shown earlier there are at least two expressions one might use, Eq. (95) or Eq. 114. We shall for this comparison use Eq. (95); the rationale for this choice will be seen shortly. To use Eq. (95) we must have a value for $b/a$, the height to wire radius ratio. Here we assume the value used by Schmitt and Palciauskas that of $b/a = 904$, which when substituted into Eq. (95) gives $Z_o = 408\Omega$.

The computation of the reflection coefficients is now straightforward.

The radiation reflection coefficient $\Gamma_e$ is thus found to be

$$
(123) \quad |\Gamma_e| = \left| \frac{408 - 41.5}{408 - 41.5} \right| = |0.81|
$$

while the three source reflection coefficients $\Gamma_s$ for the three source resistances $Z_s$ of Fig. 32 are computed from Eq. (116) to be $\Gamma_{Z_s=0\Omega} = 1.0$, $\Gamma_{Z_s=50\Omega} = 0.76$ and $\Gamma_{Z_s=300\Omega} = 0.15$. We are now ready to compare results.

We will look first at the curves of Fig. 32 as computed by Palciauskas et al. [30] Here the author has assumed that the linear radiator of Fig. 30a is excited by a train of pulses $T = 0.4(b/c)$ wide and occurring every $T = m(b/c)$ seconds for $m = 25$. The pulses are far enough apart so that the ringing effects of one pulse have disappeared before the next pulse arrives.
The wave shape of one pulse of the exciting voltage is shown in Fig. 31. It is very nearly a raised cosine function. This pulse train can be approximated by a finite Fourier series, the harmonics being determined from \( \omega_n = \frac{2\pi n}{T} \). For this example the highest harmonic used is \( n = N = 100 \) so the cutoff frequency is \( \omega_c = 8\pi b/c \).

![Waveform graph](image)

**Fig. 31.** Waveform of the source voltage, \( V_s(t) \), which generated the radiated fields shown in Figs. 32 and 33 (Palciauskas[30]).

An interesting aside at this point; if this value of \( \omega_c \) is substituted into Eq. (114) for \( Z_0(\omega) \) we obtain

\[
(114') \quad \frac{Z_0(\omega)}{2\pi} = \frac{k_o}{\ln(b/a)} + \text{const.}
\]

The constant is a function of \( m, N \) and the spectrum \( F(\omega) \) of the pulse train. We see then that \( Z_0(\omega) \) (Eq. (114)) can as well be written in terms of \( \ln(b/a) \), i.e., in the same form as Eq. (95).

The electric field \( E_\theta(t) \) at a specific angle \( \theta \) is then found by evaluating the electric field radiated by each Fourier component using Hallen's expression[40] and then synthesizing to get the total
Fig. 32. -- The time history of the calculated radiation field $E(t)$ at $\theta = \pi/2$ for several source impedances and for the excitation shown in Fig. 31.
Fig. 33.--The time history of the calculated radiation field $E_r(t)$ along several ($\theta$) look angles, for $Z_s = 50\Omega$ and the excitation shown in Fig. 31.
\( E_0(t) \) (see Figs. 32 and 33). This is also the procedure used by Schmitt, et al. [29] in obtaining Fig. 30b.

Looking now at these figures let us see how closely they match what is expected from our theory using Eq. (107) which is written below for the case \( \theta = \pi/2 \)

\[
E_0(\pi/2, t) = \frac{L_0}{4\pi r} \left[ \begin{array}{c}
(1) \\
(2) \\
(3) \\
(4) \\
(5) \\
(6)
\end{array} \right] \begin{array}{c}
2J(t) - 2(1 - \Gamma_e)J(t - b/c) \\
\Gamma_e(1 - \Gamma_s)J(t - 2b/c) - 2\Gamma_e\Gamma_s(1 - \Gamma_e)J(t - 3b/c) - \\
- 2\Gamma_e^2\Gamma_s(1 - \Gamma_s)J(t - 4b/c) - \\
- 2\Gamma_e^2\Gamma_s(1 - \Gamma_e)J(t - 5b/c) + \ldots \end{array}
\]

We consider then the three cases pictured in Fig. 32, which shows the far electric field \( E_0(t) \) radiated in the \( \theta = \pi/2 \) direction as computed for three different source impedances \( Z_s \). From the curves three things are apparent. First, pulses (1) and (2) in each of the three cases appear to be independent of \( Z_s \). Second, the shape of all the pulses is very nearly that of the exciting wave shape Fig. 31. And third, the spacing between pulses is, except for the top curve, \( \Delta t = b/c \).
Looking back at Eq. (107') we see that this is exactly what should be expected. For example, using the value of \( \Gamma_e = -0.81 \) as obtained in Eq. (123) for the radiation loss reflection coefficient to compute the ratio of pulse (1) to pulse (2), we obtain the ratio 1: -1.81 (the minus sign for \( \Gamma_e \) is used since the current waveform reflected from an open circuit is the negative of the incident waveform). This ratio compares quite favorably with the values measured from the three curves. To further illustrate Eq. (107') we have computed the ratios of the height of each of the first six pulses as compared to pulse (1) for the \( Z_s \)'s of Fig. 32. These ratios were then compared with the ratios as taken directly from the curves of Fig. 32 and Fig. 30b and c are presented in tabular form in Table 1.

Table 1 shows very good agreement between the results obtained from Eq. (107') and those obtained using Fourier Techniques. Agreement is particularly good when theoretical data are compared to the experimental data.

The strength of our analysis lies in the fact that results can be much more quickly obtained than by existing methods and the effect of the variable parameters can be easily pictured. For instance, from Eq. (107') and Fig. 30a it is seen that the first


**TABLE 1**

A COMPARISON OF THE MEASURED AND THEORETICAL RADIATION RESPONSE OF THE PULSE EXCITED END-FED DIPOLE MOUNTED VERTICALLY OVER A GROUND PLANE

<table>
<thead>
<tr>
<th>Pulse Curve Number by Impedance</th>
<th>Ratio of height of pulse number N to pulse number one</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td><strong>$Z_g = 0\Omega$</strong></td>
<td></td>
</tr>
<tr>
<td>a) Eq. 107'</td>
<td>1</td>
</tr>
<tr>
<td>b) Fig. 32</td>
<td>1</td>
</tr>
<tr>
<td><strong>$Z_g = 50\Omega$</strong></td>
<td></td>
</tr>
<tr>
<td>a) Eq. 107'</td>
<td>1</td>
</tr>
<tr>
<td>b) Fig. 32</td>
<td>1</td>
</tr>
<tr>
<td>c) Fig. 30b</td>
<td>1</td>
</tr>
<tr>
<td>d) Fig. 30c</td>
<td>1</td>
</tr>
<tr>
<td><strong>$Z_g = 300\Omega$</strong></td>
<td></td>
</tr>
<tr>
<td>a) Eq. 107'</td>
<td>1</td>
</tr>
<tr>
<td>b) Fig. 32</td>
<td>1</td>
</tr>
</tbody>
</table>

-Note-

a) Data obtained from Eq. 107'
b) Data obtained from Fig. 32 (Palciauskas et.al.)
c) Data obtained from Fig. 30b theoretical results (Schmitt et.al.)
d) Data obtained from Fig. 30c experimental results (Schmitt, et.al) with .66 correction factor included.

(Measurements taken from highest point on each pulse.)
pulse radiated after single-pulse excitation arises as the charge flows into the antenna from the source; the second radiated pulse is due to the reversal of charge flow at the end of the antenna; the third is caused by the second reversal at the impedance discontinuity between antenna and its source; etc. Thus, the second, fourth, etc., radiation pulses in each trace have their source at the free end of the monopole while the first, third, etc., originate at the junction of the monopole and driving source. (See also Fig. 26). For example, in the top curve of Fig. 32 where \( Z_s = 0 \), and thus \( \Gamma_s = 1 \), we find from Eq. (107') that \( (1 - \Gamma_s) = 0 \) and thus all of the odd numbered pulses except the first are missing. The loss in pulse strength is due to radiation. As \( Z_s \) is increased, and thus \( \Gamma_s \) is reduced, all of the pulses begin to appear as is seen in the lower two curves. Here loss occurs in \( Z_s \) as well as by radiation causing the pulse train to damp out more quickly. This is quite apparent in the bottom curve for \( Z_s = 300\Omega \). Here the ringing has nearly died out after only four passes. When \( Z_s = Z_0 \), the matched condition for which \( \Gamma_s = 0 \), the pulse train pictured in the S-T plot of Fig. 28a for \( \Gamma_e = -1 \), dies out after only one round trip, being totally absorbed when it reaches \( Z_s \).

We next look in some detail at the case of \( Z_s = 50\Omega \). For this case we have experimental data (Fig. 30) taken at \( \theta = \pi/2 \) and computer
results at four different look angles (Fig. 33) which we can compare with the S-T plot of Fig. 28a.

The theoretical results of Figs. 30b, 32 and 33 and of Eq. (107') were all based on the assumption that the point \((r, \pi/2)\) where the field was calculated was so far from the antenna that the distances to the observation point from either end of the antenna were essentially equal. Looking at the experimental set up used by Schmitt (Fig. 30a) we see that this is far from the case. The two distances differ by 0.22 meter which represents a difference in travel time of 0.74 ns. This accounts quite well for the rightward displacement in the delay time between pulse (1) and (2) and between pulse (3) and (4) of Fig. 30c.

The amplitude of the second pulse with respect to the first is affected by three factors:

1) It travels a distance that is greater by a factor of 1.14 (see Fig. 30a); since the field varies as \(1/r\), this reduces the even-numbered pulses by the factor of 1/1.14.

2) The pulse leaving the end of the antenna in the direction of the probe travels in a direction that is 29.2° from the normal (see Fig. 30a); this provides a factor of \(\cos 29.2° = 0.87\).

3) The effective field at the probe is also reduced by the factor \(\cos 29.2° = 0.87\) because the probe is not normal to the direction of propagation (see Fig. 30a). (The probe
is very short so that the wavefront excites it throughout its length almost simultaneously. Hence, the factor $\cos 29.2^\circ$ may be used as a first approximation.)

The amplitude of the even-numbered radiation pulses as measured in the experiment should, therefore, be $(1/1.14)(0.86)(0.87) = 0.66$ times the calculated value. This closely accounts for the lessened amplitude of the even-numbered pulses in Fig. 30c. A look at Table 1 shows that the corrected experimental data compares quite favorably with the data as obtained from Eq. (107').

We consider finally the waveshape of the electric field $E_0(t)$ radiated in off-broadside directions, i.e., $\theta \neq \pi/2$. To aid in our discussion we shall compare the four curves of Fig. 33 with the $S$-$T$ plot of Fig. 28a which, although drawn for the $Z_s = Z_0$ case, will illustrate the point. The directions of the four curves of Fig. 33 have been marked on Fig. 28a.

Looking at Fig. 28a and also Fig. 26 we see that as we move away from the $\theta = \pi/2$ direction the even-numbered pulses split up into two pulses as the curve for $\theta = 60^\circ$ Fig. 33 shows. In the case of pulse (2) one part moves toward pulse (1) and the other moves toward pulse (3). By the $\theta = 45^\circ$ direction pulses (1) and (2) are starting to overlap and in the $\theta = 20^\circ$ direction we begin to see what appears to be the derivative of the exciting waveform. This is exactly what
we have. The complete evolution of this is, of course, shown in
Fig. 28a. As was discussed earlier, the wire radiator with the
current velocity \( v = c \) radiates the signal derivative in the \( \theta = 0^\circ \)
direction but since the field strength \( E_\theta(t) \) is proportional to \( \sin \theta \)
the signal magnitude in that direction is zero. However, a good
approximation to this waveshape is seen at angles close to \( \theta = 0^\circ \)
which is what we see in the top trace of Fig. 33 for \( \theta = 20^\circ \).

Thus our very simple model does indeed yield amazingly
good first order results which can be presented graphically and
quite easily for all \( \theta \) allowing the reader to picture the mechanisms
involved in generating the pattern.

With the introduction of a parameter which allows us to include
in our analysis the effects of radiation damping, we can extend the
usefulness of our procedure to the scattering problem. We can show
the scattered field in all directions from a straight wire radiator and
can then quite easily deduce what type of loading, pulse train wave-
form or pulse spacing is necessary to create nulls or lobes in a
particular direction or at a particular time. For example, from
Eqs. (122) and (123) it is quite easy to determine that the magnitude
of the signal should have dropped to \( 1/e^2 \sim 1 \) after nine passes.
Bulgakov[35] has shown experimentally that Eq. (123) is quite
accurate.
CHAPTER IV
SUMMARY AND CONCLUSIONS

In the previous chapters we looked at several aspects of radiation using time domain analysis and from this investigation we have gained new insight into some of the concepts of antenna radiation. We wish here to recapitulate the findings of the study, delineating our conclusions and then speculate as to what new areas and problems might next be explored using the time domain approach.

To review briefly the nature of the study, we looked at three different types of radiating structures: the rectangular aperture, the one dimensional array, and the end- and center-fed dipoles, in an effort to determine how these structures radiate when excited by a general time dependent current excitation. In particular we looked at the far-field space-time response when these structures were excited by an assumed current distribution of known but arbitrary time dependence. Noting that electromagnetic radiation results whenever a charged particle is accelerated, i.e., there is a time rate of change of current, we were interested in determining in which direction, if any, an antenna radiates a signal whose waveshape
is the time derivative of the input excitation waveform. In this study no regard was given to the impedance effects (either input or mutual) except in the case of the linear radiators where resistive but not reactive effects were included. We look then at those conclusions which may be drawn from our investigation.

A. **Rectangular Aperture**

Here we considered the radiating properties of the rectangular aperture obliquely illuminated by a plane wave of arbitrary time variation. Thus the phase velocity \( v \) of the equivalent surface current was restricted to the values \( \omega \geq v \geq c \) (c.f., Fig. 4).

1. The radiated electric field \( E(\theta, \phi, t) \) can be considered to result from the interference of four waves radiating from the corners of the aperture (Eq. (35)). In the equatorial plane this reduces to two waves (Eq. (39)) and in the main beam direction this further reduces to one wave whose wave shape is the exact derivative of the exciting function (Eq. (42)). Thus a planar aperture radiates in one direction only, the direction in which the plane wave would have gone had the aperture not been there (the axis of the main beam), a waveform which is the exact derivative of the exciting current, i.e., \( E(t) \sim \frac{dJ(t)}{dt} \).

In the equatorial plane, \( \theta = 0 \), this direction is determined
from the relationship $\phi = \sin^{-1} \frac{C}{v}$ (c.f., Eq. (52) and also Fig. 6). Thus a discontinuity in the exciting function, such as switching when the signal is not passing through zero, would generate a spike in the main beam (c.f., Eq. (55d)).

2. Delta function excitation was considered (c.f., Fig. 7), this was approximated analytically by one cycle of the normalized raised cosine function. It radiates as a doublet in the $\phi = \sin^{-1} \frac{C}{v}$, $\theta = 0$ direction and as two distinct delta functions of opposite polarity at all other look angles (c.f., Fig. 8). Thus an excitation of sufficiently short duration is received at large look angles as two distinct pulses of opposite polarity whose waveforms are that of the excitation function.

3. Excitation by trains of two or more closely separated pulses is seen to generate lobes and/or nulls in the S-T plot whose locations can be easily computed (c.f., Eq. (61)). These lobes/nulls can be placed, within certain limits, anywhere in the S-T plot by properly specifying the polarity and spacing of the pulses.

4. A method was shown whereby the energy/power distribution pattern for a uniformly illuminated aperture with
any excitation function can be easily calculated (Eq. (16)). Patterns were drawn for four different representative excitation functions (the raised cosine, the harmonic of finite duration, the Gaussian modulated harmonic, and the full-wave rectified pulse train) and each was compared with the uniform harmonic excitation power pattern \((\sin x/x)^2\). The width of the main beam at the half power points was found to be narrower than for the harmonic in all cases but one (the raised cosine excitation). A look at the harmonic spectra of the excitation waveforms showed why this should be expected (see Section III-C). Beam narrowing was most apparent for the full-wave rectified pulse-train excitation, and here the side lobes were also lower. Further investigation showed that any pulsed periodic function should generate a main beam which is narrower than that for a harmonic signal of the same periodicity.

5. The possibility of amplitude modulating arbitrary periodic functions was considered and found to be possible with the restrictions being primarily technical (Eq. (74)).
B. **Array of Planar Apertures**

Here the radiation properties of arbitrarily excited one dimensional arrays of planar apertures were investigated in an effort to determine those parameters which are necessary to determine the existence and location of nulls and lobes in the array pattern. Thus we were investigating the fundamental properties involved with pattern multiplication. Mathematical relationships for various forms of high speed scanning were also developed.

1. There are two parameters which determine nulls and lobes in the radiation pattern of an array of uniformly spaced and equally weighted point sources -- the number and spacing of the elements and the harmonic content of the periodic excitation. Specifically, the frequency spectrum of a multifrequenced periodic excitation must contain only odd harmonics and the array must have an even number of elements. The harmonic content restriction assures the symmetry property $f(t) = -f(t + \Delta/2)$ necessary for nulls in the point source array (see Fig. 19). Nulls and lobes for other exciting functions and array configurations are not precluded but the analysis here becomes unwieldy.
2. A general expression was developed for the one dimensional array of finite apertures in order to study the effects of various types of scanning and element phasing and weighting on the S-T plot (Eq. (89)). To illustrate, the S-T plots of a raised cosine excited two-element array phased as an array of slots (Fig. 20b) and then as an array of separately excited planar apertures (Fig. 20a) were compared.

C. **Linear Antenna With Arbitrary Excitation**

The radiating characteristics of the arbitrarily excited center- and end-fed dipole, i.e., the open-ended single wire transmission line, were investigated. The current phase velocity \( v \) for this class of structures generally lies in the range \( c/v \geq 1 \). For this analysis it was necessary to included the effects of three parameters which previously had been ignored: the input transfer function \( Z(t) \) relating the input excitation voltage and the resulting current waveform \( i(t) \) as

\[
v(t) = \int_{-\infty}^{\infty} Z(\tau-t) i(t) \, dt
\]

the effect of radiation loss, and the fact that the current wave will in general be reflected from both the open end and the source.
This particular problem had been considered by other investigators and thus both theoretical and experimental data were available for comparison. In all but one of these analyses the previous investigators handled the problem in the frequency domain employing Fourier analysis which would yield exact results providing enough Fourier components are included. This approach, however, has the disadvantage of becoming quite unwieldy and time consuming for slowly converging functions, when a large number of look angles are to be considered or the effects on the radiation characteristics of variations in the parameters such as the source impedance are to be studied. A comparison of these data with the results obtained by our approximate but much simpler method showed excellent agreement (see Table 1). The conclusions drawn from this study are enumerated below.

1. The end- and center-fed dipoles can be treated analytically as an open-ended transmission line. A signal is launched into space each time the current wave passes a discontinuity or is reflected (see Figs. 22 and 26). The magnitude of the radiation is determined by the reflection coefficient at the discontinuity (see Eq. (98)). Radiation initially occurs from the input terminals as the wave is launched on the line and its waveshape is that of the
exciting wave (see Fig. 23). There is no distinction between the near and far field regions until the current wave reaches the dipole tip and is reflected. For the dipole the relationship $\sin^{-1} \frac{c}{v} = \theta$, still determines the direction in which the derivative of the signal will be radiated. Here however $c/v \geq 1$. Thus the electrically long dipole, i.e., one for which $T \leq \ell v$, where $T =$ the duration of the excitation $f(t)$ and $\ell =$ dipole length, will not radiate the derivative since there is no radiation along the dipole axis, i.e., $\sin^{-1} \frac{c}{v} = 90^\circ$ for $v = c$ and is undefined otherwise, (see Figs. 23 and 33). It is interesting, however, to note that the electrically short center-fed dipole ( $T >> \ell v$) radiates the second derivative of the exciting waveform (see Fig. 29).

2. The literature contains a number of studies which address themselves to the theoretical problem of determining the transfer function relating the input voltage waveform $v(t)$ to the resulting current $i(t)$ for a delta function excited dipole of infinite length. After studying this literature it was determined that to a very good approximation, for most of the excitation we would expect to use, this relationship is linear. That is,
\( v(t) = Z_o i(t) \) with \( Z_o = \text{constant} \). The relationship for the
input impedance \( Z_o \) which was found to yield results most
accurately matching experiment was given by Eq. (95).

3. Once a current wave is launched on the dipole structure
it bounces back and forth reflecting alternately from the
tip and the source and decaying in amplitude because of
radiation loss and power loss in the source impedance
(see Eqs. (98) and (106)). After a relationship for the
impedance \( Z_o \) had been developed, a computation of
the source reflection coefficient \( \Gamma_s \) was straightforward.
The effect of the source impedance \( Z_s \) on the radiation is
shown in Fig. 32. The effect of radiation loss on the
current wave was accounted for by assuming it to occur
entirely at the tips and thus it was possible to lump this
effect into the tip reflection coefficient \( \Gamma_e \leq 1 \). The
accuracy of this assumption may be seen from Table 1.

4. A comparison of the computed dipole radiation character-
istics from Eq. (106) with existing experimental and
theoretical results showed excellent agreement. The
advantage of the time domain method is that results are
easily and quickly calculated and can be presented in
graphic form showing the radiation waveform at all
look angles simultaneously (see Fig. 28).

The time domain method of analysis developed in this study is
quite flexible and may be used to rapidly and conveniently obtain
first order calculations of radiated fields from aperiodically excited
structures. The results, while not exact, are generally quite
accurate and quite amenable to graphic display illustrating the
fields in both space and time. The primary limitations at this
point is the difficulty of including in the calculations impedance
and scattering effects.

As to new areas of investigation the first which comes to mind
is "Pattern Synthesis" in the time domain. Here with the flexibility
of being able to independently select both the aperture distribution
and the excitation waveform it should be possible to shape and
place the fields in the S-T plane like a sculpture. Tseng and Cheng
[4] touch on the subject of non-uniform aperture distribution with
arbitrary excitation waveforms and it would appear that in many
cases a closed form relationship for the E and H fields should be
possible.

The time response of the receiving antenna should be investi-
gated, particularly the obliquely illuminated end-fed and center-fed
dipoles. Here resonant phenomena (i.e., the ringing of the current
wave on the structure) will surely become important and the mechanics governing this for finite dipoles should become apparent. This will logically lead to the concept of characteristic or natural modes for more complex structures. With this development the scattering problem can then be considered. Cherhousov [6] has investigated pulse scattering by the circular disk but with no thought given to the ringing effects.

It would appear that possibly the most effective way to analyze the time domain radiation problem is to consider the complete transmitting and receiving antenna system. That is, we should develop a transfer function $H(t, \theta, \phi)$ between the input terminals of the radiating antenna and the output terminals of the receiving antenna. This transfer function would, of course, be angle dependent and obey reciprocity. Tai and Foster[41] have formulated a possible approach to this problem.

It would appear that there is no lack of interesting and challenging problems still to be investigated, the solution of any of which should prove quite rewarding.
APPENDIX

FUNDAMENTAL PRINCIPLES OF ELECTROMAGNETIC RADIATION

A. Maxwell’s Equations and Associated Relationships

The four Maxwell field equations in differential form follow quite logically from three fundamental physical laws:

Faraday’s Induction Law

\begin{equation}
\oint_l \mathbf{E}(\mathbf{r},t) \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{E}(\mathbf{r},t) \cdot d\mathbf{s} = - \int_S \frac{\partial}{\partial t} \mathbf{B}(\mathbf{r},t) \cdot d\mathbf{s},
\end{equation}

Ampere’s Current Law

\begin{equation}
\oint_l \mathbf{H}(\mathbf{r},t) \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{H}(\mathbf{r},t) \cdot d\mathbf{s} = \int_S \mathbf{J}(\mathbf{r},t) \cdot d\mathbf{s} + \int_S \frac{\partial}{\partial t} \mathbf{D}(\mathbf{r},t) \cdot d\mathbf{s},
\end{equation}

and the Equation of Continuity

\begin{equation}
\oint_S \mathbf{J}(\mathbf{r},t) \cdot d\mathbf{s} = \int_V \nabla \cdot \mathbf{J}(\mathbf{r},t) dv = - \int_V \frac{\partial}{\partial t} \rho(\mathbf{r},t) dv.
\end{equation}

The middle and right hand sides of Eqs. (124) and (125) are two of Maxwell’s equations in integral form. In differential form, applicable at
a point in space, they may be written

\begin{equation}
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}
\end{equation}

(127)

\begin{equation}
\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}
\end{equation}

(128)

where \( \vec{J} \) is the total current, both source current (\( \vec{J}_s \)) and induced

current (\( \vec{J}_i \)),

\[ \vec{J} = \vec{J}_s + \vec{J}_i \]

In Eqs. (127) and (128) we have introduced the shorthand notation

\( \vec{J}(r, t) = \vec{J}(t) = \vec{J} \), etc., which often will be used throughout with the spatial

and/or temporal dependence understood.

Taking the divergence of Eq. (127) and noting that the divergence of
the curl is zero, we obtain

\begin{equation}
\nabla \cdot (\nabla \times \vec{E}) = -\nabla \cdot \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} \nabla \cdot \vec{B} = 0
\end{equation}

(129)

Integrating the right hand side of Eq. (129) in time from (\( t_0 \)), a time

early enough so that the system has not yet been turned on, to (\( t \)), the

present, yields

\begin{equation}
\nabla \cdot \vec{B}(r, t) \bigg|_{t_0}^{t} = \nabla \cdot \vec{B}(r, t) = 0
\end{equation}

(130)

a third member of Maxwell's equations.
The final equation is obtained in a similar manner from Eq. (128) with the aid of Eq. (126) in differential form:

\[ \nabla \cdot \overrightarrow{J} = -\frac{\partial \rho}{\partial t} \quad . \]

**Conservation of Charge**

Taking the divergence of Eq. (128) and substituting from Eq. (131) yields

\[ \nabla \cdot (\nabla \times \overrightarrow{H}) = -\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t} \nabla \cdot \overrightarrow{D} = 0 \quad . \]

Integrating the above as in Eq. (129) and equating the limits we obtain the final member of Maxwell's equations:

\[ \nabla \cdot \overrightarrow{D}(r,t) = \rho(r,t) \quad . \]

We see that the four Maxwell equations are interdependent, the divergence equations following logically from the curl equations. These four equations form the basis of electromagnetic field theory.

In order to use the Maxwell equations it is necessary to establish the relationships between \( \overrightarrow{B}(t) \) and \( \overrightarrow{H}(t) \), \( \overrightarrow{D}(t) \) and \( \overrightarrow{E}(t) \), and \( \overrightarrow{J}(t) \) and \( \overrightarrow{E}(t) \). If the media is dispersive, it is not possible to relate these functions by a simple constant as in cases of static or simple harmonic time dependent fields. That is, for dispersive media no simple relations exist between
$\overline{B}(t)$ and $\overline{H}(t)$ as $\overline{B} = \mu \overline{H}$. The relations must be expressed by the following convolution integrals:

\begin{equation}
\overline{D}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \left[ \delta(\lambda) + \overline{X}_e(\lambda) \right] \overline{E}(t - \lambda) d\lambda = \int_{-\infty}^{\infty} \overline{\varepsilon}(\lambda) \overline{E}(t - \lambda) d\lambda
\end{equation}

\begin{equation}
\overline{B}(t) = \mu_0 \int_{-\infty}^{\infty} \left[ \delta(\lambda) + \overline{X}_m(\lambda) \right] \overline{H}(t - \lambda) d\lambda = \int_{-\infty}^{\infty} \overline{\mu}(\lambda) \overline{H}(t - \lambda) d\lambda
\end{equation}

\begin{equation}
\overline{J}_f(t) = \int_{-\infty}^{\infty} \overline{\sigma}(\lambda) \overline{E}(t - \lambda) d\lambda
\end{equation}

where $\overline{\mu}(\lambda) = \overline{\varepsilon}(\lambda) = \overline{\sigma}(\lambda) = 0$, for $\lambda < 0$, and $\overline{X}_e(\lambda)$, $\overline{X}_m(\lambda)$ and $\overline{\sigma}(\lambda)$ are the electric and magnetic susceptibility and conductivity, respectively. Equations (134) and (135) are the constitutive relations appropriate to the time domain.

We will not prove Eqs. (134), (135) and (136) here since our study, with the exception of Section F on Reciprocity, will be limited to non-dispersive media in which $\mu$, $\varepsilon$ and $\sigma$ are linear, isotropic, homogeneous and time independent. With these assumptions the equations reduce to the familiar forms, i.e., $\overline{B} = \mu \overline{H}$. For their proof the interested reader is directed to Reference 9. Suffice it to say, that Eqs. (134), (135), and (136) are applicable for media which may be nonhomogeneous, dissipative and dispersive. And when the coefficients $\mu$, $\varepsilon$ or $\sigma$ are represented by tensors, they become valid for anisotropic but nongyrotropic ($\overline{\mu}_{ij} = \overline{\mu}_{ji}$) media as well. Notice also from
Eq. (135), for example, that in spite of the fact that the medium is isotropic, \( \mathbf{B}(t) \) and \( \mathbf{H}(t) \) in general are not colinear unless they are linearly polarized.

Since Eqs. (134), (135), and (136) were developed in the time domain, they are applicable to fields whose time dependence may not be Fourier transformable. The proof requires only the past histories of the fields; their behavior at \( t = +\infty \) is immaterial.

B. Plane Electromagnetic Waves
   In Free Space

Consider now the Maxwell equations in free space, i.e., in a vacuum infinitely remote from matter. It will be seen that these equations imply plane electromagnetic waves in the region and, further, that these are transverse electromagnetic (TEM) waves. No mention will be made at this point of the manner in which electromagnetic waves can be generated; this subject will be treated in Section E. Restricting the discussion to phenomena occurring in vacuum infinitely remote from matter will have the advantage of simplifying the theory, while conserving many of its basic aspects.

For vacuum then, the following are set to zero:

\[ X_e = X_m = \sigma = \rho = \mathbf{J} = 0 \]

and the Maxwell equations become
(1) \begin{align*}
(a) \quad \nabla \cdot \mathbf{D} &= 0 \\
(b) \quad \nabla \cdot \mathbf{B} &= 0 \\
(c) \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\
(d) \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= 0
\end{align*}

where

(2) \begin{align*}
(a) \quad \mathbf{D}(t) &= \varepsilon_0 \mathbf{E}(t) \\
(b) \quad \mathbf{B}(t) &= \mu_0 \mathbf{H}(t)
\end{align*}

To find an equation in terms of \( \mathbf{E} \) (or \( \mathbf{H} \)) alone the curl of Eqs. (1c) and (d) are taken yielding:

(137) \begin{align*}
(a) \quad \nabla \times (\nabla \times \mathbf{E}) + \nabla \times \frac{\partial \mathbf{B}}{\partial t} &= 0 \\
(b) \quad \nabla \times (\nabla \times \mathbf{H}) - \nabla \times \frac{\partial \mathbf{D}}{\partial t} &= 0
\end{align*}

Substituting into Eq. (137a) or (b) first, Eq. (2b) (a), then Eq. (1d) (c), and finally Eq. (2a) (b), we obtain for \( \mathbf{E} \) (and \( \mathbf{H} \))

(138) \begin{align*}
(a) \quad \nabla \times (\nabla \times \mathbf{E}) + \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial^2 t} &= 0 \\
(b) \quad \nabla \times (\nabla \times \mathbf{H}) + \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{H}}{\partial^2 t} &= 0
\end{align*}

These are vector wave equations for transverse waves under the restrictions

(139) \begin{align*}
(a) \quad \nabla \cdot \mathbf{E} &= 0 \\
(b) \quad \nabla \cdot \mathbf{H} &= 0
\end{align*}

which follow from Eqs. (1) and (2). Employing the vector identity

(140) \quad \nabla \cdot (\nabla \mathbf{A}) = \nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})
and noting Eqs. (139a) and (b) the vector wave equations of Eq. (138) may be written in a more familiar and useful form:

\[
\begin{align*}
(141) \quad & \nabla^2 \overline{E} - \epsilon_0 \mu_0 \frac{\partial \overline{E}}{\partial t^2} = 0 \\
& \nabla^2 \overline{H} - \epsilon_0 \mu_0 \frac{\partial \overline{H}}{\partial t^2} = 0.
\end{align*}
\]

Note that the constitutive quantities $\mu_0$ and $\epsilon_0$ appear in the place of $1/v^2$ in Eq. (141) and when combined in the above manner give

\[
(\epsilon_0 \mu_0)^{-\frac{1}{2}} = c \sim 3 \times 10^8 \text{ meters/sec.}
\]

The velocity of the waves is the velocity of light in vacuum.

For illustration let us consider the relatively simple case of a plane wave propagating along the z-axis. The electric field intensity $\overline{E}$ then varies only in the direction of $z$, that is, $\overline{E} = \overline{E}(z, t)$ a function of $z$ and $t$ only. Thus:

\[
(142) \quad \frac{\partial \overline{E}}{\partial x} = \frac{\partial \overline{E}}{\partial y} = 0.
\]

The divergence of $\overline{E}$ then reduces to

\[
(143) \quad \nabla \cdot \frac{\partial \overline{E}_z}{\partial z} = 0.
\]

Thus, $E_z$ cannot be a function of $z$ so we shall set

\[
E_z = 0,
\]

since we are interested in a traveling wave solution and not in a uniform field.
From Eq. (143) the vector \( \mathbf{E}(z, t) \) can have only x- and y- components, i.e., it has no longitudinal component. The same arguments, of course, apply to the \( \mathbf{H} \) vector. This constitutes a uniform plane electromagnetic (TEM) wave.

For simplicity, we choose our system of axes such that the x-axis is parallel to the vector \( \mathbf{E} \):

\[
\mathbf{E} = x^o \mathbf{E}_x(z, t) = x^o \mathbf{E}_x \psi(z, t)
\]

Substituting the above value of \( \mathbf{E} \) into Eq. (1c) we find

\[
B_x = H_x = B_z = H_z = 0
\]

and that

\[
\frac{\partial E_x}{\partial z} = - \frac{\partial B_y}{\partial t}
\] (144)

It is obvious then that an electromagnetic field with components \( E_x \) and \( H_y \) and variation with respect to \( z \) and \( t \) is consistent with Maxwell's equations. The traveling wave nature of the field can be shown by examining the wave equation

\[
\frac{\partial^2 \psi}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0
\] (145)

A possible solution is a function \( \psi(z, t) \) of the form

\[
\psi(z, t) = F(z - ct) = f(t - z/c)
\] (146)
i.e., some function of a single variable \((z - ct)\) or \((t - z/c)\). The function \(F(z - ct)\) represents a "rigid" pattern in \(z\) which travels toward positive \(z\) at the velocity \(c\). To simplify calculations we shall consider only the wave traveling in the positive direction of the \(z\)-axis.

Let us see if \(f(t - z/c)\) is indeed a solution of the wave equation. Since it is a function of only one variable, the variable \([t] = (t - z/c)\) defined as the \textit{retarded time}, we will let \(\dot{f}\) represent the derivative of \(f\) with respect to its variable and \(\ddot{f}\) represent the second derivative of \(f\). The retarded time \([t]\) will be discussed in more detail later.

Differentiating Eq. (146) with respect to \(t\) or \([t]\) we have

\[
(147) \quad \frac{\partial \psi}{\partial t} = \frac{\partial f}{\partial [t]} = \dot{f}([t])
\]

since the derivative of \([t]\) with respect to \(t\) is unity. The second derivative of \(\psi\) with respect to \(t\) is clearly

\[
(148) \quad \frac{\partial^2 \psi}{\partial t^2} = \ddot{f}([t])
\]

Taking derivative of \(\psi\) with respect to \(z\), we find

\[
(149) \quad \frac{\partial \psi}{\partial z} = -\frac{1}{c} \dot{f}([t])
\]

and

\[
(150) \quad \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{c^2} \ddot{f}([t])
\]
We see that \( \psi \) defined as in Eq. (146) does indeed satisfy the one-dimensional wave equation.

Using Eqs. (147) and (149) with Eq. (144) gives

\[
(143') \quad \frac{\partial E_x}{\partial z} = c \frac{\partial B_y}{\partial z}
\]

or

\[
(151) \quad \frac{E_x}{B_y} = c
\]

and

\[
(152) \quad \frac{E_x}{H_y} \equiv \mu_0 c = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{\varepsilon_0}{\mu_0} \sim 120 \pi \text{ ohms (free space impedance).}
\]

The \( \overrightarrow{E} \) and \( \overrightarrow{H} \) vectors are, therefore, mutually perpendicular and are oriented such that their vector product, the Poynting vector,

\[
(14) \quad \overrightarrow{E} \times \overrightarrow{H} = \overrightarrow{S}
\]

points in the direction of energy propagation. The \( \overrightarrow{E} \) and \( \overrightarrow{H} \) vectors are in phase since \( \frac{E_x}{H_y} \) is a real constant.

The electric and magnetic energy densities are in phase and are equal since

\[
(153) \quad \frac{1/2 \varepsilon_0 E^2}{1/2 \mu_0 H^2} = \mu_0 \varepsilon_0 c^2 = 1.
\]
At any instant the total energy density is distributed as \( f(t - z/c) \)^2.

C. Boundary Conditions

The boundary conditions on \( \vec{E} \), \( \vec{D} \), \( \vec{B} \) and \( \vec{H} \) at the interface between two dissimilar media are determined by applications of Maxwell's equations in integral form at the interface. We shall briefly review this illustrating the similarity between \( \vec{B} \) and \( \vec{D} \), and between \( \vec{E} \) and \( \vec{H} \).

Consider first the field vectors \( \vec{B} \) and \( \vec{D} \) as they pass from medium (1) to medium (2) as shown in Fig. 34a. By applying Maxwell's divergence equations (Eqs. (130) and (133)) to \( \vec{B} \) and \( \vec{D} \), respectively, at the interface and taking a volume integral which extends into both regions (Gauss's Law) we have

\[
\begin{align*}
(154) & \quad \int_V \nabla \cdot \vec{D} \, dv = \oint_S \vec{D} \cdot \, ds = \int_V \rho \, dv \\
& \quad \int_V \nabla \cdot \vec{B} \, dv = \oint_S \vec{B} \cdot \, ds = 0
\end{align*}
\]

If, as shown in Fig. 34a, the volume of integration is a coin-shaped region extending partly into both media, and if its thickness \( \Delta \ell \) is allowed to become infinitesimally thin, then the fraction of \( \vec{D} \) or \( \vec{B} \) passing through the rim of the volume is negligible and Eqs. (154(a) and (b)) may be written
(155) (a) \[ \lim_{\Delta \ell \to 0} \{ (\overrightarrow{D_2} - \overrightarrow{D_1}) \cdot \overrightarrow{n} \Delta A + \text{rim} = \rho \Delta \ell \Delta A \} \]

(b) \[ \lim_{\Delta \ell \to 0} \{ (\overrightarrow{B_2} - \overrightarrow{B_1}) \cdot \overrightarrow{n} \Delta A + \text{rim} = 0 \} \]

(156) (a) \[ (\overrightarrow{D_2} - \overrightarrow{D_1}) \cdot \overrightarrow{n} = \rho_S \]

(b) \[ (\overrightarrow{B_2} - \overrightarrow{B_1}) \cdot \overrightarrow{n} = 0 \]

where we indicate the positive normal to the surface S by a unit vector \( \overrightarrow{n} \) drawn from (1) to (2), the accumulated surface change density at the interface by \( \rho_S \), and the \( \overrightarrow{D_1} \) and \( \overrightarrow{B_1} \) vectors in regions (1) and (2) are indicated by \( \overrightarrow{D_1} \) and \( \overrightarrow{B_1} \) and by \( \overrightarrow{D_2} \) and \( \overrightarrow{B_2} \), respectively.

Equations (156(a) and (b)) show that the normal components of \( \overrightarrow{D_1} \) and \( \overrightarrow{D_2} \) differ at the interface by the free surface charge density \( \rho_S \) while the normal components \( \overrightarrow{B_1} \) and \( \overrightarrow{B_2} \) at the interface are equal.

Fig. 34a -- For the normal boundary condition.
Looking next at field vectors $\mathbf{E}$ and $\mathbf{H}$ we apply the remaining two Maxwell equations (Eqs. (124) and (125)) at the interface to $\mathbf{E}$ and $\mathbf{H}$, respectively. Taking the closed line integral so that it passes through both media as shown in Fig. 34b, we have

$$\int_{\gamma} \mathbf{E} \cdot d\mathbf{l} = - \int_{S} \left( \mathbf{M} + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{s} \tag{157a}$$

$$\int_{\gamma} \mathbf{H} \cdot d\mathbf{l} = \int_{S} \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s} \tag{157b}$$

In Eq. (157a) we have postulated a fictitious magnetic current density $\mathbf{M}$, a flow of magnetic charge. This magnetic current $\mathbf{M}$, although not physically realizable, will be found mathematically useful later when we assume arbitrary discontinuities in tangential $\mathbf{E}$ and normal $\mathbf{H}$, discontinuities which are, in fact, physically impossible but which would be generated by surface distributions of magnetic currents and charges were they to exist.

Fig. 34b—For the tangential boundary condition.
From Fig. 34b we see that Eqs. (157(a) and (b) can be written

\begin{equation}
(158) \quad (a) \quad \int_{\ell} \overrightarrow{E} \cdot (\overrightarrow{s}^o \times \overrightarrow{n}^o) d\ell = - \int_{S} \left( \overrightarrow{M} + \frac{\partial \overrightarrow{B}}{\partial t} \right) \cdot \overrightarrow{s}^o \; ds
\end{equation}

\begin{equation}
(b) \quad \int_{\ell} \overrightarrow{H} \cdot (\overrightarrow{s}^o \times \overrightarrow{n}^o) d\ell = \int_{S} \left( \overrightarrow{J} + \frac{\partial \overrightarrow{D}}{\partial t} \right) \cdot \overrightarrow{s}^o \; ds
\end{equation}

In the limit as $\Delta \ell \to 0$ the contributions of the components of $\overrightarrow{E}$ and $\overrightarrow{H}$ normal to the interface become negligible. Noting the vector identity $\overrightarrow{E} \{ \text{or} \overrightarrow{H} \} \cdot (\overrightarrow{s}^o \times \overrightarrow{n}^o) = (\overrightarrow{s}^o \times \overrightarrow{n}^o) \cdot \overrightarrow{E} \{ \text{or} \overrightarrow{H} \} = \overrightarrow{s}^o \cdot (\overrightarrow{n}^o \times \overrightarrow{E} \{ \text{or} \overrightarrow{H} \})$,

Eqs. (158(a) and (b)) upon integration may be written

\begin{equation}
(159) \quad (a) \quad \overrightarrow{s}^o \cdot \{ \overrightarrow{n}^o \times (\overrightarrow{E}_2 - \overrightarrow{E}_1) \} + \lim_{\Delta \ell \to 0} \left( \overrightarrow{M} + \frac{\partial \overrightarrow{B}}{\partial t} \right) \Delta \ell = 0
\end{equation}

\begin{equation}
(b) \quad \overrightarrow{s}^o \cdot \{ \overrightarrow{n}^o \times (\overrightarrow{H}_2 - \overrightarrow{H}_1) \} - \lim_{\Delta \ell \to 0} \left( \overrightarrow{J} + \frac{\partial \overrightarrow{D}}{\partial t} \right) \Delta \ell = 0
\end{equation}

The orientation of the rectangle, and hence also of $\overrightarrow{s}^o$, is entirely arbitrary from which it follows that the bracket in Eqs. (159(a) and (b)) must equal zero or

\begin{equation}
(160) \quad (a) \quad \overrightarrow{n}^o \times (\overrightarrow{E}_2 - \overrightarrow{E}_1) = - \lim_{\Delta \ell \to 0} \left( \overrightarrow{M} + \frac{\partial \overrightarrow{B}}{\partial t} \right) \Delta \ell
\end{equation}

\begin{equation}
(b) \quad \overrightarrow{n}^o \times (\overrightarrow{H}_2 - \overrightarrow{H}_1) = \lim_{\Delta \ell \to 0} \left( \overrightarrow{J} + \frac{\partial \overrightarrow{D}}{\partial t} \right) \Delta \ell
\end{equation}
The first term on the right of Eqs. (160) (a) and (b) vanishes as \( \Delta l \to 0 \) because \( \mathbf{D} (\mathbf{B}) \) and its derivatives are bounded. If the current density \( \mathbf{J} (\mathbf{M}) \) is finite, the second term vanishes as well. It may happen, however, that the current \( I (L_m) = \mathbf{J} (\mathbf{M}) \cdot \mathbf{s}^0 \Delta W \Delta l \) through the rectangle is squeezed into an infinitesimal layer on the surface as the sides are brought together. It is convenient to represent this surface current by a surface density \( J_s (M_s) \) defined as the limit of the product \( \mathbf{J} (\mathbf{M}) \Delta t \) as \( \Delta l \to 0 \) and \( \mathbf{J} (\mathbf{M}) \to \infty \). Then

\[
(161) \quad \begin{align*}
(a) & \quad \mathbf{n}^0 \times (\mathbf{E}_2 - \mathbf{E}_1) = - M_s, \\
(b) & \quad \mathbf{n}^0 \times (\mathbf{H}_2 - \mathbf{H}_1) = J_s.
\end{align*}
\]

When the conductivities of the contiguous media are finite, there can be no surface current, for \( \mathbf{E} (\mathbf{H}) \) is bounded and hence the product \( \sigma \mathbf{E} \Delta l \), for example, vanishes with \( \Delta l \). In this case, which is the usual one,

\[
(162) \quad \begin{align*}
(a) & \quad \mathbf{n}^0 \times (\mathbf{E}_2 - \mathbf{E}_1) = 0, \\
(b) & \quad \mathbf{n}^0 \times (\mathbf{H}_2 - \mathbf{H}_1) = 0, \quad \text{(finite conductivity)}.
\end{align*}
\]

Often, as will be seen in Section E, it is necessary to assume the conductivity of a current carrying body to be infinite in order to
simplify the analysis of its field. We must then apply Eqs. (161a) and (b) as boundary conditions rather than Eqs. (162a) and (b).

From Eqs. (161) and (162) then we note that the tangential components of \( \mathbf{H} (\mathbf{E}) \) at the interface are either equal or that the difference is equal to the surface current \( J_s \) \((-M_s)\). Note the sign difference between \( J_s \) and \( M_s \) indicating that the surface magnetic current \( M_s \) flows in the opposite direction to the corresponding surface electric current \( J_s \).

D. **Vector and Scalar Potentials**

1. **The relationship of the potential functions to \( \mathbf{E} \) and \( \mathbf{B} \)**

The Maxwell equations imply the existence of vector and scalar potentials whose properties are quite useful when we investigate the methods by which electromagnetic waves are launched into space. We will briefly outline the relationship of those potentials to the fields \( \mathbf{E} \) and \( \mathbf{B} \).

Restating the Maxwell equations and the constitutive relations again for convenience:

\[
\begin{align*}
(163) & \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \\
(a) & \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}
\end{align*}
\]
we begin by looking at $\nabla \cdot \vec{B} = 0$ and recall the vector identities:

\begin{equation}
\nabla \cdot (\nabla \times \vec{A}) = \nabla \times (\nabla \phi) = 0
\end{equation}

It is obvious that by setting

\begin{equation}
\vec{B} = \nabla \times \vec{A}
\end{equation}

Eq. (164b) is satisfied and $\vec{E}$, may be thought of as resulting from the curl of another vector $\vec{A}$ defined as the vector magnetic potential.

Placing this relationship for $\vec{B}$ into the equation for the curl of $\vec{E}$ results in

\begin{equation}
\nabla \times \vec{E} + \frac{\partial}{\partial t} (\nabla \times \vec{A}) = \nabla \times \left( \vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0
\end{equation}

From the vector identities of Eq. (166) it is obvious that $(\vec{E} + \partial \vec{A}/\partial t)$ is the gradient of some scalar. From statics we remember that $\vec{E} = -\nabla \phi$, where $\phi$ was the scalar potential, so we shall define $\phi$ by the equation

\begin{equation}
\vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}
\end{equation}
Substituting these equations for $A$ and $\phi$ into the equations for the curl of $\vec{H}$ (and remembering that $\mu$ and $\epsilon$ are assumed constant), we obtain

$$\frac{1}{\mu} (\nabla \times \nabla \times \vec{A}) = \vec{J} - \epsilon \frac{\partial^2 A}{\partial t^2} + \frac{\partial}{\partial t} (\nabla \phi)$$

or

$$\nabla^2 \vec{A} - \epsilon \mu \frac{\partial^2 A}{\partial t^2} = \nabla \left[ \nabla \cdot \vec{A} + \epsilon \mu \frac{\partial \phi}{\partial t} \right] - \mu \vec{J}$$

We have employed the vector identity of Eq. (140) to the left-hand side of Eq. (170) to obtain Eq. (171). It is obvious that this would be an equation in $\vec{A}$ alone if the term in the brackets were zero. Since only the part of $\vec{A}$ having non-zero curl is as yet defined, it is always possible to adjust the divergence of $\vec{A}$ so that

$$\nabla \cdot \vec{A} = - \epsilon \mu \frac{\partial \phi}{\partial t}$$

which is termed the Lorentz Condition.

Inserting Eqs. (169) and (172) into the equation for the divergence of $\vec{D}$ we have an equation for the scalar potential which together with Eq. (171) for $\vec{A}$, gives

$$\nabla^2 \phi - \epsilon \mu \frac{\partial^2 \phi}{\partial t^2} = - \frac{\rho}{\epsilon} \quad \text{(a)} \quad \nabla^2 \vec{A} - \epsilon \mu \frac{\partial^2 A}{\partial t^2} = - \mu \vec{J} \quad \text{(b)}$$
Equation (173) serves to determine the scalar and vector potentials in terms of \( \rho \) and \( \vec{J} \). When \( \rho \) and \( \vec{J} \) are zero, we have the wave equations for a wave with velocity equal to \((\epsilon\mu)^{-\frac{1}{2}}\) as in Section B. We note from these equations that the Lorentz condition is a consequence of the Equation of Continuity Eq. (126) (cf. Stratton[42]). By analogy with waves in elastic solids we call the waves in \( \phi \) longitudinal waves and the waves in \( \vec{A} \) transverse waves[43].

2. Retarded potentials

A solution of the above requires the integrating of an inhomogeneous differential equation and this will not be done here. For details the reader is directed to Reference 44. The resulting solutions for \( \phi (x, y, z, t) \) and \( \vec{A} (x, y, z, t) \) in infinite space caused by a finite distribution of \( \rho (x, y, z, t) \) and \( \vec{J} (x, y, z, t) \) are however

\[
\begin{align*}
\text{(a)} & & \phi(\vec{r}, t) = \frac{1}{4\pi\epsilon} \int_V \frac{\rho(\vec{r}', t - (|\vec{r} - \vec{r}'|/c))}{|\vec{r} - \vec{r}'|} \, dv' \\
\text{(b)} & & \vec{A}(\vec{r}, t) = \frac{\mu}{4\pi} \int_V \frac{\vec{J}(\vec{r}', t - (|\vec{r} - \vec{r}'|/c))}{|\vec{r} - \vec{r}'|} \, dv'
\end{align*}
\]

where \( \vec{r}' \) represents the coordinates of the charge or current density and \( \vec{r} \) the coordinates of the measured potentials. Thus

\[ |\vec{r} - \vec{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \]

and \( c = (\epsilon\mu)^{-\frac{1}{2}} \) is the velocity of a plane wave in the medium \( \epsilon, \mu \). These solutions can, of course,
be generalized from the static case where $\phi(\vec{r})$ obtains directly from Coulomb's Law and $\vec{A}(\vec{r})$ from the Biot-Savart Law or from the scalar and vector Poisson equations for $\phi$ and $\vec{A}$.

These integral solutions show that the effect on the potentials at the point $\vec{r}$ due to the charge and current at $\vec{r}'$ is felt at time $|\vec{r} - \vec{r}'|/c$ later in time. The potential has been retarded in its effect by the time required for a wave with speed $c = (\varepsilon\mu)^{-\frac{1}{2}}$ to travel from $\vec{r}'$ to $\vec{r}$. Since the wave equation is symmetrical with respect to reversal of the direction of time, another solution can be obtained by replacing $t - |\vec{r} - \vec{r}'|/c$ by $t + |\vec{r} - \vec{r}'|/c$ in the integrand. Physical observation and the "Principle of Causality" have traditionally dictated that only the retarded potential be retained. More recently, theories on the origin of the universe by cosmologists have attempted to lend theoretical justification to the apparent absence of advanced potentials.

It should be noted that more than one set of potentials has the same fields. The solutions given in Eq. (5) can be modified by adding to each a different function so related that the Lorentz condition still holds. In other words, we can take any arbitrary function $\psi$ of $\vec{r}$ and $t$ satisfying appropriate boundary conditions, and form new solutions of Eq. (173) by setting

\begin{align}
(174) \\
(a) \quad \vec{A}^0 &= \vec{A} - \nabla \psi \\
(b) \quad \phi^0 &= \phi + \frac{\partial \psi}{\partial t}
\end{align}
Then $\nabla \cdot \vec{A}^0 + \mu \varepsilon (\partial \phi^0 / \partial t) = -\nabla^2 \psi + \mu \varepsilon (\partial \psi / \partial t^2)$, so that the equations for the new $\vec{A}^0$ and $\phi^0$ may differ from the forms given in Eq. (173). The electric and magnetic fields, however, (which are the measurable quantities), are not affected by the choice of $\psi$. In the process of taking $\nabla \times \vec{A}^0$ to obtain $\vec{B}$, the $\nabla \psi$ term vanishes and in the process of computing $\vec{E}$ from Eq. (169) the terms involving $(\partial \psi / \partial t)$ cancel. This invariance of the actual fields to changes in the potentials, which leave Eq. (172) invariant, is called gauge invariance, and a brief comment on gauge might be in order here. Choosing the value for $\nabla \cdot \vec{A}$ is called "choosing a gauge", changing $\vec{A}$ by adding $\nabla \psi$ is called "gauge transformation", and Eq. (172) is also referred to as "the Lorentz gauge". The Lorentz gauge is the one gauge for which the electromagnetic fields remain invariant under Lorentz transformation. Under some conditions the lack of Lorentz invariance is not important, then other gauges may be more suitable as when $\phi = 0$ (c.f. Morse and Feshbach[46]).

3. Duality

Let us now postulate a dual system in which the sources have magnetic current density $\vec{M}$ (volts/square meter) as introduced in Section C and magnetic charge density $\rho_m$ (magnetic charge/cubic meter). Although no known magnetic currents or isolated magnetic charges are known to exist in nature, and since the magnetic field
seems to be a relativistic phenomenon[47], search for magnetic sources would appear futile; the mathematical concept will, however, prove quite useful. In our dual system then, Maxwell's equations become

\begin{align}
(175) & \quad (a) \nabla \times \vec{H}' - \frac{\partial \vec{D}'}{\partial t} = 0 \quad \quad (b) \nabla \times \vec{E}' + \frac{\partial \vec{B}'}{\partial t} = -\vec{M} \\
(176) & \quad (a) \nabla \cdot \vec{B}' = \rho_m \quad \quad (b) \nabla \cdot \vec{D}' = 0
\end{align}

It is at once obvious that we have vector and scalar potentials related to \(\vec{D}'\) and \(\vec{H}'\) as

\begin{align}
(177) & \quad \vec{D}' = -\nabla \times \vec{F} \\
(178) & \quad \vec{H}' = -\nabla \phi_m - \frac{\partial \vec{F}}{\partial t}
\end{align}

The new potentials are subject to conditions similar to those for \(\vec{A}\) and \(\phi\), namely

\begin{align}
(179) & \quad \nabla \cdot \vec{F} + \epsilon \mu \frac{\partial \phi_m}{\partial t} = 0 \\
(180) & \quad (a) \nabla^2 \phi_m - \epsilon \mu \frac{\partial^2 \phi_m}{\partial t^2} = \frac{\rho_m}{\mu} \\
& \quad (b) \nabla^2 \vec{F} - \epsilon \mu \frac{\partial^2 \vec{F}}{\partial t^2} = -\epsilon \vec{M}
\end{align}

and it follows that
(6) (a) \[ \phi_m(r, t) = \frac{1}{4\pi \mu} \int_V \frac{\rho_m(r', t - (|r - r'|/c))}{|r - r'|} \, dv' \]

(b) \[ \overline{E}(r, t) = \frac{\epsilon}{4\pi} \int_V \frac{\overline{M}(r', t - (|r - r'|/c))}{|r - r'|} \, dv' \]

We see, therefore, that to be completely general about the electric and magnetic fields at a point, we must consider fields resulting from both electric and magnetic vector and scalar potentials. \( \overline{E} \) and \( \overline{H} \) are thus written:

(181) \[ \overline{E} = \overline{E}^e + \overline{E}^m = - \nabla \phi - \frac{\partial \overline{A}}{\partial t} - \frac{1}{\epsilon} \nabla \times \overline{F} \]

(182) \[ \overline{H} = \overline{H}^e + \overline{H}^m = \frac{1}{\mu} \nabla \times \overline{A} - \nabla \phi_m - \frac{\partial \overline{F}}{\partial t} \]

where the superscript \( e \) indicates those fields resulting from electric currents and charges and the superscript \( m \) those fields resulting from magnetic currents and charges.

Equations (181) and (182) can be somewhat simplified from functions of three variables to functions of only two variables by application of the Lorentz condition relating the vector and scalar potentials. Integrating Eq. (172) from \( t_o \), a time early enough that \( \phi = 0 \), to \( t \), the present time, we have

(183) \[ \frac{1}{\mu \varepsilon} \int_{t_o}^{t} \nabla \cdot \overline{A} = - \int_{\phi_o}^{\phi} d\phi' = -\phi \]
Taking the divergence we can substitute back into Eqs. (181) and (182) giving

\[
\begin{align*}
\mathbf{E} &= -\frac{1}{\epsilon} \nabla \times \mathbf{F} + \frac{1}{\mu \epsilon} \nabla \int_{t_0}^{t} \nabla \cdot \mathbf{A} \, dt - \frac{\partial \mathbf{A}}{\partial t} \\
\mathbf{H} &= \frac{1}{\mu} \nabla \times \mathbf{A} + \frac{1}{\mu \epsilon} \nabla \int_{t_0}^{t} \nabla \cdot \mathbf{F} \, dt - \frac{\partial \mathbf{F}}{\partial t}
\end{align*}
\]

It is possible to further simplify Eqs. (184) and (185) by combining the last two terms through use of the homogeneous wave equation

\[
\nabla^2 \mathbf{A} = -\mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}
\]

for source free regions.

Substituting for the left-hand side in Eq. (186) from the vector identity Eq. (140) then integrating from \( t_0 \) to \( t \) and rearranging yields

\[
\begin{align*}
\frac{1}{\mu \epsilon} \nabla \int_{t_0}^{t} \nabla \cdot \mathbf{A} \, dt - \frac{\partial \mathbf{A}}{\partial t} &= -\frac{1}{\mu \epsilon} \int_{t_0}^{t} \nabla \times \nabla \times \mathbf{A} \, dt \\
\mathbf{E} &= -\frac{1}{\epsilon} \nabla \times \mathbf{F} - \frac{1}{\mu \epsilon} \int_{t_0}^{t} \nabla \times \nabla \times \mathbf{A} \, dt
\end{align*}
\]

thus

\[
\mathbf{E} = -\frac{1}{\epsilon} \nabla \times \mathbf{F} - \frac{1}{\mu \epsilon} \int_{t_0}^{t} \nabla \times \nabla \times \mathbf{A} \, dt
\]

and

\[
\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} - \frac{1}{\mu \epsilon} \int_{t_0}^{t} \nabla \times \nabla \times \mathbf{F} \, dt
\]
Equations (184) and (185) or Eqs. (3) and (4) give us a means of determining the fields $\vec{E}$ and $\vec{H}$ in a region resulting from source currents $\vec{J}$ and $\vec{M}$ outside the region through the vector potentials $\vec{A}$ and $\vec{F}$. This is the classical approach for finding the fields of an antenna.

It may by now be apparent that Eqs. (3) and (4) might just as easily have been obtained directly from the free space homogeneous Maxwell equations (Eqs. (1a, b, c, and d)). The two divergence equations give relationships for $\vec{H}^e$ in terms of $\vec{A}$ and $\vec{E}^m$ in terms of $\vec{F}$ (Eqs. (167) and (177)) while substitution of these relationships back into the respective curl equations with the aid of Eqs. (2a and b) and integrating from $t_o$ and $t$ gives $\vec{E}^e$ in terms of $\vec{A}$ and $\vec{H}^m$ in terms of $\vec{F}$ (the second terms of Eqs. (3) and (4)). Although this method would not show as clearly the source of these potentials, it does indicate that the existence of fields $\vec{E}^m$ and $\vec{H}^m$ is inherent in Maxwell's equations and need not be predicated on the assumption of magnetic sources.

Because of the duality that exists between the fields of the electric currents and the fields of the magnetic currents with similar time variation, we need solve only for the fields of one type of source. The fields of the other type are obtained immediately by interchanging the dual quantities given in Table 2.
TABLE 2

DUAL RELATIONSHIPS IN ELECTRIC AND MAGNETIC SYSTEMS WHEN $f^e(r, t) = f^m(r, t)$

<table>
<thead>
<tr>
<th>Electric System</th>
<th>Magnetic System</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>$\rho_m$</td>
</tr>
<tr>
<td>$\bar{J}$</td>
<td>$\bar{M}$</td>
</tr>
<tr>
<td>$\bar{J}_s$</td>
<td>$\bar{M}_s$</td>
</tr>
<tr>
<td>$\phi$</td>
<td>$\phi_m$</td>
</tr>
<tr>
<td>$\bar{A}$</td>
<td>$\bar{F}$</td>
</tr>
<tr>
<td>$\bar{E}^e$</td>
<td>$\bar{H}^m$</td>
</tr>
<tr>
<td>$\bar{H}^e$</td>
<td>$-\bar{E}^m$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$\epsilon$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>$\mu$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$\sigma_m$</td>
</tr>
<tr>
<td>Impedance</td>
<td>Admittance</td>
</tr>
<tr>
<td>Admittance</td>
<td>Impedance</td>
</tr>
</tbody>
</table>
E. Radiation of Electromagnetic Waves

1. The short dipole

We shall begin our study of the application of Eqs. (184) and (185) to radiation of arbitrary time functions by first examining the short dipole. Since any radiating device may be regarded as a properly weighted aggregate of short dipoles, a knowledge of the latter is useful in determining the properties of longer dipoles or radiating structures of arbitrary shape.

For the general time case we define the short dipole as a dipole whose length \( l \) is short enough and/or the variation of the exciting time function \( f(t) \) is slow enough that \( f(t) \approx f(t \pm l/c) \) where \( l/c = \Delta t \) the time for the signal to travel the length of the dipole. That is, radiation from the extreme ends of the dipole is received at a distance \( r >> l \) essentially simultaneously or in phase at all look angles \( \theta \) (see Fig. 35). It is also assumed that the diameter \( d \) of the dipole is very small compared with its length (\( d << l \)). Thus, our dipole consists simply of a thin conductor of length \( l \) with a uniform current \( I \) and point charges \( q \) at the ends[48].

By analogy we can at this point replace our dipole by an oscillating electric dipole or Hertzian dipole and will at the same time introduce the Hertzian vector or polarization potential[49, 50]. Although Hertzian potentials are not usually used in antenna work,
they are particularly well suited for this example as they will clearly illustrate the sources of the field components and eliminate the integrations with respect to time. The Hertzian dipole or electric polarization is obtained if we combine a moving charge +q with a neighboring stationary charge -q to form a dipole moment \( \mathbf{P}(t) = q \mathbf{f} \) varying with time, where \( \mathbf{f} \) signifies the separation of the two charges.

We substitute \( \mathbf{j} = \rho \mathbf{v} \) in Eq. (5b), denoting the space density of the moving charge by \( \rho \) and its velocity by \( \mathbf{v} \); and carrying out the integration where \( |\mathbf{r} - \mathbf{r}'| \propto r \), and \( r \) and \( v \) may be regarded as constant in the integration, we obtain
\begin{equation}
\oint \frac{[\mathbf{J}]}{r} \, dv = \oint \frac{[\mathbf{V}]}{r} \, dv = \frac{q}{r} \oint \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{r} \left[ \frac{\partial \mathbf{P}}{\partial t} \right]
\end{equation}

We thus obtain from Eq. (5b), where the bracket indicates the value at the retarded time \([t] = t - r/c\),

\begin{equation}
4\pi \mathbf{A} = \frac{\mu}{r} \frac{\partial}{\partial t} \mathbf{P}(t - r/c)
\end{equation}

At this point we introduce, in place of the vector potential \(\mathbf{A}\), the Hertzian vector \(\mathbf{\pi}\), by writing

\begin{equation}
\mathbf{A} = \epsilon \mu \mathbf{\pi}
\end{equation}

where

\begin{equation}
4\pi \mathbf{\pi} = \frac{1}{\epsilon r} \mathbf{P}(r - r/c)
\end{equation}

In all of space except at the source \(\mathbf{\pi}\) satisfies, by Eq. (173b), the differential equation

\begin{equation}
\nabla^2 \mathbf{\pi} - \epsilon \mu \frac{\partial^2 \mathbf{\pi}}{\partial t^2} = 0
\end{equation}

which can also readily be verified from the explicit representation of \(\mathbf{\pi}\) given in Eq. (191). By Eq. (172) the corresponding value of \(\phi\) becomes

\begin{equation}
\phi = -\nabla \cdot \mathbf{\pi}
\end{equation}
From Eqs. (181) and (182) we obtain then as the representation of the electromagnetic field, since there are no magnetic sources

\begin{equation}
\overrightarrow{H} = \epsilon \nabla \times \left( \frac{\partial \overrightarrow{\pi}}{\partial t} \right),
\end{equation}

\begin{equation}
\overrightarrow{E} = \nabla \nabla \cdot \overrightarrow{\pi} - \epsilon \mu \frac{\partial^2 \overrightarrow{\pi}}{\partial t^2}.
\end{equation}

As an example, we let the dipole and thus the direction of \( \overrightarrow{\pi} \) lie along the polar axis of a spherical coördinant system \( r, \theta, \phi \). We then have (see Fig. 35)

\begin{equation}
\pi_r = \cos \theta \cdot \pi, \quad \pi_\theta = -\sin \theta \cdot \pi, \quad \pi_\phi = 0
\end{equation}

where, according to Eq. (191), \( \overrightarrow{\pi} \) depends only on \( t \) and \( r \), i.e., \( \overrightarrow{\pi} \) is independent of \( \theta \) and \( \phi \). In these coördinates we obtain,

\[ \nabla_\phi \times \overrightarrow{\pi} = -\overrightarrow{\phi^o} \sin \theta \frac{\partial}{\partial r} \left( \frac{\partial (r \pi)}{\partial r} - \pi \right) = -\overrightarrow{\phi^o} \sin \theta \frac{\partial \pi}{\partial r}, \]

\[ \nabla_r \times \overrightarrow{\pi} = \nabla_\theta \times \overrightarrow{\pi} = 0, \]

\[ \nabla \cdot \overrightarrow{\pi} = \frac{\cos \theta}{r^2} \frac{\partial}{\partial r} (r^2 \pi) - \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin^2 \theta \pi) = \cos \theta \frac{\partial \pi}{\partial r}, \]

\[ \nabla_r (\nabla \cdot \overrightarrow{\pi}) = r^0 \cos \theta \frac{\partial^2 \pi}{\partial r^2}, \quad \nabla_\theta (\nabla \cdot \overrightarrow{\pi}) = -\overrightarrow{\phi^o} \sin \theta \frac{\partial \pi}{\partial r}, \]

\[ \nabla_\phi (\nabla \cdot \overrightarrow{\pi}) = 0. \]

Hence by Eqs. (194) and (195)

\begin{equation}
H_r = H_\theta = E_\phi = 0.
\end{equation}
and by Eq. (191)

\begin{equation}
H_\phi = -\frac{\sin \theta}{4\pi r} \left( \frac{\partial}{\partial r} \hat{r} - \frac{1}{r} \hat{\rho} \right),
\end{equation}

\begin{equation}
E_r = \frac{\cos \theta}{4\pi \epsilon_0 r} \left( \frac{\partial^2}{\partial r^2} \mathbf{P} - \frac{2}{r} \frac{\partial}{\partial r} \mathbf{P} + \frac{2}{r^2} \mathbf{P} - \frac{1}{c^2} \mathbf{\dddot{P}} \right),
\end{equation}

\begin{equation}
E_\theta = -\frac{\sin \theta}{4\pi \epsilon_0 r} \left( \frac{1}{r} \frac{\partial}{\partial r} \mathbf{P} - \frac{1}{r^2} \mathbf{P} - \frac{1}{c^2} \mathbf{\dddot{P}} \right).
\end{equation}

From Eq. (197) we conclude that the magnetic lines of force are circles about the direction of \( \mathbf{P} \), while the electric lines of force lie in the meridional planes through this direction.

Because of the argument (\( t - r/c \)) of \( \mathbf{P} \) it is possible in Eqs. (198) through (200) to transform the differentiation with respect to \( r \) into one with respect to \( t \) as was done in Section B

\begin{equation}
\frac{\partial \mathbf{P}}{\partial r} = -\frac{1}{c} \hat{r}, \quad \frac{\partial^2 \mathbf{P}}{\partial r^2} = \frac{1}{c^2} \mathbf{\dddot{P}}.
\end{equation}

Then Eqs. (198) through (200) may be written

\begin{equation}
H_\phi = -\frac{\sin \theta}{4\pi r} \left( -\frac{1}{c} \hat{r} - \frac{1}{r} \hat{\rho} \right)
\end{equation}

\begin{equation}
E_r = \frac{\cos \theta}{4\pi \epsilon_0 r} \left( \frac{1}{c^2} \hat{r} \sqrt{\mathbf{P} + \frac{2}{r} \frac{\partial}{\partial r} \mathbf{P} + \frac{2}{r^2} \mathbf{P} - \frac{1}{c^2} \mathbf{\dddot{P}}} \right)
\end{equation}

\begin{equation}
E_\theta = -\frac{\sin \theta}{4\pi \epsilon_0 r} \left( -\frac{1}{r c} \hat{r} - \frac{1}{r^2} \mathbf{P} - \frac{1}{c^2} \mathbf{\dddot{P}} \right)
\end{equation}
All but the last terms of \( E_T \) and \( E_\theta \) are due to the \( \nabla \phi \) function and here we note some interesting things. In \( E_T \) the \( \partial^2 \overline{P} / \partial r^2 \) term cancels with \( \dddot{\overline{P}} \) as was required by the application of the Lorentz Condition, Eq. (172) or (191), and the propagating longitudinal component of \( \overline{E} \) is eliminated. Thus \( E_T \) contains only terms of order higher than \( 1/r \). \( E_T \) and \( E_\theta \) contain, however, two higher order terms: one, the \( 1/r^3 \) term associated directly with the fields from a static dipole, and another, the \( 1/r^2 \) term which is coupled to the changing dipole moment and whose origin is more difficult to specify. In the case of harmonic excitation, this term would be 90° out of phase with the static term. A very good explanation for the harmonic case of what happens in the region near the dipole where these higher order terms predominate is given by Lorrain et al. [51] and some interesting phenomena are pointed out which carry over into the aperiodic case. This will not be detailed here.

We shall from here on limit our study to the "far field" (large distances from the origin, i.e., set \( r \to \infty \)). More precisely, for the short dipole, this requires that the following inequalities be satisfied:

\[
(202) \quad \frac{1}{c} |\dddot{\overline{P}}| >> \left| \frac{\dddot{\overline{P}}}{r} \right|, \quad \frac{1}{c^2} |\dddot{\overline{P}}| >> \left| \frac{\dddot{\overline{P}}}{cr} \right| >> \left| \frac{\dddot{\overline{P}}}{r^2} \right|.
\]
For large radiating surfaces where the assumptions made for the short dipole do not apply, inequalities Eq. (202) serve only to define the far limit of the "near-field". In Section E the "intermediate zone" or "Fresnel Zone" will be defined and then the near limits of the "far field" for a general source can be set. For the short dipole, however, the inequalities of Eq. (202) sufficiently define the far field region and we obtain:

\begin{align}
E_r &= 0 \\
E_\theta &= \frac{\sin \theta}{4 \pi \varepsilon_0 c^2 r} \hat{P}(t - r/c) = \frac{\sin \theta \ell}{4 \pi \varepsilon_0 c^2 r} \hat{I}([t]) \\
H_\phi &= \frac{\sin \theta}{4 \pi cr} \hat{P}(t - r/c) = \frac{\sin \theta \ell}{4 \pi cr} \hat{I}([t])
\end{align}

where since \( \hat{P} = q\hat{r} \) (\( \ell \) = separation of the mobile and the stationary charge and in our analogy is the dipole length) we have \( \hat{P} = q\hat{v} = \ell[\hat{I}] \).

The vectors \( \bar{H} \) and \( \bar{E} \) are perpendicular to each other and to the radius vector \( \bar{r} \) from the origin (see Fig. 35), and their magnitudes depend not on the magnitude of the current but on the time rate of change of the current. For more complex radiating structures this will not be so obvious. Both \( \bar{H} \) and \( \bar{E} \) vanish on the axis \( \theta = 0 \) and \( \theta = \pi \); the \( \bar{H} \) and \( \bar{E} \) fields have their maxima in the equatorial plane \( \theta = \pi / 2 \).
From Eqs. (204) and (205) we calculate

\[
\frac{E_\theta}{H_\phi} = \frac{1}{\varepsilon_0 c} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \xi_0
\]

This is the same ratio as that which was obtained in Section B, Eq. (152), for the ratio \( E_x/H_y \). In the vicinity of the observation point the structure of the radiated electromagnetic field is thus that of a transverse plane wave.

The amount of energy radiated per unit area and per unit time becomes

\[
\bar{S} = \bar{E} \times \bar{H} = (E_\theta H_\phi)_r = \frac{\ell^2}{16\pi^2 \varepsilon_0 c^2} \frac{\sin^2 \theta}{r^2} \left[ \dot{I} \right]^2
\]

When this equation is multiplied by \( r^2 \) and averaged over the duration of the radiation or over one period for periodic functions one obtains, respectively,

\[
W(\theta, \phi) = \int_{T_1}^{T_2} r^2 S(\theta, \phi, r) dt \quad \text{and} \quad P(\theta, \phi) = \frac{1}{T} W(\theta, \phi)
\]

where \( f = 1/T \) is the fundamental frequency of our periodic function.

Equation (16) would correspond to the usual antenna "power pattern". The rms value of the spatial variation of the electric (or magnetic) field, Eqs. (204) or (205), on the other hand, is called the "field pattern".
Figure 36 shows the radiation density $S$ as a function of $\theta$. It is simply the polar diagram of $\sin^2 \theta$, isotopic in $\phi$, and is the usual radiation pattern from the short and infinitesimal dipole found in any standard book on antennas[48]. It should be noted that the radiation pattern here is independent of the type of excitation. This will, in general, be true only for the electrically short dipole.

In place of a single dipole $\vec{P}$ we may, of course, also consider a discrete or contiguous string of dipoles. In the latter case we write in place of Eq. (191):

$$4\pi \bar{\pi} = \int_S \frac{d\vec{P}(t-r/c)}{r}.$$

Here the integration is to be extended over the surface or curve $S$ and the difference in direction of the vectors $d\vec{P}$ must be considered. The evaluation of an equation similar to Eq. (208) will be discussed in the next section.

![Diagram showing radiation pattern from short dipole](image-url)

**Fig. 36--Radiation pattern from short dipole.**
The usual excitation of a dipole or, for that matter, most radiating structures, is harmonic time dependence. This is obtained by assuming, in the case of the short dipole, that the electric moment \( \mathbf{P} \) oscillates monochromatically with an angular frequency \( \omega \). For example, we set

\[
\mathbf{P}(t) = A \cos \omega t = A \Re \{ \exp j\omega t \}.
\]

If, as is the usual practice, we introduce the wave number \( k = \omega / c \) and omit the sign indicating the real part, we find

\[
\frac{1}{r} \mathbf{P} \left( t - \frac{r}{c} \right) = \frac{A}{r} \frac{e^{-jkr}}{r} e^{j\omega t}.
\]

We have thus arrived at the familiar representation of the spherical wave. Suppressing the time factor in Eq. (209) we obtain from Eqs. (204) and (205) the following familiar representation of the time harmonic electromagnetic field:

\[
E = E_\theta = -\frac{Ak^2}{4\pi\epsilon_0} \frac{e^{-ikr}}{r} \sin \theta,
\]

\[
H = H_\phi = -\frac{Ak\omega}{4\pi} \frac{e^{-ikr}}{r} \sin \theta
\]

where it is remembered the excitation is \( \mathbf{P}(t) \) so that \( \mathbf{P}(t) = I(t) \). This applies to the "far field" region which for the harmonic case can alternatively be defined in the following way. If we let the wavelength
\[ \lambda = \frac{2\pi}{\omega} c \]

correspond to the angular frequency \( \omega \), the far field region includes all distances for which

\[ r \gg \lambda \]

i.e., only the immediate neighborhood of the radiation source is excluded. Again the reader is reminded that for larger radiators this would define the far end of the "near field" region, but for the short dipole the far field begins here as well.

2. **Huygens-Kirchoff Principle for obtaining the electromagnetic field**
   (the aperture integral)

We have now to consider the following problem: given the values of the electric and magnetic field vectors over a closed surface, how can we determine the field vectors at a specific point outside the surface. We follow the work of Chernousov[5].

The solution to this problem is, in fact, contained in the general field equations, Eqs. (184) and (185) of Section D. In accordance with Huygens' principle, the electromagnetic field of an antenna at the point of observation can be expressed as a result of adding waves which are radiated by sources located on the antenna surface or by the
appropriate sources on any surface $S$ enclosing the antenna. Such secondary sources are equivalent surface electric currents $\overline{J}_S(\overline{r}_S, t)$ and magnetic currents $\overline{M}_S(\overline{r}_S, t)$, induced on such surfaces by the actual sources of the electromagnetic field of radiation. This is just a statement of the equivalence theorem\[52, 53\] generally stated for harmonic time dependence, but it should be apparent from Section C that it is also true for any time varying field.

As is stated in the equivalence theorem the equivalent electrical and magnetic currents are associated with the true fields $\overline{E}(\overline{r}_S, t)$ and $\overline{H}(\overline{r}_S, t)$ on the surface enclosing the antenna through the relationships developed in the section on Boundary Conditions (Eqs. (161a) and (b) by

\begin{align*}
7) \quad \overline{J}_S(\overline{r}_S, t) &= \overline{n}^0 \times \overline{H}(\overline{r}_S, t) \\
8) \quad \overline{M}_S(\overline{r}_S, t) &= \overline{E}(\overline{r}_S, t) \times \overline{n}^0
\end{align*}

where $\overline{n}^0$ is the unit vector normal to the external surface and $\overline{r}_S$ is the distance from the origin of the coordinates to the considered element of the surface (see Fig. 37).

![Fig. 37--Relationship between vectors.](image-url)
The retarded vector potentials are from Eqs. (5b) and (6b):

\[(5b') \quad \overline{A}(r, t) = \frac{\mu}{4\pi} \int_{V} \frac{\overline{J}(\overline{r}_s, t - (R/c))}{R} \, dv\]

\[(6b') \quad \overline{F}(r, t) = \frac{\epsilon}{4\pi} \int_{V} \frac{\overline{M}(\overline{r}_s, t - (R/c))}{R} \, dv\]

which become, when we restrict ourselves to surface currents and substitute from Eqs. (7) and (8),

\[(9) \quad \overline{A}(r, t) = \frac{\mu}{4\pi} \oint_{S_a} \frac{n^\circ \times \overline{H}(\overline{r}_s, t - (R/c))}{R} \, ds\]

\[(10) \quad \overline{F}(r, t) = -\frac{\epsilon}{4\pi} \oint_{S_a} \frac{n^\circ \times \overline{E}(\overline{r}_s, t - (R/c))}{R} \, ds\]

where \( R = |\overline{r} - \overline{r}_s| \), \( r \) is the distance from the origin of coordinates to the point of observation \( O(\overline{r}) \) at which the antenna radiation field is to be determined and \( \overline{r}_s \) is the distance from the origin to a point in \( ds \) on the source (Fig. 37).

Knowing \( \overline{A}(r, t) \) and \( \overline{F}(r, t) \) we can readily find the total electromagnetic field radiated from our antenna by substituting Eqs. (9) and (10) into Eqs. (184) and (185) obtaining
\[
\mathbf{E}(\vec{r}, t) = \frac{1}{4\pi} \nabla \times \oint_{S_a} \frac{\mathbf{n}_o \times \mathbf{E}(\vec{r}_S, t - R/c)}{R} \, ds \\
+ \frac{1}{4\pi\varepsilon} \nabla \left( \nabla \cdot \oint_{S_a} \left\{ \frac{1}{R} \int_0^{t-R/c} n_o \times \mathbf{H}(\vec{r}_S, t) \, dt \right\} \, ds \right) \\
- \frac{\mu}{4\pi} \oint_{S_a} \frac{1}{R} \frac{\partial (\mathbf{n}_o \times \mathbf{H}(\vec{r}_S, t - R/c))}{\partial t} \, ds,
\]

\[
\mathbf{H}(\vec{r}, t) = \frac{1}{4\pi} \nabla \times \oint_{S_a} \frac{(\mathbf{n}_o \times \mathbf{H}(\vec{r}_S, t - R/c))}{R} \, ds \\
- \frac{1}{4\pi\mu} \nabla \left( \nabla \cdot \oint_{S_a} \left\{ \frac{1}{R} \int_0^{t-R/c} \mathbf{n}_o \times \mathbf{E}(\vec{r}_S, t) \, dt \right\} \, ds \right) \\
+ \frac{\varepsilon}{4\pi} \oint_{S_a} \frac{1}{R} \frac{\partial (\mathbf{n}_o \times \mathbf{E}(\vec{r}_S, t - R/c))}{\partial t} \, ds.
\]

We see that the fields at \(O(\vec{r}, t)\) are determined by currents which were on the surface \(S_a\) at an earlier time \((t - R/c)\) and whose effects are summed at point \(O(\vec{r}, t)\) to give the fields.

It is Eqs. (213) and (214) which are of importance to us and will be the subject of our investigations throughout the remainder of this work. If the order of integration and of vector differentiation is
interchanged, which is permissible since they are performed at
different points in space, and a few simple rearrangements per-
formed, we obtain

\[
E(r, t) = -\frac{1}{4\pi\epsilon} \oint_{S_a} \left\{ \frac{1}{c^2 R} \frac{\partial H(r_S, t - R/c)}{\partial t} \vec{n} \times \vec{h}^o + \epsilon \left[ \vec{n}^o \times \vec{e}^o \right] \times \nabla \left( \frac{E(r_S, t - R/c)}{R} \right) - \left( \vec{n}^o \times \vec{h}^o \right) \cdot \nabla \right\} ds
\]

\[
H(r, t) = \frac{1}{4\pi\mu} \oint_{S_a} \left\{ \frac{1}{c^2 R} \frac{\partial E(r_S, t - R/c)}{\partial t} \vec{n} \times \vec{e}^o - \mu \left[ \vec{n}^o \times \vec{h}^o \right] \times \nabla \left( \frac{H(r_S, t - R/c)}{R} \right) - \left[ \vec{n}^o \times \vec{e}^o \right] \cdot \nabla \right\} ds
\]

where $\vec{e}^o$ and $\vec{h}^o$ are the unit vectors of the field vectors $\vec{E}$ and $\vec{H}$, respectively. Equations (215) and (216) present the mathematical
formulation of the electrodynamic Huygens principle as applied to
vector fields with arbitrary time variation.
Using the rules of vector algebra and of the differentiation of a
definite integral with respect to the upper limit, the integrand expres-
sions in Eqs. (215) and (216) can be made to take the following form

\[ \frac{1}{c^2 R} \frac{\partial}{\partial t} \frac{U(r_s, t-R/c)}{R} \left[ \overrightarrow{n} \times \overrightarrow{u} \right] + \left( \overrightarrow{n} \times \overrightarrow{u} \right) \cdot \nabla \left( \int_0^{t-R/c} \frac{U(r_s, t)dt}{R} \right) = \]

\[ = \left\{ \frac{1}{c^2 R} \left[ \overrightarrow{R} \times \left[ \overrightarrow{R} \times \left( \overrightarrow{n} \times \frac{\partial U(r_s, t)}{\partial t} \right) \right] \right\} + \]

\[ + \frac{1}{c^2 R} \left( 3 \overrightarrow{R} \cdot \left[ \overrightarrow{n} \times \overline{U(r_s, t)} \right] \right) - \left[ \overrightarrow{n} \times \overline{U(r_s, t)} \right] + \]

\[ + \frac{1}{R^3} \left( 3 \overrightarrow{R} \cdot \left[ \overrightarrow{n} \times \left( \int_0^t \overline{U(r_s, t)} \, dt \right) \right] \right) - \]

\[ - \left[ \overrightarrow{n} \times \left( \int_0^t \overline{U(r_s, t)} \, dt \right) \right] \]

\[ t = t - R/c \]

which may be written
\[
\begin{align*}
2 \overline{R}^0 \left( \frac{\overline{n}^0 \times \overline{U}(\overline{r}_S, t)}{cR^2} + \frac{\overline{n}^0 \times \int_0^t \overline{U}(\overline{r}_S, t) dt}{R^3} \right) \right) + \\
\left[ \overline{R}^0 \cdot \left( \left[ \frac{\overline{n}^0 \times \overline{U}(\overline{r}_S, t)}{cR^2} \right] + \frac{\overline{n}^0 \times \int_0^t \overline{U}(\overline{r}_S, t) dt}{R^3} \right) \right] \right) \\
+ \left[ \frac{\overline{n}^0 \times \int_0^t \overline{U}(\overline{r}_S, t) dt}{R^3} \right] \right) \right) \\
= t = t - R/c \\
\end{align*}
\]

where also

\[(218) \quad \left[ \overline{n}^0 \times \overline{u}^0 \right] \times \nabla \left( \frac{\overline{U}(\overline{r}_S, t - R/c)}{R} \right) \]

\[
= \left\{ \frac{1}{cR} \left[ \overline{R}^0 \times \left[ \overline{n}^0 \times \frac{\partial \overline{U}(\overline{r}_S, t)}{\partial t} \right] \right] \right) \\
+ \frac{1}{R^2} \left[ \overline{R}^0 \times \left[ \overline{n}^0 \times \overline{U}(\overline{r}_S, t) \right] \right] \right) \\
= t = t - R/c \\
\]

In expressions (217) and (128) \( \overline{U} \) denotes either \( \overline{E} \) or \( \overline{H} \). It is seen, therefore, that both \( \overline{E} \) and \( \overline{H} \) fields in general have components along all three directions \( \overline{R}^0 \), \( \overline{\theta}^0 \) and \( \overline{\phi}^0 \). The general relationship for \( \overline{E} \), for example, may then be written:
\[ (219) \quad \overline{E}(\overline{r}, t) = \frac{1}{4\pi \varepsilon} \oint_{S_a} \left\{ \begin{array}{c} 2 \overline{R}^0 \left( \overline{R}^0 \cdot \left[ \frac{n^o \times \overline{H}(\overline{r}_S, t)}{cR^2} \right] \\
^o \times \int_0^t \overline{H}(\overline{r}_S, t) \, dt \right) \end{array} \right\} + \\
+ \left[ \begin{array}{c} \overline{R}^0 \times \left[ \overline{R}^0 \times \left[ \frac{1}{c^2 R} \left[ n^o \times \frac{\partial \overline{H}(\overline{r}_S, t)}{\partial t} \right] \right] + \frac{n^o \times \overline{H}(\overline{r}_S, t)}{cR^2} \right] \\
\overline{R}^0 \times \int_0^t \overline{H}(\overline{r}_S, t) \, dt \right] \right\} \\
(\theta) - \varepsilon \left[ \overline{R}^0 \times \left[ \frac{1}{cR} \left[ n^o \times \frac{\partial \overline{E}(\overline{r}_S, t)}{\partial t} \right] \right] + \\
+ \frac{1}{R^2} \left[ n^o \times \overline{E}(\overline{r}_S, t) \right] \right] \right\} \, ds \\
t = t - \frac{R}{c} \]

The components of the above have been grouped by vector operations and it is of interest to compare \( r, \theta, \phi \) groups above with Eqs. \((198')\) through \((200')\) for radiation from the short dipole and note the one to one correspondence. Note, also, that \( \overline{E}(t - R/c) \) in the above corresponds to \( \overline{E}(t - R/c) \) in Eqs. \((198'), (199')\) and \((200')\).

Thus we have in Eq. \((219)\) a quite general relationship for \( \overline{E} \) in spherical coordinates and in the time domain for radiation from an arbitrary radiator. In order to apply the equations we must first
specify either equivalent electric and magnetic currents or fields on the isolated antenna surface. The latter are usually unknown and their exact determination is generally very difficult if not impossible to obtain. Fortunately, however, a satisfactory solution to the problem can usually be obtained by using approximate values obtained from the idealized conditions of the problem.

Such approximate values can be introduced in different problems using a variety of methods. For example, the distribution of current over the surface of an antenna reflector or the distribution of the field in an aperture of an antenna can be obtained using the method of geometrical optics; the distribution of currents in dipole antennas can be determined on the basis of the transient theory of a long transmission line. The calculation of the radiation field using those formulas leads, in the general case, however, to rather involved mathematics. Depending on the conditions of the problem the solutions for the radiating fields can be considerably simplified for points which are sufficiently distant from the antenna system. At distances which considerably exceed the linear dimensions of the antenna we can, as in the short dipole case, again neglect all terms which contain $1/R$ in the second degree and higher. Sufficient conditions for this were presented by the inequalities of Eq. (202) which may be rewritten as
\( (220) \quad \frac{1}{c} \left| \frac{\partial U(\mathbf{r}_s, t)}{\partial t} \right| > \frac{1}{R} \left| U(\mathbf{r}_s, t) \right| \)

and

\( (221) \quad \frac{1}{c^2} \left| \frac{\partial U(\mathbf{r}_s, t)}{\partial t} \right| > \frac{1}{cR} \left| U(\mathbf{r}_s, t) \right| > \frac{1}{R^2} \left| \int_0^t U(\mathbf{r}_s, t) dt \right| \).

The expressions for the intensity of the electric and magnetic fields thus take on the form (with an accuracy of the quantities of the order of 1/R^2)

\( (221) \quad \overline{E}(\mathbf{r}, t) = \frac{1}{4\pi c} \oint_{S_a} \frac{1}{R} \left\{ \zeta_{\mu} \left[ \overline{R}^o \times \left[ \overline{E}^o \times \left[ \overline{n}^o \times \frac{\partial \overline{H}(\mathbf{r}_s, t - R/c)}{\partial t} \right] \right] \right] \right\} \)

\[- \left[ \overline{R}^o \times \left[ \overline{n}^o \times \frac{\partial \overline{E}(\mathbf{r}_s, t - R/c)}{\partial t} \right] \right] \} \) ds,

\( (222) \quad \overline{H}(\mathbf{r}, t) = \frac{1}{4\pi c} \oint_{S_a} \frac{1}{R} \left\{ - \zeta_{\epsilon} \left[ \overline{R}^o \times \left[ \overline{H}^o \times \left[ \overline{n}^o \times \frac{\partial \overline{E}(\mathbf{r}_s, t - R/c)}{\partial t} \right] \right] \right] \right\} \)

\[- \left[ \overline{R}^o \times \left[ \overline{n}^o \times \frac{\partial \overline{H}(\mathbf{r}_s, t - R/c)}{\partial t} \right] \right] \} \) ds,

where \( \zeta = \sqrt{\mu/\epsilon} \) is the impedance of the medium (vacuum). Equations (221) and (222) correspond to Eqs. (204) and (205) for the dipole and are valid everywhere outside what in the harmonic case is referred to as the near-field region. We shall use the same definition in the
anharmonic case. We note that again the radiation fields are related to the time derivatives of the currents or fields on the radiator. This, of course, means that the waveform received generally is not the same as the waveform sent except in the special case of harmonic time dependence. More will be said about this in the next section on reciprocity.

The above expressions can be simplified still more by replacing $R$ with its approximate value. At large distances from the antenna the quantity $R = \sqrt{r^2 + r_s^2 - 2r_s \cdot \bar{r}}$ (see Fig. 37) can be expanded into a series of terms in increasing powers of $r_s/r$, using the Binomial expansion, and retaining only those terms to the first power in $1/r$.

Thus, $R$ may be written

\begin{equation}
R = (r^2 + r_s^2 - 2r_s \cdot \bar{r})^{1/2} = r \left[1 + 1/2 \left(\frac{r_s^2}{r^2} - 2 \frac{r_s \cdot \bar{r}}{r^2}\right)\right] - \frac{1}{8r^4} \left(4(r_s \cdot \bar{r})^2 + \ldots\right) + \ldots
\end{equation}

\begin{align*}
&= r + \frac{r_s^2}{2r} - r_s \cdot \bar{r}^o - \frac{(r_s \cdot \bar{r}^o)^2}{2r} + \ldots \\
&\sim r - r_s \cdot \bar{r}^o + \frac{(r_s \times \bar{r}^o)^2}{2r}
\end{align*}

Then $1/R$ may be replaced in the integrand expressions by $1/r$ with an accuracy of the order of $r_s^2/r$. We can, as well, neglect the angle between the unit vectors $\bar{R}^o$ and $\bar{r}^o$ replacing $\bar{R}^o$ by $\bar{r}^o$. Considering
that in the integration \( r \) is constant and can, therefore, be taken outside the integral, we obtain for the fields at large distances from the source the following expressions:

\[
\mathbf{E}(r, t) = -\frac{1}{4\pi cr} \left[ \mathbf{r}^0 \times \oint_{\mathcal{S}_a} \left\{ \left[ -\mathbf{n}^0 \times \frac{\partial \mathbf{E}(\mathbf{r}_S, [t]) + \mathbf{r}^0 \cdot \mathbf{r}_S / c - \psi(\mathbf{r}_S)}{\partial [t]} \right] - \zeta_0 \left[ \mathbf{r}^0 \times \left[ -\mathbf{n}^0 \times \frac{\partial \mathbf{H}(\mathbf{r}_S, [t]) + \mathbf{r}^0 \cdot \mathbf{r}_S / c - \psi(\mathbf{r}_S)}{\partial [t]} \right] \right] \right\} ds \right]
\]

\[
\mathbf{H}(r, t) = -\frac{1}{4\pi cr} \left[ \mathbf{r}^0 \times \oint_{\mathcal{S}_a} \left\{ \left[ -\mathbf{n}^0 \times \frac{\partial \mathbf{E}(\mathbf{r}_S, [t]) + \mathbf{r}^0 \cdot \mathbf{r}_S / c - \psi(\mathbf{r}_S)}{\partial [t]} \right] + \left[ -\mathbf{n}^0 \times \frac{\partial \mathbf{H}(\mathbf{r}_S, [t]) + \mathbf{r}^0 \cdot \mathbf{r}_S / c - \psi(\mathbf{r}_S)}{\partial [t]} \right] \right\} ds \right]
\]

where

\[
\psi(\mathbf{r}_S) = \left\{ -\frac{\mathbf{r}_S \times \mathbf{r}^0}{2rc} + \phi(\mathbf{r}_S) \right\}
\]

The quantity \( \phi(\mathbf{r}_S) \) is an arbitrary delay function and retarded time \([t] = t - r/c\) is a parameter which does not depend on the coordinates with respect to which the integration was made.

There is, of course, a distance beyond which the term \((\mathbf{r}_S \times \mathbf{r}^0)^2/2r\) can be neglected. This region is referred to as the far-field or Fraunhofer region. The criteria that determines the minimum distance beyond which this term need be considered significant is, for the harmonic case, subject to debate; but the distance \( r = 2D^2/\lambda \) is most
often used\cite{54} where $D$ is the largest dimension of the radiator. For the anharmonic case an even less clear-cut definition is available and we must limit ourselves to the rather vague criteria

\begin{equation}
\frac{(\bar{r}_s \times \bar{r}_0)^2}{2r(\bar{r}_s \cdot \bar{r}_0)} << 1
\end{equation}

to define the minimum distance to the far-field region.

The intermediate region between the near and far fields where inequality \eqref{224} is not valid is the Fresnel region and integrals \eqref{11} and \eqref{12} are, of course, more difficult to solve here. In the harmonic case, we find that the field pattern varies with distance in this region\cite{55,56}.

We have limited our study of the anharmonic case to the far-field region, and it was the subject of this treatise to apply expressions \eqref{11} and \eqref{12} to specific problems.

One final observation is in order. It is evident from relationships \eqref{11} and \eqref{12} that outside the near field region the electromagnetic wave is transverse and the magnetic field can be very simply expressed in terms of the electric vector

\begin{equation}
\bar{H}(\bar{r}, t) = \frac{1}{\bar{r}_0} [\bar{r}_0 \times \bar{E}(\bar{r}, t)]
\end{equation}

This is an extremely useful relationship as it allows one to obtain either field if the other is known.
F. Reciprocity Theorem for Electromagnetic Fields with General Time Dependence

The reciprocity theorem, an expression of a basic property of electromagnetic fields, has proven to be one of the important analytical tools in the simplification and solution of various problems in electromagnetic theory. The theorem was first derived by Lorentz in 1895 for fields with simple harmonic dependence. The solution to the problem of arbitrary time dependence could be expressed in terms of the solution with harmonic time dependence through the use of the Fourier transform[57, 58, 59]. Cheo[9], however, has derived the result completely in the time domain, and, thus, it is applicable to fields whose time dependence may not be Fourier transformable. The proof requires only the past histories of the fields; their behavior beyond the present is unimportant. We will outline his theorem and proof below.

Theorem

Consider a certain environment containing an arbitrary distribution of linear time-independent media (see Fig. 38), and let \( J_a(\vec{r}, t) \) and \( J_b(\vec{r}, t) \) be two independent source current distributions with prescribed time and spatial dependences satisfying the following two conditions:

\[
(A) \quad J_a(\vec{r}, t) = J_b(\vec{r}, t) = 0 \quad \text{for } t < t_0 \quad (t_0 > -\infty) \quad \text{and all } \vec{r} .
\]
(B) \( J_a(\mathbf{r}, t) = J_b(\mathbf{r}, t) = 0 \), for \( |\mathbf{r}| > r_o \) \( (r_o < \infty) \) and all \( t < \infty \).

In other words, we assume that our sources were turned on at some finite time \( t_o \) in the past and are of finite size \( r \).

Fig. 38--Relationship between expanding wavefronts from reciprical elements and the expanding surface of integration.
If $\mathbf{E}_a(\mathbf{r}, t)$ and $\mathbf{E}_b(\mathbf{r}, t)$ denote the electric fields corresponding to the sources $\mathbf{J}_a(\mathbf{r}, t)$ and $\mathbf{J}_b(\mathbf{r}, t)$, respectively, then for all $\tau$,

$$
(17) \quad \int_{-\infty}^{\infty} dt \int_V \mathbf{J}_a(\mathbf{r}, \tau - t) \cdot \mathbf{E}_b(\mathbf{r}, t) \nonumber
$$

$$
\neq \int_{-\infty}^{\infty} dt \int_V \mathbf{J}_b(\mathbf{r}, t) \cdot \mathbf{E}_a(\mathbf{r}, \tau - t) \nonumber
$$

where $V$ is any volume containing both $\mathbf{J}_a$ and $\mathbf{J}_b$.

**Proof**

The fields $\mathbf{E}_a(t)$ satisfy Maxwell's equations:

4. $\nabla \times \mathbf{E}(t) = -\frac{\partial}{\partial t} \mathbf{B}(t)$

5. $\nabla \times \mathbf{H}(t) = \mathbf{J}_s(t) + \mathbf{J}_i(t) + \frac{\partial}{\partial t} \mathbf{D}(t)$

where $\mathbf{J}_s(t)$ is the appropriate source current and $\mathbf{J}_i(t)$ the induced current defined in Eq. (136). Likewise,

$$
4'. \quad \nabla \times \mathbf{E}(\tau - t) = \frac{\partial}{\partial t} \mathbf{B}(\tau - t)
$$

$$
5'. \quad \nabla \times \mathbf{H}(\tau - t) = \mathbf{J}_s(\tau - t) + \mathbf{J}_i(\tau - t) - \frac{\partial}{\partial t} \mathbf{D}(\tau - t)
$$

Using the vector identity

$$
\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})
$$

and the above expressions of Maxwell's equations, one obtains after integrating with respect to $t$
\[ \int_{-\infty}^{\infty} \nabla \cdot [\overline{E_a(\tau-t)} \times \overline{H_b(t)} - \overline{E_b(t)} \times \overline{H_a(\tau-t)}] \, dt = \]
\[ = \int_{-\infty}^{\infty} \left\{ \overline{H_b(t)} \cdot \frac{\partial}{\partial t} \overline{B_a(\tau-t)} + \right. \]
\[ + \overline{H_a(\tau-t)} \cdot \frac{\partial}{\partial t} \overline{B_b(t)} - \overline{E_a(\tau-t)} \cdot \overline{J_{sb}(t)} - \overline{E_a(\tau-t)} \cdot \overline{J_{ib}(t)} - \]
\[ - \overline{E_a(\tau-t)} \cdot \frac{\partial}{\partial t} \overline{D_b(t)} + \overline{E_b(t)} \cdot \overline{J_{sa}(\tau-t)} + \overline{E_b(t)} \cdot \overline{J_{ia}(\tau-t)} - \]
\[ - \overline{E_b(t)} \cdot \frac{\partial}{\partial t} \overline{D_a(\tau-t)} \right\} \, dt. \]

If we now substitute into the above Eqs. (134), (135) and (136) for \( \overline{B}, \overline{D}, \) and \( \overline{J_i} \) we discover that all the terms on the right-hand side vanish with the exception of the two containing \( \overline{J_{sa}(\tau-t)} \) and \( \overline{J_{sa}(t)} \). This substitution will, however, limit the applicability of the theorem to those media for which relationships (134), (135), and (136) apply. Making this substitution and noting that the integration in Eqs. (134), (135) and (136) is between finite limits, we obtain

\[ \int_{-\infty}^{\infty} \nabla \cdot [\overline{E_a(\tau-t)} \times \overline{H_b(t)} - \overline{E_b(t)} \times \overline{H_a(\tau-t)}] \, dt = \]
\[ = \int_{-\infty}^{\infty} [ -\overline{E_a(\tau-t)} \cdot \overline{J_b(t)} + \overline{E_b(t)} \cdot \overline{J_a(\tau-t)}] \, dt \]

where from our initial assumption \( \overline{E_a, b(-\infty)} = \overline{H_a, b(-\infty)} = \overline{J_{ia, b(-\infty)}} = 0 \).

Now Eq. (226) is integrated with respect to volume. After an interchange of the order of integration, there results
\[(227) \quad \int_{-\infty}^{\infty} dt \int_{V} dv \nabla \cdot \left[ \overline{E}_{a}(\tau-t) \times \overline{H}_{b}(t) - \overline{E}_{b}(t) \times \overline{H}_{a}(\tau-t) \right] = \]

\[= \int_{-\infty}^{\infty} dt \int_{V} dv \left[ \overline{J}_{a}(\tau-t) \cdot \overline{E}_{b}(t) - \overline{J}_{b}(t) \cdot \overline{E}_{a}(\tau-t) \right] \]

The volume integral on the left-hand side can be changed into a surface integral over a surface S enclosing volume V by use of the divergence theorem. With one further interchange of the order of integration, there is obtained

\[(228) \quad \oint_{S} \int_{-\infty}^{\infty} dt \left[ \overline{E}_{a}(\tau-t) \times \overline{H}_{b}(t) - \overline{E}_{b}(t) \times \overline{H}_{a}(\tau-t) \right] \cdot d\mathbf{s} = \]

\[= \int_{V} dv \int_{-\infty}^{\infty} dt \left[ \overline{J}_{a}(\tau-t) \cdot \overline{E}_{b}(t) \cdot \overline{J}_{b}(t) \cdot \overline{E}_{a}(\tau-t) \right] \]

Since both \( \overline{J}_{a}(t) \) and \( \overline{J}_{b}(t) \) have only finite extension, the volume V can be chosen sufficiently large to enclose \( \overline{J}_{a}(t) \) and \( \overline{J}_{b}(t) \) completely. Then the value of the left-hand side of Eq. (227) will remain constant for further increase of V beyond that which insures enclosure of the sources. Stated alternately, the surface integral of the left-hand side of Eq. (228) is a constant over any S which encloses both \( \overline{J}_{a} \) and \( \overline{J}_{b} \).

Furthermore, we will see next that this constant must be zero.

To illustrate this a physical condition on the behavior of the fields at a distance from the source is imposed. The condition is that
the fields propagate with a finite velocity \( v \) away from the source which generates them. In mathematical terms, this means that if \( J(t) = 0 \) for all \( t < t_0 \) \( (t_0 > -\infty) \) and has only finite extent, then for every finite \( t_1 \), however large, there exists a finite sphere of radius \( r_1 \) on which \( \overline{E}(t) = \overline{H}(t) = 0 \) for \( t \leq t_1 \). With this condition it is seen that the surface integral on the left-hand side of Eq. (228) can be made equal to zero by choosing a sphere for \( S \) with a radius sufficiently large (see Fig. 38). Thus, the surface integral must equal to zero on every \( S \) which encloses both \( J_a \) and \( J_b \). Therefore

\[
(17') \quad \int_{-\infty}^{\infty} dt \int_V dv \left[ \overline{J}_a(\tau-t) \cdot \overline{E}_b(t) - \overline{J}_b(t) \cdot \overline{E}_a(\tau-t) \right] = 0 .
\]

From Section A it is seen that it is possible to include environments containing anisotropic media. In conclusion, it is evident that the theorem holds in an environment which may contain anisotropic (but nongyrotropic), dissipative, and dispersive media, and with all these properties extending to infinity. The only requirement on the time functions is that the sources must have been zero at some time in the past.

The reciprocity theorem in this present form is a more general expression of the Lorentz reciprocity theorem in the Rayleigh-Carson form,
\[ \int_{V} \mathbf{j}_a \cdot \mathbf{E}_b \, dv = \int_{V} \mathbf{j}_b \cdot \mathbf{E}_a \, dv \]

where the latter is obtained by letting \( \mathbf{j}_a \), \( \mathbf{j}_b \) and \( \mathbf{E}_a \), \( \mathbf{E}_b \) have a time variation of the form \( e^{j\omega t} \).

It is quite important that the exact meaning and implications of Eq. \((17')\) be understood. For this reason we include the following two illustrations. Consider for example the pair of loop antennas shown in Fig. 39. The field \( \mathbf{a} \) results when the source is in \( \mathbf{a} \) and the field \( \mathbf{b} \) when the source is in \( \mathbf{b} \). For these antennas

\[ \int_{-\infty}^{\infty} dt \int_{V} \mathbf{E}_a(\tau-t) \cdot \mathbf{j}_b(t) \, dv = \int_{-\infty}^{\infty} dt \int_{V} \mathbf{E}_a(\tau-t) \cdot \mathbf{j}_b \, da, \]

\[ = \int_{-\infty}^{\infty} V_{sa}(\tau-t) I_{b, in a}(t) \, dt \]

where \( V_{sa} \) is the voltage supplied by the source in \( \mathbf{a} \), and \( I_{b, in a} \) is the current in the same loop \( \mathbf{a} \) when \( \mathbf{b} \) is energized. The other integral of Eq. \((17')\) can be expressed similarly, and

\[ \int_{-\infty}^{\infty} V_{sa}(\tau-t) I_{b, in a}(t) \, dt = \int_{-\infty}^{\infty} V_{sb}(t) I_{a, in b}(\tau-t) \, dt \]

We consider now the transmitting and receiving antennas as a system (see Fig. 40) and note that the source voltage \( V_{sa}(\tau-t) \) is
Fig. 39--Pair of loop antennas with the source in a (top) and then in b (bottom). The excitation function and the resistances are the same in both cases.

Fig. 40--Equivalent circuit representation of the antenna system shown in Fig. 39.
related to the output or received current $I_a$ in $b(\tau-t)$ through the transfer function or admittance $Y_a$ to $b(t)$[60] as,

$$I_{a \text{ in } b}(\tau) = \int_{-\infty}^{\infty} V_{sa}(\tau-t) \cdot Y_{a \text{ to } b}(t) \, dt$$

and likewise for $V_{sb}$

$$I_{b \text{ in } a}(\tau) = \int_{-\infty}^{\infty} V_{sb}(t) \cdot Y_{b \text{ to } a}(\tau-t) \, dt$$

It is obvious from Eqs. (233) and (234) that if we replace $V_{sa}(\tau-t)$ by $\delta_{sa}(\tau-t)$, an impulse function, we obtain

$$I_{a \text{ in } b}(\tau) = \int_{-\infty}^{\infty} \delta_{sa}(\tau-t) \cdot Y_{a \text{ to } b}(t) \, dt = Y_{a \text{ to } b}(\tau)$$

similarly for $V_{sb}$

$$I_{b \text{ in } a}(\tau) = \int_{-\infty}^{\infty} \delta_{sb}(t) \cdot Y_{b \text{ to } a}(\tau-t) \, dt = Y_{b \text{ to } a}(\tau)$$

and we see that the output currents for an impulse source voltage are exactly the transfer admittance. Substituting Eqs. (235) and (236) back into Eq. (232) we have

$$\int_{-\infty}^{\infty} \delta_{sa}(\tau-t) \cdot Y_{b \text{ to } a}(\tau) \, dt =$$

$$= \int_{-\infty}^{\infty} \delta_{sb}(t) \cdot Y_{a \text{ to } b}(\tau-t) \, dt$$

$$Y_{b \text{ to } a}(\tau) = Y_{a \text{ to } b}(\tau)$$
and similarly for the transfer impedance

\[(237b) \quad Z_{b \to a}(\tau) = Z_{a \to b}(\tau) \].

Thus we see that the transfer admittances (impedances) are identical; i.e., our antenna system is reciprocal for any waveform.

Physically this means that the transfer function between the induced current in the receiving antenna and the applied voltage on the transmitting antenna remains the same when the roles of the antennas are reversed, providing \(\mu\), \(\epsilon\), and \(\sigma\) remain unchanged and have the physical characteristics outlined in Section A. We have illustrated the reciprocity theorem here by referring to a pair of magnetic dipoles, but it is equally valid for any pair of antennas.

It must be kept in mind, however, that this theorem is concerned solely with the transfer function between \(V_s\) and \(I_{rec}\), in vice-versa; it says nothing about the power or energy transferred or the relationship between input and received waveforms. Both of these usually change when the source is moved from one position to another.

One important consequence for the reciprocity theorem is that in a reciprocal system the radiation energy or power pattern of an antenna must have exactly the same shape as the corresponding plot of the response of the antenna as a function of angle when it is used as a receiver. This fact has been of immeasurable aid in the single
frequency case for the determination of radiation patterns, and as is seen here can be extended to the general time case as well.

To further illustrate that the input waveform to the transmitting antenna and the waveform at the output of the terminals of the receiving antenna need not and, in general, will not be the same, consider the magnetic dipole as a receiving antenna (Fig. 41). The induced EMF is relatively easy to calculate

\begin{equation}
\oint \overline{E} \cdot d\overline{l} = -\oint \left( \frac{\partial \overline{A}}{\partial t} + \nabla \phi \right) \cdot d\overline{l}
\end{equation}

\[ = -\int_{S} \nabla \times \frac{\partial \overline{A}}{\partial t} + \nabla \times \nabla \phi \cdot ds \]

where S is any surface bounded by the circuit. The second term on the right vanishes since the curl of a gradient is identically zero.

Also, the order of the operation in the first term can be interchanged to give

\begin{equation}
\oint \overline{E} \cdot d\overline{l} = -\frac{\partial}{\partial t} \int_{S} (\nabla \times \overline{A}) \cdot ds = -\frac{\partial}{\partial t} \int_{S} \overline{B} \cdot ds
\end{equation}

Fig. 41--Magnetic dipole used as a receiving antenna.
The EMF induced in the loop is, therefore, equal to the rate of change of the flux linking the loop. It is maximum when the normal to the loop is parallel to the local $\mathbf{B}$.

The voltage $V_{\text{rec}}$, when $r_1 \to \infty$ would be expected to equal the induced EMF

$$V_{\text{rec}} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{s}$$

except for the fact that the circuit may also be excited in the electric dipole mode. For example, with a symmetrical loop such as that shown in Fig. 41, if the $\mathbf{E}$ vector is parallel to the wires leading to $\mathbf{R}$, charge oscillates from one end of the circuit to the other, and $V_{\text{rec}}$ is not affected by the electric dipole oscillation, thus the above relation is correct. On the other hand, if the $\mathbf{E}$ vector is in the plane of the loop but is perpendicular to the pair of wires, an extra voltage appears on $\mathbf{R}$ which comes from the dipole excitation and adds to the above induced EMF. In any case, the received voltage $V_{\text{rec}}$ is a function of the time derivative of the signal input waveform, not the waveform itself. By duality, it should be obvious that a similar result would obtain for the electric dipole. The calculations are, however, quite elaborate and will not be presented here, c.f., King [6].
BIBLIOGRAPHY


37. Stratton, p. 440.


44. Stratton, p. 424.


51. Lorrain, et.al., Ch. 14, in particular Sect. 14.1.6.


