MULTIPLICITY-ONE RESULTS AND
EXPLICIT FORMULAS FOR QUASI-SPLIT
\( p \)-ADIC UNITARY GROUPS

DISSERTATION

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ABSTRACT

In 2001, Kato, Murase and Sugano published a paper describing certain multiplicity one results for special orthogonal groups $G$ over local fields of odd residue degree. In that paper, the authors define Whittaker-Shintani functions, complex-valued functions on $G$ which possess desirable symmetry properties on the right and left by various subgroups of $G$ and prove that the space of all such functions is at most one-dimensional for each choice of parameters, and generically exactly one.

In this project, we carry an analogous study for quasi-split unitary groups, so that instead of a single local field we have a quadratic extension of local fields of odd residue degree. Gelfand-Graev functions are complex-valued functions on a unitary group characterized by the following collection of symmetry conditions.

- Invariance under the maximal compact of the full group on the right and the maximal compact of a smaller unitary group on the left.

- Transformation on the left by a particular unipotent subgroup acts by a specified character.

- Transformation on the right by the Hecke algebra of the full group and the left by the Hecke algebra of a smaller unitary group on the left acts by specified Hecke characters.

We show that, for each choice of characters, the space of these functions is one-dimensional. Furthermore we give an explicit formula for typical such functions, which turns out to be rational with respect to the Hecke characters chosen.
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# TABLE OF CONTENTS

Abstract .................................................................................................................. ii
Acknowledgments .................................................................................................... iii
Vita ........................................................................................................................... iv

Chapter 1: Notation and Setup: Quasi-Split Unitary Groups ......................... 1
Chapter 2: Notation and Setup: Gelfand-Graev Functions ......................... 12
Chapter 3: Unramified Principal Series Rep’ns and Equivariant Linear Forms .... 17
Chapter 4: Orbits ..................................................................................................... 21
Chapter 5: Double Coset Decomposition .............................................................. 25
Chapter 6: Support of Gelfand-Graev Functions ............................................. 29
Chapter 7: Uniqueness of Gelfand-Graev Functions ........................................ 33
Chapter 8: Regional Construction ....................................................................... 38
Chapter 9: Rank 1 Calculations .......................................................................... 43
Chapter 10: Rationality of Gelfand-Graev Functions ....................................... 71
Chapter 11: Generic Explicit Formula ................................................................. 75
Chapter 12: Residual Cases ................................................................................. 81
Chapter 13: Concluding Remarks ....................................................................... 86

Bibliography ........................................................................................................... 88
CHAPTER 1
Notation and Setup: Quasi-Split Unitary Groups

Let $F$ be a nonarchimedean local field with odd residue characteristic $p$ and uniformizer $\pi$. Let $E$ be nonarchimedean local field which is a quadratic extension of $F$. We assume that $E/F$ is unramified, which means in particular that we can take $\pi$ as uniformizer for $E$ and also that $E = F(\eta)$ for some $\eta$, where $\eta^2$ is a unit in $F$. Then we can write $E = F + F\eta$. Somewhat abusively, we will call elements of $F$ real and elements of $F\eta$ pure imaginary. We denote the nontrivial Galois automorphism by $x \mapsto \bar{x}$ and call it conjugation, so that if $a, b \in F$, we have $a + b\eta = a - b\eta$. Since the extension is unramified, an element and its conjugate always differ by a unit. It will be useful at times to refer to the real and imaginary parts of an element of $E$,

$$
\Re(x) = \frac{x + \bar{x}}{2}, \quad \Im(x) = \frac{x - \bar{x}}{2\eta}.
$$

Write $\mathfrak{O}_F$ and $\mathfrak{O}_E$ for the rings of integers in $F, E$ respectively, and write $q_F = |\mathfrak{O}_F/\pi\mathfrak{O}_F|$ and $q_E = |\mathfrak{O}_E/\pi\mathfrak{O}_E|$ for the orders of the residue fields. By assumption $q_E = q_F^2$. We will denote the valuations and absolute value by $v_F, v_E, | \cdot |_F, | \cdot |_E$, respectively. When no subscript is used, the default interpretation is $E$. By the remark at the end of the previous paragraph, we have $\mathfrak{O}_E = \mathfrak{O}_F + \mathfrak{O}_F\eta$, where we are

\footnote{It is a typical phenomenon in the representation of $p$-adic groups that the prime 2 must be excluded, but all odd primes can be examined in a uniform way. Surely most of the explicit formulas and quantitative results will break down when $p = 2$, but it is less clear whether the qualitative results of this project (existence and uniqueness of Gelfand-Graev functions) should generalize. The author conjectures that multiplicity one will hold in that case too, at least generically, but the arguments certainly would look quite different than that presented here.}
making explicit use of the fact that 2 and \( \eta \) are units in our setup. This statement can certainly fail over fields of even residue characteristic.

We will be interested in the quasisplit unitary groups \( U(n + 1, n) \) and \( U(n, n) \) over \( E/F \). In order to unify as much as possible of the argument, we will adopt the notation

\[
G_{2n+1} = U(n + 1, n) \quad G_{2n} = U(n, n),
\]

so that \( G_n \) will consist of \( n \times n \) matrices. For convenience, we will write \( n_+ = \lceil n/2 \rceil, n_- = \lfloor n/2 \rfloor \); this lets us write \( G_n = U(n_+, n_-) \) in a uniform way.

We define

\[
G_n = \begin{cases} 
    g \in \text{GL}_n(E) : g \begin{pmatrix} 0 & 0 & J_{n_-} \\ 0 & 2 & 0 \\ J_{n_-} & 0 & 0 \end{pmatrix} g^* = \begin{pmatrix} 0 & 0 & J_{n_-} \\ 0 & 2 & 0 \\ J_{n_-} & 0 & 0 \end{pmatrix}, & n \text{ odd} \\
    g \in \text{GL}_{2n}(E) : g \begin{pmatrix} 0 & J_{n_-} \\ J_{n_-} & 0 \end{pmatrix} g^* = \begin{pmatrix} 0 & J_{n_-} \\ J_{n_-} & 0 \end{pmatrix}, & n \text{ even}.
\end{cases}
\]

Here \( J_n \) is the matrix with 1’s on the upper-right-to-lower-left diagonal and zeroes elsewhere and \( g^* \) is the adjoint (conjugate transpose). In other words, elements of \( G_n \) preserve the form given by the matrix with 1’s along the antidiagonal except for a 2 in the center position (if there is a center position).

This form is convenient for us because \( G_n \) has a standard Borel subgroup \( P_n \) consisting of precisely the upper-triangular matrices in \( G_n \) and a maximal torus \( T_n \) consisting of the diagonal matrices in \( G_n \). In the context of algebraic groups, quasisplit means that there is a Borel subgroup which can be defined over the base field. It is evident that \( P_n \) is defined over the base field. Let \( N_n \) be the unipotent subgroup
of upper-triangular matrices with 1’s along the main diagonal, and $N_n^-$ the opposite subgroup. Also, we can take $K_n$ to be the maximal compact subgroup consisting of matrices with integer entries.

In the even case, we have a surjection $d_n : (E^\times)^{n-} \to T_n$, given by

$$d_n(t_1, t_2, \ldots, t_{n-}) = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{n-} \\ \frac{1}{t_{n-}} \\ \ddots \\ \frac{1}{t_2} \\ \frac{1}{t_1} \end{pmatrix};$$

in the odd case, the map $d_n : (E^\times)^{n-} \times E^1 \to T$ is given by

$$d_n(t_1, t_2, \ldots, t_{n-}, t_{n+}) = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_{n-} \\ t_{n+} \\ \frac{1}{t_{n-}} \\ \ddots \\ \frac{1}{t_2} \\ \frac{1}{t_1} \end{pmatrix},$$

where $E^1$ consists of the elements of $E$ with norm 1, since we must have $t_{n+} = \frac{1}{t_{n+}}$.

Furthermore, if $t \in T_n$ is given, the $t_i$ are determined. Indeed, given an element $p$ of
$P_n = N_n T_n$, we can consider the entries $t_i$, which are read off the diagonal and which are the $t_i$ for the unique $t$ such that $p = tn = \hat{n}t$ ($n, \hat{n} \in N$). Writing $\Lambda_n = \mathbb{Z}^{n-}$, we have a map $t_n : \Lambda_n \to T_n$,

$$t_n(\lambda_1, \ldots, \lambda_{n-}) = \begin{cases} d_n(\pi^{\lambda_1}, \ldots, \pi^{\lambda_{n-}}) & \text{even case} \\ d_n(\pi^{\lambda_1}, \ldots, \pi^{\lambda_{n-}}, 1) & \text{odd case} \end{cases},$$

and every element of the torus is expressible uniquely as an element in the image of $t$ and a diagonal matrix of units. That is, $T_n = T_{n,(0)} t_n(\Lambda) \cong T_{n,(0)} \times t_n(\Lambda_n)$. Also, the image of $t_n$ appears naturally in the Iwasawa decomposition

$$G_n = \bigsqcup_{\lambda \in \Lambda_n} N_n t_n(\lambda) K_n.$$

Also, let $T_n^{++}$ be the subsemigroup of $T_n$ which consists of those $t$ such that $|\alpha(t)| \leq 1$ for all positive roots $\alpha$. Equivalently, we want

$$|t_1| \geq |t_2| \geq |t_3| \geq \cdots \geq |t_n|.$$

This semigroup will be of paramount importance in constructing a fruitful double coset decomposition of our group.

We normalize measures on $K_n, N_n, T_n, G_n$ with respect to $K_n$. That is, choose $dk$, $dn$, $dt$, and $dg$ to be Haar measures on the suitable spaces so that

$$\int_{K_n} dk = \int_{N_n \cap K_n} dn = \int_{T_n \cap K_n} dt = \int_{K_n} dg = 1.$$

Let $\delta_n$ be the modulus character on $T_n$ (extended to $P_n$ by 1 along $N_n$) given by

$$\delta_n(t) = \frac{d(tnt^{-1})}{dn}.$$
Explicitly, it will turn out that we have

\[ \delta_n(t) = \prod_{i=1}^{n-1} |t_i|^{n+1-2i} , \]

but this is far from obvious. A proof of that fact depends on the following proposition, which is of independent interest.

**Proposition 1.1** An element of \( N_n \) is uniquely determined by the entries above both the diagonal and the antidiagonal, and the imaginary parts of the entries along the antidiagonal but above the diagonal. Furthermore, Haar measure on \( N_n \) is given by the product measure on those entries (using the measure on \( E \) for entries above both diagonals and the measure on \( F \) for imaginary parts).

**Proof.** Consider the group \( N \) of upper-triangular unipotent matrices in \( G \). Write a typical element \( g \in N \) as a matrix \((a_{ij})\), so that necessarily \( a_{ij} = 0 \) if \( i > j \) and \( a_{ij} = 1 \) if \( i = j \). Such a matrix is in \( G \) iff

\[ gSg^* = S. \]

Here \( S \) is the matrix appearing in the definition of the group \( G_n \) whose \( i, j \) entry is given by \( \delta_{i,n+1-j}(1 + \delta_{2i,n+1}) \), where the second factor accounts for the 2 appear in the central position (whenever there is a central position).

Then the \( i, j \) entry of the two sides of the matrix equation gives the following family of equations.

\[ \sum_{k=1}^{n} a_{ik}(1 + \delta_{2k,n+1})a_{j,n+1-k} = \delta_{i,n+1-j}(1 + \delta_{2i,n+1}) \]

Since \( a_{ik} = 0 \) when \( i > k \) and \( a_{j,n+1-k} \) when \( j > n + 1 - k \), we can simplify this.
to obtain the following.

\[ \sum_{k=1}^{n+1-j} a_{ik}(1 + \delta_{2k,n+1})a_{j,n+1-k} = \delta_{i,n+1-j}(1 + \delta_{2i,n+1}) \]

If \( i + j > n + 1 \), then the summation is empty and this is vacuously true. Likewise, if \( i + j = n + 1 \), the summation includes only the term \( a_{ii}a_{jj}\delta_{i,n+1-j}(1 + \delta_{2i,n+1}) = \delta_{i,n+1-j}(1 + \delta_{2i,n+1}) \).

The nontrivial conditions occur when \( i + j < n + 1 \). First consider the cases where \( i + j = n \), so that \( a_{i,i+1} + \overline{a_{j,j+1}} = 0 \). These conditions will be satisfied if each term on the superdiagonal is the negative of the conjugate of the corresponding other term on the superdiagonal. In particular, each entry on the superdiagonal which lies above the antidiagonal can be chosen freely. If \( i = j \), which is to say that there is an entry on the superdiagonal which lies on the antidiagonal, then its real part must be zero (it must be its own negative conjugate). We can continue this argument to successively higher superdiagonals by looking at \( i + j = n - 1, i + j = n - 2, \ldots \); in each case we obtain equations of the form \( a_{i,i+1} + \overline{a_{j,j+1}} + x = 0 \), where \( x \) is an expression in terms of entries \( a_{kl} \) on lower superdiagonals.

The upshot of this is that we can completely specify an element of \( N_n \) by giving the real and imaginary parts of all the entries which lie above the diagonal and above the antidiagonal and by giving the imaginary parts of all the entries lying above the diagonal and along the antidiagonal. Taken together, this gives a parametrization of \( N_n \) by an \( n(n-1)/2 \)-tuple of elements of \( F \) which is in fact a homeomorphism.

We want this to be a map of topological groups, so we define the group operation on tuples of elements of \( F \) as induced by the parametrization. The important thing to notice is that, if \( (a_{ij}) \) and \( (b_{ij}) \) are elements of \( N_n \), then we can write their product as \( (c_{ij}) \), and the upper-triangular unipotent nature of the matrices gives for each \( i < j \) the condition \( c_{ij} = a_{ij} + b_{ij} + x_{ij} \), where \( x_{ij} \) consists entirely of terms from...
lower superdiagonals. (If \(i + 1 = j\), so that there are no lower superdiagonals, then \(x_{ij} = 0\).) That is, as long as we write our \(n(n-1)/2\)-tuples so that all entries coming from one superdiagonal are listed before all entries coming from the next, then this group operation is unipotent in the dynamical sense. In particular, Haar measure on \(F^{n(n-1)/2}\) with this operation is just the product measure.

This, together with the trivial fact that Haar measure on \(E\) is given by the product measure from Haar measure on \(F\), gives what we claimed.

**Proposition 1.2**

\[
\delta(t) = \prod_{i=1}^{n} |t_i|^{n+1-2i}
\]

**Proof.** Continue to let \(g = (a_{ij})\) be a typical element of \(N\), and write \(t = d_n(t_1, \ldots, t_{n+})\). First we notice that the \(i, j\) entry of \(tnt^{-1}\) is given by \(t_ia_{ij}t^{-1}_j = t_ia_{ij}t_{n+1-j}\).

By the previous proposition, we need only compute the effect on the entries seen by our formula for the Haar measure. Since \(T(0)\) does not affect measure, without loss of generality suppose that \(t_i = \pi^{ri}\). The entry \(t_i\) appears in the \(n - 2i\) entries in the \(i\)-th row seen by Haar measure (of which one is on the antidiagonal), contributing \(q^{-(n-2i-1/2)ri}\). The entry \(t^{-1}_i\) appears in the \(i - 1\) entries in the \(i\)-th column seen by Haar measure, contributing \(q^{(i-1)ri}\). Finally, the factor \(t^{-1}_{n+1-i}\), which is equal to \(\pi^{ri}\), appears in the \(i\) entries seen by the Haar measure (of which one is on the antidiagonal), contributing a further \(q^{-(i+1/2)ri}\).

Combining all these contributions, \(t_i\) will cause a total change in measure of a factor of

\[
q^{-(n+1-2i)ri} = |t_i|^{n+1-2i},
\]

just as claimed. Combining this for all \(i\) gives what we desired. (Also, notices that \(|t_{n+}| = 1\), so it does not matter whether it is formally included in the product, where
it would appear with power 0 anyway.)

Since $K_n$ is compact, it is unimodular, so this is the only modulus character we need to consider. In light of these normalizations and the Iwasawa decomposition $G_n = N_n T_n K_n$, we have

$$dg = \delta_n(t) \, dk \, dt \, dn.$$ 

The unramified characters (those which are trivial on the units) of $E^\times$ can be identified with $\mathbb{C}^\times$. If $\chi$ is a character, then

$$\chi(u \pi^n) = \chi(\pi)^n,$$

so we identify $\chi$ with $\chi(\pi)$. In this setting, if we view $| \cdot |$ as a character on $E^\times$, it is identified with $q^{-1}$. Similarly, we let $X_n$ be the set of unramified characters on $T_n$. In this setting, “unramified” means vanishing on the toral units $T_{n,(0)}$, so such characters are really functions on $t_n(\Lambda_n)$, which we identify with $\Lambda_n$. Any $\chi \in X_n$ must thus have the form

$$\chi(t_n(\lambda_1, \ldots, \lambda_{n-1})) = \chi_1^{\lambda_1} \cdots \chi_{n-1}^{\lambda_{n-1}}.$$ 

Thus we identify the unramified character $\chi$ with the $n$-tuple $(\chi_1, \ldots, \chi_{n-1}) \in (\mathbb{C}^\times)^{n-1}$.

The Weyl group $W_n = N_G(T_n)/T_n$ of $G_n$ turns out to have a very simple description for our groups. The Weyl group consists of all matrices which have at most one (hence exactly one) nonzero entry in each row and each column, and the fact that we are looking at $T$-cosets means that we identify matrices with the same configuration of nonzero entries. For concreteness we choose for each Weyl element the representative consisting of only 1’s and 0’s, and (only mildly abusively) regard $W$ as a subgroup of $K_n$. The matrix with 1’s along the antidiagonal and 0’s elsewhere is then $w_\ell$, the representative of the long element of the Weyl group. The condition that the matrix
actually belong to \( G \) can be visualized as the condition that the 0, 1-matrix \( w \) satisfies
\[ w_{i,j} = w_{n+1-j,n+1-i}, \]
in other words that the matrix has “half-turn” symmetry.

For any subgroup \( V \subseteq G_n \), we write \( V_{(k)} \) for the subgroup
\[ \{ g \in K \cap V : g \equiv I_N \pmod{\pi^k} \}. \]

The Iwahori subgroup \( B_n \) is then given by
\[ B_n = N_{n,\pi}^{-}T_{n,0}N_{n,0}, \]
and we recall the decomposition \( B_nW_nB_n \).

We note that \( W_n \) acts in a natural way on \( \Lambda_n \) (by permutation and negation) and \( X_n \) (by permutation and inversion).

We write \( \Sigma_n \) for the set of roots of \( G_n \), \( \Sigma_n^+ \) for the positive roots (according to the choice \( P_n \) of Borel subgroup), and \( \Delta_n \) for the simple roots. We denote the roots \( \varepsilon_i - \varepsilon_j \) according to the convention for the full group \( GL_n \), with the understanding that there is always lurking in the background an involution (which we denote by \( \alpha \mapsto \overline{\alpha} \)) on the root system.

For each root \( \alpha \), we associate a root subgroup \( X_\alpha \). Writing \( E_{i,j} \) for the matrix with 1 at entry \( i,j \) and 0 elsewhere, we define the group by
\[ X_\alpha = \{ x_\alpha(t) : t \in E \}. \]

\[
x_{\varepsilon_i - \varepsilon_j}(t) = \begin{cases} 
I_n + (t - t)E_{i,j} & \text{even case, } i + j = n + 1 \\
I_n + tE_{i,j} - \overline{t}E_{n+1-j,n+1-i} & \text{even case, } i + j \neq n + 1 \\
I_n + (t - \overline{t})E_{i,j} & \text{odd case, } i + j = n + 1 \\
I_n + tE_{i,j} - \overline{t}E_{n+1-j,n+1-i} & \text{odd case, } i + j \neq n + 1, i,j \neq n + 1 \\
I_n + 2tE_{n+1,j} - \overline{t}E_{n+1-j,n+1} - \overline{t}E_{n+1-j,j} & \text{odd case, } i = n + 1 \neq j \\
I_n + tE_{i,n+1} - \overline{2t}E_{n+1,n+1-j} - \overline{t}E_{i,n+1-i} & \text{odd case, } i \neq n + 1 = j
\end{cases}
\]

We notice of course that roots related by the involution correspond to the same root.
subgroup. While all these subgroup are one-parameter subgroups and the embeddings are defined on $E$, in the cases where the root is fixed by the involution, the group is really parameterized by one $F$-parameter (in the language of $\mathbb{C}/\mathbb{R}$, the root subgroup only sees the “imaginary part”). In such cases, we write

$$x^*_{\varepsilon_i - \varepsilon_j}(s) = I_n + s\eta E_{i,j} = x_{\varepsilon_i - \varepsilon_j}(s\eta/2) \quad s \in F$$

when we wish to emphasize that the parameter comes from $F$.

For any set of roots $S$, we put

$$N_{n,S} = \prod_{\alpha \in S} X_{w_{\alpha}},$$

so that in particular $N_{n,\Sigma^+} = N_n$ and $N_{n,-\Sigma^+} = N_n^-$. To each root $\alpha$, we associate the simple reflection $w_{\alpha}$. The natural length function $\ell$ on $W_n$ is then the number of positive root subgroups which are sent to negative root subgroups (which is not quite the same as the number of positive roots which are sent to negative roots).

Let $\mathcal{H}_n$ be the Hecke algebra of $G_n$ relative to the maximal compact $K_n$, which consists of those functions on $G_n$ which are left- and right-invariant under $K_n$. For each unramified character $\chi \in X_n$, we define $\phi_{\chi}$ by $\phi_{\chi}(ntk) = (\chi\delta^{1/2})(t)$. This is evidently a $K_n$-fixed element of the unramified principal series representation $I(\chi) = \{f \in C^\infty(G_n) : f(pg) = (\chi\delta^{1/2})(p)f(g)\}$. On the other hand, the Iwasawa decomposition guarantees that $I(\chi)^{K_n}$ is one dimensional (since specifying transformation laws for $P_n$ on the left and $K_n$ on the right reduces consideration to

$$P_n \backslash G_n / K_n = P_n \backslash (P_n K_n) / K_n,$$
i.e. a single point. Then Satake gives us a $\mathbb{C}$-algebra homomorphism

$$\omega_\chi : \mathcal{H}_n \to \mathbb{C}$$

defined by integration against $\phi_\chi$,

$$\omega_\chi(\phi) = \int \phi_\chi(g)\phi(g) \, dg.$$  

This map $\chi \mapsto \omega_\chi$ is a bijection identifying unramified characters of the torus (up to permutation by the Weyl group) with characters of the Hecke algebra. In particular, any Hecke character can be completely described by a character $\xi \in X_n$, which we can in turn identify with an $n$-tuple of complex numbers. Then $\xi$ (or its entries $\xi_i$) is the Satake parameter for the Hecke character.
CHAPTER 2

Notation and Setup: Gelfand-Graev Functions

Fix natural numbers $n' < n$ of opposite parity, $n = n' + 2r + 1$. Then we will be studying the group $G = G_n$ and a certain subgroup isomorphic to $G_{n'}$, which will arise as the stabilizer of an anisotropic vector in a copy of $G_{n'+1}$. Since $n', n$ are fixed for everything that follows, we can safely dispense with the subscripts. Write

$$G = G_n, \quad G^\sharp = G_{n'+1}, \quad G' = G_{n'}.$$  

When we do not use any subscript on $T, P, N, K, W, \mathcal{H}$, etc., we mean $T_n, P_n, N_n, K_n, W_n, \mathcal{H}_n$, etc. Likewise, we use primed symbols to indicate the corresponding objects for $G'$ and sharped symbols for $G^\sharp$ (though the latter will not be used very often; the important groups for us are $G$ and $G'$).

We wish to regard these groups as a chain of nested subgroups; to that end we will specify particular embeddings $G' \hookrightarrow G^\sharp \hookrightarrow G$. The embedding $G^\sharp \hookrightarrow G$, which connects groups of the same parity, is the natural central embedding

$$g \mapsto \begin{pmatrix} I_r & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & I_r \end{pmatrix}.$$  

The embedding $G' \hookrightarrow G^\sharp$ depends on the parity of $n$. If $n$ is odd, embed $G'$ into $G^\sharp$
via

$$\iota : g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix},$$

where the block matrices are written with respect to the partitions $n'_-, n'_-$ and $n'_-, 1, n'_-$. It is clear that $\iota$ is a homomorphism and that its image is the stabilizer in $G^2$ of $^t(0, \ldots, 0, 1, 0, \ldots, 0)$. We embed $G'$ into $G^2$ via

$$\iota : g = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} a & b/2 & b/2 & c \\ d & (e + 1)/2 & (e - 1)/2 & f \\ d & (e - 1)/2 & (e + 1)/2 & f \\ g & h/2 & h/2 & i \end{pmatrix},$$

where the block matrices are written with respect to the partitions $n'_-, 1, n'_-$ and $n'_-, 1, 1, n'_-$. It is less obvious in this case, but still straightforward to check that $\iota$ is a homomorphism and that its image is precisely the stabilizer in $G_{m}^{even}$ of the vector $^t(0, \ldots, 0, 1, -1, 0, \ldots, 0)$. We will always consider $G', G^2$ and all associated objects as identified with their images in $G$ with respect to these embeddings.

We remark that, in the case where $n$ is even, the Borel subgroup $P'$ does not consist exclusively of upper triangular matrices in $G$, so $P' \not\subset P$ in that case. Likewise $T' \not\subset T$. However, we always do have $N' \subset N$ and $K' \subset K$, with the latter depending crucially on the fact that $F$ has odd residue characteristic.\(^\text{2}\)

We let $Q^\circ$ be the standard maximal parabolic for the root system obtained by deleting $\varepsilon_r - \varepsilon_{r+1}$ (and its conjugate) from the simple root system. (If $r = 0$, $Q^\circ$ is

\(^2\text{By now we should no longer be surprised that things do not work as we might wish when } p = 2. \text{ This feature in particular, } K' \not\subset K, \text{ makes the } p = 2 \text{ case impervious to the methods used here.}\)
all of $G$ and this construction is degenerate.) Then $Q^\circ$ has a Levi decomposition

$$Q^\circ = M^\circ U^\circ,$$

where $U^\circ$ is the unipotent radical and $M^\circ$, the Levi part, is isomorphic to $GL_r \times G^\sharp$. It is convenient to notice that we have surjections $\mu : GL_r(E) \times G^\sharp \to M^\circ$ and $\nu : Mat_{n-2r,r}(E) \times Alt^*_r(E) \to U^\circ$ defined by

$$\mu(a, g) = \begin{pmatrix} a & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & J_r(a^*)^{-1}J_r \end{pmatrix},$$

and

$$\nu(x, y) = \begin{pmatrix} I_r & J_r x^* S_{n-2r} & J_r(y - \frac{1}{2}S_{n-2r} x^* x) \\ 0 & I_{n-2r} & -x \\ 0 & 0 & I_r \end{pmatrix},$$

where $S_{n-2r}$ is the matrix preserved by $G_{n-2r}$ which has 1’s along the off diagonal except for a 2 in the center position (if there is a center position), and $Alt^*_r(E)$ consists of the matrices satisfying $J_r g^* J_r = -g$.

Because $G'$ sits inside $G^\sharp$ as a stabilizer of a vector $v$, it will be useful to have a family of elements of $G$ (in fact, of $N \cap G^\sharp$) which move $v$ around as much as possible and leave the rest of the space relatively unchanged.

In the case where $n$ is odd, for each column vector $y \in E_{n'}^-,$ put

$$g^\circ(y) = \mu \begin{pmatrix} I_{n'} & 2y & -J_{n'}yy^* \\ 0 & 1 & -y^*J_{n'} \\ 0 & 0 & I_{n'} \end{pmatrix}.$$
In the case where \( n \) is even, for each column vector \( y \in E^{n'} \), put

\[
g^\circ(y) = \mu \begin{pmatrix} I_{n'} & y & 0 \\ 0 & 1 & y^* \\ 0 & 0 & I_{n'} \end{pmatrix}.
\]

Then the family \( g^\circ(y) \) exactly compensates for the discrepancy between \( N_H \) and \( N \) in the sense that any \( n \in N \) is uniquely expressible in the form \( g^\circ(y)n' \) for some vector \( y \) and \( n' \in N_H := UN' \).

We are most interested in the case where \( y = ^t(1,1,\ldots,1) \), and we simply write \( g^\circ \) for

\[
g^\circ\left(^t(1,1,\ldots,1)\right) \in N \cap K \cap G^\sharp.
\]

We also consider an intermediate parabolic \( Q \) between \( P \) and \( Q^\circ \) with Levi part \( GL_1(E)^r \times G^\sharp \) and unipotent radical \( U \). We can realize \( U \) as

\[
\{ \mu(z, I_{n-2r}) : z \in GL_r(E), z \text{ upper triangular unipotent} \}.
\]

This subgroup \( U \) is normalized by \( G' \), and we denoted their semidirect product \( G'U \).

Let \( \psi \) be an additive character of \( E \) with conductor \( o_E \). Then we can define a \( G' \)-invariant (additive) character on \( U \), which we abusively also call \( \psi \), by

\[
\psi(\nu(x,y)\mu(z,1)) = \begin{cases} 
\psi(z_{1,2} + z_{2,3} + \cdots + z_{r-1,r} + x_{m+1,1}) & \text{odd case} \\
\psi(z_{1,2} + z_{2,3} + \cdots + z_{r-1,r} + x_{m+1,1} - x_{m+2,1}) & \text{even case}
\end{cases},
\]

where \( z \) ranges over the upper triangular unipotent matrices in \( GL_r(E) \). (In fact we can show that every \( G' \)-invariant character on \( U \) has that form for some character \( \psi \), and we merely stipulate that \( \psi \) has conductor \( o_E \).)
Definition 2.1 (Gelfand-Graev function) Fix a character $\psi$ on $U$ as in the above discussion, and let $\omega_\xi$ and $\omega_{\xi'}$ be Hecke characters on $\mathcal{H}, \mathcal{H}'$ with Satake parameters $\xi \in X, \xi' \in X'$. Then a complex-valued function $F \in C^\infty(G)$ is a Gelfand-Graev function for $(\xi, \xi')$ (or, abusively, for $(\omega_\xi, \omega_{\xi'})$) if it satisfies

$$L(uk')R(k)F = \psi(u)F \quad \forall u \in U, k \in K, k' \in K',$$  \hspace{1cm} (1)

$$L(\varphi')R(\varphi)F = \omega_\xi(\varphi)\omega_{\xi'}(\varphi')F \quad \forall \varphi' \in \mathcal{H}', \varphi \in \mathcal{H},$$  \hspace{1cm} (2)

where $L, R$ are the left and right regular representations.

We denote the space of Gelfand-Graev functions attached to $(\xi, \xi')$ by $GG(\xi, \xi')$.

Here the subgroups of $G$ act on $F$ by $[L(g_1)R(g_2)F](g) = F(g_1gg_2^{-1})$, and the Hecke algebras act by left and right convolution.

(We consider $\psi$ to be fixed throughout the whole discussion, so the dependence of $GG(\xi, \xi')$ on $\psi$ is suppressed in the notation. It may seem arbitrary that we think of $\xi, \xi'$ as varying while $\psi$ is not, but here we are paralleling Kato, Sugano, and Murase. For one, the data $\xi, \xi'$ is merely a collection of complex numbers, which are rather natural sorts of parameters. A less superficial answer will come over the course of the calculations—we first prove our main result generically, for almost all $\xi, \xi'$, and the values of $\xi, \xi'$ which are excluded is independent of $\psi$.)
CHAPTER 3

Unramified Principal Series Representations and Equivariant Linear Forms

The following results are taken more or less directly from [KMS], §1, where they are treated at a level of generality more than sufficient for our purposes.

Let $G$ be a quasi-split unitary group $U(n + 1, n)$ or $U(n, n)$ as described in §1, and keep all the notation from that section. Then we have the Iwasawa decomposition

$$G = PK = NTK$$

and the Cartan decomposition

$$G = KT^{++}K.$$

We also have Bruhat-like decompositions involving the Iwahori subgroup:

$$G = PW P = PW B = BW T B, \quad K = BW B.$$

As before, $X$ consists of the unramified characters of the torus, which we extend to all of $P$ by considering them to be trivial on $N$.

For each $\chi \in X$, define

$$I(\chi) = \{f \in C^\infty(G) : f(pg) = (\chi \delta^{1/2})(p)f(g)\}.$$

$G$ (and subgroups of $G$) acts on $I(\chi)$ by the right regular action. The Iwasawa decomposition gives us $I(\chi) \cong C_c^\infty(P \cap K \setminus K)$ as $K$-modules.
For each $\chi$, there is a natural projection $\mathcal{P}_\chi$ from the larger space $C^\infty_c(G)$ to $I(\chi)$.

$$\mathcal{P}_\chi(f)(g) = \int_P (\chi^{-1}\delta^{1/2})(p)f(pg)\,dp$$

Let $P^b$ be another subgroup of $G$. For our purposes we will be interested in the situations $P^b = P$ and $P^b = P_H$. If $\mathcal{U}$ is any locally closed subset of $G$ which is left $P$-invariant and right $P^b$-invariant (which we will call $P,P^b$-invariant in the sequel), then we can define a representation on $\mathcal{U}$,

$$I(\chi;\mathcal{U}) = \{f \in C^\infty(\mathcal{U}) : f \text{ cpt. supp. mod } P; f(px) = (\chi\delta^{1/2})(p)f(x) \forall p \in P, x \in \mathcal{U}\}.$$ If we assume furthermore that $\mathcal{U}$ is open in $G$, then $I(\chi;\mathcal{U})$ can be seen as a $P^b$-submodule of $I(\chi)$ via extension by zero.

**Proposition 3.1** If $\mathcal{U} \supseteq \mathcal{V}$ are both $P,P^b$-invariant open subsets, then we have an exact sequence of $P^b$-modules

$$0 \longrightarrow I(\chi; \mathcal{V}) \longrightarrow I(\chi; \mathcal{U}) \longrightarrow I(\chi; \mathcal{U} - \mathcal{V}) \longrightarrow 0,$$

where the injection is extension by zero and the surjection is the natural restriction.

For what follows, put $P = P^b$. Then we can define

$$G_w = (PwP) \cup \left( \bigcup_{\ell(y) > \ell(w)} PyP \right).$$

Then $PwP$ is a closed $P,P$-invariant subset of $G_w$, and $G_w$ is an open $P,P$-invariant subset of $G$. Then we can take $\mathcal{U} = G_w$ and $\mathcal{V} = G_w - (PwP)$ in the above proposition,
giving an exact sequence

\[
0 \longrightarrow I \left( \chi; \bigcup_{\ell(y) > \ell(w)} P_y P \right) \longrightarrow I(\chi; G_w) \longrightarrow I(\chi; P_w P) \longrightarrow 0
\]

for each \( w \in W \).

We will need the following collection of facts about equivariant linear forms later on, when we wish to demonstrate that a particular form is rational. Again, we merely cite the propositions from [KSM].

Let \( Q \) be an algebraic subgroup of \( G \) such that \( P \backslash G/Q \) is finite, and let \( \rho_\sigma \) be a family of one-dimensional representations parametrized by some Zariski open subset of \( \mathbb{C}^s \).

**Lemma 3.2** Let \( \mathcal{O} \in P \backslash G/Q \). Then \( \dim \text{Hom}_Q(I(\chi; \mathcal{O}), \rho) \leq 1 \).

**Proposition 3.3** Suppose the following.

- There is a unique open \( P, Q \)-orbit \( \mathcal{O} \).
- There is an open dense subset of \( X \times Y \) on which

\[
\text{Hom}_Q((U(\chi; \mathcal{O}), \rho_\sigma) = 0
\]

for all orbits other than the open orbit.

Then the restriction map \( \text{Hom}_Q(I(\chi), \rho_\sigma) \to \text{Hom}_Q(I(\chi; \mathcal{O}_0), \rho_\sigma) \) is injective in the open dense subset. If \( U \) is any \( P, Q \) invariant open subset,

\[
\dim \text{Hom}_Q(I(\chi), \rho_\sigma) \leq 1.
\]
Proposition 3.4 Suppose that $Q \subset P$ and suppose that for some open subset $Z^+$, there is a family of non-zero forms

$$l_{\chi,\sigma} \in \text{Hom}_Q(I(\chi), \rho_\sigma).$$

Suppose further that the restrictions of $l_{\xi,\sigma}$ to $l(\chi; Pw_{\ell}P)$ depend rationally on the parameters. Now let $T_w : I(w^{-1} \chi) \to I(\chi)$ be the standard intertwining operator. By uniqueness, there is a scalar factor $a(w, \chi, \sigma)$ such that

$$T_w^* l_{\chi,\sigma}|_{I(w^{-1} \chi; Pw_{\ell}P)} = a(w, \chi, \sigma)l_{\chi,\sigma}|_{I(w^{-1} \chi; Pw_{\ell}P)}.$$

Assume that $a(w_\alpha, \chi, \sigma)$ depends rationally on the parameters in the case where $\alpha$ is a simple root.

Then $l_{\chi,\sigma}$ depends rationally on $(\chi, \sigma)$. In particular, for generic parameters, $l_{\chi,\sigma}$ is defined and spans the space $\text{Hom}_Q(I(\chi), \rho_\sigma)$. 

20
CHAPTER 4
Orbits

Write $P_H = P'U$, which is a Borel subgroup of $H = G'U$. Then $P_H$ has unipotent radical $N_H = N'U$, and so $P_H = T'N_H = N_HT'$. We will be interested in the $P \times P_H$ orbits, which is to say $P \backslash G/P_H$. We will pass (via $g \mapsto g^{-1}$) back and forth between $P \backslash G/P_H$ and $P_H \backslash G/P$ as convenience dictates.

**Proposition 4.1**

$$G = \bigcup_{w \in W} \bigcup_{y \in \{0, 1\}^{n-}} Pwg^\circ(y)P_H.$$  

**Proof.** This would be trivial if we allowed $y \in E^{n-}$; in that case it is just Bruhat. Thus the only thing to notice is that the coordinates of $y$ can be adjusted by scalars via conjugation by diagonal matrices (which are then absorbed into $P, P_H$). Thus the orbit depends only on which coordinates are zero and which are not.

**Proposition 4.2** The orbit $O_0 = Pwfg^\circ P_H$ is open and dense in $G$.

**Proof.** $PwFP$ is open and dense in $G$. By the previous proof and the fact that $(E^\times)^{n-}$ is open and dense in $E^{n-}$, $O_0$ is open and dense in $PwFP$.

Actually, rather more than this is true. Notice that the $P, P_H$-double cosets are indexed by Weyl group elements and binary sequences. The Weyl elements have a natural partial order based on the length function, while binary sequences can be ordered by the product order, starting from $0 < 1$. Then the closure of $Pwg^\circ(y)P_H$ is
precisely \( \bigcup P w' g^\circ (y') P_H \), with the union over all \( w', y' \) which are smaller than \( w, y \) in the sense of this ordering. Thus for example \( P g^\circ (0) P_H = P P_H = PU = P \) is closed.

**Proposition 4.3**

\[ \mathcal{O}_0 \cong P \times P_H \cong P \times P' \times U. \]

**Proof.** In the light of the preceding, this is almost obvious. It is easy to check that commuting a nontrivial element of \( P \) past \( w_{\ell} g^\circ \) never lands in \( P_H \). (Notice that the off-diagonal 1’s in \( g^\circ \) prevent nontrivial elements of \( T' \) from commuting past, and the rest is obvious.)

Given two subsets \( I = \{ i_1, \ldots, i_s \} \) and \( J = \{ j_1, \ldots, j_s \} \) of \( \{ 1, \ldots, n \} \) of the same size \( s \) and a matrix \( g \in \text{Mat}_{n \times n}(E) \), we write \( g_{I,J} \) for the \( s \times s \) matrix obtained by considering only the rows indexed by \( I \) and the columns indexed by \( J \). Then

\[ \Delta_{I,J}(g) = \det(g_{I,J}). \]

We allow the case \( s = 0 \), and then \( \Delta_{\emptyset, \emptyset}(g) \equiv 1. \)

Then we define a collection of polynomial functions on \( G \) according to

\[ \alpha_i(g) = \Delta_{\{1, \ldots, i\}, \{1, \ldots, i\}}(w_{\ell} g) \]

Then in particular we have

\[ \alpha_i(g^\circ w_{\ell}) = 1 \]

and

\[ \alpha_i(p^{(1)} g p^{(2)}) = \left( t^{(1)}_1 \cdots t^{(1)}_i \right)^{-1} \left( t^{(2)}_1 \cdots t^{(2)}_i \right) \alpha_i(g) \]

for \( p^{(i)} \in P \), writing as before \( t_i \) for the diagonal entries of a triangular matrix.

In terms of the dominant weights \( \varpi_i = \varepsilon_1 + \cdots + \varepsilon_i \) and \( \varpi_j = \varepsilon'_1 + \cdots + \varepsilon'_j \), this formula says that \( \alpha_i \) is a highest weight vector with weight \( (\varpi_i, \varpi'_{i-r}) \) for \( P \times P' \) (where \( \varpi'_k = 0 \) for \( k \leq 0 \)).
We also define
\[ \beta_j(g) = \Delta_{\{1, \ldots, r+j-1, n_+\}, \{1, \ldots, r+j\}}(w_\ell g) \]
if \( n \) is odd and
\[ \beta_j(g) = \Delta_{\{1, \ldots, r+j-1, n_+\}, \{1, \ldots, r+j\}}(w_\ell g) - \Delta_{\{1, \ldots, r+j-1, n_++1\}, \{1, \ldots, r+j\}}(w_\ell g) \]
if \( n \) is even.

Then in particular we have
\[ \beta_j(p^{(1)}g^{(2)}) = \left(\bar{t}^{(1)}_1 \cdots \bar{t}^{(1)}_{j-1}\right)^{-1} \left(t^{(2)}_1 \cdots t^{(2)}_{r+j}\right) \beta_j(g) \]
for \( p^{(1)} \in P_H \) and \( p^{(2)} \in P \), writing as before \( t_i \) for the diagonal entries of a triangular matrix.

(By linearity of the determinant, we can visualize the even case version not as a difference of determinants but as a single determinant of a matrix, one of whose columns is a difference of columns of \( w_\ell g \).)

Then the unique open orbit in \( P_H \backslash G/P \) is picked out by the \( \alpha_i \) and \( \beta_j \) in the following way.

**Proposition 4.4** For \( g \in G \), \( g \) is in the open orbit \( P_H g^\circ w_\ell P \) iff \( \alpha_i(g), \beta_j(g) \neq 0 \) for all \( i, j \).

**Proof.** We know that \( G = \coprod_{y \in \{0, 1\}} \coprod_{w \in W} P_H g^\circ(y) w P \). Suppose \( g \in P_H g^\circ(y) w P \).

Then \( \alpha_i(g) = 0 \) for some \( i \) iff \( w \) leaves some positive root \( \epsilon_i - \epsilon_{i+1} \) positive, so \( w \neq w_\ell \).

The \( \alpha_i \), taken together, keep track of which Weyl element is in the coset. If all the \( \alpha_i(g) \) are nonzero, so that \( w = w_\ell \), then \( \beta_j = 0 \) iff the appropriate coordinate of \( y \) is 0. Thus the \( \beta_j \), taken together, keep track of which \( g^\circ(y) \) is used.
Now suppose we know \textit{a priori} that \( g \) is in the open orbit,

\[
g \in P_H g^w P = N_H T^g w T N,
\]

and write

\[
g = n_H d'(t'_1, \ldots, t'_{n_1}) g^w d(t_1, \ldots, t_{n_1}) n.
\]

Then the various \( t_i, t'_j \) can be expressed in terms of the \( \alpha_i(g), \beta_j(g) \), at least up to units. This is a simple consequence of the transformation laws by \( P \) and \( P_H \) given above.

**Proposition 4.5** If \( g = n_H d'(t'_1, \ldots, t'_{n_1}) g^w d(t_1, \ldots, t_{n_1}) n \) for some \( n \in N, n_H \in N_H \). Then we have the following.

\[
|t_i| = \left\lvert \frac{\alpha_i(g)}{\alpha_{i-1}(g)} \right\rvert \quad i \leq r
\]

\[
|t_i| = \left\lvert \frac{\beta_{i-r}(g)}{\alpha_{i-1}(g)} \right\rvert \quad i > r
\]

\[
|t'_j| = \left\lvert \frac{\beta_j(g)}{\alpha_{r+j}(g)} \right\rvert
\]
CHAPTER 5
Double Coset Decomposition

Let us temporarily assume that \( n' + 1 = n, r = 0 \), so that \( G^n = G \). Then we can describe the double coset decomposition \((UK') \backslash G / K\), taking as our starting point the Iwahori-type decomposition \( BWT^{++} K \) and moving by stages to our goal.

Let \( \mathcal{V} = K' \cdot \{ g^c(y) : y_i \in \mathfrak{o}_E \} \) and

\[
N^{-}_{w,(1)} = \prod_{\alpha < 0, w^{-1} \alpha > 0} X_{\alpha,(1)},
\]

where \( X_{\alpha,(1)} \) is by definition the subgroup of \( X_\alpha \) which is the identity modulo \( \pi \). In this case we can be more explicit and say \( X_{\alpha,(1)} = \{ x_\alpha(t) : t \in \pi \mathfrak{o}_E \} \).

Lemma 5.1

\[
G = \bigcup_{w \in W} \mathcal{V} N^{-}_{w,(1)} w T^{++} K
\]

Proof. We consider individually the various \( S_w = BwT^{++} K \) which occur in the Iwahori decomposition. Since \( B = N_{(0)} T_{(0)} N_{(1)}^{-} \),

\[
S_w = N_{(0)} T_{(0)} N_{(1)}^{-} w T^{++} K.
\]

Decompose

\[
N_{(1)}^{-} = \prod_{\alpha < 0} X_{\alpha,(1)} = \left( \prod_{\alpha < 0, w^{-1} \alpha > 0} X_{\alpha,(1)} \right) \left( \prod_{\alpha < 0, w^{-1} \alpha < 0} X_{\alpha,(1)} \right).
\]
The second part of this product contains unipotents which will remain lower triangular when commuted past \( w \). Commuting a lower-triangular matrix (from left to right) across an element of \( T^{++} \) will only introduce higher powers of \( \pi \), and in particular it will leave the matrix integral. Thus the second part of the product can be commuted through \( w \) and \( T^{++} \) and absorbed into \( K \). Thus

\[
S_w = N(0)T(0) \left( \prod_{\alpha < 0; w^{-1}\alpha > 0} X_{\alpha,1} \right) wT^{++}K.
\]

However, \( N(0) \) and \( T(0) \) are evidently contained in \( V \) (recall that \( r = 0 \), which is necessary here), so

\[
S_w \subseteq V \left( \prod_{\alpha < 0; w^{-1}\alpha > 0} X_{\alpha,1} \right) wT^{++}K =: U_w.
\]

Thus we can replace the \( S_w \) from the Iwahori decomposition by the (larger in general) \( U_w \), which appears in the lemma.

**Lemma 5.2**

\[
G = \bigcup_{y_i \in \Theta_E} K'g^\phi(y)T^{++}K
\]

**Proof.** The content of this lemma is that we can actually replace \( \bigcup U_w \) by the single construct \( U_1 \). The proof, then, is by induction on \( \ell(w) \); we show that, unless \( w = 1 \), \( U_w \) is contained in \( U_y \) for some shorter \( y \).

Suppose \( \ell(w) > 0 \); then there is a simple root \( \alpha > 0 \) so that \( w^{-1}(\alpha) < 0 \). Then \( w = w_\alpha w' \) with \( \ell(w') < \ell(w) \). Since \( \alpha \) is simple, \( N_{w_\alpha,1} = X_{-\alpha,1} \). Also,

\[
N_{w,1}^- = N_{w_\alpha,1}^- w_\alpha N_{w',1}^- w'.
\]

Also, by the choice of \( \alpha \), \( X_{-\alpha,0} \) commutes (from left to right) through \( N_{w',1}^- w' \) into
(actually we could be more precise about this, but already this gives us sufficient control).

First, if \( \alpha = \varepsilon_i - \varepsilon_{i+1} \) for \( i < n \), then we have \( \mathcal{V}X_{-\alpha,(1)}w_\alpha \subseteq \mathcal{V} \) for more or less trivial reasons. \( X_{-\alpha,(1)}w_\alpha \) consists of matrices in \( K' \) which normalize the set \( \{ g^\circ(y) : y_i \in \mathfrak{o}_E \} \).

The root \( \alpha = \varepsilon_n - \varepsilon_{n+1} \) (up to the involution this is the only case remaining) is only slightly trickier. The idea is to use the commutation relation mentioned above to commute \( X_{-\alpha,(1)} \), which is not in \( G' \), to the right, giving a lower triangular matrix which can be commuted through \( T^{++} \) and absorbed into \( K \).

**Lemma 5.3**

\[
G = \bigcup_{(\lambda,\lambda') \in \Lambda^+ \times \Lambda'^+} UK'(\lambda')g^\circ t(\lambda)K
\]

**Proof.** Now we replace \( g^\circ(y) \) with the single representative element \( g^\circ \) at the expense of introducing the torus \( T' \) and the unipotent \( U \).

We only have to compute that for integral \( y, g^\circ(g^\circ(y))^{-1} \in UT^{++} \), which is simple, and to recall that \( G' \) normalizes \( U \) so that we can pull the \( U \) to the far left.

Now we drop the assumption that \( r = 0 \) and return to the general case. Let \( \Lambda_1^+ \) be the analogue of \( \Lambda^+ \) for \( G^\circ \). Let \( \Lambda^0 \subset \Lambda \) be defined by \( \mathbb{Z}^r \times \Lambda_1^+ \). That is, if we ignore the first \( r \) entries from \( \lambda \in \Lambda^0 \), then the remaining entries satisfy the usual nonincreasing property, but there is no condition whatsoever on the the first \( r \) entries.

**Theorem 5.4**

\[
G = \bigcup_{(\lambda,\lambda') \in \Lambda^0 \times \Lambda'^+} UK'(\lambda')g^\circ t(\lambda)K
\]

**Proof.** We reduce to the case of the previous lemma. We have an Iwasawa-type decomposition

\[
G = Q^\circ K = U^\circ M^\circ K.
\]
The map $\mu$ realizes an isomorphism $M^\circ \cong GL_r(E) \times G^\sharp$ which respects integer matrices, so that furthermore

$$M^\circ/(M^\circ_0) \cong (GL_r(E)/GL_r(o_E)) \times (G^\sharp/G^\sharp_0).$$

Then we can apply the previous lemma to $G^\sharp$. For $GL_r(E)$ we have the usual Iwasawa decomposition

$$GL_r(E) = \bigcup_{(k_1, \ldots, k_r) \in \mathbb{Z}^r} Z \left( \begin{array}{cccc} \pi^{k_1} & & & \\ & \pi^{k_2} & & \\ & & \ddots & \\ & & & \pi^{k_r} \end{array} \right) GL_r(o_E),$$

where we write $Z$ for the upper-triangular unipotent matrices in $GL_r(E)$. Then we use $\mu$ to patch these decompositions into a decomposition for all of $M^\circ/(M^\circ_0)$. Then we can insert that data into $G = U^\circ M^\circ K$, giving exactly the decomposition required.

Then all that remains is the issue of disjointness of the double cosets. If we unravel what it means for the $\alpha_i$ and the $\beta_j$ to be highest weight vectors for the $G$ and $G'$ actions with the appropriate weights, then the decomposition above yields more information. We see that $g \in UK't'(\lambda')g^\circ t(\lambda)K$ if and only if

$$\min_{k',k} v(f(k'gk)) = \begin{cases} -\langle \varpi_i, \lambda \rangle & f = \alpha_i, 1 \leq i \leq r \\ -\langle \varpi_r+j, \lambda \rangle - \langle \varpi'_j, \lambda' \rangle & f = \alpha_{r+j}, 1 \leq j \leq n'_- \\ -\langle \varpi_r+j, \lambda \rangle - \langle \varpi'_{j-1}, \lambda' \rangle & f = \beta_j \end{cases}$$

In particular, the double cosets given are disjoint.

28
We will (abusively) write $F(\lambda, \lambda')$ for $F(t'(\lambda')g^\varphi t(\lambda))$. The previous theorem has an immediate application.

**Corollary 6.1** A Gelfand-Graev function $F \in GG(\xi, \xi')$ is completely determined by the values $F(\lambda, \lambda')$ for $\lambda \in \Lambda^0, \lambda' \in \Lambda^+$. 

This is progress toward uniqueness but still a bit too crude. Now we will show that if $\lambda \notin \Lambda^+$, the invariance properties of $F$ guarantee that $F(\lambda, \lambda') = 0$. (This is the step where we crucially use the conductor assumption on $\psi$.)

If $\lambda \in \Lambda^0$ but $\lambda \notin \Lambda^+$, then we must have $\lambda_i < \lambda_{i+1}$ for some $i \leq r$. We consider the cases $i < r$ (easy) and $i = r$ (harder) separately.

**Case** $i < r$. Let $\alpha = \varepsilon_i - \varepsilon_{i+1}$ be the simple root corresponding to the place where the $\lambda_j$ fails to be nondecreasing. Then

$$t'(\lambda')g^\varphi t(\lambda)x_\alpha(1) = t'(\lambda')g^\varphi x_\alpha(\pi^{\lambda_i-\lambda_{i+1}})t(\lambda) = x_\alpha(\pi^{\lambda_i-\lambda_{i+1}})t'(\lambda')g^\varphi t(\lambda),$$

where the first equality is easy (just commuting a diagonal matrix past a very sparse matrix) and the second comes because $X_\alpha$ does not interact with the central block where $G^d$ lives, so the matrices commute. Since $x_\alpha(1) \in K$, we have

$$F(\lambda, \lambda') = F(t'(\lambda')g^\varphi t(\lambda)x_\alpha(1));$$
on the other hand, \( x_\alpha \) maps into \( U \) (and the \((i, i + 1)\) entry is one of the entries seen by \( \psi \)), so we can use the transformation law for \( U \) to see

\[
\psi(\pi^{\lambda_i-\lambda_{i+1}}) F(\lambda, \lambda') = F(x_\alpha(\pi^{\lambda_i-\lambda_{i+1}})t'(\lambda')g^\circ t(\lambda)).
\]

Putting this together,

\[
F(\lambda, \lambda') = \psi(\pi^{\lambda_i-\lambda_{i+1}}) F(\lambda, \lambda').
\]

Since \( \lambda_i - \lambda_{i+1} < 0 \), we are evaluating \( \psi \) at a non-integer; \( \psi \) has conductor \( \mathfrak{o}_E \), so the character is not trivial there. Thus \( F(\lambda, \lambda') = 0 \).

**Case** \( i = r \). Now write \( \alpha = \varepsilon_r - \varepsilon_{r+1} \). This case is trickier than the previous because \( X_\alpha \) does not commute with \( G^2 \), and the \( r, r + 1 \) entry is not seen by \( U \). We compute carefully. Write

\[
x = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \\
0 & 0 & 0 & 0 \\
1 & 0 & \cdots & 0
\end{pmatrix} \in \text{Mat}_{n-2r, r}
\]

\[
x' = \begin{pmatrix}
-1 & 0 & \cdots & 0 \\
-1 & 0 & \cdots & 0 \\
\vdots & \vdots & & \\
-1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & & \\
0 & 0 & 0 & 0 \\
1 & 0 & \cdots & 0
\end{pmatrix} \in \text{Mat}_{n-2r, r},
\]
where $x'$ has $-1$ in the first $n'_+$ of its $n'$ rows. Now we can commute $x_\alpha(1)$ past one factor at a time to obtain

$$t'(\lambda')g^\circ t(\lambda)x_\alpha(1) = t'(\lambda')g^\circ x_\alpha(\pi^{\lambda_r - \lambda_{r+1}})t(\lambda) = t'(\lambda')\nu(\pi^{\lambda_r - \lambda_{r+1}}x, 0)t(\lambda) = t'(\lambda')\nu(\pi^{\lambda_r - \lambda_{r+1}}x', 0)g^\circ t(\lambda) = \nu(\pi^{\lambda_r - \lambda_{r+1}}d'(\pi^{\lambda_1}, \ldots, \pi^{\lambda_m})x', 0)t'(\lambda')g^\circ t(\lambda).$$

As before, $x_\alpha(1) \in K$, so we have

$$F(\lambda, \lambda') = F(t'(\lambda')g^\circ t(\lambda)x_\alpha(1)).$$

On the other hand, we see that the only nonzero entry of

$$\nu(\pi^{\lambda_r - \lambda_{r+1}}d'(\pi^{\lambda_1}, \ldots, \pi^{\lambda_m})x', 0)$$

which can be seen by $\psi$ is the $n'_+ + 1$, 1 entry, which is $-\pi^{\lambda_r - \lambda_{r+1}}$. Thus we can again invoke the $U$ transformation law to obtain

$$F(\lambda, \lambda') = \psi(-\pi^{\lambda_r - \lambda_{r+1}})F(\lambda, \lambda').$$

As before, by assumption $\psi(-\pi^{\lambda_r - \lambda_{r+1}}) \neq 1$, so again $F(\lambda, \lambda') = 0$.

Thus we have proven the following refinement of the previous proposition.
Proposition 6.2 Suppose $F \in GG(\xi, \xi')$. Then $F$ is supported on

$$\bigcup_{(\lambda, \lambda') \in \Lambda^+ \times \Lambda'^+} UKt'(\lambda')g^o t(\lambda)K.$$ 

The function $F$ is completely determined by the values $F(\lambda, \lambda')$ for $\lambda \in \Lambda^+$ and $\lambda' \in \Lambda'^+$. 

32
CHAPTER 7
Uniqueness of Gelfand-Graev Functions

We have used the transformation/invariance laws for $U, K', K$ to treat Gelfand-
Graev functions as functions on $\Lambda^+ \times \Lambda'^+$, and we have now exhausted the information
we can obtain from this condition on Gelfand-Graev functions. To cut down the possi-
bilities further (and ultimately establish uniqueness), we must use the transformation
laws for the Hecke algebras.

First we need two technical lemmas that allow us to control the intersections of
various kinds of double cosets that will arise in the integrals associated with computing
the action of the algebras.

We define an order $\succeq$ on $\Lambda^+ \times \Lambda'^+$. We write $(\mu, \mu') \succeq (\lambda, \lambda')$ iff all of the following
hold.

$$\mu_i = \lambda_i \quad i \leq r$$

$$\sum_{s=1}^{j} \mu_{r+s} + \sum_{t=1}^{j} \mu'_t \geq \sum_{s=1}^{j} \lambda_{r+s} + \sum_{t=1}^{j} \lambda'_t \quad j \leq n'_-$$

$$\sum_{s=1}^{j} \mu_{r+s} + \sum_{t=1}^{j-1} \mu'_t \geq \sum_{s=1}^{j} \lambda_{r+s} + \sum_{t=1}^{j-1} \lambda'_t \quad j \leq n'_-$$

Notice that this is a well-ordering, so that it will make sense to induct on $(\lambda, \lambda')$. 

33
Lemma 7.1 Suppose $\lambda, \mu \in \Lambda^+$ and $\lambda', \mu' \in \Lambda'^+$. If

$$K't'(\mu')Kt(\mu)^{-1}K \cap U K't'(\lambda')g^\omega w_t(\lambda)^{-1}K \neq \emptyset,$$

then $\mu \succcurlyeq \lambda$.

The proof of this lemma is just a technical unravelling of the statement of the highest weights of $\alpha_i$ and $\beta_j$ in the context of the bialgebra structure of $\mathfrak{o}[G]$. Writing that out formally gives a set of conditions on $\lambda, \lambda', \mu, \mu'$. The ordering $\succcurlyeq$ was defined specifically by those conditions.

Lemma 7.2 Suppose $\mu \in \Lambda^+$ and $\mu' \in \Lambda'^+$. If $u \in U$ is such that

$$K't'(\mu')Kt(\mu)^{-1}K \cap uK't'(\mu')g^\omega w_t(\mu)^{-1}K \neq \emptyset,$$

then $\psi(u) = 1$.

Proof. We need to show that if $t'(\mu')kt(\mu) = uk't'(\mu')g^\omega t(\mu)k_1$ with $u \in U$, $k, k_1 \in K$, $k' \in K'$, then $\psi_U(u) = 1$. We follow the model in [KMS].

Prove this by induction on $r$. If $r = 0$, then $U$ is the trivial subgroup and there is nothing to prove, so assume $r \geq 1$. Let $g = t'(\mu')kt(\mu) = uk't(\mu)k_1$. Consider the last row of this matrix. Let $x_1, x$ be the respective last rows of $k_1, k$. Then on the one hand, the last row of $g$ is $xt(\mu)$; on the other, it is $\pi^{-\mu_1}x_1$. Thus the row vector $v = \pi^{\mu_1}xt(\mu) = x_1$ is primitive in the sense that it has integer entries not all divisible by $\pi$. (The entries are integers because $x_1$ is part of a matrix with integer entries, and if they were all divisible by $\pi$, then the inverse of $k_1$ would have non-integer entries in its last column, a contradiction.) Suppose that the first $\alpha$ entries of $\mu$ are equal to $\mu_1$. Writing $v = (v_1, v_2, \ldots, v_n)$, one of the last $\alpha$ entries must be in $\mathfrak{o}^\times$ (since the others are certainly divisible by $\pi$), so suppose it is $v_{n+1-i}$. Choose an element
$w$ of the Weyl group interchanging $1, i$. Make the following notation, moving toward invoking the inductive hypothesis. Let $w_\ell^\dagger$ be the long element of the Weyl group for $G_\dagger = G_{n-2}$, $K_\dagger = K_{n-2}$, viewed as “centrally” embedded in $G$ in the same way that $G'$ is. In this argument, $3 \times 3$ matrices are interpreted as block matrices according to the partition $(1, n-2, 1)$.

Now the Bruhat decomposition of $K$ modulo $\pi$ guarantees that $kw^{-1}$ is written as

$$kw^{-1} = n_1 \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & k_\ell^\dagger & 0 \\ 0 & 0 & \epsilon^{-1} \end{pmatrix} w_\ell n_2$$

where $n_1, n_2$ are integral and “block unipotent” according to this partition, $\epsilon$ is a unit, and $k_\ell^\dagger \in K_\dagger$.

Now

$$g = t'(\mu')n_1 \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & k_\ell^\dagger & 0 \\ 0 & 0 & \epsilon^{-1} \end{pmatrix} w_\ell n_2 t(\mu)$$

$$= [t'(\mu')n_1 t'(\mu')^{-1}] t'(\mu') \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & k_\ell^\dagger & 0 \\ 0 & 0 & \epsilon^{-1} \end{pmatrix} t(\mu) [t(\mu)^{-1} w_\ell n_2 w_\ell t(\mu)].$$

Notice that $\psi_U(n_1) = 1$ by our characterization of $\psi_U$, and also, by easy computation, that $[t(\mu)^{-1} w_\ell n_2 w_\ell t(\mu)] \in K$. But now, since

$$kt(\mu) = uk' t(\mu) k_1,$$

we have

$$k_\ell^\dagger t(\mu_\ell^\dagger) = u_1^\dagger k_1' t(\mu_\ell^\dagger) k_1^\dagger,$$
where $\mu^\dagger$ is just $(\mu_2, \ldots, \mu_n)$. Apply the induction hypothesis to see that $\psi_U(u^\dagger) = 1$, and then we are done because $u = t'(\mu')n_1(t'(\mu')^{-1}u^\dagger$ and $\psi_U$ is a quasi-character.

**Theorem 7.3 (Uniqueness)** For any $\xi \in X, \xi' \in X'$, we have

$$\dim \mathcal{GG}(\xi, \xi') \leq 1.$$ 

Equivalently, if $F \in \mathcal{GG}(\xi, \xi')$ and $F(0,0) = F(g^\circ) = F(I_n) = 0$, $F$ is identically zero.

**Proof.** Assume that $F \in \mathcal{GG}(\xi, \xi')$ and $F(0,0) = 0$. For each $\mu, \mu'$, we have

$$\int_{K't'(\mu')Kt(\mu)K} F(g) \, dg = (L(ch_{K't'(\mu')^{-1}K'})R(ch_{Kt(\mu)K})F)(I_n) = \omega_\xi(ch_{Kt(\mu)K}) \omega'_{\xi'}(ch_{K't'(\mu')^{-1}K'}) F(I_n) = 0$$

by the Hecke character transformation law. On the other hand, $K't'(\mu')Kt(\mu)K$ is left $K'$- and right $K$-invariant, so by our main double coset decomposition we can write it as a union of simple cosets

$$K't'(\mu')Kt(\mu)K = \bigsqcup_i u(i)K't'(\lambda'_i)g^\circ t(\lambda_i)K.$$ 

Since $F$ is constant on these cosets, we have

$$\int_{K't'(\mu')Kt(\mu)K} F(g) \, dg = \sum_i \psi(u(i)) \text{vol}(K't'(\lambda'_i)g^\circ t(\lambda_i)K) F(\lambda_i, \lambda'_i)$$

$$= \sum_{(\lambda, \lambda')} \left( \sum_{\ell: \lambda(\ell) = \lambda, \lambda'_i(\ell) = \lambda'} \psi(u(i)) \text{vol}(K't'(\lambda')g^\circ t(\lambda)K) \right) F(\lambda, \lambda').$$

We write $c_{\lambda, \lambda', \mu, \mu'}$ for the inner summation. The two lemmas of the previous section give that $c_{\lambda, \lambda', \mu, \mu'} = 0$ unless $(\mu, \mu') \succ (\lambda, \lambda')$, and that $c_{\mu, \mu', \mu, \mu'}$ is a positive real
number. Thus we obtain a family of recursive relations

$$\sum_{(\mu, \mu') \geq (\lambda, \lambda')} c_{\lambda, \lambda', \mu, \mu'} F(\lambda, \lambda') = 0,$$

which can be recursively solved to obtain $F(\mu, \mu') = 0$ for each $\mu \in \Lambda^+, \mu' \in \Lambda'^+$. 
CHAPTER 8
Regional Construction

Consider $\xi, \xi', \psi$ fixed for the time being.

Recall the unramified principal series representation $I(\xi)$ and the group $H = G'U$ (a semidirect product). We let $H$ act on $I(\xi)$ via $H \mapsto H/U = G'$. Then we have a representation of $H$

$$I(\xi, \psi) = \text{Ind}_H^P (\xi\delta^{1/2} \otimes \psi),$$

where $U$ acts by the character and $G'$ acts by the right regular action. We also have a natural $G'$-invariant pairing on $I(\xi', \psi) \times I(\xi^{-1}, \psi^{-1})$ given by the formula

$$\langle \phi_1', \phi_2' \rangle = \int_{K'} \phi_1'(k') \phi_2'(k') \, dk'.$$

Then [KMS] gives the following method of generating Gelfand-Graev functions.

**Proposition 8.1** Any nonzero homomorphism $T \in \text{Hom}_H(I(\xi), I(\xi^{-1}, \psi^{-1}))$ yields a nonzero Gelfand-Graev function

$$S_T(g) = \langle \phi_{\xi'}, T(R(g)\phi_{\xi}) \rangle \in GG(\xi, \xi').$$

Suppose that we have a continuous function\(^3\) $Y$ satisfying

$$Y(pgp'u) = (\xi^{-1}\delta^{1/2})(p)(\xi'\delta'^{-1/2})(p')\psi(u)Y(g) \quad \forall p \in P, p' \in P', u \in U. \quad (\star)$$

\(^3\)In fact it would be good enough to have a continuous distribution, but in our case we will be able to find a *bona fide* function.
Then integration against $Y$ gives an equivariant linear form
\[ l = l_{\xi,\xi'} \in \text{Hom}_{P_H}(I(\xi),\xi'^{-1}\delta^{1/2} \otimes \psi^{-1}) \]
defined by
\[ l_{\xi,\xi'}(P_{\xi}(f)) = \int_G f(g)Y(g) \, dg. \]

Here $P_{\xi}$ is the projection from the larger space $C_c^\infty(G)$ to $I(\xi)$.

\[ P_{\xi}(f)(g) = \int_{P}(\xi^{-1}\delta^{1/2})(p)f(pg) \, dp. \]

Then by Frobenius reciprocity there is an intertwining operator
\[ T = T_{\xi,\xi'} \in \text{Hom}_H(I(\xi),I(\xi'^{-1},\psi^{-1})) \]
corresponding to $l$. Then $T$ is given by
\[ T_{\xi,\xi'}(P_{\xi}(f))(g') = l_{\xi,\xi'}(R(g')P_{\xi}(f)) = \int_G f(gg')Y(g) \, dg \]
and as already noted, $S = S_T = S_{T_{\xi,\xi'}}$ will be a Gelfand-Graev function. In terms of $Y$, the Gelfand-Graev function is given by
\[ S(g) = \int_{K' \times K} Y(kg^{-1}k') \, dk' \, dk. \]

The issue now is the construction of a function $Y$ with the correct transformation law under $P$ on the left and $P'U$ on the right. Equivalently, writing $\Upsilon(g) = Y(g^{-1})$, we will find a function that transforms by $UP'$ on the left and $P$ on the right.
Actually existence is not a problem, since Proposition 4.3 guarantees that

$$\mathcal{O}_0 = UP'w_\ell g^\circ P$$

does not have any “interaction” between the left and right actions. This orbit will support the desired transformation law for any $\xi, \xi'$. It is not hard to see that no other orbit in $UP'\backslash G/P$ is isomorphic to $\mathcal{O}_0$ and therefore for generic characters $\xi, \xi'$ $\Upsilon$ will be supported on the open orbit.\(^4\) Up to scalar, the only real candidate is given by

$$\Upsilon(g) = \begin{cases} 
\psi^{-1}(u)(\xi \delta^{-1/2})(p)(\xi' \delta^{1/2})(p') & g = up'g^\circ w_\ell p \in \mathcal{O}_0 \\
0 & g \not\in \mathcal{O}_0
\end{cases}$$

Since each element of the open orbit has a unique representation, there is no question of whether $\Upsilon$ is well-defined. Rather, the issue is whether $\Upsilon$ is continuous. To better understand this we “massage” the definition of $\Upsilon$ to reveal a form more naturally related to the open orbit structure.

**Proposition 8.2** If $g \in uP'w_\ell g^\circ P$, then

$$\Upsilon(g) = \psi(u)^{-1} \left( \prod_{i=1}^{r} (\xi_i \xi_{i+1}^{-1} \cdot |\cdot |^{-1})(\alpha_i(g)) \right) \left( \prod_{j=1}^{n'} (\xi'_j \xi^{-1}_{r+j} \cdot |\cdot |^{-1/2})(\alpha_{r+j}(g)) \right) \times \xi_{n'}' (\alpha_{n}(g)) \left( \prod_{k=1}^{n'} (\xi_{r+k}^{-1} \cdot |\cdot |^{1/2})(\beta_k(g)) \right)$$

\(^4\)That is, if we are not in the open orbit, we can find matrix relations $up'wg^\circ(y) = wg^\circ(y)p$. Then the left and right transformation laws taken together force either $\Upsilon(wg^\circ(y))$ or a relation among $\xi_i, \xi'_j$. The upshot of this is that other orbits can support the transformation law $\star$ *only* for suitably chosen $\xi, \xi'$. Strategic use of this observation will be valuable later on.
in the odd case, and

\[ \Upsilon(g) = \psi(u)^{-1} \left( \prod_{i=1}^{r} (\xi_i \xi_{i+1}^{-1} \cdot |^{-1}) (\alpha_i(g)) \right) \left( \prod_{j=1}^{n'} (\xi_j' \xi_{r+j+1}^{-1} \cdot |^{-1/2}) (\alpha_{r+j}(g)) \right) \]

\[ \times \left( \prod_{k=1}^{n'} (\xi_k'^{-1} \xi_{r+k} \cdot |^{-1/2}) (\beta_k(g)) \right) \xi_{n'} (\beta_{n'}(g)) \]

in the even case.

The proof is just an invocation of the modulus character and Proposition 5.5, followed by routine algebra. (In fact the technique is something like Abel’s “partial summation” or integration by parts, except that here we are operating multiplicatively; we exchange partial products for ratios and vice versa.) We focus on the immediate application of this “factorization”.

Within the orbit, there is evidently no problem. Likewise, \( \Upsilon \) is constant outside \( O_0 \). Since this orbit is open, the only possible counterexample to continuity would be for a sequence of \( g_k \in O_0 \) which approach a point \( g \) outside the orbit. If we can show that \( \Upsilon(g_k) \to 0 \) in all such cases, continuity will be established. However, we now invoke Proposition 4.4 to conclude that this can happen only if some \( \alpha_i(g_k) \to 0 \) or \( \beta_j(g_k) \to 0 \). The previous proposition expresses \( \Upsilon \) as a product including factors for each of the \( \alpha_i, \beta_j \). If the characters evaluated at the \( \alpha_i, \beta_j \) are all chosen so that they take elements near 0 to complex numbers near 0, continuity will be forced. It is not hard to see that this will happen in \( Z \), where \( Z \) is a domain in complex space.
$X \times X'$ defined by

$$Z = \begin{cases} (\xi, \xi') \in X \times X' : & \\
|\xi_i \xi_{i+1}^{-1}| < q^{-1} & 1 \leq i \leq r \\
|\xi_j' \xi_{r+j+1}^{-1}| < q^{-1/2} & 1 \leq j \leq n' \\
|\xi_k' \xi_{r+k}^{-1}| < q^{-1/2} & 1 \leq k \leq n' \\
|\xi_{n'}' | < q^{-1/2} & 
\end{cases}$$

in the odd case and

$$Z = \begin{cases} (\xi, \xi') \in X \times X' : & \\
|\xi_i \xi_{i+1}^{-1}| < q^{-1} & 1 \leq i \leq r \\
|\xi_j' \xi_{r+j+1}^{-1}| < q^{-1/2} & 1 \leq j \leq n' \\
|\xi_k' \xi_{r+k}^{-1}| < q^{-1/2} & 1 \leq k \leq n' \\
|\xi_{n'}' | < q^{-1/2} & 
\end{cases}$$

in the even case. We have proven the following theorem.

**Theorem 8.3 (Weak Existence)** For $(\xi, \xi') \in Z$, $GG(\xi, \xi')$ is nontrivial.
In this section we evaluate various integrals which will be important for the rationality argument and also for the explicit formula for Gelfand-Graev functions with generic data. For the purposes of this section, we assume that the data \((\xi, \xi')\) is in \(Z\), so that we can reference the construction of the previous section.

The Bruhat decomposition \(K = BWB\) leads to a natural basis of the \(B\)-fixed vectors of the unramified principal series representation, \(I(\xi)^B\), which is given by \(\{\phi_w = P_\xi(ch_{BwB})\}_{w \in W}\). Likewise, we have a basis for \(I(\xi')^B'\) given by \(\{\phi_w' = P_{\xi'}(ch_{B'w'B'})\}_{w' \in W'}\), where we maintain our “prime everything for the subgroup” notational convention.

For simple roots \(\alpha\) of \(G\) (resp. \(\beta\) of \(G'\)), we need to evaluate

\[
I_\alpha = \text{vol}(B)^{-1}\text{vol}(B')^{-1}\Omega\left(\phi_1', R(g^\circ w_\ell)(\phi_1 + \phi_{w_\alpha})\right)
\]

and

\[
J_\beta = \text{vol}(B)^{-1}\text{vol}(B')^{-1}\Omega\left(\phi_1' + \phi_{w_\beta}', R(g^\circ w_\ell)(\phi_1)\right).
\]

Here \(\Omega\) is a a bilinear form from \(I(\xi', \psi) \times I(\xi)\) to \(\mathbb{C}\) defined in terms of \(Y\) via the following formula.

\[
\Omega(P_{\xi'}(f'), P_{\xi}(f)) = \int_{G' \times G} f'(x')f(x)Y(xx'^{-1}) \, dx' \, dx \quad f' \in I(\xi', \psi), f \in I(\xi)
\]
Recall
\[ Y(tn_\ell g^\circ t' u) = (\xi^{-1} \delta^{1/2})(t)(\xi \delta'^{-1/2})(t') \psi(u), \]
which is a continuous function by our standing assumptions on \( \xi, \xi' \).

In order to effectively evaluate these integrals, we need a the following lemma.

**Lemma 9.1**

\[
\begin{align*}
N(1) g^\circ &\subseteq T(0) g^\circ T'(0) N'_U(1), \\
N(1) w_\ell g^\circ &\subseteq T(0) w_\ell g^\circ T'(0) N'_U(1), \\
w_\ell g^\circ N'_(U) &\subseteq N w_\ell g^\circ.
\end{align*}
\]

**Proof.** To see the first inclusion, if \( n \in N(1) \), then
\[ ng^\circ = g^\circ(y) n'u, \]
for some \( y, n' \in N'_(1) \), and \( u \in U(1) \). Since \( n \) is congruent to the identity modulo \( \pi \), the coordinates of \( y \) must in fact be *units*. As noted previously, we can adjust \( y \) by scalars via conjugation by diagonal matrices. Since \( y \) consists of units, these diagonal matrices are in \( T(0) \cap T'(0) \). This confirms the first inclusion.

The second inclusion is really the same as the first, since conjugation by \( w_\ell \) exchanges a group with its opposite and fixes \( T(0) \).

We check the third inclusion in two parts. Commuting \( N' \) past \( w_\ell g^\circ \) lands in the opposite unipotent subgroup of \( G \), so that
\[ w_\ell g^\circ N' \subseteq N' w_\ell g^\circ. \]
Since \( w_\ell, g^\circ \) consist of units, this inclusion will still be valid when we restrict to integer matrices which are the identity modulo \( \pi \).
Lemma 9.2

\[ \text{vol}(B)^{-1}\text{vol}(B')^{-1}\Omega(\phi'_1, R(g^\circ w_t\phi_1)) = 1. \]

Proof. By definition,

\[ \Omega(\phi'_1, R(g^\circ w_t\phi_1)) = \Omega(\phi'_1, \mathcal{P}_\xi(ch_{B(g^\circ w_t)})^{-1}) \]

\[ = \int_{B' \times B} Y(x(g^\circ w_t)^{-1}x') \; dx' \; dx \]

\[ = \int_{B' \times B} Y(xw_tg^\circ -1x') \; dx' \; dx. \]

However, notice at this point that \(g^\circ(1,\ldots,1)g^\circ(-1,\ldots,-1) \in N' \cap K. \) Since \(g^\circ(-1,\ldots,-1)\) has unit coefficients, we have already seen that it can be conjugated to \(g^\circ\) via elements in \(T \cap T' \cap K\), so that \(g^\circ -1 \in T_0g^\circ T'_0 N'_0\) in the above. Thus, by a change of variables \(x\) and \(x'\),

\[ \text{vol}(B)^{-1}\text{vol}(B')^{-1}\Omega(\phi'_1, R(g^\circ w_t\phi_1)) = \text{vol}(B)^{-1}\text{vol}(B')^{-1} \int_{B' \times B} Y(xw_tg^\circ -1x') \; dx' \; dx. \]

On the other other hand, we have by the previous lemma

\[ Bw_tg^\circ B' = T_0N_0N_{(1)}^-w_tg^\circ N'^{-1}(1)N_{(0)}T'_0 \]

\[ = T_0N_0N_{(1)}^-w_tg^\circ N'_0T'_0 \]

\[ \subset T_0N_0w_tg^\circ T'_0N_0U_{(1)} \]

\[ = P_0w_tg^\circ P_{H,(0)}. \]

The point is that when we evaluate \(Y\) at elements of this set, the result is trivial. (In controlling the support of Gelfand-Graev functions, we used that the conductor of \(\psi\) was no larger than \(\mathfrak{o}\); here is the first time we are using that \(\psi\) is trivial on \(\mathfrak{o}\).) Thus
we are averaging 1 over \( B' \times B \), and we are done.

Notice that each \( I_\alpha \) and \( J_\beta \) is naturally a sum of two integrals, one of which is what we just evaluated. In order to extend this type of analysis to double cosets \( BwB \) with \( w \neq 1 \), a necessary ingredient in evaluating the “other” pieces, we need to understand the relative volumes of various such sets.

**Lemma 9.3** If \( w = w_\alpha \) is a transposition corresponding to a simple positive root \( \alpha = \varepsilon_i - \varepsilon_{i+1} \),

\[
\text{vol}(BwB) = \begin{cases} 
q_F \text{vol}(B) & n = 2i \\
q_E \text{vol}(B) & \text{otherwise}
\end{cases}
\]

**Proof.** This is standard for orthogonal groups (where there is no distinction between cases, of course), and the reasoning here is the same. The right \( B \)-cosets of \( BwB \) are parameterized by the integer points of the root subgroup \( X_{-\alpha} \), which is to say by the integers, with two integers giving the same coset iff they differ by a multiple of \( \pi \). Thus \( BwB \) is a union of translates of \( B \), corresponding to the elements of the residue field. The two cases occur because the exceptional root subgroup is parameterized by \( F \) rather than \( E \).

**Lemma 9.4** If \( \alpha = \varepsilon_i - \varepsilon_{i+1} \) is a simple root for \( G \), then

\[
I_\alpha = 1 + q_E \int_{E} (\xi \delta^{-1/2})(a^{v(t)}_\alpha)Y(w_t x_\alpha(t^{-1})g^\circ) \, dt
\]

except in the case where \( n = 2i \), in which case

\[
I_\alpha + q_F \int_{F} (\xi \delta^{-1/2})(a^{v(s)}_\alpha)Y(w_t x_\alpha^*(s^{-1})g^\circ) \, ds.
\]
Proof. By Lemma 9.2,

\[ I_\alpha + \text{vol}(B)^{-1}\text{vol}(B')^{-1}\Omega(\phi'_1, R(g^\omega w_\ell)\phi_{w_\alpha}) = 1, \]

and by the same argument as in that lemma,

\[ \Omega(\phi'_1, R(g^\omega w_\ell)\phi_{w_\alpha}) = \int_{B'\times(Bw_\alpha B)} Y(x(g^\omega w_\ell)^{-1}x') \, dx' \, dx \]

\[ = \int_{B'\times(Bw_\alpha B)} Y(wx_\ell g^\omega x') \, dx' \, dx \]

Next notice that the integrand is constant as a function of \( x' \in B' \), so we have a single integral

\[ \text{vol}(B)^{-1}\text{vol}(B')^{-1}\Omega(\phi'_1, R(g^\omega w_\ell)\phi_{w_\alpha})) = \text{vol}(B)^{-1}\int_{Bw_\alpha B} Y(wx_\ell g^\omega) \, dx. \]

We have a decomposition \( Bw_\alpha B = T_{(0)}N_{(0)}w_\alpha X_{\alpha,(0)}N_{(1)}^- \), so that this integral can be reduced to an integral ranging over \( X_\alpha \cap K = \{x_\alpha(t) : t \in \sigma\} \), and so in turn an integral over \( \sigma \).

\[ \text{vol}(B)^{-1}\text{vol}(B')^{-1}\Omega(\phi'_1, R(g^\omega w_\ell)\phi_{w_\alpha})) = \frac{\text{vol}(Bw_\alpha B)}{\text{vol}(B)} \int_{\sigma} Y(w_\alpha x_\alpha(t)w_\ell g^\omega) \, dt. \]

Let \( a_\alpha = t(0, 0, 0, \ldots, 0, 1, -1, 0, \ldots, 0) \) for \( i < n \) and \( a_{\epsilon_n-\epsilon_{n+1}} = t(0, \ldots, 0, 1) \).

Then

\[ x_\alpha(t) = x_{-\alpha}(t^{-1})w_\alpha a_\alpha^{-v(t)}hx_{-\alpha}(-t^{-1}) \]
for some \( h \) in the unit torus. This is just the \( GL_2 \) fact

\[
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
t^{-1} & 1
\end{pmatrix} \begin{pmatrix}
\pi^{-k} & 0 \\
0 & \pi^k
\end{pmatrix} \begin{pmatrix}
u^{-1} & 0 \\
0 & u
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-t^{-1} & 1
\end{pmatrix},
\]

where \( t = u \pi^k \) for a unit \( u \), embedded in \( G \) in a suitable place corresponding to \( \alpha \).

Making use of this,

\[
Y(w_\alpha x_\alpha(t) w_\ell g^\circ) = (\xi^{-1} \delta^{1/2})(a_\alpha^{-v(t)}) Y(w_\ell x_\alpha(-t^{-1}) g^\circ),
\]

and we get the desired formula (taking into account the lemma on relative volume and making a change of variable \( t \mapsto -t \)). To make this all work in the exceptional case, where we are working over \( F \), we have to notice that, although \((s \eta)^{-1}\) is not equal to \( s^{-1} \eta \), they differ by a unit \( \eta^2 \), so a suitable change of variable can be made.

The same argument at the level of \( G' \) gives the corresponding statement for the \( J_\beta \) (but note that the exceptional case now occurs for the odd groups).

**Lemma 9.5** If \( \beta = \varepsilon'_i - \varepsilon'_{i+1} \) is a simple root for \( G' \), then

\[
J_\beta = 1 + q_E \int_{o_E} (\xi' \delta'^{-1/2})(a'_{\beta}^{v(t)}) Y(w_\ell g^\circ x'_{-\beta}(t^{-1})) dt
\]

except in the case where \( 2i = n' \), in which case

\[
J_\beta = 1 + q_F \int_{o_F} (\xi' \delta'^{-1/2})(a'_{\beta}^{v(s)}) Y(w_\ell g^\circ x'_{-\beta}(s^{-1})) ds.
\]

We will have to consider several cases in order to evaluate all the possible integrals that arise. In each case, there will be two stages: first, the unramified toral character evaluated at \( a_\alpha \) or \( a'_\beta \) translates to an unramified character of \( E^\times \) evaluated at \( t \); second, we express the argument of \( Y \) in the form \( pw_\ell g^\circ p' u \) and evaluate \( Y \) directly.
from the definition, which will potentially give a different character. In situations in which two different characters arise, the following elementary fact about integrating two characters against one another will be useful.

**Lemma 9.6 (Stated, not proven in [KMS], Lemma 8.6)** If \( \chi, \chi' \) are unramified characters of \( E \) and \(|\chi|, |\chi'| < q\), then for any unit \( u \),

\[
1 + q \int_\mathfrak{O} \chi(t)\chi'(u + t) \, dt = (q - 1) \frac{1 - q^{-2}\chi\chi'}{(1 - q^{-1}\chi)(1 - q^{-1}\chi')}.
\]

**Proof.** First, the integrand depends only on which powers of \( \pi \) divide \( t \) and \( u + t \), so we can decompose \( \mathfrak{O} \) into a countable family of measurable sets on which the integrand is constant. Define the following sets:

\[
A_i = \{ t \in \mathfrak{O} : v(t) = i \} \quad B_i = \{ t \in \mathfrak{O} : v(u + t) = i \} \quad C = \mathfrak{O}^\times \cap (\mathfrak{O}^\times - u).
\]

Then \( \mathfrak{O} \) is a disjoint union of \( A_1, A_2, \ldots, B_1, B_2, \ldots, C \). (Here we are using the fact that \( t, u + t \) cannot both be divisible by \( \pi \).) Also, \[
\bigcup_{i=k}^{\infty} A_i = \pi^k \mathfrak{O}
\]

has measure \( q^{-k} \), so \( A_k = (\bigcup_{i=k}^{\infty} A_i) - (\bigcup_{i=k+1}^{\infty} A_i) \) has measure \( q^{-k} - q^{-k-1} \). Likewise \( B_k \) has measure \( q^{-k} - q^{-k-1} \). Also \( C = \mathfrak{O} - (\bigcup_{i=1}^{\infty} A_i) - (\bigcup_{i=1}^{\infty} B_i) \) has measure \( 1 - 2q^{-1} \).
Then

\[
\int_0 \chi(t)\chi'(u + t) \, dt = \sum_{k=1}^{\infty} \int_{A_k} \chi(t)\chi'(u + t) \, dt + \sum_{k=1}^{\infty} \int_{B_k} \chi(t)\chi'(u + t) \, dt + \int_C \chi(t)\chi'(u + t) \, dt
\]

\[
= \sum_{k=1}^{\infty} \int_{A_k} \chi^k \, dt + \sum_{k=1}^{\infty} \int_{B_k} \chi'^k \, dt + \int_C \, dt
\]

\[
= \sum_{k=1}^{\infty} \chi^k(q^{-k} - q^{-k-1}) + \sum_{k=1}^{\infty} \chi'^k(q^{-k} - q^{-k-1}) + (1 - 2q^{-1})
\]

\[
= (1 - 2q^{-1}) + (1 - q^{-1}) \left[ \sum_{k=1}^{\infty} (\chi q^{-1})^k + \sum_{k=1}^{\infty} (\chi' q^{-1})^k \right].
\]

By hypothesis on \(\chi, \chi'\), these geometric series converge.

\[
\int_0 \chi(t)\chi'(u + t) \, dt = (1 - 2q^{-1}) + (1 - q^{-1}) \left[ \frac{\chi q^{-1}}{1 - \chi q^{-1}} + \frac{\chi' q^{-1}}{1 - \chi' q^{-1}} \right]
\]

\[
= (1 - 2q^{-1}) + (1 - q^{-1}) \left[ \frac{\chi q^{-1} + \chi' q^{-1} - 2\chi \chi' q^{-2}}{(1 - \chi q^{-1})(1 - \chi' q^{-1})} \right]
\]

\[
= -q^{-1} + (1 - q^{-1}) \left[ 1 + \frac{\chi q^{-1} + \chi' q^{-1} - 2\chi \chi' q^{-2}}{(1 - \chi q^{-1})(1 - \chi' q^{-1})} \right]
\]

\[
= -q^{-1} + (1 - q^{-1}) \left[ \frac{1 - \chi \chi' q^{-2}}{(1 - \chi q^{-1})(1 - \chi' q^{-1})} \right].
\]

Multiplying by \(q\) and adding 1 gives exactly what was desired.

(Whenever we apply this lemma in this section, the \(|\chi|, |\chi'| < q\) condition will turn out to be a restatement of one of the conditions defining the region \(Z\); no further mention will be made of this.)

It is worth making special mention of the case where we are integrating a single character \(\chi\). Since \(1 < q\), we can obtain this result by simply taking \(\chi' = 1\).
Corollary 9.7 If $\chi$ is an unramified character of $E$ and $|\chi| < q$, then

$$1 + q \int_0 \chi(t) \, dt = (q - 1) \frac{1 - q^{-2} \chi}{(1 - q^{-1} \chi)(1 - q^{-1})} = q \frac{1 - q^{-2} \chi}{1 - q^{-1} \chi}.$$ 

In cases where $\psi$ is involved, it is useful to recall another fact from [KMS].

Lemma 9.8 ([KMS], §8.8) Let $du$ be Haar measure on $E^\times$, normalized so that

$$\int_{\mathfrak{o}_E^\times} du = 1,$$

and let $\psi$ be an additive character with conductor $\mathfrak{o}_E$. Then

$$\int_{\mathfrak{o}_E^\times} \psi(\pi^{-k} u) \, du = \begin{cases} 1 & k = 0 \\ -(q - 1)^{-1} & k = 1 \\ 0 & k > 1 \end{cases}.$$ 

Lemma 9.9 (n even, $\alpha = \varepsilon_i - \varepsilon_{i+1}, i < r$)

$$I_\alpha = q \left( 1 - q^{-1} \xi_i \xi_{i+1}^{-1} \right)$$

Proof. First,

$$(\xi^{-1} \delta^{1/2})(a_{\alpha}^{-v(t)}) = \xi_i^{v(t)} \xi_{i+1}^{-v(t)} q^{v(t)} = (\xi_i \xi_{i+1}^{-1} | \cdot |^{-1})(t).$$

Evaluating $Y$ is easy in this case because $X_\alpha$ commutes with $g^\circ$. (If we write $G$ in terms of block matrices by the partition $r, n - 2r, r$, $X_\alpha$ lives in the upper and lower diagonal blocks, and $g^\circ$ is in the center block.)

$$Y \left( w_t x_\alpha (t^{-1}) g^\circ \right) = Y \left( w_t g^\circ x_\alpha (t^{-1}) \right) = \psi \left( x_\alpha (t^{-1}) \right) = \psi (t^{-1}),$$
with the last equality coming from the definition of \( \psi \) as a character on \( U \). Then by Lemma 9.4,

\[
I_\alpha = 1 + q \int_{\phi_E} (\xi \delta^{-1/2})(a_{\alpha}^{-v(t)})Y(w_{x_{\alpha}}(t^{-1})g^o) \, dt \\
= 1 + q \int_{\phi_E} (\xi \xi_{i+1}^{-1} \cdot |^{-1})(t)\psi(t^{-1}) \, dt \\
= 1 + q \sum_{k=0}^{\infty} (1 - q^{-1}) q^{-k} (\xi \xi_{i+1}^{-1})^k q^k \int_{\phi_E} \psi(\pi^{-k}u) \, du \\
= q \left(1 - q^{-1} \xi_i \xi_{i+1}^{-1}\right).
\]

with the last step invoking Lemma 9.8.

**Lemma 9.10** \((n \text{ even}, \alpha = \varepsilon_r - \varepsilon_{r+1})\)

\[
I_\alpha = q \left(1 - q^{-1} \xi_r \xi_{r+1}^{-1}\right)
\]

**Proof.** First,

\[
(\xi^{-1} \delta^{1/2})(a_{\alpha}^{-v(t)}) = \xi_r \xi_{r+1}^{-1} q^v(t) = (\xi \xi_{r+1}^{-1} \cdot |^{-1})(t).
\]

In order to evaluate \( Y \), we must study how \( x_{\alpha}(t^{-1}) \) commutes past \( g^o \). By direct computation,

\[
g^o^{-1} x_{\alpha}(t^{-1}) g^o = x_{\alpha}(t^{-1}) x_{\varepsilon_r - \varepsilon_{r}}(t^{-1}),
\]

so that

\[
Y(\omega_{x_{\alpha}}(t^{-1})g^o) = Y \left( w_{x_{\alpha}}(t^{-1}) x_{\varepsilon_r - \varepsilon_{r+1}}(t^{-1}) \right) \\
= \psi(x_{\alpha}(t^{-1}) x_{\varepsilon_r - \varepsilon_{r+1}}(t^{-1})) = \psi(t^{-1}).
\]
For the last equality, see that \( \psi \) does not see the \((r,r+1)\) entry, but it does see the \((r,n+)\) entry. Then by Lemma 9.4,

\[
I_\alpha = 1 + q E \int_{\phi_E} (\xi^{-1/2}) (a_{\alpha}^{v(t)}) Y (w_E x_\alpha (t^{-1}) g^\circ) \, dt \\
= 1 + q E \int_{\phi_E} (\xi r \xi^{-1}_{r+1}) \cdot |^{-1} (t) \psi (t^{-1}) \, dt \\
= 1 + q \sum_{k=0}^{\infty} (1 - q^{-1}) q^{-k} (\xi r \xi^{-1}_{r+1})^k q^k \int_{\phi_E} \psi (\pi^{-k} u) \, du \\
= q (1 - q^{-1} \xi r \xi^{-1}_{r+1}).
\]

with the last step invoking Lemma 10.7.

**Lemma 9.11** (*n* even, \( \alpha = \varepsilon_i - \varepsilon_{i+1}, r < i < n_+ \))

\[
I_\alpha = (q - 1) \frac{1 - q^{-1} \xi_i \xi^{-1}_{i+1}}{(1 - q^{-1/2} \xi^{-1}_{i-r+1} \xi_i) (1 - q^{-1/2} \xi^{-1}_{i-r} \xi^{-1}_{i+1})}.
\]

**Proof.** First,

\[
(\xi^{-1/2}) (a_{\alpha}^{-v(t)}) = \xi_i^{-v(t)} \xi_{i+1}^{-v(t)} q^{v(t)} = (\xi_i \xi_{i+1}) \cdot |^{-1}) (t).
\]

To evaluate \( Y \), consider the matrix

\[
x = d \left( \begin{array}{c} 1, 1, \ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots, 1 \\ i - 1 \text{ terms} \end{array} \right) g^\circ \times \ldots \\
\ldots d \left( \begin{array}{c} 1, 1, \ldots, 1, 1 + t^{-1}, 1, \ldots, 1 \\ i - 1 \text{ terms} \end{array} \right) x_\alpha (t^{-1}) g^\circ.
\]

Every matrix in the product is an upper-triangular matrix which is trivial outside the center block. The only pieces which are not unipotent as the diagonal matrices, which are inverse. Thus we see immediately that \( x \in N \cap G^2 \). Furthermore, this
matrix fixes the vector \( t(0, \ldots, 0, 1, -1, 0, \ldots, 0) \) which arises in the definition of \( G' \), so actually \( x \in G' \). Thus \( x = n' \in N' \), so that

\[
x_\alpha (t^{-1}) g^\circ = d \left( \underbrace{1, 1, \ldots, 1}_{i - 1 \text{ terms}}, 1 + t^{-1}, 1, \ldots, 1 \right) g^\circ \times \ldots d \left( \underbrace{1, 1, \ldots, 1}_{i - 1 \text{ terms}}, (1 + t^{-1})^{-1}, 1, \ldots, 1 \right) n'.
\]

This gives

\[
Y (w_\ell x_\alpha (t^{-1}) g^\circ) = Y (w_\ell d(1, \ldots, 1, 1 + t^{-1}, 1, \ldots, 1) g^\circ \times \ldots d(1, \ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots, 1) n') = Y \left( d(1, \ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots, 1) w_\ell g^\circ \times \ldots d(1, \ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots, 1) n' \right) = (\xi^{-1} \delta^{1/2}) \left( d(\ldots, (1 + t^{-1})^{-1}, \ldots) \right) \times \ldots (\xi' \delta'^{-1/2}) \left( d'(\ldots, (1 + t^{-1})^{-1}, \ldots) \right) = \xi_i (1 + t^{-1}) \xi'_{i-r} (1 + t^{-1}) |1 + t^{-1}|^{-1/2} = \left( \xi_i \xi'_{i-r} | \cdot |^{-1/2} \right) (1 + t^{-1}).
\]

Finally,

\[
I_\alpha = 1 + q_E \int_{\partial E} (\xi \delta^{-1/2})(a^{n(t)}_\alpha) Y (w_\ell x_\alpha (t^{-1}) g^\circ) \, dt = 1 + q_E \int_{\partial E} (\xi_i \xi'_{i+1} | \cdot |^{-1}) (t) \left( \xi_i \xi'_{i-r} | \cdot |^{-1/2} \right) (1 + t^{-1}) \, dt = 1 + q_E \int_{\partial E} (\xi'_{i-r} \xi_{i+1} | \cdot |^{-1/2}) (t) \left( \xi_i \xi'_{i-r} | \cdot |^{-1/2} \right) (1 + t) \, dt,
\]

54
by the simple observation $1 + t^{-1} = (1 + t)t^{-1}$. Now we are in the situation of Lemma 10.6, so that

\[ I_\alpha = (q - 1) \frac{1 - q^{-\xi_1}}{(1 - q^{-1/2} \xi_1^{i_1})(1 - q^{-1/2} \xi_1^{i_1 + 1})}. \]

**Lemma 9.12** ($n$ even, $\alpha = \varepsilon_{n+} - \varepsilon_{n+1}$)

\[ I_\alpha = q^{1/2} \frac{1 - q^{-\xi_{n+}}}{1 - q^{-1/2} \xi_{n+}}. \]

**Proof.** First,

\[ (\xi^{-1} \delta^{1/2})(a_\alpha^{-v(s)}) = \xi_{n+}^{v(s)} q^{-v(s)} = (\xi_{n+} | · |^{-1/2})(s). \]

This case is rather special.

\[ x_\alpha^*(s^{-1}) = \begin{pmatrix} 1 \\ \vdots \\ 1 & s^{-1} \eta \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \]

Recall that $s^{-1} \eta = -s^{-1} \eta$. Furthermore $1 + s^{-1} \eta$ and $1 - s^{-1} \eta$ are conjugates, so that $\frac{1 + s^{-1} \eta}{1 - s^{-1} \eta}$ is a unit (we use here that the extension $E/F$ is unramified). Based on the
2 \times 2 \text{ matrix computation}

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & s^{-1}\eta \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
1 & s^{-1}\eta
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{1+s^{-1}\eta} & \frac{s^{-1}\eta}{s^{-1}\eta+1} \\
0 & 1-s^{-1}\eta
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
\frac{1}{1-s^{-1}\eta} & \frac{-s^{-1}\eta}{1-s^{-1}\eta}
\end{pmatrix}
\begin{pmatrix}
1 & s^{-1}\eta \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & s^{-1}\eta \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{1+s^{-1}\eta} & \frac{s^{-1}\eta}{s^{-1}\eta+1} \\
0 & 1-s^{-1}\eta
\end{pmatrix}
\]

we see that

\[
d' \left(1, \ldots, 1, \frac{1-s^{-1}\eta}{1+s^{-1}\eta} g^o w_\ell \right) d(1, \ldots, 1, 1 + s^{-1}\eta) w_\ell x_\alpha^* (t^{-1}) g^o \in N'.
\]

Thus

\[
Y(w_\ell x_\alpha^* (s^{-1}) g^o) = Y \left( d \left(1, \ldots, 1, (1 + s^{-1}\eta)^{-1} \right) w_\ell g^o d' \left(1, \ldots, 1, \frac{1+s^{-1}\eta}{1-s^{-1}\eta} \right) u' \right)
\]

\[
= (\xi^{-1} \delta^{1/2}) \left(d \left(1, \ldots, (1 + s^{-1}\eta)^{-1} \right) \right) (\xi' \delta'^{-1/2}) \left(d' \left(1, \ldots, \frac{s-\eta}{s+\eta} \right) \right)
\]

\[
= (\xi_{n+} | \cdot |^{-1/2})(1 + s^{-1}\eta).
\]

Finally,

\[
I_\alpha = 1 + q_F \int_{\delta_F} (\xi^{-1/2}) (a_\alpha^{(s)}) Y(w_\ell x_\alpha (s^{-1}) g^o) ds
\]

\[
= 1 + q_F \int_{\delta_F} (\xi_{n+} | \cdot |^{-1/2}) (s) (\xi_{n+} | \cdot |^{-1/2}) (1 + s^{-1}\eta) \ ds
\]

\[
= 1 + q_F \int_{\delta_F} (\xi_{n+} | \cdot |^{-1/2}) (s + \eta) \ ds,
\]

where we are in the situation of the corollary to Lemma 9.6 after a change of variable.
Then, recalling that we are working over $F$ for the moment, and $q_F = q^{1/2}$,

$$I_\alpha = q^{1/2} \frac{1 - q^{-1} \xi_{n+}}{1 - q^{-1/2} \xi_{n+}}.$$ 

Lemma 9.13 (n even, $\beta = \varepsilon'_i - \varepsilon'_{i+1}, i < n'$)

$$J_\beta = (q - 1) \frac{1 - q^{-1} \xi'_i \xi'^{-1}_{i+1}}{(1 - q^{-1/2} \xi'_i \xi'^{-1}_{r+i+1})(1 - q^{-1/2} \xi'^{-1}_{i+1} \xi_{r+i+1})}.$$ 

Proof. First,

$$(\xi'_i \xi'^{-1/2} \xi^{-1}) (a^v_{\beta}) = \xi'^v_i \xi'^{-v}_{i+1} q^v = (\xi'_i \xi'^{-1}_{i+1} \cdot |^{-1})(t).$$

To evaluate $Y$, consider the matrix

$$x = d \left( \begin{array}{c} 1, 1, \ldots, 1, (1 + t^{-1}), 1, \ldots, 1 \\ r + i \text{ terms} \end{array} \right) g^{\circ -1} \times$$

$$\cdots d \left( \begin{array}{c} 1, 1, \ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots, 1 \\ r + i \text{ terms} \end{array} \right) x_{-\beta} (-t^{-1}) \ g^{\circ} x_{-\beta} (t^{-1}).$$

Every matrix in the product is trivial outside the center block, so $x \in G^2$. Though the $x_{-\beta}(\cdot)$ are not upper-triangular, the product of the rightmost three factors is upper-triangular unipotent. The diagonal matrices are mutually inverse, so the product of the leftmost three factors is upper-triangular unipotent. Thus $x \in N$ also. Furthermore, this matrix fixes the vector $^t(0, \ldots, 0, 1, -1, 0, \ldots, 0)$ which arises in the
definition of $G'$, so actually $x \in G'$. Thus $x = n' \in N'$, so that

$$g^o x_{-\beta} (t^{-1}) = x_{-\beta} (t^{-1}) \cdot \left( \begin{array}{c} 1, 1, \ldots, 1, 1 + t^{-1}, 1, \ldots, 1 \\ r + i \text{ terms} \end{array} \right) g^o \times$$

$$\cdots \cdot \left( \begin{array}{c} 1, 1, \ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots, 1 \\ r + i \text{ terms} \end{array} \right) n'.$$

This gives

$$Y (w_{\ell} g^o x_{-\beta} (t^{-1})) = Y (w_{\ell} x_{-\beta} (t^{-1}) d(\ldots, 1, 1 + t^{-1}, 1, \ldots) g^o \times$$

$$\cdots d(\ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots) n')$$

$$= Y \left( d(\ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots) x_{\beta}(\cdot) w_{\ell} g^o \times$$

$$\cdots d(\ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots) n')$$

$$= (\xi^{-1} \delta^{1/2}) \left( d(\ldots, (1 + t^{-1})^{-1}, \ldots) \right) \times$$

$$\cdots (\xi' \delta'^{-1/2}) \left( d'(\ldots, (1 + t^{-1})^{-1}, \ldots) \right)$$

$$= \xi_{r+i+1} (1 + t^{-1}) \xi'^{-1}_{i+1} (1 + t^{-1}) |1 + t^{-1}|^{-1/2}$$

$$= \left( \xi_{r+i+1} \xi'^{-1}_{i+1} | \cdot |^{-1/2} \right) (1 + t^{-1}).$$

Finally,

$$J_{\beta} = 1 + q_{E} \int_{a_{\beta}}^{b_{\beta}} (\xi' \delta'^{-1/2})(a_{\beta}^{n(t)}) Y (w_{\ell} g^o x_{-\beta} (t^{-1})) dt$$

$$= 1 + q_{E} \int_{a_{\beta}}^{b_{\beta}} \left( \xi' \xi'^{-1}_{i+1} | \cdot |^{-1/2} \right) (t) \left( \xi_{r+i+1} \xi'^{-1}_{i+1} | \cdot |^{-1/2} \right) (1 + t^{-1}) dt$$

$$= 1 + q_{E} \int_{a_{\beta}}^{b_{\beta}} \left( \xi' \xi'^{-1}_{r+i+1} | \cdot |^{-1/2} \right) (t) \left( \xi_{r+i+1} \xi'^{-1}_{i+1} | \cdot |^{-1/2} \right) (1 + t) dt,$$

by the simple observation $1 + t^{-1} = (1 + t)t^{-1}$. Now we are in the situation of
Lemma 10.6, so that

\[ J_\beta = (q - 1) \frac{1 - q^{-1} \xi_1 \xi_{i+1}^{-1}}{(1 - q^{-1/2} \xi_1 \xi_{r+i+1}^{-1}) (1 - q^{-1/2} \xi_{r+i+1} \xi_{r+i}^{-1})}. \]

Lemma 9.14 \((n \text{ even, } \beta = \varepsilon_{n_-}' - \varepsilon_{n_+}')\)

\[ J_\beta = (q - 1) \frac{1 - q^{-1} \xi_1'}{\left(1 - q^{-1/2} \xi_{2n}^2 \xi_{n_+}^{-1}\right) \left(1 - q^{-1/2} \xi_{n_-}' \xi_{n_-}^{-1}\right)}. \]

**Proof.** First,

\[(\xi' a_{n_+'}^{-1/2} v(t)) = \xi_{n_-}' q^n v(t) = (\xi_{n_-}' | \cdot |^{-1}) (t).\]
To evaluate $Y$, for this root, we apply the following $4 \times 4$ matrix computation.

\[
\begin{pmatrix}
1 & -\bar{t}^{-1} & 0 \\
0 & 1 + \bar{t}^{-1} & (1 + t)^{-1} \\
0 & 0 & (1 + t^{-1})^{-1} \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\times
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 + t^{-1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 + \bar{t}^{-1}
\end{pmatrix}
\]

we obtain in general

\[
w_{t\ell}g^\circ x_{-\beta}(t^{-1}) = d\left(1, \ldots, 1, 1 + \bar{t}^{-1}\right) n_{w_{t\ell}g^\circ n'd'}(1, \ldots, (1 + t^{-1}) , 1)
\]
for suitable $n \in N, n' \in N'$.

\[
Y (w_\ell g^\omega x_{-\beta} (t^{-1})) = Y \left( d(1, \ldots, 1, 1 + \bar{t}^{-1}) \, nw_\ell g^\omega n' \, d' \left( 1, \ldots, (1 + t^{-1})^{-1}, 1 \right) \right)
\]
\[
= (\xi^{-1} \delta^{1/2}) \left( d(\ldots, (1 + t^{-1})^{-1}) \right) \times
\]
\[
\ldots (\xi' \delta'^{-1/2}) \left( d'(\ldots, (1 + t^{-1})^{-1}, 1) \right)
\]
\[
= \xi_{n-} (1 + t^{-1}) \xi'_{n'} (1 + t^{-1}) |1 + t^{-1}|^{-1/2}
\]
\[
= \left( \xi_{n-} \xi'_{n'} | \cdot |^{-1/2} \right) (1 + t^{-1}) .
\]

Finally,

\[
J_\beta = 1 + q_E \int_{\rho_E} (\xi' \delta'^{-1/2}) (a'^{n(t)}_\beta) Y(w_\ell g^\omega x_{-\beta} (t^{-1})) \, dt
\]
\[
= 1 + q_E \int_{\rho_E} \left( \xi'_{n'} | \cdot |^{-1} \right) (t) \left( \xi_{n-} \xi'^{-1}_{n'} | \cdot |^{-1/2} \right) (1 + t^{-1}) \, dt
\]
\[
= 1 + q_E \int_{\rho_E} \left( \xi'^2 \xi_{n-} | \cdot |^{-1/2} \right) (t) \left( \xi_{n-} \xi'^{-1}_{n'} | \cdot |^{-1/2} \right) (1 + t) \, dt,
\]

by the simple observation $1 + t^{-1} = (1 + t)t^{-1}$. Now we are in the situation of Lemma 9.6, so that

\[
J_\beta = (q - 1) \frac{1 - q^{-1} \xi'_{n'}}{\left( 1 - q^{-1/2} \xi'_{n'} \xi_{n-} \right) \left( 1 - q^{-1/2} \xi'_{n'} \xi_{n-} \right)}.
\]

**Lemma 9.15** ($n$ odd, $\alpha = \varepsilon_i - \varepsilon_{i+1}, i < r$)

\[
I_\alpha = q \left( 1 - q^{-1} \xi \xi_{i+1}^{-1} \right) .
\]
Proof. First,

\[ (\xi^{-1} \delta^{1/2})(a_{\alpha}^{-v(t)}) = \xi^{-v(t)} \xi^{-v(t)} q^{v(t)} = (\xi_i \xi_{i+1} \cdot |^{-1}) (t). \]

Evaluating \( Y \) is easy in this case because \( X_\alpha \) commutes with \( g^\circ \). (If we write \( G \) in terms of block matrices by the partition \( r, n-2r, r \), \( X_\alpha \) lives in the upper and lower diagonal blocks, and \( g^\circ \) is in the center block.)

\[ Y( w_t x_\alpha (t^{-1}) g^\circ ) = Y( w_t g^\circ x_\alpha (t^{-1}) ) = \psi( x_\alpha (t^{-1}) ) = \psi( t^{-1} ), \]

with the last equality coming from the definition of \( \psi \) as a character on \( U \). Then by Lemma 9.4,

\[
I_\alpha = 1 + qE \int_{\phi_E} (\xi^{-1} \delta^{1/2})(a_{\alpha}^{-v(t)})Y( w_t x_\alpha (t^{-1}) g^\circ ) \, dt \\
= 1 + qE \int_{\phi_E} (\xi_i \xi_{i+1} \cdot |^{-1}) (t) \psi( t^{-1} ) \, dt \\
= 1 + q \sum_{k=0}^{\infty} (1 - q^{-1}) q^{-k} (\xi_i \xi_{i+1})^k q^k \int_{\phi_E} \psi( \pi^{-k} u ) \, du \\
= q \left( 1 - q^{-1} \xi_i \xi_{i+1} \right),
\]

with the last step invoking Lemma 9.6.

Lemma 9.16 (\( n \) odd, \( \alpha = \varepsilon_r - \varepsilon_{r+1} \))

\[
I_\alpha = q(1 - q^{-1} \xi_i \xi_{i+1})
\]

Proof. First,

\[ (\xi^{-1} \delta^{1/2})(a_{\alpha}^{-v(t)}) = \xi_i^{-v(t)} \xi_{i+1}^{-v(t)} q^{v(t)} = (\xi_i \xi_{i+1} \cdot |^{-1}) (t). \]
In order to evaluate $Y$, we must study how $x_\alpha(t^{-1})$ commutes past $g^\circ$. By direct computation,

$$g^\circ - 1 x_\alpha(t^{-1}) g^\circ = I_n + t^{-1} e_{r,r+1} + 2 t^{-1} e_{r,n_+} - t^{-1} \sum_{k=n_+ + 1}^{n-r} e_{r,k} + \cdots$$

$$- e_{n_+,n+1-r} + \sum_{k=n_+ + 1}^{n-r} e_{k,n+1-r}.$$

Writing $x$ for this whole right hand side, we see that $x = n' u$ is in $N' U$, and moreover that the only entry seen by $\psi$ is again $t^{-1}$. Then

$$Y(w_t x_\alpha(t^{-1}) g^\circ) = Y(w_t g^\circ n' u) = \psi(u) = \psi(t^{-1}).$$

Then by Lemma 9.4,

$$I_\alpha = 1 + q E \int_{\delta E} (\xi \delta^{-1/2}) (a_\alpha v(t)) Y(w_t x_\alpha(t^{-1}) g^\circ) \, dt$$

$$= 1 + q E \int_{\delta E} (\xi_\iota \xi_{i+1}^{-1} | \cdot |^{-1}) (t) \psi(t^{-1}) \, dt$$

$$= 1 + q \sum_{k=0}^{\infty} (1 - q^{-1}) q^{-k} (\xi_\iota \xi_{i+1}^{-1})^k q^k \int_{\delta E} \psi(\pi^{-k} u) \, du$$

$$= q (1 - q^{-1} \xi_\iota \xi_{i+1}^{-1}).$$

with the last step invoking Lemma 9.8.

**Lemma 9.17** $(n$ odd, $\alpha = \varepsilon_i - \varepsilon_{i+1}$, $r < i \leq n_-)$

$$I_\alpha = (q - 1) \frac{1 - q^{-1} \xi_\iota \xi_{i+1}^{-1}}{(1 - q^{-1/2} \xi_{i-r}^{-1} \xi_i) (1 - q^{-1/2} \xi_{r-i}^{-1} \xi_{i+1})}. $$

**Proof.** First,

$$(\xi^{-1} \delta^{1/2})(a_\alpha^{-v(t)}) = \xi_i v(t) \xi_{i+1}^{-v(t)} q^v(t) = (\xi_i \xi_{i+1}^{-1} | \cdot |^{-1})(t).$$
To evaluate $Y$, consider the matrix

$$
x = d \left( \begin{array}{c,cccc,cccc} 1, & 1, & \ldots, & 1, & (1 + t^{-1})^{-1}, & 1, & \ldots, & 1 \\
 & (i-1 \text{ terms}) 
\end{array} \right) g^\circ - 1 \times \ldots \times d \left( \begin{array}{c,cccc,cccc} 1, & 1, & \ldots, & 1, & 1 + t^{-1}, & 1, & \ldots, & 1 \\
 & (i-1 \text{ terms}) 
\end{array} \right) x_\alpha (t^{-1}) g^\circ.
$$

Every matrix in the product is an upper-triangular matrix which is trivial outside the center block. The only pieces which are not unipotent are the diagonal matrices, which are inverse. Thus we see immediately that $x \in N \cap G^\#$. Furthermore, this matrix fixes the vector $^t(0, \ldots, 0, 1, 0, \ldots, 0)$ which arises in the definition of $G'$, so actually $x \in G'$. (At first it might appear that special measures need to be taken for $i = n$, but this is not the case; recall that the definition of $x_\alpha(t)$ is different for $i = n$, which accounts for the apparent discrepancy.) Thus $x = n' \in N'$, so that

$$
x_\alpha (t^{-1}) g^\circ = d \left( \begin{array}{c,cccc,cccc} 1, & 1, & \ldots, & 1, & 1 + t^{-1}, & 1, & \ldots, & 1 \\
 & (i-1 \text{ terms}) 
\end{array} \right) g^\circ \times \ldots \times d \left( \begin{array}{c,cccc,cccc} 1, & 1, & \ldots, & 1, & (1 + t^{-1})^{-1}, & 1, & \ldots, & 1 \\
 & (i-1 \text{ terms}) 
\end{array} \right) n'.
$$
This gives

\[
Y \left( w_t x_\alpha \left( t^{-1} \right) g^\circ \right) = Y \left( w_t d(1, \ldots, 1, 1 + t^{-1}, 1, \ldots, 1) g^\circ \right) \ldots \\
\ldots d(1, \ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots, 1) n' \right)
\]

\[
= Y \left( d(1, \ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots, 1) w_t g^\circ \right) \ldots \\
d(1, \ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots, 1) n' \right)
\]

\[
= (\xi^{-1} \delta^{1/2}) \left( d(\ldots, (1 + t^{-1})^{-1}, \ldots) \right) \times \\
\ldots (\xi' \delta'^{-1/2}) \left( d'(\ldots, (1 + t^{-1})^{-1}, \ldots) \right)
\]

\[
= \xi_i (1 + t^{-1}) \xi_i'^{-1} (1 + t^{-1}) |1 + t^{-1}|^{-1/2}
\]

\[
= \left( \xi_i \xi_i'^{-1} \right) \left( 1 + t^{-1} \right).
\]

Finally,

\[
I_\alpha = 1 + q_E \int_{0_\alpha} (\xi \delta^{-1/2})(a^{\nu(t)}) Y(w_t x_\alpha \left( t^{-1} \right) g^\circ) \, dt
\]

\[
= 1 + q_E \int_{0_\alpha} \xi_i \xi_i' \left( t \right) \xi_i \xi_i'^{-1} \left( t \right) \left( 1 + t^{-1} \right) \, dt
\]

\[
= 1 + q_E \int_{0_\alpha} \xi_i' \xi_i+1 \left( t \right) \xi_i \xi_i'^{-1} \left( t \right) \left( 1 + t \right) \, dt,
\]

by the simple observation \( 1 + t^{-1} = (1 + t)t^{-1} \). Now we are in the situation of Lemma 9.6, so that

\[
I_\alpha = (q - 1) \frac{1 - q^{-1} \xi_i \xi_i^{-1}}{(1 - q^{-1/2} \xi_{i+1}^\prime \xi_i) \left( 1 - q^{-1/2} \xi_{i+1}^\prime \xi_{i+1}^{-1} \right)}.
\]

Lemma 9.18 (n odd, \( \beta = \varepsilon_i^\prime - \varepsilon_{i+1}^\prime, i < n' \_ \))

\[
J_\beta = (q - 1) \frac{1 - q^{-1} \xi_i \xi_i'^{-1}}{(1 - q^{-1/2} \xi_i \xi_{i+1}) \left( 1 - q^{-1/2} \xi_i \xi_{i+1}^\prime \xi_{r+i} \right)}.
\]

65
Proof. First,

\[(\xi_i' \delta_i' - \frac{1}{2})(a'_{\beta} v(t)) = \xi_i' v(t) = \xi_i' v(t)i \xi_i' v(t)i + 1 q - v(t) = (\xi_i' \xi_i' - 1 i i + 1 | \cdot | - 1) (t).\]

To evaluate \(Y\), consider the matrix

\[x = d \left( \frac{1,1,\ldots,1, (1 + t^{-1}), 1, \ldots, 1}{r + i \text{ terms}} \right) g^0 \times \]

\[\cdots d \left( \frac{1,1,\ldots,1, (1 + t^{-1})^{-1}, 1, \ldots, 1}{r + i \text{ terms}} \right) x_{-\beta} (t^{-1}) g^0 x_{-\beta} (t^{-1}). \]

Every matrix in the product is trivial outside the center block, so \(x \in G^\sharp\). Though the \(x_{-\beta}(\cdot)\) are not upper-triangular, the product of the rightmost three factors is upper-triangular unipotent. The diagonal matrices are mutually inverse, so the product of the leftmost three factors is upper-triangular unipotent. Thus \(x \in N\) also. Furthermore, this matrix fixes the vector \(t^i(0, \ldots, 0, 1, -1, 0, \ldots, 0)\) which arises in the definition of \(G'\), so actually \(x \in G'\). Thus \(x = n' \in N'\), so that

\[g^0 x_{-\beta} (t^{-1}) = x_{-\beta} (t^{-1}) d \left( \frac{1,1,\ldots,1, 1 + t^{-1}, 1, \ldots, 1}{r + i \text{ terms}} \right) g^0 \times \]

\[\cdots d \left( \frac{1,1,\ldots,1, (1 + t^{-1})^{-1}, 1, \ldots, 1}{r + i \text{ terms}} \right) n'. \]
This gives

\[
Y \left( w \ell g^x_{-\beta} (t^{-1}) \right) = Y \left( w \ell x_{-\beta} (t^{-1}) d(1, \ldots, 1, 1 + t^{-1}, 1, \ldots, 1) g^{\cdot} \right. \\
\left. \cdots d(1, \ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots, 1) n' \right) \\
= Y \left( d(1, \ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots, 1) x_{\beta} (\cdot) w \ell g^{\cdot} \right. \\
\left. \cdots d(1, \ldots, 1, (1 + t^{-1})^{-1}, 1, \ldots, 1) n' \right) \\
= \left( \xi^{-1} \delta^{1/2} \right) \left( d(\ldots, (1 + t^{-1})^{-1}, \ldots) \right) \times \\
\left. \cdots \left( \xi' \delta'^{1/2} \right) \left( d'(\ldots, (1 + t^{-1})^{-1}, \ldots) \right) \right) \\
= \xi_{r+i+1} (1 + t^{-1}) \xi'_{i+1} (1 + t^{-1}) |1 + t^{-1}|^{-1/2} \\
= \left( \xi_{r+i+1} \xi'_{i+1} \cdot |^{-1/2} \right) (1 + t^{-1}).
\]

Finally,

\[
J_\beta = 1 + q_E \int_{o_E} (\xi' \delta'^{-1/2}) (a_{\beta}^{\cdot}(t)) Y (w \ell g^x_{-\beta} (t^{-1})) \, dt \\
= 1 + q_E \int_{o_E} \left( \xi' \xi'^{-1}_{i+1} \cdot |^{-1} \right) (t) \left( \xi_{r+i+1} \xi'^{-1}_{i+1} \cdot |^{-1/2} \right) (1 + t^{-1}) \, dt \\
= 1 + q_E \int_{o_E} \left( \xi' \xi'^{-1}_{i+1} \cdot |^{-1/2} \right) (t) \left( \xi_{r+i+1} \xi'^{-1}_{i+1} \cdot |^{-1/2} \right) (1 + t) \, dt,
\]

by the simple observation \(1 + t^{-1} = (1 + t)t^{-1}\). Now we are in the situation of Lemma 10.6, so that

\[
J_\beta = (q - 1) \frac{1 - q^{-1} \xi' \xi'^{-1}_{i+1}}{\left(1 - q^{-1/2} \xi' \xi'^{-1}_{i+1}\right) \left(1 - q^{-1/2} \xi' \xi'^{-1}_{i+1}\right)}.
\]

67
Lemma 9.19 \((n \text{ odd }, \beta = \varepsilon'_{n'_-} - \varepsilon'_{n'_-+1})\)

\[
J_\beta = (q^{1/2} - 1) \frac{1 - q^{-2} \xi'_{n'_-}}{1 - q^{-1} \xi'_{n'_-}}.
\]

First,

\[
(\xi' \delta^{r-1/2})(a^{r_v(s)}_{\beta}) = \xi'^{r_v(s)} \xi'^{r-v(s)} q^{v(s)} = (\xi'^{2} \mid \cdot -1) (s).
\]

To evaluate \(Y\), for this root, we apply the following \(3 \times 3\) matrix computation.

\[
\begin{pmatrix}
1 - \frac{2\eta}{s - \eta} & \frac{\eta}{s - \eta} \\
0 & \frac{s + \eta}{s - \eta} - \frac{\eta}{s - \eta} \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 2 & -1 \\
1 + s^{-1} \eta & 0 & \frac{\eta}{s + \eta} \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 + s^{-1} \eta & 0 & \frac{\eta}{s + \eta} \\
0 & 1 & 0 \\
0 & 0 & (1 - s^{-1} \eta)^{-1}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 2 & -1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
s^{-1} \eta & 0 & 1 \\
s^{-1} \eta & 1 & -1 \\
1 - s & 2 & -1
\end{pmatrix}
\]

we obtain in general

\[
\omega_t g^{x_{-\beta}}(s^{-1}) = d(1, \ldots, 1, (s + \eta)/(s - \eta)) n \omega_t g^{n'} d'(1, \ldots, (1 + s^{-1} \eta))
\]
for suitable $n \in N, n' \in N'$.

\[
Y \left( w \ell g \circ x_{-\beta} \left( t^{-1} \right) \right) = Y \left( d \left( \ldots, \frac{s + \eta}{s - \eta} \right) nw \ell g \circ n' \circ d' \left( 1, \ldots, \left( 1 + s^{-1} \eta \right)^{-1} \right) \right) = (\xi^{-1} \delta^{1/2}) \left( d \left( \ldots, \frac{s + \eta}{s - \eta} \right) \right) (\xi' \delta'^{-1/2}) (d' \left( \ldots, 1 + s^{-1} \eta \right))
\]

\[
= \xi'_{n'} (1 + s^{-1} \eta) \left| 1 + s^{-1} \right|^{-1/2}
\]

\[
\left( \xi_{n'} \left| 1 \right|^{-1/2} \right) (1 + s^{-1} \eta).
\]

Finally,

\[
J_{\beta} = 1 + q_F \int_{\sigma_F} (\xi' \delta'^{-1/2})(a'_{\beta}(s)) Y \left( w \ell g \circ x^{*}_{-\beta}(s^{-1}) \right) ds
\]

\[
= 1 + q_F \int_{\sigma_F} (\xi_{n'}^{2} \left| 1 \right|^{-1/2}) (s) \left( \xi_{n'} \left| 1 \right|^{-1/2} \right) (1 + s^{-1} \eta) ds
\]

\[
= 1 + q_F \int_{\sigma_F} (\xi_{n'}^{2} \left| 1 \right|^{-1/2}) (s) \left( \xi_{n'} \left| 1 \right|^{-1/2} \right) (\eta + s) ds.
\]

Now we are in the situation of Lemma 10.6 (as it applies to $\sigma_F$), so that

\[
J_{\beta} = \left( q^{1/2} - 1 \right) \frac{1 - q^{-2} \xi_{n'}^{2}}{\left( 1 - q^{-1} \xi_{n'}^{2} \right)^2}.
\]

For convenience we collect the results of these calculations in one place.
Proposition 9.20 (Summary of Calculations)

\[ I_\alpha = \begin{cases}
q \left(1 - q^{-1} \xi_i \xi_{i+1}^{-1}\right) & n \text{ even}, \alpha = \varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq r \\
(q-1) \frac{1-q^{-1} \xi_i \xi_{i+1}^{-1}}{1-q^{-1} \xi_{i+1} \xi_i^{-1}} & n \text{ even}, \alpha = \varepsilon_i - \varepsilon_{i+1}, r < i \leq n - 1 \\
q \left(1 - q^{-1} \xi_i \xi_{i+1}^{-1}\right) & n \text{ odd}, \alpha = \varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq r \\
(q-1) \frac{1-q^{-1} \xi_i \xi_{i+1}^{-1}}{1-q^{-1} \xi_{i+1} \xi_i^{-1}} & n \text{ odd}, \alpha = \varepsilon_i - \varepsilon_{i+1}, r < i \leq n
\end{cases} \]

\[ J_\beta = \begin{cases}
(q-1) \frac{1-q^{-1} \xi_i \xi_{i+1}^{-1}}{1-q^{-1} \xi_{i+1} \xi_i^{-1}}(1-q^{-1} \xi_{i+1} \xi_i^{-1}) & n \text{ even}, \beta = \varepsilon_i' - \varepsilon_{i+1}', 1 \leq i < n' - 1 \\
(q-1) \frac{1-q^{-1} \xi_i' \xi_{i+1}'^{-1}}{1-q^{-1} \xi_{i+1}' \xi_i'^{-1}}(1-q^{-1} \xi_{i+1}' \xi_i'^{-1}) & n \text{ even}, \beta = \varepsilon_i' - \varepsilon_{i+1}' \\
(q-1) \frac{1-q^{-1} \xi_i \xi_{i+1}^{-1}}{1-q^{-1} \xi_{i+1} \xi_i^{-1}}(1-q^{-1} \xi_{i+1} \xi_i^{-1}) & n \text{ odd}, \beta = \varepsilon_i' - \varepsilon_{i+1}', 1 \leq i < n' \\
(q^{1/2} - 1) \frac{1-q^{-2} \xi_{i+1}' \xi_i'^{-1}}{1-q^{-1} \xi_{i+1}' \xi_i'^{-1}} & n \text{ odd}, \beta = \varepsilon_i' - \varepsilon_{i+1}' + 1
\end{cases} \]
Here we will show that the linear form $l_{\xi,\xi'}$ is rational with respect to the parameter space, which will lead us directly to the desired conclusions about $F$ itself. (Indeed, armed with this result, we will in the next section be able to give the formula explicitly, and then see directly that it is expressed rationally.) Here we follow the recipe given in [KSM].

**Proposition 10.1** For all $\xi, \xi'$, we have

$$\dim \text{Hom}_{PH}(I(\xi; O_0), \xi'^{-1}\delta'^{-1/2} \otimes \psi) = 1$$

**Proof.** This is an immediate consequence of the isomorphism $O_0 \equiv PH \times P$.

**Proposition 10.2** For generic $\xi, \xi'$, we have

$$\dim \text{Hom}_{PH}(I(\xi; \mathcal{O}), \xi'^{-1}\delta'^{-1/2} \otimes \psi) = 0$$

whenever $\mathcal{O} \neq \mathcal{O}_0$

**Proof.** Intuitively, this is a reflection of the fact that other orbits have interaction between the $P$-action and the $PH$-action, so that they cannot support independent actions. The former proof is identical to the special orthogonal version presented in [KSM] and is omitted.

We are now in the situation of Proposition 3.2, and we conclude the following.
Corollary 10.3 If $U$ in open subset of $G$ stable until $P \times P_H$, then

$$\dim \text{Hom}_{P_H}(I(\xi; U), \xi'^{-1} \delta'^{-1/2} \otimes \psi) \leq 1.$$

for generic $\xi, \xi'$. In particular, for generic data,

$$\dim \text{Hom}_{P_H}(I(\xi), \xi'^{-1} \delta'^{-1/2} \otimes \psi) \leq 1$$

With $Y$ as before, we define

$$l_{\xi, \xi'}(P_{\xi}(f)) = \int_G f(g)Y(g)dg \quad f \in C_c^\infty(G)$$

in $Z$. Notice that it makes sense to restrict $l$ to $I(\xi, \mathcal{O}_0)$, and that this function is defined everywhere.

Proposition 10.4 The form $l_{\xi, \xi'}$ restricts to a rational function on $I(\xi; P w_{\xi} P)$.

Proof. Again, the proof is identical to the special orthogonal case in [KSM].

Based on the previous proposition, we know that $l_{\xi, \xi'}|_{I(\xi; P w_{\xi} P)}$ can be rationally continued so that it extended to generic parameters. Furthermore, it naturally spans the one-dimensional space of forms from $I(\chi; w'P w_{\xi} P)$ to $\xi'^{-1} \delta'^{-1/2} \otimes \psi$. That is, when the natural intertwining operators act on this form, they change it by a scalar of proportionality.

Proposition 10.5 For each simple root $\alpha$, associate the simple reflection $w_\alpha$ and the corresponding intertwining operator. There are constants $a(w_\alpha, \xi, \xi')$ defined by the property

$$T_{w_\alpha}^* l_{\xi, \xi'}|_{I(w_\alpha \xi; P w_{\xi} P)} = a(w_\alpha, \xi, \xi')|_{I(w_\alpha \xi; P w_{\xi} P)}$$

These constants of proportionality are rational functions of the parameters.
Proof. By an identical argument to that of [KSM] (which is in fact due to Cas-selman), we can compute the proportionality constants in terms of quantities which we have already computed, namely the results of the Rank 1 calculations. The end result is

\[ a(w_\alpha, \xi, \xi') = (c_\alpha (\xi^{-1}) - 1 - q^{-1}) + q^{-1} \Omega(\phi_1, R(g^\circ w_\ell)(\phi_1 + \phi_{w_\alpha}). \]

We know from the table at the end of the previous chapter that for every simple root \( \alpha \), the corresponding

\[ R(g^\circ w_\ell)(\phi_1 + \phi_{w_\alpha} \]

is rational in the \( \xi, \xi' \). Thus everything in sight is rational, so we are done.

**Theorem 10.6** The equivariant linear form \( l_{\xi, \xi'} \) depends rationally on \( \xi, \xi' \). Furthermore, for generic \( \xi, \xi' \), \( l_{\xi, \xi'} \) is actually defined and spans the space

\[ \text{Hom}_{P_H}(I(\xi), \xi' \delta^{1/2} \otimes \psi). \]

Proof. The preceding propositions in this chapter have verified all the hypotheses of Proposition 3.4.

**Corollary 10.7** The space of \( H \)-invariant bilinear forms \( \Omega_{\xi, \xi'} \) from \( I(\xi', \psi) \times I(\xi) \) to \( C \) is one-dimensional for generic \( \xi, \xi' \) and depends rationally on \( \xi, \xi' \).

Proof. This relies on the canonical correspondence between \( H \)-invariant bilinear forms on \( I(\xi', \psi) \times I(\xi) \) and \( P_H \)-equivariant linear forms \( l \) from \( I(\xi) \) to \( (\xi' \delta^{1/2} \otimes \psi) \), so existence and uniqueness require no further comment. To obtain rationality, just
notice that for $f \in C_c^\infty(G)$, $f' \in C_c^\infty(G')$, if we write

\[ \hat{f} = \int_{G'} f'(g') f(xg') dg', \]

we have

\[
\Omega_{\xi,\xi'}(P_\xi(f'), P_\xi(f)) = \int_{G \times G} f'(g') f(g) Y(gg'^{-1}) dg' dg = l_{\xi,\xi}(\hat{f}),
\]

so that rationality is preserved by the correspondence.

**Corollary 10.8** The values of $I_\alpha$ and $J_\beta$ are valid generically (not only in $Z$).

**Remark.** It has been suggested to me by some that there is a more direct proof that the Gelfand-Graev function is rational, using a result known as the Bernstein Rationality Lemma, which is essentially a vast generalization of the well-known Cramer’s Rule used for solving a system of linear equations in high school. This is quite correct, but this computation-free argument does not bring us any closer to an explicit formula. Assuming that we ultimately want a formula for the Gelfand-Graev function, all this work will be necessary in either case.
CHAPTER 11
Generic Explicit Formula

At last we are in a position to give a formula for a Gelfand-Graev function $F$ defined by

$$F_{\xi,\xi'}(g) = \Omega(\phi_{K'\xi'}, R(g) \phi_{K,\xi}) = \int_{K' \times K} Y(k g^{-1} k') dk' dk,$$

which we know from the previous result to be a rational function of $\xi, \xi'$, provided the integral is rationally continued outside the region of convergence.

Now, for any root $\alpha$, define $e_{\alpha}(\xi) = 1 - q^{-1} \xi(a_{\alpha})$, where $a_{\alpha}$ is as before a coroot. Likewise define $e'_{\beta}$ for roots of $G'$. Combine all these into

$$e(\xi) = \prod_{\alpha \in \Sigma^+} e_{\alpha}(\xi) \quad e'(\xi') = \prod_{\beta \in \Sigma'^+} e_{\beta}(\xi').$$

For any root $\alpha$, define $d_{\alpha}(\xi) = 1 - \xi(a_{\alpha})$, where $a_{\alpha}$ is as before a coroot. Likewise define $d'_{\beta}$ for roots of $G'$. Combine all these into

$$d(\xi) = \prod_{\alpha \in \Sigma^+} d_{\alpha}(\xi) \quad d'(\xi') = \prod_{\beta \in \Sigma'^+} d_{\beta}(\xi').$$

Write $c_{\alpha}(\xi) = e_{\alpha}(\xi)/d_{\alpha}(\xi)$ for their ratio.

Now we can define

$$b(\xi, \xi') = \prod (1 - q^{-1/2}(\xi'^{-1}_i \xi_j)^{\eta_{i,j}})(1 - q^{-1/2}\xi'_i \xi_j),$$
where $\eta_{ij}$ is 1 if $j-i \leq r$ and $-1$ otherwise.

Finally we define

$$\zeta(\xi, \xi') = \frac{e(\xi)e'(\xi')}{b(\xi, \xi')}.$$ 

This function is specifically constructed to “summarize” the various rank 1 calculations performed in Chapter 9. The following propositions will make that clear.

**Proposition 11.1**

$$\frac{\zeta(w_{\alpha} \xi, x'i')}{\zeta(\xi, \xi')} = \frac{\Omega_{w_{\alpha} \xi, x'i'}(\phi_1, R(g^\circ w_\ell)(\phi_1 + \phi_{w_\alpha})}{\Omega_{\xi, x'i'}(\phi_1, R(g^\circ w_\ell)(\phi_1 + \phi_{w_\alpha})}$$

**Proposition 11.2**

$$\frac{\zeta(\xi, w_{\beta} \xi')}{\zeta(\xi, \xi')} = \frac{\Omega_{w_{\alpha} \xi, x'i'}(\phi_1, R(g^\circ w_\ell)(\phi_1 + \phi_{w_\alpha})}{\Omega_{\xi, x'i'}(\phi_1, R(g^\circ w_\ell)(\phi_1 + \phi_{w_\alpha})}$$

The way to understand these functions is that, in a suitable sense $\zeta$ transforms the same way the various $I_\alpha$ and $J_\beta$ do under certain actions of Weyl elements. This is just checked case by case, since the numerator and denominator of each side is available from the rank 1 calculations, the definition of $\zeta$, and the results of permuting the $\xi, \xi'$ by the Weyl elements.

It is worth remarking that, looking just at the definition of the $I_\alpha$ and $J_\beta$, there is no reason to expect that a single function can be found which transforms under each Weyl reflection in precisely the same way as the corresponding $I_\alpha$ or $J_\beta$. Remarkably, the results of the computation do have this coherence property. One wonders if there is a general principle at work here.

**Proposition 11.3** Considered as a rational function of $\xi, \xi'$, $\frac{F_{\xi, \xi'}(g)}{\zeta(\xi, \xi')}$ is invariant under the natural actions of $W$ and $W'$. 

76
Proof. We know from uniqueness principles that there is, up to scalar, a unique $H$-invariant bilinear form on $I(\xi', \psi) \times I(\xi)$. We also know that both $\Omega_{\xi,\xi'}$ and its Weyl-translation $(T_{w'\xi'} \times T_{w\xi})^* \Omega_{w\xi,w'\xi'}$ are $H$-invariant bilinear forms. Thus the ratio

$$r_{w,w'} = \frac{(T_{w'\xi'} \times T_{w\xi})^* \Omega_{w\xi,w'\xi'}(\cdot, \cdot)}{\Omega_{\xi,\xi'}(\cdot, \cdot)}$$

is well-defined (i.e., independent of the suppressed arguments). Of course

$$(w, w') \mapsto r_{w,w'}$$

is a character of $W \times W'$.

Then we can compute for each reflection $w_\alpha$,

$$r_{w_\alpha,1} = \frac{(T_{w'\xi'} \times T_{w\xi})^* \Omega_{w\xi,w'\xi'}(\phi'_1, R(g^2 w_\ell)(\phi_1 + \phi_{w_\alpha}))}{\Omega_{\xi,\xi'}(\phi'_1, R(g^2 w_\ell)(\phi_1 + \phi_{w_\alpha})}.$$  

Now, we can use the proposition to relate this to $\zeta$.

$$r_{w_\alpha,1} = \frac{\zeta(w_\alpha \xi, x' \tilde{y}')}{\zeta(\xi, \xi')}.$$  

Similarly, we have

$$r_{1,w'_\beta} = \frac{\zeta(\xi, w'_\beta \xi')}{\zeta(\xi, \xi')}$$

for the simple reflections in $W'$. Now the pairs $(w, w')$ for which $\zeta$ transforms under $r$, i.e. for which

$$r_{w,w'} = \frac{\zeta(w \xi, w' \xi')}{\zeta(\xi, \xi')},$$

clearly form a subgroup of $W \times W'$. Since the simple reflections have this property and generate the whole group, we see that $\Omega$ and $\zeta$ both transform by $r_{w,w'}$. Thus their ratio is invariant under $W \times W'$, which is precisely what was claimed.
Define
\[ c_{GG}(\xi, \xi') = \frac{e(\xi)e'(\xi')}{\zeta(\xi, \xi')} = \frac{b(\xi, \xi')}{d(\xi)d'(\xi')}. \]

**Theorem 11.4** For \( \lambda \in \Lambda^+ \) and \( \lambda' \in \Lambda'^+ \),
\[
F_{\xi\xi'}(t'(\lambda')g^\circ w \ell t(\lambda)^{-1}) = q^{l(w)+l'(w')} \text{vol}(B) \text{vol}(B') \times \ldots \sum_{w \in W, w' \in W'} c_{GG}(w\xi, w'\xi')(\frac{1}{\delta^{1/2}(t(\lambda))}(w\xi)^{-1} \delta^{1/2}(t'(\lambda'))(w'\xi')^{-1} \delta^{1/2}(t'(\lambda'))).
\]

**Proof.** We begin by showing that

\[
B't'(\lambda')B'g^\circ w \ell Bt(\lambda)^{-1} \subset U(0)K't'(\lambda')g^\circ w \ell t(\lambda)^{-1}K.
\]

To this end, we invoke the Iwahori factorizations \( B = N(1)T(0)N(0) \) and \( B'N'(0)T'(0)N'(1) \), giving
\[
B't'(\lambda')B'g^\circ w \ell Bt(\lambda)^{-1} \subset L't'(\lambda')N(1)g^\circ w \ell N(1)t(\lambda)^{-1}K.
\]

By the same arguments we used when beginning the rank 1 calculations, we have
\[
N(1)g^\circ w \ell N(1) \subset g^\circ w \ell N(1).N(1) \subset g^\circ w \ell N(1)T(0)N(1) \subset U(0)T(0)g^\circ w \ell T(0)N(1).
\]

Taken together, these given the sought-after
\[
B't'(\lambda')B'g^\circ w \ell Bt(\lambda)^{-1} \subset U(0)K't'(\lambda')g^\circ w \ell t(\lambda)^{-1}K.
\]

Then the defining properties of Gelfand-Graev functions give right away the rela-
\[ F_{\xi,\xi'}(t'(\lambda')gmrw_t(\lambda)^{-1}) = \text{vol}(B't'(\lambda')^{-1}B')\text{vol}(Bt(\lambda)^{-1}B) \times \cdots \]
\[ L(ch_{B't'(\lambda')^{-1}B'}) R(ch_{B(\lambda)^{-1}B}) S_{\xi,\xi'}(g^\circ w_t) \]

Now it is known since the work Casselman that there is a basis \( \{g_w\} \) for \( I(\xi)^B \) satisfying the following properties.

- \( R(ch_{B(\lambda)^{-1}B})g_w = \text{vol}(Bt(\lambda)B)(w\xi)^{-1}\delta^{1/2}(t(\lambda))g_w \) \( \lambda \in \Lambda^+ \)
- \( g_1 = \phi_1 \)
- \( \phi_K = q^{l(w)} \sum_{w \in W} \tilde{c}_w(\xi)g_w \)

(Here the expression \( \tilde{c}_w(\xi) \) denotes \( \prod c_\alpha(\xi) \), with the product over all positive roots which \( w \) leaves positive.)

Let us now write \( Q \) for \( F_{\xi,\xi'}(t'(\lambda')g^\circ t(\lambda)^{-1})/\zeta(\xi,\xi'). \)

Directly combining everything we have gives

\[ Q = q^{l(w_t)+l(w'_t)} \frac{b(\xi,\xi')}{e(\xi)e'(\xi')} \times \cdots \]
\[ \sum_{w \in W, w' \in W'} \tilde{c}_w(\xi)\tilde{c}_{w'}(\xi')(w\xi)^{-1}\delta^{1/2}(t(\lambda))(w'\xi')^{-1}\delta'^{1/2}(t'(\lambda'))\Omega_{\xi,\xi'}(g'_w, R(g^\circ w_t)g_w) \]

We have already computed that \( \Omega_{\xi,\xi'}(g'_1, R(g^\circ w_t)g_1) = \text{vol}(B)\text{vol}(B') \). Modulo that, in the expression for \( Q \) just given, it is easy to compute that the coefficient corresponding to \( w = 1, w' = 1 \) is given by the formula

\[ q^{l(w_t)+l(w'_t)}\text{vol}(B)\text{vol}(B')c_{GG}(\xi, \xi'). \]

However, we have already shown that \( Q \) is invariant under the action of the Weyl groups. That is, the coefficients must all have that form. More precisely, the (1,1)
coefficient of the expression obtained by applying \((w, w')\) to \(Q\) must be

\[ q^{l(w) + l(w')} \text{vol}(B) \text{vol}(B') c_{GG}(\xi, \xi'). \]

Putting all this together,

\[
\frac{F_{\xi, \xi'}(t'(\lambda) g^o t(\lambda)^{-1})}{\zeta(\xi, \xi')} = q^{l(w) + l(w')} \text{vol}(B) \text{vol}(B') \times \ldots \sum_{w \in W, w' \in W'} c_{GG}(w\xi, w'\xi')((w\xi)^{-1} t(\lambda))(w'\xi')^{-1} t'(\lambda'),
\]

completing the proof.

From this we can at last conclude our long-sought formula.

**Theorem 11.5 (Explicit Formula)** For generic \(\xi, \xi'\), a Gelfand-Graev function is given by the formula

\[
F(g) = \psi(u) \sum_{w \in W, w' \in W'} c_{GG}(w\xi, w'\xi')((w\xi)^{-1} t(\lambda))(w'\xi')^{-1} t'(\lambda'),
\]

where \(g = uk't'(\lambda') g^o t(\lambda)k\).
CHAPTER 12
Residual Case

At this point, we have been to achieve several of our main goals. We have shown that Gelfand-Graev functions are unique whenever they exist by controlling their support, then constructing a recursive system of relations which determine all the function values in terms of a single value. We then abandoned that approach to carry out a different analysis and directly construct a function, provided that certain integrals converge. We were then able to evaluate the object we constructed and demonstrate it was given by a rational function, which then was valid almost everywhere.

Then only one critical goal remains—we need to know that Gelfand-Graev functions exist even in the residual cases, that is, where the data is not generic. To do this we will get assistance from an unexpected source—the type of recursive system we originally used to prove injectivity. This should be a surprise, since actually computing the coefficients of the equations in the system as originally constructed in Chapter 7 is intractable, and we could not hope to directly compute such a function in this manner. However, we will see that the argument of Chapter 7 tells us everything we need to know about how many degrees of freedom, so to speak, are present in $G$ for attempting to create a Gelfand-Graev function. Together with what we have just done, we can extract the desired results.

First, we recall the recursive system which we already constructed. We modify it this time by assuming $F(0,0) = 1$ instead of assuming that $F(0,0)$ vanishes. Then
we have a family of equations

\[ \sum_{(\lambda,\lambda')} c_{\lambda,\lambda',\mu,\mu'} x(\lambda, \lambda') = d_{\mu,\mu'}, \]

where we have the following properties.

- \( c_{\lambda,\lambda',\mu,\mu'} \) vanishes unless the \( \lambda \)s are dominated by the \( \mu \)s. In particular this is a finite sum.

- \( c_{\lambda,\lambda',\lambda,\lambda'} \) is a positive real number.

- The \( c_{\lambda,\lambda',\mu,\mu'} \) depend only on \( \psi \), not on \( \xi, \xi' \).

- The \( d_{\mu,\mu'} \) is a polynomial in \( \xi_i^{\pm 1} \xi_j^{\pm 1} \).

Let us call this system \( N \) (because it is a necessary property of Gelfand-Graev functions). Notice that such a system can always be solved, regardless of the value of \( d_{\mu,\mu'} \).

Now suppose that I want to find a countable collection of numbers \( x(\lambda, \lambda') \) with the property that the function defined by

\[ F(uk't'(\lambda')g^\circ t(\lambda)k) = \psi(u)x(\lambda, \lambda') \]

actually is a Gelfand-Graev function. We know from earlier work that such a definition for \( F \) is well-defined, and that the only nonzero \( x(\lambda, \lambda') \) are those coming from \( T^{++}, T'^{++} \). Such a function automatically has the correct \( UK', K \) transformation properties.

Thus the only obstacle is correct behavior under the Hecke algebras. Since the various \( \text{ch}_{K t(\lambda)K} \) form a basis of \( \mathcal{H} \) and likewise with primes, it is enough to consider transformations of the type we have already been considering. On the other hand, it
is not sufficient to evaluate the acted-upon function at the identity. We must evaluate at a more general $t'(\nu')g^\alpha t(\nu)$ (we can assume that $u, k', k$ are trivial without loss of generality).

In precisely the same way as in Chapter 7, we can rewrite each of these as a relation among the $x(\lambda, \lambda')$. Now we have many more equations, however, indexed by $\mu, \mu', \nu, \nu'$!

Then we have a family of equations

$$\sum_{(\lambda, \lambda')} c_{\lambda, \lambda', \mu, \mu', \nu, \nu'} F(\lambda, \lambda') = d_{\mu, \mu', \nu, \nu'},$$

where we have the following properties. The properties are not quite as nice in this case, because we do not have the full strength of our double coset intersection lemmas behind us.

- When $\nu = \nu' = 0$, this restricts to the familiar equations.
- $c_{\lambda, \lambda', \mu, \mu', \nu, \nu'}$ vanishes unless the $\lambda$s are dominated by the sum of $\mu$s and some permutation of the $\nu$s. In particular this is a finite sum.
- We do not know $c_{\lambda, \lambda', \lambda, \lambda'}$ is a positive real number.
- The $c_{\lambda, \lambda', \mu, \mu', \nu, \nu'}$ depend only on $\psi$, not on $\xi, \xi'$.
- The $d_{\mu, \mu', \nu, \nu'}$ is a polynomial in $\xi_i^{\pm 1} \xi_j'^{\pm 1}$.

Let us call this much larger system $S$, since it is sufficient for the function $x$ to lead to a Gelfand-Graev function.

We can always solve $N$, and we would like to know that the solution is really a solution of all of $S$. That is, we want to know that $S$ is consistent. Suppose $S$ were not consistent. Since $N$ certainly is, there must be some condition in $S$ that is not a linear
combination of conditions in \( S \) (otherwise the systems would plainly have exactly the same solutions). Now the left-hand side of any condition in \( S \) is expressible in a unique way as a linear combination of left-hand sides of conditions in \( N \). Thus, if we begin with a condition in \( S \) not entailed by \( N \), we can derive two equations with the same left-hand side but two distinct right-hand sides, each of which is a polynomial in the \( \xi_i, \xi'_j \) and their inverses. Thus \( S \) can represent a consistent system only where these distinct polynomials coincide. Thus there can only be Gelfand-Graev functions for a subvariety of the parameter space. This, however, is a contradiction, since we have a generic existence theorem!

Thus \( S \) really consists only of linear combinations of conditions in \( N \), and the two systems are equivalent.

Let us summarize what we have done in the form of a lemma.

**Lemma 12.1** There exist two systems of linear equations, \( S \), and \( N \), with the following properties.

- \( N \) can always be solved.
- \( S \) can only be solved in cases where a Gelfand-Graev function exists.
- All solutions of \( N \) are also solutions of \( S \).

From this we deduce the desired result at once.

**Theorem 12.2 (Strong Existence Theorem)** The space \( GG(\xi, \xi') \) is one-dimensional for every choice of nonzero complex numbers \( \xi_i, \xi'_j \). Furthermore, \( GG(\xi, \xi') \) coincides with the solution space of the recursive system \( N \) (which is evidently one-dimensional).

Combining the explicit presentation of Gelfand-Graev functions which holds generically with the knowledge that the functions are constructed, whether generic or not,
by solving the recursions $N$ which contain $\xi_i^{\pm 1}, \xi_j'^{\pm 1}$ only as polynomials, the following corollary is evident.

**Corollary 12.3** A Gelfand-Graev formula for some fixed $\xi, \xi'$ is obtained by erasing all factors in the generic explicit formula which create poles for that choice of $\xi, \xi'$.

**Remark.** Since the $\Upsilon$ transformation and the other material on unramified principal series representations is rather deeper than the basic structure theory used to derive the system of recursive relations, and since we have now carried out our strongest existence proof using the recursive relations, it is natural to wonder if in fact we were working too hard. Our effort, however, is justified. Note that it was a crucial element of the proof that the function implicitly defined by $N$ (the only reasonable candidate for a Gelfand-Graev function) actually has the correct transformation laws in some region. The equivalence of $N$ and $S$ depends crucially on already knowing the equivalence in some region by other means. That is, the conditions in $S$ which are not in $N$ do not in general follow from conditions in $N$ via, say, invoking the $UK', K$ transformation laws. Even if we have no interest in an explicit formula, we must use the $\Upsilon$ transformation or some analogous method to establish existence somewhere before we can conclude that the systems $N$ and $S$ are equivalent. (Contrast the remark at the end of Chapter 11.)
One immediate application of these arguments will be to improve the results of Kato, Sugano, and Murase. It turns out that we can apply the results of the previous chapter to the special orthogonal case. In [KSM], the uniqueness of Whittaker-Shintani functions (their analogue of what we have here called Gelfand-Graev functions) is proven with a recursive sequence of relations between function values. (Indeed, this approach is classical, and parallels the argument originally used by Whittaker to prove something analogous for general linear groups.) No mention is made in their paper of the possibility of using this recursive system to prove uniqueness; however, precisely the same argument used here applies in that situation, and we can construct an enlarged family of linear equations which is vulnerable to our analysis. Thus we are in position to conclude that Whittaker-Shintani functions occur with multiplicity exactly one for all choices of Hecke characters.

A different sort of generalization arises from the observation that the problem of finding Gelfand-Graev functions has a “brother” problem. When attempting to evaluate $L$-functions for classical groups at unramified places, the integrals that arise naturally take the form of integration against a Gelfand-Graev coefficient or a Fourier-Jacobi coefficient. Integrals of the former sort will be more easily computed with the aid of this result. But what are the “Fourier-Jacobi functions”, the knowledge of which would apply in the other cases? They are obtained by modifying the definition presented here for Gelfand-Graev functions. Recall that $G'$ was characterized as
the stabilizer in $G^{\sharp}$ of a single vector, which turned out to be anisotropic. So instead taken $G''$ as the stabilizer of a given isotropic vector. Then $G''$ is not quite a quasisplit unitary group, but it is a semidirect product of a smaller group $G^{\flat}$ with a Heisenberg group. Then a complex-valued function on $G$ is a Fourier-Jacobi if has the following properties.

- it is invariant under $K$ on the right and $K''$ on the left.
- It transforms on the left by a particular unipotent subgroup via a given character.
- It transforms on the right by $\mathcal{H}$ and on the left by $\mathcal{H}^{\flat}$, the Hecke algebra of $G^{\flat}$ according to given Hecke characters (which we identify with Satake parameters).
- It transforms on the left by the Heisenberg group via a given character.

Generally speaking, Fourier-Jacobi functions appear amenable to the same sorts of arguments used to obtain these results. I predict that we will have corresponding multiplicity one theorems and explicit formulas, for all choices of Hecke characters, unipotent character, and Heisenberg character.
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88


