A MODERN PRESENTATION OF
“DIMENSION AND OUTER MEASURE”

THESES

Presented in Partial Fulfillment of the Requirements for
the Degree Master of Science in the Graduate
School of the Ohio State University

By

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ABSTRACT

Hausdorff dimension is a modern mathematical tool used to classify metric spaces. Presented here is a modern treatment of some of the content of “Dimension and Outer measure,” Hausdorff’s original publication defining what we now call Hausdorff dimension. Contained, also, is a construction of a linear set of given dimension.
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I would like to thank all of my instructors, colleagues, and friends at Ohio State.
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# TABLE OF CONTENTS

Abstract ......................................................................................................................... ii  
Acknowledgments ........................................................................................................ iii 
Vita ................................................................................................................................. iv  
List of Figures ................................................................................................................ vi  

## CHAPTER  

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2.1</td>
<td>3</td>
</tr>
<tr>
<td>2.2</td>
<td>4</td>
</tr>
<tr>
<td>2.3</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>3.1</td>
<td>7</td>
</tr>
<tr>
<td>3.2</td>
<td>7</td>
</tr>
<tr>
<td>3.3</td>
<td>8</td>
</tr>
<tr>
<td>3.4</td>
<td>10</td>
</tr>
<tr>
<td>3.5</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>4.1</td>
<td>12</td>
</tr>
<tr>
<td>4.2</td>
<td>13</td>
</tr>
<tr>
<td>4.3</td>
<td>20</td>
</tr>
</tbody>
</table>

Bibliography ................................................................................................................ 22
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Example with $n = 2$</td>
<td>14</td>
</tr>
<tr>
<td>4.2</td>
<td>$I_2$</td>
<td>15</td>
</tr>
<tr>
<td>4.3</td>
<td>Example with $n = 2$ and $k = 2$</td>
<td>16</td>
</tr>
</tbody>
</table>
CHAPTER 1
INTRODUCTION

Felix Hausdorff (1868-1942) was a German mathematician, philosopher and author. Hausdorff studied at Leipzig University under Heinrich Bruns and Adolph Mayer, graduating in 1891 with a doctorate in applications of mathematics to astronomy. J J O’Connor and E F Robertson wrote an excellent biography of Hausdorff, and it is available at http://www-history.mcs.st-andrews.ac.uk/Biographies/Hausdorff.html. In 1918 Hausdorff published *Dimension und äußeres Maß* (Dimension and Outer Measure) as a response to Constantin Carathéodory’s 1914 paper *Über das lineare Maß von Punkt mengen—eine Verallgemeinerung des Längenbegriffs* (On the linear Measure of Point Sets—a Generalization of the Concept of length). Carathéodory’s paper is quite readable and nearly represents a modern treatment of the axioms and basic results in measure theory. Most relevant to this paper, Carathéodory discusses what he calls “Linear Measure” as a generalization to “rectifiable curves.” Carathéodory writes,

Let $A$ be an arbitrary point set in $q$-dimensional space ($q > 1$). We consider a finite or countable sequence of point sets $U_1, U_2, \ldots$, satisfying the two conditions:

1. The original set $A$ is a subset of the union of the $U_k$’s
2. The diameter $d_k$ of each $U_k$ is smaller than a given positive number $\delta$.

We consider the lower bound of the sums

$$d_1 + d_2 + \ldots$$

of the diameters, $d_k$ for all sequences $U_1, U_2, \ldots$ satisfying conditions 1) and 2).

It is this notion that Hausdorff expands on and eventually leads to the study of what we call “Hausdorff Measure” and fractals. One phrasing of the motivation is as follows. We naturally have three measures on $\mathbb{R}^3$, length, area, and volume. Given a point set in $\mathbb{R}^3$ we may use these measures to “test” the dimension of the point set. For example, consider the unit disk imbedded in $\mathbb{R}^3$ ($S = \{(x, y, z) : x^2 + y^2 = 1, z = 0\}$).

We may measure its volume as 0 and, loosely speaking, we measure the length as $\infty$. However, the area is $\pi$. We may then say the disk has “dimension” two despite being a subset of $\mathbb{R}^3$, as it has finite, nonzero two dimensional measure. Then we pose the question “is there some sensible class of measures so that there are sets that do not have integer dimension?” Hausdorff, pushed by Carathéodory, has a positive answer.

However, Hausdorff did not live in a world of rigorous Mathematics. Consequently, the actual paper *Dimension and Outer Measure* suffers from precise thinking and imprecise language. The text is readable, but there are some mysterious passages. Also, the original text has awkward notation, and a critic might call it scattered.

I attempt to alleviate some of these woes as well as modernize the language and notation.
CHAPTER 2
SUMMARY OF CARATHÉODORY’S RESULTS

2.1 Axioms and Definitions:

1. A measure is a set function $L : \mathcal{P}(\mathbb{R}^q) \to [0, \infty]$.

2. For all $B \subseteq A \in \mathcal{P}(\mathbb{R}^q)$ we have $LB \leq LA$.

3. $L(\cup A_i) \leq \sum L(A_i)$, whenever $\{A_i\} \subseteq \mathcal{P}(\mathbb{R}^q)$ is a countable or finite collection.

4. $L(A \cup B) = L(A) + L(B)$ when $\inf\{d(a,b) : a \in A, b \in B\} > 0$ where $d(a,b)$ is the Euclidean distance from $a$ to $b$.

   **Definition 1.** We say $B$ is measurable whenever $L(W) = L(B \cap W) + L(W - B \cap W)$, for any $W \in \mathcal{P}(\mathbb{R}^q)$ with $L(W) < \infty$.

5. $L(A) = \inf\{L(B) : A \subseteq B, B$ measurable$\}$.

**Remark** The above axioms are that of Hausdorff and Carathéodory, however in modern settings we add the axiom that $L(\emptyset) = 0$ to exclude the identically infinite function. Set functions of the above type are commonly referred to as **outer measures**, but in this context there is no need to define an inner measure so the term “outer” is suppressed.
2.2 Basic Results

It follows from the above axioms that for any set $A$ with $L(A) < \infty$ there is some measurable set $\tilde{A}$ with $\tilde{A} \supseteq A$ and $L(A) = L(\tilde{A})$. Carathéodory’s proof of this fact is outlined by the following theorems which are given here without proof. For the proof see any basic book on measure theory.

Remark Theorems 1-5 rely on axioms 1-3 alone.

Theorem 1. The complement of a measurable point set is measurable.

Theorem 2. The union or intersection of finitely or countably many measurable point sets is again measurable.

Let $A = \{A_i\} \subseteq \mathcal{P}(\mathbb{R}^q)$, be a countable sequence of measurable point sets. Let $S_n = \bigcup_{i=n}^\infty A_i$ and $s_n = \bigcap_{i=n}^\infty A_i$. We write $S = \lim S_n$ and $s = \lim s_n$ for the limit superior and limit inferior of the sequence $A_i$ respectively.

Theorem 3. $S$ and $s$ are measurable.

Theorem 4. Suppose $A$ is pairwise disjoint. Then $L(\bigcup A_i) = \sum L(A_i)$

Theorem 5. Suppose $A_i$ is monotonic (with respect to containment), then $L(\bigcup A_i) = \lim L(A_i)$ when $A_i$ is increasing and $L(\bigcap A_i) = \lim L(A_i)$ when $A_i$ is decreasing.

Remark We now add axiom 4. By an interval in $\mathbb{R}^n$ we mean a set of the form

$$\prod_{j=1}^n I_j$$

Where $I_j$ is an interval in $\mathbb{R}$. Borel sets are the sets that can be constructed from intervals by repeatedly taking countable unions and intersections.
Theorem 6. *Intervals are measurable, and hence Borel sets are measurable.*

**Remark** We now add axiom 5.

Theorem 7. *For any set $A$ with $L(A) < \infty$ there is some measurable set $\overline{A}$ with $\overline{A} \supseteq A$ and $L(A) = L(\overline{A})$.***

### 2.3 Linear Measure

Now, Carathéodory generalizes the notion of a rectifiable curve. Consider some point set $A \in \mathcal{P}(\mathbb{R}^q)$ and some countable collection $U_i$ and a fixed $\delta$ such that,

1. $A \subseteq \bigcup U_i$

2. For each $i$, $\text{diam}(U_i) := \sup\{d(x, y) : x, y \in U\} < \delta$, where $d(x, y)$ is the Euclidian distance from $x$ to $y$ in $\mathbb{R}^q$.

Where the sets $U_i$ take the place of "inscribed polygons" in the definition of a rectifiable curve.

**Remark** For convenience we write $d$ for the set function diam, and we say $U_i$ is a $\delta$-cover of $A$ when 1 and 2 above are satisfied.
Definition 2. Let $H_{\delta,d}(A) = \inf \Sigma d(U_i)$ where the infimum is taken over all open covers of $A$ with diameters less than $\delta$, and define:

$$H_d(A) := \lim_{\delta \to 0} H_{\delta,d}(A)$$

We say $H_d$ is the measure associated with, or induced by, the set function $d$.

Theorem 8. The set function $H_d$ is a measure.
3.1 Dimension and Orderings of Measures

Hausdorff “restrict(s) our study to measures that are identical for congruent sets.”

With this compatibility condition we arrive at the naive notion of a $p$-dimensional measure, $L$, namely, when $L(A) = r^pL(B)$ when $A$ is similar to $B$ of proportion $r$.

Now, suppose we are given measures $L$ and $M$. We say $L$ and $M$ are of the same order when there exists constants $h, k$ so that

$$hL(A) \leq M(A) \leq kL(A)$$

Equivalently, $L$ and $M$ are simultaneously zero, positive, or infinite. We say $L$ is of lower order when, for any positive epsilon,

$$M(A) \leq \varepsilon L(A)$$

Equivalently, $M$ is zero whenever $L$ is finite.

3.2 Orderings and Systems of Sets

Notice that the only necessary restriction on the function $d$ in definition 2 is that it is positive. Suppose collections $\{U_i\}$ and $\{V_j\}$ are given with arbitrarily small
diameters, along with functions

\[ l : \{U_i\} \rightarrow [0, \infty) \]

\[ m : \{V_j\} \rightarrow [0, \infty) \]

Now, as in definition 2, let \( L \) be the measure associated with \( l \) and \( M \) the measure associated with \( m \). We seek conditions on the systems \( \{U_i\} \) and \( \{V_j\} \) so that we may deduce the ordering on \( L \) and \( M \). To this end, assume there are constants \( h \) and \( k \) so that for every \( i \) and for every \( \varepsilon > 0 \) there is some \( j \) with \( U_i \subseteq V_j \) so that

\[ d(V_j) < rd(U_i) + \varepsilon \]

and

\[ m(V_j) < kl(U_i) + \varepsilon. \]

Thus for any \( \delta \),

\[ M_{r\delta}(A) \leq kL_{\delta}(A). \]

So that,

\[ M(A) \leq kL(A). \]

In particular, if \( \{U_i\} \) is a subsystem of \( \{V_j\} \) and \( l = m \) we have,

\[ M(A) \leq L(A). \]

### 3.3 Continuous and Closeable

Hausdorff finds sufficient conditions on the function \( l \) and the collection \( \{U_i\} \) so that axiom 5 holds. Write \( \overline{U} \) for the closure of \( U \), and write \( U_\delta = \{x : |x - u| < \delta, \text{for some } u \in U\} \)
**Definition 3.** \( l \) is closeable iff \( l(U) = l(U) \)

**Definition 4.** \( l \) is continuous iff for every \( \varepsilon > 0 \) there is some \( \delta \) such that \( l(U_\delta) < l(U) + \varepsilon \)

Then \( d \) is continuous and closeable. Now, suppose \( l = m \), and

\[ \{V_j\} = \{\overline{U}_i\} \]

or,

\[ \{V_j\} = \{(U_i)_\delta : \delta > 0\} \]

In either case, for every \( \varepsilon > 0 \) and \( i \) there is some \( j \) so that

\[ d(V_j) < d(U_i) + \varepsilon \]

and

\[ m(V_j) < l(U_i) + \varepsilon \]

Hence,

\[ M(A) \leq L(A) \]

Now, if \( \{V_j\} \) is a sub-collection of \( \{U_i\} \) we have \( M(A) = L(A) \). But, in either case \( \{V_j\} \) is a collection of measurable sets. Thus,

**Theorem 9.** If \( l \) is continuous or closeable then \( L \) satisfies axiom 5
3.4 Examples

There are many natural functions that may replace $d$. Hausdorff chooses some of the following functions, that depend on the set $A \in \mathcal{P}(\mathbb{R}^q)$, as examples:

1. The upper bound of the surface area of triangles whose vertices lie in $A$ (when $q = 2$).

2. The lower bound of the “elementary geometry volume” of $q$-dimensional cubes (or spheres or parallelepipeds) that contain $A$.

3. The $q$-dimensional volume (in the Lebesgue or Jordan sense) of the smallest convex set containing $A$.

4. The upper bound of the $p$-dimensional volume of the orthogonal projection of $A$ onto a $p$-dimensional plane. ($p \leq q$)

3.5 Hausdorff Dimension

Definition 5. Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous, monotonically increasing in some neighborhood of zero, $\lim_{x \to \infty} f(x) = \infty$ and that $f(0) = 0$. We may define a measure,

$$H_f(A) = \lim_{\delta \to 0} H_{\delta,f}(A) := \lim_{\delta \to 0} \inf \{ \sum f(d(U_i)) \},$$

where $U_i$ is a $\delta$ - cover of $A$ and the infimum is taken over all possible $\delta$ - covers.

Definition 6. We partition the collection of all such functions into classes $[f]$ by the relation by $f \sim g$ iff $H_f$ has the same order as $H_g$. 

10
Definition 7. If $0 < H_f(A) < \infty$ we say the Hausdorff dimension of $A$ is $[f]$, and then dimension inherits the ordering on their measures described in section 3.1.

Theorem 10. when $f(x) = x^p$ for $p \in \mathbb{N}$ $H_f$ has the same order as $p$-dimensional Lebesgue measure

Remark Hausdorff dimension may be constructed in a more simplistic way only using functions of the type $f(x) = x^p$ instead of considering all continuous, monotonically increasing in some neighborhood of zero, functions. However, it is worth noting that Hausdorff worked with the latter level of generality. Also, for convenience, when $f(x) = x^p$ we write $[p]$ for the dimension $[f]$.

Example 1. The traditional “middle thirds” Cantor set may be covered by $2^n$ intervals of length $\frac{1}{3^n}$ and (as we shall see) has Hausdorff dimension $\left[\frac{\log 2}{\log 3}\right]$. 

11
4.1 Preliminaries to Linear Sets of a Given Dimension

We restrict our study to function \( f \) with the following properties:

- \( f : [0, \infty) \rightarrow [0, \infty) \) is increasing,
  \[
  \begin{array}{ccc}
  f(x_1) & x_1 & 1 \\
  f(x_2) & x_2 & 1 \\
  f(x_3) & x_3 & 1 \\
  \end{array}
  \]
  \(|f(x_1) - x_1| > 0 \) when \( x_1 < x_2 < x_3 \),

- \( f(x) \rightarrow 0 \) as \( x \rightarrow 0 \),

- \( f(x) \rightarrow \infty \) as \( x \rightarrow \infty \).

From these assumptions we conclude,

1. \( f \) is convex (that is, \(-f(x)\) is concave),

2. When \( x_1 < x_2 < x_3 \) we have \( f(x_1 - x_2 + x_3) > f(x_1) - f(x_2) + f(x_3) \),

3. For \( \alpha > 1 \) we have \( \alpha f(x) > f(\alpha x) \),

4. \( f(x_1 + x_2) < f(x_1) + f(x_2) \).

The proof of these facts is left to the reader.
4.2 Linear Sets of a Given Dimension

Hausdorff writes,

Now that (Theorem 8) has justified its (Hausdorff dimension) existence at least to some extent, it is tempting to extend it to non-integer positive values of \( p \). The only question is whether this is trivial, in the sense that all sets have outer measure 0 or \( \infty \). The problem therefore arises to construct sets \( A \) for which \( 0 < L_p(A) < \infty \).

For convenience, we use \( H \) instead of \( L \) and \( I \) instead of \( A \), since Hausdorff seems to have no reservations about using the same letter for different objects. We will consider the problem for an arbitrary function, \( f \), satisfying the assumptions in section 4.1 add the constraint that \( Hf \) is of order less than \([1]\) so that \( A \subseteq \mathbb{R} \). This is a good problem. Hausdorff’s solution is outlined in this section. Suppose some \([f]\) is given. Define \( x_0 \) so that \( f(x_0) = 1 \) and define \( x_n \) so that \( f(x_n) = 1/2^n \) (This is possible since \( \lim_{x \to 0} f(x) = 0 \) and \( x_n \) is well defined since \( f \) is monotone). By the properties of section 4.1,

\[
f(x_{n+1}) = 2f(x_n) > f(2x_n)
\]

so that

\[
x_n > 2x_{n+1}.
\]
Definition 8. Let $I = [a, b]$ be an interval. By the central open interval of length $l$ we mean:

\[
\left( \frac{1}{2}(a + b - l), \frac{1}{2}(a + b + l) \right).
\]

As in the construction of the Cantor set, we will construct a chain of nested intervals by recursively removing subintervals, and consider the intersection of the “remaining pieces”. The “remaining pieces” will be denoted by the letter $I$ and the “removed pieces” by the letter $J$ with end points $a$ and $b$. The intent of the construction is that the “remaining pieces” at the $n^{th}$ stage will have length $x_n$ (See Figure 4.1).

![Figure 4.1: Example with $n = 2$](image)

In the above figure (4.1) $x_n$ represents the length of the intervals. It may seem natural to associate to each pair $(k, 2^n)$ an interval to be removed, however we will associate each number of the form $\frac{k}{2^n}$ an interval to be removed, $J(\frac{k}{2^n})$. This may seem awkward, but it is in fact clever, soon we shall see why. Write $I_1 = [0, x_0]$,
and let \( J(1/2) := [a(1/2), b(1/2)] \) be the central open interval of length \( x_0 - 2x_1 \). Let \( I_2 = I_1 \setminus J(1/2) \). Then, as desired, there remains two intervals of length \( x_1 \). Let \( J(1/2^2) := [a(1/2^2), b(1/2^2)] \) be the central open interval of \([0, a(1/2)]\) of length \( x_1 - 2x_2 \), and \( J(3/2^2) := [a(3/2^2), b(3/2^2)] \) be the central open interval of \([b(1/2), x_0]\) of length \( x_1 - 2x_2 \). (See Figure 4.2)

![Figure 4.2: \( I_2 \)](image)

Notice that \( a(\frac{3}{4}) \) is undefined, however we may define \( a(\frac{3}{4}) = a(\frac{1}{2}) \) illustrating the convention of writing \( a(\frac{1}{2}) \) instead of \( a(1, 2) \). Moreover, with the latter convention, the endpoints of the intervals are uniquely defined. Repeat this process so that at the \( n^{th} \) stage there is \( 2^n \) intervals of length \( x_n \) and define

\[
I := \bigcap_{n} I_n
\]
This description of the set I is intuitive but cumbersome. The intent is to define \{a(k/2^n)\} and \{b(k/2^n)\} so that

\[ a(k/2^n) - b((k - 1)/2^n) = x_n \] \hspace{1cm} (4.1)

(See Figure 4.3)

![Diagram](image)

Figure 4.3: Example with \( n = 2 \) and \( k = 2 \)

In fact this observation serves to define \{a(k/2^n)\} and \{b(k/2^n)\} recursively, since for \( k = 2m \) equation (1) becomes

\[ b \left( \frac{2m - 1}{2n+1} \right) = x_{n+1} - a \left( \frac{m}{2^n} \right) \] \hspace{1cm} (4.2)

While for \( k = 2m - 1 \)

\[ a \left( \frac{2m - 1}{2n+1} \right) = x_{n+1} + b \left( \frac{m - 1}{2^n} \right) \] \hspace{1cm} (4.3)
Quaintly, Hausdorff calls $\frac{m-1}{2^n}$ and $\frac{m}{2^n}$ the “neighbors” of $\frac{2m-1}{2^{n+1}}$. We are now ready to prove,

**Theorem 11.** $H_f(I) = 1$

Proof: First, since $I$ can be covered by $2^n$ intervals of length $x_n$ we have

$$H_f(I) \leq (1/2^n)f(x_n) = 1.$$ 

Now, for the underestimate, consider some $s, t$ of the form $\frac{k}{2^n}$ with $s < t$. We concern ourselves with the interval $L := [b(s), a(t)]$ (see Figure 4.3)

Claim: $f(d(L)) \geq t - s$.

Suppose $s$ has denominator $2^{n+1}$. We will proceed by induction on $n$. When $n = 0$ we have $s = 0$ and $t = 1$ so that

$$f(d(L)) = f(d[a(0), b(1)] = f(d[0, x_0]) = f(x_0) = 1$$

Now suppose $s$ has reduced denominator $2^{n+1}$ so that $s = \frac{k-1}{2^n}$ and $t$ has a reduced denominator at most $2^n$. Let $s_1 = \frac{k-1}{2^n}$ and $s_2 = \frac{k}{2^n}$ be the “neighbors” of $s$, so that $s_1 < s_2 \leq t$. Then by (4.1), (4.2), and (4.3), we have

$$a(s_2) - b(s_1) = x_n$$

$$a(s_2) - b(s) = x_{n+1}$$

So that

$$b(s) = a(s_1) + x_n - x_{n+1}$$
Let \( L^* = [a(t), b(s_1)] \) so that the inductive hypothesis applies to \( L^* \). Then

\[
d(L) = a(t) - b(s) \\
= a(t) - b(s_1) - x_n + x_{n+1} \\
= d(L^*) - x_n + x_{n+1}
\]

Now by section 4.1 property 2 and the inductive hypotheses,

\[
f(d(L)) \geq f(d(L^*)) - f(x_n) + f(x_{n+1}) \\
\geq t - s_1 - \frac{1}{2^n} + \frac{1}{2^{n+1}} = t - s.
\]

Now, suppose both \( s \) and \( t \) have reduced denominators, \( 2^{n+1} \) so that \( t = \frac{2m-1}{2^{n+1}} \) for some \( m \). Then let \( t_1 = \frac{m-1}{2^n} \) and \( t_2 = \frac{m}{2^n} \) be the “neighbors” of \( t \). Then, as above,

\[
b(s) = a(s_1) + x_n - x_{n+1} \\
a(t) = a(t_2) - x_n + x_{n+1} \\
d(L) = d(L^*) - 2x_n + 2x_{n+1}
\]

Thus,

\[
f(d(L)) > f(L^*) - f(2x_n) + f(2x_{n+1}) \\
> f(L^*) - 2f(x_n) + 2f(x_{n+1}) \\
\geq (t_2 - s_1) - \frac{1}{2^{n-1}} + \frac{1}{2^n} \\
= t - s
\]
Proving the claim. Now, fix $\varepsilon > 0$. Since $H_{\delta,f}(I)$ is the infimum of all possible covers and since $I$ is compact, there is some finite cover of $I$ by open intervals, $\{\alpha_n\}$ with $d(\alpha_n) < \delta$ such that,

$$\sum f(d(\alpha_n)) < H_{\delta,f}(I) + \varepsilon$$

Since $[0, x_0]$ is compact and $I \cup \bigcup_{y} J(y)$ covers $[0, x_0]$ there is some finite collection $\{J(y_n)\}$ so that $\bigcup_{n=1}^{N} J(y_n) \cup \bigcup_{m=1}^{M} \alpha_n \supseteq [0, x_0]$. Thus by the claim,

$$f(d(\alpha_0)) \geq y_1$$

$$f(d(\alpha_1)) \geq y_2 - y_1$$

$$\ldots$$

$$f(d(\alpha_{n-1})) \geq y_n - y_{n-1}$$

$$f(d(\alpha_n)) \geq 1 - y_n.$$ 

Summing the above array we have,

$$H_{\delta,f}(I) + \varepsilon > \sum f(d(\alpha_n)) \geq \sum y_n - y_{n-1} \geq 1.$$

This holds for every $\varepsilon$ and as $\varepsilon \to 0$, $\delta \to 0$ so that

$$\lim_{\delta \to 0} H_{\delta,f}(I) \geq 1 \geq H_f(I) \geq \lim_{\delta \to 0} H_{\delta,f}(I).$$

Hence $H_f(A) = 1$ so that the Hausdorff dimension of $A$ is $[f]$. 

19
4.3 Examples and Exposition

Example 2. Let \( f(x) = x^p \) so that the sequence \( \{x_n\} \) is defined by \( 2^n x_n^p = 1 \). To construct the Cantor set we must remove the middle third of each interval, so the dimension is

\[
p = \frac{\log 2}{\log 3}
\]

Hausdorff notes,

Obviously \( L(A) (H_f) \) depends on the behavior of \( \lambda(x) (f) \) in an arbitrarily small neighborhood of the point \( x = 0 \), i.e., the dimension appears to be a characteristic of ... “order”

So we are thinking of dimension as a “growth rate”, for example,

1. If \( f(x) \leq g(x) \) for sufficiently small \( x \) then \( H_f(A) \leq H_g(A) \) for any \( A \).

2. If \( f(x) = cg(x) \) for a positive constant \( c \), then \( H_f(A) = cH_g(A) \).

3. If \( c \leq \frac{f}{g} \leq k \) for positive constants \( c \) and \( k \) then \([f]\) is of the same order as \([g]\).

4. If \( \frac{f}{g} \to 0 \) as \( x \to 0 \) then \([f]\) has higher order than \([g]\)

Example 3.

To construct sets of “infinitely small” dimension let

\[
f(x) = \frac{1}{(\log(\frac{1}{x}))^p}
\]

so that

\[
x_n = e^{-2^\frac{n}{p}}
\]
Since exponential decay is faster than the decay of any power function we may construct a set of “infinitely small” dimension.

**Example 4.**

For a set $I$, with dimension $[f]$, let $I_0 = I \cap [0, a]$, with dimension $[f_a]$. Is there a set $I$ so that $[f_a]$ is monotonic (in $a$), in the sense that $[f_a]$ is lower order than $[f_b]$ when $a < b$? Hausdorff provides a negative answer. To see this, suppose $I$ is such a set, and $a_n$ is a sequence that increases to $b$. Then $H_{f_b}(I_0^{a_n}) = 0$. But, by theorem 4, $\lim_{n \to \infty} H_{f_b}(I_0^{a_n}) = 0 = H_{f_b}(I_0^b)$ so that the dimension of $I_0^b$ is not $[f_b]$. Thus, as Hausdorff writes, “If the dimension $L(A) ([f_a])$ exists and is a function of $a$, then incomparable dimensions appear in every arbitrarily small interval.”

Hausdorff writes,

In the definition of $L_p (H_{x^p})$ we have seen that ... the $p^{th}$ power of the diameter $d$ has played the role of the set function $l$: Can one possibly replace this with a function such as $d^p \log(\frac{1}{d})$ which tends to 0 as $d \to 0$ slower than $d^p$ but faster than any smaller power, and in this sense construct sets whose dimensions fall in between the scale of positive numbers?

Having proven Theorem 11, the answer is yes. Moreover, we may consider “functions of the well known logarithmic scale”

$$f(x) = x^{p_0} \left( \frac{1}{\log(\frac{1}{x})} \right)^{p_1} \left( \frac{1}{\log \log(\frac{1}{x})} \right)^{p_2} \cdots \left( \frac{1}{\log \cdots \log \log(\frac{1}{x})} \right)^{p_n}$$

With sufficient conditions on the sequence $p_n$ we find some validity in the statement “there is no smallest dimension.” This statement is, seemingly, Hausdorff 's main philosophical motivation.
BIBLIOGRAPHY


