REDUCED ORDER MODEL DEVELOPMENT FOR FEEDBACK CONTROL OF CAVITY FLOWS

DISSERTATION
Presented in Partial Fulfillment of the Requirements for the Degree Doctor in Philosophy in the Graduate School of The Ohio State University

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The Ohio State University
2008

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Controlling the flow over aerodynamic bodies has been a challenging problem for many years. Different open loop control techniques have been used in several flow configurations with some degree of success. However, in most cases the effectiveness of the controller is limited to the design conditions. In the present work, Proper Orthogonal Decomposition (POD) is used to derive low dimensional models of the subsonic flow over a cavity, in an effort to develop a feedback control system that can control the characteristic of the flow field. The Galerkin method is used as an additional tool to capture the time evolution of the flow field, reducing the problem into a system of ordinary differential. The stochastic estimation method is then used to link the variables that can be physically measured with those involved in the model. Particle Image Velocimetry (PIV) data and surface pressure measurement for the unforced flow (baseline) and for several open loop forcing conditions are used to derive the models. Three different approaches are investigated for control input separation. Different combinations of the flow condition are used in the model derivation to determine which forced flow should be used as a general case. A feedback controller is designed and tested experimentally for each model. The results showed that the variation in the experimental SPL spectra between the different models was negligible. However, a closer look at other
factors hinted that the actuation mode separation method (M1) using the white noise forcing is the best choice. This method of separation does not require a clear identification of the control input region in the data. Also, it generates the best results in terms of reducing the tone and the OASPL while using a lower power input to achieve it. The white noise forcing helps to simplify the derivation process, as there is no need to pre-identify a specific forcing case. The multiple time estimation provides the best results in terms of the amplitude, but the implementation of this procedure is restricted by the hardware limitation.
Dedicated to my Family
ACKNOWLEDGEMENTS

I would like to thank my adviser Dr. Mo Samimy for giving me the opportunity to conduct this work. His support, help and advice have been of great value for my personal and professional development and will always be appreciated.

I want to give a special thanks to Dr Andrea Serrani, Dr. Marco DeBiasi, Dr. Onder Efe, Dr. Peng Yan, Dr. Khiwan Kim and Mr. Jesse Little, for all their help in the acquisition of the experimental data and in the implementation of the feedback controllers. To Dr. Xin Yuan and Mr. Coşku Kasnakoğlu for their help in the design of the feedback controller, and to Dr. James DeBonis and Mr. James Malone, for providing us with the numerically simulated (LES) data used in the initial stage of this work.

I want to thank Dr. Dietmar Rempfer, for all his help and comments about the separation method used in this research.

A very special recognition to my wife who has supported me during this project and through all the years together. To my Kids, Edgar Jose, Edgar Jesus and Elizabeth, who every day gave me reasons to keep improving and moving ahead. To my parents who have given me their support throughout my life.
I also want to acknowledge my friends Dr. James Hileman, Dr. Brian Thurow, Dr. Jeff Kastner and Mr. Doug Mitchell for their help and support during the development of this work, especially in the preparation of this manuscript.

The support of this research by the AFRL and the AFOSR through the Collaborative Center of Control Science (Contract F33615-01-2-3154) is greatly appreciated.
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CHAPTER 1

INTRODUCTION

The ability to control flow over aerodynamic surfaces is of great importance since the flow conditions dictate the overall behavior and efficiency of the systems. Hence, a great deal of research effort has been devoted to controlling or modifying the dynamic characteristics of various types of flows with the goal of improving the operating conditions. Examples of flow control include noise and drag modification, lift enhancement and turbulent mixing control. To accomplish these objectives, various approaches have been explored ranging from passive to active control. In passive control techniques, which are straightforward to implement and have widespread applications, control is accomplished by geometrical modifications to the flow system. Generally such types of control perform well in conditions for which they were designed, but can have negligible or adverse effects in off-design conditions. In active control techniques external energy (mass and/or momentum, heat) is added to the flow. This type of controller can be implemented in open or closed-loop fashion. In general they have been used in open-loop, where the actuation takes place based on an operator’s command or a predetermined input. This type of controller also show good results for the design conditions, but cannot adapt to dynamic flight environments. In recent years, closed-loop or feedback flow control has begun to gain popularity within the fluid dynamics
community as it has the potential to increase the efficiency of fluid dynamic devices over a range of operating conditions. In this case information from one or more sensors in the flow along with a flow model guides the actuation process.

Feedback control has been used successfully in a wide range of applications in areas such as robotics, aircraft dynamics, telecommunication, transportation systems, manufacturing systems, and chemical processes. Despite of its success, feedback control of aerodynamic flows has only received focused attention in recent years (Cattafesta et al., 1998; Gad-el-Hak, 2000; Williams et al., 2000; Kegerise et al., 2002; Rowley and Williams, 2003; Cattafesta et al., 2003; Samimy et al., 2003; Siegel et al., 2003; Gerhard et al., 2003; Glauser et al., 2004; Tadmor et al., 2004; Samimy et al. 2007). The main reason for this lack of attention is due to the complexity introduced by the Navier-Stokes equations, which are a set of highly non-linear partial differential equations. For this type of systems of equations it is difficult to implement the tools of classical control system theory, as the systems not only display spatial continuity and nonlinear behavior, but also present difficult modeling challenges due to their infinite dimensionality. In order to overcome the complexity imposed by the system and successfully implement a closed-loop control strategy, a reduced-order dynamic model of the flow needs to be obtained. This model should capture the important dynamics of the flow and actuation while remaining sufficiently simple to allow its use in model-based feedback control design.

In the process of developing feedback control laws several methods can be used. For instance, Traub et al. (2003) and Patel et al. (2003) have developed successful approaches making use of experimental data for the development of models and used them to implement feedback control algorithms. This method has the advantage that the control
law can sometimes be developed with a small amount of experimental data. On the other hand, this and other similar approaches for the design of flow-control laws has a weakness in that these laws are developed either on an ad-hoc basis or for a model that may not sufficiently capture the dynamics of the fluid/actuator interaction over a wide range of operating conditions and desired outputs. In the case of cavity resonance, a more flexible approach of system identification has been used to develop low-order models (e.g. Cattafesta et al., 2003; Rowley and Williams, 2003). This work focuses on a further step in the design of more general and robust control laws by introducing a systematic methodology for order reduction of models based on the Navier-Stokes equations. These low-dimensional flow models are then used to derive control laws for the system.

As mentioned before, one of the major problems with the design of a model based closed-loop flow control system is the complexity of the governing flow equations and processes involved in reducing them. A well-known technique for deriving low dimensional models in the fluid dynamics community is the proper orthogonal decomposition (POD). The method has been used to study the dynamic behavior of several flows including boundary layer (Aubry et al., 1988), planar mixing layers (Delville et al., 1999), axisymmetric jet mixing layer (Glauser et al., 1999; Citriniti and George, 2000) and cavity flows (Ukeiley et al., 2000; Rowley et al., 2004; Samimy et al. 2007). This technique uses the spatial correlation tensor in a flow to extract eigenmodes – the most dynamically significant features in the flow (and the only entities that can effectively be controlled), but does not preserve the time evolution of these eigenmodes. The time evolution of the eigenmodes can be obtained in different ways. The most direct
method is to project the instantaneous flow field onto the POD basis, but this requires time-resolved measurements of the flow, which are quite challenging. This can also be done with numerically simulated data. Another method is to project the Navier-Stokes equations onto the dominant POD eigenmodes using Galerkin methods in order to derive a set of ordinary differential equations that can predict, at least in an overall sense, the behavior of the flow (e.g. Holmes et al., 1996; Gordeyev and Thomas, 2000; Noack et al. 2003). This results in a set of nonlinear ordinary differential equations, which are used for controller design. The equations are autonomous and not useful for controller design purposes since the controller input is not explicit. Consequently, they must be recast in a form expressing the control input explicitly so that a feedback controller can be designed using the tools of control theory (Efe and Özbay, 2003; Caraballo et al., 2007).

The application selected in this dissertation to develop a reduced order model for control design is the flow over a shallow cavity. This is a configuration relevant to many practical applications, (i.e. landing gears and weapon bays), that have been extensively studied in the literature. Rossiter (1964) first developed an empirical formula to predict the frequencies of cavity flow resonance, referred to as Rossiter frequencies or modes. This empirical expression was later modified and improved by Heller and Bliss (1975). The concept of a dominant mode of oscillation was also investigated by Rossiter and was later observed by others to coincide with the natural longitudinal cavity acoustic mode (Rockwell and Naudascher, 1978). When this condition occurs, the cavity flow is subject to a strong single-mode resonance (Williams et al., 2000); otherwise multiple modes generally exist in the flow. The strong single-mode resonance conditions is also observed when there is interaction between Rossiter modes and the natural transversal cavity
acoustic modes (e.g. Ziada et al., 2003; DeBiasi and Samimy, 2004). When the flow is dominated by a multi-mode condition it has been observed a rapid switching between modes (Cattafesta et al., 1998; Williams et al., 2000; DeBiasi et al., 2004). This rapid and seemingly random switching between multiple modes places large bandwidth and fast time response requirements on both the actuation scheme and feedback control algorithm.

The strong single-mode resonance in cavity flows generates negative effects on the structures exposed to the flow, such as vibration of the landing gear or on the weapons support. To reduce the negative effects, extensive study of flow control over a cavity has been carried out in the past, ranging from the use of passive techniques up to closed-loop controllers. The passive techniques include rigid fixed fences, spoilers, ramps (Heller and Bliss, 1975; Sarno and Franke, 1994; Ukeiley et al., 2004) and cylinders or rods placed in the boundary layer near the leading edge of the cavity (McGrath and Shaw, 1996; Stanek et al., 2003). These are inexpensive, reliable, simple devices, but may not work well at off-design conditions since they have little or no ability for adjustment to changing flow conditions. In recent years different open-loop control strategies have been used with varying degrees of success (e.g., Shaw, 1998; Stanek et al., 2003; Grove et al., 2003; DeBiasi and Samimy, 2004). There have also been significant efforts to investigate closed-loop control approaches (Cattafesta et al., 1997; Shaw and Northcraft, 1999; Cattafesta et al., 1999; Williams et al., 2000; Kegerise et al., 2002; Williams et al., 2002; Rowley et al., 2002; Cabell et al., 2002; Caraballo et al., 2005 and 2007). Although, the results of these closed-loop endeavors are encouraging, they also indicate that many issues remain to be resolved and numerous opportunities exist for further advancement of the technology. While we have examined other control approaches in recent years
(Debiasi et al., 2004; Efe et al., 2005; Yan et al., 2006), our primary objective from the onset has been the development of control techniques based on reduced-order models of the cavity flow (Samimy et al., 2004 and 2007; Yuan et al., 2005; Caraballo et al., 2005 and 2007).

The objective of this work is to derive a low dimensional model of a shallow cavity flow. This model must capture the dynamics of the flow as well as the effect of the control input. The model is then used to design a feedback control system that can be implemented and tested experimentally. This work is an integral part of a larger effort to develop and implement model based closed-loop flow control technologies by the Collaborative Center of Control Science (CCCS) at the Ohio State University (Samimy et al., 2004).

Chapter 2 will present a brief description of the flow facility and data acquisition methods used in this work. Chapter 3 introduces the definition of the POD and its application to the experimental data. Chapter 4 will cover the application of the Galerkin methods adopted for deriving the reduced-order model. Chapter 5 will focus on the implementation of the stochastic estimation approach, used for real-time estimation of the flow model variables directly from dynamic surface pressure measurements. Chapter 6 presents the first separation method derived for feedback control design and its experimental implementation. This is followed in chapter 7 by the derivation of the additional separation methods of the control input. Finally, chapter 8 will present the discussion of the results followed by the conclusion and recommendations in chapter 9.
CHAPTER 2

EXPERIMENTAL AND NUMERICAL DATA FOR REDUCED ORDER MODELING

Derivation of reduced order models for the cavity flow requires data sets of the velocity field with good temporal and/or spatial resolution. Ideally, data sets that have good resolution in both time and space are desirable. However, it is very difficult to obtain such a set with the present experimental capabilities, or it is costly and time consuming with numerical simulations. For the current study, spatially resolved data is used and has been obtained with two different methods. The initial data set was obtained from numerical simulation of the cavity flow, based on a Large Eddy Simulation (LES) code. This numerical simulation provided extensive time-resolved spatial information of the flow but for short overall periods of time, as the computer cost and storage space increase very quickly with time. Due to some issues with the numerical simulations, which will be discussed below, the latest data sets were obtained from experimental measurements of the cavity flow facility at the GDTL. These sets are composed of surface pressure measurements at limited locations, where large time series of data is available but with limited spatial resolution. Additionally, planar velocity measurements were obtained, with good spatial resolution but no temporal information.
In this chapter the general characteristics of the flow used in the current investigation will be presented. Also, a brief explanation of the two data sources is provided, as well as its relevance to the overall model derivation process is explained.

2.1 Shallow Cavity Flow Characteristics

Development of flow control methodologies that could be applied to a variety of aerodynamic configurations was one of the main thrust of the Collaborative Center of Control Science (CCCS) when it was created. In particular, the goal of the aerodynamic flow control group was to design a model based feedback control algorithm. As mentioned in chapter 1, a subsonic flow over a shallow cavity was selected as the test bed. This flow is dominated by a self sustained resonance phenomena, with strong amplitude of oscillation. Figure 2.1 shows a schematic of the shallow cavity flow. The process begins with the natural Kelvin-Helmholtz instability. Disturbances in the flow are amplified over the cavity to form large scale coherent structures which eventually impinge on the cavity trailing edge. The impact generates an acoustic wave that travels back thru the flow and excites the flow at the leading edge of the cavity, also known to be the receptivity region. When the two effects couple together self sustained oscillations are formed. The oscillations are the main characteristic in shallow cavity flows. The same phenomenon makes the process sensitive to actuation or forcing at the receptivity region, as it can break the coupling effect that sustains the oscillation. These characteristics make this flow configuration a good candidate for reduced order modeling and for the development and implementation of feedback control.
The cavity designed for the present study has the following geometric characteristics: a square test section with a width \( W = 50.8 \text{ mm (2 in)} \), a depth \( D = 12.7 \text{ mm (1/2 in)} \) and a length \( L = 50.8 \text{ mm (2 in)} \), for an aspect ratio \( L/D = 4 \). Details of the facility can be found in Debiasi and Samimy (2004). Based on the modified empirical formula of Rossiter (Heller and Bliss, 1975) the possible resonance frequencies, which correlate with the number of structures present in the cavity flow, can be estimated as a function of some of the flow variables such as the Mach number, the cavity length (L) and the free stream velocity.

![Figure 2.1. Schematic of the cavity flow.](image)

Figure 2.2 shows the amplitude, sound pressure levels (SPL), and frequency characteristics of the cavity as a function of the Mach number. Also shown are the lines corresponding to the first four Rossiter modes (R1-R4) predicted by the modified Rossiter formula (Heller and Bliss, 1975), the 1st longitudinal acoustic mode based on the
cavity length ($L_1$), and the 1$^{st}$ and 2$^{nd}$ transversal acoustic modes based on tunnel height ($T_1$, $T_2$). In the regions near the intersections of the predicted Rossiter modes with the transversal or the longitudinal acoustic modes we observed strong resonant tones. The observation of the interaction between Rossiter and transversal acoustic modes is similar to that of Ziada et al. (2003) who explored low subsonic cavity flows.

![Spectral characteristics of cavity resonance for various Mach numbers measured with a transducer located in the middle of the cavity floor (Samimy et al., 2007).](image)

Figure 2.2. Spectral characteristics of cavity resonance for various Mach numbers measured with a transducer located in the middle of the cavity floor (Samimy et al., 2007).

Based on the results shown in Fig. 2.2, the flow conditions selected for the development of reduced order models are those for which only a strong single resonant peak was observed experimentally. The initial idea was to use data generated by
numerical simulations to develop the reduced order models. The simulated data was obtained for a Mach 0.3 flow for several cases; the baseline (non-actuated) and some open-loop forced conditions. For the experimental data, the facility allows for continuous variation of the Mach number form $M = 0.2$ up to $M = 0.7$. However, only Mach 0.30 was chosen for the model derivation, since it represents a case at which the actuator displayed significant authority. An ideal study of this type would use both numerical and experimental data. Unfortunately, the numerical data was unable to accurately capture the dynamics of the cavity flow. Consequently, this data was used only to develop the Matlab routines used for the derivation of the reduced order models and for control design. The experimental data was used to design the controllers that are implemented in real time. In the next sub-sections a description of the two sources of data with their benefits and limitations will be explained.

2.2 Numerical Simulations

The objective of the numerical simulation was to provide space and time-resolved data for the low-dimensional modeling of the cavity flow with and without actuation. The data needed to be well resolved in both time and space because the traditional method used for the modeling requires hundreds of sets of instantaneous “snapshots” of the flowfield over many convective time scales. Computational Fluid Dynamics (CFD) is uniquely suited to provide this information because the entire flowfield is solved for at every time step. Unsteady CFD solutions on large grids over many convective time scales requires an extremely large amount of computing resources and time. In order to minimize and, more importantly, to reduce turn-around time, several options were
explored. These included examining two-dimensional (2-D) versus quasi three-dimensional (3-D) simulations, reducing the amount of turbulence modeling and reducing the necessary grid size through use of advanced computational methods.

The general details of the simulation code have been presented in DeBonis and Scott (2002). In the initial stage, two-dimensional simulations were done in order to increase the turn-around time of the analyses. Large amounts of data are required to obtain reasonable spectra for comparison with the experimental results, in addition to the required simulation of the flow with different Mach numbers and control inputs. While it is recognized that the two-dimensional solutions do not accurately capture the details of the turbulent motion, the physical overall mechanisms of the cavity resonance should be properly accounted for.

Figure 2.3 shows an instantaneous image of the velocity field from the numerical simulation at $M = 0.30$. The instantaneous velocity vector field is superimposed on the absolute velocity contours. It can be clearly noticed that vortical structures are present in the cavity. The image also highlights one of the main benefits of using numerical simulated data: the detail of resolution obtained for the structures present in the flow.

The initial goal of obtaining numerical data was to reproduce the behavior of the experimental results, so that the low dimensional model and the control algorithm derived with these data could be used in the experiments. This has two advantages; first in a relatively short period of time, large sets of data with good spatial and time resolution can be obtained. Secondly, it reduces the complexity of the experimental setup for data acquisition and amount of experimental data required. Figure 2.4 shows the SPL spectrum in the middle of the cavity floor from the numerical simulations and the
experimental data. It can be noticed that, although the geometry and main flow characteristics of the code match the experiment, the simulation spectrum shows high broadband pressure levels, below 1000 Hz. In addition, numerical data did not reproduce the resonance peak seen in the experimental results. This prevented the derivation and implementation of a controller based on this data, since the numerical data does not represent the experimental observations. In order to overcome this problem and keep moving towards the final goal of developing the controller based on the low dimensional model, the experimental data acquisition system was upgraded to allow extensive simultaneous velocity and pressure field measurements to derive the low dimensional model and design the controller.

![Instantaneous velocity field from numerical simulation at Mach 0.30 flow over the cavity; vector field superimposed over absolute velocity contours.](image)

Figure 2.3. Instantaneous velocity field from numerical simulation at Mach 0.30 flow over the cavity; vector field superimposed over absolute velocity contours.
Figure 2.4. Experimental and computational spectra from a 4 blocks average, each of 512-point FFT.

It is important to mention, that although the numerical data did not match the experimental results, and the goal of developing a controller that could be implemented in the experiments was not achieved, it was very useful for the development of the MATLAB codes used in the processing of the data and in the design of a feedback controller. More details of the results obtained using the numerical data can be found in Caraballo et al. (2004).

2.3 The Experimental Facility and Techniques

As a result of the problems encountered with the numerical simulations of the cavity flow, the next step was to obtain the required data from experimental measurements. The experimental facility is described in detail in Debiasi and Samimy (2004). It is an
instrumented, optically accessible wind tunnel that operates in a blow-down fashion with atmospheric exhaust. The filtered, dried air is conditioned in a stagnation chamber before entering a smoothly contoured converging nozzle leading to the 50.8 mm by 50.8 mm test section. As mentioned above, the facility can run continuously in the subsonic range between Mach 0.20 and 0.70.

![Diagram](image)

Figure 2.5. Schematic of the experimental setup showing the incoming flow, the actuation location and other geometrical details.

For control, the flow is forced at the cavity shear layer receptivity region by a 2-D synthetic jet type actuator, issuing at 30 degrees relative to the main flow from a 1 mm slot embedded in the cavity leading edge spanning the width of the cavity, Fig. 2.5. A Selenium D3300Ti compression driver provides the mechanical oscillations necessary to create the zero net mass, non-zero net momentum flow for actuation. The actuator signals are produced by either a BK Precision 3011A function generator for open-loop forcing or
by a dSPACE 1103 digital signal processor control board in closed-loop studies and are amplified by a Crown D-150A amplifier in both cases.

The “snapshots” of the flow field, required for the development of the low dimensional models, are acquired and processed using a LaVision Inc. PIV system. Details of the PIV system, procedure, and results are presented in Little et al. (2007). The main flow is seeded with submicron (Di-Ethyl-Hexyl-Sebacat) particles using a 4-jet atomizer upstream of the stagnation chamber. This location allows homogenous dispersion of the particle seed throughout the test section. A dual-head Spectra Physics PIV-400 Nd:YAG laser operating at the 2nd harmonic (532 nm) is used in conjunction with spherical and cylindrical lenses to form a thin (~1 mm), vertical sheet spanning the length of the cavity at the middle of test section width. In order to minimize beam reflections, a small slot cut into the cavity floor allows the laser sheet to exhaust and diffuse in a sealed light-trap. The time separation between the lasers pulses used for PIV can be tuned according to the flow velocity, seeding density and interrogation window size. For Mach 0.30 flow in our facility this value is 1.8 microseconds. Two images, corresponding to the pulses from each laser head, were acquired by a 2000 by 2000 pixel CCD camera equipped with a 90 mm macro lens with a narrow band-pass optical filter. The images were divided into 32 by 32 pixel interrogation windows, which contained 6-10 seed particles each. For each image, subregions were cross-correlated using multi-pass processing with 50% overlap. The resulting vector fields were post-processed to remove any remaining spurious vectors. This setup gives a velocity vector grid of 128 by 128 over the measurement domain, which translates to each velocity vector being separated by approximately 0.4 mm.
Flush-mounted Kulite® pressure transducers were placed at various locations on the walls of the test section for dynamic pressure measurements (Fig. 2.6). The sensors have a nearly flat frequency response up to about 50 kHz and are powered by a signal conditioner that amplifies and low-pass filters the signals at 10 kHz. For state estimation, dynamic pressure measurements were recorded simultaneously with the PIV measurements using a National Instruments (NI) PCI-6143 S-Series data acquisition board mounted on a Dell Precision Workstation 650. The system allows simultaneous sampling of 8 channels with a maximum sampling frequency of 250 kHz per channel. Each pressure recording was band-pass filtered between 100 Hz and 10 kHz to remove unwanted frequency components. In the current study 1000 PIV snapshots were recorded for each flow/actuation condition explored. For each PIV snapshot, 128 samples from the laser Q-switch signal and from each of the transducers of Fig. 2.6 were acquired at 50 kHz. The laser Q-switch signal indicates the time at which the PIV data was acquired. A data acquisition board was triggered by a programmable timing unit housed in the PIV system that activated the beginning of the acquisition to allow the Q-switch TTL to fall approximately in the middle of the 128 pressure data points. The simultaneous sampling of the laser Q-switch signal with the pressure signals allows synchronization of the pressure time traces and the instantaneous velocity field. Additional, longer pressure recordings of 262,144 samples per channel acquired at 200 kHz were also used to derive SPL spectra as described in Debiasi et al. (2004).
2.4 Mach 0.30 Baseline Flow Characteristics

For the baseline Mach 0.3 cavity flow case, Fig. 2.7 presents the SPL spectrogram (a) and the corresponding spectrum (b) of the surface pressure measured by transducer 5, from Fig.2.6. Measurements from the other transducers, not shown here, provide consistent results, confirming that at this Mach number the cavity flow resonates at a frequency corresponding to the third Rossiter mode with strong time-invariant flow-acoustic coupling. At other Mach numbers the cavity oscillations can exhibit rapid switching between multiple modes creating a distribution of energy over a variety of frequencies.
Figure 2.7. SPL spectrogram and spectrum of baseline Mach 0.3 cavity flow (case B) from transducer 5 (Samimy et al., 2007).
Figure 2.8 is a phase averaged PIV image of the velocity field for the baseline flow at $M = 0.30$ which clearly shows the free-stream uniform velocity and the vortical structures inside the cavity, typical of this resonant flow. The presence of three rotating structures in the cavity is noticeable and, consistent with the resonance frequency of the flow at the 3$^{\text{rd}}$ Rossiter mode. For further experimental results and discussions see Samimy et al. (2007) and Little et al. (2007)

![Phase-averaged PIV images of the Mach 0.30 flow over the cavity; vector field superimposed on an absolute velocity contour.](image)

2.5 Controlled Flow Characteristics

The development of a reduced-order model for control design requires not only data from the baseline flow, but also some actuated flow cases, as the model needs to capture the behavior of the actuated flow as well. Currently, there is no clear way to incorporate
the forcing input into the model. This is one of the issues that will be investigated in this research. Therefore, data for several flow conditions was saved for the cavity at $M = 0.30$. Table 2.1 presents a summary of the different flows conditions acquired in the experiments. Each individual forcing frequency was selected based on its effect on the baseline flow. The final forcing case used was band limited white noise (between 1 kHz and 6 kHz). This is a rich signal containing the effect of many frequencies simultaneously with no preference. The notation on Table 2.1 is introduced to simplify the identification of each flow condition throughout the work. In the later chapters, as we derive the reduce-order model, an $M$ will be added in front of the notation to indicate the results obtained for the model, e.g. MB refers to the model based on the baseline flow case. The final three rows in Table 2.1 correspond to data sets composed of the combination of images from several flow cases and are used to address the issue of what flow conditions are required for the modeling.

Figure 2.9 shows the effect of open-loop forcing, for each case shown in table 1. In each case the resonant peak in the baseline case is significantly reduced and other modes have appeared. Figure 2.9(a) shows the effect of the F1 actuation, for which the peak is reduced by almost 20 dB with the introduction of a small peak at the forcing frequency. Figure 2.9(b) illustrates the effect of the F2 forcing, which corresponds to the frequency of the second Rossiter mode for this flow (Fig. 2.2). This forcing disrupts the natural resonance and artificially induces a resonance congruous with the 2\textsuperscript{nd} Rossiter mode. The corresponding spectrogram showed that the peak induced by forcing is time-invariant, which is typical for single-mode resonance.
<table>
<thead>
<tr>
<th>Case</th>
<th>Forcing frequency, Hz</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>--</td>
<td>Baseline flow</td>
</tr>
<tr>
<td>F1</td>
<td>1610</td>
<td>Open-loop forcing</td>
</tr>
<tr>
<td>F2</td>
<td>1830</td>
<td>Open-loop forcing</td>
</tr>
<tr>
<td>F3</td>
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</tr>
<tr>
<td>F4</td>
<td>3920</td>
<td>Open-loop forcing</td>
</tr>
<tr>
<td>Wn</td>
<td>N/A</td>
<td>Open-loop forcing with band limited white noise</td>
</tr>
<tr>
<td>BF#</td>
<td>--</td>
<td>Combination of snapshots of B and F#</td>
</tr>
<tr>
<td>BWn</td>
<td>--</td>
<td>Combination of snapshots of B and Wn</td>
</tr>
<tr>
<td>BF#F#</td>
<td>--</td>
<td>Combination of snapshots of B and the two F#</td>
</tr>
</tbody>
</table>

# = 1, 2, 3 or 4 represents the open loop set used

Table 2.1. Nomenclature for the various flow conditions used for modeling

Figure 2.9(c) illustrates the effect of the F3 forcing, which is one of the optimal forcing frequencies discussed by Debiasi and Samimy (2004). This case artificially induces a multi-mode state in the flow with rapid switching between the resonant and the forcing frequency. It should be noted that, due to the subtle geometry change introduced by the laser exhaust slot in the cavity floor, the effect of the forcing presented here differs slightly from that reported in Debiasi and Samimy (2004). Figure 2.9(d) shows the effect of the F4 forcing (near 4th Rossiter mode, Fig 2.2), another optimal sinusoidal forcing which significantly reduces the resonant peak. The forcing induces a peak at its subharmonic (close to the 2nd Rossiter mode) that competes with the baseline resonant frequency on a rapid time scale thus inducing a multi-mode state. Finally, Fig. 2.9(e) shows the effect of the Wn forcing. In this case the resonant peak is reduced by almost 20 dB with a small effect close to 2 kHz, without really enhancing any additional frequency.
Figure 2.9. SPL spectra from transducer 5 for Mach 0.3 cavity flow. Thin line is the B case; thick line is with open-loop forcing at cases: (a) F1; (b) F2; (c) F3; (d) F4; (e) Wn (Samimy et al., 2007).
2.6 Comments on the Examined Data

In the process of collecting the required data one needs to ask the following question: what is the minimum number of velocity snapshots required? The answer to this question depends on the characteristics of the flow and the data acquisition method being used, as the nature of the experimental data is very different from the numerical counterpart. To address this issue the minimum number of images required was defined based on the mean turbulent kinetic energy of the flow. Although the numerical data was also checked for convergence, due to the problems explained above, only the experimental data will be discussed. The PIV images to be used should provide convergence of the turbulent kinetic energy of the flow at different locations on the domain of observation, which was limited between \( x/D = 0 \) to \( 4 \) and \( y/D = -1 \) to \( 1.5 \). Figure 2.10 show that the mean turbulent kinetic energy of the baseline flow at several locations in the shear layer converges when approximate 700 or more images are used. Therefore, to obtain the spatial basis for the reduced-order model of this flow, at least 700 images are required to assure that the set contains sufficient statistical events to represent the flow behavior.

Overall, the numerical data showed good spatial and temporal resolution, but could not match the experimental results. This limited the benefits of the resolution as well as the possibility of implementing its output in the experiment. However, it allowed us to develop the tools required for the modeling process. The experimental data brought its own challenges, but gave a real representation of the flow under study, which lead to real time implementation of the model based feedback controller. Finally, we have defined a set of operating conditions where the original flow is noticeably modified. In most cases the final condition is completely different than the original flow. These flows will be used
to study some of the issues of deriving reduced order models. In the following chapters we will present and discuss the methodology used to derive the reduced order model, as well as some of the issues associated with the modeling process.

Figure 2.10. Mean turbulent kinetic energy of the cavity flow at different locations in the shear layer (Samimy et al., 2007).
CHAPTER 3

PROPER ORTHOGONAL DECOMPOSITION

3.1 Large Scale Structures

Before the second half of the 20th century, the study of turbulence was based solely on a statistical approach. After this time the concept of coherent structures was documented. According to Berkooz et al. (1993) “it was probably first articulated by Liepmann (1952), and thoroughly exploited by Townsend (1956) who showed how the presence of coherent structure in classical shear flows can be inferred from statistical evidence.” For decades, research in the field continued to be based on statistical approaches which ignore the presence of coherent structures in the flow field. Around 1970 the role of coherent structures in turbulence became clearer and a popular topic of study and it continues to be so today.

Although the concept of a coherent structure has been widely used, there are different definitions which depend on the interest and viewpoint of the researcher in question. For example, Hussain (1986) states “a coherent structure is a connected turbulent fluid mass with instantaneously phase-correlated vorticity over its spatial extent.” However, according to Holmes et al. (1996), “Brown and Roshko (1974) favored kinematic detectors, selecting a criterion for the presence of a coherent structure based on their conception of the dynamical behavior of the structure”. At the same time, Lumley (1967)
and his co-workers, e.g. Aubry et al. (1988), Berkooz et al. (1993), focused on the problem of unambiguous and objective identification of the presence of a coherent structure in a turbulent flow, and ignored the dynamical behavior of the coherent structures. It was at this stage that Lumley introduced Proper Orthogonal Decomposition (POD) as an objective way of identifying coherent structures in shear flows (1967). Despite the different definitions, there is agreement that a coherent structure can be defined as a large scale organized structure moving in the flow that interacts with other similar structures, and eventually disintegrates into something that is neither spatially nor temporally stationary.

Coherent structures are clearly identified with flow visualization. They play an important role in entrainment and mixing, flow noise, heat transfer, drag, etc. Due to their relevance, it has been suggested that turbulence could not be modeled, much less understood, without explicitly accounting for the existence of coherent structures (Zheng, 1992). Also, understanding how they interact with the surrounding fluid and physical structure is often essential for the purpose of controlling turbulent flows (Delville et al., 1998).

3.2 Proper Orthogonal Decomposition

Although the significance of coherent structures in different flows became more accepted by the flow community, the methods of extracting these structures (e.g. through flow visualization, velocimetry techniques or statistical analysis of hot wire data) are in most cases, subjective and strongly influenced by researcher’s intuition and purposes. As mentioned above, Lumley (1967) introduced Proper Orthogonal Decomposition (POD),
in the context of turbulence, as an objective way to extract large scale structures in a turbulent flow. The method had been used with success in other fields such as image processing, and is known as the Karhunen-Loeve expansion in pattern recognition theory and image processing, or the Principal Component analysis by statistical analysts.

The POD method is a mathematical tool that is used to decompose a complex flow field into a small number of fundamental modes that captures the dominant trends in the flow field. The method provides a spatial basis (set of eigenfunctions) for a modal decomposition of an ensemble of data, obtained from an experiment or computational simulations. These eigenfunctions, or modes, are extracted from the cross correlation tensor, and can be used as bases functions to represent the flow. There is a general acceptance that the empirical eigenfunctions obtained with the POD and the coherent structures present in the flow field are related. Lumley (1981) noted that if the first mode contains a dominant percentage of the fluctuating energy, it could represent a structure. Also, as mentioned by Gordeyev and Thomas (2000), “in some cases the POD modes give a good spatial basis for the flow decomposition but may have little to do with the physical shape of the underlying coherent structure. Therefore the summation of several energetic modes can be taken to represent a large-scale structure.”

The dynamic behavior of coherent structures can be obtained by projecting the empirical eigenfunctions, the POD modes, onto instantaneous realizations of the flow field. In doing so, the temporal behavior of the modes can be reconstructed by obtaining the instantaneous value of the modal amplitude for each of the modes. Although the POD method will give a spatial basis, in order to obtain the temporal evolution of the modal amplitude directly from the data, the temporal evolution of the flow field at many
locations must be measured simultaneously. This has been a difficult task to accomplish so far due to the limitations in the experimental work. Therefore, some additional techniques have to be used to recover the time information. Among these techniques are Galerkin projections, wavelet analysis and the complementary technique, which uses the POD method in conjunction with the Linear Stochastic Estimation (LSE).

In the following sections a brief explanation of the POD method will be given. More details of POD can be found in Berkooz et al. (1993), Holmes et al. (1996) and Delville et al. (1998). Once the method is introduced, a discussion of the results obtained from applying the procedure to our experimental data for the baseline flow and for the forced cases is presented.

3.2.1 General definition of the POD method

The main idea of POD is to identify structures with the largest mean square energy in the velocity field. The most energetic fluctuations in a random field are expressed mathematically as:

\[ u_a = \langle \langle \bar{u} \phi \rangle \rangle_{\phi \phi} \]  

(3.1)

where \( \bar{u} \) is the velocity, \( \phi \) is the candidate structure or basis, with the largest mean square energy, \( \langle \cdot \rangle \) is an averaging operation and \( \langle \cdot, \cdot \rangle \) indicates the inner product according to the Hilbert-space of \( L_2 \) functions (see appendix A for the definition), defined as

\[ \langle \bar{u}, \phi \rangle = \int_{\Omega} \bar{u}(X) \phi^*(X) \, dX = \sum_{i=1}^{n_c} \int_{\Omega} \bar{u}_i(X) \phi_i^*(X) \, dX \]  

(3.2)
where \( nc \) is the number of components, \( D \) is the integration domain, the asterisk (*) indicates complex conjugate and \( x \) represent different positions in space and time \((x, y, z, t)\).

The process of maximizing the mean square energy, \( |a|^2 \), leads by calculus of variation to a linear integral equation, a Fredholm’s equation of the second kind, homogeneous, which is an eigenvalue problem, and can be written as

\[
\sum_{j=1}^{nc} \int_{D} R_{ij}(X, X') \phi_j(X') dX' = \lambda \phi_i(X) \quad (3.3)
\]

where \( x \) and \( X \) represent different positions in space and time \((x, y, z, t)\), \( \lambda \) is the eigenvalue of the equation, \( \phi_i(X) \) are the eigenfunctions that we are trying to determine and \( R_{ij} \), the kernel of the integral, is the cross correlation tensor:

\[
R_{ij}(X, X') = \langle u_i(X) u_j(X') \rangle \quad (3.4)
\]

The Hilbert-Schmidt theory assures that for Fredholm’s type equation with symmetric kernel and finite domain, the following statements are valid:

- There is a discrete set of solutions, so the equation can be written as

\[
\sum_{n=1}^{nc} \int_{D} R_{ij}(X, X') \phi^n_j(X') dX' = \lambda^n \phi^n_i(X) \quad (3.5)
\]

\( n = 1, 2, ... \) mode number

- The set of eigenfunctions are orthogonal and, when normalized, will have the following values

\[
\sum_{i=1}^{nc} \int_{D} \phi^n_i(X) \phi^m_i(X) dX = \delta_{nm} = \begin{cases} 
0 & \text{for } n \neq m \\
1 & \text{for } n = m
\end{cases} \quad (3.6)
\]

- All the eigenvalues are real and positive
$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq 0 \quad (3.7)$$

- The sum of eigenvalues is finite and equals the total turbulent kinetic energy integrated over the domain D

$$KE = \int_D \sum_{i=1}^\infty (u_i(X)u_i(X))dX = \sum_{n=1}^\infty \lambda^n \quad (3.8)$$

- The fluctuating or time dependent velocity field can then be reconstructed using the eigenfunctions, as

$$u_i(X) = \sum_{n=0}^\infty a^n \phi_i^n(X) \quad (3.9)$$

where $a^n$ is the random modal amplitude, that can be calculated from

$$a^n = \int_D u_i(X)\phi_i^n(X)dX \quad (3.10)$$

The coefficients $a^n$ are mutually uncorrelated and satisfy

$$\langle a^n.a^m* \rangle = \delta_{nm} \lambda^n = \begin{cases} 0 \text{ for } n \neq m \\ \lambda^n \text{ for } n = m \end{cases} \quad (3.11)$$

The POD method, when compared with other methods used to identify large-scale structures, has the advantage that it does not require ‘a-priori’ knowledge of the structures and requires only a set of data from the flow field. Also it has been noted (e.g. Berkooz et al., 1993) that the POD method is optimal for modeling or reconstructing a signal, $u(x, t)$, since among all the linear decompositions for a given number of modes, the projection on the subspace used for modeling will contain the most kinetic energy possible in an average sense.
Different approaches of the POD method have been used, depending on how the data are acquired and its nature. According to Delville et al. (1998), based on the way the ensemble average ($\langle \bullet \rangle$) in equation 3.4 is used to calculate the kernel of the integral equation 3.3, we have:

- The “classical” method, as proposed by Lumley (1967), where a temporal average is used, in statically stationary flows.

- The “snapshot” method, proposed by Sirovich (1987), using a spatial average for $N$ uniformly sampled discrete times $t_n = n \tau$ for $n = 1, \ldots, N$. In this case instantaneous flow fields, $u_i(x, n \tau)$, are uncorrelated, where $\tau$ is a time scale on the order of or greater than the correlation time.

- “Extended POD”, used by Glezer et al. (1989), using the ensemble average of time realizations with different initial conditions.

- A spatial average estimated for time realizations obtained with conditional measurements (Sirovich, 1989; Sirovich et al., 1990).

- The “Biorthogonal Decomposition”, proposed by Aubry et al. (1991), where a spatial or time average is estimated with space-time realizations of the flow, $u(x,t)$, without assumptions as to the properties of the signal, such as being statistically stationary.

A brief description of the first two methods will be given below. The first method will be presented due to its importance to the field, and the second method, since it will be used in this work.
3.2.1.1 Classical method

This is the original formulation introduced by Lumley (1967), in which the ensemble average is temporal. This method requires a large time resolved data set at several spatial points; the problem becomes more complex as the number of spatial points is increased. The eigenfunctions \( \phi^n(x) \) can be obtained from the integral equation (3.5) where the time dependency has been eliminated:

\[
\sum_{j=1}^{nc} \int_{\Omega} R_{ij}(x, x') \phi^n_j(x') d(x') = \lambda^n \phi^n_i(x)
\] (3.12)

where now \( x \) and \( X \) represent different positions in space \( X = (x, y, z) \), and \( R_{ij} \) is the spatial correlation tensor calculated as:

\[
R_{ij}(x, x') = \frac{1}{T} \int_{T} u_i(x, t) u_j(x', t) dt
\] (3.13)

where \( T \) is a sufficiently long, in comparison with the integral time scale, period of time for which the space-time signal \( u_i(x, t) \) is obtained.

3.2.1.2 Snapshot method

Sirovich (1987) proposed the snapshot method as an alternative way of obtaining POD modes for highly spatially resolved data sets, which can be obtained using numerical simulations or advanced laser based flow diagnostics such as Particle Imaging Velocimetry (PIV). The method requires a sufficiently large number, \( k = 1, 2... M \), of time realizations for the instantaneous field, \( u_i(x, t_k) \), with the realization being uncorrelated for different values of \( k \). Then the POD eigenfunctions can be written as linear combinations of the instantaneous flow field.
The eigenfunctions can be obtained by solving the intermediate eigenvalue problem,

$$C(t, t_k)A = \lambda^n A$$  \hspace{1cm} (3.15)

where the matrix $A$, to be determined, is the eigenbasis of this intermediate problem, and $C(t, t_k)$ is the two-point correlation tensor of independent snapshots, integrated over the spatial domain of interest, defined as:

$$C(t, t_k) = \frac{1}{M} \int_D \overline{u_i(\tilde{x}, t) u_i(\tilde{x}, t_k)} \, d\tilde{x}$$  \hspace{1cm} (3.16)

This procedure reduces the eigenvalue problem from one that depends on the number of grid points to one that depends only on the number of snapshots or ensembles ($M$) used. This method is the natural choice for our study, since the data used satisfy the required conditions. Therefore, the snapshot method is used to obtain the POD basis of the cavity flow field.

For either of the two methods, the turbulent flow field can be reconstructed using Eqn. 3.9, and the modal amplitude from Eqn. 3.10. As noted before, to obtain the modal amplitude with this expression the instantaneous flow field has to be measured or numerically calculated simultaneously at every spatial point in the flow domain of interest.

Since the data used to obtain the POD basis is time uncorrelated, the time evolution of the modal amplitude to reconstruct the dynamical behavior of the flow cannot be obtained. However, these modal amplitudes can be found in different ways. The most direct method is to project the instantaneous flow field onto the POD basis, but this
requires time-resolved measurements of the flow which is not yet possible with PIV and high speed flows. This could be accomplished with accurate numerical simulations. However this is also not realizable for high Reynolds number flow fields with complex geometry. To circumvent these limitations, one can project the flow governing equations, the Navier-Stokes equations, onto the dominant POD eigenmodes using Galerkin methods in order to derive a set of ordinary differential equations that can predict, at least in an overall sense, the behavior of the flow (e.g. Holmes et al., 1996; Noack et al., 2003; Samimy et al., 2007; Caraballo et al., 2007).

3.2.2 Inner product definition

In most of the initial POD work the method was applied on single variables, such as streamwise velocity fluctuations. This was due to limited knowledge of the velocity field (from experimental data) or simplifications on the flow under study, i.e. incompressible flow conditions or homogeneous flow on two directions. In such cases, the norm and inner product used to obtain the POD modes were based on a scalar definition. For many studies, which focus only on capturing the spatial distribution of structures, this approach is sufficient. Some questions that can be asked at this point are what happens in 2D flows where both velocity components are important and how to treat more complex compressible flows, or how to account for density variations in compressible flow?

Rowley et al. (2001) and Freund and Colonius (2002) highlighted the importance of the norm used to compute the inner product in the POD method when calculating the spatial basis as well as the modal amplitude for compressible flows. They compared the resultant POD basis and flow field reconstruction obtained using two different definitions of the norm and inner product. They observed that the POD bases were dependent on the
norm used in its derivation. In the first approach, the scalar approach, each variable is
treated as an individual quantity and the standard inner product is obtained:

$$\langle u_1, u_2 \rangle = \int_D u_1(x)u_2(x) dx$$  \hfill (3.17)

In the second approach, the vector approach, a vector, \( q(u, v, c) \) in a two-dimensional
flow is defined, taking into account all the independent variables involved in the problem,
and the norm is define in such a way that the final quantity makes physical sense (i.e. all
the variables involved have the same dimensions). For such a case, Rowley et al. (2001)
defined the inner product for a compressible flow as:

$$\langle q_1, q_2 \rangle = \int_D \left( u_1 u_2 + v_1 v_2 + \frac{2\alpha}{\gamma - 1} c_1 c_2 \right) dx$$  \hfill (3.18)

In addition to the effect on the POD modes, Rowley et al. (2001) and Caraballo et al.
(2003) investigated the effect of the modal amplitude on the solution using the Galerkin
system, noticing that for compressible flows the vector norm showed better convergence
of the modal amplitude. Consequently, in the remainder of this work only the vector
norm will be used.

### 3.3 Application of the POD Method to the Baseline Cavity Flow

First, we applied the POD method to the baseline flow to test the procedure and to
characterize the flow. It is important to study parameters involved in the process of
obtaining the spatial basis of the flow. We determined the minimum number of snapshot
required to obtain convergence in the shape of the modes, in the energy captured by each
individual mode, as well as the total energy recovered by a predefined number of modes.
Also, we looked into the effect of the number of modes used in the reconstruction of the flow.

The first aspect to be addressed is the minimum number \((M)\) of snapshots required to obtain a representative spatial basis for the flow. This is defined based on the following considerations; the number of snapshots should be high enough to show convergence of the mean kinetic energy captured by the POD modes. Figure 3.1 shows the percent of energy recovered by the POD modes, for the baseline flow, as the number of snapshots increases in intervals of 100. It is observed that about 400 snapshots are sufficient to produce convergence in the first 40 modes, which recovers about 80% of the total kinetic energy.

![Figure 3.1. Total energy recovered by the POD modes for different size data sets.](image)
The number of snapshots ($M$) for which the shape of each mode becomes stable also needed to be checked. Figure 3.2 shows the first mode of the fluctuating normal velocity $v'$ of the baseline flow obtained using 100, 500, 700 and 1000 snapshots. It can be noticed that increasing the number of snapshots from 100 to 500 introduces large changes in the structures captured by the modes. However, increasing the number of snapshots from 500 to 700 does not change the general shape of the structures present in the modes or their organization. The larger number produces a smoother (cleaner) representation of the structures captured by the modes since the small scale effects are now captured by the higher modes incorporated into the system, as the total number of modes increases with the number of images. Additionally, there is little change in the energy content of the modes, especially at the lower modes. No appreciable shape changes were observed by using more than about 800 snapshots. Based on these analyses, and the convergence of the local mean turbulent kinetic energy, explained in the previous chapter, at least 700 images are required for the derivation of the model. Therefore, all 1000 PIV snapshots of the flow field acquired, as described in the experimental section, are used in the derivation of the modes and their modal amplitude.

Figure 3.3 shows the first 12 modes of the normal fluctuating velocity ($v'$) of the baseline flow. The presence of three sets of structures can be noticed since each set is represented by a pair of light and dark contours in a given mode. It is also noticeable how the structures present in the modes become more diffuse as we look at the higher modes that capture less energy. This is expected as the lower modes, by definition, are set to capture the most dominant structures in the flow. It is clear that for the baseline the
general behavior of three structures that characterize the third Rossiter mode, which is the dominant mode for the cavity under study at $M = 0.3$, for this cavity flow configuration.

![First POD mode of the normal fluctuating velocity ($v'$) for the cavity flow for different number of snapshots.](image)

Figure 3.2. First POD mode of the normal fluctuating velocity ($v'$) for the cavity flow for different number of snapshots.

Now that the required number of snapshots has been defined, the effect on the velocity reconstruction when a small number of modes are used is investigated. The number of modes, $N$, used to reconstruct the flow field using the POD expansion depends on the nature of the problem and the purpose of the model obtained. A nominal criterion for the POD method to accurately represent the flow is that it must retain the modes necessary to capture 99% of the mean turbulent kinetic energy (Sirovich, 1987). For our cavity flow this requires about 500 POD modes. However, with about 130 POD modes
90% of the energy is recovered, which is acceptable for a subsequent analytical study of the flow dynamics but is too high for real-time control of the flow.

Figure 3.3. First 12 POD modes of the normal fluctuating velocity ($v'$) for the baseline cavity flow.

In order to evaluate the trade-off between accuracy and simplicity, the impact of decreasing the number of modes in the overall reconstruction of the flow is analyzed. Figure 3.4 compares a PIV snapshot of the baseline case with its corresponding reconstructions obtained using 130, 30, and 4 POD modes. The main reason of using instantaneous velocities instead of phase average values is to test the ability of the method to recover the details of the fluctuating field. In addition, it is not possible to obtain phase average images for all the flow conditions tested, due to the non-resonant behavior of some of the flow conditions used. The lower number of modes yields a reconstruction that, while less accurate (filtering out the smaller scales), still captures the
main features of the flow. When the number of modes is increased, it can be observed how the finer details of the PIV images start to show up in the reconstruction even though it recovers only 90% of the total turbulent kinetic energy present in the flow.

Figure 3.4. Comparison of the normal velocity fluctuation from the PIV measurements with reconstruction using 130, 30 and 4 POD modes.

3.4 POD of Forced Flows

Now that the baseline flow is characterized, we can apply the same procedure to the individual forced flow conditions tested. As mentioned in the previous chapter, the forced conditions were selected based on the behavior of the cavity under open loop forcing by sweeping the actuator frequency at different actuator voltage input levels. A few conditions that significantly affected the fluctuating pressure in the cavity were selected as candidates for POD. These data sets will be used to address some of the issues
discussed in the literature regarding the derivation of a reduced order model, especially the derivation of a model for a flow that can be used for control.

One of the limitations of the POD method, noted by Rempfer (2003), is that it cannot reproduce the actual behavior of the flow if its conditions are different from those for which the model has been obtained. Noack et al. (2003) illustrated this using numerical simulation of the wake behind a cylinder. To offset this limitation, researchers have developed two approaches. One is to create models based on several flow conditions in order to provide a richer description of the flow dynamics. Taylor and Glauser (2004) and Glauser et al. (2004) obtained a “global” POD spatial basis from combinations of several flow conditions. Another approach is to add shift modes and control modes, (Noack et al., 2003; Siegel et al., 2005), to take into account the changes in the flow condition. For the first part of this research we elected to adopt the first technique, by creating a model for individual flow conditions as well as for the combination of some of the cases, which are later used for feedback control design. In what follows, we compare and discuss the POD bases obtained for the different cases described in Table 2.1.

Figure 3.5 shows the first four POD modes of the six individual flow conditions listed in Table 2.1. As mentioned in the previous section, it can be observed that the baseline POD modes (Fig. 3.5a), exhibit three structures, each represented by a pair of a positive (light) and a negative (dark) features, consistent with the baseline flow resonating at the third Rossiter mode. The first two forced flows cases F1 (Fig. 3.5b) and F2 (Fig. 3.5c), show two sets of structures, consistent with the effect of forcing the flow near and at the 2nd Rossiter mode, respectively. As expected, the structures present in these two forced cases are larger than those of the baseline case. The F3 flow (Fig. 3.5d) is forced at a
frequency slightly higher than the natural resonance and shows a behavior similar to that of the baseline, with structures similar in size and organization. The F4 flow is forced at a frequency just below the 4th Rossiter mode and excites a subharmonic close to the 2nd Rossiter mode. This reduces the acoustic noise by creating a multi-mode resonance between the natural and the sub harmonic frequency (Yan et al., 2006). Correspondingly, we observe two structures in the first two modes; three on the third while the fourth mode is not well defined which may be linked to the multi-mode behavior of the flow. Finally, the Wn case seems to capture modes from some of the previous cases, where the first mode looks similar to mode 1 of the F1 case and the second mode looks like mode 2 of the F4 case. The third mode is similar to mode 3 of the F2 case and the fourth mode looks similar to mode 4 of the baseline or F2 case. This behavior could be due to the nature of the forcing, as the white noise is band limited between 1 kHz and 6 kHz and thus includes all the individual frequencies tested.

The size and organization of the structures captured by the POD modes seem to correlate to the modal distribution of energy. It can be seen from Fig. 3.6(a) that more energy is recovered by modes 1-3 in the F1 and F2 cases, i.e. in flows with larger, more organized shear-layer structures. In terms of cumulative modal energy balance, Fig. 3.6(b), the larger energy recovery of these modes is not compensated by the lower recovery of the successive modes 4-6. This is particularly visible for F2. Conversely, cases F3 and F4, characterized by smaller and less organized structures, exhibit a lower energy recovery in the early modes. The Wn case showed a similar behavior as the F3 and F4 cases. The energy recovery of all the higher modes appears to be similar for all the cases explored in this work.
Figure 3.5. First 4 POD modes of the normal velocity fluctuations ($v'$) for the baseline and several forced flows.
Figure 3.6. Energy recovered by the POD modes, a) total energy and b) individual mode energy for the baseline and several forced flows.
Figures 3.7 and 3.8 compare PIV snapshots of the F4 case and Wn case respectively, with its reconstructions obtained using 130, 30, and 4 POD modes. As for the baseline case, the lower number of modes yields a reconstruction that, while less accurate with filtered out smaller scales, still captures the main features of the flow. It is also noted, that as the number of modes is increased the small scale structures present in the PIV images start to show up in the velocity reconstruction.

Figure 3.7. Comparison of the normal velocity fluctuation from the PIV measurements with reconstruction using 130, 30 and 4 POD modes, for the F4 flow.

In the next set of tests, the baseline flow (B) was combined with the forced cases, F1 thru F4, to obtain the composite cases BF1, BF2, BF1F2, BF3, BF4, BF3F4 and BF1F4. Figure 3.9 shows the first 4 modes for all these combinations. It can be seen that in each combination the modes capture some of the characteristics of the individual cases. The first and fourth modes resemble the baseline flow and the second and third modes seem
to capture the forced flow behavior. Overall, the combined cases show structures of similar shape and size. Similar to the individual cases, the energy content of the modes for the BF1 and BF2 models recover a higher amount in the first two modes. Also, when the combination of more forced flow is based on similar forcing effects, BF1F2 and BF3F4, the modes look similar to the individual flows. This is not the case for the BF1F4 case, were the resultant modes seem to be leaning towards the F1 case modes. This can be related to the characteristics of each flow, one is oscillating near the 2\textsuperscript{nd} Rossiter mode (F1) while the other is a multimode resonance switching between the 2\textsuperscript{nd} and the 3\textsuperscript{rd} mode.

![Figure 3.8. Comparison of the normal velocity fluctuation from the PIV measurements with reconstruction using 130, 30 and 4 POD modes (Wn).](image)

Figure 3.8. Comparison of the normal velocity fluctuation from the PIV measurements with reconstruction using 130, 30 and 4 POD modes (Wn).
Figure 3.9. First 4 POD modes of the normal velocity fluctuations ($v'$) for different combinations of the experimental flow conditions.

Figure 3.10 shows a comparison of a PIV snapshot of the F4 case with its reconstructions obtained using 130, 30, and 4 POD modes with the BF4 basis. It can be noticed that the result with 130 modes captures most of the main characteristics of the flow, is similar to the previous cases, even though the baseline flow was also include in the POD basis.
Based on these results the number of modes to be used in the reduced order model will be dependent on the purpose of the model. If the objective of the model is to describe the flow field dynamics it will require at least 130 modes for a good reconstruction. However if the objective of the reduced order model is to be used in the design a feedback controller, it is desirable to minimize the number of modes (states to control) since the complexity of the controller increases significantly with additional modes. Based on this requirement, only 4 modes are used to create a model that captures the dynamics of the large scale structures in the cavity shear layer.

Up to this point we have decomposed the flow into a set of spatial bases that capture the dominant features in the flow, and the modal amplitude that defines the time evolution. The next step in the process of developing a reduced order model is to derive
an expression to estimate the time evolution of the modal amplitude. This is required to study the dynamic behavior of the flow, but more importantly to prepare the model for feedback control design and implementation.
CHAPTER 4

GALERKIN PROJECTION

As briefly discussed in the previous chapter, with the POD method the flow field is decomposed into a set of spatial basis and its temporal modal amplitudes, responsible for the dynamical evolution of the flow (reconstruction of the fluctuations). It was also noted that to obtain the modal amplitudes using Eqn. 3.10 the flow field has to be known at every spatial location, used for the modeling, instantaneously. Furthermore, to capture the time evolution of the flow, the knowledge of the relevant flow field properties at very small time intervals is required. This is possible with accurate numerical data, but it is very difficult to achieve experimentally. Therefore, alternative ways to obtain the evolution of the modal amplitude need to be implemented.

One way to obtain the evolution of the modal amplitudes is with the help of the Galerkin projection method. The method transforms the original governing system of nonlinear partial differential equations, the Navier-Stokes equations, into a reduced system of nonlinear ordinary differential equations that is simpler and faster to solve. A reduced number of equations from this set is then used to estimate the modal amplitudes $a^n(t)$ required in Eqn. 3.9, based on the convergence of the solutions and the number of POD modes required to capture a predefined percentage of energy.
The idea of the method is to project the equations that represent the flow under study, the fully compressible Navier-Stokes equations in this case, onto the POD basis. The number of equations to be solved in the reduced system of equations is also dependent on the norm definition. For the vector norm as defined in 3.2.2, the system will consist only of $M$ equations. In the present work, the equations are obtained using the vector approach derived by Rowley (2002), where the compressible Navier-Stokes equations are written as:

\begin{align}
\frac{Dc}{Dt} + \frac{\gamma - 1}{2} c \nabla \cdot u &= 0 \\
\frac{Du}{Dt} + \frac{2}{\gamma - 1} c \nabla c = \frac{\mu}{\rho} \nabla^2 u
\end{align}

(4.1)

where $u = (u,v)$ is the 2D velocity vector and $c$ is the local speed of sound.

Before the Galerkin method can be applied, the following steps are required: first each flow variable is decomposed into the mean and fluctuating components, and the fluctuating components are then replaced by Eqn. 3.9, the POD expansion. Next, the flow variables in Eqn. 4.1 are replaced by the expanded expressions of mean and fluctuating components. The resultant expression of the governing equations is now projected onto the POD basis, by taking the inner product of each term with the POD basis, according to the specified norm (vector approach in our case, Eqn. 3.18). Finally, the resulting system of ordinary differential equations is truncated to the number of desired modes.

After applying the Galerkin methods to Eqn. 4.1, the resultant system of differential equations to calculate the modal amplitudes, for the vector approach, has the form:

\begin{align}
\dot{a}^k(t) = b^k + \sum_{j=1}^n (d^{jk} a^j) + \sum_{j=1}^n \sum_{m=1}^n (e^{-imk} a^l a^m)
\end{align}

(4.2)
where \( b, d \) and \( e \) are constant coefficients obtained from the Galerkin projection. The number of modes to be used defines the final number of ordinary differential equations (ODE’s). The system derived using this procedure will be referred as Galerkin system (GS). The details of the derivation are shown in appendix B.

To solve this system of equations only initial condition for the flow field is required. This can be obtained by projecting an instantaneous snapshot of the velocity field onto the POD basis, using Eqn. 3.10. Figure 4.1 shows the first modal amplitude coefficient of the baseline flow obtained for 1000 PIV snapshots. This data is uncorrelated and only the amplitude is relevant at this point. It is clear that the coefficient oscillates around zero, as noted by the mean value. The modal amplitude is bounded between the values of \( \pm 0.3 \) and has a standard deviation of 0.149. These quantities will be used to evaluate the solution of the GS in Eqn.4.2 and study its performance.

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**Figure 4.1.** First modal amplitude of the baseline flow, from PIV images.
The solution of the system of equations 4.2 obtained for the baseline case often diverged. This was observed independent of the initial condition used to solve the system. We attribute this behavior to numerical errors in calculating the derivative terms in the system of equations and to the use of a finite number, \( N \), of modes to describe the flow, which not only loses some flow details, but also fails to capture the energy transfer process between the \( N \) retained modes and the neglected ones. To overcome these issues, and specifically the energy transfer problem, Couplet et al. (2003) and Noack et al. (2004) proposed the inclusion of an additional viscous term, the modal eddy viscosity, \( \tilde{\nu}_i \).

This additional viscosity term is intended to maintain the overall flow energy balance, but it also compensates for other small errors introduced in the derivation of the model. Adding the modal eddy viscosity modifies Eqn. (4.2) to:

\[
\dot{a}_i(t) = F^i + \sum_{j=1}^{N} \left[ (\nu + \tilde{\nu}_i) L^{ij} + \overline{G}^{ij} \right] a_j(t) + \sum_{j=1}^{N} \sum_{k=1}^{N} H^{ij} a_j(t) a_k(t)
\]

(4.3)

where \( \nu L^{ij} \) accounts for the viscous contribution (from the Navier-Stokes equations) of \( F^i \) and \( G^j \) in Eqn. (4.2) and \( F^i \) and \( \overline{G}^{ij} \) are the inviscid parts of the same matrices. The value of the modal eddy viscosity \( \tilde{\nu}_i \) is obtained from an energy balance of Eqn. (4.3) (Noack et al., 2004), and has the form:

\[
\tilde{\nu}_i = \nu - \frac{\overline{G}^{ii} \lambda_i + H^{ik} \{ a_i a^k \}}{\lambda_i}
\]

(4.4)

where \( \lambda_i \) is the eigenvalue of each POD mode and the operator \( \{ \} \) (not to be confused with the inner product previously defined) represents the time average of its arguments. This new term changes for each mode and adjusts for the numerical errors and energy transfer left unaccounted for by the neglected modes.
This system of equations was solved using different numbers of modes, starting from 2 up to 30 modes. Figure 4.2 compares the solution of Eqn. (4.2) and Eqn. (4.3) for the first modal amplitudes of the baseline model using $N = 4$ and $N = 20$ modes. It can be noticed that for both cases, the system without the additional viscous term diverges very quickly (dashed line), predicting very large amplitudes that grows continuously, while the system with the modal eddy viscosity evolves in a bounded fashion between the two standard deviations (dotted line) of the modal amplitudes obtained from the PIV snapshots. This was true for most of the modes used in the solution. Only in the cases of $N = 3$ and $N = 7$ the system diverged even with the additional viscous term. Additionally, for $N = 4$ modes, the dampening effect of the modal eddy viscosity seems to somewhat reduce the resonance frequency of the Galerkin system Eqn. (4.3) (from the experimental 2800 Hz to 2300 Hz). One need to be careful when comparing the frequency response of the system with the experimental values, as the GS (4.3) solves for the modal amplitude, which in terms is an average value for the entire domain, and the experimental value are for surface pressure measurements at a single location. However, it should be expected that the frequency response of the modal amplitude to be close to the frequency response of the pressure signal. It is observed that using a larger number of modes mitigates this discrepancy, but increases the complexity of the model and yields a system intractable from a control design perspective. It can be also noted in the figure that for both cases the calculated modal amplitude for the first mode oscillates around or close to zero with amplitude close to those of the PIV images. We should also note that the system trajectories of Fig. 4.2 converged to the same values, irrespective of the initial condition used for the solution of Eqn. (4.3), indicating the occurrence of a stable limit cycle.
Now that the reduced order model for the baseline case shows reasonable results, we want to see the effect of deriving the model based on the different forced flow conditions. As mentioned before, there is no clear understanding on how to derive a model that would include different flow conditions or the effect of forcing. In the present work, we will focus on these two issues by developing a control input separation method that can be implemented with the different flow conditions. We will also study the effect that the combined flow cases have on the model and in the experimental implementation of the combined flow cases.
4.1 Galerkin Projection for the Forced Flow

The Galerkin projection method was applied to each of the forced flows as well as to some combined cases. Once the POD bases are obtained for the different cases, it is straightforward to derive the coefficients of the corresponding GS to calculate the modal amplitude. The final expression of the reduced set of ordinary differential equation for each case has the same form as Eqn. 4.3, but with different values for the coefficient matrices.

Each of the systems was solved for different number of modes, beginning with 2 and going up to 20 modes. For brevity, only the solutions for 4 and 20 modes are presented here. Figure 4.3 shows the evolution of the first modal amplitude for the model based on the F1 forcing case. Similar to the baseline, the values are bounded within two standard deviations of the corresponding values from the PIV snapshots. The amplitude of the mode is reduced as the number of modes is increased. This can be explained by the fact that increasing the number of modes adds more detail and turbulent kinetic energy to the model. This allows the modes to interact among themselves and transfer energy from the lower modes to the higher ones, analogous to the energy cascade in turbulence theory. For both solutions the amplitude of the coefficient oscillates around zero. It was also noticed that the frequency value of the dominant peak in the spectrum is lower than the experimental value of 1610 Hz for the 4 mode solution. The 20 mode solution is higher but still in a close range of the experimental value. The two solutions are able to capture the lower amplitude peak at the natural resonance frequency (2800 Hz) observed in the experimental results.
Figure 4.3. Solution of the GS for the model based on F1.

Figure 4.4 shows the solution for the model based on F4 forcing. The results show similar trends as the previous 2 cases. In this case the FFT plot of the 20 mode case does not show any clear peak. This can be linked to the effect that the forcing has on the flow, which goes from single resonant to multi resonance flow. It is clear from Figs. 4.2, 4.3 and 4.4 that the modal amplitude with 4 modes always has a larger oscillation range and tends to predict a lower frequency peak than the 20 mode solution. It seems form the results in Figs. 4.1 thru 4.4 that the models based on the individual forcing have more difficulties predicting the frequency response of the system.
Figure 4.4. Solution of the GS for the model based on F4.

Figure 4.5 shows the solution model based on the Wn forcing with 4 and 9 modes, as the system in this case did not converge for 20 modes. The same behavior as the previous models can be observed. However, the frequency response for this case does not resemble the experimental value. The GS for this case have more convergence problem than all the other cases, this is not completely unexpected due to the nature of this forced flow, where the cavity tone not only has been attenuated, but is no longer time invariant and no other dominant peak appears.
Figures 4.6 and 4.7 show the solution for the models based on the combination of the baseline flow and one forced flow, BF1 and BF4. For the BF1 case, each flow contains a strong resonant tone; however, the frequency spectrum does not resemble either of the individual flow conditions used in the model, the forced and unforced flow. This could be expected as the two flows used have a single dominant frequency, but their values are not close. This is especially true for the BF1 combination, where the two flows oscillate near the 2nd and 3rd Rossiter modes. The BF4 model shows a tendency towards the baseline flow frequency, as the number of modes is increased. This can be related to the nature of the two flows used, where F4 case is a multimode case, and when combined with the baseline flow is susceptible to the influence of the strong resonance.
Figure 4.6 Solution of the GS for the model based on BF1.

Figure 4.7 Solution of the GS for the model based on BF4.
Up to now we have looked at the POD modes for the different flow and its combinations, and solved the GS to obtain the modal amplitude for each one of them. The results so far seem to indicate that the combined models do capture the general trends of the flows used, which is desirable if one wants to reconstruct different flows using a single set of bases. However, the solution of the model does not predict the frequency of particular flows used; with the exception of those cases where the resonant frequency for all the flows used are similar. Then, it would be difficult to use a single set of basis for multiple flows. Nevertheless, we cannot evaluate completely the effectiveness of combining basis from different flow cases in the derivation of reduced order models, since the current model doesn’t involve a control input term explicitly. We will return to address this issue after the control term is explicitly shown in the system of equations for the modal amplitude.

At this point we have a system of equations that represents the reduced order model of the flow. The numerical solution of each system shows good convergence and captures the general trend of the flows used in its derivation. However, there are two additional issues that need to be addressed. First, our objective of developing a reduced order model is to design a feedback controller that can be implemented in the experiments to reduce the cavity oscillations. With the current model this is not possible since there is no control input present in the system of equations. The effect of forcing is embedded in the model since it is captured by the PIV images used to obtain the model. The second issue has to do with the implementation of the model based controller in the experiments. This requires linking the flow variables that can be measured with the flow variables involved in deriving the model. The latter will be the focus of the next chapter, as it describes the
tools used for the derivation and implementation of a controller based on reduced order model. The control separation issue will be explained in detail in the subsequent chapters since it is a significant part of this work.
CHAPTER 5

STOCHASTIC ESTIMATION

This chapter will present the methods used to correlate the variables in the reduced order model with the surface pressure measured in real-time in experiments or applications. Although, this correlation is not required for the development of the reduced order model, it is very important for the experimental implementation of the controller, as the model variables to be controlled (i.e. the values of the modal amplitudes) must be updated in real-time. The required rate for this process in real-time feedback control application has to be approximately an order of magnitude higher than the frequencies associated with the relevant dynamics in the flow. Because of this requirement, an update based on PIV diagnostics is impossible. Consequently, a means must be found to update the modal amplitudes based on a flow variable that can be reliably measured in real-time. In our particular case such a variable is represented by the pressure fluctuations in different locations of the test section. Also, in the practical implementation of such a technique, real-time surface measurements would perhaps be the best option for real-time updating. Another potential option would be surface shear stress measurements. The reduced-order model, on which the controller law is based, must use the available real time measurements to find the state of the flow and then determine the necessary actuator input. Therefore, the reduced-order model can be based on the surface pressure using
system identification techniques, or, in our case on the flow field variables (e.g. the flow velocity). In the latter case, a relation between the flow variables and the surface pressure must be established first.

The stochastic estimation (SE) was first proposed by Adrian (1979) as a method to extract coherent structures from a turbulent flow field. The method estimates flow variables at any location using statistical information about the flow at a limited number of locations ($L$). The method uses knowledge of the velocity fluctuation at a location $x$ to estimate velocity at some other point $x'$. Several researchers have used the method to study the dynamic characteristics of various flow configurations (e.g. Adrian and Moin, 1988; Cole et al., 1991; Cole and Glauser, 1998). In addition, some of these researchers have used it as a complementary technique to obtain modal amplitude of the POD expansion; for example, in subsonic jets (e.g. Picard and Delville, 2000) and in cavity flows (Murray and Ukeiley, 2003). When used with POD, the stochastic estimation helps to recreate a snapshot of the flow at the time of sensing. This snapshot can then be used to obtain the corresponding modal amplitude required to analyze the dynamic behavior of the flow. This is especially useful for experimental or practical applications, as it requires data at only a limited number of locations. Linear stochastic estimation has most often been used in the literature, but Naguib et al. (2001) showed that using both the linear and quadratic terms provides a better estimate of the flow field from wall pressure measurements. For the cavity flow, the improvement in the accuracy of the technique by using quadratic estimation was confirmed by Caraballo et al. (2004) and by Ukeiley and Murray (2005). Similar observations were presented by Ausseur et al. (2006) for the case
of flow separation control. The procedure followed here is similar to that presented by Murray and Ukeiley (2003).

5.1 Stochastic Estimation Procedure

The estimation coefficients are found using the known surface pressure and the flow field variables (velocities) from the PIV measurements. The goal was to obtain a set of coefficients that will correlate these two sets of variables and allow the estimation of the flow field variables based on the time resolved surface pressure measurements. The stochastic estimation assumes that the required flow variable can be obtained from the known variable based on the following relation:

$$\tilde{q}_{ij}(t) = C_{ijk}P_k(t) + D_{ijkm}P_m(t)P_m(t) + E_{ijprs}P_r(t)P_s(t) + \cdots$$

(5.1)

where $\tilde{q}_{ij}$ is an estimated flow variable vector (i.e. velocity) and $P_k$ is the instantaneous surface pressure. The subscript indices $i$ and $j$ represent the different spatial locations within the flow, and $k, l, m, p, r, s, \ldots$ correspond to the selected surface pressure locations (from 1 to the total number of pressure locations). The coefficient matrices ($C, D, E, \ldots$) for the stochastic estimation come from the correlation of the instantaneous velocity field with the surface pressure at zero lag by minimizing the mean square error, $e_{ij}$, between the real value $q_{ij}$ and the estimated one $\tilde{q}_{ij}$

$$e_{ij} = \langle \tilde{q}_{ij} - q_{ij} \rangle$$

(5.2)

Once the correlation coefficients are obtained the method can be used to estimate each individual velocity component to reconstruct the flow field and study its dynamic
behavior. The equations correlating the two velocity components and the speed of sound in the cavity to the measured surface pressures at a given time, $t_r$ are

$$
\tilde{u}_q(t_r) = C_{uqjk} P_k(t_r) + D_{uqjlm} P_l(t_r) P_m(t_r)
$$

$$
\tilde{v}_q(t_r) = C_{vqjk} P_k(t_r) + D_{vqjlm} P_l(t_r) P_m(t_r)
$$

$$
\tilde{c}_q(t_r) = C_{cqjk} P_k(t_r) + D_{cqjlm} P_l(t_r) P_m(t_r)
$$

(5.3)

It should be noted that in Eqn. 5.3 the (estimation) coefficient matrices differ for each of the flow variables used in the reduced-order model. This set of equations gives the instantaneous state of the flow, on a 2D plane in our case, over the entire cavity domain, based on the pressure at the sensing locations. For control purposes, the estimated velocity field can now be used to calculate the modal amplitude of each POD mode corresponding to this state by projecting the estimated velocities onto the POD basis for each variable, using Eqn. 3.10. This set of modal amplitudes can then used as the real-time control variable. These are the state variables of the reduced order model based on the control input separation. This will be discussed in detail in chapters 6 and 7.

5.2 Velocity Estimation

To evaluate the effectiveness of the method, the velocity field captured by the PIV images was recovered from the corresponding pressure measurements. Figure 5.1 shows the mean turbulent kinetic energy of the baseline flow. It compares the original value (a) with the estimated field using: linear estimation (b), quadratic estimation with single time (c) and quadratic estimation with a previous time (d) using a sample frequency of 50 kHz. Details on the derivation of this case will be presented below in section 5.3. It can be observed that the linear estimation, although it captures the general envelope of the distribution, cannot recover the average energy distribution showing small structures.
inside the shear layer where the velocity field shows the highest fluctuation. As we move to the quadratic estimation, the reconstruction recovers more of the energy levels but still shows areas of low values of energy. Finally, when the quadratic estimation is used together with previous time information, the reconstruction starts to recover the energy levels of the original velocity field. It is clear that retaining the quadratic terms in addition to the linear terms in Eqn. 5.1 significantly improved the results, especially if previous time information can be used.

Figure 5.1. Mean turbulent kinetic energy for the baseline flow; a) PIV, b) linear estimation, c) quadratic estimation (static) and d) quadratic estimation with 1 previous time.

Figure 5.2 shows a comparison of an instantaneous normal velocity field with the corresponding estimation using the linear and quadratic techniques from the previous figure. It is clear from the figure that the linear estimation captures the general trend, but as we move to the quadratic estimation there is an improvement on the details of the
estimation. Again, the location and size of the structures in the velocity field are better captured with previous time quadratic estimation. This procedure can be implemented in the experiments, but it also increases the number of operations required since the flow field needs to be estimated before the modal amplitude can be obtained. To simplify this process, by reducing the required number of operations, we can apply the estimation procedure to obtain the modal amplitude directly from the pressure measurements since this is the control variable.

Figure 5.2. Instantaneous velocity estimation for the baseline flow; a) PIV, b) linear estimation, c) quadratic estimation (static) and d) quadratic estimation with 1 previous time.

5.3 Surface Pressure Estimation

In this section we will apply the stochastic estimation procedure to the modal amplitude. In the initial stages of the research, we only used the instantaneous
measurement (static approach) of the surface pressure at each sensor. Although the system recovers the frequency of oscillation, it was observed that the estimated modal amplitude had difficulties recovering the correct magnitude, similar to the velocity estimation in Fig. 5.2. This could be related to deficiencies in the static approach when used to capture the dynamic behavior of the system and to the reduced number of sensors available. This was analyzed by testing the effect of using estimations with richer dynamics of the flow. To do so, the estimation coefficients are obtained using one or more previous time samples of the measured pressure fluctuations. This approach was developed for both the linear and quadratic estimation.

The correlation coefficients for the modal amplitude based on the surface pressure can be obtained using the same steps as the velocity estimation. To simplify the mathematical treatment of the derivation, each time delay includes previous time samples. These are considered as an additional set of sensors. The pressure fluctuations are then represented by:

$$ p_j' = p_k'(t-s\Delta t) \quad j=1\ldots \mathcal{L}(s+1), \quad k=1\ldots \mathcal{L}, \quad s=0, 1, 2, \ldots \quad (5.4) $$

where $\Delta t$ is the time delay, which is equal to the sampling time interval (50 kHz), $\mathcal{L}$ is the number of sensors and $s$ is the number of delays. For the case of linear SE, the expression used to estimate $\hat{a}_i(t)$ at any time $t$ is

$$ \hat{a}_i(t) = C_i^j p_j' \quad i=1\ldots N, \quad (5.5) $$

where $C_i$ is the matrix of the estimation coefficients obtained by minimizing the average mean square error $e_i$ between the values of $a_i(t_r)$ obtained with Eq. (3.10) at the times $t_r$. 

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of a 2D PIV snapshot, and the ones \( \hat{a}_i(t_r) \) estimated from the pressure data recorded simultaneously with the snapshots as discussed in chapter 2, that is,

\[
e_i = \left[ (\hat{a}_i(t_r) - a_i(t_r))^2 \right]^{1/2} \quad r = 1 \ldots M. \tag{5.6}
\]

Once the estimation matrix \( C_l \) is obtained, it can be used in Eq. (5.5) to estimate the modal amplitude from the surface pressure measurements. Similarly, for quadratic SE, the expression used to obtain the estimated values of \( \hat{a}_i(t) \) is

\[
\hat{a}_i(t) = C_q^{ij} p_j^r + D_q^{jk} p_j^r p_k^r \quad i=1 \ldots N, \quad j, k=1 \ldots L(s+1), \quad s=0, 1, 2, \ldots \tag{5.7}
\]

where \( C_q \) and \( D_q \) are matrices of the estimation coefficients obtained as in the linear case.

In our experimental setup, the real-time measurements of surface pressure used for estimation were taken at the locations of transducers 1-6 in the cavity test section, Fig. 2.6.

5.3.1 Linear estimation

Figure 5.3 shows the linear estimation of the first two modal amplitudes corresponding to the uncorrelated PIV images. The amplitudes were estimated using static estimation \( (s = 0) \) and dynamic (multi-time) estimation \( (s = 1, 2 \text{ or } 3) \). Also shown are the bounds for one standard deviation (stdv) of the experimental PIV data. It is clear from the figure that the linear estimation does not capture the correct amplitude of the modes, predicting values that remain far from the standard deviation. Also, there is no significant difference between the static estimation and the multi-time estimation.
Figure 5.3. Linear stochastic estimation of the mode amplitude, (s=0) single time, (s=1) 1 time back, (s=2) 2 time back and (s=3) 3 time back estimation. Also shown are the bounds for one standard deviation (stdv) of the experimental PIV data.

Figure 5.4 shows the first two modal amplitudes from the real-time surface pressure measurements of the baseline flow, and the bounds for one standard deviation (stdv) of the experimental PIV data. For both modal amplitudes, the same response noted in Fig. 5.3 is observed. Again, the effect of using up to three previous time samples is negligible. It can also be observed that the modal amplitudes oscillate at nearly the same frequency (2850 Hz) of the pressure fluctuations, i.e. the resonant frequency of the baseline flow (Fig. 2.7). In all cases, the maximum values of the amplitude are significantly smaller than the standard deviation of the corresponding modal amplitudes calculated from PIV data.
5.3.2 Quadratic estimation

Figure 5.5 show the results using the quadratic SE with the PIV data. Also shown are the bounds for one standard deviation (stdv) of the experimental PIV data. It can be observed that as more previous times are included in the estimation, the values of modal amplitudes start to approach those obtained from the PIV images. The comparison with the standard deviation of the PIV-derived modal amplitudes suggests that quadratic SE produces estimates more consistent with the results from experimental measurements. Furthermore, with quadratic SE the effect of using previous time samples is not negligible.
Figure 5.5. Quadratic stochastic estimation of the mode amplitude, (s=0) single time, (s=1) 1 time back, (s=2) 2 time back and (s=3) 3 time back estimation. Also shown are the bounds for one standard deviation (stdv) of the experimental PIV data.

Figure 5.6 shows the results for the quadratic SE when applied to the real-time surface pressure measurements of the baseline flow, and the bounds for one standard deviation (stdv) of the experimental PIV data. A similar behavior is observed for the modal amplitudes as noted in Fig, 5.5. In this case, the effect of using previous time in the estimation is clear, as the estimated value approaches the corresponding values from the PIV data. As in the case of the linear estimation, the modal amplitudes oscillate at the same frequency (2850 Hz) as the pressure fluctuations of the baseline flow. Again, as the number of previous times increases, the maximum value of the estimated modal coefficients start to approach the standard deviation of the corresponding modal amplitudes calculated from PIV data.
At this time there are two important points to note for the multi-time estimation. First, when the system is tested with more than 3 previous times, the estimation tends to overestimate the maximum values of the modal amplitude and degrades drastically after three times are used even though it is able to reproduce the modal amplitude of PIV data perfectly. Second, the physical system limits the practical implementation to 2 previous times in closed-loop control.

Figure 5.6. Quadratic stochastic estimation of the mode amplitude, \( s=0 \) single time, \( s=1 \) 1 time back, \( s=2 \) 2 time back and \( s=3 \) 3 time back estimation. Also shown are the bounds for one standard deviation (stdv) of the experimental PIV data.

Although the multiple time estimation provides the best results in terms of the amplitude of the modal coefficients, the implementation of this procedure is limited by the physical hardware. For the baseline and open-loop forcing cases, the system can run with 4 previous times, but when the system is run in closed-loop, it adds too much burden and can only be used with 3 previous times in the linear case or 1 previous time in the
quadratic case. The current sensor configuration does not allow investigating the effect of the sensor location or the number of sensors used. To address this issue, data from a new cavity facility developed at the GDTL is used and its results are presented and discussed in appendix C.

At this point all the tools required for the derivation of the reduced order model and their experimental implementations have been defined. The next step is to derive a reduced order model where the control input term appears explicitly so that the tools of control theory can be used to design a feedback controller that can be implemented in the experiments. This is the focus of the next two chapters and the main contribution of this work.
CHAPTER 6

DERIVATION OF REDUCED ORDER MODEL FOR CONTROL DESIGN

The next two chapters will focus on the main contribution of this work to the flow control field, the development of reduced order models of the flow for the design and implementation of feedback control. Before a feedback control algorithm can be developed and implemented, there are some important points that need to be addressed. As explained in the previous chapters, in the development of a low dimensional model based on POD, the projection of the Navier-Stokes equation onto the POD basis results in a system of ODE’s to predict the time evolution of the modal amplitudes, $a_i$. However, the expression does not contain an explicit form of the control input since it is contained within the basis. The resulting expression is of little use for the design of a control algorithm based on control theory tools. Consequently, some additional modifications are required.

Although reduced order models have been used extensively to describe several flows (Aubry et al., 1988; Delville et al., 1999; Glauser et al., 1999; Rowley et al., 2000;), little effort has been devoted for its implementation in control design. One of the main questions encountered is how to include explicitly the control effect into the low dimensional model. Another issue that needs to be considered during the development of the reduced order model is whether to obtain a basis for individual events or a global
basis, by combining different flow conditions into a single set (Ukeiley et al., 2001; Schmit and Glauser, 2004). This and the next chapters will focus on the development and implementation of control input separation methods that can be used for any flow application. During this study, three distinct methods for control separation have been developed and tested numerically. Based on these results, two of the methods have been used to design and experimentally implement a feedback control algorithm.

In this chapter the first attempt to develop a separation procedure will be explained for the baseline flow, followed by a brief description of the feedback control design. Finally, the procedure will be applied to the models based on different flow combinations and the controllers will be tested experimentally.

6.1 Separation of the Control Input Based on Spatial Sub Domain

The first method developed is based on spatial sub domain separation. As noted above, to design a controller, the model requires an explicit description of the input/state/output behavior of the plant to be controlled. This method was formulated to separate the effect of boundary excitation from the remaining terms of the POD based model, so that it appears in the set of ODE’s as an external input that can be manipulated by the feedback controller. The procedure has been presented in Caraballo et al. (2007), and will be discussed next. A detailed treatment of the control separation technique applied to simpler systems is discussed by Efe and Özbay (2003).

The general idea is to incorporate the control input into the system by dividing the entire flow domain into two small sub domains, as shown in Fig. 6.1; one small region, $S_2$, as the physical region at which the boundary excitation or control input is introduced
and the second, $S_1$, containing the rest of the flow field. The total flow domain can then be expressed as $S := S_1 \cup S_2$. This allows work to be done on partitioned subsets that should capture the forcing boundary condition and its subsequent effect over the spatial domain individually.

As noted in chapter 3, any property of the flow field can be expressed in terms of the mean component and the POD expansion of the fluctuating component. By introducing $q$ as a generic vector of the flow variables that are functions of space and time, the flow variables for the entire domain can be expressed as:

$$ q(x,t) = q_m(x) + \sum_{i=1}^{N_i} a_i(t) \Phi_i(x) $$

(6.1)

where $q_m$, $a_i$ and $\Phi_i$ denote the mean flow of each variable, the $i$-th modal amplitude for the vector field, and the $i$-th spatial basis for each individual variable, respectively. This expression can be introduced into the governing equation (4.3) leading to the following expression:
Following the Galerkin procedure described in chapter 4, the inner product of both sides with \( \mathbf{\phi}(x) \) is taken. Applying the corresponding simplifications, based on the orthogonal properties of the POD modes, results in the following system

\[
\sum_{i=1}^{N} \dot{a}_i(t) \mathbf{\phi}_i(x) = f \left( \mathbf{q}(x,t) \frac{\partial}{\partial x} \mathbf{q}(x,t), \cdots \right) \equiv f(x,t)
\]  

(6.2)

It is important to note that

\[
\langle \mathbf{\phi}_i(x), f(x,t) \rangle_{S_2} = \langle \mathbf{\phi}_i(x), f(x,t) \rangle_{S_1} + \langle \mathbf{\phi}_i(x), f(x,t) \rangle_{S_2}
\]  

holds true by the definition of the inner product. Clearly, the above partitioning corresponds to calculating an integral over two domains, the union of which gives the original domain of the problem while the intersection is an empty set. This significantly influences the dynamic representation of the set of ODEs, which now turns out to be

\[
\dot{a}_i(t) = \langle \mathbf{\phi}_i(x), f(x,t) \rangle_{S_1} + \langle \mathbf{\phi}_i(x), f(x,t) \rangle_{S_2}, \quad i = 1, 2, \ldots, N.
\]  

(6.3)

The term \( \langle \mathbf{\phi}_i(x), f(x,t) \rangle_{S_2} \) is computed by utilizing the boundary excitation denoted by \( \Gamma \) explicitly. Since the boundary excitation \( \Gamma(t) \) accounts for the flow characteristics in the \( S_2 \) sub-domain, can be written:

\[
\mathbf{q}_m(x)_{S_2} + \sum_{i=1}^{N} \dot{a}_i(t) \mathbf{\phi}_i(x)_{S_2} = \Gamma(t)
\]  

(6.5)

Depending on the form of the vector function \( f \), the procedure described will yield a non-autonomous set of ODEs capturing the dynamics in the following form:

\[
\dot{a}_i(t) = F^i + \sum_{j=1}^{N} G^i_j a_j(t) + \sum_{j=1}^{N} \sum_{k=1}^{N} H^{ijk} a_j(t) a_k(t) + B^i \Gamma(t) + \sum_{j=1}^{N} \bar{B}^i_j \Gamma(t) a_j(t)
\]  

(6.6)
or in control terms

\[ \dot{a} = A(a) + B(a)\Gamma \quad (6.7) \]

where A \((F, G \text{ and } H)\), and B \((B \text{ and } \bar{B})\) are constant coefficients matrices derived from the Galerkin projection once the control separation is performed. The calculation of the terms in the matrices A and B are straightforward but tedious. Equation (6.7) represents the reduced-order model of the cavity flow sought for the design of a feedback controller in terms of the modal amplitude \(a(t)\), obtained with POD from \(T\) time uncorrelated PIV data sets.

Figure 6.2 shows the numerical solution of the Galerkin system (Eqn. 6.6), for the baseline model (M0B) with and without forcing. It can be noticed that when the forcing is introduced there is no change in the solution of the system, as all the lines in the plot are overlapping, even though two different forcing frequencies were used (F1 and F4). The solution also matches that of the system with no control separation for the baseline Eqn. (4.2). In order to observe any significant change in the flow, the amplitude of the forcing term had to be increased significantly. This is linked to the small values in the coefficient matrices for the control part (B) and also to the fact that for the baseline case there is no forcing included in the image. The modal coefficients oscillate around zero. The first mode remains within the 2 standard deviation of the experimental PIV data, while the second has higher oscillation amplitude but still bounded close to the 2 standard deviation of experimental PIV data.
Figure 6.2. Modal amplitude variations with time and spectra of the first two modal amplitudes for the GS with the initial separation method with 4 POD modes using M0B; $\Gamma = 0$ (no control), $\Gamma = F1 (f = 1610 \text{ Hz})$, $\Gamma = F4 (f = 3920 \text{ Hz})$ and the bound for two standard deviation (2 stdv) of the experimental PIV data.

The system was also solved for higher number of modes to ensure that the behavior observed in Fig. 6.2 is not due to the reduced number of modes used. Figure 6.3 shows the solution of the same system (Eqn. 6.6) using 20 POD modes. The results show a similar behavior as the 4 modes case, especially for the F4 forcing, where no change is observed when the system is forced, with the F1 forcing there is a lower frequency component and its harmonics that starts to affect the solution. This is noted by the overlapping of the solution for all three flow conditions. In this case the amplitude of oscillations for both modes is bounded within the 2 standard deviation, with mode 1 slightly shifted above zero.
6.2 *Separation of the Control Input Based on Spatial Sub Domain for the Forced Flow*

The procedure explained above was also applied to each of the individual forced cases, as well as combinations of the baseline and forced cases according to table 2.1. The system for each case was solved using different numbers of modes. It should be noted that the solution of the system for the baseline model converges between 4 and 20 modes for most cases. Figures 6.4 thru 6.10 show the solution of the GS using 4 modes for various models based on the open loop forcing cases. It can be noticed that independent of the model used, the solution does not change when the forcing frequency is introduced into the system, as all the solutions are overlapping each other. This is the
same behavior observed in the baseline model. Also, in each case the solution of the system 6.6 reproduced the same results in terms of amplitude and frequency as the system 4.2, where no control separation had taken place. Similar results were observed when the system was solved using more modes. For all the models shown in the figures, the modal coefficients are oscillating around zero and close to the 2 standard deviation of the experimental PIV data.

Figure 6.4. Modal amplitude variations with time and spectra of the first two modal amplitudes for the GS with the initial separation method with 4 POD modes using M0F1; $\Gamma = 0$ (no control), $\Gamma = F1$ ($f = 1610$ Hz), $\Gamma = F4$ ($f = 3920$ Hz) and the bound for two standard deviation (2 stdv) of the experimental PIV data.
Figure 6.5. Modal amplitude variations with time and spectra of the first two modal amplitudes for the GS with the initial separation method with 4 POD modes using M0F4; $\Gamma = 0$ (no control), $\Gamma = F1$ ($f = 1610$ Hz), $\Gamma = F4$ ($f = 3920$ Hz) and the bound for two standard deviation (2 stdv) of the experimental PIV data.

Figure 6.6. Modal amplitude variations with time and spectra of the first two modal amplitudes for the GS with the initial separation method with 4 POD modes using M0BF1; $\Gamma = 0$ (no control), $\Gamma = F1$ ($f = 1610$ Hz), $\Gamma = F4$ ($f = 3920$ Hz) and the bound for two standard deviation (2 stdv) of the experimental PIV data.
Figure 6.7. Modal amplitude variations with time and spectra of the first two modal amplitudes for the GS with the initial separation method with 4 POD modes using M0BF4; $\Gamma = 0$ (no control), $\Gamma = F1$ (f = 1610 Hz), $\Gamma = F4$ (f = 3920 Hz) and the bound for two standard deviation (2 stdv) of the experimental PIV data.

Figure 6.8. Modal amplitude variations with time and spectra of the first two modal amplitudes for the GS with the initial separation method with 4 POD modes using M0Wn; $\Gamma = 0$ (no control), $\Gamma = F1$ (f = 1610 Hz), $\Gamma = F4$ (f = 3920 Hz) and the bound for two standard deviation (2 stdv) of the experimental PIV data.
Figure 6.9. Modal amplitude variations with time and spectra of the first two modal amplitudes for the GS with the initial separation method with 4 POD modes using M0BF1F4; \( \Gamma = 0 \) (no control), \( \Gamma = F1 \) (f = 1610 Hz), \( \Gamma = F4 \) (f = 3920 Hz) and the bound for two standard deviation (2 stdv) of the experimental PIV data.

These results were not surprising or unexpected since the forcing effect is embedded within the PIV snapshot used to obtain the coefficient matrices for each model. Additionally, the coefficient matrices for the control term (B) are small, two or three orders of magnitude smaller, when compared to the rest of the coefficients involved in the reduced order model. Despite the little effect of the control term on the numerical solution of the reduced order model (Eqn. 6.6), it contains the control term explicitly. This model is used together with the control theory tools to design a controller that can be implemented and tested in the experimental facility. The control design will be based on 4 POD modes models since the complexity of model and of the design process increases rapidly with the number of modes.
6.3 Controller Design and Implementation

This section presents a brief summary of the methodology for the design of the feedback controller based on a reduced-order model. The specific details of the derivation are presented in Yuan et al. (2005) and Yuan (2006). The procedure includes equilibrium computation; coordinate transformation, linear approximation of the Galerkin system, and linear-quadratic state feedback control design. The general description of the procedure will be presented. The methodology was applied to all the reduced order models. The expression for the reduced-order flow models is the same nonlinear state space model given by Eqn. (6.6), with $N = 4$, where the numerical values of the model parameters vary for each case.

6.3.1 Feedback control design

The first step in the design of a feedback control law is to shift the origin of the coordinate system of the reduced-order model, Eqn. (6.6), to the equilibrium point, $a_0$. To accomplish this goal, a Newton iterative algorithm was implemented to compute the equilibrium point for the unforced system, Eqn. (6.6) with $\Gamma = 0$ (Yuan et al., 2005). Shifting the origin of Eqn. (6.6) to the equilibrium point, removes the constant terms in the equations and results in Eqn. (6.8). The resulting simplified state space model in the new set of coordinates $\tilde{a} = a - a_0$ describes the behavior of the system around the equilibrium point. In this form, the adapted model is more convenient for controller design and stability analysis.
\[
\dot{\tilde{a}} = \tilde{G}\tilde{a} + \begin{bmatrix}
\tilde{a}^T H^1\tilde{a} \\
\vdots \\
\tilde{a}^T H^4\tilde{a}
\end{bmatrix} + \tilde{B}\Gamma + \begin{bmatrix}
(B^1)^T \tilde{a} \\
\vdots \\
(B^4)^T \tilde{a}
\end{bmatrix}
\]  

(6.8)

where

\[
\tilde{G} = G + \begin{bmatrix}
a_0^T \left(H^1 + (H^1)^T\right) \\
\vdots \\
a_0^T \left(H^4 + (H^4)^T\right)
\end{bmatrix}, \quad \tilde{B} = B + \begin{bmatrix}
(B^1)^T a_0 \\
\vdots \\
(B^4)^T a_0
\end{bmatrix}
\]  

(6.9)

The resultant model is nonlinear in terms of the modal amplitude. To apply linear control design, a linear approximation of (Eqn. 6.8) at the origin is readily obtained as:

\[
\tilde{a} = \tilde{G}\tilde{a} + \tilde{B}\Gamma
\]  

(6.10)

The eigenvalues of the system matrix \( \tilde{G} \) are computed for each of the models. It was observed that the models exhibit the same qualitative features in the linear approximation since the open-loop matrices possess two unstable complex conjugate eigenvalues and two stable eigenvalues. The presence of two unstable complex conjugate eigenvalues implies that the flow corresponding to the equilibrium, \( a_0 \), is an unstable solution for the GM (Eqn. 6.6). Since the pairs, \( (\tilde{G}, \tilde{B}) \), for all the cases are controllable, linear state-feedback design based on the linearized model (Eqn. 6.10) offers a simple approach to the design of a controller for the nonlinear model (Eqn. 6.6). The stochastic estimation method provides a way to estimate the amplitude of the modal coefficients of the GM from real-time surface pressure measurements with Eq. (5.4). The availability of real-time estimates of the state of the GM (Eqn. 6.6) allows for the use of linear state-
feedback control to globally stabilize the origin of (Eqn. 6.10). This, in turn, yields a controller that locally stabilizes the origin of the nonlinear system (Eqn. 6.6).

A convenient and well-established methodology for the state-feedback controller design is given by linear-quadratic (LQ) optimal control. The LQ design computes the gain matrix, $K$, such that the state-feedback law

$$\Gamma(t) = -K \tilde{a}(t)$$  \hspace{1cm} (6.11)

minimizes the quadratic cost function

$$J_c(\tilde{a}, \Gamma) = \int_0^{\infty} (\tilde{a}^T W_a \tilde{a} + W_\Gamma \Gamma^2) \, dt$$  \hspace{1cm} (6.12)

where $W_a > 0$ and $W_\Gamma > 0$ are positive definite weighting functions for the state vector and the control signal, respectively. Minimization of $J_c$ results in asymptotic stabilization of the origin, while keeping the control energy small. In our design, the weights were chosen as $W_a = I_{4 \times 4}$ and $W_\Gamma = 1$ for all the models, (Yuan, 2006).

When applying the LQ state-feedback control (6.11) to the linearized system (6.9) it results in mirroring all the right-half plane eigenvalues of the matrix $\tilde{G}$ to the left half plane, while the left-half plane eigenvalues are left almost unchanged, as shown in Fig 6.10. Results obtained for the nonlinear simulations of the closed-loop system (Eqns. 6.6, 6.10-6.11) showed that the trajectory of $a(t)$ converges to the corresponding equilibrium point, $a_0$, in each of the cases considered. This points out that, at least in principle, the LQ controller designed for the linear approximation (Eqn. 6.9) is successful in stabilizing the equilibrium of the 4-dimensional nonlinear GM (Eqn. 6.6).
6.3.2 Real-time feedback control

The resultant controller was implemented in the experiments with a DSpace control board. A Simulink model was developed and compiled into a DSpace experimental file. This system allows for real-time measurements of the surface pressure and the estimation of corresponding modal amplitude to be controlled. As depicted in Fig 6.11, the controller includes a stochastic estimation subsystem and feedback from the estimated states. It is also important to point out that the control input signal must be limited to the range $\pm 10V$ to prevent any damage to the actuator. When the initial closed-loop experiments were performed, constant saturation of the actuator was observed for most of the cases under investigation. Consequently, it was necessary to scale the actuator signal by introducing a factor, $0 < \alpha \leq 1$, in the state-feedback to keep the actuator below the saturation limit. The final scaled control is in the form

$$\Gamma_\alpha(t) = -\alpha K \hat{a}(t)$$  \hfill (6.13)
The specific value of $\alpha$ for each controller was chosen empirically as the highest value that kept the magnitude of the scaled control signal (6.13) within the given bound of the saturation limits, $\pm 10V$. It is important to notice that, theoretically, the closed-loop eigenvalues will be moved to the left-half plane only for $\alpha > 0.5$. For many of the experimental tests the scaling factor was around $\alpha = 0.4$, which suggests that the system is still unstable and oscillating. However, it must be kept in mind that stabilization of the origin of (6.8) corresponds to suppression of the oscillations produced by the limit cycle, only as predicted by the simple model (6.8), which can capture the complex behavior of the flow only in a limited qualitative sense. Therefore, when applying (6.13) to the experimental setup, the pressure fluctuations are not completely suppressed, even when stabilization of (6.8) is achieved by the controller.
6.4 Experimental Results

The performance of the control law (6.13) was tested experimentally for the all the models derived. In what follows, the general results obtained in controlling the baseline flow B are presented. Since the goal of our controller is to reduce the amplitude of pressure oscillations, the SPL spectra is used as the primary measurement of the controller effectiveness. Later, the models will be evaluated based on other parameters for comparison of their performance. During the experiments the spectra form sensors 1 and 5 (Fig. 2.8) were monitored to set the value of the gain $\alpha$. In the processing stages the spectra from each of the transducers of Fig. 2.8 are checked to study the effect of the controller on the system. In the initial implementation, only sensor 5 was monitored, but it was noted that in some cases, although sensor 5 showed a significant reduction of the resonant tone there was clear tone present in other sensor locations. A decision was made to include the spectrum from sensor 1 in the experimental results.

From this point on, most of the results will show the signal from transducer 5 to discuss the general observations. Fig. 6.12 shows the SPL spectra for the baseline and the closed-loop controller based on the model M0B. It can be observed that for this particular case, the effect of the controller is to some degree similar at every sensor. In general, the dominant tone has an initial amplitude of about 135 dB and with control it is reduced to 118 dB. Sensors 2 and 3 show the most overall reduction with no noticeable peaks present in the spectrum. Sensors 4, 5 and 6 show similar response, with good reduction of the dominant tone and the appearance of some small peaks at other frequencies. As mentioned above, the spectrum of sensor 1 shows a good reduction of the natural tone, but also shows the appearance of a broad peak around 2000 Hz introduced by the actuator.
signal (Debiasi et al., 2006). The result based on the model M0B will be used to compare with other models, as it provides good attenuation and requires the least information to derive, since it uses only data from the baseline flow.

![Figure 6.12. SPL for the baseline flow and for the close loop based on model M0B.](image)

Figure 6.12. SPL for the baseline flow and for the close loop based on model M0B.

The results for models based on forced flows, individual and/or combinations are now analyzed in an effort to answer questions regarding what flow conditions are required to derive a good model for control. The use of combined flow conditions for the development of reduced-order models for cavity flow control is a relatively new topic, and we attempt to provide some insight into it. When a model incorporating the baseline and one open-loop forcing case (Fig. 6.13(a-d)) is considered in place of the M0B model, the attenuation of the resonant tone is generally accompanied by the appearance of a new
significant peak whose frequency is about 2000 Hz for M0BF1 and M0BF2 (Fig. 6.13(a, b)) and about 3200 Hz for M0BF3 (Fig. 6.13(c)). The M0BF4 case (Fig. 6.13(d)) is the most successful of the combined models as only a small peak is observed in the spectrum around 3200 Hz. This case also exhibits multi-mode/peak splitting phenomenon (Debiasi and Samimy, 2004).

The results obtained with LQ controllers based on M0B (Fig. 6.12 and M0BF4 (Fig. 6.13(d)) and with the best open-loop forcing, F3 (figure 2.8(c)) shows similar behavior. In all three cases, the cavity flow has been forced into multi-mode behavior. The M0B and M0BF4 closed-loop control cases perform well by producing slightly lower amplitude peaks than the open loop-case. The SPL spectra of these LQ controlled flows resemble those previously obtained using a parallel-proportional with time delay controller (Yan et al., 2006). For the cases M0BF1F2 and M0BF3F4 (Fig. 6.13(d, e)) the response of the flow to the forcing is similar to the previous cases for the individual flows. The reduction of the natural frequency peak and the presence of a strong peak at a frequency similar to that of the individual models are noticeable in most cases. It its worth noticing that for each model the coefficient matrices for the estimation is based on the same flow condition as the model.
Figure 6.13. SPL spectra from transducer 5. The thin line is the B case and the thick line is control with LQ design based on model: (a) M0BF1; (b) M0BF2; (c) M0BF3; (d) M0BF4; (e) M0BF1F2; (f) M0BF3F4.
Overall, it appears that the M0BF4 model produces a marginally improved spectral attenuation compared to that of the M0B case. This result together with the mild effect of the other three models is somewhat surprising since it was expected that the incorporation of open-loop forced flow features should create a richer model, which in turn should be capable of delivering better results in closed-loop conditions. The unimproved behavior of the combined models compared to the M0B model was thought to be originated from one or a combination of the following two causes: (1) the particular technique for control input separation that has been employed to render the presence of the control input explicit in the model, or (2) modulation of the control signal introduced by the actuator transfer function. The impact caused by the control input separation method could be associated with the small area used for the control input separation, which generates small values of the control coefficient matrices. Also, the control input effect is embedded into the PIV images used to derive the models, which makes it difficult for the model to distinguish between the baseline flow and the forced one. This will be explored in the next chapter by developing a different approach for the control input separation.

The second element is created by the nature of the actuator and the limitations imposed by the physical space. The particular actuator employed in the experimental setup exhibits strong amplitude variations over the range of frequency of operation in closed-loop. This effect is largely due to the shape of the waveguide between the driver and the input region, DeBiasi and Samimy (2004). This behavior adds uncertainties to the control loop and could be limiting the performance of the control algorithm. This issue was addressed by incorporation of a compensator into the system that minimizes the frequency modulation of the actuator transfer function. The details of the development of
the actuator compensator as well as its effects on the feedback system are discussed in Kim et al. (2007). It is important to notice that the nature of the compensator design imposes some limitations on the initial estimation scheme. When the compensator is used is limited, due to computing resources, to single time estimation at 4 locations, (2, 4, 5 and 6 in Fig. 2.8). In this case, sensor 1 is no longer used for the LQ controller since it feedbacks the actuator input to the controller and could be causing negative effects on the control system.

Figure 6.14 compares the SPL reduction obtained by the LQ state feedback control with the compensator off (left) and on (right) for the two models that showed the best overall results, the baseline model M0B and the combined model M0BF4. It can be observed that for the M0B model, when the compensator is active there is an additional reduction of the dominant peak, for a total of about 18 dB. It is also noticeable that there is no significant amplification of peaks at any other frequency. In the case of the M0BF4 model, the effect seems to be more pronounced. In both cases, the compensator improves the behavior of the controller by further reducing the dominant peak remaining in the flow. The general effect of implementing the compensator is to reduce the uncertainties in the controller and to improve the performance of the feedback control loop.
Figure 6.14. SPL of LQ control for the initial separation method for two models: M0B (top two figure) and M0BF4 (bottom two figure), with the compensator off (a & c), with the compensator on (b & d).

The final two sets of experimental data for this control separation method were taken after the actuator compensator was added to the system. For the models based on White noise (M0Wn) and the combined case M0BF1F4, the data was only collected with the compensator on since it mitigates the frequency modulation created by the actuator arrangement. Figure 6.15 shows the SPL reduction obtained by the LQ state feedback control with the compensator on for the models M0BF1F4 and M0Wn. The reduction of
the natural resonant peak has been improved to over 20 dB. These two cases show similar overall behavior, with a flattening of the natural peak and the presence of a small peak around 2000 Hz. From this result, it appears that deriving a model from combined open-loop flow cases or band limited white noise provides a result that responds better when the system is forced at different conditions in closed-loop. However, it is difficult to draw a strong conclusion looking at the spectra alone. As mentioned above, after developing the second separation method, the quality of the models will be evaluated using other techniques.

![Figure 6.15. SPL spectra from transducer 5. The thin line is the B case and the thick line is control with LQ design based on model: (a) M0BF1F4; (b) M0Wn.](image)

The robustness of the controllers was tested by running the cavity at off design Mach numbers, ranging from $M = 0.28$ to $M = 0.32$. Within this interval the cavity still behaves similar to the baseline flow ($M = 0.3$), where only a single resonant tone is present. Figure 6.16 shows the response of the controller under $M = 0.28$ and $M = 0.32$ for the M0B and for the M0Wn models, as they represent the general behavior of the other
models tested. It is clear from the figure that the benefits of the scaled LQ controller are still retained. In these flows, the feedback controller clearly exhibits a good robustness.

Figure 6.16. SPL of LQ control for the initial separation method for two models: M0B (top two figure) and M0Wn (bottom two figure), at off design conditions’ M = 0.28 (a & c), and M = 0.32 (b & d).

The reduced order models developed here allowed the design of a set of controllers that successfully reduced the natural oscillation in the cavity flow. However, the method of separation raised some issues about its derivation procedure, and how to account for...
the effect of the control input. Also, the models based on the combination of different flows were not capable of reproducing the baseline flow when the control input is set to zero. The final issue is the required presence of the control input region on the PIV images used in the derivation of the model. This was possible for the cavity flow, but it could be a problem for more complex flows. In the next chapter these issues will be addressed by developing new control separation approaches.
CHAPTER 7

NEW CONTROL SEPARATION METHODS

The reduced order model developed in the previous chapter was the first approach in the process of developing a feedback controller. Although the experimental results were encouraging, it raised some issues regarding the procedure used to obtain the explicit form of the control input. It was observed that the resultant GS had little sensitivity to the introduction of the forcing input. In this chapter, two new control separation approaches are discussed in an attempt to improve the sensitivity of the final GS when the control input is introduced into the system. The first method is based on the weak formulation of the governing equations, and relies on mathematical simplifications to introduce the control input. The second method introduces the control input as a set of additional modes, control modes, which are added to the baseline model.

7.1 Control Separation Based on the Weak Formulation

The first approach introduces the control input explicitly in the final system of equations by means of the weak formulation of the Navier-Stokes equations (Camphouse, 2004). The idea in this case is to replace the second order derivative terms present in the governing equations so that the errors due to the discretization process are reduced. The
approach is based on the application of Green’s identity to transform the second order
derivatives into first order that can be solved numerically with more accuracy.

Figure 7.1 shows the boundary points for all the free surfaces in our cavity
configuration. These boundaries will be used later to evaluate the required integrals. The
control input location is defined as $x = 0$ and $y$ locations $a$ and $b$. The other areas are
defined by the segments 1-2, 2-3 and 3-4.

![Figure 7.1. Control separation regions for the weak formulation method.](image)

In Eqn. 4.1 (N-S), the second order derivatives only appear in the momentum
equation with two components, the mean and the fluctuating components. Green’s
identity states

$$\int_{\partial \Omega} f \Delta g dA - \int_{\Omega} f (\nabla g \cdot n) dA - \int_{\Omega} \left( \nabla f \cdot \nabla g \right) dv (x)$$

(7.1)

where the first term is an integration over the surface ($\partial \Omega$) and the second term is inside
the volume ($\Omega$), $f$ and $g$ are scalar fields, $\Delta$ is the Laplacian operator, $\nabla$ is the gradient, $n$
is the unit outward normal, $A(x)$ is the surface area and $v$ is the volume. Applying the
identity to each term, for the $u$ velocity component, we have
\begin{equation}
\int_{\Omega} \psi_{k} \Delta U dv = \int_{\Omega} \psi_{k} (\nabla U \cdot n) dA (x) - \int_{\Omega} (\nabla \psi_{k} \cdot \nabla U) dv (x)
\end{equation}

The cavity has solid boundaries, walls and floor, where the POD modes or the velocities are zero. There are three free surfaces; the streamwise flow inlet and outlet planes and the top surface, since we don’t use the total flow area for the analysis. In these three surfaces, the integral will yield values different than zero, therefore we cannot neglect them. Finally there is the control input area, where we know the forcing function which leads to the implicit control term on the final system of ODE’s.

Appendix D presents the details of the derivation procedure. After all the simplifications are carried out, the following expression is obtained where the control input appears explicitly in the solution as a linear term:

\begin{equation}
a^{k} (t) = b^{k} + \sum_{j=1}^{n} (d^{j} a^{j}) + \sum_{j=1}^{n} \sum_{m=1}^{n} (g^{j m k} a^{j} a^{m}) + e^{k} \Gamma
\end{equation}

As in the previous model, the constant coefficients $b$, $d$, $g$ and $e$ are obtained from the Galerkin projection, and $\Gamma$ is the control input applied at the forcing location. The difference with the previous approach is evident in the control term. In the first case (Eqn. 6.6) the controller input is multiplied by the modal amplitude, while in the latter (Eqn. 7.3) it is only multiplied by a constant.

As with the previous model, the effect of the forcing for this control separation on the system of equations was analyzed by comparing the results of the implicit form of the equations (Eqn. 4.2) with the results of the explicit forms (Eqn. 7.3) based on the numerical data. Three values of forcing frequency and amplitude were used for the test. First, a value of 0 Hz for the forcing frequency and 0 m/s for the amplitude was
introduced into Eqn. 7.3, as the control input ($\Gamma$), to simulate the effect of no control in the system. Then, the values of frequency and amplitude for two open-loop cases (e.g. 1610 Hz and 5 m/s, or 3920 Hz and 5 m/s, Debiasi and Samimy, 2004) were supplied as the control input ($\Gamma$). This was to isolate the effect of the forcing in the solution of the system. Figure 7.2 shows the first two modal amplitudes obtained from the system of ODEs for the baseline flow case. The solution of the explicit form (Eqn. 7.3) with and without the forced term included shows a similar behavior to the previous separation method, Fig. 6.2. Again, the three solutions are overlapping not showing the effect of the forcing introduced in the system.

Figure 7.2. Modal amplitude variations with time and spectra of the first two modes for the GS with the weak formulation method with 4 POD modes using MB; $\Gamma = 0$ (no control), $\Gamma = F1$ ($f = 1610$ Hz) and $\Gamma = F4$ ($f = 3920$ Hz). Also shown are the bounds for two standard deviations (2 stdv) of the experimental PIV data.
7.1.1 Application of weak formulation method to the forced flow cases

Following the same procedure explained above, we developed a reduced order model for each of the forced cases as well as for combinations of the baseline and forced case. As in the previous method, the form of the final expression is the same, but with different values for the coefficients. To test the effect that the forcing term has on the solution, each model was solved numerically. Figures 7.3-7.8 show the results for the different models. For each of the models, it can be observed that there is no change in the solution when the forcing input is introduced in the system, as was observed for the baseline model. Also, the results are similar to those observed for the previous model (M0), where the frequency and amplitude of the modes stay unchanged when forcing is introduced, as the three solution lie on top of each other. The similarities in the results for the two methods can be explained in the derivation process, where the only difference is on the treatment of the 2\textsuperscript{nd} derivative term. However, the effect of the forcing is embedded in the images used to obtain the POD bases, and the small area of the input region dictates the magnitude of the coefficients of the control portion of the equations.
Figure 7.3. Modal amplitude variations with time and spectra of the first two modes for the GS with the weak formulation method with 4 POD modes using MF1; $\Gamma = 0$ (no control), $\Gamma = F1$ ($f = 1610$ Hz) and $\Gamma = F4$ ($f = 3920$ Hz). Also shown are the bounds for two standard deviations (2 stdv) of the experimental PIV data.

Figure 7.4. Modal amplitude variations with time and spectra of the first two modes for the GS with the weak formulation method with 4 POD modes using MF4; $\Gamma = 0$ (no control), $\Gamma = F1$ ($f = 1610$ Hz) and $\Gamma = F4$ ($f = 3920$ Hz). Also shown are the bounds for two standard deviations (2 stdv) of the experimental PIV data.
Figure 7.5. Modal amplitude variations with time and spectra of the first two modes for the GS with the weak formulation method with 4 POD modes using MBF1; $\Gamma = 0$ (no control), $\Gamma = F_1$ ($f = 1610$ Hz) and $\Gamma = F_4$ ($f = 3920$ Hz). Also shown are the bounds for two standard deviations (2 stdv) of the experimental PIV data.

Figure 7.6. Modal amplitude variations with time and spectra of the first two modes for the GS with the weak formulation method with 4 POD modes using MBF4; $\Gamma = 0$ (no control), $\Gamma = F_1$ ($f = 1610$ Hz) and $\Gamma = F_4$ ($f = 3920$ Hz). Also shown are the bounds for two standard deviations (2 stdv) of the experimental PIV data.
Figure 7.7. Modal amplitude variations with time and spectra of the first two modes for the GS with the weak formulation method with 4 POD modes using MBWn; $\Gamma = 0$ (no control), $\Gamma = F1$ ($f = 1610$ Hz) and $\Gamma = F4$ ($f = 3920$ Hz). Also shown are the bounds for two standard deviations (2 stdv) of the experimental PIV data.

Figure 7.8. Modal amplitude variations with time and spectra of the first two modes for the GS system with the weak formulation method with 4 POD modes using MBF1F4; $\Gamma = 0$ (no control), $\Gamma = F1$ ($f = 1610$ Hz) and $\Gamma = F4$ ($f = 3920$ Hz). Also shown are the bounds for two standard deviations (2 stdv) of the experimental PIV data.
Based on the results obtained with this separation method, and its similarities with the previous case, no controller was implemented for this method. A different method was developed to address the sensitivity of the system towards forcing, a method that differs from the previous two in more ways than the handling of a term in the governing equation. In the following section the third method developed for control separation is described.

7.2 Control Separation Based on Addition of Control Modes

The two methods developed so far have the same main characteristics; first they are derived by mathematical manipulation of the governing equation (Eqn. 4.2) during the Galerkin projection procedure. Second, the images used for the models contain the forcing effect embedded in them. Finally, they require the presence of the control input region in the flow images. In an effort to derive a model with better characteristics than the previous two models, a third separation method was developed (M1). This method treats the control input as an additional set of modes that are integrated into the baseline flow model. The scheme for the method was initially given by Rempfer (2004). The idea is that by adding the control input to the solution of the baseline flow in the POD expansion, any variation of the flow due to forcing can be captured by the additional modes. Additionally, the control modes are introduced in the definition of the POD expansion for the velocity fluctuations before deriving the low dimensional model. These modes are derived from the difference between the forced flow and the projection of the forced flow onto the POD basis of the baseline flow.
It is clear from the definition of the method that it is different from the previous ones in at least two of the characteristics mentioned above. The control is added in the definition of the POD modes and not during the Galerkin projection and the images containing the control effect are not used to derive the baseline portion of the model. The steps required to obtain the reduced-order model in this fashion can be summarized as follows:

- Obtain the POD basis of the baseline case from the PIV images
- Calculate the modal amplitude of the controlled flow, by projecting the control case PIV images onto the POD basis of the baseline
- Reconstruct the flow field of the control case by using a truncated set of the baseline basis and the corresponding modal amplitudes obtained for the control case
- Calculate the difference between the experimental (PIV images) velocity field of the controlled case and the corresponding reconstructed velocity field
- Obtain a set of POD basis for the vector field difference
- Apply the Galerkin method to the governing equation, replacing the velocity fluctuations with the new POD expansion that includes the control modes
- Find a relation between the control input voltage and the amplitude of the corresponding modal amplitude of the forcing modes

In this approach, the POD expansion of the vector field fluctuations \( \mathbf{q} \) will be expressed as the summation of two sets of POD bases and their corresponding modal amplitudes. The first POD basis and its modal amplitude are based on the baseline case, derived in section 3.3, and will reproduce the portion of the flow field that doesn’t change
with forcing. The other POD basis, based on the vector field difference and its corresponding modal amplitude, represent the deviation from the baseline flow due to the forcing. The new form of the POD expansion can be represented as:

\[
q'(x,t) = \sum_{i=1}^{N} a_i(t) \phi_i(x) + \text{control modes}
\]  

(7.4)

where \( \phi_i(x) \) is the POD basis of the baseline case, \( a_i(t) \) are the modal amplitudes of the baseline portion and the control modes represent the deviation of the system from the baseline when forcing is introduced.

The first step to obtain the control modes requires the projection of the forced flow fluctuations onto the baseline POD modes, to obtain their corresponding modal amplitude. The projection is obtained using equation 3.10, and represents the portion of the controlled flow that is recovered (spanned) by the baseline basis. It is worth mentioning that since the mean of the baseline case, \( q_m \), is used as the mean of the total set, the change in the mean flow due to the forcing will be captured in the control modes. Once the baseline modes and the forced modal amplitudes are obtained, the forced flow can be reconstructed by:

\[
q_{fc}'(x,t) = \sum_{i=1}^{N} c_i(t) \phi_i(x)
\]  

(7.5)

where \( c_i \) is the modal amplitude of the forced case flow. Next, the differences between the original vector field for the forced case and the corresponding reconstructed vector are calculated for each PIV image as follows:

\[
\Delta q_{fc}'(x,t) = q_{fc}'(x,t) - q_{fc}'(x,t) = q_{fc}'(x,t) - \sum_{i=1}^{N} c_i(t) \phi_i(x) - q_m(x)
\]  

(7.6)
where $q_{fc}'$ is the total vector field of the control flow, $q_{rfc}'$ is the reconstruction of the controlled vector field fluctuations and $q_m$ is the mean of each variable of the vector field for the baseline flow.

Once the differences in the vector field fluctuations are known, the snapshot approach of the POD method is applied to Eqn. 7.6 and the corresponding bases ($\Psi$) are obtained. By construction, the control basis should be orthogonal to the baseline basis, as they expand the portion of the control flow not captured by the baseline basis. This was the case when the control modes were projected onto the baseline ones. However, the Gram-Schmidt procedure was used to make sure that the set composed of ($\phi$, $\Psi$) was orthonormal. This helps to preserves the properties of the POD method for the basis and is convenient for the derivation of the Galerkin system. The vector form of the fluctuation variables can be expressed now as:

$$q'(x,t) = \sum_{i=1}^{N} a_j(t)\phi_j(x) + \sum_{i=1}^{N} c_j(t)\psi_j(x)$$

(7.7)

This expression is introduced in the governing equation (Eqn. 4.2) and the Galerkin projection method is applied to derive the corresponding reduced order model. To derive the reduced order model for this new setup, the following considerations are taken:

- The mean flow is based on baseline flow
- For the Galerkin method the governing equations will projected onto the POD basis of the baseline flow ($\phi$)
- The control modes are orthonormal to the baseline modes preserving the POD properties of the inner product of the modes
After applying the procedure, the final form of the Galerkin system is:

$$
\dot{a}_i(t) = F^i + \sum_{j=1}^{N_j} (G^j a_j) + \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} (H^{jk} a_j a_k) + \sum_{j=1}^{N_j} (P^j c_j) + \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} (Q^{jk} a_j c_k) + \sum_{j=1}^{N_j} \sum_{k=1}^{N_k} (R^{jk} c_j c_k)
$$

(7.8)

The constant coefficients present in the equation are calculated as in the previous case, but $P$, $Q$ and $R$ depend on the control modes (based on the velocity difference). As it was done in chapter 4, the additional viscous term (Eqn. 4.4) was also included in this expression to account for the energy of the neglected modes.

Figure 7.9 shows the first 4 modes of the normal velocity fluctuations for the baseline portion of the model, the same used for the M0B model, and the control modes of the different forced flows used. Figure 7.10a shows the modes for the baseline flow, which capture the behavior of the unforced flow. The remaining images on the figure show the resultant modes of the control input for the forced flow cases F1 (Fig. 7.9b), F4 (Fig. 7.9c), Wn (Fig. 7.9d) and the combination of F1-F4 (Fig. 7.9e). It can be noticed that the all the control modes show smaller structures around the shear layer. Just as a reminder, in each mode a structure is represented by a pair of light and dark contours. For the F1 case (Fig. 7.9b) two structures can be observed especially for modes 2 and 3, consistent with a flow resonating at the 2nd Rossiter mode. For the F4 case (Fig. 7.9c) it is observed that 3 sets of structures being added to the baseline flow in modes 1 and 2 and 4 sets on modes 3 and 4. This is consistent with open-loop forcing at F4 which generates a switching between the 2nd and 4th Rossiter modes. The combination of F1-F4 (Fig. 7.9d) shows similar characteristics to the F1 case but with less energy and smaller structures. In this case the forced cases tend to push the flow towards the 2nd Rossiter mode as the
individual flows seem to have a preference for this mode. In the case of Wn, the white noise forcing (Fig. 7.9e); it is observed that in general the structures in the modes are smaller, with no specific pattern in the number of structures present in the modes. The first mode captures a higher percentage of the energy than in the rest of the cases; this can be related to a change in the mean flow with respect to the baseline flow.

At this point, it is worth noticing that for this separation method the control input region does not need to be present in the PIV images used to obtain the POD modes. The only requirement is to have images without forcing for the baseline flow and images with forcing to obtain the control modes. This is an important difference in comparison to the previous methods, in which the presence of the input region was required for the derivation of the reduced order model. Therefore, the new method could be used in the study of flow fields where it is difficult to identify the control input region or where the region of interest, for flow dynamics purposes, is away from the control input point.
Figure 7.9. First 4 POD modes of the normal velocity fluctuations for the Baseline portion (a) and for the control modes of different forced conditions: b) F1, c) F4, d) F1-F4 and e) Wn.

The model described in Eqn. 7.8 is not yet complete since the control input ($\Gamma$) is not present in the equation. The final step needed to make the model practical for experimental implementation is to correlate the forcing input with the modal amplitude...
(c) of the control modes present in the reduced-order model (Eqn. 7.8). This was not an issue with the previous models (Eqn. 6.6 and Eqn. 7.3) because the control input was introduced as the actual forcing signal at the control input region in the mathematical derivation stage. The required modal amplitude can be linked to the control input using the stochastic estimation method. The procedure can be applied using one of the following measurable variables:

1. Pressure at the control region
2. Pressure from inside the actuator
3. Voltage input to the actuator

The use of the pressure at the control region is the simplest to implement, as it can be used to estimate the instantaneous flow conditions thru the modal amplitude. However, this location is affected not only by the forcing input but also by the incoming flow.

The second variable that could be used is the internal pressure in the controller channel. This variable can be measured directly in the experiments, but requires an additional pressure sensor and increases the number of parameters used in the control system. The behavior of this pressure is strongly dominated by the actuator characteristics and by the frequency used in the forcing (Kim et al., 2007).

The third variable that can be used is the voltage input to the actuator. This variable is the natural choice as it captures the actual control signal sent into the system. This variable is dictated by the control system only and corresponds to the open-loop voltage or the signal sent back by the closed-loop system. Linear and quadratic estimation coefficients are calculated based on individual sets of open loop-forcing and for some combination of open–loop cases (i.e. F1 & F4).
The procedure followed for the estimation was described in chapter 5 and it will be used with single time information only since the control input in closed-loop changes continuously. Once the estimation coefficients are known, the modal amplitude of the control term can be replaced by the estimation expression,

\[ c^t = M^t \Gamma(t) + O^t \Gamma(t)^2 \]  

(7.9) where \( M \) and \( O \) are the estimation matrices for the linear and quadratic terms respectively, and \( \Gamma \) is the input voltage. Then the final form of the Galerkin system (Eqn. 7.6) can be expressed as:

\[
\dot{a}_i(t) = F^i + \sum_{j=1}^{N} G^i \cdot a_j + \sum_{j=1}^{N} \sum_{k=1}^{N} H^i \cdot a_j a_k + \left( F_M^i + \sum_{j=1}^{N} \Phi_M^i \cdot a_j \right) \cdot \Gamma + \left( F_O^i + \sum_{j=1}^{N} \Phi_O^i \cdot a_j + \frac{\Phi_{lat}}{\gamma^3} \cdot R^i_{lat} \right) \cdot \Gamma^2 + \frac{R_i^i}{\gamma^3} + \frac{R_{oo}}{\gamma^4} \cdot \Gamma^4
\]

(7.10) where the new coefficients \( F \), \( \Phi \) and \( R \) are obtained from the previous values of \( P \), \( Q \) and \( R \) multiplied by \( M \) and \( O \). When the linear estimation is used, all the terms dependent on \( O \) will become zero and the final expression will be dependent on the control input only up to the square term.

Figure 7.10 shows the linear part of the control mode after combining the original control modes (\( \Psi \)) with the estimation matrices (\( M, O \)) for different forced cases. In general, it is observed that the effect of the controller on both fluctuating velocity components is to modify the structures present in the shear layer. Figure 7.10a shows the presence of 2 pairs of structures in the shear layer region, consistent again with the F1 forcing (2\textsuperscript{nd} Rossiter mode). The case of F4 (Fig. 7.10b) shows the presence of four structures for both modes. The F1-F4 combination (Fig. 7.10c) resembles the F1 case, which could be expected as the F4 case is a multimode resonance that changes between
the 2nd and 3rd Rossiter mode and the F1 is dominated by the 2nd Rossiter mode, which
becomes the dominant mode in the combination. This was also noticed in the control
modes in Fig. 7.9d. For the white noise case (Fig. 7.10d) the resultant mode affects the
shear layer region close to the leading edge of the cavity and developed into a large
structure that affects the cavity itself even at the trailing edge.

Now that, the flow model variables have been linked to the control input, the system
(Eqn. 7.8) can be solved numerically to test the effect that the control input has on the
solution. Keeping in mind that the model is developed based on the baseline flow with
the control mode added on top, the solution with no forcing for all the different variations
of the model will be that of the model without control separation (Eqn.4.2) for the
baseline flow. Figures 7.11 thru 7.14 show the results of the Galerkin system (Eqn. 7.10)
for different forced flow cases, as well as the 2 standard deviation of the experimental
PIV data bounds (2 stdv). It can be observed that when the control is not present (\( \Gamma = 0 \)),
the system matches the solution of the model based on the baseline case (M0B), shown in
Fig. 6.2. It can also be noticed, that when the forcing (\( \Gamma \)) is introduced there is change in
the amplitude and frequency of the modal amplitude \( a_1 \) (F1 or F4). This is an
improvement over the previous separation methods (Eqn. 6.6 and Eqn. 7.3), where no
difference was observed in the solution when forcing was introduced, independently of
the forced cases used in its derivation (Figs. 6.2 - 6.16). It can be also notice in these
figures that the solutions the first 2 modal coefficient amplitude oscillates close to zero.
Mode 1 is always bounded within the 2 standard deviation limit of the experimental PIV
data, while mode 2 stays close to the bounds limits.
Figure 7.10. First $u$ and $v$ control modes for the M1 model when multiplied by the estimation coefficients ($M$ and $O$) for the forced cases based on: a) F1, b) F4, c) F1-F4 and d) Wn.
Figure 7.11. Modal amplitude variations with time and spectra of the first two modes for the GS with the control mode method (M1) with 4 POD modes using M1BF1; $\Gamma = 0$ (no control), $\Gamma = F1 (f = 1610 \text{ Hz})$ and $\Gamma = F4 (f = 3920 \text{ Hz})$. Also shown are the bounds for two standard deviations (2 stdv) of the experimental PIV data.

Figure 7.12. Modal amplitude variations with time and spectra of the first two modes for the GS with the control mode method (M1) with 4 POD modes using M1BF4; $\Gamma = 0$ (no control), $\Gamma = F1 (f = 1610 \text{ Hz})$ and $\Gamma = F4 (f = 3920 \text{ Hz})$. Also shown are the bounds for two standard deviations (2 stdv) of the experimental PIV data.
Figure 7.13. Modal amplitude variations with time and spectra of the first two modes for the GS with the control mode method (M1) with 4 POD modes using M1BF1F4; \( \Gamma = 0 \) (no control), \( \Gamma = F1 \) (\( f = 1610 \) Hz) and \( \Gamma = F4 \) (\( f = 3920 \) Hz). Also shown are the bounds for two standard deviations (2 stdv) of the experimental PIV data.

Figure 7.14. Modal amplitude variations with time and spectra of the first two modes for the GS with the control mode method (M1) with 4 POD modes using M1BWn; \( \Gamma = 0 \) (no control), \( \Gamma = F1 \) (\( f = 1610 \) Hz) and \( \Gamma = F4 \) (\( f = 3920 \) Hz). Also shown are the bounds for two standard deviations (2 stdv) of the experimental PIV data.
The system 7.8 was also solved numerically, using a Simulink model, to evaluate the effect of turning the controller on and off. Figure 7.15 shows the results for one of the models. The initial portion of the simulation corresponds to the baseline flow, where no control is introduced. After certain time has passed the forcing term was introduced into the system. It can be observed that when the controller is turned on the system responds immediately changing its behavior respect to the baseline flow.

Based on the results obtained from the numerical solution of the system (Eqn. 7.10), where there is a clear effect of control input on the systems; the models based on this separation method were used to obtain the corresponding gains for feedback control in real time. The following section will present the results of the controller design for this model.

7.3 Controller Design and Implementation

The controller for this separation method (M1) was designed following the same procedure employed for the initial separation method (M0), presented and discussed in
section 6.6. This is done since the structure of the two models is similar and we wish to use linear control methods to keep the controller as simple as possible. Therefore, the controller design is based only on the linear portion of the system. Furthermore, the coefficients of the baseline portion of all the M1 models are the same as those for the model M0B since they are obtained from the baseline flow images. From these observations and knowing that the controllers are based on the linear portion of the Galerkin system, we expect that they would have a similar behavior as the initial model, M0B.

For the models based on this separation method the compensator explained in chapter 6 was always used in the data collection process, since it eliminates the modulation of the control input. However, the majority of the experiments conducted for the previous model were performed without the actuator compensator, with the exceptions of the models M0B, M0BF1, M0BF4, M0BF1F4 and M0Wn. Therefore, for a fair comparison of the results all the new models will be compared against these models. Figure 7.16 shows the SPL reduction obtained by the LQ state feedback control for the models M1BF1 (a), M1BF4 (b), M1BF1F4(c) and M1BWn (d) for the design condition, M = 0.30. Again, the thin line yields the SPL of the unforced baseline flow, whereas the thick line corresponds to the SPL of the flow at the same location under state feedback control. All the models show improvements with respect to the uncontrolled flow by reducing the resonant peak more than 18 dB. There are small differences in the general effect of the forcing between the models. It can be noticed that the controllers based on the M1BF1 and M1BF1F4 models significantly reduce the natural peak in amplitude but do not change its frequency of oscillation. On the other hand, the controllers based on the other
two models seem to induce a peak splitting around the natural frequency of oscillation, with a slight preference for one of the peaks. In general it would be difficult to derive a strong conclusion about which model is the best from these results. Additionally, if we compare the spectra from these models to the M0B case in Fig. 6.16b the general response of the flow to the forcing is similar, again making it difficult to decide which control separation method is preferable. These results are not completely unexpected since the controllers are designed using the linear portion of the models and the corresponding coefficients of the linear portion have similar values, as explained above.

To verify that the controllers designed for the M1 models are also robust, the control law was tested in the closed-loop experiments for different flows conditions in the neighborhood of Mach 0.30. Figure 7.17 and shows the behavior of the controller when the control is applied to the lower Mach 0.28 for the models M1BF1 (a), M1BF4 (b), M1BF1F4(c) and M1BWn (d). It can be noticed that the system maintains the same general characteristics and benefits shown for the design conditions of Mach 0.30 flow, especially for the models M1BF4 and M1BWn. This is consistent with the previous results (Samimy et al., 2007), which show the robustness of the feedback control loop under off design conditions. The models M1BF1 and M1BF1F4 seem to be less effective in comparison.
Figure 7.16. SPL of LQ control for the separation method M1 using different open loop forcing set, a) M1BF1, b) M1BF4, c) M1BF1F4 and d) M1BWn.

Figure 7.18 shows the behavior of the system, when the control is applied to the higher Mach 0.32 case. In this case, all the controllers seem to perform similarly reducing the dominant peak without introducing any undesirable effects. They also maintain the same general characteristics and benefits shown for the design conditions of Mach 0.30 flow. This is again consistent with the results obtained for the previous model, Samimy et al. (2007).
Figure 7.17. SPL of LQ control at off design conditions $M = 0.28$; a) M1BF1, b) M1BF4, c) M1BF1F4 and d) M1BWn.
At this point, two methods have been developed and tested to explicitly incorporate the actuator input into the reduced order model. These two models are designed using the same tools and following similar steps. Also, as mentioned above, the linear portion of model based on the baseline flow (M0B) is very similar to the linear portion of all the M1 models. The various models all show good reduction of the amplitude of oscillation in the experiments. Consequently, it is difficult to suggest an optimal method for developing a reduced order model for control design purposes. However, there are two important
points that can partially address this issue. First, the M1 method does not require the
presence of the control input region in the images to incorporate its effect in the model.
This suggests that M1 should be the preferred method. Second, the white noise forcing
(Wn) contains the effect of multiple frequencies within the range of interest while
requiring only a single experiment to collect the data. In the next chapter we will attempt
to strengthen these two observations by introducing additional quantities to compare the
results obtained from the experimental implementation of each model based controller.
This chapter focuses on the discussion of the results obtained for the two control separation methods, M0 and M1, for which a controller was designed and tested experimentally. Also, an attempt will be made to answer some of the questions raised during the process of developing a reduced order model for feedback control in flow applications. Finally, the important aspects of the research that have made a contribution to the community of flow control will be discussed.

There are several aspects that are important in the development and implementation of a feedback controller based on a reduced order model will be discussed. The advantages of various separation methods and the benefits of using certain open loop forcing in the data acquisition process have been shown in the previous chapter. In this chapter, a more quantitative comparison of the different models, separation methods and/or forcing inputs will be presented and discussed.

8.1 Effect of Model on the Velocity Reconstruction

To ascertain the ability of each model to recover the forced flows, we compare the original velocity field with the reconstructed one using the control modes for several models from both separation methods. The quality of the reconstruction of the velocity
field will be evaluated by obtaining the mean square error measured relative to the original velocity fluctuations. This is done in an attempt to establish the best control case required in the model derivation process, in an effort to answer one of the lingering question in the community, i.e. what forced flow should be used to derive a reduced order model that can adapt to the changing flow field? To do so, the velocity fields for the forced cases (F1 and F4) are reconstructed using the modes for the white noise and the F1-F4 bases for both separation methods. The idea is to reconstruct the velocity fluctuations based the POD bases of each model and calculate how much it deviates from the original value.

8.1.1 Separation method M0

For this separation method, the first step is to project the velocity fluctuations onto the corresponding POD bases of the model used, i.e. BF1F4 bases. This represents how much of the original flow can be recovered by the basis used in the model. Then the modal amplitudes are obtained by:

$$a_i(t) = \int_{\Omega} \left( U_{fc}(x,t) - \bar{U}_m(x) \right) \varphi_{m,i}(x) \, dx$$

(8.1)

where, $U_{fc}$ is the forced flow velocity field from the PIV images, $U_m$ is the mean velocity of the flow for the model used, and $\varphi_{m,i}$ are the POD bases of the flow. In this case, the effect of the control input is embedded in the images and the velocity can be obtained from the POD expansion (Eqn. 3.9), where the velocity fluctuations are given by:

$$\tilde{u}(x,t) = \sum_{n=1}^{N_{\text{POD}}} a_n(t) \varphi_{m,n}(x)$$

(8.2)
8.1.2 Separation method M1

For this separation method, the first step is to obtain the modal amplitude of the flow captured by the baseline modes. This is how much of the controlled flow is recovered by the baseline bases. This is obtained by projecting the velocity field (PIV images) of the forced flow onto the baseline bases

\[ a_i(t) = \int_D (U_{fc}(x,t) - U_b(x)) \phi_i(x) \, dx \]  

(8.3)

where, \( U_{fc} \) is the forced flow velocity field, \( U_b \) is the mean velocity of the baseline flow and \( \phi_i \) are the POD bases of the baseline flow. In this case, the effect of the control input is included in the reconstruction by multiplying the control mode \( (\psi) \) by the corresponding voltage \( (\Gamma) \) which is measured at the same time that the PIV image was taken. The velocity is then obtained with the following expression:

\[ \bar{u}(x,t) = \sum_{n=1}^{N_{pod}} a^n(t) \varphi^n(x) + \Gamma \psi_1(x) + \Gamma^2 \psi_q(x) \]  

(8.4)

where the \( \psi_1 \) and \( \psi_q \) components of the control modes can be obtained from the original control modes \( (\psi) \) using the quadratic stochastic estimation described in chapter 7, as follows:

\[ \Gamma(t)\psi_1(x) + \Gamma^2(t)\psi_q(x) = \sum_{j=1}^{N_{pod}} \Gamma(t)m^j\psi^j(x) + \Gamma^2(t)\alpha^j\psi^j(x) \]  

(8.5)
8.1.3 Error in the velocity reconstruction

Finally, for both separation methods, the local reconstruction error is defined as the square root of mean value of the squared difference between the actual velocity (PIV images) and the reconstructed value (Eqn. 8.2 and Eqn. 8.4)

\[ err = \sqrt{\frac{1}{N} \sum (u - \bar{u})^2} \] (8.6)

where the fluctuating component of the velocity is obtained by:

\[ u(x, t) = U_{fc} - \bar{U}(x) \] (8.7)

Here the mean velocity depends on the separation method and the model being used, as explained above.

The procedure was used to calculate the error in the reconstruction of the velocity fluctuations for the F1 and F4 forced flows. In each case the flow was reconstructed using the basis for the models M0B, M1BF1-F4 or M1BWn. For these particular cases, the mean velocity corresponds to the baseline flow. Figures 8.1 and 8.2 show the mean square error of the reconstruction for the F1 and F4 forcing cases, respectively. In both figures the first plot (Fig. 8.1a and Fig. 8.2a) represents the mean square error when the flow is reconstructed using the Baseline flow model (M0B), the second plot (Fig. 8.1b and Fig. 8.2b) corresponds to the white noise control basis for M1, and the last plot (Fig. 8.1c and Fig. 8.2c) shows the error for the case based on the combination of the forced cases F1-F4. The figures contain the mean square error for both velocity components. The M0B case is important, as it represents the starting point of the two other reconstructions, by definition of the M1 method. It is clear from both sets of figures that
the error levels are the lowest for the Wn case (Fig. 8.1b and Fig. 8.2b). In this case the error is concentrated in the shear layer region and close to the leading edge. It is suspected that this is due to the difference in mean flow, as we are using the baseline mean flow as the overall mean. The M0B reconstruction (Fig. 8.1a and Fig. 8.2a) stands in between the other two cases. The F1 case (8.1a) shows a longer region around the leading edge of the cavity for the u component. The v component also shows some higher levels with respect to the Wn case near the trailing edge of the cavity and a large structure in the shear layer near the leading edge. For the F4 case these differences are less noticeable, but are still present. For the reconstructions based on F1-F4, it can be observed that the error spreads to a larger region from the midway point of the cavity and toward the trailing edge. It is thought that the main reason for the difference in the mean error is the nature of the structures present in the control mode basis. It was noticed in Figs. 7.9d and 7.9e that the F1-F4 case bases contain large and well organized structures, while the Wn case has more random structures. This in turn generates larger fluctuations in the velocity reconstruction when the forcing is introduced. The results for the other models showed behavior in between the M1BWn case and the M1BF1F4, depending on the model and what forcing case reconstructed.
Figure 8.1. Mean error in the velocity reconstruction for the forced case (F1); a) for the sub-domain method (M0B), and with the control mode method based on b) white noise forcing (Wn), and b) the combination of F1–F4 forcing.
Figure 8.2. Mean error in the velocity reconstruction for the forced case (F4) (F1); a) for the sub-domain method (M0B), and with the control mode method based on b) white noise forcing (Wn), and b) the combination of F1–F4 forcing.

From the velocity reconstruction point of view, using the white noise control modes generated the lowest mean square error. The error levels are lower than those obtained for the reconstruction based on the baseline (M0B) model; this suggests that adding the control modes affects the flow in the proper direction. This result supports the numerical solution of GS (Eqn. 7.10) where the flow is changed by the introduction of the forcing.
The result also suggests that the Wn case is capable of recovering different forced flow conditions, even though they were not used in the derivation of this model (M1BWn).

8.2 Effect on the Power Requirements and on the Overall Sound Pressure Level

The experimental results presented in the previous two chapters showed good control authority, reducing the dominant resonant peak for most models. However, it is difficult to identify a model that behaves better than the others, if one only considers the SPL spectrum. To further quantify the effect of the different models tested over the baseline flow behavior, some additional variables need to be checked. Two variables were selected for this; the average reduction in the overall sound pressure level (OASPL) using several surface pressure sensors and the root mean square control input voltage ($V_{rms}$) used in each case will be compared. These two quantities give an idea of how the controllers are performing (noise reduction or resonance attenuation) with what is being used (voltage-power). The OASPL is calculated using the average value over the 6 sensors in the cavity. Here, the OASPL is calculated as follows:

$$OASPL = 10 \log_{10} \int_{f_{1}}^{f_{2}} S(f) df$$

where $f$ is the frequency, $S(f)$ is the power spectrum of $p'/P_{ref}$, with $p'$ the pressure fluctuations and $P_{ref} = 20 \mu Pa$, the reference value commonly used. The limits of integrations were set in the frequency region of interest.

Figure 8.3 shows a comparison of the OASPL for several of the models used, i.e. the models that showed the best results in terms of resonant peak reduction. The solid symbols correspond to the M0 separation method, while the hollow ones correspond to the M1 method. The first important observation is that models based on the M1 separation
method always seem to require less power to achieve similar or better reduction than the M0 method for the same model. The only exception is for the model BF4, for which the behavior remains close for both separation methods. It can also be noticed that the M0B model shows the smallest reduction using the most voltage. This suggests that, even though this model is capable of reducing the oscillation in the cavity flow, adding the forced flow into the model helps to improve the overall behavior of the controller. When looking at the different forcing based models, it looks as if the model based on the white noise (M1BWN) performs the best, with the M1BF1 very close.

Figure 8.3. Mean OASPL change vs. rms voltage for several models at the design conditions (M = 0.3).

Figure 8.4 shows the comparison of the OASPL for two off design flow conditions, M = 0.28 (a) and M = 0.32 (b). In this two off design conditions similar response obtained for the M = 0.3 are observed. Again, the M0B model shows the least reduction
and uses the most voltage. The M1BWn model performs the best in both off design
conditions, showing the most reduction or close to the best, but requiring the least voltage
input. In these cases the models based on BF1 switch its behavior, responding better for
the M0 model. However, the response of both models is close to each other. For the
model based on BF4 the inverse is observed, where now the M1 model behaves better.
As explained in the previous chapters, the results showed that the designed controllers are
robust under a range of flow conditions, and this is further confirmed with these results,
as the net behavior is to reduce the OASPL of the flow.

Finally, the behaviors of the models are compared to the open loop forcing used to
generate them. Figure 8.5 compares the behavior of the open loop forcing with two of the
closed loop models, M0B as the reference model and M1BWn as the best model. Figure
8.5a shows the mean OASPL change at the design condition (M = 0.3). The results
indicate that, with the exception of the F3 forcing, the model M1BWn generates the best
reduction with the least energy. It even uses less energy than that of the Wn case used to
generate it. The response of the F1 and F2 cases are not completely surprising, as these
two force the flow to a new resonant condition at the 2\textsuperscript{nd} Rossiter mode. Figure 8.5b
shows the mean change in OASPL evaluated as the Mach number is changed; the open
loop cases have a constant $V_{\text{rms}}$ (in parentheses). It can be observed from the figure, that
only the F4 open loop case maintains a behavior similar to the M1BWn model response.
However, it uses a higher voltage input until the flow reaches $M = 0.32$. For all the other
open loop controllers, the behavior is drastically affected outside the $M = 0.3$ baseline
flow and behaving poorly compared to the closed loop controllers. This suggests, at least
for the current flow configuration, that a closed loop controller based on reduced order models is the preferable option to control the flow.

Figure 8.4. Mean OASPL change vs. rms voltage for several models at off design conditions: a) M = 0.28 and b) M = 0.32.
Figure 8.5. Comparison of open-loop and close-loop forcing, a) mean OASPL change vs. rms voltage, b) mean OASPL change vs. Mach number for different forced cases.

The results hint that the M1 separation method and the white noise forced flow are the best combination for developing a controller based on reduced order model. In
addition to the simplicity of its derivation, as mentioned in the previous chapter, it
generates the best results in terms of reducing the tone and the OASPL while using low
power to achieve it.

The advantages of the M1 separation methods can be summarized as:

- It does not require the presence of the control input region on the flow field.
- The control effect is added as a separate set of modes that influence the baseline
  flow.
- The model without forcing always recovers the baseline flow behavior,
  independent of the forcing used for the control modes.
- The experimental implementation showed the best overall results for most of the
  model derived with this method, when comparing to its counterpart based on
  method M0.

The use of the Wn forcing showed the following advantages:

- There is no need to pre-identify forcing conditions generating different flow
  conditions before capturing the flow field variables with control.
- Only one forcing case is used to develop the model.
- Due to its nature, it includes the effect of several frequencies simultaneously.
- It shows the best results, in terms of reconstruction error, when used to
  reconstruct the velocity field from a different flow condition.
- The experimental implementation showed the best reduction and lowest use of
  energy for the models based on this forced condition.
- It does not require the assembly of a large set of flow images to capture most of the possible flow conditions, therefore it reduces the number of images to capture and process.

- Computation time is reduced.

The results obtained for the cavity flow are promising and suggest that the procedure might be the preferred method to use in the derivation of reduced order models for feedback control applications. By doing so, two important issues have been addressed. First, how to incorporate the control input into the system so that the model is able to respond adequately under different flow conditions (forcing). Secondly, what flow conditions to use in the derivation and how to combine them so that the model is rich enough to have an adequate response under conditions different from those included in the model.

The final issue that needs to be discussed is the effect of the estimation procedure in the experimental implementation of the controller. This will be presented on the next section using the experimental results from different models.

8.3 Effect of the Stochastic Estimation Coefficient in the Experimental Implementation of the Controllers

In the experimental implementation of the model based controllers, the estimated modal amplitude is the variable controlled. These modal amplitudes are obtained from the surface pressure measurements and the stochastic estimation procedure, as described in chapter 5. However, the question always arises of which estimation coefficients should be used, the ones obtained for the baseline flow or the ones based on the flow conditions
used in the corresponding model. This is especially significant for the method M0, as each model was derived from a particular set of data and the matching estimation coefficients are known. For the method M1 it seems intuitive to use the baseline coefficients, as this is the base flow of the model. The importance of this issue in the experimental implementation is clear, since the flow condition will change as soon as the controller is turned on. To probe this, experimental data for the same model based controller was taken using different estimation coefficients. It is worth noticing that in some of the cases the experiments were performed before the actuator compensator was introduced on the system, and some were taken after. Therefore, no real comparison between the different models will be given.

Figure 8.6 shows the SPL spectrum of four different models using the estimation coefficients of the baseline (B) and of the corresponding forced case (BF4 or Wn). In figure 8.6a, for the model M0BF4, there is no noticeable effect of the estimation in the behavior of the controller since it attains similar levels of reduction of the dominant peak. In Fig. 8.6b, for the model M1BF4, it can be observed that even though the model is based on BF4, using this set of coefficients the system shows the presence of a peak that is not present when the estimation is based on the baseline case (B). Figures 8.6c and 8.6d, models M0B and M1BWn respectively, show the effect of using the Wn estimation coefficients instead of the baseline case (B). The results seem to indicate that it is preferable to use the baseline coefficients since in both cases there is less reduction of the dominant peak when the Wn set is used.
Figure 8.6. Effect of the changing the estimation coefficients of the modal amplitude on the LQ control implementation for different models; a) M0B, b) M0BF2, c) M0BF4, d) M0.

In addition to the SPL spectrum, the effect of the estimation coefficients on the mean reduction of OASPL was also calculated. Figure 8.7a presents the effect of the estimation coefficients on the OASPL reduction for the models in Figs. 8.6a and 8.6b, while 8.7b shows the models in Figs. 8.6c and 8.6d. It is apparent from the figures that when the baseline estimation coefficients are used the general behavior of the controller is
improved, requiring less power to achieve similar or larger reduction of the mean OASPL.

Figure 8.7. Mean reduction of the OASPL for different estimation coefficients: a) model based on BF4, b) models based on B and Wn.
The combination of the results from the two sets of figures seems to indicate that the estimation coefficients of the baseline flow (B) would be the better choice to use in the experimental estimation of the modal amplitudes, as it provides the best overall results. This could be explained from the derivation process of the estimation coefficients. By definition they are set to capture the behavior of the flow from which they are extracted, and when applied to a different condition, it tends to steer the behavior towards the original set. This is more noticeable when the estimation coefficients are from the F1 or F2 case, where there is a clear peak on the controlled flow that is carried into the estimation. In the case of the F4 flow, the cavity changed to a multi-mode resonance not showing any preferred frequency. This is similar to the white noise case, where there are no dominant frequencies present in the flow.
CHAPTER 9

SUMMARY, CONCLUSIONS AND FUTURE WORK

9.1 Summary

The main goal of this work was to derive reduced order models for the cavity flow that can be used in the design of feedback control. This required addressing several issues that have not been dealt with in the literature. One of the main issues concerns the proper way of treating the control input in the system, so that control theory tools can be applied in an appropriate manner. Another main issue is what flow conditions should be used in the derivation process and how to include control effects in the model. Throughout this work the required tools to tackle these issues were developed and implemented. This effort lead to the design and successful implementation of a model based feedback controller capable of reducing the amplitude of the natural resonant tone in a subsonic shallow cavity. This was done as an integral part of a larger effort to develop and implement model based closed-loop flow control technologies at the Collaborative Center of Control Science (CCCS) at the Ohio State University.

The research evolved from its initial steps, where simulation data was used to derive the reduced order model, to the present stage, where experimental data is used to derive the model. A series of difficulties in the numerical simulations prevented the design of a controller based on simulation data that could be implemented in the experiments.
However, this effort helped in the development of the MATLAB codes capable of obtaining the POD basis, as well as, applying the Galerkin projection and the stochastic estimation procedure.

To develop the reduced order model, several flow conditions were recorded for the Mach 0.3 flow, including snapshots from the baseline (no actuation) and, the same flow under open-loop forcing at various frequencies. The forced flows included both single-mode and multi-mode resonant regimes, and were selected after a complete sweep of the frequency range of interest. A set of data with band limited white noise forcing (Wn) was added, as it includes a range of frequencies without any preferred one and it does not require any detailed study of the flow under various open-loop forcing. The snapshot-based proper orthogonal decomposition (POD) technique is used on the Particle Image Velocimetry (PIV) data to extract the most dominant features in the flow. It was determined that in order to ensure the convergence of turbulent kinetic energy, at least 700 images are required. For the model derivation it was decided to include all the 1000 available snapshots. The time evolution of the modes is obtained with a set of nonlinear ordinary differential equations derived by Galerkin projection of the Navier-Stokes equations onto the POD modes; this represents the reduced order model of the flow. The modal coefficients are correlated to dynamic surface pressure measurements using linear and quadratic stochastic estimation enabling a real-time estimate of the state of the flow.

For all the cases tested, the POD reconstruction of the velocity field showed that with about 30 modes there is good agreement with the PIV data. But, for control purposes only 4 modes are used to capture the main characteristics of the flow. An additional viscous
term was added to the models to account for the energy balance with the neglected POD modes and to dampen small numerical oscillations arising in the derivation procedure. The modal energy distribution is similar for all the cases tested. More energy is recovered by the first few modes in the case of flows with larger, more organized shear-layer structures. Composite models, obtained by combining the PIV data of different flows, have POD modes that capture some of the characteristics of the individual flows.

To incorporate the effect of actuation in the model, three methods were developed and tested. For each method, the models for individual flows as well as combinations of flow cases were explored. In the first method, M0, the flow spatial domain was separated into two sub-domains; a small sub-domain where the effect of actuation enters into the flow and a larger one that contains the remainder of the flow. This produces a reduced-order model of the flow in terms of the modal amplitudes where the control effect appears explicitly, as a result of the mathematical handling of the equations. The numerical solution of the models obtained with this method showed that the forcing input had no influence on the behavior of the modal amplitude, as the control input coefficient matrices are small compared to the rest of the coefficients. This result suggests that the model spans the space for which it was developed, as the forcing effect is embedded into each PIV image.

The second separation method was based on the weak formulation of the governing equations. The method uses Green’s identity to eliminate the 2\textsuperscript{nd} order derivative terms. It produces a reduced-order model similar to the first method. Here the control input also appears explicitly in the equation as the result of the mathematical handling of the
equations. The numerical solution of the models showed the same behavior as the first method.

The final separation method, M1, incorporates the control input as an additional set of modes obtained from the difference between a forced flow and its projection onto the baseline flow. In this case, the control term is introduced in the definition of the POD expansion. The numerical solution of the models based on this method, show changes in the response of the system when the forcing is introduced. This result suggests that the model spans not only the space of the baseline flow, but also captures the dynamics due to the introduction of forcing.

A linear-quadratic optimal controller was designed for each of the models obtained from the first and third separation methods to attenuate the cavity flow resonance. The controllers are then evaluated experimentally. The result obtained for the models from the two methods show that in general the controllers significantly reduce the resonant peak of the single-mode Mach 0.3 flow, for which it was designed, without introducing any undesirable peaks. The controllers also show robustness, as they can control the flow over a range of Mach numbers that includes the design Mach number.

For real-time update of the model variables, linear and quadratic stochastic estimation procedures are developed that can operate statically (i.e. based on instantaneous values of the pressure) or dynamically by accounting for one or more previous pressure samples. The dynamic quadratic estimation provides a better approximation of the model variables, but the inclusion of the actuator compensator imposed a severe limitation on the system, allowing only single time estimation for the experimental tests.
The contribution of this work to the flow control community can be evaluated in different ways. First, it was a major part of the successful development and implementation of a feedback controller derived from an experimentally based reduced order model, where the model is obtained using the governing equations and not an empirical relation. Secondly, it addressed how to include the control input and what flow conditions are required. The conclusions reached as a result of this work are summarized in the next section.

9.2 Conclusions

The variation in the SPL spectra experimental results between the different models was marginal making it difficult to favor a particular model. However, a closer look at other factors hint that the M1 separation method using the white noise forced flow for the control modes is the best combination for developing a controller based on reduced order model.

The selection of the separation method M1 over the M0 seems natural from the derivation point of view, since it incorporates the control effect as an additional direction to the flow space. This effect was noted when the model was solved numerically, as it recovered the baseline flow behavior for the unforced case and showed the presence of the forcing frequency when included. Furthermore, this separation method does not require identification of the control input region in the data used to derive the reduced order model, which is not possible in many flow control applications. In addition, this selection is not only supported by the simplicity of its derivation, but also because it
generates the best results in terms of reducing the tone and the OASPL while using a lower power input to achieve it.

The models based on the white noise forcing (Wn) show the best overall results among the different forcing cases independent of the separation method. It helps to simplify the derivation process, as there is no need to pre-identify forcing conditions generating different flow conditions before capturing the flow field variables with control. There is a need for only one forcing case to develop the model. It includes the effect of several frequencies simultaneously. It also shows the best results in terms of reconstruction error, when used to reconstruct the velocity field from a different flow condition. Finally, it reduces the computation time.

For the real-time estimation of the modal coefficient, the multiple time estimation provides the best results in terms of the amplitude, but the implementation of this procedure is limited by the physical hardware. This became more stringent when the actuator compensator was added to the system as it adds too much burden and the procedure can only be used with 3 previous times in the linear case or 1 previous time in the quadratic case. The effect of the number of sensors was investigated with the help of a new high speed cavity facility. As expected, the increase from 5 sensors to 10 sensors helps to further improve the estimation, but little gain was noticed when including all 15 available sensors. However, the current sensor configuration on either cavity facility does not allow investigating the effect of the sensor location or its optimization.
9.3 Future Work

This work addresses two outstanding issues in the flow control community. The first is, how to incorporate the control input into the system so that the model is able to respond adequately under different flow conditions (forcing). The second is what flow conditions to use in the model derivation and how to combine them so that the model is rich enough to have a satisfactory behavior under conditions different from those included in the model. The results obtained for the cavity flow control are promising and suggest that the outlined procedure may be a good general method to use in the derivation of reduced order models for feedback control applications.

Although the methodology worked well for the cavity flow used in this investigation, by no means it can be considered a unique solution for all the feedback flow control problems. In fact, several of the components in the methodology could still be improved. There are some uncertainties associated with the use of Galerkin projection method in the model derivation process. It may introduce errors in the derivatives terms and it does not necessarily preserve the phase information between the modal coefficients. Even though this method was selected for the work, there are ways to avoid this step that could also be tested. For example, knowing the final form of the reduced order model one can use system identification techniques to obtain the constant coefficients in the model. The limitations of the actuator need to be taken into account in the modeling process, as not all the available options can introduce white noise signals or can perform detailed frequency sweep to study the effect on the flow. Finally, to test the validity of the method, the procedure should be applied to a different flow configuration. To be more
specific, it should be tested on a similar flow arrangement, the cavity flow, but for a more complex case of multiple resonant tones.
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APPENDIX A

HILBERT-SPACE OF L₂ FUNCTIONS

An L₂ function is a function $f : x \mapsto \mathbb{R}$ that is square integrable, real or complex valued, defined in the interval $\Omega = (a,b)$ that satisfies

$$\int_{a}^{b} |f(x)|^2 \, dx < \infty$$

(A.1)

i.e. the solution exists and is finite. The collection of L₂ functions on $x$ is called a Hilbert space, with an inner product defined by

$$\langle f, g \rangle = \int_{\Omega} f(x)g^*(x)\, dx$$

(A.2)

where $^*$ denotes the complex conjugate. Then it can be noted that $\langle f, g \rangle = \langle g, f \rangle^*$ and that the L₂-norm $|f|$ of $f$ can also be written as

$$|f| = \sqrt{\langle f, f \rangle}$$

(A.3)
APPENDIX B

DERIVATION OF THE REDUCED ORDER MODEL

In what follows the general procedure followed in the derivation of the reduced order model will be presented. Starting from the governing equations:

Continuity

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \]  \hspace{1cm} (B.1)

Momentum

\[ \frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} + \frac{\partial p}{\partial x_i} = \frac{\partial \sigma_{ij}}{\partial x_j} \]  \hspace{1cm} (B.2)

where \( \sigma_{ij} = -\frac{2}{3} \mu \delta_{ij} S_{kk} + 2 \mu S_{ij} \) and \( S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \)

Energy

\[ \frac{\partial \rho e_t}{\partial t} + \frac{\partial \rho u_i e_t}{\partial x_i} + \frac{\partial \rho u_i p}{\partial x_i} = \frac{\partial \sigma_{ij}}{\partial x_j} - \frac{\partial q_i}{\partial x_i} \]  \hspace{1cm} (B.3)

where \( e_t = e + \frac{1}{2} u_k u_k \) and \( q_i = -k \frac{\partial T}{\partial x_i} \)
Based on the cavity flow conditions, it can be assumed that there is no heat conduction (cold flow), then the density gradients will remain small and the viscous dissipation can be neglected. Then the flow can be treated as isentropic and the total energy will not change. The system can be transformed into:

Continuity

\[
\frac{\partial c}{\partial t} + u_i \frac{\partial c}{\partial x_i} + \frac{\gamma - 1}{2} c \frac{\partial u_i}{\partial x_i} = 0
\]

(Momentum)

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{2}{\gamma - 1} c \frac{\partial c}{\partial x_i} = -\nu \frac{\partial^2 u_i}{\partial x_j^2}
\]

where \( c^2 = \frac{E}{\rho} \)

The vector approach of the inner product (Eqn. 3.18) is used for the projection, defined as:

\[
\langle q_1, q_2 \rangle = \int_{\Omega} \left( \frac{2\alpha}{\gamma - 1} c_1 c_2 + u_1 u_2 + v_1 v_2 \right) dV
\]

where the flow variables are expressed as a vector \( q(u, v, c) \) and the value of \( \alpha \) was set to 1 for all the calculations.

In the process of deriving the POD modes the variables are non-dimensionalized as follows:

\[
x' = x/D; \quad y' = y/D; \quad u' = u/U_\infty; \quad v' = v/U_\infty; \quad \rho' = \rho/\rho_\infty; \\
t' = t* U_\infty/D; \quad p' = p/(\rho_\infty* U_\infty^2); \quad Re = U_\infty*D/\mu; \quad M = U_\infty/C_\infty;
\]
Each term is expanded into mean flow and fluctuating component as $q = Q + q$ (the total property vector, $u$, $v$ and $c$) or $u = U + \bar{u}$ (for the streamwise velocity component), where $\sim$ represents fluctuating component. Expressing the fluctuating components in terms of the POD expansion gives:

$$\tilde{q}(x,t) = (\tilde{c},\tilde{u},\tilde{v}) = \sum_{j=0}^{n} a^j(t) \phi_j^c(x) \phi_j^u(x) \phi_j^v(x)$$  \hspace{1cm} (B.6)

$n$ is the number of POD modes used. Then each individual velocity component can be written as

$$\tilde{c}(x,t) = \sum_{j=0}^{n} a^j(t) \phi_j^c(x), \quad \tilde{u}(x,t) = \sum_{j=0}^{n} a^j(t) \phi_j^u(x), \quad \tilde{v}(x,t) = \sum_{j=0}^{n} a^j(t) \phi_j^v(x)$$  \hspace{1cm} (B.7)

Replacing the mean and the fluctuation component of each variable in the governing equation results in:

Continuity

$$\frac{\partial \tilde{c}}{\partial t} + \frac{\partial}{\partial x_i} \left[ U_{ij} \phi_j^c \right] + \frac{\partial}{\partial x_i} \left[ \bar{u}_{ij} \phi_j^u \right] + \frac{\partial}{\partial x_i} \left[ \tilde{u}_{ij} \phi_j^u \right] + \frac{\partial}{\partial x_i} \left[ \tilde{u}_{ij} \phi_j^u \right] + \frac{\partial}{\partial x_i} \left[ \bar{v}_{ij} \phi_j^v \right] + \frac{\partial}{\partial x_i} \left[ u_{ij} \phi_j^v \right] + \frac{\partial}{\partial x_i} \left[ \tilde{v}_{ij} \phi_j^v \right] + \frac{\partial}{\partial x_i} \left[ \tilde{v}_{ij} \phi_j^v \right] + \frac{\partial}{\partial x_i} \left[ v_{ij} \phi_j^v \right] = 0$$  \hspace{1cm} (B.8)

rearranging the terms

$$\frac{\partial}{\partial t} \left[ \sum_{j=0}^{n} a^j \phi_j^c \right] + \frac{\partial}{\partial x_i} \left[ \sum_{j=0}^{n} a^j \phi_j^c \right] \left[ \sum_{j=0}^{n} a^j \phi_j^u \right] + \frac{\partial}{\partial x_i} \left[ \sum_{j=0}^{n} a^j \phi_j^u \right] \left[ \sum_{j=0}^{n} a^j \phi_j^u \right] + \frac{\partial}{\partial x_i} \left[ \sum_{j=0}^{n} a^j \phi_j^v \right] \left[ \sum_{j=0}^{n} a^j \phi_j^v \right] + \frac{\partial}{\partial x_i} \left[ \sum_{j=0}^{n} a^j \phi_j^v \right] \left[ \sum_{j=0}^{n} a^j \phi_j^v \right]$$

$$+ \frac{\partial}{\partial x_i} \left[ \sum_{j=0}^{n} a^j \phi_j^u \right] \left[ \sum_{j=0}^{n} a^j \phi_j^v \right] + \frac{\partial}{\partial x_i} \left[ \sum_{j=0}^{n} a^j \phi_j^v \right] \left[ \sum_{j=0}^{n} a^j \phi_j^v \right] = 0$$  \hspace{1cm} (B.9)

combining similar terms and using $\frac{\partial a}{\partial t} = \dot{a}$ then:
\[
\sum_{j=0}^{n} a_j \phi_j + U_j \frac{\partial \phi}{\partial x_j} + \frac{\gamma - 1}{2} C \frac{\partial U}{\partial x_j} + \sum_{j=0}^{n} a_j \left[ U_j \frac{\partial \phi_j}{\partial x_j} + \frac{\gamma - 1}{2} C \phi_j \frac{\partial \phi_j}{\partial x_j} \right] + \sum_{j=0}^{n} \sum_{m=0}^{n} a_j a_m \left[ \phi_j \frac{\partial^2 \phi}{\partial x_j^2} + \frac{\gamma - 1}{2} \phi_j \frac{\partial^2 \phi}{\partial x_j^2} \right] = 0
\]

for the non-dimensional momentum equation

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{2}{M^2 \gamma - 1} C \frac{\partial^2 u_i}{\partial x_i^2} = \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_i^2}
\]

then

\[
\frac{\partial \tilde{u}_i}{\partial t} + U_j \frac{\partial \tilde{U}_i}{\partial x_j} + U_j \frac{\partial \tilde{u}_i}{\partial x_j} + \tilde{u}_j \frac{\partial \tilde{U}_i}{\partial x_j} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \frac{2}{M^2 \gamma - 1} \left[ C \frac{\partial \tilde{C}}{\partial x_i} + C \frac{\partial \tilde{c}}{\partial x_i} + \tilde{c} \frac{\partial \tilde{C}}{\partial x_i} + \tilde{c} \frac{\partial \tilde{c}}{\partial x_i} \right] = \frac{1}{Re} \left( \frac{\partial^2 U_i}{\partial x_i^2} + \frac{\partial^2 \tilde{u}_i}{\partial x_i^2} \right)
\]

rearranging the terms

\[
\frac{\partial}{\partial t} \sum_{j=0}^{n} a_j \phi_j + U_j \frac{\partial U}{\partial x_j} + U_j \frac{\partial \sum_{j=0}^{n} a_j \phi_j}{\partial x_j} + \sum_{j=0}^{n} a_j \phi_j \frac{\partial U}{\partial x_j} + \sum_{j=0}^{n} a_j \phi_j \frac{\partial \sum_{j=0}^{n} a_j \phi_j}{\partial x_j} + \frac{2}{M^2 \gamma - 1} \left[ C \frac{\partial \sum_{j=0}^{n} a_j \phi_j}{\partial x_j} + C \frac{\partial \sum_{j=0}^{n} a_j \phi_j}{\partial x_j} + \sum_{j=0}^{n} a_j \phi_j \frac{\partial C}{\partial x_j} + \sum_{j=0}^{n} a_j \phi_j \frac{\partial \sum_{j=0}^{n} a_j \phi_j}{\partial x_j} \right] = \frac{1}{Re} \left( \frac{\partial^2 U_i}{\partial x_i^2} + \frac{\partial^2 \tilde{u}_i}{\partial x_i^2} \right)
\]

combining similar terms
expanding all the terms the following system of partial differential equations is obtained:

Continuity (E1)

\[
\sum_{j=0}^{n} \dot{\alpha}^{i} \phi_{u}^{j} = -U C_{x,x} - V C_{y,y} - \frac{\gamma - 1}{2} C \left( U_{,x} + V_{,y} \right)
\]

\[
- \sum_{j=0}^{n} \alpha^{j} \left[ U \phi_{u,xx}^{j} + V \phi_{u,xy}^{j} + \phi_{u}^{j} C_{x,x} + \phi_{u}^{j} C_{y,y} + \frac{\gamma - 1}{2} \left( C \left( \phi_{u,xx}^{j} + \phi_{u,xy}^{j} \right) + \phi_{u}^{j} \left( U_{,x} + V_{,y} \right) \right) \right]
\] (B.15)

\[
- \sum_{j=0}^{n} \sum_{m=0}^{n} \alpha^{j} a^{m} \left[ \phi_{u}^{j} \phi_{u,xx}^{m} + \phi_{u}^{j} \phi_{u,xy}^{m} + \frac{\gamma - 1}{2} \phi_{u}^{j} \left( \phi_{u,xx}^{m} + \phi_{u,xy}^{m} \right) \right]
\]

X-momentum (E2)

\[
\sum_{j=0}^{n} \dot{\alpha}^{i} \phi_{a}^{j} = -UU_{,x} - VU_{,y} - \frac{2}{M^{2} \gamma - 1} CC_{x,x}
\]

\[
- \sum_{j=0}^{n} \alpha^{j} \left[ U \phi_{a,xx}^{j} + V \phi_{a,xy}^{j} + \phi_{a}^{j} U_{,x} + \phi_{a}^{j} U_{,y} + \frac{2}{M^{2} \gamma - 1} \left( C \phi_{a,xx}^{j} + \phi_{a,xy}^{j} \right) \right]
\] (B.16)

\[
- \sum_{j=0}^{n} \sum_{m=0}^{n} \alpha^{j} a^{m} \left( \phi_{a}^{j} \phi_{a,xx}^{m} + \phi_{a}^{j} \phi_{a,xy}^{m} + \frac{2}{M^{2} \gamma - 1} \phi_{a}^{j} \phi_{a,xx}^{m} \right) + \frac{1}{\text{Re}} \left( U_{,xx} + U_{,xy} + \sum_{j=0}^{n} \alpha^{j} \left( \phi_{a,xx}^{j} + \phi_{a,xy}^{j} \right) \right)
\]

Y-momentum (E3)

\[
\sum_{j=0}^{n} \dot{\alpha}^{i} \phi_{v}^{j} = -UV_{,x} - WV_{,y} - \frac{2}{M^{2} \gamma - 1} CC_{y,y}
\]

\[
- \sum_{j=0}^{n} \alpha^{j} \left[ U \phi_{v,xx}^{j} + V \phi_{v,xy}^{j} + \phi_{v}^{j} V_{,x} + \phi_{v}^{j} V_{,y} + \frac{2}{M^{2} \gamma - 1} \left( C \phi_{v,xx}^{j} + \phi_{v,xy}^{j} \right) \right]
\] (B.17)

\[
- \sum_{j=0}^{n} \sum_{m=0}^{n} \alpha^{j} a^{m} \left( \phi_{v}^{j} \phi_{v,xx}^{m} + \phi_{v}^{j} \phi_{v,xy}^{m} + \frac{2}{M^{2} \gamma - 1} \phi_{v}^{j} \phi_{v,xx}^{m} \right) + \frac{1}{\text{Re}} \left( V_{,xx} + V_{,xy} + \sum_{j=0}^{n} \alpha^{j} \left( \phi_{v,xx}^{j} + \phi_{v,xy}^{j} \right) \right)
\]
Now the governing equations are expressed in terms of the POD expansion. To obtain the low dimensional model using the Galerkin method, these set of equations are projected onto the POD basis, using the given inner product.

\[
\langle N - S, \phi^k \rangle = \langle (E_1, E_2, E_3), (\phi^k, \phi^n, \phi^l) \rangle = \int_{\Omega} \left( \frac{2}{\gamma - 1} \phi^k E_1 + \phi^k E_2 + \phi^k E_3 \right) dV
\]  

(B.18)

Then for the left side term:

\[
\int_{\Omega} \left( \sum_{j=0}^n \frac{\partial}{\partial t} \phi^j \right) \left( \sum_{j=0}^n \frac{\partial}{\partial x} \phi^j \right) + \left( \sum_{j=0}^n \frac{\partial}{\partial y} \phi^j \right) \left( \sum_{j=0}^n \frac{\partial}{\partial y} \phi^j \right) dV = \sum_{j=0}^n \frac{\partial}{\partial \Omega_j} \phi^j = \phi^k
\]  

(B.19)

since the POD basis are orthonormal by definition.

Next the constant terms are simplified; these terms have no dependency on the time coefficient. Then

\[
\int_{\Omega} \frac{2}{\gamma - 1} \phi^k \left( - U C_{,x} - V C_{,y} - \frac{\gamma - 1}{2} C (U_{,x} + V_{,y}) \right) dV
\]

\[
+ \int_{\Omega} \phi^k \left( - U U_{,x} - V U_{,y} - \frac{2}{M^2 \gamma - 1} C C_{,x} + \frac{1}{Re} (U_{,xx} + U_{,yy}) \right) dV
\]

\[
+ \int_{\Omega} \phi^k \left( - U V_{,x} - V V_{,y} - \frac{2}{M^2 \gamma - 1} C C_{,y} + \frac{1}{Re} (V_{,xx} + V_{,yy}) \right) dV = b^k
\]  

(B.20)

For the first order terms

\[
\int_{\Omega} \frac{2}{\gamma - 1} \phi^k \left[ - U \phi^j_{,x} - V \phi^j_{,y} - \phi^j C_{,x} - \phi^j C_{,y} - \frac{\gamma - 1}{2} \left( \phi^j (U_{,x} + V_{,y}) \right) \right] dV
\]

\[
+ \int_{\Omega} \phi^k \left[ - U \phi^j_{,x} - V \phi^j_{,y} - \phi^j U_{,x} - \phi^j U_{,y} - \frac{2}{M^2 \gamma - 1} \left( C \phi^j + \phi^j C_{,x} \right) + \frac{1}{Re} \left( \phi^j_{,xx} + \phi^j_{,yy} \right) \right] dV
\]

\[
+ \int_{\Omega} \phi^k \left[ - U \phi^j_{,x} - V \phi^j_{,y} - \phi^j V_{,x} - \phi^j V_{,y} - \frac{2}{M^2 \gamma - 1} \left( C \phi^j + \phi^j C_{,y} \right) + \frac{1}{Re} \left( \phi^j_{,xx} + \phi^j_{,yy} \right) \right] dV = d^j
\]  

(B.21)

the second order terms
\[
\int_{\Omega} \frac{2}{\gamma - 1} \phi^k \left[ -\phi^i \phi^{m}_{,x} - \phi^i \phi^{m}_{,y} - \frac{\gamma - 1}{2} \phi^i \left(\phi^{m}_{,x} + \phi^{m}_{,y}\right)\right] dV
\]

\[
+ \int_{\Omega} \phi^i \left(-\phi^i \phi^{m}_{,x} - \phi^i \phi^{m}_{,y} - \frac{2}{\gamma^2 - 1} \phi^i \phi^{m}_{,x}\right) dV
\]

\[
+ \int_{\Omega} \phi^i \left(-\phi^i \phi^{m}_{,x} - \phi^i \phi^{m}_{,y} - \frac{2}{\gamma^2 - 1} \phi^i \phi^{m}_{,y}\right) dV = g^{km}
\] (B.22)

Finally, putting all the terms together gives the following expression:

\[
\dot{a}^k = b^k + \sum_{j=1}^{n} (d^j a') + \sum_{j=1}^{n} \sum_{m=1}^{n} (g^{jm} a'/a^m)
\] (B.23)

This is the form of the system of ODE’s with the control input implicit in the solution, and corresponds to Eqn. 4.2 in Chapter 4. As it can be seen the constant coefficients \(b, d\) and \(g\) are obtained from the Galerkin projection.
APPENDIX C

EFFECT OF SENSOR NUMBER AND LOCATION ON THE STOCHASTIC ESTIMATION

In this appendix, we will look at the effect of increasing the number of sensors for the stochastic estimation procedure. Since the cavity facility used in the current research was designed with only six sensors within the cavity (see Fig. 2.8), it is hard to examine this issue. To investigate sensor number and location, we will use a new cavity facility developed at the GDTL. This is a high speed cavity flow facility equipped with the same measuring tools as the low speed cavity but with an arrangement of 15 pressure sensors on the floor that can be sampled simultaneously. The general description of the new facility can be found in Mitchell (2007). The cavity was designed with a length of 1.00”, depth of 0.25”, and a width of 3.00” – which spans the entire width of the test section. The length and depth geometry gives an aspect ratio \((L/D)\) of 4.0. Figure C.1(a) shows a schematic of the bottom piece of the test section with an emphasis on the cavity floor, the Kulite pressure sensor locations, and the piece that will be used for the actuator housing. Figure C.1(b) shows the distribution of 15 Kulite pressure transducers on the cavity floor. The large number of pressure transducers is used to obtain detailed correlation information between surface pressure and velocity which is an essential element of reduced-order model based feedback flow control. The stagger distribution of Kulite
pressure transducers was intended to investigate the 3 dimensional behavior of the flow. Mitchell (2007) used this staggered arrangement to show that during strong resonance, the flow behaves in a 2D fashion. Consequently, this work will use simultaneous PIV and surface pressure measurements for a highly resonant Mach 0.69 cavity flow in an effort to draw some conclusions on the effects of sensor number and location.

Figure C.1 High speed cavity parts showing surface pressure sensors distributed on the floor of the cavity. Fifteen sensors are placed on the cavity floor to acquire higher spatial resolution pressure measurements. The slot in the middle of the cavity floor is the laser light trap which helps diffuse the light from the PIV laser.

Figure C.2 shows the SPL spectra of the pressure signal for the high speed cavity flow (M= 0.69) at the sensors closest to the centerline. This flow shows a strong resonant peak at about 3000 Hz, which corresponds to the first Rossiter mode for this particular cavity, Mitchell (2007). To analyze the effect of the number of sensors we will examine the following arrangements: 3 sensors (1, 7 and 15), 5 sensors (1, 6, 7, 12 and 13), 6 sensors (1, 2, 7, 8, 13 and 14), 10 sensors (1, 3, 4, 6, 7, 9, 10, 12, 13 and 15) and 15 (all sensors). The sensor arrangement was selected based on its location respect to the centerline of the cavity, were the visualization plane is located. This was more so for the lower number of sensors. However, for the lower number of sensors there was little
change in the estimations when a different selection was used (i.e. 3, 9 and 15 for the 3 sensor case). This is expected since the transducers in the same spanwise locations are highly correlated for this strongly resonant flow (Mitchell, 2007).

![SPL spectra](image)

Figure C.2. SPL spectra at the cavity floor for the sensor closest to the centerline; sensors 1, 6, 7, 12 and 13.

Before doing any comparison of the results for the two facilities, it is important to keep in mind that there are some fundamental differences between them. First, the sensors at the low speed cavity are located on the side wall, three close to the floor and three at the shear layer level, while all the sensors in the high speed facility are located in the floor. Although, it was shown that sensors located on the floor of the low speed cavity correlated well with those on the wall (Little, 2005), using the side wall sensors could affect the ability of the procedure to estimate the correct amplitudes. Second, the two cavities are working under very different conditions, not only from the velocity point of view (M = 0.3 vs. M = 0.69), but also from the dynamic point of view. The low speed cavity resonates at the third Rossiter mode while the other resonates at the first mode.
This is an important difference, since the size and strength of the structures are very different. For example, there is a stronger tone for the high speed cavity, which in turn should increase the effectiveness in the estimation process. Finally, the set of sensors used are different for both facilities which change the sensitivity of the measurement process.

C.1 Velocity Estimation

Based on the results obtained for the low speed cavity (M = 0.3) described earlier, the influence of the number of sensors for this case will be analyzed using the mean turbulent kinetic energy recovered with the quadratic stochastic estimation using one previous time step. Figure C.3 shows the result of the mean turbulent kinetic energy for the high speed cavity. It can be observed that as the number of sensors is increased, the energy recovered by the estimation approximates the actual levels captured in the PIV images. The effect of changing the number of sensors from 3 up to 10 produces noticeable changes in the energy distribution. Increasing up to 15 sensors helps to further improve the levels inside the shear layer region. For the single time quadratic estimation (static), the trend is similar to the earlier Mach 0.30 flow configuration (Fig. 5.1) where larger differences with respect to the original flow are observed. The small difference between 10 and 15 sensors is explained by the sensor distribution since sensors at the same spanwise location show high correlation for this strongly resonant flow.

The effect of the number of sensors can also be seen in the estimation of the instantaneous normal velocity fluctuations shown in Figs. C.4 (static) and C.5 (1 time back). It can be observed that increasing the number of sensors generates a better estimation of the fluctuating velocity field by capturing more details in the PIV image. A
further improvement in the estimation is visible in Fig. C.5 as the 15 sensor estimation captures most of the smaller details present in the original velocity field.

Figure C.3. Estimated mean turbulent kinetic energy for the M = 0.69 cavity flow for different numbers of sensors, a) PIV, b) 3 c) 5, d) 6, e) 10 and f) 15 based on quadratic estimation with 1 previous time.

Figure C.4. Instantaneous velocity estimation for the M = 0.69 cavity flow for different numbers of sensors, a) PIV, b) 3 sensors c) 5 sensors, d) 6 sensors, e) 10 sensors and f) 15 sensors based on quadratic estimation (static).
Figure C.5. Instantaneous velocity estimation for the $M=0.69$ cavity flow for different number of sensors, a) PIV, b) 3 sensors c) 5 sensors, d) 6 sensors, e) 10 sensors and f) 15 sensors based on quadratic estimation with 1 previous time.

C.2 Pressure Estimation

As in the low speed cavity, the effect of the number of sensors used for the estimation of the modal amplitude is investigated since it is the variable that is used for control purposes. Figure C.6 shows the estimation of the modal amplitude of the first two modes for the different groups of sensors based on the quadratic estimation with single time (static), Fig. C.6a, and 1 previous time, Fig. C.6b. Similar to the velocity estimation, increasing the number of sensors from 3 to 10 improves the estimation, but going from 10 to 15 has a less apparent effect particularly in the single time case. For this sensor arrangement the estimated values of the modal amplitudes seem to approximate the corresponding values obtained for the PIV images. In addition to the number of sensors, this could also be due to the sensor distribution and location. For example, the sensors on the low speed cavity are located on the side wall while the sensors in the high speed
cavity are on the cavity floor. It is also important to remember that this cavity has a stronger resonant tone at the first Rossiter mode while the low speed cavity tone is at the third mode.

Figure C.6. Estimated first two modal amplitude coefficients with different numbers of sensors, with quadratic estimation, a) single time (static) and b) 1 time back for the experimental PIV data.

Figure C.7 presents the first two modal amplitudes from the real-time surface pressure measurements based on the quadratic estimation with single time (static), Fig. C.7a, and 1 previous time, Fig. C.7b. It can be noted that for the single time the difference between the cases is almost negligible. However, 15 sensors show slightly higher values for estimation. For the estimation with 1 previous time, the results are similar with the exception that the 15 sensor case overestimates the amplitude and introduces some low frequency effects.

Figure C.8 shows the spectra of the first two modal amplitudes for the estimation. It can be noted that in all the cases the estimated modal amplitude oscillated at the natural
frequency of the M = 0.69 flow. There is a clear peak at the resonant frequency for all the cases, with the exception of the two time quadratic estimation with 15 sensors, where the broadband levels are almost at the same magnitude as the peak.

Figure C.7. Estimated first two modal amplitude coefficients with different numbers of sensors, with quadratic estimation, a) single time (static) and b) 1 time back from the instantaneous pressure measurements.

Figure C.8. Spectrum of the first two estimated modal amplitude coefficients with different numbers of sensors, with quadratic estimation, a) single time (static) and b) 1 time back from the instantaneous pressure measurements.
The results obtained for the two sets of data suggest that for the cavity flows it is necessary to use quadratic estimation, and if possible to use also the information from the previous time to improve the quality of the estimation. The number of sensors used has an important effect on the final amplitude of the estimation, but the locations have to be selected carefully to avoid overlapping information that, although helpful on the initial estimation (Fig. C.6b), can generate distortion on the amplitude and frequency of the estimated modal coefficient in real time (Fig. C.7b). Keeping in mind the differences between the two cavity facilities, the behavior of the estimation for the two configurations seems to indicate that it is probably better to use more than the 6 sensors used in the low speed cavity. Unfortunately, this could not be further investigated as there are physical limitations on both facilities, i.e. size of sensors and available space, as well as the preset sensor configuration, which requires new parts to relocate the current arrangements.
APPENDIX D

WEAK FORMULATION OF THE N-S EQUATION

In this section the derivation of a control separation based on the weak formulation approach, suggested by Camphouse (2004), will be presented. The idea is to eliminate the second derivatives terms in order to avoid errors due to the discretization process. This will produce a system of ODE’s where the control input appears explicitly in the solution. The approach is based on the application of Green’s identity to transform the second order derivatives into first order that can be obtained more accurately with the numerical data.

In the case of the Navier-Stokes equations, the second order derivatives only appear in the momentum equations (Eqn. B.16 and B.17) with mean value and fluctuating components. The first step required is to project these second order terms onto the POD basis as follows:

\[
\begin{align*}
\langle N - S, \phi^i \rangle &= \langle (E_1, E_2, E_3), (\phi^i, \phi^i, \phi^i) \rangle = \int_V \left[ \frac{2}{r - 1} \phi^i U + \nabla \Delta \left( U + \sum_{j=1}^{r} u_j \phi_j^i \right) \right] \nabla \phi^i \Delta \left( V + \sum_{j=1}^{r} u_j \phi_j^i \right) dV \\
\end{align*}
\]  

(D.1)

where E1, E2 and E3 represents Eqns. B.15, B.16 and B.17.

Green’s identity states
\[
\int_{\Omega} f \partial g dv = \int_{\partial \Omega} f (\nabla g \cdot n) dA(x) - \int_{\Omega} (\nabla f \cdot \nabla g) dv(x)
\]  
\hspace{1cm} (D.2)

where the first term on the right side is integration over the surface (\(\partial \Omega\)) and the second term is inside the volume (\(\Omega\)), \(n\) is the unit outward normal, \(A(x)\) is the surface area and \(v\) is the volume. Applying the identity to each term, we have

\[
\int_{\Omega} \phi^k \Delta U dv = \int_{\partial \Omega} \phi^k (\nabla U \cdot n) dA(x) - \int_{\Omega} (\nabla \phi^k \cdot \nabla U) dv(x)
\]  
\hspace{1cm} (D.3)

Figure 7.1 showed the schematic of the cavity used in the derivation. In the solid boundaries, walls and floor, the POD modes and the velocities are zero. There are three free surfaces; the streamwise flow inlet and outlet planes and the top surface, since the total flow area is not used for the analysis, in these three surfaces the integral will yield values different than zero, therefore they cannot be neglected. Finally there is the control input area, where the forcing function is known and will lead to the implicit control term on the final system of ODE’s.

The boundary points of all the free surfaces where the integrals will need to be evaluated are shown in figure 7.1. The input location is defined as \(x = 0\) and \(y = a\), and \(y = b\). The other areas are defined by the points 1-2, 2-3 and 3-4.

To bring the control input into the final expression, the forcing input term can be defined using a POD expansion, as:
where $x_{ct} = 0$. The next step is to look at the individual terms present in Green’s identity. To procedure will be shown on the $u$ component. Replacing the mean velocity component onto the first term in the right side of the equation Eqn. D.3,

$$\int_{\partial \Omega} \phi_u^i (\nabla U \cdot n) dA(x) = -b \int_a^b \frac{\partial U}{\partial x} dy - \frac{2}{3} \int_a^b \frac{\partial U}{\partial y} dy + \frac{3}{2} \int_a^b \frac{\partial U}{\partial x} dx + \frac{4}{3} \int_a^b \frac{\partial U}{\partial y} dy$$

(D.5)

adding the second term on the right hand side, the final expression for the mean is:

$$\int_{\partial \Omega} \phi_u^i \Delta U dv = -b \int_a^b \frac{\partial U}{\partial x} dy - \frac{2}{3} \int_a^b \frac{\partial U}{\partial y} dy + \frac{3}{2} \int_a^b \frac{\partial U}{\partial x} dx + \frac{4}{3} \int_a^b \frac{\partial U}{\partial y} dy - \int_{\partial \Omega} \left( \frac{\partial \phi_u^i}{\partial x} + \frac{\partial \phi_u^i}{\partial y} \right) dv$$

(D.6)

The same procedure is applied to the other three terms in Eqn. D.1. For the $u$ fluctuation,

$$\sum_{j=0}^n a^j \int_{\Omega} \phi_u^k \Delta \phi_u^j dv = \sum_{j=0}^n a^j \left[ -b \int_a^b \frac{\partial \phi_u^k}{\partial x} dy - \frac{2}{3} \int_a^b \frac{\partial \phi_u^k}{\partial y} dy + \frac{3}{2} \int_a^b \frac{\partial \phi_u^k}{\partial x} dx + \frac{4}{3} \int_a^b \frac{\partial \phi_u^k}{\partial y} dy - \int_{\partial \Omega} \left( \frac{\partial \phi_u^k}{\partial x} + \frac{\partial \phi_u^k}{\partial y} \right) dv \right]$$

(D.7)

for the normal velocity ($v$) mean velocity component,

$$\int_{\Omega} \phi_v^k \Delta V dv = -b \int_a^b \frac{\partial V}{\partial x} dy - \frac{2}{3} \int_a^b \frac{\partial V}{\partial y} dy + \frac{3}{2} \int_a^b \frac{\partial V}{\partial x} dx + \frac{4}{3} \int_a^b \frac{\partial V}{\partial y} dy - \int_{\partial \Omega} \left( \frac{\partial \phi_v^k}{\partial x} + \frac{\partial \phi_v^k}{\partial y} \right) dv$$

(D.8)

finally, the expression for the normal velocity fluctuations ($v$), can be expressed as:

$$\sum_{j=0}^n a^j \int_{\Omega} \phi_v^k \Delta \phi_v^j dv = \sum_{j=0}^n a^j \left[ -b \int_a^b \frac{\partial \phi_v^k}{\partial x} dy - \frac{2}{3} \int_a^b \frac{\partial \phi_v^k}{\partial y} dy + \frac{3}{2} \int_a^b \frac{\partial \phi_v^k}{\partial x} dx + \frac{4}{3} \int_a^b \frac{\partial \phi_v^k}{\partial y} dy - \int_{\partial \Omega} \left( \frac{\partial \phi_v^k}{\partial x} + \frac{\partial \phi_v^k}{\partial y} \right) dv \right]$$

(D.9)
The integrals over the control area can be grouped together. Then, for the streamwise velocity it becomes:

$$\int_{a}^{b} \phi_{u}^{k} \frac{\partial U}{\partial x} dy + \sum_{j=0}^{n} a' \int_{a}^{b} \phi_{u}^{k} \frac{\partial \phi_{u}^{j}}{\partial x} dy = \int_{a}^{b} \phi_{u}^{k} \frac{\partial \left( U + \sum_{j=0}^{n} a' \phi_{u}^{j} \right)}{\partial x} dy$$  \hspace{1cm} (D.10)

A similar expression is obtained for the normal velocity component. The derivative term can be obtained using central difference, defining \( f_{u} = U + \sum_{j=0}^{n} a' \phi_{u}^{j} \) then:

$$\frac{\partial f_{u}(0,y,t)}{\partial x} \approx \frac{f_{u}(0+h,y,t) - f_{u}(0,y,t)}{h}$$  \hspace{1cm} (D.11)

since \( x_{ct} = 0 \), knowing that \( f(0,y,t) = \Gamma_{u} \), then

$$\frac{\partial f_{u}(0,y,t)}{\partial x} \approx \frac{f_{u}(0+h,y,t) - \Gamma_{u}(t)}{h}$$  \hspace{1cm} (D.12)

similarly, for the \( v \) component

$$\frac{\partial f_{v}(0,y,t)}{\partial x} \approx \frac{f_{v}(0+h,y,t) - \Gamma_{v}(t)}{h}$$  \hspace{1cm} (D.13)

then, the integrals at the control boundary becomes

$$\int_{a}^{b} \phi_{u}^{k} \frac{\partial U}{\partial x} dy + \sum_{j=0}^{n} a' \int_{a}^{b} \phi_{u}^{k} \frac{\partial \phi_{u}^{j}}{\partial x} dy = \frac{\Gamma_{u}}{h} \int_{a}^{b} \phi_{u}^{k} dy - \frac{h}{U} \int_{a}^{b} \phi_{u}^{k} \left( U(h,y,t) - \frac{h}{\Gamma_{u}} \phi_{u}^{k}(h,y,t) \right) dy - \sum_{j=0}^{n} a' \int_{a}^{b} \phi_{u}^{k} \phi_{u}^{j}(h,y,t) dy$$

$$\int_{a}^{b} \phi_{v}^{k} \frac{\partial V}{\partial x} dy + \sum_{j=0}^{n} a' \int_{a}^{b} \phi_{v}^{k} \frac{\partial \phi_{v}^{j}}{\partial x} dy = \frac{\Gamma_{v}}{h} \int_{a}^{b} \phi_{v}^{k} dy - \frac{h}{V} \int_{a}^{b} \phi_{v}^{k} \left( V(h,y,t) - \frac{h}{\Gamma_{v}} \phi_{v}^{k}(h,y,t) \right) dy - \sum_{j=0}^{n} a' \int_{a}^{b} \phi_{v}^{k} \phi_{v}^{j}(h,y,t) dy$$  \hspace{1cm} (D.14)

these terms are now replaced into the system of ODE’s (Eqn. D1) obtained before.

Next, the Galerkin procedure is applied to the modified system. For the left side of the expression, the time dependant term, we have:
\[
\left( \sum_{j=0}^n \hat{a}^j \phi_j \right) \left( \sum_{j=0}^n \hat{b}^j \phi_j \right) \left( \sum_{j=0}^n \hat{c}^j \phi_j \right) = \sum_{j=0}^n \hat{a}^j \left( \frac{2}{\gamma - 1} \phi_j^* \phi_j + \phi_j^* \phi_j + \phi_j^* \phi_j \right) dV = \sum_{j=0}^n \hat{a}^j \delta_{ij} = \hat{a}^k
\] (D.15)

this term remains the same as the sub-domain separation method. Next, the constant terms are grouped together; these are the terms with no dependence on the modal amplitude. Then,

\[
b^k = \int_{\Omega} \left[ \frac{2}{\gamma - 1} \phi^k_L \left( -UC_{x,t} - VC_{\gamma} - \frac{\gamma - 1}{2} C(U_{x,t} + V_{\gamma}) \right) \right] dV
\]

\[
+ \int_{\Omega} \left[ \phi^k_L \left( -UU_{x,t} - VU_{\gamma} - \frac{2}{M^2 \gamma} CC_{t} \right) \right] dV
\]

\[
+ \frac{1}{\text{Re}} \left[ \int_{\Omega} \phi^k_L \left( \frac{h}{h} \frac{U(h,y,t)}{U(h,y,t)} \right) \right] dV
\]

\[
+ \int_{\Omega} \left[ \phi^k_L \left( -UV_{x,t} - VV_{\gamma} - \frac{2}{M \gamma} CC_{t} \right) \right] dV
\]

\[
+ \frac{1}{\text{Re}} \left[ \int_{\Omega} \phi^k_L \left( \frac{h}{h} \frac{V(h,y,t)}{V(h,y,t)} \right) \right] dV
\]

As it can be noted in the expression, this term shows the presence of the boundary integrals, but there are no second order derivatives. Next the first order terms are grouped,

\[
d^k = \int_{\Omega} \left[ \frac{2}{\gamma - 1} \phi^k \left[ -U\phi^k_{x,t} - V\phi^k_{\gamma} - \phi^k_{x,t} C_{x,t} - \phi^k_{\gamma} C_{\gamma} - \frac{\gamma - 1}{2} (C(U_{x,t} + V_{\gamma}) + \phi^k_{x,t} + \phi^k_{\gamma}) \right] \right] dV
\]

\[
+ \int_{\Omega} \left[ \phi^k \left( -UU_{x,t} - VU_{\gamma} - \phi^k_{x,t} U_{x,t} - \phi^k_{\gamma} U_{\gamma} - \frac{2}{M^2 \gamma} (C\phi^k_{x,t} + \phi^k_{C_{x,t}}) \right) \right] dV
\]

\[
+ \frac{1}{\text{Re}} \left[ \int_{\Omega} \phi^k \left( \frac{h}{h} \frac{\phi^k(h,y,t)}{\phi^k(h,y,t)} \right) \right] dV
\]

\[
+ \int_{\Omega} \left[ \phi^k \left( -UV_{x,t} - VV_{\gamma} - \phi^k_{x,t} V_{x,t} - \phi^k_{\gamma} V_{\gamma} - \frac{2}{M \gamma} (C\phi^k_{x,t} + \phi^k_{C_{x,t}}) \right) \right] dV
\]

\[
+ \frac{1}{\text{Re}} \left[ \int_{\Omega} \phi^k \left( \frac{h}{h} \frac{\phi^k(h,y,t)}{\phi^k(h,y,t)} \right) \right] dV
\]

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Again, the presence of the boundary elements can be noticed, with no second order derivatives. As the second order derivates are not present in the quadratic terms, the second order terms are not modified

\[
g_{km} = \int_{\Omega} \frac{2}{y-1} \varphi_c^m \left[ - \varphi_u^h \varphi_c^{m,x} - \varphi_v^h \varphi_c^{m,y} - \frac{y-1}{2} \varphi_c^l \left( \varphi_u^{m,x} + \varphi_v^{m,y} \right) \right] dV \\
+ \int_{\Omega} \varphi_u^h \left[ - \varphi_u^l \varphi_u^{m,x} - \varphi_v^l \varphi_u^{m,y} - \frac{2}{M^2 \gamma - 1} \varphi_c^l \varphi_u^{m,x} \right] dV \\
+ \int_{\Omega} \varphi_v^h \left[ - \varphi_u^l \varphi_v^{m,x} - \varphi_v^l \varphi_v^{m,y} - \frac{2}{M^2 \gamma - 1} \varphi_c^l \varphi_v^{m,y} \right] dV
\] (D.18)

Finally, the two control term present in the system are expressed as:

\[
e^k = \frac{\Gamma_u^h}{h} \int_a^b \varphi_u^k dy + \frac{\Gamma_v^h}{h} \int_a^b \varphi_v^k dy
\] (D.19)

since \(\Gamma_u\) and \(\Gamma_v\) are functions of a single input forcing, these two terms can be expressed as follow:

\[
\begin{align*}
\Gamma &= A \sin(2 \pi f t) \\
\Gamma_u &= \Gamma' \cdot \gamma_u = \cos(\alpha) \\
\Gamma_v &= \Gamma' \cdot \gamma_v = \begin{cases} 
\sin(\alpha), & \Gamma > 0 \\
0, & \Gamma \leq 0 
\end{cases}
\end{align*}
\]

where the corresponding values used are defined based on the experimental results for the actuator (Debiasi and Samimy, 2004),

- \(A = 20\) m/s (the value used in the experiment)
- \(\alpha = 30^\circ\)
- \(f =\) forcing frequency

then the control term coefficient can be expressed as
\[ e^k = \frac{r_u}{h} \int_a^b \varphi_u^k dy + \frac{r_v}{h} \int_a^b \varphi_v^k dy \]  
(D.20)

finally, putting all the terms together we have

\[ \dot{a}^k = b^k + \sum_{j=1}^n (d^j a^j) + \sum_{j=1}^n \sum_{m=1}^n (g^{jm} a^j a^m) + e^k \Gamma \]  
(D.21)

This is the expression for the control input separation based on the weak formulation, and corresponds to Eq. 7.3 in Chapter 7. This equation can be written in the control form:

\[ \dot{a} = B + Da + \begin{bmatrix} a^T G^1 a \\ a^T G^2 a \\ \vdots \\ a^T G^n a \end{bmatrix} + Eu \]  
(D.22)

where

\[
\begin{align*}
a &= \begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{bmatrix}, \\
B &= \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix}, \\
D &= \begin{bmatrix} d^{11} & d^{21} & \cdots & d^{1n} \\ d^{12} & d^{22} & \cdots & d^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d^{1n} & d^{2n} & \cdots & d^{nn} \end{bmatrix}, \\
G^k &= \begin{bmatrix} g^{1k} & g^{2k} & \cdots & g^{nk} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ g^{nk} & g^{2nk} & \cdots & g^{n^2k} \end{bmatrix}, \\
E &= \begin{bmatrix} e^1 \\ e^2 \\ \vdots \\ e^n \end{bmatrix}, \\
\Gamma &= \begin{bmatrix} \Gamma_1 \\ \vdots \\ \Gamma_n \end{bmatrix}
\end{align*}
\]  
(D.23)

This is the expression that was used to design a controller using the LQR procedure. However, the results obtained for this method were similar to the sub-domain method, as noted in Chapter 8.