BUNDLE BLOCK ADJUSTMENT
WITH 3D NATURAL CUBIC SPLINES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
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Graduate School of The Ohio State University

By

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ABSTRACT

One of the major tasks in digital photogrammetry is to determine the orientation parameters of aerial imageries correctly and quickly, which involves two primary steps of interior orientation and exterior orientation. Interior orientation defines a transformation to a 3D image coordinate system with respect to the camera’s perspective center, while a pixel coordinate system is the reference system for a digital image, using the geometric relationship between the photo coordinate system and the instrument coordinate system. While the aerial photography provides the interior orientation parameters, the problem is reduced to determine the exterior orientation with respect to the object coordinate system. Exterior orientation establishes the position of the camera projection center in the ground coordinate system and three rotation angles of the camera axis to represent the transformation between the image and the object coordinate system. Exterior orientation parameters (EOPs) of the stereo model consisting of two aerial imageries can be obtained using relative and absolute orientation. EOPs of multiple overlapping aerial imageries can be computed using bundle block adjustment. Bundle block adjustment reduces the cost of field surveying in difficult areas and verifies the accuracy of field surveying during the process of bundle block adjustment. Bundle block adjustment is a fundamental task in many applications, such as surface reconstruction, orthophoto generation, image registration and object recognition.
The stereo model consisting of two imageries with twelve EOPs is a common orientation model. Five unknowns are solved from relative orientation and seven unknowns three shifts, three rotations, and one scale factor are determined from absolute orientation. In traditional photogrammetry, all orientation procedures are performed manually with a photogrammetric operator. Fiducial marks which define the photo coordinate system while they define the pixel coordinate system in digital photogrammetry are employed in interior orientation which is image reconstruction with respect to perspective center. Matching conjugate entities play a role in relative orientation and ground control points (GCPs) are adopted in absolute orientation to calculate the object space coordinate system. Point-based procedure’s relationship between point primitives is most widely developed in traditional photogrammetry. Further application and analysis relies on a point as primary input data. The coefficients of interior, relative and absolute orientation are computed from point relationship. Interior orientation compensates for lens distortion, film shrinkage, scanner error and atmosphere refraction. Relative orientation makes the stereoscopic view possible, and the relationship between the model coordinate system and the object space coordinate system is reconstructed by absolute orientation. Using GCPs is a common method to compute the orientation parameters. However, employing a lot of GCPs is a time consuming procedure and blocks the robust and accurate automation of what research on digital photogrammetry has aimed to accomplish since development of a computer, storage capacity, photogrammetric software and a digital camera.

Point-based methods with experienced human operators are processed well in traditional photogrammetric activities but not the autonomous environment of digital
photogrammetry. To develop more robust and accurate techniques, higher level objects of straight linear features accommodating elements other than points are adopted instead of points in aerial triangulation. Even though recent advanced algorithms provide accurate and reliable linear feature extraction, extracting linear features is more difficult than extracting a discrete set of points which can consist of any form of curves. Control points which are the initial input data and break points which are end points of piecewise curves are easily obtained with manual digitizing, edge operators or interest operators. Employing high level features increase the feasibility of geometric information and provide an analytical and suitable solution for the advanced computer technology.
Dedicated to my wife, parents, and grandparents

for their support in all my life’s endeavors.
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CHAPTER 1

INTRODUCTION

1.1 Overview

One of the major tasks in digital photogrammetry is to determine the orientation parameters of aerial imageries quickly and accurately, which involves two primary steps of interior orientation and exterior orientation. Interior orientation defines a transformation to a 3D image coordinate system with respect to the camera’s perspective center, while a pixel coordinate system is the reference system for a digital image, using the geometric relationship between the photo coordinate system and the instrument coordinate system.

While the aerial photography provides the interior orientation parameters, the problem is reduced to determine the exterior orientation with respect to the object coordinate system. Exterior orientation establishes the position of the camera projection center in the ground coordinate system and three rotation angles of the camera axis to represent the transformation between the image and the object coordinate system. Exterior orientation parameters (EOPs) of stereo model consisting of two aerial imageries can be obtained using relative and absolute orientation.
EOPs of multiple overlapping aerial imageries can be computed using bundle block adjustment. The position and orientation of each exposure station are obtained by bundle block adjustment using collinearity equations which are linearized as an unknown position and orientation with the object space coordinate system. In addition, interior orientation parameters can be recovered by using self-calibration which adds the parameters into bundle block adjustment to correct systematic errors such as lens distortion, principal distance error, principal point effect and atmospheric refraction. Self-calibration is applied to individual images or flight lines, not to entire images, and additional terms can be added depending on environmental and operational conditions.

Bundle block adjustment is a fundamental task in many applications such as surface reconstruction, orthophoto generation, image registration and object recognition. The program of bundle block adjustment in most softcopy workstations employs point features as the control information. Bundle block adjustment is one of photogrammetric triangulation methods. Photogrammetric triangulation using digital photogrammetric workstations is more automated than aerial triangulation using analog instruments since the stereo model can be directly set using analytical triangulation outputs. Bundle block adjustment reduces the cost of field surveying in difficult areas and verifies the accuracy of field surveying during the processing of bundle block adjustment. Even though each stereo model requires at least two horizontal and three vertical control points, bundle block adjustment can reduce the number of control points with accurate orientation parameters. EOPs of all photographs in the target area are determined by bundle block adjustment which improves the accuracy and the reliability of photogrammetric tasks. Since object reconstruction is processed
by an intersection employing more than two images, bundle block adjustment provides the redundancy for the intersection geometry and contributes to the elimination of the gross error in the recovery of EOPs. Since in recent years the integration of the global positioning system (GPS) and inertial navigation system (INS) provides more accurate information of the exposure station, the required control information of the block adjustment is reduced.

GPS was initially designed by the U.S. Department of Defense to determine the geodetic position for military purposes accurately and quickly. GPS consists of 24 medium Earth orbit satellites at an altitude of about 20,000 kilometers and at least five satellites are detected at any place on Earth. To determine a position, GPS establishes the intersection from at least four satellites. INS records a position acceleration and angular acceleration to estimate the velocity, position and orientation information. The integration of GPS and INS provides more accurate trajectory information since INS generates the continuous positioning information to support the discrete interval GPS information. In addition, INS can provide the location and orientation information by itself in case of a malfunction in receiving GPS signals. Incorporation of the GPS positions into the block adjustment can be implemented by GPS receivers and the data collection equipments. Three main considerations of the GPS position integration into the block adjustment are the offset between the GPS antenna position in the aircraft and camera position, the airplane’s turn at the end of flight lines, which can lose the GPS position information, and the systematic errors of coordinate transformation from GPS coordinate system with respect to WGS84 ellipsoid to local datum with respect to local geoid.
The stereo model consisting of two images with twelve EOPs is a common orientation unit. The mechanism of object reconstruction from stereo model is the same with the animal and human visual system. The principle aspects of the human vision system including neurophysiology, anatomy and visual perception is well described in *Digital Photogrammetry* (Schenk, 1999 [57]). The classical orientation model is implemented in two steps with relative orientation and absolute orientation to solve twelve orientation parameters for a model of two images. Five unknowns are solved from relative orientation for a stereoscopic view to the model, and seven unknowns three shifts, three rotations, and one scale factor are determined from absolute orientation. At least three vertical and two horizontal control points are required to obtain seven parameters of absolute orientation. In traditional photogrammetry, all orientation procedures are performed manually by a photogrammetric operator. Fiducial marks which define the photo coordinate system while they define the pixel coordinate system in digital photogrammetry are used in interior orientation which is image reconstruction with respect to perspective center.

The matching of conjugate entities plays an important role in relative orientation, and ground control points (GCPs) are adopted in absolute orientation to calculate the object space coordinate system. Matching techniques can be divided into two categories, area-based matching and feature-based matching. Area-based matching methods employ a similarity property between a small image patch in a template window and an image patch in a matching window. Two well known area-based matching methods are cross-correlation and least squares matching, also gray levels play an important role in area-based matching. Feature-based matching uses features for conjugate entities such as points, edges, linear features and volume features and the
similarity of geometric properties are compared to find conjugate entities. Feature-based matching is more invariant to radiometric changes and the implementation time of feature-based matching is faster than that of area-based matching. Extracting and matching conjugate points are the first step for the autonomous space resection but the general procedure of the autonomous space resection has not been developed yet as no matching algorithm can guarantee consistent accuracy.

Conjugate entities correspond to the same point in the object space using collinearity equation that all a point on the image, a perspective center and the corresponding point in the object space are on the same straight line. For stereo model, the coplanarity condition which is established by the equation for the volume of the parallelepiped can be adopted with a mathematical formula. The volume of the parallelepiped is decided by the three vectors which are a vector between a left perspective center and an image point on the left image, a vector between a right perspective center and an image point on the right image and a vector between two perspective centers. That means that two perspective centers and two conjugate image rays lie in the one plane and vectors are the object space vectors.

The coplanarity equation is used for the relative orientation in the stereo model. Seven of twelve exterior parameters are fixed and the remaining five parameters are obtained by five or more coplanarity equations since one coplanarity equation eliminates one degree of freedom. Adding more coplanarity equations increases the possibility of the detection of the incorrect observation and the obtainment of a more precise result. However, the relative orientation of more than two images using coplanarity equations has a problem since, in general, all rays from each images do not
intersect in a point. Possible constraint for coplanarity equations is that conjugate points are somewhere on the epipolar lines for point matching.

Point-based procedure relationship between point primitives is most widely developed in traditional photogrammetry, that one measured point on an image is identified on another image. Even for linear features, data for the stereo model in softcopy workstation is collected by points so that further application and analysis relies on a point as primary input data. The coefficients of interior, relative and absolute orientation are computed from point relationship. Interior orientation compensates lens distortion, film shrinkage, scanner error and atmosphere refraction. Relative orientation makes the stereoscopic view possible, and the relationship between model coordinate system and object space coordinate system is reconstructed by absolute orientation. Using GCPs is widely employed to compute the orientation parameters. Although employing a lot of GCPs is a time consuming procedure and blocks the robust and accurate automation what research on digital photogrammetry aims, developing of a computer, storage capacity, photogrammetric software and a digital camera can reduce the computational and time complexity.

In traditional photogrammetric activities, point-based methods with experienced human operators are processed well but not the autonomous environment of digital photogrammetry. Although some reasonable approaches have been proven in digital photogrammetry, the object recognition, the feature extraction and matching procedures are a challenging task in the computer environment of the autonomous photogrammetry. To develop more robust and accurate techniques, linear objects of
straight linear features or formulated entities such as conic sections, which are accommodating elements, other than points that are adopted instead of points in aerial triangulation.

Even though recent advanced algorithms provide accurate and reliable linear feature extraction, extracting linear features is more difficult than extracting a discrete set of points which can consist of any form of curves. Control points which are the initial input data, and break points which are end points of piecewise curves, are easily obtained with manual digitizing, edge operators or interest operators. A curve can be represented analytically by seed points which are extracted automatically by the point extraction in digital imagery.

Employing high level features increases the feasibility of geometric information and provides an analytical and suitable solution for the advanced computer technology. With the development of extraction, segmentation, classification and recognition of features, the input data for feature-based photogrammetry has been expanded to increase redundancy of aerial triangulation. Since the identification, formulation and application of reasonable linear features is a crucial procedure for autonomous photogrammetry, the higher order geometric feature-based modeling plays an important role in modern digital photogrammetry. The digital image format is suited to autonomous photogrammetry’s purpose, especially the feature extraction and measurement, and it is useful for precise and rigorous modeling of features from images.

1.2 Scope of dissertation

Over the years, many researchers have investigated the feasibility of linear features to autonomous photogrammetry, but the modeling of linear features are limited into
straight features and conic sections. In this work the integrated model of the extended
collinearity equation utilizing 3D natural cubic spline and arc-length parameterization
is derived to recover the exterior orientation parameters, 3D natural cubic spline
parameters and spline location parameters. The research topics in this dissertation
are sketched below bullet items.

- 3D natural cubic spline is adopted for the 3D line expression in the object
  space to represent 3D features as parametric form. The result of this algorithm
  are the tie and control features for bundle block adjustment. This is a key
  concept of the mathematical model of linear features in the object space and
  its counterpart in the projected image space for line photogrammetry.

- Arc-length parameterization of 3D natural cubic splines using Simpson’s rule is
devolved to solve over-parameterization of 3D natural cubic splines. Additional
equation to the extended collinearity equation expands bundle block adjustment
from limited conditions such as straight lines or conic sections (circles, ellipses,
parabolas and hyperbolas) to general cases.

- Tangents of splines which are additional equations to solve the overparameter-
ization of 3D natural cubic splines are established in case linear features in the
object space are straight lines or conic sections.

- To establish the correspondence between 3D natural cubic splines in the object
  space and their associated features in the 2D projected image space, the ex-
tended collinearity equation employing the projection ray which intersects the
3D natural cubic splines is developed and linearized for least squares method.
• Bundle block adjustment by the proposed method including the extended collinearity equation and arc-length parameterization equation is developed to show the feasibility of tie splines and control splines for the estimation of exterior orientation of multiple images, spline parameters and $t$ spline location parameters with simulated and real data.

1.3 Organization of dissertation

This dissertation is divided into six chapters. The next chapter presents a review of line photogrammetry including mathematical representations of 3D curves. Chapter 3 provides the extended collinearity model and formulation using 3D natural cubic splines and presents arc-length parameterization of 3D natural cubic splines. The non-linear integrated model is followed to recover EOPs, spline parameters and spline location parameters by tie and control features of 3D natural cubic splines in chapter 4. Detailed derivations of the extended collinearity equations, arc-length parameterization and tangents of splines are provided in the appendix. Chapter 5 introduces experimental results to demonstrate the feasibility of bundle block adjustment with 3D natural cubic splines with synthetic and real data. Finally, a summary of experience and recommended future research are included in chapter 6.
CHAPTER 2

LINE PHOTOGRAMMETRY

Line photogrammetry refers to photogrammetric applications such as single photo resection, relative orientation, triangulation, image matching, image registration and surface reconstruction which are implemented using linear features and correspondence between linear features rather than point features. This chapter describes the issues of the photogrammetric development from point to linear feature based methods and different mathematical models of free-form lines are presented to expand the control feature from point and straight linear features to free-form linear features.

2.1 Overview of line photogrammetry

Interest conjugate points such as edge points, corner points and points on parking lane are operated well for determining EOPs with respect to the object space coordinate frame in traditional photogrammetry. The most well-known edge and interest point detectors are Canny[9], Förstner[19], Harris well-known as the Plessy feature point detector[31], Moravec[45], Prewitt[51], Sobel[67] and SUSAN[66] interest point detectors. Canny, Prewitt, and Sobel operators are an edge detector and Förstner, Harris and, SUSAN are a corner detector. Other well-known corner detection algorithms are Laplacian of Gaussian, the difference of Gaussians, and the determinant of
Hessian. Interest point operators which detect well-defined point, edges and corners play an important role in automated triangulation and stereo matching. For example, the Harris operator is defined as a measurement of corner strength as

\[ H(x, y) = \det(M) - \alpha(\text{trace}(M))^2 \] (2.1)

where \( M \) is the local structure matrix and \( \alpha \) is a parameter so that \( H \geq 0 \leq \alpha \leq 0.25 \). A default value is 0.04. The gradients of \( x \) and \( y \) direction are

\[ g_x = \frac{\partial I}{\partial x}, \quad g_y = \frac{\partial I}{\partial y} \] (2.2)

with \( I \) is an image. The local structure matrix \( M \) is

\[ M = \begin{bmatrix} A & C \\ C & B \end{bmatrix} \] (2.3)

with \( A = g_x^2, B = g_y^2, \) and \( C = g_xg_y \).

A corner is detected when

\[ H(x, y) > H_{\text{thr}} \] (2.4)

where \( H_{\text{thr}} \) is the threshold parameter on corner strength.

The Harris operator searches points where variations in two orthogonal directions are large using the local auto-correlation function and providing good repeatability under varying rotation, scale, and illumination. Förstner corner detector is also based on the covariance matrix for the gradient at a target point. A preliminary weight is computed as

\[ w_{\text{pre}} = \begin{cases} \frac{\det(M)}{\text{trace}(M)}, & p > p_{\text{thr}} \\ 0, & \text{otherwise} \end{cases} \] (2.5)
where $p_{thr}$ is the threshold parameter on corner strength and $p$ is a measure of isotropy as

$$p = \frac{4\det(M)}{\text{trace}(M)^2} \quad (2.6)$$

The final weight is obtained from a local-maxima filtering.

$$w_{\text{final}} = \begin{cases} w_{lc, \text{local maxima}} & \text{if local maxima} \\ 0 & \text{otherwise} \end{cases} \quad (2.7)$$

where $w_{lc}$ is the local maximum within a local neighborhood window.

Marr[41] proposed the zero-crossing edge point detector utilizing second order directional derivatives not first order directional derivatives. The maximum of first order derivatives indicates the location of an edge and the zero of second order derivatives indicates the location of an edge. Physical boundaries of objects are detected easily since the gray levels are changed abruptly in boundaries. Since no single operator exists in edge detection, several criteria are required for each specific application.

Matching point features occurs large percentages of matching errors since point features are ambiguous and the analytical solution for the point matching is not developed. Because of the geometric information and symbolic meaning of linear features, matching linear features is more reliable than matching point features in the autonomous environment of digital photogrammetry. Since using linear features does not require the point-to-point correspondence, matching linear features is more flexible than matching point features.

A number of researchers have published studies of the automatic feature extraction and its application for various photogrammetric tasks, Föstner(1986) [18], Hannah(1989) [30], Schenk et al.(1991) [61], Schickler(1992) [62], Schenk and Toth(1993) [52].
[60], Ebner and Ohlhof(1994) [16], Ackerman and Tsingas(1994) [1], Haala and Vosselman(1992) [24], Drewniok and Rohr(1996, 1997) [14][15], Zahran(1997) [77] and Schenk(1998) [56]. However, point-based photogrammetry based on the manual measurement and identification of interest points is not established successfully in the autonomous environment of digital photogrammetry but the labor-intensive interpretation since it has the limitation of occlusion, ambiguity and semantic information in view of robust and suitable automation.

Since point features do not provide explicit geometric information, the geometrical knowledge is achieved by perceptual organization [55] [23] [10] [8]. Perceptual organization derives features and structures from imagery without prior knowledge of geometric, spectral or radiometric properties of features and is a required step for object recognition. Perceptual organization is an intermediate level process for various vision tasks such as target-to-background discrimination, object recognition, target cueing, motion-based grouping, surface reconstruction, image interpretation, and change detection. Since objects can not be distinguished by one gray level pixel, an image must be investigated entirely to obtain perceptual information. Most recent researches related to perceptual organization are the 2D image implementation at signal, primitive and structural levels.

Generally grouping or segmentation has the same meaning with perceptual organization in computer vision. Segmentation in computer vision is typically divided into two approaches, model-based method (top-down approach) and data-based method (bottom-up approach), and many researchers have employed edges and regions in segmentation. In edge-based approaches, edges are liked to general forms of linear features without discontinuities and in region-based approaches iterative region
growing techniques using seed points are preferred for surface fitting. Model-based methods require domain knowledge for each specific application similar to the human visual system and data-based methods employ data properties for data recognition in a global fashion. The same invariant properties in different positions and orientations are combined into the same regions or the same features in data-based methods. One approach, however, cannot guarantee consistent quality so that combined approaches are implemented to minimize error segmentation.

Berengolts and Lindenbaum [6] classified perceptual organization into three main components; grouping cues, testable feature subsets, and cue integration method. Potential grouping cues are continuity, similarity, proximity, symmetry, tangents of features, lengths of features, area of regions, parallelism and other properties between objects. Testable features are a suitable subset of features under consideration by the computational and time complexity. The computation space can be reduced in specific properties such as continuity and proximity. Cue integration should be constructed to reduce a cost function, which is considered with statistical and mathematical properties rigorously.

The symbolic representation using distinct points is difficult since interest points contain no explicit information about the physical reality. While traditional photogrammetric techniques obtain the camera parameters from the correspondence between 2D and 3D points, the more general and reliable process is required for advanced computer technology such as adopting linear features. Line photogrammetry is superior in higher level tasks such as object recognition and automation as opposed to point-based photogrammetry; however, to select correct candidate linear features is
a complicated problem. The development of the algorithm from point-based to line-based photogrammetry chooses the both advantages. Selecting suitable features is easier than extracting straight linear features and the candidate feature can be used in higher applications. Another reason for developing curve features is that curve features will be the prior and fundamental features to the next higher features such as surfaces, areas and 3D volumes which consist of free-form linear features. Line-based photogrammetry is more suitable to develop the robust and accurate techniques for automation. If linear features are employed as control features, they provide advantages over point features for the automation of aerial triangulation. Point-based photogrammetry based on the manual measurement and identification of conjugate points is less reliable than line-based photogrammetry since it has the limitation of occlusion(visibility), ambiguity(repetitive patterns) and semantic information in view of robust and suitable automation. The manual identification of corresponding entities in two images is crucial in the automation of photogrammetric tasks. No knowledge of the point-to-point correspondence is required in line-based photogrammetry. In addition, point features do not carry the information about the scene but linear features contain the semantic information related to real object features. Thus additional information about linear features can increase the redundancy of the system.

2.2 Literature review

Related works begin with methods of the pose estimation of imagery using linear features which appear in most man-made objects such as buildings, roads and parking lots. Over the years, a number of researchers in photogrammetry and computer vision have used the line feature instead of point feature for example Masry(1981)
[42], Tommaselli and Lugnani (1988) [70], Heikkilä (1991) [32], Kubik (1992) [37], Petsa and Patias (1994) [50], Gülch (1995) [20], Wiman and Axelsson (1996) [76], Chen and Shibasaki (1998) [12], Habib (1999) [26], Heuvel (1999) [34], Tommaselli (1999) [71], Vosselman and Veldhuis (1999) [73], Föstner (2000) [17], Smith and Park (2000) [65], Schenk (2002) [58], Tangelder et al. (2003) [68] and Parian and Gruen (2005) [49]. In 1988 Mulawa and Mikhail [47] proved the feasibility of linear features for close-range photogrammetric applications such as the space intersection, the space resection, relative orientation and absolute orientation. This was the first step to employ the linear feature based method in the close-range photogrammetric applications. Mulawa [46] developed linear feature based methods for different sensors. While straight linear features and conic sections can be represented as unique mathematical expressions, free-form lines in nature cannot be described by algebraic equations. Hence Mikhail and Weerawong [44] employed splines and polylines to express free-form lines as analytical expressions. Tommaselli and Tozzi [72] proposed that the accuracy of the straight line parameter should be a sub-pixel with the representation of four degrees of freedom in an infinite line. Many researchers in photogrammetry have described straight lines as infinite lines using minimal representation to reduce unknown parameters. The main consideration of straight line expression is singularities. Habib et al. [29] performed the bundle block adjustment using 3D point set lying on control linear features instead of traditional control points. EOPs were reconstructed hierarchically employing automatic single photo resection (SPR). SPR was implemented by the relationship between image and object space points expressed as EOPs as following equation (2.8).
\[
\begin{bmatrix}
    x_i - x_p \\
    y_i - y_p \\
    -f
\end{bmatrix}
= \lambda R^T(\omega, \phi, \kappa)
\begin{bmatrix}
    X_i - X_o \\
    Y_i - Y_o \\
    Z_i - Z_o
\end{bmatrix}
\] (2.8)

where \( \lambda \) scale, \( x_i, y_i \) the image coordinates, \( X_i, Y_i, Z_i \) the object coordinates, \( x_p, y_p, f \) interior orientation parameters and \( X_o, Y_o, Z_o, \omega, \phi, \kappa \) exterior orientation parameters. \( R^T \) is the 3D orthogonal rotation matrix containing the three rotation angles \( \omega, \phi \) and \( \kappa \).

Linear features between image and object space were compared to calculate EOPs by the modified iterated Hough transform. In the result, EOPs were robustly estimated without the knowledge of the relationship between image and object space primitives. In addition, Habib et al.[28] demonstrated that straight lines in object space contributed for performing the bundle block adjustment with self-calibration. They proposed that a large number of GCPs were reduced in calibration because of linear features having four collinearity equations from two end points defining each straight line. Their experiment proposed the usefulness of linear features in orientation not only with frame aerial imagery, but also in bundle block adjustment.

Habib et al.[25] summarized linear features extracted from mobile mapping system, GIS database and maps for various photogrammetric applications such as single photo resection, triangulation, digital camera calibration, image matching, 3D reconstruction, image to image registration and surface to surface registration. Matched linear feature primitives are utilized in space intersection for reconstruction of object space features and linear features in the object space are used for control features in triangulation and digital camera calibration. Since linear feature extraction can meet sub-pixel accuracy across the direction of edge and linear features can be extracted
a lot in man-made structures and mobile mapping system in reality, they have focused on implementation with straight linear features with geometric constraints. Since many man-made environments including buildings often have straight edges and planar faces, it is advantageous to employ line photogrammetry instead of point photogrammetry when mapping polyhedral model objects.

Mikhail[43] and Habib et al.[27] accomplished the geometrical modeling and the perspective transformation of linear features within a triangulation process. Linear features were used to recover relative orientation parameters. Habib et al. proposed a free-form line in object space by a sequence of 3D points along the object space line.

Lee and Bethel[38] proposed employing both points and linear features were more accurate than using only points in ortho-rectification of airborne hyperspectral imagery. EOPs were recovered accurately and serious distortions were removed by the contribution of linear features.

Schenk[59] extended the concept of aerial triangulation from point features to linear features. The line equation of six dependent parameters replaced the point based collinearity equation.

\[
\begin{align*}
X &= X_A + t \cdot a \\
Y &= Y_A + t \cdot b \\
Z &= Z_A + t \cdot c
\end{align*}
\] (2.9)

where a real variable \(t\), the start point \((X_A, Y_A, Z_A)\) and direction vector \((a, b, c)\). Traditional point-based collinearity equation was extended to line features.
\[
x_p = -f \frac{(X_A + t \cdot a - X_C)r_{11} + (Y_A + t \cdot b - Y_C)r_{12} + (Z_A + t \cdot c - Z_C)r_{13}}{(X_A + t \cdot a - X_C)r_{31} + (Y_A + t \cdot b - Y_C)r_{32} + (Z_A + t \cdot c - Z_C)r_{33}}
\]
\[
y_p = -f \frac{(X_A + t \cdot a - X_C)r_{21} + (Y_A + t \cdot b - Y_C)r_{22} + (Z_A + t \cdot c - Z_C)r_{23}}{(X_A + t \cdot a - X_C)r_{31} + (Y_A + t \cdot b - Y_C)r_{32} + (Z_A + t \cdot c - Z_C)r_{33}}
\]

(2.10)

with \(x_p, y_p\) photo coordinates, \(f\) the focal length, \(X_C, Y_C, Z_C\) camera perspective center, and \(r_{ij}\) the elements of the 3D orthogonal rotation matrix. The extended collinearity equation with six parameters was derived as the line expression of four parameters \((\phi, \theta, x_o, y_o)\) since a 3D straight line has only four independent parameters. Two constrains are required to solve a common form of the 3D straight equations using six parameters determined by two vectors.

\[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} = \begin{bmatrix}
\cos \theta \cos \phi \cdot x_o - \sin \phi \cdot y_o + \sin \theta \cos \phi \cdot z \\
\cos \theta \sin \phi \cdot x_o + \cos \phi \cdot y_o + \sin \theta \sin \phi \cdot z \\
- \sin \theta \cdot x_o + \cos \theta \cdot z
\end{bmatrix}
\]

(2.11)

where \(z\) is a real variable. The advantage of the 3D straight line using four independent parameters is that it reduces the computation and time complexity in the adjustment processes such as a bundle block adjustment. The collinearity equation as the straight line function of four parameters was developed.

\[
x_p = -f \frac{(X - X_C)r_{11} + (Y - Y_C)r_{12} + (Z - Z_C)r_{13}}{(X - X_C)r_{31} + (Y - Y_C)r_{32} + (Z - Z_C)r_{33}}
\]
\[
y_p = -f \frac{(X - X_C)r_{21} + (Y - Y_C)r_{22} + (Z - Z_C)r_{23}}{(X - X_C)r_{31} + (Y - Y_C)r_{32} + (Z - Z_C)r_{33}}
\]

(2.12)

where \(X, Y, Z\) were defined in (2.11).

The solution of the bundle block adjustment with linear features was implemented so that the line-based aerial triangulation can provide a more robust and autonomous environment than the traditional point-based bundle block adjustment. Another mathematical model of the perspective relationship between the image and the object
space features is the coplanarity approach. Projection plane defined by the perspective center in the image space and the plane including the straight line in the object space are identical. The extended collinearity model using linear features comparing with the coplanarity method was proposed. Zielinski[80] described another straight line expression with independent parameters.

Zalmanson and Schenk[79] extended their algorithm to relative orientation using line parameters. Epipolar lines were employed to adopt linear features in relative orientation with the extended collinearity equations (2.10). Gauss-Markov model was partitioned into orientation parameters and line parameters as (2.13).

\[
y = \begin{bmatrix} A_R & A_L \end{bmatrix} \begin{bmatrix} \xi_R \\ \xi_L \end{bmatrix} + e
\]

where \( \xi_R \) relative orientation parameters, \( \xi_L \) line parameters, \( A_R \) the partial derivatives of extended collinearity equations with respect to relative orientation parameters, \( A_L \) the partial derivatives of extended collinearity equations with respect to line parameters, and \( e \) the error vector. They demonstrated that parallel lines to epipolar lines contributed to relative orientation totally and vertical lines to epipolar lines rendered only surface reconstruction. In addition, they introduced a regularization scheme with soft constraints for the general cases.

Zalmanson[78] updated EOPs using the correspondence between the parametric control free-form line in object space and the projected 2D free-form line in image space. The hierarchical approach, the modified iteratively close point (ICP) method, was developed to estimate curve parameters. The ray lies on the free-form line whose parametric equation represented with \( l \) parameter is as following. Besl and McKay[7] employed the ICP algorithm to solve matching problem of point sets, free-form curves,
surfaces and terrain models in 2D and 3D space. ICP algorithm is executed without
the prior knowledge of correspondence between points. The ICP method affected
Zalmanson’s dissertation in the development of the recovery of EOPs using 3D free-
form lines in photogrammetry. Euclidean 3D transformation is employed in the search
of the closest entity on the geometric data set. Rabbani et al.[52] utilized ICP method
in registration of Lidar point clouds to divide into four categories spheres, planes,
cylinder and torus with direct and indirect method.

\[
\Xi(l) = \begin{bmatrix} X(l) \\ Y(l) \\ Z(l) \end{bmatrix} = \begin{bmatrix} X^k_0 \\ Y^k_0 \\ Z^k_0 \end{bmatrix} + \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} l \tag{2.14}
\]

where \(X_0, Y_0, Z_0, \omega, \varphi, \kappa\) the EOPs and \(\rho_1, \rho_2, \rho_3\) the direction vector.
The parametric curve \(\Gamma(t) = [X(t) Y(t) Z(t)]^T\) was obtained by minimizing the Euclidian distance between two parametric curves.

\[
\Phi(t, l) \equiv \|\Gamma(t) - \Xi(l)\|^2 = (X(t) - X_0 - \rho_1 l)^2 + (Y(t) - Y_0 - \rho_2 l)^2 + (Z(t) - Z_0 - \rho_3 l)^2
\]

\(\Phi(t, l)\) had a minimum value at \(\partial \Phi / \partial l = \partial \Phi / \partial t = 0\) with two independent variables \(l\) and \(t\) as (2.16).

\[
\begin{align*}
\partial \Phi / \partial l &= -2\rho_1 (X(t) - X_0 - \rho_1 l) - 2\rho_2 (Y(t) - Y_0 - \rho_2 l) - 2\rho_3 (Z(t) - Z_0 - \rho_3 l) = 0 \\
\partial \Phi / \partial t &= 2X'(t) (X(t) - X_0 - \rho_1 l) + 2Y'(t) (Y(t) - Y_0 - \rho_2 l) + \\
&\quad 2Z'(t) (Z(t) - Z_0 - \rho_3 l) = 0
\end{align*}
\]

(2.16)

Akav et al.[2] employed planar free form curves for aerial triangulation with the ICP
method. Since the effect of \(Z\) parameter as compared with \(X\) and \(Y\) was large in
normal plane equation \(aX + bY + cZ = 1\), different plane representation was developed
to avoid numerical problems as

\[
\begin{bmatrix}
  n_1 \\
  n_2 \\
  n_3
\end{bmatrix} = \begin{bmatrix}
  \sin \theta \cos \varphi \\
  \sin \theta \sin \varphi \\
  \cos \varphi
\end{bmatrix}
\]

\(n_1(X - X_o) + n_2(Y - Y_o) + n_3(Z - Z_o) = 0\)

\(n_1X + n_2Y + n_3Z = D\)

with \(\theta\) angle from \(XY\) plane, \(\varphi\) angle around \(Z\) axis, \(n\) unit vector of plane normal and \(D\) the distance between the plane and the origin. Five relative orientation parameters and three planar parameters were obtained by using the homography mapping system which searched the conjugate point in an image corresponding to a point in the other image.

Lin[40] proposed the method of the autonomous recovery of exterior orientation parameters by the extension of the traditional point-based Modified Iterated Hough Transform (MIHT) to the 3D free-form linear feature based MIHT. Straight polylines were generalized for matching primitives in the pose estimation since the mathematical representation of straight lines are much clearer than the algebraic expression of conic sections and splines.

Gruen and Akca[21] matched 3D curves whose forms were defined by a cubic spline using the least squares matching. Subpixels were localized by the least squares matching and the quality of the localization was decided by the geometry of image patches. Two free-form lines were defined as (2.18).

\[
f(u) = [x(u) \ y(u) \ z(u)]^T = a_0 + a_1u + a_2u^2 + a_3u^3
\]

\[
g(u) = [x'(u) \ y'(u) \ z'(u)]^T = b_0 + b_1u + b_2u^2 + b_3u^3
\]

where \(u \in [0, 1]\), \(a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3\) variables and \(f(u), g(u) \in \mathbb{R}^3\). Taylor expansion was employed to adopt Gauss-Markov model as (2.19).
\[ f(u) - e(u) = g(u) \]
\[ f(u) - e(u) = g^0(u) + \frac{\partial g^0(u)}{\partial u} du \]
\[ f(u) - e(u) = g^0(u) + \frac{\partial g^0(u)}{\partial u} \frac{\partial u}{\partial x} dx + \frac{\partial g^0(u)}{\partial u} \frac{\partial u}{\partial y} dy + \frac{\partial g^0(u)}{\partial u} \frac{\partial u}{\partial z} dz \]  

(2.19)

2.3 Parametric representations of curves

While progress for the automatic detection, segmentation and recognition of 3D lines and objects consisting of free-form lines has become sophisticated by significant advances in computer technology, considerable techniques such as the development in segmentation and classification for digital photogrammetry have been developed during the last few decades, such as; geospatial image processing software, digital orthophoto generation software, and softcopy workstations. Digital photogrammetry is closely related to the field of computer graphics and computer vision. Free-form lines and objects are an important element of many applications in computer graphics and computer vision. A number of researchers in computer vision and artificial intelligence have used suitable constraints or assumptions to reduce the solution space in segmentation and have extracted constrained features such as contours [64], convex outlines [35], rectangles [54] and ellipses [53].

Free-form lines are one of three linear features and other linear features are straight linear features and linear features described by unique mathematical equations. A number of researchers have preferred straight lines for photogrammetric applications since straight lines have no singularity problem and straight lines are easily detected in man-made environments. A list of curves is described as follows.
• **Piecewise curves**
  - Bezier curve
  - Splines
  - Ogee
  - Lowess
  - Reuleaux triangle

• **Fractal curves**
  - Blancmange curve
  - Dragon curve
  - De Rham curve
  - Koch curve
  - Lévy C curve
  - Space-filling curve
  - Sierpinski curve

• **Transcendental curves**
  - Bowditch curve
  - Brachistochrone
  - Butterfly curve
  - Catenary
  - Clélies
  - Cochleoid
  - Curve of pursuit
  - Cycloid
  - Horopter
  - Isochrone
  - Lamé curve
  - Lissajous curve
  - Pedal curve
  - Roulette curve
  - Rhumb line
  - Sail curve
  - Spirals
  - Superellipse
  - Syntractrix
  - Tractrix
  - Trochoid
  - Viviani’s curve

• **Algebraic curves**
  - Cubic plane curve
  - Bicron
  - Bean curve
  - Astroid
  - Ampersand curve
  - Atriphtaloid
  - Sextic plane curve
  - Hippopede
  - Deltoid curve
  - Conchoid of de Sluze
  - Epicycloid
  - Bullet-nose curve
  - Cardioid
  - Bow curve
  - Cissoid of Diocles
  - Conic sections
  - Crooked egg curve
  - Cruciform curve
  - Devil’s curve
  - Quintic plane curve
  - Epitrochoid
  - Hessian curve
  - Quartic plane curve
  - Kappa curve
  - Lemniscate
  - Kampyle of Eudoxus
  - Limacon
  - Line
  - Folium of Descartes
  - Nephroid
  - Quadrifolium
  - Quintic of I’Hospital
  - Rose curve
  - Strophoid
  - Rational normal curve
  - Tricuspid curve
  - Trident curve
  - Serpentine curve
  - Trifolium
  - Twisted cubic
  - Trisectrix of Maclaurin
  - Bicuspid curve
  - Cassini oval
  - Witch of Agnesi
  - Cassinoide
  - Cubic curve
  - Serpentine curve
  - Elliptic curve
  - Watt’s curve
  - Butterfly curve
  - Fermat curve
  - Klein quartic
  - Elkies trinomial curves
  - Mandelbrot curve
  - Trott curve
  - Erdős lemniscate
  - Classical modular curve
  - Hyperelliptic curve


Free-form linear features are obtained easily from mobile mapping systems, object recognition system in computer vision and GIS database such as the digital map.
Tankovich[69] used linear features represented by straight lines for resection and intersection. 3D free-form lines in the object space can be represented by mathematical function and parametric representations of curves can be categorized into four groups, for example; piecewise constructions, fractal curves, transcendental curves and algebraic curves. Another approach of free-form line expression is using a point sequence along linear features to preserve the original information of the linear features. Parametric representations reduce the amount of information for the expression of free-form lines with generalization. A more accurate expression needs more parameters in a mathematical equation. Since any parametric representation of free-form lines has singularities, two more constraints or generalization are required for free-form line applications in photogrammetry. While most free-form lines are finite segments in reality, most representations of curves are infinite representations.

Piecewise curves are defined by each segment and are scale independently. The cubic polynomials with continuity $C^2$ is a standard piecewise curve. For smoothed display, small degree polynomial segments can be used for curve representation. The end point of one segment is the start point of the next segment with the value of $t$ on interval $[0, 1]$.

Fractal curves are random iterated function and can not be differentiated anywhere although fractal curves are continuous. Fractal curves are transformed hierarchically on smaller and smaller scale.

Transcendental curves are not algebraic that is if algebraic curve form is $f(x, y) = 0$ with a polynomial in $x$ and $y$, and transcendental curve form is $f(x, y) = 0$ without a polynomial in $x$ and $y$. 
Algebraic curves have implicit form having a polynomial equation in two variables. The graphs of a polynomial equation are algebraic curves and conic sections are algebraic curves of degree 2 called a higher plane curve.

2.3.1 Spline

The choice of the right feature model is important to develop the feature based approach since the ambiguous representation of features leads to an unstable adjustment. A spline is piecewise polynomial functions in \( n \) of vector graphics. A spline is widely used for data fitting in the computer science because of the simplicity of the curve reconstruction. Complex figures are approximated well through curve fitting and a spline has strength in the accuracy evaluation, data interpolation and curve smoothing. One of important properties of a spline is that a spline can easily be morph. A spline represents a 2D or 3D continuous line with a sequence of pixels and segmentation. The relationship between pixels and lines is applied to a bundle block adjustment or a functional representation. A spline of degree 0 is the simplest spline and a linear spline of degree 1, a quadratic spline of degree 2 and a common natural cubic spline of degree 3 with continuity \( C^2 \). The geometrical meaning of continuity \( C^2 \) is that the first and second derivatives are proportional at joint points and the parametric importance of continuity \( C^2 \) is that the first and second derivatives are equal at connected points.

A spline is defined as piecewise parametric form.

\[
F : [a, b] \rightarrow \Re \\
a = x_0 < x_1 < \cdots < x_{n-1} = b \\
F(x) = G_0(x), x \in [x_0, x_1]
\] (2.20)
Figure 2.1: Spline continuity

\[ F(x) = G_1(x), x \in [x_1, x_2] \]

\[ \ldots \]

\[ F(x) = G_{n-1}(x), x \in [x_{n-2}, x_{n-1}] \]

where \( x \) is a knot vector of a spline.

**Natural cubic spline**

The number of break points which are the determination of a set of piecewise cubic functions varies depending on spline parameters. A natural cubic degree guarantees the second-order continuity which means the first and second order derivatives of two consecutive natural cubic splines are continuous at the break point. The intervals for a natural cubic spline do not need to be the same as the distance of every two consecutive data points. The best intervals are chosen by a least squares method. In general, the total number of break points is less than that of original input points. The algorithm of a natural cubic spline is as below.

*Generate the break point (control point) set for the spline of the original input data*
Calculate the maximum distance between the approximated spline and the original input data

While (the maximum distance > the threshold of the maximum distance)

Add break point to the break point set at the location of the maximum distance

Compare the maximum distance with the threshold

The larger threshold makes the more break points with more accurate spline to the original input data.

$N$ piecewise cubic polynomial functions between two adjacent break points are defined from the $N + 1$ break points. There is a separate cubic polynomial for each segment with its own coefficients.

\[
X_0(t) = a_{00} + a_{01}t + a_{02}t^2 + a_{03}t^3, \quad t \in [0, 1] \\
X_1(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3, \quad t \in [0, 1] \\
\ldots
\]

(2.21)

\[
X_i(t) = a_{i0} + a_{i1}t + a_{i2}t^2 + a_{i3}t^3, \quad t \in [0, 1] \\
Y_i(t) = b_{i0} + b_{i1}t + b_{i2}t^2 + b_{i3}t^3, \quad t \in [0, 1] \\
Z_i(t) = c_{i0} + c_{i1}t + c_{i2}t^2 + c_{i3}t^3, \quad t \in [0, 1]
\]

The strength is that segmented lines represent a free-form line with analytical parameters. The number of break points is reduced and the input error should be absorbed by a mathematical model especially in the expression of points on a straight line. A natural cubic spline is data independent curve fitting. The disadvantage is that the entire curve shape depends on all of the passing points. Changing any one of them changes the entire curve.
**Cardinal spline**

A Cardinal spline is defined by a cubic Hermite spline, a third-degree spline. Hermite form defined by two control points and two control tangents is implied by each polynomial of a Hermite form. The basic scheme of solving the system of a cubic Hermite spline is continuity $C^1$ that the first derivatives are equal at break points. A disadvantage of a cubic Hermite spline is that tangents are always required while this information is not available for all curves.

A Cardinal spline consists of three points before and after a control point and a tension parameter. A tangent $m_i$ of a cardinal spline is as following with given $n+1$ points, $p_0, \ldots, p_n$.

$$m_i = \frac{1}{2}(1-c)(p_{i+1} - p_{i-1})$$

(2.22)

where $c$ is a tension parameter which contributes the length of the tangent. A tension parameter is between 0 and 1. If tension is 0, a Cardinal spline is referred to as a Catmull-Rom spline[11]. A Catmull-Rom spline is frequently used to interpolate smoothly between point data in mathematics and between key-frames in computer graphics. A cardinal spline represents the curve from the second point to the last second point of the input point set. The input points are the control points that define a Cardinal spline. A Cardinal spline produces a $C^1$ continuous curve, not a $C^2$ continuous curve, and the second derivatives are linearly interpolated within each segment. The advantage is no need for tangents but the disadvantage is the imprecision of the tangent approximation.
B-spline

The equation of B-spline with \( m + 1 \) knot vector \( T \), the degree \( p \equiv m - n - 1 \) which must be satisfied from equality property of B-spline since a basis function is required for each control points and \( n + 1 \) control points \( P_0, P_1, \ldots, P_n \) is as \([75]\)

\[
C(t) = \sum_{i=0}^{n} P_i N_{i,p}(t) \tag{2.23}
\]

with

\[
N_{i,0}(t) = \begin{cases} 
1 & \text{if } t_i \leq t \leq t_{i+1} \text{ and } t_i < t_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

\[
N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{i+1,p-1}(t)
\]

Each point is defined by control points affected by segments and B-splines are a piecewise curve of degree \( p \) for each segment. Advantages of B-splines are to represent complex shapes with lower degree polynomials while B-splines are drawn closer to their original control polyline as the degree decreases. B-splines have an inside convex hull property that B-splines are contained in the convex hull presented by its polyline. With an inside convex hull property, the shape of B-splines can be controlled in detail. Although B-splines need more information and more complex algorithm than other splines, a B-spline provides many important properties than other splines, such as degree independence for control points. B-splines change the degree of curves without changing control points, and control points can be changed without affecting the shape of whole B-splines. B-splines, however, cannot represent simple constrained curves such as straight lines and conic sections.
2.3.2 Fourier transform

Fourier series and transform have been widely employed to represent 2D and 3D curves as well as 3D surfaces. Fourier series is useful for periodic curves and Fourier transform is proper for non-periodic free-form lines. Parametric representations of curves have been established in many applications. Fourier transform is the generalization of the complex Fourier series with the infinite limit. Let \( w \) be the frequency domain, then time domain \( t \) is described as follow.

\[
X(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)e^{-iwt}dt
\]

Discrete Fourier transform (DFT) is commonly employed to solve a sampled signal and partial differential equations. Fast Fourier transform (FFT) which reduces the computation complexity for \( N \) points from \( 2N^2 \) to \( 2N\log_2 N \) improves the efficiency of DFT. DFT transforms \( N \) complex numbers \( x_0, \cdots, x_{N-1} \) into the sequence of \( N \) complex number \( X_0, \cdots, X_{N-1} \).

\[
X_k = \sum_{k=0}^{N-1} x_n e^{-\frac{2\pi i}{N}kn} \quad k = 0, \cdots, N - 1
\]

Shape information can be obtained with the first few coefficients of Fourier transform since the low frequency covers most shape information. In addition, Fourier transform can be developed in the smoothing of a 3D free-form line in the object space using the convolution theorem with the use of the FFT algorithm.

Boundary conditions affect the application of Fourier related transforms especially when solving partial differential equations. In some cases, discontinuities occur due to the number of Fourier terms used. The fewer terms, the smoother a function.
Fourier transform has strength in signal and data compression, filtering and smoothing. However, Fourier transform has little strength in the representation of a curve accurately. More terms to represent a free-form line exactly increase the time and the computation complexity.

2.3.3 Implicit polynomials

Implicit polynomials obtained by the computation of the least squares problem for the coefficients of the polynomial have the mathematical form:

\[
\begin{align*}
X(t) &= a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \\
Y(t) &= b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n \\
Z(t) &= c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n
\end{align*}
\] (2.26)

where \(a_0, a_1, \cdots, a_n, b_0, \cdots, b_n, c_0, \cdots, c_n \in \mathbb{R}^3\) and the value of \(n\) is the degree of the polynomial \(\mathbb{R}^n\). The coefficients \(a_0, b_0, c_0\) characterize a point in the object space \(\mathbb{R}^3\) and remaining parameters represent the direction. The number of parameters required to represent the \(n\) degree polynomial \(\mathbb{R}^n\) is \(3(n + 1)\). Implicit polynomial in a degree of 3 is analogous to a natural cubic spline.

A free-form line is formulated in a simple form geometrically independent according to a polynomial fit. However, a quality of the results is implemented in case of high degrees. A prediction of the outcome of fitting with implicit polynomials is difficult since coefficients are not geometrically correlated. The best solution is optimized by the iterative try and error method. In the worst case, the implementation seems to never end with a tight threshold. The number of trials to fit successful coefficients depends on the restriction of the search space. Implicit polynomials are originated through constraints and various minimization criteria.
For other polyline expressions, Ayache and Faugeras[4] described a line as the intersection of two planes that one plane is parallel to the X-axis and the other is parallel to the Y-axis with two constraints to extend this concept to general lines. Two constraints reduced line parameters six to four-dimensional vector.

Since standard collinearity model can be extended to adopt parametric representation of spline, 3D natural cubic splines are employed for this research. The collinearity equations play an important role in photogrammetry since each control point in the object space produces two collinearity equations for every photograph in which the point appears. A natural 3D cubic spline allows the utilization of the collinearity model for expressing orientation parameters and curve parameters.
3.1 3D natural cubic splines

In this chapter, the correspondence between the 3D curve in the object space coordinate system and its projected 2D curve in the image coordinate system is implemented using a natural cubic spline accommodating curve feature because of its boundary conditions the zero second derivatives at the end points. A natural cubic spline is composed of a sequence of cubic polynomial segments as figure 3.1 with $x_0, x_1, \ldots, x_n$ the $n + 1$ control points and $X_0, X_1, \ldots, X_{n-1}$ the ground coordinate of $n$ segments.

![Figure 3.1: Natural cubic spline](image)

Figure 3.1: Natural cubic spline
Each segment of a natural cubic spline is expressed with parametric representations as (3.1).

\[
X_0(t) = a_{00} + a_{01}t + a_{02}t^2 + a_{03}t^3, \quad t \in [0, 1]
\]
\[
X_1(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3, \quad t \in [0, 1]
\]
\[
\vdots \tag{3.1}
\]
\[
X_i(t) = a_{i0} + a_{i1}t + a_{i2}t^2 + a_{i3}t^3, \quad t \in [0, 1]
\]
\[
Y_i(t) = b_{i0} + b_{i1}t + b_{i2}t^2 + b_{i3}t^3, \quad t \in [0, 1]
\]
\[
Z_i(t) = c_{i0} + c_{i1}t + c_{i2}t^2 + c_{i3}t^3, \quad t \in [0, 1]
\]

Bartels et al.\cite{5} represented the solution of a natural cubic spline with given control points. Since four coefficients are required for each interval, the number of total parameters is \(4n\) to define the spline \(X_i\).

\[
X_i(t) = a_{i0} + a_{i1}t + a_{i2}t^2 + a_{i3}t^3
\]
\[
X_i^{(1)}(t) = a_{i1} + 2a_{i2}t + 3a_{i3}t^2 \quad \tag{3.2}
\]
\[
X_i^{(2)}(t) = 2a_{i2} + 6a_{i3}t
\]

The first and second order derivatives are equal at the joints of a spline and the relationship between the control points and the segments leads (3.3).

\[
X_{i-1}(1) = x_i
\]
\[
X_i(0) = x_i
\]
\[
X_{i-1}^{(1)}(1) = X_i^{(1)}(0) \tag{3.3}
\]
\[
X_{i-1}^{(2)}(1) = X_i^{(2)}(0) \quad [1]
\]
\[
X_0(0) = x_0
\]
\[
X_{n-1}(1) = x_n
\]

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Substituting (3.2) into (3.3 [1]) leads to

\[ 2a_{(i-1)2} + 6a_{(i-1)3} = 2a_{i2} \]  

(3.4)

and substituting (3.2) into (3.3) leads to

\[ a_{i0} = x_i \]
\[ a_{i1} = D_i \]
\[ a_{i2} = 3(x_{i+1} - x_i) - 2D_i - D_{i+1} \]  

(3.5)
\[ a_{i3} = 2(x_i - x_{i+1}) + D_i + D_{i+1} \]

where \( D \) is the derivative.

Substituting (3.5) into (3.4) leads to

\[ D_{i-1} + 4D_i + D_{i+1} = 3(x_{i+1} - x_{i-1}) \]  

(3.6)

with \( 1 \leq i \leq n - 1 \).

Since the total number of \( D_i \) unknowns is \( n \) and the total number of equations is \( n - 2 \), two more equations are required to solve the underdetermined system. In a natural cubic spline, two boundary conditions are given to complete the system of \( n - 2 \) equations. The second derivatives at the end points are set to zero as (3.7).

\[ X_0^{(2)}(0) = 0 \]
\[ X_{n-1}^{(2)}(0) = 0 \]  

(3.7)

Otherwise, the first and the last segment can be set to the 2nd order polynomial to reduce two unknowns, for which two boundary conditions are not required.

Substituting (3.5) into (3.7) can be described as followings.

\[ 2D_0 + D_1 = 3(x_1 - x_0) \]
\[ D_{n-1} + 2D_n = 3(x_n - x_{n-1}) \]  

(3.8)
\[
\begin{bmatrix}
2 & 1 \\
1 & 4 & 1 \\
& \vdots \\
\end{bmatrix}
\begin{bmatrix}
D_0 \\
D_1 \\
\vdots \\
D_{n-1} \\
D_n
\end{bmatrix}
= \begin{bmatrix}
3(x_1 - x_0) \\
3(x_2 - x_0) \\
\vdots \\
3(x_n - x_{n-1})
\end{bmatrix}
\] (3.9)

\[
D_i \text{ is obtained in the case of close curves as}
\begin{bmatrix}
4 & 1 \\
1 & 4 & 1 \\
& \vdots \\
\end{bmatrix}
\begin{bmatrix}
D_0 \\
D_1 \\
\vdots \\
D_{n-1} \\
D_n
\end{bmatrix}
= \begin{bmatrix}
3(x_1 - x_0) \\
3(x_2 - x_0) \\
\vdots \\
3(x_n - x_{n-1})
\end{bmatrix}
\] (3.10)

The normalized spline system is
\[
\begin{bmatrix}
b & 1 \\
1 & 4 & 1 \\
& \vdots \\
\end{bmatrix}
\begin{bmatrix}
D_0 \\
D_1 \\
\vdots \\
D_{n-1} \\
D_n
\end{bmatrix}
= k \begin{bmatrix}
(x_1 - x_0) \\
(x_2 - x_0) \\
\vdots \\
(x_n - x_{n-1})
\end{bmatrix}
\] (3.11)

where the value of \(b\) and \(k\) depends on the boundary conditions of a spline and \(k\) depends on the type of a spline. Normally the value of \(b\) and \(k\) in a natural cubic are 2 and 3 in an unclosed curve case and 4 and 3 in a closed curve case respectively.

The values of \(b\) and \(k\) in B-Splines are 5 and 6 respectively [13].

In case of two parameters, the corresponding relationship between two parameters can be calculated without an intermediate \(t\) parameter. \(n + 1\) point pairs, \((x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)\), have \(4n\) unknown spline parameters in \(n\) segments. \(2n\) equations from 0th continuity condition, \(n - 1\) equations from 1st continuity condition, \(n - 1\) equations from 2nd continuity condition, 2 equations from boundary conditions that the second derivatives at the end points are set to zero.
From 0th continuity,

\[ y_0 = a_{00} + a_{01} x_0 + a_{02} x_0^2 + a_{03} x_0^3 \]
\[ y_1 = a_{00} + a_{01} x_1 + a_{02} x_1^2 + a_{03} x_1^3 \]
\[ y_1 = a_{10} + a_{11} x_1 + a_{12} x_1^2 + a_{13} x_1^3 \]
\[ y_2 = a_{10} + a_{11} x_2 + a_{12} x_2^2 + a_{13} x_2^3 \]
\[ \cdots \]
\[ y_{n-1} = a_{(n-1)0} + a_{(n-1)1} x_{n-1} + a_{(n-1)2} x_{n-1}^2 + a_{(n-1)3} x_{n-1}^3 \]
\[ y_n = a_{(n-1)0} + a_{(n-1)1} x_n + a_{(n-1)2} x_n^2 + a_{(n-1)3} x_n^3 \]

From 1st continuity,

\[ a_{01} + 2a_{02} x_1 + 3a_{03} x_1^2 = a_{11} + 2a_{12} x_1 + 3a_{13} x_1^2 \]
\[ a_{11} + 2a_{12} x_2 + 3a_{13} x_2^2 = a_{21} + 2a_{22} x_2 + 3a_{23} x_2^2 \]
\[ \cdots \]
\[ a_{(n-2)1} + 2a_{(n-2)1} x_{n-1} + 3a_{(n-2)3} x_{n-1}^2 = a_{(n-1)1} + 2a_{(n-1)2} x_{n-1} + 3a_{(n-1)3} x_{n-1}^2 \]

From 2nd continuity,

\[ 2a_{02} + 6a_{03} x_1 = 2a_{12} + 6a_{13} x_1 \]
\[ 2a_{12} + 6a_{13} x_2 = 2a_{22} + 6a_{23} x_2 \]
\[ \cdots \]
\[ 2a_{(n-2)2} + 6a_{(n-2)3} x_{n-1} = 2a_{(n-1)2} + 6a_{(n-1)3} x_{n-1} \]

From boundary conditions,

\[ 2a_{01} + 6a_{03} x_0 = 0 \]
\[ 2a_{(n-1)2} + 6a_{(n-1)3} x_n = 0 \]
3.2 Extended collinearity equation model for splines

The collinearity equations are the commonly used condition equations for relative orientation, the space intersection which calculates a point location in the object space using projection ray intersection from two or more images and the space resection which determines the coordinates of a point on an image and EOPs with respect to the object space coordinate system. The space intersection and the space resection are the fundamental operations in photogrammetry for further processes such as triangulation. The basic concept of the collinearity equation is that all points on the image, a perspective center and the corresponding point in the object space are on a straight line. The relationship between the image coordinate system and the object coordinate system is expressed by three position parameters and three orientation parameters. The collinearity equations play an important role in photogrammetry since each control point in the object space produces two collinearity equations for every photograph in which the point appears. If \( m \) points appear in \( n \) images, then \( 2mn \) collinearity equations can be employed in the bundle block adjustment. The extended collinearity equations relating to a natural cubic spline in object space with ground coordinates \((X_i(t), Y_i(t), Z_i(t))\) into image space with photo coordinates \((x_{pi}, y_{pi})\) are

\[
\begin{align*}
x_{pi} &= -f \left( \frac{(X_i(t) - X_C)r_{11} + (Y_i(t) - Y_C)r_{12} + (Z_i(t) - Z_C)r_{13}}{(X_i(t) - X_C)r_{31} + (Y_i(t) - Y_C)r_{32} + (Z_i(t) - Z_C)r_{33}} \right) \\
y_{pi} &= -f \left( \frac{(X_i(t) - X_C)r_{21} + (Y_i(t) - Y_C)r_{22} + (Z_i(t) - Z_C)r_{23}}{(X_i(t) - X_C)r_{31} + (Y_i(t) - Y_C)r_{32} + (Z_i(t) - Z_C)r_{33}} \right)
\end{align*}
\] (3.12)

A natural cubic spline allows the utilization of the collinearity model for expressing orientation parameters and curve parameters.
with $x_{pi}, y_{pi}$ the photo coordinates of the $i$th segment, $f$ the focal length, $X_C, Y_C, Z_C$ the camera perspective center, and $r_{ij}$ the elements of the 3D orthogonal rotation matrix $R^T$ by angular elements $(\omega, \varphi, \kappa)$ of EOPs.

Figure 3.2: The projection of a point on a spline

The 3D orthogonal rotation matrix $R^T$ rotates the object coordinate system parallel to the photo coordinate system employing three sequential rotations: the primary rotation angle $\omega$ around the $X$-axis ($X_\omega Y_\omega Z_\omega$), the secondary rotation angle $\varphi$ around the once rotated $Y$-axis ($X_\omega Y_\omega Y_\varphi Z_\omega Z_\varphi$) and the tertiary rotation angle $\kappa$ around the twice rotated $Z$-axis ($X_\omega Y_\omega Y_\varphi Z_\omega Y_\varphi Z_\omega Y_\varphi$). Most often the consecutive rotations with the sequence $R_\omega, R_\varphi, R_\kappa$ or $R_\varphi, R_\omega, R_\kappa$ are used in photogrammetry. Since each three sequential rotations $R_\omega, R_\varphi, R_\kappa$ are orthogonal matrices, the matrix $R$ is orthogonal.
\( R^{-1} = R^T \). The matrix \( R^T (= R^{-1}) \) rotates the object coordinate system parallel to the photo coordinate system and the matrix \( R \) rotates the photo coordinate system parallel to the object coordinate system.

\[
R = R_\omega R_\varphi R_\kappa = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \omega & -\sin \omega \\
0 & \sin \omega & \cos \omega
\end{bmatrix} \begin{bmatrix}
\cos \varphi & 0 & \sin \varphi \\
0 & 1 & 0 \\
-\sin \varphi & 0 & \cos \varphi
\end{bmatrix} \begin{bmatrix}
\cos \kappa & -\sin \kappa & 0 \\
\sin \kappa & \cos \kappa & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos \varphi \cos \kappa & -\cos \varphi \sin \kappa & \sin \varphi \\
\cos \omega \sin \kappa + \sin \omega \sin \varphi \cos \kappa & \cos \omega \cos \kappa - \sin \omega \sin \varphi \sin \kappa & -\sin \omega \cos \varphi \\
\sin \omega \sin \kappa - \cos \omega \sin \varphi \cos \kappa & \sin \omega \cos \kappa + \cos \omega \sin \varphi \sin \kappa & \cos \omega \cos \varphi
\end{bmatrix}
\]

\( (3.13) \)

The extended collinearity equations can be written as follows:

\[
x_p = -f \frac{u}{w}, \quad y_p = -f \frac{v}{w}
\]

\( (3.14) \)

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} = R^T (\omega, \varphi, \kappa) \begin{bmatrix}
X_i(t) - X_C \\
Y_i(t) - Y_C \\
Z_i(t) - Z_C
\end{bmatrix}
\]

To recover the 3D natural cubic spline parameters and the exterior orientation parameters in a bundle block adjustment, a non-linear mathematical model of the extended collinearity equation is differentiated. The models of exterior orientation recovery are classified into linear and non-liner methods. While linear methods decrease the computation load, accuracy and robustness of linear algorithms are not excellent. Otherwise non-linear methods are more accurate and robust. However, non-linear methods require initial estimates and increase the computational complexity. The relationship between a point in the image space and a corresponding point in the object space is established by the extended collinearity equation. Prior knowledge
about correspondences between individual points in the 3D object space and their
projected features in the 2D image space is not required in extended collinearity
equations with 3D natural splines. One point on a cubic spline has 19 parameters
\((X_c, Y_c, Z_c, \omega, \varphi, \kappa, a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, c_0, c_1, c_2, c_3, t)\). The differentials of (3.14)
is (3.15).

\[
\begin{align*}
    dx_p &= -\frac{f}{w} du + \frac{fu}{w^2} dw, \quad dy_p = -\frac{f}{w} dv + \frac{fv}{w^2} dw
\end{align*}
\]  
(3.15)

with the differentials of \(du, dv, dw\) (3.16)

\[
\begin{bmatrix}
    du \\
    dv \\
    dw
\end{bmatrix} = \frac{\partial R^T}{\partial \omega} \begin{bmatrix}
    X_i(t) - X_C \\
    Y_i(t) - Y_C \\
    Z_i(t) - Z_C
\end{bmatrix} d\omega + \frac{\partial R^T}{\partial \varphi} \begin{bmatrix}
    X_i(t) - X_C \\
    Y_i(t) - Y_C \\
    Z_i(t) - Z_C
\end{bmatrix} d\varphi
\]
\[+ \frac{\partial R^T}{\partial \kappa} \begin{bmatrix}
    X_i(t) - X_C \\
    Y_i(t) - Y_C \\
    Z_i(t) - Z_C
\end{bmatrix} d\kappa - R^T \begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix} dx_c - R^T \begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix} dy_c - R^T \begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix} dz_c
\]
\[+ R^T \begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix} da_0 + R^T \begin{bmatrix}
    t \\
    0 \\
    0
\end{bmatrix} da_1 + R^T \begin{bmatrix}
    t^2 \\
    0 \\
    0
\end{bmatrix} da_2 + R^T \begin{bmatrix}
    t^3 \\
    0 \\
    0
\end{bmatrix} da_3 + R^T \begin{bmatrix}
    1 \\
    0 \\
    0
\end{bmatrix} db_0
\]
\[+ R^T \begin{bmatrix}
    0 \\
    t \\
    0
\end{bmatrix} db_1 + R^T \begin{bmatrix}
    0 \\
    t^2 \\
    0
\end{bmatrix} db_2 + R^T \begin{bmatrix}
    0 \\
    t^3 \\
    0
\end{bmatrix} db_3 + R^T \begin{bmatrix}
    0 \\
    1 \\
    0
\end{bmatrix} dc_0 + R^T \begin{bmatrix}
    0 \\
    0 \\
    t
\end{bmatrix} dc_1
\]
\[+ R^T \begin{bmatrix}
    0 \\
    0 \\
    t^2
\end{bmatrix} dc_2 + R^T \begin{bmatrix}
    0 \\
    0 \\
    t^3
\end{bmatrix} dc_3 + R^T \begin{bmatrix}
    a_1 + 2a_2 t + 3a_3 t^2 \\
    b_1 + 2b_2 t + 3b_3 t^2 \\
    c_1 + 2c_2 t + 3c_3 t^2
\end{bmatrix} dt
\]
\[\text{(3.16)}\]

Substituting \(du, dv, dw\) in (3.15) by the expressions found in (3.16) leads to
\[ dx_p = M_1 dX_C + M_2 dY_C + M_3 dZ_C + M_4 d\omega + M_5 d\varphi + M_6 d\kappa + M_7 da_0 + M_8 da_1 \]
\[ \quad + M_9 da_2 + M_{10} da_3 + M_{11} db_0 + M_{12} db_1 + M_{13} db_2 + M_{14} db_3 + M_{15} dc_0 \]
\[ \quad + M_{16} dc_1 + M_{17} dc_2 + M_{18} dc_3 + M_{19} dt \]
\[ dy_p = N_1 dX_C + N_2 dY_C + N_3 dZ_C + N_4 d\omega + N_5 d\varphi + N_6 d\kappa + N_7 da_0 + N_8 da_1 \]
\[ \quad + N_9 da_2 + N_{10} da_3 + N_{11} db_0 + N_{12} db_1 + N_{13} db_2 + N_{14} db_3 + N_{15} dc_0 \]
\[ \quad + N_{16} dc_1 + N_{17} dc_2 + N_{18} dc_3 + N_{19} dt \]

(3.17)

\[ M_1, \ldots, M_{19}, N_1, \ldots, N_{19} \] denote the partial derivatives of the extended collinearity equation for curves. Detailed derivations are in Appendix A.1. The linearized extended collinearity equations by Taylor expansion, ignoring 2nd and higher order terms can be written as followings.

\[ x_p + \int \frac{u^0}{w^0} = M_1 dX_C + M_2 dY_C + M_3 dZ_C + M_4 d\omega + M_5 d\varphi + M_6 d\kappa + M_7 da_0 \]
\[ \quad + M_8 da_1 + M_9 da_2 + M_{10} da_3 + M_{11} db_0 + M_{12} db_1 + M_{13} db_2 \]
\[ \quad + M_{14} db_3 + M_{15} dc_0 + M_{16} dc_1 + M_{17} dc_2 + M_{18} dc_3 + M_{19} dt \]
\[ + e_x \]
\[ y_p + \int \frac{v^0}{w^0} = N_1 dX_C + N_2 dY_C + N_3 dZ_C + N_4 d\omega + N_5 d\varphi + N_6 d\kappa + N_7 da_0 \]
\[ \quad + N_8 da_1 + N_9 da_2 + N_{10} da_3 + N_{11} db_0 + N_{12} db_1 + N_{13} db_2 \]
\[ \quad + N_{14} db_3 + N_{15} dc_0 + N_{16} dc_1 + N_{17} dc_2 + N_{18} dc_3 + N_{19} dt \]
\[ + e_y \]

(3.18)

with \( u^0, v^0, w^0 \) the approximate parameters by \((X_C^0, Y_C^0, Z_C^0, \omega^0, \varphi^0, \kappa^0, a_0^0, a_1^0, a_2^0, a_3^0, b_0^0, b_1^0, b_2^0, b_3^0, c_0^0, c_1^0, c_2^0, c_3^0, t^0)\) and \( e_x, e_y \) the stochastic errors of \( x_p, y_p \) the observed photo coordinates with zero expectation respectively. Orientation parameters including
3D natural cubic spline parameters are expected to recover correctly since extended collinearity equations with 3D natural cubic splines increase redundancy.

3.3 Arc-length parameterization of 3D natural cubic splines

The assumption in bundle block adjustment by the Gauss-Markov model is that all estimated parameters are uncorrelated. Hence design matrix of bundle block adjustment must be of full rank, the non-singular normal matrix. However, since spline parameters are not independent to spline location parameters, additional observations are required to obtain the estimations of parameters. In point-based approaches the point location relationship between image and object space is established to determine pose estimation including positions and orientations of a camera, which is a fundamental application in photogrammetry, the remote sensing and the computer vision. The coordinates of a point is the only piece of possible information to obtain the solutions of the space intersection and the space resection. To remove the rank deficiency which is caused by datum defects, constraints are adopted to estimate unknown parameters in point-based photogrammetry. The most common constraints are coplanarity, symmetry, perpendicularity, and parallelism. The minimum number of constraints is equal to the rank deficiency of the system. Inner constraints are often used in the photogrammetric network, which can be applied both the object features and the camera orientation parameters. Angle or distance condition equations provide the relative information between observations in the object space and points in the image space. The absolute information can be obtained from the fixed control points.
In this research, the arc-length parameterization is applied as an additional condition equation to solve the rank deficient problem in the extended collinearity equations using 3D natural cubic splines. The concept of differentiable parameterization is that the arc-length of a curve can be divided into tiny pieces and then the pieces can be added up so the length of each piece will be approximately linear as figure 3.3. The sum of the squares of derivatives is the same with a velocity since a parametric curve can be considered as the point trajectory. A velocity vector describes the path of a curve and moving characteristics. If the particle on a curve moves at a constant rate, the curve is parameterized by the arc-length. While the extended collinearity equation provides the only piece of information, curves contain additional geometric constraints such as the arc-length, the tangent of location, and the curvature, which are supportive to space resection under the assumption of measuring proper additional independent observations both the image space and the object space. A natural cubic spline in the object space is parameterized by a single variable $t$.

$$R(t) = [x_i(t) \ y_i(t) \ z_i(t)]^T$$  \hspace{1cm} (3.19)

The arc-length in the object space is determined by a geometric integration using the construction from the differentiable parameterization of a spline. A spline is differentiable if each of function $x_i(t), y_i(t)$ and $z_i(t)$ are differentiable respectively.

$$Arc(t)_{iO} = \int_{t_i}^{t_{i+1}} \left| \frac{dR}{dt} \right| dt$$

$$= \int_{t_i}^{t_{i+1}} \sqrt{(x'_i(t))^2 + (y'_i(t))^2 + (z'_i(t))^2} dt$$  \hspace{1cm} (3.20)
where

\[ x_i(t) = a_{i0} + a_{i1}(t - t_i) + a_{i2}(t - t_i)^2 + a_{i3}(t - t_i)^3 \]
\[ y_i(t) = b_{i0} + b_{i1}(t - t_i) + b_{i2}(t - t_i)^2 + b_{i3}(t - t_i)^3 \]
\[ z_i(t) = c_{i0} + c_{i1}(t - t_i) + c_{i2}(t - t_i)^2 + c_{i3}(t - t_i)^3 \]

with \( t \in [t_i, t_{i+1}] \), \( i = 0, 1, \ldots, n - 1 \) and \( a_{ij}, b_{ij}, c_{ij} \) spline parameters. The derivative \( R'(t) \) is a velocity and integrand of the arc-length integral so that the derivative is always positive. The arc-length in the image space is calculated by a geometric integration of the construction from the differentiable parameterization of the photo coordinates from a spline in the object space.

Figure 3.3: Arc length

\[
\operatorname{Arc}(t)_{ij} = \int_{t_i}^{t_{i+1}} \sqrt{(x'_p(t))^2 + (y'_p(t))^2} \, dt
\]
\[
= \int_{t_i}^{t_{i+1}} \sqrt{\left\{ -\frac{f u(t)}{w(t)} \right\}^2 + \left\{ -\frac{f v(t)}{w(t)} \right\}^2} \, dt
\]
(3.21)
\[
\int_{t_i}^{t_{i+1}} \sqrt{\left(-f \frac{u'(t)w(t) - u(t)w'(t)}{w^2(t)}\right)^2 + \left(-f \frac{v'(t)w(t) - v(t)w'(t)}{w^2(t)}\right)^2} \, dt
\]

where \( f \) is the focal length and

\[
\begin{bmatrix}
  u(t) \\
v(t) \\
w(t)
\end{bmatrix} = R^T(\omega, \varphi, \kappa) \begin{bmatrix}
a_0 + a_1 t + a_2 t^2 + a_3 t^3 - X_C \\
b_0 + b_1 t + b_2 t^2 + b_3 t^3 - Y_C \\
c_0 + c_1 t + c_2 t^2 + c_3 t^3 - Z_C
\end{bmatrix}
\]

\[
\begin{bmatrix}
u'(t) \\
v'(t) \\
w'(t)
\end{bmatrix} = R^T(\omega, \varphi, \kappa) \begin{bmatrix}
a_1 + 2a_2 t + 3a_3 t^2 \\
b_1 + 2b_2 t + 3b_3 t^2 \\
c_1 + 2c_2 t + 3c_3 t^2
\end{bmatrix}
\]

Since neither the problem of the arc-length parameterization of splines has an analytical solution, several numerical approximations of reparameterization techniques for splines or other curve representations have been developed. While most curves are not parameterized for the arc-length, the arc-length of a B-spline can be reparameterized by adjusting the knots of a B-spline. Wang et al.\[74\] approximated the parameterized arc-length of spline curves by generating a new curve which accurately approximated the original spline curve to reduce the computation complexity of the arc-length parameterization. They showed that the approximation of the arc-length parameterization works well in a variety of real time applications including a driving simulation.

Guenter and Parent\[22\] employed the hierarchical approach algorithm of the linear search of the arc-length subdivision table for parameterized curves to reduce the arc-length computation time. A table of the correspondence between parameter \( t \) and the arc-length can be established to accelerate the arc-length computation. After dividing the parameter range into intervals, the arc-length of each interval is computed for mapping parameters to the arc-length. A table is the reference of the arc-length for various intervals. Another method of the arc-length approximation is using explicit
function such as the Bézier curve which is advantageous because of its fast function evaluations. Adaptive Gaussian integrations employ recursive method which starts from few samples and add more samples as necessary. Adaptive Gaussian integration also uses a table which maps curves or spline parameter values to the arc-length values.

Nasri et al. [48] proposed the arc-length approximation method of circles and piecewise circular splines generated by control polygons or points using a recursive subdivision algorithm. While B-splines have various tangents over the curve depending on arc-length parameterization, circular splines have constant tangents, which tangent vectors are useful in the arc-length computation.

The simple approach for the integral approximation is using equal space values of \( x \), independent variable, from \( a \) to \( b \). If the range from \( a \) to \( b \) is divided into \( n \) intervals, the integral size is

\[
s = \frac{b - a}{n}
\]  

(3.22)

Integration by the trapezium rule calculates the approximation of integral values as

\[
\frac{s}{2} (y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)
\]  

(3.23)

where \( y_n = f(a + ns) \). While the trapezium rule uses intervals by its approximation of degree 1, Simpson’s rule uses intervals by its approximation of degree 2 to provide a more accurate integral approximation. Simpson’s one-third rule is

\[
\frac{s}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n)
\]

(3.24)

and the integral approximation with two intervals is

\[
\frac{s}{3} (y_0 + 4y_1 + y_2)
\]

(3.25)
This equation can be replaced with half interval, \((a + s/2)\) and \((b - s/2)\) as follows.

\[
\frac{s}{6}(y_0 + 4y_1 + y_1) \tag{3.26}
\]

Simpson's three-eighths rule with the integral approximation by a polynomial of degree 3 is

\[
\frac{3s}{8}(y_0 + 3y_1 + 3y_2 + 2y_3 + \cdots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n) \tag{3.27}
\]

and the integral approximation with two intervals is

\[
\frac{3s}{8}(y_0 + 3y_1 + 3y_2 + y_3) \tag{3.28}
\]

Simpson’s one-third rule is commonly used and referred to as Simpson’s rule since high-order polynomials do not improve the accuracy of the integral approximation significantly.

Andrew[3] improved the accuracy of Simpson’s one-third rule (3.24) and three-eighths rule (3.27) with homogenized one-third rule (3.29) and homogenized three-eighths rule (3.30) respectively.

\[
\frac{s}{24}(9y_0 + 28y_1 + 23y_2 + 24y_3 + \cdots + 24y_{n-3} + 23y_{n-2} + 28y_{n-1} + 9y_n) \tag{3.29}
\]

\[
\frac{s}{72}(26y_0 + 87y_1 + 66y_2 + 73y_3 + 72y_4 + \cdots + 72y_{n-4} + 73y_{n-3} + 66y_{n-2} + 87y_{n-1} + 26y_n) \tag{3.30}
\]

A proof of Simpson’s rule is as followings. Let’s assume the integral of an algebraic function.

\[
\int_b^a f(x)dx \approx \int_b^a f_2(x)dx \tag{3.31}
\]

with a second order polynomial \(f_2(x) = a_0 + a_1x + a_1x^2\).

Three points between the range from \(a\) and \(b\) can be selected as
\((a, f(a)), (\frac{a+b}{2}, f(\frac{a+b}{2})), (b, f(b))\). The three unknowns, \(a_0, a_1,\) and \(a_2\), are obtained from the following three equations:

\[
f(a) = f_2(a) = a_0 + a_1a + a_1a^2
\]

\[
f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_1\left(\frac{a+b}{2}\right)^2 \tag{3.32}
\]

\[
f(b) = f_2(b) = a_0 + a_1b + a_1b^2
\]

Three unknowns are

\[
a_0 = \frac{a^2f(b) + abf(b) - 4abf(\frac{a+b}{2}) + abf(a) + b^2f(a)}{a^2 - 2ab + b^2}
\]

\[
a_1 = \frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}
\]

\[
a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2} \tag{3.33}
\]

Thus

\[
\int_b^a f(x)dx \approx \int_b^a f_2(x)dx
\]

\[
= \int_b^a \left(a_0 + a_1x + a_2x^2\right)dx
\]

\[
= \left[a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3}\right]_a^b
\]

\[
= a_0(b-a) + a_1\frac{b^2-a^2}{2} + a_2\frac{b^3-a^3}{3}
\]

Substituting (3.33) into (3.34) leads to

\[
\int_b^a f_2(x)dx = \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \tag{3.35}
\]

In addition, Simpson’s rule can be derived by different methods such as the Lagrange polynomial, the method of coefficients and the approximation by a second
order polynomial using Newton’s divided difference polynomial. The error of Simpson’s rule is known to be

\[ e = -\frac{(b - a)^5}{2880} f^4(\xi), \quad a < \xi < b \] (3.36)

The error of multiple segments using Simpson’s rule is the sum of each error of Simpson’s rule as

\[ e = -\frac{(b - a)^5}{90n^4} \sum_{i=1}^{n/2} f^{(4)}(\xi_i) \] (3.37)

The error is proportional to \((b - a)^5\).

Simpson’s rule is the numerical approximation of definite integrals. The geometric integration of the arc-length in the image space can be calculated by Simpson’s rule as followings.

\[
\text{Arc}(t) = \int_{t_1}^{t_2} f(t)dt
\]

\[
\int_{t_1}^{t_2} f(t)dt \approx \frac{t_2 - t_1}{6} \left[ f(t_1) + 4f\left(\frac{t_2 + t_1}{2}\right) + f(t_2) \right]
\]

\[ f(t) = \sqrt{(x'_p(t))^2 + (y'_p(t))^2} \]

\[
= \sqrt{\left\{ \left( -\frac{u(t)}{w(t)} \right)' \right\}^2 + \left\{ \left( -\frac{v(t)}{w(t)} \right)' \right\}^2}
\]

\[
= \sqrt{\left( -\frac{u'(t)w(t) - u(t)w'(t)}{w^2(t)} \right)^2 + \left( -\frac{v'(t)w(t) - v(t)w'(t)}{w^2(t)} \right)^2}
\] (3.38)

\[
df(t) = \frac{1}{2} (f(t))^{-\frac{1}{2}} \left[ 2x'_p(t) \frac{v'}{w^2} du - 2x'_p(t) \frac{1}{w} du' + 2y'_p(t) \frac{w'}{w^2} dv - 2y'_p(t) \frac{1}{w} dv' \\
+ \left\{ 2x'_p(t) \frac{u'w^2 - (u'w - uw')2w}{w^4} - 2y'_p(t) \frac{v'w^2 - (v'w - vw')2w}{w^4} \right\} dw \\
+ \left\{ 2x'_p(t) \frac{u}{w^2} + 2y'_p(t) \frac{v}{w^2} \right\} dw' \right]
\]

\[
\text{Arc}(t) = \frac{t_2^0 - t_1^0}{6} \left[ f(t_1^0) + 4f\left(\frac{t_2^0 + t_1^0}{2}\right) + f(t_2^0) \right]
\] (3.39)
\[
= A_1 dX_C + A_2 dY_C + A_3 dZ_C + A_4 d\omega + A_5 d\varphi + A_6 d\kappa + A_7 d\alpha \\
+ A_8 d\alpha_1 + A_9 d\alpha_2 + A_{10} d\alpha_3 + A_{11} db_0 + A_{12} db_1 + A_{13} db_2 + A_{14} db_3 \\
+ A_{15} dc_0 + A_{16} dc_1 + A_{17} dc_2 + A_{18} dc_3 + A_{19} dt_1 + A_{20} dt_2 + e_a
\]

with \( t_1^0, t_2^0, f(t^0) \) the approximate parameters by \((X_C^0, Y_C^0, Z_C^0, \omega^0, \varphi^0, \kappa^0, \alpha_0^0, \alpha_1^0, \alpha_2^0, \alpha_3^0, b_1^0, b_2^0, b_3^0, c_0^0, c_1^0, c_2^0, c_3^0, t_i^0)\) and \( e_a \) the stochastic error of the arc-length between two locations with zero expectation. \( A_1, \cdots, A_{20} \) denote the partial derivatives of the arc-length parameterization of a 3D natural cubic spline. Detailed derivations are in Appendix A.2.

### 3.4 Tangents of spline between image and object space

Since employing spline leads to over parameterization, geometric constraints are required to solve the system, such as slope, distance, perpendicularity, coplanar features. While some of the geometric constraints, slope and distance observations, are dependent on splines, other constraints increase non-redundant information in adjustment to reduce the overall rank deficiency of the system. Tangents of splines provide additional constraints to solve the over parameterization of 3D natural cubic splines. In case linear features in the object space are straight lines or conic sections. Tangents are one of straight line constraints incorporated into bundle block adjustment using the assumption that the transformation of straight lines in the object space is straight lines as well in the image space. In linear features using collinearity equations, the relationship establishment of two corresponding properties in the image space and the object space is possible. Since tangents are independent measurements in the image space and the object space and the relationship between them is established
by collinearity equations, tangents are additional parameters to solve the over parameterization. In general, tangents are not represented mathematically except straight lines and conic sections since tangents cannot be measured in curves exactly in the image space. If EOPs are known parameters, the image space coordinates of curves projected from the 3D spline in object space are a function of the parameter \( t \) simply. The relationship of tangents in the object space and the image space is described in figure 3.4. Tangent direction is determined by the derivative \([X'(t)\ Y'(t)\ Z'(t)]^T\)

![Figure 3.4: Tangent in the object space and its counterpart in the projected image space](image)

represented by its individual components.
\[ x_p(t) = -f \frac{u(t)}{w(t)}, \quad y_p(t) = -f \frac{v(t)}{w(t)} \] (3.40)

where
\[
\begin{bmatrix}
  u(t) \\
  v(t) \\
  w(t)
\end{bmatrix} = R^T(\omega, \varphi, \kappa) \begin{bmatrix}
  X_i(t) - X_C \\
  Y_i(t) - Y_C \\
  Z_i(t) - Z_C
\end{bmatrix}
\]

Differentiating the collinearity equations with respect to parameter \( t \) leads to 2D tangent direction in the image space.

\[ x'_p(t) = -f \frac{u'(t)w(t) - u(t)w'(t)}{w^2(t)}, \quad y'_p(t) = -f \frac{v'(t)w(t) - v(t)w'(t)}{w^2(t)} \] (3.41)

\[
\begin{bmatrix}
  u'(t) \\
  v'(t) \\
  w'(t)
\end{bmatrix} = R^T(\omega, \varphi, \kappa) \begin{bmatrix}
  X'_i(t) \\
  Y'_i(t) \\
  Z'_i(t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \frac{du'}{dw'} \\
  \frac{dv'}{dw'} \\
  \frac{dw'}{dw'}
\end{bmatrix} = \frac{\partial R^T}{\partial \omega} \begin{bmatrix}
  X'_i \\
  Y'_i \\
  Z'_i
\end{bmatrix} d\omega + \frac{\partial R^T}{\partial \varphi} \begin{bmatrix}
  X'_i \\
  Y'_i \\
  Z'_i
\end{bmatrix} d\varphi + \frac{\partial R^T}{\partial \kappa} \begin{bmatrix}
  X'_i \\
  Y'_i \\
  Z'_i
\end{bmatrix} d\kappa
\]

\[
+ R^T \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  d_1 + R^T \\
  d_2 + R^T \\
  d_3 + R^T
\end{bmatrix}
\]

\[
+ R^T \begin{bmatrix}
  0 & 2t & 0 \\
  0 & 3t^2 & 0 \\
  0 & 2t & 0
\end{bmatrix} \begin{bmatrix}
  d_1 + R^T \\
  d_2 + R^T \\
  d_3 + R^T
\end{bmatrix}
\]

\[
+ R^T \begin{bmatrix}
  0 & 0 & 2a_2 + 6a_3t \\
  0 & 2b_2 + 6b_3t & 2a_2 + 6a_3t \\
  0 & 2c_2 + 6c_3t & 0
\end{bmatrix} dt
\] (3.42)

\[ \tan(\theta_t) = \frac{y'_p}{x'_p} = \frac{v'(t)w(t) - v(t)w'(t)}{u'(t)w(t) - u(t)w'(t)} \] (3.43)

where \( \tan(\theta_t) \) the tangent in terms of the angle \( \theta_t(0 \leq \theta_t \leq 2\pi) \)
\[
d\tan(\theta_t) = \frac{w'(v'w - w'v)}{(u'w - w'u)^2} du - \frac{w(v'w - w'v)}{(u'w - w'u)^2} du' - \frac{w'}{u'w - w'u} dv + \frac{w}{u'w - w'u} dv' \\
+ \frac{v'(u'w - w'u) - u'(v'w - v'w)}{(u'w - w'u)^2} dw - \frac{v(u'w - w'u) - u(v'w - vw')}{(u'w - w'u)^2} dw' 
\]

(3.44)

Linear observation equation can be obtained with incremental values, \(dX_C, dY_C, dZ_C,\)
\(d\omega, d\varphi, d\kappa, da_0, da_1, da_2, da_3, db_0, db_1, db_2, db_3, dc_0, dc_1, dc_2, dc_3, dt_j\) as (3.45).

\[
\tan(\theta_t) - \frac{v^0w^0 - w^0v^0}{u^0w^0 - w^0u^0} = \sum_{i=1}^{19} L_i dX_C + \sum_{i=1}^{19} L_i dY_C + \sum_{i=1}^{19} L_i dZ_C + \sum_{i=1}^{19} L_i d\omega + \sum_{i=1}^{19} L_i d\varphi + \sum_{i=1}^{19} L_i d\kappa \\
+ \sum_{i=1}^{19} L_i da_i + \sum_{i=1}^{19} L_i da_1 + \sum_{i=1}^{19} L_i da_2 + \sum_{i=1}^{19} L_i da_3 + \sum_{i=1}^{19} L_i db_i \\
+ \sum_{i=1}^{19} L_i db_1 + \sum_{i=1}^{19} L_i db_2 + \sum_{i=1}^{19} L_i db_3 + \sum_{i=1}^{19} L_i dc_1 + \sum_{i=1}^{19} L_i dc_2 \\
+ \sum_{i=1}^{19} L_i dc_3 + \sum_{i=1}^{19} L_i dt_j + e_t 
\]

(3.45)

with \(\tan(\theta_t)\) the tangent direction of a point in the image space, \(u^0, v^0, w^0, u^0, v^0, w^0\)
the approximate parameters by \((X^0_C, Y^0_C, Z^0_C, \omega^0, \varphi^0, \kappa^0, a_0^0, a_1^0, a_2^0, a_3^0, b_0^0, b_1^0, b_2^0, b_3^0,\)
\(c_0^0, c_1^0, c_2^0, c_3^0, t_0^0)\) and \(e_t\) the stochastic error of tangent between two locations with zero expectation. \(L_1, \cdots, L_{19}\) denote the partial derivatives of the tangent of a 3D natural cubic spline. Detailed derivations are in Appendix A.3.
CHAPTER 4

MODEL INTEGRATION

4.1 Bundle block adjustment

The objective of bundle block adjustment is twofold, namely to calculate exterior orientation parameters of a block of images and the coordinates of the ground features in the object space. In the determination of orientation parameters, additional interior orientation parameters such as lens distortion, atmospheric refraction, and principal point offset can be obtained by self-calibration. In general, orientation parameters are determined by bundle block adjustment using a large number of control points. The establishment of a large number of control points, however, is an expensive fieldwork so the economic and accurate adjustment method is required. Linear features have several advantages to complement points that they are useful features for higher level task and they are easily extracted in man-made environments. The line photogrammetric bundle adjustment in this research aims at the estimation of exterior orientation parameters and 3D natural cubic spline parameters using correspondence between splines in the object space and spline observations of multiple images in the image space. Nonlinear functions of orientation parameters, spline parameters and spline location parameters are represented by the extended collinearity equations and the arc-length parameterization equations. Five observation equations are produced
by each two points, which are four extended collinearity equations (3.18) and one arc-length parameterization equation (3.39). While tangents are modeled analytically in straight lines and conic sections, the parametric representation of the tangent of the 3D natural cubic spline cannot be generated in a global fashion. Integrated model provides not only the recovery of image orientation parameters but also surface reconstruction using 3D curves. Of course since the equation system of the integrated model has datum defects of seven, the control information about the coordinate system is required to obtain parameters. This is a step toward higher level vision tasks such as object recognition and surface reconstruction.

Figure 4.1: Concept of bundle block adjustment with splines
In the case of straight lines and conic sections, tangents are the additional observations in the integrated model. Conic sections which, like points, are good mathematical constraints since conic sections are non-singular curves of degree 2. Second-degree equations provide the information for reconstruction and transformation and conic sections are divided by the eccentricity $e$ as shown in table 4.1. Since conic sections can adopt more constraints than points and straight line features, conic sections are useful for close range photogrammetric applications. In addition, conic sections have the strength in the correspondence establishment between 3D conic sections in the object space and their counterpart features in the 2D projected image space.

<table>
<thead>
<tr>
<th>Eccentricity</th>
<th>Conic form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e=0$</td>
<td>Circle</td>
</tr>
<tr>
<td>$0 &lt; e &lt; 1$</td>
<td>Ellipse</td>
</tr>
<tr>
<td>$e = 1$</td>
<td>Parabola</td>
</tr>
<tr>
<td>$e &gt; 1$</td>
<td>Hyperbola</td>
</tr>
</tbody>
</table>

Table 4.1: Relationship between the eccentricity and the conic form

Ji et al. (2000) [36] employed conic sections for the recovery of EOPs and Heikkila (2000) [33] used them for camera calibration. Hough transformation reduces the time complexity of the conic section extraction using five parameter spaces for a single-photo resection, camera calibration and triangulation [63].

$$x_p + f \frac{w^0}{\omega^0} = M_1 dX_C + M_2 dY_C + M_3 dZ_C + M_4 d\omega + M_5 d\varphi + M_6 d\kappa + M_7 da_{i0}$$

$$+ M_8 da_{i1} + M_9 da_{i2} + M_{10} da_{i3} + M_{11} db_{i0} + M_{12} db_{i1} + M_{13} db_{i2}$$
\[ y_p + f \frac{v_0}{w_0} = N_1 dX_C + N_2 dY_C + N_3 dZ_C + N_4 d\omega + N_5 d\varphi + N_6 d\kappa + N_7 da_i \\
+ N_8 da_{i1} + N_9 da_{i2} + N_{10} da_{i3} + N_{11} db_{i0} + N_{12} db_{i1} + N_{13} db_{i2} \\
+ N_{14} db_{i3} + N_{15} dc_{i0} + N_{16} dc_{i1} + N_{17} dc_{i2} + N_{18} dc_{i3} + N_{19} dt_j \\
+ e_y \]

\[ Arc(t) = \frac{t_2^0 - t_1^0}{6} \left[ f^0(t_1^0) + 4f^0\left(\frac{t_2^0 + t_1^0}{2}\right) + f^0(t_2^0) \right] \]

\[ = A_1 dX_C + A_2 dY_C + A_3 dZ_C + A_4 d\omega + A_5 d\varphi + A_6 d\kappa + A_7 da_i \\
+ A_8 da_{i1} + A_9 da_{i2} + A_{10} da_{i3} + A_{11} db_{i0} + A_{12} db_{i1} + A_{13} db_{i2} + A_{14} db_{i3} \\
+ A_{15} dc_{i0} + A_{16} dc_{i1} + A_{17} dc_{i2} + A_{18} dc_{i3} + A_{19} dt_1 + A_{20} dt_2 + e_t \]

where \( x_p, y_p \) the photo coordinates and \( Arc(t) \) the arc-length between two locations respectively.

Parameters are linearized in the previous three sections and the Gauss-Markov model is employed for the unknown parameter estimation. Tangent observation can be added in case of straight linear features and conic sections but in this model bundle block adjustment with the extended collinearity equations for 3D natural cubic splines and the arc-length parameterization equation is described for general cases. The equation system of the integrated model is described as

\[
\begin{bmatrix}
A^k_{EOP} & A^i_{SP} & A^j_{L} \\
A^k_{AL} & A^i_{SP} & A^j_{L}
\end{bmatrix}
\begin{bmatrix}
\xi^k_{EOP} \\
\xi^i_{SP} \\
\xi^j_{L}
\end{bmatrix}
= \begin{bmatrix}
y^k_i \\
y^i_j
\end{bmatrix}
\] (4.2)
\[
A^k_{EOP} = \begin{bmatrix}
M_{i1}^{k1} & M_{i2}^{k1} & \cdots & M_{i6}^{k1} \\
\vdots & & & \vdots \\
M_{i1}^{km} & M_{i2}^{km} & \cdots & M_{i6}^{km} \\
N_{i1}^{k1} & N_{i2}^{k1} & \cdots & N_{i6}^{k1} \\
\vdots & & & \vdots \\
N_{i1}^{km} & N_{i2}^{km} & \cdots & N_{i6}^{km}
\end{bmatrix}
\]

\[
A^i_{SP} = \begin{bmatrix}
M_{i7}^1 & M_{i8}^1 & \cdots & M_{i18}^1 \\
\vdots & & & \vdots \\
M_{i7}^m & M_{i8}^m & \cdots & M_{i18}^m \\
N_{i7}^1 & N_{i8}^1 & \cdots & N_{i18}^1 \\
\vdots & & & \vdots \\
N_{i7}^m & N_{i8}^m & \cdots & N_{i18}^m
\end{bmatrix}
\]

\[
A^i_t = \begin{bmatrix}
M_{i19}^1 & M_{i20}^1 & \cdots & M_{i18+n}^1 \\
\vdots & & & \vdots \\
M_{i19}^m & M_{i20}^m & \cdots & M_{i18+n}^m \\
N_{i19}^1 & N_{i20}^1 & \cdots & N_{i18+n}^1 \\
\vdots & & & \vdots \\
N_{i19}^m & N_{i20}^m & \cdots & N_{i18+n}^m
\end{bmatrix}
\]

\[
A^{ki}_{AL} = \begin{bmatrix}
A_{i1}^{k1} & A_{i2}^{k1} & \cdots & A_{i20}^{k1} \\
\vdots & & & \vdots \\
A_{i1}^{km} & A_{i2}^{km} & \cdots & A_{i20}^{km}
\end{bmatrix}
\]

\[
\xi^k_{EOP} = \begin{bmatrix}
dx_C^k & dy_C^k & dZ_C^k & \omega^k & \varphi^k & \kappa^k \end{bmatrix}^T
\]

\[
\xi^i_{SP} = \begin{bmatrix}
da_{i0} & da_{i1} & da_{i2} & da_{i3} & db_{i0} & db_{i1} & db_{i2} & db_{i3} & dc_{i0} & dc_{i1} & dc_{i2} & dc_{i3} \end{bmatrix}^T
\]

\[
\xi^i_t = \begin{bmatrix}
dt_{i1} & dt_{i2} & \cdots & dt_{in} \end{bmatrix}^T
\]

\[
y^{ki} = \begin{bmatrix}
x_p^{ki} + f \frac{t_0^0}{w_0^0} & y_p^{ki} + f \frac{t_0^0}{w_0^0} \end{bmatrix}^T
\]

with \(Arc(t)^0 = \frac{t_{10} - t_{00}}{6} \left[ f^0(t_0^0) + 4f^0 \left( \frac{t_{00} + t_{10}}{2} \right) + f^0(t_{10}) \right] \), \(m\) the number of images, \(n\) the number of points on a spline segment, \(k\) the \(k\)th image and \(i\) the \(i\)th spline segment. The partial derivatives of symbolic representations \((M, N, A)\) of the extended
collinearity model are described in appendix A. Since the equation system of the integrated model has datum defects of seven, the control information about the coordinate system is required to obtain the seven transformation parameters. In a general photogrammetric network, the rank deficiency referred as datum defects is seven.

Estimates of the unknown parameters are obtained by the least squares solution which minimizes the sum of squared deviations. A non-linear least squares system is required in the conventional non-linear photogrammetric solution to obtain orientation parameters. Many observations in photogrammetry are random variables which are considered as different values in the case of repeated observations such as image coordinates of points on images. Each measured observation represents an estimate of random variables of the image coordinates of points on images. If image coordinates of points are measured using the digital photogrammetric workstation, the values would be measured slightly differently for each measurement. The integrated and linearized Gauss-Markov model and the least squares estimated parameter vector with its dispersion matrix are

\[
y^{ki} = A_{IM} \xi_{IM} + e
\]

\[
A_{IM} = \begin{bmatrix}
A^k_{EOP} & A^i_{SP} & A^i_t
\end{bmatrix}
\]

\[
\xi_{IM} = \begin{bmatrix}
\xi^k_{EOP} & \xi^i_{SP} & \xi^i_t
\end{bmatrix}^T
\]

\[
\hat{\xi}_{IM} = (A^T_{IM} P A_{IM})^{-1} A^T_{IM} P y^{ki}
\]

\[
D(\hat{\xi}_{IM}) = \sigma_o^2 (A^T_{IM} P A_{IM})^{-1}
\]

with \( e \sim N(0, \sigma_o^2 P^{-1}) \) the error vector with zero mean and cofactor matrix \( P^{-1} \), and variance component \( \sigma_o^2 \) which can be known or not, \( \hat{\xi}_{IM} \) the least squares estimated parameter vector and \( D(\hat{\xi}_{IM}) \) the dispersion matrix.
The optimal unbiased estimate of the variance component can be obtained as

\[ \hat{\sigma}_o^2 = \frac{\tilde{e}^T P \tilde{e}}{n - m} \]  \hspace{1cm} (4.4)

\[ \tilde{e} = y - A\hat{\xi} \]

where \( n \) is the number of equations and \( m \) is the number of parameters.

If one or more of the three estimated parameter sets \( \xi_{EOP}^k, \xi_{SP}^i, \xi_{t}^i \) are considered as stochastic constraints, the reduction of the normal equation matrix can be applied. Control information is implemented as stochastic constraints in bundle block adjustment. Distribution and quality of control features depend on the number and the density of control features, the number of tie features and the degree of the overlap of tie features. If adding stochastic constraints removes the rank deficiency of the Gauss-Markov model, bundle adjustment can be implemented employing only the extended collinearity equations for 3D natural cubic splines. Fixed exterior orientation parameters, control splines or control spline location parameters can be stochastic constraints. In addition, splines in the object space can be divided into control features and tie features so that tie spline parameters can be recovered by bundle block adjustment. Stochastic constraints assigned into \( \hat{\xi}_2 \), the integrated model can be written as
\[
\begin{bmatrix}
N_{11} & N_{12} \\
N_{12}^T & N_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2
\end{bmatrix} = \begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
\]

\[
N_{11}\dot{\xi}_1 + N_{12}\dot{\xi}_2 = c_1
\]

\[
N_{12}^T\dot{\xi}_1 + N_{22}\dot{\xi}_2 = c_2
\]

\[
\dot{\xi}_2 = N_{22}^{-1}c_2 - N_{22}^{-1}N_{12}^T\dot{\xi}_2
\]

\[
(N_{11} - N_{12}N_{22}^{-1}N_{12}^T)\dot{\xi}_1 = c_1 - N_{12}N_{22}^{-1}c_2
\]

\[
\dot{\xi}_1 = (N_{11} - N_{12}N_{22}^{-1}N_{12}^T)^{-1}(c_1 - N_{12}N_{22}^{-1}c_2)
\]

where the matrix of \( N, c, \xi \) depends on the stochastic constraints.

\[
D(\dot{\xi}_1) = \sigma_o^2 Q_{11}
\]

\[
D(\dot{\xi}_2) = \sigma_o^2 Q_{22}
\]

\[
D\begin{bmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2
\end{bmatrix} = \sigma_o^2 \begin{bmatrix}
N_{11} & N_{12} \\
N_{12}^T & N_{22}
\end{bmatrix}^{-1} = \sigma_o^2 \begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{12}^T & Q_{22}
\end{bmatrix}
\]

\[
\begin{bmatrix}
N_{11} & N_{12} \\
N_{12}^T & N_{22}
\end{bmatrix}
\begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{12}^T & Q_{22}
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\]

\[
N_{11}Q_{11} + N_{12}Q_{12}^T = I
\]

\[
N_{12}^TQ_{11} + N_{22}Q_{12}^T = 0
\]

\[
Q_{12}^T = -N_{22}^{-1}N_{12}^TQ_{11}
\]

\[
Q_{12} = -Q_{11}N_{22}N_{22}^{-1}
\]

\[
(N_{11} - N_{12}N_{22}^{-1}N_{12}^T)Q_{11} = I
\]

\[
Q_{11} = (N_{11} - N_{12}N_{22}^{-1}N_{12}^T)^{-1}
\]

63
\[ N_{11}Q_{12} + N_{12}Q_{22} = 0 \]
\[ Q_{12} = -N^{-1}11N_{12}Q_{22} \]
\[ N_{12}^TQ_{12} + N_{22}Q_{22} = I \]
\[ (N_{22} - N_{12}^T N_{11}^{-1} N_{12})Q_{22} = I \]
\[ Q_{22} = (N_{22} - N_{12}^T N_{11}^{-1} N_{12})^{-1} \]
\[ Q_{22} = N_{22}^{-1} - N_{22}^{-1} N_{12}^T Q_{12} \]

where \( Q \) is a cofactor matrix.

### 4.2 Evaluation of bundle block adjustment

Bundle block adjustment must be followed by an evaluation of bundle block adjustment, post-adjustment analysis, to check the suitability of project specifications and requirements. Iteratively reweighed least squares and least median of squares are the appropriate implementation for a statistical evaluation which removes poor observations. The important element to affect bundle block adjustment is the geometry of aerial images. Generally, the previous flight plan is adopted to obtain the suitable results. A simulation of bundle block adjustment is implemented before employing a flight plan of the new project design since simulation can reduce the effect of error measurements.

A qualitative evaluation which allows the operator to recognize the adjustment characteristics is often used after bundle block adjustment. The sizes of the residuals in images are drawn for a qualitative evaluation of bundle block adjustment. The image residuals can be points or long lines and if all image residuals have the same direction, then the image has a systematic error such as atmospheric refraction or orientation parameter error. In addition, the lack of flatness of the focal plane causes
systematic errors in the image space, which affects the accuracy of bundle block adjustment. Distortions are different from one location to another in the entire image space. The topographic measurement of the focal plane can correct the focal plane unflatness. Image coordinate errors are correlated in case of systematic image errors. A poor measurement can make the residual direction opposite or the residual large.

Three main elements in a statistical evaluation of bundle block adjustments are precision, accuracy and reliability. A precision is calculated employing the variances and the covariances of parameters since a small variance represents that the estimated values have a small range and a large variance means that the estimated values are not calculated properly. The range of the parameter variance is from zero, in case of error free parameters, to infinite, in case of completely unknown parameters. The dispersion matrix in which diagonal elements are parameter variances, and off-diagonal elements are covariances between two parameters.

The dispersion matrix in point-based bundle block adjustment is

$$
\begin{bmatrix}
\sigma^2_X & \sigma_{XY} & \sigma_{XZ} \\
\sigma_{XY} & \sigma^2_Y & \sigma_{YZ} \\
\sigma_{XZ} & \sigma_{YZ} & \sigma^2_Z
\end{bmatrix}
$$

or

$$
\begin{bmatrix}
\sigma_X & \rho_{XY} & \rho_{XZ} \\
\rho_{XY} & \sigma_Y & \rho_{YZ} \\
\rho_{XZ} & \rho_{YZ} & \sigma_Y
\end{bmatrix}
$$

(4.5)

where $\sigma^2_X, \sigma^2_Y, \sigma^2_Z$ variances of $X$, $Y$ and $Z$ coordinates of ground control points in the object space.

$\sigma_{XY}, \sigma_{XZ}, \sigma_{YZ}$ covariance between two parameters

$\rho_{XY}, \rho_{XZ}, \rho_{YZ}$ correlation coefficients between two parameters as $\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$.

Accuracy can be verified using check points which are not contained in bundle block adjustment like control points. Reliability can be confirmed from other redundant observations.
The extended collinearity equations are a mathematical model for bundle block adjustment. The mathematical model of bundle block adjustment consists of two models, a functional model and a stochastic model. The functional model represents geometrical properties and the stochastic model describes statistical properties. The repeated measurements at the same location in the image space are represented with respect to the functional model and the redundant observations of image locations in the image space are expressed with respect to the stochastic model. While the Gauss-Markov model uses indirect observations, condition equations such as coordinate transformations, and the coplanarity condition can be employed in the adjustment. The Gauss-Markov model and the condition equation can be combined into the Gauss-Helmert model. In addition, functional constraints such as points having the same height or straight railroad segments can be added into the block adjustment.

The difference between condition equations and constraint equations is that condition equations consist of observations and parameters, and constraint equations consist of only parameters. With the advances of technology, the input data in photogrammetry has been increased so adequate formulation of adjustment is required. All variables are involved in the mathematical equations and the weight matrix of variables is changed from zero to infinity depending on variances. Variables with the near to zero weight are considered as unknown parameters and variables with the near to infinite weight are considered as constants. Most actual observations are in existence between two boundary cases. Assessment of adjustment, post-adjustment analysis, is important in photogrammetry to analyze the results. One of the assessment methods is to compare the estimated variance with the two-tailed confidence
interval based on the normal distribution. The two-tailed confidence interval is computed by a reference variance $\sigma_o^2$ with $\chi^2$ distribution as

$$\frac{r\hat{\sigma}_o^2}{\chi_{r,\alpha/2}^2} < \sigma_o^2 < \frac{r\hat{\sigma}_o^2}{\chi_{r,1-\alpha/2}^2}$$

(4.6)

where $r$ is degrees of freedom and $\alpha$ is a confidence coefficient (or a confidence level). If $\sigma_o^2$ has the value outside of the interval, we can assume the mathematical model of adjustment is incorrect such as the wrong formulation, the wrong linearization, blunders or systematic errors.

4.3 Pose estimation with ICP algorithm

Unlike the previous case with spline segments which the correspondence between spline segments in the image and the object space are assumed, now it is unknown which image points belong to which spline segment. ICP algorithm can be utilized for the recovery of EOPs since the initial estimated parameters of the relative pose can be obtained from orientation data in general photogrammetric tasks. The original ICP algorithm steps are as follows. The closest point operators search the associate point by the nearest neighboring algorithm and then the transformation parameters are estimated using mean square cost function. The point is transformed by the estimated parameters and this step is iteratively established until converging into a local minimum of the mean square distance. The transformation including translation and rotation between two clouds of points is estimated iteratively converging into a global minimum. In other words, the iterative calculation of the mean square errors is terminated when a local minimum falls below a predefined threshold. The smaller global minimum or the fluctuated curve requires more memory intensive and time consuming computation. In every iteration step, a local minimum is calculated with
any transformation parameters, but the convergence into a global minimum with the

correct transformation parameters is not always obtained.

By the definition of a natural cubic spline, each parametric equation of spline

segment \((S_i(t))\) can be expressed as (4.7).

\[
S_i(t) = \begin{bmatrix} X_i(t) \\ Y_i(t) \\ Z_i(t) \end{bmatrix} = \begin{bmatrix} a_{i0} + a_{i1}t + a_{i2}t^2 + a_{i3}t^3 \\ b_{i0} + b_{i1}t + b_{i2}t^2 + b_{i3}t^3 \\ c_{i0} + c_{i1}t + c_{i2}t^2 + c_{i3}t^3 \end{bmatrix}, \quad t \in [0, 1] \tag{4.7}
\]

with \(X_i(t), Y_i(t), Z_i(t)\) the object space coordinates and \(a_i, b_i, c_i\) the coefficients of \(i\)th

spline segment.

The ray from the perspective center \((X_C, Y_C, Z_C)\) to image point \((x_p, y_p, -f)\) is

\[
\Xi(l) = \begin{bmatrix} X(l) \\ Y(l) \\ Z(l) \end{bmatrix} = \begin{bmatrix} X^k_C \\ Y^k_C \\ Z^k_C \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} l \tag{4.8}
\]

where

\[
\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = R^T(\omega^k, \varphi^k, \kappa^k) \begin{bmatrix} x_p \\ y_p \\ -f \end{bmatrix} \tag{4.9}
\]

with \(X^k_C, Y^k_C, Z^k_C, \omega^k, \varphi^k, \kappa^k\) EOPs at the \(k\)th iteration.

A point on the ray searches the closest a natural cubic spline by minimizing the

following target function for every spline segment as (4.10). Transformation para-

meters related with an image point and its closest spline segment can be established

using the least squares method.

\[
\Phi(l, t) \equiv \|\Xi(l) - S_i(t)\|^2 = stationary \ l, t \tag{4.10}
\]

The global minimum of \(\Phi(l, t)\) can be calculated by \(\nabla \Phi(l, t) = 0\) or

\(\partial \Phi/\partial l = \partial \Phi/\partial t = 0\). Substituting (4.7) and (4.8) into (4.10) and taking the deriva-

tives with respect to \(l\) and \(t\) leads
\[
\frac{1}{2} \frac{\partial \Phi}{\partial l} = (X_C + d_1 l - a_{i0} - a_{i1} t - a_{i2} t^2 - a_{i3} t^3)d_1 \\
+ (Y_C + d_2 l - b_{i0} - b_{i1} t - b_{i2} t^2 - b_{i3} t^3)d_2 \\
+ (Z_C + d_3 l - c_{i0} - c_{i1} t - c_{i2} t^2 - c_{i3} t^3)d_3 = 0 \tag{4.11}
\]

\[
\frac{1}{2} \frac{\partial \Phi}{\partial t} = (X_C + d_1 l - a_{i0} - a_{i1} t - a_{i2} t^2 - a_{i3} t^3)(-a_{i1} - 2a_{i2} t - 3a_{i3} t^2) \\
+ (Y_C + d_2 l - b_{i0} - b_{i1} t - b_{i2} t^2 - b_{i3} t^3)(-b_{i1} - 2b_{i2} t - 3b_{i3} t^2) \\
+ (Z_C + d_3 l - c_{i0} - c_{i1} t - c_{i2} t^2 - c_{i3} t^3)(-c_{i1} - 2c_{i2} t - 3c_{i3} t^2) = 0
\]

Convergence into a global minimum does not exist since equation (4.11) is not a linear system in \( l \) and \( t \). The relationship between an image space point and its corresponding spline segment can not be established with the minimization method.
CHAPTER 5

EXPERIMENTS AND RESULTS

In the last chapter, the mathematical integrated model was introduced using the extended collinearity equations with splines, tangents of splines, and the arc-length parameterization equations of splines. This chapter demonstrates the feasibility and the performance of the proposed model for the acquisition of spline parameters, spline location parameters and image orientation parameters based on control and tie splines in the object space with the simulated and real data set. In general photogrammetric tasks, the correspondence between image edge features must be established either automatically or manually but in this study correspondence between image edge features is not required. A series of experiments with the synthetic data set consists of six experiments that the first test recovers spline parameters and spline location parameters with respect to the case of error free EOPs. The second experiment recovers the partial spline parameters related with the shape of splines. The third procedure estimates the spline location parameters in the case of error free EOPs and the fourth one calculates EOPs and spline location parameters followed by the fifth step which estimates EOPs with full control splines, which the parametric curves used as control features are assumed to be error free. In the last experiment, EOPs and tie spline parameters are obtained with the control spline. Object space knowledge
about splines, their relationships, and the orientation information of images can be considered as control information. Spline parameters in partial control spline or orientation parameters can be considered as stochastic constraints in the integrated adjustment model. The starting point of a spline is considered as a known parameter in the partial control spline which $a_0, b_0$ and $c_0$ of $X, Y$ and $Z$ coordinates of a spline are known. The number of unknowns is described in table 5.1 and figure 5.1 where $n$ is the number of points in the object space, $t$ shows the number of spline location parameters and $m$ represents the number of overlapped images in the target area.

<table>
<thead>
<tr>
<th>EOP</th>
<th>Spline</th>
<th>Number of unknowns</th>
</tr>
</thead>
<tbody>
<tr>
<td>Known EOP</td>
<td>Tie spline</td>
<td>$12(n - 1) + t$</td>
</tr>
<tr>
<td></td>
<td>Partial control spline</td>
<td>$9(n - 1) + t$</td>
</tr>
<tr>
<td></td>
<td>Full control spline</td>
<td>$t$</td>
</tr>
<tr>
<td>Unknown EOP</td>
<td>Partial control spline</td>
<td>$6m + 9(n - 1) + t$</td>
</tr>
<tr>
<td></td>
<td>Full control spline</td>
<td>$6m + t$</td>
</tr>
</tbody>
</table>

Table 5.1: Number of unknowns

Four points on a spline segment in one image are the only independent observations so additional points on the same spline segment do not provide non-redundant information to reduce the overall deficiency of the EOP and spline parameter recovery. To prove the information content of an image spline, we demonstrate that any five points on a spline segment generates a dependent set of the extended collinearity equations. Any combinations of four points yielding eight collinearity equations are independent observations but five points bearing ten collinearity equations produce a dependent set of observations related with the correspondence between a natural
Figure 5.1: Different examples

(a) Known EOPs with tie splines
(b) Known EOPs with partial control splines
(c) Known EOPs with full control splines
(d) Unknown EOPs with partial control splines
(e) Unknown EOPs with full control splines

(Red: Unknown parameters, Green: Partially fixed parameters,
Blue: Fixed parameters)
cubic spline in the image and the object space. More than four point observations on an image spline segment increase the redundancy related with the accuracy but do not decrease the overall rank deficiency of the proposed adjustment system. In the same fashion, the case using a polynomial of degree 2 can be implemented. Three points on a quadratic polynomial curve in one image are the only independent sets so additional points on the same curve segment are a dependent observation. More than independent point observations on a polynomial increase the redundancy related with the accuracy but do not provide the non-redundant information. Depending on the order of a polynomial, the number of independent points for bundle block adjustment is limited as table 5.2.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Number of independent points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant polynomial</td>
<td>1</td>
</tr>
<tr>
<td>Linear polynomial</td>
<td>2</td>
</tr>
<tr>
<td>Quadratic polynomial</td>
<td>3</td>
</tr>
<tr>
<td>Cubic polynomial</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 5.2: Number of independent points for bundle block adjustment

The amount of information carried by a natural cubic spline can be calculated with the redundancy budget. Every spline segment has twelve parameters and every point measured on a spline segment increases one additional parameter. Let \( n \) be the number of points measured on one spline segment in the image space and \( m \) be the number of images which contain a tie spline. \( 2nm \) collinearity equations and \( m(n-1) \) the arc-length parameterizations are equations and 12 (the number of one spline segment parameters) + \( nm \) (the number of spline location parameters) are
unknowns. The redundancy is $2nm-m-12$ for one spline segment so that if two images ($m=2$) are used for bundle block adjustment, the redundancy is $4n-14$. Four points are required to determine spline and spline location parameters in case one spline segment and one degree of freedom to the overall redundancy budget is solved by each point measurement with the extended collinearity equation. Arc-length parameterization also contributes one degree of freedom to the overall redundancy budget. The fifth point does not provide additional information to reduce the overall deficiency but only makes spline parameters robust, which means it increases the overall precision of the estimated parameters.

This fact is an advantage of adopting splines which the number of degrees of freedom is four since in tie straight lines; only two points per line are independent. Independent information, the number of degrees of freedom, of a straight line is two from two points or a point with its tangent direction. A redundancy is $r=2m-4$ with a line expression of four parameters since equations are $2nm$ collinearity equations and unknowns are $4+nm$ [59]. Only two points ($n=2$) are available to determine four line parameters with two images ($m=2$) so at least three images must contain a tie line. The information content of $t$ tie lines on $m$ images is $t(2m-4)$. One straight line increases two degrees of freedom to the redundancy budget and at least three lines are required in the space resection. An additional point on a straight line does not provide additional information to reduce the rank deficiency of the recovery of EOPs but only contributes image line coefficients. If spline location parameters or spline parameters enter the integrated adjustment model through stochastic constraints, employing extended collinearity equations is enough for solving the system without the arc-length parameterization.
The redundancy budget of a tie point is $r=2m-3$ so tie points provide one more independent equation than tie lines. However, using tie points requires semi-automatic matching procedure to identify tie points on all images and employing linear features is more robust and accurate than point features for object recognition, pose determination, and other higher photogrammetric activities.

5.1 Synthetic data description

The standard block configuration is generated by strips of images with approximately 60% overlap in the flight direction and 20%-30% overlap in the neighbored flight strips. In bundle block adjustment, a ground feature is required at least in two images to determine three dimensional coordinates. The simulation of aerial image block with six images, a forward overlap of 60% and a side overlap of 20%, is performed to verify the feasibility of the proposed model in bundle block adjustment with synthetic data. EOPs of the simulation data set with six images are described in table 5.3 and figure 5.2 with 0.15m focal length and zero offsets from a fiducial-based origin to a perspective center origin of a camera. This means that interior orientation parameters are known and fixed. EOPs of six images are generated under the assumption of the vertical viewing condition.

To evaluate the new bundle block adjustment model using natural cubic splines, the analysis of sensitivity and robustness of the model is required. Verification of the model suitability can be assessed by the estimated parameters with the dispersion matrix including standard deviations and correlations. The accuracy of bundle block adjustment is determined by the geometry of a block of all images and the quality of the position and attitude information of a camera. For novel approaches, a simulation
Table 5.3: EOPs of six bundle block images for simulation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$X_c$ [m]</th>
<th>$Y_c$ [m]</th>
<th>$Z_c$ [m]</th>
<th>$\omega$ [deg]</th>
<th>$\varphi$ [deg]</th>
<th>$\kappa$ [deg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Image 1</td>
<td>3000.00</td>
<td>4002.00</td>
<td>503.00</td>
<td>0.1146</td>
<td>0.0573</td>
<td>5.7296</td>
</tr>
<tr>
<td>Image 2</td>
<td>3305.00</td>
<td>4005.00</td>
<td>499.00</td>
<td>0.1432</td>
<td>0.0859</td>
<td>-5.7296</td>
</tr>
<tr>
<td>Image 3</td>
<td>3610.00</td>
<td>3995.00</td>
<td>505.00</td>
<td>0.1719</td>
<td>0.4584</td>
<td>2.8648</td>
</tr>
<tr>
<td>Image 4</td>
<td>3613.00</td>
<td>4613.00</td>
<td>507.00</td>
<td>0.2865</td>
<td>-0.0573</td>
<td>185.6383</td>
</tr>
<tr>
<td>Image 5</td>
<td>3303.00</td>
<td>4617.00</td>
<td>493.00</td>
<td>-0.1432</td>
<td>0.4011</td>
<td>173.0333</td>
</tr>
<tr>
<td>Image 6</td>
<td>2997.00</td>
<td>4610.00</td>
<td>509.00</td>
<td>-0.1833</td>
<td>-0.2865</td>
<td>181.6276</td>
</tr>
</tbody>
</table>
of bundle block adjustment is required prior to the actual experiment with real data to evaluate the performance of the proposed algorithms. A simulation can control the measurement errors so random noises affect the overall geometry of a block a little. The individual observations are generated based on the general situation of bundle block adjustment to estimate the properties of the proposed algorithms. A simulation allows the adjustment for geometric problems or conditions with various experiments.

A spline is derived by three ground control points (3232, 4261, 18), (3335, 4343, 52), (3373, 4387, 34).

![Figure 5.3: Natural cubic spline](image)

In this chapter, several factors which affect the estimates of exterior orientation parameters, spline parameters, and spline location parameters are implemented using the proposed bundle block adjustment model with the simulated image block and the real image block.
5.2 Experiments with error free EOPs

Since spline parameters and spline location parameters are dependent with respect to other parameters, the unknowns can be obtained by the combined model of the extended collinearity equations and the arc-length parameterization equations of splines. Splines in the object space are considered as tie lines in the same fashion as tie points in conventional bundle block adjustment. Information about the exterior orientation parameters is considered as control information in this experiment. A well-known fact of least squares system is that the good initial estimates to true values make the system convergent to the correct solution quickly and correctly. The equation system of integrated model is described as

\[
\begin{bmatrix}
A^k_{SP} & A^i_t \\
A^i_{AL} & \xi^i_{SP}
\end{bmatrix}
\begin{bmatrix}
\xi^i_{SP} \\
\xi^i_{t}
\end{bmatrix}
= 
\begin{bmatrix}
y^{ki}
\end{bmatrix}
\tag{5.1}
\]

\[
A^i_{SP} =
\begin{bmatrix}
M^1_{i7} & M^1_{i8} & \cdots & M^1_{i18} \\
\vdots & & & \\
M^m_{i7} & M^m_{i8} & \cdots & M^m_{i18} \\
N^1_{i7} & N^1_{i8} & \cdots & N^1_{i18} \\
\vdots & & & \\
N^m_{i7} & N^m_{i8} & \cdots & N^m_{i18}
\end{bmatrix}
\]

\[
A^k_{i} =
\begin{bmatrix}
M^1_{i19} & M^1_{i20} & \cdots & M^1_{i18+n} \\
\vdots & & & \\
M^m_{i19} & M^m_{i20} & \cdots & M^m_{i18+n} \\
N^1_{i19} & N^1_{i20} & \cdots & N^1_{i18+n} \\
\vdots & & & \\
N^m_{i19} & N^m_{i20} & \cdots & N^m_{i18+n}
\end{bmatrix}
\]

\[
A^{ki}_{AL} =
\begin{bmatrix}
A^k_{i1} & A^k_{i2} & \cdots & A^k_{i20} \\
\vdots & & & \\
A^k_{i1} & A^k_{i2} & \cdots & A^k_{i20}
\end{bmatrix}
\]
\[\xi_{SP}^{ki} = \begin{bmatrix} da_{i0} & da_{i1} & da_{i2} & da_{i3} & db_{i0} & db_{i1} & db_{i2} & db_{i3} & dc_{i0} & dc_{i1} & dc_{i2} & dc_{i3} \end{bmatrix}^T\]

\[\xi_i^t = \begin{bmatrix} dt_{i1} & dt_{i2} & \cdots & dt_{in} \end{bmatrix}^T\]

\[y^{ki} = \begin{bmatrix} x_p^{ki} + f\frac{u^0}{w^0} & y_p^{ki} + f\frac{v^0}{w^0} & Arc(t)^{ki} - Arc(t)^0 \end{bmatrix}^T\]

with \(Arc(t)^0 = \frac{t_2^3 - t_1^3}{6} \left[f^0(t_1) + 4f^0\left(\frac{t_2 + t_1}{2}\right) + f^0(t_2)\right]\), \(m\) the number of images, \(n\) the number of points on a spline segment, \(k\) the \(k\)th image and \(i\) the \(i\)th spline segment. The partial derivatives of symbolic representations \((M, N, A)\) of the extended collinearity model are described in appendix A.

The estimated unknown parameters are obtained by least squares solution which minimizes the sum of squared deviations. The integrated and linearized Gauss-Markov model is

\[y^{ki} = A_{IM}\xi_{IM} + e\]

\[A_{IM} = \begin{bmatrix} A_{SP}^{ki} & A_{t}^{ki} \\ A_{AL}^{ki} & \end{bmatrix}\]

\[\xi_{IM} = \begin{bmatrix} \xi_{SP}^{ki} \\ \xi_{t}^{ki} \end{bmatrix}^T\]

\[\hat{\xi}_{IM} = (A_{IM}^T P A_{IM})^{-1} A_{IM}^T P y^{ki}\]

\[D(\hat{\xi}_{IM}) = \sigma_o^2 (A_{IM}^T P A_{IM})^{-1}\]

with \(e \sim N(0, \sigma_o^2 P^{-1})\) the error vector with zero mean and cofactor matrix \(P^{-1}\), and variance component \(\sigma_o^2\) which can be known or not, \(\hat{\xi}_{IM}\) the least squares estimated parameter vector and \(D(\hat{\xi}_{IM})\) the dispersion matrix. Normally distributed random noises are added in all experiments with a zero mean and \(\sigma = \pm 5 \mu m\) standard deviation to points in the image space coordinate system.
Generally the larger noise level the more accurate approximations are required to achieve the ultimate convergence of the results. The worst case scenario for estimating is that the large noise level leads the proposed model not to converge into the specific estimates since the convergence radius is proportional to the noise level. The parameter estimation is sensitive to the noise of the image measurement. Error propagation related with the noise error in the image space observation is one of the most important elements in the estimation theory. The proposed bundle block adjustment can be evaluated statistically using the variances and the covariances of parameters since a small variance represents that the estimated values have a small range and a large variance means that the estimated values are not calculated properly. The range of the parameter variance is from zero in case of error free parameters to infinite in case of completely unknown parameters. The result of one spline segment is expressed as table 5.4 with $\xi^0$ the initial values and $\hat{\xi}$ the estimates. The estimated spline and spline location parameters along with their standard deviations are established without the knowledge of the point-to-point correspondence.

If no random noise is added to image points, the estimates are converged to the true values. The quality of initial estimates is important in least squares system since it determines the iteration number of the system and the accuracy of the convergence. The assumption is that two points on one spline segment are measured in each image so the total number of equations are $2 \times 6$ (the number of images) $\times 2$ (the number of points) $+ 6$ (the number of the arc-length) and the total number of unknowns are $12$ (the number of spline parameters) $+ 12$ (the number of spline location parameters). The redundancy (= the number of equations - the number of parameters), the degrees of freedom, is 6. While some of geometric constraints such as
### Table 5.4: Spline parameter and spline location parameter recovery

<table>
<thead>
<tr>
<th>Image 1</th>
<th>Image 2</th>
<th>Image 3</th>
<th>Image 4</th>
<th>Image 5</th>
<th>Image 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>$t_7$</td>
<td>$t_2$</td>
<td>$t_8$</td>
<td>$t_3$</td>
<td>$t_9$</td>
</tr>
<tr>
<td>$\xi^0$</td>
<td>0.02</td>
<td>0.33</td>
<td>0.09</td>
<td>0.41</td>
<td>0.16</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.0415</td>
<td>0.3651</td>
<td>0.0917</td>
<td>0.4158</td>
<td>0.1412</td>
</tr>
<tr>
<td>±0.0046</td>
<td>±0.0016</td>
<td>±0.0017</td>
<td>±0.0032</td>
<td>±0.0043</td>
<td>±0.0135</td>
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</tbody>
</table>

<table>
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<tr>
<th>$t_4$</th>
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<th>$t_5$</th>
<th>$t_{11}$</th>
<th>$t_6$</th>
<th>$t_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi^0$</td>
<td>0.18</td>
<td>0.51</td>
<td>0.28</td>
<td>0.52</td>
<td>0.33</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.2174</td>
<td>0.4974</td>
<td>0.2647</td>
<td>0.5472</td>
<td>0.3133</td>
</tr>
<tr>
<td>±0.0098</td>
<td>±0.0079</td>
<td>±0.0817</td>
<td>±0.0317</td>
<td>±0.0127</td>
<td>±0.1115</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a_{10}$</th>
<th>$a_{11}$</th>
<th>$a_{12}$</th>
<th>$a_{13}$</th>
<th>$b_{10}$</th>
<th>$b_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi^0$</td>
<td>3322.17</td>
<td>72.16</td>
<td>-45.14</td>
<td>27.15</td>
<td>4377.33</td>
</tr>
<tr>
<td>$\xi$</td>
<td>3335.0080</td>
<td>70.4660</td>
<td>-48.8529</td>
<td>16.5634</td>
<td>4343.0712</td>
</tr>
<tr>
<td>±0.0004</td>
<td>±0.0585</td>
<td>±0.8310</td>
<td>±1.2083</td>
<td>±0.0004</td>
<td>±0.0258</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$b_{12}$</th>
<th>$b_{13}$</th>
<th>$c_{10}$</th>
<th>$c_{11}$</th>
<th>$c_{12}$</th>
<th>$c_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi^0$</td>
<td>-17.49</td>
<td>13.68</td>
<td>48.82</td>
<td>10.15</td>
<td>-27.63</td>
</tr>
<tr>
<td>±0.2193</td>
<td>±0.2067</td>
<td>±0.0006</td>
<td>±0.0589</td>
<td>±0.7139</td>
<td>±1.0103</td>
</tr>
</tbody>
</table>
slope and distance observations are dependent on the extended collinearity equations using splines, other constraints such as slope and arc-length increase non-redundant information in adjustment to reduce the overall rank deficiency of the system.

The coplanarity approach is another mathematical model of the perspective relationship between the image and the object space features. The projection plane defined by the perspective center in the image space and the plane including the straight line in the object space are identical. Since the coplanarity condition is for only straight lines, the coplanarity approach can not be extended to curves.

Object space knowledge about a starting point of a spline can be employed to bundle block adjustment. Since control information about a starting point is available for only three parameters of total twelve unknown parameters of a spline, a spline with control information about a starting point is called a partial control spline. Three spline parameters related to a starting point of a spline are set to stochastic constraints and the result is in table 5.5.

The total number of equations is \(2 \times 6\) (the number of images) \(\times 2\) (the number of points) + \(6\) (the number of the arc-length) = 30 and the total number of unknowns is \(9\) (the number of partial spline parameters) + \(12\) (the number of spline location parameters) = 21 so the redundancy is 9. A convergence of a partial spline and spline location parameters has been archived with a partial control spline.

In the next experiment, spline location parameters are estimated with known EOPs and a full control spline. Since spline parameters and spline location parameters are dependent with respect to other parameters, the unknowns can be obtained with the model of an observation equation with stochastic constraints. In this experiment spline parameters are set to stochastic constraints and the result is in table 5.6.
## Spline location parameters

<table>
<thead>
<tr>
<th>t_1</th>
<th>t_7</th>
<th>t_2</th>
<th>t_8</th>
<th>t_3</th>
<th>t_9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\xi^0)</td>
<td>0.04</td>
<td>0.36</td>
<td>0.09</td>
<td>0.40</td>
<td>0.14</td>
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<tr>
<td>(\xi)</td>
<td>0.0525</td>
<td>0.3547</td>
<td>0.1128</td>
<td>0.4158</td>
<td>0.1575</td>
</tr>
<tr>
<td>±0.0067</td>
<td>±0.0020</td>
<td>±0.0047</td>
<td>±0.0091</td>
<td>±0.0028</td>
<td>±0.0083</td>
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<table>
<thead>
<tr>
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<th>t_10</th>
<th>t_5</th>
<th>t_11</th>
<th>t_6</th>
<th>t_12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\xi^0)</td>
<td>0.21</td>
<td>0.50</td>
<td>0.27</td>
<td>0.54</td>
<td>0.31</td>
</tr>
<tr>
<td>(\xi)</td>
<td>0.1916</td>
<td>0.5128</td>
<td>0.2563</td>
<td>0.5319</td>
<td>0.2961</td>
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<td>±0.0087</td>
<td>±0.0044</td>
<td>±0.0056</td>
<td>±0.0139</td>
<td>±0.1115</td>
</tr>
</tbody>
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### Spline parameters

<table>
<thead>
<tr>
<th>a_{11}</th>
<th>a_{12}</th>
<th>a_{13}</th>
<th>b_{11}</th>
<th>b_{12}</th>
<th>b_{13}</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\xi^0)</td>
<td>75.14</td>
<td>-52.87</td>
<td>30.71</td>
<td>70.05</td>
<td>-40.33</td>
</tr>
<tr>
<td>(\xi)</td>
<td>71.7099</td>
<td>-47.2220</td>
<td>-15.8814</td>
<td>62.3703</td>
<td>-28.7260</td>
</tr>
<tr>
<td>±0.0795</td>
<td>±0.6872</td>
<td>±2.6439</td>
<td>±0.0579</td>
<td>±0.6473</td>
<td>±1.7699</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>c_{11}</th>
<th>c_{12}</th>
<th>c_{13}</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\xi^0)</td>
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<td>-30.72</td>
</tr>
<tr>
<td>(\xi)</td>
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</tr>
<tr>
<td>±0.9483</td>
<td>±1.3403</td>
<td>±3.5852</td>
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</table>

<table>
<thead>
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<td>-40.33</td>
<td>10.98</td>
</tr>
<tr>
<td>62.3703</td>
<td>-28.7260</td>
<td>7.1137</td>
</tr>
<tr>
<td>±0.0579</td>
<td>±0.6473</td>
<td>±1.7699</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>c_{13}</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.51</td>
</tr>
<tr>
<td>8.1557</td>
</tr>
<tr>
<td>±3.5852</td>
</tr>
</tbody>
</table>

Table 5.5: Partial spline parameter and spline location parameter recovery
total number of equations is $2 \times 6 \text{(the number of images)} \times 3 \text{(the number of points)} = 36$ and the total number of unknowns is $18 \text{(the number of spline location parameters)}$ so the redundancy is $18$. Since spline location parameters are independent of each other, the arc-length parameterization is not required.

<table>
<thead>
<tr>
<th>Spline location parameters</th>
<th>Image 1</th>
<th>Image 2</th>
<th>Image 3</th>
<th>Image 4</th>
<th>Image 5</th>
<th>Image 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi^0$</td>
<td>0.01</td>
<td>0.37</td>
<td>0.63</td>
<td>0.09</td>
<td>0.44</td>
<td>0.71</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.0589</td>
<td>0.3570</td>
<td>0.6712</td>
<td>0.1134</td>
<td>0.4175</td>
<td>0.7069</td>
</tr>
<tr>
<td>$\pm 0.0015$</td>
<td>$\pm 0.0076$</td>
<td>$\pm 0.0197$</td>
<td>$\pm 0.0072$</td>
<td>$\pm 0.0054$</td>
<td>$\pm 0.0080$</td>
<td></td>
</tr>
<tr>
<td>$\hat{\xi}$</td>
<td>0.0589</td>
<td>0.3570</td>
<td>0.6712</td>
<td>0.1134</td>
<td>0.4175</td>
<td>0.7069</td>
</tr>
<tr>
<td>$\pm 0.0015$</td>
<td>$\pm 0.0076$</td>
<td>$\pm 0.0197$</td>
<td>$\pm 0.0072$</td>
<td>$\pm 0.0054$</td>
<td>$\pm 0.0080$</td>
<td></td>
</tr>
<tr>
<td>$\xi^0$</td>
<td>0.17</td>
<td>0.46</td>
<td>0.74</td>
<td>0.21</td>
<td>0.49</td>
<td>0.81</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.1757</td>
<td>0.4784</td>
<td>0.7631</td>
<td>0.2039</td>
<td>0.4869</td>
<td>0.8122</td>
</tr>
<tr>
<td>$\pm 0.0031$</td>
<td>$\pm 0.0071$</td>
<td>$\pm 0.0095$</td>
<td>$\pm 0.0102$</td>
<td>$\pm 0.0030$</td>
<td>$\pm 0.0044$</td>
<td></td>
</tr>
<tr>
<td>$\hat{\xi}$</td>
<td>0.1757</td>
<td>0.4784</td>
<td>0.7631</td>
<td>0.2039</td>
<td>0.4869</td>
<td>0.8122</td>
</tr>
<tr>
<td>$\pm 0.0031$</td>
<td>$\pm 0.0071$</td>
<td>$\pm 0.0095$</td>
<td>$\pm 0.0102$</td>
<td>$\pm 0.0030$</td>
<td>$\pm 0.0044$</td>
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</tr>
<tr>
<td>$\xi^0$</td>
<td>0.26</td>
<td>0.53</td>
<td>0.84</td>
<td>0.29</td>
<td>0.61</td>
<td>0.89</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.2544</td>
<td>0.5554</td>
<td>0.8597</td>
<td>0.3151</td>
<td>0.6284</td>
<td>0.9013</td>
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<tr>
<td>$\pm 0.0050$</td>
<td>$\pm 0.0069$</td>
<td>$\pm 0.0089$</td>
<td>$\pm 0.0095$</td>
<td>$\pm 0.0052$</td>
<td>$\pm 0.0086$</td>
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</tr>
</tbody>
</table>

Table 5.6: Spline location parameter recovery

The result represents that a convergence of spline location parameters has been achieved with fixed spline parameters considered as stochastic constraints. The proposed model is robust with respect to the initial approximations of spline parameters. The uncertain information related to the representation of a natural cubic spline is described in the dispersion matrix.
5.3 Recovery of EOPs and spline parameters

The object space knowledge about splines is available to recover the exterior orientation parameters in bundle block adjustment. Control spline and partial control spline are applied to verify the feasibility of control information with splines. In both cases, equations of the arc-length parameterization are not necessary if we have enough equations to solve the system since spline parameters are independent of each other. In the experiment of full control spline, the total number of equations are $2 \times 6(\text{the number of images}) \times 4(\text{the number of points}) + 3(\text{the number of arc-length}) \times 6(\text{the number of images}) = 66$ and the total number of unknowns are $36(\text{the number of EOPs}) + 24(\text{the number of spline location parameters}) = 60$. The redundancy is 6.

In the case of the partial control spline with one spline segment, the total number of equations are $2 \times 6(\text{the number of images}) \times 4(\text{the number of points}) + 3(\text{the number of arc-length}) \times 6(\text{the number of images}) = 66$ and the total number of unknowns are $36(\text{the number of EOPs}) + 9(\text{the number of partial spline parameters}) + 24(\text{the number of spline location parameters}) = 69$. Thus one more segment is required to solve the underdetermined system. The total number of equations using two spline segments are $2 \times 6(\text{the number of images}) \times 4(\text{the number of points}) \times 2(\text{the number of spline segment}) + 3(\text{the number of arc-length}) \times 6(\text{the number of images}) \times 2(\text{the number of spline segment}) = 132$ and the total number of unknowns are $36(\text{the number of EOPs}) + 9(\text{the number of partial spline parameters}) \times 2(\text{the number of spline segment}) + 24(\text{the number of spline location parameters}) \times 2(\text{the number of spline segment}) = 102$. The redundancy is 30. A convergence of EOPs of an image block and spline parameters have been achieved in both experiments.
### EOPs

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$X_c$ [m]</th>
<th>$Y_c$ [m]</th>
<th>$Z_c$ [m]</th>
<th>$\omega$ [deg]</th>
<th>$\varphi$ [deg]</th>
<th>$\kappa$ [deg]</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Image 1</strong></td>
<td>$\xi^0$</td>
<td>3007.84</td>
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<td>501.81</td>
<td>8.7090</td>
<td>-9.7976</td>
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<td>4001.2238</td>
<td>503.2550</td>
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<td>0.3252</td>
<td>6.0148</td>
</tr>
<tr>
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<td>$\pm 0.0215$</td>
<td>$\pm 0.1386$</td>
<td>$\pm 0.3895$</td>
<td>$\pm 0.1351$</td>
<td>$\pm 0.8142$</td>
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</tr>
<tr>
<td><strong>Image 2</strong></td>
<td>$\xi^0$</td>
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<td>-0.5497</td>
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</tr>
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<td>$\pm 0.2489$</td>
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<td>$\pm 0.0798$</td>
<td>$\pm 0.4690$</td>
<td></td>
</tr>
<tr>
<td><strong>Image 3</strong></td>
<td>$\xi^0$</td>
<td>3612.68</td>
<td>3993.37</td>
<td>506.88</td>
<td>6.2571</td>
<td>-5.3482</td>
</tr>
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<td>$\xi$</td>
<td>3611.8996</td>
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<td>$\pm 0.0168$</td>
<td>$\pm 0.0794$</td>
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</tr>
<tr>
<td><strong>Image 4</strong></td>
<td>$\xi^0$</td>
<td>3301.84</td>
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<td>506.88</td>
<td>6.2571</td>
<td>-5.3482</td>
</tr>
<tr>
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### Spline location parameters

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Table 5.7: EOP and spline location parameter recovery
Table 5.7 expressed the convergence achievement of EOPs and spline location parameters. The correlation coefficient between parameter $X_c$ and $\varphi$ is high ($\rho \approx 1$) in the dispersion matrix, that is two parameters are highly correlated among EOPs. The correlation coefficient between parameter $Y_c$ and $\omega$ is approximately 0.85. Lee[39] represented that parameter $X_c$&$\varphi$ and parameter $Y_c$&$\omega$ are highly correlated in the line camera orientation parameters. In general, the correlation coefficient between parameter $X_c$ and $\varphi$ is higher than between parameter $Y_c$ and $\omega$. In addition, Lee demonstrated that image coordinates in $x$ direction are related with parameter $X_c$, $Z_c$, $\omega$, $\varphi$ and $\kappa$ and image coordinates in $y$ direction are the function of parameter $Y_c$, $Z_c$, $\omega$, $\varphi$ and $\kappa$ in a line camera.

Since a control spline provides the object space information about the coordinate system having datum defects of seven, tie spline parameters and EOPs can be recovered simultaneously. In the experiment of combined splines, the total number of equations are $2 \times 6$ (the number of images) $\times 3$ (the number of points) $\times 2$ (the number of splines) $\times 12$ (the number of the arc-length) $\times 2$ (the number of splines) $= 96$ and the total number of unknowns are $36$ (the number of EOPs) $+ 12$ (the number of tie spline parameters) $+ 18$ (the number of tie spline location parameters) $+ 18$ (the number of control spline location parameters) $= 84$. Since the knowledge of the object space information about spline referred to as a full control spline is available prior to aerial triangulation. A control spline is considered as a stochastic constraint in the proposed adjustment model and the representation of a control spline is the same with that of tie spline. The result of the combined splines which demonstrates the feasibility of tie spline and control spline for bundle block adjustment is described in table 5.8.
### Table 5.8: EOP, control and tie spline parameter recovery

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<th>Parameter</th>
<th>$X_e$ [m]</th>
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### Tie spline location parameters

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### Tie spline parameters

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Iteration with an incorrect spline segment in which a spline in the image space does not lie on the projection of a 3D spline in the object space results in a divergence of the system. A control spline is assumed to be error-free but in reality the assumption is not correct. The accuracy of control splines is propagated into the proposed bundle block adjustment algorithm since initial data such as GIS database, maps or orthophotos can not be without error.

### 5.4 Tests with real data

In this section, actual experiments with real data are implemented to verify the feasibility of the proposed bundle block adjustment algorithm using splines for the recovery of EOPs and spline parameters.

Medium scale aerial images covering the area of Jakobshavn Isbrae in West Greenland are employed for this study. The aerial photographs were obtained by Kort and Matrikelstyrelsen (KMS: Danish National Survey and Cadastre) in 1985. KMS established aerial triangulation using GPS ground control points with $\pm 1$ pixel RMS (Root Mean Square) error under favorable circumstances and images were oriented to the WGS84 reference frame. Technical information on aerial images are described in table 5.9.

The diapositive films are scanned with the RasterMaster photogrammetric precision scanner which has the maximum image resolution of $12\mu m$ and the scan dimension of $23cm \times 23cm$ to obtain digital images for softcopy workstation as figure 5.4.

The first experiment is the recovery of spline parameters with known EOPs obtained by manual operation using softcopy workstation. Spline consists of four parts
Figure 5.4: Test images

(a) Image 762 (b) Image 764 (c) Image 766 (d) Target area
and the second segment parameters are recovered. The total number of equations is $2 \times 3 \times 3 + 2 \times 3 = 24$ and the total number of unknowns is $12 + 9 = 21$ so the redundancy is 3. Table 5.10 expressed the convergence achievement of spline and spline location parameters.

Estimation of spline parameters including location parameters is established by the relationship between splines in the object space and their projection in the image space without the knowledge of the point-to-point correspondence. Since bundle block adjustment using splines does not require conjugate points generated by the knowledge of the point-to-point correspondence, the more robust and flexible matching algorithm can be adopted. Table 5.11 represents the result which the object space information without the knowledge of the point-to-point correspondence (the full control spline) is available. However, as mentioned earlier in chapter 3, the correspondence between the image location and object spline segment is not established.
## Spline location parameters

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## Spline parameters

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<td>0.00</td>
<td>-39.8003</td>
<td>27.7922</td>
</tr>
</tbody>
</table>

### Table 5.10: Spline parameter and spline location parameter recovery
All locations are assumed as on the second spline segment and the second spline segment calculated from softcopy workstation is used as control information.

| Spline location parameters | | | |
|---------------------------|--|--|--|--|--|--|
|                           | Image 762 | Image 764 | Image 766 |
| $t_1$                     | $t_4$     | $t_2$     | $t_5$     | $t_3$     | $t_6$     |
| $\xi^0$                   | 0.15      | 0.60      | 0.30      | 0.75      | 0.45      | 0.90      |
| $\xi$                     | 0.1647    | 0.6177    | 0.2872    | 0.7481    | 0.4362    | 0.9249    |
| $\pm 0.0084$              | $\pm 0.0091$ | $\pm 0.0034$ | $\pm 0.0093$ | $\pm 0.0155$ | $\pm 0.0087$ |

Table 5.11: Spline location parameter recovery

The next experiment is the recovery of EOPs with control spline. Spline control points are $(534415.91, 767199305, -18.97)$, $(535394.52, 7672045.02, 2.127)$, $(536110.66, 7672024.29, -13.897)$, and $(536654.04, 7671016.20, -2.51)$. Even though the edge detectors are often used in digital photogrammetry and remote sensing software, the control points are extracted manually since edge detection is not our main goal. Among three segments, the second spline segment is used for the EOP recovery. The information of control spline is obtained by manual operation using softcopy workstation with an estimated accuracy of $\pm 1$ pixel. The convergence radius of the proposed iterative algorithm is proportional to the estimated accuracy level. The image coordinate system is converted into the photo coordinate system using the interior orientation parameters from KMS. The association between a point on a 3D spline segment and a point on a 2D image is not established in this study. Of course 3D spline measurement in the stereo model using softcopy workstation can not be
without error so the accuracy of control spline is propagated into the recovery of EOPs. The result is described in table 5.12.

<table>
<thead>
<tr>
<th>Image</th>
<th>Xc [m]</th>
<th>Yc [m]</th>
<th>Zc [m]</th>
<th>ω [deg]</th>
<th>ϕ [deg]</th>
<th>κ [deg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>762</td>
<td>547000.00</td>
<td>7659000.00</td>
<td>14000.00</td>
<td>3.8471</td>
<td>2.1248</td>
<td>91.8101</td>
</tr>
<tr>
<td>ξ_0</td>
<td>547465.37</td>
<td>7658235.41</td>
<td>13709.25</td>
<td>0.3622</td>
<td>0.5124</td>
<td>91.5124</td>
</tr>
<tr>
<td>±15.0911</td>
<td>±13.0278</td>
<td>±5.4714</td>
<td>±0.8148</td>
<td>±0.1784</td>
<td>±0.1717</td>
<td></td>
</tr>
<tr>
<td>764</td>
<td>546500.00</td>
<td>7670000.00</td>
<td>13500.00</td>
<td>0.1125</td>
<td>0.6128</td>
<td>90.7015</td>
</tr>
<tr>
<td>ξ_0</td>
<td>546963.22</td>
<td>7672016.87</td>
<td>13708.82</td>
<td>-0.3258</td>
<td>-0.5217</td>
<td>91.1612</td>
</tr>
<tr>
<td>±12.5460</td>
<td>±17.1472</td>
<td>±7.1872</td>
<td>±0.6913</td>
<td>±0.8632</td>
<td>±1.1004</td>
<td></td>
</tr>
<tr>
<td>766</td>
<td>546000.00</td>
<td>7680000.00</td>
<td>13700.00</td>
<td>1.4871</td>
<td>5.9052</td>
<td>92.0975</td>
</tr>
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<td>ξ_0</td>
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<td>7685836.75</td>
<td>13712.20</td>
<td>1.2785</td>
<td>0.5469</td>
<td>92.9796</td>
</tr>
<tr>
<td>±13.8104</td>
<td>±12.1486</td>
<td>±8.4854</td>
<td>±1.4218</td>
<td>±1.1957</td>
<td>±0.6557</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Spline location parameters</th>
<th>Image 762</th>
<th>Image 764</th>
</tr>
</thead>
<tbody>
<tr>
<td>t_1</td>
<td>t_4</td>
<td>t_7</td>
</tr>
<tr>
<td>ξ_0</td>
<td>0.08</td>
<td>0.32</td>
</tr>
<tr>
<td>ξ</td>
<td>0.0865</td>
<td>0.3192</td>
</tr>
<tr>
<td>±0.0097</td>
<td>±0.0159</td>
<td>±0.0072</td>
</tr>
<tr>
<td>Image 764</td>
<td>Image 766</td>
<td></td>
</tr>
<tr>
<td>t_8</td>
<td>t_11</td>
<td>t_3</td>
</tr>
<tr>
<td>ξ_0</td>
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</tr>
<tr>
<td>ξ</td>
<td>0.6557</td>
<td>0.8685</td>
</tr>
<tr>
<td>±0.0131</td>
<td>±0.0092</td>
<td>±0.0086</td>
</tr>
</tbody>
</table>

Table 5.12: EOP and spline location parameter recovery

The control information about spline is utilized as stochastic constraints in the adjustment model. Since adding stochastic constraints removes the rank deficiency
of the Gauss-Markov model corresponding to spline parameters which spline parameters are dependent to spline location parameters, bundle block adjustment can be implemented using only the extended collinearity equations for natural cubic splines.
In this paper, the traditional least squares of a bundle block adjustment process have been augmented to support splines instead of conventional point features. Estimation of EOPs and spline parameters including location parameters is established by the relationship between splines in the object space and their projection in the image space without the knowledge of the point-to-point correspondence. Since bundle block adjustment using splines does not require conjugate points generated by the knowledge of the point-to-point correspondence, the more robust and flexible matching algorithm can be adopted. Point-based aerial triangulation with experienced human operators is processed well in traditional photogrammetric activities but not the autonomous environment of digital photogrammetry. Feature-based aerial triangulation is more suitable to develop the robust and accurate techniques for automation. If linear features are employed as control features, they provide advantages over point features for the automation of aerial triangulation. Point-based aerial triangulation based on the manual measurement and identification of conjugate points is less reliable than feature-based aerial triangulation since it has the limitations of visibility (occlusion), ambiguity (repetitive patterns) and semantic information in view of robust and suitable automation. Automation of aerial triangulation and the pose
estimation is obstracled by the correspondence problem, but employing splines is one way to overcome the occlusion and ambiguity problems. The manual identification of corresponding entities in two images is crucial in the automation of photogrammetric tasks. Another problem of point-based approaches is the weak geometric constraints compared to feature-based methods so the accurate initial values for the unknown parameters are required. Feature-based aerial triangulation can be implemented without conjugate points since the measured points in each image are not the conjugate points in this proposed adjustment model. Thus tie spline which does not appear in all overlapped images together can be employed in feature-based aerial triangulation. Another advantage of employing splines is that adopting high level features increases the feasibility of geometric information and provides an analytical and suitable solution to increase the redundancy of aerial triangulation.

3D linear features expressed by 3D natural cubic splines are employed as the mathematical model of linear features in the object space and its counterpart in the projected image space for bundle block adjustment. To solve over-parameterization of 3D natural cubic splines, arc-length parameterization using Simpson’s rule is developed and in case of straight lines and conic sections, tangents of spline can be additional equations to the overparameterized system. Photogrammetric triangulation by the proposed model including the extended collinearity equation and arc-length parameterization equation is developed to show the feasibility of tie splines and control splines for the estimation of exterior orientation of multiple images, spline and spline location parameters.

A useful stochastic constraint for a spline segment is examined to become a full or partial control spline such as known EOPs with a tie, partial control, and full

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control spline and unknown EOPs with a partial and full control spline. In addition, information content of an image spline is calculated and the feasibility of a tie spline and a control spline for a block adjustment is described.

A simulation of bundle block adjustment is implemented prior to the actual experiment with real data to evaluate the performance of the proposed algorithms. A simulation can control the measurement errors so random noises slightly affect the overall geometry of a block. The individual observations are generated based on the general situation of bundle block adjustment to estimate the properties of the proposed algorithms. A simulation allows the adjustment for geometric problems or conditions with various experiments.

The future research topics can be summarized as:

- In the proposed bundle block adjustment algorithm with 3D natural cubic splines, splines have been employed as control features. Future work should include an enhancement of the proposed model to triangulation and other feature-based photogrammetric tasks using linear features having an arbitrary mathematical representation.

- More investigations should be done to find out an additional analytical observation or stochastic constraints to solve over-parameterization of 3D natural cubic spline. Other constraints such as higher order derivatives in image space features can increase non-redundant information in bundle block adjustment to reduce the overall rank deficiency of the system.

- Future work should concentrate on the elaborated testing for complete bundle block adjustment including camera interior orientation parameters with splines.
Interior orientation defines a transformation to the 3D image coordinate system with respect to the camera’s perspective center, while the pixel coordinate system is the reference system for a digital image, using the geometric relationship between the photo coordinate system and the instrument coordinate system. Since we have used the data of the camera calibration provided by the aerial photography, the uncertainty of camera interior orientation parameters is not considered in bundle block adjustment.

- The proposed model can be augmented to the line scan imagery instead of the traditional frame aerial imagery to recover orientation parameters for every scan line since splines can provide independent observations for EOPs of all scan lines. Since all scan lines have their own orientation parameters, using only point features is not enough for the pose estimation.

- The automatic extraction of the most informative features in the image and object space, and matching features can be considered to recover orientation parameters automatically. It will be the next logical step towards automatic bundle block adjustment using a natural cubic spline.
APPENDIX A

DERIVATION OF THE PROPOSED MODEL

A.1 Derivation of extended collinearity equation

The partial derivatives of symbolic representation of the extended collinearity model for curves are introduced as follows.

\[
\frac{\partial R^T}{\partial \omega} = R^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}
\]

\[
\frac{\partial R^T}{\partial \varphi} = \begin{bmatrix} -\sin \varphi \cos \kappa & \sin \omega \cos \varphi \cos \kappa & -\cos \omega \cos \varphi \cos \kappa \\ \sin \varphi \sin \kappa & -\sin \omega \cos \varphi \sin \kappa & \cos \omega \cos \varphi \sin \kappa \\ \cos \varphi & \sin \omega \sin \varphi & -\cos \omega \sin \varphi \end{bmatrix}
\]

\[
\frac{\partial R^T}{\partial \kappa} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R^T
\]

\[
M_1 = \frac{f}{w} r_{11} - \frac{fu}{w^2} r_{31}
\]

\[
M_2 = \frac{f}{w} r_{12} - \frac{fu}{w^2} r_{32}
\]

\[
M_3 = \frac{f}{w} r_{13} - \frac{fu}{w^2} r_{33}
\]

\[
M_4 = -\frac{f}{w} \{ r_{12} (Z_i(t) - Z_C) - r_{13} (Y_i(t) - Y_C) \}
+ \frac{fu}{w^2} \{ r_{32} (Z_i(t) - Z_C) - r_{33} (Y_i(t) - Y_C) \}
\]

\[
M_5 = -\frac{f}{w} \{ s_{11} (X_i(t) - X_C) + s_{12} (Y_i(t) - Y_C) + s_{13} (Z_i(t) - Z_C) \}
\]
$$\begin{align*}
+ \frac{fu}{w^2} \left\{ s_{31} (X_i(t) - X_C) + s_{32} (Y_i(t) - Y_C) + s_{33} (Z_i(t) - Z_C) \right\} \\
(s_{ij} \text{ are the elements of } \frac{\partial R^T}{\partial \varphi})
\end{align*}$$

$$\begin{align*}
M_6 &= -\frac{fv}{w} \\
M_7 &= -\frac{f}{w} r_{11} + \frac{fu}{w^2} r_{31} \\
M_8 &= -\frac{f}{w} r_{11} t + \frac{fu}{w^2} r_{31} t \\
M_9 &= -\frac{f}{w} r_{11} t^2 + \frac{fu}{w^2} r_{31} t^2 \\
M_{10} &= -\frac{f}{w} r_{11} t^3 + \frac{fu}{w^2} r_{31} t^3 \\
M_{11} &= -\frac{f}{w} r_{12} + \frac{fu}{w^2} r_{32} \\
M_{12} &= -\frac{f}{w} r_{12} t + \frac{fu}{w^2} r_{32} t \\
M_{13} &= -\frac{f}{w} r_{12} t^2 + \frac{fu}{w^2} r_{32} t^2 \\
M_{14} &= -\frac{f}{w} r_{12} t^3 + \frac{fu}{w^2} r_{32} t^3 \\
M_{15} &= -\frac{f}{w} r_{13} + \frac{fu}{w^2} r_{33} \\
M_{16} &= -\frac{f}{w} r_{13} t + \frac{fu}{w^2} r_{33} t \\
M_{17} &= -\frac{f}{w} r_{13} t^2 + \frac{fu}{w^2} r_{33} t^2 \\
M_{18} &= -\frac{f}{w} r_{13} t^3 + \frac{fu}{w^2} r_{33} t^3 \\
M_{19} &= -\frac{f}{w} \left\{ \left( a_{i1} + 2a_{i2} t + 3a_{i3} t^2 \right) r_{11} + \left( b_{i1} + 2b_{i2} t + 3b_{i3} t^2 \right) r_{12} \\
&\quad + \left( c_{i1} + 2c_{i2} t + 3c_{i3} t^2 \right) r_{13} \right\} + \frac{fu}{w^2} \left\{ \left( a_{i1} + 2a_{i2} t + 3a_{i3} t^2 \right) r_{31} \\
&\quad + \left( b_{i1} + 2b_{i2} t + 3b_{i3} t^2 \right) r_{32} + \left( c_{i1} + 2c_{i2} t + 3c_{i3} t^2 \right) r_{33} \right\}
\end{align*}$$

$$\begin{align*}
N_1 &= \frac{f}{w} r_{21} - \frac{fv}{w^2} r_{31} \\
N_2 &= \frac{f}{w} r_{22} - \frac{fv}{w^2} r_{32} \\
N_3 &= \frac{f}{w} r_{23} - \frac{fv}{w^2} r_{33}
\end{align*}$$
\[ N_4 = -\frac{f}{w} \{ r_{22} (Z_i(t) - Z_C) - r_{23} (Y_i(t) - Y_C) \} \]
\[ + \frac{fv}{w} \{ r_{32} (Z_i(t) - Z_C) - r_{33} (Y_i(t) - Y_C) \} \]
\[ N_5 = -\frac{f}{w} \{ s_{21} (X_i(t) - X_C) + s_{22} (Y_i(t) - Y_C) + s_{23} (Z_i(t) - Z_C) \} \]
\[ + \frac{fv}{w^2} \{ s_{31} (X_i(t) - X_C) + s_{32} (Y_i(t) - Y_C) + s_{33} (Z_i(t) - Z_C) \} \]
\[ N_6 = \frac{fu}{w} \]
\[ N_7 = -\frac{f}{w} r_{21} + \frac{fv}{w^2} r_{31} \]
\[ N_8 = -\frac{f}{w} r_{21} t + \frac{fv}{w^2} r_{31} t \]
\[ N_9 = -\frac{f}{w} r_{21} t^2 + \frac{fv}{w^2} r_{31} t^2 \]
\[ N_{10} = -\frac{f}{w} r_{21} t^3 + \frac{fv}{w^2} r_{31} t^3 \]
\[ N_{11} = -\frac{f}{w} r_{22} + \frac{fv}{w^2} r_{32} \]
\[ N_{12} = -\frac{f}{w} r_{22} t + \frac{fv}{w^2} r_{32} t \]
\[ N_{13} = -\frac{f}{w} r_{22} t^2 + \frac{fv}{w^2} r_{32} t^2 \]
\[ N_{14} = -\frac{f}{w} r_{22} t^3 + \frac{fv}{w^2} r_{32} t^3 \]
\[ N_{15} = -\frac{f}{w} r_{23} + \frac{fv}{w^2} r_{33} \]
\[ N_{16} = -\frac{f}{w} r_{23} t + \frac{fv}{w^2} r_{33} t \]
\[ N_{17} = -\frac{f}{w} r_{23} t^2 + \frac{fv}{w^2} r_{33} t^2 \]
\[ N_{18} = -\frac{f}{w} r_{23} t^3 + \frac{fv}{w^2} r_{33} t^3 \]
\[ N_{19} = -\frac{f}{w} \{ (a_{i1} + 2a_{i2} t + 3a_{i3} t^2) r_{21} + (b_{i1} + 2b_{i2} t + 3b_{i3} t^2) r_{22} \}
\[ + (c_{i1} + 2c_{i2} t + 3c_{i3} t^2) r_{23} \} + \frac{fv}{w^2} \{ (a_{i1} + 2a_{i2} t + 3a_{i3} t^2) r_{31} \]
\[ + (b_{i1} + 2b_{i2} t + 3b_{i3} t^2) r_{32} + (c_{i1} + 2c_{i2} t + 3c_{i3} t^2) r_{33} \} \]
A.2 Derivation of arc-length parameterization

The partial derivatives of symbolic representation of the arc-length parameterization are introduced as follows.

\[
\begin{align*}
Du(t) &= 2x_p'(t)f \frac{w'}{w^2} \\
Du'(t) &= -2x_p'(t)\frac{f}{w} \\
Dv(t) &= 2y_p'(t)f \frac{w'}{w^2} \\
Dv'(t) &= -2y_p'(t)\frac{f}{w}dv' \\
Dw(t) &= \left\{2x_p'(t)\frac{u'w^2 - (u'w - uw')2w}{w^4} - 2y_p'(t)\frac{v'w^2 - (v'w - vw')2w}{w^4}\right\} \\
Dw'(t) &= \left\{2x_p'(t)\frac{u}{w^2} + 2y_p'(t)\frac{v}{w^2}\right\}
\end{align*}
\]

\[
\begin{align*}
A_1 &= \frac{t_2 - t_1}{6}\left[\frac{1}{2}f(t_1)^{-\frac{3}{2}}\{-Du(t_1)r_{11} - Dv(t_1)r_{21} - Dw(t_1)r_{31}\} \\
&+ 2f\left(\frac{t_1 + t_2}{2}\right)^{-\frac{3}{2}}\{-Du(\frac{t_1 + t_2}{2})r_{11} - Dv(\frac{t_1 + t_2}{2})r_{21} - Dw(\frac{t_1 + t_2}{2})r_{31}\}\right] \\
&+ \frac{1}{2}f(t_2)^{-\frac{3}{2}\{Du(t_2)r_{11} - Dv(t_2)r_{21} - Dw(t_2)r_{31}\}\right] \\
A_2 &= \frac{t_2 - t_1}{6}\left[\frac{1}{2}f(t_1)^{-\frac{3}{2}}\{-Du(t_1)r_{12} - Dv(t_1)r_{22} - Dw(t_1)r_{32}\} \\
&+ 2f\left(\frac{t_1 + t_2}{2}\right)^{-\frac{3}{2}}\{-Du(\frac{t_1 + t_2}{2})r_{12} - Dv(\frac{t_1 + t_2}{2})r_{22} - Dw(\frac{t_1 + t_2}{2})r_{32}\}\right] \\
&+ \frac{1}{2}f(t_2)^{-\frac{3}{2}\{Du(t_2)r_{12} - Dv(t_2)r_{22} - Dw(t_2)r_{32}\}\right] \\
A_3 &= \frac{t_2 - t_1}{6}\left[\frac{1}{2}f(t_1)^{-\frac{3}{2}}\{-Du(t_1)r_{13} - Dv(t_1)r_{23} - Dw(t_1)r_{33}\} \\
&+ 2f\left(\frac{t_1 + t_2}{2}\right)^{-\frac{3}{2}}\{-Du(\frac{t_1 + t_2}{2})r_{13} - Dv(\frac{t_1 + t_2}{2})r_{23} - Dw(\frac{t_1 + t_2}{2})r_{33}\}\right] \\
&+ \frac{1}{2}f(t_2)^{-\frac{3}{2}\{Du(t_2)r_{13} - Dv(t_2)r_{23} - Dw(t_2)r_{33}\}\right] \\
A_4 &= \frac{t_2 - t_1}{6}\left[\frac{1}{2}f(t_1)^{-\frac{3}{2}\{Du(t_1)(r_{12}(Z_i(t_1) - Z_C) - r_{13}(Y_i(t_1) - Y_C))\}\right]
\end{align*}
\]
\[ +Du'(t_1)\{r_{12}(Z'_i(t_1)) - r_{13}(Y'_i(t_1))\} + Dw(t_1)\{r_{22}(Z_i(t_1) - Z_C) \]
\[ -r_{23}(Y_i(t_1) - Y_C)\} + Du'(t_1)\{r_{22}(Z_i'(t_1)) - r_{23}(Y_i'(t_1))\} \]
\[ +Dw(t_1)\{r_{32}(Z_i(t_1) - Z_C) - r_{33}(Y_i(t_1) - Y_C)\} \]
\[ +Dw'(t_1)\{r_{32}(Z_i'(t_1)) - r_{33}(Y_i'(t_1))\} \]
\[ +2f\left(\frac{t_1 + t_2}{2}\right)^{\frac{3}{2}} \left[ Du\left(\frac{t_1 + t_2}{2}\right)\{r_{12}(Z_i\left(\frac{t_1 + t_2}{2}\right) - Z_C) \]
\[ - r_{13}(Y_i\left(\frac{t_1 + t_2}{2}\right) - Y_C)\} + Du'\left(\frac{t_1 + t_2}{2}\right)\{r_{12}(Z_i'\left(\frac{t_1 + t_2}{2}\right)) - r_{13}(Y_i'\left(\frac{t_1 + t_2}{2}\right))\} \]
\[ + Dw\left(\frac{t_1 + t_2}{2}\right)\{r_{22}(Z_i\left(\frac{t_1 + t_2}{2}\right) - Z_C) - r_{23}(Y_i\left(\frac{t_1 + t_2}{2}\right) - Y_C)\} \]
\[ + Dw'\left(\frac{t_1 + t_2}{2}\right)\{r_{22}(Z_i'\left(\frac{t_1 + t_2}{2}\right)) - r_{23}(Y_i'\left(\frac{t_1 + t_2}{2}\right))\} \]
\[ + \frac{1}{2} f(t_2)^{\frac{-3}{2}} \left[ Du(t_2)\{r_{12}(Z_i(t_2) - Z_C) - r_{13}(Y_i(t_2) - Y_C)\} \right. \]
\[ + Du'(t_2)\{r_{12}(Z_i'(t_2)) - r_{13}(Y_i'(t_2))\} + Dw(t_2)\{r_{22}(Z_i(t_2) - Z_C) \]
\[ - r_{23}(Y_i(t_2) - Y_C)\} + Dw'(t_2)\{r_{22}(Z_i'(t_2)) - r_{23}(Y_i'(t_2))\} \]
\[ + Dw(t_2)\{r_{32}(Z_i(t_2) - Z_C) - r_{33}(Y_i(t_2) - Y_C)\} \]
\[ + Dw'(t_2)\{r_{32}(Z_i'(t_2)) - r_{33}(Y_i'(t_2))\} \]

\[ A_5 = \frac{t_2 - t_1}{6} \left[ \frac{1}{2} f(t_1)^{\frac{-3}{2}} \left\{ Du(t_1)\{s_{11}(X_i(t_1) - X_C) + s_{12}(Y_i(t_1) - Y_C) \]
\[ + s_{13}(Z_i(t_1) - Z_C)\} + Du'(t_1)\{s_{11}(X_i'(t_1)) + s_{12}(Y_i'(t_1)) + s_{13}(Z_i'(t_1))\} \right. \]
\[ + Dw(t_1)\{s_{21}(X_i(t_1) - X_C) + s_{22}(Y_i(t_1) - Y_C) + s_{23}(Z_i(t_1) - Z_C)\} \]
\[ + Dw'(t_1)\{s_{21}(X_i'(t_1)) + s_{22}(Y_i'(t_1)) + s_{23}(Z_i'(t_1))\} \]
\[ + Dw(t_1)\{s_{31}(X_i(t_1) - X_C) + s_{32}(Y_i(t_1) - Y_C) + s_{33}(Z_i(t_1) - Z_C)\} \]
\[ + Dw'(t_1)\{s_{31}(X_i'(t_1)) + s_{32}(Y_i'(t_1)) + s_{33}(Z_i'(t_1))\} \]

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\[
+2f\left(\frac{t_1 + t_2}{2}\right)^{\frac{1}{2}} \left[ Du\left(\frac{t_1 + t_2}{2}\right)\{s_{11}(X_i(\frac{t_1 + t_2}{2}) - X_C)\} + s_{12}(Y_i(\frac{t_1 + t_2}{2}) - Y_C) + s_{13}(Z_i(\frac{t_1 + t_2}{2}) - Z_C)\right]
+Du'(\frac{t_1 + t_2}{2})\{s_{11}(X'_i(\frac{t_1 + t_2}{2}) + s_{12}(Y'_i(\frac{t_1 + t_2}{2}) + s_{13}(Z'_i(\frac{t_1 + t_2}{2})\right]
+Du(\frac{t_1 + t_2}{2})\{s_{21}(X_i(\frac{t_1 + t_2}{2}) - X_C)\} + s_{22}(Y_i(\frac{t_1 + t_2}{2}) - Y_C) + s_{23}(Z_i(\frac{t_1 + t_2}{2}) - Z_C)\right]
+Du'(\frac{t_1 + t_2}{2})\{s_{21}(X'_i(\frac{t_1 + t_2}{2}) + s_{22}(Y'_i(\frac{t_1 + t_2}{2}) + s_{23}(Z'_i(\frac{t_1 + t_2}{2})\right]
+Du(\frac{t_1 + t_2}{2})\{s_{31}(X_i(\frac{t_1 + t_2}{2}) - X_C)\} + s_{32}(Y_i(\frac{t_1 + t_2}{2}) - Y_C) + s_{33}(Z_i(\frac{t_1 + t_2}{2}) - Z_C)\right]
+Du'(\frac{t_1 + t_2}{2})\{s_{31}(X'_i(\frac{t_1 + t_2}{2}) + s_{32}(Y'_i(\frac{t_1 + t_2}{2}) + s_{33}(Z'_i(\frac{t_1 + t_2}{2})\right]
\]

\[
\frac{1}{2} f(t_2)^{-\frac{1}{2}} \{Du(t_2)\{s_{11}(X_i(t_2) - X_C)\} + s_{12}(Y_i(t_2) - Y_C) + s_{13}(Z_i(t_2) - Z_C)\}
+Du(t_2)\{s_{21}(X_i(t_2) - X_C)\} + s_{22}(Y_i(t_2) - Y_C) + s_{23}(Z_i(t_2) - Z_C)\}
+Du'(t_2)\{s_{21}(X'_i(t_2)) + s_{22}(Y'_i(t_2)) + s_{23}(Z'_i(t_2))\}
+Du(t_2)\{s_{31}(X_i(t_2) - X_C)\} + s_{32}(Y_i(t_2) - Y_C) + s_{33}(Z_i(t_2) - Z_C)\}
+Du'(t_2)\{s_{31}(X'_i(t_2)) + s_{32}(Y'_i(t_2)) + s_{33}(Z'_i(t_2))\}
\]

\[
A_6 = \frac{t_2 - t_1}{6} \left[ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \{Du(t_1)\{r_{21}(X_i(t_1) - X_C)\} + r_{22}(Y_i(t_1) - Y_C) + r_{23}(Z_i(t_1) - Z_C)\}
+Du'(t_1)\{r_{21}(X'_i(t_1)) + r_{22}(Y'_i(t_1)) + r_{23}(Z'_i(t_1))\}\right]
+Du(t_1)\{-r_{11}(X_i(t_1) - X_C)\} - r_{12}(Y_i(t_1) - Y_C) - r_{13}(Z_i(t_1) - Z_C)\}
+Du'(t_1)\{-r_{11}(X'_i(t_1)) - r_{12}(Y'_i(t_1)) - r_{13}(Z'_i(t_1))\}
+2f\left(\frac{t_1 + t_2}{2}\right)^{\frac{1}{2}} \left[ Du\left(\frac{t_1 + t_2}{2}\right)\{r_{21}(X_i(\frac{t_1 + t_2}{2}) - X_C)\} + r_{22}(Y_i(\frac{t_1 + t_2}{2}) - Y_C) + r_{23}(Z_i(\frac{t_1 + t_2}{2}) - Z_C)\right]
\]

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\[ A_7 = \frac{t_2 - t_1}{6} \left[ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \{ Du(t_1) r_{11} + Dv(t_1) r_{21} + Dw(t_1) r_{31} \} \right. \\
+2f \left( \frac{t_1 + t_2}{2} \right)^{-\frac{1}{2}} \left\{ Du \left( \frac{t_1 + t_2}{2} \right) r_{11} + Dv \left( \frac{t_1 + t_2}{2} \right) r_{21} + Dw \left( \frac{t_1 + t_2}{2} \right) r_{31} \right\} \\
\left. + \frac{1}{2} f(t_2)^{-\frac{1}{2}} \{ Du(t_2) r_{11} + Dv(t_2) r_{21} + Dw(t_2) r_{31} \} \right] \\
A_8 = \frac{t_2 - t_1}{6} \left[ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \{ Du(t_1) r_{11} + Dv(t_1) r_{21} + Dw(t_1) r_{31} \} \right. \\
+2f \left( \frac{t_1 + t_2}{2} \right)^{-\frac{1}{2}} \left\{ Du \left( \frac{t_1 + t_2}{2} \right) r_{11} + Dv \left( \frac{t_1 + t_2}{2} \right) r_{21} + Dw \left( \frac{t_1 + t_2}{2} \right) r_{31} \right\} \\
\left. + \frac{1}{2} f(t_2)^{-\frac{1}{2}} \{ Du(t_2) r_{11} + Dv(t_2) r_{21} + Dw(t_2) r_{31} \} \right] \\
A_9 = \frac{t_2 - t_1}{6} \left[ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \{ Du(t_1) r_{11} t^2 + Dv(t_1) r_{21} t^2 + Dw(t_1) r_{31} t^2 \} \right. \\
+Du(t_1) 2r_{11} t + Dv(t_1) 2r_{21} t + Dw(t_1) 2r_{31} t \\
\left. +2f \left( \frac{t_1 + t_2}{2} \right)^{-\frac{1}{2}} \left\{ Du \left( \frac{t_1 + t_2}{2} \right) r_{11} t^2 + Dv \left( \frac{t_1 + t_2}{2} \right) r_{21} t^2 \right\} \\
+ Dw \left( \frac{t_1 + t_2}{2} \right) r_{31} t^2 \right] \]
\[ A_{10} = \frac{t_2 - t_1}{6} \left[ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \left\{ Du(t_1)r_{11}t^3 + Du(t_1)r_{21}t^3 + Dw(t_1)r_{31}t^3 \right\} + \frac{1}{2} f(t_2)^{-\frac{1}{2}} \left\{ Du(t_2)r_{11}t^3 + Du(t_2)r_{21}t^3 + Dw(t_2)r_{31}t^3 \right\} + Du'(t_1)3r_{11}t^2 + Du'(t_1)3r_{21}t^2 + Du'(t_1)3r_{31}t^2 \right] + 2f \left( \frac{t_1 + t_2}{2} \right)^{-\frac{1}{2}} \left\{ Du\left( \frac{t_1 + t_2}{2} \right)r_{11}t^3 + Du\left( \frac{t_1 + t_2}{2} \right)r_{21}t^3 + Dw\left( \frac{t_1 + t_2}{2} \right)r_{31}t^3 \right\} + Dw\left( \frac{t_1 + t_2}{2} \right)r_{31}t^3 + Du'\left( \frac{t_1 + t_2}{2} \right)3r_{11}t^2 + Du'\left( \frac{t_1 + t_2}{2} \right)3r_{21}t^2 + Du'\left( \frac{t_1 + t_2}{2} \right)3r_{31}t^2 \right] \]

\[ A_{11} = \frac{t_2 - t_1}{6} \left[ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \left\{ Du(t_1)r_{12} + Du(t_1)r_{22} + Dw(t_1)r_{32} \right\} + 2f \left( \frac{t_1 + t_2}{2} \right)^{-\frac{1}{2}} \left\{ Du\left( \frac{t_1 + t_2}{2} \right)r_{12} + Du\left( \frac{t_1 + t_2}{2} \right)r_{22} + Dw\left( \frac{t_1 + t_2}{2} \right)r_{32} \right\} + \frac{1}{2} f(t_2)^{-\frac{1}{2}} \left\{ +Du(t_2)r_{12} + Du(t_2)r_{22} + Dw(t_2)r_{32} \right\} \right] \]

\[ A_{12} = \frac{t_2 - t_1}{6} \left[ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \left\{ Du(t_1)r_{12}t + Du(t_1)r_{22}t + Dw(t_1)r_{32}t \right\} \right. \]

\[ Du'\left( t_1 \right)r_{12} + Du'\left( t_1 \right)r_{22} + Dw'\left( t_1 \right)r_{32} \]

\[ + 2f \left( \frac{t_1 + t_2}{2} \right)^{-\frac{1}{2}} \left\{ Du\left( \frac{t_1 + t_2}{2} \right)r_{12}t + Du\left( \frac{t_1 + t_2}{2} \right)r_{22}t + Dw\left( \frac{t_1 + t_2}{2} \right)r_{32}t \right\} + Du'\left( \frac{t_1 + t_2}{2} \right)r_{12} + Du'\left( \frac{t_1 + t_2}{2} \right)r_{22} + Dw'\left( \frac{t_1 + t_2}{2} \right)r_{32} \right\} \]

\[ + \left. \frac{1}{2} f(t_2)^{-\frac{1}{2}} \left\{ +Du(t_2)r_{12}t + Du(t_2)r_{22}t + Dw(t_2)r_{32}t \right\} + Du'(t_2)r_{12} + Du'(t_2)r_{22} + Dw'(t_2)r_{32} \right\} \]

\[ A_{13} = \frac{t_2 - t_1}{6} \left[ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \left\{ Du(t_1)r_{12}t^2 + Du(t_1)r_{22}t^2 + Dw(t_1)r_{32}t^2 \right\} + Du(t_1)2r_{12}t + Du(t_1)2r_{22}t + Dw(t_1)2r_{32}t \right] \]
\[ A_{14} = \frac{t_2 - t_1}{6} \left[ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \left\{ Du(t_1)_{12}t^3 + Dv(t_1)_{22}t^3 + Dw(t_1)_{32}t^3 \right\} \\
+ Du'(t_1)_{32}t^2 + Dv'(t_1)_{32}t^2 + Dw'(t_1)_{32}t^2 \right\} \\
+ 2f \left( \frac{t_1 + t_2}{2} \right)^{-\frac{1}{2}} \left\{ Du \left( \frac{t_1 + t_2}{2} \right)_{12}t^3 + Dv \left( \frac{t_1 + t_2}{2} \right)_{22}t^3 + Dw \left( \frac{t_1 + t_2}{2} \right)_{32}t^3 \right\} \\
+ Du' \left( \frac{t_1 + t_2}{2} \right)_{32}t^2 + Dv' \left( \frac{t_1 + t_2}{2} \right)_{32}t^2 + Dw' \left( \frac{t_1 + t_2}{2} \right)_{32}t^2 \right\} \\
+ \frac{1}{2} f(t_2)^{-\frac{1}{2}} \left\{ Du(t_2)_{12}t^3 + Dv(t_2)_{22}t^3 + Dw(t_2)_{32}t^3 \right\} \\
+ Du'(t_2)_{32}t^2 + Dv'(t_2)_{32}t^2 + Dw'(t_2)_{32}t^2 \right] \]

\[ A_{15} = \frac{t_2 - t_1}{6} \left[ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \left\{ Du(t_1)_{13} + Dv(t_1)_{23} + Dw(t_1)_{33} \right\} \\
+ 2f \left( \frac{t_1 + t_2}{2} \right)^{-\frac{1}{2}} \left\{ Du \left( \frac{t_1 + t_2}{2} \right)_{13} + Dv \left( \frac{t_1 + t_2}{2} \right)_{23} + Dw \left( \frac{t_1 + t_2}{2} \right)_{33} \right\} \\
+ \frac{1}{2} f(t_2)^{-\frac{1}{2}} \left\{ Du(t_2)_{13} + Dv(t_2)_{23} + Dw(t_2)_{33} \right\} \right] \]

\[ A_{16} = \frac{t_2 - t_1}{6} \left[ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \left\{ Du(t_1)_{13}t + Dv(t_1)_{23}t + Dw(t_1)_{33}t \right\} \\
Du'(t_1)_{13} + Dv'(t_1)_{23} + Dw'(t_1)_{33} \right\} \\
+ 2f \left( \frac{t_1 + t_2}{2} \right)^{-\frac{1}{2}} \left\{ Du \left( \frac{t_1 + t_2}{2} \right)_{13}t + Dv \left( \frac{t_1 + t_2}{2} \right)_{23}t + Dw \left( \frac{t_1 + t_2}{2} \right)_{33}t \right\} \\
+ Du' \left( \frac{t_1 + t_2}{2} \right)_{13} + Dv' \left( \frac{t_1 + t_2}{2} \right)_{23} + Dw' \left( \frac{t_1 + t_2}{2} \right)_{33} \right\} \\
+ \frac{1}{2} f(t_2)^{-\frac{1}{2}} \left\{ Du(t_2)_{13}t + Dv(t_2)_{23}t + Dw(t_2)_{33}t \right\} \\
+ Du'(t_2)_{13} + Dv'(t_2)_{23} + Dw'(t_2)_{33} \right] \]
\[ A_{17} = \frac{t_2 - t_1}{6} \left\{ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \{ Du(t_1)r_{13}t^2 + Dv(t_1)r_{23}t^2 + Dw(t_1)r_{33}t^2 \\
+ Du'(t_1)2r_{13}t + Dv'(t_1)2r_{23}t + Dw'(t_1)2r_{33}t \} \\
+ 2f \left( \frac{t_1 + t_2}{2} \right)^{-\frac{1}{2}} \left\{ Du\left( \frac{t_1 + t_2}{2} \right)r_{13}t^2 + Dv\left( \frac{t_1 + t_2}{2} \right)r_{23}t^2 \\
+ Dw\left( \frac{t_1 + t_2}{2} \right)r_{33}t^2 \right\} \right\} \]

\[ A_{18} = \frac{t_2 - t_1}{6} \left\{ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \{ Du(t_1)r_{13}t^3 + Dv(t_1)r_{23}t^3 + Dw(t_1)r_{33}t^3 \\
+ Du'(t_1)3r_{13}t^2 + Dv'(t_1)3r_{23}t^2 + Dw'(t_1)3r_{33}t^2 \} \\
+ 2f \left( \frac{t_1 + t_2}{2} \right)^{-\frac{1}{2}} \left\{ Du\left( \frac{t_1 + t_2}{2} \right)r_{13}t^3 + Dv\left( \frac{t_1 + t_2}{2} \right)r_{23}t^3 \\
+ Dw\left( \frac{t_1 + t_2}{2} \right)r_{33}t^3 \right\} \right\} \]

\[ A_{19} = \frac{t_2 - t_1}{6} \left\{ \frac{1}{2} f(t_1)^{-\frac{1}{2}} \{ Du(t_1)r_{11}X'_i(t_1) + Dv(t_1)r_{12}Y'_i(t_1) + Dw(t_1)r_{13}Z'_i(t_1) \\
+ Du'(t_1)r_{11}X''_i(t_1) + Dv'(t_1)r_{23}Y''_i(t_1) + Dw'(t_1)r_{33}Z''_i(t_1) \} \\
+ 2f \left( \frac{t_1 + t_2}{2} \right)^{-\frac{1}{2}} \left\{ Du\left( \frac{t_1 + t_2}{2} \right)r_{11}X'_i\left( \frac{t_1 + t_2}{2} \right) + Dv\left( \frac{t_1 + t_2}{2} \right)r_{23}Y'_i\left( \frac{t_1 + t_2}{2} \right) \\
+ Dw\left( \frac{t_1 + t_2}{2} \right)r_{33}Z'_i\left( \frac{t_1 + t_2}{2} \right) \right\} \right\} \]

\[ A_{20} = \frac{t_2 - t_1}{6} \left\{ \frac{1}{2} f(t_2)^{-\frac{1}{2}} \{ Du(t_2)r_{11}X'_i(t_2) + Dv(t_2)r_{12}Y'_i(t_2) + Dw(t_2)r_{13}Z'_i(t_2) \\
+ Du'(t_2)r_{11}X''_i(t_2) + Dv'(t_2)r_{23}Y''_i(t_2) + Dw'(t_2)r_{33}Z''_i(t_2) \} \right\} \]
A.3 Derivation of the tangent of a spline

The partial derivatives of symbolic representation of the tangent of the spline between image and object space are illustrated as follows.

\[ \begin{align*}
L_1 &= -\frac{w'(v'w - w'v)}{(u'w - w'u)^2} r_{11} + \frac{w'}{(u'w - w'u)} r_{12} + \frac{w'}{(u'w - w'u)^2} r_{31} \\
L_2 &= -\frac{w'(v'w - w'v)}{(u'w - w'u)^2} r_{12} + \frac{w'}{(u'w - w'u)} r_{22} - \frac{w'(u'v - v'u)}{(u'w - w'u)^2} r_{32} \\
L_3 &= -\frac{w'(v'w - w'v)}{(u'w - w'u)^2} r_{13} + \frac{w'}{(u'w - w'u)} r_{23} - \frac{w'(u'v - v'u)}{(u'w - w'u)^2} r_{33} \\
L_4 &= \left\{ \frac{(v'w - w'v)}{(u'w - w'u)^2} \{w'[r_{12}(Z_i(t) - Z_C) - r_{13}(Y_i(t) - Y_C)] - w[r_{12}(Z_i(t)) - r_{13}(Y_i(t))] \} \right\} \\
&\quad - \frac{1}{u'w - w'u} \{w'[r_{22}(Z_i(t) - Z_C) - r_{23}(Y_i(t) - Y_C)] - w[r_{22}(Z_i(t)) - r_{23}(Y_i(t))] \} \\
&\quad + \frac{1}{(u'w - w'u)} \{[v' - \frac{u'(v'w - v'w)}{u'w - w'u}][r_{32}(Z_i(t) - Z_C) - r_{33}(Y_i(t) - Y_C)] - [v - \frac{u'(v'w - v'w)}{u'w - w'u}][r_{32}(Z_i'(t) - Z_C) - r_{33}(Y_i'(t))] \} \\
L_5 &= \left\{ \frac{(v'w - w'v)}{(u'w - w'u)^2} \{w'[s_{11}(X_i(t) - X_C) + s_{12}(Y_i(t) - Y_C) + s_{13}(Z_i(t) - Z_C)] - w[s_{11}(X_i(t)) + s_{12}(Y_i(t)) + s_{13}(Z_i(t))] \} \right\}
\end{align*} \]
\[-\frac{1}{w'w - w'u}\{w'[s_{21}(X_i(t) - X_C) + s_{22}(Y_i(t) - Y_C) + s_{23}(Z_i(t) - Z_C)]
- \{s_{21}(Y_i'(t)) + s_{22}(Y_i(t)) + s_{23}(Z_i(t))\}
\]
\[+ \frac{1}{(u'w - w'u)}\{[v' - u'(v'w - v'w)]\{s_{31}(X_i(t) - X_C) + s_{32}(Y_i(t) - Y_C)
+ s_{33}(Z_i(t) - Z_C)\}
- [v - u(v'w - v'w)]\{[s_{31}(X_i'(t)) + s_{32}(Y_i(t)) + s_{33}(Z_i(t))}\}\]
\[L_6 = \frac{(v'w - w'v)}{(u'w - w'u)^2}\{w'[r_{21}(X_i(t) - X_C) + r_{22}(Y_i(t) - Y_C) + r_{23}(Z_i(t) - Z_C)]
- w[r_{21}(X_i'(t)) + r_{22}(Y_i'(t)) + r_{23}(Z_i'(t))]\}
- \frac{1}{w'w - w'u}\{w'[-r_{11}(X_i(t) - X_C) - r_{12}(Y_i(t) - Y_C) - r_{13}(Z_i(t) - Z_C)]
- wr_{11}(X_i'(t)) - r_{12}(Y_i'(t)) - r_{13}(Z_i'(t))\}\]
\[L_7 = \frac{w'(v'w - w'v)}{(u'w - w'u)^2}r_{11} - \frac{w'}{u'w - w'u}r_{21} + \frac{v'(u'w - w'u) - u'(v'w - v'w)}{(u'w - w'u)^2}r_{31}\]
\[L_8 = \frac{(v'w - w'v)}{(u'w - w'u)^2}\{w'r_{11}t - w_{r11}\} - \frac{1}{u'w - w'u}\{w'r_{21}t - w_{r21}\}
+ \frac{v'(u'w - w'u) - u'(v'w - v'w)}{(u'w - w'u)^2}r_{31}t - \frac{v(u'w - w'u) - u(v'w - v'w)}{(u'w - w'u)^2}r_{31}\]
\[L_9 = \frac{(v'w - w'v)}{(u'w - w'u)^2}\{w'r_{11}t^2 - 2wr_{11}t\} - \frac{1}{u'w - w'u}\{w'r_{21}t^2 - 2wr_{21}t\}
+ \frac{v'(u'w - w'u) - u'(v'w - v'w)}{(u'w - w'u)^2}r_{31}t^2 - \frac{v(u'w - w'u) - u(v'w - v'w)}{(u'w - w'u)^2}r_{31}t\]
\[L_{10} = \frac{(v'w - w'v)}{(u'w - w'u)^2}\{w'r_{11}t^3 - 3wr_{11}t^2\} - \frac{1}{u'w - w'u}\{w'r_{21}t^3 - 3wr_{21}t^2\}
+ \frac{v'(u'w - w'u) - u'(v'w - v'w)}{(u'w - w'u)^2}r_{31}t^3 - \frac{v(u'w - w'u) - u(v'w - v'w)}{(u'w - w'u)^2}r_{31}t\]
\[L_{11} = \frac{w'(v'w - w'v)}{(u'w - w'u)^2}r_{12} - \frac{w'}{u'w - w'u}r_{22} + \frac{v'(u'w - w'u) - u'(v'w - v'w)}{(u'w - w'u)^2}r_{32}\]
\[L_{12} = \frac{(v'w - w'v)}{(u'w - w'u)^2}\{w'r_{12}t - wr_{12}\} - \frac{1}{u'w - w'u}\{w'r_{22}t - wr_{22}\}
+ \frac{v'(u'w - w'u) - u'(v'w - v'w)}{(u'w - w'u)^2}r_{32}t - \frac{v(u'w - w'u) - u(v'w - v'w)}{(u'w - w'u)^2}r_{32}\]
\[L_{13} = \frac{(v'w - w'v)}{(u'w - w'u)^2}\{w'r_{12}t^2 - 2wr_{12}t\} - \frac{1}{u'w - w'u}\{w'r_{22}t^2 - 2wr_{22}t\}\]
\[ L_{14} = \frac{\mathbf{v}'(u'w - w'u) - u'(v'w - v'w)}{(u'w - w'u)^2} r_{32} t^2 - \frac{\mathbf{v}(u'w - w'u) - u(v'w - vw')}{(u'w - w'u)^2} 2r_{32} t \]

\[ L_{15} = \frac{\mathbf{v}'(v'w - w'v)}{(u'w - w'u)^2} r_{13} - \frac{\mathbf{v}'(u'w - w'u) - u'(v'w - v'w)}{(u'w - w'u)^2} r_{33} \]

\[ L_{16} = \frac{\mathbf{v}'(v'w - w'v)}{(u'w - w'u)^2} r_{13} - \frac{\mathbf{v}(u'w - w'u) - u(v'w - vw')}{(u'w - w'u)^2} r_{33} \]

\[ L_{17} = \frac{\mathbf{v}'(v'w - w'v)}{(u'w - w'u)^2} r_{13} - \frac{\mathbf{v}(u'w - w'u) - u(v'w - vw')}{(u'w - w'u)^2} r_{33} \]

\[ L_{18} = \frac{\mathbf{v}'(v'w - w'v)}{(u'w - w'u)^2} r_{13} - \frac{\mathbf{v}(u'w - w'u) - u(v'w - vw')}{(u'w - w'u)^2} r_{33} \]

\[ L_{19} = \frac{\mathbf{v}'(v'w - w'v)}{(u'w - w'u)^2} \{w[r_{11}X'_i(t) + r_{12}Y'_i(t) + r_{13}Z'_i(t)]
- w[r_{11}X''_i(t) + r_{12}Y''_i(t) + r_{13}Z''_i(t)]
- \frac{1}{u'w - w'u} \{w[r_{21}X'_i(t) + r_{22}Y'_i(t) + r_{23}Z'_i(t)]
- w[r_{21}X''_i(t) + r_{22}Y''_i(t) + r_{23}Z''_i(t)]
+ \frac{\mathbf{v}'(u'w - w'u) - u'(v'w - v'w)}{(u'w - w'u)^2} \{w[r_{31}X'_i(t) + r_{32}Y'_i(t) + r_{33}Z'_i(t)]
- w[r_{31}X''_i(t) + r_{32}Y''_i(t) + r_{33}Z''_i(t)]\}
- \frac{\mathbf{v}(u'w - w'u) - u(v'w - vw')}{(u'w - w'u)^2} - w[r_{31}X''_i(t) + r_{32}Y''_i(t) + r_{33}Z''_i(t)]\} \]


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