A NEW FINITE ELEMENT PROCEDURE FOR FATIGUE LIFE
PREDICTION AND HIGH STRAIN RATE ASSESSMENT OF COLD
WORKED ADVANCED HIGH STRENGTH STEEL

DISSERTATION
Presented in Partial Fulfillment of the Requirement for
the Degree Doctor of Philosophy in the
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ABSTRACT

This dissertation presents a new finite element procedure for fatigue life prediction and high strain rate assessment of cold worked Advanced High Strength Steel (AHSS). The first part of this research is related to the development of a new finite element procedure from an energy-based fatigue life prediction framework previously developed for prediction of axial, bending and multi-axial fatigue life. The framework for the prediction of fatigue life via energy analysis consists of constitutive laws which correlate the cyclic energy to the amount of energy required to fracture a material. In this study, the energy expressions that construct the new constitutive laws are integrated into a minimum potential energy formulation to develop new finite elements for fatigue life prediction for structural components subjected to axial, bending and multi-axial cyclic loads. The comparison of finite element method (FEM) results to the existing experimental fatigue data verifies the new finite element method for fatigue life prediction. The final output of this finite element analysis is in the form of number of cycles to failure for each element. The performance of the fatigue finite element is demonstrated by the fatigue life predictions from Al 6061-T6 (Aluminum) and Ti-6Al-4V (Titanium Alloy). In addition to developing new fatigue finite elements, a new equivalent stress expression and a new finite element procedure for multi-axial fatigue life prediction are also proposed. In order to develop the new equivalent stress equation,
energy expressions that construct the constitutive law are equated in the form of total strain energy and the distortion energy dissipated in a fatigue cycle. The resulting equation is further evaluated to acquire the equivalent stress per cycle using energy based methodologies. The new procedure is applicable to biaxial as well as multiaxial fatigue applications. The second part of this research is related to the development of LSDYNA material model for vehicle crash simulation based on high strain rate assessment of cold worked AHSS. The performance of a vehicle during a crash is an important subject in automobile research. In order to simulate an actual crash using software like LSDYNA, it is desirable to have accurate stress/strain data for materials. The material models available in the literature ignore the effect of cold working on the material and present data only for flat sheets. In this research, the cold worked AHSS with curved cross-section is tested at strain rates of 1000 (in/in)/s and the data is used to develop a corresponding LSDYNA material model for vehicle crash simulation.
Dedicated to my mother
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CHAPTER 1

INTRODUCTION

The research presented in this dissertation has two parts. The first part of this research is related to development of a new finite element procedure from an energy-based fatigue life prediction framework previously developed for prediction of axial, bending and multi-axial fatigue life. The second part of this research is related to the development of a LSDYNA material model for vehicle crash simulation based on high strain rate assessment of cold worked AHSS.

1.1 Background, Motivation and Scope–Fatigue Life Prediction

Modern machine components and structures are designed to be failure free; however, failure does occur and is commonly linked to fatigue. For example, high cycle fatigue (HCF) is one of the main causes of failure in gas turbine engines [1.1] and airplane structures. Different design tools have been developed to analyze this issue. The most commonly such used tool is a stress versus cycles plot, or S-N curve. These curves provide fatigue strength with respect to time to failure. Other common tools for predicting fatigue properties are the Goodman diagram and the modified Goodman diagram [1.2], which are the popular choices for a failure-free machine component
design criterion. In order for designers to make an accurate assessment, the equivalent stress is calculated according to the cyclic loading conditions and compared to S-N curve data or the Goodman diagram to obtain the number of cycles to failure. There has been a search for a more realistic method for design comparison than the existing uni-axial design tools, which begins by observing the association between material failure/fracture and the energy dissipated during the process.

Scientists and engineers have tried since the 1940’s to relate cyclic energy conversion to fatigue life prediction of the material. These attempts initially resulted in minimal success [1.3]. The hypothesis used in this type of research implies: under cyclic loading, there exists a critical strain energy value for which failure occurs [1.4]. The continued research [1.5] in this area later justified this hypothesis by displaying agreement between the theoretical and the experimental results on S-N curve. Further investigation of the assumption made in [1.4] led to the introduction of a an improved correlation between the fatigue life of a material and the strain energy dissipation during the process [1.1, 1.5]. It is now understood that the strain energy required to fracture a material, monotonically, is the same as the strain energy during a cyclic fatigue procedure, thus indicating that the critical energy value for each material is the monotonic strain energy. From this understanding, an energy based fatigue life prediction framework was developed [1.6-1.12] and an improved energy-based criterion has been proposed to allow one to systematically determine fatigue life based on the amount of energy loss per fatigue cycle [1.6]. The new constitutive law for uni-axial fatigue is based on the monotonic and cyclic stress-strain representation expressed by the following Equations:
\[ \varepsilon = \frac{\sigma}{E} + \varepsilon_o \sinh \left( \frac{\sigma}{\sigma_o} \right) \]  
(1.1)

\[ \varepsilon = \frac{\sigma_{pp}}{E} + \frac{1}{C} \sinh \left( \frac{\sigma_{pp}}{\sigma_c} \right) \]  
(1.2)

Where the parameters are defined as follows: \( \sigma \) is the value for stress at the surface of the specimen (in the bending case, max stress), \( \varepsilon \) is the strain corresponding to the stress \( \sigma \), \( \sigma_{pp} \) is the peak to peak stress (2\( \sigma \) when stress ratio is -1.0), \( E \) is the modulus of elasticity, and the variables \( \sigma_c \), \( \sigma_o \), \( \varepsilon_o \), and \( C \) are curve fit parameters [1.6]. The details about this constitutive law, the curve fit parameters, material constants and the experimental procedure adopted for acquiring these constants are explained in [1.6-1.9]. These Equations are used to obtain the total fracture energy and the energy dissipated per cycle. Therefore, the Equations can be applied to the constitutive law to obtain the number of cycles to failure. These Equations are further modified for bending and multi-axial fatigue prediction.

As stated earlier, the conventional approach to fatigue life prediction is based on the S-N curve data, Goodman diagram or modified Goodman diagram. For example, a HCF turbine blading system design procedure is shown schematically in Figure 1.1. This design process usually consists of a structural modal analysis to determine natural frequencies and mode shapes at certain operating speed ranges and a stress analysis using a finite element based tool such as MSC/NASTRAN and ANSYS to calculate the dynamic stress distribution for identifying the maximum vibratory stress location. Once the maximum stresses for each vibration mode are determined through instrumented engine test, high cycle fatigue assessment can be achieved by measuring the margin
between the maximum vibratory stress and the material fatigue capability which is a straight line drawn between the mean ultimate strength at zero vibratory stress and mean fatigue strength at $10^7$ cycles (or infinite life). A typical Goodman diagram for the titanium alloy Ti-6Al-4V is shown in Figure 1.2 [1.13], usually constructed using uniaxial fatigue data.

Figure 1.1: Finite Element Analysis for Fatigue Life Assessment
The new constitutive law based on the fatigue energy dissipation mechanism, developed in [1.6], provides an opportunity to develop a new fatigue finite element for fatigue life analysis without accomplishing fatigue strength/life assessment through comparison with Goodman diagram or S-N data. This research uses the new constitutive law to derive new finite elements for fatigue life prediction for uni-axial, bending and multi-axial loads.

Though the conventional finite element fatigue analysis and design tools for fatigue life prediction make the process easier for the designer; however, the process does not incorporate fatigue mechanisms and hence can not characterize the fatigue strength without using the Goodman design diagram [1.14-1.16]. The new finite elements developed in this research are based on a fatigue constitutive law; therefore, the analysis with these elements directly incorporates the fatiguing process in life prediction. This fact establishes a difference of these new finite element developments from the existing finite
element fatigue analysis techniques. Furthermore, the new fatigue finite elements predict the crack initiation whereas most research available in this area focuses on crack propagation [1.17-1.27].

The constitutive law and the finite elements developed based on this law have the capability to capture the cyclic energy dissipated during the fatigue process. The number of cycles to failure initiation is predicted through comparison of energy dissipated per cycle to the total fracture energy of the material. Therefore, the new finite element procedures perform the fatigue life prediction by incorporating the fatigue mechanism. Due to the discrete nature of the finite elements, different number of cycles can be predicted at various locations in the structure depending upon the stress experienced by each element. This also provides a tool for plotting number of cycles on a colored contoured plot where different colors represent number of cycles for each element in the structure or component. This objective can be easily achieved by dividing the component into desired number of elements. Representation of such an analysis is shown in Figures 1.3 and 1.4.

Figure 1.3: A Representation of Beam Analysis with Higher Number of Elements
1.2 Background, Motivation and Scope – High Strain Rate Assessment

Performance of a vehicle during a crash is an important subject in the automobile industry. LSDYNA is a known software used for crash analysis of vehicles. In order to simulate a vehicle crash using software like LSDYNA, it is desirable to have accurate stress/strain data for the vehicle’s materials. Advanced high strength steel (AHSS) is used in vehicles for better crash performance and improved power to weight ratio. The current LSDYNA AHSS high strain rate material models available in the literature are based on flat sheet specimen tests. On the other hand, most components in the automobiles are cold-worked before installation, either in the form of rolling or bending. The current LSDYNA material models ignore the effect of this cold working in the materials. This raises the need to develop a material model for cold worked AHSS for closer approximation of the actual crash of a vehicle.

The high strain rate assessment and development of the LSDYNA material models involves various experimental and analytical steps. These include the design of
grips and specimens according to the test requirements, fabrication of specimens and grips according to the specified standards, high strain rate testing, development of stress strain curves, and finally the post processing of the data into a LSDYNA material model. The effect of cold working on stress/strain behavior of AHSS can be assessed by comparing the curve obtained from these experiments to those available in literature for flat sheets [1.28, 1.29-1.33].

Finite element analysis is widely used in analyzing the specimen and grip design for high strain rate testing [1.28]. The material being tested in this research is cold worked and has a curved cross-section. The detailed discussion on curved cross-section and the design steps to avoid the bending are discussed in detail in Chapter 6. Therefore the setup requires a special design to take care of shifted center in order to avoid bending and achieve a uniform stress distribution in the gauge Section. In addition to taking care of the shifted center, finite element analysis also helps in reducing the bulk in the grip yet still maintains rigidity, incorporate different fillet and angle dimensions to reduce stress concentration thus reducing probability of grip failure and improve the uniform nature of the tensile stress distribution through test specimen cross-section based on grip geometry changes.

The experimental testing for low strain rates is generally performed on mechanical test machines with reasonable test accuracy. For high strain rate testing, due to very short time available for test and the limitations of test setup, mechanical testing produces scattered test data and a ringing effect is observed. Therefore, the researchers have used
Hopkinson test bars successfully for high strain rate testing to obtain accurate stress/strain data. The following Figure shows a schematic to elaborate the concept of Hokinson bar. The detailed explanation of this equipment is presented in Chapter 6.

Figure 1.5: Hopkinson Bar Test Set-up

In this research, the cold worked AHSS with curved cross-section is tested at strain rates above 1000 (in/in) s\(^{-1}\) and data obtained from these high strain rate experiments can be used to obtain the stress strain curves. These curves are used to develop the corresponding LSDYNA material model for vehicle crash simulation. The high strain rate assessment can be made through comparison of the new data and the newly developed LSDYNA material model to the previously developed models for AHSS for flat sheets [1.28].
1.3 Dissertation Overview

Chapter 2 presents the procedures for development of a new finite element for uniaxial (Rod) and bending (Beam) fatigue life prediction. The methodology adopted for integration of the constitutive law [1.6] into a minimum potential energy expression is discussed in detail. A brief review of the previously acquired experimental data is also included. The mathematical formulation of the new finite element for uniaxial and bending fatigue life prediction is presented step by step which yields the K-matrices for the new finite element. A detailed explanation of pre- and post-processing procedures for implementation of the fatigue process into finite element analysis is also presented. The fatigue analysis is performed for Al 6061-T6 and Ti-6AL-4V materials and the results are compared to the experimental data and analytical solution [1.6]. A fatigue analysis with mean stress effect is also performed for Al 6061-T6. The comparison of these results is made to the experimental data and analytical solution. The pre- and post-processing procedures adopted for bending fatigue are generally similar to the procedure for the uniaxial fatigue. The bending fatigue analysis is performed for Al 6061-T6 and results are compared to the experimental and analytical data. The analysis is performed with varying number of elements and the convergence of these results to the analytical solution is observed. A fatigue analysis of a cantilever beam subjected to bending fatigue is performed with five element approximation and a colored contour plot for the number of cycles to failure is presented. This yields a plot with varying number of cycles to failure for each element depending upon the stress experienced by the corresponding element. An analysis with mean stress effect is also performed and the comparison is made to the experimental and analytical results.
We sometimes see instead of uniaxial data, bench test data using components or blades also includes multiaxial results. This has led to the search for a more realistic method for design comparison than the existing uniaxial design tools, which begins by observing the association between material failure/fracture and the energy dissipated during the process. Machine components and structures subjected to stress in two directions experience biaxial fatiguing. Biaxial loads can be a combination of a bending (and/or tension) and torque applied simultaneously to the component. A periodic pressurization/depressurization of cylinders also subjects these cylinders to biaxial fatigue. Besides this, all notched components under tension compression, experience biaxial fatigue at the root of the notch. In Chapter 3, energy expressions that construct the constitutive law are equated in the form of total strain energy and the distortion energy dissipated in a fatigue cycle. The resulting Equation is further evaluated to acquire the equivalent stress per cycle using energy based methodologies. The equivalent stress expressions are developed both for biaxial and multiaxial fatigue loads and are used to predict the number of cycles to failure based on previously developed prediction criterion. The equivalent stress expressions developed in this Chapter are further used in a new finite element procedure to predict the fatigue life for two and three dimensional structures. The final output of this finite element analysis is in the form of number of cycles to failure for each element on a scale. Therefore, the new finite element framework can provide the number of cycles to failure at each location in gas turbine engine structural components. In order to obtain experimental data for comparison, an Al6061-T6 plate test results [1.34] are presented. The finite element analysis is performed for Al6061-T6 aluminum and the results are compared with the experimental results. An
additional set of comparison is made with experimental data on 304 stainless steel presented in [1.35, 1.36]. A 3-D turbine blade like curved plate analysis is also performed to present an application of the new procedure to a real world problem.

Quadrilateral elements are used for the finite element analysis of the two and three dimensional problems in mechanical engineering. In order to provide a tool for more realistic fatigue analysis, a new QUAD 4 (In-Plane) element is developed and presented in Chapter 4. The methodology for developing this new finite element, pre- and post-processing procedures and mathematical steps for formulation of new K-matrix, are discussed in detail in this Chapter. A bench mark problem for verification of this new finite element is presented in order to validate the new development. The beam element developed in Chapter 2 is combined with the QUAD-4 (In-Plane) element to obtain a QUAD 4 (Plate) element. The biaxial plate used for experiments and analysis in Chapter 3, is discretized with the new plate element developed in this Chapter. The analysis results using both approaches are compared with each other as well as with experimental results. The results are presented in the form of a colored contour plot where different colors represent the number of cycles to failure for each element depending upon the varying stress at different locations.

Chapter 5 presents the development of a hexahedral element for fatigue life prediction. The mathematical steps for formulation of this finite element, the pre- and post-processing steps, the benchmark analysis and the application of this element to a three dimensional structure are discussed in detail in this Chapter.

Chapter 6 presents the high strain rate assessment of cold worked AHSS and the development of the LSDYNA material model for vehicle crash simulation. As stated
earlier, the performance of vehicle during a crash is an important subject in the automobile industry. The accuracy of crash simulations depends primarily on the high strain rate material models developed through high strain rate experiments. Finite element analysis is used for the design of the test specimen and the grips for experimental setup. The analysis procedure and results leading to the appropriate design of specimen and grips is presented in this Chapter. Hopkinson bar apparatus is used for the high strain rate experiments. The experimental setup, procedures and the data acquisition and processing are explained in detail. The data acquired from the Hopkinson bar experiments are used for obtaining the stress strain curves for this material. The high strain rate assessment of AHSS is made through comparison of the cold-worked material. Finally, the steps followed for development of the material model are presented in detail on how to make this model compatible to LSDYNA crash analysis.

Chapter 7 summarizes the contributions of this research and discusses the possible application areas. A brief of summary of each development in this research is presented followed by the conclusions. A discussion on the road map for future work is also presented.
1.4 List of References


CHAPTER 2

NEW FINITE ELEMENTS FOR UNIAXIAL
FATIGUE LIFE PREDICTION

2.1 Introduction

Chapter 2 presents the development of new finite elements for uniaxial tension/compression and bending fatigue life prediction. As stated in Chapter 1, an energy-based fatigue life prediction framework was previously developed for prediction of axial life at various stress ratios [2.1-2.4]. Based on this constitutive law, an improved energy-based criterion has been developed to allow one to systematically determine fatigue life based on the amount of energy loss per fatigue cycle [2.1]. The new constitutive law consists of the monotonic and cyclic stress-strain representation expressed by Equations (1.1) and (1.2) and reproduced below:

\[ \varepsilon = \frac{\sigma}{E} + \varepsilon_o \sinh \left( \frac{\sigma}{\sigma_o} \right) \quad (2.1) \]

\[ \varepsilon = \frac{\sigma_{pp}}{E} + \frac{1}{C} \sinh \left( \frac{\sigma_{pp}}{\sigma_c} \right) \quad (2.2) \]

Where the parameters displayed are defined as follows: \( \sigma \) is the value for stress at the surface of the specimen, \( \varepsilon \) is the strain corresponding to the stress \( \sigma \), \( \sigma_{pp} \) is the peak to peak stress (2\( \sigma \) when stress ratio is -1.0), \( E \) is the modulus of elasticity, and the variables
σ_c, σ_0, ε_0, and C are curve fit parameters [2.1]. These equations are used to obtain the total fracture energy and the energy dissipated per cycle. Therefore, the equations can be applied to the constitutive law to obtain the number of cycles to failure.

In this Chapter, the energy expressions that construct the new constitutive law are integrated into a minimum potential energy formulation to develop new finite elements for fatigue life prediction. The comparison of finite element method (FEM) results to existing experimental fatigue data, verifies the new finite element method for fatigue life prediction. The final output of this finite element analysis is in form of number of cycles to failure for each element. The performance of the fatigue finite elements is demonstrated by the fatigue life predictions from Al6061-T6 aluminum and Ti-6Al-4V. Results are compared with experimental results and analytical predictions [2.1].

The constitutive law for fatigue is integrated into a minimum potential energy formulation to develop new stiffness matrices (K-matrices) for uniaxial and bending fatigue. The new stiffness matrices are different from Due to the non-linear nature of the constitutive law, the resulting K-matrices require a non linear finite element analysis approach. The Newton-Raphson iteration method is used for numerical computation to handle the non linearity. The new K-matrices are capable of simulating the fatigue analysis based on constitutive law and Equations 2.1 & 2.2. The new stiffness matrices incorporate the fatigue mechanism in the analysis procedure in the form of fatigue constitutive law. Therefore, the stresses and strains obtained based on the analysis using new finite elements are the result of fatiguing process taking place in the component. For validation of the finite elements, a monotonic loading analysis on a 1-D rod is performed. The resulting displacements for monotonic loading are compared with the analytical solution (Equation 2.1) as well as experimental results to validate the new K-matrix.
Once the K-matrix is validated, the same displacement computation procedure is applied to the cyclic loading case. Furthermore, strain energy is acquired by evaluating the behavior of the respective load-displacement relation of the monotonic and cyclic loading processes.

The following Equation calculates the total energy to failure for a monotonic case [2.1].
The total energy to failure or fracture energy is a material parameter and procedure for obtaining this energy is explained in [2.1].

\[ W_f = \sigma_f \left( \varepsilon_f - \frac{\sigma_f}{2E} \right) - \varepsilon_0 \sigma_0 \left( \cosh \left( \frac{\sigma_f}{\sigma_0} \right) - 1 \right) \] (2.3)

\( E, \sigma_f, \) and \( \varepsilon_f \) are obtained from experimental monotonic fracture results [2.1]. Application of the acquired energy results to the constitutive law yields a fatigue life prediction method via FEM.

\[ N = \frac{W_f}{W_{cycle}} \] (2.4)

The same procedure is applied to the bending fatigue life prediction using the finite element (New K-Matrix) developed for bending loads.

The new finite elements can be applied to a structure made of any material as long as the parameters for the material being used are available. The discussed FEM fatigue prediction procedures are performed for Aluminum 6061-T6 (both uniaxial tension/compression and bending) and Titanium 6Al-4V (uniaxial tension/compression only) and the comparison is made with the experimental data and analytical solution from reference [2.1]. These results and procedures will be discussed at length in the following Sections. Analysis is also performed for a loading condition with mean stress effect. The results are compared to the experimental and analytical solution.
2.2 Brief Review of Previously Acquired Experimental Results

Experimental fatigue results have been acquired for both Al 6061-T6 and Ti-6Al-4V [2.1]. Axial results were acquired from a conventional MTS servo-hydraulic machine. The machine operated at a frequency of 40Hz, thus requiring 7 hours to accumulate $10^6$ cycles. The axially loaded fatigue results from this machine range approximately from $10^4$ to $10^6$ cycles for both materials.

The bending data is acquired using a vibration based methodology [2.2]. The thought behind the vibration-based methodology is supplying a dynamic base excitation to a specimen at a specified high resonant frequency, between 1200-1600 Hz, showing bending behavior. This testing method provides a significantly faster means for acquiring $10^6$ cycles (between 10 & 14 minutes), therefore making it a more efficient means for acquiring HCF based on uniaxial conditions.

Experimental fatigue results for Al 6061-T6 and Ti-6Al-4V are shown on Figures 2.1 and 2.2 respectively [2.1]. These results display an acceptable scatter for failure ranging from $10^4$ to over $10^6$ cycles. Therefore, the behavior of fatigue as applied load increases or decreases can easily be characterized visually.
2.3 Finite Element Procedures for Uniaxial Fatigue

The respective expression of Equations 2.1 and 2.2 consists of two parts, the linear elastic and the non-linear plastic. The main challenge in this research is to handle both parts correctly and develop a procedure which provides a best match with analytical and experimental results. The elastic and plastic parts of Equations 2.1 and 2.2 are written...
separately in Equations 2.5-2.8 respectively: where the subscripts \( em \) & \( pm \) designate the elastic & plastic cases for monotonic loading, and the subscripts \( ec \) & \( pc \) designate the elastic & plastic cases for cyclic loading.

\[
\varepsilon_{em} = \frac{\sigma}{E} \quad (2.5)
\]

\[
\varepsilon_{pm} = \varepsilon_o \sinh \left( \frac{\sigma}{\sigma_o} \right) \quad (2.6)
\]

\[
\varepsilon_{ec} = \frac{\sigma_{pp}}{E} \quad (2.7)
\]

\[
\varepsilon_{pc} = \frac{1}{C} \sinh \left( \frac{\sigma_{pp}}{\sigma_c} \right) \quad (2.8)
\]

Equations 2.9-2.12 give the corresponding stress equations to these provided strain expressions; where \( \sigma_{ec} \) and \( \sigma_{pc} \) are the corresponding peak-to-peak stresses for cyclic loading case. These Equations are integrated into minimum potential energy formulation (Equation 2.13) to develop new K-matrices.

\[
\sigma_{em} = E\varepsilon \quad (2.9)
\]

\[
\sigma_{pm} = \sigma_0 \sinh^{-1} \left( \frac{\varepsilon_p}{\varepsilon_0} \right) \quad (2.10)
\]

\[
\sigma_{ec} = E\varepsilon \quad (2.11)
\]

\[
\sigma_{pc} = \sigma_c \sinh^{-1} \left( C\varepsilon_p \right) \quad (2.12)
\]

The procedure for integration of the constitutive law into a minimum potential energy formulation is included in the following two Sections, 2.3.1 for uniaxial tension/compression loading and 2.3.2 for bending fatigue.
2.3.1 Finite Element Procedures for Uniaxial Load (Rod Element)

Integration of the elastic case into a potential energy formulation is a classical finite element problem and is available in the literature [2.5]. Integration of Equations 2.10 and 2.12 into Equation 2.14 provides the new K-matrix for the plastic part of the constitutive law (Equations 2.1 and 2.2) for uniaxial fatigue analysis.

\[ \Pi = \frac{1}{2} \sigma \varepsilon dV - \int u f dV - \int u T d\bar{x} - \sum u_i P_i \]  
\[ (2.13) \]

Where \( \Pi \) is the minimum potential energy, \( \sigma \) is the stress tensor, \( \varepsilon \) is the strain vector, \( u \) is the displacement, \( f \) is the body force, \( T \) is the traction force, and \( P_i \) is the point load. \( V \) is the volume and \( \bar{x} \) denotes the length of the element.

The strain energy term is given by

\[ U = \int \sigma_0 \sinh^{-1} \left( \frac{\varepsilon P}{\varepsilon_0} \right) \varepsilon_p dV \]  
\[ (2.14) \]

\[ U = \int \varepsilon \left( \sigma_0 \sinh^{-1} \left( \frac{\varepsilon P}{\varepsilon_0} \right) \right) \varepsilon_p dV \]  
\[ (2.15) \]

\[ U = \int \frac{d u^T}{d x} \sinh^{-1} \left( \frac{d u}{d x} \right) d u \frac{A d x}{d x} \]  
\[ (2.16) \]

The following Figure represents a uniaxial rod element. It has two nodes and each node has one degree of freedom. The displacements at each node are represented by \( d1 \) and \( d2 \) as shown in the Figure. The length of the element is \( L \) and \( x \) denotes the position on the element in uniaxial direction.
Expressing $u$ in terms of linear shape functions yields following equation.

$$u = \sum_{a=1}^{n} N_{a}d_{a}$$  \hspace{1cm} (2.17)

$N_{a}$ represents the shape function where $a$ denotes degrees of freedom. $d_{a}$ represents the nodal displacements for each node.

$$U = \left[ \frac{dN_{a}^{T}da}{dx} \right] \left[ \frac{dN_{a}^{T}da}{dx} \right] dx$$  \hspace{1cm} (2.18)

$$U = d_{a}^{T} \sigma_{0} d_{a}$$

(2.19)

$N_{1}$ and $N_{2}$ are linear 1-D shape functions shown in following equations.

$$N_{1} = 1 - \frac{x}{L}$$ \hspace{1cm} (2.20)
\[
N_2 = \frac{x}{L} \quad (2.21)
\]
\[
\frac{dN_1}{dx} = -\frac{1}{L} \quad (2.22)
\]
\[
\frac{dN_2}{dx} = \frac{1}{L} \quad (2.23)
\]

Therefore, Equation 2.19 becomes,
\[
\frac{1}{L^2} \left[ \sinh^{-1} \left( \frac{-d_1/L}{\varepsilon_0} \right) \begin{bmatrix} -1 \\ -L d_1 \end{bmatrix} + \sinh^{-1} \left( \frac{-d_1/L}{\varepsilon_0} \right) \begin{bmatrix} -1 \\ -L d_1 \end{bmatrix} \right] d_a dx = \int \left[ \sinh^{-1} \left( \frac{-d_1/L}{\varepsilon_0} \right) \begin{bmatrix} -1 \\ -L d_2 \end{bmatrix} + \sinh^{-1} \left( \frac{-d_1/L}{\varepsilon_0} \right) \begin{bmatrix} -1 \\ -L d_2 \end{bmatrix} \right] d_a dx
\]
\[
(2.24)
\]

The integration of Equation 2.16 over the element length L completes the formulation of K-matrix. The following equation provides the final form of energy equation after integration.
\[
U = d_a^T \sigma_0 A \begin{bmatrix} -\sinh^{-1} \left( \frac{-d_1/L}{\varepsilon_0} \right) & -\sinh^{-1} \left( \frac{-d_1/L}{\varepsilon_0} \right) \\
\sinh^{-1} \left( \frac{-d_1/L}{\varepsilon_0} \right) & \sinh^{-1} \left( \frac{-d_1/L}{\varepsilon_0} \right) \end{bmatrix} d_a = 0
\]
\[
(2.25)
\]

The resulting K-matrices for monotonic case with axial loading based on Equation 2.1 are shown below.
\[
K_{em-A} = \frac{AE}{L} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}
\]
\[
(2.26)
\]
\[
K_{pm-A} = \sigma_o A \begin{bmatrix}
\frac{1}{d_1} \sinh^{-1}\left(-\frac{d}{L} \frac{\varepsilon_o}{\sigma_o}\right) + \frac{1}{d_1} \sinh^{-1}\left(-\frac{d}{L} \frac{\varepsilon_o}{\sigma_o}\right) \\
\frac{1}{d_2} \sinh^{-1}\left(+\frac{d}{L} \frac{\varepsilon_o}{\sigma_o}\right) + \frac{1}{d_2} \sinh^{-1}\left(+\frac{d}{L} \frac{\varepsilon_o}{\sigma_o}\right)
\end{bmatrix}
\] (2.27)

A is the area, L is the length and d is the nodal displacement. Similar types of equations are developed for cyclic loading case according to Equation 2.2. The parameters \(\sigma_o\) changes to \(\sigma_c\), \(\varepsilon_o\) changes to C and the applied stress \(\sigma\) changes to peak-to-peak stress \(\sigma_{pp}\). The resulting K-matrices are shown in the following Equations.

\[
K_{ec} = \frac{AE}{L} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\] (2.28)

\[
K_{pc} = \sigma_c A \begin{bmatrix}
\frac{1}{d_1} \sinh^{-1}\left(-C \frac{d}{L} \frac{\varepsilon_o}{\sigma_o}\right) & \frac{1}{d_1} \sinh^{-1}\left(-C \frac{d}{L} \frac{\varepsilon_o}{\sigma_o}\right) \\
\frac{1}{d_2} \sinh^{-1}\left(+C \frac{d}{L} \frac{\varepsilon_o}{\sigma_o}\right) & \frac{1}{d_2} \sinh^{-1}\left(+C \frac{d}{L} \frac{\varepsilon_o}{\sigma_o}\right)
\end{bmatrix}
\] (2.29)

As is evident from Equations 2.27 and 2.29, a non linearity appears in the expression. This non linearity is due to an existing displacement parameter in the resulting K-matrices. As the new fatigue element is based on the fatigue constitutive law, therefore, parameters from Equations 2.1 and 2.2 appear in resulting stiffness matrices. To account for the non linear behavior, the Newton-Raphson iteration method is applied to the analysis [2.6-2.8].

Application of energy balance leads to the classical representation of load, \(F\), versus displacement, \(d\), shown by Equation 2.30. Therefore, based on a known applied load, the K matrices can be used in Equation 2.30 to determine the nodal displacements. Results from the nodal displacements and corresponding loads can be used to obtain a cyclic and monotonic stress-strain data.
The load is applied from peak-to-peak. The results, in the form of displacements, are post processed using classical FEA techniques to obtain stress and strain data. Equations 2.26 and 2.27 produce stress-strain data for the monotonic case. Equations 2.28 and 2.29 produce data for cyclic loading case. The acquired stress and strain data is further used to calculate the strain energy per cycle. Procedures for calculation of cyclic energy are explained in Section 2.4. These procedures are integrated into finite element analysis software developed for this research. The Number of cycles to failure is calculated using Equation 2.4. As stated in Chapter 1, the fatigue analysis with the current software like NASTRAN and ANSYS involves modal and frequency response analysis in order to obtain the vibratory stresses. Finite elements used by these software for the analysis consist of only the linear stiffness matrix. Therefore, this analysis procedure does not take fatigue process into account. New fatigue elements developed in this research are based on the fatigue constitutive law. These elements consist of an additional non-linear component in the stiffness matrix and the analysis with these new elements directly incorporates the fatigue mechanism in the procedure. Stresses and strains obtained from new fatigue elements analysis take the fatiguing process into account based on the fatigue constitutive law presented in Equations 2.1 and 2.2.
2.3.2 Finite Element Procedures for Bending Load (Beam Element)

This Section presents the development of a new beam fatigue element based on the fatigue constitutive law. A beam element has two degrees of freedom as shown in the following Figure.

![Degrees of Freedom for a Beam Element and Radius of Curvature](image)

Vertical displacement is represented by $V_i$ and the $\theta_i$ are the rotations at each node. The radius of curvature for a beam is also presented in Figure 2.4 and is denoted by $\rho$. The distance from the central axis is denoted by $y$.

The finite element (new K-Matrix) for bending fatigue is developed by integration of the new constitutive law into bending energy formulation. Bending stress is given by the following equation.

$$\sigma = -\frac{My}{I}$$  \hspace{1cm} (2.31)

Where $M$ is the bending moment, $y$ is the distance from the neutral axis and $I$ is the moment of inertia. The development of the K-matrix for elastic bending is a classical finite element problem and is available in the literature [2.5]. The Equation for plastic bending derived from Equation 2.10 is given by the following expression.
The potential energy of the beam is given by

\[
\Pi = \frac{1}{2} \int_0^L \sigma \varepsilon dV - \int_0^L p v dx - \sum_m P_m v_m - \sum_k M_k v'_k
\]  

(2.33)

Where \( p \) is the distributed load per unit length, \( P_m \) is the point load at point \( m \), \( M_k \) is the moment of the couple applied at point \( k \), \( v_m \) is the deflection at point \( m \) and \( v'_k \) is the slope at point \( k \).

The strain energy term is given by

\[
U = \int_0^L \frac{M y}{I \sigma_0} \varepsilon_o \sinh \left( \frac{M y}{I \sigma_0} \right) d\varepsilon_o
\]  

(2.34)

Integration of Equations 2.31 and 2.32 into Equation 2.34 provides the K-Matrix for non linear part of the constitutive law for bending loads.

\[
U = \int_0^L \sigma_o \sinh^{-1} \left( \frac{\varepsilon_p}{\varepsilon_o} \right) f_y \left( \frac{\sigma_o \sinh^{-1} \left( \frac{\varepsilon_p}{\varepsilon_o} \right) f_y}{I \sigma_0 y} \right) d\varepsilon_o
\]  

(2.35)

Equation 2.35 is simplified to following Equation.

\[
U = \int_0^L \sigma_o \sinh^{-1} \left( \frac{\varepsilon_p}{\varepsilon_o} \right) \varepsilon_p d\varepsilon_o
\]  

(2.36)

This Equation is further multiplied and divided with \( y^2 \) and using the Figure 2.4 and definitions in Equations 2.37 and 2.38, Equations 2.39 to 2.43 are obtained.

\[
\varepsilon = - \frac{dx}{ds} = - \frac{y d\phi}{ds} = - \frac{y}{\rho}
\]  

(2.37)

\[
\frac{1}{\rho} = \frac{d^2 y}{dx^2}
\]  

(2.38)
The local degrees of freedom for each element are represented by Equation 2.44, where $d_1$ and $d_3$ are transverse displacements and $d_2$ and $d_4$ are slopes.

$$d = [d_1, d_2, d_3, d_4]$$

These displacements and slopes are approximated using the Hermite shape functions which satisfy nodal value and continuity requirements. These shape functions are shown in Figure 2.5 and are given by the following expressions.

$$H_1 = \frac{1}{4} \left( 2 - 3\xi + \xi^3 \right) \quad (2.45)$$

$$H_2 = \frac{1}{4} \left( 1 - \xi - \xi^2 + \xi^3 \right) \quad (2.46)$$

$$H_3 = \frac{1}{4} \left( 2 + 3\xi - \xi^3 \right) \quad (2.47)$$

$$H_4 = \frac{1}{4} \left( -1 - \xi + \xi^2 + \xi^3 \right) \quad (2.48)$$
The Hermite shape functions can be used to write $v$ in the form

$$v(\xi) = H_1 v_1 + H_2 \left( \frac{dv}{d\xi} \right)_1 + H_3 v_2 + H_4 \left( \frac{dv}{d\xi} \right)_2$$  \hspace{1cm} (2.49)

The coordinate transformation is given by Equations 2.50 and 2.51. Equation 2.50 represents the coordinate transformation in terms of natural coordinates. Equation 2.51 represents the coordinate transform in terms of nodal coordinates. The displacements at each node are given by $x_1$ and $x_2$.

$$x = \frac{1-\xi}{2} x_1 + \frac{1+\xi}{2} x_2$$  \hspace{1cm} (2.50)

$$x = \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2} \xi$$  \hspace{1cm} (2.51)
\[ l_e = x_2 - x_1 \] 

(2.52)

Element length is \( l_e \). Therefore,

\[ dx = \frac{l_e}{2} d\xi \] 

(2.53)

and

\[ \frac{dv}{d\xi} = \frac{l_e}{2} \frac{dv}{dx} \] 

(2.54)

And the \( \frac{dv}{d\xi} \) evaluated at node 1 and 2 are \( d_2 \) and \( d_4 \) respectively,

Therefore, we obtain

\[ v(\xi) = H_1 d_1 + \frac{l_e}{2} H_2 d_2 + H_3 d_3 + \frac{l_e}{2} H_4 d_4 \] 

(2.55)

which may be denoted as

\[ v = Hd \] 

(2.56)

where

\[ H = \begin{bmatrix} H_1, \frac{l_e}{2} H_2, H_3, \frac{l_e}{2} H_4 \end{bmatrix} \] 

(2.57)

Note that

\[ \frac{dv}{dx} = \frac{2}{l_e} \frac{dv}{d\xi} \] 

(2.58)

\[ \frac{d^2v}{dx^2} = \frac{4}{l_e^2} \frac{d^2v}{d\xi^2} \] 

(2.59)

Using Equation 2.59 and then substituting Equation 2.56, we obtain

\[ \left( \frac{d^2v}{dx^2} \right)^2 = d^T \frac{16}{l_e^4} \left( \frac{d^2H}{d\xi^2} \right)^T \left( \frac{d^2H}{d\xi^2} \right) \] 

(2.60)
\[
\frac{d^2 H}{d\xi^2} = \left[ \frac{3}{2} \xi, \frac{-1 + 3\xi}{2} l_e, -\frac{3}{2} \xi, \frac{-1 + 3\xi}{2} l_e \right] \tag{2.61}
\]

On substituting Equations 2.60 and 2.61 into Equation 2.43, we obtain

\[
U = d^T \frac{8\sigma_{ol}}{l_e^3} \int \left[ \begin{array}{cccc}
k_{11}(d_1) & k_{12}(d_1) & k_{13}(d_1) & k_{14}(d_1) \\
k_{21}(d_2) & k_{22}(d_2) & k_{23}(d_2) & k_{24}(d_2) \\
k_{31}(d_3) & k_{32}(d_3) & k_{33}(d_3) & k_{34}(d_3) \\
k_{41}(d_4) & k_{42}(d_4) & k_{43}(d_4) & k_{44}(d_4) \end{array} \right] d\xi d \quad \tag{2.62}
\]

The elements of above matrix are given by Equations 2.63 to 2.78. These Equations are in the form of natural coordinates and also consist of material constants described in the first Section.

\[
k_{11} = \frac{3}{2} \xi \sinh^{-1} \left( \frac{6y\xi d_1}{\epsilon_{ol}l_e^2} \right) \quad \tag{2.63}
\]

\[
k_{12} = -\frac{(1 + 3\xi)}{4} l_e \sinh^{-1} \left( \frac{6y\xi d_1}{\epsilon_{ol}l_e^2} \right) \quad \tag{2.64}
\]

\[
k_{13} = -\frac{3}{2} \xi \sinh^{-1} \left( \frac{6y\xi d_1}{\epsilon_{ol}l_e^2} \right) \quad \tag{2.65}
\]

\[
k_{14} = \frac{(1 + 3\xi)}{4} l_e \sinh^{-1} \left( \frac{6y\xi d_1}{\epsilon_{ol}l_e^2} \right) \quad \tag{2.66}
\]

\[
k_{21} = -\frac{3}{2} \xi \sinh^{-1} \left( \frac{y(-1 + 3\xi)d_2}{\epsilon_{ol}l_e^2} \right) \quad \tag{2.67}
\]

\[
k_{22} = \frac{(1 + 3\xi)}{4} l_e \sinh^{-1} \left( \frac{y(-1 + 3\xi)d_2}{\epsilon_{ol}l_e^2} \right) \quad \tag{2.68}
\]

\[
k_{23} = -\frac{3}{2} \xi \sinh^{-1} \left( \frac{y(-1 + 3\xi)d_2}{\epsilon_{ol}l_e^2} \right) \quad \tag{2.69}
\]
On solving the integrals of Equations 2.62 to 2.78 using numerical integration with the Gaussian point approach, following expressions are obtained. These expressions provide the elements of the K-Matrix.

\[
k_{24} = \frac{(-1 + 9\xi^2)}{4(-1 + 3\xi)} \frac{d_2}{d_2} \left( \frac{y(-1 + 3\xi)d_2}{\varepsilon_{ol}} \right)
\]

\[
k_{42} = \frac{(-1 + 9\xi^2)}{4(1 + 3\xi)} \frac{d_4}{d_4} \left( \frac{y(1 + 3\xi)d_4}{\varepsilon_{ol}} \right)
\]

\[
k_{32} = \frac{(-1 + 3\xi)}{4} \frac{d_3}{d_3} \left( \frac{-6y\xi d_3}{\varepsilon_{ol}^2} \right)
\]

\[
k_{33} = \frac{3\xi}{2} \frac{d_3}{d_3} \left( \frac{-6y\xi d_3}{\varepsilon_{ol}^2} \right)
\]

\[
k_{34} = \frac{(1 + 3\xi)}{4} \frac{d_3}{d_3} \left( \frac{6y\xi d_3}{\varepsilon_{ol}^2} \right)
\]

\[
k_{41} = \frac{3\xi}{2} \frac{d_4}{d_4} \left( \frac{y(1 + 3\xi)d_4}{\varepsilon_{ol}} \right)
\]

\[
k_{42} = \frac{(-1 + 9\xi^2)}{4(1 + 3\xi)} \frac{d_4}{d_4} \left( \frac{y(1 + 3\xi)d_4}{\varepsilon_{ol}} \right)
\]

\[
k_{43} = \frac{3\xi}{2} \frac{d_4}{d_4} \left( \frac{y(1 + 3\xi)d_4}{\varepsilon_{ol}} \right)
\]

\[
k_{44} = \frac{(-1 + 3\xi)}{4} \frac{d_4}{d_4} \left( \frac{y(1 + 3\xi)d_4}{\varepsilon_{ol}} \right)
\]

On solving the integrals of Equations 2.62 to 2.78 using numerical integration with the Gaussian point approach, following expressions are obtained. These expressions provide the elements of the K-Matrix.

\[
k_{11pm} = \frac{2\sigma_{ol}}{l_e y} \left[ \frac{0.866}{d_1} \left( 3.46y_d \frac{d_1}{\varepsilon_{ol}^2} \right) - \sinh^{-1} \left( \frac{3.46y_d}{\varepsilon_{ol}^2} \right) \right]
\]
\[ k_{12 \, pm} = \frac{2\sigma_{ol}}{l_{ey}} \left[ \frac{l_e}{d_1} \left\{ 0.183 \sinh^{-1} \left( \frac{3.46 y_{d1}}{\varepsilon_{ol} l_e^2} \right) - 0.683 \sinh^{-1} \left( \frac{-3.46 y_{d1}}{\varepsilon_{ol} l_e^2} \right) \right\} \right] \]  \hspace{1cm} (2.80)

\[ k_{13 \, pm} = \frac{2\sigma_{ol}}{l_{ey}} \left[ \frac{0.866}{d_1} \left\{ \sinh^{-1} \left( \frac{3.46 y_{d1}}{\varepsilon_{ol} l_e^2} \right) + \sinh^{-1} \left( \frac{-3.46 y_{d1}}{\varepsilon_{ol} l_e^2} \right) \right\} \right] \]  \hspace{1cm} (2.81)

\[ k_{14 \, pm} = \frac{2\sigma_{ol}}{l_{ey}} \left[ \frac{l_e}{d_1} \left\{ 0.683 \sinh^{-1} \left( \frac{3.46 y_{d1}}{\varepsilon_{ol} l_e^2} \right) - 0.183 \sinh^{-1} \left( \frac{-3.46 y_{d1}}{\varepsilon_{ol} l_e^2} \right) \right\} \right] \]  \hspace{1cm} (2.82)

\[ k_{21 \, pm} = \frac{2\sigma_{ol}}{l_{ey}} \left[ \frac{0.866}{d_2} \left\{ \sinh^{-1} \left( \frac{0.732 y_{d2}}{\varepsilon_{ol} l_e^2} \right) - \sinh^{-1} \left( \frac{-2.732 y_{d2}}{\varepsilon_{ol} l_e^2} \right) \right\} \right] \]  \hspace{1cm} (2.83)

\[ k_{22 \, pm} = \frac{2\sigma_{ol}}{l_{ey}} \left[ \frac{l_e}{d_2} \left\{ 0.183 \sinh^{-1} \left( \frac{0.732 y_{d2}}{\varepsilon_{ol} l_e^2} \right) - 0.683 \sinh^{-1} \left( \frac{-2.732 y_{d2}}{\varepsilon_{ol} l_e^2} \right) \right\} \right] \]  \hspace{1cm} (2.84)

\[ k_{23 \, pm} = \frac{2\sigma_{ol}}{l_{ey}} \left[ \frac{0.866}{d_2} \left\{ -\sinh^{-1} \left( \frac{0.732 y_{d2}}{\varepsilon_{ol} l_e^2} \right) + \sinh^{-1} \left( \frac{-2.732 y_{d2}}{\varepsilon_{ol} l_e^2} \right) \right\} \right] \]  \hspace{1cm} (2.85)

\[ k_{24 \, pm} = \frac{2\sigma_{ol}}{l_{ey}} \left[ \frac{l_e}{d_2} \left\{ 0.683 \sinh^{-1} \left( \frac{0.732 y_{d2}}{\varepsilon_{ol} l_e^2} \right) - 0.183 \sinh^{-1} \left( \frac{-2.732 y_{d2}}{\varepsilon_{ol} l_e^2} \right) \right\} \right] \]  \hspace{1cm} (2.86)

\[ k_{31 \, pm} = \frac{2\sigma_{ol}}{l_{ey}} \left[ \frac{0.866}{d_3} \left\{ \sinh^{-1} \left( \frac{-3.46 y_{d3}}{\varepsilon_{ol} l_e^2} \right) - \sinh^{-1} \left( \frac{3.46 y_{d3}}{\varepsilon_{ol} l_e^2} \right) \right\} \right] \]  \hspace{1cm} (2.87)

\[ k_{32 \, pm} = \frac{2\sigma_{ol}}{l_{ey}} \left[ \frac{l_e}{d_3} \left\{ 0.183 \sinh^{-1} \left( \frac{-3.46 y_{d3}}{\varepsilon_{ol} l_e^2} \right) - 0.683 \sinh^{-1} \left( \frac{3.46 y_{d3}}{\varepsilon_{ol} l_e^2} \right) \right\} \right] \]  \hspace{1cm} (2.88)

\[ k_{33 \, pm} = \frac{2\sigma_{ol}}{l_{ey}} \left[ \frac{0.866}{d_3} \left\{ -\sinh^{-1} \left( \frac{-3.46 y_{d3}}{\varepsilon_{ol} l_e^2} \right) + \sinh^{-1} \left( \frac{3.46 y_{d3}}{\varepsilon_{ol} l_e^2} \right) \right\} \right] \]  \hspace{1cm} (2.89)

\[ k_{34 \, pm} = \frac{2\sigma_{ol}}{l_{ey}} \left[ \frac{l_e}{d_3} \left\{ 0.683 \sinh^{-1} \left( \frac{-3.46 y_{d3}}{\varepsilon_{ol} l_e^2} \right) - 0.183 \sinh^{-1} \left( \frac{3.46 y_{d3}}{\varepsilon_{ol} l_e^2} \right) \right\} \right] \]  \hspace{1cm} (2.90)
The resulting K-matrices for bending loading based on Equation 2.1 are shown below. Equation 2.95 is the K Matrix for linear elastic case. The Equation 2.96 provides the K-Matrix for plastic non-linear case.

\[
K_{em-B} = \frac{EI}{l_e^3} \begin{bmatrix}
12 & 6l_e & -12 & 6l_e \\
6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\
-12 & -6l_e & 12 & -6l_e \\
6l_e & 2l_e^2 & -6l_e & -4l_e^2
\end{bmatrix}
\]  
(2.95)

\[
K_{pm-B} = \begin{bmatrix}
k_{11pm}(d_1) & k_{12pm}(d_1) & k_{13pm}(d_1) & k_{14pm}(d_1) \\
k_{21pm}(d_2) & k_{22pm}(d_2) & k_{23pm}(d_2) & k_{24pm}(d_2) \\
k_{31pm}(d_3) & k_{32pm}(d_3) & k_{33pm}(d_3) & k_{34pm}(d_3) \\
k_{41pm}(d_4) & k_{42pm}(d_4) & k_{43pm}(d_4) & k_{44pm}(d_4)
\end{bmatrix}
\]  
(2.96)

These elements of \( K_{pm-B} \) in index notation are given by the following Equations. \( A_i, B_i, C_j \) and \( D_j \) are given in Table 2.1.

\[
k_{ij-pm} = \frac{0.866}{d_i} \left( C_j \sinh^{-1}(A_i d_i) - D_j \sinh^{-1}(B_i d_i) \right) \quad \text{for } i = 1, 2, 3, 4 \text{ and } j = 1, 3
\]  
(2.97)
\[ k_{ij-pm} = \frac{l_e}{d_i} \left( C_j \sinh^{-1}(A_id_i) - D_j \sinh^{-1}(B_id_i) \right) \quad \text{for } i = 1, 2, 3, 4 \text{ and } j = 2, 4 \]

\[ (2.98) \]

<table>
<thead>
<tr>
<th>( l )</th>
<th>( A_i )</th>
<th>( B_i )</th>
<th>( C_j )</th>
<th>( D_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{3.46y}{\varepsilon_0 l_e^2} )</td>
<td>( \frac{-3.46y}{\varepsilon_0 l_e^2} )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{0.732y}{\varepsilon_0 l_e} )</td>
<td>( \frac{-2.732y}{\varepsilon_0 l_e} )</td>
<td>0.183</td>
<td>0.683</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{-3.46y}{\varepsilon_0 l_e^2} )</td>
<td>( \frac{3.46y}{\varepsilon_0 l_e^2} )</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{2.732y}{\varepsilon_0 l_e} )</td>
<td>( \frac{-0.732y}{\varepsilon_0 l_e} )</td>
<td>0.683</td>
<td>0.183</td>
</tr>
</tbody>
</table>

Table 2.1: Constants for Equations 2.97 and 2.98

Similar Equations are developed for cyclic loading case according to Equation 2.2. The parameters \( \sigma_0 \) changes to \( \sigma_c \), \( \varepsilon_0 \) changes to \( l/C \) and the applied stress \( \sigma \) changes to peak- to -peak stress \( \sigma_{PP} \). The resulting K-matrices are shown in the following Equations.

\[ K_{ec-B} = \frac{EI}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & -4l_e^2 \end{bmatrix} \]

\[ (2.99) \]
\[ K_{pc-B} = \frac{2\sigma c}{l_{e,y}} \begin{bmatrix} k_{11pc}(d_1) & k_{12pc}(d_1) & k_{13pc}(d_1) & k_{14pc}(d_1) \\ k_{21pc}(d_2) & k_{22pc}(d_2) & k_{23pc}(d_2) & k_{24pc}(d_2) \\ k_{31pc}(d_3) & k_{32pc}(d_3) & k_{33pc}(d_3) & k_{34pc}(d_4) \\ k_{41pc}(d_4) & k_{42pc}(d_4) & k_{43pc}(d_4) & k_{44pc}(d_4) \end{bmatrix} \] (2.100)

The elements of above matrix are given by Equations 2.101 to 2.116. These Equations are in the form of natural coordinates and also consist of material constants described in the first Section.

\[ k_{11pc} = \frac{0.866}{d_1} \left\{ \sinh^{-1} \left( \frac{3.46Cyd_1}{l_e^2} \right) - \sinh^{-1} \left( \frac{-3.46Cyd_1}{l_e^2} \right) \right\} \] (2.101)

\[ k_{12pc} = \frac{l_e}{d_1} \left\{ 0.183 \sinh^{-1} \left( \frac{3.46Cyd_1}{l_e^2} \right) - 0.683 \sinh^{-1} \left( \frac{-3.46Cyd_1}{l_e^2} \right) \right\} \] (2.102)

\[ k_{13pc} = \frac{0.866}{d_1} \left\{ -\sinh^{-1} \left( \frac{3.46Cyd_1}{l_e^2} \right) + \sinh^{-1} \left( \frac{-3.46Cyd_1}{l_e^2} \right) \right\} \] (2.103)

\[ k_{14pc} = \frac{l_e}{d_1} \left\{ 0.683 \sinh^{-1} \left( \frac{3.46Cyd_1}{l_e^2} \right) - 0.183 \sinh^{-1} \left( \frac{-3.46Cyd_1}{l_e^2} \right) \right\} \] (2.104)

\[ k_{21pc} = \frac{0.866}{d_2} \left\{ \sinh^{-1} \left( \frac{0.732Cyd_2}{l_e} \right) - \sinh^{-1} \left( \frac{-2.732Cyd_2}{l_e} \right) \right\} \] (2.105)

\[ k_{22pc} = \frac{l_e}{d_2} \left\{ 0.183 \sinh^{-1} \left( \frac{0.732Cyd_2}{l_e} \right) - 0.683 \sinh^{-1} \left( \frac{-2.732Cyd_2}{l_e} \right) \right\} \] (2.106)

\[ k_{23pc} = \frac{0.866}{d_2} \left\{ -\sinh^{-1} \left( \frac{0.732Cyd_2}{l_e} \right) + \sinh^{-1} \left( \frac{-2.732Cyd_2}{l_e} \right) \right\} \] (2.107)

\[ k_{24pc} = \frac{l_e}{d_2} \left\{ 0.683 \sinh^{-1} \left( \frac{0.732Cyd_2}{l_e} \right) - 0.183 \sinh^{-1} \left( \frac{-2.732Cyd_2}{l_e} \right) \right\} \] (2.108)
These elements of $K_{pc-B}$ in index notation are given by the following Equations. $A_i$, $B_i$, $C_j$ and $D_j$ are given in Table 2.2.

\[
k_{ij-pm} = \frac{0.866}{d_i} \left( C_j \sinh^{-1} \left( A_i d_i \right) - D_j \sinh^{-1} \left( B_i d_i \right) \right) \quad \text{for } i = 1, 2, 3, 4 \text{ and } j = 1, 3
\]

(2.117)

\[
k_{ij-pm} = \frac{l_e}{d_i} \left( C_j \sinh^{-1} \left( A_i d_i \right) - D_j \sinh^{-1} \left( B_i d_i \right) \right) \quad \text{for } i = 1, 2, 3, 4 \text{ and } j = 2, 4
\]

(2.118)
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<tr>
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<th>(B_i)</th>
<th>(C_j)</th>
<th>(D_j)</th>
</tr>
</thead>
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<tr>
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<td>(\frac{-3.46Cy}{l_e^2})</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{0.732Cy}{l_e})</td>
<td>(\frac{-2.732Cy}{l_e})</td>
<td>0.183</td>
<td>0.683</td>
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<tr>
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<td>(\frac{3.46Cy}{l_e^2})</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{2.732Cy}{l_e})</td>
<td>(\frac{-0.732Cy}{l_e})</td>
<td>0.683</td>
<td>0.183</td>
</tr>
</tbody>
</table>

Table 2.2: Constants for Equations 2.117 and 2.118

Equations 2.96 and 2.100 are non-linear due to the presence of “\(ds\)” in the resulting K-matrices. To account for the non-linear behavior, the Newton-Raphson iteration method is applied to the analysis [2.6-2.8]. These K-matrices are used in Equation 2.119 to determine the unknown degrees of freedom.

\[
[K]\{d\} = \{F\}
\]

(2.119)

The load is applied from peak to peak. The results are post-processed using classical FEA techniques. The nodal displacement and rotation results can be further used to obtain moment for each element in the structure. This moment is used to calculate the bending energy dissipated per cycle and ultimately the number of cycles to failure for each element. The procedure for calculation of number of cycles is the same as followed for the analysis with the rod element described in the previous Section.
2.4 Pre- and Post Processing Of Data:

The procedures presented in this Section are primarily for 1-D tension/compression loads. However, similar pre- and post-processing procedures are valid for bending loads.

The geometric data was acquired from ASTM standard (E466) fatigue dog-bone (coupon) specimen in Figure 2.6 [2.1]. This specimen is loaded in axial tension and compression.

![Figure 2.6: Dimensions (inches) of the ASTM Fatigue Dog-Bone Specimen [2.1]](image)

This specimen is loaded in axial tension for the monotonic case and tension/compression for cyclic loading. The monotonic loading produces force vs. displacement data, which is converted to stress-strain relation in the form of Figure 2.7. The fatigue analysis is only performed below the yield point. Therefore, this relation is only analyzed up to the point of yielding for the validation of the K-Matrix. This analysis
is performed for Al 6061-T6 and Ti 6Al-4V material parameters. The resulting solution is compared with the experimental and analytical results (Equation 2.1). These results are presented and discussed in detail in Section 2.5.

The cyclic process is executed by tension/compression loading at a specified level. In other words, fully-reversed loading ranges from \(-P\) to \(+P\), where \(P\) is the parameter representing the specified loading level. This understanding leads to the conclusion that the cyclic stress-strain behavior, which is acquired from the corresponding load-displacement relation, forms the loop shown on the generalized axis of Figure 2.8. The stress-strain relation of Figure 2.8 is known as a hysteresis loop. The area inside this loop represents the cyclic strain energy density for the applied stress level. In order to calculate the number of cycles to failure, the cyclic strain energy acquired from the constitutive law is compared to the monotonic failure strain energy [2.1]. This analysis was conducted with Ti-6AL-4V and Al 6061-T6 material parameters. The results were compared with experimental data and the analytical solution [2.6].
Figure 2.8: Hysteresis Loop for Completely Reversible Loading

Figure 2.9 represents the hysteresis loop with the mean stress effect. The Figure shows that the calculation of strain energy density for this case is different from the completely reversible loading analysis. Due to a larger maximum applied load, the effect of mean stress increases the plastic deformation per cycle for a designated alternating load. Also, based on an increase in the minimum applied load, the stress-strain relation no longer is viewed as a closed loop. Therefore, the un-shaded area of Figure 2.9 represents cyclic strain energy density. This analysis is performed for Al 6061-T6 and the results are compared to experimental data and the analytical solution.
As stated earlier, the procedure presented in this Section for 1-D tension/compression loads is also valid for bending loads. The following Equations are used to obtain the bending moments for each element in the structure.

\[
M_e = \frac{EI}{l_e^2} \left[ 6\xi d_1 + (3\xi - 1)\ell ed_2 - 6\xi d_3 + (3\xi + 1)\ell e d_4 \right]
\]  
(2.118)

From equation 2.14,

\[
M_p = \frac{\sigma c L}{y} \sinh^{-1} \left( C \frac{d^2 \nu}{d\xi^2} \right)
\]  
(2.119)

Using Equations 2.56, 2.57 and 2.59, we obtain

\[
M_p = \frac{\sigma c L}{y} \sinh^{-1} \left[ \frac{C y}{l e^2} \left( 6\xi d_1 + (3\xi - 1)\ell ed_2 - 6\xi d_3 + (3\xi + 1)\ell e d_4 \right) \right]
\]  
(2.220)

where \(M_e\) and \(M_p\) are the elastic and plastic bending moments for each element. These moments are used to obtain the bending stress and strain. The stress and strain results are ultimately used to obtain the energy dissipated per cycle. The energy dissipated per cycle...
when compared to total fracture energy, yields the number of cycles to failure. Analysis is performed for Al 6061-T6 both for completely reversible and mean stress effects bending loads. The K-matrix for bending loads is capable of predicting different number of cycles for each element depending upon the stress different stress level. The procedure for acquiring the energy dissipated per cycle and the fatigue life prediction for each element is the same as followed for the rod fatigue element analysis and described in the previous Section.

2.5 Results and Discussion

The following Sections present the results and comparisons for axial bending loads respectively.

2.5.1 1-D Axial Load Rod Element Analysis

The finite element analysis results and analytical solution are obtained for a 1-D rod in order to compare and validate the results. The dimensions of this rod are provided in [2.6]. A set of experimental data from previous research [2.1] is also included in the comparison. Figure 2.10 shows the Force vs. Displacement curves for Al 6061-T6 with experimental data, analytical solution [2.1] and FEM prediction plotted for comparison. The FEM results are acquired from the analysis with the new rod fatigue elements. The codes used for this analysis are based on the new fatigue element and the pre- and post-processing procedures are inbuilt into these codes. These curves are plotted for a monotonically loaded specimen below the yield point. The FEM prediction compares well with the experimental and analytical results.
Figure 2.10: Force vs. Displacement for Al 6061T6 under Monotonic Tension

The FEM prediction tends to deviate from the analytical solution [2.1] and experimental data with increasing force. However, this deviation is very negligible and can be ignored considering that this does not affect final results for number of cycles to failure.

Figure 2.11 shows engineering stress vs. strain results for Al 6061-T6. The curves follow similar behavior as force vs. displacement plots. There is a small deviation of FEM prediction from experimental data and analytical solution.
Figure 2.11: Engineering Stress vs. Engineering Strain for Al 6061T6 under Monotonic Tension

Figure 2.12 shows the true stress vs. true strain, for FEM prediction and analytical solution [2.1] plotted in comparison to experimental data. The specimen is loaded monotonically below the yield point.

Figure 2.12: True Stress vs. True Strain for Al 6061-T6 under Monotonic Tension
Figures 2.10, 2.11 and 2.12 show a very good match of FEM prediction to the experimental data and analytical solution [2.1], thus, validating the accuracy of the newly developed K-matrix and FEM procedure.

The constitutive law in Equation 2.1 and 2.2 consists of linear and non-linear parts. The plastic part of Equations 2.1 and 2.2 induces non-linearity in the results. This non-linearity helps in producing hysteresis loop when dealing with the cyclic loading. As stated earlier, the area enclosed by this hysteresis loop provides the energy loss per cycle.

Though the constitutive law in Equations 2.1 and 2.2 has the capability to capture the plastic strain, the contribution of this strain to the total strain is very small. This makes the curves in Figures 2.10, 2.11 and 2.12 almost look linear. In order to display the non-linearity present in the data, results are normalized and are shown in Figure 2.13.

![Normalized Data Plot for Al 6061-T6 under Monotonic Tension](image)

**Figure 2.13** Normalized Data Plot for Al 6061-T6 under Monotonic Tension
Equation 2.1 parameters and the corresponding rod element K-matrix are used to construct the S-N curve for Al 6061-T6 and is shown in Figure 2.14 on a semi-log scale. Experimental data, analytical solution [2.1] and FEM prediction are plotted on the same graph for comparison. The FEM curve shows a good agreement with experimental data and analytical results [2.1].

![Figure 2.14. Stress vs. No. of Cycles for Al 6061-T6 for Completely Reversible Axial Load](image)

Table 2.3 shows a comparison of cyclic energy for analytical results [2.1] and FEM prediction. The percent difference between the two results is below 2.2%. This verifies a good match between results and also validates the new finite element.
<table>
<thead>
<tr>
<th>Stress (ksi)</th>
<th>Cyclic Energy (Analytical) (lb/in)/in³ x 10³</th>
<th>Cyclic Energy (FEM) (lb/in)/in³ x 10³</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>8.52E+00</td>
<td>8.71E+00</td>
<td>2.2</td>
</tr>
<tr>
<td>30</td>
<td>1.99E+00</td>
<td>2.02E+00</td>
<td>1.4</td>
</tr>
<tr>
<td>25</td>
<td>4.41E-01</td>
<td>4.44E-01</td>
<td>0.7</td>
</tr>
<tr>
<td>20</td>
<td>8.98E-02</td>
<td>9.00E-02</td>
<td>0.1</td>
</tr>
<tr>
<td>15</td>
<td>1.57E-02</td>
<td>1.56E-02</td>
<td>0.3</td>
</tr>
<tr>
<td>9</td>
<td>1.18E-03</td>
<td>1.20E-03</td>
<td>1.4</td>
</tr>
</tbody>
</table>

Table 2.3: Cyclic Energy Comparison for Analytical Solution and FEM Prediction for Axial Load

Table 2.4 presents a comparison between number of cycles predicted in previous research [2.1] and with new finite element. Since there is a direct correlation between cyclic energy and cycles to failure, the maximum percent difference for assorted stress levels is also below 2.2 percent. This provides another indicator for a good agreement between the two methods.
<table>
<thead>
<tr>
<th>Stress (ksi)</th>
<th>Cycles to Failure (Analytical)</th>
<th>Cycles to Failure (FEM)</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>6.28E+03</td>
<td>6.14E+03</td>
<td>2.2</td>
</tr>
<tr>
<td>30</td>
<td>2.69E+04</td>
<td>2.65E+04</td>
<td>1.4</td>
</tr>
<tr>
<td>25</td>
<td>1.21E+05</td>
<td>1.20E+05</td>
<td>0.7</td>
</tr>
<tr>
<td>20</td>
<td>5.96E+05</td>
<td>5.95E+05</td>
<td>0.1</td>
</tr>
<tr>
<td>15</td>
<td>3.42E+06</td>
<td>3.43E+06</td>
<td>0.3</td>
</tr>
<tr>
<td>9</td>
<td>4.52E+07</td>
<td>4.45E+07</td>
<td>1.4</td>
</tr>
</tbody>
</table>

Table 2.4: Number of Cycle Comparison between Analytical Solution and FEM Prediction for Axial Load

This research deals with the elastic and plastic parts of Equations 2.1 and 2.2 separately and ignores any coupling between elastic and plastic strains. However, in the real world, this may not be the case. As it transpires, this coupling becomes stronger with the increasing stress in particular above the yield point. Therefore, a deviation of FEM prediction from the analytical results at higher stress levels is observed. The fatigue analysis is performed only below the yield point. Therefore, the level of applied stress remains low enough to cause any significant error due to coupling on the final results for number of cycles to failure. The above Table presents the difference between analytical results and FEM prediction for different stress levels. This is also evident from Figures
2.10, 2.11 and 2.12 that the difference between experimental data and FE prediction is very negligible. Therefore, to avoid the complexity of the computation, the coupling between elastic and plastic parts is ignored.

Ti-6Al-4V material is also analyzed using the new finite element and results are plotted in Figure 2.15. From the experimental results of Figure 2.15, it can be observed that an endurance limit phenomenon is present. Due to uncertainties in energy behavior near the endurance limit, the analysis is performed for stress levels above 60Ksi. Nevertheless, the results show a good match between analytical solution and FEM prediction. The results show a good match between analytical solution and FEM prediction.

![Figure 2.15: Stress vs. No. of Cycles for Ti-6Al-4V for Completely Reversible Axial Load](image)

Figure 2.15: Stress vs. No. of Cycles for Ti-6Al-4V for Completely Reversible Axial Load
The cyclic loading case with mean stress effect is different from the completely reversible loading [2.9]. The calculation of cyclic energy in this case involves a different approach due to the presence of mean stress and strain. When mean stress is included in a fatigue procedure, it dissipates residual energy and increases the plastic strain per cycle, thus reducing the amount of cycles required to fatigue a specimen. When evaluating the fully reversed tension/compression cyclic behavior, two assumptions were set in place. It is considered that a significant amount of strain damage is caused by plastic deformation, and the tensile cyclic curve (from zero applied stress to peak-to-peak stress) is a slight modification of the true strain equation, which defines the stress-strain relation of a monotonic procedure [2.1]. In order to incorporate mean stress effect, both of these assumptions should still be in place. Meaning, each cycle, regardless of the mean stress value, should be plotted on the same axis and evaluated with the same cyclic tensile strain equation [2.1] as the fully reversed cycle. However, unlike the fully-reversed case, it can not be assumed that the compressive behavior of the hysteresis loop (stress strain plot for one complete cycle) is identical to the tensile curve. Meaning, the curve from zero applied stress to peak-to-peak is not the same as the curve from peak-to-peak to zero. This assertion is proven by the experimental results, where the compressive curve was shown to be reasonably linear, thus providing a rather fair assumption for analytical characterization. As stated earlier, and illustrated in Figure 2.9, the effect of the mean stress increases the amount of plastic deformation per cycle. This effect, as well as the residual mean strain energy, will reduce the fatigue life of materials with stress ratios greater than negative one.
The FE analysis is performed for Al 6061-T6 with mean stress effect included. The results are plotted in Figures 2.16 and 2.17 for 10ksi and 20ksi mean stress levels respectively. The FEM prediction curve follows the analytical results and experimental data closely.

![Figure 2.16](image1.png)  
Figure 2.16. S-N Curve with 10 ksi Mean Stress Effect for Al 6061-T6

![Figure 2.17](image2.png)  
Figure 2.17. S-N Curve with 20 ksi Mean Stress Effect for Al 6061-T6
2.5.2 Bending Load Analysis

The following Figure shows results for Al 6061-T6 for completely reversible bending load. The FEM analysis is performed for number of elements in the beam ranging from 1 to 5. The results tend to converge to the analytical solution with increasing number of elements. The FEM results show a good agreement with analytical and experimental data [2.1].

![Figure 2.18: Stress vs. No. of Cycles for Al 6061-T6 for Completely Reversible Bending Load](image)

Table 2.5 provides a comparison of FEM prediction and analytical solution for a 5 element analysis. The results show a reasonable match which provides another indicator for an agreement between analytical [2.1] and FEM predictions.
<table>
<thead>
<tr>
<th>Stress (ksi)</th>
<th>Cycles (Analytical)</th>
<th>Cycles (FEM)</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5.75E+07</td>
<td>5.13E+07</td>
<td>10.73</td>
</tr>
<tr>
<td>15</td>
<td>9.62E+06</td>
<td>8.993E+06</td>
<td>6.598</td>
</tr>
<tr>
<td>20</td>
<td>2.42E+06</td>
<td>2.340E+06</td>
<td>3.585</td>
</tr>
<tr>
<td>25</td>
<td>7.48E+05</td>
<td>7.220E+05</td>
<td>3.515</td>
</tr>
<tr>
<td>30</td>
<td>2.58E+05</td>
<td>2.500E+05</td>
<td>3.176</td>
</tr>
<tr>
<td>35</td>
<td>9.52E+04</td>
<td>9.551E+04</td>
<td>0.234</td>
</tr>
</tbody>
</table>

Table 2.5: Number of Cycle Comparison between Analytical Solution and FEM Prediction (5 Elements) under Bending Loads

As stated earlier, the new bending element (K-Matrix) has the capability to predict different number of cycles for different elements in the structure subjected to varying stress. As the stress varies along the length of the beam, therefore, the cyclic energy calculated for each element also varies. This provides different number of cycles to failure for each element in the structure. The following Figures show 5 element cantilever beam analysis for 10ksi and 20 ksi bending loads. The number of cycles is plotted using a colored contour plot where each color represents corresponding number of cycles to failure in the element according to the scale. The scale is plotted from highest number of cycles, represented with blue color, to the lowest numbers represented with red color as shown in the Figure.
Figure 2.19: No. of Cycles to Failure for Al 6061-T6 for Completely Reversible Bending Load (10Ksi) – 5 Element Analysis

Figure 2.20: No. of Cycles to Failure for Al 6061-T6 for Completely Reversible Bending Load (20Ksi) – 5 Element Analysis
Element 1 experiences the maximum stress, therefore, minimum number of cycles are predicted for this element. On the other hand, element 5 experiences the minimum stress, therefore, it can withstand the maximum number of cycles as shown in the above two Figures.

The following Figure shows a Goodman diagram where applied stress is plotted vs. mean stress. The FEM analysis is performed with 5 element approximation. The results show a good match between analytical, experimental [2.1] and FEM results.

![Goodman Diagram](image)

Figure 2.21: Goodman Diagram for Al 6061-T6 with 5 Element Analyses
2.6 Conclusion

The new finite elements (rod and beam) provide a useful tool for fatigue life prediction in structural components like gas engine turbine blades. The accurate prediction of number of cycles with new axial and bending finite elements and a good match of results to experimental data and analytical results [2.1] signifies that new finite elements provide fatigue life prediction axial and bending loads with considerable accuracy.

These new axial (rod) and bending (beam) elements are developed from a fatigue based constitutive law. The fact that these elements incorporate the fatigue mechanism into the analysis procedure differentiates these new developments from the existing finite element procedure [2.10-2.12]. The new fatigue finite elements predict the crack initiation whereas most research available in this area focuses on crack propagation [2.13-2.23]. Furthermore, the new finite element method is much more useful due to the discrete nature of the finite element method. The new finite element for bending fatigue life prediction has the capability to predict varying number of cycles in the structural component experiencing variable stress at different locations. The colored plots can be obtained where each color signifies respective fatigue life for each element present at different locations in the structure.
2.7 List of References


CHAPTER 3

A NEW ENERGY-BASED MULTIAXIAL FINITE ELEMENT ANALYSIS FATIGUE LIFE FINITE ELEMENT ANALYSIS PREDICTION PROCEDURE

3.1 Introduction

The Goodman diagram and the modified Goodman diagram are based on uniaxial fatigue test data. However, in the real world, machine components and structures experience biaxial and multiaxial stresses. In this Chapter, the energy expressions that construct the constitutive law presented by Equations 1.1 and 1.2 are equated in the form of total strain energy and the distortion energy dissipated in a fatigue cycle. The resulting Equation is further evaluated to obtain the equivalent stress per cycle using energy based methodologies. The equivalent stress expressions are developed both for biaxial and multiaxial fatigue loads and are used to predict the number of cycles to failure based on previously developed prediction criterion. The equivalent stress expressions developed in this Chapter are further used in a new finite element procedure to predict the fatigue life for two and three dimensional structures. The final output of this finite element analysis is in the form of number of cycles to failure for each element on a scale in ascending or descending order. Therefore, the new finite element framework can provide the number
of cycles to failure at each location in gas turbine engine structural components. In order to obtain experimental data for comparison, an Al 6061-T6 plate is tested using a previously developed vibration based testing framework. The finite element analysis is performed for Al 6061-T6 aluminum and the results are compared with experimental results.

As stated in Chapter 1, modern structural components like gas turbine engine blades are designed to be failure free, however, failure does occur and is commonly linked to cyclic fatigue. High cycle fatigue (HCF) is main cause of failure in gas turbine engines [3.1]. Different design tools have been developed to analyze this issue. The most commonly used such tool is a stress versus cycles plot, or S-N curve. These curves provide fatigue strength with respect to time to failure. Other common tools for predicting fatigue properties are the Goodman diagram and the modified Goodman Diagram [3.2], which are the common choices for a failure-free aircraft engine design criterion. In order for designers to make an accurate assessment, the equivalent stress is calculated according to the cyclic loading conditions and compared to a S-N curve or Goodman Diagram to obtain the number of cycles to failure. It is observed during experiments, that apart from uniaxial data, bench test data using components or blades also includes multiaxial results. This has led to the search for a more realistic method for design comparison than the existing uniaxial design tools, which begins by observing the association between material failure/fracture and the energy dissipated during the process.

The research presented in [3.3-3.6] includes a vibration based test method for fatigue life data acquisition and a new failure criterion. As stated in previous Chapters, the new failure criterion [3.3] includes stress-strain relationship both for monotonic as
well as cyclic loadings. This failure criterion is further used to develop fracture energy
and cyclic energy expression. These expressions are used to determine the failure energy
and energy dissipated per cycle. Comparison of total fracture energy to cyclic energy
dissipated per cycle yields the number of cycles to failure. In this Chapter, the failure
criterion presented in [3.3] is employed in the form of strain energy in order to derive an
equivalent stress expression using the distortion energy theory. The detailed formulation
of this criterion is presented in Section 3.3. Furthermore, a new finite element procedure
is proposed for multiaxial fatigue life prediction. This finite element procedure uses MSC
NASTRAN in order to obtain the multiaxial stress state. These stresses are converted to
equivalent stress using the newly developed equivalent stress expression and ultimately
used to predict the number of cycles to failure. Due to the discrete nature of finite
element method, the new analysis approach can provide the number of cycles to failure
for each element in the structure.

As stated earlier, the conventional approach to fatigue life prediction is based on
S-N curve data, Goodman diagram or the modified Goodman diagram. These tools are
based on uniaxial data. Uniaxial is simply the most basic of stress states a structural
component experiences, and it is also the most common experiment due to the capability
of conventional uniaxial fatigue test machines. However, turbine engine components are
subjected to biaxial and even multiaxial stress states due to their complex geometries and
the complex interaction of aerodynamic, centrifugal, thermal, and vibratory loadings, as
well as fretting at interfaces. In turn, the components are typically designed using the
Goodman Diagram which represents uniaxial test data. Therefore, designers must
convert the biaxial or multiaxial stress states the components experience into equivalent
uniaxial stresses to utilize the Goodman Diagram in their failure-free fatigue design
procedure. However, fatigue failures still occur in the test and development phase of new turbine engine design, as well as high-time in-service engines in both the military and commercial fleets. There is concern in the industry that one of the reasons for these failures is that the fatigue mechanisms are different for biaxial or multiaxial stress state than they are for a uniaxial stress state. In other words, the fatigue process due to biaxial or multiaxial stress are considerably different from that of the uniaxial stress case and therefore not adequately captured by assuming an equivalent uniaxial stress. Hence, there is an increasing recognition of the need for fatigue data under biaxial and multiaxial stress states. In order to obtain the experimental data for comparison in this study, an Aluminum 6061-T6 winger plate as shown in Figure 3.1 in Section 3.2, is tested using a previously developed vibration based fatigue testing framework [3.4].

Finite element analysis using the equivalent stress approach is performed for the winger and the results are compared to experimental results in order to validate the analysis procedure. The analysis is also performed for a curved plate, a turbine engine blade like structure, to demonstrate application of this new procedure to a real world problem. An additional set of comparison is presented with an experimental data for biaxial fatigue prediction for type 304 stainless steel. [3.7, 3.8]

3.2 Brief Overview of Experimental Results

The specimen development and biaxial fatigue test data discussed in the following Sections are a brief overview of work presented in [3.7-3.9].

3.2.1 Specimen Development

Fatigue data under biaxial or multiaxial stress is difficult and expensive to obtain using conventional machines, and nearly impossible to obtain at high frequencies. Fortunately, fully reversed bending biaxial data can be obtained using vibration based
testing [3.4]. It is a matter of finding a plate geometry that has a mode shape with a biaxial stress state located away from the clamp or boundary, while at the same time, the von Mises stress in this biaxial stress region is at least a factor of two higher than the von Mises stress elsewhere in the plate. This ensures the fatigue crack will develop in the desired location. The design methodology adopted for optimized design of the specimen is explained in [3.10]. A cantilevered geometry named Winger, shown in Figure 3.1, was identified as the most suitable geometry for these vibration based experiments.

Figure 3.1: Biaxial Specimen Winger [3.9]
Figure 3.2: FEM Results (Clockwise from Top Left, Displacement, von Mises, X-axis and Y-Axis Stresses) for Winger (0.125” thick Al 6061-T6)

The FEM analysis of the Winger geometry, when constructed from 0.125” Al 6061-T6, is shown in Figure 3.2. The von Mises stress plot at the upper right of Figure 3.2 shows that this geometry has significantly reduced stresses along the clamped edge and it is approximately a factor of 2 less than that in the fatigue region which meets the previously stated goal. A biaxial stress field with a ratio of $\sigma_Y / \sigma_X \approx 0.59$ is developed in the upper center of the plate as shown by the x and y stress plots in the lower left and right, respectively, of Figure 3.2. The scanning laser vibrometry results in Figure 3.3 represent mode 7 of a 0.125” thick Al 6061-T6 specimen machined using the winger geometry. The vibrometry results compare favorably with the FEM mode shape in the top left quadrant of Figure 3.2.
3.2.2 Biaxial Fatigue Tests

Biaxial fatigue experiments were conducted on the winger specimen made of Al 6061-T6, 0.125” thick aluminum sheet. The step test method developed by Maxwell and Nicholas [3.11] were used to obtain fatigue data. To allow for comparison with uniaxial stress fatigue results the biaxial fatigue strength results are given in the form of the von Mises equivalent stress given by

$$\sigma_{equiv} = \frac{1}{\sqrt{2}} \sqrt{\left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]}$$

(3.1)

where $\sigma_1$, $\sigma_2$, and $\sigma_3$ are the principal stresses [3.12]. In the biaxial case in the fatigue zone of the Alumwinger3 geometry, $\sigma_3 = 0$ and $\sigma_1$ and $\sigma_2$ are nonzero, and experimentally determined from laser data calibrated to data from a delta rosette strain.
gage applied at the fatigue zone. After accumulating the desired number of cycles, the stress level was increased incrementally until failure occurred at which point the fatigue strength was determined using the following Equation from [3.3].

\[ \sigma_A = \sigma_{pr} + \frac{N_f}{10^6} \left( \sigma_f - \sigma_{pr} \right) \]

(3.2)

where in the biaxial case the stress terms in Equation 3.2 are substituted with the corresponding von Mises equivalent stresses using Equation 3.1. The vibration-based fatigue results at $10^6$ cycles for two Al 6061-T6 biaxial test specimens are given in Table 3.1.

<table>
<thead>
<tr>
<th>Plate #</th>
<th>$\sigma_{pr}$ (ksi)</th>
<th>$\sigma_f$ (ksi)</th>
<th>$(\sigma_A)_{equiv}$ (ksi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>#4</td>
<td>15</td>
<td>20</td>
<td>18.8</td>
</tr>
<tr>
<td>#5</td>
<td>15</td>
<td>20</td>
<td>19.4</td>
</tr>
</tbody>
</table>

Table 3.1: Biaxial Fatigue Experimental Results for Al 6061-T6 for $10^6$ Cycles [3.9]

The averaged von Mises equivalent stress biaxial fatigue results are compared to the previous uniaxial fatigue results average for Al 6061-T6 [3.3] and are presented in Table 3.2.
Table 3.2: Comparison of Uniaxial and Biaxial Al 6061-T6 Data for $10^6$ Cycles [3.9]

Table 3.2 shows that the von Mises equivalent fatigue strength results for a biaxial stress state are not consistent and significantly less than those obtained for uniaxial stress. This is typical of many aerospace materials, and represents a significant challenge to designers who must rely on uniaxial fatigue data and indicates the need of obtaining biaxial fatigue data sets suitable for design.

### 3.2.3 Biaxial Data for Fatigue Tests on Type 304 Stainless Steel

References [3.7, 3.8] present biaxial fatigue data on a specimen shown in Figure 3.4. The material of the specimen is type 304 stainless steel. The experimental procedures are discussed in [3.7, 3.8] in detail. The stress-strain and S-N data is shown in Figures 3.5 and 3.6.

![Figure 3.4: The Geometry of the Tubular Specimen [3.7, 3.8]](image)
Figure 3.5: Stress-Strain Experimental Data for Type 304 Stainless Steel [3.7, 3.8]

Figure 3.6: Fatigue Life Prediction Experimental Data for Type 304 Stainless Steel [3.7, 3.8]
Data presented in Figure 3.5 is curve-fit using the new constitutive law presented in [3.3] and the material parameters $\sigma_c$, $\sigma_o$, $\varepsilon_o$, and C are obtained for type 304 Stainless Steel using the procedures from [3.3]. These material constants are used in analysis for the stainless steel using the new equivalent stress expression presented in the next Section and fatigue life prediction comparison is made to data presented in Figure 3.6.

3.3 Formulation of Equivalent Stress Expressions

This Section presents the formulation of an equivalent stress expression based on the distortion energy theory. As stated in Section 3.1, the research presented in [3.3-3.6] includes a new failure criterion. The new failure criterion is based on the monotonic and cyclic stress-strain representation expressed by the following Equations:

\[
\varepsilon = \frac{\sigma}{E} + \varepsilon_o \sinh\left(\frac{\sigma}{\sigma_o}\right) \quad (3.3)
\]

\[
\varepsilon = \frac{\sigma_{pp}}{E} + \frac{1}{C} \sinh\left(\frac{\sigma_{pp}}{\sigma_c}\right) \quad (3.4)
\]

Where the parameters displayed are defined as follows: $\sigma$ is the value for stress at the surface of the specimen (in the bending case, max stress), $\varepsilon$ is the strain corresponding to the stress $\sigma$, $\sigma_{pp}$ is the peak to peak stress (2$\sigma$ when stress ratio is -1.0), E is the modulus of elasticity, and the variables $\sigma_c$, $\sigma_o$, $\varepsilon_o$, and C are curve fit parameters. These curve fit parameters are obtained through statistical analysis of experimental data [3.3]. Equations 3.3 and 3.4 are used to obtain the total fracture energy and the energy dissipated per cycle. Therefore, the Equations can be applied to the constitutive law to obtain the number of cycles to failure. The above Equations are further used to develop a failure energy expression. Following Equation calculates total energy to failure for a monotonic case [3.3].
\[ W_f = \sigma_n \left( \varepsilon_n - \frac{\varepsilon_n}{2E} \right) + \varepsilon_o \sigma_o \left[ \cosh \left( \frac{\sigma_n}{\sigma_o} \right) - 1 \right] + \frac{\beta_1}{2} (\varepsilon_f^2 - \varepsilon_n^2) + \beta_o (\varepsilon_f - \varepsilon_n) \]  \hspace{1cm} (3.5)

\[ E, \sigma_f, \text{ and } \varepsilon_f \text{ are obtained from experimental monotonic fracture results, } \varepsilon_n, \sigma_n, \beta_1 \text{ and } \beta_o \text{ are curve fit parameters explained in [3.3]. The expression for cyclic energy is given by the following Equation [3.3]. } \sigma_A \text{ is the applied cyclic stress.} \]

\[ W_{\text{cycle}} = \frac{2\sigma_c}{C} \left[ \frac{\sigma_A}{\sigma_c} \sinh \left( \frac{2\sigma_A}{\sigma_c} \right) - \left( \cosh \left( \frac{2\sigma_A}{\sigma_c} \right) - 1 \right) \right] \]  \hspace{1cm} (3.6)

Then number of cycles to failure \( N \) is obtained through comparison of total fracture energy and energy dissipated per cycle during fatigue process [3.3] and is shown in Equation 3.7.

\[ N = \frac{W_f}{W_{\text{cycle}}} \]  \hspace{1cm} (3.7)

Distortion energy is stored in two parts, an energy from change in volume and energy due to the shape change. Equation 3.8 displays the expression for this understanding.

\[ U_T = U_v + U_D \]  \hspace{1cm} (3.8)

The parameters for Equation 3.8 are defined as follows: \( U_T \) is the total strain energy, \( U_v \) is the strain energy of the change in volume, and \( U_D \) is the strain energy for the change in shape.

The strain energy of the change in volume (\( U_v \)) is better known as strain energy caused by hydrostatic stress. The effect of this stress is even volumetric expansion of an infinitesimal element on all sides. The hysteresis loop formed by Equation 3.4 has loading and unloading curves which are a mirror image of each others. Application of this
Equation to the cyclic loading process yields a net zero hydrostatic. Therefore, during one cycle of a cyclic loading procedure, the hydrostatic stress returns to zero and the corresponding strain energy \(U_v\) becomes zero. In other words, only the deviatoric part of stress contributes towards failure \([3.9, 3.13]\). This minimizes Equation 3.8 to Equation 3.9 and shows that the total energy within a hysteresis loop is purely distortion energy.

\[ U_T = U_D \] (3.9)

Furthermore, if it can be justified that the total monotonic strain energy density of a volume subjected to multi-axial loading is the summation of the six areas under the curves formed by the respective stress-strain relationships, it can then be implied that the total strain energy density in one multi-axial cycle is the summation of the directional (3 for Principle-axes and 6 for Cartesian-axes) hysteresis loops acting on the volume. Therefore, to acquire an equivalent stress value from Equation 3.9, this implication is displayed for principle axes by Equation 3.10: where \(\varepsilon_E\) & \(\varepsilon_i\) are functions of stress shown by Equations 3.14 and 3.23 respectively, \(\sigma_{pp,E}\) & \(\sigma_{pp,i}\) are the generalized stress values corresponding to the generalized cyclic strains \(\varepsilon_E\) and \(\varepsilon_i\) (i.e. the minimum fully reversed point is observed as the origin), and \(i\) denotes the principle direction, which are values from 1-3.

\[ \sigma_{pp,E} \varepsilon_E - 2 \int_0^{\sigma_{pp,E}} \varepsilon_E d\sigma_{pp,E} = \sum_{i=1}^3 \sigma_{pp,i} \varepsilon_i - 2 \int_0^{\sigma_{pp,i}} \varepsilon_i d\sigma_{pp,i} \] (3.10)

Equation 3.10 can further be modified to include the Poisson’s ratio effect using Hook’s Law relations \([3.14]\).

\[ \varepsilon_1 = \frac{1}{E} \left[ \sigma_1 - \nu (\sigma_2 + \sigma_3) \right] \] (3.11)
\[ \varepsilon_2 = \frac{1}{E} \left[ \sigma_2 - \nu(\sigma_1 + \sigma_3) \right] \]  
(3.12)

\[ \varepsilon_3 = \frac{1}{E} \left[ \sigma_3 - \nu(\sigma_1 + \sigma_2) \right] \]  
(3.13)

Proceeding evaluation of Equation 3.10 and including the Poisson’s ratio effect, the expression for Equation 3.14 is acquired. The components of this Equation are given in Equation 3.15 to 3.22.

\[ \frac{\sigma_{PP,E}}{C} \sinh \left( \frac{\sigma_{PP,E}}{\sigma_c} \right) - \frac{\sigma_c}{C} \int_0^1 \sinh \left( \frac{\sigma_{PP,E}}{\sigma_c} \right) d\sigma_{PP,E} = A - B \]  
(3.14)

Where

\[ A = \sum_{i=1}^3 \frac{1}{C} \left[ A_i - \nu(A_2 + A_3) \right] \]  
(3.15)

\[ A_1 = \sigma_{PP,j} \sinh \left( \frac{\sigma_{PP,j}}{\sigma_c} \right) \]  
(3.16)

\[ A_2 = \sigma_{PP,j} \sinh \left( \frac{\sigma_{PP,j}}{\sigma_c} \right) \]  
(3.17)

\[ A_3 = \sigma_{PP,k} \sinh \left( \frac{\sigma_{PP,k}}{\sigma_c} \right) \]  
(3.18)

\[ B = \sum_{i=1}^2 \frac{2}{C} \left[ B_1 - \nu(B_2 + B_3) \right] \]  
(3.19)

\[ B_1 = \int_0^1 \sinh \left( \frac{\sigma_{PP,j}}{\sigma_c} \right) d\sigma_{PP,j} \]  
(3.20)

\[ B_2 = \int_0^1 \sinh \left( \frac{\sigma_{PP,j}}{\sigma_c} \right) d\sigma_{PP,j} \]  
(3.21)

\[ B_3 = \int_0^1 \sinh \left( \frac{\sigma_{PP,k}}{\sigma_c} \right) d\sigma_{PP,k} \]  
(3.22)
To simplify this expression for use in determination of equivalent stress, the summation of the integrated expression on the right side of Equation 3.14 is assumed to be equal to the integrated portion of the equivalent stress calculation on the left side. Therefore, following integration of both sides and replacing hyperbolic trigonometry with exponential functions, the expression in Equation 3.23 is acquired.

\[
\frac{2\sigma_{E}}{\sigma_{c}} + e^{-\frac{2\sigma_{E}}{\sigma_{c}}} = 1 + \frac{1}{3} \sum_{i=1}^{3} \left( e^{-\frac{2\sigma_{E}}{\sigma_{c}}} \left( \sigma_{i} - \nu \left( \sigma_{j} + \sigma_{k} \right) \right) + e^{-\frac{2\sigma_{E}}{\sigma_{c}}} \left( \sigma_{i} - \nu \left( \sigma_{j} + \sigma_{k} \right) \right) \right)
\]  

(3.23)

Upon further analysis of Equation 3.23, two assumptions are made: (1) the numerical portion has a minimal effect in the Equation, and the exponential functions of similar signs on both sides of the Equation are equivalent. In other words, Equation 3.24 is assumed to be a true expression.

\[
\frac{2\sigma_{E}}{\sigma_{c}} = \frac{1}{3} \sum_{i=1}^{3} \left( e^{-\frac{2\sigma_{E}}{\sigma_{c}}} \left( \sigma_{i} - \nu \left( \sigma_{j} + \sigma_{k} \right) \right) \right)
\]  

(3.24)

The resulting expression for calculating equivalent stress using the energy-based methodology is displayed by Equation 3.25.

\[
\sigma_{E} = \frac{\sigma_{c}}{2} \ln \left( \frac{1}{3} \sum_{i=1}^{3} e^{-\frac{2\sigma_{E}}{\sigma_{c}}} \left( \sigma_{i} - \nu \left( \sigma_{j} + \sigma_{k} \right) \right) \right)
\]  

(3.25)

The number of cycles to failure can be acquired by inserting the equivalent stress expression into Equation 3.7. The new equivalent stress expression appears in the denominator in Equation 3.26.
\[
N = C \frac{\sigma_n \left( \frac{\varepsilon_n - \sigma_n}{2E} \right) + \varepsilon_0 \sigma_0 \left[ \cosh \left( \frac{\sigma_n}{\sigma_0} \right) - 1 \right] + \frac{\beta_1}{2} \left( \varepsilon_f^2 - \varepsilon_n^2 \right) + \beta_0 \left( \varepsilon_f - \varepsilon_n \right)}{2\sigma_c \left[ \frac{\sigma}{\sigma_c} - \left( \frac{2\sigma}{\sigma_c} - \cosh \left( \frac{2\sigma}{\sigma_c} \right) - 1 \right) \right]}
\] (3.26)

### 3.4 Finite Element Procedures

A traditional HCF gas turbine blading system design procedure based on conventional fatigue life prediction approach as shown schematically in Figure 1.1 is reproduced here in Figure 3.7. As discussed in Chapter 1, this design process usually consists of a structural dynamics analysis to determine natural frequencies and mode shapes at certain operating speed ranges and a stress analysis using a finite element based tool such as MSC NASTRAN and ANSYS [3.15-3.17] to calculate the dynamic stress distribution for identifying the maximum vibratory stress location or area under a series of given excitations. Once the maximum stresses for each vibration mode are determined, high cycle fatigue assessment can be achieved by measuring the margin between the maximum vibratory stress and the material fatigue capability which is approximated by a straight line drawn between the mean ultimate strength at zero vibratory stress and mean fatigue strength at \(10^7\) cycles (or infinite life). A typical Goodman diagram for the titanium alloy Ti-6Al-4V is shown in Figure 3.8 [3.10], constructed using uniaxial fatigue data.
Figure 3.7: Conventional Finite Element Analysis Approach to Fatigue Life Prediction

Figure 3.8: Typical Goodman (or Haigh) Diagram for Ti-6Al-4V for $10^7$ cycles [3.10]
The finite element procedure for this research consists of performing a free vibration analysis using MSC NASTRAN in order to identify the suitable natural mode for excitation. The requirements for selection of the suitable mode are the same as stated in Section 2 while selecting the suitable geometry for experiments. The next step is to excite the identified mode through a frequency response finite element analysis in MSC NASTRAN and obtain the vibratory stresses for each element present at various locations in the structure. These stresses are converted to equivalent stress using newly developed equivalent stress expression in Equation 3.25. Finally, the number of cycles to failure is obtained from Equation 3.26 providing fatigue life for every location in the structure. The analysis generates two kinds of output files. The text format output file contains the number of cycles to failure data for each element in the structure. The binary output file contains failure cycles data which is plotted in the form of colored contour where different colors represent the respective fatigue life for the structure. In this finite element analysis procedure, Altair Hypermesh is used for pre-processing, MSC NASTRAN is used as a finite element solver, Altair Hyperview and Matlab are used as post-processors.

The biaxial fatigue experimental results generated for an Al6061-T6 wing plate are already discussed in Section 3.2. A finite element model of this plate is shown in Figure 3.9.
As discussed in Section 3.2, the free vibration analysis of Winger plate is performed and mode 7 is identified as the appropriate mode for this analysis. In the next step, the forced response analysis is performed and vibratory stresses are obtained for the same mode at the respective frequency. The structural damping is used as the damping factor for the analysis and inertial and thermal effects are considered to be negligible. These stresses are converted to equivalent stress and ultimately to number of cycles to failure for the Winger plate analysis. The results from this analysis are presented in the following Section. These fatigue life results are compared to experimental results presented in Section 3.2 for biaxial fatigue life for AL6061-T6.

In order to apply this new approach to more real world applications, a turbine engine blade like curved plate structure is also analyzed. The profile, geometry and a finite element model of this curved plate are shown in the following Figures.
Figure 3.10: Outer Profile of Curved Plate (Al 6061-T6)

Figure 3.11: Geometry of Curved Plate (Al 6061-T6)

Figure 3.12: A Finite Element Mesh of Curved Plate (Al 6061-T6)
The finite element analysis results for curved plate are presented in the following Section. Another set of analysis is performed to obtain fatigue life prediction data for comparison with type 304 stainless steel data as discussed in Section 3.2.3. The results are presented in the following Section.

3.5 Results and Discussion

Figure 3.10 shows mode shape 7 for Winger Plate (Al6061-T6). The results are obtained from MSC NASTRAN free vibration finite element analysis. Mode 7 frequency is 1312 Hz.

Figure 3.13: Displacement (Mode #7, Frequency=1312Hz) Results for Winger Plate (Al 6061-T6)

Figure 3.14 shows the vibratory stress results from the MSC NASTRAN forced response finite element analysis results. The Winger Plate has been excited to a maximum stress level of 25 ksi.
Figure 3.14: Vibratory Stress Results for Winger Plate (Al 6061-T6)

The fatigue results for the Winger plate are shown in Figure 3.15. The cycles are plotted on a reverse scale with minimum cycles to failure shown at the top of the scale in red color and the maximum cycles at the bottom of the scale in blue color. The high stresses in the winger plate are concentrated away from the fixed end as shown in Figure 3.14. Therefore, the low cycles is also observed in the same area.
Figure 3.15: Numbers of Cycle to Failure Results for Winger Plate (Al 6061-T6)

Table 3.3 shows the comparison of experimental and finite element analysis results for Al6061-T6 for $10^6$ cycles. These results are obtained from the text output file generated by the analysis. The FEA fatigue life predictions compare well with the experimental results thus validating the newly developed equivalent stress expression and the new analysis approach.

<table>
<thead>
<tr>
<th></th>
<th>Von Mises Stress (ksi)</th>
<th>Experiment Life Cycles</th>
<th>FE Analysis Life Cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winger Plate</td>
<td>19.1</td>
<td>$10^6$</td>
<td>$1.26 \times 10^6$</td>
</tr>
</tbody>
</table>

Table 3.3: Biaxial Fatigue Results for Al 6061-T6 for $10^6$ Cycles
In order to present an application of the new equivalent stress approach to a three-dimensional structure, the curved plate discussed in Section 3.4 is analyzed using the new analysis approach. Figure 3.16 shows the desired mode shape for the curved plate.

![Curved Plate Analysis](image)

Figure 3.16: Displacement (Frequency=12444Hz) Results for Curved Plate (Al 6061-T6)

The vibratory stress results are shown in Figure 3.17. The stresses are concentrated in the area away from the fixed end of the plate. The red color represents the locations with high stress.
Figure 3.17: Vibratory Stress Results for Curved Plate (Al 6061-T6)

Figure 3.18 shows the fatigue prediction results from the finite element analysis of the curved plate. The low cycles are observed in the same area where the maximum stresses are present.

Figure 3.18: Numbers of Cycle Results for Curved Plate (Al 6061-T6)
Figure 3.19 shows the fatigue life prediction comparison between type 304 stainless steel data discussed in Section 3.2.3. The results compare well with the experimental data [3.7, 3.8] and are also presented in Table 3.4. The data shows a closer comparison to the experimental results as compared to the data presented in Table 3.3. The Type 304 stainless steel specimen is analyzed for low cycle fatigue. As stated in Chapter 1, the error for finite element procedures developed in this research is large for the high cycle fatigue as compared to the low cycle fatigue.

Figure 3.19: Numbers of Cycle Results for Type 304 Stainless Steel [3.7, 3.8]
<table>
<thead>
<tr>
<th></th>
<th>Stress (ksi)</th>
<th>Experiment Life Cycles</th>
<th>FE Analysis Life Cycles</th>
<th>Percent Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>43.5</td>
<td>5044</td>
<td>5250</td>
<td>4.1</td>
</tr>
<tr>
<td>2</td>
<td>46.4</td>
<td>4234</td>
<td>4401</td>
<td>3.95</td>
</tr>
<tr>
<td>3</td>
<td>50.75</td>
<td>3767</td>
<td>3915</td>
<td>3.94</td>
</tr>
</tbody>
</table>

Table 3.4: Biaxial Fatigue Results for Type 304 Stainless Steel

3.6 Conclusions

The development of the equivalent stress expression based on an improved fatigue life prediction criterion and application of this new expression in a finite element procedure provides a useful tool for fatigue life prediction in gas turbine engine structural components. The prediction of number of cycles with the new finite element procedure and a comparison of results to experimental data signifies that the new finite element procedure provides encouraging estimation of number of cycles for biaxial fatigue loading.

Furthermore, the new finite element procedure is much more useful due to the discrete nature of the finite element method. The capability of this new approach to predict fatigue life for each location in the gas turbine engine structural component provides a complete visual picture of the fatiguing process in the component.

The equivalent stress expression developed in this research is applicable to multiaxial fatigue as well. Therefore, the finite element procedure applied to biaxial fatigue can easily be extended to multiaxial fatigue process. However, this requires a generation of multiaxial experimental fatigue data in order to validate the results.
3.7 List of References


CHAPTER 4

NEW QUADRILATERAL IN-PLANE AND PLATE ELEMENTS FOR FATIGUE LIFE PREDICTION

4.1 Introduction

This Chapter presents the development of a four-node Quadrilateral (QUAD-4) fatigue element. The new fatigue element is further combined with the beam element developed in Chapter 2 in order to obtain a plate element. The constitutive Equations presented by energy-based framework developed in [4.1-4.5] and discussed in previous Chapters are integrated into a minimum potential energy expression to develop the four-node QUAD element. This element has the capability to predict the number of cycles to failure for plates subjected to in-plane stresses. The element is benchmarked with the previously developed uniaxial tension/compression problem in order to verify the new development. The benchmarking procedures are discussed in detail in Section 4.3.
The newly developed QUAD-4 element is further modified by adding the extra degrees of freedom of a beam element to obtain the plate element. With added bending and rotational degrees of freedom, the plate element can be used to model plates subjected to biaxial fatigue including bending loads.

The analysis of the Winger Plate discussed in Chapter 3, is performed using the new QUAD-4 plate element. The results are compared to experimental data and the equivalent stress approach analysis results presented in Sections 3.2 and 3.5 in Chapter 3.

4.2 Finite Element Procedures for QUAD-4 (In-Plane) Element

The following Equations present the uniaxial and shear constitutive laws [4.1, 4.5] both for uniaxial monotonic and cyclic loadings respectively.

\[
\varepsilon_{\text{monotonic}} = \frac{\sigma}{E} + \varepsilon_0 \sinh \left( \frac{\sigma}{\sigma_0} \right) \tag{4.1}
\]

\[
\varepsilon_{\text{cyclic}} = \frac{\sigma_{PP}}{E} + \frac{1}{C} \sinh \left( \frac{\sigma_{PP}}{\sigma_c} \right) \tag{4.2}
\]

Equations 4.3 and 4.4 represent stress-strain relationships for shear monotonic and cyclic loads respectively.

\[
\gamma_{\text{monotonic}} = \frac{\tau}{G} + \gamma_0 \sinh \left( \frac{\tau}{\tau_0} \right) \tag{4.3}
\]

\[
\gamma_{\text{cyclic}} = \frac{\tau_{PP}}{G} + \frac{1}{C_s} \sinh \left( \frac{\tau_{PP}}{\tau_c} \right) \tag{4.4}
\]

The respective inverse relationships for stress are given by the following Equations,

\[
\varepsilon_{\text{em}} = \frac{\sigma}{E} \tag{4.5}
\]
Where the parameters displayed in Equations 4.6 to 4.13 are defined as follows: $\sigma$ is the value for stress at the surface of the specimen (in the bending case, max stress), $\varepsilon$ and $\gamma$ are the strain corresponding to the stress $\sigma$ and $\tau$, $\sigma_{pp}$ and $\tau_{pp}$ is the peak to peak stress ($2\sigma$ when stress ratio is -1.0), $E$ is the modulus of elasticity, and the variables $\sigma_c$, $\sigma_o$, $\varepsilon_o$, $\tau_c$, $\tau_o$, $\gamma_o$ and $C_s$ are curve fit parameters [4.1]. The subscripts em & pm designate the elastic & plastic cases for monotonic loading, and the subscripts ec & pc designate the elastic & plastic cases for cyclic loading.

The following Equations present the elastic constitutive laws used for the QUAD4 fatigue element development. The elastic constitutive law is a classical Equation available in literature [4.6, 4.7].
\{\sigma\} = \[D]\{\varepsilon\} \quad (4.14)

Where \(\sigma\) is stress given by Equation 4.15 with x-axis, y-axis and shear components, \(\varepsilon\) is strain given by Equation 4.16 with \(\varepsilon_x\) and \(\varepsilon_y\) the x-axis and y-axis components and \(\gamma_{xy}\) the shear component.

\[
\sigma = \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix} \quad (4.15)
\]

and

\[
\varepsilon = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} \quad (4.16)
\]

and \(D\) is given by Equations 4.17 and 4.18 for plane stress and plane strain conditions respectively.

\[
D = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix} \quad (4.17)
\]

\[
D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 \\
\nu & 1-\nu & 0 \\
0 & 0 & \frac{1}{2}-\nu
\end{bmatrix} \quad (4.18)
\]

If a two dimensional (2-D) stress tensor is defined as following,

\[
\{\sigma_{ij}\} = \begin{bmatrix}
\sigma_{xx} & \tau_{xy} \\
\tau_{yx} & \sigma_{yy}
\end{bmatrix} \quad (4.19)
\]
The corresponding stress elements for plastic part of Equations 4.1 to 4.4 are given by the following Equations. Equations 4.15 and 4.19 to 4.22 are integrated into minimum potential energy formulation to obtain the new K-matrices for fatigue life prediction.

\[
\sigma_{pm-xx} = \sigma_0 \sinh^{-1} \left( \frac{\varepsilon_{px}}{\varepsilon_0} \right) \quad (4.20)
\]

\[
\sigma_{pm-yy} = \sigma_0 \sinh^{-1} \left( \frac{\varepsilon_{py}}{\varepsilon_0} \right) \quad (4.21)
\]

\[
\tau_{pm-xy} = \tau_0 \sinh^{-1} \left( \frac{\gamma_{pxy}}{\gamma_0} \right) \quad (4.22)
\]

A four-node QUAD element is shown in the following Figure. The element has four nodes with each node having two degrees of freedom, displacements in x and y directions.

![Figure 4.1: A Four Node QUAD Element](image-url)
Nodal displacement vector is denoted by $d$ and $d_s$ are x and y displacements at each node. The displacement at any point within the element is denoted by $u= [u(x,y), v(x,y)]^T$.

$$d = [d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8]^T$$  \hspace{1cm} (4.23)

Figure 4.2 shows the QUAD-4 element defined in $\xi - \eta$ coordinates or natural coordinates and is squared shape. These displacements are approximated using the Lagrange shape functions which satisfy nodal value and continuity requirements.

![QUAD-4 Element in $\xi - \eta$ Space](image)

Figure 4.2: QUAD-4 Element in $\xi - \eta$ Space

The shape functions for this element are given by the following Equations. $N_i$ are the shape functions at each node with $i$ denoting the node numbers as 1, 2, 3 and 4. These shape functions are defined such that $N_i$ is equal to 1 at node $i$ and zero at other nodes.
\[
N_1 = \frac{1}{4}(1-\xi)(1-\eta) \quad (4.24)
\]
\[
N_2 = \frac{1}{4}(1+\xi)(1-\eta) \quad (4.25)
\]
\[
N_3 = \frac{1}{4}(1+\xi)(1+\eta) \quad (4.26)
\]
\[
N_4 = \frac{1}{4}(1-\xi)(1+\eta) \quad (4.27)
\]

The following Equations present the displacement field within the element in terms of nodal values. Equation 4.28 provides the horizontal displacement and equitation 4.29 provides the vertical displacement.

\[
u = N_1d_1 + N_2d_3 + N_3d_5 + N_4d_7 \quad (4.28)
\]
\[
v = N_1d_2 + N_2d_4 + N_3d_6 + N_4d_8 \quad (4.29)
\]

The above Equations can be re-written in matrix form as

\[
u = Nd \quad (4.30)
\]

Where

\[
N = \begin{bmatrix}
N_1, 0, N_2, 0, N_3, 0, N_4, 0 \\
0, N_1, 0, N_2, 0, N_3, 0, N_4 \\
\end{bmatrix} \quad (4.31)
\]

In the isoparametric formulation, we use the same shape functions to express the coordinates of a point within the element in terms of nodal coordinates. Thus,

\[
x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 \quad (4.32)
\]
\[
y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 \quad (4.33)
\]
In order to map the derivatives of a function in x-y coordinates in terms of its derivatives in \( \xi - \eta \) coordinates, following steps are used. Equation 4.34 defines a function \( f \).

\[
f = f(x, y)
\] (4.34)

In view of Equations 4.32 and 4.33, it can be considered as implicit function of \( \xi - \eta \).

\[
f = f[x(\xi, \eta), y(\xi, \eta)]
\] (4.35)

By chain rule of differentiation,

\[
\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \xi}
\]

(4.36)

\[
\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \eta}
\]

(4.37)

Equations 4.36 and 4.37 can be re-written as

\[
\begin{bmatrix}
\frac{\partial f}{\partial \xi} \\
\frac{\partial f}{\partial \eta}
\end{bmatrix} = J \begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial y}
\end{bmatrix}
\]

(4.38)

Where \( J \) is the Jacobian matrix,

\[
J = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}
\end{bmatrix}
\]

(4.39)

In view of Equations 4.23 to 4.27, 4.32 and 4.33, \( J \) is given by Equation 4.40. The elements of this Jacobean matrix are the derivates of these Equations as defined by Equation 4.39.
Equation 4.38 can be written as

\[
\begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial y}
\end{bmatrix}
= J^{-1}
\begin{bmatrix}
\frac{\partial f}{\partial \xi} \\
\frac{\partial f}{\partial \eta}
\end{bmatrix}
\] (4.42)

or

\[
\begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial y}
\end{bmatrix}
= \frac{1}{\det J}
\begin{bmatrix}
J_{11} & -J_{12} \\
-J_{21} & J_{22}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial f}{\partial \xi} \\
\frac{\partial f}{\partial \eta}
\end{bmatrix}
\] (4.43)

An additional result required for derivation of the QUAD 4 element stiffness matrix along with the above expressions, is given below.

\[
dx\,d\,y = \det J\,d\xi\,d\eta
\] (4.44)

This result is used to express the area of the element and the proof for this result is given in [4.7].

Integration of the elastic case, Equation 4.14 into potential energy formulation is a classical finite element problem and is already available in literature [4.6]. Integration of Equation 4.19 into Equation 4.45 provides a new K-matrix for the plastic part of the constitutive law fatigue analysis.

\[
\Pi = \int \sigma : \varepsilon \,dV - \int u^T dV - \int u^T \,dx - \sum u_i P_i
\] (4.45)
Where $\Pi$ is the minimum potential energy, $\sigma$ is the stress tensor, $\varepsilon$ is the strain vector, $u$ is the displacement, $f$ is the body force, $T$ is the traction force, and $P_i$ is the point load. $V$ is the volume and $x$ denotes the length of the element.

The strain energy term is given by

$$U = \int \sigma^T \varepsilon dV$$

$$U = \sum t_e \int \sigma^T \varepsilon dA$$

where $t_e$ is the thickness of element.

The strain displacement relations are given by

$$\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix}$$

From Equation 4.43 we have,

$$\frac{\partial u}{\partial x} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \frac{\partial u}{\partial \xi} = \frac{1}{\det J} \begin{bmatrix} \partial u \\ \partial v \end{bmatrix}$$

Similarly,

$$\frac{\partial v}{\partial x} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \frac{\partial v}{\partial \xi} = \frac{1}{\det J} \begin{bmatrix} \partial v \\ \partial u \end{bmatrix}$$
\[ \varepsilon = A \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} \] (4.51)

Where \( A \) is given by

\[ A = \frac{1}{\det J} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \] (4.52)

Now from Equations 4.32 and 4.33,

\[ \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = Gd \] (4.53)

Where

\[ G = \frac{1}{4} \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix} \] (4.54)

Equations 4.51 and 4.53 yield

\[ \varepsilon = Bd \] (4.55)

\[ B = AG \] (4.56)
Equation 4.19 can be written as Equation 4.57 in the tensor form. The three elements of this Tensor are the stresses in the \( xx \), \( yy \) and \( xy \) directions. These stresses consist of the non linear part of the fatigue constitutive law.

\[
\{\sigma\} = \begin{bmatrix}
\sigma_0 \left( \sinh^{-1} \left( \frac{\varepsilon_x}{\varepsilon_0} \right) \right) \\
\sigma_0 \left( \sinh^{-1} \left( \frac{\varepsilon_y}{\varepsilon_0} \right) \right) \\
\tau_0 \sinh^{-1} \left( \frac{\gamma_{xy}}{\gamma_0} \right)
\end{bmatrix}^T (4.57)
\]

From Equations 4.40 and 4.41 for a rectangular element we get,

\[
J_{11} = \frac{L}{2} (4.58)
\]
\[
J_{12} = 0 (4.59)
\]
\[
J_{21} = 0 (4.60)
\]
\[
J_{22} = \frac{W}{2} (4.61)
\]
\[
\text{det} J = \frac{L \times W}{4} (4.62)
\]

Where \( L \) and \( W \) are the length and width of the rectangular element.

The strain displacement relationships become

\[
\varepsilon_x = \frac{1}{2L} \left[ -d_1 + d_3 + d_5 - d_7 + (d_1 - d_3 + d_5 - d_7)\eta \right] (4.63)
\]
\[
\varepsilon_y = \frac{1}{2W} \left[ -d_2 - d_4 + d_6 + d_8 + (d_2 - d_4 + d_6 - d_8)\xi \right] (4.64)
\]
\[
\gamma_{xy} = \frac{1}{2W} \left[ -d_1 - d_3 + d_5 + d_7 + (d_1 - d_3 + d_5 - d_7)\zeta \right] + \frac{1}{2L} \left[ -d_2 + d_4 + d_6 - d_8 + (d_2 - d_4 + d_6 - d_8)\eta \right] (4.65)
\]
Rewriting Equation 4.47 by using results from Equations 4.48 to 4.62 yields Equation 4.66. Further simplification of Equation 4.66 to Equation 4.67, 4.68 and 4.69 provide the stiffness matrix.

\[
U = \sum t_e \frac{d^T B^T}{d^T B^T} \left[ \int \int \sigma^T B \det J d\xi d\eta \right] d \tag{4.66}
\]

\[
U = \sum d^T \left[ t_e \int \int \frac{B^T \sigma^T B \det J d\xi d\eta}{d^T B^T} \right] d \tag{4.67}
\]

\[
U = \sum d^T k^e d \tag{4.68}
\]

\[
k^e = t_e \int \int \frac{B^T \sigma^T B \det J d\xi d\eta}{d^T B^T} \tag{4.69}
\]

Numerical integration of Equation 4.69 yields the nonlinear K-Matrix for the QUAD-4 fatigue element. The QUAD 4 K-matrix for linear elastic case is a standard finite element formulation problem and is available in the literature. The QUAD-4 K-Matrices for linear elastic and non-linear plastic parts are represented by the following Equations.

\[
K_{em-Plane} = \frac{1}{J_1} \int \int \left[ \begin{array}{cccccccc}
k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} \\
k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} \\
k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} \\
k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} \\
k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} & k_{57} & k_{58} \\
k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} & k_{67} & k_{68} \\
k_{71} & k_{72} & k_{73} & k_{74} & k_{75} & k_{76} & k_{77} & k_{78} \\
k_{81} & k_{82} & k_{83} & k_{84} & k_{85} & k_{86} & k_{87} & k_{88}
\end{array} \right] d\xi d\eta \tag{4.70}
\]
Where subscripts \( em \) and \( pm \) denote the elastic and plastic cases respectively. The elements of \( \mathbf{K}_{pm-Plane} \) are given in the following Equations. The elements \( Q_i, Q_j, S_{ij}, R_{11} \) and \( R_{12} \) are given in Table 4.1.

\[
\begin{bmatrix}
k_{11_{pm}}(d) & k_{12_{pm}}(d) & k_{13_{pm}}(d) & k_{14_{pm}}(d) & k_{15_{pm}}(d) & k_{16_{pm}}(d) & k_{17_{pm}}(d) & k_{18_{pm}}(d) \\
k_{21_{pm}}(d) & k_{22_{pm}}(d) & k_{23_{pm}}(d) & k_{24_{pm}}(d) & k_{25_{pm}}(d) & k_{26_{pm}}(d) & k_{27_{pm}}(d) & k_{28_{pm}}(d) \\
k_{31_{pm}}(d) & k_{32_{pm}}(d) & k_{33_{pm}}(d) & k_{34_{pm}}(d) & k_{35_{pm}}(d) & k_{36_{pm}}(d) & k_{37_{pm}}(d) & k_{38_{pm}}(d) \\
k_{41_{pm}}(d) & k_{42_{pm}}(d) & k_{43_{pm}}(d) & k_{44_{pm}}(d) & k_{45_{pm}}(d) & k_{46_{pm}}(d) & k_{47_{pm}}(d) & k_{48_{pm}}(d) \\
k_{51_{pm}}(d) & k_{52_{pm}}(d) & k_{53_{pm}}(d) & k_{54_{pm}}(d) & k_{55_{pm}}(d) & k_{56_{pm}}(d) & k_{57_{pm}}(d) & k_{58_{pm}}(d) \\
k_{61_{pm}}(d) & k_{62_{pm}}(d) & k_{63_{pm}}(d) & k_{64_{pm}}(d) & k_{65_{pm}}(d) & k_{66_{pm}}(d) & k_{67_{pm}}(d) & k_{68_{pm}}(d) \\
k_{71_{pm}}(d) & k_{72_{pm}}(d) & k_{73_{pm}}(d) & k_{74_{pm}}(d) & k_{75_{pm}}(d) & k_{76_{pm}}(d) & k_{77_{pm}}(d) & k_{78_{pm}}(d) \\
k_{81_{pm}}(d) & k_{82_{pm}}(d) & k_{83_{pm}}(d) & k_{84_{pm}}(d) & k_{85_{pm}}(d) & k_{86_{pm}}(d) & k_{87_{pm}}(d) & k_{88_{pm}}(d)
\end{bmatrix}
\begin{bmatrix}d\xi d\eta\end{bmatrix}
\begin{align}
\mathbf{K}_{pm-Plane} &= \frac{1}{\mu} \int \int d\xi d\eta
\end{align}

(4.71)

\[
k_{ij_{pm}} = \frac{1}{4} \left[ \frac{Q_i S_{ij} Q_j}{R_{11}} \right]
\quad \text{for } i = 1, 3, 5, 7 \quad \text{and } j = 1, 3, 5, 7
\]

(4.72)

\[
k_{ij_{pm}} = \frac{1}{4} \left[ \frac{Q_i S_{ij} Q_j}{R_{12}} \right]
\quad \text{for } i = 1, 3, 5, 7 \quad \text{and } j = 2, 4, 6, 8
\]

(4.73)

\[
k_{ij_{pm}} = \frac{1}{4} \left[ \frac{Q_i S_{ij} Q_j}{R_{12}} \right]
\quad \text{for } i = 2, 4, 6, 8 \quad \text{and } j = 2, 4, 6, 8
\]

(4.74)

\[
k_{ij_{pm}} = \frac{1}{4} \left[ \frac{Q_i S_{ij} Q_j}{R_{11}} \right]
\quad \text{for } i = 2, 4, 6, 8 \quad \text{and } j = 1, 3, 5, 7
\]

(4.75)
The similar Equations are developed for cyclic loads. The resulting K-matrices are shown in Equations 4.76 and 4.77. The elements of $K_{pc-Plane}$ are the same as given in Equations 4.72 to 4.75 except that the parameters $\sigma_o$ changes to $\sigma_c$, $\varepsilon_o$ changes to $1/C$, $\tau_o$ changes to $\tau_c$, $\gamma_o$ changes to $1/Cs$ and the applied stress $\sigma$ and $\tau$ changes to peak to peak stress $\sigma_{pp}$ and $\tau_{pp}$.

Table 4.1: Constants for Equations 4.72 to 4.75

<table>
<thead>
<tr>
<th>$i/j$</th>
<th>$Q_{i/j}$</th>
<th>$S_{ij}$</th>
<th>$R_{11}$ and $R_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\frac{W}{2}(1-\eta)$</td>
<td>$S_{ij} = \sigma_0 \sinh^{-1}\left(\frac{\varepsilon_{px}}{\varepsilon_0}\right)$</td>
<td>$R_{11} = \frac{W}{2}\left(-d_{1+4}d_5-d_{7}\right)$</td>
</tr>
<tr>
<td>2</td>
<td>$-\frac{L}{2}(1-\xi)$</td>
<td>$S_{ij} = \sigma_0 \sinh^{-1}\left(\frac{\varepsilon_{py}}{\varepsilon_0}\right)$</td>
<td>$R_{12} = \frac{L}{2}\left(+d_{1-4}d_4-d_{8}\right)$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{W}{2}(1-\eta)$</td>
<td>$S_{ij} = \sigma_0 \sinh^{-1}\left(\frac{\gamma_{pzy}}{\gamma_0}\right)$</td>
<td>$R_{11} = \frac{W}{2}\left(-d_{1+4}d_5-d_{7}\right)$</td>
</tr>
<tr>
<td>4</td>
<td>$-\frac{L}{2}(1+\xi)$</td>
<td>$S_{ij} = \sigma_0 \sinh^{-1}\left(\frac{\gamma_{pzy}}{\gamma_0}\right)$</td>
<td>$R_{12} = \frac{L}{2}\left(+d_{1-4}d_4-d_{8}\right)$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{W}{2}(1+\eta)$</td>
<td>$S_{ij} = \tau_0 \sinh^{-1}\left(\frac{\gamma_{pzy}}{\gamma_0}\right)$</td>
<td>$R_{11} = \frac{W}{2}\left(-d_{1+4}d_5-d_{7}\right)$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{L}{2}(1+\xi)$</td>
<td>$S_{ij} = \tau_0 \sinh^{-1}\left(\frac{\gamma_{pzy}}{\gamma_0}\right)$</td>
<td>$R_{12} = \frac{L}{2}\left(+d_{1-4}d_4-d_{8}\right)$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{W}{2}(1+\eta)$</td>
<td>$S_{ij} = \tau_0 \sinh^{-1}\left(\frac{\gamma_{pzy}}{\gamma_0}\right)$</td>
<td>$R_{11} = \frac{W}{2}\left(-d_{1+4}d_5-d_{7}\right)$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{L}{2}(1-\xi)$</td>
<td>$S_{ij} = \tau_0 \sinh^{-1}\left(\frac{\gamma_{pzy}}{\gamma_0}\right)$</td>
<td>$R_{12} = \frac{L}{2}\left(+d_{1-4}d_4-d_{8}\right)$</td>
</tr>
</tbody>
</table>
Equations 4.71 and 4.77 are non-linear due to presence of “ds” in the resulting K-matrices. To account for the non-linear behavior, the Newton-Raphson iteration method is applied to the analysis [4.8-4.10]. These K-matrices are used in Equation 4.78 to determine the unknown degrees of freedom.

\[
[K] \{d\} = \{F\} \tag{4.78}
\]

The loads are applied from 0 to peak to peak. The results are post-processed using classical FEA techniques. The nodal displacement results are further used to obtain stresses and strains for each element in the structure. The 2-D stresses and strains are converted to equivalent von-Mises stress and strain. These stresses and strains are used to calculate the energy dissipated per cycle and ultimately the number of cycles to failure for each element. The calculations of energy procedures are the same as detailed in Sections 2.3 and 2.4 in Chapter 2. The number of cycles to failure is determined using Equations 2.3 and 2.4 from Chapter 2.
In order to validate the QUAD-4 element, this element is benchmarked against a uniaxial rod element developed in Chapter 2. A 2-D plate is meshed with the new QUAD-4 elements and subjected to uniaxial tension in x-direction in the form of displacement. The mesh discretization of this plate is shown in Figure 4.3. The displacement solution is compared to the solution of a 1-D bar meshed with the rod element and subjected to uniaxial tension. The plate and rod are fixed at left end/side and a unit displacement is applied at the right most end of rod and right most edge of plate. The linear QUAD-4 Fatigue K-matrix results are compared to the linear 1-D rod solution as well as an ANSYS solution of the same problem. The displacement results for linear verification from the QUAD-4 element are shown in the following Figures and are compared in Table 4.2.

Figure 4.3: 6-Element Mesh for 2-D Plate (Dimensions in Inches)
Figure 4.4: Displacement Results using QUAD-4 Element (Dimensions in Inches)

Figure 4.5: Displacement Results using QUAD-4 Element in ANSYS

(Dimensions in Inches)
As stated earlier, the non-linear analysis requires an iterative approach. The non-linear QUAD-4 Fatigue K-matrix results are compared to the linear 1-D rod solution. The results for non-linear analysis are shown in the following Figure and a tabular comparison is made in Table 4.3.

Table 4.2: Displacements-Linear Analysis Benchmarking of QUAD-4 Element.

<table>
<thead>
<tr>
<th></th>
<th>Node 2</th>
<th></th>
<th>Node 3</th>
<th></th>
<th>Node 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-D Bar (in)</td>
<td>0.333</td>
<td>Linear Code</td>
<td>0.333</td>
<td>0.666</td>
<td>0.6666</td>
</tr>
<tr>
<td>ANSYS (in)</td>
<td>0.333</td>
<td>QUAD-4 (in)</td>
<td>0.666</td>
<td>Linear Code</td>
<td>ANSYS (in)</td>
</tr>
<tr>
<td>Iteration No.</td>
<td>Node 2</td>
<td>Node 3</td>
<td>Node 4</td>
<td>Node 5</td>
<td></td>
</tr>
<tr>
<td>--------------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td>--------</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Non-Linear Axial (in)</td>
<td>Non-Linear 2D QUAD-4 (in)</td>
<td>Non-Linear Axial (in)</td>
<td>Non-Linear 2D QUAD-4 (in)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.0636458</td>
<td>0.0636469</td>
<td>0.1351291</td>
<td>0.1351315</td>
<td>1.0000</td>
</tr>
<tr>
<td>2</td>
<td>0.0815309</td>
<td>0.0815353</td>
<td>0.1731019</td>
<td>0.1731112</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.1003474</td>
<td>0.1003337</td>
<td>0.2130519</td>
<td>0.2130229</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.1102611</td>
<td>0.1102664</td>
<td>0.2341001</td>
<td>0.2341113</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.1182828</td>
<td>0.1182878</td>
<td>0.2511314</td>
<td>0.2511419</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.1277408</td>
<td>0.1277448</td>
<td>0.2712119</td>
<td>0.2712204</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>0.2125675</td>
<td>0.2125580</td>
<td>0.4513112</td>
<td>0.4512910</td>
<td></td>
</tr>
<tr>
<td>49</td>
<td>0.3333354</td>
<td>0.3333794</td>
<td>0.6666702</td>
<td>0.6665912</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.3: Displacements – Non Linear Analysis Benchmarking of QUAD-4 Element.

As is evident from the results of Figure 4.4, 4.5 and 4.6 and the Tables 4.2 and 4.3, the QUAD-4 fatigue element analysis compares exactly with the 1-D rod fatigue element. These results successfully complete the benchmarking of the new QUAD-4 fatigue element.
4.3 Finite Element Procedures for QUAD-4 (Plate) Element

This Section presents the development of a new QUAD-4 (Plate) element for biaxial fatigue life prediction. In order to obtain the QUAD-4 (Plate) element, the QUAD-4 (In-Plane) element developed in Section 4.2 and Beam element developed in Chapter 2 are combined with added degrees of freedom to model plate bending as well. Figure 4.7 shows a schematic of a beam element an extra rotational degree of freedom (DOF) required for the plate element. The vertical displacement is given by $v$ and $\theta_y$ are the rotations at each node.

![Figure 4.7: Beam Element with Added DOF (Rotation):](image)
Figure 4.8 shows a schematic diagram of addition of Beam and QUAD-4 elements to form a new plate element. For a beam element, each node has two degrees of freedom. The local degrees of freedom for each element are represented by the following Equation.
\[ d = [d_1, d_2, d_3, d_4] \] (4.79)

The non-linear fatigue Beam element developed in Chapter 2 is shown in Equation 4.80.

\[
K_{pm-B} = \begin{bmatrix}
  k_{11,pm}(d_1) & k_{12,pm}(d_1) & k_{13,pm}(d_1) & k_{14,pm}(d_1) \\
  k_{21,pm}(d_2) & k_{22,pm}(d_2) & k_{23,pm}(d_2) & k_{24,pm}(d_2) \\
  k_{31,pm}(d_3) & k_{32,pm}(d_3) & k_{33,pm}(d_3) & k_{34,pm}(d_4) \\
  k_{41,pm}(d_4) & k_{42,pm}(d_4) & k_{43,pm}(d_4) & k_{44,pm}(d_4)
\end{bmatrix}
\] (4.80)

These elements of \( K_{pm-B} \) in index notation are given by the following Equations.

The constants \( A_i, B_i, C_j \) and \( D_j \) for these Equations are given in Table 4.4.

\[
k_{ij,pm} = \frac{0.866}{d_i} \left( C_j \sinh^{-1}(A_id_i) - D_j \sinh^{-1}(B_id_i) \right) \quad \text{for } i = 1, 2, 3, 4 \text{ and } j = 1, 3
\] (4.81)

\[
k_{ij,pm} = \frac{t_e}{d_i} \left( C_j \sinh^{-1}(A_id_i) - D_j \sinh^{-1}(B_id_i) \right) \quad \text{for } i = 1, 2, 3, 4 \text{ and } j = 2, 4
\] (4.82)
Table 4.4: Constants for Equations 4.81 and 4.82.

For modified beam element, each node is assigned three degrees of freedom. The local degrees of freedom for each element are represented by Equation 4.83. The energy Equation for this element is shown in Equation 4.84.

\[
d = [d_1, d_2, d_3, d_4, d_5, d_6]
\]

(4.83)
The modified beam element with an added degree of freedom is shown in Equation 4.85.

\[
K_{pm-BM} = \begin{bmatrix}
    k_{11pm}(d_1) & k_{12pm}(d_1) & k_{13pm}(d_1) & k_{14pm}(d_1) & k_{15pm}(d_1) & k_{16pm}(d_1) \\
    k_{21pm}(d_2) & k_{22pm}(d_2) & k_{23pm}(d_2) & k_{24pm}(d_2) & k_{25pm}(d_2) & k_{26pm}(d_2) \\
    k_{31pm}(d_3) & k_{32pm}(d_3) & k_{33pm}(d_3) & k_{34pm}(d_3) & k_{35pm}(d_3) & k_{36pm}(d_3) \\
    k_{41pm}(d_4) & k_{42pm}(d_4) & k_{43pm}(d_4) & k_{44pm}(d_4) & k_{45pm}(d_4) & k_{46pm}(d_4) \\
    k_{51pm}(d_5) & k_{52pm}(d_5) & k_{53pm}(d_5) & k_{54pm}(d_5) & k_{55pm}(d_5) & k_{56pm}(d_5) \\
    k_{61pm}(d_6) & k_{62pm}(d_6) & k_{63pm}(d_6) & k_{64pm}(d_6) & k_{65pm}(d_6) & k_{66pm}(d_6)
\end{bmatrix}
\] (4.85)

These elements of \(K_{pm-B}\) in index notation are given by the following Equations.

\[
k_{ij-pm} = \frac{0.866}{d_i} \left( C_j \sinh^{-1}(A_i d_i) - D_j \sinh^{-1}(B_i d_i) \right) \quad \text{for } i = 1,2,3,4,5,6 \text{ and } j = 1,3,5
\] (4.86)

\[
k_{ij-pm} = \frac{le}{d_i} \left( C_j \sinh^{-1}(A_i d_i) - D_j \sinh^{-1}(B_i d_i) \right) \quad \text{for } i = 1,2,3,4,5,6 \text{ and } j = 2,4,6
\] (4.87)
<table>
<thead>
<tr>
<th>i</th>
<th>( A_i )</th>
<th>( B_i )</th>
<th>( C_j )</th>
<th>( D_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{3.46\gamma}{\varepsilon \cdot l_e} )</td>
<td>( \frac{-3.46\gamma}{\varepsilon \cdot l_e} )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{0.732\gamma}{\varepsilon \cdot l_e} )</td>
<td>( \frac{-2.732\gamma}{\varepsilon \cdot l_e} )</td>
<td>0.183</td>
<td>0.683</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{0.732\gamma}{\varepsilon \cdot l_e} )</td>
<td>( \frac{-2.732\gamma}{\varepsilon \cdot l_e} )</td>
<td>0.183</td>
<td>0.683</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{-3.46\gamma}{\varepsilon \cdot l_e} )</td>
<td>( \frac{3.46\gamma}{\varepsilon \cdot l_e} )</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{2.732\gamma}{\varepsilon \cdot l_e} )</td>
<td>( \frac{-0.732\gamma}{\varepsilon \cdot l_e} )</td>
<td>0.683</td>
<td>0.183</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{2.732\gamma}{\varepsilon \cdot l_e} )</td>
<td>( \frac{-0.732\gamma}{\varepsilon \cdot l_e} )</td>
<td>0.683</td>
<td>0.183</td>
</tr>
</tbody>
</table>

Table 4.5: Constants for Equations 4.86 and 4.87.

Total degrees of freedom for the QUAD-4 (Plate) fatigue element obtained by addition of the modified beam and QUAD-4 (In-Plane) elements are given by Equation 4.88.
\[ d = \begin{bmatrix} d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, \\ v_1, \theta_{x1}, \theta_{y1}, v_2, \theta_{x2}, \theta_{y2}, v_2, \theta_{x2}, \theta_{y2}, v_1, \theta_{x1}, \theta_{y1} \end{bmatrix} \]  

(4.88)

The energy Equation for this new QUAD-4 (Plate) element is given in Equation 4.89 and the new QUAD-4 (Plate) element is given by Equation 4.90.

\[ U = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \\ v_1 \\ \theta_{x1} \\ \theta_{y1} \\ v_2 \\ \theta_{x2} \\ \theta_{y2} \\ v_2 \\ \theta_{x2} \\ \theta_{y2} \\ v_1 \\ \theta_{x1} \\ \theta_{y1} \end{bmatrix}^T \begin{bmatrix} K_{pm-Plate} \end{bmatrix} \]  

(4.89)

\[ K_{pm-Plate} = \begin{bmatrix} K_{pm-PlanePart} & O \\ O & K_{pm-BeamPart} \end{bmatrix} \]  

(4.90)
\[
K_{pm-PlanePart} = [A_1 | B_1]
\]  

\[
A_1 = \begin{bmatrix}
  k_{11_{pm}}(d) & k_{12_{pm}}(d) & k_{13_{pm}}(d) & k_{14_{pm}}(d) \\
  k_{21_{pm}}(d) & k_{22_{pm}}(d) & k_{23_{pm}}(d) & k_{24_{pm}}(d) \\
  k_{31_{pm}}(d) & k_{32_{pm}}(d) & k_{33_{pm}}(d) & k_{34_{pm}}(d) \\
  k_{41_{pm}}(d) & k_{42_{pm}}(d) & k_{43_{pm}}(d) & k_{44_{pm}}(d) \\
  k_{51_{pm}}(d) & k_{52_{pm}}(d) & k_{53_{pm}}(d) & k_{54_{pm}}(d) \\
  k_{61_{pm}}(d) & k_{62_{pm}}(d) & k_{63_{pm}}(d) & k_{64_{pm}}(d) \\
  k_{71_{pm}}(d) & k_{72_{pm}}(d) & k_{73_{pm}}(d) & k_{74_{pm}}(d) \\
  k_{81_{pm}}(d) & k_{82_{pm}}(d) & k_{83_{pm}}(d) & k_{84_{pm}}(d)
\end{bmatrix}
\]

\[
B_1 = \begin{bmatrix}
  k_{15_{pm}}(d) & k_{16_{pm}}(d) & k_{17_{pm}}(d) & k_{18_{pm}}(d) \\
  k_{25_{pm}}(d) & k_{26_{pm}}(d) & k_{27_{pm}}(d) & k_{28_{pm}}(d) \\
  k_{35_{pm}}(d) & k_{36_{pm}}(d) & k_{37_{pm}}(d) & k_{38_{pm}}(d) \\
  k_{45_{pm}}(d) & k_{46_{pm}}(d) & k_{47_{pm}}(d) & k_{48_{pm}}(d) \\
  k_{55_{pm}}(d) & k_{56_{pm}}(d) & k_{57_{pm}}(d) & k_{58_{pm}}(d) \\
  k_{65_{pm}}(d) & k_{66_{pm}}(d) & k_{67_{pm}}(d) & k_{68_{pm}}(d) \\
  k_{75_{pm}}(d) & k_{76_{pm}}(d) & k_{77_{pm}}(d) & k_{78_{pm}}(d) \\
  k_{85_{pm}}(d) & k_{86_{pm}}(d) & k_{87_{pm}}(d) & k_{88_{pm}}(d)
\end{bmatrix}
\]

\[
K_{pm-BeamPart} = [A_2 | B_2]
\]
The new QUAD-4 (Plate) fatigue element has the capability to model plate bending problems as well. The analysis results performed using this element are presented in the following Section.
4.4 Results and Discussion

The Winger plate discussed in Chapter 3 is shown in the Figure below. This plate is discretized with QUAD-4 (Plate) element. The plate is fixed at one end and is subjected to a uniaxial tension/compression and a bending fatigue load at the other end.

![Figure 4.9: Winger Plate Meshed with QUAD-4 (Plate) Element](image)

The plate is stressed to a stress level of 25 ksi. The stress results are shown in the Figure below. The stresses are high near the clamp and decrease for the locations away from the clamped end. This verifies that the analysis with the new QUAD-4 Plate fatigue element is capturing the stress pattern as expected for the plate subjected to the bending load.
The fatigue life prediction results for the Winger plate are shown in the following Figure. The minimum number of cycles is predicted at the locations where maximum stress is present.
In order to have a closer comparison, the Winger plate meshed with new plate element is excited to stress level of 25 ksi for 7th mode and the stress and prediction results are compared to the equivalent stress approach prediction performed in Chapter 3. Equation 4.78 is modified for this analysis into a dynamic analysis Equation [4.11] as given by Equation 4.97.

\[
[M]\{d\} + [K]\{d\} = \{Fcos\omega t\} 
\]  

(4.97)

Where \( M \) is the mass matrix and \( \omega \) is the frequency. The following Figures show the plate element discretization of the Winger plate, 7th mode shape results, the vibratory stress and the fatigue life prediction results for this analysis.
The displacement and stress contours from both the analysis match with each other and the maximum stresses are located away from the fixed end.

Figure 4.13: 7th Mode Displacement Results (QUAD-4 (Plate) Element)
Figure 4.14: 7th Mode Stress Results (QUAD-4 (Plate) Element)

Figure 4.15: Fatigue Life Prediction (QUAD-4 (Plate) Element)
The following Table shows a comparison of the results from the vibration analysis performed on the Winger plate meshed with the new QUAD-4 (Plate) element. The results match well with the experimental results as well as the equivalent stress approach discussed in Chapter 3.

<table>
<thead>
<tr>
<th>Stress (ksi)</th>
<th>Experiment Life Cycles</th>
<th>FE Analysis Equivalent Stress Approach</th>
<th>FE Analysis Plate Element</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winger Plate</td>
<td>19.1 10^6</td>
<td>1.26 x E6</td>
<td>1.16 x E6</td>
</tr>
</tbody>
</table>

Table 4.6: Comparison of Plate Element Results with Experimental and Equivalent Stress Approach

4.5 Conclusion

The newly developed QUAD-4 (In-plane) and Modified QUAD-4 (Plate) elements provide a useful tool for biaxial fatigue life prediction. The results presented in Section 4.4 show that new QUAD-4 element can predict the fatigue life of a structural component with sufficient accuracy. As these elements are developed from an energy-based constitutive law for fatigue life prediction, analysis with these elements incorporates the fatigue mechanism into fatigue analysis. Due to the discrete nature of finite element analysis, new fatigue elements can predict the number of cycles to failure
at each location in the 2-D structures. The equivalent stress fatigue analysis discussed in Chapter 3 follows part of the conventional analysis approach [4.12-4.14] as it obtains the stresses through dynamic analysis using ordinary QUAD-4 elements. When performing the analysis with the new QUAD-4 element, the mode shapes and stresses are obtained from direct analysis of the component meshed with the QUAD-4 fatigue elements. Therefore, this analysis in comparison to equivalent stress approach skips multiple analysis steps and makes the prediction a one step procedure. Furthermore, the prediction results with the QUAD-4 fatigue elements show closer comparison to experimental results as compared to the equivalent stress approach. As in the case of uniaxial elements, the new QUAD-4 elements developed in this Chapter predict crack initiation whereas most research in this area [4.15-4.24] is related to crack propagation.
4.6 List of References


CHAPTER 5

NEW HEXAHEDRAL FOR FATIGUE LIFE PREDICTION

5.1 Introduction

This Chapter presents the development of an eight-node Hexahedral (HEX-8 Brick) fatigue element. This element has 8 nodes and each node has three degrees of freedom (DOF) in x, y and z directions. The new fatigue element is further modified to add the rotational and bending DOFs for application to real world three dimensional (3D) structures. The constitutive Equations presented by the energy-based framework developed in [5.1-5.5] and discussed in previous Chapters are integrated into a minimum potential energy expression to develop the eight node HEX element. This element has the capability to predict the number of cycles to failure for 3-D objects subjected to stresses in x, y and z directions. The element is benchmarked with the previously developed uniaxial tension/compression problem in order to verify the new development. The benchmarking procedures are discussed in detail in Section 5.3.

The newly developed HEX-8 Brick element is further modified by adding the extra degrees of freedom of a beam element to obtain the rotational capability.
With these added rotational degrees of freedom, the HEX-8 Brick element can be used to model the 3-D structures subjected to multiaxial fatigue loads. The analysis of a Curved Plate discussed in Chapter 3, is performed using the new HEX-8 element. The results are compared to the equivalent stress approach analysis results presented in Sections 3.5 in Chapter 3. Another set of comparison is made to results for type 304 stainless steel [4.6, 4.7] presented in Section 3.2 and 3.5 in Chapter 3.

5.2 Finite Element Procedures for HEX-8 Element

The following Equations already discussed in the previous Chapters and listed below present the uniaxial and shear constitutive laws [5.1, 5.5] both for uniaxial monotonic and cyclic loadings respectively.

\[
\varepsilon_{\text{monotonic}} = \frac{\sigma}{E} + \varepsilon_0 \sinh \left( \frac{\sigma}{\sigma_0} \right) \tag{5.1}
\]

\[
\varepsilon_{\text{cyclic}} = \frac{\sigma_{PP}}{E} + \frac{1}{C} \sinh \left( \frac{\sigma_{PP}}{\sigma_c} \right) \tag{5.2}
\]

Equations 5.3 and 5.4 represent stress-strain relationships for shear monotonic and cyclic loads respectively.

\[
\gamma_{\text{monotonic}} = \frac{\tau}{G} + \gamma_0 \sinh \left( \frac{\tau}{\tau_0} \right) \tag{5.3}
\]

\[
\gamma_{\text{cyclic}} = \frac{\tau_{PP}}{G} + \frac{1}{C_S} \sinh \left( \frac{\tau_{PP}}{\tau_c} \right) \tag{5.4}
\]

The respective inverse relationships for stress are given by the following Equations,

\[
\varepsilon_{\text{em}} = \frac{\sigma}{E} \tag{5.5}
\]
\[
\sigma_{pm} = \sigma_0 \sinh^{-1}\left( \frac{\varepsilon_p}{\varepsilon_0} \right) \quad (5.6)
\]

\[
\varepsilon_{ec} = \frac{\sigma_{PP}}{E} \quad (5.7)
\]

\[
\sigma_{pc} = \sigma_c \sinh^{-1}\left( C\varepsilon_p \right) \quad (5.8)
\]

\[
\gamma_{em} = \frac{\tau}{G} \quad (5.9)
\]

\[
\tau_{pm} = \tau_0 \sinh^{-1}\left( \frac{\gamma_p}{\gamma_0} \right) \quad (5.10)
\]

\[
\gamma_{ec} = \frac{\tau_{PP}}{G} \quad (5.11)
\]

\[
\tau_{pc} = \tau_c \sinh^{-1}\left( C_c\gamma_p \right) \quad (5.12)
\]

Where the parameters displayed in Equations 5.6 to 5.13 are defined as follows: \(\sigma\) is the value for stress at the surface of the specimen (in the bending case, max stress), \(\varepsilon\) and \(\gamma\) are the strain corresponding to the stress \(\sigma\) and \(\tau\), \(\sigma_{pp}\) and \(\tau_{pp}\) is the peak to peak stress (2\(\sigma\) when stress ratio is -1.0), \(E\) is the modulus of elasticity, and the variables \(\sigma_c, \sigma_o, \varepsilon_o, C, \tau_c, \tau_o, \gamma_o\) and \(C_s\) are curve fit parameters [5.1]. The subscripts \(em\) & \(pm\) designate the elastic & plastic cases for monotonic loading, and the subscripts \(ec\) & \(pc\) designate the elastic & plastic cases for cyclic loading.

The following Equations present the elastic constitutive laws used for HEX-8 fatigue element development. The elastic constitutive law is a classical Equation available in literature [5.6, 5.7].

\[
\{\sigma\} = [D]\{\varepsilon\} \quad (5.14)
\]
Where \( \sigma \) is stress given by Equation 5.15 with \( x \)-axis, \( y \)-axis, \( z \)-axis and shear components, \( \varepsilon \) is strain given by Equation 5.16 with \( \varepsilon_x \), \( \varepsilon_y \) and \( \varepsilon_z \) the \( x \)-axis, \( y \)-axis and \( z \)-axis components and \( \gamma_{yz} \), \( \gamma_{xz} \) and \( \gamma_{xy} \) the shear component.

\[
\sigma = \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yz} \\
\tau_{xz} \\
\tau_{xy}
\end{bmatrix}
\]

(5.15)

and

\[
\varepsilon = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{bmatrix}
\]

(5.16)

and \( D \) is given by Equations 5.17.

\[
D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 1-\nu & 0 & 0 \\
0 & 0 & 0 & 0.5-\nu & 0 \\
0 & 0 & 0 & 0 & 0.5-\nu
\end{bmatrix}
\]

(5.17)

If a three dimensional (3-D) stress tensor is defined as following,

\[
\{\sigma_{ij}\} = \begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{bmatrix}
\]

(5.18)
Corresponding stress elements for plastic part of Equations 5.1 to 5.4 are given by the Equations 5.19 to 5.24. Equations 5.15 and 5.18 to 5.24 are integrated into minimum potential energy formulation to obtain the new K-matrices for fatigue life prediction.

\[
\sigma_{pm-xx} = \sigma_0 \sinh^{-1}\left(\frac{\varepsilon_{px}}{\varepsilon_0}\right)
\]

(5.19)

\[
\sigma_{pm-yy} = \sigma_0 \sinh^{-1}\left(\frac{\varepsilon_{py}}{\varepsilon_0}\right)
\]

(5.20)

\[
\sigma_{pm-zz} = \sigma_0 \sinh^{-1}\left(\frac{\varepsilon_{pz}}{\varepsilon_0}\right)
\]

(5.21)

\[
\tau_{pm-yz} = \tau_0 \sinh^{-1}\left(\frac{\gamma_{pyz}}{\gamma_0}\right)
\]

(5.22)

\[
\tau_{pm-xz} = \tau_0 \sinh^{-1}\left(\frac{\gamma_{pzx}}{\gamma_0}\right)
\]

(5.23)

\[
\tau_{pm-xy} = \tau_0 \sinh^{-1}\left(\frac{\gamma_{pxy}}{\gamma_0}\right)
\]

(5.24)

An eight-node HEX element is shown in the following Figure. The element has eight nodes with each node having three degrees of freedom, displacements in x, y and z directions.
The nodal displacement vector is denoted by $d$ and $d_s$ are $x$, $y$ and $z$ displacements at each node. The displacement at any point within the element is denoted by $u = [u(x,y,z), v(x,y,z), w(x,y,z)]^T$.

$$d = [d_1, d_2, d_3, \ldots, d_{22}, d_{23}, d_{24}]^T$$  \hspace{1cm} (5.25)

Figure 5.2 shows HEX-8 element defined in $\xi-\eta-\zeta$ coordinates or natural coordinates and is brick shaped. These displacements are approximated using the Lagrange shape functions which satisfy nodal value and continuity requirements.
The shape functions for this element are given by the following Equations. \( N_i \) are the shape functions at each node with \( i \) denoting the node numbers as 1, 2, 3 …8. These shape functions are defined such that \( N_i \) is equal to 1 at node \( i \) and zero at other nodes.

\[
N_1 = \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta) 
\]

\[
N_2 = \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta) 
\]

\[
N_3 = \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta) 
\]

\[
N_4 = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta) 
\]
\[
N_5 = \frac{1}{8}(1-\xi)(1-\eta)(1+\zeta) \quad (5.30)
\]

\[
N_6 = \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta) \quad (5.31)
\]

\[
N_7 = \frac{1}{8}(1+\xi)(1+\eta)(1+\zeta) \quad (5.32)
\]

\[
N_8 = \frac{1}{8}(1-\xi)(1+\eta)(1+\zeta) \quad (5.33)
\]

The following Equations presents the displacement field within the element in terms of nodal values.

\[
\begin{align*}
    u &= N_1 d_1 + N_2 d_4 + \cdots + N_8 d_{22} \\
    v &= N_1 d_2 + N_2 d_5 + \cdots + N_8 d_{23} \\
    w &= N_1 d_3 + N_2 d_6 + \cdots + N_8 d_{24}
\end{align*} \quad (5.34-5.36)
\]

The above Equations can be re-written in matrix form as

\[
u = Nd \quad (5.37)
\]

Where

\[
N = \begin{bmatrix}
N_1, & 0, & 0, & N_2, & 0, & 0, & \cdots \cdots, & N_8, & 0, & 0 \\
0, & N_1, & 0, & 0, & N_2, & 0, & \cdots \cdots, & 0, & N_8, & 0 \\
0, & 0, & N_1, & 0, & 0, & N_2, & \cdots \cdots, & 0, & 0, & N_8
\end{bmatrix} \quad (5.38)
\]

In the isoparametric formulation, we use the same shape functions to express the coordinates of a point within the element in terms of nodal coordinates. Thus,

\[
\begin{align*}
x &= N_1 x_1 + N_2 x_2 + \cdots + N_8 x_8 \\
y &= N_1 y_1 + N_2 y_2 + \cdots + N_8 y_8 \\
z &= N_1 z_1 + N_2 z_2 + \cdots + N_8 z_8
\end{align*} \quad (5.39-5.41)
In order to map the derivatives of a function in x-y-z coordinates in terms of its derivatives in \( \xi - \eta - \varsigma \) coordinates, following steps are used. A function \( f \) is defined as follows,

\[
f = f(x, y, z)
\] (5.42)

In view of Equations 5.39, 5.40 and 5.41, it can be considered as implicit function of \( \xi - \eta - \varsigma \).

\[
f = f[x(\xi, \eta, \varsigma), \ y(\xi, \eta, \varsigma), \ z(\xi, \eta, \varsigma)]
\] (5.43)

By chain rule of differentiation,

\[
\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \xi} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \xi}
\] (5.44)

\[
\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \eta} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \eta}
\] (5.45)

\[
\frac{\partial f}{\partial \varsigma} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \varsigma} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \varsigma} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial \varsigma}
\] (5.46)

Equations 5.44, 5.45 and 5.46 can be re-written as Equation 5.47. \( J \) is the Jacobean matrix given by Equation 5.48.

\[
\begin{bmatrix}
\frac{\partial f}{\partial \xi} \\
\frac{\partial f}{\partial \eta} \\
\frac{\partial f}{\partial \varsigma}
\end{bmatrix}
= J
\begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial z}
\end{bmatrix}
\] (5.47)
\[
J = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}
\end{bmatrix}
\] (5.48)

In view of Equations 5.25 to 5.33, and Equations 5.39 to 5.41, \( J \) is given by

\[
J = \begin{bmatrix}
J_{11} & J_{12} & J_{13} \\
J_{21} & J_{22} & J_{23} \\
J_{31} & J_{32} & J_{33}
\end{bmatrix}
\] (5.49)

Where for a Hexahedral brick element of length \( L \), width \( W \) and height \( H \),

\[
J_{11} = \frac{L}{2}
\] (5.50)

\[
J_{12} = 0
\] (5.51)

\[
J_{13} = 0
\] (5.52)

\[
J_{21} = 0
\] (5.53)

\[
J_{22} = \frac{W}{2}
\] (5.54)

\[
J_{23} = 0
\] (5.55)

\[
J_{31} = 0
\] (5.56)

\[
J_{32} = 0
\] (5.57)

\[
J_{33} = \frac{H}{2}
\] (5.58)

In order to obtain the transformation of a function given in \( x, y, z \) coordinates, to natural coordinates. Equation 5.47 can be written as Equation 5.59. The inverse of Jacobean is given in Equation 5.60.
\[
\begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial z}
\end{bmatrix} = J^{-1} \begin{bmatrix}
\frac{\partial f}{\partial \xi} \\
\frac{\partial f}{\partial \eta} \\
\frac{\partial f}{\partial \zeta}
\end{bmatrix}
\]  \quad (5.59)

or

\[
\begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial z}
\end{bmatrix} = \frac{1}{\det J} \begin{bmatrix}
J_{33}J_{22} & 0 & 0 \\
0 & J_{33}J_{11} & 0 \\
0 & 0 & J_{22}J_{11}
\end{bmatrix} \begin{bmatrix}
\frac{\partial f}{\partial \xi} \\
\frac{\partial f}{\partial \eta} \\
\frac{\partial f}{\partial \zeta}
\end{bmatrix}
\]  \quad (5.60)

\[
\det J = \frac{1}{8} LWH
\]  \quad (5.61)

\[
J^{-1} = \begin{bmatrix}
\frac{2}{L} & 0 & 0 \\
0 & \frac{2}{W} & 0 \\
0 & 0 & \frac{2}{H}
\end{bmatrix}
\]  \quad (5.62)

An additional result required for derivation of HEX-8 element stiffness matrix along with the above expressions, is given below.

\[
dx\,dy\,dz = \det J d\xi d\eta d\zeta
\]  \quad (5.63)

This result is used to express the volume of the element and the proof for this result is given in [12]. Integration of the elastic case, Equation 5.14 into potential energy formulation is a classical finite element problem and is already available in literature [5.6]. Integration of Equations 5.18 into Equation 5.64 provides the new K-matrix for the plastic part of the constitutive law fatigue analysis.
\[
\Pi = \int \sigma^T \varepsilon dV - \int uf dV - \int uT dx - \sum u_i P_i \quad (5.64)
\]

Where \(\Pi\) is the minimum potential energy, \(\sigma\) is the stress tensor, \(\varepsilon\) is the strain vector, \(u\) is the displacement, \(f\) is the body force, \(T\) is the traction force, and \(P_i\) is the point load. \(V\) is the volume and \(x\) denotes the length of the element.

The strain energy term is given by

\[
U = \int \sigma^T \varepsilon dV \quad (5.65)
\]

\[
U = \int \int \int \sigma^T \varepsilon dV = \int \int \int \sigma^T \varepsilon \det J d\eta d\zeta \quad (5.66)
\]

The strain displacement relations are given by

\[
\varepsilon = \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{xz} \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \\
\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} + \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}
\end{bmatrix} \quad (5.67)
\]

From Equation 5.60 we have,

\[
\begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} \\
\frac{\partial u}{\partial z}
\end{bmatrix} = \frac{1}{\det J} \begin{bmatrix}
J_{33}J_{22} & 0 & 0 \\
0 & J_{33}J_{11} & 0 \\
0 & 0 & J_{22}J_{11}
\end{bmatrix} \begin{bmatrix}
\frac{\partial f}{\partial \xi} \\
\frac{\partial f}{\partial \eta} \\
\frac{\partial f}{\partial \zeta}
\end{bmatrix} \quad (5.68)
\]
\[
\begin{align*}
\frac{\partial v}{\partial x} & = \frac{1}{\det J} \begin{bmatrix} J_{33} J_{22} & 0 & 0 \\ 0 & J_{33} J_{11} & 0 \\ 0 & 0 & J_{22} J_{11} \end{bmatrix} \frac{\partial f}{\partial \xi} \\
\frac{\partial v}{\partial y} & = \frac{1}{\det J} \begin{bmatrix} J_{33} J_{22} & 0 & 0 \\ 0 & J_{33} J_{11} & 0 \\ 0 & 0 & J_{22} J_{11} \end{bmatrix} \frac{\partial f}{\partial \eta} \\
\frac{\partial v}{\partial z} & = \frac{1}{\det J} \begin{bmatrix} J_{33} J_{22} & 0 & 0 \\ 0 & J_{33} J_{11} & 0 \\ 0 & 0 & J_{22} J_{11} \end{bmatrix} \frac{\partial f}{\partial \zeta}
\end{align*}
\] (5.69)

\[
\begin{align*}
\frac{\partial w}{\partial x} & = \frac{1}{\det J} \begin{bmatrix} J_{33} J_{22} & 0 & 0 \\ 0 & J_{33} J_{11} & 0 \\ 0 & 0 & J_{22} J_{11} \end{bmatrix} \frac{\partial f}{\partial \xi} \\
\frac{\partial w}{\partial y} & = \frac{1}{\det J} \begin{bmatrix} J_{33} J_{22} & 0 & 0 \\ 0 & J_{33} J_{11} & 0 \\ 0 & 0 & J_{22} J_{11} \end{bmatrix} \frac{\partial f}{\partial \eta} \\
\frac{\partial w}{\partial z} & = \frac{1}{\det J} \begin{bmatrix} J_{33} J_{22} & 0 & 0 \\ 0 & J_{33} J_{11} & 0 \\ 0 & 0 & J_{22} J_{11} \end{bmatrix} \frac{\partial f}{\partial \zeta}
\end{align*}
\] (5.70)

\[
\begin{align*}
\varepsilon = A \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial \xi}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial \xi}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} \\ \frac{\partial \xi}{\partial \zeta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial \eta}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \\ \frac{\partial \eta}{\partial \eta} \\ \frac{\partial v}{\partial \zeta} \\ \frac{\partial \eta}{\partial \zeta} \\ \frac{\partial w}{\partial \xi} \\ \frac{\partial \xi}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \\ \frac{\partial \xi}{\partial \eta} \\ \frac{\partial w}{\partial \zeta} \\ \frac{\partial \xi}{\partial \zeta} \end{bmatrix}
\end{align*}
\] (5.71)
\[
A = \begin{bmatrix}
\frac{2}{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2}{W} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{H} & 0 \\
0 & 0 & 0 & 0 & \frac{2}{H} & 0 & \frac{2}{W} & 0 \\
0 & \frac{2}{H} & 0 & 0 & 0 & \frac{2}{L} & 0 & 0 \\
0 & \frac{2}{W} & 0 & \frac{2}{L} & 0 & 0 & 0 & 0
\end{bmatrix}
\]  \hspace{1cm} (5.72)

Now from Equations 5.39, 5.40 and 5.41,

\[
\begin{bmatrix}
\frac{\partial u}{\partial \xi} \\
\frac{\partial u}{\partial \eta} \\
\frac{\partial u}{\partial \zeta} \\
\frac{\partial v}{\partial \xi} \\
\frac{\partial v}{\partial \eta} \\
\frac{\partial v}{\partial \zeta} \\
\frac{\partial w}{\partial \xi} \\
\frac{\partial w}{\partial \eta} \\
\frac{\partial w}{\partial \zeta}
\end{bmatrix} = Gd
\]  \hspace{1cm} (5.73)

Where \( G \) is a 9 x 24 matrix obtained from differentiation of Equations 5.39, 5.40 and 5.41 with respect to \( \xi - \eta - \zeta \) coordinates and \( d \) is separated as an 8 x 1 displacement vector. Equations 5.71 and 5.73 yield
\[ \varepsilon = Bd \]  \hspace{1cm} (5.74)

\[ B = AG \]  \hspace{1cm} (5.75)

Equation 5.18 can be written as

\[
\{ \sigma \} = 
\begin{bmatrix}
\sigma_0 \left( \sinh^{-1} \left( \frac{\varepsilon_x}{\varepsilon_0} \right) \right)
\sigma_0 \left( \sinh^{-1} \left( \frac{\varepsilon_y}{\varepsilon_0} \right) \right)
\sigma_0 \left( \sinh^{-1} \left( \frac{\varepsilon_z}{\varepsilon_0} \right) \right)
\tau_0 \sinh^{-1} \left( \frac{\gamma_{yz}}{\gamma_0} \right)
\tau_0 \sinh^{-1} \left( \frac{\gamma_{xz}}{\gamma_0} \right)
\tau_0 \sinh^{-1} \left( \frac{\gamma_{xy}}{\gamma_0} \right)
\end{bmatrix}
\]  \hspace{1cm} (5.76)

Rewriting Equation 5.66 by using results from Equations 5.67 to 5.75.

\[
U = \frac{d^T B^T}{d^T B^T} \left[ \iiint_{-1-1-1} \sigma^T B \det J d\xi d\eta d\zeta \right] d 
\]  \hspace{1cm} (5.77)

\[
U = d^T \left[ \iiint_{-1-1-1} B^T \sigma^T B \frac{\det J d\xi d\eta d\zeta}{d^T B^T} \right] d 
\]  \hspace{1cm} (5.78)

\[
k^e = \frac{1}{d^T B^T} \left[ \iiint_{-1-1-1} B^T \sigma^T B \det J d\xi d\eta d\zeta \right] d 
\]  \hspace{1cm} (5.79)
Numerical integration of Equation 5.79 yields the non linear K-Matrix for the HEX-8 fatigue element. The HEX-8 K-matrix for the linear elastic case is a standard finite element formulation problem and is available in the literature. The HEX-8 K-Matrices for linear elastic and non-linear plastic parts are represented by the following Equations.

\[
K_{em-Brick} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \begin{bmatrix}
k_{11} & k_{12} & k_{13} & \cdots & k_{124} \\
k_{21} & k_{22} & k_{23} & \cdots & k_{224} \\
& \ddots & \ddots & \ddots & \vdots \\
& & & \ddots & \ddots \\
& & & & k_{241} \\
& & & & & k_{242} \\
& & & & & & k_{243} \\
& & & & & & & k_{244}
\end{bmatrix}
\frac{d\xi d\eta d\zeta}{d\xi d\eta d\zeta}
\]

(5.80)

\[
K_{pm-Brick} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \begin{bmatrix}
k_{11}(d) & k_{12}(d) & k_{13}(d) & \cdots & k_{124}(d) \\
k_{21}(d) & k_{22}(d) & k_{23}(d) & \cdots & k_{224}(d) \\
& \ddots & \ddots & \ddots & \vdots \\
& & & \ddots & \ddots \\
& & & & k_{241}(d) \\
& & & & & k_{242}(d) \\
& & & & & & k_{243}(d) \\
& & & & & & & k_{244}(d)
\end{bmatrix}
\frac{d\xi d\eta d\zeta}{d\xi d\eta d\zeta}
\]

(5.81)

Where subscripts \(em\) and \(pm\) denote the elastic and plastic cases respectively. The elements of \(K_{pm-Brick}\) are given in the following Equations. The elements \(Q_i, Q_j, S_{ij}, R_{11}\) and \(R_{12}\) are given in Table 5.1.
\[ k_{ij-pm} = \frac{1}{8} \left[ \frac{Q_i S_j Q_j}{R_{11}} \right] \quad \text{for } i = 1, 4, 7\ldots 22 \text{ and } j = 1, 4, 7\ldots 22 \quad (5.82) \]

\[ k_{ij-pm} = \frac{1}{8} \left[ \frac{Q_i S_j Q_j}{R_{12}} \right] \quad \text{for } i = 2, 5, 8\ldots 23 \text{ and } j = 2, 5, 8\ldots 23 \quad (5.83) \]

\[ k_{ij-pm} = \frac{1}{8} \left[ \frac{Q_i S_j Q_j}{R_{12}} \right] \quad \text{for } i = 2, 5, 8\ldots 23 \text{ and } j = 2, 5, 8\ldots 23 \quad (5.84) \]

\[ k_{ij-pm} = \frac{1}{8} \left[ \frac{Q_i S_j Q_j}{R_{12}} \right] \quad \text{for } i = 1, 4, 7\ldots 22 \text{ and } j = 2, 5, 8\ldots 23 \quad (5.85) \]

\[ k_{ij-pm} = \frac{1}{8} \left[ \frac{Q_i S_j Q_j}{R_{13}} \right] \quad \text{for } i = 1, 4, 7\ldots 22 \text{ and } j = 3, 6, 9\ldots 24 \quad (5.86) \]

\[ k_{ij-pm} = \frac{1}{8} \left[ \frac{Q_i S_j Q_j}{R_{11}} \right] \quad \text{for } i = 2, 5, 8\ldots 23 \text{ and } j = 1, 4, 7\ldots 22 \quad (5.87) \]

\[ k_{ij-pm} = \frac{1}{8} \left[ \frac{Q_i S_j Q_j}{R_{13}} \right] \quad \text{for } i = 2, 5, 8\ldots 23 \text{ and } j = 3, 6, 9\ldots 24 \quad (5.88) \]

\[ k_{ij-pm} = \frac{1}{8} \left[ \frac{Q_i S_j Q_j}{R_{11}} \right] \quad \text{for } i = 3, 6, 9\ldots 24 \text{ and } j = 1, 4, 7\ldots 22 \quad (5.89) \]

\[ k_{ij-pm} = \frac{1}{8} \left[ \frac{Q_i S_j Q_j}{R_{12}} \right] \quad \text{for } i = 3, 6, 9\ldots 24 \text{ and } j = 2, 5, 8\ldots 23 \quad (5.90) \]
<table>
<thead>
<tr>
<th>$i/j$</th>
<th>$Q_{i/j}$</th>
<th>$S_{ij}$</th>
<th>$R_{11}, R_{12}$ and $R_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\frac{2}{L}(1-\eta)(1-\xi)$</td>
<td>$S_{ij} = \sigma_0 \sinh^{-1}\left(\frac{\frac{\epsilon p_x}{\epsilon_0}}{\xi}\right)$</td>
<td>$R_{11} = \frac{2}{L}\left(\frac{-(1-\eta)(1-\xi)d_1 + (1-\eta)(1-\xi)d_4 + (1-\eta)(1-\xi)d_7}{-(1-\eta)(1-\xi)d_{10} + (1-\eta)(1+\xi)d_{13} + (1+\eta)(1+\xi)d_{16} + (1+\eta)(1+\xi)d_{19}}\right)$</td>
</tr>
<tr>
<td>2</td>
<td>$-\frac{2}{W}(1-\xi)(1-\xi)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$-\frac{2}{H}(1-\xi)(1-\eta)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\frac{2}{L}(1-\eta)(1-\xi)$</td>
<td>$S_{ij} = \sigma_0 \sinh^{-1}\left(\frac{\frac{\epsilon p_y}{\epsilon_0}}{\xi}\right)$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$-\frac{2}{W}(1+\xi)(1-\xi)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$-\frac{2}{H}(1+\xi)(1-\eta)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$\frac{2}{L}(1+\eta)(1-\xi)$</td>
<td>$S_{ij} = \sigma_0 \sinh^{-1}\left(\frac{\frac{\epsilon p_z}{\epsilon_0}}{\xi}\right)$</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$\frac{2}{W}(1+\xi)(1-\xi)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Continued

Table 5.1: Constants for Equations 5.82 to 5.90
Table 5.1 continued

<table>
<thead>
<tr>
<th>$i / j$</th>
<th>$Q_{i/j}$</th>
<th>$S_{ij}$</th>
<th>$R_{11}, R_{12}$ and $R_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>$-\frac{2}{H}(1 + \xi)(1 + \eta)$</td>
<td>$S_{ij} = \tau_0 \sinh^{-1}\left(\frac{\varepsilon_{pxy}}{\varepsilon_0}\right)$</td>
<td>$R_{12} = \frac{2}{W}$</td>
</tr>
<tr>
<td>10</td>
<td>$-\frac{2}{L}(1 + \eta)(1 - \xi)$</td>
<td>for $i = 1, 4, 7, \ldots, 22$ and $j = 2, 5, 8, \ldots, 23$</td>
<td>$-\frac{(1 - \xi)(1 - \eta)d_2}{d_1}$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{2}{W}(1 - \xi)(1 - \eta)$</td>
<td></td>
<td>$-\frac{(1 + \xi)(1 - \eta)d_5}{d_1}$</td>
</tr>
<tr>
<td>12</td>
<td>$-\frac{2}{H}(1 - \xi)(1 + \eta)$</td>
<td>$S_{ij} = \tau_0 \sinh^{-1}\left(\frac{\varepsilon_{pxz}}{\varepsilon_0}\right)$</td>
<td>$+\frac{(1 + \xi)(1 - \eta)d_8}{d_1}$</td>
</tr>
<tr>
<td>13</td>
<td>$-\frac{2}{L}(1 - \eta)(1 + \xi)$</td>
<td>for $i = 1, 4, 7, \ldots, 22$ and $j = 3, 6, 9, \ldots, 24$</td>
<td>$-\frac{(1 - \xi)(1 + \eta)d_{14}}{d_1}$</td>
</tr>
<tr>
<td>14</td>
<td>$-\frac{2}{W}(1 - \xi)(1 + \xi)$</td>
<td></td>
<td>$-\frac{(1 + \xi)(1 + \eta)d_{17}}{d_1}$</td>
</tr>
<tr>
<td>15</td>
<td>$\frac{2}{H}(1 - \xi)(1 - \eta)$</td>
<td>$S_{ij} = \tau_0 \sinh^{-1}\left(\frac{\varepsilon_{pxz}}{\varepsilon_0}\right)$</td>
<td>$+\frac{(1 + \xi)(1 + \eta)d_{20}}{d_1}$</td>
</tr>
<tr>
<td>16</td>
<td>$\frac{2}{L}(1 - \eta)(1 + \xi)$</td>
<td>for $i = 2, 5, 8, \ldots, 23$ and $j = 1, 4, 7, \ldots, 22$</td>
<td>$+\frac{(1 - \xi)(1 + \eta)d_{23}}{d_1}$</td>
</tr>
</tbody>
</table>

Continued
Table 5.1 continued

<table>
<thead>
<tr>
<th>(i / j)</th>
<th>(Q_{i/j})</th>
<th>(S_{ij})</th>
<th>(R_{11}, R_{12} \text{ and } R_{13})</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>(-\frac{2}{W} (1 + \xi)(1 + \varsigma))</td>
<td>(S_{ij} = \tau_0 \sinh^{-1}\left(\frac{\varepsilon_{p\nu\gamma}}{\varepsilon_0}\right))</td>
<td>(\frac{-{(1 - \xi)}{(1 - \eta)}d_3}{H})</td>
</tr>
<tr>
<td>18</td>
<td>(\frac{2}{H} (1 + \xi)(1 - \eta))</td>
<td>for (i = 2, 5, 8, \ldots, 23) and (j = 3, 6, 9, \ldots, 24) (\frac{-{(1 + \xi)}{(1 - \eta)}d_6}{H})</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>(\frac{2}{L} (1 + \eta)(1 + \varsigma))</td>
<td></td>
<td>(\frac{-{(1 - \xi)}{(1 + \eta)}d_9}{H})</td>
</tr>
<tr>
<td>20</td>
<td>(\frac{2}{W} (1 + \xi)(1 + \varsigma))</td>
<td>(S_{ij} = \tau_0 \sinh^{-1}\left(\frac{\varepsilon_{p\nu\gamma}}{\varepsilon_0}\right))</td>
<td>(\frac{-{(1 - \xi)}{(1 + \eta)}d_{12}}{H})</td>
</tr>
<tr>
<td>21</td>
<td>(\frac{2}{H} (1 + \xi)(1 + \eta))</td>
<td>for (i = 3, 6, 9, \ldots, 24) and (j = 1, 4, 7, \ldots, 22) (\frac{+{(1 + \xi)}{(1 - \eta)}d_{15}}{H})</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>(-\frac{2}{L} (1 + \eta)(1 + \varsigma))</td>
<td>(R_{13} = \frac{2}{H})</td>
<td>(+{(1 + \xi)}{(1 - \eta)}d_{18})</td>
</tr>
<tr>
<td>23</td>
<td>(\frac{2}{W} (1 - \xi)(1 + \varsigma))</td>
<td>(S_{ij} = \tau_0 \sinh^{-1}\left(\frac{\varepsilon_{p\nu\gamma}}{\varepsilon_0}\right))</td>
<td>(+{(1 - \xi)}{(1 + \eta)}d_{21})</td>
</tr>
<tr>
<td>24</td>
<td>(\frac{2}{H} (1 - \xi)(1 + \eta))</td>
<td>for (i = 3, 6, 9, \ldots, 24) and (j = 2, 5, 8, \ldots, 23)</td>
<td>(+{(1 - \xi)}{(1 + \eta)}d_{24})</td>
</tr>
</tbody>
</table>
Similar Equations are developed for cyclic fatigue loads using the constitutive law for cyclic fatigue and the corresponding curve-fit parameters. The resulting K-matrices are shown in Equations 5.91 and 5.92.

\[
\begin{bmatrix}
k_{11} & k_{12} & k_{13} & \cdots & k_{14} \\
k_{21} & k_{22} & k_{23} & \cdots & k_{24} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
k_{231} & k_{232} & k_{233} & \cdots & k_{244}
\end{bmatrix}
\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} d\xi d\eta d\zeta
\]

\[K_{ec-Brick} = \begin{bmatrix}
k_{11} & k_{12} & k_{13} & \cdots & k_{14} \\
k_{21} & k_{22} & k_{23} & \cdots & k_{24} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
k_{231} & k_{232} & k_{233} & \cdots & k_{244}
\end{bmatrix}
\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} d\xi d\eta d\zeta
\]

(5.91)

\[
\begin{bmatrix}
k_{11}(d) & k_{12}(d) & k_{13}(d) & \cdots & k_{14}(d) \\
k_{21}(d) & k_{22}(d) & k_{23}(d) & \cdots & k_{24}(d) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
k_{231}(d) & k_{232}(d) & k_{233}(d) & \cdots & k_{244}(d)
\end{bmatrix}
\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} d\xi d\eta d\zeta
\]

\[K_{pc-Brick} = \begin{bmatrix}
k_{11}(d) & k_{12}(d) & k_{13}(d) & \cdots & k_{14}(d) \\
k_{21}(d) & k_{22}(d) & k_{23}(d) & \cdots & k_{24}(d) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
k_{231}(d) & k_{232}(d) & k_{233}(d) & \cdots & k_{244}(d)
\end{bmatrix}
\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} d\xi d\eta d\zeta
\]

(5.92)

The elements of \(K_{pc-Brick}\) are the same as given in Equations 5.82 to 5.90 except that the parameters \(\sigma_o\) changes to \(\sigma_e\), \(\varepsilon_o\) changes to \(1/C\), \(\tau_o\) changes to \(\tau_c\), \(\gamma_o\) changes to \(1/C_s\) and the applied stress \(\sigma\) and \(\tau\) changes to peak to peak stress \(\sigma_{pp}\) and \(\tau_{pp}\).
Equations 5.81 and 5.92 are non-linear due to presence of “$ds$” in the resulting K-matrices. To account for the non-linear behavior, the Newton-Raphson iteration method is applied to the analysis [5.8-5.10]. These K-matrices are used in Equation 5.93 to determine the unknown degrees of freedom.

\[
[K]\{d\} = \{F\}
\]  \hspace{1cm} (5.93)

The loads are applied from peak- to -peak. The results are post-processed using classical FEA techniques. The nodal displacement results are further used to obtain stresses and strains for each element in the structure. The 3-D stresses and strains are converted to equivalent von-Mises stress and strain. These stresses and strains are used to calculate the energy dissipated per cycle and ultimately the number of cycles to failure for each element. The calculations of the energy procedures are the same as detailed in Sections 2.3 and 2.4 in Chapter 2. The number of cycles to failure is determined using Equations 2.3 and 2.4 from Chapter 2.

In order to verify the HEX-8 element, this element is benchmarked against the uniaxial rod element developed in Chapter 2. A 3-D beam is meshed with the new HEX-8 elements and subjected to uniaxial tension in the x-direction in the form of displacement. The mesh discretization of this beam is shown in Figure 5.3. The displacement solution is compared to the solution of a 1-D bar meshed with the rod element and subjected to uniaxial tension. The beam and rod are fixed at left end and a unit displacement is applied at the right most end of rod and beam. The linear HEX-8 fatigue K-matrix results are compared to the linear 1-D rod solution as well as an ANSYS solution of the same problem.
The displacement results for linear verification from the HEX-8 element are shown in the following Figure and are compared in Table 5.2.
Figure 5.5: Displacement Results using HEX-8 Element in ANSYS

(Dimensions in Inches)

<table>
<thead>
<tr>
<th>Node 2</th>
<th>Node 3</th>
<th>Node 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-D Bar</td>
<td>ANSYS</td>
<td>Linear Code</td>
</tr>
<tr>
<td>0.333</td>
<td>0.333</td>
<td>0.333</td>
</tr>
</tbody>
</table>

Table 5.2: Linear Analysis Benchmarking of HEX-8 Element.

As stated earlier, the non-linear analysis requires an iterative approach. The non-linear HEX-8 Fatigue K-matrix results are compared to the linear 1-D rod solution. The results for non-linear analysis are shown in the following Figure and a tabular comparison is made in Table 5.3.
Figure 5.6: Displacement Results using Non-linear HEX-8 Fatigue Element

(Dimensions in Inches)
Table 5.3: Non-Linear Analysis Benchmarking of HEX-8 Element.

As is evident from the results of Figure 5.4, 5.5 and 5.6 and the Tables 5.2 and 5.3, the HEX-8 fatigue element analysis compares exactly with the 1-D rod fatigue element. These results successfully complete the benchmarking of the new HEX-8 fatigue element.
5.3  Finite Element Procedures for HEX-8 Element with Rotational Degrees of Freedom

This Section presents the modification of the HEX-8 Brick element to include rotational degrees of freedom [5.11] at each node. The procedure is the same as presented in Section 4.3 for development of the new Quad-4 (Plate) element. The Brick element developed in Section 4.2 has 8 nodes with three translational degrees of freedom assigned to each node. The modified element is obtained by adding three rotational DOFs in x, y and z direction to each node. This modification provides the enhanced HEX-8 Brick element with translational as well as rotational capability. Figure 5.7 shows the modified HEX-8 Brick element with 6 DOFs per node.

![Figure 5.7: Modified HEX-8 Brick Element](image-url)
The beam element developed in Chapter 2 has two degrees of freedom per node. This element is modified by assigning only three rotational DOFs to each node. For the modified beam element, each node is assigned three degrees of freedom. The local degrees of freedom for each element are represented by

\[
d = [d_1, d_2, d_3, d_4, d_5, d_6]
\]

The energy Equation for this element is shown in Equation 4.84. The degrees of freedom for the modified element increase and are shown in the vector form in Equation 4.84.

\[
U = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix}^T \begin{bmatrix} K_{pm-Beam-Modified} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix}
\]

The modified beam element with an added degree of freedom is shown in Equation 5.96.

\[
K_{pm-Beam-Modified} = \begin{bmatrix}
    k_{11_{pm}}(d_1) & k_{12_{pm}}(d_1) & k_{13_{pm}}(d_1) & k_{14_{pm}}(d_2) & k_{15_{pm}}(d_2) & k_{16_{pm}}(d_2) \\
    k_{21_{pm}}(d_2) & k_{22_{pm}}(d_2) & k_{23_{pm}}(d_2) & k_{24_{pm}}(d_2) & k_{25_{pm}}(d_2) & k_{26_{pm}}(d_2) \\
    k_{31_{pm}}(d_3) & k_{32_{pm}}(d_3) & k_{33_{pm}}(d_3) & k_{34_{pm}}(d_3) & k_{35_{pm}}(d_3) & k_{36_{pm}}(d_3) \\
    k_{41_{pm}}(d_4) & k_{42_{pm}}(d_4) & k_{43_{pm}}(d_4) & k_{44_{pm}}(d_4) & k_{45_{pm}}(d_4) & k_{46_{pm}}(d_4) \\
    k_{51_{pm}}(d_5) & k_{52_{pm}}(d_5) & k_{53_{pm}}(d_5) & k_{54_{pm}}(d_5) & k_{55_{pm}}(d_5) & k_{56_{pm}}(d_5) \\
    k_{61_{pm}}(d_6) & k_{62_{pm}}(d_6) & k_{63_{pm}}(d_6) & k_{64_{pm}}(d_6) & k_{65_{pm}}(d_6) & k_{66_{pm}}(d_6)
\end{bmatrix}
\]
These elements of $K_{pm,B}$ in index notation are given by the Equations 5.97 and 5.98. The constants $A_i$, $B_i$, $C_j$ and $D_j$ are given in Table 5.4. Subscript $i$ and $j$ denote the node numbers.

$$k_{ij \cdot pm} = \frac{0.866}{d_i} \left( C_j \sinh^{-1} \left( A_i d_i \right) - D_j \sinh^{-1} \left( B_i d_i \right) \right) \quad \text{for } i = 1,2,3,4,5,6 \text{ and } j = 1,3,5$$

(5.97)

$$k_{ij \cdot pm} = \frac{l_e}{d_i} \left( C_j \sinh^{-1} \left( A_i d_i \right) - D_j \sinh^{-1} \left( B_i d_i \right) \right) \quad \text{for } i = 1,2,3,4,5,6 \text{ and } j = 2,4,6$$

(5.98)

<table>
<thead>
<tr>
<th>$i$</th>
<th>$A_i$</th>
<th>$B_i$</th>
<th>$C_j$</th>
<th>$D_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.732 \frac{y}{e_o l e}$</td>
<td>$-2.732 \frac{y}{e_o l e}$</td>
<td>0.183</td>
<td>0.683</td>
</tr>
<tr>
<td>2</td>
<td>$0.732 \frac{y}{e_o l e}$</td>
<td>$-2.732 \frac{y}{e_o l e}$</td>
<td>0.183</td>
<td>0.683</td>
</tr>
<tr>
<td>3</td>
<td>$0.732 \frac{y}{e_o l e}$</td>
<td>$-2.732 \frac{y}{e_o l e}$</td>
<td>0.183</td>
<td>0.683</td>
</tr>
<tr>
<td>4</td>
<td>$2.732 \frac{y}{e_o l e}$</td>
<td>$-0.732 \frac{y}{e_o l e}$</td>
<td>0.683</td>
<td>0.183</td>
</tr>
<tr>
<td>5</td>
<td>$2.732 \frac{y}{e_o l e}$</td>
<td>$-0.732 \frac{y}{e_o l e}$</td>
<td>0.683</td>
<td>0.183</td>
</tr>
<tr>
<td>6</td>
<td>$2.732 \frac{y}{e_o l e}$</td>
<td>$-0.732 \frac{y}{e_o l e}$</td>
<td>0.683</td>
<td>0.183</td>
</tr>
</tbody>
</table>

Table 5.4: Constants for Equations 5.97 and 5.98.
The total degrees of freedom for the Modified Brick fatigue element obtained by addition of the modified beam and HEX-8 elements are given by Equation 5.99. The energy Equation for this new HEX-8 Modified Brick element is given in Equation 5.100.

\[
d = \begin{bmatrix}
d_1, d_2, d_3, \ldots, d_{22}, d_{23}, d_{24} \\
\theta_{x1}, \theta_{y1}, \theta_{z1}, \ldots, \theta_{x8}, \theta_{y8}, \theta_{z8}
\end{bmatrix}^T
\]  
(5.99)

\[
U = \begin{bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_{24} \\
\theta_{x1} \\
\theta_{y1} \\
\theta_{z1} \\
\vdots \\
\theta_{x8} \\
\theta_{y8} \\
\theta_{z8}
\end{bmatrix} - \begin{bmatrix}
K_{pm-Brick-Mod}
\end{bmatrix} \begin{bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_{24} \\
\theta_{x1} \\
\theta_{y1} \\
\theta_{z1} \\
\vdots \\
\theta_{x8} \\
\theta_{y8} \\
\theta_{z8}
\end{bmatrix}
\]  
(5.100)

The Modified element is given by Equation 5.101.

\[
K_{pm-Brick-Mod} = \begin{bmatrix} K_{pm-BrickPart} \\ O \\ O \\ K_{pm-BeamPart} \end{bmatrix}
\]  
(5.101)
The new HEX-8 Modified Brick element has the capability to model three
dimensional structures with all the translational and rotational degrees of freedoms. The
analysis results performed using this element are presented in the following Section.
5.4 Results and Discussion

The curved plate discussed in Chapter 3 is shown in the Figure below. This plate is discretized with HEX-8 Modified Brick elements. The plate is fixed at one end and is subjected to bending load at the other end. The analysis is performed to verify that the stress results and the fatigue life prediction with the new HEX-8 element correlate with each others.

![Curved Plate Meshed with Modified Brick Elements](image)

Figure 5.8: Curved Plate Meshed with Modified Brick Elements

The following Figures present the stress and fatigue life analysis results. The stresses are high in the area closer to the fixed end. The low fatigue life is predicted at the same locations where high stresses are present.
Figure 5.9: Stress Results with Modified Brick Elements

Figure 5.10: Fatigue Life Prediction with Modified Brick Elements
In order to have a comparison of results from analysis with the new brick element to those presented in Chapter 3, the curved plate meshed with the new brick element is excited to a stress level of 25 ksi for 3rd stripe mode and the stress and prediction results are compared to the equivalent stress approach prediction performed in Chapter 3. Equation 5.93 is modified for this analysis into a dynamic analysis Equation [5.12] as given by Equation 5.104.

\[
[M][\dot{d}] + [K][d] = \{F_{\cos \omega t}\}
\]  

(5.104)

Where \( M \) is the mass matrix and \( \omega \) is the frequency. The displacement and stress contours from both the analysis match with each other and the maximum stresses are located away from the fixed end. The following Figures show the displacement, vibratory stress and the fatigue life prediction results for this analysis.

![Figure 5.11: 3rd Stripe Mode Shape Results (Brick Element)](image-url)
Figure 5.12: 3rd Stripe Mode Shape Stress Results (Brick Element)

Figure 5.13: 3rd Stripe Mode Shape Stress Results (Brick Element)
The following Table shows a comparison of results from the vibration analysis performed on the curved plate meshed with the new Brick element. The results match well with the equivalent stress approach discussed in Chapter 3.

<table>
<thead>
<tr>
<th>Stress (ksi)</th>
<th>FE Analysis Equivalent Stress Approach</th>
<th>FE Analysis Brick Element</th>
</tr>
</thead>
<tbody>
<tr>
<td>Curved Plate</td>
<td>20</td>
<td>9.16 x E5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8.99 x E5</td>
</tr>
</tbody>
</table>

Table 5.5: Comparison of Brick Element Results with Equivalent Stress Approach

Another set of comparison is made for type 304 stainless steel experimental data [5.6, 5.7] discussed in Chapter 3, the equivalent stress fatigue life prediction performed in Chapter 3 and the fatigue analysis results performed using the new HEX-8 element. The results are shown in Figure 5.14 and a comparative data is presented in Table 5.6. The life prediction results performed using the HEX-8 element show a good match to the experiential data as well as equivalent stress approach.
Figure 5.14: Comparison for Fatigue Life Prediction Results for Type 304 Stainless Steel

<table>
<thead>
<tr>
<th></th>
<th>Stress (ksi)</th>
<th>Experiment Life Cycles</th>
<th>Life Cycles Eqv. Stress</th>
<th>Life Cycles HEX-8</th>
<th>Percent Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>43.5</td>
<td>5044</td>
<td>5250</td>
<td>5235</td>
<td>3.79</td>
</tr>
<tr>
<td>2</td>
<td>46.4</td>
<td>4234</td>
<td>4401</td>
<td>4388</td>
<td>3.65</td>
</tr>
<tr>
<td>3</td>
<td>50.75</td>
<td>3767</td>
<td>3915</td>
<td>3904</td>
<td>3.64</td>
</tr>
</tbody>
</table>

Table 5.6: Fatigue Results for Type 304 Stainless Steel
5.5 Conclusion

The newly developed HEX-8 (Brick) and Modified Brick elements provide useful tool for multiaxial fatigue life prediction. The results presented in Section 5.4 show that the new Brick element can predict the fatigue life of structural component with improved accuracy. As these elements are developed from the energy-based constitutive law for fatigue life prediction, analysis with these elements incorporates the fatigue mechanism into fatigue analysis. Due to the discrete nature of finite element analysis, new fatigue elements can predict number of cycles to failure at each location in the 3-D structures. The equivalent stress fatigue analysis discussed in Chapter 3 follows part of the conventional analysis approach [5.13-5.15] as it obtains the stresses through dynamic analysis using ordinary HEX-8 elements. While in the analysis with the new HEX-8 element, the mode shapes and stresses are obtained from direct analysis of the component meshed with the HEX-8 fatigue elements. Therefore, this analysis approach skips multiple analysis steps and makes the prediction a one step procedure. As in the case of uniaxial and QUAD-4 elements, the new HEX-8 elements developed in this Chapter predict crack initiation whereas most research in this area [5.16-5.25] is related to crack propagation. This fact, along with integration of the fatigue constitutive law into the fatigue analysis mechanism, establishes the difference of the newly developed HEX-8 element from the existing fatigue FEA software and related research.
5.6 List of References


CHAPTER 6

HIGH STRAIN RATE ASSESSMENT OF COLD WORKED
ADVANCED HIGH STRENGTH STEEL

6.1 Introduction

Chapter 6 presents an assessment of cold-worked AHSS through high strain rate experiments. The experimental data is further used to develop an accurate LSDYNA material model for crash simulations.

Figure 6.1: A Vehicle Crash Simulation Model
The assessment is made through comparisons of new data and the newly developed LSDYNA material model to previously developed models for AHSS for flat sheets. The new model aims at better approximation of performance of a vehicle during crash.

The performance of the vehicle during a crash is an important subject in automobile research. In order to simulate an actual crash using software like LSDYNA, it is desirable to have accurate stress/strain data for materials. The material models available in literature ignore the effect of cold working on the material and present data only for flat sheets. The current models are based on high strain experiments on specimens cut from the flat sheets [6.1-6.4]. In a vehicle, most components are cold-worked either in the form of rolling or bending. It is well known that the cold-working changes the properties of the material, but, the cold-worked materials have never been tested before to study the effect of cold-working on the high strain rate behavior of the AHSS. In this research, cold worked AHSS with curved cross-sections is tested at strain rates of 1000 (in/in) s\(^{-1}\) and data is used to develop a corresponding LSDYNA material model for vehicle crash simulation.

This research involves design of the specimen for the cold-worked (rolled) material using finite element analysis, high strain rate testing using a split Hopkinson bar technique and finally development of the LSDYNA material model based on the experimental data.

Finite element analysis is widely used in analyzing the specimen and grip design for high strain rate testing [6.5]. As stated earlier, the material being tested in this research is cold worked and has a curved cross section. Therefore, the setup requires a special
design to take care of the shifted center in order to avoid bending and achieve a uniform stress distribution in the gauge section. The design steps followed to reduce the bending in the curved specimen are discussed in detailed in Section 6.1.4. In addition to taking care of shifted center, finite element analysis also helps in reducing the bulk in the grip, yet still maintaining rigidity, incorporate different fillet and angle dimensions to reduce stress concentration thus reducing the probability of grip failure and improve the uniform nature of the tensile stress distribution through test specimen cross-section based on grip geometry changes.

Experimental testing for low strain rates can be performed on mechanical test machines with reasonable test accuracy. However, for high strain rate testing, due to the very short time available for test and the limitations of test setup, mechanical testing produces scattered test data and a ringing effect is observed. Therefore, the researchers have used Hopkinson test bars successfully for high strain rate testing to obtain accurate stress/strain data [6.3]. The following Figure shows schematic to elaborate the concept of Hopkinson bar.

Figure 6.2: The Split Hopkinson Bar Schematic
6.1.1 Hopkinson Bar Apparatus

As stated earlier, Hopkinson bar is used for testing the AHSS at high strain rates in excess of 1000 (in/in) s\(^{-1}\). The Split Hopkinson bar apparatus consists of a striker bar, an incident bar, the test specimen, and the output bar. A rectangular compression wave of well defined amplitude and length is generated in the incident bar when it is struck by the striker bar powered by a hydraulic pump. The striker bar strikes the incident bar as if a bullet is fired onto a circular bar. When the wave reaches the specimen some of it transmits through it and some of it reflects back through the incident bar. One dimensional wave propagation analysis determines high strain rate stress-strain curves from measurements of strain in the incident and output bars. Strain gages are mounted on the incident bar, transmission bar, and the specimen, respectively. Assuming that wave propagation in the bars is nondispersive, the force and displacement at the contact between the bar and specimen can be derived from the measured strains. The strain rates produced by this device depend on the length of the specimens. An oscilloscope is used for data acquisition along with a data conditioning box. The data is transferred to a PC and is further processed to obtain the stress strain curve.

![Figure 6.3: The Hopkinson Bar Apparatus](image)
6.1.2 Stress Strain Curve

The Equations used to create the stress-strain curve from a Split Hopkinson Bar (SHPB) test follow from dynamic wave propagation theory [6.6]. A schematic of split Hopkinson bar is shown in the following Figure.

![Hopkinson Bar Schematics](image)

Figure 6.4: The Hopkinson Bar Schematics

As stated in Section 6.1.1, a part of wave is transmitted through the specimen and the rest is reflected. The strain rate is found from the strain in the incident bar caused by the reflected wave:

\[ \varepsilon^o (t) = \frac{2C_o}{L} \varepsilon_R (t) \quad (6.1) \]

Where \( \varepsilon^o (t) \) is specimen strain rate, \( \varepsilon_R (t) \) is strain caused by the reflected wave in the incident bar, \( L \) is the length of specimen before impact, \( C_o \) is infinite wave length
velocity in incident bar and is given by $C_o = \sqrt{\frac{E}{\rho}}$, $E$ is modulus of elasticity of incident bar and $\rho$ is density of incident bar. The strain is found by direct integration of the above Equation:

$$\varepsilon_s (t) = \frac{2C_o}{L} \int_0^t \varepsilon_R (t) dt$$  \hspace{1cm} (6.2)

Where $\varepsilon_s (t)$ is specimen strain.

The stress values can be found by 1D wave analysis, using only the transmitted wave, as in following Equation,

$$\sigma_s (t) = E_o \frac{A_o}{A} \varepsilon_T (t)$$  \hspace{1cm} (6.3)

Where $\sigma_s (t)$ is stress in the specimen, $A_o$ is area of output bar, $A$ is area of specimen and $E_o$ is modulus of elasticity of output bar.

6.1.3 Development of LSDYNA Material Model:

The data acquired from the experiment is used to develop a LSDYNA material model for vehicle crash simulations. The experimental test data obtained from the strain rate testing is used to obtain stress strain curves for the material being tested. The following Figure shows a stress strain curve.
The Equations for engineering stress and engineering strain are given below.

\[ S = \frac{P}{A_0} \]  \hspace{1cm} (6.4)

\[ \varepsilon = \frac{\Delta l}{l_0} \]  \hspace{1cm} (6.5)

These curves are converted to true stress vs. true strain using following Equation.

\[ \varepsilon_t = \ln(1 + \varepsilon) \]  \hspace{1cm} (6.6)

\[ S_t = S(1 + \varepsilon) \]  \hspace{1cm} (6.7)

The true stress vs. true strain curves obtained, are averaged into a single curve. Usually, for ductile materials, the strength drops a bit at yield point. However, materials do not fail after yielding and go through a strain hardening process. The power law is used to represent this behavior.

\[ \sigma_{true} = K(\varepsilon_{true})^n \] \hspace{1cm} (6.8)
Where “n” is the strain hardening exponent and the “K” is the strength coefficient. Finding “K” and “n” and then using power law Equations, allows extrapolation of true strain values greater than what can be measured during the tensile test. In the next step, plastic stress vs. strain model is created for LS-DYNA using following Equation.

\[
\varepsilon_{plastic} = \varepsilon_{total} - \frac{\sigma_{true}}{E}
\]  

(6.9)

For creation of the LSDYNA model, it is required that the data is evenly spaced into 100 points to be compatible with the software. This is done using Altair’s Hyper view software by using the “resample” function.

6.1.4 Flat Specimen vs. Curved Specimen

The following Figures show a difference between a curved (cold-worked material) specimen and a flat specimen.
The flat specimen comes from a flat sheet of material. The curved specimen comes from a material rolled into a hollow shaft. The specimen retains the curvature in its cross-section and is not symmetric. When this specimen is subjected to tension, it experiences a bending which means that the experiment is no more a tension experiment and needs a different approach to obtain a tensile data. The following Figure shows finite element results from an analysis of a curved specimen subjected to tension. It can be seen that the stress distribution is not uniform through the cross section of the specimen due to presence of bending stress.
6.2 Design of Grips (Adapters) and Specimen using Finite Element Analysis

Finite element results for flat specimen and a curved specimen are used to design a curved specimen and adapters suitable for these experiments.

As stated earlier, the finite element analysis is widely used in analyzing the specimen and grip design for high strain rate testing. The type of test specimen used for analysis is based on ASTM standard D638, which provides standard test specimen geometry and dimension for plastics and metallic metals respectively. Pre- and post-processing for this FEM analysis on the test setup is conducted in Hypermesh and IDEAS, where as the actual FEM analysis is run in ANSYS. A cylindrical shaped grip is chosen due to symmetric geometry. A 10 Node Tetrahedron Mesh is used for the analysis and specimen is fixed to the grip using a glue element available in ANSYS. Pins are used for additional safety. The setup is fixed on one side while a tensile force is applied on the other side. Figure 6.9 shows the geometry and mesh for the analysis. The stress results (von-Mises) are shown in Figure 6.10.
The cross-section for a flat specimen is symmetric. Therefore, the stress distribution across the cross-section is even and results obtained from the analysis verify that it is a tensile experiment.

The cross-section for the curved specimen is no longer symmetric. Therefore, the curved specimen when subjected to tension undergoes bending. In order to eliminate the
effect of bending in the specimen, the shear center of the curved cross-section is calculated and the specimen is placed inside the grips such that the load line passes through the shear center of the specimen and the centroid of the grip. This new solution works and is verified through finite element analysis before performing the actual experiments. The following Figures show the finite element model and results for the curved specimen placed inside the grip with its shear center to match with the load line and center of the grip.

Figure 6.11: Geometry and Finite Element Model of Curved Specimen Test Setup
The results shown in the above Figure indicate that the effect of bending on the curved specimen is reduced by placing the specimen with its shear center aligned with the load line and the center of the grip.

The following Figure shows the axial, in-plane and out of plane stresses on the specimen subjected to tension.
These results are compared to the flat specimen results in the following Table to show that shear center solution for curved specimen works well and eliminates the bending effect on the specimen thus making it a uniaxial tension experiment.

<table>
<thead>
<tr>
<th></th>
<th>Flat Specimen</th>
<th>Curved Specimen</th>
</tr>
</thead>
<tbody>
<tr>
<td>Von-Mises Stress (Ksi)</td>
<td>61.2</td>
<td>65.3</td>
</tr>
<tr>
<td>Displacement (In)</td>
<td>8.2 x 10-3</td>
<td>1.04 x 10-2</td>
</tr>
<tr>
<td>Y-Axis Stress (Tension)</td>
<td>65.4</td>
<td>61.6</td>
</tr>
<tr>
<td>X-Axis Stress (In-Plane)</td>
<td>8.8</td>
<td>11.4</td>
</tr>
<tr>
<td>Z-Axis Stress (Out of Plane)</td>
<td>9.4</td>
<td>19.3</td>
</tr>
</tbody>
</table>

Table 6.1: Comparison of Finite Element Results for Flat and Curved Specimens

The FEM results (Figure 6.12) provide a reasonable idea for grip and specimen design for high strain rate testing. The results also help in finding the center of specimen with respect to grip where the bending effect is minimal. Based on these results, a customized specimen and grips (adapters) are designed for experiments for this project and are shown in the following Figures.
Figure 6.14: Design for Curved Specimen

Figure 6.15: Design for Grips
6.3 High Strain-Rate Experiments

The specimen is cut from a hollow shaft of AHSS using an EDM machine in order to retain the curvature. The following Figures show the specimen and grips used for these experiments.

Figure 6.16: Grips and Specimen

The following experiments have been performed for this research.

Experiment 1: Performed as a trial experiment to verify the setup. Only one strain gage is used for this experiment.

Experiment 2-5: Two strain gages each on outer and inner curvature are used for these experiments to check for bending.

Experiment 4-5: A high speed video recorded.
The following Figure shows some pre- and post-experiment images. Two strain gauges, one on each side of the specimen as shown in the middle picture in the following Figure, are pasted to check for the bending. The specimen is glued inside the grips and the grips are glued to the bar.

![Experimental Set-up](image)

Figure 6.17: Experimental Set-up

6.4 Experimental Results

The results obtained from these experiments are presented in the following Sections.

6.4.1 Experiment 1

Experiment 1 was performed as a trial experiment to test the experimental set-up. Only one strain gage was used on the specimen for this experiment. The results obtained are shown in the following Figures. These results are not used for any further comparisons or LSDYNA model preparation because the only purpose was to test and verify the equipment.
Figure 6.18: Stress vs. Strain Results for Experiment 1

Figure 6.19: Multi-Scale Results for Experiment 1
6.4.2 Experiment 2

Experiment 2 was performed with two strain gages, one on the outside curve and other on the inside curve of the specimen. The two strain gages were used to check for any bending going on in the specimen. The results obtained are shown in the following Figure. The results from both the strain gages show a good match and verify that no significant bending is taking place in the specimen. The strain gage results also match well with engineering stress vs. engineering strain curve. The stress strain curves for both strain gages, exactly match thus proving that no bending is taking place in the specimen.

Figure 6.20: Stress vs. Strain Results for Experiment 2

Figure 6.21 shows a multi-scale plot with stress vs. strain data and strain rate vs. time data plotted on a single chart to provide another type of comparison for the data. Strain obtained from strain gages match well with the engineering strain, thus further validating that no bending is going on.
Figure 6.21: Multi-Scale Results for Experiment 2

Figure 6.22 shows a time history plot for the experiment and provides information about the health of the experiment. The location of gages A, B and C on the Hopkinson bar is shown in the Figure 6.22. The data for gage A, gage B and gage C match well which shows that all the test equipment was well balanced and synchronized. The forces for gage B and C become equal at the time during the experiment when the wave has propagated through the specimen. This holds the assumption made for the Hopkinson bar calculations for the strain rate.
6.4.3 Experiment 3

Experiment 3 was also performed with two strain gages, one on the outside curve and other on the inside curve of the specimen. The results obtained are shown in the following Figures. The results from this experiment for both the strain gages show a good match. The strain gage results also match well with engineering stress vs. engineering strain curve. The stress strain curves for both strain gages, exactly match thus proving that no bending is taking place in the specimen.
Figure 6.23: Stress vs. Strain Results for Experiment 3

Figure 6.24: Multi-Scale Results for Experiment 3
6.4.4 Experiment 4

Experiment 4 was also performed with two strain gages, one on the outside curve and other on the inside curve of the specimen. The results obtained are shown in the following Figures. The results from this experiment for both the strain gages show a good match. The stress strain curves for both strain gages, exactly match thus proving that no bending is taking place in the specimen. The strain gage results also match well with engineering stress vs. engineering strain curve.

A high speed video was also added to record the failure event. The set-up worked well and can be further improved with two or three cameras if more accurate results are desired.
Figure 6.26: Stress vs. Strain Results for Experiment 4

Figure 6.27: Multi-Scale Results for Experiment 4
6.4.5 Experiment 5

Experiment 5, the last experiment was performed to obtain an additional set of data. The results are shown in the following Figures.
Figure 6.30: Multi-Scale Results for Experiment 5

Figure 6.31: Time History Plot for Experiment 5
6.4.6 Comparison of Experiment 2, 3, 4 and 5

Figure 6.32 shows a comparison of results from experiment 2, 3, 4 and 5. The curves match well which further validates the repeatability of the experiment for producing good results. The stress vs. strain curves from all the experiments sit on the top of each other. The maximum variation in stress at the yield point is 7ksi. The maximum variation in the ultimate strength is 3ksi.

![Comparison of Results for Experiments 2, 3, 4 and 5](image)

Figure 6.32: Comparison of Results for Experiments 2, 3, 4 and 5

The following Figure shows the comparison of time-history plots for these experiments. The data shows that the waves for different experiments at locations A, B and C match with each others. Furthermore, the forces at location B and C for all the experiments are equal and hold the Hopkinson bar assumptions.
6.5 Development of a LSDYNA Material Model

The final objective of this research is to use the experimental data to develop a LSDYNA material model for vehicle crash simulation. In order to obtain the LSDYNA material curve, the steps detailed in Section 6.1.3 are used to develop the LSDYNA material model.

6.5.1 Converted Data Curve

The first step is to convert engineering stress/strain data to true stress-strain data. Equations 6.6 and 6.7 from Section 6.1.3 are used to obtain the curve shown in the following Figure.

Figure 6.33: Comparison of Time History Results for Experiments 2, 3, 4 and 5
6.5.2 The Average Data Curve

The following Figure shows an average data curve for all the experimental data. The curve is obtained as an average of all the experimental data in order to use for development of the LADYNA material model.
6.5.3 Strain Hardening Exponent (n) and Strength Coefficient (K)

The following Figure shows log of true stress vs. log of true strain plotted to get the slope of the curve and the y-intercept. These parameters provide the strain hardening exponent (n), and the strength coefficient (K).

Figure 6.35: Average Data Curve

Figure 6.36: Strain Hardening Exponent and Strength Coefficient
6.5.4 Power Law Fit

The following Figure shows a power law fit to the data using the parameters obtained from previous Section. The data is extended to 100% strain to ensure compatibility with the LSDYNA.

![Figure 6.37: Power Law Fit](image)

6.5.5 Plastic Curve:

The following Figure shows the plastic strain curve obtained using the Equation (6.9) from the Section 6.1.3. The data is extended to 100% strain in order to be compatible with the LSDYNA format.
This new model, when compared to flat sheet experiments, shows that both the models differ from each other. The cold-working not only changes the strength of the material but the behavior of the material is also changes under high strain rate experiments. The following Figure shows a comparison of flat sheet model and the cold-worked sheet model.
6.6 Conclusion

A need for the high strain rate experiments for cold-worked high strain rate experiments and subsequent development of a new LSDYNA material model are discussed in detail in this Chapter. The design of a curved specimen with uniform stress distribution across the thickness and compatible grips arrangement presented in this Chapter provide a useful procedure for design of curved specimen via finite element analysis. The data obtained from high strain experiments show that specimen experiences uniform stresses across the thickness of the material. The comparison of test results from various experiments show repeatability of these experiments. Data is further used to
develop a LSDYNA material model. The model is compared to flat sheet material model for advanced strength high strength steel and the difference between the two models signifies that the cold-worked simulation must be performed with the new model based on curved specimen experiments.
6.7 List of References


CHAPTER 7

CONCLUSIONS

Previous Chapters 2, 3, 4 and 5 cover the development of new finite elements for fatigue life prediction. Validation of these new finite elements through benchmarking, application of these elements to fatigue life prediction problems and comparison of results to analytical and experimental data [7.1, 7.2] has been discussed in detail in these Chapters. Chapter 6 details the development of a new experimental method for high strain rate testing of cold worked advanced high strength steel and development of a new LSDYNA material model. Comprehensive discussion of the new developments and results are included at the end of each Chapter. This Chapter presents the brief summary of these developments and sums up the results and conclusions.

7.1 Conclusions

Section 2.3.1 presents the development of a new finite element for uniaxial (Tension/Compression) fatigue life prediction. Equations 2.22 to 2.25 show the new K-Matrices for the uniaxial (Rod) element. The fatigue life prediction results for the new rod element are presented and discussed in Section 2.5.1. The new element is applied to completely reversible loading and is also capable of handling the mean stress effect. The analysis for Al 6061-T6 is performed for Al 6061-T6 and Ti-6Al-4V. The results are
compared to analytical as well as experimental results [7.1]. The results show that new fatigue element (rod) can predict the life prediction for uniaxial tension/compression fatigue problems with considerable accuracy. The following Figure shows the life prediction results for Al 6061-T6 and compares it to experimental data and analytical prediction [6]. The FEM results show a good match to analytical and experimental data.

![Figure 7.1: Stress vs. No. of Cycles for Al 6061-T6 for Completely Reversible Axial Load](image)

Section 2.3.2 presents the development of new finite element for bending fatigue (Beam Element). Equations 2.89, 2.90, 2.93 and 2.94 provide the new K-matrices for bending fatigue. Analysis is performed for Al 6061-T6, both for completely reversible as well as loads with mean effect stress. The results are presented and discussed in Section 2.5.2. Application of this new beam element to Al 6061-T6 and comparison to analytical...
as well as experimental results [7.1] show that the beam element has the capability to predict fatigue life with considerable accuracy. The following Figure shows life prediction results for Al 6061-T6 for completely reversible loading.

![Figure 7.2: Stress vs. No. of Cycles for Al 6061-T6 for Completely Reversible Bending Load](image)

As discussed in Section 2.6, the new finite elements (rod and beam) provide a useful tool for fatigue life prediction in structural components like gas engine turbine blades. The accurate prediction of number of cycles with new axial and bending finite elements and a good match of results to experimental data and analytical results [7.1] signifies that new finite elements provide sufficient estimation of number of cycles for axial and bending loads.

These new axial (rod) and bending (beam) elements are developed from a fatigue based constitutive law. The fact that these elements incorporate the fatigue mechanism into the analysis procedure differentiates these new developments from the existing finite elements.
element procedure [7.5-7.7]. The new fatigue finite elements predict the crack initiation, whereas most research available in this area focuses on crack propagation [7.6-7.16]. Furthermore, the new finite element method is more useful due to the discrete nature of the finite element method. The new finite element for bending fatigue life prediction has the capability to predict varying number of cycles in the structural component experiencing variable stress at different locations. The colored plots can be obtained where each color signifies respective fatigue life for each element present at different locations in the structure. The results from rod and beam element analysis are published in [7.17, 7.18].

This research deals with the elastic and plastic parts of Equations 2.1 and 2.2 separately and ignores any coupling between elastic and plastic strains. However, in the real world, this may not be the case. As it transpires, this coupling becomes stronger with the increasing stress in particular above the yield point. The fatigue analysis is performed only below the yield point. Therefore, the level of applied stress remains low enough to cause any significant error due to coupling on the final results for number of cycles to failure. This is also evident from Figures 2.10, 2.11 and 2.12 that the difference between experimental data and FE prediction is very negligible. Therefore, to avoid the complexity of the computation, the coupling between elastic and plastic parts is ignored.

Chapter 3 presents a new energy-based equivalent stress approach and a finite element procedure for multiaxial fatigue life prediction. The need for generating a biaxial fatigue data for biaxial loads is well demonstrated through experiments. The development of the equivalent stress expression based on an improved fatigue life prediction criterion and application of this new expression in a finite element procedure provides a useful tool
for fatigue life prediction in gas turbine engine structural components. The equivalent stress expression for fatigue life prediction is represented by Equation 3.26. The finite element procedure based on this new equivalent stress expression is discussed in Section 3.4. This procedure consists of steps to perform normal mode analysis and forced response analysis in order to obtain maximum vibratory stresses and then application of these stresses to Equation 3.26 in order to obtain the number of cycles to failure. The prediction of the number of cycles with the new finite element procedure and a comparison of results to experimental data signifies that the new finite element procedure provides encouraging estimation of the number of cycles for biaxial fatigue loading. These results are published in [7.19].

Furthermore, the new finite element procedure is more useful due to the discrete nature of the finite element method. The capability of this new approach to predict fatigue life for each location in the structural component provides a complete visual picture of the fatiguing process in the component. The following Figures show application of this new energy-based equivalent stress expression and subsequent finite element procedure to predict fatigue life for biaxial and multiaxial configurations.
Figure 7.3: Numbers of Cycle to Failure Results for Winger Plate (Al 6061-T6)

Figure 7.4: Numbers of Cycle to Failure Results for Curved Plate (Al 6061-T6)
New QUAD-4 (In-Plane) and Plate elements are developed in Chapter 4. The QUAD-4 element is benchmarked using the standard benchmarking techniques for validation and is further modified to obtain a QUAD-4 plate element for fatigue life prediction. The benchmarking results are presented in Tables 4.2 and 4.3. The plate element is obtained by combining a beam element developed in Chapter 2 and the newly developed QUAD-4 (In-Plane) element. The newly developed QUAD-4 (In-plane) and Modified QUAD-4 (Plate) elements provide a useful tool for biaxial fatigue life prediction. The results presented in Section 4.4 show that new QUAD-4 element can predict the fatigue life of structural component with considerable accuracy. As these elements are developed from energy-based constitutive law for fatigue life prediction, these elements incorporate the fatigue mechanism into fatigue analysis. Due to the discrete nature of finite element analysis, new fatigue elements can predict number of cycles to failure at each location in the 2-D structures. The equivalent stress fatigue analysis discussed in Chapter 3 follows part of the conventional analysis approach [7.3-7.5] as it obtains the stresses through dynamic analysis using ordinary QUAD-4 elements. When performing the analysis with the new QUAD-4 element, the mode shapes and stresses are obtained from direct analysis of the component meshed with the QUAD-4 fatigue elements. Therefore, this analysis skips multiple analysis steps and makes the prediction one step analysis. Furthermore, the prediction results with QUAD-4 fatigue elements show closer comparison to experimental results as compared to the equivalent stress approach. The comparison of the new fatigue QUAD-4 element analysis results to experimental and equivalent stress approach is presented in Tables 4.7.
The newly developed HEX-8 (Brick) and Modified Brick elements in Chapter 5 provide a useful tool for multiaxial fatigue life prediction. The new brick element is also benchmarked using the standard techniques and the results are presented in Tables 5.2 and 5.3. The modified brick element is developed by adding the three rotational degrees of freedom (DOF) to each node for the brick element. Therefore, the total DOFs per node for brick element become 6 for the modified brick element. The results presented in Section 5.4 show that new Brick element can predict the fatigue life of structural component with sufficient accuracy. As these elements are developed from energy-based constitutive law for fatigue life prediction, analysis incorporates the fatigue mechanism into fatigue analysis. Due to the discrete nature of finite element analysis, new fatigue elements can predict number of cycles to failure at each location in the 3-D structures. The equivalent stress fatigue analysis discussed in Chapter 3 follows part of the conventional analysis approach [7.3-7.5] as it obtains the stresses through dynamic analysis using ordinary HEX-8 elements. While in the analysis with the new HEX-8 element, the mode shapes and stresses are obtained from direct analysis of the component meshed with the HEX-8 fatigue elements. Therefore, this analysis in comparison to equivalent stress approach skips multiple analysis steps and makes the prediction a one step procedure. The comparison of results between equivalent stress approach and new brick element analysis is presented in Table 5.6 in Chapter 5. The following Figure show prediction of fatigue life for a three dimensional structure using new brick (modified) element.
Chapter 6 presents a high strain rate assessment of advanced high strength steel (AHSS) through experiments and further develops a LSDYNA material model for vehicle crash simulation. As stated in Chapter 6, the current material models available for vehicle crash simulation are based on strain rate experiments on flat sheets of materials [7.20-7.23]. On the other hand, the most components installed in the automobiles are cold-worked. Therefore, in order to obtain a closer approximation of a vehicle crash event, it is desired that a cold worked material be assessed for high strain rate experiments and new materials models be developed for LSDYNA simulations. Testing of cold-worked materials involves high strain rate experiments using the Hopkinson Bar apparatus discussed in detail in Chapter 6. Due to the curved geometry of the specimen, the design of the specimen and grips require a different approach.
The following Figure shows that when a curved specimen is subjected to uniaxial tension, it experiences a bending thus, indicating that the experiment is no longer a uniaxial tension experiment.

![FEM Results for Curved Specimen Subjected to Uni-Axial Tension](image)

Figure 7.6: FEM Results for Curved Specimen Subjected to Uni-Axial Tension

The new approach for design of a specimen for high strain rate experiments is presented in Chapter 6. This involves finite element analysis [7.24] to obtain an appropriate geometry of the specimen which produces uniform stresses across the thickness of the specimen when subjected to tension. The new setup which includes a curved specimen and grips as fixtures provides the bending free results and uniform stresses across the thickness of the specimen.

The experimental results performed using the Hopkinson bar apparatus are discussed in detail in Section 6.4. The stresses are measured on both sides of the specimen and the results obtained signify that the specimen has gone through uniaxial tension. Furthermore, the results obtained from various experiments are superposed and
further validate the repeatability of the experiments. Figure 7.7 shows a comparison of the results from experiment 2, 3, 4 and 5. The curves match well which further validates the repeatability of the experiment for producing good results. The stress vs. strain curves from all the experiments sit on the top of each other. The maximum variation in stress at the yield point is 7ksi. The maximum variation in the ultimate strength is 3ksi.

![Figure 7.7: Comparison of Converted Data for Experiments 2, 3, 4 and 5](image)

The test data is used further to develop a LSDYNA material model for vehicle crash simulation. The following Figure presents the final material model developed for advanced high strength steel compatible to LSDYNA for vehicle crash simulation.
This new model, when compared to flat sheet experiments shows that the new model differs from the flat sheet model. The following Figure shows a comparison of flat sheet model and the cold-worked sheet model. The cold-working not only changes the strength of the material but the high strain rate behavior of the material is also changed.
Future Work

Fatigue life prediction is a vast area of research and demands continued research in order to improve upon the existing prediction methods. Furthermore, due to lack of appropriate constitutive Equations, the finite element analysis tools available for fatigue analysis are limited in number as well as effectiveness.

The newly developed finite elements discussed in this dissertation provide a platform for development of a complete computer aided engineering (CAE) package for fatigue analysis. Integration of these new finite elements with CAD and finite element pre- and post-processing software will be a step forward in order to obtain a state of the art tool for fatigue life prediction.
The finite elements developed in this dissertation include a rod (2-Node) with one degree of freedom per node, a beam (2-Node) with two degrees of freedom, a QUAD-4 (In-Plane) and plate element with four degrees of freedom per node and finally a brick element with 48 degrees of freedom per node. These elements can further be modified to adapt to different analysis needs for various applications. For example, modification of QUAD-4 element to QUAD-8 element will provide another tool for applications where QUAD-8 element is required.

Development of triangular and tetrahedral elements has not been discussed in this dissertation. These elements can be developed using the same procedures adopted for QUAD-4 and Brick elements in this dissertation. This will provide another set of elements applicable to analysis needs which require meshing with triangular and tetrahedral elements.

The newly developed elements are applied to static and dynamic analysis in this dissertation. Applications of these elements to other types of analysis and applications like welds can be explored in the future.

An extension of the constitutive law presented in [6] to include thermo-mechanical fatigue (TMF) will provide an opportunity to modify the finite element developed in this dissertation to include temperature effects.

The discrete nature of fatigue analysis performed using the newly developed finite elements in this dissertation provides the opportunity to integrate this analysis into a structural health monitoring system. The system will be able to determine the real time state of the structures undergoing fatiguing under various loads and conditions.
A High Strain Rate assessment has been made for Advanced High Strength steel and a LSDYNA material model is proposed. The research can be extended to include other materials in order to develop matrix of material models used in different real world applications.

The high strain rate experiments can be extended to develop a rate sensitive model, like Johnson-Cook model, for the advanced high strength steel. This will require performing high strain experiments at different strain rates using the already verified set-up and processing the data to obtain rate sensitive model.
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