ROUTE LEVEL BUS TRANSIT PASSENGER ORIGIN-DESTINATION FLOW ESTIMATION USING APC DATA: NUMERICAL AND EMPIRICAL INVESTIGATIONS

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ABSTRACT

Understanding the origin-destination (OD) flow patterns of passengers is essential to transit planning. The passenger OD flows are usually estimated by on-board surveys in the transit industry. However, using on-board surveys to estimate OD flows is time consuming and labor intensive, and can suffer from response bias. Because of increased availability of automated data collection technologies and their increased acceptance by transit agencies, boarding and alighting data are now much more available to transit authorities than in the past. In this thesis, methods to estimate bus route-level transit passenger OD flows are reviewed and tested. Boarding and alighting counts at each bus stop and base OD flows are used as inputs to the estimation methods. The estimated route-level transit passenger OD matrix provides stop-to-stop passenger flows for all possible stop pairs along the transit route. The estimation methods are illustrated on a small hypothetical transit route with a specified set of input values. The estimated OD flows are compared, and all the methods yield very similar estimated matrices. Several of the methods are also applied on a full-scale transit bus route of the Central Ohio Transit Authority (COTA) bus transit network. Boarding and alighting counts for each bus stop are obtained from COTA’s Automatic Passenger Counting (APC) system. Simulation analysis is also conducted on the COTA route. The empirical and simulated results show that OD matrices estimated by different methods are found to be very similar to each
other, and the quality of the base OD matrix, a necessary input for several methods, has a marked effect on the quality of the estimated OD matrices. The implications on the choice of the base matrix are discussed.
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CHAPTER 1

INTRODUCTION

1.1 Background

An origin-destination (OD) matrix consists of travel flows from each origin to each destination in the considered network. The OD flows in the matrices could represent flows by any mode of transportation, or even multimodal flows. This thesis considers a transit passenger OD matrix or trip table (Ben-Akiva, Macke and Hsu 1985), indicating passenger flows for each stop pair in a transit network. If only a single transit route is considered in the transit network, the OD matrix is referred to as a route-level transit passenger OD matrix. Route-level transit passenger OD matrices are important inputs to transit planning. They are required for setting headways, for evaluating alternatives (expressing and short turning), and for forecasting revenues (Ben-Akiva, Macke and Hsu 1985).

In the public transit industry, passenger OD matrices are usually estimated by conducting on-board surveys. In the surveys, questionnaires are distributed to passengers on-board transit vehicles, asking them for origin and destination information. The OD matrix is then generated from the responses. This method has several shortcomings (Ben-
Akiva, Macke and Hsu 1985). Personnel are needed to distribute and process the questionnaires to passengers on-board. Thus, the on-board survey is expensive to conduct, and requires long processing time. Also, because of limited human resources, only a small portion of the on-board passengers can be surveyed; and not all distributed questionnaires will be returned. In addition, the results may be biased. For example, passengers making short trips tend to not respond to the questions.

Transit agencies are increasingly using automated data collection (ADC) devices in their transit systems. Large quantities and varieties of spatial and temporal data can be archived. Boarding and alighting counts are among the data recorded by ADC systems. These counts provide indirect, but potentially useful information on passenger OD flows. This thesis reviews and analyzes various methods for estimating passenger OD matrices using these boarding and alighting counts.

1.2 Benefits of using boarding and alighting counts

As mentioned above, there are several shortcomings associated with the on-board survey estimation method. Using boarding and alighting counts can reduce some of these shortcomings. (Surveys can still be used in conjunction with the boarding and alighting counts in OD estimation.) Compared to the on-board survey, boarding and alighting counts are comparatively less costly to collect. These data are already being collected by automatic passenger count (APC) systems, one of the ADC systems installed on transit vehicles. Therefore, the data already exist, and only the marginal cost of processing and using the data appropriately are incurred.
Unlike on-board surveys, APC systems collect data continuously. Therefore, a much larger sample size can be obtained. The increase in sample size should theoretically reduce the sampling errors and biases. The APC data are collected in an electronic form, and the time requirements for the processing of the raw data would also be much shorter than that involved with surveys.

Using the on-board survey method, OD matrices are estimated very infrequently, typically every 5 to 10 years (Cui 2006). Any changes between two surveys can only be interpolated, and seasonal, or other temporal changes in the OD patterns cannot be captured. Given the low marginal cost of using APC data for OD estimation, the continuous nature of the data collection, and the quick processing time, passenger OD matrices can be estimated much more frequently.

1.3 Research Objectives

The goal of this thesis is to review, compare, and analyze various existing algorithms that can be used to estimate route-level OD matrices using boarding and alighting counts. Inputs, outputs, and the general logic of the algorithms are explained. The algorithms are illustrated on a simple network, and a subset of the algorithms is applied to an actual bus route. In addition simulation studies are conducted.
1.4 Thesis Organization

This thesis is divided into five chapters. The second chapter reviews eight route level transit passenger OD estimation algorithms that use boarding and alighting counts and other inputs to estimate a route-level transit passenger OD matrix. The algorithms are divided into two main categories depending on whether the inputs to the algorithms include base OD flows or not. The third chapter illustrates the algorithms presented in the second chapter by applying them to a hypothetical bus route with a set of boarding and alighting counts. In the fourth chapter, a subset of the algorithms is applied to a Central Ohio Transit Authority (COTA) bus route using COTA APC data. The results are compared and it is seen that the OD matrices estimated by different algorithms are very similar to each other. In addition, a simulation study is conducted in the fourth chapter, and it is found that the quality of the base OD matrix affects the quality of the estimated OD matrix for those methods that require a base matrix. Finally, the fifth chapter provides a summary of the comparisons and recommendations for future research.
CHAPTER 2

LITERATURE REVIEW

2.1. Overview

This literature review focuses on the problem of route-level passenger origin-destination (OD) estimation for public transit using boarding and alighting counts. The literature reviewed here focuses on transit OD estimation. Boarding and alighting counts are used as inputs. Many other articles focus on highway OD estimation or when focused on transit do not use boarding or alighting counts. These articles are not included in this literature review.

The estimated route-level passenger OD matrix indicates the passenger flows between all possible stop pairs on a transit route. The OD matrix gives stop-level passenger flows. Origin stops on the considered route may not be the starting points of passenger trips. Passengers may transfer from another route or drive from their home to this route. Similarly, destination stops on the considered route may not be the final destinations of passenger trips. Passengers may continue their transit trips by transferring to another route. An estimated passenger route-level OD matrix simply indicates the origin stop and the destination stop on the considered route.
All route-level estimation methods require terminals at both ends of a transit route. If the transit route is a loop or the transit route ends in loops, terminal stops are formed by cutting loops. Several methods to form terminal stops from loop are discussed by Furth et al. (2006). In this thesis, the routes discussed have obvious terminals at both ends. Therefore, the methods forming terminal stops are not discussed in this thesis. The inputs for these route-level estimation methods usually consist of boarding and alighting counts and a base matrix. The output is usually a route-level OD matrix. Some methods also estimate an alighting probability matrix (APM). An APM tabulates the probabilities that a passenger alights at stops downstream of the boarding stop conditional on his or her boarding at a specified stop. These probabilities characterize passenger OD patterns on the route, and they are more likely to remain stable across transit trips than are the OD matrices themselves. When coupled with boarding counts (or distributions of the boarding counts), estimated APMs are generally better suited than estimated OD flow matrices for predicting ODs in future transit trips. An APM is generated by dividing each element in the OD matrix by its row total.

Boarding and alighting counts by themselves cannot determine a unique OD matrix. That is, multiple OD matrices can produce the same set of boarding and alighting counts. Therefore, supplemental information must be used in the estimation. Based on the type of supplemental information, different methods could be classified into two categories: methods that use a base matrix or base information, and methods that do not use base information but use additional assumptions. Section 2.2 reviews the methods with base information, while section 2.3 reviews the methods without base information.
2.2. Methods with base information

The inputs to methods that use base information consist of boarding and alighting counts and a base matrix. A base matrix could come from different sources. It could be an outdated OD matrix, be generated from a small sample on-board survey, be the output of a demand model, or come from some expert knowledge. Different methods require different types of base matrix specified as their inputs. More detailed discussion about the base matrix is included in the following sub-sections.

It may seem to be a large requirement to obtain a base matrix. However, most methods allow the use of what is called the null base matrix. A null base matrix assumes every permissible stop OD pair has equal passenger flows. The issue then is how sensitive the results are to the base matrix.

In this section, six methods are discussed: the iterative proportional fitting method, the constrained generalized least square method, the constrained maximum likelihood method, the modified maximum likelihood method, the Kikuchi and Perincherry method, and the Gur and Elia method.

2.2.1 Iterative proportional fitting method

The iterative proportional fitting (IPF) method has been widely used in transportation and other fields. It has also been referred to as the bi-proportional method,
the Furness or Fratar iterative procedure, the Kruithof algorithm, or the Bregman’s balancing method (Ben-Akiva et al. 1985).

The inputs to this method are the boarding and alighting counts for every stop along the bus route and a base matrix. The output is a route level OD matrix that satisfies the given boarding and alighting counts.

In general, IPF estimates for a two-dimensional matrix are proportional to base matrix entries with a constant of proportionality for each row and each column. IPF is an iterative method. For iteration k, a row factor \( \alpha_i^k \) is determined for each origin stop \( i \) as the ratio of the observed numbers of boardings to the estimated numbers of boardings given by the OD matrix produced in the previous iteration or by the base OD matrix in the first iteration. Each cell entry is multiplied by the factor of the corresponding row. Therefore, estimated numbers of boardings will be equal to the observed numbers of boardings. However, the estimated numbers of alightings will generally not equal the observed numbers of alightings. Therefore, a column factor \( \beta_j^k \) is computed for each destination stop \( j \) as the ratio of the observed numbers of alightings to the estimated numbers of alightings obtained after multiplying the cell entries by the row factor. Each cell entry is multiplied by the factor of the corresponding column. Therefore, estimated numbers of alightings will be equal to the observed numbers of alightings. Then, the estimated numbers of boardings will generally not equal the observed numbers of boardings. A measure of difference between the estimated and observed numbers of boardings is computed. If this measure is larger than a convergence criterion, which is
provided as an input, the process begins again with calculation of a row factor in iteration
k+1. Otherwise, the estimated OD matrix is taken as the final estimation results.

By the iterative procedure described above, the IPF estimator for OD pair (i, j) is
given by (Ben-Akiva, Macke and Hsu 1985)

\[ \hat{t}_{ij} = a_i b_j t_{ij}^0 \]  \hspace{1cm} \text{(2.1)}

where \( \hat{t}_{ij} \) = the estimated passenger flow between origin stop i and destination stop j;

\( t_{ij}^0 \) = the base matrix flow between origin stop i and destination stop j;

\( a_i, b_j \) = row and column factors which are determined as the product of the factors
at all iterations. I.e. \( a_i = \prod_k a_i^k \) and \( b_j = \prod_k b_j^k \).

Based on the method by which \( a_i \) and \( b_j \) are determined, if an infinite number of
iterations is conducted, the estimated values satisfy the row and column constraints as
equation 2.2 and 2.3

\[ \sum_{j=1}^N \hat{t}_{ij} = P_i, i = 1,2,...,N \]  \hspace{1cm} \text{(2.2)}

\[ \sum_{i=1}^I \hat{t}_{ij} = Q_j, j = 1,2,...,N \]  \hspace{1cm} \text{(2.3)}

where \( P_i \) = the observed boarding count at stop i;

\( Q_j \) = the observed alighting count at stop j;
\[ N = \text{number of stops along the considered route.} \]

The IPF method is computationally straightforward (Navick and Furth 1994), but problems arise when there are non-structural zeros in the base matrix. Usually, a large number of OD pairs on a bus route have low or no flows between them. Since it is hard to observe the OD information for all travelers on all trips, a sample of passengers is often taken to form the base OD matrix used in the IPF method. OD pairs with no flow will return observations of zero flow. OD pairs with low flow will often produce observations of zero flow, as well. The analyst cannot tell the differences between pairs with no flow and pairs with low flow that are sampled as zero flow cells. Matrix cells having zero OD flows because no passengers travel between the origin and destination pairs are called structural zeros. Those resulting from the low sampling rate are called non-structural zeros. From equation 2.1, the IPF method is seen to produce estimated flows of zero for cells whose base flow is zero. Therefore, cells with non-structural zeros will lead to biased estimates.

### 2.2.2 Constrained generalized least square method

The constrained generalized least square (CGLS) method (Ben-Akiva, Macke and Hsu 1985) estimates entries in the route level OD matrix that are considered close to the estimates produced by a simple expansion of the base matrix, while satisfying the boarding and alighting counts and accounting for different levels of confidence in the base OD flows. As with the IPF method, the inputs of this method are the boarding and
alighting counts for every stop along the bus route and a base matrix, and the output is a route level OD matrix that satisfies the given boarding and alighting counts.

The underlying assumption driving the CGLS method is that the base OD flows expanded by a single factor provide unbiased estimates of the true OD flow and that the error terms are normally distributed with expected values of zero. That is,

\[ f t_{ij}^0 = t_{ij} + \epsilon_{ij}, \forall i, j \]

where \(f\) = simple expansion factor computed as the ratio of the observed total number of trips and the total number of trips in the base matrix;

\[ t_{ij}^0 = \text{the base matrix flow between origin stop i and destination stop j;} \]

\[ t_{ij} = \text{the true passenger flow between origin stop i and destination stop j;} \]

\[ \epsilon_{ij} = \text{is the error term of OD pair (i,j), and } E(\epsilon_{ij}) = 0. \]

The objective is to minimize the sum, taken across all OD pairs, of the weighted squared differences between the expanded base OD flows and the estimated OD flows, while constraining the estimated OD flows to satisfy the boarding and alighting counts. The weights applied to each difference are obtained from the variance-covariance matrix of the base OD flows. The variance-covariance matrix is assumed to be obtained from the properties of a survey instrument. It assigns higher variances to base OD flows associated with less confidence. When used to weight the squared differences between the expanded base and estimated flows, the base flows obtained with lower variance (more confidence)
produce higher weights. Using matrix notation, the CGLS method problem can be expressed as

\[
\begin{align*}
\text{Min } & (ft^0 - t)'V^{-1}(ft^0 - t) \\
\text{s.t. } & Rt = r
\end{align*}
\]

where \( f = \) simple expansion factor computed as the ratio of the observed total number of trips and the total number of trips in the base matrix;

\( t^0 = \) vector of OD flows in the base matrix;

\( t = \) vector of true OD flows with the same OD pair order as \( t^0 \);

\( V = \) variance-covariance matrix of \( t^0 \);

\( r = \) vector of linear independent boarding and alighting counts;

\( R = \) constraint incidence matrix whose elements are either 0 or 1. The number of columns in \( R \) is the same as the number of entries in vector \( t \), and the number of rows in \( R \) is the same as the number of entries in vector \( r \). The sum of elements in vector \( t \), whose corresponding value in a particular row of \( R \) equal one, is constrained by the corresponding entry in vector \( r \).

A closed-form solution of the point estimates of OD flows can be derived from the constrained minimization formulation of equation 2.5 and 2.6 (Ben-Akiva, Macke and Hsu 1985):

\[
\hat{t} = ft^0 + VR'(RVR')^{-1}(r - Rft^0)
\]

where \( \hat{t} = \) vector of estimated OD flows.
Since all the components in equation 2.7 can be easily derived from the observed
data and equation 2.7 is easy to compute, this method is easy to implement. This point
estimate is a summation of two components. The first component consists of the base OD
flow expanded by the factor \( f \). The second component corresponds to the error term \( \epsilon \) in
equation 2.4. From the structure of the point estimate, the non-structural zero problem
can be easily avoided when implementing the method. For the structural zeros, the
对应的 observations of the boarding and alighting counts are zeros because there
is no passenger flows between these origins and destinations. Thus, these OD pairs are
not selected to put in the OD flows vector \( t \). That is, the estimates \( \hat{\epsilon} \) will not include these
OD pairs, which results in zero estimates for these OD pairs. For the non-structural zeros,
which is caused by the low sample rate for the base OD matrix, the corresponding
observed boarding and alighting counts are not zeros. These OD pairs are selected to put
in the OD flow vector \( t \) even if the base OD flows are zeros. The second component of
equation 2.7 will result in a non-zero value if the observed boarding and alighting counts
are not zeros, and the estimates for those non-structural zeros are non-zeros values. Thus,
the structural zeros and non-structural zeros are differentiated. That is, this method does
not suffer from the non-structural zero problem.

Although this method does not suffer from the non-structural zero problem, it
may yield negative estimates for some OD flows (Ben-Akiva, Macke and Hsu 1985). The
negative estimates are caused by the second component in equation 2.7. The inverse and
subtraction operations could both lead to a negative result in the estimated OD flow
vector \( \hat{\epsilon} \).
2.2.3 Constrained maximum likelihood estimation method

In the constrained maximum likelihood estimation (CMLE) method, entries in the base OD matrix are assumed to be realizations of an assumed distribution. The likelihood function is formulated by multiplying the probabilities of observing the entries in the base OD matrix. By maximizing the likelihood function subject to the boarding and alighting constraints, an estimated matrix is obtained.

The inputs of this method are the boarding and alighting counts for every stop along the bus route and a base matrix. The output is a route level OD matrix that satisfies the given boarding and alighting counts.

Under the Poisson distribution assumption, the entries in the base OD are assumed to be realizations of independent Poisson distributions. The probability of observing $t_{ij}^0$ in the base matrix is then

$$P(t_{ij} = t_{ij}^0) = \frac{t_{ij}^{0}}{t_{ij}} e^{-t_{ij}}$$  \hspace{1cm}  2.8

where $t_{ij} = \text{the true passenger flow between origin stop i and destination stop j};$

$t_{ij}^0 = \text{the base matrix flow between origin stop i and destination stop j}.$
The likelihood function is

\[ L = \prod_i \prod_j \left( \frac{t_{ij}^0}{t_{ij}^!} e^{-t_{ij}} \right) \] 2.9

By maximizing the likelihood function 2.9 subject to the boarding and alighting constraints (see equations 2.2 and 2.3), the maximum-likelihood estimator is found. It is often easier to maximize the logarithm of the likelihood function than the likelihood function. The constraints can be incorporated by forming the Lagrangian.

\[ \ln L^0 = \sum_i \sum_j \left[ -t_{ij} + t_{ij}^0 \ln(t_{ij}) - \ln(t_{ij}^0!) - a_i (\sum_j t_{ij} - P_i) - b_j (\sum_i t_{ij} - Q_j) \right] \] 2.10

The first-order conditions are then

\[ \frac{\partial \ln L^0}{\partial t_{ij}} = -1 + \left( \frac{t_{ij}^0}{t_{ij}} \right) - a_i - b_j = 0 \] 2.11

and the constrained maximum-likelihood estimator is

\[ \hat{t}_{ij} = \frac{t_{ij}^0}{1 + a_i + b_j} \] 2.12

where \( \hat{t}_{ij} \) = the estimated passenger flow between origin stop i and destination stop j;

\( a_i \) and \( b_j \) = the balancing factors as well as the Lagrangian multipliers. An algorithm described in Appendix A is used to determine the values of \( a_i \) and \( b_j \).

Like the IPF method, this method also suffers from the non-structural zero problem. Based on equation 2.12, the estimated OD flows are computed as the base OD flows.
flows divided by summation of some parameters. Therefore, if the base OD flow has a zero value, which could be caused by no passengers traveling or by a low sample rate, the CMLE method will retain this zero value.

2.2.4 Modified maximum likelihood method

The modified maximum likelihood method (Cui 2006) extends the application of the maximum likelihood estimation principle used in the CMLE method by considering the observed boarding and alighting counts to be random variables, rather than constants. These observed counts are assumed to follow the same type of distribution as that used to model the base OD flows.

The inputs of this method are the boarding and alighting counts for every stop along the bus route and a base OD matrix. The output is a route level OD matrix that satisfies the given boarding and alighting counts.

Because each entry is assumed to follow an independent Poisson distribution, the boarding and alighting values $P_i$ and $Q_j$ are also Poisson distributed with mean value of $t_i = \sum_{j=1}^{N} t_{ij}, i = 1,2, ..., N$ and $t_j = \sum_{i=1}^{J} t_{ij}, j = 1,2, ..., N$ respectively. As in the original CMLE method, the entries in the base matrix are assumed be realizations of independent Poisson probability distributions. Cui (2006) proposed that the objective function could be written as

$$L = \prod_i \prod_j \left( \frac{(t_{ij}p_{ij})^{t_{ij}}}{t_{ij}!} e^{-(t_{ij}p_{ij})} \right) \prod_i \left( \frac{t_{ij}p_i}{p_i!} e^{-t_i} \right) \prod_j \left( \frac{t_{ij}q_j}{q_j!} e^{-t_j} \right)$$  \hspace{1cm} (2.13)
where \( t_{ij} \) = the true passenger flow between origin stop i and destination stop j;

\[ p_{ij} = \text{the proportion of passenger flow between stop i and stop j captured by the base matrix}; \]

\[ t_{ij}^0 = \text{the base matrix flow between origin stop i and destination stop j}; \]

By determining the values of \( t_{ij} \) and \( p_{ij} \) that maximize this objective function, estimated OD flows are obtained.

The likelihood function should be used with caution. Based on the formulation of the method, \( t_{ij}, P_i \) and \( Q_j \) are assumed to be independent Poisson variables. To ensure this independence, the base matrix, boarding counts and alighting counts should come from different sources. For example, the base matrix could be generated from historical information, the boarding counts could come from observation on certain bus trips, while the alighting counts could come from observation on other bus trips not overlapping with the bus trips that boarding counts used. Cui (2006) appears to have used the same data source to develop his base matrix, and his boarding and alighting counts. However, he notes that there should be little impact on the point estimates of the OD flows. The independent data sources issue could likely be ignored if only point estimates are considered.
2.2.5 Kikuchi and Perincherry method

Kikuchi and Perincherry (1992) present a method that incorporates the information on OD flows that the experts may be able to provide. Usually, the OD flow information provided by the experts is in the form of ranges. For example, 10 to 30 percent of passengers boarding at stop A travel to stop B, or less than 30 percent of the passengers boarding at stop A should travel to stop B. The effective use of such information requires a model that can incorporate approximate base OD flows for some OD pairs in addition to boarding and alighting counts. In this method, it is assumed that the true value for any OD pair is equally likely to fall within the ranges formed by the lower and upper bounds.

The inputs of this method are the boarding and alighting counts for every stop along the bus route and a base matrix consisting of ranges on all OD pairs. These ranges may be specified by analysts who are familiar with the OD pattern or derived from other OD estimation methods. If only a subset of the ranges is available from these sources above, the remaining ranges can be derived from the on and off counts with a method presented in Kikuchi and Perincherry (1992). The method deriving the ranges from the boarding and alighting counts is presented in Appendix B. The output is a route level OD matrix that satisfies the given boarding and alighting counts with OD flow falling within the input ranges.

In this method, it is assumed that the true value for any OD pair is equally likely to fall anywhere within the range of values formed by the input upper and lower bounds.
Thus, the expected error, which is the expected difference between the true and estimated value, is lower when the estimated OD flow is closer to the midpoint of the input bounds. However, the boarding and alighting constraints must also be satisfied, so the optimal solution will not necessarily fall at the midpoint.

For cell (i,j) in the matrix, the lower bound \( s_{1(i,j)} \) and upper bound \( s_{2(i,j)} \) are given as inputs. Considering \( c_{ij} \) as the distance of the true OD flow from the lower bound

\[
t_{ij} = s_{1(i,j)} + c_{ij}, \quad \text{2.14}
\]

\[
0 \leq c_{ij} \leq z_{ij} = s_{2(i,j)} - s_{1(i,j)} \quad \text{2.15}
\]

where \( s_{1(i,j)} = \) lower bound of the range for OD (i, j);

\( s_{2(i,j)} = \) upper bound of the range for OD (i, j);

\( z_{ij} = \) range for OD (i, j);

\( c_{ij} = \) distance of the true OD flow from the lower bound of the range for OD (i, j);

Considering \( h_{ij} \) as a proxy measure of closeness of \( t_{ij} \) to the midpoint of the specified, \( h_{ij} \) is computed as

\[
h_{ij} = \min \left\{ \frac{2\epsilon_{ij}}{z_{ij}}, 2 - \frac{2\epsilon_{ij}}{z_{ij}} \right\} \quad \text{2.16}
\]

Based on 2.16, the closer \( t_{ij} \) is to the midpoint of the feasible range, the larger the value of \( h_{ij} \). The method assumes that the objective is to maximize the sum of the \( h_{ij} \) values, resulting a maximization program
max \sum h_{ij} \quad 2.17 \\
\text{s.t. } \sum_i s_{1(i,j)} + c_{ij} = Q_j, \quad 2.18 \\
\sum_j s_{1(i,j)} + c_{ij} = P_i, \quad 2.19 \\
h_{ij} \geq h_x, \quad 2.20 \\
h_{ij}, c_{ij} \geq 0 \quad 2.21 \\

where \( s_{1(i,j)}, c_{ij}, Q_j, P_i, \) and \( h_{ij} \) are defined above;

\( h_x = \) a threshold value on all \( h_{ij} \).

By determining the values of \( c_{ij} \) that maximize this objective function, estimated OD flows are obtained by

\[ \hat{t}_{ij} = s_{1(i,j)} + \hat{c}_{ij}, \quad 2.22 \]

where \( s_{1(i,j)} \) and \( \hat{t}_{ij} \) are defined above;

\( \hat{c}_{ij} = \) estimated distance of the true OD flow from the lower bound of the range.

The final OD matrix is estimated iteratively. First, an OD matrix is estimated by solving the constrained maximization problem, which is a linear program, using initial OD bounds and the boarding and alighting counts. In the application on a hypothetical example in this thesis, the constrained minimization program is solved using a function in MATLAB. The approach is summarized in Appendix C. The estimated OD values are then inspected by the analyst, and the pairs that appear unrealistic are identified. The unrealistic estimates could result from unrealistic initial OD bounds. The bounds for these unrealistic OD estimates would then be updated. A new matrix is produced by
solving the constrained maximization problem with the boarding and alighting counts and updated OD bounds. This procedure is repeated until a satisfactory OD matrix is generated.

### 2.2.6 Gur and Elia method

Gur and Elia (1997) present a method that incorporates information theory in the estimation process.

The inputs to this method are the numbers of passengers boarding and alighting at each stop on each bus trip along the route during a specific time period and a base APM. The output is a route-level APM for the specific time period.

The APM is estimated by formulating and solving a constrained minimization program.

\[
\begin{align*}
\text{Min } Z(X) &= A(X) + \alpha B(X) \\
\text{s.t. } \sum_t y_j(t) &= \sum_t q_j(t) \\
\sum_j x_{ij} &= 1.0, \text{for } \forall i \\
\text{if } f_{ij} > 0, x_{ij} > 0; \text{if } f_{ij} = 0, x_{ij} = 0
\end{align*}
\]

where \( A(X) = \sum_j \sum_t [q_j(t) - y_j(t)]^2 \)

\[
y_j(t) = \sum_i [p_i(t)x_{ij}];
\]

\[
B(X) = \beta \sum_i \sum_j \left\{ p_i x_{ij} \left[ \log \left( \frac{x_{ij}}{f_{ij}} \right) - 1 \right] \right\}
\]

\( f_{ij} \) = base APM entry for OD pair (i, j);
\(x_{ij} = \text{APM entry to be estimated for OD pair } (i, j);\)

\(t = \text{bus trip number; }\)

\(p_i(t) = \text{observed boardings at stop } i \text{ of bus trip } t;\)

\(q_j(t) = \text{observed alightings at stop } j \text{ of bus trip } t;\)

\(y_j(t) = \text{estimated alightings at stop } j \text{ of bus trip } t;\)

\(\alpha, \beta = \text{parameters.}\)

The objective function 2.23 is a weighted sum of two components. The first component 2.27 is the sum of the squared differences between the observed and estimated number of the run-level alightings over all stops and all bus runs during the considered time period. The second component 2.28 is a measure of divergence between the base and estimated APM derived from the concept of information content. The measure is a function of observed numbers of passengers boarding, as well as the base APM.

The constraints of this minimization program require that the estimated and observed total numbers of alightings at each stop, when aggregated across the time period considered, are equal (constraint 2.24), and that the summation of the probabilities across any row in an APM equals one (constraint 2.25). Equations 2.26 specify the structure of
the estimated APM. The estimated APM entries have non-zero values only if the corresponding entries in the base APM have non-zero values.

By determining the values of the $x_{ij}$ that minimize equation 2.23 under the constraints 2.24, 2.25, and 2.26, an estimated APM is obtained. In the application on a hypothetical example in this thesis, the constrained minimization program is solved using a function in MATLAB. A summery is provided in Appendix C.

2.3 Methods without base information

Unlike methods that use base information, the inputs for methods in this section consist of boarding and alighting counts and assumptions that provide supplemental structure to the problem. Usually, these assumptions relate to the travel behavior of the on-board passenger that leads to implication on the alighting probabilities.

In this section, two methods, the Tsygalnitsky method and Li and Cassidy’s method, are reviewed.

2.3.1 Tsygalnitsky method

The Tsygalnitsky method (Tsygalnitsky 1977) is also referred to as the intervening-opportunity method (Ben-Akiva, Macke and Hsu 1985) and the fluid analogy method (Simon and Furth 1985).
The inputs of this method are the numbers of passengers boarding and alighting at each stop along a bus route for bus runs during a specific time period and a threshold trip distance. This threshold trip distance is a distance under which a passenger is assumed not to ride the bus. The output of the Tsygalnitsky method is a route-level OD matrix, aggregated across the time period considered, that satisfies the boarding and alighting counts for every stop.

The Tsygalnitsky method assumes that a passenger will not alight if he or she has not traveled the threshold distance specified as input, and that every passenger who has traveled at least the threshold distance is equally likely to alight at a given stop. This assumption leads to a straightforward computation of a unique OD matrix. For each bus run, a bus run-specific OD matrix is estimated by using the run-level boarding and alighting counts with the algorithm presented in Appendix D. The computations are performed for each bus run in the specified period, and the bus run-specific results are then aggregated into a single OD matrix for the period.

The estimation results by the Tsygalnitsky method are identical to the estimation results by the IPF method when the IPF method using the null base OD matrix and the threshold distance in the Tsygalnitsky method is set to be one stop. Navick and Furth (1994) prove the conclusion analytically, and the hypothetical example and the empirical results in Chapter 3 and 4 also confirmed the conclusion.
2.3.2 Li and Cassidy method

Li and Cassidy (2007) modified the equally likely alighting assumption made in the Tsygalnitsky method (Tsygalnitsky, 1977) to estimate a route-level OD matrix. As in the Tsygalnitsky method, a threshold trip distance and the numbers of passengers boarding and alighting at each stop along the route for each bus run during a specific time period are used as inputs to the Li and Cassidy method. Li and Cassidy’s method differs from the original Tsygalnitsky method by requiring bus stops to be classified as either major stops or minor stops. Major stops are considered to be stops serving activity centers, such as commuter train stations or large business centers, along the route. Other stops are treated as minor stops.

Like the Tsygalnitsky method, the output of the Li and Cassidy method is a route-level OD matrix that satisfies on and off counts for every stop for each bus run. The authors also propose producing a conditional APM based on an OD matrix that is aggregated across the individual bus run OD matrices.

Instead of using the equally likely to alight assumption, Li and Cassidy assign different alighting probabilities to the eligible passengers based on the type of their boarding stop. For a stop $s$, an onboard passenger $a$ who boarded at a major stop is randomly selected. The probability that this passenger alights at stop $s$ is denoted $p_{asm}$. An onboard passenger $b$ who boarded at a minor stop is randomly selected. The probability that this passenger alights at stop $s$ is denoted $p_{bsm}$. For stop $s$, the ratio of two alighting probability is assumed as
where $\alpha \in (0,1)$, and it is assumed that $\alpha = \alpha_a$ for all stops that are considered major stops; and $\alpha = \alpha_b$ for all stops that are considered minor stops. That is, at a given stop $s$, the ratio of $p_{as}$ to $p_{bs}$ is assumed to be constant for any given major alighting stop and constant, but possibly equal to a different value, for any minor alighting stop. For example, Li and Cassidy (2007) have calibrated the values of $\alpha_a$ and $\alpha_b$ for a bus route in California’s San Francisco Bay Area, and the most suitable values for the observed data are $\alpha_a = 0.1$ and $\alpha_b = 0.3$.

Based on this assumption, for a given stop, the expectation of the number of alighting passengers who boarded at major stops can be derived as

\[
E(n_a) = \frac{(1-\alpha)N_a}{(1-\alpha)N_a + \alpha N_b} n
\]

where $N_a = \text{eligible onboard passengers who boarded at major stops}$;

$N_b = \text{eligible onboard passengers who boarded at minor stops}$;

$n = \text{number of alighting passengers}$.

This expectation of the number of alighting passengers who boarded at major stops $E(n_a)$ is used as a point estimate $\widehat{n}_a$ of the number of alighting passengers who boarded at upstream major stops, and the rest of the alighting passengers are assumed to board at minor stops. That is, $\widehat{n}_a = E(n_a)$, and $\widehat{n}_b = n - \widehat{n}_a$. Then $\widehat{n}_a$ is split...
proportionally for each of the major stops upstream, such that the origin contributing most to $N_a$ gives that greatest contribution to $\bar{n}_a$. Similarly, $\bar{n}_b$ is split in the same fashion from all minor stops upstream.

Like the Tsygalnitsky method, the computations are performed for each bus run in the specified period, and the bus run-specific results are then aggregated into a single OD matrix for the period. For a given set of alpha values and threshold distance, a bus run-specific OD matrix is estimated by using the run-level boarding and alighting counts with the algorithm presented in Appendix E. After all the bus runs in the considered time period are considered, the estimated run-level OD matrices are averaged into a single OD matrix for the period, and a corresponding APM is calculated. That is, for the given set of alpha values and threshold distance, the observed boarding and alighting counts will result in a corresponding APM. A distance weighted measure is then computed using the observed boarding and alighting counts and estimated APM. That is, a given set of alpha values and threshold distance will have a corresponding distance weight measure. The $\alpha_a$ and $\alpha_b$ are jointly selected through all possible values, and the ones have the lowest value of the distance weight measure is chosen as the best fit values. The corresponding APM are chosen as estimation results. A detailed calculation method is presented in Appendix F.
CHAPTER 3

ILLUSTRATION OF METHODS USING SIMPLE EXAMPLE ROUTE

3.1 General approach to estimate a route-level bus OD matrix

The method to be chosen to estimate a route-level OD matrix will depend on the type and quality of data available. However, there is a general approach which will be described in this section. This general approach assumes that a set of boarding and alighting counts for all stops are available. The general approach consists of the following two steps:

(1) Data preparation: Process the raw boarding and alighting counts, which can be provided by an Automatic Passenger Counter (APC) system, to obtain equal total boarding and total alighting counts for each bus trip. Summarize other data sources to obtain a base OD matrix if required. A base OD matrix can be obtained from data provided by an Automatic Fare Collection (AFC) system, a survey of sampled passenger trips with known boarding and alighting stops, or outputs of some transportation demand model. This first step is the most difficult to generalize, since it is based on available data (Cui, 2006). In Section 3.2, the major factors to be considered are discussed. More detailed discussion in the context of empirical study of COTA bus route is presented in Chapter 4.
(2) Method implementation: Combine the boarding and alighting counts and base OD matrix using a method to produce a single route OD matrix for the bus route-direction considered. This step is discussed in Section 3.3.

Although the general procedure is similar, different methods may require different input data and generate different types of outputs. For example, the base OD matrix could consist of point value entries in some method or range entries in some other method. The output could be point estimation of the OD flows or an alighting probability matrix.

3.2 Data preparation

There are three key components in preparing data for OD estimation: route structure, boarding and alighting counts, and base OD matrix. A hypothetical example will be presented to illustrate these components.

The route structure describes the physical layout and the Automated Data Collection (ADC) systems installed on the considered route. The elements contained in the route structure usually do not change within a short term, for example one year. The key elements of the physical layout include:

- The direction of the considered route;
- The number of stops on this route;
- The distances between stop pairs;
- The land use characteristics of each stop.
The ADC system will provide data used to estimate an OD matrix for the considered route. If there is no ADC system installed, data required in the OD estimation methods could come from manual collection. Three ADC systems usually considered are: APC, AFC, and Automatic Vehicle Location (AVL) systems (Cui, 2006).

To illustrate the components described above, a route with five stops is presented in Figure 3.1. The buses on this route are assumed to travel from stop A to stop E. The route length is 8 km, and the stops are evenly distributed on the route. That is, the distances between all consecutive stops are equal to 2 km. Stop A and stop C are located at two business centers on this route, and the others are located at residential areas. It is assumed that an APC system is installed for all the buses running on this route so that the boarding and alighting counts for all the buses running on this route are available, and an AVL system is installed so that the stops where boarding and alighting counts are collected are known.

![Figure 3.1: Route structure of the hypothetical example](image)

Two bus trips are considered in this example. The data assumed to be generated from APC data are presented in Table 3.1

Here, it is assumed that there are no errors in APC records and the loads are zero before the bus enters the first stop and after it leaves the last stop. Thus, the total number
of boardings for a given bus trip equals the total number of alightings across all stops for the bus trip. In actual APC data sets, counts are not always balanced in this way. An imbalance between the total number of boardings and the total number of alightings could result from measurement errors, where the measurements are obtained from an APC system or from manual counts. An imbalance could also arise from the fact that some passengers board near the end of a bus trip to remain on the bus for the reverse trip to save some walking or to secure a seat. Thus, the load at the beginning or end will not equal to zero, thus, the boarding and alighting counts will not be balanced. To balance the boarding and alighting counts, data cleaning rules would typically be required in practice. This issue will be discussed further in Chapter 4.

Table 3-1: Boarding and alighting counts at stops assumed for hypothetical example

<table>
<thead>
<tr>
<th>Stop Number</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trip 1 boarding</td>
<td>15</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Trip 1 alighting</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
</tr>
<tr>
<td>Trip 2 boarding</td>
<td>15</td>
<td>9</td>
<td>8</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Trip 2 alighting</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>17</td>
<td>19</td>
</tr>
<tr>
<td>Total boarding</td>
<td>30</td>
<td>19</td>
<td>15</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td>Total alighting</td>
<td>0</td>
<td>2</td>
<td>9</td>
<td>34</td>
<td>35</td>
</tr>
</tbody>
</table>

Base matrices can be generated from the AFC records, results of surveys, outputs of some transportation demand model, or expert opinion. The base matrix represents a preliminary estimate of the OD pattern of the considered route.

In the hypothetical example, it assumed that there is no information available to form the base OD matrix. A null base OD matrix can then be used for methods that
require a base matrix as input. Recall from Chapter 2 that a null base matrix means that a random passenger is equally likely to travel between any OD pair considered. For this example, the null base is given in Table 3-2.

Table 3-2: Null base OD matrix assumed for hypothetical example

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

3.3 Illustration of methods

Based on the given data, we could implement all the methods discussed in Chapter 2 on this hypothetical example. The methods implemented in this section are the IPF method, the CGLS method, the CMLE method, the modified MLE method, the Kikuchi and Perincherry method, the Gur and Elia method, the Tsygalnitsky method, and the Li and Cassidy method.

Since there are two bus trips in the given hypothetical example, there could be two ways to estimate an OD matrix. One could aggregate the boarding and alighting counts across all trips for each stop on the considered route, and then apply an estimation method to estimate an OD matrix. Or, one could apply the estimation method to the boarding and alighting counts on each bus trip before aggregating, and then aggregate the
estimated trip-level OD matrices into a single OD matrix. The second procedure, i.e., estimating ODs first at the bus trip level and then aggregating, is used here.

The outputs of the different methods may vary. Some of the methods will estimate an OD matrix, while some methods will yield an APM. To compare the estimated results from different methods, the estimated APM is transformed into an OD matrix by coupling it with the observed boarding counts.

### 3.3.1 IPF method

As discussed in Chapter 2, the inputs of the IPF method are the boarding and alighting counts for every stop along the bus route, and a base OD matrix. The estimated OD matrix can be obtained by following the procedure described in the section 2.2.1.

For the first bus trip, the boarding and alighting counts and the base OD matrix are obtained from Table 3-1 and 3-2. The IPF method is conducted iteratively. For the first iteration, row factors $\alpha_i$'s are determined as the ratio of the observed numbers of boardings to the estimated numbers of boardings given by the base OD matrix. The values for $\alpha_i$ are shown in Table 3-3. The entries in the stop-to-stop cells are those in the base OD matrix.
Table 3-3: Row factors in the first iteration using the IPF method for bus trip one of the hypothetical example

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
<th>Obs. Boarding</th>
<th>( a_i^1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>15</td>
<td>15/4 = 3.75</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>10</td>
<td>10/3 = 3.33</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>7/2 = 3.5</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>8/1 = 8</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then, each call entry in the base OD matrix is multiplied by the row factor of the corresponding row computed in Table 3-3. For example in the cell corresponding to trips from stop A to stop B, one obtains 3.75 * 1 = 3.75. An estimated OD matrix after using the row factors is obtained as Table 3-4.

Table 3-4: The estimated OD matrix after using the row factors in the first iteration

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
<th>Obs. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>3.75</td>
<td>3.75</td>
<td>3.75</td>
<td>3.75</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3.50</td>
<td>3.50</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The results in Table 3-4 are then used to compute column factors \( b_j^1 \) as the ratio of the observed numbers of alightings to the estimated numbers of alightings. The values for \( b_j^1 \) are shown in Table 3-5.
Table 3-5: Column factors in the first iteration using the IPF method for bus trip one of the hypothetical example

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>3.75</td>
<td>3.75</td>
<td>3.75</td>
<td>3.75</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>3.33</td>
<td>3.33</td>
<td>3.33</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3.50</td>
<td>3.50</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Est. Alighting</th>
<th>Obs. Alighting</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3.75</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>7.08</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>10.58</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>18.58</td>
<td>16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>bj</th>
<th>0 1/3.75 = 0.27</th>
<th>6/7.08 = 0.85</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>17/10.58 = 1.61</td>
<td>16/18.58 = 0.86</td>
</tr>
</tbody>
</table>

Then, the estimated OD matrix of the first iteration is obtained by multiplying each entry by the column factor of the corresponding column computed in Table 3-5. For stop A to stop B, one obtains 3.75 * 0.27 = 1.00. The estimated OD matrix of the first iteration, which is shown in Table 3-6, is then used to check the convergence by computing differences between the estimated boardings and observed boardings. The difference for each stop is calculated as

\[ dif_i^k = \frac{|\hat{p}_i^k - p_i|}{p_i} \]

where \( dif_i^k \) is the difference of stop i in iteration k;

\( \hat{p}_i^k \) is the estimated number of boardings at stop i in iteration k;

\( p_i \) is the observed number of boardings at stop;

The maximum difference is found for the considered route, and if this maximum difference is less than a criterion value, which is given as an input, the IPF method stops.
at this iteration, and the estimated OD matrix at this iteration is taken as final estimated OD matrix. Otherwise, the next iteration is conducted.

**Table 3-6: The estimated OD matrix after first iteration**

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>1.00</td>
<td>3.18</td>
<td>6.03</td>
<td>3.23</td>
<td>13.43</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>2.82</td>
<td>5.35</td>
<td>2.87</td>
<td>11.04</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5.62</td>
<td>3.01</td>
<td>8.64</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6.89</td>
<td>6.89</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0</td>
<td>1.00</td>
<td>6.00</td>
<td>17.00</td>
<td>16.00</td>
<td></td>
</tr>
</tbody>
</table>

The differences after the first iteration are calculated in Table 3-7. The criterion is set to be 0.001 for this example. As the maximum difference is 0.23, which is greater than 0.001, the next iteration is conducted.

**Table 3-7: The differences for all stops after first iteration**

<table>
<thead>
<tr>
<th></th>
<th>Estimated boarding</th>
<th>Observed boarding</th>
<th>$d_{i}^{1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>13.43</td>
<td>15</td>
<td>0.10</td>
</tr>
<tr>
<td>Stop B</td>
<td>11.04</td>
<td>10</td>
<td>0.10</td>
</tr>
<tr>
<td>Stop C</td>
<td>8.64</td>
<td>7</td>
<td>0.23</td>
</tr>
<tr>
<td>Stop D</td>
<td>6.89</td>
<td>8</td>
<td>0.14</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

After eight iterations, the convergence criterion is met and the estimated OD matrix for bus trip one is obtained and shown in Table 3-8. Similarly, the estimated OD matrix for bus trip 2 is estimated, and the estimated OD matrix for the sum of the two bus trips is obtained by adding up the two estimated OD matrices. The result is shown in Table 3-9.
### Table 3-8: Estimated OD matrix for bus trip 1 using the IPF method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>1.00</td>
<td>3.50</td>
<td>7.14</td>
<td>3.36</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>2.50</td>
<td>5.10</td>
<td>2.40</td>
<td>10.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>4.76</td>
<td>2.24</td>
<td>7.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
<td>8.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>1.00</td>
<td>6.00</td>
<td>17.00</td>
<td>16.00</td>
<td>40.00</td>
</tr>
</tbody>
</table>

### Table 3-9: Estimated OD matrix for the sum of bus trips using the IPF method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>2.00</td>
<td>5.33</td>
<td>14.53</td>
<td>8.14</td>
<td>30.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>3.67</td>
<td>9.85</td>
<td>5.48</td>
<td>19.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>9.62</td>
<td>5.38</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>16.00</td>
<td>16.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>2.00</td>
<td>9.00</td>
<td>34.00</td>
<td>35.00</td>
<td>80.00</td>
</tr>
</tbody>
</table>

### 3.3.2 CGLS method

The CGLS method requires the same type of inputs as the IPF method. The estimated OD matrix can be obtained by following the procedure described in the section 2.2.2.

To conduct the CGLS method, the base OD matrix, shown in Table 3-2, is arranged into vector form $t^0$ as shown in equation 3.2. This vector contains all possible OD pairs in the base OD matrix. Similarly, the true OD matrix, which is to be estimated, and estimated OD matrix are also arranged, in the same order as $t^0$, into vector form $t$.
and $\hat{t}$, respectively. That is, the first entry in $t^0$, for example, indicates the same OD pair as the first entry in $t$ and $\hat{t}$.

$t^0 = [1 1 1 1 1 1 1 1 1 1]'; \tag{3.2}$

Boarding and alighting counts are also arranged into a vector $r$, shown in equation 3.3. $r$ is a vector of linearly independent boarding and alighting counts. Since the total number of boardings equals the total number of alightings, the alighting count for stop E is excluded to obtain the linear independence.

$r = [15 10 7 8 1 6 17]'; \tag{3.3}$

An incidence matrix $R$ is formed by indicating the relationship between the true OD flow vector $t$ and boarding and alighting vector $r$, shown in equation 3.4. The elements in the incidence matrix are either 0 or 1. Each row in the incidence matrix has the same length and order as vector $t$ in terms of the number of OD pairs. For the first row in the incidence matrix, for example, the elements with value of 1 indicate the OD pairs whose summation should equal the first value in the boarding and alighting vector $r$.

$$R = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} \tag{3.4}$$

The estimated OD flows using the CGLS method are computed using equation 2.5.
\[ \hat{t} = f \, t^0 + V R'(RR')^{-1} (r - Rf t^0) \]

The single expansion factor \( f \) is computed as the ratio of observed total number of trips and total number of trips in the base matrix. Thus, for this example, the single expansion factor is computed in equation 3.5.

\[
f = \frac{(15+10+7+8)}{(1+1+1+1+1+1+1+1+1+1)} = \frac{40}{10} = 4 \quad 3.5
\]

The variance-covariance matrix is assumed to be obtained from the properties of a survey instrument. It assigns higher variances to base OD flows associated with less confidence. For this example, it is assumed that the diagonal elements of the variance-covariance matrix equal one, and the off-diagonal elements equal zero.

With all the variables given above, the estimated OD flows are computed using (2.5).

\[
\hat{t} = [1.00 \quad 3.67 \quad 6.67 \quad 3.67 \quad 2.33 \quad 5.33 \quad 2.33 \quad 5.00 \quad 2.00 \quad 8.00];
\]

The estimated OD flows in vector form are put back into the matrix form, and the results are presented in Table 3-10.

Similarly, the estimated OD matrix for bus trip 2 is estimated, and the estimated OD matrix for the sum of the two bus trips is obtained by adding up the two estimated OD matrices. The result is shown in Table 3-11.
### Table 3-10: Estimated OD matrix for the first bus trip using the CGLS method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>1.00</td>
<td>3.67</td>
<td>6.67</td>
<td>3.67</td>
<td>15.01</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>2.33</td>
<td>5.33</td>
<td>2.33</td>
<td>9.99</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>5.00</td>
<td>2.00</td>
<td>7.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
<td>8.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>1.00</td>
<td>6.00</td>
<td>17.00</td>
<td>16.00</td>
<td>40.00</td>
</tr>
</tbody>
</table>

### Table 3-11: Estimated OD matrix using the CGLS method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>2.00</td>
<td>6.00</td>
<td>13.50</td>
<td>8.50</td>
<td>30.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>3.00</td>
<td>10.50</td>
<td>5.50</td>
<td>19.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>10.00</td>
<td>5.00</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>16.00</td>
<td>16.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>2.00</td>
<td>9.00</td>
<td>34.00</td>
<td>35.00</td>
<td>80.00</td>
</tr>
</tbody>
</table>

### 3.3.3 CMLE method

The CMLE method requires the same type of inputs as the IPF method and an assumption about underlying distribution of each entry in the OD matrix. In this implementation, Poisson distributions are used. It is assumed that each entry in the OD matrix has a Poisson distribution and is independent from other entries in the OD matrix. As described in equation 2.8, the entries in the base OD matrix are assumed to be realizations of independent Poisson distributions, and the product of probabilities of observing the base OD entries are taken as the likelihood function. Based on equation 2.9, the likelihood function for the example used in this section is
where $t_{ij}$ = the true passenger flow between origin stop i and destination stop j;

$t_{ij}^0$ = the base passenger flow between origin stop i and destination stop j, and $t_{ij}^0 = 1$ for all possible OD pairs because of the null base OD matrix.

A constrained maximization problem is considered using the logarithm of this likelihood function as the objective function. The Lagrangian is formed as equation 3.7

$L^0 = \sum_i \sum_j \left[ -t_{ij} + \ln(t_{ij}) \right] - a_i \left( \sum_j t_{ij} - P_i \right) - b_j \left( \sum_i t_{ij} - Q_j \right)  \quad 3.7$

where $P_i$ = the observed boarding count at stop i;

$Q_j$ = the observed alighting count at stop j;

$a_i$ and $b_j$ are the Lagrangian multipliers.

The first order conditions, which are obtained by setting the derivative of the Lagrangian with respect to the parameters (the mean OD flow $t_{ij}$) and the Lagrangian multipliers equal to zero, are presented in equation 3.8. By solving this system of equations, mean OD flows $t_{ij}$, $a_i$ and $b_j$ are determined such that the constrained likelihood function is maximized.
\[
\frac{\partial L}{\partial t_{ij}} = -1 + \frac{1}{t_{ij}} - a_i - b_j = 0
\]
\[
\frac{\partial L}{\partial a_i} = \sum_j t_{ij} - P_i = 0
\]
\[
\frac{\partial L}{\partial b_j} = \sum_i t_{ij} - Q_j = 0
\]

3.8

For this example, there are 10 unknown OD flows and 8 Lagrangian multipliers. That is, there are 18 unknowns in the system of equations. There are 18 linearly independent equations in 3.8. Therefore, there will be unique solution for 3.8. However, it is hard to solve the system of equations even for the example used in this section. To solve this system of equations, an iterative algorithm is developed and presented in Appendix A. Using this algorithm, the system of equations is solved, and \(t_{ij}\)s are estimated. In Table 3-12, the estimated mean OD flows, the \(a_i\)’s, and the \(b_j\)’s are presented.

Table 3-12: Estimated OD matrix for the first bus trip using the CMLE method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
<th>(a_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>1.00</td>
<td>3.33</td>
<td>7.59</td>
<td>3.08</td>
<td>15.00</td>
<td>-0.9914</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>2.67</td>
<td>4.84</td>
<td>2.50</td>
<td>10.00</td>
<td>-0.9163</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>4.57</td>
<td>2.43</td>
<td>7.00</td>
<td>-0.9045</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
<td>8.00</td>
<td>-1.1916</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>1.00</td>
<td>6.00</td>
<td>17.00</td>
<td>16.00</td>
<td>40.00</td>
<td>0.9914</td>
</tr>
</tbody>
</table>

Similarly, the estimated OD matrix for bus trip 2 is estimated, and the estimated OD matrix for the sum of the two bus trips is obtained by adding up the two estimated OD matrices. The result is shown in Table 3-13.
Table 3-13: Estimated OD matrix using the CMLE method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>2.00</td>
<td>4.94</td>
<td>15.43</td>
<td>7.63</td>
<td>30.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>4.06</td>
<td>9.29</td>
<td>5.66</td>
<td>19.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>9.29</td>
<td>5.71</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>16.00</td>
<td>16.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>2.00</td>
<td>9.00</td>
<td>34.00</td>
<td>35.00</td>
<td>80.00</td>
</tr>
</tbody>
</table>

3.3.4 Modified MLE method

The modified MLE method includes the relationships between the sum of the OD flows and the boarding and alighting counts into the objective function. The modified MLE method requires the same type of inputs as the IPF method, and an assumption about the underlying distribution of each entry in the OD matrix. Like the CMLE method, Poisson distributions are used in this implementation. As described in equation 2.13, the entries in the base OD matrix are assumed to be realizations of independent Poisson distributions, and the observed boarding and alighting counts are also assumed to be realizations of independent Poisson distributions. The product of probabilities of observing the base OD matrix entries and the observed boarding and alighting counts are taken as the likelihood function. The likelihood function for the example used in this section is

\[ L = \prod_i \prod_j \left( \frac{(t_{ij}p_{ij})^{q_{ij}}}{e^{-(t_{ij}p_{ij})}} \right) \prod_i \left( \frac{t_i^{q_i}}{p_i} e^{-t_i} \right) \prod_j \left( \frac{t_j^{q_j}}{q_j} e^{-t_j} \right) = \]

\[ \prod_i \prod_j \left( (t_{ij}p_{ij})e^{-(t_{ij}p_{ij})} \right) \prod_i \left( t_i^{q_i} \frac{p_i}{p_i} e^{-t_i} \right) \prod_j \left( t_j^{q_j} \frac{q_j}{q_j} e^{-t_j} \right) \]

\[ 3.9 \]
where \( t_i = \sum_{j=1}^{N} t_{ij} \) for stop \( i \), and \( t_j = \sum_{i=1}^{j} t_{ij} \) for stop \( j \);

\[ t_{ij} = \text{the true passenger flow between origin stop } i \text{ and destination stop } j; \]

\[ p_{ij} = \text{the proportion of passenger flow between stop } i \text{ and stop } j \text{ captured by the base matrix}; \]

\[ t_{ij}^0 = \text{the base matrix flow between origin stop } i \text{ and destination stop } j, \text{ and } t_{ij}^0 = 1 \]

for all possible OD pairs because of the null base OD matrix assumption.

\[ P_i = \text{the observed boarding count at stop } i; \]

\[ Q_j = \text{the observed alighting count at stop } j. \]

An optimization function in MATLAB is used. Equation 3.9 is used as the objective function, and no constraint is considered in this optimization problem. A description of the procedure of this optimization function is presented in Appendix B. By conducting the procedure presented in Appendix B, an estimated OD matrix for bus trip 1 is obtained and shown in Table 3-14.

Similarly, the estimated OD matrix for bus trip 2 is estimated, and the estimated OD matrix for the sum of the two bus trips is obtained by adding up the two estimated OD matrices. The result is shown in Table 3-15.
Table 3-14: Estimated OD matrix for the first bus trip using the modified MLE method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>1.00</td>
<td>3.69</td>
<td>6.65</td>
<td>3.66</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>2.31</td>
<td>5.35</td>
<td>2.34</td>
<td>10.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>5.00</td>
<td>2.00</td>
<td>7.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
<td>8.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>1.00</td>
<td>6.00</td>
<td>17.00</td>
<td>16.00</td>
<td>40.00</td>
</tr>
</tbody>
</table>

Table 3-15: Estimated OD matrix using the modified MLE method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>2.00</td>
<td>5.91</td>
<td>13.53</td>
<td>8.55</td>
<td>30.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>3.09</td>
<td>10.46</td>
<td>5.45</td>
<td>19.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>10.00</td>
<td>5.00</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>16.00</td>
<td>16.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>2.00</td>
<td>9.00</td>
<td>34.00</td>
<td>35.00</td>
<td>80.00</td>
</tr>
</tbody>
</table>

3.3.5 Kikuchi and Perincherry method

Unlike the inputs of the methods considered above, the Kikuchi and Perincherry method (Kikuchi & Perincherry, 1992) requires lower and upper bounds to complete the estimation. As discussed in section 2.2.5, the required bounds could come from analyst’s expert knowledge or from boarding and alighting counts. In this example, it is assumed that no analyst expert knowledge on the ranges is available. Thus, the lower and upper bounds are derived from the boarding and alighting counts. Using the algorithm presented in Appendix C, the bounds can be derived from the boarding and alighting counts, and results are shown in Table 3-16 and Table 3-17.
Table 3-16: Lower Bounds $s_{1(i,j)}$ for the first bus trip in the example

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 3-17: Upper Bounds $s_{2(i,j)}$ for the first bus trip in the example

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>1.00</td>
<td>6.00</td>
<td>15.00</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>6.00</td>
<td>10.00</td>
<td>10.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>7.00</td>
<td>7.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

The range $z_{ij}$ for each OD pair can then be derived from the upper and lower bounds by subtracting the lower bound from the corresponding upper bound. The proxy measure $h_{ij}$ considered in this method is then computed using equation 2.16. For OD pair Stop A to Stop D, for example, the range $z_{AD}$ is computed as

$$z_{AD} = s_2(A,D) - s_1(A,D) = 15.00 - 0.00 = 15.00$$

3.10

The proxy measure $h_{AD}$ is then computed as

$$h_{AD} = \min \left\{ \frac{2c_{AD}}{z_{AD}}, 2 - \frac{2c_{AD}}{z_{AD}} \right\} = \min \left\{ \frac{c_{AD}}{15}, 2 - \frac{c_{AD}}{15} \right\}$$

3.11

where $c_{AD}$ is distance of the true OD flow from the lower bound of the range for OD (A,D).
The sum of proxies for all possible OD pairs is taken as the objective function as described in equation 2.17. That is, for this example, the objective function is

$$\max \sum \sum h_{ij} = \min \left\{ \frac{2c_{AB}}{z_{AB}}, 2 - \frac{2c_{AB}}{z_{AB}} \right\} + \min \left\{ \frac{2c_{AC}}{z_{AC}}, 2 - \frac{2c_{AC}}{z_{AC}} \right\} + \cdots + \min \left\{ \frac{2c_{DE}}{z_{DE}}, 2 - \frac{2c_{DE}}{z_{DE}} \right\}$$

3.12

s.t. \( \sum_i s_{1(i,j)} + c_{ij} = Q_j \),

3.13

\( \sum_j s_{1(i,j)} + c_{ij} = P_i \),

3.14

\( h_{ij} \geq h_z \),

3.15

\( h_{ij}, c_{ij} \geq 0 \)

3.16

where \( s_{1(i,j)}, c_{ij}, Q_j, P_i, \) and \( h_{ij} \) are the same notations as those presented above;

\( h_z \) is a threshold value on all \( h_{ij} \);

c_{ij}s are decision variables for the objective function. The estimated \( c_{ij} \) is obtained by using the same optimization procedure presented in Appendix C. The optimization results are shown in Table 3-18.

Table 3-18: Estimated \( c_{ij}s \) for the first bus trip of the example

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>1.00</td>
<td>3.00</td>
<td>8.50</td>
<td>2.50</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>3.00</td>
<td>5.00</td>
<td>2.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>3.50</td>
<td>3.50</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
The estimated OD flows are then computed as equation 3.17, and the estimated OD matrix is shown in Table 3-19.

\[ \hat{t}_{ij} = s_{1(i,j)} + \hat{c}_{ij}, \quad 3.17 \]

where \( s_{1(i,j)} \) = the lower bound contained in Table 3-16;

\[ \hat{c}_{ij} = \text{the estimated distance of the OD flow from the lower bound for OD pair } (i,j); \]

\[ \hat{t}_{ij} = \text{the estimated OD flow for OD pair } (i,j); \]

Table 3-19: Estimated OD matrix for the first bus trip

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>1.00</td>
<td>3.00</td>
<td>8.50</td>
<td>2.50</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>3.00</td>
<td>5.00</td>
<td>2.00</td>
<td>10.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>3.50</td>
<td>3.50</td>
<td>7.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
<td>8.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>1.00</td>
<td>6.00</td>
<td>17.00</td>
<td>16.00</td>
<td>40.00</td>
</tr>
</tbody>
</table>

Similarly, the estimated OD matrix for bus trip 2 is estimated, and the estimated OD matrix for the sum of the two bus trips is obtained by adding up the two estimated OD matrices. The result is shown in Table 3-20.

Table 3-20: Estimated OD matrix using the analyst knowledge incorporation method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>2.00</td>
<td>4.50</td>
<td>17.00</td>
<td>6.50</td>
<td>30.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>4.50</td>
<td>9.50</td>
<td>5.00</td>
<td>19.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>7.50</td>
<td>7.50</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>16.00</td>
<td>16.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Sum</td>
<td>0.00</td>
<td>2.00</td>
<td>9.00</td>
<td>34.00</td>
<td>35.00</td>
<td>80.00</td>
</tr>
</tbody>
</table>
3.3.6 Gur and Elia method

Gur and Elia (1997) proposed a method to estimate a route level APM. This method requires the boarding and alighting counts and a base APM as inputs. In this example, it is assumed that no base information is available. Thus, a null base APM is used in this method. The null base APM is generated by evenly distributing the alighting probabilities to the boarding stops upstream of the alighting stop. The null base APM used in this example is shown in Table 3-21.

Table 3-21: The null base APM used in this example

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>1.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>0.33</td>
<td>0.33</td>
<td>0.33</td>
<td>1.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.50</td>
<td>0.50</td>
<td>1.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Based on the equation 2.23 through 2.29, the objective function and constraints for the example used in this chapter could be formulated.

\[
\begin{align*}
\text{Min } Z(X) &= A(X) + \alpha B(X) \\
&= \sum_j \sum_t [Q_j(t) - \sum_i [P_i(t)x_{ij}]]^2 + \alpha \beta \sum_i \sum_j \left\{P_i x_{ij} \left[\log \left(\frac{x_{ij}}{x_{ij}^*}\right) - 1\right]\right\} \\
\text{s.t. } \sum_j \sum_i [P_i(t)x_{ij}] &= \sum_t Q_j(t), \\
\sum_j x_{ij} &= 1.0, \text{ for } \forall i, \\
\text{if } f_{ij} > 0, x_{ij} > 0; \text{ if } f_{ij} = 0, x_{ij} = 0
\end{align*}
\]
where $x_{ij}^0$ = the entry in base APM for OD pair (i, j);

$x_{ij}$ = the true APM entry for OD pair (i, j);

t = the bus trip number, $t = 1, 2$;

$P_i(t)$ = the observed number of boardings at stop i for bus trip t;

$Q_j(t)$ = the observed number of alightings at stop j for bus trip t;

$\alpha$ and $\beta$ are parameters. $\alpha$ is a weighting parameter, assigning different weight to the two components based on the quality of the base APM and boarding and alighting counts. $\beta$ is a parameter used to normalize the effect of the total number of passengers per bus trip and the number of bus trips. In this example, it is assumed that both parameters equal one.

$x_{ij}$s are decision variables for the objective function described in equation 3.18. The estimated $x_{ij}$ values are obtained by using the same optimization procedure presented in Appendix C. The optimization results are shown in Table 3-22.

The estimated OD matrix for the first bus trip is computed by multiplying the observed boarding counts of the first bus trip to each APM cell value in the corresponding row. The estimated OD matrix for bus trip 1 is shown in Table 3-23.
Table 3-22: Estimated APM using the Gur and Elia method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.000</td>
<td>0.067</td>
<td>0.300</td>
<td>0.176</td>
<td>0.457</td>
<td>1.000</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.778</td>
<td>0.222</td>
<td>1.000</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.930</td>
<td>0.070</td>
<td>1.000</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Table 3-23: Estimated OD matrix for the first bus trip

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>1.01</td>
<td>4.50</td>
<td>2.64</td>
<td>6.86</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>7.78</td>
<td>2.22</td>
<td>10.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>6.51</td>
<td>0.49</td>
<td>7.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
<td>8.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>1.01</td>
<td>4.50</td>
<td>16.93</td>
<td>17.57</td>
<td>40.00</td>
</tr>
</tbody>
</table>

Similarly, the estimated OD matrix for bus trip 2 is computed by multiplying the observed boarding counts of the second bus trip to each APM cell value in the corresponding row, and the estimated OD matrix for the sum of the two bus trips is obtained by adding up the two estimated OD matrices. The result is shown in Table 3-24.

Table 3-24: Estimated OD matrix using the Gur and Elia method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>2.00</td>
<td>9.00</td>
<td>5.28</td>
<td>13.72</td>
<td>30.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>14.77</td>
<td>4.23</td>
<td>19.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>13.95</td>
<td>1.05</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>16.00</td>
<td>16.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>2.00</td>
<td>9.00</td>
<td>34.00</td>
<td>35.00</td>
<td>80.00</td>
</tr>
</tbody>
</table>
From the estimation results, it can be found that the trip level alighting counts are not the same as the observed alighting counts, but the sum of the trip level alighting counts across all bus trips, as described in the constraint of the optimization problem, are equal to sum of the observed trip level alighting counts across all bus trips considered.

3.3.7 Tsygalnitsky method

The Tsygalnitsky method (Tsygalnitsky, 1977) requires a threshold distance and the boarding and alighting counts for every stop along the bus route. It does not require a base OD matrix. In this example, the threshold distance is specified to be two kilometers. That is, no passenger is assumed to alight if he or she travels less than two kilometers.

For the first bus trip, the trip level OD matrix is estimated by the algorithm described in Appendix D, and the estimation results are shown in Table 3-25. To illustrate the procedure to estimate the stop-to-stop passenger OD flows, the flows of stop A to B, stop A to C, and stop B to C are estimated. There are 15 passengers boarding at stop A, and one passenger alighting at stop B. Therefore, the OD flows between stop A and B is 1, and there are $15 - 1 = 14$ passengers who boarded at stop A still on board after stop B. There are 10 passengers boarding at stop B, and there are 6 passengers alighting at stop C. Therefore, there are $14 + 10 = 24$ on board passengers: 14 of them boarded at stop A, and 10 of them boarded at stop B. Based on the equally likely assumption, each of the 24 on board passengers is equally likely to alight at stop C. So, the alighting probability is $6 / 24 = 0.25$. Therefore, of the 14 passengers who boarded at stop A, there
are 14 * 0.25 = 3.50 passengers estimated to alight at stop C, and of the 10 passengers who boarded at stop B, there are 10 * 0.25 = 2.50 passengers estimated to alight at stop C. That is, the OD flow between stop A and stop C is estimated to be 3.50, and the OD flow between stop B and stop C is estimated to be 2.50. Similarly, the other OD flows for trip 1 can be derived. The results are presented in Table 3-25.

Table 3-25: Estimated OD matrix for the first bus trip using the Tsygalnitsky method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>1.00</td>
<td>3.50</td>
<td>7.14</td>
<td>3.36</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>2.50</td>
<td>5.10</td>
<td>2.40</td>
<td>10.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>4.76</td>
<td>2.24</td>
<td>7.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
<td>8.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>1.00</td>
<td>6.00</td>
<td>17.00</td>
<td>16.00</td>
<td>40.00</td>
</tr>
</tbody>
</table>

Similarly, the estimated OD matrix for bus trip 2 is estimated using the same procedure, and the estimated OD matrix for the two bus trips is obtained by adding up the two estimated OD matrices. The result is shown in Table 3-26.

Table 3-26: Estimated OD matrix using the Tsygalnitsky method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>2.00</td>
<td>5.33</td>
<td>14.53</td>
<td>8.14</td>
<td>30.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>3.67</td>
<td>9.85</td>
<td>5.48</td>
<td>19.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>9.62</td>
<td>5.38</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>16.00</td>
<td>16.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>2.00</td>
<td>9.00</td>
<td>34.00</td>
<td>35.00</td>
<td>80.00</td>
</tr>
</tbody>
</table>
Comparing Table 3-26 with Table 3-9, which is the estimation results by the IPF method using the null base OD matrix, we can find that the two matrices are identical. The equivalence between the Tsygalnitsky method and the IPF method is confirmed.

### 3.3.8 Li and Cassidy method

The Li and Cassidy method (Li & Cassidy, 2007) requires the same inputs as the Tsygalnitsky method, in addition to specification of stops as being either major stops or minor stops. Major and minor stops imply a specification of parameters of the alighting probabilities. Major stops are considered to be stops serving activity centers, such as commuter train stations or large business centers, along the route. Other stops are treated as minor stops. Based on the structure of the example route, the first and third stop are considered to serve the business centers, and they are specified as major stops. In this example, the threshold distance is specified to be two kilometers. That is, no passenger is assumed to alight if he or she travels less than two kilometers.

For a given set of alphas for major stops and minor stops, estimated OD matrices for all bus trips are computed by the algorithm described in Appendix E. These estimated OD matrices are then used to generate an APM. The APM and trip level boarding counts are then used to compute a distance weighted measure D by an algorithm described in Appendix F. Therefore, one set of alphas will have a corresponding distance weighted measure D. The set of alphas that produces the smallest D is selected as the parameter for these bus trips to estimate OD matrices.
In the example used in this chapter, alphas for major stops, for example, are set to be 0.5, and alphas for minor stops are set to be 0.3. Given these alphas, boarding and alighting counts, and the threshold distance, bus trip level OD matrices are estimated using the algorithm in Appendix E for both bus trips. To illustrate the procedure to estimate the stop-to-stop passenger OD flows, the flows of stop A to B, stop A to C, and stop B to C are estimated. There are 15 passengers who boarded at stop A, and one passenger who alighted at stop B. Therefore, the OD flows between stop A and B is 1, and there are 15 – 1 = 14 passengers who boarded at stop A still on board after stop B. There are 10 passengers who boarded at stop B, and there are 6 passengers who alighted at stop C. Therefore, there are 14 + 10 = 24 on board passengers; 14 of them boarded at stop A and 10 of them boarded at stop B. Based on the specification, stop A and stop C are major stops. Thus, the alpha for stop C is 0.5. In addition, 14 on board passengers boarded at a major stop (stop A), and 10 on board passengers boarded at a minor stop (stop B). Using equation 2.31 presented in Chapter 2, the expected number of the 6 alighting passengers who boarded at major stops is

\[ E(n_a) = \frac{(1 - \alpha)N_a}{(1 - \alpha)N_a + \alpha N_b}n = \frac{(1 - 0.5) * 14}{(1 - 0.5) * 14 + 0.5 * 10} * 6 = 3.50 \]

where \( \alpha \) is the parameter current stop C, which is 0.5 as specified;

\( N_a \) is onboard passengers who boarded at major stops;

\( N_b \) is onboard passengers who boarded at minor stops;
\( n \) is number of alighting passengers.

Similarly, the expected number of the 6 alighting passenger who boarded at minor
stops is \( 6 - 3.50 = 2.50 \). Stop A is the only one major stop upstream of stop C. Thus, the
OD flow between stop A and stop C is estimated to be 3.50. Similarly, stop B is the only
minor stop upstream of stop C. Thus, the OD flow between stop B and Stop C is
estimated to be 2.50. Similarly, the other OD flows can be derived. The estimated results
for each of the two bus trips are shown in Table 3-27 and Table 3-28.

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>1.00</td>
<td>3.50</td>
<td>8.62</td>
<td>1.88</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>2.50</td>
<td>2.64</td>
<td>4.86</td>
<td>10.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>5.74</td>
<td>1.26</td>
<td>7.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
<td>8.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>1.00</td>
<td>6.00</td>
<td>17.00</td>
<td>16.00</td>
<td>40.00</td>
</tr>
</tbody>
</table>

The two estimated OD matrices are then used to form an aggregated OD matrix
by adding up the two OD matrices. The aggregated OD matrix is then used to derive an
APM by dividing each entry value in the aggregated OD matrix by its row total. The
APM for the two bus trips is shown in Table 3-29.
Table 3-28: Estimated OD matrix for the second bus trip using the Li and Cassidy method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>1.00</td>
<td>1.83</td>
<td>8.80</td>
<td>3.38</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>1.17</td>
<td>2.42</td>
<td>5.40</td>
<td>9.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>5.78</td>
<td>2.22</td>
<td>8.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
<td>8.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>1.00</td>
<td>3.00</td>
<td>17.00</td>
<td>19.00</td>
<td>40.00</td>
</tr>
</tbody>
</table>

Table 3-29: Estimated APM using the Li and Cassidy method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.000</td>
<td>0.067</td>
<td>0.178</td>
<td>0.581</td>
<td>0.175</td>
<td>1.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.000</td>
<td>0.000</td>
<td>0.193</td>
<td>0.266</td>
<td>0.541</td>
<td>1.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.768</td>
<td>0.232</td>
<td>1.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>1.000</td>
<td>1.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.00</td>
</tr>
</tbody>
</table>

The estimated APM in Table 3-29 and the trip level boarding counts are then used as inputs to the algorithm described in Appendix F, and the distance weight measure D is computed as 1.5288. That is, for the observed boarding and alighting counts, setting alphas for major stops to be 0.5 and alphas for minor stops to be 0.3 will give the distance weight measure D of 1.5288.

Using the same procedure, other values of alphas are used and the corresponding distance weight measure D is computed. The smallest D is 1.3619. The APM producing this value of D is shown in Table 3-30. The alphas that produced this APM are 0.9 for major stops and 0.5 for minor stops.
Table 3-30: The estimated APM producing the smallest D using the Li and Cassidy method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.000</td>
<td>0.067</td>
<td>0.042</td>
<td>0.573</td>
<td>0.318</td>
<td>1.000</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.000</td>
<td>0.000</td>
<td>0.408</td>
<td>0.378</td>
<td>0.214</td>
<td>1.000</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.000</td>
<td>0.000</td>
<td>0.641</td>
<td>0.359</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>Stop D</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>1.000</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>Stop E</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td></td>
</tr>
</tbody>
</table>

To be comparable to other estimation methods, the aggregated OD matrix producing the APM in Table 3-30 is used as the estimated OD matrix produced by the Li and Cassidy method. This OD matrix is shown in Table 3-31.

Table 3-31: Estimated OD matrix using the Li and Cassidy method

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0.00</td>
<td>2.00</td>
<td>1.25</td>
<td>17.20</td>
<td>9.55</td>
<td>30.00</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>0.00</td>
<td>7.75</td>
<td>7.18</td>
<td>4.07</td>
<td>19.00</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>9.62</td>
<td>5.38</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>16.00</td>
<td>16.00</td>
</tr>
<tr>
<td>Stop E</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. Alighting</td>
<td>0.00</td>
<td>2.00</td>
<td>9.00</td>
<td>34.00</td>
<td>35.00</td>
<td>80.00</td>
</tr>
</tbody>
</table>

3.4 Results comparison

The OD matrices estimated by the different methods are different, but similar. Except for the modified MLE method and the Gur and Elia method, all methods produce the same estimated trip level boarding and alighting counts as the observed trip level boarding and alighting counts. However, the estimated OD values in each cell differ from method to method.
To measure the difference between these matrices, we could compute the distance between the two estimated OD flows by different methods. Such a metric, however, has little physical meaning. Instead, the estimated OD matrix could be translated into a passenger trip length distribution, and the difference between the OD matrices can be assessed by comparing the trip length distributions.

Every transit passenger trip has a corresponding trip length. Combining the OD flows and these trip lengths, one can produce a passenger trip length distribution. This passenger trip length distribution can then be used to compare the results of the different methods. Other measures, the total or the average passenger distance traveled, cannot be used to compare the difference between different estimation results because these measures are dominated by the boarding and alighting counts. That is, if the estimated boarding and alighting counts are the same for the different methods, the estimated total or average passenger distances will be the same. Since most route-level estimation methods retain the same boarding and alighting counts, the total or average passenger distance traveled cannot be used to measure the difference between different methods.

The trip length distributions calculated from the different methods are shown in Table 3-32. Each entry in the table presents the proportion of passengers whose trip lengths are within the distance range indicated by the corresponding row. For example, the value under IPF and 2 km is 0.391. This number means that, when using the IPF method, 39.1% of the passengers are estimated to have trip lengths of two kilometers.
Using the values in Table 3-32, the cumulative trip length distribution can be then obtained by computing the cumulative proportions. The cumulative trip length distributions are shown in Table 3-33.

<table>
<thead>
<tr>
<th></th>
<th>IPF</th>
<th>CGLS</th>
<th>CMLE</th>
<th>Modifed MLE</th>
<th>Kik. and Perin.</th>
<th>Gur and Elia</th>
<th>Tsyg.</th>
<th>Li and Cassidy</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 km</td>
<td>0.391</td>
<td>0.388</td>
<td>0.392</td>
<td>0.389</td>
<td>0.375</td>
<td>0.399</td>
<td>0.391</td>
<td>0.420</td>
</tr>
<tr>
<td>4 km</td>
<td>0.257</td>
<td>0.269</td>
<td>0.249</td>
<td>0.267</td>
<td>0.269</td>
<td>0.310</td>
<td>0.257</td>
<td>0.156</td>
</tr>
<tr>
<td>6 km</td>
<td>0.250</td>
<td>0.238</td>
<td>0.264</td>
<td>0.237</td>
<td>0.275</td>
<td>0.119</td>
<td>0.250</td>
<td>0.366</td>
</tr>
<tr>
<td>8 km</td>
<td>0.102</td>
<td>0.106</td>
<td>0.095</td>
<td>0.107</td>
<td>0.081</td>
<td>0.172</td>
<td>0.102</td>
<td>0.058</td>
</tr>
</tbody>
</table>

The entries in the cumulative trip length distribution present the cumulative proportion of passengers whose trips length are less than the distance indicated by the corresponding row. For example, the number 0.898 under the IPF method means that 89.8% passengers have estimated trip length less than or equal to 6 km using the IPF method. These cumulative trip length distributions are plotted in the Figure 3.2.

60
From Figure 3.2, the trip length distributions produced by different methods are close to each other. The IPF, CGLS, CMLE, modified MLE and Tsygalnitsky methods produce similar results in terms of the trip length distribution as the curves are almost overlapping. The other three methods, the Kikuchi and Perincherry, the Gur and Elia and the Li and Cassidy methods seem to produce different results as the curves produced by these three methods are farther apart from other curves. By looking at the trip length distributions produced by different methods, the differences among methods can be compared. To further compare the difference between methods using trip length distributions, a scalar measure determined as the areas between a pair of curves can be computed. For example, to compute the areas between the curves produced by the IPF
and CGLS methods, the absolute differences of the proportions between the two curves are first computed and presented in Table 3-34.

Table 3-34: Computation of the area between two trip length distribution curves

<table>
<thead>
<tr>
<th>Distance</th>
<th>IPF</th>
<th>CGLS</th>
<th>Absolute difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 km</td>
<td>0.391</td>
<td>0.388</td>
<td>0.006</td>
</tr>
<tr>
<td>4 km</td>
<td>0.648</td>
<td>0.657</td>
<td>0.018</td>
</tr>
<tr>
<td>6 km</td>
<td>0.898</td>
<td>0.895</td>
<td>0.006</td>
</tr>
<tr>
<td>8 km</td>
<td>1.000</td>
<td>1.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Using the absolute differences presented in Table 3-34 and the distances between stops, the areas between the curves of the IPF and CGLS can be then computed as

$$0.006 \times 2 + 0.018 \times 2 + 0.006 \times 2 = 0.060$$

Similarly, the scalar differences between any two cumulative trip length distributions can be computed. The results are shown in Table 3-35.

Table 3-35: Difference between cumulative trip length distribution curves using scale measure

<table>
<thead>
<tr>
<th></th>
<th>IPF</th>
<th>CGLS</th>
<th>CMLE</th>
<th>Modify MLE</th>
<th>Kik. and Perin.</th>
<th>Gur and Elia</th>
<th>Tsyg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>CGLS</td>
<td>0.060</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CMLE</td>
<td>0.060</td>
<td>0.120</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Modified MLE</td>
<td>0.056</td>
<td>0.044</td>
<td>0.100</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kik. and Perin.</td>
<td>0.164</td>
<td>0.200</td>
<td>0.136</td>
<td>0.212</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gur and Elia</td>
<td>0.556</td>
<td>0.520</td>
<td>0.608</td>
<td>0.508</td>
<td>0.720</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tsyg.</td>
<td>0.000</td>
<td>0.060</td>
<td>0.060</td>
<td>0.056</td>
<td>0.164</td>
<td>0.556</td>
<td></td>
</tr>
<tr>
<td>Li and Cassidy</td>
<td>0.580</td>
<td>0.640</td>
<td>0.520</td>
<td>0.604</td>
<td>0.544</td>
<td>1.072</td>
<td>0.580</td>
</tr>
</tbody>
</table>
As we observed by looking at Figure 3.2, the IPF, CGLS, CMLE, modified MLE and Tsygalnitsky methods produce similar results because they have small areas between the cumulative trip length distribution curves. The other three methods -- namely, the Kikuchi and Perincherry, the Gur and Elia and the Li and Cassidy methods -- seem to produce different results.
This chapter introduces a case study where different methods are applied to a full scale transit route. One of the routes of the Central Ohio Transit Authority (COTA) is selected, and is used to implement some of the methods discussed in the previous chapter.

4.1 Introduction to COTA system

The Central Ohio Transit Authority (COTA) is formed in 1971 by the local governments of Bexley, Columbus, Gahanna, Grandview Heights, Grove City, Hilliard, Reynoldsburg, Upper Arlington, Westerville, Whitehall, Worthington and Franklin County. COTA began providing service in Franklin County on January 1, 1974.

COTA is the only transit agency in the Columbus metropolitan area. With a fleet of 234 buses, COTA travels throughout central Ohio on 53 routes, with 4,214 bus stops, 380 bus shelters, 24 park & ride locations, and over 1 million calls annually to the customer information center (COTA 2006). COTA also provides service for people with disabilities who can’t use fixed-route service. In 2005, COTA provided approximately 49,000 weekday bus rides. The COTA network follows approximately a radial structure. A typically inbound transit route starts at a suburban location, travels through the
downtown area, and ends at another suburban location of the metropolitan area. The following map shows the layout of all the COTA routes. Each color represents one route. The area with the highest density of routes is downtown Columbus.

![COTA system map](image)

**Figure 4.1: COTA system map**

### 4.2 Data preparation for COTA case study

Based on the general method described in chapter 3, three types of data need to be prepared: route structure, boarding and alighting counts, and a base matrix. In this case study, one of the routes among 53 transit routes is selected, and the corresponding data prepared are discussed in the following sections.
Two types of data are obtained from COTA, geographic data and APC data. The geographic data contains route structure information. Different routes are described in separated geographic files. Each geographic file includes all the stop information of the corresponding route, such as stop ID, stop latitude and longitude, and direction of the route. The APC data contain APC records from January 2007 to July 2007. Each APC record indicates a bus serving a bus stop. The boarding and alighting counts, time stamp, stop ID, and bus trip number constitute the record. The two types of data are processed and integrated, so that the data required by the different methods are compiled for use as inputs.

4.2.1 Route structure

Route 2 SB&EB is selected among all 53 COTA routes. This route starts from the Crosswood park & ride station, which is located at the north side of Columbus. Buses on this route travel along High Street southbound to downtown Columbus, and then turns east traveling on Main Street. This route ends at Main Street and Hanson Street, which is located at the east side of Columbus. The following map highlights route 2 to show its geographical layout.
The buses on this route travel southbound and eastbound. There are 208 stops on this route. The location of each stop is given by its latitude and longitude from COTA. The total route length is 43.95 km. Thus, the average distance between two consecutive stops is 211 m.

This route traverses park and ride stations, OSU campus area, and downtown Columbus. These are important passenger originations and attractions. These areas correspond to the time points given by the COTA schedule, which are shown in Table 3-1. For example, the first and last time point stop of the route are park and ride stations. The
7th time point stop, which is N. High St. & 11th, is the stop located at OSU campus area.

And the 8th and 9th time point stops are in downtown.

Table 4-1: Time points of Route 2 SB&EB

<table>
<thead>
<tr>
<th>Index</th>
<th>Stop Name</th>
<th>Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Crosswoods Park &amp; Ride</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>High St. &amp; Larrimer Ave.</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>Graceland shopping center</td>
<td>58</td>
</tr>
<tr>
<td>4</td>
<td>N. High St. &amp; Morse Rd.</td>
<td>62</td>
</tr>
<tr>
<td>5</td>
<td>N. High St. &amp; N. W. Broadway</td>
<td>76</td>
</tr>
<tr>
<td>6</td>
<td>N. High St. &amp; Hudson</td>
<td>85</td>
</tr>
<tr>
<td>7</td>
<td>N. High St. &amp; 11th</td>
<td>95</td>
</tr>
<tr>
<td>8</td>
<td>N. High St. &amp; Nationwide</td>
<td>109</td>
</tr>
<tr>
<td>9</td>
<td>N. High St. &amp; W. Broad</td>
<td>113</td>
</tr>
<tr>
<td>10</td>
<td>E. Main &amp; Ohio Ave.</td>
<td>129</td>
</tr>
<tr>
<td>11</td>
<td>E. Main &amp; College Ave.</td>
<td>142</td>
</tr>
<tr>
<td>12</td>
<td>E. Main &amp; Weyant Ave.</td>
<td>155</td>
</tr>
<tr>
<td>13</td>
<td>E. Main &amp; Hamilton Rd.</td>
<td>167</td>
</tr>
<tr>
<td>14</td>
<td>Fountain land &amp; E. Main St.</td>
<td>173</td>
</tr>
<tr>
<td>15</td>
<td>Consumer square east</td>
<td>196</td>
</tr>
<tr>
<td>16</td>
<td>E. Main &amp; Hanson</td>
<td>208</td>
</tr>
</tbody>
</table>

An APC system is available for COTA. The APC sensors are installed for some of the buses running on this route. Thus, the boarding and alighting counts could be obtained from these APC data. The data provided by COTA has the format shown in table 4-2.

Table 4-2: Format of the COTA APC data

<table>
<thead>
<tr>
<th>Seq.</th>
<th>ID</th>
<th>Time</th>
<th>On</th>
<th>Off</th>
<th>Day</th>
<th>Route</th>
<th>Trip No.</th>
<th>Direction</th>
<th>Bus No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>109</td>
<td>4100</td>
<td>19656</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>526</td>
<td>0</td>
<td>2138</td>
</tr>
<tr>
<td>110</td>
<td>4101</td>
<td>19666</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>526</td>
<td>0</td>
<td>2138</td>
</tr>
<tr>
<td>111</td>
<td>4102</td>
<td>19727</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>526</td>
<td>0</td>
<td>2138</td>
</tr>
</tbody>
</table>
where Seq. = sequence number of stops for each bus trip., which is unique for each route;

ID = unique stop ID for stops in the COTA network;

Time = time in second starting from midnight of current day;

On, Off = boarding and alighting counts;

Day = day sequence starting from January 1, 2007;

Route = route number;

Trip No. = bus trip number, and each bus trip has a unique trip number;

Direction = direction index, 0 presents SB&EB, and 1 presents WB&NB;

Bus No. = bus ID.

### 4.2.2 Boarding and alighting counts

Boarding and alighting counts are derived from APC data. Each bus trip is determined by the bus trip number and day. That is, on a particular day, a unique trip number represents the same bus trip. For a particular trip number on a day, by sorting the time field, the on and off fields will give boarding and alighting counts for each stop of this bus trip.

From the derived boarding and alighting counts, we notice the imbalance problem. That is, the total number of boardings is not equal to the total number of alightings. Table 4-3 shows example bus trips exhibiting such a problem.
Table 4-3: imbalance problem examples

<table>
<thead>
<tr>
<th></th>
<th>Observed total number of boarding passengers</th>
<th>Observed total number of alighting passengers</th>
<th>Absolute difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bus trip 1</td>
<td>52</td>
<td>47</td>
<td>5</td>
</tr>
<tr>
<td>Bus trip 2</td>
<td>26</td>
<td>25</td>
<td>1</td>
</tr>
<tr>
<td>Bus trip 3</td>
<td>64</td>
<td>54</td>
<td>10</td>
</tr>
<tr>
<td>Bus trip 4</td>
<td>65</td>
<td>66</td>
<td>1</td>
</tr>
<tr>
<td>Bus trip 5</td>
<td>77</td>
<td>80</td>
<td>3</td>
</tr>
</tbody>
</table>

Two situations could lead to the imbalance between total boarding and total alighting counts. One is the errors from the APC devices, and the other is the ‘passenger carry-over’, which means that there are some passengers boarding in the reverse direction close to one of the terminals to save some walking, or to secure a seat.

Several methods are considered to balance the boarding and alighting counts to obtain equal total number of boarding and total number of alighting for a particular bus trip. By using these methods, the total number of boarding and the total number of alighting are adjusted to achieve equality so that various passenger OD estimation methods can be applied.

*Factor to average method*

If the total number of boarding is not equal to the total number of alighting for a particular bus trip, an average number of the total number of boarding and total number of alighting is computed as a target total number of passengers for the bus trip. A boarding factor and an alighting factor are computed as the ratio of the target total number of passengers to the total number of boarding and total number of alighting,
respectively. The boarding and alighting counts at each stop for the bus trip are then multiplied by the boarding and alighting factor, respectively. Therefore, the factored boarding and alighting counts will result in balanced total number of boarding and total number of alighting, which are equal to the target total number of passengers.

**Factor to the higher count method**

Cui (2006), in dealing with the transit OD estimation problem, also encountered the imbalance between the total number of boarding and total number of alighting problem. The author argued that the APC system tends to under-count than over-count. Therefore, the author proposed to raise the lower counts, which could either be the total boarding or total alighting, to match the higher one. That is, the higher total counts, either boarding or alighting, is selected as the target total number of passengers for that bus trip, and a factor is computed as the ratio of the target total number of passengers to the other total counts. The factor is then multiplied to the stop counts associated with the lower total count for each bus trip. Therefore, the factored counts will result in a balanced total number of boarding and total number of alighting counts, which are equal to the target total number of passengers.

**Pseudo stops method**

Furth et al. (2006) mentioned that the imbalance problem can also be explained by passenger carry-over. Therefore the authors discussed that pseudo stops can then be used to capture the carry-over passengers. A pseudo stop is not an actual stop on the bus
route. It represents a virtual stop capturing the carry-over passengers. If a trip has more total alightings than total boardings, the difference is assumed as passengers inherited from the previous bus trip of the opposite direction, and these passengers are assumed to board at the pseudo stop at the beginning of the bus trip. If there are more boardings than alightings, the difference is assumed to be due to passengers remaining on bus to travel on the next bus trip in the opposite direction. These passengers are assumed to alighting at the pseudo stop at the end of the bus trip.

All methods could address the imbalance problem effectively, but they cannot solve the negative load problem, which could cause some of the OD estimation methods to break down. The negative load problem arises when computing the bus load using the boarding and alighting counts at each stop where negative values are found for some bus load. The load could be further separated as the stop arrival load and through load. The arrival load is calculated as the number of passengers before the bus entering the stop, and the through load is calculated as the arrival load minus the alighting counts. When calculating the loads, it is assumed that there is no passenger on board at the beginning of the route. The through load is a more meaningful measure to look at because it could detect the negative load problem more strictly. For example, table 4-4 shows partial APC data for a bus trip. The arrival loads for all stops are positive, but for stop 19, it is impossible for two passengers to get off when there is only one passenger onboard.

The imbalance correction method where the lower counts are factored up to match the higher total counts cannot eliminate the negative load problem in the data. For the same example used in table 4-4, we could first balance the boarding and alighting counts.
by factoring up the lower total count, which is the boarding count, to match the higher total count. Then, both the arrival loads and through loads are calculated. In this example, the negative load is still present.

Table 4-4: Negative load example

<table>
<thead>
<tr>
<th>Stop Sequence</th>
<th>Observed boarding</th>
<th>Observed alighting</th>
<th>Calculated arrival load</th>
<th>Calculated through load</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>18</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>21</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Total</td>
<td>28</td>
<td>30</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4-5: Negative load example

<table>
<thead>
<tr>
<th>Stop Sequence</th>
<th>Observed boarding</th>
<th>Observed alighting</th>
<th>Calculated arrival load</th>
<th>Calculated through load</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0714</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1.0714</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>18</td>
<td>0</td>
<td>0</td>
<td>1.0714</td>
<td>1</td>
</tr>
<tr>
<td>19</td>
<td>2.1429</td>
<td>2</td>
<td>1.0714</td>
<td>-0.9286</td>
</tr>
<tr>
<td>20</td>
<td>1.0714</td>
<td>0</td>
<td>1.2142</td>
<td>1.2142</td>
</tr>
<tr>
<td>21</td>
<td>0</td>
<td>0</td>
<td>2.2857</td>
<td>2.2857</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Total</td>
<td>30</td>
<td>30</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To address the negative load problem, a method is needed to eliminate all the negative load. The following procedure is used to eliminate the negative load problem.
Step 1: Pseudo stops, which represent boarding and alighting at the previous and subsequent bus trip in the opposite direction, are added at the beginning and end of the considered route.

Step 2: Assume that there is no passenger on board at the beginning of the route. Calculate the through load, and find the maximum absolute negative load. If there is no negative through load, go to step 4.

Step 3: Add the absolute value of the maximum negative load from step 2 as passengers boarding at the pseudo stop at the beginning of the route, and calculate the through load again.

Step 4: If the through load is positive at the end of route, add this number as the alighting count at the pseudo stop at the end of the route.

By using any of the balancing methods and the procedure to eliminate the negative load, the boarding and alighting counts for each stop on a particular bus trip are balanced in terms of the total number of count, and there will be no negative loads through the bus trip. The boarding and alighting counts for all stops, together with the pseudo stops, are then used as the boarding and alighting counts to estimate the passenger OD matrix.
4.2.3 Base OD matrix

COTA does not employ AFC technology and historical OD survey results are not available. That is, no information provides OD base matrix for the estimation. Thus, the null base matrix is used as the base OD matrix for the methods requiring a base matrix.

4.3 Implement methods on the selected routes

In the previous section, all the data required in the estimation process are prepared. For route 2 SB&EB, five AM and five PM bus trips are somewhat arbitrarily selected from the APC data based on the time stamps of the data. These bus trips are selected so that the difference between the total boardings and total alightings for the selected bus trips are less than 15% of the average of the total boarding and alighting counts. The AM peak time period is defined as 7:00am to 9:00am and the PM peak time period is defined as 5:00pm to 7:00pm.

Five methods are implemented in this section. They are the IPF method, the CGLS method, the CMLE method, the Tsygalnitsky method, and the Li and Cassidy method. For all the methods above, the bus trip level OD matrices are first estimated, and then a time period OD matrix is generated by adding up all the trip level OD matrices. The estimation results are shown using 3D plots for period OD matrices, since the estimated OD matrices are too large to show using text. The x-axis and y-axis represent the origin and destination stops respectively, and the z-axis represents the OD flows between that stop pair.
4.3.1 IPF

The IPF method is applied using the null base matrix and the boarding and alighting data from five AM and five PM bus trips. The estimation results are shown in the Figure 4.3 and Figure 4.4.

Figure 4.3: Estimated OD matrix for AM period using IPF
4.3.2 CGLS

The CGLS method is applied using the null base matrix and the boarding and alighting data from five AM and five PM bus trips. The estimation results are shown in the Figure 4.5 and Figure 4.6.
Figure 4.5: Estimated OD matrix for AM period using CGLS

Figure 4.6: Estimated OD matrix for PM period using CGLS
4.3.3 CMLE

The CGLS method is applied using the null base matrix and the boarding and alighting data from five AM and five PM bus trips. The estimation results are shown in the Figure 4.7 and Figure 4.8.

Figure 4.7: Estimated OD matrix for AM period using CMLE
4.3.4 Tsyganitsky

The Tsyganitsky method is applied using the boarding and alighting data from five AM and five PM bus trips. The estimation results are shown in the Figure 4.9 and Figure 4.10.
Figure 4.9: Estimated OD matrix for AM period using Tsygalnitsky

Figure 4.10: Estimated OD matrix for PM period using Tsygalnitsky
4.3.5 Li and Cassidy

The Tsygalnitsky method is applied using the boarding and alighting data from five AM and five PM bus trips. The parameters, \( \alpha_{major} \) and \( \alpha_{minor} \), are calibrated from the data. The calibration method is presented in Appendix F. For the AM bus trips, \( \alpha_{major} = 0.2 \), and \( \alpha_{minor} = 0.9 \). For the PM bus trips, \( \alpha_{major} = 0.4 \), and \( \alpha_{minor} = 0.1 \). The estimation results are shown in the Figure 4.11 and Figure 4.12.

Figure 4.11: Estimated OD matrix for AM period using Li can Cassidy
4.4 Results analysis

Although the estimated results could be easily shown, it is hard to tell the difference or similarity between the estimated results by looking at the plots presented above. Some manner is needed to measure the difference between the estimated results. In this section, the trip length distribution is used to assess the difference between the estimated results. The differences produced by different methods are analyzed.

As defined in chapter 3, the trip length distribution is generated from the OD flow and the distance traveled across all the passengers. Given an OD matrix, the passenger flows between every origin and destination is known. Based on the information from the route structure, the distance between every origin and destination is known. A passenger
OD matrix could be easily translated into a trip length cumulative distribution. To eliminate difference due to the varying total number of trips, each bus trip-level OD matrix is normalized by its total number of passengers.

For each estimated period OD matrix, a passenger trip length distribution is calculated. These cumulative distributions are shown in figure 4-13 and figure 4-14.

From the two figures, we could find some of the curves are pretty close. For example, the cumulative distribution functions (CDFs) produced by the IPF method and the Tsygalnitsky method are overlapping as expected (recall the discussion in section 2.3.1). Some curves are different from one another. To further quantify the differences between the curves, areas between any two curves are used. The area between two curves is calculated by integrating over the absolute difference between them. Table 4-6 shows the results for all possible methods and time periods.
Figure 4.13: CDF of passenger distance traveled for AM period

Figure 4.14: CDF of passenger distance traveled for PM period

85
From Table 4-6, we could find that the calculated areas in the red and green zones are smaller than those in the yellow zone. That is, the difference produced by using different methods is smaller than the difference produced by different periods. This may indicate that these estimation results by different methods within the same time period are similar.

To further test the similarity of the estimation results from different method, the period OD matrices estimated by different methods are further aggregated into a single OD matrix by adding up all the estimation results produced by the different methods. Therefore, a single period OD matrix is generated. The CDFs of the passenger distance traveled for each of the two periods OD matrices are shown in Figure 4-15.
A formal test, Pearson’s chi-square goodness of fit test, is conducted to assess the difference between methods. Pearson’s chi-square goodness of fit tests the null hypothesis that the frequency distribution observed in a sample is the same as another frequency distribution. For each estimation method, the estimated period trip length distribution is tested against the aggregated period trip length distribution. That is, for one time period (AM or PM), the aggregated period trip length distribution is compared with the trip length distributions estimated by the five methods. Each comparison gives a test statistic calculated as (Rice 2003)

\[ \chi^2_{stat} = \sum_i \frac{(p_i - p_i^*)^2}{p_i^*} \]  
4.1

where \( p_i \) and \( p_i^* \) is passenger proportions for each discretized interval of two different passenger distance travel distributions.
This test statistic in equation 4.1 is then used to decide whether to reject the null hypothesis or not. Based on Pearson’s chi-square goodness of fit test, the decision rule is: reject the null hypothesis if the test statistic is greater or equal to $\chi_{1-\alpha}^2(n)$, where $\alpha$ is a parameter determined by the confident level of the test; $n$ is the degree of freedom of this test.

In our test, $\alpha$ is chosen to be 0.05, and the route length discretized based on a 50-meter interval. For each discretized distance, the proportion of passengers whose trip length belongs to this distance is calculated. If the proportion of a certain distance is zero, this distance is combined with the next larger discretized distance to avoid zero values. Therefore, the degree of freedom is the number of discretized distances with non-zero trips minus one. Table 4-7 shows the test statistics and the critical values for the decision rule. It can be seen that the statistics are less than the corresponding critical value. Therefore, the null hypothesis cannot be rejected. That is, the estimated OD matrices by different methods are not statistically different from one another.

Table 4-7: Chi-square test for the empirical trip length distributions

<table>
<thead>
<tr>
<th>AM period</th>
<th>Test Statistic</th>
<th>$\chi_{1-\alpha}^2(n)$</th>
<th>PM period</th>
<th>Test Statistic</th>
<th>$\chi_{1-\alpha}^2(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IPF</td>
<td>1.803</td>
<td>368.04</td>
<td>IPF</td>
<td>0.660</td>
<td>531.02</td>
</tr>
<tr>
<td>CGLS</td>
<td>7.290</td>
<td>368.04</td>
<td>CGLS</td>
<td>5.629</td>
<td>531.02</td>
</tr>
<tr>
<td>CMLE</td>
<td>49.982</td>
<td>368.04</td>
<td>CMLE</td>
<td>6.535</td>
<td>531.02</td>
</tr>
<tr>
<td>Tsygalnitsky</td>
<td>1.804</td>
<td>368.04</td>
<td>Tsygalnitsky</td>
<td>0.663</td>
<td>531.02</td>
</tr>
<tr>
<td>Li and Cassidy</td>
<td>7.225</td>
<td>368.04</td>
<td>Li and Cassidy</td>
<td>5.189</td>
<td>531.02</td>
</tr>
</tbody>
</table>
Using the same procedure, we can test whether the AM and PM trip length distributions are the same. Table 4-8 shows the comparison between time periods with the same estimation method. That is, the estimated OD matrices using the same estimation method are first aggregated by time period, and the two time period OD matrices are compared to each other.

Table 4-8: Chi-square test for different time period trip length distributions

<table>
<thead>
<tr>
<th>Method</th>
<th>IPF</th>
<th>CGLS</th>
<th>CMLE</th>
<th>Tsygalnitsky</th>
<th>Li and Cassidy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test Statistic</td>
<td>938.7</td>
<td>1680.5</td>
<td>625.5</td>
<td>938.3</td>
<td>1066.3</td>
</tr>
<tr>
<td>$\chi^2_{1-\alpha}(n)$</td>
<td>415.8</td>
<td>440.2</td>
<td>415.8</td>
<td>415.8</td>
<td>415.8</td>
</tr>
</tbody>
</table>

Table 4-8 shows the test statistics and the critical values for the decision rule. It can be seen that the statistics are greater than the corresponding critical value. Therefore, the null hypothesis is rejected. That is, the estimated OD matrices for different time period are different. Therefore, in summary we could conclude that these estimation results by different methods within the same time period are not different in terms of trip length distribution, and that the estimation results across time periods are different.

4.5 Simulation

In section 4.4, the results estimated by different methods are compared and analyzed. However, there are no ‘true’ observations to compare to. Thus, it is not possible to truly assess the performance of each estimation method. That is, it is not clear how
close the estimated OD matrix is to the ‘true’ OD matrix. This difficulty could be overcome by simulation.

The general procedure of a simulation is one where an OD matrix is generated as the truth, boarding and alighting counts are derived from the ‘true’ OD matrix, an estimation method is then applied using the derived boarding and alighting counts, and the estimated OD matrix is compared with the generated truth. Therefore, simulation allows the estimated results to be compared with the observed truth, so the performance of each estimation method could be directly assessed.

4.5.1 Performance of different methods using null base

In section 4.4.1, we compared the estimated trip length distribution with the aggregated time period trip length distribution to assess the difference caused by different methods. This comparison could also be done by simulating the true time period OD matrix.

It is assume that the entries in the true passenger OD matrix follow Poisson distributions. The aggregated AM and PM period OD matrices across different methods are used as the true mean value of the Poisson distributions. Using these assumptions, ‘true’ passenger OD matrices could be randomly generated. In this simulation, 100 bus trip OD matrices for each time period are generated, and the boarding and alighting counts for each bus trip are derived from the corresponding OD matrix. The simulated ‘true’ OD matrices are aggregated to form a period OD matrix. The CDFs for the distance traveled for the simulated ‘true’ period OD matrices are shown in figure 4-16.
Using the bus trip simulated boarding and alighting counts and the null base matrix, the five methods discussed above are applied to estimate trip-level OD matrices, which are aggregated to period OD matrices. These period OD matrices are then translated into the trip length distributions shown in Figure 4-17 and Figure 4-18 for each method.

Since the ‘true’ period trip length distributions are obtained by simulated period OD matrices, the estimated trip length distribution could be compared to the truth. Areas between the estimated and true period trip length distributions are summarized in Table 4-9.
Table 4-9: Areas between true and estimated trip length distributions

<table>
<thead>
<tr>
<th></th>
<th>IPF</th>
<th>CGLS</th>
<th>CMLE</th>
<th>Tsygalnitsky</th>
<th>L &amp; C</th>
<th>Null Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>AM</td>
<td>73.8</td>
<td>695.6</td>
<td>550.3</td>
<td>73.8</td>
<td>113.9</td>
<td>11748</td>
</tr>
<tr>
<td>PM</td>
<td>70.0</td>
<td>495.3</td>
<td>314.8</td>
<td>70.1</td>
<td>87.4</td>
<td>10865</td>
</tr>
</tbody>
</table>

Figure 4.17: CDFs of distance traveled for the simulated OD matrices in AM period
Figure 4.18: CDFs of distance traveled for the simulated OD matrices in PM period

The area between the two time period true trip length CDFs is 1636.8. Therefore, the differences produced by different methods are still smaller than the differences between different time periods.

Similarly, we could conduct a formal test to assess these differences. Using the same Pearson’s chi-square goodness of fit test applied earlier, the test results are summarized in Table 4-10.
Table 4-10: Chi-square test for the simulated trip length distributions

<table>
<thead>
<tr>
<th>Method</th>
<th>AM period</th>
<th>PM period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Test Statistic</td>
<td>$\chi^2_{1-a}(n)$</td>
</tr>
<tr>
<td>IPF</td>
<td>9.131</td>
<td>317.89</td>
</tr>
<tr>
<td>CGLS</td>
<td>78.227</td>
<td>317.89</td>
</tr>
<tr>
<td>CMLE</td>
<td>67.969</td>
<td>317.89</td>
</tr>
<tr>
<td>Tsygalnitsky</td>
<td>9.128</td>
<td>317.89</td>
</tr>
<tr>
<td>Li and Cassidy</td>
<td>11.001</td>
<td>317.89</td>
</tr>
</tbody>
</table>

Based on the results in the Table 4-10, the null hypothesis cannot be rejected for all the comparisons. Using the same procedure, we could test whether the aggregated AM and PM trip length distributions are the same. The test statistic is 653.70, and the null hypothesis is rejected because $\chi^2_{1-a}(n) = 317.89$. That is, the trip length distributions for aggregated AM and PM are different.

Therefore, from the above analysis, we could conclude that the estimated OD matrices by different methods are not different and fairly close to the truth. Thus, when the null base OD matrix is used, the estimated results are close to the truth in terms of the passenger trip length distribution.

### 4.5.2 Effect of base matrix

For all the analysis above, we use the null base as inputs to these methods requiring a base OD matrix, and the estimated results are close to the truth in terms of the passenger trip length distribution. However, base OD matrices may be available from survey results or outputs of demand models. These base OD matrices provide
supplemental information to the estimation for the methods that use base OD matrices. However, the quality of the base, reflecting the confidence in this information, should affect the estimated results. This section exams the effect of base matrices on the estimation.

The same 100 simulated bus trip OD matrices are used in this section. That is, 100 bus trips for each time period with known stop level OD matrices are simulated. For each time period, two base matrices are generated. One is close to the true period OD matrix, which is referred to as a high quality base, and the other is far from the true period OD matrix, which is referred to as a low quality base. For the AM period, yet another OD matrix is randomly generated based on the assumed true mean values of section 4.5.1. This generated OD matrix is then used as the good base OD matrix for 100 bus trips in the AM. Similarly, an OD matrix is generated based on the true mean values in the PM, and this OD matrix is then used as the bad base OD matrix for 100 bus trips in the AM. The good and bad base OD matrices used for the 100 PM bus trips are generated in a similar manner. Figures 4-19 and 4-20 show the estimated results using the high quality base for both time periods.
Figure 4.19: Estimation results for AM high quality base

Figure 4.20: Estimation results for PM high quality base
From the plots, we could see that the estimated trip length distributions are very close to that produced by the true OD matrix, and the areas between the estimated CDFs and the true CDF could be calculated and are shown in Table 4-11.

Table 4-11: Area between the curves associated with the estimated and true OD matrices for the high quality base

<table>
<thead>
<tr>
<th></th>
<th>IPF</th>
<th>CGLS</th>
<th>CMLE</th>
<th>Good Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>AM</td>
<td>84.3</td>
<td>476.4</td>
<td>101.6</td>
<td>196.4</td>
</tr>
<tr>
<td>PM</td>
<td>104.6</td>
<td>356.1</td>
<td>100.2</td>
<td>136.4</td>
</tr>
</tbody>
</table>

Figures 4-21 and 4-22 show the estimated results using the low quality base for both time periods.

Figure 4.21: Estimation results for AM low quality base
Figure 4.22: Estimation results for PM low quality base

From the plots, we could see that the estimated trip length distributions are more spread out and are quite far from that of the true OD matrix. The areas between the estimated CDFs and the true CDF are calculated and shown in Table 4-12.

Table 4-12: Area between the curves associated with the estimated and true OD matrices for the low quality base

<table>
<thead>
<tr>
<th></th>
<th>IPF</th>
<th>CGLS</th>
<th>CMLE</th>
<th>Bad Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>AM</td>
<td>411.9</td>
<td>1480</td>
<td>632.2</td>
<td>1172.4</td>
</tr>
<tr>
<td>PM</td>
<td>1824</td>
<td>1202</td>
<td>529.5</td>
<td>1054.4</td>
</tr>
</tbody>
</table>

From Table 4-11 and 4-12, we could see that the quality of the base matrix affects the estimation results substantially in terms of the trip length distribution. Also, we could compare these results with the estimation results using the null base matrix as input. The results are shown in Table 4-13. From Table 4-13, when using the good base OD matrix,
the difference between the estimated results and simulated truth reduces from 366.7 to 203.9 as using the null base OD matrix. However, when using the bad base OD matrix, the difference between the estimated results and simulated truth increases from 366.7 to 1013.3 as using the null base OD matrix. The latter change is much bigger than the first one. Notice though that for the IPF method the performance with a good based is slightly worse than that with the null base. This could possibly be explained by the fact that the simulated true OD matrices are generated based on mean values derived from the estimation results using the null base OD matrix. That is, the simulated true OD matrices are not independent from the null base OD matrix but rather are positively associated with it.

Table 4-13: Area between the curves associated with the estimated and true OD matrices for all the situations

<table>
<thead>
<tr>
<th></th>
<th>AM truth</th>
<th>PM truth</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Null Base</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IPF</td>
<td>73.8</td>
<td>70.0</td>
<td>366.7</td>
</tr>
<tr>
<td>CGLS</td>
<td>695.6</td>
<td>495.3</td>
<td></td>
</tr>
<tr>
<td>CMLE</td>
<td>550.3</td>
<td>314.8</td>
<td></td>
</tr>
<tr>
<td><strong>Good Base</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IPF</td>
<td>84.3</td>
<td>104.6</td>
<td>203.9</td>
</tr>
<tr>
<td>CGLS</td>
<td>476.4</td>
<td>356.1</td>
<td></td>
</tr>
<tr>
<td>CMLE</td>
<td>101.6</td>
<td>100.2</td>
<td></td>
</tr>
<tr>
<td><strong>Bad Base</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IPF</td>
<td>411.9</td>
<td>1824</td>
<td>1013.3</td>
</tr>
<tr>
<td>CGLS</td>
<td>1480</td>
<td>1202</td>
<td></td>
</tr>
<tr>
<td>CMLE</td>
<td>632.2</td>
<td>529.5</td>
<td></td>
</tr>
<tr>
<td>Tsygalnitsky</td>
<td>73.8</td>
<td>70.1</td>
<td>71.9</td>
</tr>
<tr>
<td>LC</td>
<td>113.9</td>
<td>87.4</td>
<td>100.7</td>
</tr>
</tbody>
</table>

In summary, from the empirical results and the simulation results, the IPF, CGLS, and CMLE methods produce similar results in terms of trip length distribution when
using a null base OD matrix, as the Tsygalnitsky and Li and Cassidy methods. When using the null base, the IPF method produces the same estimates as the Tsygalnitsky method as expected. The base OD matrices have significant impacts on the estimation results. Good base OD matrices will yield better estimation results than null base OD matrices, but the improvements are limited. Bad base OD matrices will lead to estimations much worse than the null base OD matrices. Therefore, only a base OD matrix of high quality should be used; otherwise, the estimation results can be far from the truth. In the absence of a high quality base matrix, the null base would be the next best option.
CHAPTER 5

SUMMARY AND RECOMMENDATIONS

5.1 Summary

In this thesis, eight methods to estimate a transit route-level passenger OD matrix using boarding and alighting counts are reviewed, compared, and analyzed. The methods are considered to fall into one of two categories. In one category, a base OD matrix, which expresses an initial estimate of the matrix, is used as input. A null base matrix, in which the flows between all OD pairs are considered equal, can be used in all of the methods that require a based matrix when no other information is available to produce an initial OD estimate. In the second category, no base matrix is required, but additional assumptions are used. OD matrices are estimated with each of the methods on a hypothetical route and a given set of assumed boarding and alighting counts. A null base OD matrix is used for those methods requiring a base OD matrix. All the methods yield very similar estimated matrices.

Five of the estimation methods are also applied using boarding and alighting counts obtained from an Automatic Passenger Counting (APC) system on a bus route operated by the Central Ohio Transit Authority (COTA) network. The estimated OD
matrices are again found to be very similar. Simulated data are also generated for bus trips on the COTA route used in the empirical study. The simulated data consist of boarding and alighting counts derived from a simulated “true” OD matrix. Different base OD matrices are considered as input for the methods that require a base OD matrix. The estimation results show that the quality of the base OD matrix affects the quality of the results. When using a high quality base OD matrix, which means the base OD matrix is close to the simulated true OD matrix, as input to different methods, the estimated OD matrices are similar to the simulated true OD matrix and are similar to each other. When using a low quality base OD matrix, which means the base OD matrix is very different from the simulated true OD matrix, the OD matrices estimated by different methods are different from the simulated true OD matrix. The estimated OD matrix using a high quality base OD matrix is better than the matrix estimated when using a null base OD matrix. If a low quality base OD matrix is used as input, the estimated OD matrices are poorer than the OD matrices estimated from the null base OD matrices. The improvement obtained by changing the null base OD matrix to the high quality base is much less than the improvement obtained by changing the low quality base to the null base OD matrix.

5.2 Recommendation

Since a null base could be used when there is no base OD matrix available, all methods could be applied using boarding and alighting counts. The empirical results obtained using COTA APC data show that the methods produce very similar estimates of the OD matrix. Therefore, one could use any of the procedures.
When a base OD matrix other than a null matrix is available, the use of the base OD matrix should be assessed with caution because the quality of the base OD matrix will affect the estimation results. Based on the simulation analysis, when the quality of the base OD matrix is high, the methods that require a base matrix provide better estimates than when a null base is used. However, when the quality of the base is bad, the methods that require a base matrix provide much worse estimates than when a null base is used. That is, using the high quality base matrix can improve the estimation results, compared to using the null base matrix, but using a low quality base matrix will produce worse results. Therefore, whether or not one should use a non-null base matrix should depend on the confidence one has that the base matrix is a good preliminary estimate of the OD matrix. Determining what constitutes a “good preliminary estimate” does not seem to have been addressed in the literature.

5.3 Future research

The results of this thesis also call for future research.

The analysis in this thesis is based on the estimation results on a hypothetical transit bus route and a single COTA transit bus route. It would be helpful to replicate the results in larger scale studies. Different types of routes and different type of data could be considered.

The sensitivity of the estimation results to different input parameters used needs to be considered. Methods for estimating OD matrices require input parameters in the
estimation process. For example, two of the methods considered require a threshold
distance, and another method requires a variance-covariance matrix for the base OD
flows. By changing the values of these parameters, the estimated OD flows would likely
change. The sensitivity of the estimated results to different parameter values needs to be
analyzed.

Finally, the methods were all applied on a linear route in which loads are balanced
and in which there are no negative loads. This will not always be the case as seen in
Chapter 4. As described in Chapter 4, different methods to resolve the imbalance and
negative load problem can be used. The sensitivity of the OD estimation results to these
methods should also be analyzed.


APPENDIX A

ALGORITHM TO SOLVE THE SYSTEM OF EQUATIONS IN THE CMLE METHOD
This algorithm presents an iterative method to solve the system of equations in the CMLE method. As described in equation 2.12, the system of equations is

\[
\begin{align*}
\frac{\partial L^0}{\partial t_{ij}} &= -1 + \frac{t^0_{ij}}{t_{ij}} - a_i - b_j = 0 \\
\frac{\partial L^0}{\partial a_i} &= \sum_j t_{ij} - P_i = 0 \\
\frac{\partial L^0}{\partial b_j} &= \sum_i t_{ij} - Q_j = 0
\end{align*}
\]  

(A.1)

where \( L^0 \) is the Lagrangian of the logarithm of the likelihood function;

\( t_{ij} \) is the true passenger flow between origin stop \( i \) and destination stop \( j \);

\( t^0_{ij} \) is the base passenger flow between origin stop \( i \) and destination stop \( j \);

\( P_i \) is the observed boarding count at stop \( i \);

\( Q_j \) is the observed alighting count at stop \( j \);

\( a_i \) and \( b_j \) are the Lagrangian multipliers.

Re-arrange the equations in (A.1), the estimated OD flows \( \hat{t}_{ij} \) can be written as the equation (A.2)

\[
\begin{align*}
\hat{t}_{ij} &= \frac{t^0_{ij}}{1 + a_i + b_j} \\
\sum_j \hat{t}_{ij} &= P_i \\
\sum_i \hat{t}_{ij} &= Q_j
\end{align*}
\]  

(A.2)
The problem of solving system of equations (A.1) is then transformed to the problem of solving system of equations (A.2). The variables to be determined are \( \hat{t}_{ij} \), \( a_i \) and \( b_j \). To illustrate the procedure of solving (A.2), a hypothetical example is used. The boarding and alighting counts are presented in Table A-1, and the base matrix is present in Table A-2.

### Table A-1: Boarding and alighting counts used in the example

<table>
<thead>
<tr>
<th>Stop Number</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boarding ( P_i )</td>
<td>15</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Alighting ( Q_j )</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
</tr>
</tbody>
</table>

### Table A-2: Base matrix \( t_{ij}^0 \) used in the example

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Zero boarding or alighting counts are first remove from the original boarding and alighting counts, and the corresponding row or column in the base matrix also removed. So, the row presenting stop E and the column presenting stop A are removed from the base OD matrix. To initialize \( a_i \) and \( b_j \), let the estimated boarding and alighting counts divided by the observed boarding and alighting counts. For example, the estimated boarding counts for stop A is the sum of the first row in the base OD matrix, which is 4. Thus, \( a_A^0 \) is computed as \( 4/15 = 0.2667 \).
Table A-3: Initialized values for $a_i$ and $b_j$

<table>
<thead>
<tr>
<th></th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. On</th>
<th>Obs. On</th>
<th>$a_i^0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>15</td>
<td>4/15=0.2667</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>10</td>
<td>3/10=0.3000</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>2/7=0.2857</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>1/8=0.1250</td>
</tr>
<tr>
<td>Est. Off</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Obs. Off</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_i^0$</td>
<td>1.0000</td>
<td>0.3333</td>
<td>0.1765</td>
<td>0.2500</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

After initializing $a_i$ and $b_j$, the first iteration of estimating $a_i$ and $b_j$ is conducted.

For a given set of $b_i^0$, $a_i^1$'s can be determined uniquely. Firstly, upper and lower bounds for all $a_i^1$'s are computed. Based on the first two equations in system of equations in (A.2),

\[ p_i = \sum_j \hat{t}_{ij} = \sum_j \frac{t_{ij}^0}{1 + a_i + b_j} \leq \sum_j \frac{t_{ij}^0}{1 + a_i + \min_{k=i+1,N} b_k} = \frac{\sum_j t_{ij}^0}{1 + a_i + \min_{k=i+1,N} b_k} \quad (A.3a) \]

Therefore,

\[ a_i \leq \frac{\sum_j t_{ij}^0}{p_i} - 1 - \min_{k=i+1,N} b_k \quad (A.3b) \]

Similarly, the low bound can be determined.

\[ p_i = \sum_j \hat{t}_{ij} = \sum_j \frac{t_{ij}^0}{1 + a_i + b_j} \geq \sum_j \frac{t_{ij}^0}{1 + a_i + \max_{k=i+1,N} b_k} = \frac{\sum_j t_{ij}^0}{1 + a_i + \max_{k=i+1,N} b_k} \quad (A.3c) \]

Therefore,

\[ a_i \geq \frac{\sum_j t_{ij}^0}{p_i} - 1 - \max_{k=i+1,N} b_k \quad (A.3d) \]

Also, based on the physical constraints of OD flows, all the estimated OD flows should be non-negative. Thus,

\[ 1 + a_i + b_j \geq 0, \forall i, j \quad (A.3e) \]
Therefore, another lower bound for $a_i$ could be determined as

$$a_i \geq -1 - \min_{k=i+1,N}(b_k) \quad (A.3f)$$

Thus, the upper and lower bounds for $a_i$ could be determined as

$$\max \left( -1 - \min_{k=i+1,N}(b_k), \frac{\Sigma_i t_{ij}}{p_i} - 1 - \max_{k=i+1,N} b_k \right) < a_i < \frac{\Sigma_i t_{ij}}{p_i} - 1 - \min_{k=i+1,N} b_k \quad (A.3g)$$

Take stop A for example, the upper and lower bounds for $a_1^A$ given $b_j^0$ for all j are

$$\max \left( -1 - 0.1765, \frac{4}{15} - 1 - 1.000 \right) < a_1^A < \frac{4}{15} - 1 - 0.2500$$

$$-1.1765 < a_1^A < -0.9830$$

Therefore, the upper and lower bounds for all $a_i^j$ are determined. Based on equation (A.2) $\hat{t}_{ij}$ is monotonous function of $a_i^1$. The value of $a_i^1$ could determined using bisection method for each stop since the upper and lower bound are determined by (A.3g)

Take stop A for example again. The values of $b_j^0$ are read from Table A-3. The value of $a_1^A$ in the first iteration of the bisection method is determined as

$$a_1^A = \frac{1}{2}(-1.1765 - 0.9830) = -1.0798$$

Thus, the estimated OD flows originating at stop A are computed using the first equation in (A.2). The results are shown in Table A-4.
Table A-4: Estimated OD flows for stop A given the value of $a_A^1$

<table>
<thead>
<tr>
<th>OD pair (A, B)</th>
<th>OD pair (A, C)</th>
<th>OD pair (A, D)</th>
<th>OD pair (A, E)</th>
<th>Est. Boarding counts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_i^0$</td>
<td>1.0000</td>
<td>0.3333</td>
<td>0.1765</td>
<td>0.2500</td>
</tr>
<tr>
<td>$v_{Aj}^0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{v_{Aj}^0}{1 + a_A^1 + b_j^0}$</td>
<td>1.1</td>
<td>3.9</td>
<td>10.3</td>
<td>5.9</td>
</tr>
</tbody>
</table>

Check the difference between the estimated and observed boarding counts by equation (A.4). If the difference is less than 1%, take the current $a_A^1$ as the estimated value; otherwise, conduct another iteration of the bisection method.

\[
\frac{\text{abs}(\text{Est.} - \text{Obs.})}{\text{Obs.}} \times 100\% \tag{A.4}
\]

Table A-5 presents four iterations of the bisection method for stop A. After seven iterations, the difference is less than 1%. Thus, the value for $a_A^1$ is -1.0208.

Table A-5: Estimated OD flows for stop A given the value of $a_A^1$

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Upper bound</th>
<th>Lower bound</th>
<th>$a_A^1$</th>
<th>Est. Boarding $\sum_{j} \frac{v_{Aj}^0}{1 + a_A^1 + b_j^0}$</th>
<th>Obs. Boarding</th>
<th>Abs (Est. - Obs.) / Obs. *100%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.9830</td>
<td>-1.1765</td>
<td>-1.0798</td>
<td>21.2</td>
<td>15</td>
<td>41.6%</td>
</tr>
<tr>
<td>2</td>
<td>-0.9830</td>
<td>-1.0798</td>
<td>-1.0314</td>
<td>15.8</td>
<td>15</td>
<td>5.4%</td>
</tr>
<tr>
<td>3</td>
<td>-0.9830</td>
<td>-1.0314</td>
<td>-1.0072</td>
<td>14.1</td>
<td>15</td>
<td>6.0%</td>
</tr>
<tr>
<td>4</td>
<td>-1.0072</td>
<td>-1.0314</td>
<td>-1.0193</td>
<td>14.9</td>
<td>15</td>
<td>1.7%</td>
</tr>
<tr>
<td>5</td>
<td>-1.0193</td>
<td>-1.0314</td>
<td>-1.0253</td>
<td>15.3</td>
<td>15</td>
<td>2.3%</td>
</tr>
<tr>
<td>6</td>
<td>-1.0193</td>
<td>-1.0253</td>
<td>-1.0223</td>
<td>15.1</td>
<td>15</td>
<td>0.8%</td>
</tr>
<tr>
<td>7</td>
<td>-1.0193</td>
<td>-1.0223</td>
<td>-1.0208</td>
<td>15.0</td>
<td>15</td>
<td>0.0%</td>
</tr>
</tbody>
</table>

Similarly, the values of all $a_i^1$s are determined, and shown in Table A-6.
Table A-6: Estimated $a_i^1$ given $b_j^0$

<table>
<thead>
<tr>
<th></th>
<th>$a_i^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>-1.0207</td>
</tr>
<tr>
<td>Stop B</td>
<td>-0.9401</td>
</tr>
<tr>
<td>Stop C</td>
<td>-0.9229</td>
</tr>
<tr>
<td>Stop D</td>
<td>-1.125</td>
</tr>
</tbody>
</table>

Then, the values of $b_j^1$s are determined using the observed alighting counts and estimated $a_i^1$. Thus, the first iteration is completed, and the estimated OD flows, $a_i^1$ and $b_j^1$ are shown in Table A-7.

Table A-7: The estimated OD flows after first iteration and $a_i^1$ and $b_j^1$

<table>
<thead>
<tr>
<th></th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. On</th>
<th>Obs. On</th>
<th>$a_i^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>1.00</td>
<td>3.36</td>
<td>7.79</td>
<td>3.85</td>
<td>16.00</td>
<td>15</td>
<td>-1.0207</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>2.64</td>
<td>4.79</td>
<td>2.94</td>
<td>10.36</td>
<td>10</td>
<td>-0.9401</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>4.42</td>
<td>2.79</td>
<td>7.22</td>
<td>7</td>
<td>-0.9229</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>6.42</td>
<td>6.42</td>
<td>8</td>
<td>-1.1250</td>
</tr>
<tr>
<td>Est. Off</td>
<td>1.00</td>
<td>6.00</td>
<td>17.00</td>
<td>16.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Obs. Off</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_j^1$</td>
<td>1.0207</td>
<td>0.3185</td>
<td>0.1490</td>
<td>0.2807</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Check the differences between the estimated boarding counts and the observed boarding counts using equation (A.4). If the difference is small enough, which means the maximum difference is less than a critical value specified as an input, the estimated OD flows are used as the final estimated OD flow by this method. Otherwise, another iteration is conducted.
After 23 iterations, this maximum difference computed by equation (A.4) is less than 0.0001, which is set as an input. The estimated OD flows, $a_i^{23}$ and $b_j^{23}$ are shown in Table A-8.

Table A-8: The final estimated OD flows, $a_i^{23}$ and $b_j^{23}$

<table>
<thead>
<tr>
<th></th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Est. On</th>
<th>Obs. On</th>
<th>$a_i^{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>1.00</td>
<td>3.33</td>
<td>7.59</td>
<td>3.08</td>
<td>15.00</td>
<td>15</td>
<td>-0.9914</td>
</tr>
<tr>
<td>Stop B</td>
<td>0.00</td>
<td>2.67</td>
<td>4.84</td>
<td>2.50</td>
<td>10.00</td>
<td>10</td>
<td>-0.9163</td>
</tr>
<tr>
<td>Stop C</td>
<td>0.00</td>
<td>0.00</td>
<td>4.57</td>
<td>2.43</td>
<td>7.00</td>
<td>7</td>
<td>-0.9045</td>
</tr>
<tr>
<td>Stop D</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
<td>8.00</td>
<td>8</td>
<td>-1.1916</td>
</tr>
<tr>
<td>Est. Off</td>
<td>1.00</td>
<td>6.00</td>
<td>17.00</td>
<td>16.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Obs. Off</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_j^{23}$</td>
<td>0.9914</td>
<td>0.2914</td>
<td>0.1231</td>
<td>0.3166</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX B

THE ALGORITHM FOR DERIVING RANGES FROM BOARDING AND ALIGHTING COUNTS IN THE KIKUCHI AND PERINCHERRY METHOD
In the Kikuchi and Perincherry method (Kikuchi and Perincherry 1992), the inputs of this method are the boarding and alighting counts for every stop along the bus route and a base matrix consisting of ranges on all OD pairs. These ranges may be specified by analysts who are familiar with the OD pattern or derived from other OD estimation methods. If only a subset of the ranges are available from these sources above, the remaining ranges can be derived from the on and off counts with a method presented in Kikuchi and Perincherry 1992.

The system of equations (B.1) are used to derived the lower bounds $s_{1(i,j)}$ and upper bounds $s_{2(i,j)}$ for OD pair $(i, j)$. The inputs of this algorithm are the numbers of passengers boarding and alighting at each stop along a bus route for a specific bus.

$$\begin{align*}
{s_{2(i,j)}} &= \min\left(p_i, q_j\right) \\
{s_{1(i,j)}} &= \max\left(p_i - \sum_{k=i+1}^{n} s_{2(i,k)}, q_j - \sum_{k=1}^{j} s_{2(k,j)}, 0\right)
\end{align*}$$

(B.1)

where $p_i$ is the observed boarding count at stop $i$;

$Q_j$ is the observed alighting count at stop $j$;

From the equation above, the upper bound is just the minimum boarding and alighting number. The lower bound is determined by maximizing three items. The first item means that the boardings in the downstream stops are all at the upper bound and the sum of them are still not satisfied the total boardings. Similarly, the second item means
that the alightings in the upstream stops are all at the upper bound and the sum of them are still not satisfied the total alightings.

An example is used to illustrate the results of estimating the lower and upper bounds from the boarding and alighting counts. Table B-1 presents the boarding and alighting counts used in the example.

Table B-1: Boarding and alighting counts used in the example

<table>
<thead>
<tr>
<th>Stop Number</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boarding $P_i$</td>
<td>15</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Alighting $Q_j$</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
</tr>
</tbody>
</table>

Using the first equation in (B.1), and the upper bounds for all OD pairs can be derived. For example, for OD pair (A, B), the boarding count for stop A is 15, and the alighting counts for stop B is 1. Therefore, the upper bound for OD pair (A, B) is $\max(15, 1) = 15$. The results for all the upper bounds are shown in Table B-2.

Table B-2: Upper bound matrix $S_2$ for all OD pairs in the example

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Obs. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>-</td>
<td>1</td>
<td>6</td>
<td>15</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>Stop B</td>
<td>-</td>
<td>-</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Stop C</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Stop D</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Stop E</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>Obs. Alighting</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>

Using the second equation in (B.1), and the lower bounds for all OD pairs can be derived. For example, for OD pair (A, B), the boarding count for stop A is 15, and the
alighting counts for stop B is 1. In addition, the upper bounds for all OD pairs are known. Therefore, the lower bound for OD pair (A, B) is

\[ s_{1(AB)} = \max \left( p_A - \sum_{k=i+1(k \neq j)}^{n} s_{2(i,k)}, q_B - \sum_{k=1(k \neq l)}^{j-1} s_{2(k,j)}, 0 \right) \]

\[ = \max(15 - (6 + 15 + 15), 1 - 0, 0) = 1 \]

The results for all the lower bounds are shown in Table B-3.

Table B-3: Lower bound matrix S₁ for all OD pairs in the example

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Obs. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>-</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>15</td>
</tr>
<tr>
<td>Stop B</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>Stop C</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>Stop D</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Stop E</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>Obs. Alighting</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX C

OPTIMIZATION FUNCTION USED TO OPTIMIZE THE OBJECTIVE FUNCTION IN THE MODIFIED MLE METHOD, THE KIKUCHI AND PERINCHERRY METHOD, AND THE GUR AND ELIA METHOD
The modified MLE method (Cui 2006), the Kikuchi and Perincherry method (Kikuchi and Perincherry 1992), and the Gur and Elia method (Gur and Elia 1997) are require to optimize an objective function under certain constraints. An optimization function in MATLAB is used to solve the optimization program.

To illustrate the use the optimization function, a hypothetical bus trip is used. The boarding and alighting counts are presented in Table C-1, and the base OD matrix is presented in Table C-2.

Table C-1: Boarding and alighting counts used in the example

<table>
<thead>
<tr>
<th>Stop Number</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boarding $P_i$</td>
<td>15</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Alighting $Q_j$</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
</tr>
</tbody>
</table>

Table C-2: Base matrix $t_{ij}^0$ used in the example

<table>
<thead>
<tr>
<th>Stop 1</th>
<th>Stop 2</th>
<th>Stop 3</th>
<th>Stop 4</th>
<th>Stop 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop 1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Stop 2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Stop 3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Stop 4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Stop 5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The modified MLE method is used as an example to present the method of utilizing the optimization package in MATLAB. The other two methods, Kikuchi and Perincherry method, and the Gur and Elia method, can be conducted in the similar way.

As shown in equation 2.13, the objective function for the modified MLE method is
where \( t_{ij} \) is the true passenger flow between origin stop i and destination stop j;

\( t_{ij}^0 \) is the base passenger flow between origin stop i and destination stop j;

\( p_{ij} \) is the proportion of passenger flow between stop i and stop j captured by the base matrix;

\( P_i \) is the observed boarding count at stop i;

\( Q_j \) is the observed alighting count at stop j;

For the example considered, there are ten unknown OD flows. To use the optimization function, the twenty unknowns are first ranged into a vector with consecutive index. That is, the first ten unknowns are the true passenger flow between all OD pairs, and the second ten unknowns are the proportions of passenger flow between each OD pair captured by the base OD matrix.

Then, the objective function (C.1) is re-written with the ten unknowns and numerical values of other variables as in equation (C.2).
Initialize the ten unknown variables with values of one. That is, the initialized values for all $x_i$s are one. The optimization function – fminunc is then used to solve the optimization problem. Two parameters are required as inputs to this function, objective function and initialized values for the unknowns. The objective function is presented in equation (C.2), and initialized values for all unknowns are one. The function ‘fminunc’ returns the estimated values contained in Table C-3.

Table C-3: Estimated values by ‘fminunc’

<table>
<thead>
<tr>
<th>OD flows</th>
<th>$x_1$</th>
<th>1.00</th>
<th>$x_6$</th>
<th>5.35</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_2$</td>
<td>3.69</td>
<td>$x_7$</td>
<td>2.34</td>
</tr>
<tr>
<td></td>
<td>$x_3$</td>
<td>6.65</td>
<td>$x_8$</td>
<td>5.00</td>
</tr>
<tr>
<td></td>
<td>$x_4$</td>
<td>3.66</td>
<td>$x_9$</td>
<td>2.00</td>
</tr>
<tr>
<td></td>
<td>$x_5$</td>
<td>2.31</td>
<td>$x_{10}$</td>
<td>8.00</td>
</tr>
</tbody>
</table>

| Proportions | $x_{11}$ | 1.000 | $x_{16}$ | 0.187 |
|             | $x_{12}$ | 0.271 | $x_{17}$ | 0.427 |
|             | $x_{13}$ | 0.150 | $x_{18}$ | 0.200 |
|             | $x_{14}$ | 0.273 | $x_{19}$ | 0.501 |
|             | $x_{15}$ | 0.433 | $x_{20}$ | 0.125 |

These $x_i$s are then put back to the OD matrix. Therefore, the estimated OD matrix is obtained. The estimation results are presented in Table C-4.
### Table C-4: Estimated OD matrix

<table>
<thead>
<tr>
<th></th>
<th>Stop 1</th>
<th>Stop 2</th>
<th>Stop 3</th>
<th>Stop 4</th>
<th>Stop 5</th>
<th>Est. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop 1</td>
<td>0.00</td>
<td>1.00</td>
<td>3.69</td>
<td>6.65</td>
<td>3.66</td>
<td>15.00</td>
</tr>
<tr>
<td>Stop 2</td>
<td>0.00</td>
<td>0.00</td>
<td>2.31</td>
<td>5.35</td>
<td>2.34</td>
<td>10.00</td>
</tr>
<tr>
<td>Stop 3</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>5.00</td>
<td>2.00</td>
<td>7.00</td>
</tr>
<tr>
<td>Stop 4</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>8.00</td>
<td>8.00</td>
</tr>
<tr>
<td>Stop 5</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Est. alighting</td>
<td>0.00</td>
<td>1.00</td>
<td>6.00</td>
<td>17.00</td>
<td>16.00</td>
<td>40.00</td>
</tr>
</tbody>
</table>
APPENDIX D

ALGORITHM TO ESTIMATE BUS TRIP LEVEL OD MATRIX IN THE TSYGALNITSKY METHOD
The inputs of this method are the numbers of passengers boarding and alighting at each stop along a bus route for a specific bus run and a threshold trip distance. The threshold trip distance is defined as the number of stops, under which a passenger is assumed not to ride the bus. This algorithm assigns boarding and alighting counts to all the possible cells of bus run level OD matrix based on the assumptions of Tsygalnitsky method.

Assume a bus route with \( n \) stops, and the threshold trip distance is specified as \( m \) stops. The algorithm could be described as following steps.

Step 1: Initialize the OD matrix

\[
v_{ij} = 0, e_{ij} = 0 \text{ for all } i, j \text{ for which } j < i + m
\]

\[
e_{i,i+m} = P_i \text{ for all } i = 1, \ldots, n - m
\]

where \( v_{ij} \) is volume of passengers going from stop \( i \) to stop \( j \);

\( e_{ij} \) is number of passengers who boarded at stop \( i \) that are eligible to alight at stop \( j \);

\( P_i \) is the observed boarding count at stop \( i \).

Step 2: Compute the OD flows for column \( j \) using the equally likelihood assumption. Start from \( j = m+1 \)

\[
e_j = \sum e_{ij}
\]
\[ f_j = Q_j / e_j \quad \text{(D.4)} \]

\[ v_{ij} = f_j e_j \text{ for } i = 1, \ldots, j - m \quad \text{(D.5)} \]

\[ e_{i,j+1} = e_{ij} - v_{ij} \quad \text{(D.6)} \]

where \( f_j \) is the alighting probability for stop \( j \);

\( Q_j \) is the observed alighting count at stop \( j \).

Step 3: Check whether it reaches the last column. If not, let \( j = j + 1 \) and compute step 2 again.

A numerical example is employed to illustrate this algorithm. Table D-1 shows the boarding and alighting counts used in the example. The threshold distance is chosen as one stop.

<table>
<thead>
<tr>
<th>Stop Number</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boarding</td>
<td>15</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Alighting</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
</tr>
</tbody>
</table>

Step 1 is to initialize the OD matrix. The results are shown in table D-2, and the numbers in parentheses are \( e_{ij} \)
Table D-2: Step 1 results

<table>
<thead>
<tr>
<th>Stop</th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>(15)</td>
<td></td>
<td></td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>(10)</td>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(7)</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(8)</td>
<td>8</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sum</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td>40</td>
</tr>
</tbody>
</table>

Step 2 is to compute the OD flows for column j using the equally likelihood assumption. Start from \( j = m+1 = 1+1 = 2 \) column. The results are shown in Table D-3.

Table D-3: Step 2 results

<table>
<thead>
<tr>
<th>Stop</th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>15*(f_i = 1) (15)</td>
<td></td>
<td></td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>(10)</td>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(7)</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(8)</td>
<td>8</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sum</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td>40</td>
</tr>
</tbody>
</table>

\( f_i \) 1/15

Step 3 is to check whether \( j \) equals the number of stops. For now, \( j = 2 < 5 \). Thus, let \( j = j+1 = 3 \), and compute step 2 again. The results of step 2 when \( j = 3 \) are shown in Table D-4.

Table D-3: Step 2 results

<table>
<thead>
<tr>
<th>Stop</th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>15*(f_i = 1) (15)</td>
<td></td>
<td></td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>(10)</td>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(7)</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(8)</td>
<td>8</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sum</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td>40</td>
</tr>
</tbody>
</table>

\( f_i \) 1/15
Table D-4: Step 2 results

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>1 (15)</td>
<td>$14*\frac{1}{15}=3.5$</td>
<td>$\frac{15-1}{1}=14$</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>$10*\frac{1}{10}=2.5$</td>
<td>$\frac{10}{10}=2.5$</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{7}{7}=1$</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sum</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td>40</td>
</tr>
</tbody>
</table>

$\frac{1}{15}$ $\frac{6}{14+10}$

Conduct the same procedure for the 4th and 5th column, and the final results are shown in table D-5.

Table D-5: Estimated bus trip level OD matrix

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>1 (15)</td>
<td>3.5 (14)</td>
<td>7.14 (10.5)</td>
<td>3.36 (3.36)</td>
<td>15</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>2.5 (10)</td>
<td>5.1 (7.5)</td>
<td>2.4 (2.4)</td>
<td>10</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4.76 (7)</td>
<td>2.24 (2.24)</td>
<td>7</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8 (8)</td>
<td>8</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sum</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td>40</td>
</tr>
</tbody>
</table>

$\frac{1}{15}$ $\frac{6}{24}$ $\frac{17}{25}$ $\frac{16}{16}$
APPENDIX E

ALGORITHM TO ESTIMATE BUS TRIP LEVEL OD MATRIX IN THE LI AND CASSIDY METHOD
The inputs of this method are the numbers of passengers boarding and alighting at each stop along a bus route for a specific bus run, specification and corresponding alphas for major and minor stops, and a threshold trip distance. This algorithm assigns boarding and alighting counts to all the possible cells of bus run level OD matrix based on the assumptions of Li and Cassidy method.

A numerical example is employed to illustrate this algorithm. Table E-1 shows the boarding and alighting counts used in the example. Stop A and stop C are major stops, and other stops are minor stops. For this example, $\alpha_{major} = 0.25$ and $\alpha_{minor} = 0.50$. The threshold distance is chosen as one stop.

Table D-1: Boarding and alighting at each stop

<table>
<thead>
<tr>
<th>Stop Number</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boarding</td>
<td>15</td>
<td>10</td>
<td>7</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Alighting</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
</tr>
</tbody>
</table>

Since the threshold distance is one stop, no passenger will alight at stop A. Start to estimated the OD flows from stop B. First, the number of passengers who board at major and minor stops are calculated. Before the bus entering stop B, there are 15 passengers on-board and all of them are from stop A, a minor stop. That is, the number of passengers who boarded at minor stops is $N_a = 15$, and the number of passengers who boarded at major stops is $N_b = 0$. From Table D-1, the number of alighting passengers $n = 1$. Using equation 2.19, the estimated number of passenger who boarded at major stops is computed.
Table D-2: Step 1 computation

<table>
<thead>
<tr>
<th>$N_a$</th>
<th>$N_b$</th>
<th>$n$</th>
<th>$\hat{n}_a = \frac{(1-\alpha)N_a}{(1-\alpha)N_a + \alpha N_b} \cdot n$</th>
<th>$\hat{n}_b = n - \hat{n}_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>0</td>
<td>1</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

Therefore, the column for stop B of the estimated OD matrix is known.

Table D-3: Step 1 results

<table>
<thead>
<tr>
<th>Stop</th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Sum</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td>40</td>
</tr>
</tbody>
</table>

Then compute OD flows for the column of stop C similar as stop B.

Table D-4: Step 2 computation

<table>
<thead>
<tr>
<th>$N_a$</th>
<th>$N_b$</th>
<th>$n$</th>
<th>$\hat{n}_a = \frac{(1-\alpha)N_a}{(1-\alpha)N_a + \alpha N_b} \cdot n$</th>
<th>$\hat{n}_b = n - \hat{n}_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 – 1</td>
<td>10</td>
<td>6</td>
<td>4.85</td>
<td>6 – 4.85 = 1.15</td>
</tr>
</tbody>
</table>

Table D-5: Step 2 results

<table>
<thead>
<tr>
<th>Stop</th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>1</td>
<td>4.85</td>
<td></td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>1.15</td>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Sum</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td>40</td>
</tr>
</tbody>
</table>
Then compute OD flows for the column of stop D. For stop D, there are two upstream major stops. \( \hat{n}_a \) is then split based on the boarding numbers at each major stop. That is, there are \((15 - 1 - 4.85 =) 9.15\) passengers boarding at stop A, and there are 7 passengers boarding are stop C. Therefore, for the 10.98 alighting passengers, \( \frac{9.15}{7 + 9.15} \times 10.98 \) are from stop A, and the others are boarding at stop C.

Table D-6: Step 3 computation

<table>
<thead>
<tr>
<th>Current Stop: C ((\alpha = \alpha_{\text{minor}} = 0.50))</th>
<th>(N_a)</th>
<th>(N_b)</th>
<th>(n)</th>
<th>(\hat{n}_a = \frac{(1-\alpha)N_a}{1-\alpha + \alpha N_b} - n)</th>
<th>(\hat{n}_b = n - \hat{n}_a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 + 7 - 1 - 4.85 = 16.15</td>
<td>10 - 1.15 = 8.85</td>
<td>17</td>
<td>10.98</td>
<td>17 - 10.98 = 6.02</td>
<td></td>
</tr>
</tbody>
</table>

Table D-7: Step 3 results

<table>
<thead>
<tr>
<th>Stop</th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>1</td>
<td>4.85</td>
<td>6.22</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>1.15</td>
<td>6.02</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4.76</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Sum</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td>40</td>
</tr>
</tbody>
</table>

Then compute OD flows for the column of stop E.

Table D-8: Step 4 computation

<table>
<thead>
<tr>
<th>Current Stop: E ((\alpha = \alpha_{\text{minor}} = 0.50))</th>
<th>(N_a)</th>
<th>(N_b)</th>
<th>(n)</th>
<th>(\hat{n}_a = \frac{(1-\alpha)N_a}{1-\alpha + \alpha N_b} - n)</th>
<th>(\hat{n}_b = n - \hat{n}_a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 + 7 - 1 - 4.85 - 10.98 = 5.17</td>
<td>10 + 8 - 1.15 - 6.02 = 10.83</td>
<td>16</td>
<td>5.17</td>
<td>10.83</td>
<td></td>
</tr>
</tbody>
</table>
Table D-9: Step 4 results

<table>
<thead>
<tr>
<th></th>
<th>Stop A</th>
<th>Stop B</th>
<th>Stop C</th>
<th>Stop D</th>
<th>Stop E</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop A</td>
<td>0</td>
<td>1</td>
<td>4.85</td>
<td>6.22</td>
<td>2.93</td>
<td>15</td>
</tr>
<tr>
<td>Stop B</td>
<td>0</td>
<td>0</td>
<td>1.15</td>
<td>6.02</td>
<td>2.83</td>
<td>10</td>
</tr>
<tr>
<td>Stop C</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4.76</td>
<td>2.24</td>
<td>7</td>
</tr>
<tr>
<td>Stop D</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8.00</td>
<td>8</td>
</tr>
<tr>
<td>Stop E</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sum</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>17</td>
<td>16</td>
<td>40</td>
</tr>
</tbody>
</table>

Table D-9 is the estimated OD matrix for the given boarding and alighting counts using Li and Cassidy method.
APPENDIX F

ALGORITHM TO COMPUTE THE DISTANCE WEIGHT MEASURE IN THE LI
AND CASSIDY METHOD
The inputs of this method are the numbers of passengers boarding and alighting at each stop along a bus route for a specific bus run, estimated bus trip level OD matrices for each bus trip, and the distance weight measure D is computed.

A numerical example is employed to illustrate this algorithm. Table F-1 shows the boarding and alighting counts used in the example. There are two bus trips.

Table F-1: Boarding and alighting at each stop

<table>
<thead>
<tr>
<th>Stop Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trip 1 Boarding</td>
<td>2</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Trip 1 Alighting</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>Trip 2 Boarding</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Trip 2 Alighting</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>

Thus, the passenger load can be calculated by the boarding and alighting counts for bus trip shown in Table F-2.

Table F-2: Observed passenger load for each bus trip

<table>
<thead>
<tr>
<th>Stop 1 – Stop 2</th>
<th>Stop 2 – Stop 3</th>
<th>Stop 3 – Stop 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trip 1</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>Trip 2</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

Suppose the distances between the stops are all 1.0 km. Thus, the distance weighted average load for trip 1 is

\[ x^1 = \frac{2 \times 1 + 8 \times 1 + 6 \times 1}{1 + 1 + 1} = 5.33 \]

Similarly, the distance weighted average load for trip 2 is

\[ x^2 = 5.33 \]
In this example, suppose the parameters are $\alpha_{major} = 0.25$ and $\alpha_{minor} = 0.50$.

Follow the procedure in Appendix E, the estimated APM for the two bus trips is shown in Table F-3.

Table F-3: Estimated APM for the two bus trips

<table>
<thead>
<tr>
<th></th>
<th>Stop 1</th>
<th>Stop 2</th>
<th>Stop 3</th>
<th>Stop 4</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop 1</td>
<td>--</td>
<td>0</td>
<td>0.8</td>
<td>0.2</td>
<td>1</td>
</tr>
<tr>
<td>Stop 2</td>
<td>--</td>
<td>--</td>
<td>0.2</td>
<td>0.8</td>
<td>1</td>
</tr>
<tr>
<td>Stop 3</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Stop 4</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>0</td>
</tr>
</tbody>
</table>

Using the estimated APM and the boarding counts of the bus trips, the alighting counts for each bus trip can be estimation. For example, for bus trip 1, the boarding counts are 2, 6, 0 and 0 for stop 1 through stop 4. Use the estimated APM to multiply the boarding counts, an estimated OD matrix for bus trip 1 is obtained shown in Table F-4.

Table F-4: Estimated OD matrix for bus trip 1

<table>
<thead>
<tr>
<th></th>
<th>Stop 1</th>
<th>Stop 2</th>
<th>Stop 3</th>
<th>Stop 4</th>
<th>Obs. Boarding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stop 1</td>
<td>--</td>
<td>0</td>
<td>1.6</td>
<td>0.4</td>
<td>2</td>
</tr>
<tr>
<td>Stop 2</td>
<td>--</td>
<td>--</td>
<td>1.2</td>
<td>4.8</td>
<td>6</td>
</tr>
<tr>
<td>Stop 3</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Stop 4</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>0</td>
</tr>
<tr>
<td>Est. alighting</td>
<td>--</td>
<td>0</td>
<td>2.8</td>
<td>5.2</td>
<td>8</td>
</tr>
</tbody>
</table>

Using the observed boarding counts and estimated alighting counts, another set of passenger loads are calculated in Table F-5.

Table F-5: Estimated passenger load for each bus trip

<table>
<thead>
<tr>
<th></th>
<th>Stop 1 – Stop 2</th>
<th>Stop 2 – Stop 3</th>
<th>Stop 3 – Stop 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trip 1</td>
<td>2</td>
<td>8</td>
<td>5.2</td>
</tr>
<tr>
<td>Trip 2</td>
<td>6</td>
<td>8</td>
<td>2.8</td>
</tr>
</tbody>
</table>

The estimated distance weighted average load for trip 1 is 136.
\[ \hat{x}^1 = \frac{2 \times 1 + 8 \times 1 + 5.2 \times 1}{1 + 1 + 1} = 5.07 \]

The estimated distance weighted average load for trip 2 is

\[ \hat{x}^2 = \frac{6 \times 1 + 8 \times 1 + 2.8 \times 1}{1 + 1 + 1} = 5.60 \]

Thus the distance weight measure \( D \) of the estimated APM defined by the authors would be

\[ D = \sqrt{\frac{1}{J} \sum_{j=1}^{J} (\hat{x}^j - x_j)^2} = \sqrt{\frac{1}{2}[(5.07 - 5.33)^2 + (5.60 - 5.33)^2]} = 0.27 \]

Therefore, the distance weight measure \( D \) of the estimated APM is 0.27 for the parameters \( \alpha_{major} = 0.25 \) and \( \alpha_{minor} = 0.50 \).
APPENDIX G

CODES USED IN DIFFERENT METHODS
% input
stopNo=208; % number of stops on the route
loadBase=0; % 0: null base; 1: load base
balancing=1; % balancing method: 1: factor to average; 2: factor to the higher one; 3: pseudo stops;
negL=1; % deal the negative load: 1: deal; 0: not deal

% load base matrix
if loadBase==0
    % NULL base
    base=zeros(stopNo);
    for i=1:stopNo
        for j=i+1:stopNo
            base(i,j)=1;
        end
    end
elseif loadBase==1
    load('BaseB.txt');
    base=BaseB;
end
if balancing==3 | negL==1
    base=[ones(1,stopNo);base;zeros(1,stopNo)];
    base=[zeros(stopNo+2,1),base,ones(stopNo+2,1)];
    base(stopNo+2,stopNo+2)=0;
end

% load on and off
OnOff=load('T_R2_AM5.txt');
error1=0;
convergence=0.0001;
time=1000;
run=[0;find(diff(OnOff(:,1))<0);size(OnOff,1)];
run_summary=[];
for i=1:size(run,1)-1
    % for each run
    subOnOff=OnOff(run(i)+1:run(i+1),:);
    Ton=sum(subOnOff(:,3));
    Toff=sum(subOnOff(:,4));
    run_summary(i,1)=sum(subOnOff(:,3));
    run_summary(i,2)=sum(subOnOff(:,4));
    if balancing==1
        avg=(Ton+Toff)/2;
Fon=avg/Ton;
Foff=avg/Toff;
subOnOff(:,3)=Fon*subOnOff(:,3);
subOnOff(:,4)=Foff*subOnOff(:,4);
run_summary(i,3)=Fon; % on factor
run_summary(i,4)=Foff; % off factor
elseif balancing ==2
    high=max(Ton,Toff);
    Fon=high/Ton;
    Foff=high/Toff;
    subOnOff(:,3)=Fon*subOnOff(:,3);
    subOnOff(:,4)=Foff*subOnOff(:,4);
    run_summary(i,3)=Fon; % on factor
    run_summary(i,4)=Foff; % off factor
elseif balancing==3
    diff=Ton-Toff;
    if diff>0
        subOnOff=[999 999 0 0;subOnOff;-999 999 0 diff];
    elseif diff<0
        subOnOff=[999 999 abs(diff) 0;subOnOff;-999 999 0 0];
    elseif diff==0
        subOnOff=[999 999 0 0;subOnOff;-999 999 0 0];
    end
    run_summary(i,3)=subOnOff(1,3);
    run_summary(i,4)=subOnOff(length(subOnOff),4);
end
if negL==1 & balancing<3
    subOnOff=pse(subOnOff);
    run_summary(i,6)=subOnOff(1,3);
    run_summary(i,7)=subOnOff(length(subOnOff),4);
elseif negL==1 & balancing==3
    subOnOff=pse1(subOnOff);
    run_summary(i,6)=subOnOff(1,3);
    run_summary(i,7)=subOnOff(length(subOnOff),4);
end
if size(subOnOff,1)<stopNo+2        % check if the some of the stops are omitted in the raw data
    temp=zeros(stopNo,4);
    temp1=zeros(1,4);
    temp2=temp1;
    for j=1:length(subOnOff)
        if subOnOff(j,1)==999
            temp1=subOnOff(j,:);
        elseif subOnOff(j,1)==-999
            temp2=subOnOff(j,:);
        else
            temp(subOnOff(j,1),:)=subOnOff(j,:);
        end
    end

    if size(temp,1)<stopNo+2
        temp1= zeros(1,4);
        temp2= temp1;
        for j=1:length(temp)
            if temp(j,1)==999
                temp1(temp1(j,:));
            elseif temp(j,1)==-999
                temp2= temp2(temp2(j,:));
            else
                temp(temp(j,:))=temp(j,:);
            end
        end
    end
end
```matlab
if balancing==3 | negL==1
    temp=[temp1;temp;temp2];
else
    % if all the stops are in the raw data
    on=subOnOff(:,3);
    off=subOnOff(:,4);
end

if (sum(on)==0 | sum(off)==0)
    continue
end

[od loop_time(i)]=ipf(base,on,off,convergence,time);
if balancing==3 | negL==1
    temp=sum(od(:,1:2),2);
    od(:,1)=temp;
    od(:,2)=[];

    temp=sum(od(:,size(od,2)-1:size(od,2)),2);
    od(:,size(od,2))=temp;
    od(:,size(od,2)-1)=[];

    temp=sum(od(1:2,:),1);
    od(1,:)=temp;
    od(2,:)=[];

    temp=sum(od(size(od,1)-1:size(od,1),1);
    od(size(od,1),)=temp;
    od(size(od,1)-1,:)=[];
end

run_summary(i,5)=sum(sum(od));
od_hat{i,1}=od;
end

*************

ipf.m
*************

function [X,con_time]=ipf(base,on,off,cri,time)
n=size(base,1);
X=base;
% cri=0.01;
con_time=0;
done=0;
```
while ~done
    % for the row
    for j=1:size(base,1)
        rMargin=sum(X(j,:));
        if rMargin~=0
            rf=on(j)/rMargin;
            X(j,:)=rf*X(j,:);
        else
            X(j,:)=X(j,:);
        end
    end
    % for the column
    for i=1:size(base,2)
        cMargin=sum(X(:,i));
        if cMargin~=0
            cf=off(i)/cMargin;
            X(:,i)=cf*X(:,i);
        else
            X(:,i)=X(:,i);
        end
    end
    for i=1:n
        rMargin=sum(X(i,:));
        if rMargin~=0
            rf(i,1)=on(i)/rMargin;
        else
            rf(i,1)=1;
        end
    end
    judge1=rf>1+cri*ones(n,1);
    judge2=rf<1-cri*ones(n,1);
    % check the converse
    if sum(judge1)+sum(judge2)<1
        done=1;
    elseif con_time>time
        done = 1;
    else
        con_time=con_time+1;
    end
end

*************
pse.m
*************

%%% add pseudo stop
function pseudoOnOff=pse(OnOff)
%%% calculate the load
on=OnOff(:,3);
off=OnOff(:,4);
front_pse=0;
end_pse=0;

on=[front_pse;on;0];
off=[0;off;end_pse];
N=length(on);

time=load=zeros(N,1);
for i=2:N
    load(i,1)=load(i-1,1)+on(i-1,1)-off(i-1,1);
end

done=0;
while ~done
    elig=load-off;
    id=find(abs(elig)<0.0000001);
    if isempty(id)
        done=1;
    else
        temp=elig(id,:);
        mm=max(abs(temp));
        front_pse=front_pse+mm;
        on(1,1)=front_pse;
        for i=2:N
            load(i,1)=load(i-1,1)+on(i-1,1)-off(i-1,1);
        end
    end
end

done=0;
while ~done
    elig=load-off;
    id=find(abs(elig)<0.0000001);
    if isempty(id)
        done=1;
    else
        temp=elig(id,:);
        mm=max(abs(temp));
        front_pse=front_pse+mm;
        on(1,1)=front_pse;
        for i=2:N
            load(i,1)=load(i-1,1)+on(i-1,1)-off(i-1,1);
        end
    end
end

time=load(N,1);
done=1;
end_pse=load(N,1);
off(N,1)=end_pse;

front_pse=[999 999 front_pse 0];
end_pse=[-999 999 0 end_pse];
pseudoOnOff=[front_pse; OnOff; end_pse];

**********************
pse1.m
**********************

%% add pseudo stop
function pseudoOnOff=pse1(OnOff)
%% calculate the load
on=OnOff(:,3);
off=OnOff(:,4);
N=length(on);
load=zeros(N,1);
for i=2:N
    load(i,1)=load(i-1,1)+on(i-1,1)-off(i-1,1);
end
load(i,1)=load(i-1,1)+on(i-1,1)-off(i-1,1);
end

done=0;
while ~done
    elig=load-off;
    id=find(elig<-0.0000001);
    if isempty(id)
        done=1;
    else
        temp=elig(id,:);
        mm=max(abs(temp));
        on(1,1)=on(1,1)+mm;
        for i=2:N
            load(i,1)=load(i-1,1)+on(i-1,1)-off(i-1,1);
        end
    end
end
end_pse=load(N,1);
off(N,1)=load(N,1);
OnOff(:,3)=on;
OnOff(:,4)=off;
pseudoOnOff=OnOff;

*************
cgls_main.m
*************

%%%%%%
% input
stopNo=208;   % number of stops on the route
loadBase=0;   % 0: null base; 1: load base
balancing=1;  % balancing method: 1: factor to average; 2: factor to the higher one; 3: pseudo stops;
negL=1; % deal the negtive load: 1:deal; 0: not deal

%%%%%%
% load base matrix
if loadBase==0
    % NULL base
    base=zeros(stopNo);
    for i=1:stopNo
        for j=i+1:stopNo
            base(i,j)=1;
        end
    end
elseif loadBase==1
    load('base_high_AM');
    base=base1;
end
if balancing==3 | negL==1
    base=[ones(1,stopNo);base;zeros(1,stopNo)];
    base=[zeros(stopNo+2,1),base,ones(stopNo+2,1)];
    base(stopNo+2,stopNo+2)=0;
end

%%%%%%%%
% load on and off
OnOff=load('T_R2_PM5.txt'); % the format is [stop number; time; on counts; off counts]

error1=0;
run=[0;find(diff(OnOff(:,1))<0);size(OnOff,1)];
run_summary=[];
for i=1:size(run,1)-1
    % for each run
    subOnOff=OnOff(run(i)+1:run(i+1),:);
    Ton=sum(subOnOff(:,3));
    Toff=sum(subOnOff(:,4));
    run_summary(i,1)=sum(subOnOff(:,3));
    run_summary(i,2)=sum(subOnOff(:,4));
    if balancing==1
        avg=(Ton+Toff)/2;
        Fon=avg/Ton;
        Foff=avg/Toff;
        subOnOff(:,3)=Fon*subOnOff(:,3);
        subOnOff(:,4)=Foff*subOnOff(:,4);
        run_summary(i,3)=Fon;   % on factor
        run_summary(i,4)=Foff;  % off factor
    elseif balancing ==2
        high=max(Ton,Toff);
        Fon=high/Ton;
        Foff=high/Toff;
        subOnOff(:,3)=Fon*subOnOff(:,3);
        subOnOff(:,4)=Foff*subOnOff(:,4);
        run_summary(i,3)=Fon;   % on factor
        run_summary(i,4)=Foff;  % off factor
    elseif balancing==3
        diff=Ton-Toff;
        if diff>0
            subOnOff=[999 999 0 0;subOnOff;-999 999 0 diff];
        elseif diff<0
            subOnOff=[999 999 abs(diff) 0;subOnOff;-999 999 0 0];
        elseif diff==0
            subOnOff=[999 999 0 0;subOnOff;-999 999 0 0];
        end
        run_summary(i,3)=subOnOff(1,3);
        run_summary(i,4)=subOnOff(length(subOnOff),4);
if negL==1 & balancing<3
    subOnOff=pse(subOnOff);
    run_summary(i,6)=subOnOff(1,3);
    run_summary(i,7)=subOnOff(length(subOnOff),4);
elseif negL==1 & balancing==3
    subOnOff=pse1(subOnOff);
    run_summary(i,6)=subOnOff(1,3);
    run_summary(i,7)=subOnOff(length(subOnOff),4);
end

if size(subOnOff,1)<stopNo % check if the some of the stops are omitted in the raw data
    temp=zeros(stopNo,4);
    temp1=zeros(1,4);
    temp2=temp1;
    for j=1:length(subOnOff)
        if subOnOff(j,1)==999
            temp1=subOnOff(j,:);
        elseif subOnOff(j,1)==-999
            temp2=subOnOff(j,:);
        else
            temp(subOnOff(j,1),:)=subOnOff(j,:);
        end
    end
    if balancing==3
        temp=[temp1;temp;temp2];
    end
    on=temp(:,3);
    off=temp(:,4);
else % if all the stops are in the raw data
    on=subOnOff(:,3);
    off=subOnOff(:,4);
end

if (sum(on)==0 | sum(off)==0)
    continue
end

[od err(i)]=cgls(base,on,off);
if balancing==3
    od=od(2:stopNo+1,2:stopNo+1);
end

run_summary(i,5)=sum(sum(od));
od_hat{i,1}=od;
end
function [X err]=cgls(base, on, off)

stopNo=length(on);
% seq
seq=zeros(stopNo);

k=0;
for i=1:stopNo
    for j=i+1:stopNo
        k=k+1;
        seq(i,j)=k;
    end
end

org_seq=seq;

p=on;
q=off;

id=find(p==0);
seq(id,:)=[];
id=find(q==0);
seq(:,id)=[];
p=p(find(p));
q=q(find(q));

id=find(seq);
if isempty(id)
    X=[];
    err=999;
else
    iid=[];
    mm=0;

    for i=1:size(seq,1)
        kk=sum(seq(i,:));
        if kk==0
            mm=mm+1;
            iid(mm)=i;
        end
    end
    if ~isempty(iid)
        seq(iid,:)=[];
        p(iid)=[];
    end
end
end

iid=[]; mm=0;
for i=1:size(seq,2)
    kk=sum(seq(:,i));
    if kk==0
        mm=mm+1;
        iid(mm)=i;
    end
end
if ~isempty(iid)
    seq(:,iid)=[];
    q(iid)=[];
end

% define the constraints
N=length(find(seq)); % # of variables
k=0;
ind=zeros(size(seq));
for i=1:size(seq,1)
    for j=1:size(seq,2)
        if seq(i,j)
            k=k+1;
            ind(i,j)=k;
        end
    end
end
% for the on
k=0;
A=zeros(1,N);
for i=1:size(ind,1)
    lin=ind(i,:);
    k=k+1;
    for j=1:size(ind,2)
        if lin(j)==0
            A(k,lin(j))=1;
        end
    end
end
% for the off
for i=1:size(ind,2)
    lin=ind(:,i);
    k=k+1;
    for j=1:size(ind,1)
        if lin(j)==0
            A(k,lin(j))=1;
        end
    end
end
\begin{verbatim}
end
end

b=[p,q];
%%%%%%%%
% remove the sum(on)=sum(off)
K=N;
C=size(b,1);
b(C,:)=[];
A(C,:)=[];
r=b;
R=A;
% V=zeros(K);
% for i=1:K
%    V(i,i)=1;
% end
t0=zeros(K,1);
base_k=0;
for i=1:size(seq,1)
    for j=1:size(seq,2)
        if seq(i,j)
            base_k=base_k+1;
            id_base=find(org_seq==seq(i,j));
            t0(base_k)=base(id_base);
        end
    end
end

t0=zeros(K,1);
base_k=0;
for i=1:size(seq,1)
    for j=1:size(seq,2)
        if seq(i,j)
            base_k=base_k+1;
            id_base=find(org_seq==seq(i,j));
            t0(base_k)=base(id_base);
        end
    end
end

f=sum(q)/sum(t0);
t=(f*t0+R'*inv(R*R')*(r-R*t0*f));
if det(R*R')<0.1
    err=999;
else
    err=0;
end
od_hat=zeros(stopNo);
k=0;
for i=1:size(seq,1)
    for j=1:size(seq,2)
        if seq(i,j)
            k=k+1;
        end
    end
end
\end{verbatim}
id_base = find(org_seq == seq(i,j));
od_hat(id_base) = t(k);
end
end
end
X = od_hat;
end

cmle_main.m
*************
tic
clear
cle

%%% % input
stopNo = 208; % number of stops on the route
loadBase = 0; % 0: null base; 1: load base
balancing = 1; % balancing method: 1: factor to average; 2: factor to the higher one; 3: pseudo stops;
negL = 1; % deal the negative load: 1: deal; 0: not deal
max_loop = 500;
Dist = 0; % 0 is Poisson; 1 is Multinomial
Conv = 0.00001;

%%% % load on and off
OnOff = load('T_R2_PM5.txt'); % the format is [stop number; time; on counts; off counts]
run = [0; find(diff(OnOff(:,1)) < 0); size(OnOff,1)];
run_summary = [];
for i = 1:size(run,1)-1
  % load base matrix
  if loadBase == 0
    % NULL base
    base = zeros(stopNo);
    for ii = 1:stopNo
      for jj = ii+1:stopNo
        base(ii,jj) = 1;
      end
    end
  elseif loadBase == 1
  % base
load('base_high_AM');
base=base1/100+0.000001;
end

if balancing==3 | negL==1
    base=[ones(1,stopNo);base;zeros(1,stopNo)];
    base=[zeros(stopNo+2,1),base,ones(stopNo+2,1)];
    base(stopNo+2,stopNo+2)=0;
end

% for each run
subOnOff=OnOff(run(i)+1:run(i+1),:);
Ton=sum(subOnOff(:,3));
Toff=sum(subOnOff(:,4));
run_summary(i,1)=sum(subOnOff(:,3));
run_summary(i,2)=sum(subOnOff(:,4));
if balancing==1
    avg=(Ton+Toff)/2;
    Fon=avg/Ton;
    Foff=avg/Toff;
    subOnOff(:,3)=Fon*subOnOff(:,3);
    subOnOff(:,4)=Foff*subOnOff(:,4);
    run_summary(i,3)=Fon; % on factor
    run_summary(i,4)=Foff; % off factor
elseif balancing ==2
    high=max(Ton,Toff);
    Fon=high/Ton;
    Foff=high/Toff;
    subOnOff(:,3)=Fon*subOnOff(:,3);
    subOnOff(:,4)=Foff*subOnOff(:,4);
    run_summary(i,3)=Fon; % on factor
    run_summary(i,4)=Foff; % off factor
elseif balancing==3
    diff=Ton-Toff;
    if diff>0
        subOnOff=[999 999 0 0;subOnOff;999 999 0 diff];
    elseif diff<0
        subOnOff=[999 999 abs(diff) 0;subOnOff;-999 999 0 0];
    elseif diff==0
        subOnOff=[999 999 0 0;subOnOff;-999 999 0 0];
    end
    run_summary(i,3)=subOnOff(1,3);
    run_summary(i,4)=subOnOff(length(subOnOff),4);
end

if negL==1 & balancing<3
    subOnOff=pse(subOnOff);
    run_summary(i,6)=subOnOff(1,3);
run_summary(i,7)=subOnOff(length(subOnOff),4);
elseif negL==1 & balancing==3
    subOnOff=pse1(subOnOff);
    run_summary(i,6)=subOnOff(1,3);
    run_summary(i,7)=subOnOff(length(subOnOff),4);
end

if size(subOnOff,1)<stopNo % check if the some of the stops are omitted in the raw data
    temp=zeros(stopNo,4);
    temp1=zeros(1,4);
    temp2=temp1;
    for j=1:length(subOnOff)
        if subOnOff(j,1)==999
            temp1=subOnOff(j,:);
        elseif subOnOff(j,1)==-999
            temp2=subOnOff(j,:);
        else
            temp(subOnOff(j,1),:)=subOnOff(j,:);
        end
    end
    if balancing==3 | negL==1
        temp=[temp1;temp;temp2];
    end
    on=temp(:,3);
    off=temp(:,4);
else % if all the stops are in the raw data
    on=subOnOff(:,3);
    off=subOnOff(:,4);
end

id=find(on<10^(-6));
on(id,:)=0;
id=find(off<10^(-6));
off(id,:)=0;

if (sum(on)==0 | sum(off)==0)
    continue
end

id_on=find(on);
id_off=find(off);

id=find(on==0);
on(id,:)=[];
base(id,:)=[];
id=find(off==0);
base(:,id)=[];
off(id,:)=[];
\[ [M,N] = \text{size}(\text{base}); \]
\[ T_{\text{inter}} = \text{base}/1000; \]
\[ B_{\text{inter}} = \text{sum}(T_{\text{inter}},2); \]
\[ a0 = B_{\text{inter}}./\text{on}; \]
\[ a0(\text{isnan}(a0)) = 0; \]
\[ A_{\text{inter}} = \text{sum}(T_{\text{inter}}); \]
\[ b0 = A_{\text{inter}}./\text{off}; \]
\[ b0(\text{isnan}(b0)) = 0; \]
\[ T_{\text{est}} = []; \]
\[ \text{for} \ iii = 1:\text{max}_\text{loop} \]
\[ \quad \text{a}_\text{inter} = \text{CMLE}_\text{Iter}(T_{\text{inter}}, \text{on}, b0, \text{Dist}); \% \text{Row balance} \]
\[ \quad \text{b}_\text{inter} = \text{CMLE}_\text{Iter}(T_{\text{inter}}', \text{off}, a_\text{inter}, \text{Dist}); \% \text{Column balance} \]
\[ \]
\[ \quad \text{if} \ \text{Dist} = 0 \]
\[ \quad \quad T_{\text{est}} = T_{\text{inter}}./((\text{ones}(M,1)\ast(b_\text{inter}+1))+a_\text{inter}\ast\text{ones}(1,N)); \]
\[ \quad \text{else} \]
\[ \quad \quad T_{\text{est}} = T_{\text{inter}}./((\text{ones}(M,1)\ast(b_\text{inter}))+a_\text{inter}\ast\text{ones}(1,N)); \]
\[ \quad \text{end} \]
\[ \]
\[ \text{Marg} = \text{sum}(T_{\text{est}},2); \]
\[ \text{if} \ \text{max(abs(Marg-on))} < \text{Conv} \]
\[ \quad \text{break}; \]
\[ \text{else} \]
\[ \quad a0 = a_\text{inter}; \]
\[ \quad b0 = b_\text{inter}; \]
\[ \text{end} \]
\[ \text{end} \]
\[ \text{if} \ \sim\text{isempty}(T_{\text{est}}) \]
\[ \quad \text{loop}\_\text{time}(i,1) = iii; \]
\[ \quad \text{od} = \text{zeros}(\text{stopNo}); \]
\[ \quad \text{for} \ kkk = 1:\text{size(id}\_\text{on},1) \]
\[ \quad \quad \text{for} \ kkkk = 1:\text{size(id}\_\text{off},1) \]
\[ \quad \quad \quad \text{od}(\text{id}\_\text{on}(kkk,1), \text{id}\_\text{off}(kkkk)) = T_{\text{est}}(kkk,kkkk); \]
\[ \quad \quad \text{end} \]
\[ \quad \text{end} \]
\[ \quad \text{od}\_\text{hat}(i,1) = \text{od}; \]
\[ \quad \text{run}\_\text{summary}(i,5) = \text{sum}(\text{sum}(\text{od})); \]
\[ \text{end} \]
\[ \text{end} \]

***************

CMLE_Iter.m

***************

function Para_est=CMLE_Iter(T0,TOT,Alpha,Dist,MaxLoop) \%Dist=0 Possion,Dist=1 Multinomial
if nargin<6

153
MaxLoop=100;
end
Len=length(TOT);
Len2=length(Alpha);
Temp=-(T0>0)*diag(Alpha);
Temp(Temp==0)=-10000;
Tempt_max=max(Temp,[],2);

if Dist==0
    Lx=Tempt_max-1;
    %Ux=min(10-1-(Alpha))*ones(Len,1);
    Ux=ones(Len,1);
    while 1
        T_test=T0./(ones(Len,1)*(Alpha+1)+Ux*ones(1,Len2));
        T_test(isnan(T_test))=0;
        y0=sum(T_test,2)-TOT;
        y_F=find(y0>0);
        if ~isempty(y_F)
            Ux(y_F)=Ux(y_F)+1;
        else
            break;
        end
    end
elseif Dist==1
    Lx=Tempt_max;
    %Ux=min(10-(Alpha))*ones(Len,1);
    Ux=ones(Len,1);
    while 1
        T_test=T0./(ones(Len,1)*Alpha+Ux*ones(1,Len2));
        T_test(isnan(T_test))=0;
        y0=sum(T_test,2)-TOT;
        y_F=find(y0>0);
        if ~isempty(y_F)
            Ux(y_F)=Ux(y_F)+1;
        else
            break;
        end
    end
end
x0=(Lx+Ux)/2;
y=[];
x=[];
L=[];
U=[];
for i=1:MaxLoop
if Dist==0
    T_temp=T0./(ones(Len,1)*(Alpha+1)'+x0*ones(1,Len2));
elseif Dist==1
    T_temp=T0./(ones(Len,1)*Alpha'+x0*ones(1,Len2));
end
T_temp(isnan(T_temp))=0;
y0=sum(T_temp,2)-TOT;
%     y=[y,y0];
%     x=[x,x0];
%     L=[L,Lx];
%     U=[U,Ux];
if max(abs(y0))<0.00001
    Para_est=x0;
    return;
else
    Ux(y0<0)=x0(y0<0);
    Lx(y0>0)=x0(y0>0);
    x0=(Ux+Lx)/2;
    Para_est=x0;
end
*************
Mcmle_main.m
*************
clear
clc
N=5;
% NULL base
base=zeros(N);
for i=1:N
    for j=i+1:N
        base(i,j)=1;
    end
end
global on off;
% load on and off
OnOff=load('Exb.txt'); % the format is [stop number; time; on counts; off counts]
run=[0;find(diff(OnOff(:,1))<0);size(OnOff,1)];
for ii=1:size(run,1)-1
    % on=[30 19 15 16 0];
    % off=[0 2 9 34 35];
    subOnOff=OnOff(run(ii)+1:run(ii+1),:);
    on=subOnOff(:,3);
off=subOnOff(:,4)';
t0=ones(20,1)*0.5;

t = fminunc (@Mcmle,t0);
k=0;
for i=1:N
    for j=i+1:N
        k=k+1;
        X(i,j)=t(k);
    end
end
od_hat{ii,1}=X;
end

**********
Mcmle
**********

function loglik=Mcmle(x)
    global on off;
    loglik=(-((-x(1)*x(11)+log(x(1)*x(11)))+(-x(2)*x(12)+log(x(2)*x(12)))+(-
    x(3)*x(13)+log(x(3)*x(13)))+
    (-x(4)*x(14)+log(x(4)*x(14)))+(-x(5)*x(15)+log(x(5)*x(15)))+(-
    x(6)*x(16)+log(x(6)*x(16)))+
    (-x(7)*x(17)+log(x(7)*x(17)))+(-x(8)*x(18)+log(x(8)*x(18)))+(-
    x(9)*x(19)+log(x(9)*x(19)))+
    (-x(10)*x(20)+log(x(10)*x(20)))+
    (-x(1)+x(2)+x(3)+x(4))+on(1)*log(x(1)+x(2)+x(3)+x(4))-log(factorial(on(1))))+...
    (-x(5)+x(6)+x(7))+on(2)*log(x(5)+x(6)+x(7))-log(factorial(on(2))))+...
    (-x(8)+x(9))+on(3)*log(x(8)+x(9))-log(factorial(on(3))))+...
    (-x(10)+on(4)*log(x(10))-log(factorial(on(4))))+...
    (-x(1)+off(2)*log(x(1))-log(factorial(off(2))))+...
    (-x(2)+x(5)+off(3)*log(x(2)+x(5))-log(factorial(off(3))))+...
    (-x(3)+x(6)+x(8)+off(4)*log(x(3)+x(6)+x(8))-log(factorial(off(4))))+...
    (-x(4)+x(7)+x(9)+x(10)+off(5)*log(x(4)+x(7)+x(9)+x(10))-log(factorial(off(5)))));

**********
kiku_main.m
**********

clear
clc

on=[15 10 7 8 0];
off=[0 1 6 17 16];
N=5;
global z;
z=[1 6 15 15 6 10 10 7 7 8];
ind=zeros(N);
k=0;
for i=1:N
    for j=i+1:N
        k=k+1;
        ind(i,j)=k;
    end
end
k=0;
A=zeros(1,N);
for i=1:size(ind,1)
    lin=ind(i,:);
    k=k+1;
    for j=1:size(ind,2)
        if lin(j)==0
            A(k,lin(j))=1;
        end
    end
end

% for the off
for i=1:size(ind,2)
    lin=ind(:,i);
    k=k+1;
    for j=1:size(ind,1)
        if lin(j)==0
            A(k,lin(j))=1;
        end
    end
end

Aeq=A;
beq=[on off];
t0=ones(10,1);
% # of variables 10

t = fmincon (@kiku,t0,[],[],Aeq,beq);
k=0;
for i=1:N
    for j=i+1:N
        k=k+1;
        X(i,j)=t(k);
    end
end
***************
kiku.m
***************

function f=kiku(x)
global z;
f=0;
for i=1:10
    f=f+min(2*x(i)/z(i),2-2*x(i)/z(i));
end
f=-f;

***************
yehuda_main.m
***************

clear
clc

N=5;

global on off re_base A;
% NULL base
base=zeros(N);
for i=1:N
    for j=i+1:N
        base(i,j)=1/(N-i);
    end
end
re_base=zeros(10,1);
k=0;
for i=1:N
    for j=1+i:N
        k=k+1;
        re_base(k)=base(i,j);
    end
end
on=[15 10 7 8 0;15 9 8 8 0];
off=[0 1 6 17 16;0 1 3 17 19];
t0=ones(10,1);

ind=zeros(N);
k=0;
for i=1:N
    for j=i+1:N
        k=k+1;
        ind(i,j)=k;
    end
end
function f=yehuda(x)
global on off re_base A;

% runNo=2;
% stopNo=5;
% apm=zeros(stopNo);
% k=0;
% for i=1:stopNo
%   for j=i+1:stopNo
%       k=k+1;
%       apm(i,j)=x(k);
%   end
% end

% calculate the component of A
dif_off=0;
for t=1:runNo
    for i=(1+size(A,1)/2):size(A,1)
        est_off=A(i,:)*x;
        dif_off=(est_off-off(t,i-5))^2+dif_off;
    end
end
com_A=dif_off;

% calculate component of B
q=sum(on);
re_q=[ones(4,1)*q(1);ones(3,1)*q(2);ones(2,1)*q(3);ones(1,1)*q(4)];
B=0;
for i=1:10
    B=re_q(i,1)*x(i)*(log(x(i)/re_base(i))-1)+B;
end
com_B=B;

alpha=1;
beta=1;
f=com_A+alpha*beta*com_B;

tic
clear
clear
%% input
stopNo=208;  % number of stops on the route
balancing=1; % balancing method: 1: factor to average; 2: factor to the higher one; 3: pseudo stops;
negL=1; % deal the negative load: 1: deal; 0: not deal

% load on and off
OnOff=load('T_R2_AM5.txt'); % the format is [stop number; time; on counts; off counts]
load cumuDistR2SB
dist=diff(dist)/1000;
% dist=[2;2;2;2];
major=[1;25;58;62;76;85;95;109;113;129;142;155;167;173;196;208]; % for R2 SB
% major=[1;15;27;32;61;72;81;91;105;110;125;127;137;142;164]; %for R10 WB
% major=[1;3];
n_alpha=10;
alpha_step=1/n_alpha;
for d_alpha=1:n_alpha-1
    alpha_major=alpha_step*d_alpha;
end
for dd_alpha=1:n_alpha-1
    alpha_minor=alpha_step*dd_alpha;
    % alpha_major=0.9;
    % alpha_minor=0.5;
L=1; % threshold distance in km
alpha=[alpha_major,alpha_minor];

% function
error1=0;

% OnOff=sortrows(OnOff,2);
run=[0;find(diff(OnOff(:,1))<0);size(OnOff,1)];
run_summary=[];
for i=1:size(run,1)-1
    % for each run
    subOnOff=OnOff(run(i)+1:run(i+1),:);
    Ton=sum(subOnOff(:,3));
    Toff=sum(subOnOff(:,4));
    run_summary(i,1)=sum(subOnOff(:,3));
    run_summary(i,2)=sum(subOnOff(:,4));
    if balancing==1
        avg=(Ton+Toff)/2;
        Fon=avg/Ton;
        Foff=avg/Toff;
        subOnOff(:,3)=Fon*subOnOff(:,3);
        subOnOff(:,4)=Foff*subOnOff(:,4);
    end
end
run_summary(i,3)=Fon; % on factor
run_summary(i,4)=Foff; % off factor
elseif balancing == 2
    high=max(Ton,Toff);
    Fon=high/Ton;
    Foff=high/Toff;
    subOnOff(:,3)=Fon*subOnOff(:,3);
    subOnOff(:,4)=Foff*subOnOff(:,4);
    run_summary(i,3)=Fon; % on factor
    run_summary(i,4)=Foff; % off factor
elseif balancing==3
    diff=Ton-Toff;
    if diff>0
        subOnOff=[999 999 0 0;subOnOff;-999 999 0 diff];
    elseif diff<0
        subOnOff=[999 999 abs(diff) 0;subOnOff;-999 999 0 0];
    elseif diff==0
        subOnOff=[999 999 0 0;subOnOff;-999 999 0 0];
    end
    run_summary(i,3)=subOnOff(1,3);
    run_summary(i,4)=subOnOff(length(subOnOff),4);
end
if negL==1 & balancing<3
    subOnOff=pse(subOnOff);
    run_summary(i,6)=subOnOff(1,3);
    run_summary(i,7)=subOnOff(length(subOnOff),4);
elseif negL==1 & balancing==3
    subOnOff=pse1(subOnOff);
    run_summary(i,6)=subOnOff(1,3);
    run_summary(i,7)=subOnOff(length(subOnOff),4);
end
if negL==1 | balancing==3
    subOnOff(2,3)=sum(subOnOff(1:2,3));
    subOnOff(2,4)=sum(subOnOff(1:2,4));
    subOnOff(1,:)=[];
    subOnOff(size(subOnOff,1)-1,3)=sum(subOnOff(size(subOnOff,1)-1:1:size(subOnOff,1),3));
    subOnOff(size(subOnOff,1)-1,4)=sum(subOnOff(size(subOnOff,1)-1:1:size(subOnOff,1),4));
    subOnOff(size(subOnOff,1),:)=[];
end
if size(subOnOff,1)<stopNo % check if the some of the stops are omitted in the raw data
temp=zeros(stopNo,4);
temp1=zeros(1,4);
temp2=temp1;
for j=1:length(subOnOff)
    if subOnOff(j,1)==999
        temp(j,:)=subOnOff(j,:);
        temp1(4)=temp1(4)+subOnOff(j,4);
        temp2=999;
    end
end
if size(temp,1)>0
    temp2=sum(temp1(4));
    temp2=temp2/size(temp,1);
    subOnOff(1,:)=[temp1(1:3) temp2]
end
if size(subOnOff,1)<stopNo
temp=zeros(stopNo,4);
temp1=zeros(1,4);
temp2=temp1;
for j=1:length(subOnOff)
    if subOnOff(j,1)==999
        temp(j,:)=subOnOff(j,:);
        temp1(4)=temp1(4)+subOnOff(j,4);
        temp2=999;
    end
end
if size(temp,1)>0
    temp2=sum(temp1(4));
    temp2=temp2/size(temp,1);
    subOnOff(1,:)=[temp1(1:3) temp2]
end
end
temp1 = subOnOff(:,);
elseif subOnOff(j,1) == -999
    temp2 = subOnOff(:,);
else
    temp(subOnOff(j,1),:) = subOnOff(j,);
end
end
if balancing == 3
    temp = [temp1; temp; temp2];
end
on = temp(:,3);
off = temp(:,4);
else
    % if all the stops are in the raw data
    on = subOnOff(:,3);
off = subOnOff(:,4);
end
if (sum(on)==0 | sum(off)==0)
    continue
end
od = licassi(on, off, alpha, L, major);
% if balancing == 3
%    od = od(2:stopNo+1,2:stopNo+1);
% end
run_summary(i,5) = sum(sum(od));
end_hat{i,1} = od;
%%%%%%%
% output
runNo = size(run,1)-1;
avgOD = zeros(size(od_hat{1}));
APM = avgOD;
stopNo = size(avgOD,1);
k = 0;
id = [];
for i = 1:size(od_hat,2)
    if isempty(od_hat{i})
        k = k + 1;
        id(k) = i;
    end
end
matrix = avgOD;
for i = 1:size(od_hat,1)
if ~isempty(od_hat{i})
    matrix=matrix+od_hat{i};
end
end
avgOD=matrix/runNo;

for i=1:stopNo
    marginal=sum(avgOD(i,:));
    if marginal~=0
        APM(i,:)=avgOD(i,:)/marginal;
    end
end

% fitness assessment of alighting probabilities
obs_load=zeros(runNo-1,1);
pre_load=zeros(runNo-1,1);
for i=1:size(run,1)-1
    subOnOff=OnOff(run(i)+1:run(i+1),:);
    if size(subOnOff,1)<stopNo        % check if the some of the stops are omitted in the raw data
        temp=zeros(stopNo,4);
        for j=1:length(subOnOff)
            temp(subOnOff(j,1),:)=subOnOff(j,:);
        end
        on=temp(:,3);
        off=temp(:,4);
    else                            % if all the stops are in the raw data
        on=subOnOff(:,3);
        off=subOnOff(:,4);
    end
    pre_off=APM'*on;
    obs_load(i,1)=busLoad(on,off,dist);
    pre_load(i,1)=busLoad(on,pre_off,dist);
end
D(d_alpha,dd_alpha)=(sum((obs_load-pre_load).^2)/runNo)^0.5;
% D=(sum((obs_load-pre_load).^2)/runNo)^0.5;
end
end
fix_D=min(min(D));
for i=1:n_alpha-1
    for j=1:n_alpha-1
        if fix_D==D(i,j)
            alpha_major=alpha_step*i;
            alpha_minor=alpha_step*j;
        end
    end
end
function od=licassi(on,off,alpha,L,major)
nStop=length(on);
od=zeros(nStop);
eOD=od;

% Na_off=0;
% Nb_off=0;

for i=2:nStop
    if 1+L<=i
        eStop=(1:i-L)';
        if isempty(eStop)
            continue
        end
        eStopNo=size(eStop,1);
        kk=0;
        eMajor=[];
        for j=1:size(major,1)
            if ~isempty(find(eStop-major(j)==0))
                kk=kk+1;
                eMajor(kk,1)=find(eStop-major(j)==0);
            end
        end
        temp=eStop;
        if ~isempty(eMajor)
            temp(eMajor)=[];
        end
        eMinor=temp;
        if size(eStop,1)==1
            eOD(eStop,i)=on(eStop);
        end
        % if i==104
        %  i
        % end

    end
    % end
    Na=sum(eOD(eStop(eMajor),i));
    Nb=sum(eOD(eStop(eMinor),i));
    % if Na<0.000001 & Nb<0.000001
    %  if on(i-1)==0

end
end

alpha=[alpha_major,alpha_minor]
fix_D
toc
% od(i-1,i-1)=on(i-1);
% continue
% else
% continue
% end

if find(i==major)
    i_alpha=alpha(1);  % major stop
else
    i_alpha=alpha(2);  % minor stop
end

n=off(i);
if Na~=0
    na_hat=n*(1-i_alpha)*Na/((1-i_alpha)*Na+i_alpha*Nb);
else
    na_hat=0;
end

nb_hat=n-na_hat;
if na_hat>Na
    na_hat=Na;
    nb_hat=n-na_hat;
end

majorNo=size(eMajor,1);
minorNo=size(eMinor,1);

if Na~=0
    od(eStop(eMajor),i)=na_hat*(eOD(eStop(eMajor),i)/Na);
else
    od(eStop(eMajor),i)=0;
end
if Nb~=0
    od(eStop(eMinor),i)=nb_hat*(eOD(eStop(eMinor),i)/Nb);
else
    od(eStop(eMinor),i)=0;
end

% Na_off=Na_off+na_hat;
% Nb_off=Nb_off+nb_hat;
if i<nStop
    eOD(:,i+1)=eOD(:,i)-od(:,i);
    eOD(i-L+1,i+1)=on(i-L+1);
    id=find(eOD<10^-10);
    eOD(id)=0;
end
end
end
function f=busLoad(on, off, distance)

N=length(on);
callLoad=zeros(N-1,1);

callLoad(1,1)=on(1);
for j=2:N-1
    callLoad(j,1)=callLoad(j-1,1)+(on(j)-off(j));
end

disLoad=zeros(N-1,1);
for i=1:N-1
    disLoad(i,1)=callLoad(i,1)*distance(i);
end

f=mean(disLoad);