DEVELOPMENT OF A TIME DOMAIN HYBRID FINITE DIFFERENCE/FINITE ELEMENT METHOD FOR SOLUTIONS TO MAXWELL’S EQUATIONS IN ANISOTROPIC MEDIA

DISSERTATION

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* * * * *

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ABSTRACT

The finite difference time domain (FDTD) and finite element numerical methods are two popular time domain computational methods in electromagnetics, but the two numerical methods have certain tradeoffs. FDTD is a fast explicit method with second order accuracy, but the method’s accuracy is reduced when analyzing structures that are not conforming to a Cartesian grid. The finite element method on the other hand excels at examining domains with non-conforming structures, but its method of solution usually requires a matrix inverse operation, which is computationally expensive. Fortunately, research in hybrid methods have shown that the FDTD method for isotropic materials can be viewed upon as a subset of finite elements, and from this viewpoint, the FDTD and finite element method in the time domain can be hybridized together to the advantages of both methods while mitigating the disadvantages.

With the recent rise in the study of metamaterials, which contain anisotropic media, having a hybridized method to study anisotropic media is a desirable tool as, for example, the effects of these materials combined with antennas are being examined. However, the hybridization approach combining the FDTD and finite element method for isotropic media does not extend to anisotropic media since the anisotropic FDTD equation cannot be recovered from the finite element formulation in this fashion. In this
dissertation, a hybridized FDTD/finite element method for anisotropic materials will be
developed. In the derivation of the hybridized method, a new finite element method will
be formulated which incorporates the constitutive relation in a finite element point of
view. This new finite element method will also be used to construct new anisotropic
FDTD stencils in a systematic manner for certain interface and boundary conditions that
the traditional anisotropic FDTD update fails to handle. Numerical tests will be
performed to demonstrate the accuracy of the both the hybridized anisotropic
FDTD/finite element method as well as the new FDTD stencils that are derived from the
new finite element method.
Dedicated to mom, dad, and Dan
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CHAPTER 1

INTRODUCTION

In the electromagnetic’s community, the finite difference time domain (FDTD) method and the finite element method are two of the most popular time domains for examining electromagnetic phenomena. The FDTD method is the most popular electromagnetic time domain method due to its ease of implementation based on the central differencing approximation to the derivatives [1, 2]. This implementation is second order accurate, and it results an efficient explicit updating scheme that requires no matrix inversion. It has been well studied and proven for use in examining a wide variety of electromagnetic applications. For the regular Yee update, the FDTD computation domain is split into Cartesian cubes, which are adequate for modeling surfaces that conform to the Cartesian coordinate system. However, if the computation domain contains volumes with non-conforming curved surfaces, the cubes are “stair-cased” around the curve as an approximation, and this approximation introduces a significant error into the computation [3, 4]. A finer discretization of cubes can be used to compensate against the effects of the stair-casing error, but the finer discretization will require higher memory usage due to the introduction of additional cubes. Also, the Courant stability criterion dictates that the FDTD time step is proportional to the
discretization, so the allotted time step to guarantee stability will become smaller, which will result in longer simulation runs. Conversely, the computational domain for the time domain finite element method can be formulated using an unstructured grid, which allows the method to accurately model curved surfaces without the stair-casing error in FDTD [5-13]. Also, depending on the time update scheme chosen, the finite element method can be formulated to be unconditionally stable without any restrictions on the time step being based on the discretization. However, the unconditional stability comes as result of the time domain finite element method being an implicit updating scheme, which requires a matrix inversion at every time step, and depending on the size of the domain, the cost of inverting this matrix can be exorbitantly expensive. And overall, the finite element method is more expensive memory-wise compared to FDTD.

Upon examining the advantages and disadvantages of the FDTD and finite element methods, one can conclude that a hybrid methodology that combines these two numerical methods will be able to tackle the deficiencies of each method when used alone. Hybridizing the two methods can address both the FDTD stair-casing error and the finite element matrix inversion issue and memory issue by appropriately choosing an appropriate method for different parts of the domain. The FDTD method can model the large, homogenous regions that conform to cubes, and the finite element method can be restricted to modeling volumes with surfaces that do not easily conform to cubes. This results in finite element matrices that have a smaller memory footprint and are less computationally expensive to invert, and the fast explicit FDTD update can be used for large portions of the homogenous domain, which results in a more efficient numerical
analysis compared to using each method by itself. Therefore, the hybrid method will combine the speed and accuracy advantages of both methods while reducing the deficiencies exhibited by each method.

There has been a great deal of research into developing a hybridized method for isotropic materials that combine both the FDTD and finite element methods, and the biggest key to hybridizing the method is to develop the appropriate interface conditions and updates between the FDTD and finite element domains. Wu and Itoh developed an interface method by inserting an edge along the diagonal of the cube faces which interfaced with the tetrahedral in the finite element domain, and the additional edge was updated based on a linear combination of the FDTD edges around it [14]. Another approach was developed by Yee et al using a two grid method where the results from a FDTD Cartesian grid and a finite element curvilinear grid were interpolated where the two grids overlapped [15]. Unfortunately, these methods suffered from late-time instabilities, which opened up research avenues of filtering methods to suppressing these late-time instabilities [16]. Later, Rylander and Bondeson developed a method to tackle the problem completely within a finite element viewpoint [17, 18]. They showed that the FDTD method can be derived from finite elements with a particular choice of integration method to evaluate the integral resulting after a weighted-residual method was applied, thus providing a hybrid scheme which can be viewed entirely from a finite element point of view. To interface the cube region with the tetrahedral region, a pyramid element serves as an intermediate element [19, 20]. This particular hybrid method was shown to
be stable without any need of filtering methods, and the hybrid method has been successfully applied to antenna studies in large domains [21-26].

Recently, there has been great interest in the academic and engineering community in examining the use of metamaterials; i.e., materials that can be synthesized to produce remarkable and interesting electromagnetic responses in a variety of applications. Photonic crystals, which are periodic structures of contrasting dielectrics, are an example of metamaterials being investigated since their properties are well understood, and their photonic band gap properties can be applied for applications such as waveguides and resonators. Figontin et al. proposed a magnetic photonic crystal (MPC) that consists of two mis-aligned anisotropic layers and a gyrotropic ferrite layer that has a unique spectral property in that it exhibits a stationary inflection point (SIP) in the dispersion curve [27, 28]. The SIP allows for uni-directional propagation of electromagnetic fields with much slower group velocities and increased amplitude within the MPC.

With the interest surrounding metamaterials, there comes a need for mature electromagnetic tools to be able to analyze these materials. In the past few years, there’re many numerical methods that have been successfully adapted to analyze the electromagnetic phenomena with metamaterials. One dimensional and two dimensional metamaterials have been analyzed using finite difference techniques [29-31], and several numerical studies have been done to examine the effects of metamaterials on the properties of antennas [32, 33]. But to efficiently examine large complicated problems with metamaterials, many of which are constructed with anisotropic media, a hybrid
method for anisotropic media would be an indispensible tool. However, the isotropic hybrid method described in [17] and [18] cannot be simply generalized to anisotropic materials. By attempting to derive the anisotropic hybrid methodology from the isotropic hybrid, one would find that the resulting finite element system would remain an implicit method. Even if the matrix inversion is performed to effectively create an “explicit” method from this finite element formulation, the explicit update does not recover the anisotropic FDTD update equation. Thus, there is no clear path to properly interface the anisotropic FDTD update and the finite element update from this methodology, and attempts to interface the two methods as so would likely result in an unstable method.

This dissertation will show a similar but slightly modified methodology from the isotropic hybrid can be used to derive the anisotropic hybrid method for FDTD and time domain finite elements. The key step lies in examining the constitutive relations and using the relation in a finite element point of view to derive the anisotropic time domain FDTD update from the finite element method consisting of cube elements. This new finite element method which uses the constitutive relation also has the capability to construct and derive anisotropic FDTD updates and stencils in a methodical manner that could not easily be obtained from an FDTD approach. For example, the anisotropic FDTD update for electric fields normal to a perfect electric conductor (PEC) wall can be derived from a finite element point of view. Such an update could not easily be derived just from the FDTD method as the anisotropic FDTD stencil extends one Yee cell ahead and behind in the direction of the electric field being updated. However, this and other anisotropic problems that could not easily be examined with anisotropic FDTD can now
be tackled with this new finite element method, and new stencils can be derived to properly interface with the anisotropic FDTD method.

In this dissertation, a derivation of the hybrid method for anisotropic materials will be shown for the second order vector wave equation and for first order Maxwell’s equations, and it will be shown that the second order and first order FDTD update can be derived respectively from these equations using the new finite element methodology in order to create the hybrid method. Afterwards, several numerical examples will be presented to show that the new finite element will result in the construction of an accurate and stable hybrid method capable of examining a variety of electromagnetic problems.
CHAPTER 2

ANISOTROPIC HYBRID FDTD/FINITE ELEMENT DERIVATION

For the isotropic case, it has been shown that hybridization of the FDTD/finite element time domain method was found by expressing the FDTD update equation as a time domain finite element update equation. In this section, a hybridization methodology will be applied in a similar manner to derive the hybridized anisotropic FDTD/finite element method from the second order wave equation and from first order Maxwell’s equations.

Section 2.1: Second Order Anisotropic FDTD Derivation from Finite Elements

The derivation starts with the second order vector wave equation in a source-free domain consisting of a lossless, homogenous, non-dispersive material with an anisotropic permittivity and isotropic permeability.

\[
\nabla \times \frac{1}{\mu_r} \nabla \times \vec{E} + \frac{1}{c^2} \frac{\partial^2 \vec{D}}{\partial t^2} = 0
\]

(1)

Next, the domain is broken into cubes, and cube edge basis functions are chosen for the expansion of the electric field \( \vec{E} \) and electric flux density \( \vec{D} \).
\[ E(\vec{r}, t) = \sum_{i=1}^{N} E_i(t) \vec{W}_i(\vec{r}) \]  

(2)

\[ D(\vec{r}, t) = \sum_{i=1}^{N} D_i(t) \vec{W}_i(\vec{r}) \]  

(3)

\( N \) represents the total number of edges in the domain. To formulate the weak solution to (1), the expansions in (2) and (3) are substituted into (1), and the weighted residual method is applied by testing the expanded equation with edge basis function \( \vec{W}_j \).

\[
\int_V \vec{W}_j \cdot \sum_{i=1}^{N} \left( \nabla \times \frac{1}{\mu_r} \nabla \times E_i(t) \vec{W}_i(\vec{r}) \right) dV = 0
\]

(4)

To separate the \textit{curl-curl} term in (4) for ease in evaluation, the following vector identity is applied.

\[
\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})
\]

(5)

Using (5) and the divergence theorem, (4) can be rewritten as the following.

\[
\int_V \left( \nabla \times \vec{W}_j(\vec{r}) \cdot \frac{1}{\mu_r} \nabla \times \sum_{i=1}^{N} E_i(t) \vec{W}_i(\vec{r}) \right) dV
\]

\[+ \frac{1}{c^2} \int_V \left( \vec{W}_j(\vec{r}) \cdot \frac{\partial^2}{\partial t^2} \sum_{i=1}^{N} D_i(t) \vec{W}_i(\vec{r}) \right) dV
\]

\[+ \oint_S \frac{1}{c} \vec{W}_j(\vec{r}) \cdot \left( \frac{\partial}{\partial t} \sum_{i=1}^{N} \left( E_i(t) \vec{W}_i(\vec{r}) \right) \times \hat{n} \right) dS = 0
\]

(6)
To simplify the formulation in (6), it is assumed that the domain extends outward indefinitely, so Sommerfeld radiation condition will come in play to eliminate the surface integral from (6). From here, a system of equations can be constructed as

$$\frac{\partial^2}{\partial t^2} [T\{D_i\}] + [S]E_i = 0$$

where \([T]\) is the mass matrix, \([S]\) is the stiffness matrix, and the components of each respective matrix is defined by

$$T_{ij} = \frac{1}{c^2} \int_V \vec{W}_i \cdot \vec{W}_j dV \quad (8.1)$$

$$S_{ij} = \int_V (\nabla \times \vec{W}_i) \cdot \frac{1}{\mu_r} (\nabla \times \vec{W}_j) dV \quad (8.2)$$

Although the integrals in (8.1) and (8.2) can be evaluated analytically, the authors in [17] found that by choosing an alternative method of integration to evaluate the mass and stiffness matrix terms, the FDTD update can be derived from finite elements. This method of integration is simply an extension of trapezoidal integration into three dimensions, and this integration will diagonalize the mass matrix when applied to the mass matrix integral and result in a sparser stiffness matrix when applied to the stiffness matrix integral [17, 18, 34]. The integration method is also referred to as mass lumping since the matrix terms are being “lumped” into the diagonal. It is important to note that while this integration is an approximation, the approximation is second order in accuracy, which is the same level of accuracy resulting from using the first order basis functions.

For a cube element, trapezoidal integration is defined as
\[ \int_V f(r) dV_e \approx \frac{V_e}{8} \sum_{k=1}^{8} f(r_k) \]  

where \( r_k \) indicates the coordinates of the \( k \)th node of the cube, \( V_e \) is the volume of the cube, and the summation sums together the value of the function \( f(r) \) at the corners of each cube. Trapezoidal integration will be applied to the terms in the mass and stiffness matrices in (7). The mass matrix term in (8.1) can be rewritten as the following over a cube element.

\[
T_{ij} = \frac{1}{c^2} \int_V \overline{W}_i(r) \cdot \overline{W}_j(r) \, dV \approx \frac{\Delta x^3}{8c^2} \sum_{k=1}^{8} \overline{W}_i(x_k, y_k, z_k) \cdot \overline{W}_j(x_k, y_k, z_k)
\]  

Figure 1: Reference cube element with edge numbering
Application of trapezoidal integration on the stiffness matrix term in (8.2) results in the following.

\[ S_{i,j} = \int \nabla \times \hat{W}_i(r) \cdot \frac{1}{\mu_r} \left( \nabla \times \hat{W}_j(r) \right) dV \]

\[ \approx \frac{\Delta x^3}{8\mu_r} \sum_{k=1}^{8} \left( \nabla \times \hat{W}_i(x_k, y_k, z_k) \right) \cdot \left( \nabla \times \hat{W}_j(x_k, y_k, z_k) \right) \]  

The evaluation of (12) results in the following base stiffness matrix for a cube element.

\[ S = \frac{\Delta x^3}{\mu_r} \begin{bmatrix} S_1 & S_2 & S_3 \\ S_2 & S_1 & S_2 \\ S_3 & S_2 & S_1 \end{bmatrix} \]  

\[ S_1 = \begin{bmatrix} \frac{1}{\Delta x^2} & -\frac{1}{2\Delta x^2} & -\frac{1}{2\Delta x^2} & 0 \\ -\frac{1}{2\Delta x^2} & \frac{1}{\Delta x^2} & 0 & -\frac{1}{2\Delta x^2} \\ -\frac{1}{2\Delta x^2} & 0 & \frac{1}{\Delta x} & -\frac{2\Delta x}{\Delta x^2} \\ 0 & -\frac{1}{2\Delta x^2} & -\frac{2\Delta x}{\Delta x^2} & \frac{1}{\Delta x^2} \end{bmatrix} \]  

\[ (13.1) \]
With the finite element expansion of the vector wave equation written in terms of two variables in (7), there needs to be a method to relate the electric flux back to the electric field, and the constitutive relation is utilized to enforce this relation.

\[
\bar{E}(\bar{r}, t) = \frac{1}{\varepsilon_0} \bar{\varepsilon}^{-1} \cdot \bar{D}(\bar{r}, t) \quad (14.1)
\]

\[
\bar{\varepsilon}^{-1} = \begin{bmatrix}
\varepsilon_{xx}^{-1} & \varepsilon_{xy}^{-1} & \varepsilon_{xz}^{-1} \\
\varepsilon_{yx}^{-1} & \varepsilon_{yy}^{-1} & \varepsilon_{yz}^{-1} \\
\varepsilon_{zx}^{-1} & \varepsilon_{zy}^{-1} & \varepsilon_{zz}^{-1}
\end{bmatrix} \quad (14.2)
\]

(14.1) is expanded using (2) and (3) and tested with edge basis function \( \bar{W}_j \) to create a matrix relation for the constitutive relation

\[
[T_E][E] = [\ast_{\varepsilon^{-1}}][D] \quad (15)
\]

where

\[
T_{E,ij} = \int_V \bar{W}_i \cdot \bar{W}_j \, dV \quad (16.1)
\]

\[
\ast_{\varepsilon^{-1},ij} = \int_V (\bar{\varepsilon}^{-1} \cdot \bar{W}_i) \cdot \bar{W}_j \, dV \quad (16.2)
\]
The integrals in (16.1) and (16.2) will be evaluated using trapezoidal integration, resulting in \([T_E]\) to be a diagonal matrix similar to (11.1) and \([*_{ε^{-1}}]\) to be a sparse banded matrix. Evaluating (16.2) over a single cube element, the matrix \([*_{ε^{-1}}]\) looks as follows. The matrix \(T_1\) is the same as in (11.2).

\[
[*_{ε^{-1}}] = \Delta x^3 \begin{bmatrix}
ε_{xx}^{-1}T_1 & ε_{xy}^{-1}T_2 & ε_{xz}^{-1}T_4 \\
ε_{yx}^{-1}T_2 & ε_{yy}^{-1}T_1 & ε_{yz}^{-1}T_2 \\
ε_{zx}^{-1}T_2 & ε_{zy}^{-1}T_2 & ε_{zz}^{-1}T_1 
\end{bmatrix}
\]  \hspace{1cm} (17.1)

\[
T_2 = \begin{bmatrix}
1/8 & 0 & 1/8 & 0 \\
1/8 & 0 & 1/8 & 0 \\
0 & 1/8 & 0 & 1/8 \\
0 & 1/8 & 0 & 1/8 
\end{bmatrix}
\]  \hspace{1cm} (17.2)

(15) can be substituted into (7) to result in the following matrix equation

\[
\frac{∂^2}{∂t^2} [T_T]\{E_i\} + [S]\{E_i\} = 0
\]  \hspace{1cm} (18)

where

\[
[T_T] = [T][*_{ε^{-1}}]^{-1}[T_E]
\]  \hspace{1cm} (19)

Given (19), time stepping can be implemented using the Newmark method of time discretization [5, 35].

\[
([T_T] + 2\Delta t^2θ[S])\{E\}^{n+1} = (2[T_T] + \Delta t^2(4θ - 1)[S])\{E\}^n - ([T_T] + 2\Delta t^2θ[S])\{E\}^{n-1}
\]  \hspace{1cm} (20)

Various choices of \(θ\) will result in different time-stepping schemes. By setting \(θ = 0\) and inverting \([T_T]\) to the right hand side of (20), the explicit anisotropic FDTD update
equation for the second order wave equation is recovered. This will be visually verified in the section after the derivation of the first order anisotropic FDTD update from finite elements by examining the stencils from the above expression.

2.2: First Order Anisotropic FDTD Derivation from Finite Elements

The first order anisotropic FDTD update can be derived using finite elements on first order Maxwell’s equations using a similar process described in [36]. In a homogenous, lossless, non-dispersive anisotropic region with no sources, Maxwell’s equations can be written as follows.

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (21.1) \]

\[ \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad (21.2) \]

For the basis function expansion and assuming the domain is divided into cube elements, the electric field \( \vec{E} \) and electric flux density \( \vec{D} \) will be expanded using a vector edge basis function, which is the Whitney 1-form basis function for cubes, and the magnetic flux density \( \vec{B} \) will be expanded using a Whitney 2-form facet basis function for cubes [37-41].

\[ \vec{E} = \sum_e E_e(t) \vec{W}_e^1(\vec{r}) \quad (22.1) \]

\[ \vec{D} = \sum_e D_e(t) \vec{W}_e^1(\vec{r}) \quad (22.2) \]
\[ \vec{B} = \sum_{f} B_f(t) \hat{W}^2_f(\vec{r}) \]  

(22.3)

In (22.1) through (22.3), \(e\) stands for the number of edges, and \(f\) stands for the number of faces.

Before the basis function expansions are substituted into Maxwell’s equations, a relation between Whitney 1-form basis functions and Whitney 2-form basis functions in a cube element will first be examined. The Whitney 1-form basis functions for the edges are written as follows.

![Cube reference figure with node numbering](image)

**Figure 2:** Cube reference figure with node numbering

\[ \vec{e}_{12} \equiv \vec{e}_1 = \frac{(\Delta y - y)(\Delta z - z)}{\Delta y \Delta z} \hat{x} \]  

(22.4)
\[ \vec{e}_{34} \equiv \vec{e}_2 = \frac{y(\Delta z - z)}{\Delta y \Delta z} \hat{x} \]  
(22.5)

\[ \vec{e}_{56} = \vec{e}_3 = \frac{(\Delta y - y)z}{\Delta y \Delta z} \hat{x} \]  
(22.6)

\[ \vec{e}_{78} = \vec{e}_4 = \frac{yz}{\Delta y \Delta z} \hat{x} \]  
(22.7)

\[ \vec{e}_{13} = \vec{e}_5 = \frac{(\Delta x - x)(\Delta z - z)}{\Delta x \Delta z} \hat{y} \]  
(22.8)

\[ \vec{e}_{57} = \vec{e}_6 = \frac{(\Delta x - x)z}{\Delta x \Delta z} \hat{y} \]  
(22.9)

\[ \vec{e}_{24} = \vec{e}_7 = \frac{x(\Delta z - z)}{\Delta x \Delta z} \hat{y} \]  
(22.10)

\[ \vec{e}_{68} = \vec{e}_8 = \frac{xz}{\Delta x \Delta z} \hat{y} \]  
(22.11)

\[ \vec{e}_{15} = \vec{e}_9 = \frac{(\Delta x - x)(\Delta y - y)}{\Delta x \Delta y} \hat{z} \]  
(22.12)

\[ \vec{e}_{26} = \vec{e}_{10} = \frac{x(\Delta y - y)}{\Delta x \Delta y} \hat{z} \]  
(22.13)

\[ \vec{e}_{37} = \vec{e}_{11} = \frac{(\Delta x - x)y}{\Delta x \Delta y} \hat{z} \]  
(22.14)

\[ \vec{e}_{48} = \vec{e}_{12} = \frac{xy}{\Delta x \Delta y} \hat{z} \]  
(22.15)

The Whitney 2-form basis functions for the cube facets based upon figure 2 are written as follows.
\[ \tilde{f}_{1375} = \frac{\Delta x - x}{\Delta x} \hat{x} \]  

(23.1)

\[ \tilde{f}_{2486} = \frac{x}{\Delta x} \hat{x} \]  

(23.2)

\[ \tilde{f}_{1562} = \frac{\Delta y - y}{\Delta y} \hat{y} \]  

(23.3)

\[ \tilde{f}_{3784} = \frac{y}{\Delta y} \hat{y} \]  

(23.4)

\[ \tilde{f}_{1243} = \frac{\Delta z - z}{\Delta z} \hat{z} \]  

(23.5)

\[ \tilde{f}_{5687} = \frac{z}{\Delta z} \hat{z} \]  

(23.6)

Using \( \tilde{e}_{12} \) as an example, when the curl of \( \tilde{e}_{12} \) is taken, it can be shown to be equal to a summation of Whitney 2-forms.

\[
\nabla \times \tilde{e}_{12} = \nabla \times \left( \frac{(\Delta y - y)(\Delta z - z)}{\Delta y \Delta z} \hat{x} \right)
\]

\[= \hat{y} \frac{\partial}{\partial z} \left( \frac{(\Delta y - y)(\Delta z - z)}{\Delta y \Delta z} \right) + \hat{z} \frac{\partial}{\partial y} \left( \frac{(\Delta y - y)(\Delta z - z)}{\Delta y \Delta z} \right)\]

\[= -\hat{y} \frac{\Delta y - y}{\Delta y \Delta z} + \hat{z} \frac{\Delta z - z}{\Delta y \Delta z}\]

\[= -\frac{1}{\Delta z} \tilde{f}_{1562} + \frac{1}{\Delta y} \tilde{f}_{1243}\]  

(24)

The remaining Whitney 1-form edges of the block can also be rewritten as a sum of Whitney 2-form facet basis functions.
\[ \nabla \times \mathbf{e}_{34} = -\frac{1}{\Delta z} f_{3784} - \frac{1}{\Delta y} f_{1243} \quad (25.1) \]

\[ \nabla \times \mathbf{e}_{56} = \frac{1}{\Delta y} f_{5687} + \frac{1}{\Delta z} f_{1562} \quad (25.2) \]

\[ \nabla \times \mathbf{e}_{78} = -\frac{1}{\Delta y} f_{5687} + \frac{1}{\Delta z} f_{3784} \quad (25.3) \]

\[ \nabla \times \mathbf{e}_{13} = -\frac{1}{\Delta x} f_{1243} + \frac{1}{\Delta z} f_{1375} \quad (25.4) \]

\[ \nabla \times \mathbf{e}_{57} = -\frac{1}{\Delta z} f_{1375} - \frac{1}{\Delta x} f_{5687} \quad (25.5) \]

\[ \nabla \times \mathbf{e}_{24} = \frac{1}{\Delta x} f_{1243} + \frac{1}{\Delta z} f_{2486} \quad (25.6) \]

\[ \nabla \times \mathbf{e}_{68} = -\frac{1}{\Delta z} f_{2486} + \frac{1}{\Delta x} f_{5687} \quad (25.7) \]

\[ \nabla \times \mathbf{e}_{15} = \frac{1}{\Delta x} f_{1562} - \frac{1}{\Delta y} f_{1375} \quad (25.8) \]

\[ \nabla \times \mathbf{e}_{26} = -\frac{1}{\Delta x} f_{1562} - \frac{1}{\Delta y} f_{2486} \quad (25.9) \]

\[ \nabla \times \mathbf{e}_{37} = \frac{1}{\Delta y} f_{1375} + \frac{1}{\Delta x} f_{3784} \quad (25.10) \]

\[ \nabla \times \mathbf{e}_{48} = \frac{1}{\Delta y} f_{2486} - \frac{1}{\Delta x} f_{3784} \quad (25.11) \]

Now the expansion of (21.1) will be examined. Substituting (22.1) and (22.3) into (21.1) results in the following.
\[ \nabla \times \sum_e E_e(t) \vec{W}^1_e(\vec{r}) = -\frac{\partial}{\partial t} \sum_f B_f(t) \vec{W}^2_f(\vec{r}) \]  

(26)

In (26), two different basis functions are used in the expansion; however, since it has been shown that the curl of a Whitney-1 form can be written as a sum of Whitney-2 forms, the terms on the left hand side of (26) can be rewritten.

\[ \sum_e E_e(t) \sum_k \vec{W}^2_{(k,e)} = -\frac{\partial}{\partial t} \sum_f B_f(t) \vec{W}^2_f(\vec{r}) \]  

(27)

In (27), \( k \) refers to the face indices that make up the sum of the Whitney 1-form basis of edge \( e \). Applying the weighted residual method on both sides of (27) using the Whitney 2-basis function as the testing function and evaluating the integral using trapezoidal integration, (27) can be rewritten in matrix form.

\[ \frac{1}{\Delta x} [C] \{E\} = -\frac{\partial}{\partial t} [T_B] \{B\} \]  

(28.1)

\[ C_{f,e} = \int_V \left( \sum_e \sum_k \vec{W}^2_{(e,k)} \right) \cdot \vec{W}^2_f \, dV \]  

(28.2)

\[ T_{B_{i,j}} = \int_V \vec{W}^2_i \cdot \vec{W}^2_j \, dV \]  

(28.3)

Matrix \([C]\) consists of ±1 and has a size of \( f \) (number of faces) by \( e \) (number of edges). Matrix \([T_B]\) is of size \( f \) by \( f \). It is assumed that \( \Delta x = \Delta y = \Delta z \). Since \([T_B]\) is a diagonal matrix, applying a central difference equation to replace the partial derivative with respect to time will result in an explicit update for \( B \).
The expansion of (21.2) will now be examined. After the substitution of (22.2) and (22.3) into (21.2) and applying the weighted residual method with the Whitney 1-form basis function as the testing function, the result is the following.

\[
\int \nabla \times \left( \frac{1}{\mu} \sum_f B_f(t) \mathbf{W}_f^2(\mathbf{r}) \right) \cdot \mathbf{W}_t^1(\mathbf{r}) \, dV = \frac{\partial}{\partial t} \int \nabla \cdot \left( \sum_e D_e(t) \mathbf{W}_e^1(\mathbf{r}) \right) \cdot \mathbf{W}_t^1 \, dV
\]

(S29)

Since the curl of a Whitney 2-form is equal to zero, the left hand side of (29) must be rewritten in order to use this expansion. To rewrite the curl term on the left hand side, the vector identity from (5) is applied to take the curl term off of the Whitney 2-form basis, resulting in the follow weak form.

\[
\int \nabla \cdot \left( \frac{1}{\mu} \sum_f B_f W_f^2 \times W_t^1 \right) \, dV + \int \nabla \times \left( \frac{1}{\mu} \sum_f B_f W_f^2 \right) \cdot (\nabla \times W_t^1) \, dV = \frac{\partial}{\partial t} \int \nabla \cdot \left( \sum_e D_e W_e^1 \right) \cdot W_t^1 \, dV
\]

(S30)

The first term of (30) can be rewritten using the divergence theorem.
\[
\int_V \nabla \cdot \left( \frac{1}{\mu} \sum_f B_f W_f^2 \times W_t^1 \right) dV
\]

\[
= \oint_S \left( \frac{1}{\mu} \sum_f B_f W_f^2 \times W_t^1 \right) \cdot \hat{n} dS
\]  

As with previous derivations, it’s assumed that the region being examined extends infinitely, so from the Sommerfeld Radiation condition, the surface integral term on the right hand side of (30) will go to zero, leaving only the volume integral terms.

\[
\int_V \left( \frac{1}{\mu} \sum_f B_f W_f^2 \right) \cdot (\nabla \times W_t^1) dV
\]

\[
= \frac{\partial}{\partial t} \int_V \left( \sum_e D_e W_e^1 \right) \cdot W_t^1 dV
\]  

Since the curl of a Whitney 1-form is a summation of Whitney 2-forms, the curl term on the left hand side of (32) can be replaced with an appropriate summation.

\[
\int_V \left( \frac{1}{\mu} \sum_f B_f W_f^2 \right) \cdot \left( \sum_f W_f^2, t \right) dV
\]

\[
= \frac{\partial}{\partial t} \int_V \left( \sum_e D_e W_e^1 \right) \cdot W_t^1 dV
\]  

The integrals in (33) are evaluated using trapezoidal integration, resulting in the following matrix expression.
\[
\frac{1}{\mu \Delta x} [C]'_t \{B\} = \frac{\partial}{\partial t} [T]\{D\}
\]  
(34.1)

\[
C_{f',f,t} = \int_V W_f^2 \cdot \sum_{f'} W_{f',t}^2 \, dV \quad \text{(34.2)}
\]

It’s interesting to note that the matrix resulting on the left hand side of (34.2) is simply the transpose of matrix C from (28.1). To be able to recover the electric field, the constitutive relation expansion derived from the second order anisotropic hybrid expansion can also be applied in this particular situation. Substituting the matrix expansion of \{D\} into (34.1) results in the following.

\[
\frac{1}{\mu \Delta x} [C]' \{B\} = \frac{\partial}{\partial t} [T][^* e^{-1}]^{-1}[T_E]\{E\}
\]  
(35)

After inverting the resulting matrix on the right hand side of (35) to the left hand side and applying a central differencing expansion on the partial derivative in time, the final first order FDTD update for Ampere’s law is derived.

It’s difficult to visualize in matrix form that the derivation ultimately results in the first order FDTD update. However, upon examining the stencils that results from Ampere’s Law and from the electric constituent relation expansion with the basis function, one can see that this formulation will recover the anisotropic stencil of the electric field update.

For example, taking a single line from Ampere’s Law in (34.1), the stencil to update the electric flux \(D_x\) is represented as follows. The stencil for \(D_y\) and \(D_z\) are drawn similarly.
Figure 3: Stencil for $D_x$

From the matrix expression (15) of the constitutive relation, a stencil can be constructed for $E_x$ that consists of edges of $D_x$, $D_y$, and $D_z$. 
Figure 4: Constitutive relation stencil for $E_x$. Note that $E_x$ and $D_x$ share an edge in the figure.

If each $D$ edge in the figure 4 has its corresponding Ampere’s Law stencil substituted in place of the edge, the full anisotropic FDTD stencil for the electric field can be constructed and match the magnetic fields involved in the full anisotropic FDTD stencil displayed in figure 6.
Figure 5: Substitution of Ampere’s Law Stencils into Constitutive Relation Stencil
Thus from a stencil point of view, we see that the first order anisotropic electric field FDTD update can be recreated from this alternative finite element formulation. In addition, the stencil of the second-order FDTD anisotropic update can be found by replacing the magnetic fields in figure 3 with the appropriate electric field stencil derived from Faraday’s Law.

The use of the constitutive relation in a finite element point of view is a first in the finite element community, and it is the key to deriving the anisotropic FDTD from the finite element methods.
Section 2.3: Restriction Operator

Although the new finite element methodology is capable of deriving the anisotropic FDTD equation, the assumption made in the derivation was that the equations were evaluated in a homogenous domain. When there are multiple different anisotropic regions in the domain, the new finite element formulation incorrectly enforces that the tangential electric flux density is continuous across an interface between two different materials due to the expansion of the electric flux density using vector edge basis functions. However, this issue can be dealt with by carefully constructing the finite element formulation and by introducing a restriction operator to correctly enforce the tangential continuity condition at the interface.

Let’s examine a case where there’re two different materials in the domain of interest.

Figure 7: Constitutive Relation stencil at the material interface
In order to correctly formulate the finite element matrices, the edges that lie tangentially along the material interface will be doubly counted. In figure 7, this means that edges 1, 6, 7, 8, and 9 will initially have two different coefficients associated with the electric field and electric flux density. Generalized to matrix form, the finite element expression associated with the constitutive relation based on figure 7 will be written as follows.

\[ [T_E] \{E_{\text{raw}}\} = [\ast e^{-1}] \{D_{\text{raw}}\} \]  \hspace{1cm} (36.1)

\[ \{E_{\text{raw}}\} = \begin{pmatrix} E_{\text{int}_1} \\ E_{b_1} \\ E_{\text{int}_2} \\ E_{b_2} \end{pmatrix} \]  \hspace{1cm} (36.2)

\[ \{D_{\text{raw}}\} = \begin{pmatrix} D_{\text{int}_1} \\ D_{b_1} \\ D_{\text{int}_2} \\ D_{b_2} \end{pmatrix} \]  \hspace{1cm} (36.3)

In (36.2) and (36.3), \(E_{\text{int}_1}, E_{\text{int}_2}, D_{\text{int}_1},\) and \(D_{\text{int}_2}\) represent the electric field and electric flux density unknowns that lie completely in either region one or region two but do not lie along the material interface. \(E_{b_1}\) and \(D_{b_1}\) represent the tangential electric field and electric flux density unknowns on the region 1 side of the interface, and \(E_{b_2}\) and \(D_{b_2}\) represent the tangential electric field and electric flux density unknowns on the region 2 side of the interface.

Ampere’s Law is also written to represent the doubly counted tangential edges for the electric flux density and doubly counted normal magnetic fields along the interface.
\[
\frac{1}{\mu \Delta x} \{C\} \{B_{\text{raw}}\} = \frac{\partial}{\partial t} \{T\} \{D_{\text{raw}}\} \quad \text{(37)}
\]

Inverting \([\epsilon^{-1}]\) in (36.1) to the left hand side and substituting into (37) creates the following.

\[
\frac{1}{\mu \Delta x} \{C\} \{B_{\text{raw}}\} = \frac{\partial}{\partial t} \{T\} [\epsilon^{-1}]^{-1} \{T_E\} \{E_{\text{raw}}\} \quad \text{(38)}
\]

The matrix on the right hand side of (38) is inverted to the left hand side.

\[
\frac{1}{\mu \Delta x} \{T_E\}^{-1} [\epsilon^{-1}] \{T\}^{-1} \{C\} \{B_{\text{raw}}\} = \frac{\partial}{\partial t} \{E_{\text{raw}}\} \quad \text{(39)}
\]

With (39) completely defined by electric field edges on the right hand side, the doubly counted electric field edges along the interface can be summed together using a restriction operator \([R]\) of size \(e\) (number of edges) by \(d\) (number of single and doubly counted edges). For this particular example, the restriction operator can be written as follows.

\[
[R] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \quad \text{(40)}
\]

The restriction operator enforces the tangential continuity of the electric field at the material interface by setting the coefficients of the doubly counted edges within material 2 to be equal to the coefficients of the doubly counted edges within material 1.

\[
\{E_{b_1}\} = \{E_{b_2}\} \quad \text{(41)}
\]
With two of the same coefficients on the right side of (39) representing the electric field edge along the interface, the restriction operator summing the doubly counted electric field edges must be modified with a factor of $\frac{1}{2}$ to account for the doubly summed terms in the constitutive relation matrix and the electric field vector.

$$[R_m] = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & \frac{1}{2}I & 0 & \frac{1}{2}I \\ 0 & 0 & I & 0 \end{bmatrix}$$ \hspace{1cm} (42)

However, one should note that the factor of $\frac{1}{2}$ is for this particular interface case between two interfaces. In the most general case where an electric field edge would interface between four different materials, a factor of $\frac{1}{4}$ would be used in the restriction operator to sum four coefficients for one edge.

Applying the restriction operator based on (40) to the stiffness matrix term $[C]^t$ and the restriction operator based on (42) to the constitutive relation matrix and the electric field vector results in the following.

$$\frac{1}{\mu} [R_m][T_e]^{-1}[*_{\varepsilon^{-1}}][T]^{-1}[R_m]^t[R][C]^t\{B_{raw}\} = \frac{\partial}{\partial t}[R_m]\{E_{raw}\}$$ \hspace{1cm} (43)

(43) represents the proper formulation to correctly enforce the tangential continuity of the electric field at a material interface. Although the formulation may look computationally expensive, it is not because the matrix inversions involved in (43) are inversions of diagonal matrices, which are easy to calculate.
This formulation can also be examined more closely using stencils for a better understanding of the mechanics of the manipulation. For example, the equations for the constitutive relation for electric field edge 1 in figure 7 can be written as follows.

\[
\frac{1}{2} E_1 = \frac{1}{2} \epsilon_{xx}^{-1} D_{11} + \frac{1}{4} \epsilon_{xy}^{-1} (D_2 + D_4) + \frac{1}{8} \epsilon_{xz}^{-1} (D_{61} + D_{71} + D_{81} + D_{91})
\]

(44.1)

\[
\frac{1}{2} E_2 = \frac{1}{2} \epsilon_{xx}^{-1} D_{12} + \frac{1}{4} \epsilon_{xy}^{-1} (D_3 + D_5) + \frac{1}{8} \epsilon_{xz}^{-1} (D_{62} + D_{72} + D_{82} + D_{92})
\]

(44.2)

From the matrix expression of Ampere’s law in (34.1), the discrete equations for the flux density along edge 1 can be written as follows after central differencing has been applied on the time derivative.

Figure 8: Ampere's Law Stencil
\[
\frac{D_{11}^{n+1} - D_{11}^n}{2\Delta t} = \frac{-2H_1^{n+\frac{1}{2}} - H_3^{n+\frac{1}{2}} + H_4^{n+\frac{1}{2}}}{2\Delta x} \tag{45.1}
\]

\[
\frac{D_{12}^{n+1} - D_{12}^n}{2\Delta t} = \frac{2H_2^{n+\frac{1}{2}} + H_4^{n+\frac{1}{2}} - H_3^{n+\frac{1}{2}}}{2\Delta x} \tag{45.2}
\]

(45.1) and (45.2) can be substituted into (44.1) and (44.2) in place of \(D_{11}\) and \(D_{12}\) to populate the equation in terms of magnetic fields. For the remaining \(D\) terms, expressions similar to (45) can be substituted. Next, (44.1) and (44.2) are summed together. The \(D^n\) terms that remain can all be grouped together and equated to \(E^n\), which is the last step to constructing the anisotropic electric field FDTD update for edge 1.

In this section, it has been shown that by doubly counting the edges that lie along a material interface and by using a properly constructed restriction operator, the tangential continuity of the electric field at the material interface can be maintained even though the basis function expansion of the electric flux uses an edge basis function.

**Section 2.4: Alternative Anisotropic FDTD Stencils**

With the capability to derive the anisotropic FDTD update using finite elements, new stencils can be developed in a methodic manner for situations that FDTD cannot easily address. Since the anisotropic FDTD stencil extends one cell ahead and behind in the vector direction of the edge being updated, it presents issues when the normal FDTD electric field update is considered at a boundary. In this section, the derivation of the anisotropic FDTD update that is normal and half a cell away from a PEC boundary will be examined. Shown in figure 9, the regular anisotropic FDTD stencil extends into the
PEC region, making it unclear how the boundary conditions of the PEC should be properly enforced.

![Diagram of 2D Anisotropic FDTD Stencil for $E_x$ normal to PEC interface]

Figure 9: 2D Anisotropic FDTD Stencil for $E_x$ normal to PEC interface

With the new finite element formulation, the boundary conditions can simply be enforced in (35) by removing the rows and columns of $[T][ε^{-1}]^{-1}[T_E]$ associated with the electric field edges that are tangential along the PEC interface, effectively shorting these edges to zero. After this adjustment, the resulting matrix is inverted to the opposite side of the equation to create the final first-order explicit update. If the permittivity tensor happens to be full, the inversion of the matrix results in a full matrix; however, in cases where there exists off-diagonal components in the permittivity tensor but the tensor isn’t full, there can be several different FDTD stencils derived for this particular case.

For the following example, the permittivity tensor has the following form, and the matrix expression for Ampere’s law is rewritten for ease of reference.
\[
\vec{\varepsilon}^{-1} = \begin{bmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & f \end{bmatrix}
\]  
\hspace{1cm} (46.1)

\[
[S] \{B\} = \frac{\partial}{\partial t} \{E\}
\]  
\hspace{1cm} (46.2)

\[
[S] = [T_e]^{-1}[*\varepsilon^{-1}][T]^{-1}[C]^t
\]  
\hspace{1cm} (46.3)

The situation will be examined where the $E_x$ field update edge is pointing normal and located half a cell away from a PEC boundary. This will break down into six different cases.

1. The vector direction of $E_x$ is away from the PEC boundary, and there are no tangential PEC boundaries less than $2\Delta x$ away in either the $+y$ or $-y$ direction.

2. The vector direction of $E_x$ is away from the PEC boundary, and there is a tangentially oriented PEC boundary located $-\Delta x$ away in the $y$-direction.

3. The vector direction of $E_x$ is away from the PEC boundary, and there is a tangentially oriented PEC boundary located $\Delta x$ away in the $y$-direction.

4. The vector direction of $E_x$ is towards the PEC boundary, and there are no tangential PEC boundaries less than $2\Delta x$ away in either the $+y$ or $-y$ direction.

5. The vector direction of $E_x$ is towards the PEC boundary, and there is a tangentially oriented PEC boundary located $-\Delta x$ away in the $y$-direction.

6. The vector direction of $E_x$ is toward the PEC boundary, and there is a tangentially oriented PEC boundary located $\Delta x$ away in the $y$-direction.
Note that is only a sampling of the cases that can be found, but it will give the reader an idea of how to approach the construction of such stencils.

For case 1, the electric field stencil will incorporate the following magnetic fields.

\[ H_x \]

\[ E_x \]

Figure 10: $H_x$ Magnetic Fields involved in $E_x$ stencil in Case 1
Figure 11: $H_y$ Magnetic Fields involved in $E_x$ Stencil in Case 1

Figure 12: $H_z$ Magnetic Fields involved in $E_x$ Stencil in Case 1
The following are the coefficients that are multiplied with magnetic fields via matrix \([S]\) in (46.3). The index of each coefficient is associated with the indexing of the magnetic field in figures 10 through 12 and is not based on row or column numbering of \([S]\).

\[
S_1 = -\frac{b}{4} \quad (47.1)
\]

\[
S_2 = -\frac{b}{4} \quad (47.2)
\]

\[
S_3 = \frac{b}{4} \quad (47.3)
\]

\[
S_4 = \frac{b}{4} \quad (47.4)
\]

\[
S_5 = -\frac{b^2}{8d} \quad (47.5)
\]

\[
S_6 = \frac{4da - b^2}{4d} \quad (47.6)
\]

\[
S_7 = -\frac{b^2}{8d} \quad (47.7)
\]

\[
S_8 = \frac{b^2}{8d} \quad (47.8)
\]

\[
S_9 = -\frac{4da - b^2}{4d} \quad (47.9)
\]

\[
S_{10} = \frac{b^2}{8d} \quad (47.10)
\]
\[ S_{11} = \frac{b^2}{8d} \]  
(47.11)

\[ S_{12} = \frac{b}{4} - \frac{4da - b^2}{4d} - \frac{b^2}{8d} \]  
(47.12)

\[ S_{13} = \frac{b}{4} + \frac{4da - b^2}{4d} + \frac{b^2}{8d} \]  
(47.13)

\[ S_{14} = -\frac{b^2}{8d} \]  
(47.14)

\[ S_{15} = -\frac{b}{4} \]  
(47.15)

\[ S_{16} = -\frac{b}{4} \]  
(47.16)

For the second case in consideration, the magnetic fields involved in the stencil look as follows.
Figure 13: $H_x$ Magnetic fields involved in $E_x$ stencil for Case 2
Figure 14: $H_y$ Magnetic Fields involved in $E_x$ stencil for Case 2
Figure 15: $H_z$ Magnetic Fields involved in $E_x$ stencil for Case 2

The following are the coefficients that are associated with the magnetic fields, which are calculated from matrix $[S]$ in (46.3). As with case 1, the index of each coefficient is associated with the indices of the magnetic field in figures 13 through 15 and is not based upon row or column numbering in $[S]$.

\[
S_1 = -\frac{dab}{4da - b^2} \tag{48.1}
\]

\[
S_2 = -\frac{b}{4} \tag{48.2}
\]

\[
S_3 = \frac{dab}{4da - b^2} \tag{48.3}
\]
\[ S_4 = \frac{b}{4} \quad (48.4) \]

\[ S_5 = \frac{32d^2a^2 - 16dab^2 + b^4}{8(4d^2a - b^2d)} \quad (48.5) \]

\[ S_6 = -\frac{b^2}{8d} \quad (48.6) \]

\[ S_7 = -\frac{32d^2a^2 - 16dab^2 + b^4}{8(4d^2a - b^2d)} \quad (48.7) \]

\[ S_8 = \frac{b^2}{8d} \quad (48.8) \]

\[ S_9 = -\frac{32d^2a^2 - 16dab^2 + b^4}{8(4d^2a - b^2d)} + \frac{dab}{4da - b^2} \quad (48.9) \]

\[ S_{10} = \frac{32d^2a^2 - 16dab^2 + b^4}{8(4d^2a - b^2d)} + \frac{b}{4} + \frac{b^2}{8d} \quad (48.10) \]

\[ S_{11} = -\frac{b^2}{8d} \quad (48.11) \]

\[ S_{12} = -\frac{dab}{4da - b^2} \quad (48.12) \]

\[ S_{13} = -\frac{b}{4} \quad (48.13) \]

Case 3 is similar to case 2 except the tangential PEC boundary is on the opposite side.
Figure 16: $H_x$ Magnetic Fields involved in $E_x$ stencil for case 3
Figure 17: $H_y$ Magnetic Fields involved in $E_x$ stencil for case 3
Figure 18: $H_z$ Magnetic Fields involved in $E_x$ stencil for case 3

The coefficients for the magnetic fields are the following for case 3.

\[ S_1 = -\frac{b}{4} \quad (49.1) \]

\[ S_2 = -\frac{dab}{4da - b^2} \quad (49.2) \]

\[ S_3 = \frac{b}{4} \quad (49.3) \]

\[ S_4 = \frac{dab}{4da - b^2} \quad (49.4) \]

\[ S_5 = -\frac{b^2}{8d} \quad (49.5) \]
The magnetic fields involved with the stencil for case 4 looks as follows.

$$S_6 = \frac{32d^2a^2 - 16dab^2 + b^4}{8(4d^2a - b^2d)}$$  \hspace{2cm} (49.6)

$$S_7 = \frac{b^2}{8d}$$  \hspace{2cm} (49.7)

$$S_8 = -\frac{32d^2a^2 - 16dab^2 + b^4}{8(4d^2a - b^2d)}$$  \hspace{2cm} (49.8)

$$S_9 = \frac{b^2}{8d}$$  \hspace{2cm} (49.9)

$$S_{10} = -\frac{32d^2a^2 - 16dab^2 + b^4}{8(4d^2a - b^2d)} + \frac{b}{4} - \frac{b^2}{8d}$$  \hspace{2cm} (49.10)

$$S_{11} = \frac{32d^2a^2 - 16dab^2 + b^4}{8(4d^2a - b^2d)} + \frac{dab}{4da - b^2}$$  \hspace{2cm} (49.11)

$$S_{12} = -\frac{b}{4}$$  \hspace{2cm} (49.12)

$$S_{13} = -\frac{dab}{4da - b^2}$$  \hspace{2cm} (49.13)
Figure 19: $H_x$ Magnetic Fields involved in $E_x$ stencil for case 4
Figure 20: $H_y$ Magnetic Fields involved in $E_x$ stencil for case 4
Figure 21: $H_z$ Magnetic Fields involved in $E_x$ stencil for case 4

Most of the coefficients for the magnetic fields are the same as in case 1, so those will not be repeated; however, the following coefficients are different.

$$S_{12} = -\frac{b}{4} - \frac{4da - b^2}{4d} - \frac{b^2}{8d}$$  \hspace{1cm} (50.1)

$$S_{13} = -\frac{b}{4} + \frac{4da - b^2}{4d} + \frac{b^2}{8d}$$  \hspace{1cm} (50.2)

$$S_{15} = -\frac{b}{4}$$  \hspace{1cm} (50.3)

$$S_{16} = -\frac{b}{4}$$  \hspace{1cm} (50.4)

The magnetic fields involved in the stencil for case 5 look as follows.
Figure 22: $H_x$ Magnetic Fields involved in $E_x$ stencil for case 5
Figure 23: $H_y$ Magnetic Fields involved in $E_x$ stencil for case 5
The coefficients for the magnetic fields in case 5 are the same as the coefficients found in case 2 with a few exceptions, which are the following.

\[ S_9 = -\frac{32d^2a^2 - 16dab^2 + b^4}{8(4d^2a - b^2d)} - \frac{dab}{4da - b^2} \]  \hspace{1cm} (51.1)

\[ S_{10} = \frac{32d^2a^2 - 16dab^2 + b^4}{8(4d^2a - b^2d)} - \frac{b}{4} + \frac{b^2}{8d} \]  \hspace{1cm} (51.2)

\[ S_{12} = \frac{dab}{4da - b^2} \]  \hspace{1cm} (51.3)

\[ S_{13} = \frac{b}{4} \]  \hspace{1cm} (51.4)

For case 6, the magnetic fields making up the stencil look as follows.
Figure 25: $H_z$ Magnetic Fields involved in $E_x$ stencil for case 6

Figure 26: $H_y$ Magnetic Fields involved in $E_x$ stencil for case 6
Case 6 shares many of the coefficients from case 3 with the following exceptions.

\[ S_{10} = -\frac{32d^2a^2 - 16dab^2 + b^4}{8(4d^2a - b^2d)} - \frac{b}{4} - \frac{b^2}{8d} \]  

(52.1)

\[ S_{11} = \frac{32d^2a^2 - 16dab^2 + b^4}{8(4d^2a - b^2d)} - \frac{dab}{4da - b^2} \]  

(52.2)

\[ S_{12} = \frac{b}{4} \]  

(52.3)

\[ S_{13} = \frac{dab}{4da - b^2} \]  

(52.4)

These are examples of the stencils that can be derived from the finite element formulation. In Chapter 3, an example of the new stencil will be tested and compared
with previously used FDTD updates for electric field edges that are half a cell away from a PEC boundary.
3.1: Parallel Plate Waveguide Experiment with Alternative FDTD Stencils

This section will examine a numerical test in a parallel plate waveguide to test the new anisotropic FDTD stencils derived in section 2.4 for the normal edges along a PEC boundary. The problem is simulated in three dimensions with periodic boundary condition applied in the y-direction. The problem domain looks as follows.

Figure 28: Parallel Plate Waveguide Domain Setup
The domain consists of a freespace region and an anisotropic region on the far end of the waveguide. The permittivity tensor of the anisotropic region has the following properties.

\[
\bar{\varepsilon}_a = \varepsilon_0 \begin{bmatrix}
2.5 & -0.866 & 0 \\
-0.866 & 1.5 & 0 \\
0 & 0 & 1.0
\end{bmatrix}
\]

(53)

In the freespace region, the guide is excited with a soft source excitation consisting of a twice-differentiated Gaussian TM\(_x\) plane wave with a center frequency of 752 MHz. The chosen discretization \(\Delta x = 0.0125\) meter, and the height \(h\) of the waveguide is 40 \(\Delta x\). Inside the anisotropic region, \(d_1 = 10\Delta x\), \(d_2 = 20\Delta x\), and point 1 is offset in the \(z\)-direction by \(h/2\). To determine the time step, the Courant number is chosen to be 0.9999. In this particular setup, the \(x\)-directed edges normal to the PEC wall at the far end of the guide in the anisotropic region will receive the modified FDTD update derived using the new finite element formulation. For comparison purposes, the discrete solution measured at point 1 will be Fourier transformed and compared with the analytical solution of the parallel plate waveguide in the frequency domain.

To start the derivation of the analytical solution for the parallel plate waveguide for comparison purposes, source-free Maxwell’s equations are examined in the frequency domain.

\[
\nabla \times \overrightarrow{H} = j\omega \varepsilon_0 \overrightarrow{\varepsilon_r} \cdot \overrightarrow{E}
\]

(54.1)

\[
\nabla \times \overrightarrow{E} = -j\omega \mu_0 \overrightarrow{H}
\]

(54.2)
Since the waveguide is infinite in the y-direction, the curl operator reduces to the following.

$$\nabla \times \vec{A} = -\hat{x} \frac{\partial}{\partial z} A_y + \hat{y} \left( \frac{\partial}{\partial z} A_z - \frac{\partial}{\partial x} A_x \right) + \hat{z} \frac{\partial}{\partial x} A_y$$  \hspace{1cm} (55)

In the isotropic region of the waveguide, TM and TE modes both exist as solutions to the waveguide. With the curl operator as given above, the TM mode solution is derived from the following equations.

$$-\frac{\partial}{\partial z} H_y = j\omega\varepsilon_0 E_x$$  \hspace{1cm} (56.1)

$$\frac{\partial}{\partial x} H_y = j\omega\varepsilon_0 E_z$$  \hspace{1cm} (56.2)

$$\frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z = -j\omega\mu_0 H_y$$  \hspace{1cm} (56.3)

The TE mode solution is dictated by the following equations.

$$-\frac{\partial}{\partial z} E_y = -j\omega\mu_0 H_x$$  \hspace{1cm} (57.1)

$$\frac{\partial}{\partial x} E_y = -j\omega\mu_0 H_z$$  \hspace{1cm} (57.2)

$$\frac{\partial}{\partial z} H_x - \frac{\partial}{\partial x} H_z = j\omega\varepsilon_0 E_y$$  \hspace{1cm} (57.3)

In the anisotropic region, the solutions to the electromagnetic waves are governed by the following equations.
\[-\frac{\partial}{\partial z} H_y = j\omega \epsilon_0 \epsilon_{xx} E_x + j\omega \epsilon_0 \epsilon_{xy} E_y \quad (58.1)\]

\[\frac{\partial}{\partial z} H_x - \frac{\partial}{\partial x} H_z = j\omega \epsilon_0 \epsilon_{yx} E_x + j\omega \epsilon_0 \epsilon_{yy} E_y \quad (58.2)\]

\[\frac{\partial}{\partial x} H_y = j\omega \epsilon_0 \epsilon_{zz} E_z \quad (58.3)\]

\[-\frac{\partial}{\partial z} E_y = -j\omega \mu_0 H_x \quad (58.4)\]

\[\frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z = -j\omega \mu_0 H_y \quad (58.5)\]

\[\frac{\partial}{\partial x} E_y = -j\omega \mu_0 H_z \quad (58.6)\]

For this particular problem, the source will generate a TM mode wave travelling in the +x direction. The waves travelling in the –x direction from the source will be absorbed by the perfectly matched layer (PML) and not interfere with the solution. To find a solution for this particular problem, an initial guess is used for \(E_x\) in the freespace region for the TM mode case, which satisfies the boundary conditions at the “top” and “bottom” of the waveguide in that the tangential electric field at a PEC surface is equal to zero. The superscript ‘i’ indicates a field in the isotropic region.

\[E^{i,+x}_x = A \sin \left(\frac{\pi z}{d}\right) e^{-j k_i x} \quad (59.1)\]

\[k_i = \sqrt{\omega^2 \mu_0 \epsilon_0 - \left(\frac{\pi}{h}\right)^2} \quad (59.2)\]

Applying (59.1) into (56.1) and (56.2), the equations for \(E_z\) and \(H_y\) can be derived.
Once the initial TM wave impinges on the freespace/anisotropic interface, the reflected waves back into the freespace region will consist of both TM and TE modes due to the anisotropy of the second region. The waves travelling in the $-x$ direction are given by the following six equations. The first three equations are the reflected TM modes with $R_{TM}$ as the TM mode reflection coefficient, and the second three equations are the reflected TE modes with $R_{TE}$ as the TE mode reflection coefficient.

\[
E_{x_i,-x} = R_{TM} \sin \left( \frac{\pi Z}{d} \right) e^{j k_i x} \quad (60.1)
\]

\[
E_{z_i,-x} = - \frac{R_{TM} (d k_i)}{j \pi} \cos \left( \frac{\pi Z}{d} \right) e^{j k_i x} \quad (60.2)
\]

\[
H_{y_i,-x} = \frac{j \omega \varepsilon_0 d}{\pi} R_{TM} \cos \left( \frac{\pi Z}{d} \right) e^{j k_i x} \quad (60.3)
\]

\[
H_{x_i,-x} = R_{TE} \cos \left( \frac{\pi Z}{d} \right) e^{j k_i x} \quad (61.1)
\]

\[
H_{z_i,-x} = \frac{k_i d}{j \pi} R_{TE} \sin \left( \frac{\pi Z}{d} \right) e^{j k_i x} \quad (61.2)
\]

\[
E_{y_i,-x} = \frac{j \omega \mu_0 d}{\pi} R_{TE} \sin \left( \frac{\pi Z}{d} \right) \quad (61.3)
\]

In the anisotropic region, an initial guess will also be used to derive the solution in this region. The initial guess for the solution of $E_y$ travelling in the $+x$ direction in the
anisotropic region are given as follows. The superscript ‘a’ indicates a field in the anisotropic region

\[ E_{y}^{a,+x} = T_{a1} \sin \left( \frac{\pi z}{d} \right) e^{-j k_{a1} x} + T_{a2} \sin \left( \frac{\pi z}{d} \right) e^{-j k_{a2} x} \]  

From this solution and (58.1) through (58.6), the solutions for the fields travelling in the +x direction inside the anisotropic region can be derived.

\[ H_{x}^{a,+x} = \frac{1}{j \omega \mu_{0}} \left[ \frac{\pi}{d} T_{a1} \cos \left( \frac{\pi z}{d} \right) e^{-j k_{a1} x} \right] + \frac{1}{j \omega \mu_{0}} \left[ \frac{\pi}{d} T_{a2} \cos \left( \frac{\pi z}{d} \right) e^{-j k_{a2} x} \right] \]  

\[ H_{z}^{a,+x} = \frac{k_{a1}}{\omega \mu_{0}} T_{a1} \sin \left( \frac{\pi z}{d} \right) e^{-j k_{a1} x} + \frac{k_{a2}}{\omega \mu_{0}} T_{a2} \sin \left( \frac{\pi z}{d} \right) e^{-j k_{a2} x} \]  

\[ E_{x}^{a,+x} = T_{a1} \sin \left( \frac{\pi z}{d} \right) e^{-j k_{a1} x} \left[ \frac{\pi^{2}}{d^{2} k_{0}^{2} \epsilon_{yx}} + \frac{k_{a1}^{2}}{k_{0}^{2} \epsilon_{yx}} - \epsilon_{yy} \right] + T_{a2} \sin \left( \frac{\pi z}{d} \right) e^{-j k_{a2} x} \left[ \frac{\pi^{2}}{d^{2} k_{0}^{2} \epsilon_{yx}} + \frac{k_{a2}^{2}}{k_{0}^{2} \epsilon_{yx}} - \epsilon_{yy} \right] \]  

\[ H_{y}^{a,+x} = C_{1} \frac{j \omega \epsilon_{0} d}{\pi} T_{a1} \cos \left( \frac{\pi z}{d} \right) e^{-j k_{a1} x} + C_{2} \frac{j \omega \epsilon_{0} d}{\pi} T_{a2} \cos \left( \frac{\pi z}{d} \right) e^{-j k_{a2} x} \]  

\[ E_{z}^{a,+x} = \frac{d}{j \epsilon_{zz} \pi} \cos \left( \frac{\pi z}{d} \right) \left( k_{a1} T_{a1} C_{1} e^{-j k_{a1} x} + k_{a2} T_{a2} C_{2} e^{-j k_{a2} x} \right) \]  

The anisotropic solution travelling in the –x direction inside the anisotropic region can be similarly written as follows.
\[ H_x^{a,-x} = \frac{1}{j \omega \mu_0} \left[ d R_{a1} \cos \left( \frac{\pi z}{d} \right) e^{j k_{a1} x} \right] + \frac{1}{j \omega \mu_0} \left[ d R_{a2} \cos \left( \frac{\pi z}{d} \right) e^{j k_{a2} x} \right] \]  
\[ (65.1) \]

\[ H_z^{a,-x} = -\frac{k_{a1}}{\omega \mu_0} R_{a1} \sin \left( \frac{\pi z}{d} \right) e^{j k_{a1} x} \]
\[ - \frac{k_{a2}}{\omega \mu_0} R_{a2} \sin \left( \frac{\pi z}{d} \right) e^{j k_{a2} x} \]  
\[ (65.2) \]

\[ E_x^{a,-x} = R_{a1} \sin \left( \frac{\pi z}{d} \right) e^{j k_{a1} x} \left[ \frac{\pi^2}{d^2 k_0^2 \epsilon_{yxx}} + \frac{k_{a1}^2}{k_0^2 \epsilon_{xxy}} - \frac{\epsilon_{yy}}{\epsilon_{yxx}} \right] \]
\[ + R_{a2} \sin \left( \frac{\pi z}{d} \right) e^{j k_{a2} x} \left[ \frac{\pi^2}{d^2 k_0^2 \epsilon_{yxx}} + \frac{k_{a2}^2}{k_0^2 \epsilon_{xxy}} - \frac{\epsilon_{yy}}{\epsilon_{yxx}} \right] \]  
\[ (65.3) \]

\[ H_y^{a,-x} = C_1 \frac{j \omega \epsilon_0 d}{\pi} R_{a1} \cos \left( \frac{\pi z}{d} \right) e^{j k_{a1} x} \]
\[ + C_2 \frac{j \omega \epsilon_0 d}{\pi} R_{a2} \cos \left( \frac{\pi z}{d} \right) e^{j k_{a2} x} \]  
\[ (65.4) \]

\[ E_z^{a,+x} = \frac{d}{j \epsilon_{zz} \pi} \cos \left( \frac{\pi z}{d} \right) \left( \frac{-k_{a1} R_{a1} C_1 e^{-j k_{a1} x}}{k_{a2} R_{a2} C_2 e^{-j k_{a2} x}} \right) \]  
\[ (65.5) \]

The constants \( C_1 \) and \( C_2 \) are defined by the following.

\[ C_1 = \frac{\pi^2}{d^2 k_0^2 \epsilon_{yxx}} + \frac{k_{a1}^2}{k_0^2 \epsilon_{xxy}} - \frac{\epsilon_{xx} \epsilon_{yy}}{\epsilon_{yxx}} + \epsilon_{xy} \]  
\[ (66.1) \]

\[ C_2 = \frac{\pi^2}{d^2 k_0^2 \epsilon_{yxx}} + \frac{k_{a2}^2}{k_0^2 \epsilon_{xxy}} - \frac{\epsilon_{xx} \epsilon_{yy}}{\epsilon_{yxx}} + \epsilon_{xy} \]  
\[ (66.2) \]

With all the solutions defined in the anisotropic region, the wave numbers \( k_{a1} \) and \( k_{a2} \) can be solved starting with the following equation.
\[ \frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z = -j\omega\mu_0 H_y \]  

(67)

Substituting (64.3) through (64.5) into (67) and reducing the expression results in the following.

\[ k_a^4 \left( \frac{\epsilon_{xx}}{\epsilon_{zz}} \right) + k_a^2 \left( \frac{\pi^2}{d^2} + \frac{\pi^2}{d^2} \frac{\epsilon_{xx}}{\epsilon_{zz}} - \frac{k_0^2 \epsilon_{xx} \epsilon_{yy}}{\epsilon_{zz}} \right) \]
\[ + \left( \frac{\pi^4}{d^4} - \frac{\pi^2}{d^2} k_0^2 \epsilon_{xx} + k_0^4 \epsilon_{xx} \epsilon_{yy} - k_0^4 \epsilon_{xy} \epsilon_{yx} - \frac{\pi^2}{d^2} k_0^2 \epsilon_{yy} \right) = 0 \]

(68)

With (68), the two wave numbers \( k_{a1} \) and \( k_{a2} \) can be solved by using the quadratic equation.

\[ ak_a^4 + bk_a^2 + c = 0 \]  

(69.1)

\[ a = \frac{\epsilon_{xx}}{\epsilon_{zz}} \]  

(69.2)

\[ b = \frac{\pi^2}{d^2} + \frac{\pi^2}{d^2} \frac{\epsilon_{xx}}{\epsilon_{zz}} - \frac{k_0^2 \epsilon_{xx} \epsilon_{yy}}{\epsilon_{zz}} + \frac{k_0^2 \epsilon_{xy} \epsilon_{yx}}{\epsilon_{zz}} - k_0^2 \epsilon_{xx} \]  

(69.3)

\[ c = \frac{\pi^4}{d^4} - \frac{\pi^2}{d^2} k_0^2 \epsilon_{xx} + k_0^4 \epsilon_{xx} \epsilon_{yy} - k_0^4 \epsilon_{xy} \epsilon_{yx} - \frac{\pi^2}{d^2} k_0^2 \epsilon_{yy} \]  

(69.4)

Solving for \( k_a^2 \) with the quadratic equation, both the wave numbers travelling in the \(+x\) and \(-x\) directions in the anisotropic region can be accounted for.

\[ \pm k_{a1} = \pm \sqrt{\frac{-b + \sqrt{b^2 - 4ac}}{2a}} \]  

(70.1)
\[\pm k_{a2} = \pm \sqrt{-b - \sqrt{b^2 - 4ac}} \quad \frac{2a}{70.2}\]

To solve for the six unknowns \((R_{TM}, R_{TE}, T_{a1}, T_{a2}, R_{a1}, R_{a2})\), the tangential fields at the freespace/anisotropic interface are set equal to each other at \(x = x'\) where \(x'\) is the distance from the source to the freespace/anisotropic interface, and the tangential electric fields to the PEC wall at the far end of the parallel plate waveguide in the anisotropic region are set equal to zero at \(x = x''\) where \(x''\) is the distance from the source to the PEC wall at the far end of the waveguide. This creates six equations to solve for the six unknowns. The four equations at the freespace/anisotropic interface are the following.

\[
E_z: \quad \frac{k_i d}{j \pi} A \cos \left(\frac{\pi z}{d}\right) e^{-j k_i x'} - \frac{k_i d}{j \pi} \cos \left(\frac{\pi z}{d}\right) e^{j k_i x'} = \frac{1}{j \varepsilon_{zz}} \left[ \frac{k_{a1} T_{a1} d}{\pi} \cos \left(\frac{\pi z}{d}\right) e^{-j k_{a1} x'} C_1 \right] \\
+ \frac{1}{j \varepsilon_{zz}} \left[ -\frac{k_{a1} R_{a1} d}{\pi} \cos \left(\frac{\pi z}{d}\right) e^{-j k_{a1} x'} - \frac{k_{a2} R_{a2} d}{\pi} \cos \left(\frac{\pi z}{d}\right) e^{j k_{a2} x'} C_2 \right]
\]

\[
\rightarrow k_i A = k_i R_{TM} e^{2j k_i x'} + \frac{k_{a1} T_{a1} C_1}{\varepsilon_{zz}} e^{-j (k_{a1} - k_i) x'} \\
+ \frac{k_{a2} T_{a2} C_2}{\varepsilon_{zz}} e^{-j (k_{a2} - k_i) x'} - \frac{k_{a1} R_{a1} C_1}{\varepsilon_{zz}} e^{j (k_{a1} + k_i) x'} \\
- \frac{k_{a2} R_{a2} C_2}{\varepsilon_{zz}} e^{j (k_{a2} + k_i) x'}
\]

64
$$E_y: \frac{j\omega_0 d}{\pi} R_{TE} \sin \left( \frac{\pi z}{d} \right) e^{jk_x x'} = T_{a1} \sin \left( \frac{\pi z}{d} \right) e^{-jk_{a1} x'} + T_{a2} \sin \left( \frac{\pi z}{d} \right) e^{-jk_{a2} x'}$$

$$+ R_{a1} \sin \left( \frac{\pi z}{d} \right) e^{jk_{a1} x'} + R_{a2} \sin \left( \frac{\pi z}{d} \right) e^{jk_{a2} x'}$$

$$\rightarrow - \frac{j\omega_0 d}{\pi} e^{jk_x x'} R_{TE} + T_{a1} e^{-jk_{a1} x'}$$

$$+ T_{a2} e^{-jk_{a2} x'} + R_{a1} e^{jk_{a1} x'} + R_{a2} e^{jk_{a2} x'} = 0$$

$$H_y: \frac{j\omega_0 d}{\pi} A \cos \left( \frac{\pi z}{d} \right) e^{-jk_x x'} + \frac{j\omega_0 d}{\pi} R_{TM} \cos \left( \frac{\pi z}{d} \right) e^{jk_x x'}$$

$$= \frac{j\omega_0 T_{a1}}{\pi} \cos \left( \frac{\pi z}{d} \right) e^{-jk_{a1} x'} C_1$$

$$+ \frac{j\omega_0 T_{a2}}{\pi} \cos \left( \frac{\pi z}{d} \right) e^{-jk_{a2} x'} C_2$$

$$+ \frac{j\omega_0 R_{a1}}{\pi} \cos \left( \frac{\pi z}{d} \right) e^{jk_{a1} x'} C_1$$

$$+ \frac{j\omega_0 R_{a2}}{\pi} \cos \left( \frac{\pi z}{d} \right) e^{jk_{a2} x'} C_2$$

$$\rightarrow A = -R_{TM} e^{jk_x x'} + T_{a1} C_1 e^{-jk_{a1} x'}$$

$$+ T_{a2} C_2 e^{-jk_{a2} x'}$$

$$+ R_{a1} C_1 e^{jk_{a1} x'} + R_{a2} C_2 e^{jk_{a2} x'}$$

$$H_x: - \frac{kd}{j\pi} R_{TE} \sin \left( \frac{\pi z}{d} \right) e^{jk_x x'}$$

$$= \frac{T_{a1} k_{a1}}{\omega \mu_0} \sin \left( \frac{\pi z}{d} \right) e^{-jk_{a1} x'} + \frac{T_{a2} k_{a2}}{\omega \mu_0} \sin \left( \frac{\pi z}{d} \right) e^{-jk_{a2} x'}$$

$$- \frac{R_{a1} k_{a1}}{\omega \mu_0} \sin \left( \frac{\pi z}{d} \right) e^{jk_{a1} x'} - \frac{R_{a2} k_{a2}}{\omega \mu_0} \sin \left( \frac{\pi z}{d} \right) e^{jk_{a2} x'}$$

$$\rightarrow - \frac{kd}{j\pi} R_{TE} e^{jk_x x'} + \frac{T_{a1} k_{a1}}{\omega \mu_0} e^{-jk_{a1} x'} + \frac{T_{a2} k_{a2}}{\omega \mu_0} e^{-jk_{a2} x'}$$

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\[-\frac{R_{a1}k_a}{\omega \mu_0}e^{jk_{a1}x'} - \frac{R_{a2}k_a}{\omega \mu_0}e^{jk_{a2}x'} = 0\]

At the PEC boundary at the far end of the waveguide in the anisotropic region, the tangential $E_y$ and $E_z$ fields are shorted to zero.

\[
E_y: \quad T_{a1}e^{-jk_{a1}x''} + T_{a2}e^{-jk_{a2}x''} + R_{a1}e^{jk_{a1}x''} + R_{a2}e^{jk_{a2}x''} = 0
\]

\[
E_z: \quad k_{a1}T_{a1}C_1e^{-jk_{a1}x''} + k_{a2}T_{a2}C_2e^{-jk_{a2}x''} - k_{a1}R_{a1}C_1e^{jk_{a1}x''} - k_{a2}R_{a2}C_2e^{jk_{a2}x''} = 0
\]

To solve for the six unknowns, (71.1) through (71.4) and (72.1) and (72.2) are rewritten in a matrix form where the solutions can be solved over the frequencies of interest.

\[
\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} R_{TM} \\ R_{TE} \\ T_{a1} \\ T_{a2} \\ R_{a1} \\ R_{a2} \end{bmatrix} \begin{bmatrix} k_iA \\ 0 \\ A \\ 0 \\ 0 \end{bmatrix}
\]

\[
\{V_1\} = \begin{bmatrix} ke^{2k_i x} \\ 0 \\ -e^{2k_i x} \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
\{V_2\} = \begin{bmatrix} 0 \\ \frac{j\omega \mu_0 d}{\pi} e^{ik_{a1} x} \\ 0 \\ kd e^{i k_{a1} x} \\ \frac{j}{\pi} \\ 0 \end{bmatrix}
\]
\[
\{V_3\} = \begin{pmatrix}
\frac{k_{a1}}{\epsilon_{zz}} C_1 e^{-j(k_{a1} - k)z} \\
e^{-j\epsilon_{zz}} \\
k_{a1} e^{-j\epsilon_{zz}} \\
k_{a1} C_1 e^{-j\epsilon_{zz}} \\
\end{pmatrix}
\]

(73.4)

\[
\{V_4\} = \begin{pmatrix}
\frac{k_{a2}}{\epsilon_{zz}} C_2 e^{-j(k_{a2} - k)z} \\
e^{-j\epsilon_{zz}} \\
k_{a2} e^{-j\epsilon_{zz}} \\
k_{a2} C_2 e^{-j\epsilon_{zz}} \\
\end{pmatrix}
\]

(73.5)

\[
\{V_5\} = \begin{pmatrix}
-\frac{k_{a1}}{\epsilon_{zz}} C_1 e^{j(k_{a1} + k)z} \\
e^{j\epsilon_{zz}} \\
k_{a1} e^{j\epsilon_{zz}} \\
k_{a1} C_1 e^{j\epsilon_{zz}} \\
\end{pmatrix}
\]

(73.6)

\[
\{V_6\} = \begin{pmatrix}
\frac{k_{a2}}{\epsilon_{zz}} C_2 e^{j(k_{a2} - k)z} \\
e^{j\epsilon_{zz}} \\
k_{a2} e^{j\epsilon_{zz}} \\
k_{a2} C_2 e^{j\epsilon_{zz}} \\
\end{pmatrix}
\]

(73.7)
When making the comparison between the analytical solution and the discrete solution, the discrete solution is formally the frequency response of the parallel plate waveguide system, so the Fourier transform of the sampled time domain solution at point 1 must be divided by the frequency content of the source function. In this particular numerical experiment, a soft source is enforced on a YZ-surface of $E_z$ fields in the isotropic region, and therefore, to properly find the impulse response of the system, the frequency content at point 1 must be divided by the frequency content of the integral of the soft source function since Ampere’s Law has a time derivative on the electric field component. The solutions of 25,000 time steps at point 1 were gathered and used for the frequency domain solution.

Comparisons between the analytical solution and the discrete solution at point 1 in are plotted below.
Figure 29: Comparison between the Analytical and Discrete solution for the $E_x$ field
Figure 30: Comparison between the Analytical and Discrete solution for the $E_y$ field
Figure 31: Comparison between the Analytical and Discrete solution for the $E_z$ field

As seen from figures 29 through 31, the new stencil update for the normal edges to the PEC accurately simulates the fields of the parallel plate waveguide problem. Although the solution seems to diverge at the higher frequencies, it is believe that this is due to the discretization being unable to properly resolve the solution at those higher frequencies. At 20 discretization points per wavelength, the frequency resolved is 1.2 GHz, and as the plots show, the discrete solution accurately matches the analytical solution up to that frequency.
Another comparison was done versus another implementation that also addresses the issues with electric field edges normal and half a cell away from the PEC [42], and the comparison of the errors between the implementation in [42] versus the implementation in this dissertation at point 1 is shown below. The error from the implementation in [42] is indicated as the FDTD error, and the error from the implementation using the stencil derived from the finite element method is indicated by the FEM error.

Figure 32: Comparison of errors between FDTD implementation and FEM implementation for $E_x$ field
Figure 33: Comparison of errors between FDTD implementation and FEM implementation for $E_y$ field
Figure 34: Comparison of errors between FDTD implementation and FEM implementation for $E_z$ field

The errors are comparable for $E_y$ and $E_z$ since the implementation in both cases are the same; however, for the $E_x$ solution, the new FDTD update derived using finite elements has a much lower error compared to the method in [42] throughout most of the frequency band and especially below 1.2 GHz frequency which indicates the $\lambda/20$ discretization.

In this numerical parallel plate experiment, it was verified that the new anisotropic FDTD stencils derived from the new finite element method for electric fields
that are normal and half a cell away from a PEC boundary can accurately capture the solution, and this new stencil predicts the fields more accurately than previous methods.

### 3.2: Resonator Experiment

The second numerical test of the new anisotropic finite element method performed was to find the resonating modes of a PEC cavity resonator filled with an anisotropic material. In this particular example, a 10 Δx by 10 Δx by 10 Δx domain of cube elements is created where the outermost surface is set to be a PEC, and the inside of the domain is enforced with an anisotropic material with the following properties.

\[
\varepsilon = \varepsilon_0 \begin{bmatrix} 1.25 & -0.354 & 0.250 \\ -0.354 & 1.50 & -0.354 \\ 0.250 & -0.354 & 1.25 \end{bmatrix} \tag{73.8}
\]

\[
\mu = \mu_0 \tag{73.9}
\]

In this case, there is not an analytical solution for the resonator, so for purposes of comparison, the resonate modes found from using the new finite element derivation will be compared to resonate modes found from the traditional implicit finite element method using cube elements. The chosen discretization for this test problem is Δx = 0.5 meter, and the time step is chosen based on the Courant number of 0.9999. The excitation for the resonator was a unit amplitude pulse with a time width of 2Δt to provide a wide frequency band response to excite the resonator, and an edge was chosen at random to excite the resonator. In the tables below, the numerical resonant frequencies are displayed for both methods.

<table>
<thead>
<tr>
<th>Resonance</th>
<th>Traditional FEM: (E_x f)</th>
<th>New FEM: (E_x f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>31.11 MHz</td>
<td>31.11 MHz</td>
</tr>
<tr>
<td>2</td>
<td>41.04 MHz</td>
<td>40.98 MHz</td>
</tr>
</tbody>
</table>
Table 1: Comparison of $E_x$ Field resonances

<table>
<thead>
<tr>
<th>Resonance</th>
<th>Traditional FEM: $E_y f$</th>
<th>New FEM: $E_y f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>31.11 MHz</td>
<td>31.11 MHz</td>
</tr>
<tr>
<td>2</td>
<td>41.04 MHz</td>
<td>40.98 MHz</td>
</tr>
<tr>
<td>3</td>
<td>41.88 MHz</td>
<td>42.00 MHz</td>
</tr>
</tbody>
</table>

Table 2: Comparison of $E_y$ field resonances

<table>
<thead>
<tr>
<th>Resonance</th>
<th>Traditional FEM: $E_z f$</th>
<th>New FEM: $E_z f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>31.11 MHz</td>
<td>31.11 MHz</td>
</tr>
<tr>
<td>2</td>
<td>41.04 MHz</td>
<td>40.98 MHz</td>
</tr>
<tr>
<td>3</td>
<td>41.88 MHz</td>
<td>42.00 MHz</td>
</tr>
</tbody>
</table>

Table 3: Comparison of $E_z$ Field Resonances

As seen from the tables 1 through 3, the resonant frequencies from the traditional finite element method and new finite element method are closely matched with each other for all field directions, and this experiment also shows that the new finite element method can accurately simulate full permittivity tensors.

Section 3.3: Magnetic Photonic Crystal (MPC) Experiment

The third numerical test will be a transient analysis of an MPC to show that the new method can handle both anisotropic and gyrotropic materials, and in this numerical experiment, the desired effect to be seen is the reduced group velocity of the electromagnetic fields within the MPC as well as amplification of the field amplitude.
This particular MPC is constructed as a periodic arrangement of two misaligned anisotropic layers and a ferrite layer. The anisotropic layer has the following properties.

\[
\bar{\varepsilon}_A = \varepsilon_0 \begin{bmatrix} 
\varepsilon_{xx} & 0 & 0 \\
0 & \varepsilon_A + \delta_A \cos(2\theta_A) & \delta_A \sin(2\theta_A) \\
0 & \delta_A \sin(2\theta_A) & \varepsilon_A - \delta_A \cos(2\theta_A) 
\end{bmatrix}
\] (74.1)

\[
\mu_A = \mu_r \mu_0
\] (74.2)

The ferrite layer has the following properties.

\[
\bar{\varepsilon}_F = \varepsilon_r \varepsilon_0
\] (75.1)

\[
\bar{\mu}_F = \mu_0 \begin{bmatrix} 
1 & 0 & 0 \\
0 & 1 + \chi_{yy}(\omega) & \chi_{yz}(\omega) \\
0 & \chi_{xy}(\omega) & 1 + \chi_{zz}(\omega) 
\end{bmatrix}
\] (75.2)

\[
\chi_{yy}(\omega) = \chi_{zz}(\omega) = \frac{(\omega_0 + j\omega\alpha)\omega_m}{(\omega_0 + j\omega\alpha)^2 + (j\omega)^2}
\] (75.3)

\[
\chi_{yz}(\omega) = -\chi_{zy}(\omega) = \frac{j\omega\omega_m}{(\omega_0 + j\omega\alpha)^2 + (j\omega)^2}
\] (75.4)

For the material properties in (130.3) and (130.4), \(\alpha\) is the damping or loss constant, \(\omega_0 = \gamma H_0\), and \(\omega_m = \gamma 4\pi M_s\). \(\gamma\) is the gyromagnetic ratio, \(H_0\) is the dc biasing magnetic field magnitude, and \(M_s\) is the DC saturation magnetization.
The approach to derive the update for the ferrite layers is similar to the derivation of the anisotropic FDTD update via finite elements. Assuming that there are no losses in the ferrite ($\alpha = 0$), the inverse of the permeability tensor in (75.2) can be written as followed.

$$\mu_F^{-1} = \frac{1}{\mu_0 d} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{bmatrix}$$ (76.1)

$$a = \omega_0^2 + (j\omega)^2 + \omega_0 \omega_m$$ (76.2)

$$b = j\omega \omega_m$$ (76.3)

$$d = \omega_0^2 + 2\omega_0 \omega_m + (j\omega)^2 + \omega_m^2$$ (76.4)

The magnetic constitutive relation will be examined to convert the magnetic flux density back into the magnetic field.
\[ \vec{H} = \frac{1}{\mu_0} \vec{\mu}_r^{-1} \cdot \vec{B} \]  

(77)

In (77), the magnetic field and magnetic flux density are both expanded using the Whitney 2-form basis function for the cube element.

\[ \vec{H} = \sum_j H(t) \vec{W}_j^2(r) \]  

(78.1)

\[ \vec{B} = \sum_j B(t) \vec{W}_j^2(r) \]  

(78.2)

Substituting (78.1) and (78.2) into (77) and testing the resulting equation with a Whitney 2-form basis results in the following.

\[ \int_V \left( \sum_j H(t) \vec{W}_j^2(r) \right) \cdot \vec{W}_k^2(r) \, dV \]

(79)

\[ = \frac{1}{\mu_0} \int_V \left( [\vec{\mu}_r]^{-1} \cdot \sum_j B(t) \vec{W}_j^2(r) \right) \cdot \vec{W}_k^2(r) \, dV \]

As with the integrals in developing the anisotropic FDTD update via finite elements, the integrals in (79) are evaluated using trapezoidal integration. Applied over a single cube element, the integrals in (79) create the following matrices. The facet numbering is also defined below.
Figure 36: Reference cube for facet numbering

\[ f_1 \equiv f_{1375} \quad (80.1) \]

\[ f_2 \equiv f_{2486} \quad (80.2) \]

\[ f_3 \equiv f_{1562} \quad (80.3) \]

\[ f_4 \equiv f_{3784} \quad (80.4) \]

\[ f_5 \equiv f_{1243} \quad (80.5) \]

\[ f_6 \equiv f_{5687} \quad (80.6) \]

\[ T_{H_{ij,k}} = \int_V \overline{W}_j^2(r) \cdot \overline{W}_k^2(r) \, dV \quad (81.1) \]
\[ T_H = \Delta x^3 \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix} \] (81.2)

\[ *_{\mu^{-1},(j,k)} = \int_{\mathcal{V}} \left( [\mu_r]^{-1} \cdot \bar{W}_j^2 (r) \right) \cdot \bar{W}_k^2 (r) \, dV \] (81.3)

\[ *_{\mu^{-1}} = \Delta x^3 \begin{bmatrix} 1/2 & 0 & 1/4 & \mu_{xy}^{-1} & 1/4 & \mu_{xz}^{-1} & 1/4 & \mu_{xz}^{-1} \\ 0 & 1/2 & \mu_{xx}^{-1} & 1/4 & \mu_{xy}^{-1} & 1/4 & \mu_{xz}^{-1} & 1/4 \mu_{xz}^{-1} \\ 1 & 1 & 1 & 1/2 & \mu_{yy}^{-1} & 1 & 1 & 1/4 \mu_{yz}^{-1} \\ 1/4 & \mu_{yx}^{-1} & 1 & 1/2 & \mu_{yy}^{-1} & 1 & 1 & 1/4 \mu_{yz}^{-1} \\ 1 & 1 & 1/4 & \mu_{yx}^{-1} & 1 & 1 & 1 & 1/4 \mu_{yz}^{-1} \\ 1/4 & \mu_{yx}^{-1} & 1 & 1/4 & \mu_{zy}^{-1} & 1 & 1 & 1/2 \mu_{zz}^{-1} \\ 1 & 1 & 1/4 & \mu_{zx}^{-1} & 1 & 1 & 1 & 0 \end{bmatrix} \] (81.4)

With the integral values defined, the equation for the permeability stencil for \( H_x \) can be written as follows with facet references from figure 37. Although not written, the stencils for \( H_y \) and \( H_z \) can be similarly derived.
Figure 37: Indexing of facet basis functions

\[ H_1 = \mu_{xx}^{-1} B_1 + \frac{1}{4} \mu_{xy}^{-1} (B_2 + B_3 + B_4 + B_5) \]
\[ + \frac{1}{4} \mu_{xz}^{-1} (B_6 + B_7 + B_8 + B_9) \]

Using (82) as a starting point, the time-domain ferrite update to convert the magnetic flux density into the magnetic field can be derived. For brevity, the derivation of the time-domain update for \( H_y \) will be examined, but the derivation for \( H_z \) follows similarly. Using the permeability tensor of the ferrite, the equation for the constitutive relation is the following.

\[ H_y = a_1 B_y - a_2 (j\omega) B_{z, all} \]
\[ a_1 = \frac{\omega_0^2 + (j\omega)^2 + \omega_0 \omega_m}{\mu_0 d} \]
\[ a_2 = \frac{\omega_m}{4\mu_0 d} \]  \hspace{1cm} (83.3)

\( B_{z,all} \) encompasses all the \( B_z \) terms in the permeability stencil. Performing an inverse Fourier transform on (83.1), the \( j\omega \) term becomes a derivative in time, and this partial derivative is replaced with a central differencing equation.

\[ H_{y,\frac{n+3}{2}} = a_1 B_{y,\frac{n+3}{2}} - \frac{a_2 \left( B_{z,\text{all}}^{n+\frac{3}{2}} - B_{z,\text{all}}^{n-\frac{1}{2}} \right)}{2\Delta t} \]  \hspace{1cm} (84)

Since the time indexing in the central differencing terms \( B_{z,\text{all}} \) are off by half a step, these terms are averaged to be indexed at half time steps.

\[ H_{y,\frac{n+1}{2}} = a_1 B_{y,\frac{n+1}{2}} - \frac{a_2 \left( B_{z,\text{all}}^{n+\frac{1}{2}} - B_{z,\text{all}}^{n-\frac{1}{2}} \right)}{2\Delta t} \]  \hspace{1cm} (85)

For stability purposes, the \( H \) and \( B \) terms at time step \( n + \frac{1}{2} \) are double averaged.

\[ \frac{H_{y,\frac{n+3}{2}} + 2H_{y,\frac{n+1}{2}} + H_{y,\frac{n-1}{2}}}{4} = a_1 \frac{B_{y,\frac{n+3}{2}} + 2B_{y,\frac{n+1}{2}} + B_{y,\frac{n-1}{2}}}{4} - \frac{a_2 \left( B_{z,\text{all}}^{n+\frac{3}{2}} - B_{z,\text{all}}^{n-\frac{1}{2}} \right)}{2\Delta t} \]  \hspace{1cm} (86)

Rearranging (86) to solve for \( H_{y,\frac{n+3}{2}} \), the magnetic field update for \( H_y \) is the following.

\[ H_{y,\frac{n+3}{2}} = -2H_{y,\frac{n+1}{2}} - H_{y,\frac{n+1}{2}} + a_1 \left( B_{y,\frac{n+3}{2}} + 2B_{y,\frac{n+1}{2}} + B_{y,\frac{n-1}{2}} \right) - \frac{2a_2 \left( B_{z,\text{all}}^{n+\frac{3}{2}} - B_{z,\text{all}}^{n-\frac{1}{2}} \right)}{\Delta t} \]  \hspace{1cm} (87)
(87) will be the magnetic field update used in the ferrite region for $H_y$. The update for $H_z$ can be derived and written similarly.

\[
H_{z}^{n+\frac{3}{2}} = -2H_{z}^{n+\frac{1}{2}} - H_{z}^{n-\frac{1}{2}} \\
+ a_1 \left( B_{z}^{n+\frac{3}{2}} + 2B_{z}^{n+\frac{1}{2}} + B_{z}^{n-\frac{1}{2}} \right) + \frac{2a_2}{\Delta t} \left( B_{y}^{n+\frac{3}{2}} - B_{y}^{n-\frac{1}{2}} \right)
\]

(88)

The parameters of the anisotropic and ferrite layers are similar to that from [29]. The dimensions of the anisotropic layers are 5 mm in the x-direction and 5 meters in the y and z directions for both $A_1$ and $A_2$. For the material properties of the anisotropic layers, the parameters are as follows.

\[
\epsilon_{A1} = \epsilon_{A2} = 7 \\
\delta_{A1} = \delta_{A2} = 6 \\
\theta_{A1} = 0^\circ \\
\theta_{A2} = 36.0963^\circ \\
\epsilon_{xx,A1} = \epsilon_{xx,A2} = 1 \\
\mu_{r,A1} = \mu_{r,A2} = 1
\]

(89)

The dimensions of the ferrite layer are 1 mm in the x-direction and 5 meters in the y and z direction, and the material properties of the ferrite are as follows.

\[
\epsilon_r = 5 \\
\omega_0 = 36.504 \times 10^9 \\
\omega_m = 73.006 \times 10^9
\]

(90)

In this numerical example, the MPC consists of 237 unit cells suspended inside the freespace region, which is surrounded by PML. The narrow layer of freespace
between the MPC and PML region in the XY and XZ plane has a thickness of $\Delta x$. A narrow bandwidth soft-source Gaussian pulse modulated with a sine wave of frequency 4.0298 GHz excites a YZ plane wave with a peak amplitude of 15 mV in the freespace region and impinges upon the MPC. To reduce the effects of any spurious reflections from the ferrite/PML interface at the far end of the domain, loss ($\sigma = 0.01$) was inserted into the first anisotropic layer of the unit cells in the latter half of the MPC. The discretization used for this experiment was $\Delta y = \Delta z = 0.25$ m and $\Delta x = 0.08333$ mm. The Courant number for the time step is 0.9999.

With the given discretizations, each unit MPC block consisting of 2 misaligned anisotropic layers and a single ferrite layer will contain 132 Yee cells in the x-direction and 18 Yee cells in the y and z-direction. Given that there’re 237 of these unit cells, the computational cost of examining this problem on a single personal computer would be exorbitantly time consuming, so the resources at Ohio Supercomputing were utilized to take advantage of a parallel implementation of this numerical experiment. With the computing capacity of 128 processor cores, this particular setup was simulated for 1 million time steps. The solution was examined on an XY-plane of edges that cut the MPC in half.
Figure 38: Setup of the MPC Problem

\[
\epsilon_{r,A_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (91.1)
\]

\[
\epsilon_{r,A_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8.83 & 5.71 \\ 0 & 5.71 & 5.17 \end{bmatrix} \quad (91.2)
\]

\[
\mu_{r,F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4.85 & j2.67 \\ 0 & -j2.67 & 4.85 \end{bmatrix} \quad (91.3)
\]
Figure 39: Cross-section of $E_y$ field propagation in the MPC at 500,000 time steps
Figure 40: Cross-section of $E_y$ field propagation in the MPC at 800,000 time steps
Figure 41: 2D Quantitative Plot of $E_y$ fields in MPC at 500,000 time steps
As seen in figure 41 and 42, the overall amplification of the field amplitude seems to be only by a factor of two from 15 mV to 30 mV. However, there is a mismatch between the freespace interface and the anisotropic interface, which means that the fields amplified in the MPC is likely more than a factor of two since only a fraction of the fields of the source is transmitted into the MPC. However, what is apparent in this numerical experiment is the reduced group velocity of the fields as it propagates through the MPC. In figure 39 and 41, there are remnants of the electric field that is propagating through the thin freespace region that surrounds the MPC; however, in figures 40 and 42, it’s clear that all the energy in the freespace region has passed by while there’s still energy
propagating slowly in the MPC. The group velocity slowdown is by 2 orders of magnitude since the fields in the MPC travelled approximately 1.2 meters over a time of 834 ns, which is $1.43 \times 10^6$ meters per second.

In this experiment, it has been shown that the field amplification and reduced group velocity can exist in a three dimensional MPC based on the material properties of a one dimensional MPC, and it was also shown that the new modeling capabilities developed for FDTD at the anisotropic material interfaces will properly update the electric fields. In addition, the finite element approach to derive the anisotropic FDTD update equations from finite elements has been successfully applied to derive FDTD equations magnetic materials with anisotropic permeability.

Section 3.4: Parallel Plate Waveguide Experiment using Anisotropic FDTD/Finite Element Hybrid

To demonstrate that the new finite element method developed can interface with traditionally derived finite elements so that the anisotropic FDTD and traditional anisotropic finite element method can exist in a hybrid, the parallel plate waveguide problem is reexamined with a slight change. The anisotropic region will be modeled with both anisotropic FDTD and anisotropic finite elements to show that the hybridized method can produce an accurate solution.

In the $x$-direction, the first 5 cell layers in the anisotropic region will be updated using FDTD. The next 10 cell layers will be updated using anisotropic finite elements, and the last 5 layers before the PEC boundary will be updated using FDTD.
Figure 43: Division of numerical methods in anisotropic region

To interface with the finite element region, a layer of cubes composed completely of pyramids lies between the FDTD cube region and the finite element tetrahedral region since pyramid elements were used successfully before to interface the isotropic FDTD updates with isotropic finite elements. The finite element matrices, however, have to be constructed carefully due to the anisotropy of the materials. Near the FDTD/finite element interface where the pyramid elements will interface with the FDTD cubes, there is a region of 2 layers where the FDTD fields and finite element fields will overlap and pass fields values between each other. This is visually shown in figure 44.
Figure 44: Update Methodology at FDTD/FEM Interface

In the overlap region, there are two layers of cube elements where the new FEM formulation derived in chapter 2 for cube elements is used to interface properly with the anisotropic FDTD. The stiffness matrix terms created in the overlap region are the same as the regularly derived stiffness matrix terms with cube elements from traditional FEM with integral lumping. However, the mass matrix terms are derived using the new FEM method and can be written as follows.

\[ [T]_{\text{overlap}} = [T][\varepsilon^{-1}][T_E] \]

(92)
In addition, the mass and stiffness matrices must be incorporated into the Newmark time-updating scheme in the following manner.

\[
[T + \Delta t^2 (2\theta)S][E^{n+1}] + [-2T - \Delta t^2 (4\theta - 1)S][E^n] \\
+ [T + \Delta t^2 (2\theta)S][E^{n-1}]
\]  \hspace{1cm} (93)

For the mass and stiffness matrices derived with traditional finite elements, these matrices are incorporated into (93) with \(\theta = 1/8\). For the mass and stiffness matrix derived using the new finite element method, these matrices are incorporated into (93) with \(\theta = 0\). This summation of the mass and stiffness matrices into (93) completes the hybrid formulation.

The plots below compare the discrete solution measured at point 1 versus the analytical solution.
Figure 45: Comparison between the Analytical and Discrete solution for the $E_x$ field
Figure 46: Comparison between the Analytical and Discrete solution for the $E_y$ field
As seen from the plots above, the hybridized method combining the anisotropic FDTD update and the anisotropic FEM update can accurately predict the analytical solution of the waveguide up to 1.2 GHz, which is the frequency based on 20 points per wavelength discretization, and the large discrepancies between the solutions are due to the discretization being unable to resolve the solution at the higher frequencies.

Section 3.5: Dielectric Sphere Scattering Experiment using Anisotropic FDTD/Finite Element Hybrid
For the last numerical study, the scattering from an anisotropic dielectric sphere is examined, and the time domain solutions of the far-fields based on three different cases are compared.

1. FDTD for the entire domain.
2. A hybrid FDTD/finite elements where the anisotropic sphere region is simulated only with finite elements
3. A hybrid FDTD/finite element hybrid where the anisotropic sphere region is simulated using a hybrid of finite elements and FDTD.

It will be shown that the anisotropic hybrid can provide a solution comparable to a simulation using only finite elements while performing faster and utilizing less memory, and it will also be shown that the hybrid method performs faster than the FDTD solution.

The setup of the scattering problem is as follows.
In this setup, the radius of the sphere is .6 meters.

In case one where both the anisotropic sphere region and freespace surrounding the sphere are simulated using FDTD, the curved surface of the sphere will be stair-stepped to approximate the surface curvature. The sphere is gridded using a discretization of $\Delta x = 0.0375$ m, the radius of the sphere is $16\Delta x$, and the size of the FDTD domain containing the sphere is $128\Delta x$ by $128\Delta x$ by $128\Delta x$. This particular discretization was chosen for comparison purposes since the solution resulting from using this discretization most closely matches the finite element solution. The stair-stepped sphere looks as follows in figure 49.
In the second case, there is a finite element region in the shape of a box that contains the anisotropic sphere, and pyramid elements will line the surface of this box so that the FDTD cubes can transition into the finite element region without any issues as indicated in figure 50. The discretization in the FDTD region is $\Delta x = 0.075$ m, and the size of the finite element box region measures $64\Delta x$ by $64\Delta x$ by $64\Delta x$. The sphere and surround freespace finite element region consists of 43059 tetrahedral and pyramid elements.
The third case will be similar to the second case except that there will be a FDTD region of 8 by 8 by 8 cubes inside the sphere that will be simulated using the anisotropic FDTD update to demonstrate the transition from an anisotropic finite element region into an anisotropic FDTD region. As in case 2, a layer of pyramid elements will transition the tetrahedral finite element region into the FDTD cube region making up the box region as shown in figure 51. The discretization used in the FDTD region in this case is $\Delta x = 101$. 

Figure 50: Cross-section of finite element region for case 2
0.075 m, and the FEM sphere is enclosed in a discrete region of size $64\Delta x$ by $64\Delta x$ by $64\Delta x$. There are 38276 tetrahedral and pyramid elements making up the sphere and surrounding freespace finite element region.

Figure 51: Cross-section of transition region inside sphere between finite elements and FDTD

The source will be a plane wave in the isotropic freespace region that impinges on the sphere from the $-x$ direction using the three-dimensional total field/scattered field formulation in FDTD. The shape of the source is modeled as a first derivative Gaussian pulse, and the polarization of the field will be at 45 degrees in the $+\theta$ direction. In the
scatter region, an FDTD near-to-far field transformation will be performed to recreate the
time domain scattered field formulation. The tensor representing the anisotropic
permittivity is given as follows, and the figure below indicates the locations of the far-
fields being examined.

\[
\bar{\varepsilon}_a = \epsilon_0 \begin{bmatrix}
3.0 & -0.98 & 0 \\
-0.98 & 4.0 & 0 \\
0 & 0 & 5.0
\end{bmatrix}
\]

(94.1)

\[
\mu_a = \mu_0
\]

(94.2)

Figure 52: Diagram of incoming source and location of far field samples
For ease of comparison, the far-field time domain solutions sampled at each point are Fourier transformed into the frequency domain to compare between the three cases of the numerical $E_\theta$ and $E_\phi$ far fields.

Figure 53: Normalized Far Field $E_\theta$ at Point 1
Figure 54: Normalized Far Field $E_\phi$ at Point 1
Figure 55: Normalized Far Field $E_\theta$ at Point 2
Figure 56: Normalized Far Field $E_\phi$ at Point 2
Figure 57: Normalized Far Field $E_\theta$ at Point 3
Figure 58: Normalized Far Field $E_{\phi}$ at Point 3
Figure 59: Normalized Far Field $E_\theta$ at Point 4
From the frequency domain plots, the hybridized anisotropic method solution closely matched the anisotropic FEM solution, but the memory usage for the hybridized method was 12% lower, which is consistent with the reduced element count. Although the FDTD version is more memory efficient than the FEM methods, the reduced discretization to achieve the same level of accuracy as the FEM method results in a much smaller time step being used, resulting in the FDTD version taking over twice as long to simulate. Thus, the hybridized anisotropic method demonstrates that it can maintain a high level of
accuracy while giving the user the flexibility to choose a larger time step to examine complex objects. The table below summarizes the memory and program timings.

<table>
<thead>
<tr>
<th></th>
<th>Anisotropic FDTD</th>
<th>Anisotropic FEM</th>
<th>Anisotropic FDTD/FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Memory Usage</td>
<td>318 MB</td>
<td>1.31 GB</td>
<td>1.15 GB</td>
</tr>
<tr>
<td>Simulation Time</td>
<td>34 min, 38 seconds</td>
<td>13 min, 8 seconds</td>
<td>12 min, 21 seconds</td>
</tr>
</tbody>
</table>

Table 4: Program Run Statistics

Although the memory savings and gains in time are relatively small in this example, it can be imagined that the savings and gains can be greatly appreciated in an extremely large domain filled with complex-shaped objects. In such an example, the computational cost of finite elements would become exorbitantly expensive memory-wise, and the depending on the complexity of the objects, using only FDTD would likely take an unreasonable amount of time to simulate the problem due to having to use a small discretization to properly model the most complex object.
CHAPTER 4

CONCLUSION AND FUTURE DIRECTION

In this dissertation, the derivation of a hybridized time domain anisotropic FDTD/finite element method was presented using a new finite element method that utilized the constitutive relation in a finite element manner. The new finite element method also allows for the proper derivation of interface conditions for the anisotropic FDTD method, and it also provides a systematic approach to derive stencils for certain cases where the traditional anisotropic FDTD update could not be applied, such as in the update of the electric field normal and half a cell away to a PEC boundary. The new anisotropic FDTD stencils derived from the new finite element method were tested and verified to provide more accurate solutions compared to previously derived methods. Also, the hybridized anisotropic FDTD/finite element method was tested and shown to provide solutions comparable to using the finite element by itself with less memory and faster execution time.

For future work, an interesting topic would be to see whether a higher-order methodology can be developed for the anisotropic hybrid formulation. The key to this approach would be to find an appropriate basis function in finite elements that would be able to recover the FDTD update, and an integration method similar to trapezoidal
integration but with a higher order of accuracy would be needed. There has also been research into developing a higher order isotropic hybrid method in which the finite element region uses higher-order spatial basis functions [43]. This has been demonstrated to accurately predict antenna patterns and properties, and it’s believed that the adaption of this higher order method may be straightforward in the anisotropic case since only the finite element region is affected. Another topic of interest would be to explore different methods of transitioning the region between the FDTD and finite element region [44, 45]. Although the pyramid element provides a useable solution to this issue, the pyramid elements adds an additional layer of complexity in the construction of the mesh. As mentioned before in the introduction there has been some research done to directly connect the FDTD cubes and finite element tetrahedral, but those methods have been shown to be unstable without filtering the solution. More recently however, Degerfeldt and Rylander have demonstrated by using Nitche’s method to treat the tangential discontinuity of the fields [46], a direct interface between FDTD cubes and finite element tetrahedral can be realized, so applying this methodology with the anisotropic hybrid developed in this dissertation may produce similar results.
APPENDIX

ANISOTROPIC TIME DOMAIN METHODS

This section will examine the derivation the first order anisotropic FDTD method and traditional time domain finite element method with anisotropic materials. Both the FDTD and finite element updates will be later referenced in the hybrid derivation for purposes of comparison.

A.1 Anisotropic FDTD Derivation

In the upcoming section the first order anisotropic FDTD equations will be derived from first order Maxwell’s equations (first order meaning that the partial derivatives in time and space are of order one). But first, an overview of the central difference equation, an essential equation used throughout FDTD, will be provided.

A.1.1 Derivation of the central difference equation

The FDTD update equations for solving partial differential equations are derived by replacing the partial derivatives with a discrete equation known as the central differencing equation. The central differencing equation is a second order accurate approximation of the partial derivative, and in electromagnetic, the second order aspect of the equation relates to the discretization spacing of time and space used in the FDTD domain. This discrete update is derived by a combination of two other discrete
equations: the forward differencing equation and the backwards differencing equation. Both of these respective equations are also approximations to partial derivatives, but the accuracy from both equations is first order if used by themselves. These discrete differencing equations originate from reordering the Taylor Series expansion of a function, and the upcoming example will outline the derivation of the spatial central difference equation, but the derivation can be generally applied to other variables such as time.

Given a function $f(x)$, the Taylor series expansion of a function $f(x + \Delta x)$ can be expressed by the following.

$$f(x + \Delta x) = \frac{1}{0!}(\Delta x)^0 f(x) + \frac{1}{1!}(\Delta x)^1 \frac{\partial f(x)}{\partial x} + \frac{1}{2!}(\Delta x)^2 \frac{\partial^2 f(x)}{\partial x^2} + \ldots$$

$$= \sum_{c=0}^{\infty} \frac{1}{c!}(\Delta x)^c \frac{\partial^c f(x)}{\partial x^c} \quad (95)$$

Similarly, a Taylor series expansion can be written for $f(x - \Delta x)$.

$$f(x - \Delta x) = \frac{1}{0!}(-\Delta x)^0 f(x)$$

$$+ \frac{1}{1!}(-\Delta x)^1 \frac{\partial f(x)}{\partial x} + \frac{1}{2!}(-\Delta x)^2 \frac{\partial^2 f(x)}{\partial x^2} + \ldots$$

$$= \sum_{c=0}^{\infty} \frac{1}{c!}(-\Delta x)^c \frac{\partial^c f(x)}{\partial x^c} \quad (96)$$

The forward finite difference equation for the partial derivative of $f(x)$ is derived by isolating the first partial derivative term on the right hand side of (95) by itself while
subtracting and multiplying the remaining terms on the right hand side to the left hand side of the equation.

\[
\frac{\partial f(x)}{\partial x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} + O(x)_f \tag{97}
\]

\[
O(x)_f = -\frac{1}{2!} \Delta x \frac{\partial^2 f(x)}{\partial x^2} - \frac{1}{3!} (\Delta x)^2 \frac{\partial^3 f(x)}{\partial x^3} \ldots \tag{98}
\]

The \( O(x)_f \) term in (97) is the residual error resulting from the forward differencing equation, and this residual error is commonly referred as having an error of order one since the first term of \( O(x)_f \) is scaled by the term \( \Delta x \). Applying the same methodology as with the forward difference equation, the backward finite difference equation for the first partial derivative of \( f(x) \) can be derived from (96).

\[
\frac{\partial f(x)}{\partial x} = \frac{f(x) - f(x + \Delta x)}{\Delta x} + O(\Delta x)_b \tag{99}
\]

\[
O(\Delta x)_b = \frac{1}{2!} \Delta x \frac{\partial^2 f(x)}{\partial x^2} - \frac{1}{3!} (\Delta x)^2 \frac{\partial^3 f(x)}{\partial x^3} \ldots \tag{100}
\]

The \( O(\Delta x)_b \) term is the residual error term for the backwards differencing equation. To derive the center differencing equation, (97) and (99) are summed together to result in the following.

\[
\frac{\partial f(x)}{\partial x} = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} + O(x) \tag{101}
\]

Summing the forward and backward difference equations also affects the order of the error in a beneficial fashion by cancelling out the terms with odd powers of \( \Delta x \).
\[ O(x) = -\frac{\Delta x^2}{3!} \frac{\partial^3 f}{dx^3} - \frac{\Delta x^4}{5!} \frac{\partial^5 f}{dx^5} - \ldots \tag{102} \]

This cancellation results in the largest error term to be on the order of \( \Delta x^2 \), meaning the central difference equation is second order accurate. With the improvement in accuracy and ease of implementation, the central difference equation is the cornerstone of all finite differencing schemes. In the next section, the central difference equation will be applied to derive the first order anisotropic FDTD update for Maxwell’s equations.

**Section A.1.2: First Order Anisotropic FDTD Method**

With the knowledge of the central difference equation approximation for partial derivatives, we can examine how to derive the anisotropic FDTD update for Maxwell’s equations. In the derivation, it is assume that Maxwell’s equations are examined in a homogenous, source-free region.

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{103.1} \]

\[ \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \tag{103.2} \]

\[ \vec{D} = \varepsilon_0 \varepsilon_r \cdot \vec{E} \tag{103.3} \]

\[ \varepsilon_r = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{103.4} \]

\[ \vec{B} = \mu_0 \mu_r \vec{H} \tag{103.5} \]
\( \varepsilon_0 \) and \( \mu_0 \) are the permittivity and permeability constants respectively, and \( \varepsilon_r \) and \( \mu_r \) are the relative permittivity and permeability. The anisotropy of the material is indicated by the tensor form of \( \bar{\varepsilon}_r \).

The curl can be expanded into its respective vector components via a determinant.

\[
\nabla \times \vec{A} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
A_x & A_y & A_z
\end{vmatrix}
\]

\[
= \hat{x} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - \hat{y} \left( \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) + \hat{z} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)
\]

From (103.1), the FDTD update equations for the magnetic field update will be derived. For brevity, the \( H_x \) magnetic field update will be derived, but the remaining field component updates for \( H_y \) and \( H_z \) can be derived using the same methodology. Substituting (103.5) and (104) into (103.1) results in the following.

\[
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu_r \mu_0 \frac{\partial H_x}{\partial t}
\]

In the FDTD literature, it is common to refer to the spatial and temporal discretization in integer terms based off of a chosen discretization in space and time, so the FDTD referencing method will now be introduced. In the Cartesian coordinate system where FDTD is most commonly used, assume that the location of \( H_x \) is at Cartesian coordinate location \( (i \Delta x, (j + \frac{1}{2}) \Delta y, (k + \frac{1}{2}) \Delta z) \), and the magnetic field value of interest is at time \( n \Delta t \). Note that \( i, j, k, \) and \( n \) are integer indexes, \( \Delta x, \Delta y, \) and \( \Delta z \) are
the chosen discretizations in the $x$, $y$, and $z$ directions, and $\Delta t$ is the time discretization.

Applying this convention to the fields in (105) results in the following.

$$\frac{\partial E_z^n_{i,j,k+\frac{1}{2}}}{\partial y} - \frac{\partial E_y^n_{i,j,k+\frac{1}{2}}}{\partial z} = -\mu_r \mu_0 \frac{\partial H_x^n_{i,j,k+\frac{1}{2}}}{\partial t}$$  \hspace{1cm} (106)

The center differencing equation is applied to the spatial and temporal partial derivatives, and (106) becomes the following.

$$\frac{E_z^n_{i,j+1,k+\frac{1}{2}} - E_z^n_{i,j,k+\frac{1}{2}}}{\Delta y} - \frac{E_y^n_{i,j,k+1} - E_y^n_{i,j,k+\frac{1}{2}}}{\Delta z} = -\mu_r \mu_0 \frac{H_{x,i,j,k}^{n+\frac{1}{2}} - H_{x,i,j,k}^{n-\frac{1}{2}}}{\Delta t}$$  \hspace{1cm} (107)

Moving terms on the right hand side of (107) to the left hand side, the future update of $H_x$ can be written as the following.

$$H_{x,i,j+1,k+\frac{1}{2}}^{n+\frac{1}{2}} = H_{x,i,j,k+\frac{1}{2}}^{n-\frac{1}{2}} + \frac{\Delta t}{\mu_r \mu_0} \left( \frac{E_y^n_{i,j,k+\frac{1}{2}} - E_y^n_{i,j,k+1}}{\Delta z} - \frac{E_z^n_{i,j,k+1} - E_z^n_{i,j,k+\frac{1}{2}}}{\Delta y} \right)$$  \hspace{1cm} (108)
We see in (108) that the future value of magnetic field $H_x$ at time step $n + \frac{1}{2}$ can be calculated based upon the magnetic field value $H_x$ at time step $n - \frac{1}{2}$ and surrounding electric field values at half-spaced discretizations in the $y$ and $z$ directions and at time step $n$. The FDTD update for the magnetic fields in the $y$ and $z$ directions are also derived in a similar fashion. Visually, the $H_x$ update stencil can be drawn as in figure 61.

The anisotropic electric field update is derived in a similar fashion as the isotropic magnetic field update, but there are several additional steps to account for the anisotropy of the permittivity. First, (103.3) and (103.4) are substituted into (103.2) and the permittivity tensor is inverted to the left hand side.

$$\frac{1}{\epsilon_0} \varepsilon_r^{-1} \cdot (\nabla \times \vec{H}) = \frac{\partial \vec{E}}{\partial t}$$  \hspace{1cm} (109)
Expanding the curl of the magnetic fields results in the following equations.

\[
\frac{1}{\varepsilon_0} \begin{bmatrix}
\varepsilon_{xx}^{-1} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \\
-\varepsilon_{xy}^{-1} \left( \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right) \\
+\varepsilon_{xz}^{-1} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)
\end{bmatrix} = \frac{\partial E_x}{\partial t} \quad (110.1)
\]

\[
\frac{1}{\varepsilon_0} \begin{bmatrix}
\varepsilon_{yx}^{-1} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \\
-\varepsilon_{yy}^{-1} \left( \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right) \\
+\varepsilon_{yz}^{-1} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)
\end{bmatrix} = \frac{\partial E_y}{\partial t} \quad (110.2)
\]

\[
\frac{1}{\varepsilon_0} \begin{bmatrix}
\varepsilon_{zx}^{-1} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \\
-\varepsilon_{zy}^{-1} \left( \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} \right) \\
+\varepsilon_{zz}^{-1} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)
\end{bmatrix} = \frac{\partial E_z}{\partial t} \quad (110.3)
\]

Again for brevity, we will examine the expansion of (110.1) as a finite difference equation, but the finite difference version of (110.2) and (110.3) can be derived in a similar manner.

In applying the FDTD convention, it’s assumed that the location of \(E_x\) is at Cartesian coordinate location \((i + \frac{1}{2})\Delta x, j\Delta y, k\Delta z\) and the current time step index is \((n + \frac{1}{2})\Delta x\).
\[
\frac{e_{xx}^{-1}}{\varepsilon_0} \left( \frac{\partial H_z^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k}}{\partial y} - \frac{\partial H_z^{n+\frac{1}{2}}_{i,j,k}}{\partial z} \right) - \frac{e_{xy}^{-1}}{\varepsilon_0} \left( \frac{\partial H_y^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k}}{\partial x} - \frac{\partial H_z^{n+\frac{1}{2}}_{i,j,k}}{\partial z} \right) \\
+ \frac{e_{xz}^{-1}}{\varepsilon_0} \left( \frac{\partial H_y^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k}}{\partial x} - \frac{\partial H_x^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k}}{\partial y} \right) = \frac{\partial E_x^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k}}{\partial t} 
\]

Next, the spatial and temporal partial derivatives in (111) are expanded via central differencing.

\[
\frac{e_{xx}^{-1}}{\varepsilon_0} \left( \frac{H_z^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k} - H_z^{n+\frac{1}{2}}_{i,j,k}}{\Delta y} - \frac{H_y^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k} - H_y^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k}}{\Delta z} \right) \\
- \frac{e_{xy}^{-1}}{\varepsilon_0} \left( \frac{H_x^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k} - H_x^{n+\frac{1}{2}}_{i,j,k}}{\Delta x} - \frac{H_y^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k} - H_y^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k}}{\Delta z} \right) \\
+ \frac{e_{xz}^{-1}}{\varepsilon_0} \left( \frac{H_x^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k} - H_x^{n+\frac{1}{2}}_{i,j,k}}{\Delta x} - \frac{H_y^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k} - H_y^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k}}{\Delta y} \right) \\
= \frac{E_x^{n+\frac{1}{2}}_{i+\frac{1}{2},j,k} - E_x^n_{i+\frac{1}{2},j,k}}{\Delta t} 
\]

At this point, the magnetic fields associated with the permittivity component \( e_{xx}^{-1} \) will conform to the Cartesian/Yee grid as shown in . This stencil exactly visualizes the isotropic FDTD stencil for \( E_x \).
Figure 62: Magnetic fields associated with the $\epsilon_{xx}^{-1}$ term of $E_x$ update in (112)

However, when the magnetic fields associated with the off-diagonal permittivity components $\epsilon_{xy}^{-1}$ and $\epsilon_{xz}^{-1}$ are examined, these magnetic fields are not located normal to the faces on the Cartesian grid like the magnetic fields associated with $\epsilon_{xx}^{-1}$. Visually, the stencil for the magnetic fields associated with $\epsilon_{xy}^{-1}$ and $\epsilon_{xz}^{-1}$ look as follows based on (112).
Figure 63: Magnetic Fields associated with the $\epsilon_{x'y}^{-1}$ term of $E_x$ update in (112)

Figure 64: Magnetic Fields associated with the $\epsilon_{xz}^{-1}$ term of $E_x$ update in (112)
To create a stencil that conform to the Cartesian grid, each of the non-conforming magnetic fields in (112) are approximated using an average of four surrounding magnetic fields of identical direction that already conform to the Yee grid. For the magnetic fields associated with the $\epsilon_{xy}^{-1}$ term, they’re modified as follows.

\[
H_{z,i+1,j,k}^n \approx \frac{H_z^{n}{}_{i+\frac{1}{2},j+\frac{1}{2},k} + H_z^{n}{}_{i+\frac{3}{2},j+\frac{1}{2},k} + H_z^{n}{}_{i+\frac{1}{2},j-\frac{1}{2},k} + H_z^{n}{}_{i+\frac{3}{2},j-\frac{1}{2},k}}{4} \quad (113.1)
\]

\[
H_{z,i,j,k}^n \approx \frac{H_z^{n}{}_{i-\frac{1}{2},j+\frac{1}{2},k} + H_z^{n}{}_{i+\frac{1}{2},j+\frac{1}{2},k} + H_z^{n}{}_{i-\frac{1}{2},j-\frac{1}{2},k} + H_z^{n}{}_{i+\frac{1}{2},j-\frac{1}{2},k}}{4} \quad (113.2)
\]

\[
H_{x,i+\frac{1}{2},j,k}^n \approx \frac{H_x^{n}{}_{i+1,j+\frac{1}{2},k} + H_x^{n}{}_{i+1,j-\frac{1}{2},k+\frac{1}{2}} + H_x^{n}{}_{i,j+\frac{1}{2},k+1} + H_x^{n}{}_{i,j-\frac{1}{2},k+\frac{1}{2}}}{4} \quad (113.3)
\]

\[
H_{x,i-\frac{1}{2},j,k}^n \approx \frac{H_x^{n}{}_{i+1,j+\frac{1}{2},k-\frac{1}{2}} + H_x^{n}{}_{i+1,j-\frac{1}{2},k-\frac{1}{2}} + H_x^{n}{}_{i,j+\frac{1}{2},k-\frac{1}{2}} + H_x^{n}{}_{i,j-\frac{1}{2},k-\frac{1}{2}}}{4} \quad (113.4)
\]

Figure 65: Visualization of averaged magnetic fields used to represent the non-conforming magnetic field associated with $\epsilon_{xy}^{-1}$ term
Similarly, the magnetic field components associated with $\epsilon_{xz}^{-1}$ are averaged as follows with magnetic fields conforming to the Cartesian grid.

\begin{align}
H_{y,i+1,j,k}^n & \approx \frac{H_y^n_{i+\frac{1}{2},j,k+\frac{1}{2}} + H_y^n_{i+\frac{1}{2},j,k-\frac{1}{2}} + H_y^n_{i+\frac{3}{2},j,k+\frac{1}{2}} + H_y^n_{i+\frac{3}{2},j,k-\frac{1}{2}}}{4} 
& \quad (114.1) \\
H_{y,i,j,k}^n & \approx \frac{H_y^n_{i-\frac{1}{2},j,k+\frac{1}{2}} + H_y^n_{i-\frac{1}{2},j,k-\frac{1}{2}} + H_y^n_{i+\frac{1}{2},j,k+\frac{1}{2}} + H_y^n_{i+\frac{1}{2},j,k-\frac{1}{2}}}{4} 
& \quad (114.2) \\
H_{x,i+\frac{1}{2},j,k}^n & \approx \frac{H_x^n_{i,j+\frac{1}{2},k+\frac{1}{2}} + H_x^n_{i,j+\frac{1}{2},k-\frac{1}{2}} + H_x^n_{i+1,j+\frac{1}{2},k+\frac{1}{2}} + H_x^n_{i+1,j+\frac{1}{2},k-\frac{1}{2}}}{4} 
& \quad (114.3) \\
H_{x,i+\frac{1}{2},j,k}^n & \approx \frac{H_x^n_{i,j-\frac{1}{2},k+\frac{1}{2}} + H_x^n_{i,j-\frac{1}{2},k-\frac{1}{2}} + H_x^n_{i+1,j-\frac{1}{2},k+\frac{1}{2}} + H_x^n_{i+1,j-\frac{1}{2},k-\frac{1}{2}}}{4} 
& \quad (114.4)
\end{align}
Figure 66: Visualization of averaged magnetic field used to represent the non-conforming magnetic field associated with $\varepsilon_{xx}^{-1}$ term

Substituting (113) and (114) into (112) and rearranging terms to solve for the future electric field value will result in the appropriate anisotropic FDTD update for the $E_x$ field.

$$
\frac{\varepsilon_{xx}^{-1}}{\varepsilon_0} \left( \frac{H^+_{x, i+\frac{1}{2}, j+\frac{1}{2}, k} - H^-_{x, i+\frac{1}{2}, j+\frac{1}{2}, k}}{\Delta y} - \frac{H^+_{y, i+\frac{1}{2}, j+\frac{1}{2}, k} - H^-_{y, i+\frac{1}{2}, j+\frac{1}{2}, k}}{\Delta z} \right) \quad (115)
$$
The $E_y$ and $E_z$ field can be derived in a similar fashion. The stencil of magnetic fields looks as follows for the anisotropic $E_x$ stencil.
Section A.2: Traditional Time Domain Finite Element Derivations with Anisotropic Materials

This section will examine the general derivation of traditional finite elements and examine the effects of anisotropy on cube elements, tetrahedral elements, and pyramid elements. Typically, the time domain finite element update starts with the 2nd order vector wave equation. In a source-free, isotropic domain, this equation is written as follows.
\[ \nabla \times \left( \frac{1}{\mu_r} \nabla \times \vec{E} \right) + \varepsilon_r \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \]  

(116)

The first step is to take the continuous equation and rewrite it in a discrete form, but unlike FDTD where the partial derivatives are replaced with the central difference equation, the solution to the electric field is represented by choosing a basis function expansion to approximate the electric field \( \vec{E} \) throughout the entire domain.

\[
\vec{E}(r,t) = \sum_{i=0}^{N-1} \overline{W}_i(r) E_i(t)
\]  

(117)

In (117), \( \overline{W}_i(r) \) represents the spatial basis function, and \( E_i(t) \) is a coefficient representing the electric field value, although it’s written as a function of time. Substituting (117) into (116) results in the following.

\[
\nabla \times \left( \frac{1}{\mu_r} \nabla \times \sum_{i=0}^{N-1} \overline{W}(r) E(t) \right) + \varepsilon_r \frac{\partial^2 \sum_{i=0}^{N-1} \overline{W}(r) E(t)}{\partial t^2} = R(r,t)
\]  

(118)

Since this expansion is a discrete approximation, (118) will have a residual term \( R \) since not every electric field values will be exactly known. However, this residual can be made small by choosing a set of basis functions that are orthogonal to it via an inner product. Enforcing this inner product is the key behind the weighted residual method, or Galerkin testing, and typically, the choice of basis functions for the testing is the same as the expansion basis. Therefore, (118) is tested with the same spatial basis function used in the expansion in (116), and the result of the testing is integrated over the entire volume of the domain to complete the inner product calculation.
\[
\int_V \left( \nabla \times \frac{1}{\mu_r} \nabla \times \sum_{i=0}^{N-1} \vec{W}_i(r) E_i(t) \right) \cdot \vec{W}_j(r) \, dV 
\]

\[
+ \int_V \left( \varepsilon_r \frac{\partial^2}{c^2 \partial t^2} \sum_{i=0}^{N-1} \vec{W}_i(r) E_i(t) \right) \cdot \vec{W}_j(r) \, dV = 0 
\]

To simplify the evaluation of the integral of the curl-curl aspect of (119), the following vector identity is applied to split up the curl-curl.

\[
\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) 
\]

(120)

Substituting (120) into (119) and applying the divergence theorem will result in the following.

\[
\int_{\partial S} \left[ \left( \frac{1}{\mu_r} \nabla \times \sum_{i=0}^{N-1} \vec{W}_i(r) E_i(t) \right) \times \vec{W}_j(r) \right] \, dS 
\]

\[
+ \int_V \left[ \left( \frac{1}{\mu_r} \nabla \times \sum_{i=0}^{N-1} \vec{W}_i(r) E_i(t) \right) \cdot (\nabla \times \vec{W}_j(r)) \right] \, dV 
\]

(121)

\[
+ \int_V \frac{\varepsilon_r}{c^2} \left( \frac{\partial^2}{\partial t^2} \sum_{i=0}^{N-1} \vec{W}(r) E(t) \right) \cdot \vec{W}_j(r) \, dV = 0 
\]

For simplicity, it’s assumed that the homogenous domain examined extends to infinity, so the integral over the surface of the domain will go to zero via the Sommerfeld Radiation Condition. This leaves (121) with two terms.
\[
\int_V \left[ \left( \frac{1}{\mu_r} \nabla \times \sum_{i=0}^{N-1} \overline{W}_i(r) E_i(t) \right) \cdot \left( \nabla \times \overline{W}_j(r) \right) \right] dV \\
+ \int_V \left( \frac{\partial^2}{\partial t^2} \sum_{i=0}^{N-1} \overline{W}_i(r) E(t) \right) \cdot \overline{W}_j(r) dV = 0
\] (122)

With certain choices of basis function, two matrices commonly known as the mass and stiffness matrix in the finite element community can be constructed from the two parts of the integration in (122). Terms for the mass matrix, denoted by \( T \), can be written from the integral of the dot product between the two spatial basis functions.

\[
T_{ij} = \int_V \overline{W}_i(r) \cdot \overline{W}_j(r) dV
\] (123)

Terms for the stiffness matrix, usually by \( S \), is written integrating the dot product of the curl of the spatial basis functions.

\[
S_{ij} = \int_V \left( \nabla \times \overline{W}_i(r) \right) \cdot \left( \nabla \times \overline{W}_j(r) \right) dV
\] (124)

In matrix representation, (122) can simply be written as the following.

\[
\frac{\varepsilon_r}{c^2} T \frac{\partial^2}{\partial t^2} \{E_i(t)\} + \frac{1}{\mu_r} S \{E_i(t)\} = 0
\] (125)

Up to this point, the spatial discretization has been addressed, but not the temporal discretization. To make (125) a time-domain equation, the Newmark-time stepping method is commonly applied, which modifies (125) to the following [35].
\[
[T + \Delta t^2 (2\theta)S][E^{n+1}] + [-2T - \Delta t^2 (4\theta - 1)S][E^n] \\
+ [T + \Delta t^2 (2\theta)S][E^{n-1}] = 0
\]  

(126)

Several different time stepping schemes can be derived by setting the values of \( \theta \), but the most popular value is to set \( \theta = 1/8 \), which result in an unconditionally stable implicit time stepping scheme, meaning that the stability of the time domain update is insensitive to the choice of the size of the time step.

The equations derived up to this point assumed that the finite element method was derived in isotropic freespace. To derive the finite element method in a material with an anisotropic permittivity, the relative permittivity \( \varepsilon_r \) in (116) is expressed as a tensor, and the mass matrix calculation is modified as follows.

\[
T_{i,j} = \int_V \left( [\varepsilon_r] \cdot \overrightarrow{W}_i(r) \right) \cdot \overrightarrow{W}_j(r) dV
\]  

(127.1)

\[
\varepsilon_r = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{bmatrix}
\]  

(127.2)

The next section will overview commonly used first order spatial basis functions in finite elements, and it will be examined how the anisotropy of the permittivity works into the formulation in each case.

**A.2.1: Cube Element Anisotropy**

A first order cube or hexahedral element will contain 12 linear basis functions, one for each edge. For each coordinate direction, the value of the basis functions along each edge in the cube is set equal one and linearly tapers to zero towards the other edges.
of the same coordinate direction. For example, as shown in figure 68, the basis function defined on edge \{1,2\} has a magnitude of 1 along edge \{1,2\} and linearly tapers to 0 along the remaining edges that are parallel to it.

Figure 68: Vector Plot of Edge Basis Function

Assuming that the coordinates at node 1 are (0,0,0), the vector basis functions for each of the edges can be defined as follows. \( \Delta x, \Delta y, \) and \( \Delta z \) represent the length of the edges in the x, y, and z coordinate directions respectively.
Figure 69: Cube reference figure with node numbering

\[
\tilde{e}_{12} \equiv \tilde{e}_1 = \frac{(\Delta y - y)(\Delta z - z)}{\Delta y \Delta z} \hat{x}
\]  
(128.1)

\[
\tilde{e}_{34} \equiv \tilde{e}_2 = \frac{y(\Delta z - z)}{\Delta y \Delta z} \hat{x}
\]  
(128.2)

\[
\tilde{e}_{56} = \tilde{e}_3 = \frac{(\Delta y - y)z}{\Delta y \Delta z} \hat{x}
\]  
(128.3)

\[
\tilde{e}_{78} \equiv \tilde{e}_4 = \frac{yz}{\Delta y \Delta z} \hat{x}
\]  
(128.4)

\[
\tilde{e}_{13} \equiv \tilde{e}_5 = \frac{(\Delta x - x)(\Delta z - z)}{\Delta x \Delta z} \hat{y}
\]  
(128.5)

\[
\tilde{e}_{57} \equiv \tilde{e}_6 = \frac{(\Delta x - x)z}{\Delta x \Delta z} \hat{y}
\]  
(128.6)
\[ \hat{e}_{24} \equiv \hat{e}_7 = \frac{x(\Delta z - z)}{\Delta x \Delta z} \hat{y} \]  
(128.7)

\[ \hat{e}_{68} \equiv \hat{e}_8 = \frac{xz}{\Delta x \Delta z} \hat{y} \]  
(128.8)

\[ \hat{e}_{15} \equiv \hat{e}_9 = \frac{(\Delta x - x)(\Delta y - y)}{\Delta x \Delta y} \hat{z} \]  
(128.9)

\[ \hat{e}_{26} \equiv \hat{e}_{10} = \frac{x(\Delta y - y)}{\Delta x \Delta y} \hat{z} \]  
(128.10)

\[ \hat{e}_{37} \equiv \hat{e}_{11} = \frac{(\Delta x - x)y}{\Delta x \Delta y} \hat{z} \]  
(128.11)

\[ \hat{e}_{48} \equiv \hat{e}_{12} = \frac{xy}{\Delta x \Delta y} \hat{z} \]  
(128.12)

Assuming that \( \Delta x = \Delta y = \Delta z \), substituting (128.1) through (128.12) into (127.1) and analytically integrating over the volume of the cube results in the following values for the cube mass matrix.

\[
T = \frac{\Delta x^3}{72} \begin{bmatrix}
\epsilon_{xx} T_1 & \epsilon_{xy} T_2 & \epsilon_{xz} T_2^t \\
\epsilon_{yx} T_2^t & \epsilon_{yy} T_1 & \epsilon_{yz} T_2 \\
\epsilon_{zx} T_2 & \epsilon_{zy} T_2^t & \epsilon_{zz} T_1
\end{bmatrix}
\]  
(129.1)

\[
T_1 = \begin{bmatrix}
8 & 4 & 4 & 2 \\
4 & 8 & 2 & 4 \\
4 & 2 & 8 & 4 \\
2 & 4 & 4 & 8
\end{bmatrix}
\]  
(129.2)

\[
T_2 = \begin{bmatrix}
6 & 3 & 6 & 3 \\
6 & 3 & 6 & 3 \\
3 & 6 & 3 & 6 \\
3 & 6 & 3 & 6
\end{bmatrix}
\]  
(129.3)
In (129.1), $T^t$ represents the transpose of the matrix $T$. Substituting (128.1) through (128.12) into (124), the values in the stiffness matrix for a cube looks as follows after the integration over the cube volume is analytically evaluated.

$$S = \Delta x \begin{bmatrix} S_1 & S_2 & S_2^t \\ S_2^t & S_1 & S_2 \\ S_2 & S_2^t & S_1 \end{bmatrix}$$

(130.1)

$$S_1 = \frac{1}{6} \begin{bmatrix} 4 & -1 & -1 & -2 \\ -1 & 4 & -2 & -1 \\ -1 & -2 & 4 & -1 \\ -2 & -1 & -1 & 4 \end{bmatrix}$$

(130.2)

$$S_2 = \frac{1}{6} \begin{bmatrix} -2 & -1 & 2 & 1 \\ 2 & 1 & -2 & -1 \\ -1 & -2 & 1 & 2 \\ 1 & 2 & -1 & -2 \end{bmatrix}$$

(130.3)

### Section A.2.2: Tetrahedral Element Anisotropy

The edge basis function for the tetrahedral is derived from the Whitney form idealology where the edge basis functions are derived from nodal basis functions [37-41]. The nodal basis function can generally be defined as followed for a tetrahedral.

$$N = a^e + b^e x + c^e y + d^e z$$

(131)

Assuming that $N_i$ is the value of $N$ at the $i^{th}$ node, a system of equations can be written to solve for the coefficients $a$, $b$, $c$, and $d$.

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{bmatrix} \begin{bmatrix} a^e \\ b^e \\ c^e \\ d^e \end{bmatrix}$$

(132)
Figure 70: Tetrahedral with node indexing

From (131), the values of $a^e$, $b^e$, $c^e$, and $d^e$ can be analytically evaluated.

$$a^e = \frac{1}{6V^e} \begin{vmatrix} N_1^e & N_2^e & N_3^e & N_4^e \\ x_1^e & x_2^e & x_3^e & x_4^e \\ y_1^e & y_2^e & y_3^e & y_4^e \\ z_1^e & z_2^e & z_3^e & z_4^e \end{vmatrix}$$

$$= \frac{1}{6V^e} (a_1^e N_1^e + a_2^e N_2^e + a_3^e N_3^e + a_4^e N_4^e)$$

$$b^e = \frac{1}{6V^e} \begin{vmatrix} 1 & 1 & 1 & 1 \\ N_1^e & N_2^e & N_3^e & N_4^e \\ y_1^e & y_2^e & y_3^e & y_4^e \\ z_1^e & z_2^e & z_3^e & z_4^e \end{vmatrix}$$

$$= \frac{1}{6V^e} (b_1^e N_1^e + b_2^e N_2^e + b_3^e N_3^e + b_4^e N_4^e)$$

$$c^e = \frac{1}{6V^e} \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1^e & x_2^e & x_3^e & x_4^e \\ N_1^e & N_2^e & N_3^e & N_4^e \\ z_1^e & z_2^e & z_3^e & z_4^e \end{vmatrix}$$

$$= \frac{1}{6V^e} (c_1^e N_1^e + c_2^e N_2^e + c_3^e N_3^e + c_4^e N_4^e)$$

(133.1)
\[
= \frac{1}{6V^e} (c^e_1 N_1^e + c^e_2 N_2^e + c^e_3 N_3^e + c^e_4 N_4^e)
\]

\[
d^e = \frac{1}{6V^e} \begin{vmatrix}
1 & 1 & 1 & 1 \\
x_1^e & x_2^e & x_3^e & x_4^e \\
y_1^e & y_2^e & y_3^e & y_4^e \\
N_1^e & N_2^e & N_3^e & N_4^e
\end{vmatrix}
\]

\[
= \frac{1}{6V^e} (d^e_1 N_1^e + d^e_2 N_2^e + d^e_3 N_3^e + d^e_4 N_4^e)
\]

\[
V^e = \frac{1}{6} \begin{vmatrix}
x_1^e & x_2^e & x_3^e & x_4^e \\
y_1^e & y_2^e & y_3^e & y_4^e \\
z_1^e & z_2^e & z_3^e & z_4^e
\end{vmatrix}
\]

Substituting (133.1) through (133.5) into (131), we get that

\[
N^e = \sum_{i=1}^{4} \lambda_i N^e_i
\]

\[
\lambda_i = \frac{1}{6V^e} (a^e_i x + b^e_i y + c^e_i z)
\]

\(\lambda_i\) can be viewed as the barycentric function for the \(i^{th}\) node.

To define the edge basis function, the barycentric function in (134.2) is used as follows to define a Whitney-1 form vector equation which only has tangential components over a particular edge and is orthogonal to the remaining edges of the tetrahedral. The Whitney 1-form is written as follows.

\[
\bar{W}_i = l_i (\lambda_{i_1} \nabla \lambda_{i_2} - \lambda_{i_2} \nabla \lambda_{i_1})
\]

\[
\nabla \lambda_i = \frac{1}{6V^e} (b^e_i \hat{x} + c^e_i \hat{y} + d^e_i \hat{z})
\]
(135.1) represents the spatial basis function of the tetrahedral edge with nodes \( i_1 \) and \( i_2 \) as its endpoints, the “direction” associated with the edge points from node \( i_1 \) to node \( i_2 \), and the length of the edge being examined is \( l_i \). From these edge basis functions, the mass and stiffness matrices can be defined. For the stiffness matrix, the following vector identity is applied to transform the curl of the spatial basis function (135.1) to mitigate evaluation of the integration in the stiffness matrix calculation.

\[
\nabla \times \bar{W}_i = 2 \left( \nabla \lambda_{i_1} \times \nabla \lambda_{i_2} \right)
\]

(136)

With this, each stiffness matrix entry can be calculated by the following.

\[
S_{i,j} = \frac{4l_i l_j V^e}{(6V^e)^2} \begin{bmatrix}
(c_{i_1} d_{i_2}^e - d_{i_1}^e c_{i_2}^e)(c_{j_1}^e d_{j_2}^e - d_{j_1}^e c_{j_2}^e)
+ (d_{i_1}^e b_{i_2}^e - b_{i_1}^e d_{i_2}^e)(d_{j_1}^e b_{j_2}^e - b_{j_1}^e d_{j_2}^e)
+ (b_{i_1}^e c_{i_2}^e - c_{i_1}^e b_{i_2}^e)(b_{j_1}^e c_{j_2}^e - c_{j_1}^e b_{j_2}^e)
\end{bmatrix}
\]

(137)

\( b_{i_1}, c_{i_1}, \) and \( d_{i_1} \) are constants in (137) associated with the node in edge \( i \) that the edge direction is pointing away from, and \( b_{i_2}, c_{i_2}, \) and \( d_{i_2} \) are constants associated with the node in edge \( i \) that the edge direction is pointing towards.

To examine how the mass matrix entries are calculated, (135.2) is substituted into (135.1).

\[
\bar{W}_i = l_i \left( \lambda_{i_1} \nabla \lambda_{i_2} - \lambda_{i_2} \nabla \lambda_{i_1} \right)
\]

\[
= l_i \left( \frac{1}{6V^e} \left( b_{i_2}^e \hat{x} + c_{i_2}^e \hat{y} + d_{i_2}^e \hat{z} \right) \right)
\]

(138)

Next, a dot product is taken with the result in (138) and the permittivity tensor.
\[
\varepsilon_r \cdot \bar{W}_i = \frac{l_i}{6V^e} \left[ \begin{array}{c}
\lambda_{i1} b_{i1} \varepsilon_{xx} + c_{i1} \varepsilon_{xy} + d_{i1} \varepsilon_{xz} \\
-\lambda_{i2} b_{i1} \varepsilon_{xx} + c_{i1} \varepsilon_{xy} + d_{i1} \varepsilon_{xz} \\
\lambda_{i1} b_{i2} \varepsilon_{yx} + c_{i2} \varepsilon_{yy} + d_{i2} \varepsilon_{yz} \\
-\lambda_{i2} b_{i2} \varepsilon_{yx} + c_{i2} \varepsilon_{yy} + d_{i2} \varepsilon_{yz} \\
\lambda_{i1} b_{i1} \varepsilon_{zx} + c_{i1} \varepsilon_{zy} + d_{i1} \varepsilon_{zz} \\
-\lambda_{i2} b_{i1} \varepsilon_{zx} + c_{i1} \varepsilon_{zy} + d_{i1} \varepsilon_{zz} \\
\end{array} \right]
\]

The integrand in the mass matrix for the anisotropic case becomes the following.

\[
\bar{W}_i \cdot ([\varepsilon_r] \cdot \bar{W}_j) = \frac{l_i l_j}{36V^2} \left[ \begin{array}{c}
\lambda_{i1} \lambda_{j1} b_{i2} (b_{i2} \varepsilon_{xx} + c_{i2} \varepsilon_{xy} + d_{i2} \varepsilon_{xz}) \\
-\lambda_{i1} \lambda_{j2} b_{i2} (b_{i1} \varepsilon_{xx} + c_{i1} \varepsilon_{xy} + d_{i1} \varepsilon_{xz}) \\
-\lambda_{i2} \lambda_{j1} b_{i1} (b_{j2} \varepsilon_{yx} + c_{j2} \varepsilon_{yy} + d_{j2} \varepsilon_{yz}) \\
+\lambda_{i2} \lambda_{j2} b_{i1} (b_{j1} \varepsilon_{xx} + c_{j1} \varepsilon_{xy} + d_{j1} \varepsilon_{xz}) \\
+\lambda_{i1} \lambda_{j1} c_{i2} (b_{j2} \varepsilon_{yx} + c_{j2} \varepsilon_{yy} + d_{j2} \varepsilon_{yz}) \\
-\lambda_{i1} \lambda_{j2} c_{i2} (b_{j1} \varepsilon_{yx} + c_{j1} \varepsilon_{yy} + d_{j1} \varepsilon_{yz}) \\
-\lambda_{i2} \lambda_{j1} c_{i1} (b_{j2} \varepsilon_{yy} + c_{j2} \varepsilon_{yy} + d_{j2} \varepsilon_{yz}) \\
+\lambda_{i2} \lambda_{j2} c_{i1} (b_{j1} \varepsilon_{yy} + c_{j1} \varepsilon_{yy} + d_{j1} \varepsilon_{yz}) \\
+\lambda_{i1} \lambda_{j1} d_{i2} (b_{j2} \varepsilon_{zx} + c_{j2} \varepsilon_{zy} + d_{j2} \varepsilon_{zz}) \\
-\lambda_{i1} \lambda_{j2} d_{i2} (b_{j1} \varepsilon_{zx} + c_{j1} \varepsilon_{zy} + d_{j1} \varepsilon_{zz}) \\
-\lambda_{i2} \lambda_{j1} d_{i1} (b_{j2} \varepsilon_{zx} + c_{j2} \varepsilon_{zy} + d_{j2} \varepsilon_{zz}) \\
+\lambda_{i2} \lambda_{j2} d_{i1} (b_{j1} \varepsilon_{zx} + c_{j1} \varepsilon_{zy} + d_{j1} \varepsilon_{zz}) \\
\end{array} \right] \tag{140}
\]

Since the variables \(b, c, \text{ and } d\) are constants, the only part of (140) that needs to be evaluated is the volume integral of the product between the two nodal functions, which can be evaluated analytically [6].

\[
\int_V (\lambda_1^{k} \lambda_2^{l} \lambda_3^{m} \lambda_4^{n} \, dV = \frac{k! \, l! \, m! \, n!}{(k + l + m + n + 3)!} 6V^e \tag{141}
\]
With (141), the integral of the product between two barycentric functions can easily be solved.

\[ \int_{V} \lambda_i \lambda_j \, dV = \int_{V} (N_1)^2 \, dV = \frac{2!}{5!} 6V^e = \frac{1}{10} V^e \quad \text{(142.1)} \]

\[ \int_{V} \lambda_i \lambda_j^2 \, dV = \int_{V} N_1 N_2 \, dV = \frac{1!}{5!} 6V^e = \frac{1}{20} V^e \quad \text{(142.2)} \]

With this, the derivation of the mass and stiffness matrices involved in the anisotropic finite element formulation for tetrahedral is completed, and these formulations will be used in the numerical examples presented later in this document.

**Section A.2.3: Pyramid Element Anisotropy**

The edge basis functions for a pyramid will be examined in this section. First, a “unit” pyramid element will be defined.

![Figure 71: Unit Pyramid](image)
The coordinates are bounded as follows for the pyramid.

\[-(1 - z) \leq x \leq (1 - z)\]
\[-(1 - z) \leq y \leq (1 - z)\]
\[0 \leq z \leq 1\]  

(143)

Each node has a basis function associated with it which is equal to one at node \(N_i\) and zero at \(N_j\) where \(i \neq j\).

\[N_1 = \frac{1}{4}[(1 + x)(1 + y) - z + \frac{xyz}{1 - z}]\]  
(144.1)

\[N_2 = \frac{1}{4}[(1 - x)(1 + y) - z - \frac{xyz}{1 - z}]\]  
(144.2)

\[N_3 = \frac{1}{4}[(1 - x)(1 - y) - z + \frac{xyz}{1 - z}]\]  
(144.3)

\[N_4 = \frac{1}{4}[(1 + x)(1 - y) - z - \frac{xyz}{1 - z}]\]  
(144.4)

\[N_5 = z\]  
(144.5)

The basis functions for the edges along the square face are defined as followed.

\[\vec{W}_{ij} = N_i \nabla (N_j + N_k) - N_j \nabla (N_i + N_l)\]  
(145)

\(N_k\) is a node that lies on an edge with \(N_j\) but not with \(N_i\). Also, \(N_l\) is a node that lies on an edge with \(N_i\) but not with \(N_j\).

For the edges that contain node 5 in figure 71, the edge basis function is the same as the tetrahedral edge basis function.
\[ \vec{W}_{i,j} = N_i \nabla N_j - N_j \nabla N_i \] (146)

To find the mass and stiffness matrices, the integral can be analytically evaluated over the following bounds over the volume of the pyramid.

\[
\int_0^1 \int_{-(1-z)}^{(1-z)} \int_{-(1-z)}^{(1-z)} dx dy dz \quad (147)
\]

With this derivation, the unit pyramid can simply be scale appropriately meshing purposes. Modifying the node basis functions of the unit sized pyramid to scale with the length of the edges of the pyramid’s square base results in the following nodal basis functions.

\[
N_1 = \frac{1}{L^2} \left[ \left( \frac{L}{2} + x \right) \left( \frac{L}{2} + y \right) - \frac{L}{2} z + \frac{xyz}{L} \right] 
\] (148.1)

\[
N_2 = \frac{1}{L^2} \left[ \left( \frac{L}{2} - x \right) \left( \frac{L}{2} + y \right) - \frac{L}{2} z - \frac{xyz}{L} \right] 
\] (148.2)

\[
N_3 = \frac{1}{L^2} \left[ \left( \frac{L}{2} - x \right) \left( \frac{L}{2} - y \right) - \frac{L}{2} z + \frac{xyz}{L} \right] 
\] (148.3)

\[
N_4 = \frac{1}{L^2} \left[ \left( \frac{L}{2} + x \right) \left( \frac{L}{2} - y \right) - \frac{L}{2} z - \frac{xyz}{L} \right] 
\] (148.4)

\[
N_5 = \frac{2}{L} 
\] (148.5)
With (148.1) through (148.5) and use of (145) and (146) defining the basis functions of all the pyramid edges, the finite element matrices for a pyramid can be evaluated and found. The stiffness matrix can be written by the following.

\[
S = L \begin{bmatrix}
S_1^t & S_2
\end{bmatrix}
\]  

(149.1)

\[
S_1 = \begin{bmatrix}
17/18 & 7/18 & -1/18 & 1/18 \\
7/18 & 17/18 & 1/18 & -1/18 \\
-1/18 & 1/18 & 17/18 & 7/18 \\
1/18 & -1/18 & 7/18 & 17/18
\end{bmatrix}
\]  

(149.2)

\[
S_2 = \begin{bmatrix}
\frac{4\sqrt{3}}{9} & \frac{2\sqrt{3}}{9} & \frac{4\sqrt{3}}{9} & \frac{2\sqrt{3}}{9} \\
\frac{2\sqrt{3}}{9} & \frac{4\sqrt{3}}{9} & \frac{2\sqrt{3}}{9} & \frac{4\sqrt{3}}{9} \\
\frac{4\sqrt{3}}{9} & \frac{4\sqrt{3}}{9} & \frac{2\sqrt{3}}{9} & \frac{2\sqrt{3}}{9} \\
\frac{2\sqrt{3}}{9} & \frac{2\sqrt{3}}{9} & \frac{4\sqrt{3}}{9} & \frac{4\sqrt{3}}{9}
\end{bmatrix}
\]  

(149.3)
Note that the stiffness matrix derived above is for pyramids of all orientations where the point directed by node 5 can be pointing in any Cartesian coordinate direction.

Evaluation of the mass matrix integral results in the following values for the mass matrix. Note that these values are only valid for the pyramid that is pointing in the $+z$ coordinate direction.

\[
T_{11} = L^3 \left( \frac{\varepsilon_{xx}}{30} + \frac{\varepsilon_{zz}}{90} \right) \quad (150.1)
\]

\[
T_{12} = L^3 \left( \frac{\varepsilon_{xx}}{60} + \frac{\varepsilon_{zz}}{180} \right) \quad (150.2)
\]

\[
T_{13} = L^3 \left( -\frac{\varepsilon_{xz}}{120} + \frac{\varepsilon_{zz}}{360} + \frac{\varepsilon_{xy}}{40} - \frac{\varepsilon_{zy}}{120} \right) \quad (150.3)
\]

\[
T_{14} = L^3 \left( -\frac{\varepsilon_{xz}}{120} - \frac{\varepsilon_{zz}}{360} + \frac{\varepsilon_{xy}}{40} + \frac{\varepsilon_{zy}}{120} \right) \quad (150.4)
\]

\[
T_{15} = \sqrt{3}L^3 \left( \frac{\varepsilon_{xx}}{240} - \frac{\varepsilon_{xz}}{192} - \frac{\varepsilon_{yy}}{960} + \frac{19\varepsilon_{xz}}{960} + \frac{\varepsilon_{yy}}{320} \right) \quad (150.5)
\]

\[
T_{16} = \sqrt{3}L^3 \left( -\frac{\varepsilon_{xz}}{320} + \frac{\varepsilon_{xx}}{480} - \frac{\varepsilon_{zz}}{320} + \frac{\varepsilon_{yz}}{960} + \frac{11\varepsilon_{xz}}{960} \right) \quad (150.6)
\]

\[
T_{17} = \sqrt{3}L^3 \left( \frac{\varepsilon_{zz}}{192} - \frac{\varepsilon_{xx}}{240} + \frac{\varepsilon_{yy}}{960} + \frac{19\varepsilon_{xz}}{320} + \frac{\varepsilon_{yy}}{960} \right) \quad (150.7)
\]
\[ T_{18} = \sqrt{3}L^3 \left( -\frac{\varepsilon_{xx}}{480} - \frac{\varepsilon_{xy}}{320} + \frac{\varepsilon_{yx}}{320} - \frac{\varepsilon_{yy}}{960} + \frac{11\varepsilon_{xz}}{960} \right) \]  
(150.8)

\[ T_{21} = L^3 \left( \frac{\varepsilon_{xx}}{60} + \frac{\varepsilon_{zz}}{180} \right) \]  
(150.9)

\[ T_{22} = L^3 \left( \frac{\varepsilon_{xx}}{30} + \frac{\varepsilon_{zz}}{90} \right) \]  
(150.10)

\[ T_{23} = L^3 \left( \frac{\varepsilon_{xx}}{120} - \frac{\varepsilon_{xz}}{360} + \frac{\varepsilon_{xy}}{40} - \frac{\varepsilon_{yz}}{120} \right) \]  
(150.11)

\[ T_{24} = L^3 \left( \frac{\varepsilon_{xx}}{120} + \frac{\varepsilon_{xz}}{360} + \frac{\varepsilon_{xy}}{40} + \frac{\varepsilon_{yz}}{120} \right) \]  
(150.12)

\[ T_{25} = \sqrt{3}L^3 \left( -\frac{\varepsilon_{zz}}{320} + \frac{\varepsilon_{xz}}{320} - \frac{\varepsilon_{yz}}{960} + \frac{\varepsilon_{xx}}{480} + \frac{11\varepsilon_{xz}}{960} \right) \]  
(150.13)

\[ T_{26} = \sqrt{3}L^3 \left( -\frac{\varepsilon_{xy}}{320} + \frac{19\varepsilon_{xz}}{960} - \frac{\varepsilon_{zz}}{192} + \frac{\varepsilon_{xx}}{240} + \frac{\varepsilon_{xy}}{960} \right) \]  
(150.14)

\[ T_{27} = \sqrt{3}L^3 \left( -\frac{\varepsilon_{xx}}{480} + \frac{\varepsilon_{zz}}{320} + \frac{\varepsilon_{xy}}{960} + \frac{11\varepsilon_{xz}}{960} + \frac{\varepsilon_{xy}}{320} \right) \]  
(150.15)

\[ T_{28} = \sqrt{3}L^3 \left( \frac{\varepsilon_{zz}}{192} + \frac{19\varepsilon_{xz}}{960} - \frac{\varepsilon_{xx}}{240} - \frac{\varepsilon_{xy}}{320} - \frac{\varepsilon_{yz}}{960} \right) \]  
(150.16)

\[ T_{31} = L^3 \left( \frac{\varepsilon_{zz}}{360} - \frac{\varepsilon_{zx}}{120} + \frac{\varepsilon_{yx}}{40} - \frac{\varepsilon_{yz}}{120} \right) \]  
(150.17)

\[ T_{32} = L^3 \left( -\frac{\varepsilon_{zz}}{360} + \frac{\varepsilon_{zx}}{120} + \frac{\varepsilon_{yx}}{40} - \frac{\varepsilon_{yz}}{120} \right) \]  
(150.18)

\[ T_{33} = L^3 \left( \frac{\varepsilon_{zz}}{90} + \frac{\varepsilon_{yy}}{30} \right) \]  
(150.19)

\[ T_{34} = L^3 \left( \frac{\varepsilon_{zz}}{180} + \frac{\varepsilon_{yy}}{60} \right) \]  
(150.20)
\[ T_{35} = \sqrt{3}L^3 \left( \frac{\varepsilon_{yx}}{320} + \frac{\varepsilon_{yy}}{240} - \frac{\varepsilon_{zz}}{192} + \frac{19\varepsilon_{yz}}{960} - \frac{\varepsilon_{zx}}{960} \right) \]  

(150.21)

\[ T_{36} = \sqrt{3}L^3 \left( \frac{19\varepsilon_{yz}}{960} + \frac{\varepsilon_{yx}}{320} - \frac{\varepsilon_{yy}}{240} + \frac{\varepsilon_{zx}}{192} \right) \]  

(150.22)

\[ T_{37} = \sqrt{3}L^3 \left( \frac{\varepsilon_{yy}}{480} - \frac{\varepsilon_{zz}}{320} + \frac{\varepsilon_{zx}}{960} - \frac{11\varepsilon_{yz}}{960} \right) \]  

(150.23)

\[ T_{38} = \sqrt{3}L^3 \left( -\frac{\varepsilon_{yx}}{320} - \frac{\varepsilon_{yy}}{480} + \frac{\varepsilon_{zx}}{320} - \frac{11\varepsilon_{yz}}{960} - \frac{\varepsilon_{zx}}{960} \right) \]  

(150.24)

\[ T_{41} = L^3 \left( -\frac{\varepsilon_{zx}}{120} - \frac{\varepsilon_{zz}}{360} + \frac{\varepsilon_{yx}}{40} + \frac{\varepsilon_{yz}}{120} \right) \]  

(150.25)

\[ T_{42} = L^3 \left( \frac{\varepsilon_{zz}}{360} + \frac{\varepsilon_{zx}}{120} + \frac{\varepsilon_{yx}}{40} + \frac{\varepsilon_{yz}}{120} \right) \]  

(150.26)

\[ T_{43} = L^3 \left( \frac{\varepsilon_{yy}}{180} + \frac{\varepsilon_{yx}}{60} \right) \]  

(150.27)

\[ T_{44} = L^3 \left( \frac{\varepsilon_{zx}}{90} + \frac{\varepsilon_{yy}}{30} \right) \]  

(150.28)

\[ T_{45} = \sqrt{3}L^3 \left( -\frac{\varepsilon_{zz}}{320} + \frac{11\varepsilon_{yz}}{960} + \frac{\varepsilon_{yx}}{320} + \frac{\varepsilon_{yy}}{480} - \frac{\varepsilon_{zx}}{960} \right) \]  

(150.29)

\[ T_{46} = \sqrt{3}L^3 \left( \frac{\varepsilon_{yx}}{320} + \frac{\varepsilon_{zz}}{320} + \frac{11\varepsilon_{yz}}{960} - \frac{\varepsilon_{yy}}{480} + \frac{\varepsilon_{zx}}{960} \right) \]  

(150.30)

\[ T_{47} = \sqrt{3}L^3 \left( -\frac{\varepsilon_{zz}}{192} - \frac{\varepsilon_{yx}}{320} + \frac{\varepsilon_{yy}}{240} + \frac{\varepsilon_{zx}}{960} + \frac{19\varepsilon_{yz}}{960} \right) \]  

(150.31)

\[ T_{48} = \sqrt{3}L^3 \left( -\frac{\varepsilon_{yx}}{320} - \frac{\varepsilon_{yy}}{240} + \frac{\varepsilon_{zx}}{192} + \frac{19\varepsilon_{yz}}{960} - \frac{\varepsilon_{zx}}{960} \right) \]  

(150.32)

\[ T_{51} = \sqrt{3}L^3 \left( \frac{\varepsilon_{yx}}{320} + \frac{\varepsilon_{xx}}{240} - \frac{\varepsilon_{zz}}{192} - \frac{\varepsilon_{yz}}{960} + \frac{19\varepsilon_{zx}}{960} \right) \]  

(150.33)
\[ T_{52} = \sqrt{3}L^3 \left( \frac{\epsilon_{yx}}{320} - \frac{\epsilon_{zz}}{320} + \frac{11\epsilon_{xz}}{960} + \frac{\epsilon_{xx}}{480} - \frac{\epsilon_{yz}}{960} \right) \] (150.34)

\[ T_{53} = \sqrt{3}L^3 \left( \frac{19\epsilon_{zy}}{960} - \frac{\epsilon_{zz}}{192} + \frac{\epsilon_{yx}}{320} + \frac{\epsilon_{yy}}{240} - \frac{\epsilon_{xx}}{960} \right) \] (150.35)

\[ T_{54} = \sqrt{3}L^3 \left( -\frac{\epsilon_{xz}}{960} + \frac{\epsilon_{xy}}{320} + \frac{\epsilon_{yy}}{480} - \frac{\epsilon_{zz}}{320} + \frac{11\epsilon_{zy}}{960} \right) \] (150.36)

\[ T_{55} = L^3 \left( \frac{3\epsilon_{xz}}{320} + \frac{\epsilon_{yx}}{320} + \frac{\epsilon_{yy}}{240} + \frac{13\epsilon_{zz}}{320} + \frac{3\epsilon_{zx}}{320} \right) \] (150.37)

\[ T_{56} = L^3 \left( \frac{\epsilon_{xz}}{160} + \frac{3\epsilon_{yz}}{320} + \frac{43\epsilon_{zz}}{1440} - \frac{\epsilon_{xy}}{320} + \frac{\epsilon_{xx}}{160} \right) \] (150.38)

\[ T_{57} = L^3 \left( \frac{3\epsilon_{xz}}{320} - \frac{\epsilon_{xx}}{240} - \frac{3\epsilon_{zx}}{320} + \frac{\epsilon_{xy}}{320} + \frac{\epsilon_{yy}}{480} \right) \] (150.39)

\[ T_{58} = L^3 \left( \frac{\epsilon_{xz}}{160} + \frac{29\epsilon_{zz}}{320} - \frac{\epsilon_{yx}}{320} - \frac{\epsilon_{yy}}{1440} + \frac{\epsilon_{xy}}{160} \right) \] (150.40)

\[ T_{61} = \sqrt{3}L^3 \left( \frac{\epsilon_{xx}}{480} - \frac{\epsilon_{zz}}{320} + \frac{\epsilon_{yz}}{960} - \frac{\epsilon_{yx}}{320} + \frac{11\epsilon_{zx}}{960} \right) \] (150.41)

\[ T_{62} = \sqrt{3}L^3 \left( \frac{\epsilon_{xx}}{240} - \frac{\epsilon_{yx}}{320} - \frac{\epsilon_{zz}}{192} + \frac{\epsilon_{yz}}{960} + \frac{19\epsilon_{zx}}{960} \right) \] (150.42)

\[ T_{63} = \sqrt{3}L^3 \left( \frac{\epsilon_{zz}}{192} + \frac{19\epsilon_{zx}}{960} + \frac{\epsilon_{xy}}{320} - \frac{\epsilon_{yy}}{240} \right) \] (150.43)
\[ T_{64} = \sqrt{3}L^3 \left( \frac{\varepsilon_{zz}}{320} + \frac{\varepsilon_{xy}}{320} + \frac{\varepsilon_{xz}}{960} + \frac{11\varepsilon_{zy}}{960} - \frac{\varepsilon_{yy}}{480} \right) \] (150.44)

\[ T_{65} = L^3 \left( \frac{\varepsilon_{xx}}{160} - \frac{\varepsilon_{yy}}{240} + \frac{\varepsilon_{xy}}{480} + \frac{\varepsilon_{xz}}{320} + \frac{3\varepsilon_{zy}}{320} \right) \] (150.45)

\[ T_{66} = L^3 \left( \frac{\varepsilon_{xz}}{320} + \frac{13\varepsilon_{zz}}{288} + \frac{\varepsilon_{xx}}{240} - \frac{3\varepsilon_{yz}}{320} + \frac{3\varepsilon_{zy}}{320} \right) \] (150.46)

\[ T_{67} = L^3 \left( \frac{\varepsilon_{xy}}{160} + \frac{29\varepsilon_{zz}}{1440} - \frac{\varepsilon_{yz}}{160} - \frac{\varepsilon_{xy}}{320} - \frac{\varepsilon_{xx}}{480} \right) \] (150.47)

\[ T_{68} = L^3 \left( \frac{\varepsilon_{yy}}{480} + \frac{3\varepsilon_{xz}}{320} + \frac{43\varepsilon_{zz}}{1440} - \frac{\varepsilon_{xy}}{320} - \frac{3\varepsilon_{yy}}{320} + \frac{3\varepsilon_{zy}}{320} \right) \] (150.48)

\[ T_{71} = \sqrt{3}L^3 \left( \frac{\varepsilon_{yx}}{320} - \frac{\varepsilon_{xx}}{240} + \frac{\varepsilon_{xz}}{192} + \frac{\varepsilon_{yz}}{960} + \frac{19\varepsilon_{zx}}{960} \right) \] (150.49)

\[ T_{72} = \sqrt{3}L^3 \left( -\frac{\varepsilon_{xx}}{480} + \frac{\varepsilon_{yz}}{320} + \frac{\varepsilon_{xy}}{960} + \frac{\varepsilon_{yx}}{320} + \frac{11\varepsilon_{zx}}{960} \right) \] (150.50)

\[ T_{73} = \sqrt{3}L^3 \left( \frac{\varepsilon_{xz}}{960} - \frac{\varepsilon_{yy}}{320} + \frac{\varepsilon_{yz}}{320} + \frac{11\varepsilon_{yy}}{480} + \frac{11\varepsilon_{zy}}{960} \right) \] (150.51)

\[ T_{74} = \sqrt{3}L^3 \left( -\frac{\varepsilon_{zz}}{192} - \frac{\varepsilon_{xy}}{320} + \frac{\varepsilon_{xz}}{960} + \frac{\varepsilon_{yy}}{240} + \frac{19\varepsilon_{zy}}{960} \right) \] (150.52)

\[ T_{75} = L^3 \left( -\frac{3\varepsilon_{xz}}{320} + \frac{\varepsilon_{yx}}{320} + \frac{\varepsilon_{yy}}{480} - \frac{\varepsilon_{xx}}{240} + \frac{3\varepsilon_{zx}}{320} \right) \] (150.53)
\[ T_{76} = L^3 \left( \frac{-\varepsilon_{xz}}{160} - \frac{\varepsilon_{xy}}{320} + \frac{\varepsilon_{yx}}{160} + \frac{29\varepsilon_{zz} + \varepsilon_{xx}}{1440} + \frac{\varepsilon_{yy}}{160} \right) \] (150.54)

\[ T_{77} = L^3 \left( \frac{-3\varepsilon_{xz}}{320} + \frac{13\varepsilon_{zz}}{288} + \frac{\varepsilon_{yy}}{240} + \frac{\varepsilon_{xy}}{320} - \frac{3\varepsilon_{x}}{320} \right) \] (150.55)

\[ T_{78} = L^3 \left( \frac{-\varepsilon_{xz}}{160} - \frac{\varepsilon_{xy}}{320} - \frac{\varepsilon_{xx}}{160} + \frac{3\varepsilon_{yy}}{320} + \frac{43\varepsilon_{zz}}{1440} \right) \] (150.56)

\[ T_{81} = \sqrt{3}L^3 \left( \frac{-\varepsilon_{yz}}{960} + \frac{\varepsilon_{zz}}{320} + \frac{11\varepsilon_{xz}}{960} + \frac{\varepsilon_{yx}}{320} - \frac{\varepsilon_{xx}}{480} \right) \] (150.57)

\[ T_{82} = \sqrt{3}L^3 \left( \frac{-\varepsilon_{xz}}{240} + \frac{\varepsilon_{yy}}{192} - \frac{\varepsilon_{yz}}{960} - \frac{\varepsilon_{yx}}{320} + \frac{19\varepsilon_{xx}}{960} \right) \] (150.58)

\[ T_{83} = \sqrt{3}L^3 \left( \frac{-\varepsilon_{yy}}{480} - \frac{\varepsilon_{xz}}{960} + \frac{\varepsilon_{zz}}{320} - \frac{\varepsilon_{xy}}{320} + \frac{11\varepsilon_{zy}}{960} \right) \] (150.59)

\[ T_{84} = \sqrt{3}L^3 \left( \frac{\varepsilon_{zz}}{192} - \frac{\varepsilon_{xy}}{320} - \frac{\varepsilon_{yy}}{240} - \frac{\varepsilon_{xz}}{960} + \frac{19\varepsilon_{zy}}{960} \right) \] (150.60)

\[ T_{85} = L^3 \left( \frac{-\varepsilon_{xz}}{160} + \frac{\varepsilon_{xx}}{480} + \frac{29\varepsilon_{zz}}{1440} + \frac{\varepsilon_{yz}}{160} - \frac{\varepsilon_{xy}}{320} \right) \] (150.61)

\[ T_{86} = L^3 \left( \frac{-3\varepsilon_{xz}}{320} - \frac{\varepsilon_{yy}}{320} + \frac{43\varepsilon_{zz}}{240} + \frac{\varepsilon_{xz}}{320} - \frac{3\varepsilon_{yz}}{320} \right) \] (150.62)
\[ T_{87} = L^3 \left( -\frac{\epsilon_{xz}}{160} - \frac{\epsilon_{zx}}{160} - \frac{\epsilon_{xy}}{320} + \frac{43\epsilon_{zz}}{1440} - \frac{3\epsilon_{yz}}{320} \right) + \left( \frac{3\epsilon_{zy}}{320} + \frac{\epsilon_{yx}}{320} + \frac{\epsilon_{xx}}{480} - \frac{\epsilon_{yy}}{240} \right) \] (150.63)

\[ T_{88} = L^3 \left( -\frac{3\epsilon_{xz}}{320} + \frac{\epsilon_{yx}}{240} + \frac{\epsilon_{zx}}{240} + \frac{3\epsilon_{xx}}{320} - \frac{3\epsilon_{xz}}{320} \right) + \left( \frac{13\epsilon_{zy}}{288} + \frac{3\epsilon_{zy}}{320} + \frac{\epsilon_{xy}}{320} - \frac{3\epsilon_{yz}}{320} \right) \] (150.64)

The mass matrix terms for pyramids pointing in the y and z coordinate directions are similarly defined, but not written for brevity.
BIBLIOGRAPHY


