BROWNIAN MOTION OF A PARTICLE IMMERSED IN A VISCOUS, INCOMPRESSIBLE, THERMALLY FLUCTUATING SOLVENT

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

By

Wei Xiong, B.Sc., M.Sc., M.A.Sc.

* * * * *

The Ohio State University
2008

Dissertation Committee:
Prof. Peter March, Advisor
Prof. Avner Friedman
Prof. Saleh Tanveer

Approved by

Advisor
Graduate Program in Mathematics
ABSTRACT

My current work concerns the hydrodynamics of particles immersed in a thermally fluctuating, viscous, incompressible solvent. The governing equations stipulate conservation of momentum in the fluid, conservation of linear and angular momentum of the particle, and no-slip boundary conditions on the boundary of the particle. Is there existence and uniqueness for the solution? What are the limit theorems when time goes to infinity? These problems not only provide more detailed study of physical Brownian motions but also give a testing ground for the techniques in stochastic partial differential equations.

This thesis is a first step to answer these questions. We analyze parts of the system: stochastic Stokes equations in the whole space, passive point particle, passive particle with finite size. We characterize the regularity properties and statistical behaviors of the solution $u(t, x)$ and $p(t, x)$ to the stochastic Stokes equations in the whole space. We give existence and uniqueness results for passive particles (a point particles as well as a finite size particle), and we give limit theorems for a point particle when the time goes to infinity.
ACKNOWLEDGMENTS

I would like to thank my advisor, Prof. Peter March, for his patience, guidance and constant encouragement which made this work possible. His help can never be overstated, not only in my academic growth, but also in the development of my personality.

I would like to thank Prof. Saleh Tanveer for numerous discussions and suggestions as well as the intellectual support he has provided.

I also would like to thank Profs. Avner Friedman, Chiu-Yen Kao, Ovidiu Costin, Ed Overman for their enlightening discussions as well as helpful comments.

Last but not the least, I would like to thank my wife Jian Li and my parents for their constant support and encouragement.
VITA

October 1, 1974 ..................... Born in Nanfeng, China

1996 ............................ B.Sc. in Mathematics, Wuhan University, China

2001 ............................. M.Sc. in Mathematics, The Ohio State University

2007 ............................. M.A.Sc in Statistics, The Ohio State University

2000-Present ..................... Graduate Teaching Associate, The Ohio State University

FIELDS OF STUDY

Major Field: Mathematics
# TABLE OF CONTENTS

Abstract ............................................................... ii
Acknowledgments ...................................................... iii
Vita ................................................................. iv

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>5</td>
</tr>
<tr>
<td>1.3</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>2.1</td>
<td>11</td>
</tr>
<tr>
<td>2.2</td>
<td>20</td>
</tr>
<tr>
<td>2.3</td>
<td>22</td>
</tr>
<tr>
<td>2.4</td>
<td>27</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>3.1</td>
<td>30</td>
</tr>
<tr>
<td>3.2</td>
<td>35</td>
</tr>
<tr>
<td>3.3</td>
<td>42</td>
</tr>
<tr>
<td>3.4</td>
<td>46</td>
</tr>
<tr>
<td>3.5</td>
<td>52</td>
</tr>
<tr>
<td>4</td>
<td>57</td>
</tr>
<tr>
<td>4.1</td>
<td>57</td>
</tr>
<tr>
<td>4.2</td>
<td>64</td>
</tr>
<tr>
<td>5</td>
<td>75</td>
</tr>
<tr>
<td>5.1</td>
<td>75</td>
</tr>
</tbody>
</table>

v
5.2 Central Limit Theorems ........................................ 76

6 Future Directions .................................................. 99

6.1 Limit Theorem for Passive Finite Size Particle .......... 99
6.2 Full Active Model - Analytics ............................... 102
6.3 Full Active Model - Numerics ............................... 104

Bibliography ......................................................... 106
1.1 Motivations and Modeling Issues

Let a particle immersed in a thermally fluctuating, viscous, incompressible solvent. The governing equations stipulate conservation of momentum in the fluid, conservation of linear and angular momentum of the particle, and no-slip boundary conditions on the boundary of the particle. Is there existence and uniqueness for the solution? What are the limit theorems when time goes to infinity? These problems not only give a testing ground for the techniques in stochastic partial differential equations, which are a relative new field in mathematics, but also provide more detailed study of physical Brownian motions. Physical problems motivate the development of mathematics, and mathematics conversely brings physical problems rigor, effectiveness and beauty.

We are working in an intermediate regime of length and time scales, which are sufficiently large such that the continuum approximation is valid, and are sufficiently short such that the stochastic effects from thermally induced fluctuations are appreciable. Below we do a simple scaling analysis to find the constraints.

Let $l_m$ be the mean free path length for solvent molecules, say in unit of meter. Let $t_m$ be the average elapse time, say in second, between collisions for solvent molecules. Let $t_r$ be the time scale of motion for the rigid body, i.e. the macro distance for
the body. Let $v_j$ be the velocity for $j$th molecule of the solvent and $< v >$ be their average. Let $d$ be the diameter of this rigid body. Denote the density of this rigid body by $\rho_r$ and let $\rho_s$ be the density of solvent. We write $m_r$ as the mass of the rigid body, $m$ as the mass of the solvent molecule. Finally, let $T$ and $p$ be the solvent temperature and pressure, while $K_1$ be the constant in the equation of state $p = \frac{3}{2}K_1\rho_sT$.

We start by considering the lower bound on $d$ and $t_r$. Clearly, assumption requires that $d >> l_f, t_r >> t_m$. We need to find expressions for $t_r, t_m$ and $l_m$ in terms of $d, T, \rho_r, \rho_s$ and $m$.

By Newton’s second law, force equals mass times acceleration. The mass of rigid body $m_r$ scales as $\rho_r d^3$ and force scales as $K_1\rho_sTd^2$, which is the product of pressure and area. Hence acceleration scales as $\frac{K_1\rho_sTd^2}{\rho_r d^3}$; On the other hand, acceleration must scale as $\frac{d}{t_r^2}$. Thus,

$$\frac{d}{t_r^2} \approx \frac{K_1\rho_sTd^2}{\rho_r d^3}.$$

From which, we have:

$$t_r \approx \left(\frac{d^2\rho_r}{K_1\rho_sT}\right)^{\frac{1}{2}}.$$

Since inter-solvent molecule distance must scale as $N^{-1/3}$, $N$ being the number of solvent molecules per unit volume. It follows that

$$l_m \approx \left(\frac{m}{\rho_s}\right)^{\frac{1}{3}}.$$

From equilibrium Boltzmann velocity distribution,

$$\left(\frac{l_m}{t_m}\right)^2 \approx K_2T.$$

Therefore,

$$t_m \approx \frac{l_m}{(K_2T)^{\frac{1}{2}}} \approx \left(\frac{m}{\rho_s}\right)^{\frac{1}{3}} \frac{1}{(K_2T)^{\frac{1}{2}}}$$
Hence the requirement \( d >> l_m, t_r >> t_m \) translates into:

\[
d >> \left( \frac{m}{\rho_s} \right)^{\frac{1}{3}}, \quad t_r^2 = \frac{d^2 \rho_r}{K_1 T \rho_s} >> \left( \frac{m}{\rho_s} \right)^{\frac{1}{3}} \frac{1}{K_2 T},
\]

i.e.,

\[
d >> \left( \frac{m}{\rho_s} \right)^{\frac{1}{3}} \quad \text{and} \quad d >> \left( \frac{m}{\rho_s} \right)^{\frac{1}{3}} \left( \frac{\rho_s}{\rho_r} \right)^{\frac{1}{3}}.
\]

Additionally, it is natural to require that \( \rho_s \approx \rho_r \). Thus, we have the following constraint for the lower bound on \( d \):

\[
d >> \left( \frac{m}{\rho_s} \right)^{\frac{1}{3}}. \tag{1.1}
\]

Next, let us discuss the upper bound for \( d \) and \( t_r \). Fluid approximation deals with averaged quantities over a volume containing sufficient number of solvent molecules for averages to be well defined. Let \( N_c \) be the number of molecules of the control volume, which is a small fraction of the rigid body. We have

\[
\rho_s \approx \frac{N_c m}{d^3}.
\]

In other word,

\[
N_c^{-1/2} \approx \sqrt{\frac{m}{\rho_s}} d^{-3/2}.
\]

Since the probability distribution function for \( X = \frac{1}{N_c} \sum_{j=1}^{N_c} |v_j - < v > |^2 \) only involves the parameter \( KT \) (from the assumed equilibrium Boltzmann distribution for each \( v_j \)), it follows that the variance of \( X \) only depends on \( KT \). From dimensional consideration, the random part of the force per unit volume scales as

\[
\frac{\rho_s}{d} N_c^{-1/2} K_1 T \approx \sqrt{m \rho_s K_1 T} d^{-5/2}.
\]

It has to be comparable to any other forces in order for randomness from thermal fluctuation to be important. Thus, from here we can deduce the upper bound on \( d \). Note we can not write an explicit formula for this bound in general since it depends on the particular problems, such as the solvent molecule mass, other forces and etc.
In the problem we are going to consider in this thesis, the only other force is gravity force and we want to neglect it. Hence we have:

\[ \rho_r d^3 g << \sqrt{m \rho_s K_1 T} d^{-5/2}, \]
i.e.,

\[ d^{11/2} << \frac{\sqrt{m \rho_s K_1 T}}{\rho_r g}, \]
where \( g \) is the gravity constant. This gives the upper bound on \( d \). Thus,

\[ d >> \left( \frac{m}{\rho_s} \right)^{\frac{1}{3}} \quad \text{and} \quad d^{11/2} << \frac{\sqrt{m \rho_s K_1 T}}{\rho_r g}. \]

This completes the discussion for the length and time scales.

An example is a pollen in the water. The size of the pollen is typically \( 10^{-7} \) meters and the density is \( 10^3 \) kilograms/cubic meter. The mass for a water molecule is of the order \( 10^{-26} \) kilograms and the density for water is \( 10^3 \) kilograms/cubic meter. We have \( \rho_s \approx \rho_r \),

\[ d \approx 10^{-7} >> \left( \frac{m}{\rho_s} \right)^{\frac{1}{3}} \approx \left( \frac{10^{-26}}{10^3} \right)^{\frac{1}{3}} \approx 10^{-10}. \]

Also, the only other force the pollen has is gravity force. Since

\[ \frac{\sqrt{m \rho_s K_1 T}}{\rho_r g} \approx 10^{-13} \frac{10^{3/2} 10^{-23} 10^2}{10^4} = 10^{-36.5}, \]

and

\[ d^{11/2} \approx 10^{-38.5}, \]

we have

\[ d^{11/2} << \frac{\sqrt{m \rho_s K_1 T}}{\rho_r g}. \]

On this length and time scales, the pollen performs the famous Brownian motion.

**Remark 1.1.** From the discussion of the length and time scales, we can give a rough range for the size of \( d \). We can assume the solvent molecule mass is between \( 10^{-26} \)
and $10^{-20}$ kg and the density of solvent and particle is between $10^3$ and $10^6$. By (1.1), roughly,

$$d \geq 10^{-10}.$$ 

Although the upper bound for $d$ depends on the particular problems, we can still give a rough upper bound. The random forces have to be at least comparable to the gravity forces,

$$\rho_r d^6 g \approx \sqrt{m \rho_s K_1} T d^{-5/2}.$$ 

By direct calculations, we have

$$d \approx 10^{-5}.$$ 

Therefore, roughly speaking,

$$10^{-10} \leq d \leq 10^{-5}.$$ 

Note that the unit here is meter.

### 1.2 Mathematical Model

On the length and time scales we discussed in last section, we will build up our model. We start with some notation.

We write $\rho_s$ as the density of solvent, $\eta$ as the viscosity coefficient of the solvent and $m_r$ as the mass of the particle. The moment of inertia matrix $I$ of the particle is defined by

$$I_{ij} = \int_D \rho_f (|x|^2 \delta_{ij} - x_i x_j) dV; \quad 1 \leq i \leq 3.$$  \hspace{1cm} (1.2)

We model the particle by a compact domain $D$ with smooth boundary and assume the center of mass of $D$ is the origin. Let $D(t)$ be the region occupied by this particle at time $t$ and $D(0) = D$. $D(t)$ performs random rotations and translations. Evidently,
\(D(t) = A(t)D(0) + c(t), A(t) \in SO(3)\) is a rotation matrix and \(c(t)\) is the center of mass of the particle.

Assume that almost surely \(A(t)\) is differentiable with respect to \(t\), then

\[
\frac{d}{dt} A(t) = B(t)A(t), \quad B(t) = \begin{pmatrix}
0 & -w_3(t) & w_2(t) \\
w_3(t) & 0 & -w_1(t) \\
-w_2(t) & w_1(t) & 0
\end{pmatrix}.
\] (1.3)

Here \(w(t)\) is the angular velocity. Geometrically, it means that the rotation is in the plane perpendicular to \(w(t)\) and the magnitude is the length of \(w(t)\).

Exterior to the particle, conservation of momentum in the thermally fluctuating solvent and the incompressibility of the solvent yield the following stochastic Navier-Stokes equations:

\[
\rho_s du(t,x) + \rho_s u(t,x) \cdot \nabla u(t,x) dt = -\nabla p(t,x) dt + \eta \Delta u(t,x) dt + F(dt,x),
\]

\[
\nabla \cdot u(t,x) = 0, \quad x \in D(t)^c.
\] (1.4)

Since the solvent is also viscous, we are in a low Reynolds number regime. Thus, we can use stochastic Stokes equations to approximate stochastic Navier-Stokes equations:

\[
\rho_s du(t,x) = -\nabla p(t,x) dt + \eta \Delta u(t,x) dt + F(dt,x), \quad x \in D(t)^c,
\] (1.4)

\[
\nabla \cdot u(t,x) = 0, \quad x \in D(t)^c.
\] (1.5)

Here \(F(t,x)\) is a stochastic forcing, modeled by a generalized Brownian martingale measure.

For the particle, Newton’s second law and Newton’s second law for rotation give:

\[
m_r \frac{dv(t)}{dt} = -\int_{\partial D(t)} S(t,x)n(t,x) S(dx),
\] (1.6)

\[
I \frac{dw}{dt} = -\int_{\partial D(t)} (x - c(t)) \times (S(t,x)n(t,x)) S(dx).
\] (1.7)
Here, the hydrodynamic stress tensor $S(t, x)$ is defined by

$$S(t, x) = -p(t, x)I + \frac{\eta}{2}(\nabla_x u(t, x) + \nabla_x u^T(t, x)). \quad (1.8)$$

The particle and solvent are coupled together by no-slip boundary conditions:

$$u(t, x) = v(t) + w(t) \times (x - c(t)), \quad x \in \partial D(t), \ t > 0. \quad (1.9)$$

For initial conditions, we have $c(0) = 0$ since we assume that the center of mass of $D$ is origin. We let $u(0, x) = U(0, x)$ almost surely, where $U(t, x)$ is a stationary solution to the equations (1.4) and (1.5) in free space, i.e., a stationary solution to the undisturbed solvent. Also we let $w(0) = w_0, A(0) = I$.

Finally, we impose the boundary condition (finite energy condition) at infinity. When a particle is put in the solvent, it perturbs the solvent. But in the far field, this effect is negligible. It then is physically reasonable to assume that

$$E\left(\int_{D(t)^c} |u(t, x) - U(t, x)|^2 dx\right) < \infty, \text{ for every } t > 0, \quad (1.10)$$

i.e. the perturbation has finite energy.
In summary, we thus have the following coupled system of equations:

\[ \rho_s du(t, x) = -\nabla p(t, x) + \eta \Delta u(t, x) + F(dt, x), \quad x \in D(t)^c, \]  

(1.11)

\[ \nabla \cdot u(t, x) = 0, \quad x \in D(t)^c, \]  

(1.12)

\[ \frac{d}{dt} A(t) = B(t) A(t), \]  

(1.14)

\[ \frac{d}{dt} c(t) = v(t), \]  

(1.15)

\[ \frac{m_r}{d} dv(t) = \int_{\partial D(t)} (x - c(t)) \times (S(t, x)n(t, x)) S(dx), \]  

(1.16)

\[ u(0, x) = U(0, x), \quad x \in D^c \]  

(1.17)

\[ w(0) = w_0 \]  

(1.18)

\[ A(0) = I \]  

(1.19)

\[ c(0) = 0 \]  

(1.20)

\[ u(t, x) = v(t) + w(t) \times (x - c(t)), \quad x \in \partial D(t), \quad t > 0, \]  

(1.21)

\[ E(\int_{D(t)^c} |u(t, x) - U(t, x)|^2 dx) < \infty, \quad \text{for every } t > 0. \]  

(1.22)

**Remark 1.2.** Throughout this thesis, unless stated otherwise, all the equations we talk about hold in distribution sense. In other word, the equations hold when we multiply the test function on both sides of the equations and integrate with respect to spatial variable. Also, \( S'(\mathbb{R}^3) \)-valued process \( u, p \) and etc are written as \( u(t, x), p(t, x) \) and etc in a common abuse of notation.
1.3 Main Results

The full system is still out of reach analytically. This thesis is a first step. We analyze parts of the system: stochastic Stokes equations in the whole space, passive point particle and passive particle with finite size. Here the ”passive” means we only consider the effect of solvent on the particles but neglect the effects of particles on the solvent.

We study the stochastic Stokes equations in the whole space

\[ \rho_s du(t, x) = -\nabla p(t, x) dt + \eta \Delta u(t, x) dt + F(dt, x), \quad x \in \mathbb{R}^3 \]  
(1.23)
\[ \nabla \cdot u(t, x) = 0, \quad x \in \mathbb{R}^3, \]  
(1.24)

and characterize the regularity properties and statistical behaviors of the solution \( u(t, x) \) and \( p(t, x) \).

For a passive particle without the finite size, we have the following governing equation

\[ z_t = z_0 + \int_0^t U(s, z_s) ds, \quad t \in [0, T], \ a.s. \]  
(1.25)

Here and hereafter, \( U(t, x) \) is a stationary solution of stochastic Stokes equations (1.23) and (1.24). We prove that there is a unique adapted process solution to (1.25) under some mild conditions. Also, after rescale \( z_\lambda(t) \) as \( \frac{1}{\sqrt{\lambda}} z(\lambda t) \), we prove \( z_\lambda(t) \) converges weakly to a Brownian motion.

In the case of a passive particle with the finite size, the governing equations are:

\[ \frac{dc(t)}{dt} = v(t) \]  
(1.26)
\[ m_r \frac{dv}{dt} = -\int_{\partial D(t)} S(t, x) n(t, x) \ S(dx), \]  
(1.27)
\[ \frac{dA(t)}{dt} = B(t) A(t) \]  
(1.28)
\[ \mathbf{I} \frac{dw}{dt} = -\int_{\partial D(t)} (x - c(t)) \times (S(t, x)n(t, x)) S(dx), \]  
(1.29)
where

\[ S(t, x) = -p(t, x)I + \frac{n}{2}(\nabla U(t, x) + \nabla U(t, x)^T). \]

We prove there is a unique adapted process solution to (1.26)-(1.29).
CHAPTER 2
BACKGROUND

Here and hereafter, we let \((\Omega, \mathcal{F}, P)\) be a complete probability space, \(\{\mathcal{F}_t, t \geq 0\}\) be an increasing filtration of \(\sigma\)-fields \(\mathcal{F}_t \subset \mathcal{F}\) containing all \(P\)-null subsets of \(\Omega\). Also, in this chapter, unless stated explicitly, the generalized random field will be in one dimension.

2.1 Spectral representation for generalized random field \(F\)

Let us start with Bochner-Schwartz theorem.

**Theorem 2.1.** Every positive definite distribution \(L \in S'(\mathbb{R}^d)\) is the Fourier transform of a positive tempered measure, i.e., for every \(\phi \in S(\mathbb{R}^d)\)

\[
L(\phi) = \int_{\mathbb{R}^d} \hat{\phi}(\xi)d\mu(\xi)
\]  

(2.1)

From the above theorem we can derive the following theorem which gives a representation of the correlation functional of homogeneous generalized random field. Recall first

**Definition 2.2.** A mean-zero generalized random field is said to be homogeneous if, for all \(\phi, \psi \in S(\mathbb{R}^d)\),

\[
E(\tau_h F(\phi) \overline{\tau_h F(\psi)}) = E(F(\phi)\overline{F(\psi)}),
\]

where \(\tau_h F(\phi) := F(\tau_{-h}\phi)\) and \(\tau_{-h}\phi(x) = f(x + h)\).
Theorem 2.3. Let \( F = \{ F(\phi); \phi \in \mathcal{S}(R^d) \} \) be a mean zero homogeneous generalized \( L_2(\Omega) \) valued random field. Then there exists an associated nonnegative, tempered Radon measure \( \mu \), such that for every \( \phi, \psi \in \mathcal{S}(R^d) \), the correlation functional \( J(\phi, \psi) \equiv E(F(\phi) \overline{F(\psi)}) \) can be represented in the form

\[
J(\phi, \psi) = \int_{R^d} \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)} d\mu(\xi). \tag{2.2}
\]

\( \mu \) is called the spectral measure of the random field \( F \).

Proof. This is a well-known theorem. Interested readers are referred to [39] for more details. \( \square \)

This theorem suggests a spectral representation for our generalized random field \( F \), i.e. a stochastic Bochner-Schwartz theorem. Before we do this, we introduce the definition of random orthogonal measure (cf. Krylov [14]).

Definition 2.4. Let \((\Omega, \mathcal{F}, P)\) be a probability space and \((X, \mathcal{U}, \mu)\) be a measurable space. Denote \( \Pi_0 \) be the subset of \( X \) such that \( \mu(\Delta) < \infty \) for \( \Delta \in X \). Suppose that for every \( \Delta \) there is assigned a random variable \( \zeta(\Delta) \). \( \zeta \) is called a random orthogonal measure with reference measure \( \mu \) if:

1. \( E|\zeta|^2 < \infty \) for every \( \Delta \in \Pi_0 \).

2. \( E(\zeta(\Delta_1)\overline{\zeta(\Delta_2)}) = \mu(\Delta_1 \cap \Delta_2) \) for all \( \Delta_1, \Delta_2 \in \Pi_0 \).

3. \( \zeta \) is countably additive, i.e. if \( \Delta = \bigcup_{i=1}^{\infty} \Delta_i \), where \( \Delta \in \Pi_0, \Delta_i \in \Pi_0 \), and \( \Delta_i \cap \Delta_j = 0 \) if \( i \neq j \), then \( \zeta(\Delta) = \sum_{i=1}^{\infty} \zeta(\Delta_i) \).

Remark 2.1. (A) The term ”measure” comes from property (3). Yet, the exceptional set depends on \( \Delta \) and on the sequence \( \Delta_i \). Thus in general there does not exist a real measure \( \nu(\omega, \cdot) \) which depends on \( \omega \) such at for any \( \Delta \in \Pi_0 \), \( \zeta(\Delta)(\omega) = \nu(\omega, \Delta) \).
(B) The convergence of the series $\sum_{i=1}^{\infty} \zeta(\Delta_i)$ is in $L_2(\Omega)$.

Now we turn to the stochastic analog for Bochner-Schwartz theorem.

**Theorem 2.5.** Let $F = \{F(\phi); \phi \in S(R^d)\}$ be a mean zero homogeneous generalized $L_2(\Omega)$ valued random field. Then there exists a random orthogonal measure $M$ such that

$$F(\phi) = \int_{R^d} \hat{\phi}(\xi) M(d\xi), \quad (2.3)$$

where the integral on right hand side is a Bochner integral. Also, the spectral measure $\mu$ of $F$ is the reference measure for $M$. i.e.,

$$E(M(A_1)\overline{M(A_2)}) = \mu(A_1 \cap A_2). \quad (2.4)$$

where $A_1, A_2 \in \mathcal{B}(R^d)$ are $\mu$ finite.

This theorem is also a basic fact and the proof is included for the completeness of the thesis. To prove this theorem, we need the following lemma.

**Lemma 2.6.** Suppose $H$ is a Hilbert space and $L$ is a dense subspace of $H$. Let $F$ be a continuous random linear functional on $L$ and denote $F(L) = \{F(\phi) : \phi \in L\}$. Then we can define an inner product on $F(L)$ by setting $< F(\phi), F(\psi) >_F = < \phi, \psi >$ and the completion of $F(L)$ under this inner product is isometrically isomorphic to $H$.

**Proof.** We first verify that for $\phi, \psi \in L$, $< F(\phi), F(\psi) >_F = < \phi, \psi >$ does define an inner product. We have:

1. $< F(\phi), F(\phi) >_F = < \phi, \phi > \geq 0$.

2. $< F(\phi), F(\psi) >_F = < \phi, \psi > = < \psi, \phi > = < F(\psi), F(\phi) >$. 

13
\[ <aF(\phi) + bF(\psi), F(v) >_F = < F(a\phi + b\psi), F(v) >_F = < a\phi + b\psi, v > = a < \phi, v > + b < \psi, v > = a < F(\phi), F(v) >_F + b < F(\psi), F(v) >_F. \]

Thus \(< F(\phi), F(\psi) >_F \) is an inner product.

Next denote \( \overline{F(L)} \) as the completion of \( F(L) \) under the above inner product, let us show that \( \overline{F(L)} \) and \( H \) are isometrically isomorphic.

For \( \phi \in H \), since \( L \) is dense in \( H \), there exists \( \phi_n \in L \) such that \( \lim_{n \to \infty} \phi_n = \phi \). Define an operator \( T \) from \( H \) to \( \overline{F(L)} \) by letting \( T(\phi) = \lim_{n \to \infty} F(\phi_n) \). Since \( ||F(\phi_n) - F(\phi_m)|| = ||F(\phi_n - \phi_m)|| \), by the continuity of \( F \), \( F(\phi_n) \) is a cauchy sequence in \( \overline{F(L)} \). Thus \( \lim_{n \to \infty} F(\phi_n) \) exists; Now if \( \psi_n \) is another sequence such that \( \lim_{n \to \infty} \psi_n = \phi \), then again by the continuity of \( F \), \( \lim_{n \to \infty} (F(\phi_n) - F(\psi_n)) = \lim_{n \to \infty} F(\phi_n - \psi_n) = 0 \). Thus \( T(\phi) \) does not depend on the choice of \( \phi_n \). Therefore \( T \) is well defined. Finally for \( \phi, \psi \in H \), \( \phi_n, \psi_n \in L \) and \( \phi_n \to \phi, \psi_n \to \psi \), we have

\[ < T(\phi), T(\psi) >_F = < \lim_{n \to \infty} T(\phi_n), \lim_{n \to \infty} T(\psi_n) >_F \]
\[ = \lim_{n \to \infty} < T(\phi_n), T(\psi_n) >_F \]
\[ = \lim_{n \to \infty} < F(\phi_n), F(\psi_n) >_F \]
\[ = \lim_{n \to \infty} (\phi_n, \psi_n) \]
\[ = (\phi, \psi). \]

We thus have that \( H \) and \( \overline{F(L)} \) are isometric.

Now, \( \forall \psi \in \overline{F(L)}, \exists \psi_n \in F(L), \) such that \( \psi_n \to \psi \) in \( \overline{F(L)} \). Correspondingly, there exists \( \phi_n \in L \) such that \( F(\phi_n) = \psi_n \). Since \( \psi_n \) is cauchy in \( F(L) \), by isometry, \( \phi_n \) is cauchy in \( L \). Thus \( \phi_n \to \phi \in H \). By continuity of \( F \), \( F(\phi) = \psi \). Hence \( F \) is surjective.

Therefore \( F(L) \) is isometrically isomorphic to \( H \). Lemma is proved. \( \square \)

Now back to the proof of theorem 2.5.
Proof of Theorem 2.5. Let $J(\phi, \psi) \equiv E(F(\phi)F(\psi))$ be the correlational functional of $F$. Since $J(\phi, \psi)$ is positive definite bilinear functional, we can define a scalar product in $S(R^d)$:

$$<\phi, \psi>_S = J(\phi, \psi).$$

Denote the completion of $S(R^d)$ under this scalar product as $H_S$. Next, let $R$ be the space of all the random variables $F(\phi), \phi \in S(R^d)$, we define:

$$<F(\phi), F(\psi)>_R = <\phi, \psi>.$$

By lemma 2.6 $<F(\phi), F(\psi)>_R$ is an inner product in the space of $R$. Denote the completion of $R$ under this scalar product as $H_R$.

By lemma 2.6 $H_S$ and $H_R$ are isometrically isomorphic.

Also, by theorem 2.3, we have:

$$J(\phi, \psi) = \int_{R^d} \hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}d\mu(\xi),$$  (2.5)

where $\mu$ is a nonnegative tempered Radon measure. Hence we have:

$$<\phi, \psi>_S = <\hat{\phi}, \hat{\psi}>_\mu.$$

Notice the inner product on the right hand side is the normal inner product in $L_2(\mu)$. Since $F(S(R^d)) = S(R^d)$ and $S(R^d)$ is dense in $L_2(\mu)$, by lemma 2.6 again, we have $H_S$ and $L_2(\mu)$ are isometrically isomorphic.

Hence we have an isometric isomorphism between $L_2(\mu)$ and $H_R$, i.e. there exists a bijective invertible continuous operator $T$ from $L_2(\mu)$ to $H_R$. Now, for any $\mu$ finite set $A$, we have $1_A \in L_2(\mu)$. Define the corresponding element in $H_R$ as $M(A)$, i.e. $M(A) = T(1_A)$. Now let us check that $M$ defined in this way is indeed a random measure with reference measure $\mu$.

(1) By isometry between $H_R$ and $L_2(\mu)$, $EM(A)^2 < \infty$. 

15
(2) If $A = \bigcup_{i=1}^{\infty} A_i$, where $A, A_i$ are $\mu$ finite, and $A_i \cap A_j = 0$ if $i \neq j$, then

$$1_A = 1_{\bigcup_{i=1}^{\infty} A_i} = \sum_{i=1}^{\infty} 1_{A_i}. \text{ By the continuity of } T, \text{ we have } M(A) = T(1_A) = T(\sum_{i=1}^{\infty} 1_{A_i}) = \sum_{i=1}^{\infty} T(1_{A_i}) = \sum_{i=1}^{\infty} M(A_i).$$

(3) For $A_1, A_2 \in \mathcal{B}(R^d)$ which are $\mu$ finite,

$$E(M(A_1)M(A_2)) = \int_{R^d} 1_{A_1}(\xi)1_{A_2}(\xi)\mu(d\xi) = \mu(A_1 \cap A_2).$$

Thus $M(A)$ is a random orthogonal measure with reference measure $\mu$.

Finally, let us show that

$$F(\phi) = \int_{R^d} \hat{\phi}(\xi)M(d\xi). \quad (2.7)$$

Clearly, for $\phi \in \mathcal{S}(R^3), T(\hat{\phi}) = F(\phi)$. Since $\hat{\phi} \in \mathcal{S}(R^3)$ and indicator functions are dense in $\mathcal{S}(R^3)$, we have: $\hat{\phi}(\xi) = \lim_{n \to \infty} f_n(\xi)$, where $f_n(\xi)$ are simple functions. Thus,

$$T(\hat{\phi}) = \lim_{n \to \infty} T(f_n).$$

But $\lim_{n \to \infty} T(f_n)$ is nothing but the Bochner integral $\int_{R^d} \hat{\phi}(\xi)M(d\xi)$. Thus as desired,

$$F(\phi) = \int_{R^d} \hat{\phi}(\xi)M(d\xi). \quad (2.8)$$

This completes the proof of theorem.

Thus we have the stochastic representation theorem for generalized random field. A natural question is: for a time-indexed homogeneous generalized random field, do we still have a similar representation? The answer is affirmative if this time-indexed homogenous generalized random field is a generalized Brownian martingale measure.

**Definition 2.7.** A process $\{F_t(\phi), \mathcal{F}_t; t \geq 0, \phi \in \mathcal{S}(R^d)\}$ is generalized martingale measure if for all $\phi \in \mathcal{S}(R^d)$
1. $F_0(\phi) = 0$;

2. for all $t > 0$, $F_t(\cdot)$ is a $L_2(\Omega)$-valued homogeneous generalized random field;

3. for fixed $\phi$, $\{F_t(\phi), F_t; t \geq 0\}$ is a martingale.

A martingale measure $F_t$ is called a generalized Brownian martingale measure if for each $\phi$, $\{F_t(\phi), F_t; t \geq 0\}$ is a Brownian motion.

We have the following corollary:

**Corollary 2.8.** Let $\{F_t(\phi), F_t; \phi \in \mathcal{S}(R^3), t \geq 0\}$ be a generalized Brownian martingale measure with correlation functional

$$B(\phi, \psi) = E\{F_t(\phi) F_s(\psi)\} = t \wedge s \int_{R^3} \hat{\phi}(\xi) \hat{\psi}(\xi) \mu(d\xi).$$

Then there exists an orthogonal martingale measure $\{M_t(\Gamma), F_t; t \geq 0, \Gamma \in \mathcal{B}(R^3), \mu(\Gamma) < \infty\}$ such that $< M(\Gamma_1), M(\Gamma_2) >_t = t \mu(\Gamma_1 \cap \Gamma_2)$ and almost surely, for every $t \geq 0$, for every $\phi \in \mathcal{S}(R^3)$, we have

$$F_t(\phi) = \int_0^t \int_{R^3} \hat{\phi}(\xi) M_t(d\xi).$$

**Proof.** In the spirit of the proof of theorem 2.5, we have for every $t \geq 0$, there exists a random orthogonal measure $M_t$ such that for every $\phi \in \mathcal{S}(R^3)$, almost surely,

$$F_t(\phi) = \int_{R^3} \hat{\phi}(\xi) M_t(d\xi),$$

and for $\Gamma_1, \Gamma_2 \in \mathcal{B}(R^3)$, $\mu(\Gamma_1) < \infty$, $\mu(\Gamma_2) < \infty$,

$$E(M_t(\Gamma_1) M_t(\Gamma_2)) = t \mu(\Gamma_1 \cap \Gamma_2).$$

Next, let us show that for $\Gamma \in \mathcal{B}(R^3)$, $M_t(\Gamma)$ is a martingale. By the proof of theorem 2.5 again, we have:

$$M_t(\Gamma) = \lim_{n \to \infty} F_t(\phi_n),$$
where the convergence on the right hand side is mean square convergence and $\hat{\phi}_n \to 1_\Gamma$. Thus it is obvious that $M_0(\Gamma) = 0$ and $M_t(\Gamma) \in \mathcal{F}_t$. Also, for $s \leq t$, since

$$E(M_t(\Gamma)|\mathcal{F}_s) = E(\lim_{n \to \infty} F_t(\phi_n)|\mathcal{F}_s) = E(\lim_{n \to \infty} F_t(\phi_n) - \lim_{n \to \infty} F_s(\phi_n) + \lim_{n \to \infty} F_s(\phi_n)|\mathcal{F}_s) = E(\lim_{n \to \infty} (F_t(\phi_n) - F_s(\phi_n))) + \lim_{n \to \infty} F_s(\phi_n)$$

$$= E(\lim_{n \to \infty} (F_{t-s}(\phi_n)) + M_s(\Gamma)) = \lim_{n \to \infty} E(F_{t-s}(\phi_n)) + M_s(\Gamma) = M_s(\Gamma).$$

Thus $M_t(\Gamma)$ is a martingale. Notice that since

$$\lim_{n \to \infty} E((F_t(\phi_n) - M_t(\Gamma))^2) = 0,$$

the interchange of the order of limit from the third equality to the fourth equality is legitimate.

Hence $\{M_t(\Gamma), \mathcal{F}_t; t \geq 0, \Gamma \in \mathcal{B}(R^3), \text{and } \mu(\Gamma) < \infty\}$ is a martingale measure.

Thus, we have

$$< M(\Gamma_1), M(\Gamma_2) >_t = t \mu(\Gamma_1 \cap \Gamma_2).$$

Also by the definition of integral with respect to martingale measure, for every $t \geq 0$, for every $\phi \in \mathcal{S}(R^3)$, almost surely, we have

$$F_t(\phi) = \int_{R^3} \hat{\phi}(\xi)M_t(d\xi)$$

$$= \int_0^t \int_{R^3} \hat{\phi}(\xi)M(ds,d\xi).$$

Finally we need to show that almost surely, for every $t \geq 0$, for every $\phi \in \mathcal{S}(R^3)$, we have

$$F_t(\phi) = \int_0^t \int_{R^3} \hat{\phi}(\xi)M(ds,d\xi),$$
i.e. if we let random variable \( G_t(\phi) = \int_0^t \int_{R^3} \hat{\phi}(\xi)M(ds,d\xi) \), we need to show that

\[
P(F_t(\phi) = G_t(\phi), \forall t \geq 0, \forall \phi \in S(R^3)) = 1.
\]

Fix \( t \geq 0 \). Since \( S(R^3) \) is separable, there is a dense subset \( \mathcal{D} \subset S(R^3) \) such that for every \( \phi \), there is a sequence \( \{\psi_n, n = 1, 2, \ldots\} \subset \mathcal{D} \) such that \( \psi_n \to \phi, n \to \infty \). Thus on the one hand, we have:

\[
P(F_t(\phi) = G_t(\phi), \forall \phi \in \mathcal{D}) = 1, \text{ for every } t \geq 0.
\]

On the other hand, if \( \phi \to 0 \) in \( S(R^3) \), then \( \hat{\phi} \to 0 \) in \( S(R^3) \). Since \( \mu \) is tempered, \( \int_{R^3} \hat{\phi}(\xi)\bar{\phi}(\xi)\mu(d\xi) \to 0 \). Thus

\[
E\{|G_t(\phi)|^2\} = t \int_{R^3} \hat{\phi} \bar{\phi}(d\xi) \to 0.
\]

Hence \( G_t(\phi) \to 0 \) in probability as \( \phi \to 0 \), therefore the mapping \( \phi \to G_t(\phi) \) is continuous in probability. Thus for every \( \phi \), there exist a subsequence \( \{\psi_{n_k}\} \) of \( \{\psi_n\} \) such that when \( k \to \infty \), \( \psi_{n_k} \to \phi \) and \( \lim G_t(\psi_{n_k}) \to G_t(\phi) \). Of course we also have \( \lim F_t(\psi_{n_k}) \to F_t(\phi) \) since the mapping \( \phi \to F_t(\phi) \) is continuous. Thus, for every \( t \geq 0 \), almost surely, for every \( \phi \in S(R^3) \), we have \( F_t(\phi) = G_t(\phi) \), i.e.

\[
P(F_t(\phi) = G_t(\phi), \forall \phi \in S(R^3)) = 1, \text{ for every } t \geq 0.
\]

Since \( R_+ \) is also separable, we have

\[
P(F_t(\phi) = G_t(\phi), \forall \text{ rational } t \geq 0, \forall \phi \in S(R^3)) = 1.
\]

Similar to the above argument, if we fix \( \phi \in S(R^3) \), we have the mapping \( t \to G_t(\phi) \) is continuous in probability. Together with the continuity of the mapping \( t \to F_t(\phi) \), we have:

\[
P(F_t(\phi) = G_t(\phi), \forall t \geq 0, \forall \phi \in S(R^3)) = 1.
\]

This completes the proof of this corollary.
An interesting consequence from this corollary is that the stochastic integral with respect to a generalized Brownian martingale measure is nothing but the stochastic integral with respect to an orthogonal martingale measure, which is in fact a spectral representation for this generalized Brownian martingale measure.

Before we end this section, let us introduce some notations that will be used later.

**Definition 2.9.** A $d$-dimensional homogeneous generalized random field $F$ is said to in the class $\mathcal{M}(q_1,q_2)$ if its (complex) spectral measure matrix $\mathcal{M}$ satisfies for $1 \leq i,j \leq d$,

$$
\max_{1 \leq i,j \leq d} \int_{\mathbb{R}^d} |\xi|^{|q_1|} (1 + |\xi|^2)^{\frac{1}{2} q_2} |\mu^{ij}| (d\xi) < \infty, \tag{2.9}
$$

where $\mu^{ij}$ is the $(i,j)$th entry of $\mathcal{M}$ and $p,q$ are real numbers.

Note for spectral class $\mathcal{M}(0,q)$, we have the following monotoncity: if $q_1 \leq q_2$, then $\mathcal{M}(0,q_1) \subset \mathcal{M}(0,q_2)$; While for spectral class $\mathcal{M}(q,0)$, it is more complicated. If $0 \leq q_1 \leq q_2$, then $\mathcal{M}(q_1,0) \subset \mathcal{M}(q_2,0)$; If $q_1 \leq q_2 \leq 0$, then $\mathcal{M}(q_2,0) \subset \mathcal{M}(q_1,0)$. The indexes are some kinds of indicator of the correlation of the field.

### 2.2 Stochastic Integral and Stochastic Convolution with respect to Generalized Martingale Measure

Stochastic integral and stochastic convolution with respect to generalized martingale measure is a generalization to stochastic integral and stochastic convolution with respect to martingale measure. In this section, we will give some brief summaries of the properties of this kind of stochastic integral and stochastic convolution. For more details, see McKinley [23].

For an elementary process $f$, i.e., $f_t(x,\omega) = \phi(x)1_{(a,b]}X(\omega)$, we define for $\Phi \in \mathcal{S}(\mathbb{R}^d)$,

$$
(f \cdot F)_t(\Phi) = (F_{t\wedge b} (\phi\Phi) - F_{t\wedge a} (\phi\Phi))X(\omega). \tag{2.10}
$$
Passing this to \(f \in \mathcal{H}_\mu\), where \(\mathcal{H}_\mu\) is the space of all \(S'(R^3)\)-valued predictable processes \(f_t\) whose spatial Fourier transform \(\hat{f}_t\) satisfying

\[
E\left\{ \int_0^T \int_{R^d} |\hat{f}_t(\xi)|^2 \mu(d\xi) dt \right\} < \infty, \quad \forall T > 0,
\]

we can define the stochastic integral \(f \cdot F\). Furthermore, we have for \(f, g \in \mathcal{H}_\mu\),

\[
E\{(f \cdot F)_t(\Phi)(g \cdot F)_t(\Psi)\} = E\left\{ \int_0^t \int_{R^d} \hat{f}_s \Phi(\xi) \hat{g}_s \Psi(\xi) \mu(d\xi) ds \right\}
\]

If \(f \in \mathcal{H}_\mu\), for a \(\mu\) finite set \(A\), take \(\{\phi_n\} \subset \mathcal{H}_\mu\) that approximates \(1_A\), we can see that \((f \cdot F)_t(1_A)\) is well defined. Now we can define

\[
\int_0^t \int_A f_s(x; \omega) F(dx, ds) := (f \cdot F)_t(1_A)
\]

we thus have for \(f, g \in \mathcal{H}_\mu\):

\[
E\{(f \cdot F)_t(1_A)(g \cdot F)_t(1_B)\} = E\left\{ \int_0^t \int_{R^d} (f1_A)^\wedge (g1_B)^\wedge \mu(d\xi) ds \right\}
\]

\[
= E\left\{ \int_0^t \int_A \int_B f(x, s; \cdot) \Gamma(x - y) g(y, s; \cdot) dxdyds \right\}
\]

\[
= E\{(f \cdot M)_t(A)(g \cdot M)_t(B)\}
\]

where \(\Gamma\) is the Fourier transform of the spectral measure \(\mu\) and \((f \cdot M)_t, (g \cdot M)_t\) are Dalang’s stochastic integral built up from worthy martingale measure \(M\). Note that for elementary function of the form:

\[
f(x, t; \omega) = 1_A(x)1_{(a,b)}(t)X(\omega).
\]

Dalang’s stochastic integrals integral is defined as

\[
(f \cdot M)_t(B) = (M_{t\wedge b}(A \cap B) - M_{t\wedge a}(A \cap B))X(\omega).
\]

Thus we see that these two stochastic integral are essentially the same.
Now, let us talk about the stochastic convolution. We will only deal with deterministic integrand. For an elementary function $f$, i.e., $f_t(x) = \phi(x)1_{(a,b]}$, we define for $\Phi \in \mathcal{S}(\mathbb{R}^d)$,

$$(f * F)_t(\Phi) = (F_{t-t\wedge a} * \phi(\Phi) - F_{t-t\wedge b} * \phi(\Phi)).$$

Passing this to $f \in \mathcal{H}_\mu$, we can define the stochastic convolution $f * F$. Furthermore, we have for $f, g \in \mathcal{H}_\mu$,

$$E\left\{ (f * F)_t(\Phi)(g * F)_t(\Psi) \right\} = E\left\{ \int_0^t \int_{\mathbb{R}^d} \hat{f}_{t-s} \hat{\Phi}(\xi) \hat{g}_{t-s} \hat{\Psi}(\xi) \mu(d\xi) ds \right\}$$

(2.14)

2.3 Other notion of stochastic integrals

Stochastic integral with respect to the generalized martingale measure is the intrinsic characterization of the stochastic integral. But sometimes it is not convenient to maneuver. In this section, following Gyongy and Krylov [10], we will show that stochastic integrals with respect to generalized martingale measure is equivalent to a series of usual stochastic integrals. The series representation is some kind of coordinate representation of the stochastic integral, not intrinsic, but comparatively convenient to deal with.

First, assume that $\{F_t(\phi), \mathcal{F}_t; \phi \in \mathcal{S}(\mathbb{R}^3), t \geq 0\}$ is a generalized Brownian martingale measure with correlation functional

$$B(\phi, \psi) = E\{F_t(\phi)\bar{F}_s(\psi)\} = t \wedge s \int_{\mathbb{R}^3} \hat{\phi}(\xi)\bar{\psi}(\xi) \mu(d\xi),$$

let us represent $F_t(\phi)$ as a series.

**Proposition 2.10.** Let $F_t$ be a generalized Brownian martingale measure as above, then there exists a family of independent one dimensional Brownian motions $w^k_t$ such that almost surely, for every $\phi \in \mathcal{S}(\mathbb{R}^3)$ and for every $t \geq 0$, we have $F_t(\phi) =$
$\sum_{k=1}^{\infty} w^k_t(g_k, \phi)$, where $g_k \in S'(R^3)$, $(g_k, \phi) = \langle \eta_k, \phi \rangle > \mu$, $k = 1, 2, \ldots$, and $\{\eta_k, k = 1, 2, \ldots\}$ is an orthonormal basis for $L_2(\mu)$ on which the inner product is defined as:

$$< \phi, \psi >_\mu = \int_{R^3} \hat{\phi}(\xi) \hat{\psi}(\xi) \mu(\xi)$$

(2.15)

for $\phi, \psi \in S(R^3)$, and then extended to $L_2(\mu)$ as in theorem 2.5.

Proof. By Corollary 2.8, almost surely, for every $\phi \in S(R^3)$, for every $t \geq 0$, we have

$$F_t(\phi) = \int_{0}^{t} \int_{R^3} \phi(x) F(ds, dx)$$

$$= \int_{0}^{t} \int_{R^3} \hat{\phi}(\xi) M(ds, d\xi),$$

where $M$ is the random orthogonal martingale measure, which serves as a spectral representation of the generalized Brownian martingale measure $F_t$, and we have

$$< M(\Gamma) >_t = t \mu(\Gamma).$$

Thus, by Gyongy and Krylov [10], we have

$$F_t(\phi) = \sum_{k=1}^{\infty} \int_{0}^{t} f_k dw^k_t,$$

where

$$w^k_t = \int_{0}^{t} \int_{R^3} \eta_k(s) W(ds, dx),$$

and

$$f_k = \int_{R^3} \hat{\eta}_k(\xi) \hat{\phi}(\xi) \mu(d\xi) = \langle \eta_k, \phi \rangle > \mu.$$ 

Hence,

$$F_t(\phi) = \sum_{k=1}^{\infty} w^k_t < \eta_k, \phi > \mu.$$

For each $k$, define $(g_k, \phi) = \langle \eta_k, \phi \rangle>$. Notice that $\mu$ is a tempered measure, it is easy to check that $g_k \in S'(R^3)$ for $k = 1, 2, \ldots$. Thus,

$$F_t(\phi) = \sum_{k=1}^{\infty} w^k_t(\eta_k, \phi).$$
Now, fix integer $k$.

For any $0 \leq s \leq t$,

$$
E\{w^k_t - w^k_s\} = E\left\{\int_0^t \int_{R^3} \eta_k(l) F(dl, dx) - \int_0^s \int_{R^3} \eta_k(l) F(dl, dx)\right\}
$$

$$
= E\left\{\int_s^t \int_{R^3} \eta_k(l) F(dl, dx)\right\}
$$

$$
= 0,
$$

$$
E\{(w^k_t - w^k_s)^2\} = E\left\{\int_s^t \int_{R^3} \eta_k(l) F(dl, dx)\right\}^2
$$

$$
= (t-s) \int_{R^3} \hat{\eta}_k(\xi) \hat{\eta}_k(\xi) \mu(d\xi)
$$

$$
= t-s,
$$

also, notice $w^k_t$ is Gaussian for any $t$ since the integrand is deterministic and independent of $t$, therefore we have $w^k_t - w^k_s \sim N(0, t-s)$.

For any $t_1 \leq \cdots \leq t_n$, $t_i \in [0, \infty)$, $w^k_{t_1}, w^k_{t_2} - w^k_{t_1}, \cdots, w_{t_n} - w_{t_{n-1}}$ are independent due to the facts that for any $\phi \in S(R^3)$, $F_{t_1}(\phi), F_{t_2}(\phi) - F_{t_1}(\phi), \cdots, F_{t_n}(\phi) - F_{t_{n-1}}(\phi)$ are independent and $w^k_{t_i} - w^k_{t_{i-1}}$ is the limit of $F_{t_i}(\phi_n) - F_{t_{i-1}}(\phi_n)$ as $n \to \infty$.

So $w^k_t$ is a family of independent one dimensional Brownian motions, which is desired and proposition is proved.

Now let us show how to represent stochastic integral with respect to generalized martingale measure as a series of usual one-dimensional stochastic integrals.

Instead of trying to do this representation to as broad a class as possible, we restrict ourselves to the case when the integrand is a $S'(R^3)$-valued function on $[0, \infty)$.

On the one hand, this will suffice for our needs. On the other hand, one can easily extend these ideas to the corresponding generalized martingale measure integrals. We have the following two propositions.

**Proposition 2.11.** Assume that $f(t, x) \in H_\mu$. Also assume $\{\eta_k, k = 1, 2, \ldots\}$ is an orthonormal basis for $L_2(\mu)$ on which the inner product is defined as proposition
2.10. Then, there exists a family of independent one dimensional Brownian motions
\( w^k_t \) such that almost surely, for every \( \Phi \in \mathcal{S}(\mathbb{R}^3) \) and for every \( t \geq 0 \), we have
\[
\left( \int_0^t \int_{\mathbb{R}^3} f(t-s, x-y) F(dy, ds), \Phi \right) = \sum_{k=1}^{\infty} \int_0^t \left( g_k, \Phi \right) dw^k_s
\]
where \( g_k \in \mathcal{S}'(\mathbb{R}^3) \) and \( (g_k, \Phi) = f_{t-s} \ast \eta_k, \Phi > \mu, k = 1, 2, \ldots \).

Proof. First we prove an elementary fact: \( \langle \eta_k, \Phi \ast \tilde{\phi} \rangle > \mu = \langle \eta_k, \Phi \rangle > \mu \), where \( \tilde{\phi}(x) := \phi(-x) \). Indeed,
\[
\langle \eta_k, \Phi \ast \tilde{\phi} \rangle = \int_{\mathbb{R}^3} \hat{\eta}_k(\xi) \tilde{\phi}(\xi) \hat{\Phi}(\xi) \hat{\mu}(d\xi)
= \int_{\mathbb{R}^3} \hat{\eta}_k(\xi) \hat{\phi}(\xi) \hat{\Phi}(\xi) \hat{\mu}(d\xi)
= \langle \eta_k, \Phi \rangle > \mu.
\]

Next, by proposition 2.10, almost surely, for every \( \phi \in \mathcal{S}(\mathbb{R}^3) \) and for every \( t \geq 0 \),
\[
F_t(\phi) = \sum_{k=1}^{\infty} w^k_t < \eta_k, \phi > \mu.
\]
Let \( f(x, t) = \phi(x)1_{(a,b)}(t) \), we have
\[
\left( \int_0^t \int_{\mathbb{R}^3} f(t-s, x-y) F(dy, ds), \Phi \right) = F_{t-t\wedge a} * \phi(\Phi) - F_{t-t\wedge b} * \phi(\Phi)
= F_{t-t\wedge a}(\Phi * \tilde{\phi}) - F_{t-t\wedge b}(\Phi * \tilde{\phi})
= \sum_{k=1}^{\infty} w^k_{t-t\wedge a} < \eta_k, \Phi \ast \tilde{\phi} > \mu - \sum_{k=1}^{\infty} w^k_{t-t\wedge b} < \eta_k, \Phi \ast \tilde{\phi} > \mu
= \sum_{k=1}^{\infty} \left( w^k_{t-t\wedge a} - w^k_{t-t\wedge b} \right) < \eta_k, \Phi > \mu
= \sum_{k=1}^{\infty} \int_0^t 1_{(a,b)}(t-s) < \eta_k, \Phi > \mu \, dw^k_s
= \sum_{k=1}^{\infty} \int_0^t < f_{t-s} \ast \eta_k, \Phi > \mu \, dw^k_s.
\]
Now, for \( f(t, x) \in \mathcal{S}'(\mathbb{R}^3) \) on \([0, \infty)\), we use the dense arguments and take the limits. We have:

\[
(\int_0^t \int_{\mathbb{R}^3} f(t - s, x - y)F(dy, ds), \Phi) = \sum_{k=1}^{\infty} \int_0^t <f_{t-s} * \eta_k, \Phi > \delta^k_s.
\]

Finally, for \( k = 1, 2, \ldots \), define \((g_k, \Phi) = < f_{t-s} * \eta_k, \Phi >\), it is easy to check that \( g_k \in \mathcal{S}'(\mathbb{R}^3) \). Thus, we have

\[
(\int_0^t \int_{\mathbb{R}^3} f(t - s, x - y)F(dy, ds), \Phi) = \sum_{k=1}^{\infty} \int_0^t (g_k, \Phi) dw^k_s.
\]

The proposition is proved. \( \square \)

**Proposition 2.12.** Assume \( f(t, x) \in \mathcal{H}_\mu \). Also assume \( \{\eta_k, k = 1, 2, \ldots \} \) is an orthonormal basis for \( L_2(\mu) \) on which the inner product is defined as proposition 2.10. Then, there exists a family of independent one dimensional Brownian motions \( w^k_t \) such that almost surely, for every \( \Phi \in \mathcal{S}(\mathbb{R}^3) \) and for every \( t \geq 0 \), we have

\[
(\int_0^t \int_{\mathbb{R}^3} f(s, x)F(dx, ds), \Phi) = \sum_{k=1}^{\infty} \int_0^t (g_k, \Phi) dw^k_s,
\]

where \( g_k \in \mathcal{S}'(\mathbb{R}^3) \) and \((g_k, \Phi) = < f_s \eta_k, \Phi > \mu, k = 1, 2, \ldots \).

**Proof.** Similar to proposition 2.11, we prove first the representation is true for \( f(t, x) = \phi(x)1_{(a,b]}(t) \), where \( \phi \in \mathcal{S}(\mathbb{R}^3), 0 \leq a < b \)

By proposition 2.10, almost surely, for every \( \phi \in \mathcal{S}(\mathbb{R}^3) \) and for every \( t \geq 0 \),
\[ F_t(\phi) = \sum_{k=1}^{\infty} w_k^t < \eta_k, \phi >. \] Thus,
\[
(\int_0^t \int_{\mathbb{R}^3} f(s, x) F(dx, ds), \Phi) = F_{t \wedge b}(\phi \Phi) - F_{t \wedge a}(\phi \Phi)
\]
\[
= \sum_{k=1}^{\infty} w_{t \wedge b}^k < \eta_k, \phi \Phi > - \sum_{k=1}^{\infty} w_{t \wedge a}^k < \eta_k, \phi \Phi >
\]
\[
= \sum_{k=1}^{\infty} (w_{t \wedge b}^k - w_{t \wedge a}^k) < \eta_k, \phi \Phi >
\]
\[
= \sum_{k=1}^{\infty} \int_0^t < \eta_k, f_s \Phi >, dw_s^k
\]
\[
= \sum_{k=1}^{\infty} \int_0^t < f_s \eta_k, \Phi >, dw_s^k.
\]

Next, for \( f(t, x) \in S'(\mathbb{R}^3) \) on \([0, \infty)\), we use the dense arguments and take the limits. We have:
\[
(\int_0^t \int_{\mathbb{R}^3} f(s, x) F(dx, ds), \Phi) = \sum_{k=1}^{\infty} \int_0^t f_s \eta_k, \Phi >, dw_s^k.
\]

Finally for \( k = 1, 2, \ldots \), define \((g_k, \Phi) = < f_s \eta_k, \Phi >\), it is easy to check that \( g_k \in S'(\mathbb{R}^3) \). Thus, we have
\[
(\int_0^t \int_{\mathbb{R}^3} f(s, x) F(dx, ds), \Phi) = \sum_{k=1}^{\infty} \int_0^t (g_k, \Phi) dw_s^k.
\]

The proposition is proved. \( \square \)

### 2.4 Krylov’s Embedding Theorem

In this section, we will introduce the definitions of Krylov’s stochastic Banach space and give some embedding theorems without proofs. For more details, see Krylov [16], [18]. These machineries will be used in the proofs of our theorems.

If \( s > 0 \) is not an integer, the holder space \( C^s(\mathbb{R}^d) \) is defined as
\[
C^s(\mathbb{R}^d) = \{ f | f \in C^{[s]}(\mathbb{R}^d),
\]
\[
|| f ||_{C^s(\mathbb{R}^d)} = || f ||_{C^{[s]}(\mathbb{R}^d)} + \sum_{|\alpha|=s} \sup_{x \neq y, x, y \in \mathbb{R}^d} \frac{|\partial^{\alpha} f(x) - \partial^{\alpha} f(y)|}{|x - y|^s},
\]

27
where \( \alpha \) is a multi-index, \( s = [s] + \{s\} \) with \([s]\) integer and \(0 < \{s\} < 1\).

For given \( q \in (1, \infty) \) and \( n \in (-\infty, \infty) \), the space \( H_q^n = H_q^n(R^d) \) is defined as the space of all generalized functions \( u \) such that \((1 - \triangle)^{n/2} u \in L_q = L_q(R^d)\). The norm for \( u \in H_q^n \) is defined as

\[
\|u\|_{n,q} := \|(1 - \triangle)^{n/2} u\|_q.
\]

We apply the same definitions to \( l_2 \)-valued functions \( g \), where \( l_2 \) is the set of all real-valued sequences \( g = \{g^k; k = 1, 2, \ldots\} \) with the norm defined by \( \|g\|_{l_2}^2 = \sum_k |g^k|^2 \).

Specially,

\[
\|g\|_q := \|g\|_2, \quad \|g\|_{n,q} := \|(1 - \triangle)^{n/2} g\|_q.
\]

Finally, for stopping times \( \tau \), we denote \((0, \tau] = \{\omega, t : 0 < t \leq \tau(\omega)\}\),

\[
\mathbb{H}_q^n(\tau) = L_q((0, \tau], H_q^n), \quad \mathbb{H}_q^n = \mathbb{H}_q^n(\infty),
\]

\[
\mathbb{H}_q^n(\tau, l_2) = L_q((0, \tau], H_q^n(R^d, l_2)).
\]

For \( n \in R \) and

\[
(f, g) \in \mathcal{F}_q^n(\tau) := \mathbb{H}_q^n(\tau) \times \mathbb{H}_q^{n+1}(\tau, l_2),
\]

set

\[
\|(f, g)\|_{\mathcal{F}_q^n(\tau)} := \|f\|_{\mathbb{H}_q^n(\tau)} + \|g\|_{\mathbb{H}_q^{n+1}(\tau, l_2)}.
\]

**Definition 2.13.** For a \( S' \)-valued function \( u \in \cap_{T > 0} \mathbb{H}_q^{-n}(\tau \wedge T) \), we write \( u \in \mathcal{H}_q^n(\tau) \) if \( u_{xx} \in \mathbb{H}_q^{n-2}(\tau), u(0, \cdot) \in L_p(\Omega, \mathcal{F}_0, H_q^{n-2/q}) \), and there exists \((f, g) \in \mathcal{F}_p^{n-2}(\tau)\) such that, for any \( \phi \in S(R^3) \), the equality

\[
(u(t, \cdot), \phi) = (u(0, \cdot), \phi) + \int_0^t (f(s, \cdot), \phi) ds + \sum_{k=1}^{\infty} \int_0^t (g^k(s, \cdot), \phi) dw_s^k \quad (2.16)
\]

holds for all \( t \leq \tau \) with probability 1. We also define \( \mathcal{H}_q^{n,0}(\tau) = \mathcal{H}_q^n(\tau) \cap \{u : u(0, \cdot) = 0\} \),

\[
\|u\|_{\mathcal{H}_q^n(\tau)} = \|u_{xx}\|_{\mathcal{H}_q^{n-2}(\tau)} + \|(f, g)\|_{\mathcal{F}_p^{n-2}(\tau)} + (E\|u(0, \cdot)\|_{H_q^{n-2/q}}^{n/q})^{1/q}. \quad (2.17)
\]

We drop \( \tau \) in \( \mathcal{H}_q^n(\tau) \) and \( \mathcal{F}_q^n(\tau) \) if \( \tau = \infty \).
Theorem 2.14. (i) For any function \( u \in \mathcal{H}_q^n(\tau) \), we have \( u \in C([0, \tau], H_q^{n-1})(a.s.) \) and

\[
E \sup_{t \leq \tau} |u(t, \cdot)|^q_{H_q^n(\tau)} \leq N |u|_{\mathcal{H}_q^n(\tau)}^q,
\]

where \( q \geq 2 \) and \( N \) is a constant depending on \( d, n, q, T \).

(ii) If \( q > 2, 1/2 > \beta > \alpha > 1/q \), then for any function \( u \in \mathcal{H}_q^n(\tau) \), we have \( u \in C^{\alpha-1/q}([0, \tau], H_q^{n-2\beta})(a.s.) \) and for any stopping time \( \eta \leq \tau \),

\[
E |u(t, \cdot)|^q_{C^{\alpha-1/q}([0, \tau], H_q^{n-2\beta})} \leq N |u|_{\mathcal{H}_q^n(\tau)}^q,
\]

where \( N \) is a constant depending on \( d, \beta, \alpha, q, T \).
Although a lot of work has been done in stochastic Navier-Stokes equations, see for instance Rozovskii [33] and [34], we will focus on stochastic Stokes equations. It turns out that the stochastic Stokes equations define an infinite dimensional OU process, and we will be concerned with the stationary version of that process. We will start by reviewing the finite dimensional situation, which maybe helpful for purpose of comparison.

3.1 Ornstein-Uhlenbeck process

In 1930, Ornstein and Uhlenbeck [27] studied a free particle moving in a rarefied gas and affected by a friction force proportional to the pressure. The velocity process $X_t$ is know as Ornstein-Uhlenbeck process.

The thermal noise perturbs the velocity from their equilibrium values so that $X_t$ describes the process of relaxation back to equilibrium. One simplest model consistent with these assumptions is the solution of the classic Langevin’s equation:

$$dX_t = -\nabla V(X_t)dt + dF_t$$

The physical meaning of this equation is the following: the friction force works to maintain the particle at a stable equilibrium by a balance between relatively slow, systematic processes and relatively fast, residual processes that appear as a single
fluctuating effect when lumped together. The systematic part of the dynamics is given by the gradient of a potential $U$ which is quadratic in the deviation, i.e

$$V(x) = \frac{1}{2}(x - m)^T \zeta (x - m)$$

Since the equilibrium is stable, we have $V(x) \geq 0$, thus $\zeta$ must be positive definite and can be assumed to be symmetric. The fluctuating part of the dynamics

$$\dot{F}_t = \frac{dF_t}{dt}$$

has mean 0 and is delta-correlated in time, i.e

$$\langle \dot{F}_t \rangle = 0 \text{ and } \langle \dot{F}_t, \dot{F}_s \rangle = \alpha \delta(t - s)$$

for some positive definite matrix $\alpha$ describing the correlation between components of $F$. Express $\alpha$ as $\sigma \sigma^T$, we have $F_t = \sigma B_t$, which leads to the following Ornstein-Uhlenbeck process:

$$d(X_t - m) = -\zeta (X_t - m) dt + \sigma dB_t$$

where $\{B_t\}_{t \geq 0}$ is a standard Brownian motion, $m \in \mathbb{R}^d$, $\sigma \in \mathbb{R}^{d \times d}$, $\zeta \in \mathbb{R}^{d \times d}$, and $\zeta$ is symmetric positive definite.

Now, consider the above stochastic differential equation with initial condition $X_0$, explicitly solve this equation, we have

$$X_t = m + e^{-t\zeta} (X_0 - m) + \int_0^t e^{-(t-s)\zeta} \sigma \, dB_s.$$  

(3.2)

There are two important properties for OU processes. The first one is that no matter what the initial condition $X_0$ is, $X_t$ is asymptotically stationary. More specifically, $X_t$ converges weakly to a normal random variable $N(m, a)$ as $t \to \infty$, $a$ is a positive definite square matrix. The second is that if the initial condition $X_0$ is distributed as the limiting normal random variable $N(m, a)$ as above, then $X_t$ is a stationary process.
We need an elementary lemma in matrix algebra to prove these two properties. Recall that the Kronecker product for a $m \times n$ matrix $A$ and a $p \times q$ matrix $B$.

$$A \otimes_k B = \begin{pmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \ldots & a_{mn}B \end{pmatrix}.$$  \hspace{1cm} (3.3)

The vec operator for a $m \times n$ matrix $A$ with $A_j$ as its $j$-th column is defined as

$$\text{vec}(A) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}.$$  \hspace{1cm} (3.4)

**Lemma 3.1.** Let $\alpha, \zeta$ be symmetric positive definite $d \times d$ matrices, then there exists one and only one positive definite matrix such that the following equation holds:

$$\alpha = a\zeta + \zeta a.$$  \hspace{1cm} (3.5)

**Proof.** The idea of the proof for existence and uniqueness of a square matrix $a$ such that (3.5) holds is rather simple. If we treat $\alpha, a$ as vectors with length $d^2$, then (3.5) is a linear equation for $a$. Therefore if we can show that the coefficient matrix is nonsingular, then we prove that there is one and only solution to (3.5).

Take vec operator on both sides of equation (3.5), by direct calculation we have

$$\text{vec}(\alpha) = \text{vec}(a\zeta) + \text{vec}(\zeta a)$$

$$= (\zeta' \otimes_k I_d)\text{vec}(a) + (I_d \otimes_k \zeta)\text{vec}(a)$$

$$= (\zeta \otimes_k I_d + I_d \otimes_k \zeta)\text{vec}(a).$$

Let $\lambda_i, 1 \leq i \leq d$ be the eigenvalues of $\zeta$. By theorem 1 in section 3, chapter 2 of [22], $\zeta \otimes_k I_d$ and $I_d \otimes_k \zeta$ have $\lambda_i, 1 \leq i \leq d$ as their eigenvalues (with multiplicity $d$).
Thus, $\zeta \otimes_k I_d$ and $I_d \otimes_k \zeta$ are positive definite. Therefore by theorem 22 in section 18 of chapter 1 and theorem 1 in section 3 of chapter 2 of [22],

$$
\det(\zeta \otimes_k I_d + I_d \otimes_k \zeta) \geq \det(I_d \otimes_k \zeta) = \det(I_d)^d \times \det(\zeta)^d = \det(\zeta)^d > 0.
$$

Hence there is a unique $d \times d$ matrix $A$ satisfying (3.5). Since $a'$ also satisfies (3.5), $a = a'$, i.e. $a$ is symmetric.

Finally, let us prove $a$ is positive definite. Need to show that for $x \in R^d, x \neq 0$, $x'ax > 0$. It reduces to show that for any nonzero eigenvector $x$ of $\zeta$, $x'ax > 0$. Left multiply (3.5) by $x'$ and right multiply by $x$, we have

$$
x'\alpha x = x'\zeta ax + x'a\zeta x
= \lambda x'ax + \lambda x'a x
= 2\lambda x'ax,
$$

where $\lambda$ is some eigenvalues of $\zeta$. Now since $x'ax > 0$ and $\lambda > 0$, we have $x'ax > 0$. Lemma is proved.

\begin{proof}
\end{proof}

**Proposition 3.2.** Let $X_t$ be a solution to OU process (3.2) with arbitrary initial normal distribution $X_0$ (independent of $\{B_t\}_{t \geq 0}$), then $X_t$ converges weakly to a normal random variable $N(m,a)$, $a$ is positive definite. Furthermore, if $X_0$ is distributed as the limiting normal random variable $N(m,a)$ as above, then $X_t$ is stationary.

\begin{proof}
In order to show that $X_t$ converges weakly to a normal random variable $N(m,a)$, we only need to show that $\lim_{t \to \infty} E(X_t) = m$ and $\lim_{t \to \infty} \text{var}(X_t) = a$ since $X_t$ is normal.

33
Clearly, \( E(X_t) = m \), thus \( \lim_{t \to \infty} E(X_t) = m \). For the variance of \( X_t \), we have

\[
\begin{align*}
\text{var}(X_t) &= E\{(X_t - m)(X_t - m)^T\} \\
&= E\{(e^{-t\kappa}(X_0 - m) + \int_0^t e^{-(t-s)\kappa} \sigma dB_s)(e^{-t\kappa}(X_0 - m) + \int_0^t e^{-(t-s)\kappa} \sigma dB_s)^T\} \\
&= E\{e^{-t\kappa}(X_0 - m)(X_0 - m)^T e^{-t\kappa}\} + E\{e^{-t\kappa}(X_0 - m)(\int_0^t e^{-(t-s)\kappa} \sigma dB_s)^T\} \\
&+ E\{\int_0^t e^{-(t-s)\kappa} \sigma dB_s (X_0 - m)^T e^{-t\kappa}\} \\
&+ E\{\int_0^t e^{-(t-s)\kappa} \sigma dB_s (\int_0^t e^{-(t-s)\kappa} \sigma dB_s)^T\}
\end{align*}
\]

\[
= e^{-t\kappa} \text{var}(X_0)e^{-t\kappa} + \int_0^t e^{-(t-s)\kappa} \sigma \sigma^T e^{-(t-s)\kappa} ds
\]

\[
= e^{-t\kappa} \left( \text{var}(X_0) + \int_0^t e^{s\kappa} \sigma \sigma^T e^{s\kappa} ds \right) e^{-t\kappa},
\]

Thus, as \( t \to \infty \),

\[
\lim_{t \to \infty} \text{var}(X_t) = \lim_{t \to \infty} e^{-t\kappa} \left( \int_0^t e^{s\kappa} \sigma \sigma^T e^{s\kappa} ds \right) e^{-t\kappa} \tag{3.6}
\]

On the other hand, by lemma (3.1), there exist only one positive definite matrix which satisfies

\[
\sigma \sigma^T = a\zeta + \zeta a,
\]

denote it as \( a \). We thus have

\[
e^{t\kappa}ae^{t\kappa} = \zeta e^{t\kappa}ae^{t\kappa} + e^{t\kappa}ae^{t\kappa} \zeta. \tag{3.7}
\]

Integrate from 0 to \( t \) on both sides of (3.7), we get

\[
a + \int_0^t e^{s\kappa} \sigma \sigma^T e^{s\kappa} ds = e^{t\kappa} ae^{t\kappa}, \tag{3.8}
\]

i.e.

\[
a = e^{-t\kappa} \left( a + \int_0^t e^{s\kappa} \sigma \sigma^T e^{s\kappa} ds \right) e^{-t\kappa}. \tag{3.9}
\]

Let \( t \to \infty \), we have:

\[
a = \lim_{t \to \infty} e^{-t\kappa} \left( \int_0^t e^{s\kappa} \sigma \sigma^T e^{s\kappa} ds \right) e^{-t\kappa}. \tag{3.10}
\]
Combined this with (3.6), we have \( \lim_{t \to \infty} \text{var}(X_t) = a \). The first half of this proposition is proved. Now let us prove if \( X_0 \) is distributed as \( N(m,a) \), where \( a = \lim_{t \to \infty} \text{var}(X_t) \), then \( X_t \) is stationary. Since

\[
\text{var}(X_t) = e^{-t\zeta} \left( a + \int_0^t e^{s\zeta} \sigma \sigma^T e^{s\zeta} \, ds \right) e^{-t\zeta},
\]

and \( \text{var}(X_0) = a \), we only need to prove that \( \frac{d}{dt} \text{var}(X_t) = 0 \). By (3.8) and (3.7),

\[
\begin{align*}
\frac{d}{dt} \text{var}(X_t) &= -\zeta e^{-t\zeta} ae^{-t\zeta} - e^{-t\zeta} ae^{-t\zeta} - \zeta e^{-t\zeta} \left( \int_0^t e^{s\zeta} \sigma \sigma^T ds \right) e^{-t\zeta} \\
&= -e^{-t\zeta} \left( \int_0^t e^{s\zeta} \sigma \sigma^T ds \right) e^{-t\zeta} \zeta + e^{-t\zeta} \left( e^{t\zeta} \sigma \sigma^T e^{t\zeta} \right) e^{-t\zeta} \\
&= e^{-t\zeta} \left[ (e^{-t\zeta} ae^{-t\zeta} - a + \int_0^t e^{s\zeta} \sigma \sigma^T e^{s\zeta} \, ds) \zeta e^{-t\zeta} \right] \\
&= 0.
\end{align*}
\]

Thus proposition is proved. \( \square \)

In particular, when \( d = 1, m = 0 \), by proposition (3.2), we have for one dimensional (stationary) Ornstein-Uhlenbeck process \( X_t \), we have \( X_t \sim N(0, \sigma^2 / \zeta) \).

### 3.2 Stochastic Stokes system

Stochastic Stokes system is essentially an infinite dimensional Ornstein-Uhlenbeck process and we will see this shortly. Here and hereafter, unless stated explicitly, the generalized random field \((\mathcal{S}(R^3)\)-valued process\) for velocity \( u \) and generalized martingale measure \( F \) will be in three dimensions, the generalized random field \((\mathcal{S}(R^3)\)-valued process\) for pressure will be in one dimension.
First, let \( u, p, F \) be \( S'(\mathbb{R}^3) \) valued processes, consider the following stochastic Stokes equations (in ordinary differential equation form).

\[
\rho_s du^i_t(\phi) - \eta \Delta u^i_t(\phi)dt + (\nabla p)_i(\phi)dt = dF^i_t(\phi), \quad 1 \leq i \leq 3 \tag{3.11}
\]

\[
\sum_{i=1}^{3} \partial^i u^i(\phi) = 0, \tag{3.12}
\]

**Proposition 3.3.** Let \( u, p \) be \( S'(\mathbb{R}^3) \) valued processes and \( F \) be a generalized martingale measure. Then (3.11) and (3.12) are equivalent to

\[
\rho_s du^i_t(\phi) - \eta u^i(\phi)dt = \Xi dF^i_t(\phi), \tag{3.13}
\]

where \( \Xi \) is the second order Oseen tensor defined by

\[
\Xi^{ij} = \delta_{ij} + R_i R_j, \tag{3.14}
\]

\( R_i \) is the Riesz transform with respect to the \( i \)-th coordinate. \( \Xi F_t(\phi) := F_t(\Xi \phi) \). Here the equivalence means that \( u_t(\phi) \) satisfies (3.11) \( \text{iff} \) and \( u_t(\phi) \) satisfies (3.12).

**Proof.** Take Fourier transforms, we have the following:

\[
\rho_s du^i_t(\hat{\phi}) - \eta u^i(\xi^2 \hat{\phi})dt + 2\pi i \xi_i p_t(\hat{\phi})dt = dF^i_t(\hat{\phi}), \quad 1 \leq i \leq 3, \tag{3.15}
\]

\[
\sum_{i=1}^{3} u^i(\xi_i \hat{\phi}) = 0, \tag{3.16}
\]

where we use \( i \) to represent the imaginary number \( i \) in this section in order to avoid the notation confusion between the imaginary number \( i \) and integer \( i \). Multiply the first equation by \( \xi_i \) and use the second equation, we easily get:

\[
p_t(\hat{\phi}) = \sum_{i=1}^{3} F^i_t(\frac{\xi_i}{2\pi i |\xi|^2} \hat{\phi})
= \frac{\xi \cdot F_t}{2\pi i |\xi|^2}(\hat{\phi}),
\]

Plug back to original equation, we thus have:

\[
\rho_s du_t(\hat{\phi}) + \eta u_t(|\xi|^2 \hat{\phi})dt = dF_t(P_\xi \hat{\phi}), \tag{3.17}
\]
where the second order tensor $P_\xi$ is defined by $P_\xi^{ij} = \delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}$, $\delta_{ij}$ is the Kronecker delta function.

Thus, our Stokes equations are equivalent to the following equations:

$$
\rho_s du_t(\phi) - \eta \triangle u_t(\phi) dt = \Xi dF_t(\phi), \quad (3.18)
$$

From the above equation (3.13), it is obvious the Stokes system is an infinite dimensional Ornstein-Uhlenbeck process. In section 3.3, we also will see there is a nice analog between the correlation(covariance) functional of the infinite dimensional process and the correlation(covariance) of one dimensional Ornstein-Uhlenbeck process.

As suggested in section 2.3, it is sometimes helpful to write (3.13) in series form.

**Proposition 3.4.** Let $u, p$ be $S'(\mathbb{R}^3)$ valued processes and $F$ be a generalized Brownian martingale measure with spectral matrix $\mathcal{M}$. Then (3.13) is equivalent to the following series form

$$
du^i_t(\phi) = \triangle u^i_t(\phi) dt + \sum_{k=1}^{\infty} (g^i_k, \phi) dw^k_t, \quad 1 \leq i \leq 3, \quad (3.19)
$$

where $g^i_k \in S'(\mathbb{R}^3)$ is defined by $(g^i_k, \phi) = \langle \Xi \eta^i_k, \phi >_{\mu^i}, \langle \cdot, \cdot >_{\mu^i}$ is the inner product in $L_2(\mu^i)$ as defined in proposition 2.10 and $\{\eta^i_k, k = 1, 2, \ldots\}$ is an orthonormal basis for $L_2(\mu^i)$.

**Proof.** Follow the series representation of $F_t(\phi)$ in section 2.3, we have for $1 \leq i \leq 3$:

$$
du^i_t(\phi) = \triangle u^i_t(\phi) dt + \Xi dF^i_t(\phi) \\
= \triangle u^i_t(\phi) dt + dF^i_t(\Xi \phi) \\
= \triangle u^i_t(\phi) dt + \sum_{k=1}^{\infty} < \eta^i_k, \Xi \phi >_{\mu^i} dw^k_t \\
= \triangle u^i_t(\phi) dt + \sum_{k=1}^{\infty} < \Xi \eta^i_k, \phi >_{\mu^i} dw^k_t.
$$
where \( \{\eta_k^i, k = 1, 2, \ldots \} \) is an orthonormal basis for \( L_2(\mu^i) \). For \( \phi \in S(R^3) \), define 
\[
(g_k^i, \phi) := \langle \Xi \eta_k^i, \phi \rangle_{\mu^i}, k = 1, 2, \ldots \ 
\]
Clearly \( g_k^i \in S'(R^3) \). Thus, we have:
\[
du^i_t(\phi) = \Delta u^i_t(\phi)dt + \sum_{k=1}^{\infty} (g_k^i, \phi)dw^k_t.
\]

Proposition is proved.

Now let us turn to the solution of the distribution valued stochastic Stokes equation (3.13).

Let
\[
T(t, x) = \int_{R^3} e^{-4\pi^2 \frac{2}{\nu s} |\xi|^2 t} e^{2\pi i \xi \cdot x} d\xi, \quad H(t, x) = \int_{R^3} e^{2\pi i \xi \cdot x} e^{-4\pi^2 \frac{2}{\nu s} |\xi|^2 t} P_\xi d\xi.
\]

For the deterministic Stokes equation
\[
\rho_s du(t, x) - \eta \Delta u(t, x) dt = \Xi df(t, x), \quad (3.20)
\]
where \( f(t, x) \) is a smooth enough function, it is well known that \( u = T*u_0 + H*f \) is the unique solution for (3.20) with initial condition \( u(0, x) = u_0(x) \). Naturally, we would guess that (3.13) has a unique solution and it could be written as \( T*u_0(\phi) + H*F(\phi) \).

The following propositions confirm these conjectures.

**Proposition 3.5.** Let \( u, p \) be \( S'(R^3) \) valued processes on \([0, S]\), \( F \) be a generalized Brownian martingale measure and \( M \) be its corresponding orthogonal martingale measure as in corollary 2.8. (3.13) admits a continuous semi-martingale \( S'(R^3) \) valued solution given by
\[
u_t(\phi) = T*u_0(\phi) + H*F(\phi), \phi \in S(R^3), \quad t \in [0, S]. \quad (3.21)
\]

Here \( u_0(\phi) \) is an arbitrary homogeneous Gaussian random field. Also, \( F_t(\phi) \) and \( u_0(\phi) + \int_0^t \Delta u(\phi) ds \) is its Doob decomposition.
Proof. For notational simplicity, but without loss of generality, we assume that \( \eta = \rho_s = 1 \). Since the Fourier transform of \( H(t, x) \) has exponential decay in spatial variable, \( E\{\int_0^{\infty} \int_{\mathbb{R}^3} |\hat{H}(t, \xi)|^2 \mu(d\xi) dt\} < \infty \) for any tempered measure \( \mu \). Therefore \( H*F \) is well defined for any generalized Brownian martingale measure. By corollary 2.8,

\[
\begin{align*}
 u_t(\phi) &= T*u_0(\phi) + H*F(\phi) \\
 &= u_0(\tilde{T} * \phi) + \int_0^t \int_{\mathbb{R}^3} \hat{\phi}(\xi)e^{-4\pi^2|\xi|^2(t-s)}P_\xi M(ds, d\xi),
\end{align*}
\]

where \( \tilde{T}(t, x) := T(t, -x) \). Therefore,

\[
\begin{align*}
 du_t(\phi) &= u_0(d(\tilde{T} * \phi)) - \int_0^t \int_{\mathbb{R}^3} 4\pi^2|\xi|^2\hat{\phi}(\xi)e^{-4\pi^2|\xi|^2(t-s)}P_\xi M(ds, d\xi) \\
&\quad + \int_{\mathbb{R}^3} \hat{\phi}(\xi)P_\xi M_t(d\xi),
\end{align*}
\]

\[
\begin{align*}
 \triangle u_t(\phi) dt &= u_t(\triangle \phi) dt \\
&= u_0(\tilde{T} * (\triangle \phi)) dt - \int_0^t \int_{\mathbb{R}^3} 4\pi^2|\xi|^2\hat{\phi}(\xi)e^{-4\pi^2|\xi|^2(t-s)}P_\xi M(ds, d\xi),
\end{align*}
\]

and

\[
\begin{align*}
 \Xi dF_t(\phi) &= dF_t(\Xi \phi) \\
&= \int_{\mathbb{R}^3} \hat{\phi}(\xi)P_\xi M_t(d\xi).
\end{align*}
\]

Notice that \( \tilde{T}(t, x) = T(t, -x) = T(t, x) \) is the heat kernel, thus

\[
\begin{align*}
 d(\tilde{T} * \phi) - \tilde{T} * (\triangle \phi) dt &= d(\tilde{T} * \phi) - \triangle(\tilde{T} * \phi) dt \\
&= d(T * \phi) - \triangle(T * \phi) dt \\
&= 0.
\end{align*}
\]

Therefore

\[
\begin{align*}
 du_t(\phi) = \triangle u_t(\phi) dt + \Xi dF_t(\phi).
\end{align*}
\]

Up to here, we showed that for every \( \phi \in S(\mathbb{R}^3) \), almost surely, \( u_t(\phi) \) is a solution to (3.13). By using almost the same arguments as in corollary 2.8, we can show that
almost surely, for every $\phi \in \mathcal{S}(\mathbb{R}^3)$, $u_t(\phi)$ defined by (3.21) solves (3.13). Thus $u_t(\phi)$ is a $\mathcal{S}'(\mathbb{R}^3)$ valued process.

Next, let us prove that $u_t(\phi)$ defined by (3.21) is a semi-martingale. For mathematical simplicity, but without loss of generality, we can assume that $\forall \phi, u_0(\phi) = 0$. By proposition 2.11, we then have

$$u_t(\phi) = \left( \int_0^t \int_{\mathbb{R}^3} H(t-s, x-y) F(dy, ds), \phi \right)$$

$$= \sum_{k=0}^{\infty} \int_0^t (T_{t-s} * g_k, \phi) dw_s^k$$

$$= \sum_{k=0}^{\infty} \int_0^t (g_k, T_{t-s} * \phi) dw_s^k$$

where $g = \{g_k; k = 1, 2, \ldots \} \in H^2_n(\mathbb{R}^3, l^2)$.

Let us prove that this series of stochastic integrals converges in $L^2(\Omega)$ uniformly in $t$ on $[0, S]$. Denote $\psi = (1 - \triangle)^{\frac{n-1}{2}} \phi$, it is obvious that $\psi \in \mathcal{S}(\mathbb{R}^3)$. Also denote $(1 - \triangle)^{\frac{n-1}{2}} g = f$, then $f \in L^2(\mathbb{R}^3, l^2)$. Almost surely the quadratic variations of these stochastic integrals satisfy

$$\sum_{k=0}^{\infty} \int_0^S (g_k, T_{S-s} * \phi)^2 ds = \sum_{k=0}^{\infty} \int_0^S ((1 - \triangle)^{\frac{n-1}{2}} g_k, (1 - \triangle)^{\frac{n-1}{2}} T_{S-s} * \phi)^2 ds$$

$$= \sum_{k=0}^{\infty} \int_0^S \left( \int_{\mathbb{R}^3} f_k(x) T_{S-s} * \psi(x) \right)^2 dx ds$$

$$\leq \sum_{k=0}^{\infty} \int_0^S \int_{\mathbb{R}^3} |f_k(x)|^2 |T_{S-s} * \psi(x)|^2 dx ds$$

$$\leq K_1 \sum_{k=0}^{\infty} \int_0^S \int_{\mathbb{R}^3} |f_k(x)|^2 |T_{S-s} * \psi(x)| dx ds$$

$$= K_1 \int_0^S \int_{\mathbb{R}^3} (\sum_{k=0}^{\infty} |f_k(x)|^2) |T_{S-s} * \psi(x)| dx ds$$

$$\leq S K_1 K_2 \left( \int_{\mathbb{R}^3} (\sum_{k=0}^{\infty} |f_k(x)|^2)^2 dx \right)^{\frac{1}{2}}$$

$$= S K_1 K_2 ||g||_{n-1,2}$$

$$< \infty,$
since
\[ K_1 := \sup_x |T_{S-s} \ast \psi(x)| \leq \sup_x \psi(x) < \infty, \]
and
\[
K_2 := \int_0^S \left( \int_{\mathbb{R}^3} |T_{S-s} \ast \psi(x)|^2 dx \right)^{\frac{1}{2}} ds \\
\leq KS \left( \int_{\mathbb{R}^3} \psi(x)^2 dx \right)^{\frac{1}{2}} \\
< \infty,
\]
where \( K \) is some constant. Therefore this series of stochastic integrals converges in \( L_2(\Omega) \) uniformly in \( t \) on \([0, S]\).

Since \( \int_0^t (g_k, T_{t-s} \ast \phi) dw_s^k \) is a semi-martingale for each \( k \), we can write \( \int_0^t (g_k, T_{t-s} \ast \phi) dw_s^k \) as \( M_t^k(\phi) + A_t^k(\phi) \), where \( M_t^k(\phi) \) is a martingale and \( A_t^k(\phi) \) is a process with bounded variation. By proposition (1.18) in Revuz [31],
\[
< M_t(\phi), M_t(\phi) > = < \int_0^t (g_k, T_{t-s} \ast \phi) dw_s^k, \int_0^t (g_k, T_{t-s} \ast \phi) dw_s^k >
\]
we have \( \sum_{k=1}^n M_t^k(\phi) \) converges in \( L_2(\Omega) \) uniformly in \( t \) on \([0, S]\). For \( 0 \leq s < t \leq S \) and \( A \in \mathcal{F}_s \), we have from uniform \( L_2 \)-convergence and the Cauchy-Schwartz inequality that \( \lim_{n \to \infty} E(1_A (M_s^k(\phi) - M_s(\phi))) = 0 \) and \( \lim_{n \to \infty} E(1_A (M_t^k(\phi) - M_t(\phi))) = 0 \). Therefore \( E(1_A M_t^k(\phi)) = E(1_A M_s^k(\phi)) \) implies that \( E(1_A M_t(\phi)) = E(1_A M_s(\phi)) \) and \( M_t(\phi) \) is seen to be a martingale. Thus, \( u \) is a semimartingale. Finally, the continuity of \( u_t(\phi) \) follows easily from the uniform convergence.

Therefore, \( u \) is a continuous semi-martingale. Finally, the Doob decomposition is obvious since \( F_t(\phi) \) is a martingale.

Under some further conditions on the spectral measure matrix for \( F \), we have that the solution is unique.

**Proposition 3.6.** Let \( u, p \) be \( S'(\mathbb{R}^3) \) valued processes, \( F \) be a generalized Brownian martingale measure with spectral matrix \( \mathcal{M} \) in spectral class \( \mathcal{M}(q_1, 2q_2 - 2), q_1 \leq \)
−2, q₂ ∈ R. (3.13) admits a unique solution in \((H^q_2)^3, q ≥ 2\), which is given by (3.21).

Proof. By theorem 4.10 in [16], (3.19) admits a unique solution in \((H^q_2)^3, q ≥ 2\). So does (3.13). But by proposition 3.16 in section 3.5, \(u\) defined by (3.21) is in \((H^q_2)^3, q ≥ 2\). Proposition is proved.

\[\square\]

3.3 Correlation Functional of \(u(t, x)\)

In this section, we will find the correlation functional for \(u(t, x)\). Recall that for a time-indexed mean 0 generalized random field \(u_t\), its correlation functional \(B_t(\phi, \psi)\) is defined as

\[B_t(\phi, \psi) = E\{u_t(\phi)u_t(\psi)\}\]

Proposition 3.7. Let \(F\) be a generalized Brownian martingale measure with spectral measure matrix \(\mathcal{M}\). Suppose the initial condition \(u_0 \in S'(\mathbb{R}^3)\), and for each \(\phi \in S(\mathbb{R})^3\), \(u_0(\phi) \in L^2(\Omega)\). Also assume that for each \(\phi \in S(\mathbb{R})^3\), \(u_0(\phi)\) is independent of the filtration \(\{\mathcal{F}_t, t \geq 0\}\). Then the correlation functional \(B_t(\phi, \psi)\) for \(u\) is:

\[
E\{u_t(\phi)v_s(\psi)\} = E\{u_0(\tilde{T} \ast \phi)u_0(\tilde{T} \ast \psi)) + E\{v_t(\phi)v_s(\psi)\}. \tag{3.25}
\]

Here \(\tilde{T}(t, x) := T(t, \mathbf{x})\) and \(A \ast B\) is defined as:

\[
(A \ast B)_{ij} = A^iBA_j \tag{3.23}
\]

with \(A^i\) and \(A_j\) denoting the \(i\)th row vector and \(j\)th column vector of \(A\), respectively.

Proof. Let

\[
v_t(\phi) = \left(\int_0^t H(t, s, x - y)F(dy, ds, \phi)\right), \tag{3.24}
\]

then we have:

\[
E\{u_t(\phi)v_s(\psi)\} = E\{u_0(\tilde{T} \ast \phi)u_0(\tilde{T} \ast \psi)) + E\{v_t(\phi)v_s(\psi)\}. \tag{3.25}
\]
We proceed with the calculation coordinate-wise.

\[
E\{u_i^t(\phi)\overline{u_j^t(\psi)}\} = \frac{1}{\rho^2_s} \sum_{k,l=1}^{3} \int_0^{t+s} \int_{R^3} \Omega^\phi_i (t-r, \xi) \Lambda^\psi_j (s-r, \xi) \mu_{kl}(d\xi) dr
\]

\[
= \frac{1}{\rho^2_s} \sum_{k,l=1}^{3} \int_0^{t+s} \int_{R^3} \Omega^\phi_i e^{-4\pi^2 \frac{\rho_s}{\sigma^2} |\xi|^2 (t-r)} F_{k\xi} e^{-4\pi^2 \frac{\rho_s}{\sigma^2} |\xi|^2 (s-r)} F_{l\xi} \mu_{kl}(d\xi) dr
\]

Thus, proposition is proved. \(\square\)

Now let us find the explicit formula for the correlation functional of \(u(t, x)\) for equilibrium.

**Proposition 3.8.** Let \(F\) be a generalized Brownian martingale measure with spectral measure matrix \(\mathcal{M}\) in spectral class \(\mathcal{M}(-2, 0)\). Suppose the initial condition \(u_0 \in S'(R^3)\), and for each \(\phi \in S(R)^3\), \(u_0(\phi) \in L_2(\Omega)\). Also assume that for each \(\phi \in S(R^3)\), \(u_0(\phi)\) is independent the filtration \(\{\mathcal{F}_t, t \geq 0\}\). Then \(u_t\) converges weakly to a generalized vector field \(u_\infty\) as \(t \to \infty\), and

\[
E\{u_i^\infty(\phi)\overline{u_j^\infty(\psi)}\} := \lim_{t \to \infty} E\{u_i^t(\phi)\overline{u_j^t(\psi)}\} = \int_{R^3} \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)} \frac{1}{8\eta \rho_s \pi^2 |\xi|^2} (P_\xi \ast \mathcal{M})^{ij}(d\xi)
\]

**Proof.** First, notice that almost surely, uniformly with respect to \(\phi\),

\[
\lim_{t \to \infty} u_0(T \ast \phi) = 0,
\]

thus, the correlational functional does not depend on the initial conditions. Therefore,

\[
E\{u_i^\infty(\phi)\overline{u_j^\infty(\psi)}\} := \lim_{t \to \infty} E\{u_i^t(\phi)\overline{u_j^t(\psi)}\}
\]

\[
= \lim_{t \to \infty} \frac{1}{\rho^2_s} \int_0^t \int_{R^3} \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)} e^{-4\pi^2 \frac{\rho_s}{\sigma^2} |\xi|^2 (2t-2r)} (P_\xi \ast \mathcal{M})^{ij}(d\xi) dr
\]

\[
= \lim_{t \to \infty} \frac{1}{\rho^2_s} \int_{R^3} \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)} \int_0^t e^{-4\pi^2 \frac{\rho_s}{\sigma^2} |\xi|^2 (2t-2r)} dr (P_\xi \ast \mathcal{M})^{ij}(d\xi)
\]
Since
\[
\int_0^t e^{-4\pi^2 \frac{\rho_s}{\eta} |\xi|^2 (2t-2r)} dr = \frac{\rho_s}{8\pi^2 \eta |\xi|^2} (1 - e^{-8\pi^2 \frac{\rho_s}{\eta} |\xi|^2 t}),
\]
\[
\leq \frac{\rho_s}{8\pi^2 \eta |\xi|^2}
\]
and for \(1 \leq i, j \leq 3\)
\[
\int_{R^3} \hat{\phi}(\xi) \hat{\psi}(\xi) \frac{1}{|\xi|^2} \mu^{ij}(d\xi) < \infty,
\]
thus by Lebesgue’s dominated convergence theorem,
\[
E\{u^i_\infty(\phi) u^j_\infty(\psi)\} = \frac{1}{\rho_s} \int_{R^3} \hat{\phi}(\xi) \hat{\psi}(\xi) \frac{1}{8\pi^2 |\xi|^2} (P_\xi * \mathcal{M})^{ij}(d\xi)
\]
Finally, the weak convergence of \(u_t\) to \(u_\infty\) is obvious since \(u_t\) is a Gaussian random field. This completes the proof of this proposition.

\[\square\]

**Remark 3.1.** As we see from section 3.1, for one dimensional Ornstein-Uhlenbeck process
\[
dX_t = -\xi X_t dt + \sigma dB_t,
\]
we have \(X_\infty \sim N(0, \frac{\sigma^2}{\xi})\). The covariance for \(X_\infty\) is \(\frac{\sigma^2}{\xi}\). Now for the stochastic Stokes equation
\[
\rho_s du_t = \eta \Delta u_t dt + \Xi dF_t,
\]
if we neglect the physical constant, the covariance by the above proposition is
\[
E\{u_\infty(x) u_\infty(y)\} = \int_{R^3} e^{2\pi i \xi \cdot (y-x)} \frac{1}{8\pi^2 |\xi|^2} (P_\xi * \mathcal{M})(d\xi).
\]
In other word, the covariance in stochastic Stokes equation is the inverse Fourier transform of the Fourier transform of the operator \(\Xi(-2\Delta)^{-1} = \Xi^2(-2\Delta)^{-1}\) on the spectral measure. Here we used the fact that \(\Xi\) is idempotent. Notice that \(-\Delta\) is a positive operator, the analog between the stochastic Stokes equation and OU process is now obvious.

44
In finite dimensional OU process, we see that if the OU process starts with the initial velocity field from the equilibrium, then it is stationary. Similarly, we have the same result in our stochastic Stokes equations.

**Proposition 3.9.** Let $F$ be a generalized Brownian martingale measure with spectral measure matrix $\mathcal{M}$ in spectral class $\mathcal{M}(-2,0)$. Suppose the initial condition $u_0$ is distribution valued Gaussian random variable with mean 0 and covariance matrix

$$E(u_0(\phi)u_0(\psi)) = \int_{\mathbb{R}^3} \hat{\phi}(\xi)(\psi(\xi)) \frac{1}{8\eta \rho_s \pi^2 |\xi|^2} (P_\xi * \mathcal{M})(d\xi),$$

then for each fixed $\phi$, $u_t(\phi)$ is a stationary process.

**Proof.** For notational simplicity, but without loss of mathematical simplicity, we assume $\rho_s = \eta = 1$. Need to show that for each fixed $\phi$, $u_t(\phi)$ has the same variance matrix for any $t$. In other word, we need to show that $\frac{d}{dt} E(u_t^i(\phi)u_t^j(\phi)) = 0$. By proposition 3.7,

$$E(u_t^i(\phi)u_t^j(\phi))$$

$$= E(u_0^i(\hat{T} * \phi)u_0^j(\hat{T} * \phi)) + \int_0^t \int_{\mathbb{R}^3} |\hat{\phi}(\xi)|^2 e^{-8\pi^2 |\xi|^2(t-s)} (P_\xi * \mathcal{M})^{ij}(d\xi) ds$$

$$= \int_{\mathbb{R}^3} |\hat{H}_0(t,\xi)\hat{\phi}(\xi)|^2 \frac{1}{8\pi^2 |\xi|^2} (P_\xi * \mathcal{M})^{ij}(d\xi)$$

$$+ \int_0^t \int_{\mathbb{R}^3} |\hat{\phi}(\xi)|^2 e^{-8\pi^2 |\xi|^2(t-s)} (P_\xi * \mathcal{M})^{ij}(d\xi) ds$$

$$= \int_{\mathbb{R}^3} e^{-8\pi^2 |\xi|^2 t} |\hat{\phi}(\xi)|^2 \frac{1}{8\pi^2 |\xi|^2} (P_\xi * \mathcal{M})^{ij}(d\xi)$$

$$+ \int_0^t \int_{\mathbb{R}^3} |\hat{\phi}(\xi)|^2 e^{-8\pi^2 |\xi|^2(t-s)} (P_\xi * \mathcal{M})^{ij}(d\xi) ds. $$
Thus,
\[
\frac{d}{dt} E(u_i^j(\phi)u_i^j(\phi)) \\
= - \int_{\mathbb{R}^3} e^{-8\pi^2|\xi|^2 t}|\hat{\phi}(\xi)|^2 (P_\xi \ast \mathcal{M})^{ij}(d\xi) + \int_{\mathbb{R}^3} |\hat{\phi}(\xi)|^2 (P_\xi \ast \mathcal{M})^{ij}(d\xi) \\
- \int_0^t \int_{\mathbb{R}^3} 4\pi^2|\xi|^2|\hat{\phi}(\xi)|^2 e^{-8\pi^2|\xi|^2 (t-s)} (P_\xi \ast \mathcal{M})^{ij}(d\xi) \, ds.
\]

Now, since
\[
\int_0^t \int_{\mathbb{R}^3} 4\pi^2|\xi|^2|\hat{\phi}(\xi)|^2 e^{-8\pi^2|\xi|^2 (t-s)} (P_\xi \ast \mathcal{M})^{ij}(d\xi) \, ds \\
= \int_{\mathbb{R}^3} \int_0^t 4\pi^2|\xi|^2|\hat{\phi}(\xi)|^2 e^{-8\pi^2|\xi|^2 (t-s)} ds (P_\xi \ast \mathcal{M})^{ij}(d\xi) \\
= \int_{\mathbb{R}^3} (1 - e^{-8\pi^2|\xi|^2 t}) |\hat{\phi}(\xi)|^2 (P_\xi \ast \mathcal{M})^{ij}(d\xi),
\]
we have
\[
\frac{d}{dt} E(u_i^j(\phi)u_i^j(\phi)) = 0.
\]

Therefore, proposition is proved. \(\square\)

Finally, the following question remains: when is \(u\) well-defined as a function? We will address these questions in next section.

### 3.4 \(u(t, x)\) as a Pointwise Process

**Definition 3.10.** A stochastic process \(u(t, x, \omega) : \mathbb{R}_+ \times \mathbb{R}^3 \times \Omega \rightarrow C^3\) is a mild solution to the stochastic Stokes equations (3.13) with initial condition \(u_0(x, \omega)\) if there exists a generalized Brownian martingale measure \(F\) such that for all \(x \in \mathbb{R}^3\) we have
\[
u(t, x) = \int_{\mathbb{R}^3} T(t, x - y)u_0(y)dy + \frac{1}{\rho_s} \int_0^t H(t - s, x - y)F(dy, ds).
\]

**Proposition 3.11.** Suppose \(F\) is a generalized Brownian martingale measure with spectral measure matrix \(\mathcal{M}\) in spectral class \(\mathbb{M}(0, -2)\). Also assume for any \(x, u_0(x) \in\)
\[ L_2(\Omega \times R^3), \text{Eu}_0(x) = 0 \text{ and for any } x_1, \ldots, x_n \in R^3, n \geq 1, (u_0(x_1), \ldots, u_0(x_n)) \text{ is a joint normal random variable independent with the filtration } \{\mathcal{F}_t, t \geq 0\} \text{ of the given generalized Brownian martingale measure } F. \text{ Then } u(x,t) \text{ defined by (3.26) is a continuous (in time and space) complex valued process solution with initial value } u_0(x), \text{ and } u(t,x) \in L_2(\Omega) \text{ for almost every } x,t. \text{ The covariance of } u^i \text{ and } u^j \text{ is given by the following:}

\[ E\{u^i(t,x)\overline{w^j(s,y)}\} = \int_{R^3} \int_{R^3} T(t,x-z)T(s,y-w)E(u_0^i(z)\overline{u_0^j(w)})dzdw \]

\[ + \frac{1}{\rho_s^2} \int_0^{t+s} \int_{R^3} e^{2\pi i \xi \cdot (y-x)} e^{-4\pi^2 \frac{2}{\rho_s^2} |\xi|^2 (t+s-2r)} (P_{\xi} * M(d\xi))^{ij} dr \]  

\[(3.27)\]

**Proof.** Denote \( C_c^\infty(R^3) \) as the space of all \( C^\infty \) functions on \( R^3 \) whose support is compact. Let \( \phi \in C_c^\infty(R^3) \) and \( \int_{R^3} \phi(x)dx = 1 \). Put \( \phi_\epsilon(x) = \epsilon^{-3}\phi(\epsilon^{-1}(x-x_0)) \), \( x_0 \in E, E \) is a countable dense subset of \( R^3 \). Define

\[ v_t(\phi) = (\int_0^t H(t-s,x-y)F(dy,ds), \phi). \]

Let us show that \( \{v_t(\phi_\epsilon)\}_\epsilon \) is a Cauchy sequence in \( L_2(\Omega) \).

Let \( \gamma \) be an arbitrary positive number. We will show that \( \exists \epsilon > 0 \), such that if \( \epsilon', \epsilon'' < \epsilon \), then

\[ E|v_t(\phi_{\epsilon'}) - v_t(\phi_{\epsilon''})|^2 < \gamma. \]

By direct computation,

\[ \hat{\phi}_\epsilon = \frac{1}{\epsilon^3} \int_{R^3} e^{-2\pi i \xi \cdot x} \phi\left(\frac{1}{\epsilon}(x-x_0)\right) \]

\[ = e^{-2\pi i \xi \cdot x_0} \int_{R^3} e^{-2\pi i (\epsilon\xi) \cdot x} \phi(x)dx \]

\[ = e^{-2\pi i \xi \cdot x_0} \hat{\phi}(\epsilon \xi). \]
Thus,
\[
E|v_t^i(\phi_{\epsilon'}) - v_t^i(\phi_{\epsilon''})|^2
= E|v_t^i(\phi_{\epsilon'} - \phi_{\epsilon''})|^2
= \frac{1}{\rho_s^2} \int_0^t \int_{\mathbb{R}^3} |\hat{\phi}_{\epsilon'} - \hat{\phi}_{\epsilon''}|^2 e^{-4\pi^2 \frac{\eta}{\rho_s^2} \zeta^2(2t-2r)} (P_\xi \ast \mathcal{M})(d\xi) dr
= \frac{1}{\rho_s^2} \int_0^t \int_{\mathbb{R}^3} |\hat{\phi}(\epsilon') - \hat{\phi}(\epsilon'')|^2 e^{-4\pi^2 \frac{\eta}{\rho_s^2} \zeta^2(2t-2r)} (P_\xi \ast \mathcal{M})(d\xi) dr
\]

Since \(||P_\xi|| \leq 2||, we have:
\[
\left| \frac{1}{\rho_s^2} \int_0^t \int_{\mathbb{R}^3} e^{-4\pi^2 \frac{\eta}{\rho_s^2} \zeta^2(2t-2r)} (P_\xi \ast \mathcal{M}(d\xi))^{ij} dr \right|
\leq K_1 \int_0^t \int_{\mathbb{R}^3} e^{-4\pi^2 \frac{\eta}{\rho_s^2} \zeta^2(2t-2r)} |\mu^{ij}|(d\xi) dr
\leq K_2 \int_{\mathbb{R}^3} \frac{1 - e^{-8\pi^2 \frac{\eta}{\rho_s^2} \zeta^2 t}}{\zeta^2} |\mu^{ij}|(d\xi)
\leq K_3 \int_{\mathbb{R}^3} (1 \wedge \frac{1}{\zeta^2}) |\mu^{ij}|(d\xi).
\]

If each entry of the spectral matrix satisfies
\[
\int_{\mathbb{R}^3} \frac{1}{1 + |\zeta|^2} |\mu^{ij}|(d\xi) < \infty,
\] (3.28)
then,
\[
\left| \frac{1}{\rho_s^2} \int_0^{t+s} \int_{\mathbb{R}^3} e^{-4\pi^2 \frac{\eta}{\rho_s^2} \zeta^2(t+s-2r)} (P_\xi \ast \mathcal{M}(d\zeta))^{ij} dr \right| < \infty.
\] (3.29)

By Lebesgue dominated theorem, \(\exists \epsilon > 0\), such that if \(\epsilon', \epsilon'' < \epsilon\), then
\[
E|v_t(\phi_{\epsilon'}) - v_t(\phi_{\epsilon''})|^2 < \gamma.
\]

Therefore, \(v_t(\phi_{\epsilon})\) converges in \(L^2(\Omega)\) and hence converges in probability. So we can find a subsequence of \(v_t(\phi_{\epsilon_{k}})\) such that \(v_t(\phi_{\epsilon_k})\) converges almost surely. In a common abuse of notation, define \(v(t, x_0) := \lim_{\epsilon \to 0} v_t(\phi_{\epsilon})\). Thus, we can define a point process \(u(t, x)\), almost surely, for every \(x \in E\), by
\[
u(t, x) = \int_{\mathbb{R}^3} T(t, x - y)u_0(y)dy + v(t, x).
\]
Notice $u(t, x)$ is a Gaussian random field, thus for $t \geq 0, x_0 \in E$,

$$
E|u^i(t, x_0)|^2 = \lim_{c \to 0} E|u^i_c(\phi_e)|^2
= \int_{R^3} \int_{R^3} T(t, x_0 - z)T(t, y_0 - w)E(u^i_0(z)u^i_0(w))dzdw
+ \frac{1}{\rho^2} \int_0^t \int_{R^3} e^{-4\pi^2 \frac{2n}{\rho^2} |\zeta|^2(t-s-2r)}(P_\zeta * M(d\zeta))^i_j dr
$$

Similarly, for $t \geq 0, x, y \in E$,

$$
E\{u^i(t, x)\overline{u^j(s, y)}\}
= \int_{R^3} \int_{R^3} T(t, x - z)T(s, y - w)E(u^i_0(z)u^j_0(w))dzdw
+ \frac{1}{\rho^2} \int_0^{t\wedge s} \int_{R^3} e^{2\pi i \xi \cdot (y-x)}e^{-4\pi^2 \frac{2n}{\rho^2} |\zeta|^2(t+s-2r)}(P_\zeta * M(d\zeta))^i_j dr.
$$

By Corollary 3.2.8 in Mckinley [23], we have for $t, s \geq 0, x, y \in E$,

$$
E|u(t, x) - u(s, y)|^4 \leq |x - y|^4 + |t - s|^2.
$$

Thus, $u(t, x)$ has a uniform continuous modification in $[0, S] \times B$, where $S$ is an arbitrary positive number and $B$ is any bounded subset of $E$. See for instance, theorem 1.4.7 of Kunita [19] or theorem 2.8 of Karatzas [12]. Hence, $u(t, x)$ can be continuously extended to $[0, \infty) \times R^3$. Clearly,

$$
E\{u^i(t, x)\overline{u^j(s, y)}\}
= \int_{R^3} \int_{R^3} T(t, x - z)T(s, y - w)E(u^i_0(z)u^j_0(w))dzdw
+ \frac{1}{\rho^2} \int_0^{t\wedge s} \int_{R^3} e^{2\pi i \xi \cdot (y-x)}e^{-4\pi^2 \frac{2n}{\rho^2} |\zeta|^2(t+s-2r)}(P_\zeta * M(d\zeta))^i_j dr.
$$

Also, for every $\phi, \psi \in S(R^3)$, almost surely,

$$
E \left( \int_{R^3} \int_{R^3} u(t, x)\overline{u(s, y)}\phi(x)\psi(y)dxdy \right)
= E(u_0(\tilde{T} * \phi)u_0(\tilde{T} * \psi))
+ \frac{1}{\rho^2} \int_0^{t\wedge s} \int_{R^3} e^{2\pi i \xi \cdot (y-x)}e^{-4\pi^2 \frac{2n}{\rho^2} |\zeta|^2(t+s-2r)}(P_\zeta * M)(d\zeta)dr
= E(u_t(\phi)u_s(\psi)).
$$
By the continuity of \( u(t, x) \) and \( u_t(\phi) \), in the same argument as in corollary 2.8, we have almost surely, for every \( \phi, \psi \in S(R^3) \), (3.33) holds. Thus the extended \( u(t, x) \) is a realization of \( u_t(\phi) \). This completes the proof of the proposition.

**Remark 3.2.** So what is the mathematical insight in this long complex formula? It is in fact simple. If we do not consider the physical constants \( \rho_s, \eta \), i.e. we let \( \rho_s = \eta = 1 \), then the covariance matrix of \( u(t, x) \) is nothing but the inverse Fourier transform of the Fourier transform of \( P\Xi \) on spectral measure, where \( P \) is the fundamental solution of heat equation and \( \Xi \) is the Oseen tensor.

Let us find the explicit formula for the correlation function of \( u_\infty(x) \), the weak limit of \( u(t, x) \) as \( t \to \infty \).

**Proposition 3.12.** Suppose \( F \) is a Brownian martingale measure with spectral measure matrix \( \mathcal{M} \), where each entry of matrix is in spectral class \( \mathbb{M}(-2, 0) \), i.e. each entry of satisfies

\[
\int_{R^3} \frac{1}{|\xi|^2} \mu_{ij}(d\xi) < \infty, \tag{3.34}
\]

then \( u_\infty(x) \) is a complex valued process and for every \( x, u_\infty(x) \in L_2(\Omega) \). Also we have

\[
E\{u_\infty^i(x)u_\infty^j(y)\} = \frac{1}{\rho_s} \int_{R^3} e^{2\pi i \xi \cdot (y-x)} \frac{1}{8\eta \pi^2 |\xi|^2} (P_\xi \ast \mathcal{M})^{ij}(d\xi) \tag{3.35}
\]

**Proof.** By proposition 3.8, we have

\[
E\{u_\infty^i(\phi)u_\infty^j(\psi)\} = \int_{R^3} \hat{\phi}(\xi)\overline{\hat{\psi}(\xi)} \frac{1}{8\eta \pi^2 |\xi|^2} (P_\xi \ast \mathcal{M})(d\xi)
\]

\[
= \int_{R^3} \int_{R^3} \int_{R^3} \phi(x)\psi(y)e^{2\pi i (y-x) \cdot \xi} \frac{1}{8\eta \pi^2 |\xi|^2} (P_\xi \ast \mathcal{M})(d\xi)dxdy.
\]

Since

\[
\int_{R^3} e^{2\pi i (y-x) \cdot \xi} \frac{1}{8\eta \pi^2 |\xi|^2} (P_\xi \ast \mathcal{M})(d\xi)dxdy < \infty,
\]

\( u_\infty(x) \) is a complex valued process and for every \( x, u_\infty(x) \in L_2(\Omega) \). Also,

\[
E\{u_\infty^i(x)u_\infty^j(y)\} = \frac{1}{\rho_s} \int_{R^3} e^{2\pi i \xi \cdot (y-x)} \frac{1}{8\eta \pi^2 |\xi|^2} (P_\xi \ast \mathcal{M})^{ij}(d\xi).
\]
This completes the proof. \( \square \)

**Remark 3.3.** Again neglect the physical constants. (3.35) is the inverse Fourier transform of the Fourier transform of \( K(x) := N\Xi \) of the spectral measure, where \( N(x) \) is the Newtonian potential. This is consistent with Remark 3.2 since it is well known that when \( t \to \infty \), \( P_t(x) \to N(x) \). When the spectral measure is diagonal with Lebesgue measures on diagonal entries, i.e.

\[
\mathcal{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}
\]  

(3.36)

we will recover \( K(x) \) as is shown in the following. We have

\[
E(u^i_{\infty}(x)u^j_{\infty}(y)) = \int_{R^3} e^{2\pi i \xi \cdot (y-x)} \frac{1}{8\pi^2 |\xi|^2} (\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}) d\xi
\]

\[
= \frac{1}{8\pi^2} \left( \frac{4\pi^3}{|x-y|} \delta_{ij} - \frac{\partial^2}{\partial x_i \partial x_j} (2\pi^3 |x-y|) \right)
\]

\[
= \frac{1}{8\pi^2} \left( \frac{4\pi^3}{|x-y|} \delta_{ij} - 2\pi^3 \left( \frac{\delta_{ij}}{|x-y|} - \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3} \right) \right)
\]

\[
= \pi \left( \frac{\delta_{ij}}{|x-y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^3} \right)
\]

\[
= N\Xi.
\]

Next proposition is for the estimate for the spacial correlation of \( u(t, x) \), which is sometimes helpful.

**Proposition 3.13.** If \( F \) is in spectral class \( \mathcal{M}(0, 0) \), then we have the following estimate for spatial correlation of \( u \):

\[
E(|u(t, x) - u(t, y)|^2) \leq K|x - y|^2,
\]

(3.37)

where \( K \) is some constant.
Proof. Without loss of generality, we can assume that \( u_0 = 0 \) since \( T(t, x) \) satisfies the following Lipschitz condition:

\[
|T(t, x) - T(t, y)| \leq N|x - y|,
\]

where constant \( N \) does not depend on \( t \).

We proceed by direct calculation. We have:

\[
E\{|u(t, x) - u(t, y)|^2\} = E\{(u(t, x) - u(t, y))(u(t, x) - u(t, y))\}
\]

\[
= \sum_{k=1}^{3} E\{u^i(t, x)\overline{u^i(t, x)} - u^i(t, x)\overline{u^i(t, y)} - u^i(t, y)\overline{u^i(t, x)} + u^i(t, y)\overline{u^i(t, y)}\}
\]

\[
= \sum_{k=1}^{3} \int_{R^3} (1 - e^{2\pi i \xi (y-x)} - e^{2\pi i \xi (x-y)} + 1) \frac{\rho_s(1 - e^{-8\pi^2|\xi|^2 t})}{8\eta\pi^2|\xi|^2} (P_\xi * M)^{ii}(d\xi)
\]

\[
= \sum_{k=1}^{3} \int_{R^3} -e^{\pi i \xi (x-y)} - e^{\pi i \xi (y-x)} 2 \frac{\rho_s(1 - e^{-8\pi^2|\xi|^2 t})}{8\eta\pi^2|\xi|^2} (P_\xi * M)^{ii}(d\xi)
\]

\[
= \sum_{k=1}^{3} \int_{R^3} 2 \sin^2(\pi \xi (x-y)) \frac{\rho_s(1 - e^{-8\pi^2|\xi|^2 t})}{8\eta\pi^2|\xi|^2} (P_\xi * M)^{ii}(d\xi)
\]

Thus,

\[
E\{|u(t, x) - u(t, y)|^2\} \leq K|x - y|^2 \sum_{k=1}^{3} \int_{R^3} (P_\xi * M)^{ii}(d\xi)
\]

\[
\leq K|x - y|^2 (\max_{1 \leq i, j \leq 3} \int_{R^4} |\mu^{ij}|(d\xi))
\]

\[
\leq K|x - y|^2
\]

since \( \max_{1 \leq i, j \leq 3} \int_{R^4} |\mu^{ij}|(d\xi) < \infty \). Proposition is proved. \( \square \)

### 3.5 Regularity Properties for \( u(t, x) \) and \( p(t, x) \)

In this section, we will discuss the regularity properties for the solution of stochastic Stokes equation. We identify the Krylov’s stochastic Banach space to which \( u(t, x) \)
and \( p(t, x) \) belong, then apply Krylov’s stochastic Banach space theory to discuss the regularity for \( u(t, x) \) and \( p(t, x) \). For mathematical simplicity but without loss of generality, we let \( \rho_s = \eta = 1 \) in the discussions below. Also, here and hereafter, in the spirit of proposition 3.11, we do not have to distinct \( u(t, x) \) and \( u_t(\phi) \) in the calculations. We begin with \( u(t, x) \).

**Lemma 3.14.** Assume the spectral measure is in the spectral class \( \mathbb{M}(q_1, 2q_2 - 2) \), where \( q_1 \leq -2 \) and \( q_2 \) is an arbitrary real number. Then the solution \( u(t, x) \) to the equation (3.13) with initial condition \( u_\infty(x) \) from equilibrium is in \( (H^{q_2}_{2}(T))^3 \) for any \( T \geq 0 \).

**Proof.** It suffices to show that \( u^i(t, x) \in H^{q_2}_{2}(T), 1 \leq i \leq 3 \). Notice that since the spectral measure in class \( \mathbb{M}(q_1, 2q_2 - 2) \), \( q_1 \leq -2 \), we have:

\[
\int_{\mathbb{R}^3} |\xi|^{-2}(1 + |\xi|^2)^{q_2-1}d\xi < \infty. \tag{3.38}
\]

We proceed by direct calculations.

\[
|E\{\int_{\mathbb{R}^3} ((1 - \triangle)^{\frac{q_2}{2}} u^i(t, x)) ((1 - \triangle)^{\frac{q_2}{2}} u^i(t, x)) dx\}| = |\int_0^t \int_{\mathbb{R}^3} e^{-4\pi^2|\xi|^2(2t-2r)}(1 + |\xi|^2)^{q_2} (P_\xi * \mathcal{M})^{ii}(d\xi)dr|
\]

\[
\leq K_1 max_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} \frac{1 - e^{-8\pi^2|\xi|^2t}}{|\xi|^2}(1 + |\xi|^2)^{q_2} |\mu^{ij}|(d\xi)
\]

\[
\leq K_1 max_{1 \leq i, j \leq 3} \int_{\mathbb{R}^3} (1 + |\xi|^2)^{q_2} |\mu^{ij}|(d\xi)
\]

\[
< \infty.
\]

Hence \( \forall T > 0, u^i \in H^{q_2}_{2}(T) \).
Next, similarly, for $1 \leq k, l \leq 3$ we have:

$$|E\{\int_{R^3} \left( (1 - \triangle)^{q_2-2} u_{x_k x_l}(t, x) \right) \left( (1 - \triangle)^{q_2-2} u_{x_k x_l}(t, x) \right) dx \}|$$

$$= |\int_0^t \int_{R^3} e^{-4\pi^2|\xi|^2(2t-2r)} \xi_i^2 \xi_l^2 (1 + |\xi|^2)^{q_2-2} (P_\xi \ast \mathcal{M})^{ii}(d\xi)dr|$$

$$\leq K_{2max1 \leq i,j \leq 3} \int_0^t \int_{R^3} \frac{1 - e^{-8\pi^2|\xi|^2t}}{|\xi|^2} (1 + |\xi|^2)^{q_2} |\mu^{ij}|(d\xi)$$

$$\leq K_{2max1 \leq i,j \leq 3} \int_{R^3} (1 + |\xi|^2)^{q_2} |\mu^{ij}|(d\xi)$$

$$< \infty,$$

We thus have $u^i_{x_k x_l} \in \mathbb{H}^{q_2-1}_{2}(T)$ for $1 \leq k, l \leq 3$.

Now, let us check that $u^i_\infty(\cdot) \in L_2(\Omega, \mathcal{F}_0, H^{q_2-1}_{2})$. Similar to the calculations before, by (3.38),

$$|E\{\int_{R^3} \left( (1 - \triangle)^{q_2-1} u_\infty(x) \right) \left( (1 - \triangle)^{q_2-1} u_\infty(x) \right) dx \}|$$

$$= |\int_{R^3} \frac{1}{8\pi^2|\xi|^2} (1 + |\xi|^2)^{q_2-1} (P_\xi \ast \mathcal{M})^{ii}(d\xi)|$$

$$\leq K_{3max1 \leq i,j \leq 3} \int_{R^3} \frac{1}{8\pi^2|\xi|^2} (1 + |\xi|^2)^{q_2-1} |\mu^{ij}|(d\xi)$$

$$< \infty.$$

Thus, indeed $u_\infty(\cdot) \in L_2(\Omega, \mathcal{F}_0, H^{q_2-1}_{2})$.

Finally, By proposition 3.4, $u^i$ satisfies (2.16). Since $\mathcal{M}$ is tempered measure matrix, $\mathcal{S}(R^3)$ is dense in $L_2(\mu^{ii})$. With the help of separability of $\mathcal{S}(R^3)$, we can take $\{\eta^i_k, k = 1, 2, \ldots \} \subset \mathcal{S}(R^3)$. Thus $g^i = \{g^i_k, k = 1, 2, \ldots \}$ is in $\mathbb{H}^{q_2-1}_{2}(T, l_2)$. Thus we have $u^i \in \mathcal{H}^{q_2}_{2}(T)$ for any $T \geq 0$. Lemma is proved.

Next, let us discuss the regularity of $p(t, x)$. 

54
Lemma 3.15. If the spectral measure is in class $\mathcal{M}(q_1, 2q_2), q_1 \geq 0$, then $p(t, x)$ is in $\mathcal{H}^{q_2}_2(T)$ for any $T \geq 0$.

Proof. Recall that we have:

$$\triangle_x p(t, x) = -\nabla_x \cdot F(t, x). \quad (3.39)$$

Note that the above equation is understood in distribution sense, i.e. for any test function $\phi \in \mathcal{S}(\mathbb{R}^3)$, we have:

$$\triangle_p(t, \phi) = -\nabla \cdot F(t, \phi). \quad (3.40)$$

Since

$$E(\int_{\mathbb{R}^3} (1 - \triangle)^{\frac{q_2-1}{2}} \triangle p(t, x)(1 - \triangle)^{\frac{q_2-1}{2}} \triangle p(t, x) dx)$$

$$= \sum_{i,j=1}^{3} E(\int_{\mathbb{R}^3} \xi_i \xi_j (1 + |\xi|^2)^{q_2-1} \hat{F}^i(t, \xi) \hat{F}^j(t, \xi) dx)$$

$$= \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \xi_i \xi_j (1 + |\xi|^2)^{q_2-1} E(\hat{F}^i(t, \xi) \hat{F}^j(t, \xi)) dx$$

$$= \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \xi_i \xi_j (1 + |\xi|^2)^{q_2-1} E\left(\int_{\mathbb{R}^3} e^{-2\pi i \xi \cdot \zeta} F^i(t, d\zeta) \int_{\mathbb{R}^3} e^{-2\pi i \xi \cdot \zeta} F^j(t, d\zeta)\right) dx$$

Since in distribution sense,

$$E\left(\int_{\mathbb{R}^3} e^{-2\pi i \xi \cdot \zeta} F^i(t, d\zeta) \int_{\mathbb{R}^3} e^{-2\pi i \xi \cdot \zeta} F^j(t, d\zeta)\right) = t \int_{\mathbb{R}^3} \delta(\xi + \zeta) \mu^{ij}(d\zeta) \quad (3.41)$$

Thus,

$$E(\int_{\mathbb{R}^3} (1 - \triangle)^{\frac{q_2-1}{2}} \triangle p(t, x)(1 - \triangle)^{\frac{q_2-1}{2}} \triangle p(t, x) dx)$$

$$= \sum_{i,j=1}^{3} t \int_{\mathbb{R}^3} \xi_i \xi_j (1 + |\xi|^2)^{q_2-1} \mu^{ij}(d\xi),$$

where in order to distinct from the index $i$, we use $i$ to represent the imaginary number $i$. Thus we have $\triangle p(t, x) \in \mathcal{H}^{q_2-2}_2(T)$. Therefore we have $\triangle p(t, x) \in H^{q_2-2}_2$, thus $p \in H^{q_2}_2$. Notice that $\triangle^{-1}$ is a bounder linear operator from $H^{q_2-2}_2$ to $H^{q_2}_2$. 55
Also since for \( \phi \in \mathcal{S}(R^3) \), \( \Delta^{-1}\phi \in \mathcal{S}(R^3) \), we have \( p(t, x) \in \mathcal{H}_2^{q_2}(T) \). The lemma is proved.

We thus have the conditions for which \( u(t, x) \in (\mathcal{H}_2^{q_2}(T))^3 \) and \( p(t, x) \in \mathcal{H}_2^{q_2}(T) \), \( q_2 \) is an arbitrary real number. Now, let us extend the space \((\mathcal{H}_2^{q_2}(T))^3\) to \((\mathcal{H}_q^{q_2}(T))^3\) and \( \mathcal{H}_2^{q_2}(T) \) to \( \mathcal{H}_q^{q_2}(T) \) where \( q \geq 2 \).

**Proposition 3.16.** Let \( T \geq 0, q \geq 2 \). If the spectral measure is in the spectral class \( \mathcal{M}(q_1, 2q_2 - 2) \), where \( q_1 \leq -2 \) and \( q_2 \) is an arbitrary real number, then \( u \in (\mathcal{H}_q^{q_2}(T))^3 \); if the spectral measure is in class \( \mathcal{M}(q_1, 2q_2), q_1 \geq 0 \), then \( p(t, x) \) is in \( \mathcal{H}_q^{q_2}(T) \).

**Proof.** The proof is trivial. For \( 1 \leq i \leq 3 \), since \((1 - \triangle)^{\frac{q_2}{2}} u^i \) is Gaussian, we have:

\[
E\{ \int_0^T \int_{R^3} |(1 - \triangle)^{\frac{q_2}{2}} u^i|^q dx dt\} = \int_0^T \int_{R^3} \left| (1 - \triangle)^{\frac{q_2}{2}} u^i \right|^q dx dt \\
\leq \int_0^T K(q) (\int_{R^3} \left| (1 - \triangle)^{\frac{q_2}{2}} u^i \right|^2 dx)^{\frac{q}{2}} dt \\
< \infty
\]

where \( K(q) \) is a constant depending on \( q \). Thus \( u \in \mathcal{H}_q^{q_2}(T) \). Therefore, \( u^i \in (\mathcal{H}_q^{q_2}(T))^3 \). Similarly, we have \( p(t, x) \) is in \( \mathcal{H}_q^{q_2}(T) \). Proposition is proved.

**Corollary 3.17.** Let \( k \) be an positive integer. Suppose that the spectral measure \( \mu \in \mathcal{M}(q_1, q_2), q_1 \leq -2, q_2 > 2k - 2 \). Then \( u^i(t, x) \in C([0, T], C_0^k), 1 \leq i \leq 3 \). Furthermore, if \( q_1 = -2 \), then \( p(t, x) \in C([0, T], C_0^2) \).

**Proof.** By proposition 3.16, \( u^i(t, x) \in \mathcal{H}_q^{1 + \frac{q_2}{2}}(T) \), for any \( q \geq 2 \). Choose \( q \) big enough and choose \( \beta > \frac{1}{q} \) such that \( q \geq 2 \) and \( 1 + \frac{q_2}{2} - 2\beta - \frac{3}{q} > k \). By theorem 2.14, there exists a \( \gamma > 0 \) such that \( u^i \in C^\gamma([0, T], H_q^{1 + \frac{q_2}{2} - 2\beta}) \). Thus by Sobolev embedding theorem \( u^i(t, x) \in C([0, T], C_0^k) \). If \( q_1 = -2 \), then since \( \mathcal{M}(-2, q_2) \subset \mathcal{M}(0, q_2 + 2) \), by lemma 3.14, \( p(t, x) \) is also in \( \mathcal{H}_q^{1 + \frac{q_2}{2}}(T) \). Thus also \( p \in C([0, T], C_0^k) \) in the same reasoning as above.
CHAPTER 4
EXISTENCE AND UNIQUENESS FOR PASSIVE PARTICLES

In this chapter, we will prove the existence and uniqueness for passive particles, with and without finite size. We start by proving the existence and uniqueness for a particle without finite size. Then we proceed to prove the existence and uniqueness for particles with finite size.

4.1 Existence and Uniqueness for a Point Particle

The governing equation for a tracer particle is:

$$dz_t = U(t, z_t)dt$$

with initial condition $z_0 = x_0$, where $x_0$ is an arbitrary fixed point in $\mathbb{R}^3$, i.e. the starting position of this point particle. We will seek a continuous adapted process solution. Recall that a continuous adapted process $\{z_t, \mathcal{F}_t; t \in [0, T]\}$ is called a solution to equation (4.1) if

$$z_t = z_0 + \int_0^t U(s, z_s)ds, \ t \in [0, T], \ a.s.$$ 

First let us show there exists a unique continuous adapted local process solution. We will apply localization method, which is called truncation method by Kunita in [19].
Let us introduce some definitions (cf. Kunita [19]).

**Definition 4.1.** A stopping time $\tau$ with values in $[0, \infty]$ is called accessible if there exists an increasing sequence $\{\tau_n\}$ of stopping times such that $\tau_n < \tau$ and $\lim_{n \to \infty}$ hold almost surely. A family of random variables $X_t$ with the random time parameter $t \in [0, \tau)$ is called a local process.

Note that if $\{X_t, t \in [0, \tau)\}$ is a continuous local process and $\{\tau_n\}$ is the sequence of stopping time mentioned above, the stopped process $X_{t \wedge \tau_n} = X_{t \wedge \tau}$ is a usual (global) process for every $n$. The local process is called $\{\mathcal{F}_t\}$-adapted if $X_{t \wedge \tau_n}$ is $\{\mathcal{F}_t\}$-adapted for every $n$.

**Definition 4.2.** Let $\sigma_\infty$ be an accessible stopping time. A continuous local process $\{z_t, t \in [0, \sigma_\infty)\}$ adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ is called a local solution of equation (4.1) if

$$z_{t \wedge \sigma_N} = z_0 + \int_0^{t \wedge \sigma_N} U(s, z_s)ds$$

is satisfied for any positive integer $N$ where $\{\sigma_N\}$ is a sequence of stopping times such that $\sigma_N < \sigma_\infty$ and $\sigma_N \uparrow \sigma_\infty$. Furthermore if $\lim_{t \uparrow \sigma_\infty} z_t = \infty$ is satisfied when $\sigma_\infty < T$, it is called a maximal solution and $\sigma_\infty$ is called the explosion time.

We need the following lemmas.

**Lemma 4.3.** Suppose that the spectral measure matrix $\mathcal{M} \in \mathbb{M}(q_1, q_2), q_1 \leq -2, q_2 > 0$. For each positive integer $N$, take a function $\psi_N(x), x \in \mathbb{R}^3$ in $\mathcal{S}(\mathbb{R}^3)$ such that

$$\psi_N(x) = \begin{cases} 
1 & \text{if } |x| \leq N \\
\in [0, 1] & \text{if } N \leq |x| \leq N + 1 \\
0 & \text{if } |x| \geq N + 1
\end{cases}$$

Define $U^N(t, x) \equiv U(t, x)\psi_N(x)$. Then there exists a unique solution to the following equation:

$$dz_t = U^N(t \wedge \tau_R, z_t)dt$$ (4.3)
with initial condition $z_0 = \psi_N(x_0)x_0$, where $\tau_R = \inf\{t \in [0, T] : \sup_s |\nabla U^N(t, x)| > R\} (= T$ if $\{\ldots\} = \emptyset$), $R$ is an arbitrary number which is bigger than 0.

Proof. By corollary 3.17, $U \in C([0, T], (C^1_0)^3)$. Hence on $[[0, \tau_R]] = \{(\omega, t) : 0 \leq t \leq \tau_R(\omega)\}$, we have:

$$|U(t, x) - U(t, y)| \leq K(R)|x - y|.$$  \hspace{0.5cm} (4.4)$$

where $K(R)$ is a constant depending on $R$. We will use Picard iteration method. Let $z^0_t = \psi_N(x_0)x_0$, $z^{k+1}_t = z^0_t + \int_0^t U^N(s \wedge \tau_R, z^k_s)ds$. Then,

$$z^{k+1}_t - z^k_t = \int_0^t (U^N(s \wedge \tau_R, z^k_s) - U^N(s \wedge \tau_R, z^{k-1}_s))ds$$

Thus, for $0 \leq t \leq T$,

$$E\{\max_{0 \leq s \leq t}|z^{k+1}_s - z^k_s|^2\}$$

$$= E\{\max_{0 \leq s \leq t}\left|\int_0^s (U^N(l \wedge \tau_R, z^k_l) - U^N(l \wedge \tau_R, z^{k-1}_l))dl\right|^2\}$$

$$\leq K_1(T)E\{\max_{0 \leq s \leq t}\left|\int_0^s (U^N(l \wedge \tau_R, z^k_l) - U^N(l \wedge \tau_R, z^{k-1}_l))dl\right|^2\}$$

$$\leq K_1(T)\cdot K(R)\int_0^t E|z^k_s - z^{k-1}_s|^2ds$$

$$\leq C^* \frac{(NK(R)t)^k}{k!}$$

where $C^* = \max_{0 \leq t \leq T}E|z^1_t - z^0_t|^2$, which is seen to be a finite number. Thus by Chebyshev’s theorem,

$$P(\max_{0 \leq s \leq t}|z^{k+1}_s - z^k_s| > \frac{1}{2^{k+1}}) \leq 4C^* \frac{(4K_1(T)K(R)t)^k}{k!}.$$  \hspace{0.5cm} (4.9)$$

Since $\sum_{k=0}^\infty 4C^* \frac{(4K_1(T)K(R)t)^k}{k!} < \infty$, we have by Borel-Cantelli lemma, almost surely, there exists an integer-valued random variable $I(\omega)$ such that

$$\max_{0 \leq s \leq t}|z^{k+1}_s - z^k_s| \leq 2^{-(k+1)}, \ \forall k \geq I(\omega),$$

59
Consequently,

\[ \max_{0 \leq s \leq t} |z_{s}^{k+m} - z_{s}^{k}| \leq 2^{-k}, \ \forall m \geq 1, \ \forall k \geq I(\omega). \]

Thus, almost surely, \( \{z_{t}^{k}; 0 \leq t \leq T\} \) converges uniformly with respect to \( t \in [0, T] \).

Denote this continuous limit by \( z_{t} \). It is easy to check that \( z_{t} \) is the solution to the equation (4.3).

Finally, let us show the uniqueness of the solution. Let \( v_{t} \) be another solution. Since

\[
E|z_{t} - v_{t}|^{2} = E\left\{ \int_{0}^{t} U(s \wedge \tau_{R}, z_{s}) - U(s \wedge \tau_{R}, v_{s}) ds \right\}^{2} \leq tE\left\{ \int_{0}^{t} |U(s \wedge \tau_{R}, z_{s}) - U(s \wedge \tau_{R}, v_{s})|^{2} ds \right\} \leq TK(R)^{2} \int_{0}^{t} E|z_{s} - v_{s}|^{2} ds,
\]

Thus by Gronwall inequality, almost surely, \( v_{t} = z_{t}, \ t \in [0, T] \). This completes the proof.

Lemma 4.4. Let \( U^{N}(t, x) \) be defined as in lemma 4.3. Suppose that the spectral measure matrix \( \mathcal{M} \in \mathcal{M}(q_{1}, q_{2}), q_{1} \leq -2, q_{2} > 0 \), then there exists a unique solution for equation

\[
dz_{t} = U^{N}(t, z_{t}) dt
\]

with initial condition \( z_{0} = \psi_{N}(x_{0})x_{0} \), where \( x_{0} \) is an arbitrary fixed point in \( R^{3} \).

Proof. First let us prove the uniqueness. Suppose \( z_{t} \) are \( v_{t} \) are two solutions for equation (4.5), need to show that almost surely, for each \( t \in [0, T] \), we have \( z_{t} = v_{t} \).

By corollary 3.17 \( u \) is almost surely Lipschitz, i.e. almost surely we have

\[ |U(t, x) - U(t, y)| \leq K_{1}(\omega, t)|x - y|. \]
where $K_1(\omega, t)$ is a constant depending on $\omega, t$. Thus on compact set $[0, T] \times \{|x| \leq N\}$, almost surely, we have:

$$|U^N(t, x) - U^N(t, y)| \leq K_2(\omega)|x - y|,$$

where $K_2(\omega)$ is a constant depending on $\omega$ and $T$.

Now exclude a null set, $\omega$ by $\omega$, we have:

$$|z_t - v_t| = \left| \int_0^t U^N(s, z_s) - U^N(s, v_s) ds \right| \leq \int_0^t |U^N(s, z_s) - U^N(s, v_s)| ds \leq K_2(\omega) \int_0^t |z_s - v_s| ds.$$

Thus by Gronwall inequality, almost surely, for each $t \in [0, T]$, we have $z_t = v_t$.

Next let us prove the existence of a solution. Let

$$\tau_R = \inf \{ t \in [0, T] : \sup_x |\nabla U^N(t, x)| > R \} = T \text{ if } \{ \ldots \} = \emptyset.$$

Since $U \in C([0, T], C^2_0)$, thus almost surely, $\lim_{R \to \infty} \tau_R = T$. By lemma 4.3, for each $R$, there exists a unique solution to equation (4.3), denote it as $z^R_t$. Now define $z_t = z^R_t$ on $[0, \tau_R]$. There is no ambiguity in the definition since for $R_1 \leq R_2$, $\tau_{R_1} \leq \tau_{R_2}$ and $z^{R_1}_t$ and $z^{R_2}_t$ coincide on $[0, \tau_{R_1}]$ by uniqueness. Now that $z_t$ is a solution to equation 4.5 follows easily. The proof is complete.

With these two lemmas, we can derive the following key lemma in this section:

**Lemma 4.5.** Suppose that the spectral measure matrix $\mathcal{M} \in \mathcal{M}(q_1, q_2), q_1 \leq -2, q_2 > 0$, then there exists a unique maximal solution for equation (4.1) with initial condition $z_0 = x_0$, where $x_0$ is an arbitrary fixed point in $R^3$.

**Proof.** The following arguments is from Kunita [19]. By lemma 4.4, equation (4.5) has a unique solution starting at $x_0$ for every $N$. Denote it by $z^N_t$. Set

$$\sigma_N = \inf \{ t \in [0, T] : |z^N_t| > N \} = T \text{ if } \{ \ldots \} = \emptyset.$$
Now if \( M < N \), \( z_t^M = z_t^N \) holds for \( t < \sigma_M \) which implies that \( \{\sigma_N\} \) increases with \( N \). Set \( \sigma_\infty = \lim \sigma_N \) and define \( z_t, t < \sigma_\infty \) by \( z_t = z_t^N \) if \( t < \sigma_N \). Then \( \lim_{t \to \sigma_\infty} = \infty \) holds if \( \sigma_\infty < T \). Hence it is a maximal solution of equation (4.1).

Next, we need to prove the uniqueness of the solution. Let \( v_t, t \in [0, \gamma_\infty) \) be any maximal solution of equation (4.1). Obviously lemma 4.4 is valid for a local process \( v_t \). Therefore we have \( z_t^N = v_t \) for \( t < \sigma_N \). This proves \( z_t = v_t \) for \( t < \sigma_\infty \) and hence \( \sigma_\infty = \gamma_\infty \). The proof is complete. \( \square \)

Finally, we remove the restriction to “local solution” and prove the the existence and uniqueness of a continuous adapted process solution. Let us state and prove the existence and uniqueness theorem for this section.

**Theorem 4.6.** Suppose that the spectral measure matrix \( M \in \mathcal{M}(q_1, q_2), q_1 \leq -2, q_2 > 0 \), then there exist a unique continuous adapted process solution to equation (4.1) with initial condition \( z_0 = x_0 \), where \( x_0 \) is an arbitrary fixed point in \( \mathbb{R}^3 \).

**Proof.** Let \( z_t^N, \sigma_N \) be defined as above lemma, i.e. \( z_t^N \) is the unique solution starting at \( x_0 \) to equation (4.5) and

\[
\sigma_N = \inf \{ t \in [0, T] : |z_t^N| > N \} \quad (= T \text{ if } \{\ldots\} = \emptyset).
\]

Set \( \sigma_\infty = \lim \sigma_N \). We already know from the above lemma that there exist a unique local solution to equation (4.1). If we can show that almost surely, \( \sigma_\infty = T \). Then we have a unique solution to equation (4.1).

For each positive integer \( N \) and \( 0 < t < T \), we have:

\[
P(\sigma_N \leq t) = P(\sup_{s \leq t}|z_s^N| \geq N) \\
\leq \sup_m P(\sup_{s \leq t}|z_s^m| \geq N) \\
\leq \sup_m (P(|U(0, x_0)| \geq \frac{N}{2}) + P(\sup_{s \leq t} | \int_0^s U^m(s, z_s^m)ds | \geq \frac{N}{2})) \\
\leq P(|U(0, x_0)| \geq \frac{N}{2}) + \sup_m P(\sup_{s \leq t, x} | U^m(t, x) | \geq \frac{N}{2t}).
\]

62
Now, for the second term on the right hand side of the last inequality, we have for any $q > 0$:

\[
P(\sup_{s \leq t,x}|U^m(t,x)| \geq \frac{N}{2t}) \leq P(\sup_{s \leq t,x}|U(t,x)| \geq \frac{N}{2t}) \leq \frac{E(\sup_{s \leq t,x}|U(t,x)|^q)}{\left(\frac{2t}{q}\right)^q} \leq \frac{(2T)^qE(\sup_{s \leq T,x}|U(t,x)|^q)}{Nq},
\]

By proposition 3.16, $U(t,x) \in (H^{1+\frac{q q}{2}}_q(T))^3$, for any $q \geq 2$. Choose $q$ big enough and choose $\beta > \frac{1}{q}$ such that $q \geq 2$ and $1 + \frac{q q}{2} - 2\beta - \frac{3}{q} > 1$. By theorem 2.14, there exists a $\gamma > 0$ such that $u \in C^\gamma([0,T],(H^{1+\frac{q q}{2}-2\beta}_q(T))^3)$. By theorem 2.14 and Sobolev embedding theorem, we have

\[
E(\sup_{s \leq T,x}|U(t,x)|^q) \leq K||U||^{q}_{(H^{1+\frac{q q}{2}}_q(T))^3},
\]

where $K$ is some constant independent of $N$. Thus,

\[
P(\sup_{s \leq t,x}|U^m(t,x)| \geq \frac{N}{2t}) \leq \frac{K(2T)^q||U||^{q}_{(H^{1+\frac{q q}{2}}_q(T))^3}}{Nq}.
\]

Therefore,

\[
\sup_m P(\sup_{s \leq t,x}|U^m(t,x)| \geq \frac{N}{2t}) \leq \frac{K(2T)^q||U||^{q}_{(H^{1+\frac{q q}{2}}_q(T))^3}}{Nq}.
\]

Hence

\[
P(\sigma_N \leq t) \leq P(\|U(0,x_0)\| \geq \frac{N}{2}) + \frac{K(2T)^q||U||^{q}_{(H^{1+\frac{q q}{2}}_q(T))^3}}{Nq}.
\]

Letting $N \to \infty$, we have $\lim_{N \to \infty} P(\sigma_N \leq t) = 0$. Thus we have almost surely, $\sigma_\infty = \lim_{N \to \infty} \sigma_N = T$. This completes the proof. \(\square\)
4.2 Existence and uniqueness problem for a particle with finite size

We are now ready to prove the existence and uniqueness results articles with finite size. Notice that the system of equations (1.26)-(1.29) are equivalent to

\[ m_r \frac{dv}{dt} = - \int_{\partial D(t)} S(t, x)n(t, x) S(dx), \]  
\[ I \frac{dw}{dt} = - \int_{\partial D(t)} (x - c(t)) \times (S(t, x)n(t, x)) S(dx), \]

where

\[ S(t, x) = -P(t, x)I + \frac{\eta}{2}(\nabla_x U(t, x) + \nabla_x U^T(t, x)). \]

Thus below, instead of proving the existence and uniqueness of the solution to equations (1.26)-(1.29), we prove the existence and uniqueness of the solution to (4.7)-(4.8) with initial conditions \( v(0) = v_0 \in \mathbb{R}^3 \) and \( w(0) = w_0 \in \mathbb{R}^3 \).

The proof is similar to the proof for tracer particle without finite size. We first use localization techniques and prove the existence and uniqueness for a local solution, and then we remove the restriction of "local solution" and get the existence and uniqueness for a global solution.

We start from the following lemma, which is elementary but plays an important rule in our estimations.

**Lemma 4.7.** Let \( A_i(t) \in SO(3), i = 1, 2 \) is the rotation matrixs which satisfy

\[ \frac{d}{dt} A_i(t) = B_i(t)A_i(t), \]  

where \( B_i(t), i = 1, 2 \) are skew-symmetric and \( A_i(0) = I, i = 1, 2 \). Then we have:

\[ |A_1(t) - A_2(t)| \leq \int_0^t |B_1(s) - B_2(s)|ds \]
Proof. Since $A_i(t) \in SO(3)$, thus $|A_i(t)u| = |A_i^T(t)u| = |u^TA_i(t)| = |u^TA_i^T(t)| = |u|$, for any $u \in R^3$, $i = 1, 2$. Let $A(t) = A_2^T(t)A_1(t)$. Then the left hand side of (4.11) is:

$$
LHS = |A_2^T(t)(A_1(t) - A_2(t))|
$$

while the right hand side of (4.11) is:

$$
\int_0^t |A_2^T(t)B_1(s) - B_2(s))|ds = \int_0^t |A_2^T(t) \dot{A}_1(t)A_1^T(t) - A_2^T(t) \dot{A}_2(t)A_2^T(t)|ds
$$

Thus to show (4.11) is equivalent to show that

$$
|A(t) - I| \leq \int_0^t |\dot{A}(s)|ds \tag{4.12}
$$

Let us first show the following inequality:

$$
\sqrt{|A(t) - I|^2 + \epsilon^2} - \epsilon \leq \int_0^t |\dot{A}(s)|ds \tag{4.13}
$$

where $\epsilon$ is an arbitrary positive real number. Since when $t = 0$, (4.13) is obvious true, therefore we only need to show that for any $t \geq 0$,

$$
\frac{d}{dt}(\sqrt{|A(t) - I|^2 + \epsilon^2} - \epsilon) \leq |\dot{A}(t)| \tag{4.14}
$$

65
Now, by Cauchy inequality, we have:

\[
\frac{d}{dt}( \sqrt{|A(t) - I|^2 + \epsilon^2} - \epsilon) = \frac{\langle A(t) - I, \dot{A}(t) \rangle}{\sqrt{|A(t) - I|^2 + \epsilon^2}} \\
\leq \frac{|A(t) - I|}{\sqrt{|A(t) - I|^2 + \epsilon^2}} |\dot{A}(t)| \\
\leq |\dot{A}(t)|
\]

where \( \langle C, D \rangle := \sum_{i=1}^{3} \sum_{j=1}^{3} C_{ij} D_{ij} \), for \( C = (C_{ij})_{3x3}, D = (D_{ij})_{3x3} \).

Thus we proved inequality (4.12). Now (4.12) obviously implies (4.11). Lemma is proved.

We now follow the same line of arguments as previous section.

**Lemma 4.8.** Suppose that the spectral measure matrix \( M \in \mathbb{M}(-2, q), q > 2 \). For each positive integer \( N \), take a function \( \psi_N(x), x \in \mathbb{R}^3 \) in \( S(\mathbb{R}^3) \) such that

\[
\psi_N(x) = \begin{cases} 
1 & \text{if } |x| \leq N \\
\in [0, 1] & \text{if } N \leq |x| \leq N + 1 \\
0 & \text{if } |x| \geq N + 1
\end{cases}
\]

Define \( U^N(t, x) := U(t, x)\psi_N(x) \) and \( P^N(t, x) := P(t, x)\psi_N(x) \). Then there exists a unique solution to the following equations:

\[
m_r \frac{dv}{dt} = -\int_{\partial D(t)} S^N(t \wedge \tau_R, x)n(t, x) S(dx), \\
I \frac{dw}{dt} = -\int_{\partial D(t)} (x - c(t)) \times (S^N(t \wedge \tau_R, x)n(t, x)) S(dx),
\]

with initial condition \( v(0) = v_0 \in \mathbb{R}^3 \) and \( w(0) = w_0 \in \mathbb{R}^3 \), where \( S^N(t, x) = -P^N(t, x)I + \frac{q}{2} (\nabla_x U^N(t, x) + \nabla_x (U^N)^T(t, x)) \) and \( \tau_R = \inf \{t \in [0, T] : \sup_x \{ |U^N_{xx}(t, x)| + P^N(t, x)| \} > R \} (= T \text{ if } \{ \ldots \} = \emptyset ) \), \( U^N_{xx} \) is the matrix of second-order derivatives of \( U^N \) with respect to \( x \) and \( R \) is an arbitrary number which is bigger than 0.
Proof. For mathematical simplicity, but without loss of generality, we can assume that \( m_r = 1, I = I \) and \( \eta = 1 \). Make a change of coordinate, let \( t = t, y = A(t)x + c(t) \), then equations (4.15) and (4.16) become

\[
\begin{align*}
\frac{dv}{dt} &= -\int_{\partial D} S^N(s, x)|_{s=t \wedge \tau_R, x=A(s)y+c(s)} A(t)n(y)S(dy), \\
\frac{dw}{dt} &= -\int_{\partial D} (A(s)y) \times (S^N(s, x)|_{s=t \wedge \tau_R, x=A(s)y+c(s)}) A(t)n(y)S(dy),
\end{align*}
\] (4.17) (4.18)

By corollary 3.17, \( U \in C([0, T], C^2_0) \) and \( P(t, x) \in C([0, T], C^2_0) \). Thus on \([0, \tau_R] = \{(\omega, t) : 0 \leq t \leq \tau_R(\omega)\}\), we have:

\[|\nabla U(t, x) - \nabla U(t, y)| \leq K_1(R)|x - y|,\]
\[|P(t, x) - P(t, y)| \leq K_2(R)|x - y|\]

where \( K_1(R), K_2(R) \) are constants depending only on \( R \).

Let

\[
\begin{align*}
v_t^0 &= 0, \\
v_t^{k+1} &= -\int_0^t \int_{\partial D} S^N(l, x)|_{l=s \wedge \tau_R, x=A^k(l)y+c^k(l)} A^k(s)n(y)S(dy)ds, \\
w_t^0 &= 0, \\
w_t^{k+1} &= -\int_0^t \int_{\partial D} (A^k(s)y) \times (S^N(l, x)|_{l=s \wedge \tau_R, x=A^k(l)y+c^k(l)} A^k(s)n(y)S(dy)ds,
\end{align*}
\]

where \( c^k(l) = \int_0^l v_t^k dl \), \( A^k(s) \) is the solution for

\[
\frac{d}{dt} A^k(s) = \Omega^k(s)A(s)
\]

with initial condition \( A^k(0) = I \) and

\[
\Omega^k(s) = \begin{pmatrix}
0 & -w^k_3(t) & w^k_2(t) \\
w^k_3(t) & 0 & -w^k_1(t) \\
-w^k_2(t) & w^k_1(t) & 0
\end{pmatrix}.
\]
Let
\[ z_t^0 = \begin{pmatrix} v_t^0 \\ w_t^0 \end{pmatrix}, \quad z_t^k = \begin{pmatrix} v_t^k \\ w_t^k \end{pmatrix} \]

Need to show that \( z_t^k \) converges uniformly to a continuous adapted process \( z_t \) with respect to \( t \in [0, T] \), such that the first three components of \( z_t \) satisfy (4.17) while the last three components satisfy (4.18).

Now,
\[
E\{\max_{0 \leq s \leq t}|v^{k+1}_s - v^k_s|^2\} \\
\leq NE|v^{k+1}_t - v^k_t|^2 \\
\leq \int_0^t \int_{\partial D} E|S^{N}_{k+1}(s \land T_R, y)A^{k+1}(s) - S^{N}_{k}(s \land T_R, y)A^k(s)|^2 S(dy)ds,
\]

where \( S^{N}_{k}(s \land T_R, y) = S^N(l, x)|_{l=s \land T_R, x=\omega^k(l) + c^k(l)} \).

Since
\[
|S^{N}_{k+1}(s \land T_R, y) - S^{N}_{k}(s \land T_R, y)| \\
\leq |P(s \land T_R, A^{k+1}(s \land T_R)y + c^{k+1}(s \land T_R)) - P(s \land T_R, A^k(s \land T_R)y + c^k(s \land T_R))| \\
+ \frac{1}{2}|(\nabla_x U(s \land T_R, A^{k+1}(s \land T_R)y + c^{k+1}(s \land T_R))^T| \\
+ \nabla_x U(s \land T_R, A^{k+1}(s \land T_R)y + c^{k+1}(s \land T_R)) \\
+ \nabla_x U(s \land T_R, A^k(s \land T_R)y + c^k(s \land T_R))^T| \\
\leq K_1(R)|A^{k+1}(s \land T_R)y + c^{k+1}(s \land T_R) - (A^k(s \land T_R)y + c^k(s \land T_R))| \\
+ K_2(R)|A^{k+1}(s \land T_R)y + c^{k+1}(s \land T_R) - (A^k(s \land T_R)y + c^k(s \land T_R))| \\
\leq K_3(R)(|A^{k+1}(s \land T_R) - A^k(s \land T_R)||y| + |c^{k+1}(s \land T_R) - c^k(s \land T_R)|) \\
\leq K_3(R)(|y| \int_{s \land T_R}^t |w_{l+1}^k - w_l^k|dl + \int_{s \land T_R}^t |v_{l+1}^k - v_l^k|dl) \\
\leq K_3(R)(|y| \int_0^t |w_{l+1}^k - w_l^k|dl + \int_0^t |v_{l+1}^k - v_l^k|dl)
\]
since \( |A^{k+1}(l) - A^k(l)| \leq \int_0^l |w_{j+1}^k - w_j^k|dj \) for any \( l \geq 0 \) by lemma 4.7.
Thus,

\[ |S_{k+1}^N(s \wedge \tau_R, y) - S_k^N(s \wedge \tau_R, y)|^2 \leq K_4(R, T)(|y|^2 \int_0^t |w_i^{k+1} - w_i^k|^2 dl + \int_0^t |v_i^{k+1} - v_i^k|^2 dl). \]

Hence,

\[
E|S_{k+1}^N(s \wedge \tau_R, y)A^{k+1}(s) - S_k^N(s \wedge \tau_R, y)A^k(s)|^2 \\
\leq E(|S_{k+1}^N(s \wedge \tau_R, y) - S_k^N(s \wedge \tau_R, y)||A^{k+1}(s)| \\
+|S_k^N(s \wedge \tau_R, y)||A^{k+1}(s) - A^k(s)||^2 \\
\leq K_5E(|S_{k+1}^N(s \wedge \tau_R, y) - S_k^N(s \wedge \tau_R, y)|^2|A^{k+1}(s)|^2 \\
+|S_k^N(s \wedge \tau_R, y)|^2|A^{k+1}(s) - A^k(s)|^2 \\
\leq K_6(R, T)(|y|^2 \int_0^t |w_i^{k+1} - w_i^k|^2 dl + \int_0^t |v_i^{k+1} - v_i^k|^2 dl) \\
+K_7(R, T) \int_0^t |w_i^{k+1} - w_i^k|^2 dl \\
\leq K_8(R, T)(\int_0^t E(|v_i^{k+1} - v_i^k|^2 + |w_i^{k+1} - w_i^k|^2) dl) \\
\]  

since \(|A^k(l)| \leq 3\) for any \(k\) and \(l \geq 0\), and \(|S_k^N(s \wedge \tau_R, y)| \leq K_8\) for any \(\omega, s, y\), here \(K_8\) is constant depending only on \(R\).

Thus,

\[
E\{\max_{0 \leq s \leq t}|v_s^{k+1} - v_s^k|^2\} \\
\leq \int_0^t \int_{\partial D} E|S_{k+1}^N(s \wedge \tau_R, y)A^{k+1}(s) - S_k^N(s \wedge \tau_R, y)A^k(s)|^2 S(dy)ds \\
\leq \int_0^t \int_{\partial D} (K_6(R, T)(|y|^2 \int_0^t |w_i^{k+1} - w_i^k|^2 dl + \int_0^t |v_i^{k+1} - v_i^k|^2 dl) \\
+K_7(R, T) \int_0^t |w_i^{k+1} - w_i^k|^2 dl) S(dy)ds \\
\leq K_9(R, T)(\int_0^t E(|v_i^{k+1} - v_i^k|^2 + |w_i^{k+1} - w_i^k|^2) dl) \\
\] (4.19)
Next, let us consider \( E\{max_{0 \leq s \leq t}|w_s^{k+1} - w_s^k|^2\} \). Similarly we have:

\[
E\{max_{0 \leq s \leq t}|w_s^{k+1} - w_s^k|^2\} \leq NE|w_t^{k+1} - w_t^k|^2
\]

\[
\leq \int_0^t \int_{\partial D} E|((A^{k+1}(s)y) \times S_{k+1}^N(s \wedge \tau_R, y))A^{k+1}(s)
\]

\[
-((A^k(s)y) \times S_k^N(s \wedge \tau_R, y))A^k(s)|^2 S(dy)ds.
\]

Since

\[
E|((A^{k+1}(s)y) \times S_{k+1}^N(s \wedge \tau_R, y))A^{k+1}(s) - ((A^k(s)y) \times S_k^N(s \wedge \tau_R, y))A^k(s)|^2
\]

\[
\leq K_{10} E|((A^{k+1}(s)y) \times S_{k+1}^N(s \wedge \tau_R, y))(A^{k+1}(s) - A^k(s))|^2
\]

\[+ |(A^{k+1}(s)y) \times (S_{k+1}^N(s \wedge \tau_R, y) - S_k^N(s \wedge \tau_R, y))|^2
\]

\[+ |(A^{k+1}(s)y - A^k(s)y) \times S_k^N|^2 \]

\[
\leq K_{11}(R)|A^{k+1}(s) - A^k(s)|^2|y|^2
\]

\[+ K_{12}(R, T)|y|^2 \int_0^t |w_t^{k+1} - w_t^k|^2 dl + \int_0^t |v_t^{k+1} - v_t^k|^2 dl |y|^2
\]

\[+ K_{13}(R)|A^{k+1}(s) - A^k(s)|^2|y|^2
\]

\[
\leq K_{11}(R)|y|^2 (\int_0^t |w_t^{k+1} - w_t^k|^2 dl)
\]

\[+ K_{12}(R, T)|y|^2 \int_0^t |w_t^{k+1} - w_t^k|^2 dl + \int_0^t |v_t^{k+1} - v_t^k|^2 dl |y|^2
\]

\[+ K_{13}(R)|y|^2 (\int_0^t |w_t^{k+1} - w_t^k|^2)
\]

Hence,

\[
E\{max_{0 \leq s \leq t}|w_s^{k+1} - w_s^k|^2\}
\]

\[
\leq K_{14}(R, T) \int_0^t E(|v_t^{k+1} - v_t^k|^2 + |w_t^{k+1} - w_t^k|^2) dl
\]

\[
(4.20)
\]

Therefore, by (4.19) and (4.20), we have:

\[
E\{max_{0 \leq s \leq t}|z_s^{k+1} - z_s^k|^2\} \leq K_{15}(R, T) \int_0^t E|z_t^{k+1} - z_t^k|^2 dl
\]

\[
(4.21)
\]

Thus similar to the proof of the existence of a (global) solution to particle tracer problem in Lemma 4.3, there exist a (global) solution to equations (4.15) and (4.16),
namely, a continuous adapted process $z_t = \begin{pmatrix} v_t \\ w_t \end{pmatrix}$ such that $v_t$ satisfies (4.15) and $w_t$ satisfies (4.16).

Finally, let us prove the uniqueness of the solution. Let $a_t$ be another solution. Then in the same reasoning as above, we have:

$$E|z_t - a_t|^2 \leq K_{16}(R, T) \int_0^t E|z_s - a_s|^2 ds$$

By Gronwall inequality, almost surely, $z_t = a_t$. This completes the proof.

Lemma 4.9. Let $U^N(t, x)$ be defined as in lemma 4.8. Suppose that the spectral measure matrix $M \in \mathcal{M}(-2, q), q > 2$, then there exists a unique solution for the following equations:

\begin{align*}
\text{Inertia } & \frac{dw}{dt} = -\int_{\partial D(t)} (x - c(t)) \times (S^N(t, x)n(t, x)) \, S(dx), \\
\text{Inertia } & \frac{dv}{dt} = -\int_{\partial D(t)} S^N(t, x)n(t, x) \, S(dx),
\end{align*}

with initial condition $v(0) = v_0 \in \mathbb{R}^3, w(0) = w_0 \in \mathbb{R}^3$.

Proof. First, let us prove the existence of a solution. Let

$$\tau_R = \inf\{t \in [0, T] : \sup_x \{|U^N_{xx}(t, x) + P^N(t, x)|\} > R\} = T \text{ if } \{\ldots\} = \emptyset.$$ 

By corollary 3.17, $U \in C([-T, T], (C^2_0)^3)$ and $P \in C([-T, T], C^2_0)$. Hence, almost surely, $\lim_{R \to \infty} \tau_R = T$. By lemma 4.8, for each $R$, there exists a unique solution to equations (4.15) and (4.16), denote it as $z^R_t$. Now define $z_t = z^R_t$ on $[0, \tau_R]$. There is no ambiguity in the definition since for $R_1 \leq R_2$, $\tau_{R_1} \leq \tau_{R_2}$ and $z^R_{t_1}$ and $z^R_{t_2}$ coincide on $[0, \tau_{R_1}]$ by uniqueness. Now that $z_t$ is a solution to equations (4.22) and (4.23) follows easily.

Next, let us prove the uniqueness. Suppose $a_t$ is another solution to equations (4.22) and (4.23). Notice that $z_{t \wedge \tau_R}$ and $a_{t \wedge \tau_R}$ are both the solutions for equations (4.15) and (4.16) on $[0, \tau_R]$, by the uniqueness in lemma 4.8, we have $z_{t \wedge \tau_R} = a_{t \wedge \tau_R}$.

Letting $R \to \infty$, we have $z_t = a_t$. The proof is complete. \qed
With these two lemmas, we can derive the following key lemma in this section:

**Lemma 4.10.** Suppose that the spectral measure matrix $\mathcal{M} \in \mathcal{M}(-2,q), q > 2$, then there exists a unique maximal solution for equations (4.7) and (4.8) with initial condition $v_0 = 0, w_0 = 0$.

**Proof.** The following arguments are similar to lemma 4.5. By lemma 4.9, equations (4.22) and (4.23) has a unique solution starting at $v_0 = 0$ and $w_0 = 0$ for every $N$. Denote it by $z_t^N$. Set

$$
\sigma_N = \inf \{ t \in [0,T] : |z_t^N| > N \} \quad (= T \text{ if } \{ \ldots \} = \emptyset).
$$

Now if $M < N$, $z_t^M = z_t^N$ holds for $t < \sigma_M$ which implies that $\{ \sigma_N \}$ increases with $N$. Set $\sigma_\infty = \lim \sigma_N$ and define $z_t, t < \sigma_\infty$ by $z_t = z_t^N$ if $t < \sigma_N$. Then $\lim_{t \to \sigma_\infty} z_t = \infty$ holds if $\sigma_\infty < T$. Hence it is a maximal solution of equations (4.7) and (4.8).

Next, we need to prove the uniqueness of the solution. Let $a_t, t \in [0, \gamma_\infty)$ be any maximal solution of equations (4.7) and (4.8). Obviously lemma 4.9 is valid for a local process $a_t$. Therefore we have $z_t^N = a_t$ for $t < \sigma_N$. This proves $z_t = a_t$ for $t < \sigma_\infty$ and hence $\sigma_\infty = \gamma_\infty$. The proof is complete. \qed

Finally, we remove the restriction to “local solution” and prove the the existence and uniqueness of a continuous adapted process solution. Let us state and prove the existence and uniqueness theorem for this chapter.

**Theorem 4.11.** Suppose that the spectral measure matrix $\mathcal{M} \in \mathcal{M}(-2,q), q > 2$, then there exist a unique continuous adapted process solution to equations (4.7) and (4.8) with initial condition $v(0) = v_0 \in \mathbb{R}^3, w(0) = w_0 \in \mathbb{R}^3$.

**Proof.** Let $z_t^N, \sigma_N$ be defined as above lemma, i.e. $z_t^N$ is the unique solution starting at $v_0$ and $w_0$ to equations (4.7) and (4.8) and

$$
\sigma_N = \inf \{ t \in [0,T] : |z_t^N| > N \} \quad (= T \text{ if } \{ \ldots \} = \emptyset).
$$
Set $\sigma_\infty = \lim \sigma_N$. We already know from the above lemma that there exist a unique local solution to equations (4.7) and (4.8). If we can show that almost surely, $\sigma_\infty = T$. Then we have a unique solution to equations (4.7) and (4.8).

For each positive integer $N$ and $0 < t < T$, we have:

$$P(\sigma_N \leq t) = P(\sup_{s \leq t}|z_s^N| \geq N)$$

$$\leq \sup_m P(\sup_{s \leq t}|z_s^m| \geq N)$$

Since there exist constants $K_1$ and $K_2$ such that

$$P(\sup_{s \leq t}|z_s^m| \geq N) \leq P(\sup_{s \leq t,x}|P^m(t,x)| \geq \frac{NK_1}{t}) + P(\sup_{s \leq t,x}|U^m(t,x)| \geq \frac{NK_2}{t}),$$

Thus,

$$P(\sigma_N \leq t) \leq \sup_m \{P(\sup_{s \leq t,x}|P^m(t,x)| \geq \frac{NK_1}{t}) + P(\sup_{s \leq t,x}|U^m(t,x)| \geq \frac{NK_2}{t})\}.$$ 

Now, for the second term on the right hand of the last inequality, we have for any $q_1 > 0$:

$$P(\sup_{s \leq t,x}|U^m(t,x)| \geq \frac{NK_2}{t}) \leq P(\sup_{s \leq t,x}|U(t,x)| \geq \frac{NK_2}{t})$$

$$\leq \frac{E(\sup_{s \leq t,x}|U(t,x)|^{q_1})}{N^{q_1}K_2^{q_1}}$$

$$\leq \frac{T^{q_1}E(\sup_{s \leq T,x}|U(t,x)|^{q_1})}{N^{q_1}K_2^{q_1}}$$

By proposition 3.16, $U(t,x) \in (H_{q_1}^{1+\frac{2}{q_1}}(T))^3$, for any $q_1 \geq 2$. Choose $q_1$ big enough and choose $\beta > \frac{1}{q_1}$ such that $q_1 \geq 2$ and $n - 2\beta - \frac{3}{q_1} > 2$. By theorem 2.14, there exists a $\gamma > 0$ such that $u \in C^\gamma([0,T],(H_{q_1}^{1+\frac{2}{q_1}})^3)$. Thus by theorem 2.14 and Sobolev embedding theorem, we have

$$E(\sup_{s \leq T,x}|U(t,x)|^{q_1}) \leq K_3||U||_{(H_{q_1}^{1+\frac{2}{q_1}}(T))^3}^{q_1},$$

where $K_3$ is some constant independent of $N$. Hence

$$P(\sup_{s \leq t,x}|U^m(t,x)| \geq \frac{NK_2}{t}) \leq \frac{K_3T^{q_1}||U||_{(H_{q_1}^{1+\frac{2}{q_1}}(T))^3}^{q_1}}{N^{q_1}K_2^{q_1}}$$

73
Therefore,

$$\sup_m P(\sup_{s \leq t,x} |U^m(t,x)| \geq \frac{NK_2}{t}) \leq \frac{K_3 T^{q_1} ||U||_{(H^{1+\frac{q_1}{2}}_q(T))}^q}{N^{q_1} K_2^{q_1}}.$$ 

Since $\mathbb{M}(-2, q_1) \subset \mathbb{M}(0, q_1 + 2)$, $P(t, x) \in H^{1+\frac{q_1}{2}}_q(T)$ by proposition 3.16. Hence we have:

$$\sup_m P(\sup_{s \leq t,x} |P^m(t,x)| \geq \frac{NK_1}{t}) \leq \frac{K_4 T^{q_1} ||P||_{H^{1+\frac{q_1}{2}}_q(T)}^q}{N^{q_1} K_1^{q_1}},$$

Therefore,

$$P(\sigma_N \leq t) \leq \frac{K_4 T^{q_1} ||U||_{(H^{1+\frac{q_1}{2}}_q(T))}^q}{N^{q_1} K_1^{q_1}} + \frac{K_3 T^{q_1} ||P||_{H^{1+\frac{q_1}{2}}_q(T)}^q}{N^{q_1} K_2^{q_1}},$$

Letting $N \to \infty$, we have $\lim_{N \to \infty} P(\sigma_N \leq t) = 0$. Thus we have almost surely, $\sigma_\infty = \lim \sigma_N = T$. This completes the proof. 

$\square$
5.1 Background

In this section, we will study the conditions under which the particle has classical diffusive behavior and compute the drift and diffusion coefficients. In other word, we will study the central limit theorems for turbulent diffusions. A lot work has been done in this field. For instance, see [9], [19], [24], [32], and [13]. In their pioneering paper, Kesten, and Papanicolau [9] studied the conditions under which the motion of a particle in a random forcing field approximates a diffusion; While in [19], Kunita studied the central limit theorems for turbulent diffusions as a special case of weak convergence of stochastic flows to diffusions, under his novel treatment of stochastic differential equations based on stochastic integral with respect to semimartingale with spatial parameters.

Despite the rich literature, it does not fit easily in our settings. The settings we are working on are sufficiently different from the literature. Firstly, the starting points are different. In the literature, the scaling is done on the random forcing field,
i.e. speeding up the time. In terms of the covariance structure, we have:

\[
\frac{1}{\epsilon^2} E\{U^i(t, x) U^j(s, y)\} = \int_{R^3} \int_{R^3} H_0(\frac{t}{\epsilon^2}, x-z) H_0(\frac{s}{\epsilon^2}, y-w) E(U^i(0, z) U^j(0, w) dz dw
\]

\[
+ \frac{1}{\epsilon^2} \int_0^{t+s} \int_{R^3} e^{2\pi i \xi (y-x)} e^{-4\pi^2 |\xi|^2 \frac{m}{\rho_s \epsilon^2} (t+s-2r)} (P_\xi \ast M(d\xi))^{ij} dr
\]

Now the physical meaning for the scaling is equivalent to saying that the ratio of
viscosity and the density for solvent is infinite.

While in our case, the scaling is done on the position of the particle, i.e. we observe
the particle as the time goes to infinity. Physically it seems to be more natural.

Secondly, in these literature, strong conditions have been assumed to achieve
some general convergence theorems. Yet, in our case, we can directly prove the
central limit theorems and calculate the diffusion and drift coefficients, only under
some mild conditions on spectral measure matrix.

5.2 Central Limit Theorems

In this section, we assume that the physical constants \( \eta = \rho_s = 1 \). To approach the
central limit theorems for \( z(t) \), we first speed up the time, then choose the corre-
sponding scale in the space. We have

\[
z_\lambda(t) = k z(\lambda t)
\]

\[
= k \int_0^{\lambda t} U(s, z(s)) ds
\]

\[
= k \lambda \int_0^{t} U(\lambda s, z(\lambda s)) ds
\]

\[
= k \lambda \int_0^{t} U(\lambda s, \frac{1}{k} z_\lambda(s)) ds.
\]
It turns out later that \( k = \lambda^{-1/2} \) would be the correct scaling for space. It is expected that \( z_\lambda(t) \) converges to a Brownian motion. To illustrate the rough idea, we assume that \( t = 1 \) and \( \lambda \) is an integer. We thus have

\[
z_\lambda(1) = \lambda^{-1/2} \sum_{i=1}^{\lambda} z'_i,
\]

where \( z'_i = \int_{i-1}^{i} U(s, z(s)) ds \). Since \( \{z'_1, z'_2, \ldots\} \) are weakly dependent random variables, it is natural that \( z_\lambda(1) \) converges weakly to a normal random variable under some conditions of the mixing rates.

The procedure of the proof is routine. We first prove that \( z_\lambda(t) \) is tight, then we proceed to prove that every convergent subsequence has the same limiting distribution. Since the limiting distributions are Gaussian, it suffices to show that they have the same mean vector and covariance matrix. To compute the mean vector, we need to estimate

\[
Ez'_1(t) = \sqrt{\lambda} E(\int_0^t U^i(\lambda s, \sqrt{\lambda} z_\lambda(s)) ds)
\]

\[
= \sqrt{\lambda} E(\int_0^t U^i(\lambda s, \sqrt{\lambda} z_\lambda(s)) ds).
\]

Next, to compute the covariance matrix, we need to compute

\[
E[(z'_1(t) - E(z'_1(t)))(z'_j(t) - E(z'_j(t))))
\]

\[
= E(z'_1(t)z'_j(t)) - E(z'_1(t))E(z'_j(t)).
\]

Since

\[
E(z'_1(t)z'_j(t)) = \lambda \int_0^t \int_0^t E(U^i(\lambda s', \sqrt{\lambda} z_\lambda(s')))U^j(\lambda s'', \sqrt{\lambda} z_\lambda(s''))) ds' ds'',
\]

we have to estimate \( \lambda \int_0^t \int_0^t E(U^i(\lambda s', \sqrt{\lambda} z_\lambda(s')))U^j(\lambda s'', \sqrt{\lambda} z_\lambda(s''))) ds' ds''.

Now, let us start our journey of the proof of the central limit theorem for turbulent diffusion. We begin with some lemmas involving mixing conditions and the strong mixing rate.
Lemma 5.1. For mathematical simplicity but without loss of generality, we can assume the initial $U(0, x) = 0$. For $1 \leq i, j \leq 3, m, n \geq 1$, let

$$\rho_{ij, mn}(t) := \sup_{s \geq 0} \sup \{|P(A \cap B) - P(A)P(B)| : A \in \sigma(U^i(s_1, x_1), \ldots, U^i(s_m, x_m)), s_k \leq s, x_k \in R^3, 1 \leq k \leq m, B \in \sigma(U^j(t_1, y_1), \ldots, U^j(t_n, y_n)), t_l \geq t + s, y_l \in R^3, 1 \leq l \leq n \}.$$  

(5.1)

Assume that the spectral measure matrix is in class $\mathbb{M}(q, 0), q \leq -4$, then for $\alpha > 1/2$, \exists K, K does not depend on $t, s$, such that

$$\int_0^\infty \rho_{ij, mn}(t)^\alpha dt < K < \infty.$$  

(5.2)

Proof. Let $f_m(x_1, x_2, \ldots, x_m)$ be the density function for $U^i(s_k, x_k), 1 \leq k \leq m$, $f_n(y_1, y_2, \ldots, y_n)$ be the density function of $U^j(t_l, y_l), 1 \leq l \leq n$ and $f_{mn}(x, y)$ be the joint density function for $U^i(s_k, x_k)$ and $U^j(t_l, y_l)$. Since $\{U(t, x), t \in R^+, x \in R^3\}$ is a Gaussian random field, we thus have:

$$f_m(x_1, \ldots, x_m) = \frac{1}{(2\pi)^{m/2}\Sigma_{mm}^{1/2}} \exp(-\frac{1}{2}(x_1, \ldots, x_m)\Sigma_{mm}^{-1}(x_1, \ldots, x_m)^T),$$

$$f_n(y_1, \ldots, y_n) = \frac{1}{(2\pi)^{n/2}\Sigma_{nn}^{1/2}} \exp(-\frac{1}{2}(y_1, \ldots, y_n)\Sigma_{nn}^{-1}(y_1, \ldots, y_n)^T),$$

and

$$f(x_1, \ldots, x_m, y_1, \ldots, y_n) = \frac{1}{(2\pi)^{(m+n)/2}\Sigma^{1/2}} \exp(-\frac{1}{2}(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

$$\Sigma^{-1}(x_1, \ldots, x_m, y_1, \ldots, y_n)^T).$$

Here, $\Sigma_{mm}$ is the covariance matrix for $\{U^i(s_k, x_k), 1 \leq k \leq m\}$, $\Sigma_{nn}$ is the covariance matrix for $\{U^j(t_l, y_l), 1 \leq l \leq n\}$, $\Sigma_{mn}$ is the covariance matrix for $\{U^i(s_k, x_k), 1 \leq k \leq m\}$ and $\Sigma_{nm}$ is the covariance matrix for $\{U^j(t_l, y_l), 1 \leq l \leq n\}$.
\[ k \leq m \} \text{ and } \{ U^j(t_l, y_l), 1 \leq l \leq n \} \), and \( \Sigma \) is the covariance matrix for \( \{ U^j(s_k, x_k), 1 \leq k \leq m, U^j(t_l, y_l), 1 \leq l \leq n \} \). We have

\[ \Sigma_{mn} = \Sigma'_{nm}, \quad \Sigma = \begin{pmatrix} \Sigma_{mm} & \Sigma_{mn} \\ \Sigma_{nm} & \Sigma_{nn} \end{pmatrix}. \]

Also, let

\[ \Sigma_{mm} = (\sigma_{mm}^{k_1k_2})_{m \times m}, \quad \Sigma_{nn} = (\sigma_{nn}^{l_1l_2})_{n \times n}, \quad \Sigma_{mn} = (\sigma_{mn}^{k_l})_{m \times n}, \]

and

\[ (\Sigma_{mm})^{-1} = (\sigma_{mm}^{k_1k_2})_{m \times m}, \quad (\Sigma_{nn})^{-1} = (\sigma_{nn}^{l_1l_2})_{n \times n}. \]

Notice that \( \Sigma \) is uniformly bounded with respect to \( t \) and \( s \). Also, as \( t \to \infty \),

\[ \lim_{t \to \infty} \Sigma_{mn} = \lim_{t \to \infty} \Sigma_{mn}(t) = 0. \]

We have

\[
\begin{align*}
& f(x_1, \ldots, x_m, y_1, \ldots, y_n) - f_m(x_1, \ldots, x_m)f_n(y_1, \ldots, y_n) \\
& = \frac{1}{2\pi^{(m+n)/2} |\Sigma|^{1/2}} \exp \left( -\frac{1}{2}(x_1, \ldots, x_m, y_1, \ldots, y_n)^T \Sigma^{-1}(x_1, \ldots, x_m, y_1, \ldots, y_n)^T \right) \\
& - \frac{1}{2\pi^{(m+n)/2} |\Sigma_{mm}|^{1/2} |\Sigma_{nn}|^{1/2}} \exp \left( -\frac{1}{2}(x_1, \ldots, x_m)^T \Sigma_{mm}^{-1}(x_1, \ldots, x_m)^T \\
& + (y_1, \ldots, y_n)^T \Sigma_{nn}^{-1}(y_1, \ldots, y_n)^T \right) \\
& + \frac{\left(\frac{1}{2}(x_1, \ldots, x_m, y_1, \ldots, y_n)^T \Sigma_{mm}^{-1}(x_1, \ldots, x_m, y_1, \ldots, y_n)^T \right)}{2\pi^{(m+n)/2} |\Sigma_{mm}|^{1/2} |\Sigma_{nn}|^{1/2}} \exp \left( -\frac{1}{2}(x_1, \ldots, x_m, y_1, \ldots, y_n)^T \Sigma_{mm}^{-1}(x_1, \ldots, x_m, y_1, \ldots, y_n)^T \right). \\
& \text{where} \\
& I_1 = \frac{1}{2\pi^{(m+n)/2} \left( |\Sigma|^{1/2} - |\Sigma_{mm}|^{1/2} |\Sigma_{nn}|^{1/2} \right)}, \\
& I_2 = 1 - \exp \left( -\frac{1}{2}(x_1, \ldots, x_m)^T \Sigma_{mm}^{-1}(x_1, \ldots, x_m)^T + (y_1, \ldots, y_n)^T \Sigma_{nn}^{-1}(y_1, \ldots, y_n)^T \\
& - (x_1, \ldots, x_m, y_1, \ldots, y_n)^T \Sigma_{mm}^{-1}(x_1, \ldots, x_m, y_1, \ldots, y_n)^T \right). \\
& \text{and}
\end{align*}
\]
By direct computation,

\[
|\Sigma| = \begin{vmatrix}
\Sigma_{mm} & \Sigma_{mn} \\
\Sigma_{nm} & \Sigma_{nn}
\end{vmatrix}\]

\[
= |\Sigma_{mm}||\Sigma_{nn}|I - \Sigma_{mm}^{-1}\Sigma_{mn}\Sigma_{nn}^{-1}\Sigma_{nm}|
\]

and for \(1 \leq k_1, k_5 \leq m\), the \(k_1k_5\)-th entry for \(\Sigma_{mm}^{-1}\Sigma_{mn}\Sigma_{nn}^{-1}\Sigma_{nm}\) satisfies

\[
(\Sigma_{mm}^{-1}\Sigma_{mn}\Sigma_{nn}^{-1}\Sigma_{nm})^{k_1k_5} = \sum_{k_2=1}^{m} \sum_{k_3=1}^{n} \sum_{k_4=1}^{n} a_{k_1k_2} b_{k_2k_3} c_{k_3k_4} d_{k_4k_5}
\]

\[
= o(\sum_{k=1}^{m} \sum_{l=1}^{n} (\sigma_{mn}^{kl}(t))^2).
\]

Note here

\[
(\Sigma_{mm}^{-1}\Sigma_{mn}\Sigma_{nn}^{-1}\Sigma_{nm})^{k_1k_5} = o(\sum_{k=1}^{m} \sum_{l=1}^{n} (\sigma_{mn}^{kl}(t))^2)
\]

means that

\[
\lim_{t \to \infty} \frac{(\Sigma_{mm}^{-1}\Sigma_{mn}\Sigma_{nn}^{-1}\Sigma_{nm})^{k_1k_5}}{\sum_{k=1}^{m} \sum_{l=1}^{n} (\sigma_{mn}^{kl}(t))^2} = 0.
\]

Thus, for \(I_1\), we have

\[
I_1 = \frac{1}{2\pi^{(m+n)/2} |\Sigma_{mm}|^{1/2}|\Sigma_{nn}|^{1/2}} \frac{1}{\sqrt{1 + \sum_{k=1}^{m} \sum_{l=1}^{n} o(\sigma_{mn}^{kl}(t))}} - 1
\]

\[
= o(\sum_{k=1}^{m} \sum_{l=1}^{n} (\sigma_{mn}^{kl}(t))^2).
\]

By section 11 in chapter 1 of Magnus [22],

\[
\Sigma^{-1} = \begin{pmatrix}
\Sigma_{mm}^{-1} + \Sigma_{mm}^{-1}\Sigma_{mn}D^{-1}\Sigma_{nm}\Sigma_{mm}^{-1} - \Sigma_{mm}^{-1}\Sigma_{mn}D^{-1} \\
-D^{-1}\Sigma_{mm}\Sigma_{mm}^{-1} & D^{-1}
\end{pmatrix},
\]

where \(D = \Sigma_{nn} - \Sigma_{nn}^{-1}\Sigma_{mm}^{-1}\Sigma_{mm}\). Also, by theorem 3 of chapter 8 of Magnus [22],

\[
D^{-1} = \Sigma_{nn}^{-1} - \Sigma_{nn}^{-1}\Sigma_{mm}^{-1}\Sigma_{mm}^{-1}\Sigma_{mm}^{-1}.
\]
Hence, by direct calculations, we have

\[-\frac{1}{2}((x_1, \ldots, x_m)\Sigma_{mm}^{-1}(x_1, \ldots, x_m)^T + (y_1, \ldots, y_n)\Sigma_{nn}^{-1}(y_1, \ldots, y_n)^T - (x_1, \ldots, x_m, y_1, \ldots, y_n)\Sigma^{-1}(x_1, \ldots, x_m, y_1, \ldots, y_n)^T)\]

\[= o\left(\sum_{k=1}^{m} \sum_{l=1}^{n} (\sigma_{mn}^{kl}(t))^2\right).\]

Therefore,

\[I_2 = 1 - \exp(o\left(\sum_{k=1}^{m} \sum_{l=1}^{n} (\sigma_{mn}^{kl}(t))^2\right))\]

\[= o\left(\sum_{k=1}^{m} \sum_{l=1}^{n} (\sigma_{mn}^{kl}(t))^2\right).\]

Hence,

\[\int_{R^3} \cdots \int_{R^3} |f(x_1, \ldots, x_m, y_1, \ldots, y_m) - f_m(x_1, \ldots, x_m)f_n(y_1, \ldots, y_m)| dx_1 \ldots dx_m dy_1 \ldots dy_n \]

\[= o\left(\sum_{k=1}^{m} \sum_{l=1}^{n} (\sigma_{mn}^{kl}(t))^2\right).\]

Since

\[\rho_{ij,mn}(t) \leq \int_{R^3} \cdots \int_{R^3} |f(x_1, \ldots, x_m, y_1, \ldots, y_m) - f_m(x_1, \ldots, x_m)f_n(y_1, \ldots, y_m)| dx_1 \ldots dx_m dy_1 \ldots dy_n.
\]

we have \(\rho_{ij,mn}(t) = o\left(\sum_{k=1}^{m} \sum_{l=1}^{n} (\sigma_{mn}^{kl}(t))^2\right).\)

Now, let \(h(t) = \max_{1 \leq k \leq m, 1 \leq l \leq n} \int_{R^3} \frac{e^{-4\pi^2|x|^2t}}{4\pi^2|x|^2} (P_\xi \star \mathcal{M})^{kl}(d\xi), t > 0.\) Since

\[\sigma_{mn}^{kl} = \int_{0}^{s_k} \int_{R^3} e^{2\pi i \xi (x_1 - y_1)} e^{-4\pi^2 |\xi|^2(t_l + s_k - 2r)} (P_\xi \star \mathcal{M})^{kl}(d\xi) dr\]

\[= \int_{R^3} e^{2\pi i \xi (x_1 - y_1)} e^{-4\pi^2 |\xi|^2(t_l + s_k)} - e^{-4\pi^2 |\xi|^2(t_l + s_k)} \frac{4\pi^2 |\xi|^2}{4\pi^2 |\xi|^2} (P_\xi \star \mathcal{M})^{kl}(d\xi),\]

we have for every \(1 \leq k \leq m, 1 \leq l \leq n,\)

\[\frac{\sigma_{mn}^{kl}(t)}{h(t)} = O(1), \quad \text{i.e.,} \quad \lim_{t \to \infty} \frac{\sigma_{mn}^{kl}(t)}{h(t)} \quad \text{is finite.}\]
Hence,
\[
\lim_{t \to \infty} \frac{\rho_{ij,mn}(t)}{h^2(t)} = 0.
\] (5.3)

Therefore, for $\alpha > 1/2$, $\exists K$, such that
\[
\int_0^\infty h(t)^{2\alpha} dt \leq \max_{1 \leq k \leq m, 1 \leq l \leq n} \int_{R^3} \frac{1}{32 \pi^4 |\xi|^4} (P_\xi * \mathcal{M})^{kl}(d\xi)
\]
\[
< K < \infty,
\]
by ratio test, (5.2) follows. \qed

Now, let us introduce the definition of the strong mixing rate and then prove some useful estimates on the strong mixing rate, which will be used to prove that \{\(z_\lambda(t)\)\}_{\lambda > 0} is tight.

**Definition 5.2.** The strong mixing rate $\rho$ associate with $G_{s,t} := \sigma(U(s', \cdot) : s \leq s' \leq t)$ is defined as
\[
\rho(t) = \sup_{s \geq 0} \sup \{|P(A \cap B) - P(A)P(B)| : A \in G_{0,s}, B \in G_{s+t,\infty}\}.
\] (5.4)

**Lemma 5.3.** Assume that the spectral measure matrix is in class $\mathcal{M}(q,0), q \leq -4$, then for $\alpha > 1/2$,
\[
\int_0^\infty \rho(s)^\alpha ds < \infty.
\] (5.5)

**Proof.** First, let us prove for fixed $B \in \sigma(U^3(t_1,y_1), \ldots, U^3(t_n,y_n)), t_i \geq t + s, y_l \in R^3, 1 \leq l \leq n, 1 \leq j \leq 3$, we have that $\forall A \in G_{0,s}$,
\[
\int_0^\infty (\rho^A_{ij}(t))^\alpha dt < K < \infty,
\] (5.6)
where $\rho_j^{AB}(t) := |P(A \cap B) - P(A)P(B)|$, and $K$ is a constant which does not depend on $s$.

Let

$$\mathcal{H} = \{ A : A \subset G_{0,s}, \int_0^\infty \rho_j^{AB}(t) \, dt < K < \infty \}. $$

Also, let

$$\mathcal{L} = \{ A : A \in \sigma(U^i(s_1, x_1), \ldots, U^i(s_m, x_m), s_k \leq s, x_k \in \mathbb{R}^3, 1 \leq k \leq m, m \geq 1, 1 \leq i \leq 3 \}. $$

By lemma 5.1, $\mathcal{L} \subset \mathcal{H}$. Clearly, $\mathcal{L}$ is an algebra. Also, since $\sigma(U^i(s_k, x_k)) \subset \sigma(\mathcal{L}), \forall s_k \leq s, x_k \in \mathbb{R}^3, 1 \leq i \leq 3$, we have $G_{0,s} = \sigma(\mathcal{L})$. Thus, by monotone class theorem, $\mathcal{H} = G_{0,s}$.

Hence for fixed $B \in \sigma(U^j(t_1, y_1), \ldots, U^j(t_n, y_n)), t_i \geq t + s, y_l \in \mathbb{R}^3, 1 \leq l \leq n, 1 \leq j \leq 3, \forall A \in G_{0,s}$, we have

$$\int_0^\infty (\rho_j^{AB}(t) \alpha) \, dt < K < \infty, \quad (5.7)$$

where $\rho_j^{AB}(t) := |P(A \cap B) - P(A)P(B)|$, and $K$ is a constant which does not depend on $s$.

By applying monotone class theorem again, we thus have $\forall A \in G_{0,s}, \forall B \in G_{s+t, \infty},$

$$\int_0^\infty (\rho_j^{AB}(t) \alpha) \, dt < K < \infty. \quad (5.8)$$

i.e.

$$\int_0^\infty \rho(s) \alpha \, ds < \infty. \quad (5.9)$$

This completes the proof of the lemma. \hfill \Box

**Lemma 5.4.** If $X \in G_{0,s}$ and $Y \in G_{s+t, \infty}$ with $|X| \leq C_1$ and $|Y| \leq C_2$. Then

$$|E(XY) - EXEY| \leq 4C_1C_2 \rho(t). \quad (5.10)$$
Furthermore, let $t < s$ and $X,Y$ be $G_{s,s}$-measurable and $G_{t,t}$-measurable random variables respectively such that $E|X|^{2q} < \infty, E|Y|^{2q} < \infty$, where $q > 1$. Then we have:

$$|E(XY) - EXEY| \leq K\rho(s-t)^{1-\frac{1}{q}}, \quad (5.11)$$

Here $K$ is some constant depending on $E|X|^{2q}$ and $E|Y|^{2q}$.

Proof. These are standard results for mixing conditions. See, for instance, Ibragimov [8]. For completeness, we include the proofs. Let $\xi = \text{sgn}(E(Y|G_0,s) - EY)$. We have:

$$|E(XY) - EXEY| = |E\{X(E(Y|G_0,s) - EY)\}|$$

$$\leq C_1E|E(Y|G_0,s) - EY|$$

$$= C_1E\{\xi(E(Y|G_0,s) - EY)\}$$

$$= C_1E\{\xi(E(Y|G_0,s)) - E\xi EY\}$$

$$= C_1(E(\xi Y) - E\xi EY),$$

since $\xi$ is measurable with respect to $G_{0,s}$. Now, let $\eta = \text{sgn}(E(\xi|G_{s+t,\infty}) - E\xi)$, similar to the above argument, we have:

$$|E(XY) - EXEY| \leq C_1C_2|E(\xi) - E\xi E\eta|.$$

Let $A = \{\xi = 1\}, B = \{\eta = 1\}$. The strong mixing condition (5.4) gives:

$$|E(\xi) - E\xi E\eta|$$

$$= |P(AB) + P(A^cB^c) - P(A^cB) - P(AB^c) - (P(A) - P(A^c))(P(B) - P(B^c))|$$

$$\leq 4\rho(t)$$

(5.10) is thus proved.
Next, let us prove (5.11). For mathematical simplicity, but without loss of generality, we may assume that \( \rho(t) > 0, \forall t \geq 0 \). For \( N \geq 1 \), let

\[
X_N = XI(|X| \leq N), \quad X'_N = X - X_N,
\]

\[
Y_N = YI(|Y| \leq N), \quad Y'_N = Y - Y_N.
\]

Then,

\[
|E(XY) - EXEY| = |E\{(X_N + X')(Y_N + Y') - (EX_N + EX')(EY_N + EY')\}|
\]

\[
\leq |E(X_NY_N) - EX_NEY_N| + |E(X_NY'_N) - EX_NEY'_N| + |E(X'Y_N) - EX'EY_N| + |E(X'Y'_N) - EX'EY'_N|
\]

\[
=: I_1 + I_2 + I_3 + I_4.
\]

By part (1) of this lemma,

\[
I_1 \leq 4N^2 \rho(s - t). \tag{5.12}
\]

Now, let us estimate \( I_2, I_3, I_4 \). By Holder’s inequality, \( E|X_N| \leq (E|X|^{2q})^{1/2q} \) and \( E|Y_N| \leq (E|Y|^{2q})^{1/2q} \). Also we have:

\[
E|X'| \leq N^{-(2q-2)}E|X'|^{1+2q-2}
\]

\[
\leq N^{-(2q-2)}(E|X'|^{(2q-1)\frac{2q}{2q-1}})^{\frac{2q-1}{2q}}
\]

\[
\leq N^{-(2q-2)}(E|X'|^{2q})^{1-\frac{1}{q}}
\]

\[
\leq N^{2-2q}(E|X|^{2q})^{1-\frac{1}{q}}.
\]

Similarly,

\[
E|Y'| \leq N^{2-2q}(E|Y|^{2q})^{1-\frac{1}{q}}.
\]
Furthermore, we have:

\[
E|X'Y_N| \leq (E|X'|^{2q})^{1-\frac{1}{2q}} (E|Y_N|^{2q})^{\frac{1}{2q}}
\]

\[
\leq \left( \frac{E|X'|^{2q} + 2q - 1 - \frac{1}{2q-1}}{N^{2q - 1 - \frac{1}{2q-1}}} \right) (E|Y'|^{2q})^{\frac{1}{2q}}
\]

\[
\leq (N^{\frac{(2-2q)2q}{2q-1}})^{1-\frac{1}{2q}} (E|X'|^{2q})^{1-\frac{1}{2q}} (E|Y'|^{2q})^{\frac{1}{2q}}
\]

\[
\leq N^{2-2q} (E|X'|^{2q})^{1-\frac{1}{2q}} (E|Y'|^{2q})^{\frac{1}{2q}}.
\]

Similarly, we have:

\[
E|X_NY'| \leq N^{2-2q} (E|X|^{2q})^{\frac{1}{2q}} (E|Y|^{2q})^{1-\frac{1}{2q}},
\]

\[
E|X'Y'| \leq N^{2-2q} (E|X|^{2q})^{1-\frac{1}{2q}} (E|Y|^{2q})^{\frac{1}{2q}}.
\]

Therefore,

\[
I_2 \leq E|X_NY'| + E|X_N|E|Y'|
\]

\[
\leq N^{2-2q} (E|X|^{2q})^{1-\frac{1}{2q}} (E|Y|^{2q})^{\frac{1}{2q}} + (E|X|^{2q})^{1/2q} N^{2-2q} (E|Y|^{2q})^{1-\frac{1}{q}}
\]

\[
\leq N^{2-2q} \{ (E|X|^{2q})^{1-\frac{1}{2q}} (E|Y|^{2q})^{\frac{1}{2q}} + (E|X|^{2q})^{1/2q} (E|Y|^{2q})^{1-\frac{1}{q}} \}.
\]

\[
I_3 \leq E|X'Y_N| + E|X'|E|Y_N|
\]

\[
\leq N^{2-2q} (E|X|^{2q})^{1-\frac{1}{2q}} (E|Y|^{2q})^{\frac{1}{2q}} + N^{2-2q} (E|X|^{2q})^{1-\frac{1}{q}} (E|Y|^{2q})^{1/2q}
\]

\[
\leq N^{2-2q} \{ (E|X|^{2q})^{1-\frac{1}{2q}} (E|Y|^{2q})^{\frac{1}{2q}} + (E|X|^{2q})^{1-\frac{1}{q}} (E|Y|^{2q})^{1/2q} \}.
\]

\[
I_4 \leq E|X'Y'| + E|X'|E|Y'|
\]

\[
\leq N^{2-2q} (E|X|^{2q})^{1-\frac{1}{2q}} (E|Y|^{2q})^{\frac{1}{2q}} + N^{2-2q} (E|X|^{2q})^{1-\frac{1}{q}} N^{2-2q} (E|Y|^{2q})^{1-\frac{1}{q}}
\]

\[
\leq N^{2-2q} (E|X|^{2q})^{1-\frac{1}{2q}} (E|Y|^{2q})^{\frac{1}{2q}} + N^{2-2q} (E|X|^{2q})^{1-\frac{1}{q}} (E|Y|^{2q})^{1-\frac{1}{q}}
\]

\[
\leq N^{2-2q} \{ (E|X|^{2q})^{1-\frac{1}{2q}} (E|Y|^{2q})^{\frac{1}{2q}} + (E|X|^{2q})^{1-\frac{1}{q}} (E|Y|^{2q})^{1-\frac{1}{q}} \}.
\]
Hence,

\[ |E(XY) - EXEY| \]
\[ \leq I_1 + I_2 + I_3 + I_4 \]
\[ \leq 4N^2 \rho(s - t) + N^{2-2q} \left\{ (|X|^{2q})^{1-\frac{1}{2q}}(|Y|^{2q})^{1-\frac{1}{2q}} + (|X|^{2q})^{1-\frac{1}{2q}}(|Y|^{2q})^{1-\frac{1}{2q}} \right\} + N^{2-2q} \left\{ (|X|^{2q})^{1-\frac{1}{2q}}(|Y|^{2q})^{1-\frac{1}{2q}} + (|X|^{2q})^{1-\frac{1}{2q}}(|Y|^{2q})^{1-\frac{1}{2q}} \right\}.

Let \( N = \rho(s - t)^{-\frac{1}{2q}} \), then

\[ N^2 \rho(s - t) = \rho(s - t)^{-\frac{1}{2q}} \cdot \rho(s - t) = \rho(s - t)^{1-\frac{1}{2q}}. \]

Also,

\[ N^{2-2q} = (\rho(s - t)^{-\frac{1}{2q}})^{2-2q} = \rho(s - t)^{1-\frac{1}{2q}}. \]

Hence, we have:

\[ |E(XY) - EXEY| \leq K \rho(s - t)^{1-\frac{1}{2q}}, \tag{5.13} \]

where \( K \) is a constant depending on \( E|X|^{2q} \) and \( E|Y|^{2q} \). Lemma is proved. \( \square \)

The following is a key lemma for proving the tightness. It also will be used to compute the drift and diffusion coefficients.

**Lemma 5.5.** Let \( T \) be any positive number and \( p_1 > 1 \). Suppose that the spectral measure matrix is in class \( \mathcal{M}(q_1, q_2), q_1 \leq -4, q_2 \geq 2 \). For \( 1 \leq i, j \leq 3 \), and multi-index \( \alpha, \beta \) such that \( |\alpha| \leq 1, |\beta| \leq 1 \), we have:

\[ \sup_{x \in \mathbb{R}^3, \lambda > 0, T \geq t \geq \sigma > 0} \lambda \int_t^\sigma \left\{ E\left| \frac{\partial^\alpha}{\partial x^\alpha} U^i(\lambda s, \sqrt{\lambda} x) \frac{\partial^\beta}{\partial x^\beta} U^j(\lambda \sigma, \sqrt{\lambda} x) |x|^{p_1} \right|^q \right\}^{1/q} ds \leq C, \tag{5.14} \]

where \( C \) is some constant only depending on \( p_1, T \).
Proof. For any $s > 0$, $\lambda > 0$, $1 \leq k \leq 3$, multi-index $\gamma$ with $|\gamma| \leq 1$, we have:

$$
E|\frac{\partial^\gamma}{\partial x}U^k(\lambda s, \sqrt{\lambda}x)|^{2p_1} \leq K_1(E|\frac{\partial^\gamma}{\partial x}U^k(\lambda s, \sqrt{\lambda}x)|^{2p_1} \leq K_2(\int_0^\lambda \int_{R^3} e^{-8\pi^2|\xi|^2(\lambda s-r)} \xi^{2\gamma}(P_\xi * M)^{kk}(d\xi)dr)^{p_1} \leq K_3(\int_{R^3} \xi^{2\gamma}|\mu|(d\xi))^{p_1} \leq K_4,
$$

(5.15)

where $K_4$ is a constant which only depends on $p_1$. By lemma 5.4, we have:

$$
\lambda\{E|\frac{\partial^\gamma}{\partial x}U^i(\lambda s, \sqrt{\lambda}x)\frac{\partial^\beta}{\partial x}U^j(\lambda \sigma, \sqrt{\lambda}x)|^{p_1}\}^{1/p_1} \leq K_5\lambda_\rho(\lambda(s - \sigma))^{1/p_2},
$$

where $p_2$ is $p_1$’s conjugate, i.e. $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Since

$$
\int_\sigma^t \rho(\lambda(s - \sigma))^{1/p_2}dl \leq \frac{1}{\lambda} \int_0^\infty \rho(s)^{1/p_2}ds,
$$

by (5.9), we have:

$$
\sup_{x \in R^3, \lambda > 0, T \geq t \geq \sigma > 0, \lambda} \int_\sigma^t \{E|\frac{\partial^\alpha}{\partial x}U^i(\lambda s, \sqrt{\lambda}x)\frac{\partial^\beta}{\partial x}U^j(\lambda \sigma, \sqrt{\lambda}x)|^{p_1}\}^{1/p_1}ds
\leq K_5 \int_0^\infty \rho(s)^{1/p_2}ds
\leq C,
$$

where $C$ is some constant only depending on $p_1, T$. Lemma is proved.

We are ready to prove the tightness now.

Lemma 5.6 (Tightness). Let

$$
z_\lambda(t) = x_0 + \sqrt{\lambda} \int_0^t U(\lambda s, \sqrt{\lambda}z_\lambda(s))ds,
$$

(5.17)

$0 \leq t \leq T$, $T$ is some arbitrary positive number. If we assume that the spectral measure matrix is in class $M(q_1, q_2), q_1 \leq -4, q_2 \geq 2$, then $\{z_\lambda(t)\}_{\lambda > 0}$ is tight.
Proof. First, by theorem 4.6, there exist a unique solution to (5.17). It suffices to show that there exist constants \(C\) such that

\[
E|z_\lambda(t) - z_\lambda(s)|^4 \leq C|t - s|^2.
\] (5.18)

where \(0 \leq s, t \leq T\) and \(C\) only depends on \(T\).

Notice that (5.18) is equivalent to: for any \(1 \leq k \leq 3\),

\[
E|z_\lambda^k(t) - z_\lambda^k(s)|^4 \leq C_k|t - s|^2.
\] (5.19)

where \(0 \leq s, t \leq T\) and \(C_k\) only depends on \(T\). We proceed by proving (5.19).

Since

\[
z_\lambda(t) - z_\lambda(s) = \sqrt{\lambda} \int_s^t \sqrt{\lambda} z_\lambda(s')ds',
\] (5.20)

we thus have:

\[
E|z_\lambda^k(t) - z_\lambda^k(s)|^4
\]

\[= E(\sqrt{\lambda} \int_s^t \lambda^k(\lambda s', \sqrt{\lambda}z_\lambda(s'))ds')^4\]

\[= \lambda^2 E(\int_s^t \lambda^{k'}(\lambda s', \sqrt{\lambda}z_\lambda(s'))ds')^4 \int_s^t \lambda^{k''}(\lambda s', \sqrt{\lambda}z_\lambda(s'))ds' \int_s^t \lambda^{k'''}(\lambda s', \sqrt{\lambda}z_\lambda(s'))ds''
\]

\[\leq \lambda \int_s^t \int_s^t \int_s^t \int_s^t E([U^k(\lambda s'_1, \sqrt{\lambda}z_\lambda(s'_1))U^k(\lambda s''_1, \sqrt{\lambda}z_\lambda(s''_1))])^2 \cdot
\]

\[\cdot E([U^k(\lambda s'_2, \sqrt{\lambda}z_\lambda(s'_2))U^k(\lambda s''_2, \sqrt{\lambda}z_\lambda(s''_2))])^2 \int_s^t \int_s^t \int_s^t \int_s^t d\lambda s'_1 ds''_1 d\lambda s''_1 d\lambda s''_2 d\lambda s''_2
\]

\[= \lambda \int_s^t \int_s^t \int_s^t \int_s^t [E(U^k(\lambda s'_1, \sqrt{\lambda}z_\lambda(s'_1))U^k(\lambda s''_1, \sqrt{\lambda}z_\lambda(s''_1))])^2 \cdot
\]

\[\cdot E(U^k(\lambda s'_2, \sqrt{\lambda}z_\lambda(s'_2))U^k(\lambda s''_2, \sqrt{\lambda}z_\lambda(s''_2))])^2 \int_s^t \int_s^t d\lambda s'_1 ds''_1 d\lambda s''_1 d\lambda s''_2 d\lambda s''_2.
\]
By lemma 5.5,
\[
\lambda \int_s^t \int_s^{s'} \left[ E(U^k(\lambda s'_1, \sqrt{\lambda} z_\lambda(s'_1))) U^k(\lambda s''_1, \sqrt{\lambda} z_\lambda(s''_1))) \right]^{1/2} ds'_1 ds''_1 = \lambda \int_s^t \int_s^{s'} \left[ E(U^k(\lambda s'_1, \sqrt{\lambda} z_\lambda(s'_1))) U^k(\lambda s''_1, \sqrt{\lambda} z_\lambda(s''_1))) \right]^{1/2} ds''_1 ds'_1 \leq C_1 |t - s|,
\]
where \(C_1\) is some constant only depending on \(T\). Similarly, we have:
\[
\lambda \int_s^t \int_s^{s'} \left[ E(U^k(\lambda s'_2, \sqrt{\lambda} z_\lambda(s'_2))) U^k(\lambda s''_2, \sqrt{\lambda} z_\lambda(s''_2))) \right]^{1/2} ds''_2 ds'_2 \leq C_2 |t - s|.
\]
Therefore,
\[
E|z^k_\lambda(t) - z^k_\lambda(s)|^2 \leq C_k |t - s|^2, \tag{5.21}
\]
where \(C_k\) is some constant only depending on \(T\). Lemma is proved.

Next, we need to prove that every convergent subsequence has the same limiting distribution. We will show that the limiting distribution has the same mean and variance.

**Lemma 5.7.** Assume that the spectral measure matrix is in class \(M(q_1, q_2)\), where \(q_1 \leq -4\) and \(q_2 \geq 2\). Also assume that for any \(1 \leq i, j \leq 3\), the Fourier transform of the measure \(\frac{1}{|\xi|^4} \mu^{ij}(d\xi)\) goes to 0 in the infinity, i.e.
\[
\lim_{|x| \to \infty} \int_{R^3} e^{-2\pi x \cdot \xi} \frac{1}{|\xi|^4} \mu^{ij}(d\xi) = 0. \tag{5.22}
\]
Let \(1 \leq i \leq 3\) and \(s < t\). Set
\[
A^{ij}_\lambda(t, s, x, y) = \lambda E[\int_s^t U^i(\lambda s', \sqrt{\lambda} x) U^j(\lambda s, \sqrt{\lambda} y) ds'|G_{0, \lambda s}] \tag{5.23}
\]
then
\[
\lim_{\lambda \to \infty} \int_s^t E[A^{ij}_\lambda(t, s', x, y)|G_{0, \lambda s}] ds' = \int_s^t A^{ij}(s', x, y) ds', \tag{5.24}
\]
where
\[
\lambda \int_s^t \int_s^{s'} \left[ E(U^k(\lambda s'_1, \sqrt{\lambda} z_\lambda(s'_1))) U^k(\lambda s''_1, \sqrt{\lambda} z_\lambda(s''_1))) \right]^{1/2} ds'_1 ds''_1 = \lambda \int_s^t \int_s^{s'} \left[ E(U^k(\lambda s'_2, \sqrt{\lambda} z_\lambda(s'_2))) U^k(\lambda s''_2, \sqrt{\lambda} z_\lambda(s''_2))) \right]^{1/2} ds''_2 ds'_1 \leq C_1 |t - s|,
\]
where \(C_1\) is some constant only depending on \(T\). Similarly, we have:
\[
\lambda \int_s^t \int_s^{s'} \left[ E(U^k(\lambda s'_2, \sqrt{\lambda} z_\lambda(s'_2))) U^k(\lambda s''_2, \sqrt{\lambda} z_\lambda(s''_2))) \right]^{1/2} ds''_2 ds'_2 \leq C_2 |t - s|.
\]
Therefore,
\[
E|z^k_\lambda(t) - z^k_\lambda(s)|^2 \leq C_k |t - s|^2, \tag{5.21}
\]
where \(C_k\) is some constant only depending on \(T\). Lemma is proved.
where

\[ A_{ij}(t, x, y) := \lim_{\lambda \to \infty} \lambda^{-3/2} \int_t^{t+1/\sqrt{\lambda}} d\tau \int_t^\tau d\sigma E(U_i(\lambda\sigma, \sqrt{\lambda}x)U_j(\lambda\tau, \sqrt{\lambda}y)) \]  

(5.25)

\[ = \begin{cases} 
\int_{R^3} \frac{1}{8\pi^2 |\xi|^2} (P_\xi \ast M)^{ij}(d\xi) & \text{if } x = y \\
0 & \text{if } x \neq y
\end{cases}. \]  

(5.26)

The convergence in (5.24) is in $L_1$ and uniform with respect to spatial variable $x, y \in R^3, 0 \leq s < t \leq T$; while the convergence in (5.25) is uniform with respect to spatial variable $x, y \in R^3$ and temporal variable $t \in [0, T]$.

Proof. When $\sigma \leq \tau$, we have:

\[ E(U_i(\lambda\sigma, \sqrt{\lambda}x)U_j(\lambda\tau, \sqrt{\lambda}y)) \]

\[ = \int_{\lambda\sigma}^{\tau} \int_{R^3} e^{2\pi i \xi \cdot \sqrt{\lambda}(y-x)} e^{-4\pi^2|\xi|^2(\lambda\sigma+\lambda\tau-2\tau)}(P_\xi \ast M)^{ij}(d\xi)dr \\
= \int_{R^3} e^{2\pi i \xi \cdot \sqrt{\lambda}(y-x)} \cdot \frac{1}{8\pi^2 |\xi|^2}(e^{-4\pi^2|\xi|^2\lambda(\tau-\sigma)} - e^{-4\pi^2|\xi|^2\lambda(\tau+\sigma)})(P_\xi \ast M)^{ij}(d\xi). \]

Since

\[ \int_t^\tau e^{-4\pi^2|\xi|^2\lambda(\tau-\sigma)}d\sigma = \frac{1}{4\pi^2 |\xi|^2 \lambda(1 - e^{-4\pi^2|\xi|^2\lambda(\tau-t)})}, \]

we have

\[ \int_t^{t+1/\sqrt{\lambda}} \int_t^\tau e^{-4\pi^2|\xi|^2\lambda(\tau-\sigma)}d\sigma d\tau = \frac{1}{4\pi^2 |\xi|^2 \lambda^{3/2}} - \frac{1}{4\pi^2 |\xi|^2 \lambda} \int_t^{t+1/\sqrt{\lambda}} e^{-4\pi^2|\xi|^2\lambda(\tau-t)}d\tau. \]
Thus,
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda^{3/2}} \int_t^{t+1/\sqrt{\lambda}} d\tau \int_{t}^{\tau} E(U^i(\lambda \sigma, \sqrt{\lambda} x)U^j(\lambda \tau, \sqrt{\lambda} y))d\sigma
\]
\[
= \lim_{\lambda \to \infty} \frac{1}{\lambda^{3/2}} \int_t^{t+1/\sqrt{\lambda}} d\tau \int_{t}^{\tau} \int_{\mathbb{R}^3} e^{2\pi i \xi \cdot \sqrt{\lambda}(y-x)} \cdot \frac{1}{8\pi^2 |\xi|^2} \cdot (e^{-4\pi^2 |\xi|^2 (\tau-\sigma)} - e^{-4\pi^2 |\xi|^2 (\tau+\sigma)})(P_\xi \ast M)^{ij}(d\xi)d\sigma
\]
\[
= \lim_{\lambda \to \infty} \int_{\mathbb{R}^3} \frac{e^{2\pi i \xi \cdot \sqrt{\lambda}(y-x)}}{32\pi^4 |\xi|^4} (P_\xi \ast M)^{ij}(d\xi)
\]
\[
- \lim_{\lambda \to \infty} \sqrt{\lambda} \int_t^{t+1/\sqrt{\lambda}} \int_{\mathbb{R}^3} \frac{e^{2\pi i \xi \cdot \sqrt{\lambda}(y-x)}}{32\pi^4 |\xi|^4} e^{-4\pi^2 |\xi|^2 \frac{\tau-\sigma}{\sigma}} (P_\xi \ast M)^{ij}(d\xi)d\tau.
\]
Since
\[
|(P_\xi \ast M)^{ij}|
\]
\[
= \sum_{k,l=1}^{3} (\delta_{ik} - \frac{\xi_i \xi_k}{|\xi|^2}) \mu^{kl} (\delta_{lj} - \frac{\xi_l \xi_j}{|\xi|^2})
\]
\[
= \sum_{k,l=1}^{3} (\delta_{ik} \delta_{lj} - \delta_{ik} \frac{\xi_l \xi_k}{|\xi|^2} - \delta_{lj} \frac{\xi_i \xi_j}{|\xi|^2} + \frac{\xi_i \xi_k \xi_l \xi_j}{|\xi|^4}) \mu^{kl},
\]
and each term above in the parenthesis is a bounded function of \(\xi\), thus, by assumption,
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda^{3/2}} \int_t^{t+1/\sqrt{\lambda}} d\tau \int_{t}^{\tau} E(U^i(\lambda \sigma, \sqrt{\lambda} x)U^j(\lambda \tau, \sqrt{\lambda} y))d\sigma
\]
\[
= \begin{cases} 
\int_{\mathbb{R}^3} \frac{1}{32\pi^4 |\xi|^4} (P_\xi \ast M)^{ij}(d\xi) & \text{if } x = y \\
0 & \text{if } x \neq y
\end{cases}
\]
Hence, we have proved (5.25) and (5.26). The convergence uniformly with respect to spatial variable \(x, y\) and temporal variable \(s'\) is obvious.

Now let us prove the uniform convergence in (5.24). Notice that on the one hand, by (5.14),
\[
\lim_{\lambda \to \infty} \lambda \int_t^{t+1/\sqrt{\lambda}} \int_{s}^{t} E(U^i(\lambda s', x)U^j(\lambda \sigma, x))d\sigma ds' = 0
\]
uniformly in $x, t, s$, thus
\[
\lambda \int_t^{t+1/\sqrt{\lambda}} \int_s^{s'} E\left(U^i(\lambda s', x)U^j(\lambda \sigma, x)\right) d\sigma ds' \\
= \lambda \int_t^{t+1/\sqrt{\lambda}} \int_t^{s'} E\left(U^i(\lambda s', x)U^j(\lambda \sigma, x)\right) d\sigma ds' + o(\epsilon) \\
= \frac{1}{\sqrt{\lambda}} A^{ij}(t, x) + o\left(\frac{1}{\sqrt{\lambda}}\right).
\]

Therefore
\[
\lim_{\lambda \to \infty} \lambda \int_s^t E\left(U^i(\lambda t, x)U^j(\lambda \sigma, x)\right) d\sigma = A^{ij}(t, x) \\
(5.29)
\]
uniformly in $x, t, s$. Thus,
\[
\lim_{\lambda \to \infty} \lambda \int_s^t \int_s^{s'} E\left(U^i(\lambda s', x)U^j(\lambda \sigma, x)\right) d\sigma ds' = \int_s^t A^{ij}(s', x)ds' \\
(5.30)
\]
uniformly in $x, t, s$.

On the other hand, we have:
\[
\int_s^t E\left(A^{ij}_\lambda(t, s', x)|G_{0, \lambda s}\right) ds' \\
= \lambda \int_s^t \int_s^{s'} E\left(U^i(\lambda \tau, x)U^j(\lambda s', x)|G_{0, \lambda s}\right) d\sigma ds' d\tau \\
= \lambda \int_s^t \int_s^{s'} E\left(U^i(\lambda s', x)U^j(\lambda \sigma, x)|G_{0, \lambda s}\right) d\sigma ds', \\
(5.31)
\]
after changing integration order and rewrite the integral, for $p > 1$ such that lemma 5.3 holds, we have:
\[
\left(\mathbb{E}\int_s^t \int_s^{s'} E\left(\lambda U^i(\lambda s', x)U^j(\lambda \sigma, x) - E\left(\lambda U^i(\lambda s', x)U^j(\lambda \sigma, x)\right)|G_{0, \lambda s}\right) d\sigma ds'|p\right)^{1/p} \\
\leq K \lambda \int_s^t \int_{s'}^{s'} \rho(\lambda (s' - \sigma))^{1-1/p}\left(\mathbb{E}|U^i(\lambda s', x)U^j(\lambda \sigma, x)|^{2p}\right)^{1/p} d\sigma ds' \\
= K \lambda \int_s^t \int_{\lambda s}^{\lambda s'} \rho(\lambda s' - \sigma)\left(\mathbb{E}|U^i(\lambda s', x)U^j(\lambda \sigma, x)|^{2p}\right)^{1/p} d\sigma ds' \\
\leq C \int_{\lambda s}^{\infty} \rho(s')^{1-1/p} ds'. \\
(5.32)
\]
Thus
\[
\left(\mathbb{E}\int_s^t \int_s^{s'} E\left(\lambda U^i(\lambda s', x)U^j(\lambda \sigma, x) - E\left(\lambda U^i(\lambda s', x)U^j(\lambda \sigma, x)\right)|G_{0, \lambda s}\right) d\sigma ds'|p\right)^{1/p}
\]
converges to 0 uniformly in $x, s, t$ in $L_p$ and hence in $L_1$. This completes the proof of the lemma.

Now we show that the limiting distribution has the same mean and variance.

**Lemma 5.8.** Let

$$z_\lambda(t) = x_0 + \sqrt{\lambda} \int_0^t U(\lambda s, \sqrt{\lambda} z_\lambda(s)) ds,$$  \hspace{1cm} (5.33)

$0 \leq t \leq T$, $T$ is some arbitrary positive number. If we assume that the spectral measure matrix is in class $\mathcal{M}(q_1, q_2)$, where $q_1 \leq -4$ and $q_2 \geq 2$. Then

$$\lim_{\lambda \to \infty} E(z_\lambda^i(t)) = x_0$$  \hspace{1cm} (5.34)

and

$$\lim_{\lambda \to \infty} E[(z_\lambda^i(t) - E(z_\lambda^i(t)))(z_\lambda^j(t) - E(z_\lambda^j(t)))] = t \int_{\mathbb{R}^3} \frac{1}{32\pi^4|\zeta|^4} (P_\zeta \ast \mathcal{M})^{ij}(d\zeta).$$  \hspace{1cm} (5.35)

**Proof.** They follow directly from lemma 5.7 and lemma 5.2.5 of Kunita [19].

We now prove that the limiting distribution is normal, then since normal random variable is determined by its mean and variance, we have for each fixed $t$, $z_\lambda(t)$ converges weakly to a normal distribution.

**Lemma 5.9.** Assume that the spectral measure matrix is in class $\mathcal{M}(q_1, q_2)$, where $q_1 \leq -4$ and $q_2 \geq 2$. Let

$$z_\lambda(t) = x_0 + \sqrt{\lambda} \int_0^t U(\lambda s, \sqrt{\lambda} z_\lambda(s)) ds.$$  \hspace{1cm} (5.36)

Then, $z_\lambda(t)$ converges weakly to a normal random variable $\mathcal{N}$ with mean $x_0$ and variance matrix $t \int_{\mathbb{R}^3} \frac{1}{32\pi^4|\zeta|^4} (P_\zeta \ast \mathcal{M})^{ij}(d\zeta)$.

**Proof.** By previous lemma, $z_\lambda(t)$ has mean $x_0$ and variance matrix $t \int_{\mathbb{R}^3} \frac{1}{32\pi^4|\zeta|^4} (P_\zeta \ast \mathcal{M})^{ij}(d\zeta)$. We thus only need to show the normality for the limit variable.

94
For simplicity of notations, but without loss of generality, take $t = 1$ and $\lambda = n$ an integer. Let $W_n = \sum_{i=1}^{3} k_i z_n^i$. By the previous lemmas, the limit variance of $S_n$ exists. Denote the standard deviation of it as $\tau$. By Cramer-Wold theorem, we only need to show that for any $(k_1, k_2, k_3)$, $\frac{1}{\tau \sqrt{n} W_n}$ converges weakly to $\sum_{i=1}^{3} k_i N_i$. Put $S_n = \sum_{j=1}^{n} W_j$. Clearly, $E(S_n^4) \leq Kn^2$.

Next, split the sum $W_1 + \cdots + W_n$ into blocks of length $b_n$ and $l_n$ (big blocks and little blocks). Let

$$U_{ni} = W_{(i-1)(b_n+l_n)+1} + \cdots + W_{(i-1)(b_n+l_n)+b_n}, 1 \leq i \leq r_n,$$  \hspace{1cm} (5.37)

where $r_n$ is the largest $i$ such that $(i-1)(b_n+l_n) + b_n < n$. Also, let

$$V_{ni} = W_{(i-1)(b_n+l_n)+b_n+1} + \cdots + W_n.$$  \hspace{1cm} (5.38)

Then $S_n = \sum_{i=1}^{r_n} U_{ni} + \sum_{i=1}^{r_n} V_{ni}$.

Let $b_n \sim n^{3/4}$, $l_n \sim n^{1/4}$, then $r_n \sim n^{1/4}$. Since

$$P\left[\left| \frac{1}{\tau \sqrt{n}} \sum_{i=1}^{r_n-1} \frac{V_{ni}}{\text{var}(V_{ni})} \geq \delta \right| \right]$$

$$\leq \sum_{i=1}^{r_n-1} P\left[|V_{ni}| \geq \frac{\epsilon \tau \sqrt{n}}{r_n} \right]$$

$$\sim \frac{r_n^4}{\epsilon^4 \tau^4 l_n^2} r_n K l_n^2$$

$$\sim \frac{K}{\epsilon^4 \tau^4 n^{1/2}}$$

$$\to 0.$$  

Thus $\sum_{i=1}^{r_n} \frac{V_{ni}}{\tau \sqrt{n}}$ converges to 0 weakly. Next let us prove that $\frac{U_{ni}}{\tau \sqrt{n}}$ converges weakly to standard normal.

Let $U'_{ni}, 1 \leq i \leq r_n$, be independent random variables having the distribution common to $U_{ni}$. By 5.10, we have that the characteristic functions of $\sum_{i=1}^{r_n} \frac{U'_{ni}}{\tau \sqrt{n}}$ and of $\sum_{i=1}^{r_n} \frac{U_{ni}}{\tau \sqrt{n}}$ differ by at most $16r_n \rho(l_n)$. Thus the characteristic function of $\sum_{i=1}^{r_n} \frac{U'_{ni}}{\tau \sqrt{n}}$ will approach $e^{-\frac{t^2}{2}}$ if that of $\sum_{i=1}^{r_n} \frac{U_{ni}}{\tau \sqrt{n}}$ does.
But $E[|U_n'|^2] \sim b_n \tau^2$, also $E[|U_n'|^4] \leq K b_n^2$, thus by Lyapounov’s theorem, 
\[
\sum_{i=1}^{r_n} \frac{U_n'}{\tau^{1/2}} \sim b_n \tau^2, \text{ also } E[|U_n'|^4] \leq K b_n^2, \text{ thus by Lyapounov's theorem, }
\]
\[
\sum_{i=1}^{r_n} \frac{U_n'}{\tau^{1/2}} \text{ converges to a standard normal variable weakly. This completes the proof of lemma.}
\]

Finally, we are ready for the main theorem, we will show that $z_{\epsilon}(t)$ converges weakly to a Brownian motion.

**Theorem 5.10.** Assume that the spectral measure matrix is in class $\mathcal{M}(q_1, q_2)$, where $q_1 \leq -4$ and $q_2 \geq 2$. Also assume that for any $1 \leq i, j \leq 3$, the Fourier transform of the measure $\frac{1}{|\xi|^4} \mu^{ij}(d\xi)$ goes to 0 in the infinity, i.e.

\[
\lim_{|x| \to \infty} \int_{\mathbb{R}^3} e^{-2\pi x \cdot \xi} \frac{1}{|\xi|^4} \mu^{ij}(d\xi) = 0. \tag{5.39}
\]

Let

\[
z_{\lambda}(t) = x_0 + \sqrt{\lambda} \int_0^t U(\lambda s, \sqrt{\lambda} z_{\lambda}(s)) ds. \tag{5.40}
\]

Then, $\sigma^{-1}(z_{\lambda}(t) - E(z_{\lambda}))$ converges weakly to a 3d Brownian motion. Here $\sigma^{ij} = (t \int_{\mathbb{R}^3} \frac{1}{32\pi^4 |\xi|^2} (P_\xi * \mathcal{M})^{ij}(d\xi))^{1/2}$.

**Proof.** Since $\{z_{\lambda}(t)\}_{\epsilon>0}$ is tight, if we can show that $(z_{\lambda}(t_1), \ldots, z_{\lambda}(t_n)), 0 \leq t_1 \leq \cdots \leq t_n$, converges weakly to desired normal distribution, then by lemma 5.7 and 5.8, $\{z_{\lambda}(t)\}$ converges weakly to a martingale with deterministic quadratic variation. Thus the normalized process converges to a 3d Brownian motion.

Now let us prove the convergence of finite dimensional distribution. We only need to take $n = 2$; the other case differ from this one only by being notationally more complicated. We wish to show that weakly

\[
(\sigma^{-1}(z_{\lambda}(s) - E(z_{\lambda}(s))), \sigma^{-1}(z_{\lambda}(t) - E(z_{\lambda}(t)))) \to (B(s), B(t)), \tag{5.41}
\]

where $\{B(t)\}$ is a standard 3d Brownian motion.
This is equivalent to prove weakly
\[
(\sigma^{-1}(z_{\lambda}(s) - E(z_{\lambda}(s))), \sigma^{-1}(z_{\lambda}(t) - E(z_{\lambda}(\lambda))) - \sigma^{-1}(z_{\lambda}(s) - E(z_{\lambda}(s))))
\rightarrow (B(s), B(t) - B(s)).
\]
Similar to the argument in lemma 5.9, \((\sigma^{-1}(z_{\lambda}(s) - m - x_0)\) and \(\sigma^{-1}(z_{\lambda}(t) - E(z_{\lambda}(\lambda))) - \sigma^{-1}(z_{\lambda}(s) - E(z_{\lambda}(s))))\) are independent as \(\lambda \to \infty\). The results then follow and the proof is completed. \(\square\)

An interesting question is under what condition the limiting variance matrix is a diagonal matrix. The following proposition answers this question.

**Proposition 5.11.** Assume that the generalized martingale measure \(F\) is real. Assume the spectral measure matrix is in class \(\mathcal{M}(0, q)\), where \(q \leq -4\). The covariance matrix
\[
\int_{\mathbb{R}^3} \frac{1}{32\pi^4|\xi|^4} (P_{\xi} \ast \mathcal{M})(d\xi)\]
is diagonal iff the entries of the spectral matrix satisfies the following condition: \(\mu_{ij} = 0\) if \(i \neq j\) and \(\int_{\mathbb{R}^3} \frac{1}{32\pi^4|\xi|^4} (1 - \xi^2) \mu_{ii}(d\xi) = 1.\)

**Proof.** First notice that
\[
(P_{\xi} \ast \mathcal{M})^{ij} = \sum_{k,l=1}^3 \left( \delta_{ik} \frac{\xi_l \xi_k}{|\xi|^2} - \delta_{ij} \frac{\xi_l \xi_k}{|\xi|^2} - \delta_{ij} \frac{\xi_l \xi_k}{|\xi|^2} + \frac{\xi_l \xi_k \xi_i \xi_j}{|\xi|^4} \right) \mu^{kl}.
\]
Thus
\[
\int_{\mathbb{R}^3} \frac{1}{32\pi^4|\xi|^4} (P_{\xi} \ast \mathcal{M})^{ij}(d\xi)
= \int_{\mathbb{R}^3} \frac{1}{32\pi^4|\xi|^4} \sum_{k,l=1}^3 \left( \delta_{ik} \delta_{lj} - \delta_{ik} \frac{\xi_l \xi_k}{|\xi|^2} - \delta_{ij} \frac{\xi_l \xi_k}{|\xi|^2} + \frac{\xi_l \xi_k \xi_i \xi_j}{|\xi|^4} \right) \mu^{kl}(d\xi).
\]
Notice that \(u\) is a real random field, thus \(\mu\) is symmetric in the sense that \(\mu(-A) = \mu(A), A \subset \mathbb{R}^3.\)
Therefore, when $i = j$, 
\[
\int_{R^3} \frac{1}{32\pi^4|\xi|^4} (P_\xi \ast \mathcal{M})^{ij}(d\xi) = \int_{R^3} \frac{1}{32\pi^4|\xi|^4} (1 - \frac{\xi_i^2}{|\xi|^2}) \mu^{ii}(d\xi), \tag{5.43}
\]
and when $i \neq j$,
\[
\int_{R^3} \frac{1}{32\pi^4|\xi|^4} (P_\xi \ast \mathcal{M})^{ij}(d\xi) = \int_{R^3} \frac{1}{32\pi^4|\xi|^4} (1 - \frac{\xi_i^2}{|\xi|^2} - \frac{\xi_j^2}{|\xi|^2} + \frac{\xi_i^2 \xi_j^2}{|\xi|^4}) \mu^{ii}(d\xi) \\
\geq \int_{R^3} \frac{\xi_i^2 \xi_j^2}{32\pi^4|\xi|^4} \mu^{ii}(d\xi)
\]

Hence \( \int_{R^3} \frac{1}{32\pi^4|\xi|^4} (P_\xi \ast \mathcal{M})(d\xi) \) is identity matrix iff the entries of the spectral matrix satisfies the following condition: \( \mu^{ij} = 0 \) if $i \neq j$ and \( \int_{R^3} \frac{1}{32\pi^4|\xi|^4} (1 - \frac{\xi_i^2}{|\xi|^2}) \mu^{ii}(d\xi) = 1 \), which is as desired. \( \square \)
CHAPTER 6
FUTURE DIRECTIONS

The research presented in this thesis opens doors to the rooms of novel mathematical questions.

6.1 Limit Theorem for Passive Finite Size Particle

Recall that for passive finite size particle, we have the following equations:

\[
\frac{dc(t)}{dt} = v(t) \quad (6.1)
\]

\[
m_r \frac{dv}{dt} = -\int_{\partial D(t)} S(t, x)n(t, x) \; S(dx), \quad (6.2)
\]

\[
\frac{d}{dt}A(t) = B(t)A(t) \quad (6.3)
\]

\[
I \frac{dw}{dt} = -\int_{\partial D(t)} (x - c(t)) \times (S(t, x)n(t, x)) \; S(dx), \quad (6.4)
\]

with initial conditions \( u(0, x) = U(0, x), c(0) = 0, A(0) = I, v(0) = v_0 \in \mathbb{R}^3 \) and \( w(0) = w_0 \in \mathbb{R}^3 \). Here

\[
S(t, x) = -P(t, x)I + \frac{\eta}{2}(\nabla_x U(t, x) + \nabla_x U^T(t, x)).
\]

For mathematical simplicity, but without loss of generality, we can assume that
\( m_r = 1 \) and \( \mathbf{I} = I \). Make a change of coordinate, let \( t = t, x = A(t)y + c(t) \), then we have the following equivalent equations:

\[
\begin{align*}
\dot{c}(t) &= v(t) \\
\dot{v}(t) &= f(t, c(t), A(t)) \quad (6.5) \\
\dot{A}(t) &= B(t)A(t) \quad (6.6) \\
\dot{w}(t) &= g(t, c(t), A(t)), \quad (6.7)
\end{align*}
\]

where for \( c \in \mathbb{R}^3, A \in SO(3), \)

\[
\begin{align*}
f(t, c, A) &= -\int_{\partial D} S(t, Ay + c)A\mathbf{n}(y)S(dy), \quad (6.9) \\
g(t, c, A) &= -\int_{\partial D} (Ay) \times (S(t, Ay + c)A\mathbf{n}(y))S(dy), \quad (6.10)
\end{align*}
\]

and

\[
B(t) = \begin{pmatrix}
0 & -w_3(t) & w_2(t) \\
-3/2 & 0 & -w_1(t) \\
-w_2(t) & w_1(t) & 0
\end{pmatrix}. \quad (6.11)
\]

Now let us scale the problems in the usual way. Let

\[
\begin{align*}
c_\lambda(t) &= \frac{1}{\sqrt{\lambda}} c(\lambda t), \\
v_\lambda(t) &= \sqrt{\lambda} v(\lambda t), \\
A_\lambda(t) &= A(\lambda t), \\
w_\lambda(t) &= \lambda w(\lambda t), \quad B_\lambda(t) = \lambda B(\lambda t) \\
f_\lambda(t, c, A) &= -\lambda^{3/2} \int_{\partial D} S(\lambda t, Ay + \sqrt{\lambda} c)A\mathbf{n}(y) dS(y), \\
g_\lambda(t, c, A) &= -\lambda^2 \int_{\partial D} (Ay) \times (S(\lambda t, Ay + \sqrt{\lambda} c)A\mathbf{n}(y)) dS(y),
\end{align*}
\]
then
\[ \begin{align*}
\dot{c}_\lambda(t) &= v_\lambda(t) \quad (6.12) \\
\dot{v}_\lambda(t) &= f_\lambda(t, c_\lambda(t), A_\lambda(t)) \quad (6.13) \\
\dot{A}_\lambda(t) &= B_\lambda(t)A_\lambda(t) \quad (6.14) \\
\dot{w}_\lambda(t) &= g_\lambda(t, c_\lambda(t), A_\lambda(t)) \quad (6.15)
\end{align*} \]

Do we have the limit theorems for the pair \((c_\lambda(t), A_\lambda(t))\)?

Let
\[
\begin{align*}
h_\lambda(s, c_\lambda(s), A_\lambda(s)) &=: \int_0^s f_\lambda(c_\lambda(s'), A_\lambda(s'), s')ds' \\
&= -\lambda^{3/2} \int_0^s \int_{\partial D} \mathbf{S}(\lambda s', A_\lambda(s')y + \lambda^{1/2}c_\lambda(s'))A_\lambda(s')n(y)dS(y)ds',
\end{align*}
\]

and
\[
g_\lambda(c_\lambda(s), A_\lambda(s), s) = -\lambda^2 \int_{\partial D} (A_\lambda(s)y) \times \mathbf{S}(\lambda s, A_\lambda(s)y + \lambda^{1/2}c_\lambda(s))A_\lambda(s)n(y)dS(y).
\]

Clearly \(h_\lambda(s, c_\lambda(s), A_\lambda(s))\) and \(g_\lambda(c_\lambda(s), A_\lambda(s), s)\) play the same role as \(\sqrt{\lambda}U(\lambda s, \sqrt{\lambda}z_\lambda(s))\) in point particle case. But new methods must be developed to get good estimates for \(h_\lambda\) and \(g_\lambda\).
6.2 Full Active Model - Analytics

For the full, active model of particle immersed in an incompressible, viscous, thermally fluctuating solvent, we have the following equations:

\[
\begin{align*}
\rho_s \frac{du(t, x)}{dt} &= -\nabla p(t, x) dt + \eta \Delta u(t, x) dt + F(dt, x), & x \in D(t)^c, \\
\nabla \cdot u(t, x) &= 0, \quad x \in D(t)^c, \\
\int \frac{dw}{dt} &= -\int_{\partial D(t)} (x - c(t)) \times (\mathbf{S}(t, x)n(t, x)) S(dx), \\
\frac{d}{dt} A(t) &= B(t)A(t), \\
\frac{dc(t)}{dt} &= v(t), \\
\frac{m_r}{dt} \frac{dv(t)}{dt} &= -\int_{\partial D(t)} \mathbf{S}(t, x)n(t, x) S(dx), \\
u(0, x) &= U(0, x), \quad x \in D^c \\
w(0) &= w_0 \\
A(0) &= I \\
c(0) &= 0 \\
u(t, x) &= v(t) + w(t) \times (x - c(t)), \quad x \in \partial D(t), \quad t > 0,
\end{align*}
\]

\[
E(\int_{D(t)^c} |u(t, x) - U(t, x)|^2 dx) < \infty, \quad \text{for every } t > 0. \quad (6.27)
\]
For simplicity, let \( w_0 = 0 \). Put
\[
\begin{align*}
t &= t \\
y &= A(t)^T(x - c(t)) \\
h(t, y) &= A(t)^T u(t, x) \\
W(t, y) &= A(t)^T U(t, x) \\
\xi(t) &= A(t)^T v(t) \\
\eta(t) &= A(t)^T w(t)
\end{align*}
\]

We also change the form of pressure and noise. Let
\[
\begin{align*}
p(t, x) &= q(t, y) \\
F(t, x) &= G(t, y)
\end{align*}
\]

Thus our new system of equations are:
\[
\begin{align*}
\rho_s \frac{dh(t, y)}{dt} &= \eta \nabla y h(t, y) dt - \rho_s ((\eta(t) \times y + \xi(t)) \cdot \nabla y h(t, y) \\
-\eta(t) \times h(t, y)) dt - \nabla y q(t, y) dt + G(dt, y) & \text{for } y \in D^c \\
\nabla y \cdot h(t, y) &= 0 & \text{for } y \in D^c \\
\int_{\partial D} y \times (S'(t, y)n) \ S(dy) - \eta(t) \times (I\eta(t)) \\
\frac{dA(t)}{dt} &= B(t) A(t) \\
\frac{d\xi}{dt} &= -\int_{\partial D} S'(t, y)n S(dy) - \eta(t) \times \xi(t) \\
\frac{dm}{dt} &= -\int_{\partial D} S'(t, y)n S(dy) - \eta(t) \times \xi(t) \\
h(0, y) &= W(0, y) & \text{for } y \in D^c \\
h(t, y) &= \xi(t) + \eta(t) \times y & \text{for } y \in \partial D, \ t > 0 \\
E(\int_{D^c} |A(t)h(t, y) - A(t)W(t, y)|^2 dy) & < \infty & \text{for every } t > 0,
\end{align*}
\]

where \( S'(t, y) = qI + \frac{1}{2}(\nabla h + \nabla h^T) \).

Do we have existence and uniqueness for this system of stochastic partial differential equations? When the characteristic length of the body shrinks to zero, will the
motion of this particle converge to some diffusion process? What is the long time behavior of the motion of this particle?

There are two major difficulties in these nonlinear problems. The first one is that the coefficient of $\nabla_y h(t,y)$ is unbounded in spacial variable $y$. This makes problem very difficult even in deterministic case. To the best of our knowledge, the existence and uniqueness for the solution of the above system of equations with only deterministic force has not been proved yet. The other difficulty is here we are dealing with stochastic boundary value problems. Even for zero Dirichlet boundary value problem we need weighted Sobolev spaces (cf. [17]) or compatibility conditions (cf. [5]).

### 6.3 Full Active Model - Numerics

One other interesting problem is to solve this system of stochastic partial differential equations numerically. We propose a numerical method which combines level set method (cf. [25], [26]), rigid fluid method (cf. [1]) and stochastic Galerkin method. Level set method is to track propagating interfaces, rigid fluid method is to maintain the rigidity of the rigid particle, while stochastic Galerkin method is to approximate the noise.

By using level set method and rigid fluid method, we are able to solve the system of equations when the forcing is deterministic. Below is the algorithm.

Let $C$ be the computational domain, $F$ be the region filled with solvent and $D$ be the region occupied by the particle. Assume we are in time step $t^n$, and have the velocity field $u^n$, level set function $\phi^n$.

First we solve Navier-Stokes equations (cf. [3]) for the whole computational field $C = F \cup D$, i.e we advance the solution $u^n$ of Navier-Stokes equations at time $t^n$ to $t^{n+1}$ and get an intermediate velocity field $u^*$ at $t^{n+1}$.  

104
Then we integrate the intermediate $u^*$ inside the given rigid particle $D$ with the following equations to get the velocity of the center of mass and angular velocity:

$$m_r v^n = \int_D \rho u^* dx, \quad (6.36)$$

$$\text{Inertia} \omega^n = \int_D r \times u^* dx \quad (6.37)$$

where $m_r, I$ and $\rho$ are the mass, momentum of inertia and density of the rigid particle, $r = x - c$ is the vector between the current point in $D$ that is under integration and the center of mass $c$, and $dx$ means the integral is volume integral. The equations (6.36)-(6.37) ensure that the momentum is conserved (cf. [30]).

After we get $v^n, \omega^n$ from (6.36) and (6.37), we compute the velocity for the rigid particle:

$$u^n_R = v^n + \omega \times r, \quad (6.38)$$

then we correct the fluid velocity by distributing the rigid particle velocity over the body to get the final velocity $u^{n+1}$:

$$u^{n+1} = (1 - \alpha)u^* + \alpha u^n_D, \quad (6.39)$$

where $\alpha$ is the portion of the body in solvent.

Finally, we advance the computational domain by level set method, using third-order accurate TVD Runge-Kutta in time discretization and third-order accurate ENO in spatial discretization (cf. [35], [36]). We then iterate until the solution converges.

For the case when the forcing is stochastic, we are still working on the stochastic Galerkin method to approximate the noise.
BIBLIOGRAPHY


