JOINT ENHANCEMENT OF MULTICHANNEL SYNTHETIC APERTURE RADAR DATA

A Thesis

Presented in Partial Fulfillment of the Requirements for
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Graduate School of The Ohio State University

By

Naveen Ramakrishnan, B.E.

* * * * *

The Ohio State University

2008

Master’s Examination Committee:

Dr. Randolph L. Moses, Adviser

Dr. Lee Potter

Approved by

Adviser

Graduate Program In
Electrical Engineering
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In this thesis, we consider the problem of joint image enhancement of multi-channel Synthetic Aperture Radar (SAR) data. Previous work by Çetin and Karl introduced nonquadratic regularization methods for image enhancement using sparsity enforcing penalty terms. For multi-channel data, we show that independent enhancement of each channel does not enforce a common support and degrades the relative phase information across channels that is useful for 3D reconstruction of targets.

We formulate a joint enhancement problem to simultaneously enhance multi-channel SAR data while preserving the common support and the cross-channel phase information. We pose this problem as a joint optimization problem with constraints on pixel magnitudes and propose three methods for solving it. We develop both a gradient-based method and a Lagrange-Newton-based method for solving the joint reconstruction problem, and demonstrate the performance of our approach on the Interferometric SAR (IFSAR) height extraction problem from multi-elevation data. We also propose the Dual Descent method, which has reduced complexity, for solving the joint optimization problem.

The proposed algorithms are applied to both synthetic data and Xpatch synthetic radar scattering prediction data of a backhoe. The resulting images are found to preserve the relative phase information required for 3D reconstruction and are also sparse. Finally we derive the Cramér-Rao bound for the relative phase error and
compare the accuracy of the independent and the joint enhancement approaches to that bound.
to my family
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VITA

November 23, 1983 ....................... Born - Chennai, India

2005 ................................. B.E. Electronics and Communications Engineering, Anna University

2005-2006 .............................. Graduate Fellow, The Ohio State University.

2006-present ............................. Graduate Research Associate, The Ohio State University.

PUBLICATIONS

Research Publications


FIELDS OF STUDY

Major Field: Electrical and Computer Engineering
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CHAPTER 1

INTRODUCTION

Radar, which stands for RAdio Detection And Ranging, has been around since the 1940s and has been widely used not only for defense purposes but also for many other applications, including navigation and mapping. Although radars were mostly used for ranging purposes initially, their applicability in mapping terrains and identifying targets were soon realized after the invention of what is now called the Synthetic Aperture Radar (SAR). SAR systems operate by sending a sequence of radar pulses to a scene from a moving platform (usually an aircraft) and reconstructing an image of the scene from the received radar returns. These images are often used for target detection or identification, and for scene visualization. The use of radars for imaging, instead of photometric techniques, enables all-weather and day or night imaging capabilities possible. Moreover, SAR has made very high resolutions (on the order of 25cm) achievable without using huge antennas.

1.1 Motivation

Synthetic Aperture Radar is one of the most widely-used sensing technologies for object and scene recognition, due to its day/night and all-weather performance
capabilities. Traditional SAR image formation entails Fourier processing of limited-angle, limited-bandwidth measurements to form a reflectivity image of the scene \cite{1}. Fourier processing has many advantages, which include linear processing, and computational speed. However, imaging of limited-extent data has inherent disadvantages like reduced resolution limits \cite{1}; in addition, Fourier processing generates significant sidelobe artifacts when some frequencies or measurement angles are denied or blocked \cite{2}. This kind of blocking of a certain band of frequencies occurs when one does not have permission to radiate radar energy within certain frequency ranges.

To address these resolution limits and artifact terms, nonlinear reconstruction techniques have been proposed \cite{3, 4, 5} and these are reviewed in Chapter \cite{2}. In this thesis we focus on a sparse-signal enhancement approach \cite{3, 2} whose basic idea is to reconstruct an image that simultaneously is in good agreement with the measured data and is regularized by using some prior information. The most common prior information used is that the reconstruction is sparse in some domain. SAR image reconstruction techniques of targets, using a sparse image regularization term, has been shown to provide enhanced resolution \cite{2} and also significant robustness to missing data \cite{2}.

The sparseness term, when included in the objective function to be minimized, aims to find a solution vector with minimum number of nonzero terms. When used for reconstructing images, this term results in a sparse number of large-amplitude pixels; this accounts for the simple prior knowledge that at high frequencies, most large-amplitude backscattering from objects occurs at discrete scattering centers on the object \cite{6}. 

2
The main objective of this work is to develop techniques for SAR image enhancement for the multi-channel case while preserving the interchannel information (relative phase) across images.

Multi-channel SAR may arise from multiple polarization channel measurements or from multiple phase center measurements, such as in Interferometric SAR (IFSAR). For many multi-channel data sets, the common support across all channels is expected to be maximized; for example, in IFSAR the high-amplitude pixels are expected to be in the same locations for all channels and is important for achieving super-resolution. In addition, the relative phase of the images across channels often contains important information, and it is important to preserve this in the reconstruction process. For example, both the polarimetric properties of a scattering center in multi-polarization data and the 3D location of a scattering center from two IFSAR images depend on the relative phase.

1.2 Our Contribution

For multi-channel data, one approach to reconstruction is to apply existing techniques independently for each channel. Such an approach neither enforces common sparsity support nor preserves relative phase. Simulation results presented in Chapter 4 demonstrate that for IFSAR, independent reconstruction results in different large-amplitude pixels in each channel and degrades relative phase across reconstructed images. To address these issues, we formulate a joint reconstruction approach that enhances the multi-channel images jointly. This is achieved by including multi-channel regularization terms, in the objective function (in addition to sparsity terms), that represent prior information about the underlying multi-channel process.
for stable inversion. By constraining all reconstructed images to have the same large-amplitude pixels, the relative phase of the reconstructed images appears to be much better preserved than in the independent reconstruction case.

We consider three numerical methods for solving the joint reconstruction problem. One method is a gradient-based approach and another one is a Lagrange-Newton based approach. The Lagrange-Newton approach, which is a second order method, has significant time complexity; thus, we also develop the Dual Descent method, which has reduced time complexity. Finally we compare all three techniques by applying them to synthetic data and show that the joint approach leads to improved performance over the independent enhancement technique. Apart from this, we also derive the CRLB and the ML estimate for the relative phase term using a generalized point scatterer model and compare the performances of the joint and the independent enhancement techniques using the CRLB. We find through the CRLB analysis that windowing of the phase-history data degrades the relative phase estimation error by approximately 2-3 dB while the performance without windowing achieves the CRLB.

1.3 Organization of the Thesis

Chapter 2 reviews SAR and IFSAR concepts, and briefly discusses a nonlinear SAR image reconstruction technique based on sparsity regularization. We give an outline of the multi-channel SAR image enhancement problem and illustrate the problems associated with applying the existing techniques for the multi-channel case using numerical examples, thereby motivating the need for joint enhancement.

Chapter 3 formulates the joint enhancement problem and describes the approach towards solving it. Here we derive the three numerical methods for solving the joint
enhancement problem. We discuss the varying orders and complexities of these optimization methods, along with the inherent assumptions embedded in each method.

Chapter 4 presents the simulation results of these techniques. We first validate our joint enhancement approach with the aid of synthetic data generated using Matlab. Then we present a comparison (complexity and accuracy of final solution) of the independent enhancement method and the three approaches for solving the joint optimization problem from the simulation results obtained using the XpatchT Backhoe Dataset. Finally we present 3D reconstructions of the Block “O” and the Backhoe (using XpatchF Dataset) by enhancing the images jointly and independently.

Chapter 5 summarizes the main contributions of this work. We conclude by discussing future research directions in this area.
CHAPTER 2

BACKGROUND

In this chapter, we introduce basic SAR, IFSAR concepts and then explain the multi-channel enhancement problem for SAR. We also provide some background on the previous work done in SAR image enhancement and motivate the need to enhance the images jointly for multi-channel SAR data.

2.1 SAR System Model

Synthetic Aperture Radar has gained significant attention in recent times owing to its increased resolution compared to other photometric imaging techniques and also because of its day-night and all weather capabilities. A typical SAR system consists of a radar set mounted on an aircraft which flies around the scene of interest (ground patch), transmitting pulsed waveforms and receiving backscattered energy. There are two modes of operation for SAR systems - strip-map mode and spotlight mode. In the strip-map-mode SAR, the electromagnetic signals illuminate different parts of the ground patch as the aircraft moves, whereas in the spotlight-mode SAR, the electromagnetic signals are made to illuminate the same ground patch at all times. A point scatterer, or in general, a target, present at the scene of interest reflects or scatters the electromagnetic energy which is received by the antenna mounted on the
aircraft. These received backscatters from the targets are processed to form a SAR image. The most commonly used signal for illuminating the ground patch is the linear FM chirp signal given by,

\[ s(t) = \begin{cases} 
  e^{j(\omega_0 t + \alpha_0 t^2)}, & |t| \leq \frac{T_p}{2} \\
  0, & \text{otherwise}
\end{cases} \]  

(2.1)

where \( \omega_0 \) is the center carrier frequency, \( 2\alpha_0 \) is the chirp rate and \( T_p \) is the total pulse duration. The delay between the transmitted and the received signals encodes the distance between the transmitter and the target. In the commonly-employed case that the radar illuminates a scene perpendicular to the aircraft flight path, the direction perpendicular to the flight path is called the range, or downrange direction, and the one parallel to it is called the cross-range direction.

Figure 2.1 shows the ground plane geometry of the spotlight-mode SAR alongwith downrange and cross-range directions. The received signal for the azimuth angle \( \theta \), and the grazing angle \( \psi \) (under plane-wave assumption), is given by

\[ r_{\theta,\psi}(t) = A \Re \left[ \int_{-u_1}^{u_1} p_{\theta,\psi}(u) s \left( t - \frac{2(R + u)}{c} \right) du \right] \]  

(2.2)

where \( A \) is the scale factor that accounts for propagation attenuation, \( \Re \) denotes the real part of ., \( R \) is the range from the scene center to the radar, \( u \) is the slant range, \( u_1 \) is the maximum slant range and \( c \) is the speed of light. The projection function \( p_{\theta,\psi}(u) \) is obtained by two-dimensional integration of the three-dimensional (radar) complex reflectivity function \( f(x, y, z) \) after a simple rotational transformation to the \((u, v, w)\) co-ordinates, which are shown in Figure 2.1. The received signal, given by equation (2.2), is first mixed with a reference chirp signal and low-pass filtered after which it can be interpreted as the Fourier transform of the projection function.

\[ \tilde{r}_{\theta,\psi}(t) = \frac{A}{2} \tilde{p}_{\theta,\psi} \left[ \frac{2}{c} (\omega_0 + 2\alpha(t - \tau_0)) \right] \]  

(2.3)
where $P_{\theta,\psi}(U)$ is the Fourier transform of the projection function $p_{\theta,\psi}(u)$ and $\tau_0$ is the delay term given by $\tau_0 = \frac{2R}{c}$. According to the Projection Slice Theorem \[1\], the Fourier transform of the projection of a function is a radial slice from the Fourier transform of the original function. In practice, the received signal, after deramping and low-pass filtering, is sampled to yield the samples $\bar{r}_{\theta,\psi}(t_j), j = 1, \ldots, l$. In our case, the samples represent samples on a radial slice of the Fourier transform of the radar reflectivity function. A number of such radar pulses are transmitted from the observation angles $(\theta_i, \psi_i), i = 1, \ldots, P$ and the received signals are sampled in each case and represented as a vector $\bar{r}_{\theta_i,\psi_i}(t_j)$ \[1\]. Thus each value of $\theta_i$ and the corresponding $\psi_i$ determine the line segment in the three-dimensional Fourier transform of $f(x, y, z)$ along which the received samples lie. The size of the data collection surface is determined by the bandwidth of the transmitted pulse and the azimuth angle swept. The spatial-frequency domain in which this data collection surface exists is commonly referred to as the \textit{phase-history domain} \[1\].

Given the phase-history data, the corresponding SAR image representing the radar reflectivity function can be obtained simply by inverse Fourier transforming the collected samples after appropriate windowing, say using a Taylor or Hamming window, to attenuate the high sidelobes resulting from limited bandwidth in the spatial-frequency domain. Taking inverse Fourier transform of the phase-history data forms the image typically on the surface on which the data is present. This surface, which is determined by the flight path, assumed to be straight in this thesis, is a plane called the \textit{slant plane}. So the inverse Fourier transform of the phase-history data, present

\[1\]It should be noted that the variables in bold case, throughout this thesis, refer only to vectors/matrices and those not in bold are scalars (either real or complex). Particularly, the lower case bold variables refer to vectors and the upper case bold variables to matrices.
Figure 2.1: Spotlight SAR Observation Model.

in this plane, would form an image on the slant plane, i.e. the 3D targets would be projected onto this plane and represented in the SAR image formed.

The received samples usually lie on a polar raster and require interpolation to a rectangular grid before using algorithms like the Fast Fourier Transform (FFT) for taking the inverse Fourier transform. This method, which is commonly used for SAR image formation, is called the Polar Reformatting Algorithm. Another method for SAR image formation is the Filtered Backprojection technique [1], which is beyond the scope of this thesis.
The resolution of the SAR image (in the absence of windowing) in the range and cross-range dimensions are \( x_{\text{res}} = \frac{c}{2B_c} \) and \( y_{\text{res}} = \frac{\lambda_0}{2\Delta\theta} \) respectively, where \( B_c \) is the bandwidth of the transmitted pulse, \( \lambda_0 \) is the center wavelength and \( \Delta\theta \) is the angular extent of the annulus in the spatial-frequency domain. A resolution cell, centered at the pixel \((x_k, y_l)\) in the ground-plane image, is defined as all the points \((x, y)\) on the ground plane, such that, \( |x - x_k| \leq \frac{x_{\text{res}}}{2} \) and \( |y - y_l| \leq \frac{y_{\text{res}}}{2} \).

Alternately, the SAR image can be viewed as the phase-history data multiplied with the inverse Fourier matrix. The bandlimited data in the spatial-frequency domain can be interpreted as a windowing operation performed on the two-dimensional Fourier transform of the radar reflectivity function. The rectangular window used must be nonzero only on the collection surface, i.e. slant plane in our case. This is equivalent to the convolution of the actual radar reflectivity function, which can be modeled using a matrix operation as follows,

\[
g = T f + w
\]

where the \( N \times 1 \) complex vector \( g \) represents the observed SAR image, \( N \times N \) matrix \( T \) is the forward linear operator used in forming \( g \), the \( N \times 1 \) complex vector \( f \) represents the discretized radar reflectivity function. We include a noise term in (2.4), where \( w \) is the \( N \times 1 \) measurement noise vector, which is assumed to be complex additive white Gaussian noise; this models measured error. The \( N \times 1 \) complex vectors \( g, f \) are formed by stacking the \( N_r \times N_c \) observed and enhanced SAR images as vectors respectively, where \( N = N_r \cdot N_c \) and where \( N_r \) and \( N_c \) are the number of rows and columns of the SAR image. A point scatterer, represented by a single nonzero element in \( f \), when multiplied with \( T \), would result in what is called the Point Spread Function (PSF) represented by \( h \). Due to the convolution operation, a single point scatterer
present in the scene is spread both in the range and the cross-range directions; hence the name point-spread function. The relation between \( g, f \) and \( h \), when there is no noise, is as follows,

\[
Im(g) = Im(f) \otimes Im(h) \iff g = Tf
\]

where \( Im(\cdot) \) denotes reshaping of an \( N \times 1 \) vector into an \( N_r \times N_c \) image and \( \otimes \) represents 2D circular convolution. Instead of performing the convolution, \( Im(g) \) can be found first by elementwise multiplication of the 2D Fourier transforms of the reshaped \( f \) and \( h \) and then taking the 2D inverse Fourier transform. Efficient implementation of \( Tf \) typically stores the 2D FFT of \( Im(h) \) and is carried out in the frequency domain. The PSF is very important in many SAR systems as it completely determines the matrix \( T \) and helps in modeling sensor errors such as non-ideal frequency response in the radar front-end [3].

In the next section, we briefly introduce IFSAR which is a widely used method for 3D target reconstruction from SAR images.

2.2 Interferometric SAR

Multi-channel SAR may arise from multiple polarization channel measurements or from multiple phase center measurements, such as in IFSAR. IFSAR is a technique used for 3D target reconstruction, by using multiple phase center measurements of the same ground patch. Usually the multiple phase centers, assumed to be two in our case, correspond to closely spaced grazing (elevation) angles. After collecting the phase-history data from the two closely spaced elevation angles, with the same center azimuth angle and the azimuth extent, the corresponding SAR images are formed. Assuming that the flight path for both the passes is linear, the SAR images formed
would lie on the slant planes corresponding to the elevation angles $\psi_1$ and $\psi_2$ which are separated by $\Delta \psi \ll 1$. Since the difference in the elevation angles is very small, the corresponding slant planes can be taken to be the same. We assume the slant plane corresponding to the average of the two elevation angles to be the reference slant plane orthogonal to which the height of the target is estimated.

In IFSAR, the height of a scattering center (as measured from the reference slant plane) can be estimated from the phase difference between the corresponding pixels of the two phase-coherent, high resolution SAR images with some elevation offset \[.\] The height estimate $\hat{z}_i$ of the scattering center, as measured orthogonal to the reference slant plane is given by,

$$\hat{z}_j = \frac{\lambda_c}{4\pi \Delta \psi} \arg((f_2)_j(f_1)_j^*) \quad j = 1, \ldots, N$$ (2.6)

where $\lambda_c$ is the center wavelength and $(f_1)_j, (f_2)_j$ refer to the $j^{th}$ pixel in the corresponding SAR image. We also show, in Section 4.2, that $\arg((f_2)_j(f_1)_j^*)$ is the ML estimate of the relative phase from which the height estimate is obtained. It must be noted that the height estimate is linearly dependent on the relative phase between the image pixels. Thus, it is important that the phase difference across images be preserved during any processing of the multi-channel SAR images.

For many multi-channel data sets, the common support across all channels is expected to be maximized; for example, in IFSAR the high-amplitude pixels are expected to be in the same locations for all channels and is important for achieving super-resolution. In addition, the relative phase of the images across channels often contains important information, and it is important to preserve this in the reconstruction process. Moreover for applications like terrain mapping, where the height
of the terrain is piecewise constant, the height estimate can be improved by averaging across many pixels, whereas the 3D target-reconstruction applications typically span a few pixels and hence demand accurate height estimates. So this work focuses mainly on enhancing SAR images for accurate 3D target-reconstruction applications by preserving their height estimates during the enhancement process.

2.3 Enhancement Problem for Multi-channel SAR

In this section, we explain the multi-channel SAR enhancement problem and its significance in enhanced 3D target-reconstructions. The mathematical formulation of the simplified (two-channel) version of this problem is presented and solved using optimization techniques, in Chapter 3.

The main reasons for the recent surge in SAR image enhancement techniques include the resolution limitations of SAR images formed using the polar format algorithm [3] and also the high sidelobes this algorithm produces. Some 2D spectral estimation methods which estimate the spatial-frequency samples to improve resolution have been proposed [4], but these techniques have been reported to have degraded performance for real-world targets [8]. Although other techniques, including ones based on regularization and entropy methods [9], have been proposed for the real-valued image enhancement and medical tomography applications, solutions for the SAR image enhancement problem, which involves complex-valued reflectivities, have been presented only recently using nonquadratic regularization techniques [3, 2] and also using concepts of optimization transfer or MM algorithms [10]. These, unlike some of the previously developed methods, are robust to missing phase-history data.
and perform equally well for both synthetic and real-world scenarios. These techniques have been proposed only for 2D reflectivity functions and their extension for the 3D case has not yet been studied. Spectral estimation based techniques [11] have also been proposed to address the issue of feature extraction for 3D complex-valued SAR problem.

The main objective of this work is to develop techniques for joint enhancement of multi-channel (multi-elevation) SAR images while preserving the interchannel information (relative phase) across them.

The problem stated above is for multiple SAR measurement apertures at closely-spaced elevations as used for IFSAR, although we consider a simplified version with only two SAR channels while solving for the same.

We consider a linear path SAR at each of the $n$ elevations with the same center azimuth angle. The reflected field from the ground patch is collected and processed to form the complex reflectivity function $f(u, v, w)$ projected onto the slant plane, where $u$ and $v$ are the downrange and crossrange dimensions on the slant plane respectively. The observation model for multiple spotlight-mode SAR measurements is represented as,

$$g_i = T_i f_i + w_i, \quad i = 1, 2, .., n$$

(2.7)

In this thesis, we assume $n = 2$ and that $g_1$ and $g_2$ are images formed at elevation angles $\psi_1$ and $\psi_2$ that are sufficiently closely-spaced so that IFSAR processing can be used to obtain the 3D reconstruction of objects in the scene.

In the next section, we shall discuss why the existing enhancement techniques, particularly the one proposed in [3], will result in degraded performance even for the two elevation-angle IFSAR case.
2.4 Independent Enhancement

In this section we shall briefly describe a nonlinear SAR image enhancement technique and explain why it cannot be used directly for the multi-channel enhancement problem, thereby making the case for joint enhancement of multi-channel SAR images.

In [3], Çetin and Karl presented an iterative nonlinear algorithm for SAR image enhancement using nonquadratic regularization techniques. In the paper, image enhancement is achieved by setting up and solving an optimization problem whose objective function includes a quadratic error term for data fidelity and a sparsity term for sparseness in the final image. A fundamental assumption used here is that at high frequencies, the strong reflectors in the scene are discrete and sparsely located. The optimization problem [3] considers is given by,

$$\hat{f} = \arg \min_f J(f)$$  \hspace{1cm} (2.8)

where $J(f)$ is given by,

$$J(f) = \|g - Tf\|_2^2 + \lambda^2 \|f\|_p^p$$  \hspace{1cm} (2.9)

In order to avoid the difficulty of differentiating the $\ell_p$-norm term near zero, we approximate it as in [3] by,

$$\|f\|_p^p \approx \sum_{i=1}^{N} (|f(i)|^2 + \epsilon)^{\frac{p}{2}}$$  \hspace{1cm} (2.10)

where $\epsilon$ is a small positive constant. The above approximation is also used in all the three techniques that have been developed for solving the joint enhancement problem in Chapter 3. The gradient of $J(f)$ with respect to the complex vector $f$ has to be
taken separately with respect to the real and imaginary parts of $f$ and then combined to form a single complex vector, which is given by

$$\nabla J(f) = \tilde{H}(f)f - 2T^Hg$$  \hspace{1cm} (2.11)

where $\tilde{H}(f)$ is the approximated Hessian matrix given by

$$\tilde{H}(f) = 2T^H + p\lambda^2 \Lambda(f)$$ \hspace{1cm} (2.12)

and $\Lambda(f) = \text{diag}\left\{ \frac{1}{(\|f\|)^{2+\epsilon}} \right\}$ with $\text{diag}$ representing a diagonal matrix whose $i^{th}$ diagonal element is given by the term inside the braces. The paper [3] discusses a robust numerical solution for solving the above problem using the approximated Hessian matrix $\tilde{H}(f)$. The quasi-Newton update equation is given by,

$$f^{n+1} = f^n - \gamma_1 [\tilde{H}(f^n)]^{-1}\nabla J(f^n)$$ \hspace{1cm} (2.13)

The conjugate gradient algorithm [12] is used at each iteration in order to find the updated vector $f^{n+1}$ by first multiplying out equation (2.13) with $\tilde{H}(f^n)$ and then using (2.11) to solve the following matrix equation

$$\tilde{H}(f^n)f^{n+1} = \gamma_1 2T^Hg.$$ \hspace{1cm} (2.14)

This saves computations by not finding the inverse of the approximated Hessian matrix for each updated vector $f^{n+1}$. Typically $\gamma_1$ is taken to be equal to 1.

We shall now apply this enhancement technique for the multi-channel case with two elevation angles and observe its performance. We assume that a pair of SAR images taken from two elevations $\psi_1$ and $\psi_2$ separated by $\Delta \psi \ll 1$ and from the same center azimuth angle is formed using the Polar Formatting Algorithm. In [2], a method for enhancing the SAR images by introducing nonquadratic sparsity inducing
terms has been discussed, where the point-enhanced SAR imaging is achieved by solving the following optimization problem,

$$f_i = \arg \min_f J_i(f) \quad i = 1, 2$$

(2.15)

where $J_i(f)$ is given by,

$$J_i(f) = \| g_i - T_i f \|_2^2 + \lambda^2 \| f \|_p^p$$

(2.16)

Each cost function $J_i$ represents a tradeoff between choosing $f_i$ to match the measured data $g_i$ in an $\ell_2$-norm sense, and sparsity in the resulting image by minimizing the $\ell_p$-norm of $f_i$ for $p \leq 1$. The parameter $\lambda^2$ is chosen to weigh the data-fitting and solution sparseness terms in the optimization problem. Typically, $p$ is chosen between 0.7 and 1 [2].

For independent enhancement, the above optimization problem is solved for each of the images to form two point-enhanced SAR images corresponding to $\psi_1$ and $\psi_2$. Each enhancement involves solving an unconstrained optimization problem with $2N$ real variables $(\Re(f_i), \Im(f_i))$. For IFSAR, one expects that the noiseless image has the property that large-amplitude pixels have the same amplitude, i.e $|(f_1)_i| \approx |(f_2)_i|, \quad i = 1, ..., N$ and differing phase. Since the height estimate is linearly dependent on the relative phase between the image pixels as described in Section 2.2, it is important that the phase difference across images be preserved during the enhancement process.

The independent enhancement technique has no component designed to preserve either amplitude or (relative) phase information across individual images. In fact, since the methods produce sparse output images, it is often the case that scattering terms represented by several adjacent large-amplitude pixels in the conventional
image are enhanced to produce fewer (possibly one) large-amplitude pixels in the enhanced image. Due to variations, noise and independent processing of the individual images, it may well be that a different subset of large-amplitude pixels result from each enhancement process. In the worst case, none of the large-amplitude pixels in \( f_1 \) may have a corresponding large-amplitude pixel in image \( f_2 \). Here, IFSAR processing of independently-enhanced images is not possible. The simulation results of the independent enhancement technique for synthetic multi-elevation data with the corresponding SAR images are provided in Chapter 4, where we demonstrate using simulation results that such processing of the SAR multi-channel data does not preserve the needed inter-channel phase information.

An alternative method of enhancement would be to perform a direct 3D reconstruction. This could be implemented by letting \( f \) in (2.4) be a discretized grid of 3D sample points and letting \( h \) be the corresponding 3D PSF. Although this seems to be a natural setting for 3D reconstruction, both the storage and the computational requirements for such a 3D enhancement technique are significantly higher than those for the 2D case. This is because the enhancement technique must do 3D convolution at each step and has to solve \( N^{\frac{3}{2}} \) vector equations where \( N \) is the number of pixels in a SAR image. Since the memory and computational requirements for the 2D case is much lower, we focus on developing effective enhancement techniques in the 2D domain while retaining the information required for 3D reconstruction at a later stage.

Thus, we seek to modify the optimization problem in (2.15) to preserve the common large-amplitude pixels and the relative phase information thereby maximizing the common support across the images. In the following chapter, we shall pose the
joint enhancement problem as an optimization problem and provide three different numerical methods for solving it with varying orders and complexities.
In this chapter, we first formulate the problem of joint enhancement as a (constrained) joint optimization problem and then propose numerical methods for jointly processing both images.

We consider a simplified version of the multi-channel enhancement problem in this thesis by setting $n = 2$ in equation (2.4) of Chapter 2. We first form the two images corresponding to the two elevation angles and perform joint enhancement processing. This joint processing of the two SAR images uses common sparsity inducing terms and also appropriate constraints which result in the reduction of the errors in the height estimates. Specifically, we impose an equal magnitude constraint in the resulting images, in which we jointly enhance the two SAR images such that their magnitudes are equal. The equal magnitude constraint is a more appropriate one for the two channel IFSAR problem as the large-magnitude pixels are expected to be at the same position in both the SAR images. Mathematically this can be expressed as an optimization problem given by,

$$\arg \min_{f_1, f_2} L(f_1, f_2)$$

(3.1)
subject to the constraint $|(f_1)_i| = |(f_2)_i|$ for $i = 1, \ldots, N$, where $L(f_1, f_2)$ is given by,

$$L(f_1, f_2) = \|g_1 - T_1 f_1\|^2 + \|g_2 - T_2 f_2\|^2 + \lambda_2^2 \|f_1\|_p^p + \lambda_2^2 \|f_2\|_p^p \quad (3.2)$$

The values of $\lambda_1^2$ and $\lambda_2^2$ are usually taken to be equal (say $\lambda^2$) since both images are expected to be identically sparse. As before, the $\lambda_i^2$ parameters provide a relative weight to the data-fitting and sparseness terms in the optimization problem. The $N \times N$ matrices $T_1$ and $T_2$ are the forward linear operators representing the point spread function and $g_i$, $f_i$ for $i = 1, 2$ are $N \times 1$ vectors representing the observed and enhanced images respectively for the two elevation angles, where the image pixels are stacked into a vector.

The above joint formulation leads to a constrained optimization problem with $4N$ real variables ($|f_1|, |f_2|, \phi_1$ and $\phi_2$), where $|f_i|, |\phi_i|$ for $i = 1, 2$, are $N \times 1$ vectors of element magnitudes and phases of the complex vector $f_i$:

$$f_1 = e^{j\phi_1} |f_1| \quad f_2 = e^{j\phi_2} |f_2|. \quad (3.3)$$

we will impose the constraint

$$|(f_1)_i| = |(f_2)_i| \quad i = 1, \ldots, N. \quad (3.4)$$

We define,

$$|f_1| = \begin{bmatrix} |(f_1)_1| \\ |(f_1)_2| \\ \vdots \\ |(f_1)_N| \end{bmatrix} \quad |f_2| = \begin{bmatrix} |(f_2)_1| \\ |(f_2)_2| \\ \vdots \\ |(f_2)_N| \end{bmatrix} \quad (3.5)$$

$$e^{j\phi_1} = diag \{ e^{j(\phi_1)_i} \}, \quad e^{j\phi_2} = diag \{ e^{j(\phi_2)_i} \}, \quad i = 1, 2, \ldots, N$$

where $(\phi_1)_i, (\phi_2)_i$ for $i = 1, 2, \ldots, N$ are the set of angles at each pixel of $f_1$ and $f_2$ respectively. This joint optimization is compared to two unconstrained optimizations with $2N$ real variables each resulting from equation (2.15).
The following sections develop three methods to solve this joint optimization problem.

### 3.1 Gradient Descent Method

One solution to the optimization problem defined in (3.1) is obtained by explicitly including the equal magnitude constraint (3.4) in the cost function itself. This is accomplished by forcing the enhanced images to have the same magnitudes, i.e. $(f_1)_i = (f_2)_i = (f)_i$ for $i = 1, \ldots, N$.

$$L(f_1, f_2) = \|g_1 - T_1 e^{j\phi_1} f\|_2^2 + \|g_2 - T_2 e^{j\phi_2} f\|_2^2 + 2\lambda^2 \|f\|_p^p$$  \hspace{1cm} (3.6)

With this formulation, the number of unknowns in the cost function is decreased from $4N$ to $3N$ real variables, i.e., we minimize the cost function $L(f_1, f_2)$ with respect to the $3N \times 1$ real-valued vector $\Theta$ given by,

$$\Theta = \begin{bmatrix} |f| \\ \phi_1 \\ \phi_2 \end{bmatrix}$$

We propose to use the steepest descent method [13] for cost function minimization and hence find the gradient of $L(f_1, f_2)$ with respect to $\Theta$. The gradient vector of a scalar function, with respect to a vector, points in the direction of greatest rate of change of the scalar function. The $3N \times 1$ gradient vector, for our case, after sufficient simplifications is given by,

$$\nabla L_\Theta = \begin{bmatrix} \nabla L_{|f|} \\
abla L_{\phi_1} \\
abla L_{\phi_2} \end{bmatrix}$$  \hspace{1cm} (3.7)
where,

\[
\nabla L_{|f|} = 2\Re[-S_1^H g_1 - S_2^H g_2 + (S_1^H S_1 + S_2^H S_2)|f|] + 2\lambda^2 A|f|
\]

\[
\nabla L_{\phi_1} = 2\Re[j F_1^H T_1^H g_1 - j F_1^H T_1^H T_1 f_1]
\]

\[
\nabla L_{\phi_2} = 2\Re[j F_2^H T_2^H g_2 - j F_2^H T_2^H T_2 f_2]
\]

and where,

\[
S_1 = T_1 e^{i\phi_1}, \quad S_2 = T_2 e^{i\phi_2} \quad (N \times N)
\]

\[
F_1 = \text{diag}\{(f_1)_i\}, \quad F_2 = \text{diag}\{(f_2)_i\}, \quad (N \times N)
\]

\[
\Lambda = \text{diag}\left\{\frac{1}{(|(f)|^2 + \epsilon)^{1/2}}\right\} \quad i = 1, 2, \ldots, N \quad (N \times N)
\]

The superscript \(H\) represents the Hermitian of a matrix or conjugate transpose for vectors. The gradient vector is used to update the value of \(\Theta\).

\[
\Theta^{k+1} = \Theta^k - \gamma \nabla L_{\Theta^k}
\]

where \(\gamma > 0\) is a user-selected step size. The iteration is initialized by \(|f| = \frac{|g_1| + |g_2|}{2}\) and the initial phase angles are set to be the same as the observed phase angles, i.e. those of \(g_1\) and \(g_2\). These iterations are carried out until the convergence criteria

\[
\frac{\|f_1^{k+1} - f_1^k\|}{\|f_1^k\|} < \delta_{GD} \quad \text{and} \quad \frac{\|f_2^{k+1} - f_2^k\|}{\|f_2^k\|} < \delta_{GD}
\]

are met, for some user-selected \(\delta_{GD} > 0\).

One main advantage of the Gradient Descent method is that it ensures that the final images \(f_1\) and \(f_2\) have pixels with exactly the same magnitudes, since the constraint is directly included in the cost function.
Pseudocode For The Gradient Descent Method

**Input:** Two SAR images $g_1, g_2$, stepsize $\gamma$, sparseness parameters $p, \lambda^2$ and convergence parameter $\delta_{GD}$.

**Output:** Two jointly enhanced images $f_1, f_2$.

**Given:** The Point Spread Functions $h_1, h_2$.

**Store:** $H_1, H_2, Hh_1, Hh_2, T^H_1 g_1, T^H_2 g_2$.

**Initialize:** $|f_1^0| = |f_2^0| = \frac{|g_1| + |g_2|}{2}$, Phase angles of $f_1^0, f_2^0$ same as those of $g_1, g_2$ respectively.

**The Algorithm**

1: while Relative change in both $f_1^k, f_2^k > \delta_{GD}$ do
2: Form diagonal matrices $\Lambda, F_1, F_2, e^{j\phi_1}, e^{j\phi_2}$ using vectors $f_1^k, f_2^k$ and equation (3.9)
3: Evaluate the vectors $T^H_1 T_1 f_1^k$ and $T^H_2 T_2 f_2^k$
4: Calculate the $3N \times 1$ gradient vector $\nabla L_{\Theta^k}$ using equation (3.8)
5: Use the gradient vector to update $\Theta^k$ and obtain $\Theta^{k+1}$
6: $f_1^{k+1} = e^{j\phi_1^{k+1}} |f_1^{k+1}|, f_2^{k+1} = e^{j\phi_2^{k+1}} |f_2^{k+1}|$
7: Relative change in $f_1^k, f_2^k : err1 = \frac{\|f_1^{k+1} - f_1^k\|}{\|f_1^k\|}, err2 = \frac{\|f_2^{k+1} - f_2^k\|}{\|f_2^k\|}$
8: $f_1^k = f_1^{k+1}, f_2^k = f_2^{k+1}$
9: end while
10: $f_1 = f_1^{k+1}$ and $f_2 = f_2^{k+1}$

**Table 3.1: Pseudocode For The Gradient Descent Method**

The pseudocode for the Gradient Descent method implementation is given in Table 3.1. All the matrix multiplications involving $T_1$ and $T_2$, which represent convolutions, are performed as elementwise multiplications in the frequency domain, by taking 2D FFT of the Point Spread Function, defined in Chapter 2, and also of the updated vectors, $f_1^{k+1}$ and $f_2^{k+1}$. The stored $(N_r \times N_c)$ matrices $H_1, H_2, Hh_1, Hh_2$ are
the 2D FFTs of the point spread functions corresponding to the \((N \times N)\) forward operators \(T_1, T_2, T_1^H\) and \(T_2^H\) respectively. Such an implementation reduces memory and computational requirements by using only \(4N\) elements instead of \(4N^2\). All these conventions, which have been used for this method, are also used for other methods to be developed in this chapter.

3.2 Lagrange-Newton Method

The Lagrange-Newton method [13] is a constrained nonlinear optimization method in which the constraints are included in the cost function using the Lagrangian and the solution to the resulting minimization problem is found using Newton’s method. The optimization problem in this case is,

\[
\min_{f_1, f_2} \max_\beta L(f_1, f_2, \beta) \tag{3.11}
\]

where, \(\beta = [\beta_1, \ldots, \beta_N]^T\) and

\[
L(f_1, f_2, \beta) = \|g_1 - T_1 f_1\|_2^2 + \|g_2 - T_2 f_2\|_2^2 + \lambda_1^2 \|f_1\|_p^p + \lambda_2^2 \|f_2\|_p^p + \sum_{i=1}^{N} \beta_i (|(f_1)_i|^2 - |(f_2)_i|^2) \tag{3.12}
\]

The constraint \(|f_1| = |f_2|\) has been included in the cost function; the \(\beta_i\)s are the Lagrange multipliers, which are additional parameters to be optimized. The cost function can also be written in matrix form as follows,

\[
L(f_1, f_2, \beta) = \|g_1 - T_1 f_1\|_2^2 + \|g_2 - T_2 f_2\|_2^2 + \lambda_1^2 \|f_1\|_p^p + \lambda_2^2 \|f_2\|_p^p + f_1^H B f_1 - f_2^H B f_2 \tag{3.13}
\]

where the \(N \times N\) matrix \(B\) is defined as \(B = \text{diag}\{\beta_i\}\). The goal is to minimize the function given by the equation (3.13). The partial derivatives of \(L(f_1, f_2)\) are with respect to two complex vectors and hence have to be taken with respect to the real
and imaginary components separately. By doing this and simplifying the first partial derivatives with respect to $f_1, f_2$, we find

$$
\nabla L(f_1, f_2)_{f_1} = [2T_1^HT_1 + p\lambda_1^2 \Lambda_1 + 2B]f_1 - 2T_1^Hg_1
$$

$$
\nabla L(f_1, f_2)_{f_2} = [2T_2^HT_2 + p\lambda_2^2 \Lambda_2 - 2B]f_2 - 2T_2^Hg_2
$$

(3.14)

where the matrices $\Lambda_1$ and $\Lambda_2$ are given by,

$$
\Lambda_1 = \text{diag}\left\{\frac{1}{(||(f_1)_i|^2 + \epsilon)^{1-\frac{1}{2}}}\right\}
$$

$$
\Lambda_2 = \text{diag}\left\{\frac{1}{(||(f_2)_i|^2 + \epsilon)^{1-\frac{1}{2}}}\right\} \quad i = 1, 2, ..., N
$$

(3.15)

The gradient with respect to each $\beta_i$ gives the constraint $|(f_1)_i| = |(f_2)_i|$ which, when substituted into equation (3.15), yields $\Lambda_1 = \Lambda_2 = \Lambda$. Thus, equations (3.14) become

$$
\nabla L(f_1, f_2)_{f_1} = [2T_1^HT_1 + p\lambda^2 \Lambda + 2B]f_1 - 2T_1^Hg_1
$$

$$
\nabla L(f_1, f_2)_{f_2} = [2T_2^HT_2 + p\lambda^2 \Lambda - 2B]f_2 - 2T_2^Hg_2
$$

(3.16)

To find vectors $f_1, f_2$ which minimize (3.13), we set (3.16) to zero which further leads to the following set of equations, with the additional constraint that both $f_1$ and $f_2$ must have the same magnitude in each of their components.

$$
[2T_1^HT_1 + p\lambda^2 \Lambda + 2B]f_1 = 2T_1^Hg_1
$$

$$
[2T_2^HT_2 + p\lambda^2 \Lambda - 2B]f_2 = 2T_2^Hg_2
$$

(3.17)

For solving these set of equations, along with the $\beta_i$s as parameters, iterative methods such as the conjugate gradient method can be used [12]. But with direct implementation, it is not possible to ensure that the $\beta_i$s remain real-valued. Instead, we implement the real and imaginary parts of equations (3.17) separately, along with the
(real-valued) equation for \( \beta \). This gives:

\[
M \Delta x = a
\]  

(3.18)

where the matrix,

\[
M = 
\begin{bmatrix}
M_{11} & 0 \\
0 & M_{22}
\end{bmatrix}
\]

\[
\text{diag}[2 \Re (f_1)] \quad \text{diag}[2 \Im (f_1)] \\
\text{diag}[2 \Re (f_2)] \quad \text{diag}[2 \Im (f_2)]
\]

\[
= 
\begin{bmatrix}
\Delta \Re (f_1) \\
\Delta \Im (f_1) \\
\Delta \Re (f_1) \\
\Delta \Im (f_1) \\
\Delta \beta
\end{bmatrix}
\]

\[
\text{diag}[2 \Re (f_1)] \quad \text{diag}[2 \Im (f_1)] \\
\text{diag}[-2 \Re (f_2)] \quad \text{diag}[-2 \Im (f_2)]
\]

and the matrices \( R_1, R_2 \) are given by,

\[
R_1 = \begin{bmatrix} \Re(T_1) & \Im(T_1) \\ -\Im(T_1) & \Re(T_1) \end{bmatrix} \quad R_2 = \begin{bmatrix} \Re(T_2) & \Im(T_2) \\ -\Im(T_2) & \Re(T_2) \end{bmatrix}
\]  

(2N \times 2N)  

(3.19)

Thus, the \( 3N \) complex equations have been rewritten as \( 5N \) real equations so that the constraint on the \( \beta_i \)s that they must be real is enforced. The matrix equation \( (3.18) \)
cannot be iteratively solved using the conjugate gradient method since the matrix $M$ is not guaranteed to be positive definite. We instead solve this equation using the Minimum Residual method [14], which requires the matrix $M$ to be symmetric but not positive definite. The minimum residual method solves a matrix equation by iteratively minimizing the relative residual vector norm, which in our case is $r_{MR} = \frac{\|M\Delta x - a\|}{\|a\|}$. We run the minimum residual method until $r_{MR}$ is less than a predefined value $\delta_{MR} > 0$.

The implementation of the Lagrange-Newton method requires two loops which are nested, an outer loop for updating the $5N$ variables and an inner loop for solving the matrix equation (3.18) using the minimum residual method. This is done iteratively until the stopping criterion

$$\frac{\|f_1^{k+1} - f_1^k\|}{\|f_1^k\|} < \delta_{LN1}, \quad \frac{\|f_2^{k+1} - f_2^k\|}{\|f_2^k\|} < \delta_{LN1} \quad \text{and} \quad \frac{\|\beta^{k+1} - \beta^k\|}{\|\beta^k\|} < \delta_{LN2}$$

is met for some user-selected $\delta_{LN1} > 0$ and $\delta_{LN2} > 0$. The pseudocode given in Table 3.2 for the Lagrange-Newton method contains the implementation details. It must be noted that the vectors $T_1^Hg_1, T_2^Hg_2$ are computed only once.

The primary advantage of the Lagrange-Newton method, over the Gradient Descent method discussed previously, is that it has quadratic convergence. On the other hand, the computation involves a larger number of parameters which slows down this algorithm dramatically. It is for this reason that we developed another method which is computationally faster than the Lagrange-Newton method, since it has a fewer number of equations to be solved at each step for updating the vectors.
Pseudocode For The Lagrange-Newton Method

**Input:** Two SAR images \( g_1, g_2 \), sparseness parameters \( p, \lambda^2 \) and convergence parameters \( \delta_{LN1} \) and \( \delta_{LN2} \).

**Output:** Two jointly enhanced images \( f_1, f_2 \).

**Given:** The forward operator matrices \( T_1, T_2 \).

**Store:** \( H_1, H_2, Hh_1, Hh_2, T_1^H g_1, T_2^H g_2 \).

**Initialize:** \( |f_1^0| = |f_2^0| = \frac{|g_1| + |g_2|}{2} \), Phase angles of \( f_1^0, f_2^0 \) same as those of \( g_1, g_2 \) respectively and \( \beta^0 = 0 \).

**The Algorithm**

1: while Relative change in both \( f_1^k, f_2^k \) > \( \delta_{LN1} \) and \( \beta^k > \delta_{LN2} \) do
2: Form diagonal matrices \( \Lambda_1, \Lambda_2 \) using vectors \( f_1^k, f_2^k \) and equation (3.15)
3: Solve for the vectors \( f_1^{k+1}, f_2^{k+1} \) and \( \beta^{k+1} \) in the matrix equation (3.18) iteratively, using minimum residual method
4: Relative change in \( f_1^k, f_2^k, \beta \): \( err_1 = \frac{\|f_1^{k+1} - f_1^k\|}{\|f_1^k\|} \), \( err_2 = \frac{\|f_2^{k+1} - f_2^k\|}{\|f_2^k\|} \) and \( err_3 = \frac{\|\beta^{k+1} - \beta^k\|}{\|\beta^k\|} \)
5: \( f_1^k = f_1^{k+1} \), \( f_2^k = f_2^{k+1} \)
6: end while
7: \( f_1 = f_1^{k+1} \) and \( f_2 = f_2^{k+1} \)

**Table 3.2: Pseudocode For The Lagrange-Newton Method**

3.3 Dual Descent Method

The optimization problem in this case is similar to the one in the preceding section, given by,

\[
\max_{\beta} \min_{f_1, f_2} L(f_1, f_2, \beta) \quad (3.22)
\]
where, \( \beta = [\beta_1, \ldots, \beta_N]^T \) and \( L(f_1, f_2, \beta) \) is given by (3.12). The constraint (3.4) has been included in the cost function and \( \beta \)'s are the Lagrange multipliers which will also be taken as parameters to be optimized as in the previous case.

The basic idea is to partially enhance the two SAR images independently using slight modifications to the algorithm described in Section 2.4 and then use the updated vectors \( \hat{f}_1 \) and \( \hat{f}_2 \) to estimate \( \beta \). So there will be two iterations, the inner iteration to find the enhanced images for each updated \( \beta \) and the outer iteration, which employs a gradient-based method [13], to update \( \beta \) using the refined estimates of the enhanced images. Thus, instead of simultaneously solving for all the vectors, \( f_1, f_2 \) and \( \beta \), we fix \( \beta \) and then minimize the objective function for that \( \beta \). Then we update \( \beta \) and repeat the minimization, thereby finding the surface of \( \hat{f}_1(\beta) \) and \( \hat{f}_2(\beta) \) over which \( \beta \) must be maximized. We thus make an implicit assumption that \( \beta \) remains constant throughout the inner iteration, thereby obviating the need to maximize the function simultaneously along with the minimization.

Given the vector \( \beta^k \), we need to minimize \( L(f_1, f_2) \) with respect to \( f_1, f_2 \), for which we can adopt a quasi-Newton approach similar to the one proposed in [2]. Our method is a slight modification of [2] in which we approximate the Hessian matrix as

\[
H(f_1, f_2) = \begin{bmatrix}
2T_1^H T_1 + p\lambda^2 \Lambda_1 + 2B^k & 0 \\
0 & 2T_2^H T_2 + p\lambda^2 \Lambda_2 + 2B^k
\end{bmatrix}
\]

(3.23)

where the matrix \( B^k \) is the same as in the previous section and the superscript \( k \) for a vector or matrix denotes the outer iteration number. Since the Hessian matrix is diagonal, the updated vectors \( f_1^{i+1} \) and \( f_2^{i+1} \), \( i \) being the inner iteration number, can be solved for independently of the other. The quasi-Newton update equation is similar to (2.13) and hence the updated vectors can be found by solving the following
matrix equations

\[
\begin{align*}
[2T^HT_1 + p\lambda^2\Lambda_1 + 2B^k]f_{i+1}^1 &= \gamma_D 2T^H_1 g_1 \\
[2T^HT_2 + p\lambda^2\Lambda_2 + 2B^k]f_{i+1}^2 &= \gamma_D 2T^H_2 g_2.
\end{align*}
\] (3.24)

The above equations are solved using conjugate gradient algorithm, typically with \(\gamma_D = 1\), as in the independent enhancement case. The result of each inner minimization, which is a pair of partially enhanced SAR images, say \(\hat{f}_1^k, \hat{f}_2^k\), are used for updating the vector \(\beta\) using the gradient descent algorithm. The gradient descent update equation is given by

\[
\beta^{k+1} = \beta^k + \alpha \nabla L_{\beta^k}
\] (3.25)

where \(\alpha > 0\) is the user-selected stepsize and

\[
\nabla L_{\beta^k} = |\hat{f}_1^k|^2 - |\hat{f}_2^k|^2
\] (3.26)

is the first derivative of \(L(f_1, f_2, \beta)\) with respect to \(\beta\). In this method, instead of directly solving the complicated optimization problem in (3.11), we adopt a step-by-step approach. The inner and outer loops aid each other in refining the updates and attempt to finally converge to the solution. The stopping criterion for this case was chosen to be the following,

\[
\frac{|L(\hat{f}_1^{k+1}, \hat{f}_2^{k+1}, \beta^{k+1}) - L(\hat{f}_1^k, \hat{f}_2^k, \beta^k)|}{L(\hat{f}_1^k, \hat{f}_2^k, \beta^k)} < \delta_{DD2}
\] (3.27)

as this would ensure the convergence of the inner loop too. The pseudocode for the Dual Descent method is given in Table 3.3.
Pseudocode For The Dual Descent Method

**Input:** Two SAR images $g_1, g_2$, stepsize $\alpha$, sparseness parameters $p, \lambda^2$ and convergence parameters $\delta_{DD_1}$ and $\delta_{DD_2}$.

**Output:** Two jointly enhanced images $f_1, f_2$.

**Given:** The forward operator matrices $T_1, T_2$.

**Store:** $H_1, H_2, Hh_1, Hh_2, T_1 H g_1, T_2 H g_2$.

**Initialize:** $|f_1^0| = |f_2^0| = \frac{|g_1|^2 + |g_2|^2}{2}$, Phase angles of $f_1^0, f_2^0$ same as those of $g_1, g_2$ respectively and $\beta^0 = 0$.

**The Algorithm**

```plaintext
1: while $err3 > \delta_{DD_2}$ do
2:   Update $\beta$ using its gradient : $\beta^{k+1} = \beta^k + \alpha \nabla L(\beta)$
3:   while Relative change in both $f_1^i, f_2^i > \delta_{DD_1}$ do
4:     Form diagonal matrices $\Lambda_1, \Lambda_2$ using vectors $f_1^i, f_2^i$ and equation (3.15)
5:     Solve for the vectors $f_1^{i+1}, f_2^{i+1}$ in equation (3.24) for a given $\beta^{k+1}$ using conjugate gradient method
6:     Relative change in $f_1^i, f_2^i$: $err_1 = \frac{\|f_1^{i+1} - f_1^i\|}{\|f_1^i\|}, err_2 = \frac{\|f_2^{i+1} - f_2^i\|}{\|f_2^i\|}$
7:   end while
8:   $f_1^{k+1} = f_1^{i+1}, f_2^{k+1} = f_2^{i+1}$
9:   $L(\beta) = |f_1^{k+1}|^2 - |f_2^{k+1}|^2$
10:  Relative change in $\beta$: $err3 = \frac{|L(f_1^{k+1}, f_2^{k+1}, \beta^{k+1}) - L(f_1^k, f_2^k, \beta^k)|}{L(f_1^k, f_2^k, \beta^k)}$
11: end while
12: $f_1 = f_1^{k+1}$ and $f_2 = f_2^{k+1}$
```

Table 3.3: Pseudocode For The Dual Descent Method

The main computation advantage of the Dual Descent method over the Lagrange-Newton method arises by the approximation that $\beta$ remains constant over the inner minimization, which allows us to use a modified form of the second-order method proposed in [3]. It must be noted here that the inner loop has quadratic convergence.
thereby, making the Dual Descent method converge faster as compared to Lagrange-
Newton method. Naturally this method is expected to be faster than the first order
Gradient Descent method; it is also computationally faster than the Lagrange-Newton
method, which requires solving a $5N \times 5N$ matrix equation for each update. Simu-
lation results in Chapter 4 show that this method is indeed much faster compared to
the other two methods.

Note that the Dual Descent method is closely related to bilevel programming which
belongs to a class of hierarchical optimization problems [15]. Bilevel programming
problems are two-level mathematical programs in which the set of optimal solutions
to the parameterized lower-level programs forms the feasible set of solutions for the
higher-level program.

3.4 Summary

This chapter has developed three methods for joint enhancement of multichannel
SAR data. First, we developed a direct solution to the problem using a first order
(Gradient Descent) method. Second, we derived a second order (Lagrange-Newton)
method that implicitly imposes the reconstruction magnitude constraint via a La-
grange method. The computations involved in the Lagrange-Newton method are
quite intensive, so we developed the Dual Descent method which has lower compu-
tational cost. In the next chapter, we shall present a comparative study of their
performances both with respect to computations involved and also the accuracy of
the final solution.
CHAPTER 4

SIMULATION RESULTS

In this chapter, we illustrate the performance of the proposed joint approach using synthetic, noisy two-elevation data. We compare the three algorithms derived in Chapter 3 with respect to computation time, computational complexity, and accuracy of the final solution. We also present the Cramér-Rao Lower Bound (CRLB) for the variance of the relative phase estimation error and study the performances of the independent and the joint enhancement approaches with respect to the CRLB. Finally we compare the performance of the independent and the joint enhancement techniques using 3D reconstructions of a synthetic dataset containing 40 scattering centers.

4.1 Synthetic Data Results

This section presents the preliminary results that illustrate the performance of the joint enhancement approach described in the previous chapter using two-elevation data which is generated synthetically. We simulate four point scatterers in the scene at the \((x, y, z)\) positions given by: \((0, 0, 0), (-1, 3, 0), (-2, -2, 0)\) and \((1, 1, 2)\) meters, and each with unit amplitude and zero phase shift. We simulate two linear flight path apertures, with center elevations of 30° and 30.05°; these are close enough in
elevation to permit IFSAR processing for scattering center height estimation \[7\]. Both apertures are 20° wide and centered at 0° azimuth. The center frequency and bandwidth of the SAR system are 10.16 GHz and 1 GHz respectively. Complex white Gaussian noise is added to the phase-history data such that the peak SNR (PSNR) of the scatterers in the image domain is 24 dB. The noisy phase-history data for the monostatic case (for each center elevation) can be generated by slightly modifying the equation (for bistatic SAR) given in \[16\],

\[ u_{mn} = \sum_{l=1}^{4} s_l e^{j \frac{4\pi f_m}{c} (x_l \cos \theta_n \sin \varphi_n + y_l \sin \theta_n \sin \varphi_n + z_l \sin \psi_n)} + z_{mn} \]  

where \( m = 1, \ldots, N_f \) and \( n = 1, \ldots, N_{az} \), \( s_l \) is the (complex-valued) amplitude of each scatterer located at \((x_l, y_l, z_l)\), \( f_m (m = 1, \ldots, N_f) \) are the frequencies swept by the chirp at each azimuth \( (\theta_n) \) and elevation \( (\psi_n) \) pair, \( z_{mn} \) is the complex white Gaussian noise and \( N_f, N_{az} \) are the number of frequency and azimuth samples respectively. In our simulations, we vary the azimuth angle from \(-10^\circ\) to \(10^\circ\) in steps of 0.0714 and the frequencies from 9.66 GHz to 10.66 GHz in steps of 11.55 MHz.

The slant-plane SAR images for the two elevation angles are formed using the polar format algorithm. A brief summary of the polar format algorithm is as follows: the data collection surface, which is defined in terms of the azimuth, elevation angles and frequencies used, is first projected onto the slant plane. The samples of the phase-history are multiplied with the Hamming window and then interpolated to a rectangular grid by using all the available data and zero padding wherever necessary. Finally, the inverse Fourier transform of the interpolated data gives the slant-plane image of the scene of interest. This procedure is implemented for the two available phase-history data collected from closely-spaced elevation angles and the corresponding SAR images are formed. The noisy images are first normalized using the peak
amplitude pixel value in that image before enhancing them (both independent and joint).

The relative phase angles for image domain pixels (within the top 35 dB of the peak pixel magnitude), are obtained by finding the phase difference between the corresponding pixels in the two images. These phases are compared with the relative phase angles of the noisy raw-data pixels, which we refer to as the true phase angles, in the following subsections. These relative phase angles from the image domain pixels are directly related to the height of the scatterer above or below the slant plane.

Figure 4.1 shows the resulting SAR slant-plane images found using the polar format algorithm with a Hamming window to reduce sidelobe artifacts. As can be seen, the two images which are formed from data collected from two close center elevation angles, are corrupted by noise.

4.1.1 Independent Enhancement Results

Figure 4.2 shows the result of independent image enhancement. Specifically, point-enhanced SAR images are obtained using the method proposed in [2] with $p = 1$ and $\lambda^2 = 0.8$. The magnitudes of the resulting images are shown in Figure 4.2. We have used for the termination condition, $\delta_{Ind} = 10^{-4}$, CG tolerance to be $\delta_{CG} = 10^{-3}$ and the approximation parameter in (2.10) as $\epsilon = 10^{-5}$.

Figure 4.3 shows the plots of the relative phase between large-amplitude pixels in the 30° and 30.05° elevation angles, as a function of range, for both the noisy images and the independently enhanced images. Here, large-amplitude pixels are defined as follows. The two noisy raw-data images are first masked using the mask containing only those pixels which are within the top 35 dB in the independently
Figure 4.1: Noisy SAR Slant plane images showing the top 35 dB for the two center elevation angles 30° and 30.05°.

enhanced images and then their relative phases are plotted. We see four clusters of relative phase values, corresponding to each of the four scattering centers in the images. The point scatterer at (0, 0, 0) is on the slant plane and therefore has a zero relative phase. The other three point scatterers do not lie on the slant plane and
Figure 4.2: Independently Enhanced SAR Slant plane images showing the top 35 dB for the two center elevation angles 30° and 30.05°.

hence have nonzero relative phases. It is clear from Figure 4.3 that processing the two images independently degrades the inter-channel information, compared with the polar format algorithm images in Figure 4.1. A relative phase error of say 10° causes a corresponding 0.5m error in height approximately. The height degradation due to
Figure 4.3: Relative Phase plots for the masked noisy images (top) and the corresponding Independently Enhanced images (bottom).

the relative phase estimation error has been shown through the Block “O” simulation results in Section 4.4.
4.1.2 Joint Enhancement Results

Figure 4.4 shows the resulting magnitude image for the joint enhancement case using the Gradient Descent method discussed in Section 3.1. Since the magnitudes of both the reconstructed images are equal, only a single magnitude-image is shown. The images were obtained using the Gradient Descent method, using $\lambda^2 = 0.8$ and $p = 1$ as in the independent enhancement case; we set $\delta_{GD} = 1 \times 10^{-3}$ and we have used $\gamma = 10^{-4}$. The Gradient Descent method was slower than the independent enhancement case and required about 40 times more computation than the independent enhancement technique. The convergence of the algorithm can be seen from the plot of the objective function, which is given by equation (3.2), calculated for each update in the Gradient Descent method. Figure 4.6 shows such a plot of the objective function as a function of the iteration number in the Gradient Descent method. Although the objective function appears to have nearly converged in the first few iterations, the relative change in the updated vectors $f_1^k$ and $f_2^k$ are not less than $\delta_{GD}$ which is our stopping criterion. It can be inferred from the plot that the stopping criterion can also be defined in terms of the relative change in the objective function values but with a lower value for $\delta_{GD}$ to ensure that the corresponding change in the updated vectors is also small. The relative phases corresponding to the four point scatterers for this case (see Figure 4.5) are considerably less varied than in the independent enhancement case, and hence would result in lower scattering center height estimation errors in subsequent IFSAR height estimation as the relative phases are linearly dependent on the height of the scatterer above the reference slant plane (equation (2.6)), as described in Section 2.2. This can be clearly seen from Figure 4.5 which shows plots of the relative phase from the jointly enhanced images and the corresponding masked
ones from the noisy raw-data images. The 3D reconstruction-accuracy of the joint enhancement technique is further explored in Section 4.4.

Figure 4.4: Jointly Enhanced (Gradient Descent) SAR Slant plane image showing the top 35 dB for the two center elevation angles 30° and 30.05°.

We observed from simulations that the convergence time for the Gradient Descent method was sensitive to the choice of $\delta_{GD}$. Figure 4.7 shows the logarithm of the convergence time as a function of $\delta_{GD}$. The Figure 4.7 shows that the convergence time increases drastically for a linear change in $\delta_{GD}$. The choice of $\delta_{GD}$, which is vital for the Gradient Descent method, is discussed in Subsection 4.3.3.
4.2 CRLB Analysis

This section presents the Cramér-Rao Lower Bound for the variance of the relative phase estimation error, which is derived in Appendix A, and studies the performances of the independent and the joint enhancements approaches with respect to the CRLB.
We also provide an expression for the Maximum Likelihood (ML) estimate of the relative phase in the phase-history domain and its relation to the image domain pixels.

The CRLB is a very important tool used for analyzing the estimation errors of parameters, as it helps us compare various techniques developed for estimating those parameters. The CRLB is a lower bound on the error variance for *any* unbiased estimator of the parameter. Specifically, when applied to our problem it provides a lower bound on the accuracy of the 3D-target reconstruction, against which we can compare various enhancement techniques.
We consider a generalized model of a point scatterer with unknown location \((x, y)\), amplitude \(s\), common phase \(\phi_c\) and relative phase \(\phi_\Delta\) from two closely-spaced elevation angles. The exact model for the generalized model (in the phase-history domain) of a point scatterer for the two-elevation case is as follows

\[
\begin{align*}
\mathbf{u} & = se^{j(\phi_c + ax + by + c \phi_\Delta)} + \mathbf{n}_u \\
\mathbf{v} & = se^{j(\phi_c + ax + by - c \phi_\Delta)} + \mathbf{n}_v
\end{align*}
\]

Note that the generalized model accommodates variations in the coefficients \(a, b\) and \(c\) of the location parameters and relative phase due to changes in azimuth, frequency

Figure 4.7: Convergence time for Gradient Descent Method as a function of \(\delta_{GD}\).
and elevation, which are very common in linear flight-path SAR. Moreover, no assumptions on the azimuth angle extent are made in this generalized model for a point scatterer. The derivation for the CRLB for the relative phase estimation error is provided in Appendix A. It can be seen from the Fisher Information matrix provided in Appendix A that the estimation error of the relative phase and those of the other parameters are decoupled. Although the CRLB has been derived for any \( c_k \) (see Appendix A), typically for the two-elevation linear flight-path SAR, the relative phase term, \( c_k \phi_\Delta \), in the phase-history does not vary much with azimuth and hence we can assume \( c_k = 1 \) for \( k = 1, \ldots, n \). So the CRLB for the relative phase, in this case, turns out to be,

\[
\text{CRLB}_{\phi_\Delta} = \frac{\sigma^2}{ns^2} \quad (4.4)
\]

where \( n \) is the number of samples in phase-history domain, \( \sigma^2 \) is the variance of the complex additive white Gaussian noise and \( s^2 \) is the signal power. Similarly, the Maximum Likelihood estimate of the relative phase \( \phi_\Delta \), for known location of the scatterer, which gives us the optimum estimate of the relative phase given the observations, is given by,

\[
\hat{\phi}_\Delta = \angle \left( \sum_{i=1}^{n} u_i e^{-j(a_i x + b_i y)} \right) \left( \sum_{i=1}^{n} v_i^* e^{j(a_i x + b_i y)} \right) \quad (4.5)
\]

where \( u_i \) and \( v_i \) are the noisy observations in the phase-history domain, \((x, y)\) denotes the location of the point scatterer, and where the coefficients \( a_i, b_i \) for \( k = 1, \ldots, n \) are fixed known constants computed from the azimuth, elevation and frequency used in the \( i^{th} \) measurement. An interesting connection between the ML estimate for the relative phase in the phase-history domain and that in the image domain can be obtained if we make use of the fact that the two domains are related by a Fourier
transform operation. Consider a scatterer placed at \((x, y) = (0, 0)\) in which case the ML estimate becomes 
\[ \hat{\phi}_\Delta = \angle[(\sum_{i=1}^{n} u_i)(\sum_{i=1}^{n} v_i^*)]. \]
This corresponds to finding the phase difference between the DC components of \(u_i\) and \(v_i^*\). So the maximum likelihood estimate of the relative phase, in the image domain is determined only by a single pixel in each of the images. This means, even though a point scatterer spreads in the image domain due to the limited phase-history data, the optimum estimate of the height of the point scatterer does not depend on any of the adjacent non-zero pixels but only on the one with the peak amplitude.

We next compare the variance of the height estimation errors before and after enhancement with respect to the CRLB. We place a scatterer at the location \((x, y, z) = (0, 0, 0)\) and select the elevation angles to be 0° and 0.0714° which would allow IFSAR processing. We then form the two noisy SAR images as described in Section 4.1. We assume the \((x, y)\) location of the scatterer to be known and hence the ML phase difference estimate between pixels corresponding to that pixel, which typically has the peak amplitude, would be the ML estimate for relative phase. If any unbiased estimate reaches the CRLB then it must be the ML estimate. Similarly, the two noisy images are enhanced using both independent and joint enhancement techniques (Dual Descent) and the corresponding estimates of the relative phase are found from the peak amplitude pixels after enhancement. This procedure is repeated for various PSNR values. A plot of the relative phase estimation error variance is shown in Figure 4.8 along with the CRLB. The Dual Descent method is used for the joint enhancement approach. The sparseness parameters are set at \(p = 1\) and \(\lambda^2 = 1.5\) for both the enhancement techniques. The stopping criterion for the independent enhancement is set at \(\delta_{Ind} = 10^{-4}\). The stopping criteria for the Dual Descent method
were set at $\delta_{DD2} = 3 \times 10^{-4}$ and $\delta_{DD1} = 5 \times 10^{-3}$; the inner loop criterion was set at a rather large value as it only requires partial convergence. We considered five PSNR values of 13, 19, 25, 32 and 39 dB and averaged the relative phase estimate error variance over 100 realizations for each PSNR value. The results are shown in Figure 4.8 both before and after enhancements. It can be seen clearly from Figure 4.8 that the performance of both the enhancement techniques lie very close to the CRLB, for all the PSNR values considered. This further indicates that both the enhancement techniques do not destroy the height information during the enhancement process when the scattering center location is known. It is worth mentioning that windowing
the phase-history data using, say, a Hamming window causes 2-3 dB error variance loss when compared to the CRLB. So for the CRLB simulations we have not used windowing.

In order to find if the location of the point scatterer influenced the ML estimation error variance, we placed 8 point scatterers at arbitrary locations and repeated the relative phase estimation procedure described previously. The averaged relative phase estimation error variance, found using a pair of noisy SAR images (without enhancement), over 100 realizations for each PSNR value, also was very close to the CRLB indicating that knowledge of the location of the scatterer, does not influence the relative phase estimation error.

The ML estimate considered here suggests that the height of a point scatterer must be estimated by using the relative phase of only the peak pixel. So one method for 3D reconstruction would be to find local maxima from the segmented SAR image and then find the height estimate from the relative phases of the corresponding local maxima. Although this is a good alternative in the case of point scatterers or in general for parametric models, it does not work well for realistic, complex targets in which scattering centers may not be isolated by distinct local maxima. On the other hand, for comparing various enhancement methods it is useful to consider parametric models and derive bounds on the error variance for the various parameters. We use the CRLB for this purpose, i.e. for comparing the independent and the joint methods from a parametric point of view.

Although CRLB serves as a performance metric for comparing the various techniques, it does not capture the accuracy of 3D-target reconstructions entirely using
the point scatterer model we have considered, as it depends only on the peak amplitude pixel. However, in our case a few adjacent pixels also represent the same point scatterer and hence their relative phase estimates must also be taken into account for determining the accuracy of the 3D-target that is being reconstructed. In fact that is where the independent enhancement technique fails as seen from the relative phase plots in the previous section. In Section 4.4 we show that the joint enhancement technique preserves the relative phase information even for the adjacent pixels, and this aids in an accurate 3D-target reconstruction. We further compare independent and joint enhancement techniques using another metric which takes into consideration the errors arising from inaccurate relative phase estimates of the adjacent pixels besides those resulting from the peak amplitude pixels.

4.3 Performance Comparisons of The Three Methods

In this section, we compare the performance of the three numerical methods used for solving the joint optimization problem. The computational, time complexities and accuracy of the final solutions for the three techniques are analysed in detail.

4.3.1 Computational Complexity

This subsection briefly analyses the computational time and algorithm computation complexity of the three numerical techniques derived in Chapter 3. In all the three techniques, the frequency domain representations of the Point Spread Function \((H_i, Hh_i)\), used for finding \(T_i f_i\) and \(T_i^H f_i\) for \(i = 1, 2\) are stored. For all the three methods, interestingly, the dominating terms for the computational complexity (number of multiplications per iteration of the algorithm) are \(T_i T_i^H f_i\) for \(i = 1, 2\), which are computed once for each iteration. So the computational complexity (for each
iteration) of the three methods can be approximated to be on the order of $N \log(N)$, $N$ being the number of pixels in the image, as all other multiplications required for each iteration are of a lesser order. On the other hand, the most important factor in determining the time complexity of the three methods analytically is the number of iterations the methods take to converge and its dependence on the number of pixels $N$, which is still an open research topic. Note that the computational complexity per iteration of the independent enhancement method, which involves two unconstrained optimization problems, is also $N \log(N)$.

We first compare the computational complexities of the three numerical techniques, by simulations, using the XpatchT Backhoe Dataset. The images of the XpatchT Backhoe are slant-plane images with a resolution of 3.8cm (about 1.5") in both the downrange and crossrange directions. The elevation angles, azimuth angles and aperture widths are the same as the ones used for generating synthetic data in the previous section, and the bandwidth is 3.96 GHz, which results in a 512 x 512 pixel-image at each elevation angle. Complex additive white Gaussian noise is added to the images such that the Peak Signal-to-Noise Ratio (PSNR) in the image domain is approximately 24 dB. In this case, white noise can be added in the image domain, as the images are critically sampled, and hence white noise in the phase-history domain will give white noise in the image domain. These two images are used as inputs to the three joint enhancement techniques for analyzing their computational times. The tolerance $\delta_{MR}$ for the minimum residual method running inside the Lagrange-Newton method was set to a value of $10^{-3}$, which ensures sufficient convergence for all cases. The tolerance $\delta_{CG}$ for the conjugate gradient method, running inside the Dual Descent method was set to $10^{-2}$.
We decrease the values of $\delta_{DD1}, \delta_{DD2}$ (stopping criteria) for the Dual Descent method and also the values of $\delta_{GD}$ and $\delta_{LN1}, \delta_{LN2}$ (stopping criteria) for the Gradient Descent and Lagrange-Newton methods respectively until the successive differences in their corresponding objective function values are not significant. This would also ensure that the differences in the enhanced images (top 35 dB) resulting from the three joint enhancement algorithms are not visually noticeable. The results of this experiment are shown in Table 4.1. As can be seen, the Lagrange-Newton method and the Gradient Descent method take almost the same computational time for such an experiment. The large computation time of the Lagrange-Newton method, which is a second order method, is due in part to the higher computational complexity resulting from the minimum residual method, which has to solve a $5N \times 5N$ matrix equation for every outer iteration. The fastest of the three methods is the Dual Descent method which is approximately 23 times as fast as the other two methods for our experiment.

Figure 4.13 shows the evolution of the objective function values for the three algorithms. Note that all three joint techniques have a very steep initial descent of the objective and reach very close to their final settling values within a few iterations. The rest of the computational time provides a gradual decrease in the cost function, and during this time, the algorithm reduces the sidelobes or noisy pixels to be as close to zero as possible. During this latter period, the pixels in the enhanced images within the top 35 dB do not undergo much of a change. Hence, it is difficult to precisely quantify the differences in the computational times of the three methods in general, and therefore it is a proposed topic of future research.
Figure 4.9: Noisy SAR Slant plane images of the Backhoe showing the top 35 dB for the two center elevation angles 30° and 30.05°.

4.3.2 Accuracy of Final Solutions

We next compare the accuracy of the final solutions obtained using the three numerical techniques. Although all three techniques are solving the common problem
of joint enhancement, there is a slight difference in the way the equal magnitude constraint has been included in the cost function. Moreover, the joint optimization problem defined in the beginning of Chapter 3 has non-convex feasible set, so convergence to a local minimum is possible. Since the cost functions used in the three techniques are not the same, we have to consider a common cost function for which all the three methods can be compared fairly. We take this common cost function to be,

$$L(f_1, f_2) = \frac{\|g_1 - T_1 f_1\|_2^2 + \|g_2 - T_2 f_2\|_2^2}{1} + \frac{\lambda_1^2 \|f_1\|_p^p + \lambda_2^2 \|f_2\|_p^p}{2} + \sum_{i=1}^{N} (|(f_1)_i|^2 - |(f_2)_i|^2))$$  (4.6)

This is the cost function similar to the one used for the Lagrange-Newton and Dual Descent methods (see equation (3.12)) but without the Lagrange multipliers \(\beta_i\)s. In the Gradient Descent method, the magnitudes of the two final images are exactly the same; hence the last term in (4.6) vanishes for the Gradient Descent method, and (4.6) has the same value as the cost function in (3.6). We use the XpatchT Backhoe data discussed in Subsection 4.3.1 for comparing the accuracy of the three methods. Table 4.2 summarizes the results. The Lagrange-Newton and the Gradient Descent methods result in images with many pixels having manitudes very close to zero (\(\approx 0.0001\)) as compared to the Dual Descent method which results in many small-valued pixels (\(\approx 0.001\)) which when summed over the number of pixels (\(\approx 200 \times 200\)), results in the observed differences in the values of the second term of the objective function. But the differences in the enhanced images whose top 35 dB are shown, are hardly noticeable in all the three cases. This is seen from the enhanced and noisy raw-data
The evolution of the objective function values for the three joint techniques are shown in Figure 4.13 and we can see that the three algorithms have converged. Moreover we can also notice that all the three algorithms reach very close to their final objective function values in the first few seconds and then take a long time to reach the stopping criteria, which depend on the relative change in the images. We also observe that the top 35 dB of the images do not change much after the first few seconds/iterations as discussed previously. The contribution of each of the three terms in equation (4.6) to the objective function value, for the three algorithms, is shown in Table 4.2.

Figure 4.10: Jointly Enhanced (Gradient Descent) SAR Slant plane image of the Backhoe showing the top 35 dB for the two center elevation angles 30° and 30.05°
4.3.3 Choice of Parameters for the Algorithms

This subsection discusses the choice of various user-selected parameters in the algorithms, such as the stepsize, sparseness parameters, and convergence parameter.
Figure 4.12: Jointly Enhanced (Dual Descent) SAR Slant plane images of the Backhoe showing the top 35 dB for the two center elevation angles 30° and 30.05°.

for the three algorithms developed in Chapter 3. The sparseness parameters $p$ and $\lambda^2$ are common to all the three algorithms. We choose $p = 1$ for both the independent and the joint enhancement techniques as any other value of $p$ less than one would make the
problem highly non-convex. It is worth mentioning here that the joint enhancement problem described in the beginning of Chapter 3 results in a non-convex feasible set, even for $p = 1$, because of the equal magnitude constraint. For the selection of the sparseness parameter $\lambda^2$, we studied the variation of number of pixels (both for independent and joint enhancements) in the top 35 dB for a point scatterer with respect to $\lambda^2$ for a PSNR of 25 dB. We compared the independent enhancement technique to the Dual Descent joint enhancement technique. The remaining two joint enhancement techniques give similar results as the Dual Descent method. Figure 4.14 shows the results of the simulation; it is clear that the number of pixels reaches a minimum and stays there for both the independent and the joint enhancement techniques for $\lambda^2 \geq 1.5$. We also observed that as $\lambda^2$ is increased beyond 2 the
<table>
<thead>
<tr>
<th>Method</th>
<th>Stopping Criterion</th>
<th>Time Taken (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient Descent</td>
<td>$\delta_{GD} = 3 \times 10^{-3}$</td>
<td>$161.92 \times 10^2$</td>
</tr>
<tr>
<td>Lagrange-Newton</td>
<td>$\delta_{LN1} = \delta_{LN2} = 3 \times 10^{-3}$</td>
<td>$151.26 \times 10^2$</td>
</tr>
<tr>
<td>Dual Descent</td>
<td>$\delta_{DD1} = 5 \times 10^{-3}, \delta_{DD2} = 10^{-4}$</td>
<td>$6.71 \times 10^2$</td>
</tr>
</tbody>
</table>

Table 4.1: Comparison of the Three Joint Enhancement Methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>First Term</th>
<th>Second Term</th>
<th>Third Term</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient Descent</td>
<td>$3.36 \times 10^2$</td>
<td>$1.50 \times 10^2$</td>
<td>0</td>
<td>$4.86 \times 10^2$</td>
</tr>
<tr>
<td>Lagrange-Newton</td>
<td>$4.13 \times 10^2$</td>
<td>$1.41 \times 10^2$</td>
<td>0.75</td>
<td>$5.55 \times 10^2$</td>
</tr>
<tr>
<td>Dual Descent</td>
<td>$3.25 \times 10^2$</td>
<td>$1.86 \times 10^2$</td>
<td>1.10</td>
<td>$5.12 \times 10^2$</td>
</tr>
<tr>
<td>Independent</td>
<td>$3.25 \times 10^2$</td>
<td>$2.36 \times 10^2$</td>
<td>3.60</td>
<td>$5.65 \times 10^2$</td>
</tr>
</tbody>
</table>

Table 4.2: Contribution of the Three Terms in equation (4.6) to the Objective Function Value.

algorithms takes more time to converge. For these reasons, we have selected the value of $\lambda^2$ to be between 1.5 and 2 in this thesis.

We next consider the selection of the stepsize $\gamma$. For the Gradient Descent method, the choice of the stepsize $\gamma$ and the convergence parameter $\delta_{GD}$ are interlinked — a lower value for $\delta_{GD}$ would also demand a corresponding lower value for $\gamma$, if the stepsize remains constant throughout and no line search is used in each iteration to determine the stepsize. We have not used line search for determining the stepsize at each iteration for any of the algorithms since it involves additional computation. We found empirically that, for the values of $\delta_{GD}$ between $1 \times 10^{-3}$ and $5 \times 10^{-3}$, which
typically reaches very close to the minimum of the objective function, a stepsize of $10^{-4}$ works well. So we have used $\gamma = 10^{-4}$ for the Gradient Descent method throughout this thesis.

The user-selected parameters in the Lagrange-Newton method are the convergence parameters $\delta_{LN1}$, $\delta_{LN2}$. These are taken to be equal and are set between $1 \times 10^{-3}$ and $5 \times 10^{-3}$ as in the Gradient Descent method. The two parameters are set to be equal as we expect similar convergence in both the enhanced images and the Lagrange multipliers. Secondly, since the convergence criteria are similar for the Gradient Descent and the Lagrange-Newton methods, their convergence parameters can also be the same and set between $1 \times 10^{-3}$ and $5 \times 10^{-3}$.
Finally, for the Dual Descent method, a suitable stepsize must be chosen for updating the Lagrange multipliers $\beta$, which are the weights for the squared differences between the two images. The stepsize $\alpha$ in this case is chosen to be two orders of magnitude higher than the stepsize $\gamma$ in the Gradient Descent method, thereby ensuring that the contribution of the third term is comparable to the second term in (3.12). Hence we used $\alpha = 10^{-2}$ for all our simulations involving the Dual Descent method. As mentioned in Subsection 4.1.2 if the stopping criterion depends on the relative change in the objective function value, then the convergence parameter has to be small enough to ensure that the relative change in the updated vectors are also small. Hence, in the Dual Descent method, we set the convergence parameter $\delta_{DD2}$ between $3 \times 10^{-4}$ and $10^{-4}$. However, the convergence parameter for the inner loop of the Dual Descent method, is not made as small as in the independent enhancement case ($\delta_{Ind} = 10^{-4}$), but instead set to a higher value, say between $5 \times 10^{-3}$ and $7 \times 10^{-3}$, which would ensure only partial convergence as required.

Finally a word on the initialization for the Lagrange-Newton and the Dual Descent methods. Although we have initialized the Gradient Descent method by taking the average of the magnitudes of the observed vector, we found through simulations that such an initialization results in higher computation time for the other two methods. So we initialized the other two methods using the observed vector itself (for both magnitude and phase). Note that this initialization cannot be applied for the Gradient Descent method, as there are only $3N$ variables to be optimized whereas the other two methods have $4N$ variables each.
4.4 The Block “O” Example

This section considers a 3D reconstruction example and quantifies the performance improvement that can be achieved by using the proposed joint enhancement technique over the independent enhancement technique. We simulate a set of 40 point scatterers placed at two discrete heights from the ground in such a way that they have the shape of The Ohio State University’s Block “O” icon. Twenty point scatterers form the outer ring of the Block “O”, and are located at ground height; twenty other scattering centers form the inner ring, at a height of 1m above the ground. The 3D reconstruction of the Block “O” for the noiseless case using ideal point scatterers is shown in Figure 4.15.

We simulate two linear flight-path apertures with center elevation angles of $0^\circ$ and $0.0714^\circ$, which are close enough to enable IFSAR height estimation from this data. The bandwidth and center azimuth angle of the SAR system were 1 GHz and $0^\circ$ respectively. We set the center frequency of the radar to be $10.16$ GHz. Complex additive white Gaussian noise was added in the phase-history domain such that the PSNR in the image domain is 25 dB. As described in Section 4.1, the two images $(256 \times 256)$ corresponding to the two closely-spaced elevation angles are formed using the polar format algorithm. The noisy images of the Block “O”, formed on the ground plane in this case, are shown in Figure 4.16.

The two noisy ground-plane images formed above are independently enhanced using the algorithm proposed in [3], which we have described briefly in Chapter 2. The sparseness parameters were chosen to be $\lambda^2 = 2$ and $p = 1$ respectively. We used $\delta_{Ind} = 10^{-4}$ as the stopping criterion, $\delta_{CG} = 10^{-2}$ as the CG tolerance, and the $p$-norm approximation parameter $\epsilon = 10^{-5}$. The point-enhanced images are
Figure 4.15: Noiseless 3D reconstruction of the Block “O”.

shown in Figure 4.17. The two images are first masked using a mask containing only those pixels which are within the top 25 dB of the maximum pixel in both the images. These two masked independently-enhanced images are used to form the 3D reconstruction of the Block “O” by finding the height estimate for each pixel using the IFSAR equation (2.6), which employs the ML estimate for the relative
Figure 4.16: Noisy SAR Ground-plane images of the Block “O” showing the top 35 dB for the two center elevation angles 0° and 0.0714°.

Although this enhancement technique has effectively reconstructed the point scatterers, when viewed parallel to the downrange-crossrange plane, the degradation in the relative phase, as explained in Subsection 4.1.1 has
been translated as a corresponding degradation in height from the slant plane, which in this case is the ground plane, as shown in Figure 4.18.

Figure 4.17: Independently-Enhanced SAR Ground-plane images of the Block “O” showing the top 25 dB for the two center elevation angles 0° and 0.0714°.
Similarly, the two noisy images are jointly enhanced using the Dual Descent method described in Section 3.3 using the same sparsity parameters as mentioned above. We set $\delta_{DD2} = 3 \times 10^{-4}, \delta_{DD1} = 5 \times 10^{-3}$ as the stopping criterion parameters for the outer and the inner loops respectively and the CG tolerance for the inner loop.
loop was set to $\delta_{CG} = 10^{-2}$. The enhanced images are shown in Figure 4.19. A similar 3D reconstruction of the Block “O” is shown in Figure 4.20 and the joint enhancement has effectively reconstructed the point scatterers when viewed parallel to the downrange-crossrange plane. The 3D reconstruction resulting from the jointly-enhanced images is sparser than the corresponding reconstruction from the independently-enhanced images. Moreover, the relative phase degradation is lower, as seen in Figure 4.20 compared to the one shown in Figure 4.18. In Figure 4.20 one can more readily discern the height difference between the inner ring points and the outer ring points.

In order to quantify the results obtained using the independent, we consider the following Euclidean error metric defined in terms of the estimated location of the target in 3D,

$$E_{3D} = \frac{1}{K} \sum_{i=1}^{K} \| (\hat{x}_i, \hat{y}_i, \hat{z}_i) - (x_{j(i)}, y_{j(i)}, z_{j(i)}) \|^2$$  \hspace{1cm} (4.7)

where

$$j(i) = \arg \min_{1 \leq j \leq L} \{ (x_j - \hat{x}_i)^2 + (y_j - \hat{y}_i)^2 \}$$  \hspace{1cm} (4.8)

where $K$ is the number of pixels resulting after the RCS mask is applied to both the images, $L$ is the total number of point scatterers each of whose actual location is $(x_j, y_j, z_j)$, and $(x_i, y_i, z_i)$ is the estimated location of the $i^{th}$ pixel. The minimization inside the summation term is used for clustering each of the estimated points to one of the $L$ actual point scatterers for finding the Euclidean error. This is done only in the downrange-crossrange directions since the points are well separated on that plane and clustering becomes easy. The $\frac{1}{K}$ factor in the above equation is included for finding the average RMS error per pixel and would give a fair comparison between the errors resulting from independent and joint enhancements, which result in different number
Figure 4.19: Jointly-Enhanced (Dual Descent) SAR Ground-plane images of the Block “O” showing the top 25 dB for the two center elevation angles 0° and 0.0714°.

of pixels after the enhancement process. The error values of the 3D reconstructions using the independent and joint enhancements are $E_{3D}^{ind} = 0.0173$ and $E_{3D}^{joint} = 0.0073$. Thus the proposed technique preserves the 3D location of the targets during the
Figure 4.20: 3D Reconstruction of Block “O” using Jointly-Enhanced SAR images.

enhancement process with an error slightly less than 1/2 the error obtained using the independent enhancement technique.

To see if the better performance of the joint enhancement is a result of fewer pixels after enhancement, we investigate the effect of image sparsity on 3D error. Table 4.3
shows the RMS error values as a function of $p$, which controls the sparsity of the reconstructed image. We fix the value of $\lambda^2 = 2$ and the PSNR=25 dB for this study.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$E_{3D}^{\text{ind}}$</th>
<th>$E_{3D}^{\text{joint}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0173</td>
<td>0.0073</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0106</td>
<td>0.0073</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0071</td>
<td>0.0076</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0070</td>
<td>0.0070</td>
</tr>
</tbody>
</table>

Table 4.3: The Euclidean Error Metric for the Independent and Joint Enhancement Techniques as a function of $p$ for $\lambda^2 = 2$ and PSNR = 25 dB.

The results in Table 4.3 indicate that, as more sparsity is imposed the error in both independent and joint enhancements become equal. This suggests that the independent enhancement performs as well as the joint enhancement when the solution is sparse. Another important point to note from Table 4.3 is that the joint enhancement has near minimum error even for $p = 1$ and remains unchanged as $p$ is reduced. The $p = 1$ solution is generally preferred as the optimization problem is convex and the convergence is much faster than for $p < 1$, for both the independent and the joint enhancement techniques.
Table 4.4: The Euclidean Error Metric for the Independent and Joint Enhancement Techniques as a function of $p$ for $\lambda^2 = 2$ and PSNR = 15dB.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$E_{3D}^{\text{ind}}$</th>
<th>$E_{3D}^{\text{joint}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0438</td>
<td>0.0183</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0300</td>
<td>0.0178</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0200</td>
<td>0.0180</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0190</td>
<td>0.0200</td>
</tr>
</tbody>
</table>

Table 4.4 shows the results of a similar analysis but using a lower PSNR (15 dB). The errors for independent and joint enhancements become equal for lower $p$ values, as was the case in Table 4.3. The slight difference in the error values for the high and low PSNR values, say for $p = 0.7$, is because of the same value of $\lambda^2$ used for both the cases. The choice of $\lambda^2$ depends both on sparsity required and the operating PSNR. An explicit expression relating $\lambda^2$ and PSNR is an open research topic.

An interesting way of interpreting the results would be that for low $p$ values both the techniques (independent and joint enhancement) typically result in one pixel for each point scatterer. This pixel is usually the peak pixel of that point scatterer. As discussed in Section 4.2, the ML estimate of the relative phase, which depends on the phase difference between the peak pixels, has an estimation error variance which reaches the CRLB for both the independent and joint enhancement techniques. Hence, for the case when the independent and joint enhancement result in a single pixel for each point scatterer, the height estimation errors are the same, resulting in almost the same $E_{3D}$ error values.
The choice of $p$ is critical for independent enhancement since the value of $p$ decides its performance by increasing sparsity and a very low fixed value for $p$ (say 0.1) would make the optimization problem highly non-convex and time consuming. The time taken for the independent enhancement technique to converge for $p = 0.8$ is 2.5 times as high as that for $p = 1$. Moreover, when the images are not critically sampled, finding the value of $p$ which would result in a single pixel (the sparsest solution) itself is an open problem. In such cases, joint enhancement, which typically results in a single pixel even for $p = 1$ is a good alternative, even though it requires a little more time for convergence than the independent enhancement for $p = 1$.

4.5 XPatchF Backhoe Dataset Results

This section presents simulation results of the independent and joint enhancement techniques and their subsequent 3D reconstructions using the XpatchF Backhoe Dataset. Unlike the XpatchT dataset, the XpatchF dataset contains the synthetic phase-history measurements of a backhoe using a frequency range of 7-12 GHz and for the entire $360^\circ$ azimuth. We consider two closely-spaced elevation angles of $29.5^\circ$ and $29.5714^\circ$ from the XptachF Dataset. We also set the bandwidth and center frequency to be 640 MHz and 10 GHz respectively. We form 72 slant-plane SAR images for each elevation angle, each of whose center azimuth is spaced by $5^\circ$ starting with $0^\circ$ azimuth and with a symmetric azimuth span of $10^\circ$ around the center azimuth; note that there is an overlap of $5^\circ$ between any two successive SAR images formed. Before forming the images, complex additive white Gaussian noise is added to the entire $360^\circ$ phase-history data such that the noise variance $\sigma_n^2 = 0.0039$. This noise
level corresponds to a PSNR of between 40 dB and 60 dB across the 144 images. A Facet model of a backhoe is shown in Figure 4.21.

![Facet Model of a Backhoe](image)

Figure 4.21: Facet Model of a Backhoe.

First, the 144 noisy SAR images are each independently enhanced for both the elevation angles using the sparseness parameters $p = 1$ and $\lambda^2 = 0.8$. We have chosen a lower value for $\lambda^2$ here since the SNR varies significantly across the images; a higher value of $\lambda^2$ results in convergence issues. We set the stopping criterion to be $\delta_{ind} = 10^{-4}$, the CG tolerance as $\delta_{CG} = 10^{-3}$ and the approximation parameter as $\epsilon = 10^{-5}$. These enhanced images are then used for 3D reconstruction of the backhoe by first retaining only the pixels within the top 35 dB for each pair of images having the same center azimuth but different elevation angles. A 3D reconstruction of the backhoe is obtained for each pair of point enhanced slant-plane SAR images with the same center azimuth by first finding the height corresponding to each pixel using the
IFSAR equation (2.6). Each pixel along with its height estimate represents a point in space with respect to the reference slant plane, which in our case has an elevation angle of 29.5357°, used in determining the height estimate. Hence the location of those points with respect to the ground is determined by a simple rotation of the range-height plane by 29.5357°. Each 3D reconstruction obtained is then rotated by the center azimuth of the images used for the reconstruction and hence the true location of the points with respect to a common co-ordinate system is obtained. The common co-ordinate system allows us to overlay the points obtained from 3D reconstructions corresponding to various center azimuths. The 3D reconstruction of the backhoe using the independently-enhanced slant-plane SAR images is shown in Figure 4.22.

![Figure 4.22: Side view of the Backhoe Reconstructed using Independently-Enhanced Images. Colour denotes the amplitude of the reconstructed point.](image)

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Similarly, each pair of slant-plane SAR images with the same center azimuth is jointly enhanced using the Dual Descent method, resulting in 72 pairs of jointly-enhanced images which are used for 3D reconstruction of the backhoe. The sparseness parameters were taken to be the same as in the independent enhancement but the CG tolerance $\delta_{CC}$ for the inner loop was set to $10^{-2}$. The stopping criteria for the outer and inner loops of the Dual Descent algorithm were set as $\delta_{DD2} = 3 \times 10^{-4}$ and $\delta_{DD1} = 5 \times 10^{-3}$ respectively. The jointly-enhanced images are then processed the same way as the independently-enhanced ones for forming the 3D reconstruction of the backhoe. This reconstruction is shown in Figure 4.23.

Figure 4.23: Side view of the Backhoe Reconstructed using Jointly-Enhanced Images. Colour denotes the amplitude of the reconstructed point.
For both the 3D reconstructions, the marker size used in the plots was modulated by the RCS value at that point so that points with higher RCS values are represented using large markers. In addition to this, colours were also used to modulate the RCS value at each point.

As can be seen from the 3D reconstructions of the backhoe, both the independent and joint enhancement techniques seem to have faithfully reconstructed the backhoe. The 3D reconstruction using the independent enhancement technique resulted in approximately 2 times the number of 3D points as the joint enhancement technique, since the Dual Descent method typically results in a sparser number of pixels. But from Figure 4.23 we can see that the Dual Descent method has retained the strong backscatters from the backhoe and not rejected them during its search for a sparse representation.
CHAPTER 5

CONCLUSIONS AND FUTURE DIRECTIONS

In this chapter, we first summarize the main contributions of the thesis and then conclude by mentioning some future directions that can be explored.

5.1 Summary

We considered the problem of multi-channel enhancement in this thesis. We argued that the traditional methods for SAR image enhancement are not the best options for solving this problem since these methods do not explicitly enforce common sparsity support which is needed for height estimation from IFSAR image pairs. We formulated a joint approach for image enhancement as a constrained joint optimization problem. We then developed three numerical methods for solving this joint optimization problem. One method is a gradient-based approach and another method is a second order Lagrange-Newton approach. We found that the time taken for the Lagrange-Newton method to converge was high and hence derived a third method, the Dual Descent method, which is shown to be the fastest computationally among the three methods.

In Chapter 4, we showed that the relative phase, between SAR images formed from two closely-spaced elevations is preserved better by jointly enhancing the two
images than by independently enhancing them. We showed this first by using a simple SAR scene containing four point scatterers in noise. As expected, the independent enhancement had higher error in the relative phases of high amplitude image pixels than did the joint enhancement (Gradient Descent) methods. Since relative phase is linearly related to scattering center height, this result suggests that the proposed method would result in more accurate 3D reconstruction of the scene.

A comparison of the relative phase error for the independent and the joint enhancement techniques with the Cramér-Rao Lower Bound (CRLB) shows that both the methods have statistical errors that are close to the CRLB. However, the derived ML estimation error does not include the errors in the height estimates of the adjacent pixels, for the model we have considered, and errors in these phases of the adjacent pixels lead to poor 3D reconstructions. We proposed an error metric which takes into account position estimation errors of the adjacent pixels, and showed that the independent enhancement method gave higher location errors unless the enhancements were forced to be very sparse. We have also shown that the sparsity parameter $p$ plays a central role in determining the performance of the independent enhancement technique for this 3D location error metric.

We compared the three numerical methods with respect to the computational time, computational complexity and accuracy of the final solution. The choice of various parameters for the three numerical methods has also been discussed.

Finally we presented the results of the joint enhancement technique for the backhoe dataset thereby demonstrating its applicability for real-world targets.
5.2 Future Research Directions

This section briefly discusses the possible future directions for the multi-channel enhancement problem. An interesting research direction would be to mathematically extend the joint enhancement problem for the multi-channel case \( n > 2 \) and develop effective numerical solutions for solving it. Although a gradient-based method would be relatively easy to develop, a second-order method like Lagrange-Newton requires multiple constraints to be included in the cost function, which would be difficult for the multi-channel case. Such an approach for the multi-channel case would minimize the total variation of the magnitudes across the images.

Another interesting research direction would be finding an automatic method for choosing the sparsity parameter \( p \), for the independent enhancement technique. The choice of \( p \) entails a tradeoff between the sparsest solution achievable and the time for convergence because of the increased non-convex nature of the problem. Issues such as proper initialization of the algorithms have to be addressed since varying \( p \) need not ensure convergence. Similarly developing a clear understanding, possibly an explicit expression, for how to choose \( \lambda^2 \) as a function of the SNR and \( p \) would be very useful.
APPENDIX A

CRAMÉR RAO BOUND DERIVATION

We derive the Cramér-Rao Lower Bound (CRLB) for the estimation error of the relative phase using a general model for a point scatterer. The CRLB is a lower bound on the error variance for any unbiased estimator of the parameter vector. The generalized model for a point scatterer is motivated from the phase-history data generation equation (4.1) provided in Chapter 4. In our case, we assume that there is only one point scatterer in the scene and so the equation (4.1) becomes,

\[ u_{lk} = se^{j\frac{4\pi f_l}{c}(x \cos \theta_k \cos \psi_k + y \sin \theta_k \cos \psi_k + z \sin \psi_k)} + z_{lk} \]  \hspace{1cm} (A.1)

where \( l = 1, \ldots, N_f, k = 1, \ldots, N_{az} \), and where \( N_f, N_{az} \) are the number of frequency and azimuth samples, respectively. The above model, when applied for the two-elevation case, results in two sets of phase-history data \( \{u_{lk}\} \) and \( \{v_{lk}\} \) corresponding to each elevation angle. These data can be expressed as vectors:

\[ u = se^{j(\phi_c + ax + by + c \frac{\phi_c}{\lambda})} + n_u \]  \hspace{1cm} (A.2)

\[ v = se^{j(\phi_c + ax + by - c \frac{\phi_c}{\lambda})} + n_v \]  \hspace{1cm} (A.3)

where the parameters to be estimated are the location \( (x, y) \) of the scatterer, its (real-valued) amplitude \( s \), the common phase \( \phi_c \) (including the phase shift introduced by
the scatterer) and the relative phase $\phi_{\Delta}$, resulting from its height above the slant plane. The real-valued $n \times 1$ vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ have known elements defined from the measurement geometry (see equation (A.1)). The complex-valued amplitude in equation (4.1) has been written as a real-valued amplitude $s$ and a common phase $\phi_{c}$ in our model here. $\mathbf{n}_u, \mathbf{n}_v$ represent the complex white Gaussian noise added to the phase-history data with a variance of $\frac{\sigma^2}{2}$ each on the real and imaginary parts.

We take $\mathbf{u}, \mathbf{v}, \mathbf{n}_u$ and $\mathbf{n}_v$ as $n \times 1$ complex vectors, where $n = N_f \cdot N_{az}$ is the total number of samples in the phase-history domain. We define

\[
e^{j(\phi_c + ax + by + c \frac{\Delta}{2})} = \begin{bmatrix}
e^{j(\phi_c + a_1 x + b_1 y + c_1^2 \frac{\Delta}{2})} \\
e^{j(\phi_c + a_2 x + b_2 y + c_2^2 \frac{\Delta}{2})} \\
\vdots \\
e^{j(\phi_c + a_n x + b_n y + c_n^2 \frac{\Delta}{2})}
\end{bmatrix}
\]

\[
e^{j(\phi_c + ax + by - c \frac{\Delta}{2})} = \begin{bmatrix}
e^{j(\phi_c + a_1 x + b_1 y - c_1 \frac{\Delta}{2})} \\
e^{j(\phi_c + a_2 x + b_2 y - c_2 \frac{\Delta}{2})} \\
\vdots \\
e^{j(\phi_c + a_n x + b_n y - c_n \frac{\Delta}{2})}
\end{bmatrix}
\]  

We derive the Fisher Information matrix and show that the estimation error of $\phi_{\Delta}$ is decoupled from that of the other four parameters.

We define $\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$, $\phi_p = \begin{bmatrix} x \\ y \\ \phi_c \\ s \\ \phi_{\Delta} \end{bmatrix}$ and $\mu = \begin{bmatrix} \mu_u \\ \mu_v \end{bmatrix}$ where $\mu_u = e^{j(\phi_c + ax + by + c \frac{\Delta}{2})}$ and $\mu_v = e^{j(\phi_c + ax + by - c \frac{\Delta}{2})}$. So the conditional probability of the set of observations given the vector $\phi_p$ is given by,

\[
p(x|\phi_p) = \frac{1}{C} e^{(x-s\mu)^H \Sigma^{-1}(x-s\mu)}
\]  

where $C$ is a constant independent of $\phi_p$ and $\Sigma = \sigma^2 I$, $I$ being the identity matrix of size $2n$. The second order partial derivatives of $\log(p(x|\phi_p))$ with respect to the five
parameters $x, y, s, \phi_c$ and $\phi_{\Delta}$, which are necessary for determining the CRLB on the variance of the corresponding estimation errors for the parameters, are as follows,

\[
\frac{\partial^2}{\partial x^2} \log(p(x|\phi_p)) &= -\frac{s}{\sigma^2} [\mu_u^H A^2 u + \mu_v^H A^2 v + u^H A^2 \mu_u + v^H A^2 \mu_v] \\
\frac{\partial^2}{\partial y^2} \log(p(x|\phi_p)) &= -\frac{s}{\sigma^2} [\mu_u^H B^2 u + \mu_v^H B^2 v + u^H B^2 \mu_u + v^H B^2 \mu_v] \\
\frac{\partial^2}{\partial \phi_c^2} \log(p(x|\phi_p)) &= -\frac{s}{\sigma^2} [\mu_u^H v + \mu_v^H + u^H \mu_u + v^H \mu_v] \\
\frac{\partial^2}{\partial s^2} \log(p(x|\phi_p)) &= \frac{4}{\sigma^2} \\
\frac{\partial^2}{\partial \phi_{\Delta}^2} \log(p(x|\phi_p)) &= -\frac{s}{4\sigma^2} [\mu_u^H C^2 u + \mu_v^H C^2 v + u^H C^2 \mu_u + v^H C^2 \mu_v] (A.6)
\]

where $A, B, C$ are $(n \times n)$ diagonal matrices with $a_k, b_k, c_k$ (for $k = 1, \ldots, n$) as the diagonal entries respectively. The mixed partial derivatives with respect to the parameters are also straightforward and are given by

\[
\frac{\partial^2}{\partial x \partial y} \log(p(x|\phi_p)) &= -\frac{s}{\sigma^2} [\mu_u^H AB u + \mu_v^H AB v + u^H AB \mu_u + v^H AB \mu_v] \\
\frac{\partial^2}{\partial x \partial \phi_c} \log(p(x|\phi_p)) &= -\frac{s}{\sigma^2} [\mu_u^H A v + \mu_v^H Av + u^H A \mu_u + v^H A \mu_v] \\
\frac{\partial^2}{\partial x \partial s} \log(p(x|\phi_p)) &= -\frac{j}{\sigma^2} [\mu_u^H A u + \mu_v^H A u - u^H A \mu_u - v^H A \mu_v] \\
\frac{\partial^2}{\partial x \partial \phi_{\Delta}} \log(p(x|\phi_p)) &= -\frac{s}{2\sigma^2} [\mu_u^H AC u - \mu_v^H AC v + u^H AC \mu_u - v^H AC \mu_v] \\
\frac{\partial^2}{\partial s \partial \phi_c} \log(p(x|\phi_p)) &= -\frac{j}{\sigma^2} [\mu_u^H u + \mu_v^H v - u^H \mu_u - v^H \mu_v] \\
\frac{\partial^2}{\partial s \partial \phi_{\Delta}} \log(p(x|\phi_p)) &= -\frac{j}{2\sigma^2} [\mu_u^H C u - \mu_v^H C v - u^H C \mu_u + v^H C \mu_v] \\
\frac{\partial^2}{\partial \phi_c \partial \phi_{\Delta}} \log(p(x|\phi_p)) &= -\frac{1}{\sigma^2} [\mu_u^H C u - \mu_v^H C v + u^H C \mu_u - v^H C \mu_v] (A.7)
\]

The mixed partials involving $y$ can be obtained from the ones involving $x$ by simply replacing the $A$ matrix by the $B$ matrix. The Fisher Information matrix, in our case,
is as follows

\[ I_{\phi_p} = -E \begin{bmatrix}
\frac{\partial^2}{\partial s^2} \log(p(x|\phi_p)) & \frac{\partial^2}{\partial s \partial \phi} \log(p(x|\phi_p)) & \frac{\partial^2}{\partial \phi \partial \phi} \log(p(x|\phi_p)) & \frac{\partial^2}{\partial s \partial \phi \partial \phi} \log(p(x|\phi_p)) \\
\frac{\partial^2}{\partial \phi \partial x} \log(p(x|\phi_p)) & \frac{\partial^2}{\partial \phi \partial y} \log(p(x|\phi_p)) & \frac{\partial^2}{\partial \phi \partial \phi} \log(p(x|\phi_p)) & \frac{\partial^2}{\partial \phi \partial \phi \partial \phi} \log(p(x|\phi_p)) \\
\frac{\partial^2}{\partial \phi \partial x} \log(p(x|\phi_p)) & \frac{\partial^2}{\partial \phi \partial y} \log(p(x|\phi_p)) & \frac{\partial^2}{\partial \phi \partial \phi} \log(p(x|\phi_p)) & \frac{\partial^2}{\partial \phi \partial \phi \partial \phi} \log(p(x|\phi_p)) \\
\frac{\partial^2}{\partial \phi \partial x} \log(p(x|\phi_p)) & \frac{\partial^2}{\partial \phi \partial y} \log(p(x|\phi_p)) & \frac{\partial^2}{\partial \phi \partial \phi} \log(p(x|\phi_p)) & \frac{\partial^2}{\partial \phi \partial \phi \partial \phi} \log(p(x|\phi_p))
\end{bmatrix} \]

where \( E \) stands for the expectation over \( x \). The Fisher Information matrix is thus given by,

\[ I_{\phi_p} = \begin{bmatrix}
\frac{4s^2}{\sigma^2}a^t a & \frac{4s^2}{\sigma^2}a^t b & \frac{4s^2}{\sigma^2}c^t a & 0 & 0 \\
\frac{4s^2}{\sigma^2}a^t b & \frac{4s^2}{\sigma^2}b^t b & \frac{4s^2}{\sigma^2}c^t b & 0 & 0 \\
\frac{4s^2}{\sigma^2}c^t a & \frac{4s^2}{\sigma^2}c^t b & \frac{4s^2}{\sigma^2}c^t c & 0 & 0 \\
0 & 0 & 0 & \frac{4n^2}{\sigma^2} & 0 \\
0 & 0 & 0 & 0 & \frac{2s^2}{\sigma^2}c^t c
\end{bmatrix} \]

where \( .^t \) stands for transpose of the vector and \( 1 \) is a column vector containing 1 as all its entries. Since \( CRLB = (I_{\phi_p})^{-1} \) and the inverse of a block diagonal matrix is block diagonal, the estimation error variance of \( \phi_\Delta \) is decoupled from that of others.

An explicit expression for the CRLB for estimation error variance of relative phase is given by,

\[ CRLB_{\phi_\Delta} = \frac{\sigma^2}{ns^2} \]

By comparing our model for point scatterer with a more precise model given by equation (1.1), we can see that final term in the exponent of (1.1) does not vary much with the azimuth. So we assume \( c_k \) (for \( k = 1, \ldots, n \)) to be identically 1. Moreover the location of the scatterer \((x, y)\) is assumed to be known. The Maximum Likelihood (ML) estimator for \( \phi_\Delta \) can be easily found by equating the first partial derivatives of \( \log(p(x|\phi_p)) \) with respect to \( s \), \( \phi_\Delta \) and \( \phi_c \) to zero and obtaining an expression in terms of the unknown, i.e \( \phi_\Delta \) in this case. So the ML estimator for \( \phi_\Delta \) turns out to be,

\[ \hat{\phi}_\Delta = 2\pi \left[ \frac{1}{n} \left( \sum_{i=1}^{n} u_i e^{-j(x_i a + b_i y_i)} \right) \left( \sum_{i=1}^{n} v_i e^{-j(x_i a + b_i y_i)} \right)^* \right] \]
where \* denotes complex conjugate of the variable. For the general case in which the location of the scatterer is unknown, it is very complicated to find a closed form expression for the ML estimator for \( \phi_\Delta \) only in terms of the observations.
BIBLIOGRAPHY


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