LAYERED CARTESIAN HALF-SPACE MODELS FOR EARTH’S ELASTIC RESPONSE TO CONTEMPORARY SURFACE LOADING PHENOMENA

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Hao Zhou, M.S.

*****

The Ohio State University
2008

Dissertation Committee:

Professor Mike Bevis, Advisor
Professor Terry Wilson
Professor C.K Shum
Professor Lindsay Schoenbohm

Approved by

---------------------------------------------

Professor C.K Shum
Geological Sciences Graduate Program

--------------------

Advisor

---------------------------------------------
ABSTRACT

Postglacial rebound (PGR) represents a complicated interplay of the solid Earth and ice history, through which scientists can investigate past climate changes. During the past decades, the advancement of spatial technology significantly enhanced scientists’ capability to observe PGR signals. However, in many glacial regions of the world, PGR signals are ‘contaminated’ by vertical elastic crustal deformation, which is induced by present ice mass change.

In order to estimate the magnitude and pattern of the elastic crustal deformation, a layered elastic loading model has been developed in this dissertation work. It is capable of computing surface deformation due to an arbitrary shape load. The high-speed computing capability of this model provides us possibilities to investigate elastic properties of the Earth’s crust in detail.

This elastic loading model is applied to the Patagonia ice field to eliminate elastic rebound signal due to present ice loss. Using different combinations of ice loss models and Earth’s elastic structures, we estimated that the present ice loss contributes 5-25% to the total crustal uplift. In addition, we used geodetic measurements to further confirm that the broad pattern of uplift in the Patagonia region reflects the Earth’s response to the deglaciation of the Little Ice Age (LIA) rather than that of the Last Glacial Maximum (LGM), due to the unique tectonic setting of this region.
The Inverse Bousinessq Response (IBR) effect observed by the strainmeters in California is also explained by numerical modeling using the layered elastic loading model. High Poisson’s ratio and strong stiffness contrast of the crust’s soft and hard sediment layers play an important role in the IBR’s occurrence. The thickness of the soft sediment layer controls the location and spatial extent of the IBR effect.

In Greenland, we used the elastic loading model and the most complete estimate of ice mass change (1993 - 2000) to demonstrate that by using continuous GPS measurements, we can detect and assess ice loss acceleration in Greenland faster than by using other geodetic tools.
Dedicated to my wife, Chunlian Hao
ACKNOWLEDGEMENTS

I wish to thank my adviser, Mike Bevis, for his guidance, encouragement, and patience, which made this dissertation possible.

I wish to thank my committee members: Terry Wilson, C.K. Shum, and Lindsay Schoenbohm for providing very helpful suggestions to my research. Terry and Lindsay brought my interest to geology, for which I am very grateful.

I thank Eric Kendrick, my colleague and a good friend for stimulating discussion and providing help when I ever I needed.

I also wish to thank Dana Caccamise who is also a good friend of mine, and has been very helpful in every aspect of my life in the states.

Lastly I like to thank all my lab mates: Shan Shan, David Raleigh, Abel Brown, and Jian Wang, who have always been very supportive.
VITA

2000 - 2003…………………………………. M.S. Geology & Geophysics, University of Hawaii at Manoa

2004 – 2008 ………………………………….Graduate Research Associate, The Ohio State University

PUBLICATIONS

Research Publication


FIELDS OF STUDY

Major Field: Geological Sciences
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CHAPTER 1

INTRODUCTION

1.1 Behavior of Earth material

The word ‘isostasy’ is used to describe a condition of rest to which Earth’s crust and mantle tend, in the absence of disturbing forces (Watt, 2000). It is based on the idea that the light crust floats on top of dense mantle. Ideas about isostasy originated as early as 1400, the age of Leonardo da Vinci. The following quote shows da Vinci’s thought regarding the sediment removed from a mountain top:

“The part of the surface of any heavy body will become more distant from the center of its gravity which becomes of greater lightness. The earth therefore, the element by which the rivers carry away the slopes of mountains and bear them to the sea, is the place from which such gravity is removed; it will make itself lighter …. The summits of the mountains in course of time rise continually.”

Various kinds of loads that have different spatial and temporal scales exert forces onto the earth surface, e.g., seismic waves, lakes, glaciers, river deltas, volcanic islands, etc., and Earth responds to these different loads in different ways. For example, seismic wave loading is on the duration of tens or hundreds of seconds; in this short time scale, lithosphere and lower mantle behave like elastic material to a great depth. Whereas for ice sheet loading, the duration is on the order of tens of thousands of years, and the mantle therefore flows like
viscous fluid to regain an equilibrium state. For loads that last long term, such as a volcanic island, crust and mantle behave like a thin elastic plate on top of invicid fluid.

In the past two decades, the threat of global warming has become more and more apparent. Glacier melting and sea level rise pose a danger to hundreds of millions of people’s lives by inundating densely inhabited coastal areas. Isostatic effects that are associated with past and present deglaciation become a powerful tool for scientists to monitor climate change, and to understand the complicated interaction between the solid Earth, cryosphere, and hydrosphere.

1.2 Geodetic tools for measuring deformations in glacial regions

In the past two decades, with the advancement of space technology, scientists are capable of measuring ice mass variation as well as the related Earth deformation on a large scale, and with high frequency. For example, in 1991, the Global Positioning System (GPS) was proved to be capable of measuring ice mass change by taking measurements of crustal uplift near ice caps (Hager, 1991). Since then, many researchers have conducted GPS surveys in different parts of the world to study different isostatic effects. In Greenland, Wahr et al. (2001) used measurements from five continuous GPS stations to show there was possibly an ice advancement in the western margin of the ice sheet in the past 3 ~ 4 thousand years. Khan et al. (2007) show that between 2001 and 2006, the rapid unloading of ice from the southeastern sector of the Greenland ice sheet caused an elastic uplift of 35 mm at a GPS site near Kulusuk. In Antarctica, the West Antarctic GPS Network (WAGN) measures crustal motions of the bedrock under the West Antarctic ice sheet. The Transantarctic Mountains Deformation (TAMDEF) network measures crust and mantle readjustments in response to the Last Glacial Maximum (LGM) (Willis et al., 2006). In the Patagonia region of South
America, Bevis et al. (2002) show a broad pattern of uplift with the maximum uplift rate as large as 20mm/year.

Earth’s isostatic response and mass variation affects the gravity field. To observe this effect, in March 2002, NASA launched the Gravity Recovery and Climate Experiment mission (GRACE). The twin satellites fly at an altitude of 500km, and make detailed measurements of the Earth’s gravity field by detecting minor changes of the distance between the tandem satellites. Using gravity data, Velicogna and Wahr (2007) determined that, between 2002 and 2005, the Antarctic ice sheet volume decreased at a rate of 152+/-80 km$^3$/year, which contributes 0.4+/-0.2mm/yr to global scale sea level rise. Velicogna (2005) isolated seasonal ice mass variation in Greenland from GRACE measurements. Barletta et al. (2008) isolated the Postglacial Rebound (PGR) signal from GRACE in Antarctica and Greenland by exploring solid Earth parameter space (viscosity, lithospheric thickness). Chen et al. (2007) published the present ice loss rate for the Patagonia ice field, which was the first result that was derived from GRACE’s measurements for this region. They found 27.9+/-11 km$^3$ ice demises each year.

Besides GRACE and GPS, the laser altimeter is also a tool that can measure Earth topography change. Laser altimeters can be equipped either on a satellite, such as ICEsat, or on an aircraft. Laser altimeters measure the round trip travel time of a laser pulse from the instrument to the ground, from which the distance is calculated. Satellites and aircraft utilize GPS to resolve their position, and then this position is used to determine the shooting angle of the laser as well as the location on the Earth’s surface that is illuminated. Measurements collected at different epochs allow researchers to determine the topography change on Earth. After removal of crustal uplift, Krabill et al. (1995, 1998, 2004) estimated the average ice
loss during 1993 and 1998 was about $60\,\text{km}^3/\text{yr}$, and that the rate increased to $80\,\text{km}^3/\text{yr}$ between 1997 and 2003.

Sea level is directly related to ice melting. Sea level has been rising steadily in the past decade (Cabanes et al., 2001, Miller and Douglas, 2004). Scientific studies have found sea level drops near the melting ice due to self-gravitation. Mitrovica (2001) showed it is possible to infer present ice loss rate from the non-uniform sea-level pattern. Therefore, sea level records from tide gauge stations that are equipped with GPS (to observe absolute crust motion) are valuable to estimate present ice variation.

1.3 Modeling deformation in glacial regions

Complete modeling of deformation in glacial regions requires elastic modeling and viscoelastic modeling, and each modeling process includes three major components: ice load, Earth rheology, and isostatic response of the Earth. At many places, the Earth’s mantle is still recovering from the deglaciation since the Last Glacial Maximum (LGM), therefore at these locations, the geodetic measurements are usually a mix of the Earth’s isostatic signals from two sources: Earth’s response to the present ice mass variation and Earth’s response to the past ice history. Elastic modeling concerns the former, while viscoelastic modeling concerns the latter. The mix of the two sources of signals complicates the situation. With geodetic measurements in hand, scientists who are only interested in one aspect of the GIA modeling have to apply a ‘correction’ to remove the unwanted signal from the measurements (e.g., Veclicogna and Wahr, 2004, Chen et al., 2007, Wahr and Velicogna, 2007). In fact, one of the initiatives of this dissertation work is to model the elastic crustal uplift for the Patagonian region, to obtain a better estimate of the PGR signal can be obtained.
Through the modeling process, scientists study one component of the model by constraining the other components using measurements or assumptions. Since elastic modeling is the theme of this dissertation, in the following sections we elaborate on the details of elastic modeling processes, and only briefly touch on viscoelastic modeling.

1.3.1 Elastic modeling

1.3.1.1 Seasonal elastic signal

Annual or inter-annual elastic signals are caused by seasonal inter-hemisphere water mass exchange (Blewitt et al., 2001). For example, Bevis et al. (2005) show a GPS station oscillates with a magnitude of 50-70mm annually in the Amazon basin, which is due to the seasonal water fluctuation in the Amazon river. The vertical component of the GPS time series is almost perfectly anti-correlated with the annual variations of water stage heights (Fig. 1.1). Heki (2001) observed periodic variations with annual frequencies in GPS measurements in northeast Japan. Those were due to snow accumulation each winter on nearby mountain ridges. In the case of ice sheets/ice fields, seasonal mass variations are affected by precipitation, evaporation, melt water runoff, and ice discharge (Van der Veen, 1999). Velicogna (2005) shows the rate of Greenland ice mass decrease reaches a minimum in January and increases to a maximum in April/May each year. This is because in winter, precipitation rate is at the highest, which adds mass to the ice sheet. In summer, evaporation rate and melting rate are larger with higher summer temperature. Melt water in turn increases ice discharge by enhancing basal sliding.

Because the seasonal elastic signal is characterized by periodicity, it is relatively easy to be separated from geodetic observations, or directly observed at locations where there is no significant PGR or tectonic signal. Earth’s crust ‘weighs’ load change like a bathroom
scale, however, the scale needs to be ‘calibrated’ by relating the amount of movement of the crust to the seasonal mass change. Upon doing this, scientists gain a better understanding of Earth’s elastic structure, so that they can estimate crustal movement caused by secular mass change more accurately. In fact, this is exactly what we propose to do, as we will explain in later chapters.

Figure 1.1 (a) Stage height time series $H(t)$ observed in Manaus, (b) daily solutions for the upwards component of displacement $U(t)$ at GPS station MANA (red dots), and the model prediction (solid curve), (c) and (d) geodetic measurements (red dots) and model predictions (solid curves) for the north and east components of displacement. (Bevis et al., 2005)

1.3.1.2 Secular elastic signal

Besides the seasonal elastic signal, the other component is the secular elastic signal. It represents crustal movement caused by gradual, steady, mass gain/loss in a multi-year or decadal period. The secular elastic signal does not have a character that can be easily
distinguished from the PGR signal, therefore it is usually estimated through modeling (e.g., Khan et al., 2007).

1.3.1.3 Earth’s elastic structure

Earth elastic structure controls the magnitude of the elastic signal caused by surface load. The most well accepted elastic structure is based on the Preliminary Reference Earth Model (PREM) (Dziewonski and Anderson, 1981). The PREM model is constructed by global inversion of seismic data. In PREM, Earth is divided into eight layers, which include: inner core, outer core, lower mantle, transition zone, LVZ (low velocity zone), LID (region above the low velocity zone), crust, and ocean. In addition, lower mantle, transition zone, and crust are further divided into sub layers, therefore in total PREM consists of thirteen layers. Within each layer, Earth properties (body wave velocities, density) are averaged over some depth. Vp (P-wave velocity), Vs (S-wave velocity), density (\( \rho \)) profiles are specified by three polynomials, which indicate these quantities are continuously defined within each layer.

Another well-known model is especially for the Earth’s crust: CRUST2.0. This model has a 2x2 degree resolution. It takes advantage of the recent compilation of a global sediment thickness map, which is defined on a 1x1 degree grid (Mooney et al., 1998). CRUST2.0 uses type keys to assign various types of crustal structure in each cell (such as Archean, early Proterozoic, rifts etc.). It divides the crust into seven layers: ice, water, soft sediments, hard sediments, upper crust, middle crust, and lower crust. Similar to PREM, Vp, Vs and \( \rho \) are specified for each layer.

There are many more geophysical data available since the development of PREM. Earth scientists have ongoing efforts to develop a more up to date Earth model to replace the PREM model, e.g. the work of the REM group.
However, PREM and CRUST2.0 are still the only complete Earth models available.

1.3.1.4 On the importance of shallow elastic structure for elastic modeling

In this section, we use a small synthetic example to show that the elastic response to short wavelength loads can be sensitive to the details of shallow elastic structure. Figure 1.2 depicts a vertical section through a uniform elastic half-space whose origin is at the center of a uniform circular surface load of radius $A = 1$. The contours depict the strain energy density invested in the elastic half-space as it supports this load. Note we are color-coding the logarithm (base 10) of energy density in this plot, and so the color corresponding to a value of -3 in the key represents ten times less energy density than the color corresponding to a value of -2. Clearly the strain energy is strongly localized in the near-field of the loading zone. In fact, half of the strain energy invested in the entire half-space is located in a cylindrical volume centered on, and immediately underlying the load, whose radius is $1.5A$ and whose vertical extent is $1.25A$ (in this case $A=1$). The radius of the load provides a length scale for the size of the region doing most of the work in supporting the load. Within or very close to the circular load almost all of the surface response of the load can be accounted for by the compression of elastic material located within the depth range $Z < 2A$. But note that this is not true even in the medium field of the load. For $R=5A$, for example, we can see that maximum strain energy density occurs near $Z = 4A$, and the strain energy density at this horizontal offset from the load center varies only slightly between $Z= 3A$ and $Z=5A$. As one moves away from the load the surface displacement is much less dominated by the straining of the shallow subsurface.
Figure 1.2 A vertical section depicting strain energy density in a uniform, isotropic elastic half-space subject to a uniform circular load. This surface \(z=0\) load has its center at \(R=0\) and radius \(A=1\). Note that the color-coding invokes a logarithmic scale.

PREM is widely thought to be adequate for modeling loads at the global scale, but the synthetic experiment demonstrates that, depending on the size of the load, PREM may not be sufficient due to its lack of resolution at shallow depth. For example, it does not distinguish between oceanic and continental crust, nor account for any lateral variation in elastic properties. However, the elastic structure of continental crust is known to be very laterally heterogeneous. Bevis et al. (2005) evaluated the spatial influence of the load of the Amazon River on the response at a GPS station (MANA) and showed that half of the displacement at MANA was produced by load changes imposed within 88 km of the GPS station. (Almost all of the displacement at MANA was produced by water mass changes developed less than 200 km from MANA). This horizontal scale length implies that the
loading response at MANA is sensitive to the elastic structure of the uppermost mantle and especially the continental crust.

Elastic modeling for small scale loads requires better knowledge of shallow Earth structure. While deep seismic refraction surveys, other seismological techniques, and to a limited extent, drill hole data, allow us to characterize the elastic structure of some regions in considerable detail, there are many continental areas in which these kinds of surveys have never taken place, and prior information about elastic structure within the crust is sparse or totally missing. With more and more geodetic measurements becoming available around the world, it is increasingly possible to further investigate Earth elastic structure via the elastic loading problem. A practical requirement for such an effort is to be able to solve the forward problem involving a layered half-space (LHS) with 5 or 10 layers and hundreds of surface stations, many thousands of times in a few hours, which is a prerequisite for tackling the inverse problem.

1.3.1.5 Existing mathematical models for the elastic loading problem

Many times we are interested in modeling short wavelength and general loads over apertures which are small and often very small, compared to the radius of the earth, and so the best approach (at least at first) is to create ‘flat earth’ models based on a layered Cartesian elastic half-space. Given the number of sophisticated numerical codes available for modeling surface loading on a spherical viscoelastic earth, it is rather surprising that very few codes are available for solving the Bousinessq (surface loading) problem for a layered elastic half-space (LHS), even though the general approach to a solution has been known for many years (Gilbert and Backus, 1966; Farrel, 1972). One reason for this is that spherical viscoelastic codes are used in glacial isostatic adjustment models, which are presently of great interest to
the sea level and climate change communities, so this class of development has received a great deal more attention and resources in the last 20 years.

Some of the early codes developed for layered Cartesian spaces suffered from serious numerical instabilities related to the use of propagator matrices in the Hankel transform domain (e.g. Sato and Matsu’ura, 1973). These problems have been resolved in the last decade (Pan, 1997; Fukahata & Matsu’ura, 2005a, b).

1.3.2 Viscoelastic modeling

The viscoelastic response is not the focus of this dissertation work, so it will only be briefly discussed here. The viscoelastic response of the Earth is sensitive to the past ice history, the rheology of the mantle, and lithospheric thickness (Watts, 2001). Because of the lack of data, neither ice history nor Earth’s viscosity profile are well constrained. Scientists piece together past ice history by studying past sea level records, ice core analysis, terminal moraine dating, tree ring dating etc. Earth viscosity information is mainly derived from seismic tomography inversion. As a consequence, the modeled Earth viscoelastic response can vary over a large range. For example, Ivins and James (1999) modeled vertical uplift rate in the Patagonia region using different viscosity scenarios: ‘Fennoscandian’ type, which implies a cold and stiff mantle, ‘Basin and Range’ type, which implies a very hot and weak rheology, and ‘Australian’ type, which refers to an intermediate mantle rheology. The uplift rates they predicted using different mantle rheology differed by several orders of magnitude. More detailed sensitivity tests are conducted by Wu (2005) and Wu and Van der Wal (2003) in other parts of the world.
1.4 Dissertation outline

The dissertation results are organized into five chapters after this introduction. In chapter two, a sparse evaluation and massive interpolation (SEMI) numerical technique is developed to enhance the speed of calculating displacements due to a uniform circular load on a layered half-space. Chapter 3 is a companion to chapter 2, it details how to construct a solution for a general polygon load while taking maximal computational advantage of the SEMI method. In chapter 4, we use geodetic measurements to analyze the tectonic control on the PGR signal in Patagonia, and explore the magnitude and pattern of elastic uplift in this region. Chapter 5 explains the ‘Inverse Boussinesq’ problem utilizing a LHS loading model, and defines ‘rules of thumb’ for its occurrence through numerical experiments. Chapter 5 demonstrate that, it is possible to use continues GPS stations to detect ice loss surge of an ice sheet with a much shorter delay than other geodetic means.
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CHAPTER 2

SURFACE DEFORMATION DUE TO LOADING OF A LAYERED ELASTIC HALF-SPACE I:
A SPARSE EVALUATION AND MASSIVE INTERPOLATION METHOD

2.1 Introduction

Since the development of the surface loading solution in an elastic half-space by Boussinesq in 1885 (see, Love, 1944), various extended solutions have been derived, with applications in diverse areas such as cell biology (e.g., Balaban et al., 2001), civil engineering (e.g. Graig, 1997), and earth science (Bevis et al., 2004; Becker and Bevis, 2004). One of the important extensions of the Boussinesq solution is to the corresponding multilayered elastic half-space with various solution methods proposed (Pan, 1997). The extensive studies of the layered structures come from the fact that structures with multilayers are needed broadly, e.g., composites in material science, ground foundation structures, layered pavements in highway transportation, and existing earth models have layered structure (Mooney et al., 1998, Dziewonski, A. M. & Anderson, 1981). This in turn has motivated the corresponding analytical and numerical researches aimed at both forward and inverse problems associated with layered structures (Wang et al., 2003; Pan and Han, 2005). However, even for the forward problems, namely the elastic response due to the surface loading on the surface of the layered half-space, there is still no fast algorithm available. How to quickly calculate the displacement fields on the surface remains a very challenging problem.
In the appendix section of this dissertation, we present the formulations (hereafter referred to as ‘direct method’) of a surface displacement field that is due to a uniform circular loading on the surface of a layered half-space (Figure 2.1). The formulations are in terms of the cylindrical system of vector functions combined with the propagator matrix method.

For different surface locations, these formulations compute surface displacements one location at a time, which involves substantial computation time and thus makes the calculation very slow even for a relatively small set of observations. This is particularly awkward when dealing with multi-parameters inverse problem. As such, a fast calculation is needed, which motivates the development of the following pre-calculation algorithm.

The pre-calculation method is capable of calculating the surface response due to any irregular surface loading discretized by millions of loading dots. The surface response in the r-direction is divided into three sections with certain key points in each section. Based on their different response features, three different interpolation methods were developed to interpolate displacements from key points. Below we will describe these methods in detail.
2.2 Pre-calculation algorithm of the surface displacement fields

Based on the common features of the surface displacements (as shown in Figs. 2.2 a,b and 2.3a,b for the two typical layered models given in Tables 2.1 and 2.2), the pre-calculation algorithm will be usually divided into three sections: A near-field (from 0 to 2R), a middle-field (2R to 40R), and a far-field (40R to 200R) section. As can be observed from both Figs. 2.2 and 2.3, in the near-field the displacement has the largest value with rough variation. In the middle field, the variation of the displacements is very smooth and gentle. Finally in the far-field, the displacements asymptotically approach zero. Therefore, in each section one needs to use a different pre-calculation scheme. In our scheme, in each section, the relative error between the direct and pre-calculated values is kept at 5% or less.
<table>
<thead>
<tr>
<th>Layer #</th>
<th>Thickness (km)</th>
<th>Young’s modulus $E$ (GPa)</th>
<th>Poisson’s Ratio $\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>5.0</td>
<td>0.3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>30.0</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>infinite</td>
<td>150.0</td>
<td>0.2</td>
</tr>
</tbody>
</table>

**Table 2.1** Layered model 1: Pressure $q=1$ KPa and loading radius $R=1$ km.

<table>
<thead>
<tr>
<th>Layer #</th>
<th>Thickness (km)</th>
<th>Young’s modulus $E$ (GPa)</th>
<th>Poisson’s Ratio $\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
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<tr>
<td>2</td>
<td>5</td>
<td>30.0</td>
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</tr>
<tr>
<td>3</td>
<td>infinite</td>
<td>5.0</td>
<td>0.3</td>
</tr>
</tbody>
</table>

**Table 2.2** Layered Model 2: Pressure $q=1$ KPa and loading radius $R=1$ km.

2.2.1 Near-field pre-calculation using multiplication theorem for Bessel functions

First, we rewrite Eq. (A-24) in a uniform format as for a selected pre-calculation point $r_1$:

$$u_i(r_1,0) = qR \int_0^\infty \frac{D_i}{\lambda} J_i(\lambda R)J_i(\lambda r_1) d\lambda$$

(2.1)

where $i=0$ and 1 corresponds to the displacement component $u_z$ and $u_r$, respectively.

To obtain the surface displacements, say at point $r$, from the given solution at $r_1$, we first employ the following multiplication theorem for the Bessel function $J_i(\lambda r)$ (Watson, 1996):

$$J_i(\lambda r) = \left[ \frac{r}{\lambda r_1} \right] \sum_{m=0}^\infty (-1)^m \left[ \frac{(r / r_1)^2 - 1}{m!} \right] \frac{\lambda r_1}{2} J_m(\lambda r_1)$$

(2.2)

Substitution of Eq. (2.2) in (2.1) gives

$$\frac{u_i(r,0)}{qR} = \left[ \frac{r}{\lambda r_1} \right] \sum_{m=0}^\infty (-1)^m \left[ \frac{(r / r_1)^2 - 1}{m!} \right] \frac{D_i}{\lambda} \int_0^\infty \frac{J_i(\lambda R)}{\lambda} \frac{\lambda r_1}{2} J_m(\lambda r_1) d\lambda$$

(2.3)

Therefore, we can pre-calculate the $M$ integrals in Eq. (2.3) associated with the surface point $r_1$; after that, the surface solution at any point $r$ can be obtained using Eq. (2.3) as
\[ \frac{u_i(r,0)}{qR} = \left[ \frac{r}{r_1} \right]^{i} \sum_{m=0}^{M} \frac{(-1)^m [(r/r_1)^2 - 1]^m}{m!} f_i(r_1; m) \] (2.4)

where \( f_i(r_1, m) \) are the pre-calculated \( 2(M+1) \) functions at point \( r_1 \).

\[ f_i(r_1; m) = \frac{(-1)^m [(r/r_1)^2 - 1]^m}{m!} \] (2.5)

We have found that, to achieve a relative error about or less than 5\% within the \( r \)-interval from 0 to \( 2R \), the most important interval with rough variation, we need only two points to be pre-calculated. One is inside the circle and another is outside, with the truncated \( M \) being usually around \( M=8 \). For \( r>2R \), the variation of the displacement curve becomes very smooth and we apply the B-spline to handle it, as discussed below.

### 2.2.2 Middle-field pre-calculation using B-splines

Splines are piecewise polynomials of degree \( n \) joined together at the break points with \( n-1 \) continuous derivatives. The break points of splines are called knots, and if \( n \geq 2 \), then the spline is smooth. In the B-spline algorithm, the targeted function, i.e. \( u(r) \), is expressed as (de Boor, 1979; Rice, 1983)

\[ u(r) = \sum_{j=1}^{N} a_j B_j(r) \] (2.6)

where \( B_j=B_{j,k,t} \) denotes the \( j \)-th B-spline of order \( k \) with respect to the knot sequence \( t \), and the coefficients \( a_j \) are solved from the following equation using the function values at data points \( r_i \) (i.e., requires that the B-spline passes exactly through each data point)

\[ \sum_{j=1}^{N} a_j B_j(r_i) = u(r_i) \] (2.7)
The B-spline option in the *IMSL* fits the B-spline of a specified order using a knot sequence based upon the \( r \)-values of the data, and the permissible orders range from 2 (quadratic) through order 8. In our algorithm, we fix the order at 3, and choose about 10 pre-calculated data points to approximate the interval from \( 2R \) to \( 40R \), with the distance between neighboring points increasing with the increasing \( r \)-value.

### 2.2.3 Far-field pre-calculation using asymptotical expansion

For \( r \) larger than \( 40R \), the displacement field decays to zero, inversely proportional to the distance. Therefore, for a given circle, the magnitude of the displacements in the far field will be usually negligible as compared to those in the near- and middle-fields. However, we still include this far-field response from \( r=40R \) all the way to \( r=200R \). In this interval, we propose the following asymptotical expression for both \( u_r \) and \( u_z \), as

\[
\begin{align*}
  u_i(r) &= \frac{a_{i1}}{r} + \frac{a_{i2}}{r^2} + \frac{a_{i3}}{r^3} + \frac{a_{i4}}{r^4} \\
  \text{where } i &= 0 \text{ and } 1 \text{ corresponds to the displacement component } u_r \text{ and } u_z, \text{ respectively, and the coefficients } a_{ij} \text{ are determined by pre-calculating the displacements at 4 selected points in the interval of } 40R-200R.
\end{align*}
\]

where \( i=0 \) and 1 corresponds to the displacement component \( u_r \) and \( u_z \), respectively, and the coefficients \( a_{ij} \) are determined by pre-calculating the displacements at 4 selected points in the interval of \( 40R-200R \).
2.3 Direct vs. pre-calculation methods: Numerical comparison

In this section, two typical layered models listed in Tables 1 and 2 have been selected to test our pre-calculation algorithm. While model 1 corresponds to the earth structure, model 2 is the inverse layup of the model 1 material property, which is a typical layered pavement structure.

As we discussed, we divide the $r$-interval into 3 sections with a total of 16 pre-calculation points. The 3 sections are: 0-2$R$, 2$R$-40$R$, and 40$R$-200$R$, corresponding to the near-, middle, and far-fields, respectively ($R$=1km for the two models). In section 1 from 0 to 2$R$, the pre-calculation points are 0.75$R$ and 1.65$R$ for the approximation based on the multiplication theorem of the Bessel functions, with the former inside the loading circle whilst the latter one outside the circle. In section 2 from 2$R$ to 40$R$, 11 pre-calculation points are selected in the B-spline approximation: 2.0$R$, 2.7$R$, 3.65$R$, 4.9$R$, 6.64$R$, 8.97$R$, 12.1$R$, 16.34$R$, 25.06$R$, 32.0$R$, and 40.0$R$. Notice that $r$=40.0$R$ will be also used for the far-field pre-calculation. Finally, in section 3 from 40$R$-200$R$, 4 pre-calculation points are chosen for the asymptotical expansion (8): 40.0$R$, 50.0$R$, 90.0$R$, and 180.0$R$. 
Shown in Figs. 2.2 and 2.3 are the comparison of the surface displacements based on both the direct and pre-calculated methods for both models 1 and 2, with station points starting from 0 to $50R$ at an increment of $0.1R$. The inserted figures are the zoom-in results from 0 to $2R$ to show the rapid variation of the displacement fields in the near field. As can
be observed from these figures, the pre-calculated results agree well with those calculated directly. To quantitatively evaluate the accuracy of the proposed pre-calculation algorithm, we have also estimated their relative errors between the direct and pre-calculation approaches with the result being shown in Figure 2.4. It is clear from this figure that even with less than 20 pre-selected points, the pre-calculation algorithm can be used to rapidly calculate the surface response with a relative error at about or less than 5%.

However, the most important and appealing part of our pre-calculation algorithm is not on the acceptable accuracy of the surface displacement, but on the rapid calculation. Listed in Table 2.3 are the CPU times for the two layered models based on both the direct and pre-calculation approaches for different numbers of points (or stations). As can be observed, the pre-calculation algorithm is much faster than the direct method, in particular when the station numbers are large. More specifically, for 500 points, the pre-calculation algorithm is about 25 times faster than the direct method, and for 100,000 points, our algorithm is at least 1,200 times faster than the direct method.

<table>
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<tr>
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<th>500 points</th>
<th>1000 points</th>
<th>10000 points</th>
<th>100000 points</th>
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<td>48s</td>
<td>8m6s</td>
</tr>
<tr>
<td></td>
<td>Pre-cal</td>
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<td>1s</td>
<td>2s</td>
</tr>
<tr>
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<td>51s</td>
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<tr>
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<td>Pre-cal</td>
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<td>2s</td>
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</tbody>
</table>

Table 2.3 Comparison of CPU times between direct and pre-calculation methods. Computer configuration: Dell Workstation PWS650, Xeon CPU 3.06GHz, 2.0GB RAM.
Figure 2.4 Relative errors of the surface radial and vertical displacements for layered models 1 and 2 based on the direct and pre-calculation approaches.
2.4 Conclusions

We developed a rapid numerical method for the calculation of surface displacement fields due to a uniform circular loading on the surface of the layered half-space. The solutions are first expressed in terms of the cylindrical system of vector functions combined with the propagator matrix method. Instead of numerically carrying out the integration directly point by point, a fast pre-calculation is developed, which can be applied to calculate the surface response due to any irregular surface loading discretized by load dots. In our pre-calculation, the surface response in the r-direction is divided into three sections with certain pre-selected key points in each section. Due to their different response features, three different approaches have been developed to pre-calculate the response in the three sections. In the calculation at the key points, an adaptive Gauss quadrature approach is utilized which captions the oscillation feature of the integrands for fast and accurate evaluation of the integral. For the two typical layered examples presented, it is found that in order to calculate the response at 100,000 different surface locations, the pre-calculation approach is more than 1,200 times faster than the direct method.
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CHAPTER 3

SURFACE DEFORMATION DUE TO LOADING OF A LAYERED ELASTIC HALF-SPACE, II:
CONSTRUCTING THE SOLUTION FOR A GENERAL POLYGONAL LOAD

3.1 Introduction

In chapter 2, we developed the sparse evaluation and massive interpolation (SEMI) method, which allows us to obtain an approximate solution for the displacement field at very large numbers of points with vastly less computation per point. This approach utilizes the symmetry of the problem and the fact that the radial and vertical components of surface displacement are functions only of the radial distance, \( r \), from the center of the load. We used our high accuracy but computationally expensive method to compute the displacement vectors at a limited number of \( r \) values (called control points or knots), and then we use a variety of fast interpolation methods to determine the displacements at much larger numbers of intervening points. We can trade off the computational leverage achieved with the SEMI method and the magnitude of the errors associated with its approximations by choosing to use fewer or greater numbers of knots. The computational advantage of the SEMI method increases with the ratio of the number of surface stations to the number of knots.

In this chapter we show how a circular loading element can be used to compute the surface displacement fields for an arbitrary surface load. Suppose we wish to compute the
displacement at \( n \) stations due to a load we will approximate with \( m \) circular loading elements. The essence of our technique is to reorganize all but a computationally insignificant part of this calculation into an equivalent problem: compute the displacements due to a single circular loading element at a total of \( mn \) stations (where \( mn \) is the product \( m \times n \)). This maximizes the ratio of the number of stations to the number of knots, and so takes maximum possible advantage of the SEMI algorithm.

After describing this method in detail, we present some basic numerical tests of our code. In particular we address the problem of determining an appropriate number of loading elements for approximating a given load. Here the goal is to control the amplitude of artifacts or errors associated with discretization of the load. Lastly we consider some example loading problems drawn from earth science and show how depth controlled variations in the elasticity constants can cause interesting and diagnostic features in surface displacement fields. These examples indicate that measuring the spatial development of the elastic response to known patterns of surface loading will enable us to infer information about subsurface structure. Of course inversions of this kind require the forward problem to be evaluated many times, and it is this requirement which prompted us to develop a computationally efficient means for solving the forward problem for non-trivial loading geometries.

### 3.2 Problem statement

Following chapter 2, we consider a layered half space made up of \( p \) parallel, elastic isotropic layers lying over an elastic isotropic half space. We adopt a Cartesian coordinate system in which the \( x \) and \( y \) axes lie in the surface plane \((z = 0)\), and the \( z \) axis is positive downwards into the half-space. We assume that the surface of this half space is subject to an imposed pressure field within a polygonal boundary \( B \). The pressure \( P \) is zero everywhere
outside of B, and is a known function $P(x,y)$ within and on B. The pressure $P$ can be identified with the normal stress component $\square_{zz}$, and $P$ is taken to be positive if the associated force is directed in the positive $z$ direction. We assume that no shear tractions are imposed on the surface.

We wish to represent the pressure field within B using a suite of simple loading elements or cells, with the surface pressure applied within a single element being constant. The pressure field $P(x,y)$ is most easily approximated as piece-wise constant by dividing the polygon into a regular grid of square loading cells (Fig. 1a), and assuming that the pressure everywhere within the $k$ th square cell is $P_k = P(x_i,y_i)$ where $(x_i,y_i)$ are the coordinates of the point in the center of that cell. There are two weaknesses to this approach: (i) the outer edges of the suite of square loading cells do not exactly correspond to the geometry of the polygon (Fig. 3.1a), and (ii) the actual pressure field $P(x,y)$ is not piecewise constant, and so the net force imposed by the pressure field on any square may deviate from that implied by the piece-wise constant representation described above. However, by refining the grid so as to reduce the size of the individual cells, the errors associated with these problems can be reduced until they are negligible.
Figure 3.1 Discretization of a surface pressure field $P(x,y)$ applied within a polygonal boundary $B$. (a) A square grid is developed for the loading area, and the average pressure in each square cell is approximated by the pressure at the center of that cell. (b) The load imposed within each square loading cell will actually be represented by a uniform circular load of diameter $2a$, which nominally exerts the same net force as the square load.

We cannot, in fact, use square loading cells, because square loads lack the symmetry which is essential to the SEMI method. We must use circular loading cells instead. However, it is useful to consider the circular loading cells as representing the square cells discussed
above. Let us assume that the square cells in Fig. 1a had a width of $2a$, in which case the corresponding circular cells have radius $a$, as seen in Fig. 1b. Let us suppose that the $i$ th circle lies within the $i$ th square, so both are centered at point $(x_i, y_i)$. The area of the square is $A_s = 4a^2$, and that of the circle is $A_c = \pi a^2$. Clearly the use of a circular loading element is problematic because it cannot properly tile or cover the entire polygon – there are gaps between adjacent circles (Fig. 3.1b). However, we can largely overcome this problem by appropriate choice of the pressure we will assign to each circular element. If the constant pressure applied in the $i$ th circular cell is $Q_i$, and this load is to produce the same total force on the surface as the constant pressure $P_i$ applied within the square cell, then we require $A_c Q_i = A_s P_i$. This implies that

$$Q_i = \frac{4}{\pi} P(x_i, y_i) \quad (3.1)$$

We have scaled the actual pressure at the center of each circular loading element by an amount that accounts for the gaps between the circles. There are still some minor problems associated with this piece-wise constant but discontinuous representation of the original pressure field, as we will discuss in section 4, but by making the circular elements sufficiently small, we can reduce the magnitude of these problems to any level that we need.

We can now state our problem, assuming that the decomposition of the load into circular loading elements has already been achieved. Given a multilayered elastic half-space described using the notation of chapter 2, and given a set of $n$ circular loads with the same radius ($a$) but different pressures ($Q_i$, for $i=1, 2 \ldots n$), compute the displacements at $m$ stations located on the surface of the half-space.
3.3 Description of the algorithm

In this section we describe the algorithm used to solve the problem just stated. Before developing this algorithm in a form suitable for efficient coding of the general problem, we explain the essence of our approach by considering an extremely simple example involving just two circular loads, 1 and 2, and a single station, S (Fig. 3.2).

Figure 3.2 The horizontal displacements induced at station S by uniform circular pressure loads Q1 and Q2. These problem geometry including the displacement field can be expressed in radial coordinate systems attached to the center of each load, and in a global Cartesian coordinate system \{X,Y\}.

The pressure applied in the first circle is Q1, and that in the second circle is Q2. By linear superposition we can state that the displacement vector \( \mathbf{u} \) at S is the vector sum of the displacements \( \mathbf{u}^{(1)} \) and \( \mathbf{u}^{(2)} \) due to the first and second loads, respectively. Given the symmetry of a single circular load, the displacement it causes at any station is most easily described and computed in a cylindrical coordinate system whose origin lies at the center of the load. In this coordinate system the displacement vector at any point has only two non-
zero components - the vertical component $u_z$ and the radial component $u_r$. Because each load has its own cylindrical coordinate system, we must transform vectors $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ into Cartesian coordinates in order to perform the vector summation and deliver the solution $\mathbf{u}$ in the same coordinate system in which the load and station geometry are described. Let us represent this coordinate transformation in standard matrix form $\mathbf{u}_{\text{cart}} = \mathbf{T} \mathbf{u}_{\text{cyl}}$. (Because the matrix $\mathbf{T}$ is sparse in this particular context, this is not the most efficient way in which to implement the transformation. We ignore this minor detail for now). We choose not to compute $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ explicitly, but instead compute $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$ which are the displacement vectors at $S$ produced by unit loading of our circular loading domains. Let us assume that $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$ are computed and expressed in cylindrical (or local) coordinate systems attached to each load, and that we wish to express the net displacement $\mathbf{u}$ at $S$ in the Cartesian (or global) coordinate system. Then

$$\mathbf{u} = Q_1 \mathbf{T}^{(1)} \mathbf{d}^{(1)} + Q_2 \mathbf{T}^{(2)} \mathbf{d}^{(2)} \quad (3.2)$$

where $\mathbf{T}^{(1)}$ is the transformation matrix associated with load 1, etc. There are two important points about this equation. Firstly, almost all of the computation burden involved in evaluating this equation is incurred in evaluating $\mathbf{d}^{(1)}$ and $\mathbf{d}^{(2)}$. The coordinate transformations and the summation are computationally trivial in comparison. Secondly, when considered from the perspective of their local coordinate systems the two unit loads are identical, and since the spatial variability of the surface displacement field $\mathbf{d}$ depends only on $r$ (since the symmetry of the load implies no $\theta$ dependence), then $\mathbf{d}^{(1)} = \mathbf{d}(r_1)$ and $\mathbf{d}$ view the evaluation of as solving the problem of two stations and a single unit load, rather than two unit loads and a single station.
This second insight is the crucial one. If we want to compute the displacement at a single station due to \( m \) circular loads, we can do most of the work by solving the parallel problem of determining the displacements caused at \( m \) stations by a single unit circular load. In the original problem \( r_i \) is the distance between the station and the center of the \( i \)th unit circular load. In the parallel problem \( r_i \) is the distance between the center of the single unit circular load and the \( i \)th station. Having solved this parallel problem we can construct the solution to the original problem by the obvious generalization of equ. (3.1). We can extend this trick even further: the problem of solving the displacements at \( m \) surface stations due to \( n \) circular loads (with common radius \( a \)) can be very largely transformed into the problem of finding the displacements at \( n m \) stations due to a single unit circular load. The great advantage of diverting to the parallel form of the problem is that the computational efficiency of the SEMI method increases in proportion with the total number of stations.

We are ready now to present the algorithm for the problem stated in its most general form. We wish to compute the displacements at \( m \) stations due to \( n \) circular loads. Each of these loads has identical radius \( a \). The \( i \)th circular load is centered at \((x_i^c, y_i^c)\) and is subject to uniform pressure \( Q_i \), which is considered positive if the associated force is oriented in the \( z \)-direction, i.e. into the half-space. The \( j \)th station has surface coordinates \((x_j^s, y_j^s)\).

Consider the \( k \)th combination of the \( i \)th circle and the \( j \)th station (Fig. 3.3).
Figure. 3.3 The coordinate systems used to describe the horizontal displacement at station \( j \) due to load \( i \).

We can consider \( k \) the station number in the parallel problem. The relative position vector describing the position of station \( j \) relative to circle \( i \) is

\[
r^k = r^{ij} = \sqrt{(x_j^i - x_i^c)^2 + (y_j^i - y_i^c)^2}
\]  

(3.3)

which has Euclidean length

\[
r^k = r^{ij} = \sqrt{(x_j^i - x_i^c)^2 + (y_j^i - y_i^c)^2}
\]  

(3.4)

which is simply the distance from the center of load \( i \) to station \( j \) in the original problem, or the distance from the center of the single unit load to the \( k \)th station in the parallel problem.

The unit vector which points from the center of circle \( i \) to station \( j \) is

\[
\hat{r}^{ij} = \hat{r}_x^{ij} \hat{x} + \hat{r}_y^{ij} \hat{y}
\]  

(3.5)

where

\[
\hat{r}_x^{ij} = \frac{(x_j^i - x_i^c)}{r^{ij}}
\]  

(3.6)

\[
\hat{r}_y^{ij} = \frac{(y_j^i - y_i^c)}{r^{ij}}
\]
We use the SEMI method (chapter 2) to solve for the parallel problem, finding a total of \( m n \) displacement vectors

\[
\mathbf{d}^k = d^k_r \hat{\mathbf{r}} + d^k_z \hat{\mathbf{z}} \equiv d^k_x \hat{\mathbf{x}} + d^k_y \hat{\mathbf{y}} + d^k_z \hat{\mathbf{z}}
\]  

(3.7)

produced by the single unit circular load. In order to revert to our original problem we must transform these vectors into the global Cartesian coordinate system, i.e. find

\[
\mathbf{d}^{ij} = d^{ij}_x \hat{\mathbf{x}} + d^{ij}_y \hat{\mathbf{y}} + d^{ij}_z \hat{\mathbf{z}}
\]  

(3.8)

for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \), or, equivalently, for \( k = 1, 2, \ldots, n m \). This can be done by projecting the radial component of \( \mathbf{d} \) onto unit vectors in the X and Y directions (Fig. 3.3), i.e.

\[
d^{ij}_x = d^{ij}_r \hat{\mathbf{x}}
\]

\[
d^{ij}_y = d^{ij}_r \hat{\mathbf{y}}
\]  

(3.9)

We can now express the displacement at each station due to a unit load at the position of the \( n \) non-unit loads in the original problem. To solve the displacement at a given station (\( j \)) in response to the original \( n \) circular loads, we simply scale the unit load responses (Equ. 3.8) with the appropriate loads, i.e.

\[
\mathbf{u}^j = \sum_{i=1,n} Q_i \mathbf{d}^{ij}
\]  

(3.10)

3.4 Some numerical tests and examples

We begin with a test in which we attempt to reproduce the solution presented by Becker and Bevis (2004) for a uniform rectangular load on a uniform elastic half-space (UHS), which is known as Love’s Problem. We do this using our SEMI approach by invoking a layered elastic half-space (LHS), consisting on 3 layers on a half-space, in which
all four of these layers have identical elastic properties – the general problem thus
degenerates into the special case (UHS), and the solutions should match.

The geometry associated with this problem is shown in Figure 4. The 1 km x 2 km
rectangle is an idealized representation of a lake, and the pressure loading corresponds to a
water depth of 100 meters.

![Figure 4: Geometry of the Problem](image)

**Figure 3.4** A uniform rectangular load is approximated by an array of circular loading
elements, and the resulting surface displacement field is sampled along profiles 1 and 2.

We evaluate the surface displacement field along two profiles (1 and 2) each of which
bisect the load. We follow the notation of Becker and Bevis (2004) in which the $x$, $y$ and $z$
components of displacement are called $u$, $v$ and $w$, respectively. We evaluate the UHS
solution assuming that Young’s modulus $E = 0.6 \times 10^{11} \text{ Nm}^{-2}$ and Poisson’s ratio $\nu = 0.25$
(or, equivalently, that the Lamé parameters $\lambda$ and $\mu$ are given by $\lambda = \frac{E}{2(1+\nu)} = 2.4 \times 10^{10} \text{ Nm}^{-2}$).
We set surface pressure $P = \rho gh$ where $\rho = 1000 \text{ kg m}^{-3}$, $g = 9.82 \text{ ms}^{-2}$ and $h = 100 \text{ m}$.
These exact solutions are shown by the red dots in Figure 3.5. In order to test our SEMI code,
we approximate the load using $20 \times 40 = 800$ circular loading elements (Fig. 3.4) and invoke
a LHS in which the first three layers have thicknesses of 1,500 m, 3,000 m, and 6,000 m, and
the fourth layer is semi-infinite. We set $E = 0.6 \times 10^{11} \text{ Nm}^{-2}$ and $\nu = 0.25$ in all four layers.
The resulting solution is shown by the blue curves in Figure 3.5.
Figure 3.5 The horizontal ($u$ and $v$) and vertical ($w$) components of displacement along each profile (see Fig. 3.4) computed exactly using the equations of Becker and Bevis (2004), and using the approximation techniques developed in this chapter. The dotted green lines indicate the edges of the rectangular load.
We can see that the UHS and LHS correspond very closely for all points (or stations) outside of the rectangle, and closely within the rectangle. A careful examination shows that within the rectangle the SEMI solution for the degenerate LHS oscillates around the exact solution for the UHS.

This oscillatory behavior, which we call ripple, is made more obvious in Figure 3.6 in which we difference the two sets of solutions for the horizontal component of displacement, \( u \), along a small segment of profile 2, where it crosses the left side of the rectangle.

![Figure 3.6](image)

**Figure 3.6** The deviation between the exact and the SEMI solutions for the vertical displacements near the edge of the rectangle (see Fig. 3.4) for two cases in which the load is approximated by (i) a 20 x 40 array, and (ii) a 40 x 80 array of circular loading elements.

The black curve shows the difference between the solutions when the load is approximated by 20 x 40 = 800 circles (corresponding to the SEMI solution shown in Figure 5). This oscillation has a sinusoidal or ‘egg crate’ form within the rectangle, and wavelength of this sinusoid (in the \( x \) and \( y \) directions) is the distance \( (2a) \) between the centers of adjacent circular loading elements (CLEs). Clearly the ripple in the SEMI solution manifests the discretization of the load using CLEs. The red curve shows the SEMI solution error or ripple.
when \( 40 \times 80 = 3,200 \) CLEs are used. Halving the radius of the CLE reduces the wavelength and the amplitude of the ripple by a factor of two (Figure 3.6), but at the cost of increasing the computational burden by a factor of 4. But since the SEMI algorithm is so fast, it will often be possible to reduce the magnitude of ripple to an acceptable level at an acceptable computational cost.

We now consider some examples which test the SEMI algorithm in the context of a non-degenerate LHS (i.e. the layers have different elastic constants) by developing certain special or limiting cases in which our intuition provides us with an expected value. In both of the examples that follow we consider the surface response to a single uniform circular load of radius \( a \). This load is applied to an elastic space consisting of one layer, of thickness \( T \), overlying a half-space. We shall assume that Poisson’s ratio, \( \nu \), is 0.25 for both layers, and that Young’s modulus is \( E_1 \) in the upper layer, and \( E_2 \) in the underlying half-space. Because of the symmetry of the circular load, the surface displacement vector at each point has only vertical and radial components, and both are purely a function of \( r \), the radial distance from the center of the load. We shall contrast the surface response for this 2-layer space with the surface response of a UHS subject to the same load, assuming that this UHS has \( \nu = 0.25 \) and Young’s modulus equal either to \( E_1 \) or to \( E_2 \).

We first consider the problem of a thick layer on a half-space, for which \( T >> a \). This geometry is shown in the left hand side of the diagram (a vertical section) inset into Figure 3.7a. We assume that \( E_1 = 0.1 \), \( E_2 = 1 \) and consider two values for the thickness of the upper layer: \( T= 10 \ a \) and \( T = 100 \ a \). The vertical and radial displacement profiles are shown in Figure 7 a and 7b, respectively, and can be compared to the UHS response for the case in which Young’s modulus \( E_{\text{UHS}} = E_1 = 0.1 \). By comparing these response curves we can confirm what we might have guessed intuitively: if the upper layer of the 2-layer space is
very much thicker than radius of the load, the surface response in the near and medium field is almost identical to that of a UHS whose properties are those of the upper layer.

Figure 3.7 A comparison between the (a) vertical and (b) radial displacements caused by a uniform circular load imposed on (i) a uniform half-space (UHS) with $E=0.1$, and (ii) a layer of thickness $t$ and $E=0.1$ overlying a half-space with $E=1.0$, as depicted in the inset in sub-plot (a). The comparison between the UHS and 2-layer solutions are presented for $T=t/a$ = 10 and $T=100$.

Next we consider the ‘opposite’ problem from that just discussed, in which the first layer of the 2-layered space is much thinner that the radius of the circular load (specifically $T$ = 0.1 $a$). In this case one might suspect that the influence of the thin upper layer might be restricted, and so it is useful to compare this response to the UHS response in which $E_{UHS} = E_2=1$. Comparing these response profiles we see that the vertical surface displacement profiles are very distinct within the loading domain. Since the LHS has a very compliant
upper layer, the surface beneath the load is deflected to a much greater extent than in the case of the UHS (Figure 3.8a). But for \( r > a \), the vertical response of the LHS is very nearly identical to that of the UHS. In other words, the surface response of the layered space is strongly influenced by the first layer within the loading area, but is dominated by the lower layer a short distance outside of the load. If we examine the radial component of displacement for this same problem (Fig. 3.8b) we see that in the medium field (say \( r > 3a \)) the surface response of the LHS is dominated by the lower layer (it matches the response of a UHS with the same properties as the 2nd layer of the LHS). The LHS and UHS responses differ to their greatest extent at the boundary of the circle (\( r = a \)). As we can see from the UHS response curve the load tends to pull material near the edge of the circle inwards as well as downwards. In the case of the LHS the very compliant upper layer leads to an enhanced inwards displacement, and the width of the ‘spike’ in the LHS response curve is controlled by the thickness of the first layer.

Lastly we consider the accuracy of the SEMI algorithm itself, in the primitive context of evaluating the displacements due to a single circular load of unit radius. Here the question is how accurate is the approximation delivered by the SEMI method in comparison with (much slower) direct evaluation? We investigated this issue by generating a suite of elastic structure models using a Monte Carlo approach. We generated 50 models consisting of five layers over a half-space, and 50 models consisting of 9 layers over a half-space. Each layer thickness was generated randomly, and within our ensemble thicknesses ranged between 0.070 and 8282 m. Young’s modulus and Poisson’s ratio for each layer were also randomly generated, with the former falling in the range 0.11 – 5147 Nm-2, and the latter in the range 0.01 – 0.49. In every case the load had a radius of 1 m, and the surface displacement field was evaluated at 1250 stations from \( r = 0 \) to \( r = 1000 \) m, with more than half of the stations
falling in the range 0 – 40 m. For each subsurface model the vertical and horizontal
displacement components we computed directly and using the SEMI approach, and these
quantities were compared. This amounted to a total of 125,000 comparisons for each
component of displacement. We defined the ‘relative error’ as the difference between the
SEMI and the directly computed value for displacement divided by the directly computed
value. The RMS relative errors were 5.1 x 10^{-3} for the radial component and 3.1 x 10^{-3} for
the vertical component. This result was obtained using our standard SEMI code which
employs a total of 134 knots. It should be kept in mind that when we model a general surface
load using many circular loading elements, the interpolation errors associated with the
different loading elements will tend to cancel at a specific station, particularly in the near
field of the load where the displacements are largest.

3.5 Discussion

Approximating a spatially finite but otherwise general pattern of surface loading with
a suite of circular loading elements (of equal diameter) allows us to exploit the computational
acceleration associated with the SEMI method presented in chapter 2. This chapter builds on
the results (and utilizes the computer codes) discussed in chapter 2. In turn, chapter 2 was
motivated by the application and the algorithm presented in this chapter. The key innovation
presented here is the reformulation all but a computationally minor component of the
problem of finding the displacements at \( m \) stations due to \( n \) circular loads into the ‘parallel’
problem of computing the displacement produced at \( n \) \( m \) stations in response to a single
circular load. This is precisely the problem that the SEMI method was designed to address.
Note, however, that the problem reformulation presented herein is in no way tied to the
specific interpolation strategies employed by a code that implements the SEMI algorithm for
a single circular load. Our code for computing the displacement field due to a suite of circular loading elements calls a distinct code that implements the SEMI method for solving the ‘primitive’ or elementary problem of a single circular load. This modularity makes it easy to incorporate any improvements that may be achieved in the SEMI method. The methodology described in this chapter is very simple to state and to implement. We have stated the algorithm in great, and perhaps a surprising level of detail. This is because the whole point of this chapter, and chapter 2, is computing a solution as rapidly as possible. It would be easy to present our algorithm using equations that are more compact and/or more evocative than those used above, and yet do so in a way that potentially wastes computer time. We have avoided the use of the transformation matrix $T$ that appears in equ. 3.2, for example, because 4 of its 9 elements are zero and we do not want to waste time computing products and sums that contribute nothing towards the final solution. Similarly we have avoided computing trigonometric or inverse trigonometric functions. At the risk of appearing pedantic, we have presented the computation in what seems to us a nearly optimal approach for coding the algorithm.

The computational acceleration associated with the strategy presented herein, combining problem reformulation and the SEMI method, is particularly useful in solving inverse problems, since inverse methods often involve solving the forward problem thousands or even hundreds of thousands of times. Having solved the inverse problem using the approach described in this chapter, one could use the direct method described in chapter 2, to explore the solution in more detail, for example by evaluating quantities, such as the subsurface strain field, that did not feature in the inverse problem. Perhaps the most serious drawback of the SEMI method is that it is restricted to computing the displacement field at the surface of the half-space.
Figure 3.8 A comparison between the (a) vertical and (b) radial displacements caused by a uniform circular load imposed on (i) a uniform half-space with $E = 1.0$, and (ii) a layer of thickness $t = 0.1 \ a$ and $E = 0.1$ overlying a half-space with $E = 1.0$, as depicted in the inset in sub-plot (a).
References


CHAPTER 4

MODELING GLACIALLY INDUCED DEFORMATION IN PATAGONIA

4.1 Introduction

The Northern Patagonia icefield (NPI), Chile, and the Southern Patagonia icefield (SPI), Chile and Argentina, are the largest ice masses in South America, and together they are the third largest ice reservoir after Greenland and Antarctica (Naruse and Aniya 1992, Warren and Sugden, 1993). At present, the NPI covers an area of 4200 km$^2$, and the SPI has an area of 13000km$^2$, (Van der Veen, 1999). Melted water and ice are discharged to the Pacific ocean directly on the west side, and to lakes on the east side. The ocean on the west causes a strong east-west precipitation gradient (Rignot et al., 2003).

4.1.1 Ice History

A far larger and essentially contiguous sheet of ice covered much of the Patagonian Andes during the Last Glacial Maximum (LGM) at 19 - 21 ka (Fig. 4.1), it extended about 1800km on the crest of Andes, with volume of half a million cubic kilometers, which is capable of causing 1.2m of sea level rise (Aniya, 1995, Hulton et al., 2002) Following the rapid demise of the giant ice sheets in the Northern hemisphere (15-14ka), the Patagonia ice field started to collapse around (13ka), and by ~ 10 ka nearly all of this ice had melted (McCulloch et al., 2000). Dateable moraines and tree ring records indicate that during the

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mid and late Holocene, there were at least four relatively minor neoglacial cycles of advance and retreat, the most recent being the Little Ice Age (1400-1850 AD) and a subsequent retreat that continues to the present day (Wenzens, 1999).

**Figure 4.1** The limits of ice at Last Glacial Maximum and the distribution of existing icefield, Patagonia. Present day precipitation, its seasonal distribution with latitude, and temperatures are also shown. (Hulton et al., 2002)
4.1.2 Present ice loss from glaciology studies

Glacier studies indicate that, during the past several decades, the fronts of the glaciers in the Patagonia region have been retreating rapidly. Rignot et al., (2003) is the most recent detailed study on present ice loss in Patagonia. The data and ice loss model from this article are adopted in this dissertation research. In addition, we sought help from another well-known glaciologist, Gino Casassa, who is also a major contributor to Rignot et al. (2003), to obtain the latest ice model (not published). In the next several paragraphs, the data and ice loss models from these two sources will be briefly summarized.

According to Rignot et al. (2003), between 1968 and 1975, the Patagonian mountain glacier system contributed 0.042 +/- 0.002 mm/yr equivalent sea level rise (SLR), and in the late nineties, it caused the sea level to rise about 0.105 +/- 0.001 mm/yr. Their study gathered data from several sources: topography data from the Shuttle Radar Topography Mission (SRTM), prior Digital Elevation Models (DEM) compiled from maps, aerial photographs (March 1975) and digital photogrammetry (May 1995). Supplementing these data with field measurements, they compiled the most complete ice loss model for the Patagonia Icefield (Fig. 4.3), which relates surface elevation with ice thickness change. Between 1975 and 2000, the glacier thinning rate for the NPI is 2.63 +/- 0.4 km$^3$/yr over an area of 3481 km$^2$, and with a frontal loss of 0.20 km$^3$/yr. Scaled over the entire icefield of 4200 km$^2$, the total volume loss is 3.2 +/- 0.4 km$^3$/yr. During the same period, the SPI lost 7.2 +/- 0.5 km$^3$/yr over an area of 8167 km$^2$, with frontal ice loss being 1.3 km$^3$/yr. Scaled over the entire Southern Patagonia ice field of 13,000 km$^2$, the ice loss rate is 13.5 +/- 0.8 km$^3$/yr. The total volume loss from the NPI and SPI is 16.7 +/- 0.9 km$^3$/yr. This rapid demise is only comparable to the ongoing glacial retreat in Alaska (Arendt et al., 2002). Among the glaciers,
Pio XI and Trinidad are the only glaciers in SPI that gained mass between 1975 and 2000 (Fig. 4.2), which was caused by ice mass transportation from high elevation to lower elevation (Rivera and Casassa, 1999). In addition, Rignot et al. also detected an accelerated thinning period between 1995-2000. Because data are only available for the SPI for years 1995 and 2000, they can only assume there was no accelerated thinning for NPI, and the total ice loss volume is 41.9+/−4.4 km³/yr. The accelerated thinning rate for SPI is almost three times larger than the thinning rate in 1975 – 2000. If NPI had experienced a similar accelerated thinning, its thinning rate would be about 9.6 km³/yr and the total thinning rate would increase to approximately 48.3 km³/yr.

All three ice loss scenarios (16.7 km³/yr, 41.9 km³/yr, and 48.3 km³/yr) represent the minimum, medium and maximum unloading force that the ice field could exert on the solid Earth. Computing deformations caused by these three scenarios can help us to determine the general range of the elastic uplift signal and the elastic deformation pattern.
Figure 4.2 Location of the 63 surveyed glaciers, labeled in white, overlaid on the topography of (A) NPI and (B) SPI, derived from SRTM of February 2000. Drainage boundaries between glaciers are indicated in red. Elevation contours in gray lines are separated by 100 m between 100 and 2000 m, and by 200 m above 2000 m elevation. Ice-covered areas are shown in white shaded relief. Non-ice areas are shown in colored shaded relief with illumination from the west. (C) Location of the icefields in South America. (Rignot et al., 2003)
Figure 4.3 Thickness change versus elevation of NPI (brown, 24 glaciers) and SPI (black, 31 glaciers) during the 1975–2000 period and the 1995–2000 period (SPI, red, 20 glaciers). Each plot is the average of all glaciers. The decrease in thinning at low elevations occurs because ice was completely removed at the ice front. The average of all glaciers includes too few data points above 1400 m elevation. The inset shows the thinning rate measured along the central flow line of glaciers HPS 12 (1995–2000, light blue), Jorge Montt (1975–2000, red), Upsala (1968–2000, black), and Moreno (1975–2000, brown). (Rignot et al., 2003)

Gino Casassa from Centro de Estudios Científicos (CECS), Valdivia, Chile, provided us an updated and more detailed ice loss estimate for the Patagonia ice field. He has followed a similar, but more rigorous procedure than that used in Rignot et al.’s study. Just as in Rignot et al. (2003), Casassa’s estimate is only valid for the period 1975-2000, which is constructed from a digital elevation model based on IGM 1975 1:50.000 maps and SRTM 2000 data. 356 polygons, which are defined by 30,096 vertices, were used to describe the ice volume change for NPI. For SPI, 1366 polygons were defined by 78,580 vertices. Each
polygon is assigned a mass loss type: either frontal loss or lateral loss. Different density values are then assigned according to the location. Casassa assigned different density values to the accumulation and ablation area. For the ablation areas, an ice density of 0.917 g/$cm^3$ as used, and for the accumulation areas, a firn density of 0.45 g/$cm^3$ was used. The polygon-based ice loss estimate is a major advantage of our study, since our LHS loading model is designed to compute surface deformation induced by a polygonal load, as previously described (See chapter 3).

Casassa’s estimate shows a wasting volume of 2.17 km$^3$/yr due to surface thinning, 0.36 km$^3$/year due to frontal loss, and in total 2.53 km$^3$/yr for the NPI. Following Rignot et al. (2003), if we convert this volume to an ice equivalent rate, then it is 2.38 km$^3$/yr. The total volume is about one quarter less than Rignot et al. estimated.

The total volume loss for the SPI is 10.24 km$^3$/yr (ice equivalent). Casassa’s estimation considered 37 large basins in the SPI (large outlet glaciers according to Aniya 1986, and Rignot et al. 2003) and an extra 60 smaller basins over an area of 12,925 km$^2$. Since the SPI has a total area of 13,000 km$^2$, a small correction factor for the total loss from the SPI is considered, which increased the total volume loss to 10.24 x 13,000/12,925 = 10.3 km$^3$/yr. Compared to Rignot et al. estimated 13.5 km$^3$/yr ice loss for the entire SPI, roughly 1/4 less than Rignot et al.’s value.

Neither estimate is very accurate because of the lack of data. However, Rignot et al. only estimated 37 glaciers over a total area of 8167 km$^2$, a much smaller area than Casassa did. Another cause for Rignot et al.’s overestimate is that the total volume loss in the 37 glaciers they considered all have low-lying fronts, but the rest of the SPI has higher
elevations, and therefore they over extrapolated the ice loss for those high elevation areas (Personal communication from Casassa).

Figure 4.4 Casassa’s polygons for NPI. Color represents ice/snow thickness change rate during 1975 – 2000. 356 polygons are presented.
4.1.3 Patagonia tectonic setting

The tectonic history of southern South America is characterized by the almost orthogonal subduction of the Nazca and Antarctic plates beneath the South America plate. Subsequent to their subduction, the two oceanic plates diverged and formed a “slab window” (Thorkelson, 1996). The associated Chile Rise triple junction began migrating from west of
Tierra Del Fuego (55S) to the North at 14 to 15 Ma (Fig. 4.6). After a series of collisions, the modern Chile triple junction is now about 50km north of the Taitao Peninsula (Gorring et al., 1997). On the Earth’s surface, the migration of the slab window is marked by progressive extinction of calc-alkaline volcanism, which is then replaced by “slab window volcanism”, namely alkalic and tholeiitic eruptions. After the passage of the slab window, a new calc-alkaline arc has formed, but the new arc is closer to the trench, because the Antarctic plate descends more steeply than the Nazca plate (Thorkelson, 1996).

Figure 4.6 Tectonic setting of southern South America. Location of the volcanic centers of the Andean Austral Volcanic Zone (AVZ) and the southern Southern Volcanic Zone (SSVZ) north of Chile are also shown by triangles. Times of ridge collision are shown in red. Modified from Stern and Kilian (1996) according to Thorkelson (1996).
4.2 GPS Measurements

Bevis’s group has been using the Global Positioning System (GPS) to measure crustal motion and deformation in Southern Patagonia since 1994. The GPS data used in this study have been analyzed in conjunction with those collected by the Central Andes GPS Project (Kendrick et al., 2001) in order to study plate motion and intraplate deformation associated with subduction of the Nazca plate. Our approach to fieldwork and data analysis has been described elsewhere (Kendrick et al., 2001, Bevis et al, 1997). The velocity solutions for S. Patagonia (Fig. 4.7) are in a reference frame that has been realized by simultaneously minimizing (i) the horizontal velocities of 10 stations located in the stable core of the South American plate and (ii) the vertical velocities of 24 stations located in the South American plate and several adjacent plates. Five of these stations were survey GPS (SGPS) stations with total observational time spans of 5.7 – 8.2 years, and all others were continuous GPS (CGPS) stations with time spans of 3.4 – 9.6 years (the median span being 6.7 years). The RMS vertical velocity of all 24 reference stations is 1.6 mm/yr, a factor of 2.3 larger than the RMS horizontal velocity (0.7 mm/yr) of the 10 stations located in the stable part of the South American plate.

Of the 26 SGPS stations yielding velocity solutions in S. Patagonia, 21 were established and first occupied in a one month period in January and February of 1994. These stations were reoccupied over an extended period of time, from January 1998 to October 1999. The SGPS station PARE was established earlier, in February 1993, and had been observed for a total of 105 days by the time of its last occupation in mid-late 1999. Two SGPS stations were established in early-mid 1995 and both of these were reoccupied in early-mid 2000. Two more (EPRF and WLCH) were established in early 1998 and reobserved in early 2001. Apart from this last pair of stations, the total time span of observation for the
The vertical (upward positive) velocities of the various SGPS and CGPS stations in S. Patagonia vary between -1.8 mm/yr and 20.9 mm/yr, with a mean value of 4.8 mm/yr. Despite the fact that many of the SGPS stations have been occupied only twice (usually for 1-3 days per campaign), the spatial coherence of the vertical velocity solutions (Fig. 4.7), the fact that nearly all of these stations are located in solid rock, and the reasonably long time spans involved provide us with confidence in our initial geodetic results. There is an unmistakable tendency for vertical velocity at a station to increase with its proximity to the Northern Patagonia Icefield (NPI) or the Southern Patagonia Icefield (SPI). No station located north of 46° S or south of 52° S has a vertical velocity greater than 2.2 mm/yr. The most obvious interpretation is that we are observing postglacial rebound. But since these measurements have been made in a plate boundary zone, it is necessary to address the possibility that this broad pattern of uplift manifests a tectonic process.

The horizontal velocities of the various SGPS and CGPS stations in S. Patagonia vary between 1.4 mm/yr and 10.6 mm./yr, with a mean value of 5.4 mm/yr. (All but 2 stations, which are located close to the trench, have horizontal velocities of < 7.6 mm/yr). These velocities, which are oriented roughly eastwards, tend to decline inland, consistent with the notion that in large part they manifest elastic compression of the leading edge of the South American plate in response to locking of the subduction interface (Bevis et al., 2001). We believe that this ‘elastic loading’ phenomenon (which we and others have observed in the Central and South-Central Andes) makes no significant contribution to the large uplift rates observed in S. Patagonia because: (i) those stations located at and north of the Chile triple
junction near 46.5° S, where the Nazca – South American convergence rate is > 60 mm/yr, are uplifting less rapidly than the stations between 47.5° S and 50° S where the Antarctica – Souther American convergence rate is < 20 mm/yr, (ii) between about 47.5° S and 52° S the uplift rate declines strongly to the south, even though the Antarctica – South American convergence rate in this latitude range varies by less than 1 mm/yr, and (iii) numerical models of the elastic loading process (Bevis et al., 2001) suggest that it produces peak rates of uplift of no more than ~15% of the plate convergence rate. Even if we assume that none of the Antarctica – South American convergence is occurring within the Magellan Fold and Thrust Belt, and that the main plate boundary near the Chile trench is 100% locked, then tectonic elastic loading would produce maximum uplift rates of ≤ 3 mm/yr. In light of these arguments, we attribute the extraordinary vertical velocity field of this region to glacial isostatic adjustment.
Figure 4.7  (A) The vertical velocity field of Patagonia as measured using GPS (black arrows) and predicted by Ivins and James (1999) (purple contours, labeled in mm/yr). Green circles represent survey GPS stations, and red squares indicate continuous GPS stations. Icefields are shown in white, and flood basalts in different shades of red. There are two triple junctions in Patagonia: the Chile Rise Triple Junction (CRTJ) between the South American (SoAm), Antarctic (Ant) and Nazca plates (barely visible as a sliver in the top left corner of the map), and the junction between the SoAm, Ant and Scotia plates near (52° S, 76° W).
4.3 Postglacial rebound in Patagonia

The GPS measurements of vertical crustal motion indicate a broad pattern of uplift near the Patagonian ice field, with maximum observed velocities of ~ 20 mm/yr. This rate of uplift is an order of magnitude larger than the expected level of postglacial rebound associated with the major deglaciation following the Last Glacial Maximum (LGM). It can be explained if, as anticipated by Ivins and James (1999), the unusual tectonic setting of this region equips it with a short isostatic response time and an extraordinary sensitivity to the neoglacial fluctuations caused by Late Holocene climate change.

Glacial isostatic adjustment involves interplay between the time and space history of the ice load and the mechanical properties of the solid earth (Cathles 1975, Mitrovica and Peltier, 1991). The viscoelastic response of the earth to changing surface loads is influenced by the stiffness of the lithosphere and by the viscosity of the underlying mantle. For loads of great lateral extent the flexural rigidity of the lithosphere is much less important than the viscosity profile of the underlying mantle. Both mantle viscosity and elastic thickness are strongly influenced by tectonic setting. Fennoscandia, which is a stable continental shield, has a long isostatic memory, and so even though the great ice sheet which existed at the LGM had almost completely melted by ~ 10 ka, postglacial rebound continues today with a peak rate of ~ 9 mm/yr (Johannson et al., 2002). In contrast, geologically young and volcanically hyperactive Iceland, which has a thin elastic lithosphere overlying a very low viscosity mantle, is characterized by much shorter isostatic response times. As a result when the Younger Dryas ice cap that had covered most of Iceland began melting about 10 ka, postglacial rebound was essentially completed in ~ 1000 years (Sigmundsson, 1991).

Ivins and James (Ivins and James, 1999) modeled postglacial rebound in Patagonia assuming the ICE-3G model for ice load history (Tushingham and Peltier, 1991) and a wide
range of geomechanical scenarios, including a Fennoscandia-like regime at one extreme and a Basin and Range or Iceland-like regime at the other. These simulations indicated that isostatic memory of the LGM should be minimal, producing modern uplift rates of \( \leq 1 \) mm/yr under all geomechanical scenarios. But for Basin and Range regimes with very short isostatic response times, memory of the neoglacial cycles that have occurred since 5 ka, including the Little Ice Age, could produce contemporary rates of uplift of \( \geq 10 \) mm/yr.

This ‘hot and weak’ mechanical scenario is plausible given the tectonic setting of the Patagonian Andes. The basement rocks comprise a late Paleozoic-early Mesozoic fore-arc accretionary wedge rather than a Precambrian shield. In addition, the Chile Rise triple junction associated slab window formed as the subducted portions of the Nazca and Antarctic plates diverged in response to continued divergence of the surface extensions of these plates at the mid ocean ridge (MOR) (Thorkelson, 1996). The mantle upwelling associated with the MOR has passed and continues to pass through this slab window and impinges directly on the base of the South American plate. This geologically recent encounter between the East Chile Rise and the Chile Trench has led to high temperatures, high degrees of partial melting and low mantle viscosities near the Patagonian ice field.

Of the many geomechanical scenarios considered by Ivins and James (Ivins and James, 1999), only the ‘hottest and weakest’ scenario generating the shortest isostatic response times predicted a vertical velocity field comparable to that observed using GPS. This prediction, which assumes a mantle viscosity of 5 x \( 10^{18} \) Pa s and an elastic thickness of 25 km, is shown in Figure 4.7. Under this scenario the Patagonian Andes have essentially no memory of the LGM, and present-day uplift is dominated by the neoglacial cycles of the last five thousand years, including (and especially) the Little Ice Age and the vigorous deglaciation that began in about 1850 and continues to the present day (Wenzens, 1999;
Aniya et al., 1995, Casassa et al., 2002). The suggestion is that, in a good example of earth system interactions, the peculiar tectonic setting of the Patagonian Andes has equipped this mountain belt with a short term isostatic memory and an extraordinary sensitivity to recent climate change.

A detailed comparison of the GPS results with the closest prediction of Ivins and James (1999) indicates that near the NPI and the SPI their model tends to under predict the geodetic results. Further south near the Cordillera Darwin the station geometry is rather poor, but there is a suggestion that their model is over predicting uplift rates. This might be because the model ice load history seriously overestimates the amount of ice loss in this region. Alternatively the thermal anomaly associated with low mantle viscosity and short isostatic response times might be much less pronounced this far south: we note that the large quantities of geologically recent flood basalts do not extend much further south than 52° S. If this second explanation is correct, then the next generation of glacial isostatic adjustment models (for Patagonia, at least) will need to accommodate lateral variability in geomechanical structure.

4.4 Elastic rebound due to the present ice loss

4.4.1 Elastic structure

As the input to the elastic loading model developed in chapters 2 and 3, we derived two elastic properties profiles from CRUST2.0 and PREM (Mooney et al., 1998, Dziewonski and Anderson, 1981). Figure 4.8 and figure 4.9 show the elastic properties versus depth. In addition, we combined the crustal structure from CRUST2.0 with the mantle structure from PREM to form a hybrid elastic structure that extends from the earth’s surface to the core-
mantle boundary. Hereafter we refer to this as the HYBRID model. HYBRID therefore has vastly more shallow detail than does PREM.

**Figure 4.8** Young’s modulus and Poisson’s ratio derived from the CRUST2.0 model.

**Figure 4.9** Young’s modulus and Poisson’s ratio derived from the PREM model.
4.4.2 Derive ice thickness change from Rignot’s ice loss model

Rignot et al. (2003) plotted surface elevation against ice thickness using data collected between 1975 and 2000 (Fig. 4.3). The two curves (black and brown) show almost linear relationships between elevation and ice thickness change. The curve (red) for the period (1995 - 2000) of accelerated thinning of the SPI is also shown. For simplicity, we fit three linear models to these curves. 2714 circles were filled inside the polygon that is defined by the outline of the NPI, and 7400 circles were filled in the SPI polygon. The large number of circles depicts the details of the ice field shape reasonably well. All of these circles have a radius of 500m. From SRTM data we obtained elevations for the centers of the circles, and then using the linear models that were derived earlier, we assigned each circle an ice thickness change value. Because very few data exist above 1400m, Rignot et al. (2003) assume no ice thickness change above that elevation. The results of this process are shown in figure 4.10 and 4.11. Comparing these two maps with the maps that were made from Casassa’s estimates (Fig. 4.4, 4.5), they generally agree well, although these two maps do not depict every outlet glacier. One concern is that Rivera and Casassa (1999) and Rignot et al. (2003) both observed advancing of the glacier Pio XI in the SPI between 1975 and 2000. The above procedure does not take this into account, and instead only relates elevations with ice thickness changes. Consequently, Pio XI appears to have been losing mass instead of gaining. However we do not believe this will affect our conclusion significantly because: i) The loading effect of a single glacier is very localized. Elastic signals approach maximums at the edge of the load and decrease almost exponentially as the distance increases from the load. ii) We had made sure the total ice loss volume derived from the above procedure matches the results that Rignot et al. reported, therefore the overall loading effect remains the same.
Figure 4.10 Ice thickness change rate derived from the Rignot et al., (2003) model for NPI, which corresponds to a total ice volume loss of $3.2 \, \text{km}^3/\text{year}$.
Figure 4.11 Ice thickness change rate derived from the Rignot et al., (2003) model for SPI, which corresponds to a total ice volume loss of 13.5 km$^3$/year.

In total, 12 different loading scenarios were created. The abbreviated names for these scenarios are listed in table 4.1. ‘R’ refers to ice loss scenarios that are derived from Rignot et al.(2003), and ‘C’ refers to the ice loss scenario derived from Casassa’s estimate.
Table 4.1 Abbreviated names for different loading scenarios.

<table>
<thead>
<tr>
<th>Ice Loss km³/yr</th>
<th>Earth Model</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CRUST2.0</td>
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<td></td>
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<tr>
<td></td>
<td>PREM</td>
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</tr>
<tr>
<td></td>
<td>CRUST2.0 / PREM</td>
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4.4.3 Results

We computed vertical deformation fields for each of the ice loss – elastic structure combinations (Fig. 4.12 – 4.17). With 10,114 loading disks and a 60x90 grid, the computation of each scenario usually takes less than two minutes. Our goal is to explore the range of the elastic uplift signal, and to determine whether the present ice loss is a significant contributor to the observed broad pattern of uplift.

By examining the figures, we can immediately eliminate several results that are obviously not acceptable. GPS stations indicate that the stable part of the South American plate (East of 70W) shows zero or minor uplift (<1mm/yr) (Fig. 4.7). Results from R2 CRUST and R3-CRUST both have larger than 1mm/yr uplift in the stable craton of the South American plate. This is because for the elastic structure derived from CRUST2.0, the half space below the Moho has the elastic properties of the upper mantle, which is much ‘softer’ than PREM at a similar depth. All three results that used Casassa’s ice loss model show uplift around glacier Pio XI (blue concentric contours), while the R series results do not depict this detail. But as expected, ignoring this detail does not affect the overall uplift pattern.

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Figure 4.12 Crustal uplift caused by the present ice loss. Upper: C-CRUST Lower: C-PREM
Figure 4.13. Crustal uplift caused by the present ice loss. Upper: C-HYBRID Lower: R1-CRUST
Figure 4.14. Crustal uplift caused by the present ice loss. Upper: R1-PREM Lower: R1 HYBRID
Figure 4.15 Crustal uplift caused by the present ice loss. Upper: R2-CRUST2 Lower: R2 PREM
Figure 4.16 Crustal uplift caused by the present ice loss. Upper: R2-HYBRID Lower: R3-CRUST2
Figure 4.17 Crustal uplift caused by the present ice loss. Upper: R3-PREM Lower: R3 HYBRID
If we pick any ice load scenario and compare the deformation fields that were derived using PREM and HYBRID (e.g., R2-PREM and R2-HYBRID), we notice a very similar vertical uplift pattern. It seems that the two elastic structures behave the same. Which elastic structure is superior? This question needs to be answered before we could proceed any further. We therefore conducted a synthetic numerical experiment to investigate this issue.

The Patagonia ice field spans about six degrees in the North-South direction, and about one degree in the East-West direction. We designed a rectangular shape water body, which has a length of 600km, and a width of 100km (Fig. 4.18) to mimic the size of the ice field. A two-layer elastic structure was created, with the first layer’s thickness as 100km, equal to the shorter dimension of the load, and the second layer being a half-space. We then averaged the elastic moduli of the top 100km of HYBRID, and assign them to the first layer. Similarly, the elastic parameters of the rest of HYBRID were averaged, and assigned to the bottom layer. We also created two uniform half-space structures, with elastic properties equal to that of the top and bottom layer of the two-layer structure respectively. Table 4.2 shows the details of the setup. We then computed vertical deformation on the grid using a LHS model and two uniform half-space (UHS) loading models, and three deformation fields were derived (Fig. 4.19).
Figure 4.18 A 600km x 100km large area is filled with 60x10 circular loading disks (magenta circles) to simulate the load applied by a 100m deep water body. Vertical deformation is computed on a 40 x 40 grid (blue dots).

<table>
<thead>
<tr>
<th>LHS</th>
<th>UHS-1</th>
<th>UHS-2</th>
</tr>
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<tbody>
<tr>
<td>E1, nu1</td>
<td>E1, nu1</td>
<td>E2, nu2</td>
</tr>
<tr>
<td>E2, nu2</td>
<td></td>
<td></td>
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Table 4.2 E1 = 86Gpa, nu1=0.28, E2=390Gpa, nu2=0.27
Figure 4.19 Vertical deformation fields derived from LHS (upper), and UHS-1 model (lower). Black dashed line indicates the location of the water body.
As expected, the vertical deformation field derived by the UHS-1 model has the largest magnitude, because it has the softest elastic structure, while the deformation field from UHS-2 has the smallest magnitude, and the magnitude of the LHS result is intermediate. In order to determine the similarity among these three deformation fields, we computed absolute relative error for the LHS – UHS-1 pair and the LHS – UHS-2 pair (Fig. 4.20 and fig 4.21). From 4.20, we observe the largest differences are in the far field (large relative error), and the largest similarities reside inside the load and in the area adjacent to the load. Because UHS-1 has elastic properties of the top layer of LHS, the similarities indicate that deep layer structure plays only a minor role in producing uplift in the near field, and the load mostly ‘feels’ the shallow elastic structure. Figure 4.21 shows the opposite situation, where uplift in the far field is largely dependent on deep structure. This result is a further confirmation of the theory described in chapter one (Fig. 1.2). Once again, shallow elastic
structure is shown to be particularly important when one tries to estimate elastic deformation at locations near the load.

Since 1972, when Farrell (1972) presented the general solution for solid Earth deformation induced by loading, it has become very prevalent in the geophysics community. A recent summary of the state of the art was given by Clarke et al. (2005):

“The geometric and gravimetric effects of loading may be computed by convolving models giving the gridded surface mass distributions with a Green's function, describing the unit impulse response of the Earth as a function of load and response location (Farrell, 1972). Typically an elastic model with radial structure, such as PREM (Dziewonski and Anderson, 1981) is used, in which case the Green's function depends on load-response separation only...”

Many researchers have used this method to compute elastic deformations, and most of the computations are at locations that are adjacent to the load (Wahr et al., 2001; Khan et al., 2007; Wahr and Han, 1997; van Dam et al., 2001), where geodetic measurements are usually conducted. It is well known that Earth’s crust is extremely heterogeneous, and PREM does not have good resolution vertically at shallow depth and does not account for any lateral variation, therefore the results they derived could have been highly degraded.

A further examination of figures 4.20 and 4.21 shows that, although shallow structure controls the deformation at near field, and deep structure controls the deformation at far field, neither of them can be solely responsible for the deformation in the area they dominate. This is illustrated by the 60% relative error inside the loading area in figure 4.20, and by the 10% relative error at the far field in figure 4.21. Therefore, the hybrid elastic structure (HYBRID) that takes advantages of both CRUST2.0 and PREM is a better option for the purpose of elastic modeling.
Figure 4.20 Absolute relative error of the vertical deformation field for the LHS - UHS-1 pair. White dashed line is the outline of the load.

Figure 4.21 Absolute relative error of the vertical deformation field for the LHS - UHS-2 pair. White dashed line is the outline of the load.
The conclusion drawn from the above experiment is that the results derived using HYBRID are more trustworthy. However, at the moment we do not have any data (e.g., seismic data) collected from this region to support this conclusion. Therefore we cannot simply rule out the results derived using other elastic structures. We believe all the results (except the two that have been excluded) are still valid scenarios for elastic uplift in the Patagonia region, but the results that are based on HYBRID are probably closer to the ‘truth’. More data are needed to guide us on selecting the most realistic model.

We estimated uplift at all the locations that GPS measurements exist, and computed the percentage of the modeled elastic uplift to the observed uplift (Fig. 4.22). In general, C-PREM and C-HYBRID have the smallest magnitudes of uplift, on average about 5-6% of the observed uplift. This is because Casassa’s ice loss estimates are much smaller than those of Rignot et al. (2003). R3-PREM and R3-HYBRID have the largest magnitudes of uplift among all the models. On average, the modeled uplifts from these two contribute 20-25% of the total observed uplift. From left to right, the GPS stations in figure 4.22 are ordered by the magnitude of the observations. Normally, GPS stations show a larger magnitude of deformation when closer to the ice field. In figure 4.22, we see the modeled percentage rise to the right with a very gentle slope, which means that the elastic models can explain more uplift signal in the far field than in the near field.
Figure 4.22 Percentage of the modeled vertical uplift to the total observed. From left to right, GPS stations are ordered by the magnitude of observed uplift.
4.5 Conclusions

1. Geodetic measurements in the Patagonia region indicate a broad pattern of uplift near the ice field, with maximum observed velocities of ~ 20 mm/yr. This rate of uplift is an order of magnitude larger than the expected level of postglacial rebound associated with the major deglaciation following the Last Glacial Maximum. As anticipated by Ivins and James (1999), the unusual tectonic setting of this region equips it with a short isostatic response time and an extraordinary sensitivity to the neoglacial fluctuations caused by Late Holocene climate change.

2. From the most recent estimates of secular ice loss rate, we determined that about 5-25% of the observed uplift is contributed by present ice loss. However, neither ice loss rate nor Earth’s elastic structure is well constrained. Further ice loss measurements are needed especially at high elevations (>1400m) of the ice field, in order to construct a more accurate ice loss model. Seismic surveys would be very helpful for studying Earth’s elastic structure.

3. Shallow elastic structure plays a dominating role when modeling local response to a short wavelength load. A hybrid elastic structure that incorporates detailed crust information as well as deep elastic properties is most suitable for modeling such a response.
References:


Rabassa, J., C. M., Clapperton, (1990), “Quaternary glaciations of the Southern Andes”, *Quaternary Science Review*. 9, 153-174


Wenzens, G., (1999), “Fluctuations of outlet and valley glaciers in the Southern Andes (Argentina) during the past 13,000 years”, Quaternary Research, 51, 238-247
CHAPTER 5

INVERSE BOUSSINESQ RESPONSE

5.1 Introduction

When a surface pressure load is imposed on an elastic half-space representing the earth, we nearly always find that the solid earth beneath and adjacent to the load deflects downwards, that this deflection diminishes with increasing distance from the load, and that in the region outside but in the near field of the load, the horizontal displacement is directed towards (not away from) the center of the load. This is always true for a uniform elastic half-space (UHS) and is also true for the vast majority (if not all) of the published loading response curves for a layered elastic half-space (LHS) that characterizes the earth's crust and mantle (e.g. a PREM-based profile). But at a USGS operated laser strainmeter facility located near a railroad track, it was reported that when a train passes by one strainmeter arm (oriented nearly perpendicular to the track) the signals recorded by the strainmeter imply that the ground surface near the track is displaced away from the track (load) as the train passes by and so loads the ground (Agnew and Wyatt, 2003). This is the inverse of the expected horizontal displacement signal. One possibility is that near-surface (shallow) elastic structure is anomalous, and an inverse Boussinesq response (IBR) can occur as an edge effect in the near field of the load. If this is true, this could be an important detail to understand when attempting to 'weigh' rivers or glaciers using continuous GPS stations (CGPS). It is important
to keep the inverse Bousinessq effect in mind and take it into consideration when selecting locations for CGPS stations.

In this chapter, a LHS loading model is used to study this issue, and we try to provide some rules of thumb about when and where an IBR can occur.

5.2 An extreme example

When an elastic model is used to address the earth surface loading problem, Young’s modulus \((E)\) and Poisson’s ratio \((\nu)\) are the major elastic properties that need to be considered. Young’s modulus is a measure of the stiffness of a given material, and Poisson’s ratio is used to describe the tendency of elongation of a material when it is compressed in a perpendicular direction (Turcotte and Schubert, 2002). One possible explanation of the irregular strain pattern observed in the Durmid Hill strainmeter facility could be anomalous elastic properties \((E, \nu)\) in the shallow subsurface.

Consider a two-layer flat earth model (one layer on top of a uniform half space, we ignore the earth curvature because the spatial scale of this problem is small), each layer has a different Young’s modulus \(E_i\), and Poisson’s ratio \(\nu_i\). We assume there is a relatively strong contrast of the stiffness between the two layers \((E_1 < E_2)\). We also assume the top layer has a high Poisson’s ratio, which indicates the material in the top layer does not compress much when force is exerted.

In figure 5.1, the Z axis is positive downwards into the half-space. The figure below shows the vertical and horizontal displacement along a profile when a vertical load is applied to the two-layer model described above:
Figure 5.1 Horizontal and vertical displacements evaluated along the x-axis. The circular load is centered at the origin, and the load radius is 500m. The surface pressure is \(9.8 \times 10^5\) Pa. \(E_1 = 0.1 \times 10^{11}\) Nm\(^{-2}\), and \(E_2 = 1 \times 10^{11}\) Nm\(^{-2}\). \(\nu_1 = 0.49\) and \(\nu_2 = 0.25\). The black dot indicates a ‘local’ minimum. When \(u_z(i-1) > u_z(i) < u_z(i+1)\), \(u_z(i)\) is a ‘local’ minimum. The red dot indicates the maximum value of \(u_r\). In later experiments, the positive value of \(u_r\) are used to identify the occurrence of IBR.

In the lower panel, between 0~1500m, the positive \(u_r\) values indicate IBR effect occurs, that is, surface material moves away from the load. In addition, we observe an unusual ‘low’ in the vertical component \(u_z\) (inside the dotted circle in the upper panel), however, this effect is not easy to quantify. One possible (but maybe not a reliable solution) is to use the localized lowest point to represent its occurrence (see the black dot on the \(u_z\) curve). On the Earth’s surface, this would be characterized as a ‘bump’.
In the previous case, some elastic parameters adopted may not be realistic for every place, e.g. the first layer’s Poisson’s ratio is 0.49. But a Poisson’s ratio this large exists under certain conditions, as we explain later. Figure 1 shows that using a two-layer model, it is possible to produce the situation that the ground surface moves away from the load. In the following sections, we investigate in detail the necessary conditions for IBR to take place.

5.3 Elastic structure for crust

In order for this study to be as realistic as possible, we used elastic structures based on CRUST2.0 (Mooney et al, 1998). As introduced earlier, it is the best available global crust model, which has a resolution of 2° x 2°. CRUST2.0 divides Earth’s crust into seven layers, including: ice, water, soft sediment, hard sediment, upper crust, mid crust, and lower crust. From density, compression wave, and shear wave velocities, Young’s modulus and Poisson’s ratio are computed for each layer. It is necessary to investigate the range of elastic structure parameters in the CRUST2.0 space, so that in the following experiments we can choose meaningful values to conduct sensitivity tests. Important statistics are listed in table 5.1.

In the soft sediment layer, the mean Young’s modulus is 8.0Gpa, and the mean thickness is about 500m. The minimum and maximum layer thickness are 70m and 2000m respectively. The maximum Poisson’s ratio is approximately 0.38, and the minimum of Poisson’s ratio is 0.24. For simplicity of the numerical experiments, I treat the soft sediment layer as the top layer in the elastic structure, and the rest of the crust layers form the half-space underneath.
Young’s modulus is averaged over the layers underneath the soft sediment $(E_{crust \_avg})$. The maximum value of $E_{crust \_avg}$ is close to 100Gpa. The averaged Poisson’s ratio is about 0.25.

<table>
<thead>
<tr>
<th>Variable definition</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean $E$ of the soft sediment layer</td>
<td>$E_{AVG _SOFT _SED}$</td>
<td>8.0 Gpa</td>
</tr>
<tr>
<td>Maximum value of $E_{crust _avg}$</td>
<td>$E_{MAX _AVG}$</td>
<td>89 Gpa</td>
</tr>
<tr>
<td>Maximum ratio between soft sediment $E$ and $E_{crust _avg}$</td>
<td>$E_{MAX _RATIO}$</td>
<td>13</td>
</tr>
<tr>
<td>Minimum ratio between soft sediment $E$ and $E_{crust _avg}$</td>
<td>$E_{MIN _RATIO}$</td>
<td>3</td>
</tr>
<tr>
<td>Minimum soft sediment layer thickness</td>
<td>$T_{MIN _THICKNESS}$</td>
<td>70m</td>
</tr>
<tr>
<td>Maximum soft sediment layer thickness</td>
<td>$T_{MAX _THICKNESS}$</td>
<td>2000m</td>
</tr>
<tr>
<td>Average soft sediment layer thickness</td>
<td>$T_{AVG _THICKNESS}$</td>
<td>500m</td>
</tr>
<tr>
<td>Maximum Possion’ ratio of the soft sediment layer</td>
<td>$\nu_{MAX}$</td>
<td>0.38</td>
</tr>
<tr>
<td>Minimum Possion’ ratio of the soft sediment layer</td>
<td>$\nu_{MIN}$</td>
<td>0.24</td>
</tr>
</tbody>
</table>

Table 5.1 Important quantities from CRUST2.

5.4 Numerical experiments

The purpose of this section is to explore the tunable variables in the elastic structure in order to determine which variable has the most effect on producing IBR.
5.4.1 Vary the thickness of the top layer

In this experiment, we study the relationship between IBR and the thickness of the top layer. The Young’s modulus of the top layer is assigned as 8Gpa, and that of the second layer is 100Gpa. Their ratio is about 13. In order to maximize the possibility of IBR occurrence, the top layer’s Poisson’s ratio $\nu_1$ is set to 0.49, while the half-space has a more common Poisson’s ratio of 0.25. For the load object, a circular disk with a radius of 100 m is applied at the origin, and the pressure is $4.9 \times 10^6$ Pa (This is equal to the weight of a 500m deep water body). We then sampled one hundred times between $T_{MIN\_THICKNESS}$ (70m) and $T_{MAX\_THICKNESS}$ (2000m), and assigned those values to the thickness of the first layer. On the x-axis we set a 500 nodes profile along which we computed vertical and radial displacements for the one hundred cases. Figure 5.2 shows the result of this experiment.
Figure 5.2. Gray bars represent the spatial extent over which the ur components have positive signs, namely the spatial extent of the IBR effect (read this quantity according to the left vertical axis). The red dots represent the locations of maximum ur value, black dots represent the locations of minimum uz values (use the left vertical axis), and the red and black curves are the corresponding maximum and minimum values (use the right vertical axis).

We can observe the following facts from this figure as the layer thickness increases:

1. IBR does not occur until the top layer thickness is close to 100m.
2. The spatial extent of the IBR effect increases (represented by the length of the gray bars).
3. The location of maximum ur and minimum uz values moves away from the load, and the magnitude of the IBR effect decreases. The largest IBR effect occurs when the first layer thickness is close to the load radius (represented by the location of the red and black dots).
4. The starting point of the IBR moves away from the load when the layer thickness is large enough (>450m) (See the position change as indicated by the lower ends of the gray bars).

5. The spatial extent of IBR, and the distance between the starting point of IBR and the load, as well as maximum ur and minimum uz values are strongly affected by the first layer thickness.

Figure 5.2 only represents the extreme case, since \( \nu_1 \) took a value of 0.49. To be more representative, we re-assign \( \nu_1 \) to 0.35, which is the most common value for the soft sediment layer (based on CRUST2.0), and implement the experiment again. Figure 5.3 shows the result.

**Figure 5.3** Same as Figure 1 except \( \nu_1 \) is set to 0.35.
Changing $\nu_1$ from 0.49 to 0.35 lowered the maximum value of $ur$ and $uz$ by almost an order of magnitude, and spatial extent of IBR became smaller, but all the other conclusions still hold.

5.4.2 Vary the top layer’s Poisson’s ratio $\nu_1$

In this experiment, we investigate how the Poisson’s ratio of the top layer affects the occurrence of IBR. The parameters of the loading model are the same as experiment 1 except that the top layer Poisson’s ratio $\nu_1$ takes on values that are equally sampled 100 times between 0.25 to 0.49, and the top layer thickness is set to $T_{\text{AVG,THICKNESS}} = 500$ m.

**Figure 5.4** Poisson’s ratio $\nu_1$ is equally sampled 100 times between 0.25 to 0.49, first layer thickness is set to 500m.
From figure 5.4, we can observe the followings facts:

1. Some local minimum $uz$ values (black dots) appear at infinity. Only when $\nu_1$ is large enough (>0.33), do they appear at near field of the load (see the location of the black dots). These values are very sensitive to different variable changes in the model, and their appearance may not be related to IBR. Therefore we do not treat the locations of these values as reliable indicators of IBR, instead we use the positive sign of the $ur$ component to identify IBR’s occurrence (surface materials move toward the load).

2. The starting position of the IBR effect moves closer to the load as $\nu_1$ increases. We also observe that the larger the Poisson’s ratio becomes, the larger the magnitude and spatial extent of IBR.

3. First layer Poisson’s ratio strongly affects the occurrence of IBR. IBR does not occur until $\nu_1$ is larger than 0.28. In the following two figures, we show the results of the same experiment except that the first layer thickness is changed to $T_{\min \_THICKNESS} = 70m$ and $T_{\max \_THICKNESS} = 2000m$ respectively. By doing this, we can ascertain that 0.28 is the minimum value that $\nu_1$ has to take in order for IBR to occur, and this minimum value is not affected by the top layer’s thickness change.
Figure 5.5 Poisson’s ratio $\nu_1$ is equally sampled 100 times between 0.25 ~ 0.49. First layer thickness is set to 70m.
5.4.3. Vary Young’s modulus ratio between the two layers

In this section, we vary the stiffness contrast between the two layers to study its relation to IBR. Based on CRUST2.0, $E_{\text{MIN\_RATIO}} = 3$ and $E_{\text{MAX\_RATIO}} = 13$ (see table 5.1 for definition), therefore we set the maximum ratio to be 15 to cover all the possibilities. Similar to what we did before, the Young’s modulus ratio took on values that are equally sampled 100 times between 0 and 15. The top layer thickness is set to $T_{\text{AVG\_THICKNESS}} = 500$ m. For Poisson’s ratio we set $\nu_1 = 0.49$ and $\nu_2 = 0.25$. The rest of the loading model is the same as in previous experiments.
Figure 5.7 The Young’s modulus ratio between the two layers is equally sampled 100 times between 0 and 15. \( \nu_1 = 0.49 \) and \( \nu_2 = 0.25 \).

The following facts can be observed from figure 5.7:

1. Varying stiffness contrast of the two layers does not affect the starting position of the IBR effect (the lower ends of the gray bars are almost at the same distance to the load).

2. The locations of maximum \( u_r \) and minimum \( u_z \) are almost independent of the stiffness contrast of the two layers.

3. The stiffness contrast contributes to the magnitude change of maximum \( u_r \) and local minimum \( u_z \).
We then lowered $\nu_1$ to 0.35, a common value for the soft sediment layer, to see how strong the stiffness contrast needs to be to produce IBR.

Figure 5.8 Same as figure 5.7, except $\nu_1$ is changed to 0.35.

Figure 5.8 shows that when $\nu_1$ is 0.35, only a minor increase of the stiffness contrast (an increase from 1.5 to 4.8) is needed in order to produce IBR. However, the magnitude of IBR is lowered with this lower Poisson’s ratio.
5.5 Unusual strain pattern near loads observed by strainmeters at Drumid Hill, California

5.5.1 Introduction

The Caltech-operated laser strain meter, located in Durmid Hill (DHL) at the southern end of the San Andreas Fault, started recording strain measurements in 1994 (Agnew and Wyatt, 2003). Agnew and Wyatt reported an unexpected strain pattern in response to the local load, which is only several hundred meters away, caused by the weight of passing trains. Plotting of time against strain confirms the correlation between the passing trains and the strain signal. But instead of seeing extensional strain around the load, they observed the opposite strain pattern (red dotted circle in the left panel, Fig. 5.9). They modeled the loading effect utilizing a uniform half-space (UHS) model, and they found the sign of the model results have to be switched in order to have a similar pattern as the observed strain (right panel, figure 5.9).

![Figure 5.9 Observed strain due to passing train loading (left panel) and modeled strain (right panel). Agnew and Wyatt, (2003).]

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Three strainmeters are located to the east of the Salton Sea (5.10). The railroad is shown as a green dotted line that runs from northwest to southeast. Strainmeter NE-SW is almost perpendicular to the railway, and strainmeter NS forms a 45-degree angle with the railway.

**Figure 5.10** The location and geometry of the NS, NE-SW strainmeters and the railroad. Agnew and Wyatt, (2003).

5.5.2 Numerical modeling

The geometry of the strainmeters and railroad, including location, length, and orientation are highly mimicked in this synthetic experiment. Since we do not have accurate information (e.g. the weight of the passing train) or observation data for this site, the goal is not to produce precisely the strain that the strainmeters observed, but to generate a similar
strain ‘pattern’. Through this modeling process, we try to explain the deformation mechanism involved.

First of all, following Agnew and Wyatt, we used a uniform half space model to reproduce their modeled strain. The weight of the train is assigned as $4 \times 10^6$ kg, as they used in their experiment. The averaged Young’s modulus and Poisson’s ratio, which are derived from CRUST2.0 for this area, are input elastic parameters to the uniform half space model. One hundred loading disks were used along the railroad, with one loading disk taking effect at a time to simulate a moving train. Horizontal displacements were computed at four ends of the two strainmeters, as well as on a rectangular grid that surrounds the strainmeters. Elastic strains are then computed on the strainmeters. Figure 5.11 shows the strains of the two strainmeters as the train moves from northwest to southeast. Following Agnew and Wyatt (2003)’s figure, the x-axis is the distance from the train to the southern end of the NS strainmeter.

![Figure 5.11](image)

**Figure 5.11** Strain produced by the passing trainload at the two strainmeters on a uniform half-space.

The curves in figure 5.11 have very similar shapes compare to Agnew and Wyatt’s modeled result (right panel of figure 5.9). However, notice that the position of the NS curve
is different. The two curves Agnew and Wyatt derived are apart from each other, especially
the NS curve is completely above zero, which means that the NS strainmeter always in
extension as the train passes by. This is also true even before they switch the sign of the
curves. Whereas our UHS model shows the NS strainmeter has significant compression when
the train is close (Fig. 5.12, time step 1). In figure 5.12, the ground deformation at time step 1
is shown, that is, when the NS strainmeter has the most compression. We can see it is quite
reasonable for the NS strainmeter to ‘feel’ compression.

Figure 5.12 Horizontal displacements of the ground surface on a uniform half space, when a
train passes by.

Considering the relative position of the NS strainmeter to the railroad, and its
orientation, it is not obvious why the NS strainmeter does not measure compression at all.
One possible explanation could be that the strainmeter is so close to the Salton Sea, the water
loading signal may interfere with the train loading signal to cause this effect. In addition, California is a well known tectonically active area. Tectonic signals and fault movements could also have an effect.

Similar to what Agnew and Wyatt (2003) concluded, the previous small experiment demonstrated that a UHS model is incapable of explaining the observed strain pattern. What they observed is clearly the IBR effect, and it can only be explained by using a LHS model. From our previous discussion, we concluded that three factors control the occurrence of IBR: high Poisson’s ratio of the top layer, strong stiffness contrast between the top layer and layers beneath, and top layer thickness. The table below shows crust properties of this area from CRUST2.0.

<table>
<thead>
<tr>
<th>Layers Definition</th>
<th>Young’s modulus (Gpa)</th>
<th>Poisson’s ratio</th>
<th>Layer Thickness (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>soft sediment</td>
<td>8.17</td>
<td>0.35</td>
<td>1000</td>
</tr>
<tr>
<td>hard sediment</td>
<td>39.43</td>
<td>0.26</td>
<td>500</td>
</tr>
<tr>
<td>upper crust</td>
<td>84.53</td>
<td>0.25</td>
<td>9000</td>
</tr>
<tr>
<td>middle crust</td>
<td>91.27</td>
<td>0.25</td>
<td>8500</td>
</tr>
<tr>
<td>lower crust</td>
<td>96.83</td>
<td>0.29</td>
<td>8500</td>
</tr>
<tr>
<td>mantle below Moho</td>
<td>174.99</td>
<td>0.25</td>
<td>Inf</td>
</tr>
</tbody>
</table>

Table 5.2 Crust properties at Durmid Hill from CRUST2.0 model.

Our numerical experiments (section 5.4) showed that when the top layer’s Poisson’s ratio is 0.35, the stiffness contrast ratio between the top layer and layers underneath only needs to be 4.8 for IBR to occur. At Durmid Hill, the Poisson’s ratio is fairly high, and the stiffness contrast ratio is about 4.82, therefore it is highly possible for the crust to deform in
an irregular way. After exploring different combination of model parameters, we found that when the top layer thickness is set to 250m, the modeled strain is most similar to that observed (Fig. 5.13).

Figure 5.13 Strain produced by a passing trainload at the two strainmeters on a layered half-space.

The horizontal displacements on a rectangular grid are computed at time step one and time step two. Time step one is when the NS strainmeter has largest extension while the NE-SW strainmeter has largest compression (Fig. 5.14). Time step 2 is when the NS strainmeter has largest compression (Fig. 5.15). The blue arrows in these two figures clearly show that some crust material is displaced away from the load. At places very close to the load, and places that are far away from the load, crust material moves toward the load as expected (gray arrows).

The above experiments show that for the specific orientations of the strainmeters at Durmid Hill, it is easy to cause the NS and NESW strainmeters to have extension (this is when one end of the strainmeter moves more towards the load than does the other end) or for the NS strainmeter to have compression (Fig. 5.12). But IBR must play a part in order for the NESW strainmeter to have compression. This only happens when one end of the NESW
strainmeter sits in the IBR zone (blue arrows in fig. 5.15 and 5.15), and the other end sits in the regular zone (gray arrows in fig. 5.14 and fig. 5.15). It is interesting to note that, because the location and spatial IBR are very sensitive to several properties of the crust, and considering that researchers at the time did not have a detailed knowledge of the crust of this area, it was almost pure luck for the strainmeters to have caught these interesting signals.

**Figure 5.14.** Horizontal surface displacements on a LHS model due to a passing train load at time step 1 (See fig. 5.13). IBR effect is shown. Blue arrows show the areal extent of the ground that moves away from the load, and gray arrows show ‘regular’ surface movement (toward the load center). Red arrows in the dashed square box are enlarged for better visibility.
Figure 5.15 Same as figure 5.14, but at time step 2. Red arrows in the dashed square box are enlarged for better visibility.

5.6 Geological plausibility of high Poisson’s ratio

Rock Poisson’s ratio can be calculated from shear wave velocity and compression wave velocity (Costain et al., 2004).

\[ \nu = 0.5 \times (1 - \frac{1}{(V_p/V_s - 1)}) \]  

(5.1)

Wave velocities in rocks are most conveniently measured in lab environments. Under different pressure, the arrival time of the shear wave and compression wave are measured, and from these the velocities are calculated. Poisson’s ratios are then calculated using the relationship in equation 5.1. Table 5.1 listed Vp/Vs ratio and Poisson’s ratio for common rock types (Christensen, 1996). Most of these rocks have Poisson’s ratio less than 0.30.
<table>
<thead>
<tr>
<th>Rocks</th>
<th>200MPa Poisson'</th>
<th>400MPa Poisson'</th>
<th>600MPa Poisson'</th>
<th>800MPa Poisson'</th>
<th>1000Mpa Poisson'</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Vp/Vs ratio</td>
<td>Vp/Vs ratio</td>
<td>Vp/Vs ratio</td>
<td>Vp/Vs ratio</td>
<td>Vp/Vs ratio</td>
</tr>
<tr>
<td>Andesite</td>
<td>1.823</td>
<td>0.285</td>
<td>1.844</td>
<td>0.292</td>
<td>1.858</td>
</tr>
<tr>
<td>Basalt</td>
<td>1.838</td>
<td>0.29</td>
<td>1.846</td>
<td>0.292</td>
<td>1.851</td>
</tr>
<tr>
<td>Diabase</td>
<td>1.8</td>
<td>0.277</td>
<td>1.802</td>
<td>0.278</td>
<td>1.805</td>
</tr>
<tr>
<td>Granite-granodiorite</td>
<td>1.702</td>
<td>0.237</td>
<td>1.705</td>
<td>0.238</td>
<td>1.707</td>
</tr>
<tr>
<td>Diorite</td>
<td>1.759</td>
<td>0.261</td>
<td>1.766</td>
<td>0.264</td>
<td>1.771</td>
</tr>
<tr>
<td>Slate</td>
<td>1.865</td>
<td>0.298</td>
<td>1.862</td>
<td>0.297</td>
<td>1.861</td>
</tr>
<tr>
<td>Mica quartz schist</td>
<td>1.777</td>
<td>0.268</td>
<td>1.78</td>
<td>0.269</td>
<td>1.782</td>
</tr>
<tr>
<td>Quartzite</td>
<td>1.478</td>
<td>0.077</td>
<td>1.485</td>
<td>0.085</td>
<td>1.492</td>
</tr>
<tr>
<td>Calcite marble</td>
<td>1.893</td>
<td>0.307</td>
<td>1.873</td>
<td>0.301</td>
<td>1.86</td>
</tr>
<tr>
<td>Anorthosite</td>
<td>1.91</td>
<td>0.311</td>
<td>1.913</td>
<td>0.312</td>
<td>1.914</td>
</tr>
<tr>
<td>Dunite</td>
<td>1.754</td>
<td>0.259</td>
<td>1.755</td>
<td>0.26</td>
<td>1.756</td>
</tr>
</tbody>
</table>

Table 5.3 Vp/Vs ratio and Poisson’s ratio as a function of pressure. (Christensen, 1996)

However, based on CRUST2.0, about 82% of the soft sediment layer has Poisson’s ratio larger than 0.35. This is because lab environments only represent the ‘ideal’ situation, where mostly competent rock samples are used. In fact, since the top layer Poisson’s ratio in CRUST2.0 only represents the average of any 2 x 2 degree cell, in some parts of the world, even larger Poisson’s ratios have been observed. For example, in Tarzana California, from seismic refraction data, Catchings and Lee (1999) derived Poisson’s ratios in the range of 0.477 - 0.494 at a depth of less than 30m. Fumal et al. (1981) described drill samples in this area as “shale, black, soft to firm, very close horizontal parting, close to very close fracture spacing, and fresh.” Catchings and Lee (1999) concluded that the high Poisson’s ratios are caused by highly saturated shale. Also in California, Fumal et al. (1981, 1982, 1984) collected velocity data from 68 boreholes, and they found it is very common for crust less than 30m deep to have a Poisson’s ratio between 0.45 – 0.49. Similar discoveries have also
been made in Tennessee and Missouri (Catchings, 1999), Utah (Williams et al., 1993), central Asia (Nicholson and Simpson, 1985), and Russia (Morozov et al., 2002).

Equation (5.1) indicates Poisson’s ratio is solely a function of the $V_p/V_s$ ratio: the larger the $V_p/V_s$ ratio, the larger the Poisson’s ratio. $V_p/V_s$ ratio of a geological material is determined by many factors, including lithology, pressure, porosity, cementation, depth, age, and temperature (Tatham and McCormack, 1991). Among these, water saturation in sediment/rock is a key ingredient to cause an unusually low shear wave velocity. Nur (1982) shows a figure to demonstrate water saturation’s effect on the magnitude of P-wave and S-wave velocities (Fig. 5.16). At very low water saturation, $V_p$ and $V_s$ in sedimentary rock are high, but as water saturation increases, both velocities decrease. As water saturation approaches 100%, $V_p$ increases again while $V_s$ remains low, which makes the $V_p/V_s$ ratio very high. On the Earth surface, poorly consolidated sediment can be easily saturated with surface water. For example, a GPS station operated by Bevis’s group next to a water body has experienced the IBR effect. The ground surface uplifts when the water volume increases. It has been reported that this area is very rich in volcanic ashes at the surface, which is easily saturated with water from precipitation. In the subsurface, where rocks are highly faulted and fractured, water saturation tends to be high too, which in turn generates high pore pressure. The oil industry utilizes $V_p/V_s$ ratio (or Poisson’s ratio) to identify overpressured zones which are hazardous for drilling and completion of wells (Lee, 2003).
Figure 5.16 Velocity of sandstone as a function of the percentage of water saturation. Note that velocities decrease with greater saturation until nearly 100% saturation, where P-wave velocities greatly increase. Nur (1982)

5.7 Inverse Bousinessq effect elsewhere in the world

Keeping in mind the conclusions we drew earlier in this chapter, it is worth while to further investigate in the space of CRUST2.0 to see at the 2x2 degree resolution, all the places that IBR may occur. Figure 5.17 shows the distribution of the top sediment layer properties for the entire globe.
Figure 5.17 Distributions of top layer crust properties based on CRUST2.0.

About 82% of the soft sediment layers have a Poisson’s ratio larger than 0.35, which is far larger than the threshold value 0.28. Another crucial factor for producing IBR is that the top two layers’ stiffness contrast ratio has to be approximately equal to or larger than 5. According to CRUST2.0, about 33% of the world satisfies this condition. The last requirement is that a place also needs to have the soft sediment layer thickness close to 100m. About 58% of the Earth’s surface meets this requirement. The intersection of these conditions filtered out most of the places in the world (Fig. 5.18). The areal extent of these locations is far smaller than one would expect. This is because most of the places that have a soft
sediment thickness close to 100 m are on the continents, while the places that have a stiffness contrast ratio larger than 5 are mostly in the ocean. Oceanic crust is mainly composed of mafic rock (Turcotte and Schubert, 2002), with a thin sediment layer deposited on it, therefore it is reasonable to expect that these two layers have a strong stiffness contrast. However, oceanic crust is much younger than continental crust, so only a very few parts of the oceanic crust have a sediment layer thicker than 100m. At last, we compare figure 5.18 with a world sedimentary basin map (Fig. 5.19), it is not surprising to see that these places are within the extent of major sedimentary basins.

Figure 5.18 A world map shows the locations where the IBR effect is likely to occur based on CRUST2.0 model.
5.8 Conclusions

From the previous experiments we concluded that the occurrence of IBR effect is highly determined by the Poisson’s ratio of the top layer. One can decrease the thickness of the top layer or increase the Young’s modulus ratio between the two layers to increase the likelihood of producing the IBR effect, but it would not be possible without a high enough Poisson’ ratio. Poisson’s ratio and the top layer thickness both control the location and spatial extent of the IBR effect, and the top layer thickness has a stronger influence on the spatial extent (Figs. 5.2, and 5.3). All three quantities contribute to the magnitude of the IBR effect.

Many studies have shown it is quite common to find high Poisson’s ratio in the Earth’s shallow crust. In addition, the 2” resolution CRUST2.0 model shows Poisson’s ratio in the soft sediment layer can be as large as 0.38. Since it is a crude model, even higher Poisson’s can be expected in an environment of poorly consolidated sediments.

Figures 5.4, 5.5, and 5.6 show that with a large stiffness contrast, a Poisson’s ratio as low as 0.28 can produce the IBR effect. Figures 5.7 and 5.8 show that with a Poisson’s ratio
of 0.35, a stiffness contrast as low as 5 is enough to produce the effect. Therefore, all the conditions needed are relatively easy to obtain in the Earth’s crust.

With better knowledge of the local geology, it is quite reasonable to conclude that the IBR effect in California can be easily reproduced/explained by a LHS elastic loading model.
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CHAPTER 6

ELASTIC REBOUND AND THE GEODETIC DETECTION OF ACCELERATING ICE LOSS IN GREENLAND

6.1 Introduction

In many parts of the world the last interglacial high stand of sea level is preserved at about 5 - 6 meters above present sea level (Bloom et al., 1974). One of the more pressing concerns about global warming is the prospect of increasingly high rates of mass transfer from the West Antarctic and Greenland ice sheets into the ocean, causing a sea level rise of similar or greater magnitude and inundation of many heavily populated coastal areas around the world. The rapid and apparently accelerating disappearance of tropical and temperate mountain glaciers around the world, though of lesser concern in terms of their combined potential impact on sea level, underlines the fragility of the present-day cryosphere, as does the rapid reduction in winter sea ice in the Arctic. At least some ice mass wasting processes in West Antarctica appear to be accelerating (Thomas et al., 2004) and there are also signs of increasing instability in Greenland (Krabill et al., 2004, Alley et al., 2005, Rignot and Kanagaratnam, 2006). It is now clear to most climate and sea level change researchers that it is imperative to monitor ice mass balance in both Antarctica and Greenland on an essentially continuous basis. While it may be too late to reverse global warming before sea level rise becomes seriously problematic, it is crucial to assess both the possible severity of sea level
rise, and the amount of time that our governments have to respond to, or mitigate, developments that they may be powerless to prevent. One of the challenges in assessing ice mass balance is the suggestion that the fundamental behavior of ice sheets is now capable of changing over time scales of ~ 5 years - historically this is comparable to the time scale of many individual measurement programs. In this paper we focus on the following question: if a large acceleration in ice loss is occurring within Greenland right now, or if it does occur in the near future, how could we detect and assess these mass changes with delays of much less than one year? We shall concentrate on the contribution that geodesy can make in monitoring ice loss in Greenland, anticipating that the number of continuous GPS stations installed in Greenland’s coastal regions will increase to ~ 50 stations by September 2009 as a result of International Polar Year activities by geodesists and geophysicists from the USA, Denmark, Luxembourg, Germany (Project GNET).

6.2 Recent ice mass Changes in Greenland

Greenland is one of the largest contributors to global sea level rise in the world. At higher elevations (>2000m) of Greenland, ice mass has been in balance or slightly thickening, while the lower elevations are dominated by peripheral thinning (Krabill et al., 2000, Zwally et al., 2005). Thickening in the ice field interior is mostly caused by increased precipitation in the warming climate (Thomas et al., 2006, Zwally et al., 2005), and thinning at the margin is due to summer melting and enhanced ice discharge (Velicogna et al., 2005). Using measurements from airborne laser altimeter surveys, Krabill et al. (1999) reported during the 1993-1998 period a net ice loss of 51 km$^3$/yr from the entire Greenland ice sheet, which is sufficient to raise sea level by 0.13mm/year. This ice loss rate increased to 80 km$^3$/year between 1997 and 2003 due to increased coastal thinning (Krabill, 2004).
Krabill’s group surveyed ten flight lines in South Greenland in the early summer of 1993 and 1998, and another twelve flight lines in North Greenland in 1994 and 1999. Using the measurements from these surveys, they created a dataset that covers the entire Greenland ice sheet, with a spatial resolution of ~2.2 km. The dataset represents surface elevation change. In order to convert this dataset to ice/snow thickness change, a minor (4.6 mm/yr) correction for isostatic uplift of the crust was subtracted. The resulting ice gain/loss model is shown in figure 6.1.

**Figure 6.1** Ice loss model created by Krabill et al. (2000). Black lines outline the four study zones, which are shown in detail in figure 6.3-6.6. Magenta squares indicate town/base locations. The corresponding names are listed in counter-clockwise order.
6.3 Crustal motion geodesy and the PGR Correction

The most widely discussed role for bedrock GPS networks in Greenland and West Antarctica is improving the ‘postglacial rebound (PGR) correction’ for the GRACE satellite mission (Le Meur and Huybrechts, 2001, Velicogna and Wahr, 2005, 2006). GRACE measures the earth’s gravity field every 30 days or so, thereby directly sensing near-surface mass changes. In principle, GRACE can measure ice mass changes equivalent to 10 mm in equivalent water thickness by averaging over a disc with a radius of ~ 600 km or greater (Velicogna and Wahr, 2006). GRACE is the only system ever realized that can characterize the mass changes occurring in an ice sheet as a whole, with both great sensitivity and a measurement latency of only one month. While the averaging process does result in a rather blurred image of the spatial distribution of ice loss or gain, other techniques such as satellite altimetry, interferometric SAR and airborne LIDAR, which can monitor changes in the height of the ice sheet surface, can help clarify the spatial distribution of ice loss (Rignot et al., 2001; Abdalati et al., 2002), although this clarification may be achieved with much greater delays than are incurred by GRACE. In the context of ice mass monitoring, the major problem with GRACE is not its limited spatial resolution, but rather that it measures total mass change and not just ice mass change. There is no way for GRACE to distinguish between a linear rate of change in ice mass (in time) and a linear rate of change in the underlying rock mass. The rock mass fluxes associated with PGR can produce signals comparable to, and even larger than, the signals derived only from ice mass changes. Therefore fairly modest errors in the PGR correction can lead to large errors in estimates of ice mass rates of change.

PGR is the earth’s viscoelastic response to past changes in ice mass, including the enormous and widespread patterns of ice loss that have occurred since the Last Glacial
Maximum (LGM). While the PGR velocity field is now fairly well modeled over most of the globe, these models are far more uncertain within Greenland and Antarctica. This is because contemporary rates of PGR in the near-field of these ice sheets depends on the history of loading and unloading both before and after the LGM, which is not yet known in sufficient detail, and because the viscoelastic adjustment process is sensitive to the structure of the lithosphere and especially the spatial distribution of mantle viscosity (Ivins and James, 1999, 2004), which is also not well determined for Greenland and Antarctica. The present level of uncertainty associated with contemporary PGR fields is manifested by the considerable differences apparent in the various published PGR models for Antarctica (Barletta et al.; 2008, a and Wahr, 2006; Ivins and James, 2005; Chen et al., 2006). The uncertainties attending GRACE’s ice mass change estimates due to the ice mass / rock mass ambiguity can be reduced only by improving our models for PGR. Direct local measurements of crustal rebound probably constitute the single most important basis for making such improvements in Greenland as well as Antarctica.

6.4 Elastic rebound

Any ice mass changes that have developed abruptly during the last decade, or that will develop abruptly in the next decade, will not produce a significant contemporary viscoelastic response because the timescale of this load change is or will be too short. In terms of crustal deflection, such loading or unloading will produce only an instantaneous elastic response (Bevis et al, 2005). If the secular rate of ice loss that is presently occurring near the margins of the Greenland ice sheet (Krabill et al., 2000, 2004, Johannessen et al., 2005) were suddenly to accelerate in the near future, e.g. increasing by a factor of three over several years, the incremental component of ice loss would produce an elastic signal which
was distinct from, and would add to the steady motions associated with PGR. That is, accelerations in ice loss (or gain) will be manifested by temporal changes in the rates of crustal rebound. The open issue that we address here is the magnitude of these signals, and the amount of time required for them to accumulate to the point of detectability.

Most recent work on earth’s elastic response to surface loading has focused on the oscillations of the earth’s surface driven by seasonal fluctuations in the loads imposed on the lithosphere by the atmosphere and, more importantly, by the hydrosphere (Van Dam et al., 2001; Heki, 2001; Dong et al., 2002). This dominantly vertical elastic response to environmental loading occurs at global (Blewitt et al., 2001), regional (Heki, 2001), and local (Bevis et al., 2004, 2005) scales. One significant problem in modeling these effects is that the near-field surface response to load changes is sensitive to the details of shallow elastic structure, and the elastic structure of the uppermost mantle and especially the crust is far more variable than is the elastic structure of the deeper mantle (Bevis et al., 2004, 2005). In most continental areas the uppermost crust is far less stiff than the middle or lower crust, and this compliant layer will ‘amplify’ the vertical crustal motions (i.e. the response to loading) relative to an area in which elastic stiffness does not sharply decline near the surface. Whereas it is reasonable to use global earth structure models such as PREM to model the medium- and far-field elastic response to surface loading, this will usually cause the near-field loading response to be underestimated, since PREM does not resolve the strong near-surface decline in stiffness characteristic of most continental areas.

In many continental areas the elastic structure of the crust is not well constrained by direct observations, though it is possible to make intelligent guesses about elastic structure based on the thermotectonic setting and the topography of the area, as well as any available information about the depth to basement (Mooney et al., 1998). We use the crustal structure
model CRUST 2.0 (http://mahi.ucsd.edu/Gabi/rem.html) to provide a reasonable guess for the average crustal structure beneath Greenland. CRUST 2.0 provides a layered model for the crust and upper mantle, in each 2° by 2° square, specifying both the P and S wave velocity for each layer, as well as density. Given these three parameters it is a simple matter to estimate the two elastic parameters for each layer - either the two Lamé parameters, or, equivalently, Young’s modulus and Poisson’s ratio. We averaged the results we obtained over most of Greenland and adopted the nominal structure indicated by figure 6.2 for the purpose of modeling the earth’s elastic response to changing ice loads.

Figure 6.2 Elastic structure derived from CRUST2.0.

In order to implement such a computation we need a rate of change of ice mass field (dM/dt) for Greenland as a whole. After consulting with R. Thomas and W. Krabill we
decided to use the results presented by Krabill et al. (2000) for this purpose, because many of the more recent dM/dt fields have incomplete spatial coverage or poorer spatial resolution. We appreciate that most workers believe that the spatial pattern of ice loss is changing, and that present rates of ice loss near the margins of the Greenland ice sheet are now higher than those described by Krabill et al. (2000). Nevertheless this loading field constitutes a reasonable ‘baseline’ case study. To the extent that we believe that ice loss has accelerated beyond that implied by our loading field, then the actual elastic rebound we should expect will be proportionally larger than the predictions made using the earlier results.

We computed the elastic rebound field based on a ‘flat earth’ or cartesian framework composed on N isotropic layers overlying an isotropic half-space (Pan et al., 2007). We represent the surface load as a very large number of circular loading elements. The response for each single loading element is computed using a refinement of the techniques previously described by Pan (1997), in which the axi-symmetric nature of the elementary loading problem is exploited through the use of a radial coordinate system. The solution for each loading element is framed in terms of cylindrical vector functions, Hankel transforms and propagator matrices. Adaptive Gauss quadrature is used to handle the oscillatory nature of the integrands in an optimal manner. The total elastic response to all loading elements is obtained by superposition. This code was tested by comparing its results to those obtained for a uniform half-space (Becker and Bevis, 2004), by testing that the numerical results for a single layer over half-space behaved correctly in certain asymptotic limits, and by comparing the results obtained for more general cases with those obtained using a code developed independently at the University of Hong Kong. Our algorithm and codes are described in detail by Pan et al. (2007).
6.5 Results

Our results are illustrated in Figures 6.3 – 6.6, which zoom into four different coastal segments within Greenland so that it is possible to see the spatial variation of the vertical elastic response predicted within the ice-free land between the ice sheet and the coast. It is not easy to measure absolute vertical velocities with an accuracy of ~ 1 mm/yr or better because of the difficulties of realizing a global vertical reference frame at this level of stability over many years (although this agenda is perhaps almost in reach). It is relatively easy to measure differential or relative vertical velocities of order ~ 1 mm/yr between pairs of stations located within ~ 100 km of each other, since instabilities of the global reference frame are rarely admitted (at this amplitude) into such short baselines. Therefore, while it is interesting to study the absolute values of the velocities predicted using our elastic model and ice loss/gain rates of Krabill et al. (2000), it is probably of more practical interest to examine the differential velocities between stations, especially those located close together and with a baseline vector oriented nearly perpendicular to the ice front. In many parts of the Greenland coast where the ice free coastal strip is wider than a few tens of km, it will be the tilting of the ice free land surface in a direction roughly perpendicular to the coastline (or ice front) that will provide the most robust manifestation of ice load changes near the edge of the ice sheet. In particular, it is the slope or range of the vertical displacement plotted in part (b) of figures 6.3 – 6.6 that gives the most immediate indication of detectability. The vertical velocity changes by amount on the order of 1 - 2 mm/yr given the Krabill et al. (2000) scenario for secular rates of ice loss. It would take several years of observation to even detect such differential uplift signals. But should the rate of ice loss accelerate significantly, and the loss rates exceed the Krabill et al. (2000) estimates by a factor of two or three, then it would be relatively straightforward for geodesy to measure the associated elastic response. Most likely
the single biggest problem would be distinguishing between interannual variations in the seasonal pattern of loading and an acceleration in the secular loss rate.

Figure 6.3 Vertical crustal motion in zone 1 induced by secular ice loss.
**Figure 6.4** Vertical crustal motion in zone 2 induced by secular ice loss.
Figure 6.5 Vertical crustal motion in zone 3 induced by secular ice loss.
Figure 6.6 Vertical crustal motion in zone 4 induced by secular ice loss.
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APPENDIX A

SURFACE DEFORMATION DUE TO LOADING OF A LAYERED ELASTIC HALF-SPACE
1. Problem statement and governing equations

Let us consider a layered half space made up of \( p \) parallel, elastic isotropic layers lying over an elastic isotropic half space. The layers are numbered serially with the layer at the top being layer 1 and the last layer \( p \), which is just above the half space (Chapter 2, Fig. 2.1). We place the cylindrical coordinate on the surface with the \( z \)-axis pointing into the layered half space. The \( k \)-th layer is bounded by the interfaces \( z = z_{k-1}, z_k \). As such, \( z_{k-1} \) is the coordinate of the upper interface of the \( k \)-th layer, and \( z_k \) that of the lower interface. It is obvious that the thickness of the \( k \)-th layer is \( h_k = z_k - z_{k-1} \), with \( z_0 = 0 \) and \( z_p = H \), where \( H \) is the depth of the last layer interface. While the interface between the adjacent layers are assumed to be in welded connection, the top surface is under a uniform unit pressure within a circle of radius \( R \). For a well-posed problem, the solution in the homogeneous half space of the layered system should be also finite when the physical dimension approaches infinity.

For the isotropic elastic solid, we have, in each layer, the following governing equations in the cylindrical coordinates:

1). Equilibrium equations without body force

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{r\theta}}{r \partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{r\theta}}{r} = 0
\]

\[
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta\theta}}{r \partial \theta} + \frac{\partial \sigma_{r\theta}}{z} + \frac{2\sigma_{r\theta}}{r} = 0
\]

\[
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{\theta z}}{r \partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0
\]

(a1)

where \( \sigma_{ij} \) is the stress tensor.

2). Constitutive relations

\[
\sigma_{rr} = c_{11} \gamma_{rr} + c_{12} (\gamma_{\theta\theta} + \gamma_{zz})
\]

\[
\sigma_{\theta z} = c_{44} \gamma_{\theta z}
\]

(a2)

where
\[ c_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} ; \quad c_{12} = \frac{E\nu}{(1+\nu)(1-2\nu)} ; \quad c_{44} = \frac{E}{2(1+\nu)} \]  

(a3)

The constitutive relations for the other normal and shear components can be found similarly.

While in Eq. (a2), \( \gamma_{ij} \) are the engineering strain components, in Eq. (a3), \( E \) and \( \nu \) are, respectively, the Young’s modulus and Poisson’s ratio.

3). The strain-displacement relations

\[ \gamma_{rr} = \frac{\partial u_r}{\partial r}, \quad \gamma_{\theta\theta} = \frac{\partial u_\theta}{r \partial \theta} + \frac{u_r}{r}, \quad \gamma_{zz} = \frac{\partial u_z}{\partial z} \]
\[ \gamma_{r\theta} = \frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{\partial \theta}, \quad \gamma_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} \]  

(a4)

where \( u_i \) is the displacement field.

2. General Solution in terms of Cylindrical System of vector functions

The cylindrical system of vector functions is very convenient in treating axisymmetric problem and it is defined as (Pan, 1989a,b, 1997)

\[ \mathbf{L}(r, \theta; \lambda, m) = \mathbf{e}_rS(r, \theta; \lambda, m), \]
\[ \mathbf{M}(r, \theta; \lambda, m) = (\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{\partial}{r \partial \theta})S(r, \theta; \lambda, m), \]
\[ \mathbf{N}(r, \theta; \lambda, m) = (\mathbf{e}_r \frac{\partial}{r \partial \theta} - \mathbf{e}_\theta \frac{\partial}{\partial r})S(r, \theta; \lambda, m) \]  

(a5)

with

\[ S(r, \theta; \lambda, m) = \frac{1}{\sqrt{2\pi}} J_m(\lambda r)e^{im\theta}, \]

(a6)

where \( J_m(\lambda r) \) is the Bessel function of order \( m \) with \( m = 0 \) corresponding to the axisymmetric deformation, which will be discussed in detail later on. It should be also noticed that the scalar function \( S \) in Eq. (a6) satisfies the Holmholtz equation
\[
\frac{\partial^2 S}{\partial r^2} + \frac{\partial S}{r \partial r} + \frac{\partial^2 S}{r^2 \partial \theta^2} + \lambda^2 S = 0 \quad (a7)
\]

We quickly remark that the cylindrical system of vector functions is an extension of the Hankel transform and can be directly applied to a vector function. Since this vector function system (Eq. (a5)) forms an orthogonal and complete space, any integrable vector and/scalar function can be expressed in terms of it. In particular, the displacement and traction (with \(z\)-axis as the normal) vectors can be expressed as

\[
\mathbf{u}(r, \theta, z) = \sum_m \int_0^\infty [U_L(z)L(r, \theta) + U_M(z)M(r, \theta) + U_N(z)N(r, \theta)] \lambda d\lambda \quad (a8)
\]

\[
t(r, \theta, z) = \sigma_{rr} e_r + \sigma_{r\theta} e_{\theta} + \sigma_{z} e_z
\]

\[
= \sum_m \int_0^\infty [T_L(z)L(r, \theta) + T_M(z)M(r, \theta) + T_N(z)N(r, \theta)] \lambda d\lambda
\quad (a9)
\]

Making use of these expansions along with the strain-displacement and constitutive relations, we have, in general,

\[
u_r(r, \theta, z) = \sum_m \int_0^\infty \left( U_M \frac{\partial S}{\partial r} + U_N \frac{\partial S}{r \partial \theta} \right) \lambda d\lambda \quad (a10a)
\]

\[
u_\theta(r, \theta, z) = \sum_m \int_0^\infty \left( U_M \frac{\partial S}{r \partial \theta} - U_N \frac{\partial S}{\partial r} \right) \lambda d\lambda \quad (a10b)
\]

\[
u_z(r, \theta, z) = \sum_m \int_0^\infty U_L \lambda d\lambda \quad (a10c)
\]

\[
\sigma_{rr}(r, \theta, z) = \sum_m \int_0^\infty \left[ c_{11} \left( U_M \frac{\partial^2 S}{\partial r^2} + U_N \frac{\partial^2 S}{r \partial r \partial \theta} - U_N \frac{\partial S}{r^2 \partial \theta^2} \right) + c_{12} \left( U_M \frac{\partial^2 S}{r^2 \partial \theta^2} + U_N \frac{\partial S}{r \partial r \partial \theta} + \frac{dU_L}{dz} \right) \lambda d\lambda \right]
\quad (a11a)
\]

\[
\sigma_{r\theta}(r, \theta, z) = \sum_m \int_0^\infty \left[ c_{12} \left( U_M \frac{\partial^2 S}{\partial r^2} + U_N \frac{\partial^2 S}{r \partial r \partial \theta} - U_N \frac{\partial S}{r^2 \partial \theta^2} \right) + c_{11} \left( U_M \frac{\partial^2 S}{r^2 \partial \theta^2} + U_N \frac{\partial S}{r \partial r \partial \theta} + \frac{dU_L}{dz} \right) \lambda d\lambda \right]
\quad (a11b)
\]

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The relation of the expansion coefficients between $T$ and $U$ can be found by comparing Eqs. (a9) to (a11c,d,e)

$$T_L = -\lambda^2 c_{12} U_L + c_{11} \frac{dU_L}{dz},$$

$$T_M = c_{44} (U_L + \frac{dU_M}{dz}).$$

The N-component is independent and will not be discussed here (Pan, 1997) as we will be interested in the axis-symmetric deformation only.

Substituting the stress expansion (a11) into the equilibrium equation (a1), we further found

$$\frac{dT_L}{dz} - \lambda^2 T_M = 0,$$

$$-\lambda^2 c_{12} U_M + c_{12} \frac{dU_L}{dz} + \frac{dU_M}{dz} = 0.$$
where \([K]\) is a 4×1 column matrix with its elements to be determined by the interface and/or boundary conditions, and the elements of the solution matrix \([Z(z)]\) is given in Appendix A-1.

The propagating relation for the coefficient vector \([E]\) of \(k\)-th layer at the \(z\)-level \(z_{k-1}\) and that at \(z_k\), is found to be

\[
[E(z_{k-1})] = [a_k] [E(z_k)],
\]

where \([a_k]\) is the propagator matrix for the \(k\)-th layer.

3. Solutions to circular surface loading

Assume that a circular surface loading with magnitude \(q\) is in the vertical direction and is within the circle of \(r=R\) (Chapter 2, Fig. 2.1), then the traction boundary condition on the surface \(z=0\) is expressed as:

\[
\sigma_{zz} = \begin{cases} 
-q & r < R \\
0 & r > R
\end{cases} \quad \sigma_{rz} = \sigma_{r\theta} = 0 \quad 0 \leq r \leq \infty
\]

Therefore, the corresponding expansion coefficients in the cylindrical systems of vector functions are:

\[
T_k(\lambda,0) = -\frac{q R \sqrt{2\pi}}{\lambda} J_1(\lambda R)
\]
\[
T_{m}(\lambda,0) = T_{N}(\lambda,0) = 0
\]

We now first solve the problem in transformed domain (i.e., in terms of the expansion coefficients). Propagating the propagator matrix \([a_k]\) from the top of the homogeneous half space \(z = H\) to the surface \(z=0\), we found

\[
[E(0)] = [G] [K_p],
\]
where

$$[G] = [a_1][a_2] - - [a_\rho][Z_\rho(H)], \quad \text{(a20)}$$

The unknown coefficients $[K_\rho]$ are those in the half-space. As the solution in the half space should be bounded, the first and third elements in $[K_\rho]$ should be zero (see Appendix A for the general solution in each layer). The remaining two unknown coefficients can be determined by the two boundary conditions on the surface $z=0$ as given by Eq. (18) (for the $L$- and $M$-components only).

After the unknown coefficients in $[K_\rho]$ are determined, the expansion coefficients at any depth (e.g. in the $k$-th layer with $z_{k-1} \leq z \leq z_k$) can be obtained exactly as:

$$[E(z)] = [a_k(z - z_{k-1})][a_{k+1}] - - [a_\rho][Z_\rho(H)][K_\rho]. \quad \text{(a21)}$$

In general, direct multiplication of the propagator matrix $[a_k]$ can be carried out in order to propagate the transformed domain solution from one layer to the next. However, as discussed in Pan (1997) and Yue & Yin (1998), overflow may occur from multiplication of matrices in Eqs. (a20) and (a21). Fortunately, this can be overcome by factoring out the exponentially growing factor in the elements of the propagator matrix. The resulting modified propagator matrices have no element exponentially growing, and therefore there will be no overflow problem for a multilayered half space having any number of layers with any thickness for each layer.

After solving the problem in the transformed domain, the displacement and stress solutions at any location in the physical domain can be expressed as (independent of $\theta$ because of the symmetric feature)

$$u_r(r, z) = -\frac{1}{\sqrt{2\pi}} \int_0^\infty (\lambda U_{s\lambda}) J_1(\lambda r) \lambda d\lambda \quad \text{(a22a)}$$
\[ u_{\theta}(r, z) = 0 \]  \hspace{1cm} \text{(a22b)}

\[ u_z(r, z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty (U_L) J_0(\lambda r) \lambda d\lambda \]  \hspace{1cm} \text{(a22c)}

\[ \sigma_{r z}(r, z) = -\frac{1}{\sqrt{2\pi}} \int_0^\infty (T_M) J_1(\lambda r) \lambda^2 d\lambda \]  \hspace{1cm} \text{(a23a)}

\[ \sigma_{\theta z}(r, z) = \sigma_{r \theta}(r, z) = 0 \]  \hspace{1cm} \text{(a23b)}

\[ \sigma_{zz}(r, z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \left( \frac{T}{\lambda^2} \right) J_0(\lambda r) \lambda^2 d\lambda \]  \hspace{1cm} \text{(a23c)}

\[ \sigma_{rr}(r, z) = \frac{\nu}{1-\nu} \sigma_{zz} + \frac{2\epsilon_{44}}{\sqrt{2\pi}} \int_0^\infty \left( \lambda U_M \right) \left[ -\frac{1}{1-\nu} J_0(\lambda r) + \frac{J_1(\lambda r)}{\lambda} \right] d\lambda \]  \hspace{1cm} \text{(a23d)}

\[ \sigma_{\theta \theta}(r, z) = -\frac{2\nu}{1-\nu} \sigma_{zz} - \sigma_{rr} - \frac{E}{\sqrt{2\pi}} \left( \frac{1}{1-\nu} \right) \int_0^\infty \left( \lambda U_M \right) J_0(\lambda r) \lambda^2 d\lambda \]  \hspace{1cm} \text{(a23e)}

where the expansion coefficients are functions of \( z \) and the transform variable \( \lambda \).

Of particular interests is the displacement field on the surface, which is directly connected to the GPS and InSar observation data (Bevis et al., 2004). Considered the circular uniform loading, the horizontal (radial) and vertical displacement on the surface can be expressed as:

\[ u_r(r, 0) = qR \int_0^\infty \frac{D_1}{\lambda} J_1(\lambda r) J_1(\lambda R) d\lambda \]  \hspace{1cm} \text{(24a)}

\[ u_z(r, 0) = -qR \int_0^\infty \frac{D_0}{\lambda} J_1(\lambda r) J_0(\lambda R) d\lambda \]  \hspace{1cm} \text{(24b)}

where

\[ D_0 = \frac{G_{12}G_{44} - G_{14}G_{42}}{G_{32}G_{44} - G_{34}G_{42}}; \quad D_1 = \frac{G_{22}G_{44} - G_{24}G_{42}}{G_{32}G_{44} - G_{34}G_{42}} \]  \hspace{1cm} \text{(24c)}
with \( G_{ij} \) being the elements of the matrix \([G]\) in Eq. (a20). This matrix is a function of the transform variable \( \lambda \), the elastic property, and thickness of each layer.

To find the physical domain solutions, the transformed-domain results need to be integrated numerically. It is further noted that the integrands in the infinite integrals for the displacements involve production of Bessel functions that are oscillatory and go to zero slowly when their variable approaches infinity. Thus, the common numerical integral methods, such as the trapezoidal rule or Simpson's rule, are not suitable for such integrations. While numerical integration of infinite integrals involving single Bessel function has been discussed by Chave (1983) and Lucas and Stone (1995), the corresponding numerical integration involving production of Bessel functions of different orders have been studied by Lucas (1995) based on the adaptive Gauss quadrature. As this method is very accurate and efficient, we therefore, adapt this algorithm.

Base on this algorithm, the infinite integral is expressed as a summation of:

\[
\int_0^{+\infty} f(\lambda)J_m(\lambda R)J_n(\lambda r)d\lambda = \int_0^{r_{\text{max}}} f(\lambda)J_m(\lambda R)J_n(\lambda r)d\lambda + \int_{r_{\text{max}}}^{+\infty} f(\lambda)J_m(\lambda R)J_n(\lambda r)d\lambda \quad (a25)
\]

The finite and infinite integrations on the right-hand side of (a25) are then calculated using the adaptive IMSL routines dqdag() and dqdag1(), respectively. However, in using these subroutines, the \( mW \) transform and \( \varepsilon \)-algorithm are applied for handling the oscillation feature of the integrands and for accelerating the calculation. The \( mW \) transform and \( \varepsilon \)-algorithm are also involved in finding and approximating zeros of the integrands and extrapolating the corresponding values.
References:


Pan, E., (1989a), “Static response of a transversely isotropic and layered half-space to general dislocation sources.”, Physics of Earth Planetary Interiors, 58, 103-117

