APPROXIMATION
BY MULTIVARIATE POLYNOMIALS
OF FIXED LENGTH

DISSERTATION

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By

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* * * * *

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[Signature]
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To

The Most Sacred Heart of Jesus

and

The Immaculate Heart of Mary
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CHAPTER I

INTRODUCTION

In several contexts in classical analysis we find results which can be interpreted as results about polynomials of given length. We find trigonometric polynomials of given length $m$

$$\sum_{k=1}^{m} c_k e^{i n_k x}$$  \hspace{1cm} (1.1)

where $n_1 < n_2 < \cdots < n_m$ are integers and $c_k \neq 0$, algebraic polynomials of length $m$

$$\sum_{k=1}^{m} c_k x^{s_k}$$  \hspace{1cm} (1.2)

where $s_1 < s_2 < \cdots < s_m$ are non-negative integers and $c_k \neq 0$, Dirichlet polynomials of length $m$

$$\sum_{k=1}^{m} c_k e^{-s_k t}$$  \hspace{1cm} (1.3)

where $s_1 < s_2 < \cdots < s_m$ are reals and $c_k \neq 0$ (which can by change of variables $x = e^{-t}$, be expressed as (1.2) and in that case (1.2) are known as Dirichlet polynomials)

We also find as a special case of polynomials of length $\leq m$ the so-called incomplete polynomials of G.G. Lorentz [14] i.e. polynomials of the form $x^k P_{m-1}(x)$ where $P_{m-1}$ is any polynomial of degree $m - 1$.

From the point of view of applications in analysis, it seems that the trigonometric
polynomials of given length have played particularly important role. It is easy to see
that if $\sum c_n e^{int}$ is the Fourier series of the function $f \in L_2(0, 2\pi)$, and if the non-
zero Fourier coefficients $\{c_n\}$, $c_n \neq 0$, are rearranged in the order of their decreasing
absolute value to give the sequence $\{c_{\gamma_k}\}$ i.e.

$$|c_{\gamma_1}| \geq |c_{\gamma_2}| \geq \cdots$$

then

$$T_m(t) = \sum_{k=1}^{m} c_{\gamma_k} e^{-i\gamma_k t}$$

is a trigonometric polynomial of length $m$ of best $L_2$-approximation to $f$. This
trigonometric polynomial of best $L_2$-approximation appears in two important con-
texts:

(1) in Stechkin's theorem [20] namely, the Fourier series of $f$ is absolutely convergent
if and only if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} e_n(f) < +\infty$$

where

$$e_n(f) = ||f - T_n||_{L_2} = \left( \sum_{k=n+1}^{\infty} |c_{\gamma_k}|^2 \right)^{1/2}$$

and (2) in Carleson's proof [9] of convergence almost everywhere of Fourier series of
square-integrable functions, when the trigonometric polynomial $\sum_{|c_n| \geq \gamma} e^{int}$ is formed
from a Fourier expansion $\sum c_n e^{int}$.

Another celebrated result where the trigonometric polynomials of given length appear
is Littlewood’s conjecture (1948) [12], according to which

$$\int_0^{2\pi} |e^{in_1 z} + \cdots + e^{in_k z}| \, dx \geq c \log k$$

As it was proved in [10], if $\mu(x)$ is a trigonometric polynomial of length $m$ i.e,

$$\mu(x) = \sum_{k=1}^{m} c_k e^{in_k x}$$

then,

$$\int_0^{2\pi} |\mu(x)| \, dx \geq \frac{1}{30} \sum_{k=1}^{m} \frac{|c_k|}{k}$$

One of the basic questions about polynomials of given length is how to estimate coefficients of a polynomial $P$, when the length of $P$ and some norm of $P$ are known.

This question has been considered by many authors.

Turan [22] has the following estimates: If

$$P(x) = \sum_{j=1}^{k} b_j e^{-\lambda_j x}, \quad 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$$

then

$$|b_j| \leq e^{(m+1)\lambda_j} \prod_{1 \leq l \leq k, l \neq j} \frac{1 + e^{-\lambda_l}}{e^{-\lambda_j} - e^{-\lambda_l}} \max_{m+1 \leq x \leq m+k, x \text{ an integer}} |P(x)|$$

for $j = 1, 2, \cdots, k$ and for any non-negative integer $m$.

In the case of a Dirichlet polynomial

$$P(x) = \sum_{j=1}^{m} c_j x^{\mu_j}, \quad 0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_m,$$
L. Schwartz [19] got the following estimate

$$\sum_{j=1}^{m} |c_j|^r \mu_j^{-\mu_m} \leq C(r) \|P\|_{L^r(0,1)}, \quad 0 < r < 1,$$

where $C(r)$ depends on $r$ and on the exponents $\mu_1, \mu_2, \ldots, \mu_m$.

In the case of an algebraic polynomial, if

$$P(x) = \sum_{k=1}^{m} a_k x^{n_k}$$

is an algebraic polynomial of length $\leq m$ on some interval $[c,1]$ where $0 \leq c < 1$ then Baishanski [2] got the following result.

For every $r$, $0 < r < 1$, and $1 \leq p < \infty$

$$|a_k| \leq \frac{K}{r^{n_k}} \|P\|_p$$  \hspace{1cm} (1.4)

where $K = K(c,m,r,p)$ does not depend on $n_k$ nor on the polynomial $P$.

Later, Baishanski and Bojanic [3] proved a stronger inequality, namely

$$|a_k| \leq K (n_k + 1)^{m-1/q} \|P\|_p$$  \hspace{1cm} (1.5)

where $1/p + 1/q = 1$ and $K = K(c,m)$ does not depend on $n_k$ nor on the polynomial $P$ nor on $p$. In fact, the authors have obtained an estimate under more general conditions.

If $P(x) = \sum_{k=1}^{m} a_k x^{n_k}$ is a Dirichlet polynomial where $n_k, k = 1, 2, \ldots, m$ are real numbers such that

$0 \leq n_1 < n_2 < \ldots < n_m$ and if $\min_s \{|n_k - n_s| : k \neq s\} = h$ and $h \geq h_0 > 0$
then,
\[ |a_k| \leq C(n_k + 1/p + \frac{h}{q})^{m-1/q} h^{1-m} \|P\|_{L_p(c,1)} \]  
(1.6)

where \( C = C(c, m, h_0) \) is independent of \( n_k, h, P \). The question about the estimate of coefficients of an algebraic polynomial of length \( m \) is equivalent, by duality, to the question how well \( x^n \) can be approximated by polynomials of length \( m - 1 \) (which do not contain a term with \( x^n \)). The result of Baishanski and Bojanic [3], is, for example, equivalent to the following.

There exists \( A = A(c, m, p) > 0 \) such that for every polynomial \( P \) with length \( \leq m \) which does not contain the term \( x^n \),
\[ \|x^n - P(x)\|_{L_p(c,1)} \geq A n^{-m-1/p} \]

Special cases of the above inequality were obtained previously by Borosh, Chui and Smith [8]. Their results have been significantly sharpened by Saff and Varga in [18]. Another question involving polynomials of given length was raised by G.G. Lorentz. If \( P_{m,n}(x) \) is the best approximation to \( x^n \) by polynomials of length \( m \) of degree \( < n \) (we assume \( m < n \)), Lorentz conjectured that the exponents appearing in \( P_{m,n} \) are \( n-1, n-2, \ldots, n-m \). This was proved by Borosh, Chui and Smith [7] and were further developed in Smith [21] and then by Baishanski [5].

It is well known that if \( f \) is a continuous function on \([a, b]\), it may happen that it does not possess a best approximation by exponential polynomials of length \( \leq m \). However, there exists best approximation by generalized exponential polynomials of length \( \leq m \). Rice [17], [11] and [13].
The question of the existence of best approximation by algebraic polynomials was raised and solved by Baishanski [2]. Among other questions about polynomials of given length, we mention the following result proved by B.M. Baishanski and S.A. Ali [6]:

In the case of $L_p$-approximation, $1 < p < \infty$, the approximation error strictly decreases as the length of approximating polynomial is increasing (except in the obvious trivial cases) and this strict decrease of the error does not hold in the case of uniform approximation.

Another question that was raised by Baishanski [4] was the following:

To each polynomial $P$ of degree $\leq n$ there corresponds a polynomial $Q(P)$ such that among all polynomials of length $\leq k$, $Q(P)$ is a best approximation to $P$ in $L_2$-norm (we assume that $k < n$). How large can $\deg Q(P)$ be? This upper bound depends on $n$ and $k$. If $k = 1$ the polynomials $Q$ become monomials; if one allows also quasimonomials, i.e, expressions of the form $ax^s$ where $s$ is a positive real, S. Ali [1] obtained that

$$\deg Q \leq C(n + 1)^3$$

where $C \leq 6$ and cannot be replaced by a constant $\leq 1/4$.

It is clear from above that there is a variety of questions involving polynomials of given length that have been considered in case of one variable and that have not been studied so far in case of several variables.

In this dissertation, the concept of the length of a polynomial has been generalized to several variables in two different ways. We have restricted ourselves to the question
of the existence of best approximation by polynomials of given length in $L^p(Q)$ where 
$Q$ is a cube in $\mathbb{R}^n$ and the questions which are necessary to solve this problem ( An 
Abstract Existence Theorem and the Estimate for the coefficients of the polynomials 
in two variables ). Of course, the most interesting problem ( raised by Baishanski ) would be to consider more general domains $\Omega$ rather than the cube $Q$ and to see how 
the coefficient estimate depends on $\Omega$.

The rest of the dissertation is organized as follows. In Chapter 1, we define the 
terms s-length and h-length of a polynomial in two variables and state our main 
theorems. To prove our main theorems, we need an Abstract Existence Theorem 
which we prove in Chapter 2. In Chapter 3, we prove an estimate for the coefficients 
of polynomials using the estimate for the coefficient of a polynomial in one variable. 
Also, we prove an estimate for the norm of the homogeneous part of a polynomial 
and prove that these estimates are sharp. We prove our main existence theorems 
in Chapter 4. In Chapter 5, we discuss an application of the estimates proved in 
Chapter 3. In Chapter 6, we focus our attention on polynomials of total degree $\leq n$. 
Markov has proved a set of sharp inequalities for the coefficients of a polynomial in one 
variable. We prove a multivariate generalization of these Markov's Inequalities. We 
have restricted ourselves except in Chapter 6 to stating and proving the results to 
the case of two variables since generalization to $r$ variables in each case is obvious.

1.1 Definitions

Definition : Length of a polynomial:

The number of non-zero coefficients of a polynomial $P(x)$ is called its length and it
is denoted by \( l(P) \). Baishaaski [1] has proved the following existence theorem.

**Theorem 1**:  

Let \( 1 \leq p \leq \infty \) and \( f \in L_p[a, b] \). Then for every positive integer \( m \), there exists a polynomial \( P^* \) of length \( \leq m \) such that  

\[
||f - P^*||_p = \inf \{ ||f - P||_p : P \text{ a polynomial, } l(P) \leq m \}
\]

In this chapter we generalize this theorem to polynomials in two variables. In the case of two variables, the notion of length of a polynomial can be generalized in two ways.

**Definition: s-length of a polynomial \( P(x, y) \):**

A polynomial in two variables \( P(x, y) \) is said to be of s-length (monomial length) \( m \) if \( m \) is the smallest integer such that \( P(x, y) \) can be written as a sum of \( m \) monomials. (A monomial is any expression of the form \( A x^j y^k \) where \( j \) and \( k \) are non-negative integers and \( A \), a constant.)

**Notation:** The set of all polynomials in two variables \( x \) and \( y \) and of s-length \( \leq m \) will be denoted by \( \mathcal{P}^s_m \).

**Definition: h-length of a polynomial \( P(x, y) \):**

A polynomial in two variables \( P(x, y) \) is said to be of h-length (homogeneous length) \( m \) if \( m \) is the smallest integer such that \( P(x,y) \) can be written as a sum of \( m \) homogeneous polynomials. Each such homogeneous polynomial will be called a homogeneous part of \( P(x, y) \).

**Notation:** The set of all polynomials in two variables \( x \) and \( y \) and of h-length \( \leq m \) will be denoted by \( \mathcal{P}^h_m \). Using the above notions of s-length and h-length, Theorem 1
can be generalized as follows.

1.2 Generalizations of Theorem 1

Let $1 \leq p \leq \infty$ and $f \in L_p([0, 1] \times [0, 1])$. Let $m$ be a positive integer. Then,

**Theorem 2:**

There exists a polynomial of best approximation $P^*$ from $\mathcal{P}_m^*$ to $f$.

i.e. $\|f - P^*\|_p = \inf \{\|f - P\|_p : P \in \mathcal{P}_m^*\}$

**Theorem 3:**

There exists a polynomial of best approximation $P^*$ from $\mathcal{P}_m^h$ to $f$.

i.e. $\|f - P^*\|_p = \inf \{\|f - P\|_p : P \in \mathcal{P}_m^h\}$
CHAPTER II

AN ABSTRACT EXISTENCE THEOREM

Baishanski [2] has proved Theorem 1 using the following abstract existence theorem (Theorem A) and an estimate for the coefficient of a polynomial in terms of its length which we quote in section (3.1).

2.1 Best approximant from a union of countably many subspaces of bounded dimension

Let $B$ be a normed linear space. Let $\{x_n\}$ be a linearly independent set in $B$. Let $S$ be a union of a family of subspaces of $B$, each subspace in the family satisfying the following conditions. 1) its dimension is less than or equal to $m$. 2) it has a basis consisting of elements from $\{x_n\}$. Now, the question is: Does there exist an element of best approximation in $S$ for every $z \in B$? If so, under what conditions? The question is answered in the following theorem A due to Baishanski [2].

Theorem A:

Suppose $s = \sum a_k x_k \in S$. Define the linear functional $\delta_n$ on $S$ as $\delta_n(s) = a_n$. Then, for every $z \in B$ there exists $z^* \in S$ such that

$$||z - z^*|| = \inf_{x \in S} \{||z - s||\}$$

if the following conditions are satisfied.

1) $||\delta_n||_S = \sup_{x \in S} \left\{ \frac{||\delta_n(x)||}{||x||} \right\} < \infty$ for every $n$
2) there exists a family $\Phi$ of continuous linear functionals on $B$ such that

a) $\Phi$ is normalizing i.e. $\sup_{\phi \in \Phi} \left\{ \frac{\|\phi(z)\|}{\|\phi\|} \right\} = \|z\|$ for every $z \in B$

b) $\|\delta_n\|_C \phi(x_n) \to 0$ as $n \to \infty$ for every $\phi \in \Phi$.

To generalize Theorem 1, we need a more general result than Theorem A and an estimate for the coefficients of a polynomial in two variables. The following Theorem B is a generalization of Theorem A. In this generalization, we replace the linearly independent set $\{x_n\}$ of vectors, by a linearly independent set of finite dimensional vector spaces $\{V_n\}$.

### 2.2 Generalization of Theorem A

Let $B$ be a normed linear space. Let $V_1, V_2, ..., V_k, ...$ be finite dimensional subspaces of $B$. We do not exclude the possibility that $\dim(V_k) \to \infty$ as $k \to \infty$. Let $V_1, V_2, ..., V_k, ...$ satisfy the following condition.

If for any integer $n \geq 2$, $v_1 + v_2 + v_3 + ... + v_n = 0$ where $v_j \in V_j$, $j = 1, 2, ..., n$ then $v_1 = v_2 = ... = v_n = 0$.

Let $m$ be a fixed positive integer. Let $\mathcal{S}$ denote the union of a family of subspaces of $B$ which can be represented as direct sum of $m$ spaces like $V_{i_1} \oplus V_{i_2} \oplus ... \oplus V_{i_m}$, where $1 \leq i_1 < i_2 < i_3 < ... < i_m$.

Suppose $s \in \mathcal{S}$. Then, $s \in V_{i_1} \oplus V_{i_2} \oplus ... \oplus V_{i_m}$.

If $s = v_{i_1} + v_{i_2} + ... v_{i_m}$ then there is a unique $j \leq m$ and a unique representation of the form $s = v_{i_1} + v_{i_2} + v_{i_3} + ... + v_{i_j}$ where $v_{i_q} \in V_{i_q}$ and $v_{i_q} \neq 0$ for $q = 1, 2, ..., j$. 
For every positive integer \( n \), define then

\[
v_n(s) = \begin{cases} 
  v_{ij} & \text{if } n = i_j \\
  0 & \text{otherwise}
\end{cases}
\]

and

\[
\lambda_n = \sup_{s \in \mathcal{S}} \left\{ \frac{||v_n(s)||}{||s||} \right\}
\]

Then, we have the following result.

**Theorem B:**

For every \( z \in \mathcal{B} \) there exists an element of best approximation in \( \mathcal{S} \) if the following conditions are satisfied.

1. \( \lambda_n = \sup_{s \in \mathcal{S}} \left\{ \frac{||v_n(s)||}{||s||} \right\} < \infty \) for every \( n \).

2. There exists a family \( \Phi \) of continuous linear functionals on \( \mathcal{B} \) such that
   
   (i) \( \sup_{\phi \in \Phi} \left\{ \frac{||\phi(z)||}{||\phi||} \right\} = ||z|| \) for every \( z \in \mathcal{B} \) i.e. \( \Phi \) is normalizing

   (ii) \( \lambda_n ||\phi||_{V_n} \to 0 \) as \( n \to \infty \) for every \( \phi \in \Phi \) where
   
   \[
   ||\phi||_{V_n} = \sup_{z \in V_n} \left\{ \frac{||\phi(z)||}{||z||} \right\}
   \]

**Proof:**

Let \( z \in \mathcal{B} \) and \( \inf_{s \in \mathcal{S}} \{||z - s||\} = \gamma \). Then there exists \( s_n \in \mathcal{S} \) such that

\[
||z - s_n|| \to \gamma \text{ as } n \to \infty.
\]

Evidently, \( ||s_n|| \leq M \) for all \( n \) and some \( M \).

Let \( s_n = v_{k(n,1)} + v_{k(n,2)} + \ldots + v_{k(n,m)} \) where \( v_{k(n,i)} \in V_{k(n,i)} \). Also, we can assume that the sequence \( \{s_n\} \) is such that the set of indices \( \{1,2,\ldots,m\} \) can be subdivided
into two subsets $I_1$ and $I_2$ so that

- $i \in I_1$ implies that $k(n, i)$ is constant $= k(i)$ and
- $i \in I_2$ implies that $k(n, i) \to \infty$ as $n \to \infty$.

Let us write $s_n = \sigma_n^{(1)} + \sigma_n^{(2)}$ where

$$\sigma_n^{(j)} = \sum_{i \in I_j} v_{k(n, i)}(s_n)$$

for $j = 1, 2$.

The sequence $\sigma_n^{(1)}$ lies in a finite dimensional subspace $V$ of $B$.

$$||\sigma_n^{(1)}|| \leq \sum_{i \in I_1} ||v_{k(n, i)}(s_n)||$$

\[
\leq \sum_{i \in I_1} \lambda_{k(n, i)} ||s_n||
\]

\[
\leq M \sum_{i \in I_1} \lambda_{k(n, i)}
\]

\[
\leq M \sum_{i \in I_1} \lambda_k(i)
\]

This implies that $\sigma_n^{(1)}$ is a bounded sequence and hence it has subsequence which converges to some element $\sigma \in V \subset S$. We can assume that $\sigma_n^{(1)}$ itself converges to $\sigma$.

Let $z - \sigma = y$ and $\sigma_n^{(2)} = \sigma_n$. Then,

$$||y - \sigma_n|| \to \gamma \text{ as } n \to \infty.$$  

Claim: $\phi(\sigma_n) \to 0$ as $n \to \infty$ for every $\phi \in \Phi$
\begin{align*}
|\phi(\sigma_n)| & \leq \sum_{i \in I_2} |\phi(v_{k(n,i)}(\sigma_n))| \\
& \leq \sum_{i \in I_2} ||\phi|| V_{k(n,i)} ||V_{k(n,i)}(\sigma_n)|| \\
& \leq \sum_{i \in I_2} ||\phi|| V_{k(n,i)} (\lambda_{k(n,i)} ||(\sigma_n)||) \\
& \leq M \sum_{i \in I_2} ||\phi|| V_{k(n,i)} (\lambda_{k(n,i)})
\end{align*}

Since \( i \in I_2 \) implies that \( k(n,i) \to \infty \) as \( n \to \infty \) we get \( \phi(\sigma_n) \to 0 \) as \( n \to \infty \) by (ii) of (2).

Let \( \epsilon > 0 \) be an arbitrary positive number.

Then by (2) there exists \( \phi \in \Phi \) such that

\begin{align*}
|\phi(y)| & \geq ||\phi|| (||y|| - \epsilon) \text{ and } \phi \neq 0 \\
|\phi(y - \sigma_n)| & \leq ||\phi|| ||y - \sigma_n|| \to ||\phi|| \gamma \\
|\phi(y - \sigma_n)| & \to |\phi(y)| \geq ||\phi|| (||y|| - \epsilon)
\end{align*}

Hence, \( \gamma \geq ||y|| - \epsilon \). Since \( \epsilon \) is arbitrary \( \gamma \geq ||y|| \).

On the other hand, \( ||y|| = ||z - \sigma|| \geq \gamma \).

Hence, \( ||z - \sigma|| = \gamma \). Since \( \sigma \in \mathcal{S} \), the theorem is proved.
CHAPTER III

AN ESTIMATE FOR THE COEFFICIENTS OF POLYNOMIALS IN TWO VARIABLES

In this chapter, we obtain an estimate for the coefficients of a polynomial $P(x, y)$ in two variables in terms of its s-length. Also, we obtain an estimate for the norm of the homogeneous part of $P(x, y)$ in terms of h-length of $P(x, y)$ and its norm $\|P\|.$

3.1 Estimate for the coefficients of a polynomial in one variable

Recall that the length of a polynomial $P(x)$ is defined as the number of non-zero coefficients of $P(x)$. Baishanski [2] has given an estimate for the coefficients of an algebraic polynomial of one variable of given length. More accurately, if $P(x) = \sum_{k=1}^{m} a_k x^{n_k}$ is an algebraic polynomial of length $\leq m$ on some interval $[c, 1]$ where $0 \leq c < 1$ then we have the following result.

For every $r$, $0 < r < 1$, and $1 \leq p \leq \infty$

$$|a_k| \leq \frac{K}{p^{n_k}} \|P\|_p \quad (3.1)$$

where $K = K(c, m, r, p)$ does not depend on $n_k$ nor on the polynomial $P$.

Later, Baishanski and Bojaanic [3] proved a stronger inequality, namely
\[ |a_k| \leq K (n_k + 1)^{m-1/q} p \|P\|_p \]

where \( 1/p + 1/q = 1 \) and \( K = K(c, m) \) does not depend on \( n_k \) nor on the polynomial \( P \) nor on \( p \). In fact, the authors have obtained an estimate under more general conditions.

If \( P(x) = \sum_{k=1}^{m} a_k x^{n_k} \) is a Dirichlet Polynomial where \( n_k, k = 1, 2, \ldots, m \) are real numbers such that

\[ 0 \leq n_1 < n_2 < \ldots < n_m \]

and if

\[ \min_s \{ |n_k - n_s| : k \neq s \} = h \quad \text{and} \quad h \geq h_0 > 0 \]

then,

\[ |a_k| \leq C (n_k + 1/p + h/q)^{m-1/q} h^{1-m} \|P\|_{L_p(c, 1)} \]

where \( C = C(c, m, h_0) \) is independent of \( n_k, h, P \).

In this chapter, we extend this estimate (3.3) to polynomials in two variables.

### 3.2 Notation

We use the following notation.

\( P(x, y) \) is a polynomial in two variables given by

\[ P(x, y) = \sum_{k=1}^{m} a_k x^{m_k} y^{n_k} \]

where \( m_k, n_k, k=1,2,\ldots,m \) are non-negative integers.

Here, we will assume that \( P(x, y) \) is of \( s \)-length \( m \). Also, \( p \) and \( q \) are are always
conjugate indices i.e. $1 \leq p \leq \infty$ and $1/p + 1/q = 1$. The $L_p$-norm is taken over the rectangle $[0, 1] \times [0, 1]$.

### 3.3 Estimate for the coefficients of a polynomial in two variables

**Theorem 4:**

If $P(x, y) = \sum_{k=1}^{m} a_k x^{m_k} y^{n_k}$ is a polynomial of s-length $m$, then

$$|a_k| \leq C_1 [1 + \operatorname{Max} (m_k, n_k)]^{m-1} ||P||_{\infty}$$

where $C_1$ is a constant depending only on $m$ and $||P||_{\infty}$ denotes the $L_{\infty}$-norm of $P$ on $[0, 1] \times [0, 1]$.

**Proof:**

We assume that $||P||_{\infty} = 1$. The basic idea in estimating the bounds for the coefficients of $P(x, y)$ of given s-length is to reduce $P(x, y)$ to a polynomial of a single variable by means of a suitable transformation. Replacing $x$ by $t^\alpha$ and $y$ by $t^\beta$ where $\alpha + \beta = 1$ and $0 \leq \alpha, \beta \leq 1$, $P(x, y)$ is reduced to

$$Q(t) = \sum_{k=1}^{m} a_k t^{\alpha m_k + \beta n_k}$$

Using the estimate (3.3) we get

$$|a_k| \leq C (\alpha m_k + \beta n_k + h)^{m-1} h^{1-m}$$

$$= C \left[ \frac{\alpha m_k + \beta n_k}{h} + 1 \right]^{m-1}$$
where \( h = \min_{k \neq s} |\alpha m_k + \beta n_k - (\alpha m_s + \beta n_s)| \). We shall choose \( \alpha, \beta \) so that the right-hand side above is the smallest. Thus, we have

\[
|a_k| \leq C \min_{\alpha, \beta} \left[ 1 + \frac{\alpha m_k + \beta n_k}{h} \right]^{m-1}
\]

\[
\leq C \min_{\alpha, \beta} \left[ 1 + \frac{\max (m_k, n_k)}{\min_{k \neq s} |\alpha m_k + \beta n_k - (\alpha m_s + \beta n_s)|} \right]^{m-1}
\]

\[
= C \left[ 1 + \frac{\max (m_k, n_k)}{\max_{k \neq s} \min_{\alpha, \beta} |\alpha (m_k - m_s) + \beta (n_k - n_s)|} \right]^{m-1}
\]

\[
= C \left[ 1 + \frac{\max (m_k, n_k)}{\max_{0 \leq t \leq 1} \min_{k \neq s} |(m_k - m_s - n_k + n_s) t + (n_k - n_s)|} \right]^{m-1} \tag{3.4}
\]

Let

\[
\gamma = \max_{0 \leq t \leq 1} \min_{k \neq s} |(m_k - m_s - n_k + n_s) t + (n_k - n_s)|
\]

\[
= \max_{0 \leq t \leq 1} \min_{1 \leq s \leq m-1} |(a_s - b_s) t + b_s|
\]

where \( a_s = m_k - m_s \) and \( b_s = n_k - n_s \).

Note that \( a_s \) and \( b_s \) are integers, not necessarily positive and not both zero.

We claim that \( \gamma \geq \frac{1}{m-1} \).

For, consider the lines \( L_i = (a_i - b_i) t + b_i, 0 \leq t \leq 1, i = 1 \) to \( m-1 \).

and the set \( A = \{0, \frac{1}{m-1}, \frac{2}{m-1}, \ldots, 1\} \).

There are \( m-1 \) lines ( \( L_i \)'s) each of which can have at most one zero and there are \( m \)
points in A. Hence, there exists at least one point \( x_0 = \frac{k}{m-1} \in A \) where \( 0 \leq k \leq m - 1 \) at which none of \( L_i \) vanishes by the pigeon-hole principle.

Now,

\[
\gamma \geq \min_{1 \leq s \leq m-1} \left| (a_s - b_s)x_0 + b_s \right|
\]

\[
= \min_{1 \leq s \leq m-1} \left| (a_s - b_s)\left( \frac{k}{m-1} \right) + b_s \right|
\]

\[
= \min_{1 \leq s \leq m-1} \left| \frac{(a_s - b_s)k + (m - 1)b_s}{m - 1} \right|
\]

\[
= \frac{\text{a positive integer}}{m - 1}
\]

\[
\geq \frac{1}{m - 1}
\]

Using this in (3.4) we get

\[
|a_k| \leq C \left[ 1 + (m - 1)\max (m_k, n_k) \right]^{m-1}
\]

where \( C \) depends on \( m \).

Thus in general,

\[
|a_k| \leq C \left[ 1 + (m - 1)\max (m_k, n_k) \right]^{m-1} ||P||_{\infty}
\]

\[
|a_k| \leq C_1 \left[ 1 + \max (m_k, n_k) \right]^{m-1} ||P||_{\infty}
\]  \hspace{1cm} (3.5)
3.4 Extension to $L_p$, $1 \leq p < \infty$

**Theorem 5:**

If $P(x,y) = \sum_{k=1}^{m} a_k x^{m_k} y^{n_k}$ is a polynomial of s-length $m$, then

$$|a_k| \leq C_1 \left[ 1 + \text{Max} \ (m_k, n_k) \right]^{m-1/q} \left[ 1 + \text{Min} \ (m_k, n_k) \right]^{1-1/q} \ ||P||_p$$

where $1/p + 1/q = 1$, $C_1$ is a constant depending only on $m$ and $||P||_p$ denotes the $L_p$ norm of $P$ on $[0,1] \times [0,1]$.

**Proof:**

Consider

$$Q(x,y) = \int_0^x \int_0^y P(u,v) u^s v^t \ du \ dv$$

where $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $s$ and $t$ are non-negative real numbers.

$$Q(x,y) = \int_0^x \int_0^y \left( \sum_{k=1}^{m} a_k u^{m_k} v^{n_k} \right) u^s v^t \ du \ dv$$

$$= \int_0^x \int_0^y \left( \sum_{k=1}^{m} a_k u^{m_k+s} v^{n_k+t} \right) \ dv \ du$$

$$= \sum_{k=1}^{m} a_k \int_0^x u^{m_k+s} \ du \int_0^y v^{n_k+t} \ dv$$

$$= \sum_{k=1}^{m} a_k \frac{x^{m_k+s+1}}{m_k+s+1} \frac{y^{n_k+t+1}}{n_k+t+1}$$

$$= \sum_{k=1}^{m} \frac{a_k}{(m_k+s+1)(n_k+t+1)} x^{m_k+s+1} y^{n_k+t+1}$$
Thus, \( Q(x,y) \) is a polynomial of s-length \( m \) and the coefficient of
\[
x^{m_k + s + 1} y^{n_k + t + 1} \\text{is} \frac{a_k}{(m_k + s + 1)(n_k + t + 1)}
\]

Using (3.5),

\[
|a_k| \leq C \ M_{k,s} \ N_{k,t} \ [1 + \text{Max}(M_{k,s}, N_{k,t})]^{m-1} \ \|Q(x,y)\|_\infty \quad (3.6)
\]

where \( M_{k,s} = m_k + s + 1 \) and \( N_{k,t} = n_k + t + 1 \)

\[
\|Q(x,y)\|_\infty = \max_{x, y} \left| \int_0^x \int_0^y P(u,v)u^s v^t dv \ du \right|
\]

\[
\leq \max_{x, y} \int_0^x \int_0^y |P(u,v)|u^s v^t dv \ du
\]

\[
\leq \int_0^1 \int_0^1 |P(u,v)|u^s v^t dv \ du
\]

\[
\leq \left[ \int_0^1 \int_0^1 |P(u,v)|^p dv \ du \right]^{1/p} \left[ \int_0^1 \int_0^1 |u^s v^t|^q dv \ du \right]^{1/q}
\]

\[
= \|P(u,v)\|_p \left[ \int_0^1 u^{qs} dv \ \int_0^1 v^{tq} dv \right]^{1/q}
\]

\[
= \|P(x,y)\|_p \left[ \frac{1}{(qs + 1)(tq + 1)} \right]^{1/q}
\]
\[ \leq \|P(x, y)\|_p \left[ \frac{1}{(s+1)(t+1)} \right]^{1/q} \]

Hence, (3.6) becomes

\[ |a_k| \leq C \frac{(m_k + s + 1)(n_k + t + 1)}{((s+1)(t+1))^{1/q}} [1 + \max(m_k + s + 1, n_k + t + 1)]^{m-1} \|P\|_p \]

\[ \leq C \frac{(m_k + s + 2)(n_k + t + 2)}{((s+1)(t+1))^{1/q}} [1 + \max(m_k + s + 1, n_k + t + 1)]^{m-1} \|P\|_p \]

Putting \( s = m_k \) and \( t = n_k \) we get

\[ |a_k| \leq C \frac{4(m_k + 1)(n_k + 1)}{((m_k + 1)(n_k + 1))^{1/q}} [1 + \max\{(2m_k + 1), (2n_k + 1)\}]^{m-1} \|P\|_p \]

\[ = 4C (m_k + 1)^{1-1/q}(n_k + 1)^{1-1/q} [1 + \max\{(2m_k + 1), (2n_k + 1)\}]^{m-1} \|P\|_p \]

\[ = C_1 [1 + \max(m_k, n_k)]^{m-1} (1 + m_k)^{1-1/q} (1 + n_k)^{1-1/q} \|P\|_p \]

\[ = C_1 [1 + \max(m_k, n_k)]^{m-1/q} [1 + \min(m_k, n_k)]^{1-1/q} \|P\|_p \]

Thus,

\[ |a_k| \leq C_1 [1 + \max(m_k, n_k)]^{m-1/q} [1 + \min(m_k, n_k)]^{1-1/q} \|P\|_p \quad (3.7) \]
3.5 An estimate for the norm of the homogeneous part of a polynomial

Theorem 6:

Let \( P(x, y) \in P_m^h \) and \( 1 \leq p \leq \infty \). i.e. \( P(x, y) = P_{j_1}(x, y) + P_{j_2}(x, y) + \ldots + P_{j_m}(x, y) \)

where \( P_{j_i}(x, y) = \sum_{k=0}^{j_i} a_{j_i,k} x^k y^{j_i-k} \), \( i = 1 \) to \( m \).

Then, \( ||P_{j_i}||_p \leq C_1 (1 + j_i)^{m-1} ||P||_p \) where \( C_1 \) depends only on \( m \).

Proof:

Case 1: \( p = \infty \)

Suppose \( |P_{j_i}| \) attains its maximum on \( [0, 1] \times [0, 1] \) at \( (x_0, y_0) \).

Then \( x_0 = 1 \) or \( y_0 = 1 \) since \( P_{j_i} \) is homogeneous.

Without loss of generality, we will assume that \( y_0 = 1 \).

Hence, \( ||P_{j_i}||_\infty = |P_{j_i}(x_0, 1)| \)

Consider \( P(x_0t, t) = \sum_{k=1}^{m} P_{j_k}(x_0t, t) = \sum_{k=1}^{m} t^{j_k} P_{j_k}(x_0, 1) \)

\( P(x_0t, t) \) is a polynomial in \( t \) of length \( m \). The coefficient of \( t^{j_i} \)

in \( P(x_0t, t) \) is \( P_{j_i}(x_0, 1) \) and by (3.2)

\[
|P_{j_i}(x_0, 1)| \leq C(1 + j_i)^{m-1} ||P(x_0t, t)||_{[0,1]}
\]

\[
||P_{j_i}|| \leq C(1 + j_i)^{m-1} ||P(x, y)||_{[0,1] \times [0,1]}
\]

Case 2: \( 1 \leq p < \infty \)

Split the rectangle \( [0, 1] \times [0, 1] \) into two triangles \( \Delta_1 \) and \( \Delta_2 \) where \( \Delta_1 \) is the triangle bounded by \( y = 0, y = x, x = 1 \) and \( \Delta_2 \) is the remaining triangle.
Consider $\int \int _{\Delta _{1}} |P_{j}(x,y)|^{p} \, dy \, dx$

Applying the transformation $y = tx$ we get

$\int \int _{\Delta _{1}} |P_{j}(x,y)|^{p} \, dy \, dx = \int_{0}^{1} \int_{0}^{1} |P_{j}(x,tx)|^{p} \, x \, dx \, dt$

Now,

$P_{j}(x,tx) = x^{j_{i}} \left( \sum_{k=0}^{j_{i}} a_{j_{i},k} \, t^{j_{i}+k} \right)$

$|P_{j}(x,tx)|^{p} = x^{p j_{i}} \left| \sum_{k=0}^{j_{i}} a_{j_{i},k} \, t^{j_{i}+k} \right|^{p}$

Consider $P(x,y)$ where

$P(x,y) = \sum_{i=1}^{m} P_{j_{i}}(x,y)$

$= \sum_{i=1}^{m} \left( \sum_{k=0}^{j_{i}} a_{j_{i},k} \, x^{k} \, y^{j_{i}+k} \right)$

$P(x,tx) = \sum_{i=1}^{m} x^{j_{i}} \left( \sum_{k=0}^{j_{i}} a_{j_{i},k} \, t^{j_{i}+k} \right)$

$x^{1/p} P(x,tx) = \sum_{i=1}^{m} x^{j_{i}+1/p} \left( \sum_{k=0}^{j_{i}} a_{j_{i},k} \, t^{j_{i}+k} \right)$

Thus, $\sum_{k=0}^{j_{i}} a_{j_{i},k} \, t^{j_{i}+k}$ is the coefficient of $x^{j_{i}+1/p}$ in $x^{1/p} P(x,tx)$

Consider

$\int_{0}^{1} |P_{j}(x,tx)|^{p} \, x \, dx = \int_{0}^{1} x^{p j_{i} + 1} \left| \sum_{k=0}^{j_{i}} a_{j_{i},k} \, t^{j_{i}+k} \right|^{p} \, dx$
\[ \int_0^1 \left| \sum_{k=0}^{j_i} a_{j_i,k} t^{j_i-k} \right|^p \int_0^1 x^{p(j_i+1)} dx \]

\[ = \left| \text{coefficient of } x^{j_i+1/p} \text{ in } x^{1/p} P(x, tx) \right|^p \left[ \frac{1}{p \ j_i + 2} \right] \]

\[ \leq \left[ C(1 + j_i + 1/p)^{m-1/q} \| x^{1/p} P(x, tx) \|_p \right]^p \left[ \frac{1}{p \ j_i + 2} \right] \]

\[ = \left[ C(1 + j_i + 1/p)^{m-1/q} \right]^p \int_0^1 \left| x^{1/p} P(x, tx) \right|^p dx \left[ \frac{1}{p \ j_i + 2} \right] \]

\[ \leq \left[ C(2 + j_i)^{m-1/q} \right]^p \int_0^1 x \ |P(x, tx)|^p \ dx \left[ \frac{1}{j_i + 2} \right] \]

\[ = \left[ C(2 + j_i)^{m-1} \right]^p \int_0^1 x \ |P(x, tx)|^p \ dx \]

Integrating with respect to \( t \) we get

\[ \int_0^1 \int_0^1 |P_j(x, tx)|^p \ x \ dx \ dt \leq \left[ C(2 + j_i)^{m-1} \right]^p \int_0^1 \int_0^1 x \ |P(x, tx)|^p \ dx \ dt \]

\[ \iint_{\Delta_1} |P_j(x, y)|^p \ dy \ dx \leq \left[ C(2 + j_i)^{m-1} \right]^p \iint_{\Delta_1} |P(x, y)|^p \ dy \ dx \quad (3.8) \]

Similarly,
\[ \int \int_{\Delta_2} |P_{\lambda}(x,y)|^p \, dy \, dx \leq \left[ C(2 + j_i)^{(m-1)} \right]^p \int \int_{\Delta_2} |P(x,y)|^p \, dy \, dx \] (3.9)

Adding (3.8) and (3.9) we get

\[ \int_0^1 \int_0^1 |P_{\lambda}(x,y)|^p \, dy \, dx \leq \left[ C(2 + j_i)^{(m-1)} \right]^p \int_0^1 \int_0^1 |P(x,y)|^p \, dy \, dx \] (3.10)

Hence,

\[ ||P_{\lambda}||_p \leq C(2 + j_i)^{(m-1)} ||P||_p \]

\[ ||P_{\lambda}||_p \leq C_1(1 + j_i)^{(m-1)} ||P||_p \] (3.11)

Hence the result is proved.
3.6 Sharpness of the estimate (3.7)

In this section, we prove that the estimate (3.7) for the coefficients of a polynomial obtained in section (3.4) is sharp in the following sense.

For every pair of non-negative integers \((m_1, n_1)\) and for every positive integer \(m > 1\) there exists a polynomial \(P(x, y)\) of length \(m\) where

\[
P(x, y) = P_{m,m_1,n_1}(x, y) = \sum_{k=1}^{m} a_k x^{m_k} y^{n_k}
\]

and a constant \(B_{m,p} > 0\) such that

\[
|a_1| = |a_1(P_{m,m_1,n_1})| \geq B_{m,p} \left[ 1 + \max(m_1, n_1) \right]^{m-1/q} \left[ 1 + \min(m_1, n_1) \right]^{1-1/q} ||P||_p
\]

where \(1/p + 1/q = 1\).

Without loss of generality, we can assume that \(m_1 \leq n_1\) in which case the above inequality reduces to

\[
|a_1| \geq B_{m,p}(n_1 + 1)^{m-1/q}(m_1 + 1)^{1-1/q}||P||_p
\]

(3.12)

Proof:

Case 1: \(p = \infty\)

Consider \(P_{m,m_1,n_1} = x^{m_1} y^{n_1} (1 - y)^{m-1}\) where \(m_1 < n_1\). This is a polynomial of length \(m\) and it attains its maximum at \(1, \frac{n_1}{m+n_1-1}\). Hence,

\[
||P_{m,m_1,n_1}||_\infty = \left[ \frac{n_1}{m + n_1 - 1} \right]^{n_1} \left[ 1 - \frac{n_1}{m + n_1 - 1} \right]^{m-1}
\]
\[ \left( \frac{n_1}{m + n_1 - 1} \right)^{n_1} \left( \frac{m - 1}{m + n_1 - 1} \right)^{m-1} \]

\[ \leq (m - 1)^{m-1} \frac{1}{(m + n_1 - 1)^{m-1}} \]

\[ \leq (m - 1)^{m-1} \frac{1}{(1 + n_1)^{m-1}} \]

The coefficient of \( x^{m_1} y^{n_1} \) in this polynomial is \( a_1 = 1 \). This shows that (3.12) is satisfied with \( B_{m,p} = \frac{1}{(m-1)^{m-1}} \).

**Case 2**: \( 1 \leq p < \infty \)

Consider again the polynomial \( P_{m,m_1,n_1}(x,y) = x^{m_1} y^{n_1} (1 - y)^{m-1} \) where \( m_1 < n_1 \).

This is a polynomial of length \( m \).

\[
||P_{m,m_1,n_1}||_p^p = \int_0^1 \int_0^1 x^{pn_1} y^{pn_1} (1 - y)^{(m-1)p} dy \, dx
\]

\[ = \frac{1}{pm_1 + 1} \int_0^1 y^{pn_1} (1 - y)^{(m-1)p} dy \]

\[ = \frac{1}{pm_1 + 1} B(pn_1 + 1, (m - 1)p + 1) \]

\[ = \frac{1}{pm_1 + 1} \frac{\Gamma(pn_1 + 1) \Gamma((m - 1)p + 1)}{\Gamma(pn_1 + (m - 1)p + 2)} \]

\[ = \frac{1}{pm_1 + 1} \frac{1}{(pn_1 + 1)(pn_1 + 2) \ldots (pn_1 + (m - 1)p + 1)} \Gamma((m - 1)p + 1) \]
\[
\leq \frac{1}{m_1 + 1} \frac{1}{(1 + n_1)(m-1)p+1} \Gamma((m - 1)p + 1)
\]
\[
\leq \frac{1}{m_1 + 1} \frac{1}{(1 + n_1)^{m-(p-1)}} \Gamma((m - 1)p + 1)
\]

Hence,
\[
||P||_p = ||P_{m,m_1,n_1}||_p
\]
\[
\leq \frac{1}{(m_1 + 1)^{1/p}} \frac{1}{(1 + n_1)^{m-1/q}} \left[ \Gamma((m - 1)p + 1) \right]^{1/p}
\]

Here, \(|a_1| = 1\) and hence, (3.12) is satisfied with \(B_{m,p} = \frac{1}{\left[ \Gamma((m - 1)p + 1) \right]^{1/p}}\)

### 3.7 Sharpness of the estimate (3.11)

In this section, we prove that the estimate (3.11) for the norm of the homogeneous part of a polynomial is sharp. That is, for every positive integer \(m\), there exists a positive constant \(C\) (depending on \(m\) only) such that for every positive integer \(j\), there exists a polynomial \(P \in P_m^h\) (depending on \(m\) and \(j\) only) such that
\[
||P_j||_p \geq C(1 + j)^{m-1}||P||_p
\]

(3.13)

where \(P_j\) is the homogeneous part of degree \(j\) of the polynomial \(P\). It is known [3] that for every positive integer \(m\), there exists a positive constant \(B_m\), depending on \(m\) only, such that for every positive integer \(j\) there exists a polynomial \(P\) in one variable (\(P\) depending on \(m\) and \(j\)) such that
\[
|a_j(P)| \geq B_m(1 + j)^{m-1/q}||P||_{L_p([0,1])}
\]

(3.14)
Consider $P$ as a polynomial in two variables. Then, $P \in P^h_m$ where the homogeneous part of total degree $j$ is given by $P_j = a_j x^j$.

Note that

$$\|P(x)\|_{L^p([0,1] \times [0,1])} = \|P(x)\|_{L^p([0,1])}$$

Now,

$$\|P_j(x)\|_{L^p([0,1] \times [0,1])} = \|P_j(x)\|_{L^p([0,1])}$$

$$= |a_j| \left( \int_0^1 x^{pj} \right)^{1/p}$$

$$= |a_j| \frac{1}{(pj + 1)^{1/p}}$$

Multiplying (3.14) by $\frac{1}{(pj + 1)^{1/p}}$ and using the previous step we get

$$\|P_j(x)\|_{L^p([0,1] \times [0,1])} \geq B_m \left[ \frac{(1 + j)^{m-1/q}}{(1 + pj)^{1/p}} \right] \|P\|_{L^p([0,1])}$$

$$\geq C(1 + j)^{m-1} \|P\|_{L^p([0,1] \times [0,1])}$$

where $C$ depends only on $m$. This proves the sharpness of the estimate (3.11)

### 3.8 Estimates in the case of more than two variables

In this section we generalize the estimates (3.7) and (3.11) to several variables. Since the proofs of these generalizations are very similar to those of two variable estimates, we just state these generalizations without proof.
3.8.1 Generalization of the estimate (3.7)

Let $P_{m,r}^s$ denote the set of all polynomials of $s$-length $\leq m$ in $r$ variables and $1 \leq p \leq \infty$. If $P \in P_{m,r}^s$ where

$$P(x_1, x_2, \ldots, x_r) = \sum_{i=1}^{m} a_i \prod_{j=1}^{r} x_j^{k_{i,j}}$$

then

$$|a_i| \leq C \left[ 1 + \max(k_{i,1}, k_{i,2}, \ldots, k_{i,r}) \right]^{m-1/q} \prod_{1 \leq j \leq r, j \neq j(i)} (1 + k_{i,j})^{1-1/q} ||P||_p$$

where $j(i)$ is the smallest $j$ such that $k_{i,j} = \max(k_{i,1}, k_{i,2}, \ldots, k_{i,r})$, $C = C(m)$ depends neither on the exponents $k_{i,j}$, $j = 1$ to $r$ nor on $P$ and $||P||_p$ denotes the $L_p$-norm of $P$ over $Q = [0, 1]^r$

3.8.2 Generalization of the estimate (3.11)

Let $P_{m,r}^h$ denote the set of all polynomials of $h$-length $\leq m$ and $1 \leq p \leq \infty$.

If $P \in P_{m,r}^h$ so that

$$P(x_1, x_2, \ldots, x_r) = \sum_{j=1}^{m} P_{j_i}(x_1, x_2, \ldots, x_r)$$

where $P_{j_i}$ is a homogeneous polynomial of total degree $j_i$ in $r$ variables $x_1, x_2, \ldots, x_r$ then

$$||P_{j_i}||_p \leq C_1 r^{1/p} (1 + j_i)^{m-1} ||P||_p$$

where $C_1$ depends only on $m$ and $||P||_p$ denotes the $L_p$-norm of $P$ over $Q = [0, 1]^r$
CHAPTER IV

EXISTENCE THEOREMS

In this chapter, we prove two existence theorems (Theorem 2 & Theorem 3). In Theorem 2, we prove the existence of best approximation from polynomials of given s-length. Even though this theorem can be proved using Theorem A, we prefer to use Theorem B to prove it. In Theorem 3, we prove the existence of best approximation from polynomials of given h-length. This is a deeper result and here we really need our Theorem B.

4.1 Best approximation from polynomials of given s-length:

Theorem 2:

Let $1 \leq p \leq \infty$ and $f \in L_p([0,1] \times [0,1])$. Let $m$ be a positive integer. Then, there exists a polynomial of best approximation $P^*$ from $P_m^s$ to $f$.

i.e. \[ \|f - P^*\|_p = \inf \{ \|f - P\|_p : P \in P_m^s \} \]

Proof:

We use the Abstract Existence Theorem (Theorem B) of Chapter 2 and the coefficient estimate(3.7) to prove this theorem. It is enough to verify the conditions of Theorem B. Using the notations of Theorem B, let $B = L_p([0,1] \times [0,1])$. and $V_1, V_2, V_3, V_4, V_5, \ldots$ be the spaces spanned by $\{x\}, \{y\}, \{x^2\}, \{xy\}, \{y^2\}, \ldots$
respectively.

Let \( s \in S \). Then \( s = v_{i_1} + v_{i_2} + \ldots + v_{i_m} \) where \( v_{i_j} = v_{i_j}(s) \in V_{i_j} \), \( j = 1 \) to \( m \) and \( i_1 < i_2 < \ldots < i_m \).

Then, using the estimate (3.7) we get

\[
\|v_{i_n}(s)\|_p \leq C(1 + n')^{m+1}\|s\|_p
\]

where \( n' \) is the total degree of the monomial spanning \( V_{i_n} \). Note that \( n' \leq i_n \).

Hence we can say that

\[
\gamma_n = \sup_{s \in S} \left\{ \frac{\|v_{i_n}(s)\|_p}{\|s\|_p} \right\} \leq C(1 + n')^{m+1} < \infty
\]

for every \( n \) where \( n' \) is the total degree of the monomial spanning \( V_n \). Thus, condition (1) is verified.

To every continuous function \( \phi^* \), the support of which lies in \([0, 1 - \epsilon] \times [0, 1 - \epsilon]\) for some \( \epsilon > 0 \), there corresponds a linear functional \( \phi \) defined on \( B \) given by

\[
\phi(f) = \int_0^1 \int_0^1 \phi^*(x, y) f(x, y) dy dx
\]

Let \( \Phi \) denote the family of such linear functionals. Evidently, \( \Phi \) is normalizing.

**Claim:** \( \|\gamma_n\|_S \|\phi\|_{V_n} \to 0 \) as \( n \to \infty \) for every \( \phi \in \Phi \)

Now,

\[
\|\phi\|_{V_n} = \sup \left\{ \frac{|\phi(z)|}{\|z\|_p} : z \in V_n \right\}
\]

Let \( z \in V_n \). Then, \( z = ax^{m_1}y^{n_1} \) where \( m_1 + n_1 = n' \).

Let \( \phi \in \Phi \). Then,
\[ |\phi(z)| = |\int_0^1 \int_0^1 z \phi^*(x, y) dy dx| \]

\[ \leq M \int_0^{1-\epsilon} \int_0^{1-\epsilon} |z| dy dx \] where \( M = \max_{[0,1-\epsilon] \times [0,1-\epsilon]} |\phi^*(x, y)| \)

\[ \leq M (1 - \epsilon)^{n' + 2} \int_0^1 \int_0^1 |z| dy' dx' \] where \( x = x'(1 - \epsilon) \) and \( y = y'(1 - \epsilon) \)

\[ \leq M (1 - \epsilon)^{n' + 2} ||z||_1 \]

\[ \leq M (1 - \epsilon)^{n' + 2} ||z||_p \]

\[ \frac{|\phi(z)|}{||z||_p} \leq M (1 - \epsilon)^{n' + 2} \]

\[ ||\phi||_{V_n} = \sup \left\{ \frac{|\phi(z)|}{||z||_p} : z \in V_n \right\} \leq M (1 - \epsilon)^{n' + 2} \]

\[ ||\gamma_n||_S ||\phi||_{V_n} \leq MC (1 + n')^{m-1} (1 - \epsilon)^{n' + 2} \]

where \( n' \) is the total degree of the monomial spanning \( V_n \). As \( n \to \infty, \) \( n' \to \infty, \)

and hence, \( ||\gamma_n||_S ||\phi||_{V_n} \to 0 \) as \( n \to \infty. \)

Thus, condition (ii) of (2) is verified. Hence the theorem is proved.
4.2 Best approximation from polynomials of given h-length:

Theorem 3:

Let $1 \leq p \leq \infty$ and $f \in L_p([0,1] \times [0,1])$. Let $m$ be a positive integer. Then, there exists a polynomial of best approximation $P^*$ from $\mathcal{P}_m^h$ to $f$.

i.e. $\|f - P^*\|_p = \inf \{\|f - P\|_p : P \in \mathcal{P}_m^h\}$

Proof:

To prove this theorem, we use the Abstract Existence theorem (Theorem B) of chapter [2] and the estimate(3.11) for the norm of the homogeneous part of a polynomial. It is enough to check the conditions (1) and (2) of theorem B. Using the notations of Theorem B, let $\mathcal{B} = L_p([0,1] \times [0,1])$.

Let $P_j = V_j = \text{The space of homogeneous polynomials of total degree } j$.

Then, $\dim (P_j) = j + 1$. Let $s \in S$. If $s \in V_{i_1} \oplus V_{i_2} \oplus \ldots \oplus V_{i_m}$ then,

$s = v_{i_1} + v_{i_2} + \ldots + v_{i_m}$ where $v_{i_j} = v_{i_j}(s) \in V_{i_j}, j = 1 \text{ to } m$.

Using the estimate 3.11, we get

$\|v_{i_n}(s)\|_p \leq C(1 + i_n)^{m-1}\|s\|_p$ for $n = 1 \text{ to } m$.

So, we can write

$\|v_n(s)\|_p \leq C(1 + n)^{m-1}\|s\|_p$ for every $n=1,2,\ldots$

Hence,

$\gamma_n = \sup_{s \in S} \left\{ \frac{\|v_n(s)\|_p}{\|s\|_p} \right\} \leq C(n + 1)^{m-1} < \infty$

for every $n$.

Thus condition (1) is verified.

To every continuous function $\phi^*$, the support of which lies in...
for some $\epsilon > 0$ there exists a linear functional $\phi$ defined on $B = L_p([0, 1] \times [0, 1])$ given by

$$
\phi(f) = \int_0^1 \int_0^1 \phi^*(x, y) f(x, y) dy dx
$$

Let $\Phi$ denote the family of such linear functionals. Evidently, $\Phi$ is normalizing.

**Claim:** $\|\gamma_n\|_S \|\phi\|_{V_n} \to 0$ as $n \to \infty$ for every $\phi \in \Phi$.

Now,

$$
\|\phi\|_{V_n} = \sup \left\{ \frac{|\phi(z)|}{\|z\|} : z \in V_n \right\}
$$

Let $z \in V_n$. Then,

$$
z = \alpha_1^{(n)} x^{m_1} y^{n_1} + \alpha_2^{(n)} x^{m_2} y^{n_2} + \ldots + \alpha_l^{(n)} x^{m_l} y^{n_l}
$$

where $m_i + n_i = n$ and $l \leq n + 1$. Let $\phi \in \Phi$. Then,

$$
|\phi(z)| = |\int_0^1 \int_0^1 z \phi^*(x, y) dy dx|
$$

$$
\leq M \int_0^1 \int_0^{1-\epsilon} |z| dy dx \quad \text{where} \quad M = \max_{[0,1-\epsilon] \times [0,1-\epsilon]} |\phi^*(x, y)|
$$

$$
= M (1 - \epsilon)^{n+2} \int_0^1 \int_0^1 |z| dy' dx' \quad \text{where} \quad x = x'(1 - \epsilon) \text{ and } y = y'(1 - \epsilon)
$$

$$
\leq M (1 - \epsilon)^{n+2} \|z\|_1
$$

$$
\leq M (1 - \epsilon)^{n+2} \|z\|_p
$$

$$
\frac{|\phi(z)|}{\|z\|_p} \leq M (1 - \epsilon)^{n+2}
$$

$$
\|\phi\|_{V_n} = \sup \left\{ \frac{|\phi(z)|}{\|z\|} : z \in V_n \right\} \leq M (1 - \epsilon)^{n+2}
$$

$$
\|\gamma_n\|_S \|\phi\|_{V_n} \leq MC (1 + n)^{n-1} (1 - \epsilon)^{n+2}
$$
Hence, \( ||\gamma_n||_S ||\phi||_{V_n} \to 0 \) as \( n \to \infty \)

Thus, condition (ii) of (2) is verified. Hence, the theorem is proved.
CHAPTER V
APPLICATION OF THE ESTIMATES

How well can \( x^{\alpha} \) be approximated in \( L_p(\theta, 1) \)-norm by a linear combination of \( x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_m} \)? The answer depends on three quantities: (1) how close can \( \alpha_j \)'s come to \( \alpha \) (2) the number of terms \( m \) in the linear combination (3) the number \( \alpha \) itself.

If one restricts \( \alpha_j \)'s by assuming, for example, \( |\alpha_j - \alpha| \geq 1 \) for \( j = 1, 2, \ldots, m \) where \( m \) is kept fixed, estimates for how the approximation error depends on \( \alpha \) have been obtained by several authors. In fact, as it is remarked in [3] these estimates are just dual forms of writing the estimate for the coefficients of polynomials of given length.

In this chapter, we shall obtain analogous results in two dimensional case. That is, we shall write Theorem 2 and Theorem 3 in their dual forms. Since the estimates in Theorem 2 and Theorem 3 are sharp as we have shown, the estimates given in the following theorems are also sharp.

5.1 Application of the estimate (3.7)

In this section, we apply our estimate (3.7) of the coefficient of a polynomial in two variables to find how well a monomial \( x^\lambda y^\mu \) can be approximated by a polynomial \( P \) of s-length \( \leq m \) where \( P \) does not contain a term with \( x^\lambda y^\mu \).
Theorem 2':

Let $\lambda, \mu$ be non-negative integers and $1/p + 1/q = 1$.

If $d(x^\lambda y^\mu, m)$

$$= \inf \left\{ \|x^\lambda y^\mu - \sum_{j=1}^{m} b_j x^{\lambda_j} y^{\mu_j}\|_p : b_j \in R, \lambda_j, \mu_j \text{ non-negative integers, } (\lambda_j, \mu_j) \neq (\lambda, \mu) \right\}$$

then

$$d(x^\lambda y^\mu, m) \geq K [1 + \max(\lambda, \mu)]^{-m-1/p} [1 + \min(\lambda, \mu)]^{-1/p}$$

where $K$ depends only on $m$.

Proof:

Consider the polynomial $P = x^\lambda y^\mu - \sum_{j=1}^{m} b_j x^{\lambda_j} y^{\mu_j}$

This polynomial $P$ is of length $m+1$. By applying the estimate (3.7) to $P$ and noting that the coefficient of $x^\lambda y^\mu$ in $P$ is 1, we see from (3.7) that

$$1 \leq C [1 + \max(\lambda, \mu)]^{m+1-1/q} [1 + \min(\lambda, \mu)]^{1-1/q} \|x^\lambda y^\mu - \sum_{j=1}^{m} b_j x^{\lambda_j} y^{\mu_j}\|_p$$

Thus,

$$\|x^\lambda y^\mu - \sum_{j=1}^{m} b_j x^{\lambda_j} y^{\mu_j}\|_p \geq K [1 + \max(\lambda, \mu)]^{-m-1/p} [1 + \min(\lambda, \mu)]^{-1/p}$$

where $K = 1/C$. Hence,

$$d(x^\lambda y^\mu, m) \geq K [1 + \max(\lambda, \mu)]^{-m-1/p} [1 + \min(\lambda, \mu)]^{-1/p}$$

where $K$ depends only on $m$. 
5.2 Application of the estimate (3.11)

In this section, we apply our estimate (3.11) of the homogeneous part of a polynomial in two variables to find how well a homogeneous polynomial of degree \( \lambda \) can be approximated by polynomials of h-length \( \leq m \) which contain no homogeneous part of degree \( \lambda \).

**Theorem 3':**

Let \( P \) be a homogeneous polynomial of degree \( \lambda \) and \( \|P\| = 1 \) where \( 1/p + 1/q = 1 \). If

\[
d(P, m) = \inf \{ \|P - \sum_{j=1}^{m} Q_j\| : Q_j \text{ is a homogeneous polynomial of degree } \neq \lambda \}
\]

then,

\[
d(P, m) \geq K(1 + \lambda)^{-m}
\]

where \( K \) depends only on \( m \).

**Proof:**

Consider the polynomial \( P - \sum_{j=1}^{m} Q_j \)

This is a polynomial of h-length \( m+1 \).

Applying the estimate (3.11) to this polynomial we see that

\[
1 = \|P\|_p \leq C (1 + \lambda)^m \|P - \sum_{j=1}^{m} Q_j\|_p
\]

Thus,

\[
\|P - \sum_{j=1}^{m} Q_j\|_p \geq K (1 + \lambda)^{-m}
\]

where \( K = 1/C \). Hence,

\[
d(P, m) \geq K(1 + \lambda)^{-m}
\]
CHAPTER VI

MULTIVARIATE GENERALIZATION OF MARKOV'S INEQUALITY

6.1 Introduction

Let $T_m(x) = \sum_{j=0}^{m} t_j^{(m)} x^j$ denote the $m$th Chebyshev polynomial of first kind with respect to $I = [-1, 1]$.

Let

$$M_{j,m} = \begin{cases} |t_j^{(m)}| & \text{if } j \equiv m \mod 2 \\ |t_{j-1}^{(m-1)}| & \text{if } j \equiv m-1 \mod 2 \end{cases} \quad (6.1)$$

We refer to the numbers $M_{j,m}$, $j = 0, 1, 2, \ldots, m$, as Markov Numbers.

Let $P_n(x) = \sum_{j=0}^{m} a_j x^j$ be an arbitrary real-valued univariate polynomial with norm $||P_m|| \leq 1$ where $|| \cdot ||$ denotes the uniform norm on $[-1, 1]$. Markov obtained the following set of sharp inequalities for the coefficients $a_j$ as follows.

$$|a_j| \leq \begin{cases} |t_j^{(m)}| & \text{if } j \equiv m \mod 2 \\ |t_{j-1}^{(m-1)}| & \text{if } j \equiv m-1 \mod 2 \end{cases} \quad (6.2)$$

The integers $t_j^{(m)}$ are explicitly known [15, p.32]. In particular, when $j=m$, we get

$$|a_m| \leq 2^{m-1} \quad (6.3)$$

This inequality is originally due to Chebyshev [15]. When $j = m-1$ we get

$$|a_{m-1}| \leq 2^{m-2} \quad (6.4)$$

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In this chapter, we generalize these inequalities (6.2) to polynomials in $r$ variables.

### 6.2 Notation

Let $P_n^r$ denote the polynomials of total degree $\leq n$ in $r$ variables

$$P_n^r = \sum_{|k|\leq n} a_k x^k$$

where

$$k = (k_1, k_2, \ldots, k_r) \in \mathbb{Z}_+^r$$

$$|k| = k_1 + k_2 + \cdots + k_r$$

$$x = (x_1, x_2, \ldots, x_r) \in \mathbb{R}^r$$

$$x^k = x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}$$

$$a_k \in \mathbb{R}$$

Let $H_{n,r}$ denote the homogeneous polynomials of total degree $n$ in $r$ variables

$$H_{n,r} = \sum_{|k|=n} a_k x^k$$

Evidently, $P_n^r$ can be written as

$$P_n^r = H_{n,r} + H_{n-1,r} + \cdots + H_{0,r}$$

$H_{k,r}$ will be referred to as the homogeneous part of degree $k$ of $P_n^r$.

Let $x_i = (x_{(i,1)}, x_{(i,2)}, \ldots, x_{(i,r)}) \in \mathbb{R}^r$ where $i \in \mathbb{Z}_+$.

Let $\| \cdot \|_{L^\infty(Q)}$ denote the $L^\infty$-norm on the $r$-dimensional cube $Q = [-1, 1]^r$. 

6.3 Generalizations

Visser [23] and Rack [16] have obtained the extensions of (6.3) and (6.4) to polynomials in \( r \) variables in the following theorems.

**Theorem 7: (Visser)**

\[
| \sum_{|k|=n} a_k | \leq 2^{n-1} \| P_r^n \|_{L^\infty(Q)} \tag{6.5}
\]

Equality in (6.5) occurs if

\[
P_r^n(x) = \sum_{j=1}^r T_n(x_j)
\]

where \( T_n(x) \) denotes the \( n \)th Chebyshev polynomial of first kind.

**Theorem 8: (Rack)**

\[
| \sum_{|k|=n-1} a_k | \leq 2^{n-2} \| P_r^n \|_{L^\infty(Q)} \tag{6.6}
\]

Equality in (6.6) occurs if

\[
P_r^n(x) = \sum_{j=1}^r T_{n-1}(x_j)
\]

where \( T_{n-1}(x) \) denotes the \((n-1)\)th Chebyshev polynomials of first kind.

We extend the set of inequalities (6.2) to polynomials in \( r \) variables in Theorem 9. As a result, Theorem 7 and Theorem 8 are particular cases of Theorem 9. It should be noted that the proof techniques of Visser [23] and Rack [16] involve the discrete orthogonality of exponential function and they do not provide estimates for higher coefficients. While our proof is quite elementary, it provides estimates for higher coefficients as well.
Theorem 9:

If $P_m^r$ is a polynomial of total degree $\leq m$ in $r$ variables then

$$|\sum_{|k|=j} a_k| \leq M_{j,m} \|P_m^r\|_{L^\infty(Q)}$$

(6.7)

where $M_{j,m}$ are Markov Numbers as in (6.1).

Equality in (6.7) occurs if

$$P_m^r(x) = \sum_{j=1}^{r} T_m(x_j)$$

where $T_m(x)$ is the $m$th Chebyshev polynomial of first kind.

The proof of Theorem 9 follows as a corollary to Theorem 10.

Theorem 10:

If $P_m^r$ is a polynomial of total degree $\leq m$ in $r$ variables and

$H_{j,r} = H_{j,r}(P)$ denotes the homogeneous part of degree $j$ of $P_m^r$,

then

$$\|H_{j,r}(P)\|_{L^\infty(Q)} \leq M_{j,m} \|P_m^r\|_{L^\infty(Q)}$$

where $Q$ is the $r$-dimensional cube $[-1,1]^r$.

Proof:

Suppose $H_{j,r}(x)$ attains the maximum on the cube $Q$ at $x_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,r})$.

Then,

$$|H_{j,r}(x_0)| = \|H_{j,r}\|_{L^\infty(Q)}$$
Consider the univariate polynomial $Q(t)$ where

$$Q(t) = P_m^r(x_{(0,1)} t, x_{(0,2)} t, \cdots, x_{(0,r)} t) \quad \text{where } t \in [-1, 1]$$

$$= \sum_{j=0}^{m} H_{j,r}(x_{(0,1)} t, x_{(0,2)} t, \cdots, x_{(0,r)} t)$$

$$= \sum_{j=0}^{m} t^j H_{j,r}(x_{(0,1)}, x_{(0,2)}, \cdots, x_{(0,r)})$$

$$= \sum_{j=0}^{m} C_j t^j$$

where $C_j = H_{j,r}(x_{(0,1)}, x_{(0,2)}, \cdots, x_{(0,r)})$.

Applying (6.2) to $Q(t)$ we get

$$|C_j| \leq M_{j,m} ||Q(t)||_{[-1,1]}$$

$$\leq M_{j,m} ||P_m^r||_{L^\infty(Q)}$$

$$|H_{j,r}(x_{(0,1)}, x_{(0,2)}, \cdots, x_{(0,r)})| \leq M_{j,m} ||P_m^r||_{L^\infty(Q)}$$

$$|H_{j,r}(x_0)| \leq M_{j,m} ||P_m^r||_{L^\infty(Q)}$$

$$||H_{j,r}||_{L^\infty(Q)} \leq M_{j,m} ||P_m^r||_{L^\infty(Q)}$$

**Corollary:**

$$\left| \sum_{|k| \neq j} a_k \right| \leq M_{j,m} ||P_m^r||_{L^\infty(Q)}$$

Equality occurs in the above if

$$P_m^r(x) = \sum_{j=1}^{r} T_m(x_j)$$

where $T_m(x)$ is the mth Chebyshev polynomial of first kind.
Proof:

The proof follows from the fact that

\[ \left| \sum_{|k|=j} a_k \right| = |H_{j,r}(1,1,\cdots,1)| \leq \|H_{j,r}\|_{L^\infty(Q)} \]


[23] C. Visser, A generalization of Tchebychev's inequality to polynomials in more than one variable, Indag. Math.8 (1946), 310-311.