COLLEGE STUDENTS' UNDERSTANDING
OF RATIONAL EXPONENTS:
A TEACHING EXPERIMENT

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the
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By

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ABSTRACT

The study examines the understanding college students have of the concept of rational and negative exponents and the justifications for their notions of exponents. Pre-interviews with novice and expert mathematics students suggested, that novice students had fragmented notions of exponents. In applications with rational and negative exponents novice students relied on operational procedures and the authority of teachers. Neither novice students nor expert students proposed integrated concept of exponents to explain all types of exponents.

A conjecture for transforming the teaching and learning of exponents was proposed, that the teaching and learning of exponents can be improved through the study of the concepts of rate of growth and factors of multiplication, and a thorough study of roots and powers of factors. The conjecture was tested in a teaching experiment with the novice students. The role of the laws of exponents in the formation of rational and negative exponents was examined.

The students’ construction of the concept of rational and negative exponents is described through models. The results suggest that students do not base their understanding of rational or negative exponents on patterns of the Laws of Exponents. The Common Definition of Exponents seemed the preferred lens through which the concept of exponents was viewed first, and then replaced by memorized rules, cues from notations, and teachers’ authority.
The tendency of students to use linear forms of thinking for multiplicative models of change affected their understanding of factors of multiplication and rates of growth under various conditions. A process for calculating decimal exponents that brings together all the components of the construction of rational exponents was part of the study. The zero exponent was given special attention in the context of rational exponents. Negative exponents were studied as reverse actions of multiplication equivalent to multiplication of inverses.

All students showed improvement in their understanding of rational, decimal and negative exponents. Two students presented integrated concept of exponents covering rational, decimal, and negative exponents. The other students continued to focus on the separate operations of different forms of exponents and did not propose integrated concepts.
Dedicated to my parents:

Augusta Dreischor and Bernhardus Elstak
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CHAPTER 1

INTRODUCTION

My teaching experience with high school students and student teachers in Suriname and the Dutch Antilles and my experience with college students in the USA have made me very much aware of the superficial type of "understanding" that many students develop when exponents or logarithms are involved. Rules of operation are drilled into students, but they have little understanding of whole number exponents and no understanding of rational exponents. Reliance on calculators is almost the norm. Few students seem to know how to prove the properties of exponents or even less of logarithms they use so diligently.

The wider teaching community all over the world has called for making mathematics a powerful tool for life in the contemporary world. The National Council of Teachers of Mathematics (NCTM) in the USA has stated that mathematics is important for life, culture, careers, and as a personal source of satisfaction (NCTM, 2000). This goal can only be realized if students understand concepts, are proficient in applying the knowledge and know how to transfer factual and conceptual knowledge from one area to the next. The study of rational exponents provides all the ingredients for such knowledge.

The National Council of Teachers of Mathematics also states explicitly in the Principles and Standards for School Mathematics (NCTM, 2000) that a major emphasis
in the measurement standard is the understanding what a measurable attribute of an object is and that students become familiar with the units and processes used in the measuring of the attributes. I claim that exponents are measures of multiplication with specific units and that the processes of understanding that measurement must become part of standard school knowledge.

Exponential functions are important for contemporary society in many different ways (Confrey, 1994). They appear in models for population control, disease prediction, the study of radioactive decay, growth of wildlife populations, biological growth under conditions of limited or unlimited resources, problems of inflation and the growth of debts due to interest charges, and many other fields. Exponential forms are also the foundation for studying the closely related concepts of logarithmic functions with their own wide field of applications.

Despite the importance of exponents very little empirical research has been conducted on how students develop their concepts of non-integer exponents. Existing studies (Confrey, 1991, 1994; Confrey & Smith, 1995; Rizzuti, 1991) have focused on how repeated multiplication is related to problems of percentage increase or decrease or how students conceptualize and operate with powers of numbers. These studies suggest that the transition from a fixed percentage rate of growth or decrease to an exponential form is problematic for many students. It is still an open question how students construct their concept of negative and rational exponents and how deep their understanding is of the connections between integer exponents, rational exponents, and negative forms of exponents.
Problem Statement

This study examined college students' knowledge of rational and negative exponents. Explanations and justifications of the concept of rational and negative exponents as voiced by the students were examined. How do students relate various concepts of exponents as defined for natural numbers to those defined for fractions and for negative numbers? What are their concepts of decimal exponents? Are students able to explain what the digits in a decimal representation of an exponent represent? If students' concepts do not reflect conceptual understanding, how can we improve the instruction and teaching of rational exponents for these students?

Studies on the learning of exponential functions are all quite recent. None of the studies on exponential functions (Confrey, 1991, 1994; Confrey & Smith, 1994) involve empirical research of how rational or negative exponents are constructed by students. These studies deal with students' conceptions of powers of ten (Confrey, 1991; Confrey & Smith, 1994; 1995), or the problem solving difficulties students have when they try to explain how a fixed percentage change translates into an exponential form. Solving such translation problems could be interpreted as a first step toward the formulation of the exponential function with positive integer exponents.

Based on a pilot project conducted in 2003-2004 and a review of relevant literature, I formulated the following conjecture that was to be tested in a teaching experiment using the main principles of the transformative, conjecture driven teaching experiment design proposed by Confrey and Lachance (2000).
Conjecture

The teaching and learning of rational exponents can be improved through a step by step study of, on the one hand, the concepts of rate of growth and factors of multiplication, and on the other hand a thorough study of both roots and powers of factors of multiplication. The study of rational exponents should be embedded in a format of the actual construction of an exponential function. By studying the process of the construction of the exponential function and the main justifications for its validity, including the study of the construction and meaning of decimal exponents, rational exponents and the extended laws of exponents emerge as meaningful formalizations of the main principles of exponential functions. The conjecture can only be developed in a coherent way by first investigating the actual knowledge of exponential functions by students in the group. The aim of this investigation was to answer the following questions:

1. What are novice students' concepts of rational and negative exponents?

2. What are expert mathematics students' concepts of rational and negative exponents, and how do they differ from novices' concepts?

3. What is the role of the Laws of Exponents in the process of developing rational and negative exponents, from the perspective of novice students (before the teaching experiment) and as emerging from the teaching experiment?

4. How can we model students' construction and development of the concept of rational and negative exponents as emerging from the teaching experiment and the interviews?

5. What is the impact of the teaching experiment on the knowledge of the participating students?
Expectations and questions connected to the conjecture. The conjecture aimed to provide evidence for the central role of roots and powers of roots in the construction and understanding of the exponential function in general and the rational exponents in particular. Will the rate of growth approach, combined with the concept of factors of multiplication, improve the understanding of rational exponents? Is the discussion of roots and powers of roots powerful and convincing enough to enable the students to link their early notion of exponents with the rational exponent concepts that is reinforced in the teaching experiment? Does the embedding of the instruction in problems of growth and multiplication taken from real life prototypes make the construction more meaningful to the students? Lastly, the study addressed the issue of introducing negative exponents after the introduction of rational exponents (typically negative exponents are introduced before).

Context of the Study

The development of the notion of exponential functions and in particular the construction of the concept of rational exponents by novice students in college forms the context of this study. To develop an idea of the field I conducted a pilot project on the knowledge of exponents of college students in 2003-2004.

Pilot Project on Exponents

In 2003-2004 I carried out a pilot project on the concepts of exponents and interviewed a selected group of twenty students all 18 years or older, most in their first year at the Ohio State University. The students had successfully completed a pre-calculus course I had taught during the year 2003. I also interviewed four graduate teaching
associates (GTA's) from the mathematics department.\textsuperscript{1} The transcribed interviews were
coded for forms of understanding, justifications, sources of the notions used for solving
problems, logical connections and the use of representations of different kinds. Codes
were classified into categories and preliminary conclusions were drawn from those
classifications.

The preliminary results of the pilot project suggested that most of the students did
not know the connection between negative exponents and positive exponents. Inferences
were made of how the students justified their concepts of negative and rational
exponents. I modeled my interpretation of the students' knowledge based on these
inferences. No evidence was found that the students embedded their understanding of
exponents in the laws of exponents as suggested by traditional textbooks. All students
could state the traditional concept for positive integer exponents without any help from
me. Only one student was able to recall, without help, the meaning of negative exponents
and how they relate to other positive integer exponents. None of the students referred to
the laws of exponents to justify the concept of negative exponents. Most stated from
memory that it was just a definition established by convention. A constant occurrence
with all the students was that negative exponents were discarded and replaced by
fractions as soon as they appeared in calculations or formulas. No attempt was made by
either the novice or the graduate students to use these negative exponents as exponents.

The pilot project also revealed weaknesses in understanding the decimal form of
exponents. All students could graph exponential functions (on a graphing calculator) and

\textsuperscript{1} Appendix A contains questions very similar to the ones used in this pilot project.
use the calculator to estimate the solution of exponential equations of the form $5^x = 8$.

However, explaining the meaning of decimal exponents proved problematic. Students were able to explain why $x$ had to be between 1 and 2, but were unable to explain the meaning of the decimals or the digits in the solution $x \approx 1.2920\ldots$ What was the meaning of the decimal 0.2 in the solution and how was this increase from 1 to 1.2 related to the exponential function? The best answers involved using the calculator to show that $5^{1.3} = 8.103\ldots$ was more than $5^{1.2} = 6.89\ldots$ Despite repeated questions from the interviewer and attempts by students, they could not explain how, or through which mechanism the exponential function can increase its value from $5^1 = 5$ to $5^{1.2} = 6.89\ldots$ From this I inferred that the change in value as a result of the change in exponents was not clear to the students. They were unable to explain the relation between the increase from 1 to 1.2 for the exponent and the increase of the outcome. The answer I received most frequently was that we added 5 times 0.2 to the amount of the initial 5. Some students, however, realized that this amount did not add up to what the calculator showed.

The GTA's were able to explain the increase through the use of calculus or by decomposing the exponent, applying the laws of exponents and connecting them to the roots of radical powers of ten. I interpreted the use of calculus as a sophisticated form of linear thinking on the part of the GTA's, supported by their knowledge of advanced mathematics. My interpretation was that the GTA's combined what they had learned about differentiation with the tendency of linear thinking.\(^2\)

\(^2\) The tendency of many students to apply forms of linear thinking has been identified by other researchers (De Brock, van Dooren, Janssen & Verschaffel, 2002; De Brock, Verschaffel & Janssen, 2002).
One student came up with the notion that the 0.2 in the exponent could be interpreted as two-tenths of a factor of 5. This approach to explaining exponents struck me as not only peculiar, but as innovative for its formulation and it made sense to me when I was transcribing the interview. My interpretation was that it can be viewed as an example of students' mathematics (Steffe & D'Ambrosio, 1996) because it contains the element of multiplication and the idea of using only a fraction of the base unit of 5 in the multiplication process. Furthermore it connects the initial definition of exponents based on counting of factors with the fraction form of the exponents by counting the fractions in terms of the unit of multiplication. I decided to use this construct as part of my teaching experiment. These findings also led to the research questions.

The results of the study suggested that novice students do not rely on the laws of exponents for their introduction and understanding of negative and rational exponents. I found that after the students had been exposed to all the definitions of rational and negative exponents, they were unable to explain how these concepts of exponents as formulated through the definitions are related to one another. Moreover, none of the students were able to explain the decimal form of exponents.

**Rational Exponents**

The results of the pilot study strengthened my belief (based on my previous teaching experience) that the problems that students have with logarithms could be rooted in their poor knowledge of rational exponents. This led to the general question: how are rational exponents constructed by novice students? There seems to be an assumption in the teaching of negative and rational exponents that patterns inferred from the laws of exponents for positive integer exponents by more mature students and teachers can be a
sufficient basis for convincing explanations of the content and nature of negative and rational exponents for novice learners in this field. The importance attached to the "laws of exponents" in most textbooks with the implication that students will use these laws as their foundations for understanding rational exponents, prompted the question: do students base their construction and understanding of exponents on the validity of, or the pattern they (allegedly) perceive in, the general laws of exponents? The difficulties students seem to have in explaining or relating different definitions and concepts of exponents to each other led to the question: do students develop a general concept of exponents that allows them to explain both positive integer and rational exponents with their general concept? One set of questions led to others. Can students explain what decimal exponents mean? What is the notion of exponents that students develop from the separate, case by case definitions for exponents of different kinds of numbers?

Because research suggests (Leinhardt, Zaslavsky, & Stein, 1990) that students have no problem with the global (visual) concept of the exponential function, I made the function and its graphical representation the starting point of my investigation. The introduction of graphing calculators has made it relatively easy to create visual representations of the (exponential) function and to answer many questions about exponents without knowing or understanding, the finer details of how these functions are defined or how they vary with the given variable. The global graphical approach could obscure a lack of insight into the details of the exponents and their connections to roots and power properties.

The way rational exponents are constructed by students is an open question. The definitions of rational exponents are inconsistent in language and concept compared with
the definitions for negative exponents, and the definitions for exponents one and zero. Lochhead (1991, p.77) commented on this inconsistency and characterized the lack of fit between these definitions as an example of a "selective notion of consistency because it ignores the original definition in which the superscript was the number of times the base number was to be multiplied by itself, and there is no way to multiply a number a negative number of times." In many textbooks (for example in Hall, B. & Fabricant, M., 1993, or in Smith et al. 1992) the inconsistencies in definitions and language mentioned by Lochhead (1991) are ignored. The laws of exponents are presented as rules for powers and exponents with no attempt to develop a general concept of an exponent.

Objective and Goals

The objective of my research is to contribute to a more coherent development of the notion of rational exponents. A conjecture is proposed based on a study of relevant literature and a pilot project. The conjecture is that the teaching and learning of rational exponents can be improved through a step by step study of on the one hand the concept of rate of growth and factors of multiplication and on the other hand a thorough study of both roots and powers of factors of multiplication. This study of rational exponents should be embedded in the construction of an exponential function. Other goals this research tried to achieve were: to describe a model for the learning process of rational exponents and identify some of the problems students can experience trying to make sense of the various definitions of exponents, to find out the role of the laws of exponents in the formation of the concept of rational and negative exponents. In the long term, the ways exponents are introduced in schools should be studied extensively with the aim of reforming this part of the curriculum in schools.
To develop an idea of the nature of the problem, I reviewed the studies that involved exponents, and I used the preliminary results of the 2003-2004 pilot project to understand and describe the concepts of exponential forms and functions college students have. The literature review together with my pilot project suggested the conjecture and the choice of the teaching experiment design for the major part of the investigation.

Through the instructional intervention of the teaching experiment I studied the students' way of understanding rational exponents. The design of the teaching experiment was much like the transformative, conjecture driven teaching experiment methods developed by a number of researchers (Confrey & Lachance, 2000), only on a smaller scale. In other words, I developed a model of the students' mathematics (Steffe & Thompson, 2000; Steffe & D'Ambrosio, 1996) as they construct the rational exponents in the course of the teaching experiment and as interpreted from the perspective of my theoretical framework and my observations.

**Definition of Terms**

The following terms were used throughout the study:

*Conjecture:* An inference based on inconclusive or incomplete evidence. It serves as a guide within a theoretical framework and is subject to alterations and modifications during the research (Confrey & Lachance, 2000).

*Decimal and rational numbers:* In this study the term "decimal numbers" designates rational numbers written in decimal notation.

*Decimal exponents:* These are exponents written with the decimal notation.

*Decimal roots:* The frequently used roots like \(\sqrt[10]{\ldots} \), \(\sqrt[100]{\ldots} \), \(\sqrt[1000]{\ldots} \), etc. The root index is a positive integer power of 10.
The Decimal Exponent Calculation process (DEC): The system introduced to the students in the teaching experiment, to study and analyze the structure and calculation of decimal exponents. Exponential equations are used as a problem context.

Functions: Mappings from one set, the domain, to another set called the range (Dummit & Foot, 1999). These mappings have the following properties: for every element X in the domain there is exactly one element Y in the range that is called the image of X. In symbolic notation F(X) = Y, where F is the symbol for the name of the function.

The Common Definition of Exponents (CDE): The idea that exponents represent the number of factors of the same unit or the same base multiplied together. This notion is relatively well understood by the students in this study and it functions as some form of baseline for understanding exponents.

The traditional way of introducing exponents: The method of creating exponents by repeated multiplication and a case by case approach for rational and negative exponents without explaining the connection between the various forms. The traditional way of teaching exponents is a disconnected way of presenting definitions for rational and negative numbers as exponents and ignoring the previous definitions in terms of consistency of presentations, language and meanings (Lochhead, 1991). The definition of $A^n$ (for n a natural number greater than one) states that it is a product of n factors of A, that is $A \times A \times A \times \ldots \times A$, taken n times. The symbol "\times" stands for multiplication in the traditional sense. Some texts ignore the inconsistency of this definition when applied to the case $n = 1$, that simply leaves out the multiplication when there is only one factor, and assumes that $A^1 = A$ after stating that "$A^B = A \times A \times \ldots \times A$, taken B times," (Beckmann,
2005, p. 228). No justification is provided for the inconsistency in the definitions and the absence of any multiplicative operation for the case $B = 1$.

The case by case definition of exponents: The practice of defining exponents in a case by case format presenting definitions for exponents that are negative numbers, and definitions for exponents that are fractions (see for example Beckmann, 2005; Hall & Fabricant, 1993; Smith, Charles, Dossey, Keedy & Bittinger, 1992). The connection between the definitions is not explored and the connection with the original repeated multiplication form is left to the student to figure out. I have identified five different cases:

Case 1: $A^n = A \times A \times A \times \ldots \times A$, where the number of factors is $n$, a natural number, not equal to 1. $A$ can be any number. Important here is to notice that there is an assumption of a multiplication to be carried out $(n-1)$ times.

Case 2: $A^1 = A$, for any number $A$. This is the definition for an exponent equal to 1 (one). There is no reference to or presence of any form of multiplication.

Case 3: $A^0 = 1$ for any non-zero number $A$, the definition for an exponent equal to zero (0). There is no reference to multiplication, and no explanation of why and how zero can also enter into the world of counting factors of multiplication.

Case 4: $A^{(p/q)} = q\sqrt[q]{(A^p)}$ for $p$ an integer number and $q$ a positive integer and positive values for $A$. The assumption here is that if $q$ is negative, that the fraction must be converted into a form so that an equivalent fraction $r/s$ is created, with $r/s = p/q$, and $s$ is taken to be positive and the numbers $r$ and $s$ are integers that are relative primes.

Case 5: $A^{-n} = 1/(A^n)$ for integer values of $n$. The case where the exponent is a real (non-rational) number would require the limit concept which is not studied in
this research because of the very specific nature of the limit concept involved in the definition of real exponents.

*The laws of exponents:* The basic principles and relations as derived from the multiplication and division of integer exponential forms.

\[ A^n \times A^m = A^{n+m}, \text{ the law of the addition of exponents through multiplication of powers of } A. \]

\[ A^n / A^m = A^{n-m}, \text{ the law of subtraction of exponents through division of powers of } A. \]

\[ (A^n)^m = A^{nm}, \text{ the law of powers of powers.} \]

*Exponential functions:* Functions of the form \( F(x) \) such that \( F(x + a) = k_a \times F(x) \), in which \( k_a \) is only dependent on the value of \( a \). The value \( k_a \) is called the multiplicative rate of change over the interval \([x, x + a]\). In this study the factor \( k_a \) is called the factor of multiplication. This special property of the exponential functions puts the multiplicative rate of change at the center of the concept of exponential functions. Exponential functions have equal rates of change over equal intervals. For example, if \( F(x) = 5 \times 3^x \) then

\[ F(x + a) = 5 \times 3^x \times 3^a = 3^a \times F(x). \] The number \( 3^a \) depends only on the value of \( a \) and the unit of multiplication 3. It does not depend on the choice of \( x \). The number \( 3^a \) can be thought of as the factor of multiplication over the interval \([x, x + a]\) for the function \( F(x) \). The rate of growth and the factor of multiplication are connected by: \[ \text{factor of multiplication} = 1 + \text{rate of growth}. \] The factor of multiplication over the interval \([x, x + a]\) for \( F(x) \) is \( F(x + a)/F(x) = 3^a \). So the rate of growth over the given interval is then \( 3^a - 1 \). For \( a = 2 \), the factor of multiplication is 9; for \( a = 1/5 \) the factor of multiplication is \( 3^{5 \times 3} \times 1.246 \). The rate of growth over \( a = 1/5 \) is then 0.246 or 24.6%.
The concept of Factor of Multiplication and Rate of Growth: If a population grows by p percent per year, and the population at the beginning of the year was P(0), then at the end of the year the new population will be

\[ P(1) = P(0) + \frac{p}{100} P(0) \Rightarrow P(1) = P(0) \times [1 + \frac{p}{100}] \]

The number \(1 + \frac{p}{100}\) is called the Factor of Multiplication for one year (FOM).

If we use decimal notation for the percentages we can rewrite this factor of multiplication as \(\text{FOM} = 1 + r\) with \(r = \frac{p}{100}\), dropping the percent form for the rate of growth. The rate of growth per year is written as ROG per 1 year.

Over \(n\) years the population will be \(P(n) = P(0) \times (1 + r)^n\). We could consider the number \((1 + r)^n\) as the factor of multiplication over \(n\) years. Notation: \(\text{FOM}(n) = (\text{FOM})^n\)

We can also define the ROG over \(n\) years as follows: \(P(n)/P(0) = \text{FOM} \text{ over } n \text{ years and } [P(n) - P(0)]/ P(0) = (1 + r)^n - 1 = \text{ROG over } n \text{ years.}\) In other words the ROG over \(n\) years is \((\text{FOM over } n \text{ years}) - 1 = (\text{FOM per year})^n - 1\). The ROG over many years is not linear in terms of years.

In population growth problems, we assume that the ROG is fixed not only over a year, but throughout the whole process of growth. In other words the fixed rate of growth is valid for any time period. This allows the extension of the FOM and ROG method to smaller fractions of the time period. My use of the FOM is similar to the concept of "multiplicative rate of change" used by Confrey in her splitting conjecture (Confrey & Smith, 1994, 1995). I adapted this concept for use in population problems or similar contexts.
CHAPTER 2

THEORETICAL FRAMEWORK

The concept of exponents as an indication of how much repeated multiplication an operation requires has a long history in mathematics. Extending this concept to rational numbers was an idea that developed over many decades (Dennis & Confrey, 1993; Dennis, Confrey & Smith, 1993). Teaching rational exponents seems to present certain obstacles for students that seem to reflect the problems that the mathematical community experienced. To develop a way to make this teaching more efficient should be an important objective of mathematical education research. My conjecture proposes a step towards improving this teaching.

In my research, I am testing the claim that a better conceptual understanding of rational exponents can be developed in learners that result in more consistency in developing a unified concept of exponents. A central aspect of this conjecture is that the study of rational exponents is more meaningful than the traditional approach if the concept of factors of multiplication is made the center piece of the study. This concept of factors of multiplication should be combined with the concept of the rate of growth and coordinated with the study of roots and powers of roots. After studying the procedures of constructing decimal exponents with roots and powers of those roots, the invariance of the laws of exponents are re-established for rational exponents. This exercise should also
contribute to the re-formulation of the concept of exponents in such a way that both the integer and the rational number exponents are explained in consistent language.

The theoretical framework to support this conjecture is based on theories of understanding that allow for an inclusion of perspectives that are varied and detailed. The theoretical model underlying the research is mostly (but not exclusively) based on the model developed by Goldin, Herscovics, Bergeron, and Kaput (Goldin, 1998; Goldin & Herscovics, 1991; Goldin & Kaput, 1996). The theoretical insights of Dubinsky (Dubinsky, 1994; 1991) and Sfard (Sfard, 2000, 1991) are interwoven later with those already mentioned for the purpose of accentuating the tools and models that are important for this study. I will discuss the aspects of understanding mathematical concepts as formulated in the theoretical model of Goldin & Herscovics (1991, the role of representations in both the concept of understanding and their place in the model, and the concepts of encapsulation and reification from Dubinsky and Sfard and how I incorporated them into the main model. These explanations are the building blocks leading to the description of the model of understanding that is the framework of my study.

Understanding Mathematical Concepts

Modes of Understanding

Skemp (1987) distinguished two forms of understanding in the teaching and learning of mathematics: the instrumental form, that involves knowledge of rules without too much knowledge of the reasons, and the relational form, that emphasizes knowing what to do and why. Herscovics (1996) described understanding as the outcome of meaningful learning, which corresponds roughly with Skemp's notion of relational
understanding. Herscovics and Bergeron (1983) proposed an extension of the notion of understanding with four modes (or aspects) of understanding: in addition to the modes from Skemp they introduced the modes of intuitive and formal understanding. Intuitive understanding is present when a problem is solved without much prior analysis, and formal understanding is when the ability is shown by a student to connect mathematical symbolism with relevant mathematical and logical chains of reasoning.

**Conceptual Schemes**

A fundamental notion in understanding what mathematical understanding is, is the constructivist concept of learning that claims that learning higher order concepts can only be accomplished if the new knowledge is actively constructed by the learner and a prerequisite for this is that the new knowledge finds anchor points in the learner's existing cognition (Piaget, 1970; Herscovics, 1996). To describe the formation of mathematical ideas the concept of cognitive structures or schemes was introduced and has been widely applied to the notion of learning and understanding. Fundamental mathematical ideas are built and acquired as extensive networks of related knowledge that are viewed as conceptual schemes (Bergeron & Herscovics, 1990; Herscovics, 1996).

A *conceptual scheme* is a psychological description of how and through which steps, a learner comes to form an understanding of a mathematical concept. This description uses theoretical constructs to explain observed behavior and activity and infers the construction of layers of knowledge that has the understanding of the concept as its outcome (Herscovics, 1996). The elementary, fundamental notions of mathematics all seem to be rooted in physical actions and derived forms from those actions that form the basis for the pre-concepts (Piaget, 1970, Herscovics & Bergeron, 1988). The general
characteristics of such scheme building or scheme constructions (Hatano, 1996) are that they are actively constructed by the learner, and not readily transmitted from one person to another. Forms of transmission are possible in the sense that the knowledge must be connected to existing systems of knowledge of the learner and in that process the knowledge is interpreted and restructured to fit existing concepts. Knowledge is continually restructured to incorporate new pieces of knowledge and to resolve apparent conflicts in the knowledge systems. This process of reorganization happens under the strain of solving problems and it evolves domain by domain and always in certain contexts. This means that problem-solving competence in one domain does not necessarily transfer to other less related domains. There does not seem to be something called general intelligence for all domains of knowledge (Hatano, 1996)

Model of Understanding

*The Herscovics-Bergeron model.* This model for mathematical understanding was formulated to answer questions such as: How do learners develop their mathematical notions and concepts? How do they come to understand fundamental concepts such as natural numbers, addition, subtraction, fractions, the number zero, the negative numbers etc? Where do their concepts of measuring and comparing, ordering and understanding of invariant properties of numbers come from?

To describe the construction and growth of conceptual schemes in mathematical understanding, Bergeron and Herscovics (1981, 1990) proposed their model of understanding that distinguished four levels of mathematical understanding. The first level is *intuitive understanding* that has the informal knowledge of the child as its starting point. It is based on visual perceptions, estimation, and un-quantified action. These pre-
concepts lead to the first construction of a concept. Such *initial conceptualizations* form the second level of understanding.

When the initial conceptualization takes shape and gains precision, it gradually separates from the procedure that was associated with its initial construction, and abstraction becomes possible. This *abstraction* forms the third level of understanding and comprehension. At this stage of development the separation of the concept from its procedure is considered a first phase. The next phase of development of the third (abstraction) level of understanding is characterized by generalization of the concept or by the reversibility associated with it or by some form of mathematical invariance or conservation of properties over many domains.

The detached concept can eventually be identified as a concept in its own right and a new "form" is attached to it. This is the fourth level of understanding, the level of *formalization*. This level of understanding is taken to be present if operations could be justified by formal mathematical symbols, or by the appreciation of axioms based on some form of prior abstraction of the concept, as opposed to mere rote learning of rules. New knowledge must be linked to those areas of the conceptual schemes that are within the reach of the learner, viewed from her/ his existing conceptual scheme (Herscovics (1996) labeled this the zone of proximal cognition, or ZPC). In this model of understanding, some form of communication of knowledge is possible provided it fits within an existing conceptual scheme. It is abstraction that can only be achieved by the individual learner. This form of constructivism that acknowledges and accepts different types of knowledge and different forms of acquisition of knowledge as a basic assumption, is termed rational constructivism (Herscovics, 1996). The radical
constructivist hypothesis assumes that "all knowledge is under all circumstances constructed by individual thinkers as an adaptation to their subjective experience" (von Glasersfeld, 2000, p.4) while rational constructivism accepts that transmission or communication of knowledge is possible under certain conditions.

Herscovics (1996) stated that the early learning of new mathematical concepts should start explicitly with the search for possible physical pre-concepts that are progressively connected to standard mathematics through a process of mathematization. Pre-concepts are notions like surface, a pre-concept for area, unquantified actions like augmenting a set of objects, or estimating in the form of more than, less than or equal to. Mathematization is the translation of a problem or a real life situation into mathematical objects like numbers, operations and transformations using mathematical language. The reason for this physical pre-concept is the need for intuitive grounding of notions.

*The two-tiered model of understanding by Goldin & Herscovics.* Goldin, Herscovics and Bergeron (1991) proposed a synthesized model of understanding with two levels. *First level:* the first level of understanding in this model can be summarized as a first form of understanding of preliminary physical concepts. This level starts as intuitive understanding in the form of a global perception of the notion. This form of understanding develops next into logico-physical procedural understanding where the learner can relate logico-physical procedures to her/his intuitive knowledge and is able to use this knowledge. Over time this develops into logico-physical abstraction when the learner constructs a sense of the various invariant or conservation properties of plurality or position, or when the learner constructs the notion of generality like the commutativity of the physical union of two sets.
Second level: this is the understanding of the emerging mathematical concepts, an important feature of this model that needs careful explanation. The first form of understanding is the logico-mathematical procedural understanding where the learner constructs explicit logical mathematical procedures that she/he can relate to her/ his underlying preliminary physical concepts and that she/he is able to use. For the interpretation of this model it is important to understand that the forms of understanding from the first (preliminary physical) tier need not be complete before the second form can be constructed. There is no linearity implied in the separation of the two forms of understanding. The diagram in Figure 2.1 below brings the various components of the first part of our model together.

Figure 2.1: A model for conceptual understanding with two tiers (Goldin & Herscovics, 1991)

The upper part of the model in Figure 2.1 represents the stage of development of understanding at the level of preliminary physical concepts. Understanding the
preliminary physical concept goes through three stages as described. The first box represents the notion of intuitive understanding described above. The second box in the upper part represents the next stage of the construction of logico-physical invariance or generalization of logico-physical patterns.

The lower part of the model represents the level of abstraction achieved by the learner with its three aspects of mathematical procedure related to physical concepts, mathematical abstraction related to mathematical invariants or patterns of generalization, and the formalization that represents both the form of defining concepts formally and using symbols and mathematical notation to introduce axioms and formal proofs. The arrow from intuitive understanding to logico-physical procedural understanding and then to logico-physical abstraction represents the development over time of the understanding of the preliminary physical concept. The two arrows from logico-physical abstraction to both logico-mathematical procedural understanding and logico-mathematical abstraction indicates that reflecting on actions on physical objects can lead to mathematical procedural and to mathematical abstraction. The arrow from logico-mathematical procedural understanding in the second tier straight to the formalization part indicates that formalization can also occur without the mathematical abstraction.

Goldin and Herscovics (1991) pointed out that when the constructions and formations of more advanced mathematical concepts (like the construction of rational exponents) are involved physical pre-concepts are not necessarily involved, but primarily prior mathematical concepts. For such conditions the model needed to be extended and refined.
The main step taken was the inclusion within the earlier model of understanding of forms of internal representations constructed by the learner. These forms are part of the constructs to describe the learner's problem solving competence and her/his construction of schemes for the construction of more advanced systems of knowledge.

**Internal and External Representations**

Representations are configurations of some kind that are partly or wholly associated with or stand for something else (Goldin and Kaput, 1996). They are always part of a system in which the representational relationships are generally not fixed. Interactive acts of interpretation are involved in the relation between that which is being represented and that which is representing. Such interactive acts of interpretation are called representational acts. The theory of representations distinguishes between external (observable) representations and internal, mental, inferred representations (Lesh & Doerr, 2000; Goldin & Herscovics, 1991; Goldin & Kaput, 1996). In Figure 2.2 the interaction is modeled after Lesh & Doerr, 2001. The internal part refers to mental configurations and constructions, while the external part refers to the embodied, symbolic, physical and observable representations that (together with other forms of individual or group behavior) provide the basis for the study of learning and problem solving. The model also represents the notion that the representations of both the internal and the internal varieties form one integrated system. The boundaries are fluid, shifting and ambiguous. External representations could sometimes be interpreted as expressions of internal conceptual systems and internal representation systems as internalized systems of external systems (indicated by the double arrow between the two components of the model (Lesh & Doerr, 2000).
Internal representations are possible mental cognitive systems that learners develop. They are not directly observable. The existence of such configurations in individuals is inferred from what they say or what they do. Such activity includes in particular mathematical activity. In the extended model of understanding (Goldin & Herscovics, 1991) an extensive system of different internal cognitive representations are posited to explain the variety and complexity of problem solving competencies and forms of understanding for more advanced mathematical concepts. These systems of cognitive representation that form part of wider conceptual schemes, will be discussed after the section on external representations.

External representations are tools for thinking and for conceptual development. They are physically embodied, observable configurations such as words, graphs, pictures, mathematical symbols and notations like equations and even calculator or computer displays and micro-worlds. All these external representations are part of wider structured systems and need to be interpreted as being part of such systems. The interpretations are always through persons and individuals, and as part of diverse, larger systems, are constructed over time (Goldin & Herscovics, 1991).
The Relation between Internal and External Representations

Internal representations are concepts that an observer postulates to form part of possible mental constructions. Humans develop conceptual systems that are partly embedded in conceptual tools like language, specialized symbols, diagrams, or experience-based metaphors (Lesh & Doerr, 2000). These tools are integral parts of the acts or processes of reasoning of the human or in our case, the student. Not only that, but it is important to realize that the reasoning process is not entirely in the head of the student. We as humans project our conceptual systems into our environment. These systems can range from large scale systems of communication to small scale systems like spreadsheets. There is interaction between external and internal representations (Goldin & Kaput, 1996). Acts of writing, speaking, manipulating elements of concrete systems are all forms of such interaction. When a student internalizes interactions with notational systems, or by reading or by interpreting words and sentences, interpreting graphs or diagrams this student is using mental structures of which internal representations are a part for these interactions. For example (Goldin & Kaput, 1996) when a student draws a graph of the function $Y = 2^X$ in a coordinate system the graph could be considered a representation of the function given as an equation. But the representing does not stop there. The coordinate system is itself a representation with strict rules that one must understand before the representation of the exponential function can begin. The graph can be considered an external representation of some internal image brought forward by the graph. The internal representation cannot be observed directly. In this model the internal image and the external representation are not simple copies, not pictures of each other.
For this model of mathematical understanding the way representations are constructed as part of the conceptual system is what is most relevant. Goldin described the need for hypothesized (internal) constructs as the need to allow for many perspectives like visual/spatial mental operations, planning heuristics, language and words, comparison of features during problem solving etc, to be included in the constructs. Through representational acts the learner constructs her/his internal representation system. A central feature of this model is that internal systems of representations always have an element of representational ambiguity. This ambiguity means that there are no strict boundaries between systems of internal representation (Goldin, 1998), or that relations between systems of internal representations can exist and be mediated by another system or structure. External representations include not only the tools for thinking, but also the observable products of performance or competence. Internal representations are constructs that are broad enough to enable one to explain learning and problem solving behavior.

**Components of the Model for Internal Representations**

The inclusive Goldin-Herscovics-Bergeron model of mathematical understanding (Herscovics & Bergeron, 1988; Goldin & Herscovics, 1991; Herscovics, 1996; Goldin & Kaput, 1996; Goldin, 1998) recognizes five specific types or categories of internal representational systems for problem solving competence and understanding. These internal systems are context-dependent constructs, which can only be inferred from other observations. The constructs are not necessarily "part of our brain," but descriptive of the mental structures. The systems are not separate, but can occur simultaneously, and (with
the possible exception of the formal notational system), occur universally in human beings. The five (inter-related) systems are:

A verbal-syntactic system describes the learner's capabilities for processing language, words, sentences and phrases. Definitions, word associations, grammar, and syntax information are in this representation system. Verbal-syntactic configurations can correspond to imagistic, formal or heuristic configurations that are described in words. Imagistic systems are distinct because they are not necessarily connected to words or notations. They are the visual and spatial, auditory and rhythmic, tactile and kinesthetic systems of representation. These systems are connected to the notion of "imagination," that is non-verbal encoding of relations, attributes, and transformations that influence problem-solving behavior and understanding. Imagistic systems can also harbor non-quantitative concepts or even misconceptions. The distinction between visual/spatial configurations and auditory or kinesthetic configurations is an important part of this model. The proposals of Lakoff, Johnson, Nunez and others (Lakoff & Johnson, 1980; Lakoff & Nunez, 1998; Lakoff & Nunez, 2000; Nunez, Edwards, & Matos, 1999; Sfard, 2000) on the central role of metaphor in imagistic thinking in the form of image schemata are another branch of this representational system that explains many basic concepts and mechanisms of mathematical thinking.

The (external) formal mathematical notational system suggests that there is a corresponding separate internal system of competence that constitutes the third internal component of cognitive representation. This form of internal representation is necessary for the context of mathematical problem solving. This representational system has strong connections with imagistic representations, verbal representations (in the form of
interpretations of formal descriptions) and specialized (mathematical) syntax configurations. Meaningful understanding in mathematics has very much to do with relationships that formal configurations of symbols have to different forms of internal representations (Goldin, 1998). Some cognitive processing is present when a learner stays within the formal notation system, but it is the ability to talk about notations and algorithms that is more closely associated with meaningful mathematics. The ability to move from imagistic forms and internal representations to formal notations and from the formal back to the imagistic system is one of the characteristics of competent problem solvers.

The next part of the system of internal representations that makes up this model of mathematical understanding and problem solving is the system that encodes planning, monitoring and control of the process of learning and problem solving (Goldin, 1998, Goldin and Kaput, 1996, Goldin & Herscovics, 1991). In this model, cognitive and meta-cognitive processes are intertwined, and they cannot be separated from each other across different representational systems.

The systems of internal representations can be and are used to represent information about themselves and about other internal systems. This aspect of the model is vital for explaining certain competencies. Processes of planning and execution and monitoring like "trial and error," "exploration of simplified cases," and "draw a diagram" fall under the construct of heuristic processes. The study of these heuristic processes has a special place in mathematical problem solving competencies and has many dimensions of analysis like the analysis of the reasons for advance planning, the study of the
domain–specific methods of a process, and the identification of criteria to recognize that a particular process can be applied.

Researchers have documented the affective, attitudinal, and values and belief systems that learners experience and develop when mathematical problem solving and learning occurs. In this framework, I follow the distinction that the model makes between global affect, which refers to feelings about the mathematical experience as a whole, and which is relatively stable for a given person, and the local affect, the feelings associated with the problem solving experience as such. Feelings like curiosity, bewilderment, frustration, elation, occurring during the problem solving process, can strongly influence the heuristic process (Goldin, 1998, Goldin & Herscovics, 1991). The model takes affect as a representational system that interacts with the cognitive process and forms configurations that encodes signals with powerful effects throughout the learning and problem solving constructive process. The system of affective representation that is created with and during the constructive process of learning and problem solving forms the fifth type of internal mental representation and an integral part of the conceptual system. Learners and problem solvers develop their own knowledge about the affective signals that occur during mathematical activity and problem solving. This is a meta-affective process that needs to be reformed and improved constantly.

**Developmental Stages of Growth of Internal Representational Systems**

Learning is a constructive process. The systems of internal representations that are components of this theory of understanding and problem solving competencies are conceptualized as constructed over time by the learner. This construction of the internal
systems goes through three stages of (internal) development (Goldin, 1998; Goldin & Kaput, 1996; Goldin & Herscovics, 1991). There is an initial stage called the inventive-semiotic stage, where new signs are created or learned. These signs are usually rooted in a previously established system of (internal & external) representation. The signs symbolize aspects of the existing system. The prior system gives initial meaning to the proposed (new) symbol. The new symbol externalizes certain features of the prior cognitive system and accompanies and directs the construction of the new system.

After the first inventive semiotic phase is established configurations and syntactic rules are explored within the new system. During the next stage the construction of rules and configurations is driven mainly by the structure of the previous system of representations. This stage of development is the stage of structural development of the new (internal) system of representations. Much of the meanings in this stage are tied and connected to the previous system. Only gradually the new system with its extended domain of meanings and relations takes on a life of its own and becomes autonomous relative to the previous domain. To reach this third structural, integrated stage, interaction with words and language and new mathematical experiences and constructions are extremely important. This stage is also a stage of mathematical abstraction.

The Unifying Function of the Theory of Understanding and Problem Solving Competencies

One of the explicit goals of Goldin et al. (1998, 1996; 1991) is to formulate a theory of understanding and a model for the interaction between internal and external representational systems that allows for inclusion of different theories of mathematical learning and understanding. They suggest that the notion of representational systems in
conjunction with the related ideas described provides such underpinnings for a compatible system for unification. I will now sketch the theoretical framework formulated by Dubinsky (1991; 1994) with the notion of encapsulation at its core. This description will incorporate this perspective of encapsulation within the framework of representations.

**Integrating the Notion of Encapsulation (Reification) into the Model of Understanding**

One of the constructs I needed in the framework for the study which is not explicitly presented in the model discussed so far, is the construct of encapsulation of interiorized mathematical processes into objects as proposed by Dubinsky (1994; 1991), and the very similar but more extensively explained notion of reification of interiorized processes into mathematical objects as proposed and discussed by Sfard (2000; 1994; 1991). Dubinsky researched processes of mathematical learning, while Sfard tried to explain why mathematical objects seem to retain a dual nature: on the one hand an operational, sequential, process-like feature and on the other hand, a structure-like character that makes them holistic, object-like, abstract and continuously changing and eventually becoming a targets for new processes and operations.

These constructs of encapsulation and reification can be used to describe sub-notions or partial concepts that students need to understand to develop understanding of larger concepts. They need to reach the point of experiencing the concepts and their constituent parts as meaningful mathematical objects (Sfard, 2000) before we can state that some level of understanding has been reached. It is my claim that such a device to
describe intermediate understanding in the context of a particular new concept, is missing in the framework of Goldin- Herscovics-Bergeron (and Kaput).

**Actions, Processes, and Objects**

For Dubinsky (1994) mathematical objects start their existence as physical or mental actions on already existing mathematical objects like numbers, geometric figures, sets etc., that transform those objects and establishes new objects (that could be of the same category). These actions are explicit and precise and the constructed objects are present in either physical or mental form at each step of the action. These actions are interiorized by the learner (see the arrow from action to process in Figure 2.3) when the need for a step by step approach is no longer there or when the procedure can be carried out entirely in the mind of the learner. The interiorization gives rise to a process when the learner is able to apply other processes on the interiorized actions, for example by reversing the action or by coordinating it with other actions. The stage of encapsulation has been reached for Dubinsky when the process stage has reached a point where it can be transformed by some action. The process takes on the experience and meaningfulness of a mathematical object.
Figure 2.3: Dubinsky's model of mathematical abstractions (Dubinsky, 1994)

For Dubinsky (1994) mathematical objects can only be constructed through the encapsulation of interiorized processes. Sfard (1991) proposed a similar construct to convey the notion of meaning of mathematical concepts in students in the study of mathematical processes and concepts. When the operational stage of the new concept has been condensed in the mind of the learner and she/he can oversee the whole operational process in the mind without actually carrying out the steps, then the mind constructs an object (this stage of mental development is often driven by symbols that mediate this particular process) that has meaning for the learner and becomes a mathematical object with (growing) structural features (Sfard, 2000). The actual way in which this happens is
not the focus of this study but language, metaphors, discourse and reflection are driving forces (Sfard, 2000; 1991; Lakoff & Johnson, 1980; Lakoff & Núñez, 1998; Lakoff & Núñez, 2000).

In the theoretical framework that I applied for the research, rational exponents require knowledge of both the root function \( F(X) = \sqrt[n]{X} \) and the power function \( G(X) = X^n \). These functions have to be connected to the method of creating rational exponents and their role must be understood in a meaningful way, or in other words, their role must be encapsulated in the context of the construction of exponential forms with rational exponents. This action will require extensive processing and discussions to re-formulate the notion of exponents and units of multiplication. All these new interpretations need to be fully developed to ensure that the properties of exponents can be re-discovered for rational exponents and the invariance of the laws of exponents is re-established but now for a wide domain of numbers.

Only after the encapsulation of the sub processes of incorporating roots and powers into the structure and context of exponents can the learner try the next step of encapsulating the rational exponents in the form of their general laws. Goldin and his colleagues did not discuss such steps in sufficient detail. That was my justification for not relying only on their framework. On the other hand, the explicit aim of their theory as formulated in many publications (Goldin, 1998; Goldin & Kaput, 1996; Goldin & Herscovics, 1991) was to allow precisely for such inclusive theoretical constructs in their model.
The Theoretical Framework and the Conjecture

In this section I will sketch how the theoretical framework guided and informed my conjecture. I assumed that the students in the study had knowledge of the Cartesian coordinate system, graphs, the function concept including the exponential function and its graphical representation. Beginning algebra and the corresponding knowledge of notation and problem solving was also assumed to be included in the student's knowledge system.

Intuitive and Logico-Procedural Understanding of the Emerging Mathematical Concept

To lay an intuitive foundation for rational exponents I introduced the familiar real world concept of relative rate of growth expressed in percentages, and studied in the context of a stable population growth over years. My aim was to describe what it means to have a (relative, multiplicative) rate of growth of, for example, 25 % per year that is stable or fixed over many years. By introducing the concept of rate of growth I tried to avoid an immediate reliance on exponential definitions that could obstruct the construction of new notions of exponents. It was my assumption that the concept of rate of growth was sufficiently intuitive and meaningful for the students. The lessons at this stage included the step by step calculations of the next population assuming that the rate of growth over many years remained stable.

Diagrams and Graphs

All the calculations in rate of growth problems include diagrams to illustrate the calculations and also to graph the results with only discreet natural numbers as years. This creates a beginning of external and internal representations of the connection between rates of growth and exponential functions. The strategy was to foster in the
students different ways to interpret the process of stable rates of growth. A visual representation in the form of discrete points and an operational algorithm that calculates the next population from the application of the growth rate applied to the existing population. That creates a list of what the next population is when we know the population of the previous year, repeated for a number of years. At this point, the conjecture was that the concept of factor of multiplication will not spontaneously be inferred or understood from the recursive list.

**Encapsulation of the Factor of Multiplication**

To help the student encapsulate the concept of factor of multiplication emphasis was placed on the intimate connection between the rate of growth and the factor of multiplication. The understanding I aimed for with the rate of growth approach was partly intuitive understanding (a growing population), partly an understanding based on visual and graphical representations (create a graph of the evolution of the population) and partly an understanding of notation and functions (write the generalized formula for the population growth over many years).

From the understanding of the rate of growth, the concept of the factor of multiplication was constructed and fostered to prepare the transition to the formal notation of exponents through recursive pattern recognition. The choice for the rate of growth concept as the gateway to exponents was based on the importance of the concept of the factor of multiplication in the recursive pattern. The students were asked to practice repeatedly with graphs, formulas and with diagrams, so they could understand how that recursive pattern lead to an exponential function with natural numbers as variables. By connecting the rate of growth to the factor of multiplication over a certain
period, it could be possible to make a case for a smaller or larger rate of growth and therefore for a smaller or larger factor of multiplication.

**Procedure for Finding Factors of Multiplication**

The rate of growth can be applied to any smaller time period and that fact could create a basis for the notion that the same mechanism that results in a 25% increase of the population per year can also work over one-half of a year. I used this schema for the transformation of rates of growth into factors of multiplication to introduce the concept of a rate of growth over a shorter time period and convert that to a smaller factor of multiplication. By repeating the rate of growth for the first half of a year in the second half, it becomes possible to find the actual factor of multiplication and from there the rate of growth. However, if the relation between the rate of growth, the factor of multiplication and the possibility of using the recursive formula to create an exponential general formula is not sufficiently encapsulated, the student may experience difficulties in this part of the understanding of rational exponent construction.

My conjecture was that the strategy of breaking up a time period in a rate of growth problem into two or three equal parts and use smaller factors of multiplication over each fraction part would require reorganization of the concept of factor of multiplication. I consider such reorganization an example of de-encapsulation, a return from mathematical object to process (Sfard, 1991). The concept I tried to reinforce here is the presence of a distinct rate of growth and a distinct factor of multiplication over a fraction of the time period.

**Calculation of factors of multiplication for fractions of the time unit.** To strengthen the interiorization of the construction of rates of growth and factors of
multiplication over fractions of the unit of time, I constructed the process for fractions like \( \frac{1}{2}, \frac{1}{3} \) and \( \frac{1}{5} \) of a period and discussed for each calculated rate and factor of growth what the generalized formula would be and how that compared to the earlier results.

**Formalization and Generalization**

*The incorporation of the power and root function and*

\[ X^n - Y^n = (X - Y) \left( X^{n-1} + \ldots + Y^{n-1} \right). \]

Here I differ from the traditional approach of defining rational exponents through a process of reduction of the (positive) rational exponent to its relative prime form (Tirosh & Even, 1997; Goel & Robillard, 1997). For example, the form \( 8^{\frac{2}{6}} \) is defined by first reducing \( \frac{2}{6} \) to \( \frac{1}{3} \) and then applying \( \sqrt[3]{8} = 2 \). I preferred to include a discussion of the meaning of \( 8^{\frac{2}{6}} \) in terms of factors of multiplication associated with (for example) time periods first and then argue that the factor of multiplication per part for two units of time divided into six parts is the same factor as the factor of multiplication associated with the following operation: what is the factor of multiplication per part for one unit divided into three equal parts? The study of the root function and the power function should be combined with the concept that if \( X^n = Y^n \) and \( X \) and \( Y \) are positive real numbers, then \( X^n - Y^n = 0 \). This difference of equal natural exponent powers \( X^n - Y^n \) had to be studied briefly, starting at

\[ X^2 - Y^2 = (X - Y)(X + Y) \]

and continuing with

\[ X^3 - Y^3 = (X - Y)(X^2 + XY + Y^2) \]

suggesting the general form:

\[ X^n - Y^n = (X - Y)(X^{n-1} + X^{n-2}Y + X^{n-3}Y^2 + \ldots + X^2Y^{n-3} + XY^{n-2} + Y^{n-1}). \]
This would be used to argue that if $X^n = Y^n$ then $X=Y$ (again only for positive values of $X$ and $Y$!)

The model of Goldin, Herscovics and colleagues is particularly fruitful because it incorporates the formalization of the emergent mathematical concept and the necessity of building this specific form of understanding. Without the power and root functions and the subsequent factorization of the difference of equal powers, there is no justification beyond the intuitive notion that equal powers imply equal numbers. The steps have their procedural level of understanding and competence, and their mathematical abstraction that is interwoven with the formalization of the process. Formalization should play an important role in the creation of rational exponents and it should always be connected to meaningful and convincing forms of justification.

**Verbal/Syntactic Representation System**

New words are introduced for example: a *fraction period* for the shorter time unit. The corresponding factor of multiplication is called the *multiplicative fraction* of the original factor of multiplication. The new words are part of a system of representation to broaden both the foundation of the concept of exponents and the modes of understanding (repeated multiplication of one unit is extended to repeated multiplication of factors that are roots of the original factor). The system should help the establishment of connecting links and schemas that can justify rational exponents as extensions of the natural number case.

**Summary of the Theoretical Framework**

The framework for this study is based on three sources of theory: the model on understanding developed by Goldin and colleagues on the one hand and the model of
understanding developed by Dubinsky and later by Sfard on the other. Goldin describes the first stage of understanding as an understanding of preliminary physical concepts, starting at intuitive understanding followed by the understanding of logical procedures rooted in physical understanding to understanding based on logical abstraction. The fundamental mathematical concepts are formed in the mind of the learner through this initial process of concept formation and understanding. Then (not necessarily in a linear order) understanding of the emerging mathematical concept develops from logical and mathematical procedural understanding through mathematical abstraction and formalization. In this model, formalization is a separate domain of understanding that needs to develop in close connection with other forms of understanding to create meaningful mathematics. To explain the learning and understanding of more advanced mathematical concepts the notions of internal and external representations are proposed as useful constructs to describe, explain and predict the evolution of knowledge systems in the mind. When more advanced mathematical concepts are learned, five distinct forms of internal representations are proposed to describe the process of understanding and problem solving competence that is seen as very closely related to mathematical understanding at this level. In particular the imagistic internal representations that include systems of visualization, spatial representation and auditory and kinesthetic representations, and where each system can be mobilized to encode and carry information about itself and other representation systems. These systems of representation, internal to the mind are the real pillars and sources of more advanced mathematical understanding.

Language, imagistic cognitive systems, affective representation systems and systems of formal mathematical notation and symbols all need to develop and contribute
to the process of meaning giving to achieve mathematical understanding. These representational systems do not come ready made to the learner. They develop over time through (generally) three stages of development. In the first, inventive and semiotic (meaning giving) stage the system starts out as an outgrowth of an existing knowledge system. In the structural stage there is extension and development driven by the previous system. In the fully mature stage of development, the new system takes on autonomous forms of existence and can be studied separately as a system in its own right.

Dubinsky and Sfard focus precisely on this process of development of the new systems and the psychological stages that new concepts go through. Dubinsky uses the Piagetian notion of reflective abstraction (Piaget, 1970; Vergopoulo, 2001) and extends this with the concept of (mathematical or cognitive) actions on objects (mathematical or physical) that produce cognitive processes. By interiorizing these processes, learners turn processes into entities that are encapsulated which means that they take on the character of mathematical objects in the mind of the learner. The ability to transform a process through cognitive actions into an experience where these processes take on the character of mental objects that can be treated as if they were as real as physical objects, combining them with other mathematical entities, reversing the processes, or coordinating the processes with other processes, that is when they are encapsulated. Sfard (1991) inserts one extra construct into the sequence of development by separating the interiorization stage from a stage where the whole action process is compressed into manageable chunks so that the act of thinking about the whole procedure becomes possible. This stage is also separated from the interiorization stage because it is this part of the process that seems to make the transition to an object (reification) stage possible (Sfard, 1991). In my pilot
project and through my experience I came to the conclusion that this notion of encapsulation is very useful and I prefer to give it a place in the theory explaining the development of concept formation and mathematical understanding.

The way the students actually develop their understanding of rational exponents in the instructional setting that is created during the teaching episodes is the real objective of this study. The analysis of the teaching experiment should allow us to understand how empirical data can lead to a better integration of the Goldin model of understanding and problem solving and the encapsulation and reification theories of Dubinsky and Sfard.
CHAPTER 3

REVIEW OF LITERATURE

The following areas are reviewed in relation to the development of the research for this study. First I describe the research on the slow encapsulation of the fundamental concepts of the process of multiplication. Then a discussion of the difficulties students have with developing understanding of this process of multiplication that is at the heart of the exponential function and the concept of exponents.

I argue first, that there is a similarity between the poor integration of the intuitive knowledge of multiplication taught in most curricula as repeated addition followed by a shift to multiplication of rational numbers on the one hand, and the integration of prior knowledge of exponents as repeated multiplication followed by a shift to concepts of negative and rational exponents on the other hand.

My second argument states that the development of rational exponents should precede the introduction of negative exponents. The reasons for this approach are based on the theoretical models of understanding (Herscovics & Bergeron, 1988; Goldin & Herscovics 1991) and the theories of abstraction (Piaget, 1978; Dubinsky, 1994; Vergnaud, 1998) that I apply to study how advanced mathematical concepts develop in the learning process. In particular, I argue that when the generality of a concept is expanded, as with exponents, the laws of exponents need to be re-interpreted and their
invariance re-established so that they truly become invariants for the student. Because concepts are interconnected, one change in such a field of concepts can influence many others which make the invariant properties all the more important (Vergnaud, 1988, 1998). Only after the invariance of the laws of exponents has been established for rational exponents can the teacher invoke them in a meaningful way to support any new, expanded concepts in exponents.

My third argument is that rational exponents require a focus on the unit of multiplication (Confrey & Smith, 1994). The unit of multiplication connects the act of multiplication and the concept of the exponent as a measure of how much multiplication with the given initial or original unit has occurred. By introducing (multiplicative) fractions of the units of multiplication, the prior knowledge of repeated multiplication is brought into the conception of rational exponents. The smaller units allow a more transparent counting scheme for (rational) exponents. Such a counting scheme is based on the concept of co-variation (Confrey & Smith, 1995), one of the subjects that will be reviewed later. The student can reflect on how she/he wants to reframe the system of counting the exponents, either as before, with integer exponents only or by expanding the way exponents are measured and include the use of fractions in exponents just like the introduction of rational numbers, when ordinary counting is concerned. Essential in this approach is the need to have discussions that raise awareness of, and anticipate cognitive hurdles. These discussions direct attention to the close parallel between rational and decimal numbers (based on powers of 10) and the choice of roots based on powers of 10 (I single out decimal numbers from the wider set of rational numbers in my discussion, because of the special role of these numbers for our general representation of numbers.
and their frequent use as exponents). This activity could reinforce both the understanding of the system of exponents with decimal numbers and the place value system of notation.

I review the research on multiplication first, and then discuss the research around the problems of understanding definitions of mathematical concepts in the learning process of students. Problems of understanding of formal algebra and their relation to conceptual understanding will then follow. These sections are central to my argument that the standard introduction of rational and negative exponents through the laws of exponents may not be the best way to achieve a strong conceptual understanding of the rational exponents (Lochhead, 1991) and may even contribute to a superficial procedural concept of what exponents are and how the various definitions connect together to one consistent and powerful notion.

The next section will be on the function approach and the possibilities and limitations of graphical, tabular and other forms of representations for the study of exponents. The use of co-variation as an alternative to the correspondence concept of functions and Confrey's incorporation of the co-variation approach in the splitting conjecture as an alternative formulation of exponents will be my concluding section for the review of research related to exponents. I close with a discussion of the concepts related to teaching experiments.

**Multiplicative Reasoning**

Multiplication is fundamental to the concept and understanding of exponentiation and clarity on why some students fail to recognize multiplication in some problem situations must be sought and discussed before we can understand the problems of exponentiation. Within the framework of conceptual field theory much of the
conceptualizing process lies "beneath the surface like an iceberg" (Vergnaud, 1996, p. 224). The operational invariants (concepts and theorems, embedded in the individual's schemes), form the core of an individual's conceptual or pre-conceptual representation of the world however the invariants may be (Vergnaud, 1996).

The process of understanding and encapsulation of multiplication (Breidenbach et al., 1992, Dubinsky, 1994) turned out to be slow and full of unexpected, persistent misconceptions. One misconception is now known as MMBDS, which stands for "Multiplication makes bigger and division makes smaller" and another one is "Division is always a larger integer divided by a smaller one." Af Ekenstam and Greger (1983) reported that in an experiment to measure the problem solving abilities of 12-13 year olds in Sweden, only 29% of the children from 30 classes from all across the country could correctly solve a problem like: A piece of cheese weighs 0.923 kg.; 1 kg costs 27.50 kr. Find out the price of the cheese. Because the focus was not on computational ability, the students were asked only to choose the appropriate operation for solving the problem. The choices for the students were: $27.50 + 0.923$; $27.50/0.923$; $0.923 \times 27.50$; $27.50 – 0.923$. One of the conclusions of the study was that when decimal numbers were involved the children had difficulties in finding matching problems (that they had to formulate to mirror the arithmetic given).

Bell, Fischbein and Greer (1984) reported that in a group of 12-13 year old children of above average mathematical ability in the UK who were tested for their ability to solve multiplicative word problems, three trends were visible. First, multiplication with decimals less than 1 was a source of difficulty. When the multiplication could be translated as a form of repeated addition ($0.51 \times 33$) the results
were better than when they could not (10.5 x 0.71). Secondly, division by a larger number (for example 5:8) consistently led to a reversal (of the number order). Thirdly, division by decimals less than 1 proved particularly difficult, and most children just changed the operation to multiplication. It is interesting to notice that most children could explain (in interviews) qualitatively what should be a good solution, but were less successful in indicating the correct operation for the solution.

The misconceptions are not limited to 12-13 year old children. Pre-service teachers enrolled in a University in the USA in 1986 (Tirosh & Graeber, 1986) were interviewed about their responses on statements reflecting multiplication and division. True or false questions were posed such as: In a multiplication problem the product is greater than either factor. Or: In a division problem, the quotient must be less than the dividend. Another question was: The quotient for the problem 60/0.65 is greater than 60. True or False? Eighty five percent answered the first question correctly, 11 % gave wrong answers. For the second question 45% gave the correct answer, while 52% was wrong on this. For the third question 62.5 % was right, and 20% actually calculated the answer before stating their opinion. Analysis of the responses indicated that although some answers were correct, most students used whole numbers as their primary source for choice of examples. Moreover, there was no consistent belief about numbers. When these students encountered multiplication and division problems such as 0.6 x 0.5 or $\frac{0.5}{4}$ they relied on procedural methods to navigate the problems.

An investigation of 116 students in grade 5 (12 years old) in Belgium (De Corte, Verschaffel, & Coillie, 1988) on their ability to solve multiplicative word problems
revealed that if the numbers were systematically varied to cover integers, decimals greater than 1 and decimals less than 1, it was problematic for many students to decide which operations to use in problem situations with different number types, even when the structure of the problem remained the same. All questions were solved twice, once indicating only the choice of operation, and once as free-response questions including simple multiplication. The results were consistent with the hypothesis that when the multiplier in a problem was a decimal less than 1, the success rate was the lowest. When the multiplier was a decimal greater than 1, results were better, but still worse than when the multiplier was a simple integer. (The multiplier is the number conceived of as operating on the multiplicand, in the repeated addition model of multiplication. In M x N, M is conceptualized as the number of groups of N (objects) that is counted M times. M is the multiplier on N as the multiplicand). One explanation for this phenomenon (Bell et al. 1984; Greer, 1988) is given in terms of the students' primitive models of multiplication and division based on repeated addition and subtraction which makes it extremely difficult to give meaning to problem situations where decimals less than 1 are involved in multiplication problems with multipliers with a decimal form less than 1. Students tend to prefer division to solve such problems.

Another aspect of the multiplier effect (the persistent error to use division instead of multiplication when this is the correct operation in cases where the multiplier is a decimal less than 1) was researched (Taber, 1991) with 4th and 6th graders for problems where the multiplier was of the form $\frac{1}{N}$ (with N equal to 3, 4, 5 or 6). Given a choice of strategy with word problems to reflect various types of multiplication, 6th graders had a
preference for viewing and treating these problems as division problems and they were quite successful. This happened despite the fact that the multiplication strategy and algorithm for multiplying fractions was known to them and had been taught explicitly before the investigation. Thirty percent of the 6th grade students still preferred to use division strategies when dealing with multiplication by \( \frac{1}{N} \). This was almost identical to the twenty eight percent of 4th graders in the study who used division for similar problems. The suggestion was that perhaps some of the students still thought about these problems as division problems even after two years of effort to teach multiplication strategies. The explanation for the multiplier effect, offered by this study on students' knowledge about multiplication by \( \frac{1}{N} \), is that the students seem to see such multiplication as more of a division. The student who knows to find \( \frac{1}{4} \) of a quantity by dividing by 4 may think that to find 0.4 or 0.43 of a quantity is to divide by 0.4 or by 0.43. The numeral 4 and the operation which was appropriate in the context of finding \( \frac{1}{4} \) (dividing by 4) may be more salient to them than other aspects of the number such as "1/ "or the decimal point. Learning to find \( \frac{a}{N} \) of M or 0.b of M (with a, b, N and M as whole numbers) may not follow easily from students' strategies for finding \( \frac{1}{N} \) of M (Taber, 1991).

The effects of the repeated addition model of multiplication are far reaching (Luke, 1988). It seems to be connected to the multiplier effect, and it also influences what has been termed the non-conservation of (multiplicative) operations. Moreover, this intuitive body of knowledge also seems to be quite resistant to change. One explanation (Greer, 1988) for the power of the intuitive knowledge about numbers, transferred to
rational numbers is a lack of sufficient experience with attempts to eliminate these misconceptions, and the absence of explanations by teachers that mathematical modes of thinking can be liberated from their origins.

Another side of the repeated addition model that is addressed poorly in mathematics education is the concept that all multiplication is based on counting of equal-sized groups and the structure that runs parallel to it (Mulligan & Mitchelmore, 1997; Boulet, 1998). Students may not have had extensive opportunities to develop an intuitive model for such situations. Boulet (1998) made a special case for the notion that the essence of multiplication is the counting of equal sized groups in the form of a transfer of larger units of count to a smaller one. For her the commutative law does not state that the multiplication is carried out identically when the order of the numbers is switched. A multiplication of an integer and a fraction requires some act of division before the transfer of units can continue and the operation is finalized.

How can we improve the basic knowledge and the intuitive understanding of multiplication from early on? The splitting model (Confrey, 1994) proposes one alternative to the repeated addition model for multiplication. This proposal was motivated by the concern to find a model that could clarify both multiplication and repeated multiplication as a precursor to exponential forms.

**Some Theoretical Issues: the Splitting Model of Multiplication and Exponents**

Because our body of empirical research on exponents is limited, we begin with a review of some closely related discussions of an alternative model for multiplication in the form of the splitting model of multiplication, proposed by Jere Confrey and elaborated in collaboration with other researchers (Confrey, 1991, 1994; Confrey &
Smith, 1994, 1995; Confrey & Scarano, 1995; Confrey & Lachance, 2000). The importance of the splitting model (also called the splitting conjecture) for my study lies on the one hand in the search for more diverse sources for an intuitive set of actions that makes multiplication more accessible than through the sole reliance on the method of repeated addition. On the other hand the splitting model has direct applications in the field of exponents. The notion of splitting grew out of Confrey's research (mentioned above) into students' conceptions of exponential functions. Splitting in Confrey's framework is a primitive, cognitive scheme defined as an action of creating multiple versions of an original. This action is mostly represented by a tree-diagram. Dividing a piece of paper into four equivalent sections, or breaking a candy bar in half and then in half again to share the parts among four people, or organizing a set of objects into an array are all forms of splitting, because it requires the production of groups of equal size. All these actions rely more on geometrical equivalences between parts of a structure then on counting or on the creation of a form of one-to-one correspondence or one-to-many correspondences. The act of splitting in the form of sharing, folding, and actual splitting is very familiar to young children and is one model of multiplication different from repeated addition. It creates similar copies of objects and the claim is that its operation provides a basis for the concepts of ratio, multiplication and division.

Figure 3.1: Splitting concept of change equal to 2 (From Confrey & Smith, 1994)
Figure 3.1 illustrates the splitting concept with unit of change equal to 2, represented as a tree-diagram with successive 2-splits, while Figure 3.2 (below) shows a representation of splitting as sharing (Confrey et al., 1991, 1994, 1995). Each step produces equivalent shares, the first has shares for two, and the second has shares for six parts. Both actions create copies of an original.

![Diagram of splitting concept]

Figure 3.2: Representation of splitting as sharing (From Confrey & Smith, 1994)

The justification for the use of splitting as a way to represent exponential forms lies primarily in the construction of a splitting world for the exponential function together with and next to a counting world for the exponents (in this case equivalent to the independent variable). In the splitting action each step in the split branches into N copies of the original. Each such step is conceived as a successor action. Counting the number of branching steps (not to be confused with the actual copies of the original) will produce the additive world of the exponents. By placing the geometric sequence for the exponential form next to the arithmetical sequence of the exponents, the notion of the
multiplicative rate of change is introduced. The multiplicative rate of change is directly related to the notion of the ratio between the number of copies at step N+1 to the number of copies at step N (Multiplicative Rate of Change can be defined mathematically as \( MRoC = \frac{F(N+1)}{F(N)} \)). This rate of change is different from the rate of change based on difference (mathematically, the Additive Rate of Change, \( ARoC = F(N+1) - F(N) \)).

Table 3.1 illustrates a process of a 3-split with a column for the number of splits, a column for the number of units, and a column for the multiplicative rate of change at each level. By comparing the number of splits to the powers and the number of units next to the multiplicative rate of change, the structure of this schema becomes clear and can provide the student with a model or pattern for exponential forms.

<table>
<thead>
<tr>
<th>n number of 3-splits</th>
<th>c number of units</th>
<th>u=®c unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>9.00</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>27.00</td>
<td>3.00</td>
</tr>
<tr>
<td>2.00</td>
<td>81.00</td>
<td>3.00</td>
</tr>
<tr>
<td>3.00</td>
<td>243.00</td>
<td>3.00</td>
</tr>
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<td>4.00</td>
<td>729.00</td>
<td>3.00</td>
</tr>
<tr>
<td>5.00</td>
<td>2187.00</td>
<td>3.00</td>
</tr>
<tr>
<td>6.00</td>
<td>6561.00</td>
<td>3.00</td>
</tr>
</tbody>
</table>

Table 3.1. Juxtaposed exponents and powers of the base (From Confrey & Smith, 1994)

\( u = ®c = 3 \) represents the fixed rate of (multiplicative) change. It represents the quotient of the number of units \( F(N+1) \), at the level \( N+1 \), divided by the number of units \( F(N) \) at the level \( N \). This quotient \( \frac{F(N+1)}{F(N)} \) is fixed for exponential forms \( F(x) \). In Table 3.1 this fixed value is equal to 3.00. For the column of the unit of (multiplicative) change this
means that when the number of splits changes from $N$ to $N+1$, the numbers in this column should all be 3.00.

The successor action for splitting by $n$ (Confrey & Smith, 1994, 1995) corresponds to the successor action of adding one (in terms of the order of the branch) with counting. The origin or starting point for the action of splitting is taken to be one unit, or just 1, while the origin for the action of counting is zero, or just 0. With counting we focus on differences, while the focus with splitting is on ratio. In the multiplicative splitting world, $N$-parts are created by $N$-rooting. This means that for the multiplicative world, the operation of creating equal (multiplicative) parts of a unit is the operation of finding the $N$th root of the unit. In the additive world, $N$-parts are created by an operation that is essentially $N$-splitting! The unit is split into $N$ parts that in reverse equal the unit after addition of all parts.

To study the construction of exponents with the splitting model, more emphasis is placed on the idea of co-variation of powers and exponents. This may help students understand how each variable varies with the other, particularly if the rooting is introduced into the geometric sequences and co-varied in multiple ways with the powers. The viability of the splitting conjecture was studied in a three-year teaching experiment (Confrey & Scarano, 1995). The efficacy and didactical value was studied of including the conjecture in the curriculum of 3rd to 5th graders in their learning of multiplication, division, and ratio as one related trio (meaning that the three operations and concepts are studied together using splitting) and the study of ratio and proportion. The teaching experiment was conducted using project-based teaching methods. A special feature of the three-year study was the intensive use of Venn diagrams, contingency tables of values,
drawings, graphs, and ratio-boxes (to find missing numbers for a given ratio). When ratio was studied, geometrical concepts of similarity were included, fractions and decimals were connected and the concept of transfer from lower to smaller units with fractions was part of instruction. Splitting sequences with twos, fours, fives, tens, eights, threes, sixes, nines and then sevens were used. Students from grades 3 to 5 in the experimental group showed much more appreciation for multiplicative situations than a control group of 13 to 15 years of age solving similar problems. They still struggled with the question of when to use multiplication and when to use division. The rate of solving proportion word problems was significantly better for the treatment group than that for the (control) group as was the retention of the performance. The better performance of the treatment group was partly credited to the richer use of representational variety compared to the control group. This empirical study on the potential power of the splitting conjecture, used with a rich contextual embedding with multiple representations, might provide a way to connect multiplication (with a fixed base) more fruitfully to exponential forms.

Summarizing the review for this section, the splitting concept may provide some alternatives to the difficulties with the primitive concepts of multiplication confined to repeated addition and create a more unified basis for multiplication, division and the future exponentiation. The concept of multiplication as a process of transfer from larger to smaller units of count could create a foundation for a stronger concept of integer, rational and decimal multiplication. The co-variation approach to the introduction of rational exponents is closer to the way of thinking by novice students than the abstract approach of using their unclear notion of the invariance of the laws of exponents to justify the new exponential forms.
The Role of Definitions in Mathematical Learning and the Creation of Rational and Negative Exponents

As discussed earlier, I was interested in investigating the role of the laws of exponents in the construction and understanding of rational and negative exponents. Because of the strong reliance of many textbooks on the definition only in explaining the rational exponents, I also reviewed the research on definitions in learning. What can the research tell us about the way students view and handle definitions in mathematics and how are the connections made between definitions and concepts? Are students able to validate a statement based on a general law that is unfamiliar to them, or only familiar in a limited context?

To shed some light on this issue, interviews were conducted with a junior mathematics major student (Edwards, 1997). The focus was on the student's use and understanding of formal mathematical definitions in real analysis. Two types of definitions were identified as relevant for the study: "logical" definitions and "lexical" definitions. The use of the word "logical" here is not in the mathematical sense of this word. "Logical" definitions referred to things in the real world. Such "logical" definitions analyze the object in such a way that the description is acceptable to most people. "Lexical" definitions delineate the characteristics or features of a concept so that the concept is completely determined by the definition.

The study by Edwards suggests that students had difficulties understanding lexical definitions and often did not understand that mathematical concepts are entirely defined by their definition. This is related to the question of the difference between concept-image and concept definition (Tall & Vinner, 1981). The concept-image is the cognitive
structure associated with a concept, constructed over time and based on one's experience. Edwards and Ward (2004) state that if students do not have an understanding of the role of mathematical definitions it is their concept-images that will dictate the content of the concept. Students seem to develop their understanding of mathematical definitions in phases. These phases range from intuitive responses to definitions, concept-images and stipulated definitions as used in advanced mathematics. Stipulated definitions create usage. Extracted definitions report usage. Stipulated definitions create concepts "by decree" as in mathematics, to achieve precision and avoid ambiguity. Edwards and Ward (2004) report that students even in advanced algebra or real analysis courses in the mathematics department do not understand, do not use and sometimes even ignore definitions of a concept. Students tend to extract definitions from cases and examples of the concept.

In a study of 216 sixth graders on how they drew triangles, rectangles and squares and how they judged statements made on a collection of figures and their properties, Wilson (1990) concluded that although the students were sometimes aware of their inconsistencies in the statements and properties they identified, the usual reaction was not to refer to the definition, but to construct a logic that minimized the inconsistency. Another conclusion from the study was that students did not see implications in definitions. They also possessed very strong prototypes of figures. Sometimes these prototypes were helpful, sometimes they reinforced confusion. The students' flexibility with prototypes developed with their understanding of the properties of the figures and mathematical statements. The study suggests that teachers should not expect students to
reason deductively about definitions until they become familiar with the properties of the concepts.

Analogous to the concept-image, the notion of statement-image was proposed after a study of students' understanding of decimal number sizes (Selden & Selden, 1995, 2005). Statement-images are the alternative statements, examples, counterexamples, visualizations, properties and consequences associated with a mathematical theorem or statement. They represent mental images, attached to the statement. This statement-image is important in the process of unpacking, understanding or constructing logically equivalent mathematical statements or theorems and being able to determine their correctness. The Seldens (2005) offered the example of young college students' struggles with decimals and they pointed out that the suggestion of one researcher to define the relative size of decimals (designated by X and Y) by stating that X < Y, if X - Y is negative, will not help novice students, because of the abstract analytical nature of the definition. For most novice students decimals are somewhere out there and this definition will change little in the way decimal comparisons are made.

Dahlberg and Housman (1997) added one more construct to the concept-image, concept-definition, and statement-image scheme, which is the concept-usage notion. The concept-usage is the way one operates with a concept in generating or using examples or doing proofs. Eleven mathematics majors from a small college were presented with a definition of a new mathematical object, and through interviews and problem-solving questions, it was investigated how these students tried to understand the concept and answer questions about tasks related to the new concept soon after they were introduced to the new concept. The students who used example generation (producing examples
related to the concept) and concept reformulation (trying to express the concept with pictures, symbols or words different from the definition) were the ones best able to develop a complete concept-image. Students using example generation were best able to identify the correctness of conjectures and provide examples. Students, who showed primarily concept reformulation and did not generate examples, were more easily convinced of the validity of false conjectures.

In this section I have brought together research suggesting that abstract definitions do not always lay strong foundations for the students to construct understanding of mathematical concepts. The assumption that students spontaneously unpack the logical connections and inferences from definitions is not necessarily true for many students. Instead these abstract definitions invoke concept-images and concept statements that may or may not help to build understanding of the mathematical concepts. Even the notion that the definition establishes the concept may not be in the statement-image or concept-image of the learner.

**Students' Understanding of Algebraic Expressions and General Laws in Mathematical Education**

In this section I review what research reveals about the difficulties of understanding algebraic expressions if there is no meaningful concept to support its transformation from formal meaningless symbols to a transparent mathematical object and the consequences of disconnecting procedural and conceptual knowledge. Owens and Super (1993) analyzed the experience of a student in handling decimal fraction manipulation. The interviewer and the student engaged in a conversation and although most of what the student said was correct, the way the responses were framed, suggested
that there was very little conceptual understanding of the decimals involved. At the same time the student showed quite some ease in manipulating decimal numbers. The symbols had little meaning, but there is well-rounded knowledge about the symbols themselves, with knowledge confined to knowing the rules of how to carry out all steps of a procedure. This can be seen as a different form of procedural knowledge. The researchers pointed out that if students had acquired a procedure through rote learning, it was harder for the student to connect that body of knowledge to conceptual understanding.

Hiebert and Wearne (1985) described a model of students' computation procedures in which they hypothesized that students rely solely on syntactic rules and that semantic knowledge had no effect on their performance. The model's prediction about the performance was confirmed in tests and interviews from students in grades 5-9 and groups ranging in size from 55 to 223 students. It appears that computations were done without conceptual understanding.

Sfard and Kieran (1999) conducted a 30-day teaching module in three grade 7 classes. Their approach was to use graphing before the formal introduction of algebra. Functions were introduced through tabular and graphical representations in a realistic context. The students had to discover the rules for the manipulation of functions themselves including the rules for adding, subtracting and scalar multiplication. Working with the graphs and tables seemed to help students develop meaning for the algebraic expressions and the mathematical expressions "came to life." The assumption was that the abstract mathematical forms had become objects for the students and they were able to mentally manipulate those expressions as objects. Without the treatment of giving meaning to the mathematical expressions, the students would be just outsiders watching
others play the virtual game of algebra. These studies also suggest that students can display quite some knowledge in solving problems and getting correct results, without a deeper knowledge of the meanings or the concepts used to find those results. On the other hand, if they get the opportunity to infuse meaning into their formal algebraic expressions, the mathematical content of the expressions comes to life and is internalized as real mathematical objects that can be handled as if they were real objects.

In this section I have argued that providing a definition alone without a major effort to help students unpack the logic of the definition or build knowledge of the properties of the object created may foster a limited procedural approach to mathematics rather than a growth based on conceptual understanding. The same limited procedural understanding (Skemp, 1987) will result when formal but disconnected algebraic expressions are learned that have no basis in the connected, meaningful world of students.

**The Development of a Function Concept with Co-Variation and Correspondence Aspects**

In this section the various approaches to learning the function concept are discussed. For the construction of the exponential function and in particular for the construction of the rational and negative exponents, the student needs to develop a strong sense of numbers, rate of change, intuitive continuity and the connection between variables as exponents and the resulting values (the powers). The growing role of graphing calculators (Schwarz, Dreyfus & Bruckheimer, 1990) and their power to create detailed and extensive tables of values make the study of the function aspect of exponents very important.
The development of the function concept can be approached in many different ways. The process-conception of functions, central in the studies of Schwarz, Dreyfuss and Bruckheimer (1990), Schwarz and Dreyfuss (1995), Dubinsky (1991, 1992), Breidenbach et al.(1992), Sfard and Linchevsky (1994) and Tall (1994) is one form. The process-conception of functions exists when students acquire a complete understanding of a given transformational activity performed on a function, including understanding of causal and dependency relations between dependent and independent variables (Slavit, 1997). The process approach to functions tries to create an internal notion of the function concept through a process, or a transformation that may not be explicit, but that is imagined in the mind of the learner and that can be further interiorized into a mathematical object or encapsulated. As long as the learner is at the action (or computational, algorithmic) level of the transformation, the transformation is relatively external to her/him and that separates her/him from someone who can internalize this step much better. The use of the computer in the teaching of functions to help in this process is quite striking. These studies also try to develop a more powerful way to teach the function concept with special attention to understanding the many ramifications of the mathematics involved. Most of the studies are limited to functions in the domain of numbers, although Breidenbach et al.'s (1992) computer approach specifically tries to expand this side of the function concept to include non-numbers.

Slavit (1997) studied the way students' concepts of functions develop and found that the use of graphing calculators and the multiple representations associated with those tools promoted a concept that focused on properties of functions rather than the promotion of reification, which is an encapsulation, a more abstract, structural view
(Sfard, 1991) of the general function concept. The property-oriented concept of functions deals with the gradual awareness of specific functional growth properties of a local and global nature, and the ability to recognize and analyze functions by identifying the presence or absence of these properties. In a study (Slavit, 1997) of three students from an honors high school algebra-2 class over a full school year, these students were tested for their ability to discuss similarity or differences of functions, presented on cards in various forms (tabular, graphical, and symbolic). The researcher compared the responses of the students given during interviews with the answers the students gave from the cards and concluded that students initially focused on actions when developing understanding of functions, reinforcing an action-view of functions. This action view did not necessarily contribute to an object-view of functions. An object-oriented view of functions is described as the reification of an action view of functions (Sfard, 1991; Slavit, 1997). The object-view of functions makes it possible for students to manipulate functions as if they were physical objects (Kieran & Sfard, 1999). The student can "see" the functions, talk about them, move them as if they were objects. "This ability is what we call understanding" (Kieran & Sfard, 1999, p. 15).

The correspondence or relational view of functions focuses on causal and dependency relationships between input and output pairs and generalizes to an entire set of input and output pairs. This correspondence view of functions (Slavit, 1997; Breidenbach et al., 1992) can obscure some alternative approaches like the co-variation approach. Co-variational reasoning (Carlson, Jacobs, Coe Larsen & Hsu, 2002) is the cognitive activity that involves the coordinating of two varying quantities, while attending to the ways in which they change in relation to one another. Students do not
necessarily view a graph of a function as a means of defining a co-varying relationship between two variables. The co-variation view of functions illustrated by the proposal of Confrey and Smith (1995) for studying the rate of change in exponential situations is a specific model that highlights how the function variable changes per unit change of the independent variable. Slavit (1997) considered the co-variance view of functions as a form of the property-oriented view, with special attention to growth properties. The co-variance view helps to define the nature of the function under review. For the study of exponential functions, the concept of multiplicative rate of change ($\frac{F(n+1)}{F(n)}$) for every $n$ to characterize a function $F$ can help to formulate a generalized concept of exponential forms (Confrey & Smith, 1994) independent of the nature of the variable.

This approach to the exponential function $F(x) = B^x$ may help clarify for students some of the properties of exponential functions and focus the attention of the student on the growth aspect and its connection to the multiplicative properties of the exponential function in general. (The multiplicative rate of change of $F(x)$ over the interval from $x$ to $x + a$ is the quotient $F(x + a)/ F(x)$. For the exponential function $F(x) = B^x$, the multiplicative rate of change of $F(x)$ over the interval $[x, x + a]$ would be the value $B^a$).

Working with physical and dynamic (a spool elevating system) models, Hines and Khoury (2001) studied 7 middle school students and 19 pre-service teachers who explored the models. The students used tables to explore the height reached by the system in relation to the number of turns for the handle of the device. On top of that the teachers used real world problems, like investing money, to study relationships between variables. The students were tested for their understanding of variables and of the function concept.
The testing instrument also gauged the use of strategies to link representations of functions and variables and documented the levels of understanding of functions of both students and pre-service teachers of middle school. The creation and interpretation of tables and the tabular mode as representation of functions originating from dynamic physical models or real world situations became powerful tools for interpreting functions as relationships involving systematic co-variation of variables. Reasoning with tables provided opportunities to deepen the understanding of those relationships.

Summarizing this section, I have argued that using representations in the form of tables and tabular lists, with an emphasis on the study of simultaneous variation of the variables, represents a shift in the image of the function concept from the correspondence principle to the co-variation principle. This can become a tool for conceptual thinking in the construction of the exponential functions. In particular, this approach can lay the foundation for the understanding of the isomorphism between the additive world of the exponents and the multiplicative world of the exponential function.

**The Teaching Experiment**

**The Teaching Experiment as an Investigative Tool**

In this part I will discuss briefly the general principles, the origin and some types of teaching experiments. Furthermore I review the relations of such experiments to constructivist insights, and as tools for investigating the development of mathematical knowledge of students and the growth of the knowledge of teachers about students' mathematical development. Because the teaching experiment is a tool in my study, this part will focus mainly on clarifying the notion of teaching experiments.
**General principles.** A teaching experiment consists of a series of teaching episodes and individual interviews and it can cover time periods from a few weeks to two years (Cobb & Steffe, 1983; Confrey & Lachance, 2000). The researcher acts as teacher and formulates explanations of her/his understanding of the students' mathematical constructions in the teaching experiment. The explanations consist of models and theoretical constructs formulated in the context of the interactions with the students. The number of students can vary from just one to a whole class or even groups of classes.

**Origins and types.** The origins of the teaching experiment can be found in different places. One source (Hunting, 1983) is the research conducted by Vygotsky (1978) in the Soviet Union in his study of the development of children and the role of instruction (in the widest possible sense) in this development. Human mental processes are not given like a natural built-in mechanisms, but formed. Vygotsky considered the development of humans as a living adaptation to the environment. Humans acquire new forms of behavior through acquisition, imitation, invention and other means. Vygotsky's method was to show experimentally that the acquisition of new operations was always related to some internal process that he called an internal genetic law. Every new development was dependent on previous forms. External operations are transformed into internal ones through a series of successive stages. The study of such processes was through the building of models of development (Vygotsky, 1978; El'konin, 1966).

In the first teaching experiments carried out in the Soviet Union there were two basic forms that were followed (Hunting, 1983). The *micro-scheme* was a method where single student's developments in the learning of mathematics subjects were studied. The focus was on transitions that were identified from observations and interpretations in the
conceptual development or problem solving activity of one student during the teaching of
specified subjects. The *macro-scheme* involves the study of a student's mental changes
during school activity from one age level to the next. Even entire classes can be studied
over the course of many years to document their mental development in a teaching
experiment. The instructional material was organized on principles that were not
necessarily based on experimental methods. The objective of developing models of
students' development was not the primary goal of these experiments.

The second source of the teaching experiment methodology is the clinical method
developed by Piaget in his study of the thinking of children (Piaget, 1970). Hunting
(1983) describes the clinical method used by Piaget as a way out of the unsatisfactory
methods used then in the form of standardized testing or observation to understand the
development of children and students. The clinical method is basically a dialogue or
conversation between the student and the interviewer, who poses tasks and asks
questions. Both questions and tasks are designed to give the student ample opportunities
to respond. From these responses inferences are made of the thinking processes of the
student. The responses of the student determine the next questions from the researcher
until the task has been explored as much as the interview allows, and the researcher
moves on to the next area of investigation. The teaching experiment is primarily an
exploratory tool derived from but involving more than Piaget's clinical interview
methodology (Steffe et al., 2000; Steffe & D'Ambrosio, 1996; Steffe, 1980) and aimed at
exploring the mathematics that students develop.

The approach of Piaget has also been used to create methods that lie between the
clinical interview and the full-scale teaching experiment. One of them is the Clinical
Intervention Method (Booker, 1980). This involves the same setting as a Piagetian interview, but now an intermediate factor of instruction is introduced after the interview. The initial assessment and subsequent performance of the student in relation to the task are then analyzed. The objective is not so much hypothesis testing as theory generation.

**The Teaching Experiment and its Components**

A teaching experiment involves a sequence of teaching episodes and a researcher who is usually directly involved as a teacher. Recording devices such as audio-recorders and video-recorders, document what happens during the teaching episodes. In a teaching experiment, the focus is not on how to teach a pre-determined way of operating mathematically, but it is an exploratory tool aimed at understanding what might go on in a student's mind as she or he engages in mathematical activity (Steffe & D'Ambrosio, 1996). From a constructivist point of view, the teaching experiment is considered particularly suited to study the development of the process of the construction of mathematical objects as it actually occurs in children and students (Steffe, 1980). A teaching experiment within the constructivist paradigm usually includes teaching episodes, individual interviews, and modeling activities.

**Teaching episodes.** What counts as a teaching episode depends on the purpose of the data collection. In general teaching episodes are sessions of teaching an idea or method. The content of the episodes and the tasks set for students are motivated by the model the researcher constructs for the student's mathematical activity. With a small group of students, content and tasks are designed for each particular student (Steffe, 1980). During data analysis, episodes can stretch over many sessions, or be as short as a few minutes (Lesh & Lehrer, 2000) depending on the unit of analysis for the study of
what occurs in terms of learning or understanding. The teacher and the students are the main participants in the teaching episodes (Steffe, 1980; Steffe, Thompson & Glasersfeld, 2000). A third adult as witness and support for the researcher could be part of the research team. In the constructivist perspective cause and effect relationships are not assumed. The researcher as teacher tests the model that was constructed and monitors very closely how the model may change under the teaching that is done in the teaching experiment. It is extremely important that the teacher-researcher focuses on the thinking of the student and not on his or her own concepts. It is the responsibility of the researcher to formulate tasks and questions spontaneously to be able to learn as much as possible from the teaching experiment.

**Modeling.** Modeling is the most important aspect of a teaching experiment (Steffe, 1980, Cobb & Steffe, 1983). These models are basically the researcher's interpretations of the behavior of the student - the way the student acts and communicates in the teaching episodes or during problem solving. The assumption is that human beings are self-regulating and self-organizing organisms (Potari, 2000). They may become disequilibrated (in the Piagetian sense) to the extent that their current thinking differs from what the teacher tries to engender. Signs of disequilibrium are what the teacher tries to find in the behavior of the student that can then point her/him to critical areas in the model of student's thinking that she/he wants to investigate. One aspect of a teaching episode is the instruction. The nature of instruction during a teaching experiment is different from a regular classroom context, because the teacher-researcher creates optimal conditions for the teaching and works individually and intensively with the students (Steffe, 1980; Steffe & D'Ambrosio, 1996; Potari, 2000). The number of students is kept
small to facilitate maximal interaction between teacher and student. The goal of the teaching is to observe as closely as possible, as interpreted from the standpoint of the researcher, the constructive process of the mathematical objects by the student. This close and intensive observation is essential to avoid substituting the researcher's mathematical behavior as a model for the explanation of the student's constructive process. Because a child grows or develops in mathematics mostly under the influence of instruction, the observation of this constructive process has to be done under conditions of instruction.

**Interviews with students.** The interviews can be conducted at the start, at the end, and in between the teaching episodes. Essential is the realization, that analyzing clinical interviews is critical to the construction of a model of (hidden) mental structures and processes that are grounded in observations from protocols (Clement, 2000). The goal of these models is to generate models of mathematical thinking in students. In the construction of models the viability, is the main test of its strength.

**Model viability.** The model is the theoretical construct of the researcher that explains the students' mathematical activity, their exploration of ways to understand and make sense of the mathematics involved. Model viability is roughly the explanatory power and usefulness of a model (Clement, 2000). The factors affecting this viability of a model are mainly its plausibility, the empirical support, its coherence with established models, its performance in tests over time (external viability) and its power of extension into new contexts.

**The Mathematics by (or of) the Students**

Distinctive features of the constructivist teaching experiment (Steffe & D'Ambrosio, 1996) are not only the core principle that learning is an active process but
also the principle that the researcher (and teacher) makes an effort to distinguish the
student's ways of constructing knowledge in mathematics from the concepts and ways of
thinking of the researcher. The student's activity to develop her/his understanding of
mathematics is what we call the student's mathematics.

From a constructivist point of view the way students come to know mathematics
is through their own constructive processes. The teaching experiment is an exploratory
tool to try to understand what might be going on in students' heads when they engage in
mathematical activity (Steffe & D'Ambrosio, 1996). How students solve and explain
solutions to mathematical problems is what we call students' mathematics. Another way
of stating the construct of student's mathematics and its function in a teaching experiment
is: students' mathematics is indicated by what they say and do as they engage in
mathematical activity, and a basic goal of the researchers in a teaching experiment is to
"construct models of students' mathematics" (Steffe & Thompson, 2000, p. 269).
Ignoring the possibility that students can develop their own, autonomous ways of dealing
with mathematical tasks or problems, can lead a researcher to model mathematical
development based on their own mathematical, adult behavior (Steffe, 1980).

The construct of students' mathematics is also part of the concepts that reflect the
teacher's reconstruction of her/his understanding of the mathematics involved in the
teaching experiment (Steffe & D'Ambrosio, 1996). It is not only the student who actively
constructs knowledge. The teacher-researcher has to go through similar active
reconstructions. Closely related to the concept of students' mathematics is the appearance
of constraints in the teaching and learning of the subject matter of the mathematical
goals.
Constraints in the Teaching during a Teaching Experiment

A critical area of the teaching experiment is the identification of the constraints that the researcher-teacher experiences in achieving her/his goal, or in developing the mathematics that is taught in the direction planned. The student has her/his own way of operating and the teacher-researcher studies mistakes (Steffe & Thompson, 2000) arising from the students' attempts to make sense of a new or unfamiliar situation in her own way. These mistakes help the researcher become aware of the limitations in the conceptual schemes of students. Only the student can overcome these mistakes, or limiting schemes, by reorganizing her/his schemata involved in the problem situation. These developments in the schemes or schemata of the student eventually result in the disappearance of the mistakes (Steffe & Thompson, 2000; Steffe & D'Ambrosio, 1996).

Thought Revealing Activities

In order to develop productive activities for the teaching experiment certain principles are observed to enhance the students' development and at the same time design activities that create as much as possible trails of documentation that reveal the students' mathematical thinking processes. Six principles are identified in the literature (Lesh & Lehrer, 2000) as important in varying degrees for the development of problem solving episodes. Such developments of problem solving episodes produce cycles of conceptual formation or instances of reorganizations of concepts, the refining of weak unstable conceptual systems, the linking of stable conceptual systems, and the sorting out of notions, all activities that fall under the idea of students' constructions.

The first of the six principles is the model construction principle that requires the thought revealing activities to be directed towards an explicit construction, a description,
or an explanation of the activity when students solve problems. The *meaningfulness* principle states that the student should make sense of the problem situation based on extensions of their own knowledge and experiences. The *self-assessment* principle is the idea that students should be able to judge their solutions and assess their progress. The *construct documentation* principle states that problem solving situations need to be not only thought revealing but the responses of the students should require them to be as explicit as possible. Students should be encouraged to document and externalize their learning and think about their thinking. Students should be encouraged to address their understanding of the kind of mathematical objects they used for explanations. What is the nature of the relationships you used? Was the operation you used additive or multiplicative? Was commutativity involved in the operation or in the concept? What representational systems were used and why? The principle that the construct created by students can be shared or reused by others is the *sharability* principle. This can force a student to consider if a personal method can be useful to others or can be generalized. It can help the student to focus more on the process than on the final product. The principle of the *effective prototype* states that the construct should be a useful prototype for the problem situation.

**Retrospective and Prospective Analysis and Model Building**

All the teaching episodes and all the verbal comments and interactions during the teaching experiment are recorded both by video-taping and by separate audio-taping. The notes taken by the researcher after or during the teaching episodes are part of the documents and data for the teaching experiment. Retrospective analysis (Steffe & Thompson, 2000) is the careful analysis of the audio-and video tapes to activate the
records of experience of the researcher during the teaching experiment including her/his thoughts and ideas. Notes by students are also part of the documentation. By studying the video and audio-tapes after a teaching session, the researcher has the opportunity to analyze the student's actions and mathematics and make predictions that can be tested in the next teaching session. This is the combination of retrospective and prospective (before the event) analysis (Steffe & Thompson, 2000). The aim is the creation of a model of the constructive activity of the students in the teaching experiment to understand how they build their knowledge of the subject at hand, or the solutions to the problems and tasks posed in the teaching episodes. The model that is proposed will rely on the constructs of the framework and explanatory concepts and operations that help us formulate claims about the process of the development of students' mathematics (Steffe & Thompson, 2000).

**The Conjecture-Driven Teaching Experiment**

To achieve a closer connection between the way mathematics is actually taught and educational research, the method of conjecture driven research teaching experiments was developed (Confrey & Lachance, 2000). This research design best fits the objective of direct intervention in the teaching and content of a mathematical subject.

**The Conjecture**

In this research design a conjecture is not a hypothesis (Confrey & Lachance, 2000). The focus is not to prove or disprove the content of the conjecture. A conjecture is first of all a means or a proposal to restructure the way content and pedagogy of a set of mathematical topics is taught in ordinary classrooms. Secondly the conjecture evolves in the process of being tested. During the research in the form of the teaching experiment,
the conjecture is revised constantly through guessing, critical analysis of what actually happens in the sessions, and speculation. The speculation can come from dissatisfaction with existing teaching practices or from the mathematical content of lessons. Part of the driving force of conjectures is the desire to let the student's voice and her/his mathematics be heard (Confrey & Lachance, 2000).

The conjecture must have a mathematical dimension described explicitly. There must also be a clear pedagogical content that describes how the mathematical content of the conjecture should be taught. The activities, resources and forms of class organization should be specified as much as possible.
CHAPTER 4

METHODOLOGY

The purpose of this study is to investigate the role of the definitions of rational and negative exponents as given in traditional textbooks for the development of understanding of rational exponents, and how students construct their understanding of exponents. A conjecture-driven teaching experiment methodology was followed to guide a teaching experiment to test a conjecture about the process of exponent construction. The core of the conjecture is the claim that the construction and understanding of rational exponents can be improved by placing the concepts of rates of growth, factors of multiplication and the study of the power and root functions (Goldin & Herscovics, 1991), at the center of the concept construction of the exponential forms or functions. In the conjecture the study of root and power functions was presented in the context of creating factors of multiplication and new rates of growth.

The theoretical framework guiding the investigation of the conjecture was the model of mathematical understanding and problem solving proposed by Goldin, Herscovics, and Bergeron, supplemented by the theories of mathematical learning as formulated by Dubinsky and his colleagues, and the (similar) theory on concept formation by Sfard. The method of the conjecture-driven transformative teaching experiment was based on the model proposed by Confrey and Lachance (2000).
The first part of the investigation was to find out through clinical, semi-structured interviews (Zaskis & Hazzan, 1999) what students knew and understood about rational and negative exponents. After the interviews I conducted a teaching experiment with five students over a period of 4 weeks studying rational exponents. The aim was to understand from my perspective how students constructed their concepts of rational exponents, including the quality of their constructions, and to identify the students’ own ways of making sense of rational and negative exponents.

Qualitative Methods of Research

I used a qualitative method of research for my studies because of my interest in the specific individual constructive processes of students when they develop the notion of rational exponents. Guided by the theoretical framework my focus was on the inferred details and stages of the constructive process. The preparation and experience of the teacher-researcher who conducts qualitative investigations and teaching experiments is a major factor influencing the outcome of such a qualitative endeavor. With this in mind I will describe my own experience, preferences and personal views on mathematical understanding.

The Researcher

Based on the perspective described above, it is important to sketch the experience and background of the teacher-researcher and to have an idea of the human instrument involved in this qualitative study (Lincoln & Guba, 1985). I am from the Dutch Antilles and Suriname and studied Mathematics at the State University of Leyden in the Netherlands. I graduated as doctorandus with a major in mathematics, which is roughly equivalent to a master’s degree in mathematics at a university in the USA. I taught
mathematics in the Netherlands for one year, in the Republic of Suriname for 10 years, then in the Dutch Antilles for another 12 years. I began my studies in the USA in 2001 and I taught courses like the pre-calculus course Math 148 as a Graduate Teaching Associate at The Ohio State University. During my years in Suriname I taught Calculus and Linear Algebra at the Teachers Training College, and I was also a supervisor for teacher professional development at the same College. In the educational systems in Suriname and the Netherlands the training of teachers is mostly separated from University education. During the period from 1995-2001, I was involved in the implementation of reforms in mathematical education at the secondary level in the Dutch Antilles for the Islands of Saint Maarten, Saint Eustatius and Saba (the northern part of the Dutch Antilles). I took part in professional development programs conducted by the Freudenthal Institute of the State University of Utrecht for the implementation of Realistic Mathematics Education in the Antilles and attended annual conferences of the National Council of Teachers of Mathematics in the USA. In 1999 I attended a professional development week involving the use of graphing calculators in statistics organized by the organization Teachers Teaching with Technology (T³) in Columbus, Ohio.

My personal views on rational exponents have been influenced by my experience with pilot studies on students’ conceptions of exponents conducted during my doctoral studies. The pilot studies suggested that students’ original notions of exponents were not integrated with their understanding of rational exponents. The experience with the pilot studies has reinforced my view that conceptual growth (Duit & Confrey, 1996) should include the early notions of students, their so-called intuitive preconceptions, and from
there cognitive routes should be constructed to establish a restructured and more extensive domain of understanding. The student must be made aware of the process of construction and justification of the extended domain of exponents and how these concepts were related to their original intuitive concepts (Duit & Confrey, 1996). This implies, I believe, that the student should be able to understand how notions of the positive integer exponents have been integrated and coordinated with the notions of rational and negative exponents. This was my bias going into this investigation.

**Ideological Position**

I strongly believe that mathematics is a human construction and there is no Platonic realm outside of the human spirit. Without human beings there is no mathematics (Lakoff & Nunez, 2000). Mathematics is doing science with empirical components, consisting mainly of data and discovery. Mathematics is also a social enterprise (Schoenfeld, 1994). I believe that proofs should have a prominent place in mathematical education particularly in the college teaching of mathematics. Formal mathematics must be integrated into the fabric of meaningful content in the learning going on in schools and colleges.

**Selecting the Students for Interviews and for the Teaching Experiment.**

The students taking part in the study were relatively fresh from high school. It was assumed that their knowledge of exponents reflected what they had learned in school and in the pre-calculus courses at the university. The students that were selected had either successfully completed the Math 148 pre-calculus courses or the Mathematics 106 courses for middle school teachers and had shown good communication skills. One student was fresh from high school.
The students were also selected on the basis of their availability for an extended period of time. The choice to limit the number of students to five was motivated by my objective to engage intensively with my students and create for each individual student a model of development that was as detailed and fine-tuned as possible. This required a small number of participants to have maximal space and pay as much attention as possible to and interact as closely as possible with individual students.

**Implications for the Conjecture –Driven Teaching Experiment**

**Internal and External Representations**

Goldin and Herscovics (1991) adapted their original theory of understanding specifically to account for the understanding of advanced mathematics. Tall (1992) described advanced mathematics as characterized by precise definitions and logical deductions based on those definitions. In advanced mathematics the role of physical actions becomes insufficient in understanding those concepts (Goldin et al., 1991). In the framework for emergent mathematical understanding, the constructs of internal cognitive representations of specific kinds take on increasing importance and relevance in explaining the meaning of understanding. In particular, internal systems like verbal representations, where words and sentences are central; internal visualization, spatial representations designated as imagistic representations, and formal mathematical symbolization. These are some of the concepts and constructs I use in my study of the way students try to understand what rational exponents are and how those rational exponents are related to their initial concepts of integer exponents.
Small, Exploratory Teaching Experiments with a Small Group of Students

Teaching experiments with conjectures are typically large-scale experiments for whole classes extending over one or more years. Such teaching experiments require smaller investigations that explore the power and the components of the conjecture. These smaller investigations are recommended (Confrey & Lachance, 2000) to test aspects of the conjecture and for narrowing the scope and definition of the conjecture. My investigation is situated in this domain of the small group studies related to the conjecture about rational exponents.

Development of the Conjecture

In studying the relevant literature, I was guided by my beliefs that students need to develop their own mental representations rooted in previous knowledge to allow more abstract notions to develop with meaning instead of on the purely formal level of splitting (Confrey, 1991; 1994; Confrey & Smith, 1995). I combined my belief in the value of intuition with elements of Confrey’s studies. The first element in my conjecture is a strong emphasis on the multiplicative rate of change. Confrey stated that her work on exponential functions led her to consider rate of change as a primary point of entry in her work with students (Confrey, 1994). This notion is related to her splitting conjecture where the split, as a cognitive primitive, is associated with the concept she calls the multiplicative unit as opposed to the notion of an additive unit. Also the multiplicative unit connects a predecessor quantity with its successor in a sequence (Confrey, 1995).

Furthermore, in a series of interviews, Confrey and her colleagues have documented (Confrey, 1988, 1994; Confrey & Rizzuti, 1994) how long it took for the student that was interviewed to construct the notion of a (repeatable) factor of
multiplication, when percentage increase or decrease was involved. The fact that the factor of multiplication represents a combination of the percentage increase (or decrease) and the addition to the previous value in a percentage increase problem and weaves these two operations into one as a factor for multiplication, was not easy to construct for the student in the research. In my pilot project I noticed very similar problems in my participant. These problems motivate the first element of my conjecture, namely that there should be extensive attention to how rates of change (growth) can be transformed into factors of multiplication in a sequence with consequences for repeated multiplication.

In the analysis of the long series of interviews with a student, Confrey (1994) asserted that for an understanding of exponential functions, students need to be well-grounded in the understanding of the operational character of functions. In particular students need to understand how the function varies with variations of the variable and which actions correspond with which operations on exponential functions. This led me to the second element of the conjecture, namely the power and root function and their relation to the exponential function need to be studied thoroughly and connected to the concept of repeated multiplication that was the starting point of the early concept of exponents.

The traditional approach to defining rational exponents includes defining $A^{\frac{n}{m}}$ as $p\sqrt[q]{(A)^q}$ where $n/m = p/q$ such that $p$ and $q$ are relatively prime and $A$ is a positive number (Mueller & Brent, 2006). The discussion between Tirosh and Even on the one hand and Goel and Robillard on the other on the matter of defining or not defining the expression $(-8)^{1/3}$ (Tirosh & Even 1997; Goel & Robillard, 1997) suggests that even for mature mathematicians there could be a blurring of interpretations of how to
define rational exponents and what the conditions are for evaluating their value. In their model of mathematical understanding applied to exponents, Goldin and Herscovics (1991) insisted on the necessity of including a discussion of both the power and the root functions in the explanations for the rational exponents. The discussion aforementioned and the remarks by Goldin and Herscovics (1991) strengthened my assumptions and led to my third element for the teaching experiment: the root and power functions had to be included in the discussions of rational exponents but in a style and with a perspective that was compatible with the requirements of the exponential context. In particular, the question of rates of change (or rates of growth) and their connections to roots and powers need special attention to improve the understanding of rational exponents.

I noticed that decimal exponents are poorly encapsulated by students. Few students could explain in words or in mathematical language how the digits in a decimal exponent were related to the base. In solving exponential equations students were quite capable of using the calculator to approximate the solution, but when asked to explain what caused the shift in value, or how the exponent was related to the numerical value of the answer on the calculator screen, the students did not know how to respond. This prompted me to include a specific discussion on decimal exponents in the study of rational exponents. That was my fourth element of the teaching experiment. Calculating and studying decimal exponents should help the students to see the laws of exponents in action and help them to understand that despite the rational forms, the laws still have validity and are therefore part of the invariant properties of the new expanded concept for exponents.
The fifth element of my conjecture is the creation of an acceptable convincing connection between the common definition of exponents as measures of how many factors a power contains and the description of the concept of the rational form. This description should have sufficient intuitive content to make the transition from one form to the other as smooth and gapless as possible. Lockhead (1991) noticed that the different definitions that are allowed to coexist without the slightest hint of an explanation amounts to a selective notion of consistency in presenting ideas in textbooks. If we present exponents as counting how many times a factor has been used in a multiplication, then our next definition should at least make an effort to guide the student from one formulation to the next. My conjecture was (and this really needed to be tested empirically) that we need to re-formulate exponents as measures of how much of the base is used in a multiplication. The base then becomes similar to a unit of measurement, while the exponent is the actual result of the measurement using a particular unit. Change of unit should then result in a change of the measurement.

The sixth and last element of my conjecture is that negative exponents should be introduced after rational exponents. This section of the conjecture is also based on my experience in the pilot project and my interviews with novice and expert students of mathematics. Almost all the participants mentioned only the formal definition as the justification for the interpretation of the negative exponent concept. The fundamental problem that I have with this approach is that it assumes some form of abstraction of the laws of exponents on the part of the students. The assumption was that students think about these laws as independent of the positive integers. The students are then supposed to apply these laws to negative numbers and to understand why they are valid. The
experience with counting numbers suggests that such abstractions need not be there at all. The pattern recognition needed might not be sufficient to justify the acceptance of the far reaching extension of the counting exponent into the domain of negative numbers. I suggest building the abstraction for the laws of exponents first, and then testing its maturity by introducing negative exponents.

The Teaching Experiment Methodology in this Study

I use the constructivist principles in this study and in the teaching experiment (Steffe, 1980). This means that I assume that students actively construct meaning for the mathematical objects that they create, and that mathematical knowledge cannot be transferred ready made to any person. Furthermore, I assume that students can only learn and understand new concepts based on previous schemes constructed in the past. The theoretical lens for this study of exponential construction is the perspective created by the theories stated in my theoretical framework. The essence of the teaching experiment methodology is to acquire firsthand experience with the practice of fostering, sustaining and modifying the mathematics of students with the objective to learn from them how they think mathematically (Steffe & D’Ambrosio, 1996). The mathematical concepts that we as researchers have, constitute the starting points for the inquiry, but we should be aware that the students’ concepts can be quite different from ours.

The teaching experiment was preceded by interviews of a semi-structured nature (Appendix A) to provide information first on the extent to which the students had constructed the concept of positive integer exponents, and second to find out if they were able to explain their ideas and concepts. Furthermore, the interviews were designed to provide evidence on the knowledge the students had of rational and negative exponents.
and the nature of their understanding of these rational and negative exponents. One area of interest for these interviews was how the students connected the various notions of fractional and negative integer exponents with the original concepts of positive integer exponents formulated as repeated multiplication in traditional and recent textbooks (Beckmann, S., 2005; Hall, B. & Fabricant, M., 1993; Smith et al., 1992).

The students in the teaching experiment were divided into two groups. One group had two students and the second group three students. The groups had the same instruction with one week time difference. The objective was to have an opportunity to adjust the teaching of the second group if necessary, based on experience with the first group. In actual practice the two groups became interacting entities (through the actions of the researcher) with one influencing the other.

In the teaching experiment the researcher was the teacher. There was an assistant recording what transpired during the teaching sessions through audio- and video-taping. Students had one audio-taping device to record their words and reactions. Each session lasted two hours twice a week over a period of 4 weeks. The teaching experiment took place during July and August 2006.

The instruction was delivered in the form of teaching sequences. Each episode was conducted according to a plan of instruction around a core concept (Appendix B, the original form at the beginning of the teaching experiment) and a plan of work for the students in the form of tasks and assignments prepared on worksheets (Appendix C). A diagram outlining the organization and general content of the sequences is included in a later section of this chapter. The content was subject to change depending on the
experience with the mathematics developed by the students (Steffe & D’Ambrosio, 1996; Steffe & Thompson, 2000).

At the end of the second, the third, and the fourth weeks of teaching sessions, I conducted open-ended interviews (Appendix D) with the students to create more documentation of their level of understanding. The format for these interviews was determined by the teaching experiences for the preceding week.

**Development of the Teaching Experiment based on the Conjecture**

This teaching experiment is situated between the large-scale conjecture driven teaching experiment and the small-scale constructivist teaching experiment with a small group of participants to study detailed aspects of the learning and conceptual development process. I used the conjecture-driven teaching experiment methodology, but on a small scale. I worked with five students for four weeks of instruction with a planned intervention where I was the researcher-teacher. During the intervention, I adjusted the plans and the conjecture slightly, and I changed the interpretations of observed processes based on new experiences (Confrey & Lachance, 2000).

**Components of Instruction during the Teaching Experiment**

The small scale of the conjecture driven teaching experiment determined the following components of instruction: (1) the method of instruction, (2) the teacher’s actions, and (3) the method of pre-and post-testing. The method of instruction was planned and conducted in the form of lecture, whole group discussions and individual work with worksheets. All classes were held in the same OSU classroom. The lesson plans were prepared in detail, but with an open mind for changing and responding to emerging possibilities and directions for new teaching indicated by the experience of the
teaching experiment (Meletiou, 2000). As researcher and teacher, I looked for the development of the students’ mathematics and for the development of as much conceptual development as possible. I engaged the students in active discussions to understand and foster the growth of their way of constructing the rational exponents. As the teacher I listened, stimulated discussions, and facilitated interaction.

In the next section I describe the data that were obtained from the two parts of the investigation and the data analysis used to determine what the teaching experiments have yielded and how the results have been established.

Data

The research for this study combines data from clinical interviews, conducted before, during, and after experimental episodes of teaching, with data from the teaching experiment that lasted four weeks. The aim was to investigate first the existing knowledge of rational exponents of the seven students, based on the interviews conducted before the teaching experiment, and second to create a model for the development of the knowledge and understanding of the novice students from data collected before, during, and after the teaching experiment. The questions and problems are stated in Appendices C and D.

Data Collection through Interviews

The interviews were conducted for three reasons. First they were conducted to collect data to create a model for what the students knew, and to study the nature of their understanding of the exponential concept for different numbers such as positive integers, negative integers, and rational numbers to study how the students explained the various connections between forms of the exponents. This part of the study aimed to clarify if the
student relied on the validity of the laws of exponents as suggested by the textbooks, or if they used other methods (such as the authority of teachers and/or the textbook) to explain how the new exponents could be justified.

The second objective for the interviews was collection of data on students’ knowledge and understanding of exponents to establish for those students taking part in the teaching experiment a base line, a clear picture of where the students were in their understanding of rational exponents. In the teaching experiment that followed I tried to connect as best as possible with the students’ existing notions of exponents and develop one way to reach the goals of the teaching experiment, that is an understanding of exponents that included all the layers of understanding (logico-mathematical procedural understanding, logico-mathematical abstraction, and understanding of the formalization in the form of notations and symbols) mentioned in the model by Goldin et al. (1991). Furthermore, the teaching experiment was designed to foster an understanding of how the concepts of the rational exponents could be related to the basic concepts of exponents for positive integers.

The third objective for the interviews was to document the students’ understanding of exponents during and after the teaching experiment. I conducted semi-structured, task-based interviews with the students to document the evolution of their constructs and their understanding of exponents including their way of integrating the power and root functions with their earlier ideas of exponents for the integer case. The content of the interviews or written questions (Appendix D) between sessions of the teaching experiment was adapted to the actual course taken for the teaching episodes.
The sources of data (Vithal & Jansen, 1997) were five novice students who had successfully completed the pre-calculus mathematics course or mathematics courses for middle childhood teachers, and two expert mathematics graduate students for comparison purposes, making a total of seven students. The interview sessions were conducted in my office. Each interview lasted about 1.5 hours. The students were encouraged to support their explanations with written notes. The interviews have all been transcribed and the relevant transcriptions have become part of my data.

**Data Collection in the Teaching Experiment**

In the teaching experiment the sources of data were the students and the teacher-researcher. The students and the teacher-researcher provided data in the form of audio and video tapes of the teaching sessions that were transcribed. Relevant segments and episodes have been included in my data together with notes by the researcher and worksheets produced by the students.

The transcripts of the weekly interviews with individual students in the teaching experiment (interviews two, three and four) are also part of the teaching-experiment data. Video images have been used as data to select relevant moments where students or teachers or both were engaged in episodes or gestures that recorded how they talked, acted, or formulated ideas or reacted to ideas that related to the process of understanding the mathematics (Steffe, 1983). Video segments were also selected to record possible interactions suggesting that students were engaged in finding new strategies or personal constructions to solve a problem or finding explanations for their mathematical realities.

The process of data collection during the teaching experiment occurred parallel to data analysis in the sense that the collection of data for one session influenced the data
collection activities of the next session (Lesh & Lehrer, 2000). In such cycles of data collection and analysis, the researcher formulates an interpretation of a student’s responses from one session and designs the content of the next session to test or refine the interpretation of earlier data. The Decimal Exponent Calculation process (a method to calculate decimal exponents) was given more prominence because of perceived need.

Figure 4.1 depicts the overall organization of the research activity. Starting from the top, the pre-interviews represent the beginning of the study, followed by the teaching experiment in two groups, with a one week time difference. The use of common worksheets is reflected by the box in the middle, linking the two groups. There were of course small variations in the teaching and the work produced by the students in the two groups. After two weeks of teaching interview two was conducted, followed one week later by interview three, and after the fourth week by interview four.

The post interview was conducted after completion of the teaching and the three interviews, and the results were analyzed for finding answers to the research questions.
Data Analysis

Considering that the investigation consisted of a descriptive part to create a model of the knowledge of exponents in two distinct groups of students (the expert graduate students and the novice students) on the one hand, and on the other hand a teaching experiment, two separate but related analyses were involved in this study.

I used the method of grounded theory (Glaser & Strauss, 1967) to develop a model of how exponents were integrated into the schemes of the different novice and expert students from my interview pool of students. The interviews were all audio-taped and transcribed and together with the notes and students' solutions to the questions, they became data for the analysis.
Analysis aimed at Creating a Model of the Students’ Knowledge of Rational and Negative Exponents

The analysis of the clinical interviews started with transcription of the first interviews with the students. These transcripts were analyzed for their content, paragraph by paragraph. The software program "Atlas.ti, version 5" was used to help organize the coding and reduction process. Chunks of texts that indicated concepts (Weitzman, 2000), cognitive processes, or moments of constraints for students or teacher were tagged with key words by the researcher. Relevant segments of tagged text were located, linked and formed into categories.

After categorizing the tagged chunks of texts, these categories were organized in classifications called phases of learning activity. The purpose of the classifications and phases was to help draw conclusions that explain, or model interpretations of what the researcher thought happened in the minds of the students. Models were then proposed for the interpretation and explanation of how the constructive process of exponents developed for the novice students, based on the classifications and phases abstracted from the qualitative study of the interviews. Data from different sources, in this case interview transcripts, responses to problems from the structured part of the interviews, and my own notes taken during or after each interview, were used to support conclusions or explanations of the ways concepts were integrated into the larger conceptual schemes of the students. The phases with their categories were organized in networks through diagrams. The relations between the (five) phases were depicted in a network of phases to visualize the process of learning as interpreted by the researcher.
The diagrams and the coding categories became elements in formulating an interpretation of the level and character of understanding that these students showed in my interpretation of their knowledge of rational exponents. In particular I looked for evidence to answer the questions: How are the laws of exponents situated in their concepts of such exponents? Are the concepts of rational exponents embedded in their understanding of the laws of exponents or is there evidence of a cognitive link between their concepts and the laws of exponents? Is there evidence that the expert students did embed their knowledge of rational exponents in their understanding of the laws of exponents? What is the meaning of the word exponent for the student? How do the students explain the name exponent for both their positive integer concept of exponents (based primarily on repeated multiplication) and their rational number concept of that group of objects (based on roots)?

The material of the interviews (and later of the teaching experiment and the weekly interviews) was organized according to multiple dimensions (Lesh & Lehrer, 2000). First the material was divided up per student. Each student from the first group that was interviewed had a file for transcripts, notes and coding categories. An additional analysis of the data of each student was based on concept development, encapsulation or reification, and understanding.

To display all the reduced conceptual data, case-ordered matrices, or tables were used with common displays, common data segments and common reporting formats. The objective was to produce coherent units with associated analytic texts (Miles & Huberman, 1994).
For each student a diagram of hypothetical connections for understanding and concepts was drawn and interpreted. The expert students and the novice students were compared in terms of the results for each of the research questions and the first conclusions outlined the differences and commonalities between the two groups. This section of the study was completed with a final report of the findings for each student and for the group as a whole.

**Data Analysis for the Teaching Experiment**

For the teaching experiment, the data collection process and the data analyses were conducted during and after the teaching experiment as described earlier in the data section. After completing the teaching experiment all audio-tapes of the sessions and the weekly interviews were transcribed and analyzed in the same way as the previous interviews in the first part of the investigation. However, before conclusions were drawn, or models proposed, the results of the teaching experiments and the session-to-session analyses were incorporated and integrated to create cycles of modeling.

**Procedures for Developing Cycles of Testing, Refining and Extending Interpretations of Videotaped Sessions**

For this study the method of telescoping analyses (Lesh & Lehrer, 2000) began with equal attention to all the sessions taped, and then focusing on a smaller number of selected sessions. Interpretation cycles included analyzing isolated sessions, selecting related sessions, giving equal attention to issues and events present in all sessions to focusing in on exemplary or illuminating sessions and events for the purpose of studying the processes of exponential construction. The next section outlines more details of this part of the analysis.
Cycles of Interpretation for the Teaching Experiment

The *first cycle* of interpretation (Lesh & Lehrer, 2000) involved the preparation of an on the scene brief report summarizing observations and collecting and appending all relevant documentation of the session. This report documented fresh, firsthand assessments and impressions. The reports were later used for comparisons and explanations of events. The *second cycle* of interpretation involved the production of observations and suggestions for the next session in the sequence. The second cycle also involved looking back at the videotapes and writing a report. In the *third cycle* of interpretation, transcripts of the sessions were produced together with summaries for the session. In this cycle, successive interpretations and comments were made together with comparisons of different persons or groups.

Various themes were considered, like looking for cycles of thinking that students went through, how they planned or carried out their solutions, or which problem solving strategies were used and when. The purpose of the telescoping analysis was to screen video tapes and transcripts to focus on sessions that seemed the most promising.

The *fourth cycle* of interpretations studied each problem or concept across the participating students and presented analyses from different theoretical perspectives. This cycle produced a report with columns with quotes about what students did and said. The *fifth cycle* of interpretation focused on each student and across problems to produce a report with columns with analyses. In general the focus of the videotapes was student-centered interactions.

When analyzing video or audio-tapes of events, the question arose of what counted as an episode. An episode depends on the content of an interaction (Lesh &
Lehrer, 2000) or the time that a change in, or an evolution of, a student’s epistemology took place, as can be traced from the lesson or the interaction under analysis. Episodes of learning were compared to earlier segments of the student’s recorded activity and interpreted from a historical perspective (Steffe et al., 2000). If an activity was identified that was new or original for the student (as interpreted by the researcher) an inference was made to modify an existing interpretation of the students’ mathematics or to formulate a new explanation of the student’s mathematical activity. The events that were given special attention were those that indicated meaning-making activity, for example when students found ways to link roots and fractional exponents, devised alternative formats for finding decimal exponents.

The way mathematical symbols were used and explained by students was a special area of interest. In the teaching sequences, worksheets (Appendices B and C) that fostered reflection on the logico-mathematical abstraction layer of understanding were used. This was done by presenting problem situations that reflected such questions as what it meant to raise a number to the power 0.23, and then raise that form again to the power $\frac{2}{3}$. Psychological indications of frustration or elation or indifference during discussions of concepts constituted important evidence for inferences or conclusions on learning or understanding.

Creating a Model of the Students’ Learning

One of the goals of the weekly interviews was to extract and document more moments of importance in the process of the students’ constructive and meaning-making activity for the development of rational exponents. Another important part of the data
analysis for the teaching experiment was the activity of model building and theory
generation by the researcher. This process used concepts from the theoretical framework
to create models and formulate theory.

All the problem situations and sequences posed by the researcher in the teaching
experiment form a context for understanding the mathematics of the students. They were
arranged to reflect the researcher’s anticipation of the ways of thinking of the students in
the teaching experiment. By carefully observing how the students reacted to these
sequences and situations, I formulated a model of the students’ evolution of mathematical
activity and their (hypothetical) interpretation of the mathematics of exponents.

Organization, Content and Sequencing
of the Instructional Material

The teaching began with a discussion of population growth introducing the
concept of rates of growth (ROG) and factors of multiplication (FOM). Conversions of
fixed rates of growth and of fixed factors of multiplication into repeated multiplication
were discussed. Rates of growth and factors of multiplication over multiple or fractions
of periods were used to introduce roots as parts of factors. This notion of parts of factors
was important in this teaching experiment. The question was: how would this approach
be assimilated by the students? How would their notion of this concept evolve? Or would
this be too great a jump compared to their previous notions of exponents? I refer to
Figure 4.2 for a description of the sequence of teaching.
The instruction sequence began with the rate of growth (ROG) and the factor of multiplication (FOM) in the context of modeling population growth with a fixed rate of growth per time unit. Two basic forms of the relations between the FOM and the ROG
are presented with the aim of establishing the concept of multiplicative fractions of the unit or the FOM. The upper part of Figure 4.2 visualizes this sequence. The left side of the upper part represents the general relation between the rate of growth and the factor of multiplication. On the right hand side in Figure 4.2, the relations between the ROG and the FOM for multiple periods and for fractions of a period are displayed. All the concepts of ROG and FOM are embedded in the context of population growth and the model to describe this growth.

My main objective for introducing the rate of growth and the factor of multiplication as described above was to broaden the concepts that the students had of the common definition of exponents (CDE). From the ROG and FOM the instruction proceeded with calculating decimal exponents to make the previous ideas on factors more concrete. Exponential equations were considered to be a good basis for exploring the decimal exponents’ calculation process (DEC). This activity became one of the central learning moments in the teaching experiment.

Special attention was directed to explaining how rational exponents emerge from rational actions associated with radical operations. This section was called the conversion of rational actions on radicals into rational exponents. The purpose was to raise the students’ awareness of the methodology involved in this conversion process. This was reflected in the middle section of Figure 4.2 with DEC and Rational actions.

Justifying the zero exponent was done through the process of using smaller and smaller fractions of the FOM, by using the calculator, and by showing that the FOM becomes 1.000 with ever longer chains of zero-digits if we take higher and higher order roots of numbers. Graphs were used for visual support to create a moving image of the
reverse operation associated with divisions and inverses. The last section brought all strands of the exponential concept together. The aim of the teaching experiment was encapsulating all the schemas into a generalized notion of exponents.

**Lesson Plans for the Teaching Experiment**

The detailed plan for the lessons is available in Appendix B. The teaching episodes included worksheets for the students for documenting their work and comments.

**Posttest after the Teaching Experiment**

After the teaching experiment the students were given a posttest parallel to the pretest. The values were changed, but the questions were the same as before in order to document what changes could be observed in their understanding of the problems and what changes occurred in their answers to questions about the meaning of exponents.

**Establishing Trustworthiness**

In this qualitative study of exponents, the question of triangulation and other forms of enhancing the trustworthiness and reliability of the study needed to be addressed to provide a strong base for the investigation (Lincoln & Guba, 1985). The first form of establishing trustworthiness is *prolonged engagement* with the participants. This involved on the one hand knowing the culture and environment of teaching and education and being experienced in dealing with students in general. On the other hand it involved familiarizing myself sufficiently with the students with whom I would be working and giving them time to get to know me as a teacher.

The first (novice) group of students in this study was from the pre-calculus courses or the teacher mathematics courses (for middle childhood teachers) and four of the five students were familiar with my teaching style and person for at least one full
quarter. One student was a freshman starting his studies at the University. The second group of students consisted of graduate students, one a doctoral student in mathematics and the other a doctoral student in architecture. Both students were Graduate Teaching Associates in the department of Mathematics. These two students formed my group of expert students for the study. I had known the doctoral student in architecture for more than two years in my capacity as a Graduate Teaching Associate, and the doctoral student in mathematics for more than three years from my own mathematics classes and from my work as a Graduate Teaching Associate.

The technique of triangulation (Lincoln & Guba, 1985) is a way to support observations by using multiple sources and apply different perspectives to interpret the results. I conducted audio-taped interviews, made video-recordings of teaching episodes. I collected student’s notes from their interviews and from their worksheets during the teaching episodes and my own written reflections after each teaching session.

I had pretests, posttests and interviews during the teaching experiment that included written tests and questionnaires, to verify and support conclusions by observations of different kinds. The interviews with the students at the end of the second, the third and the last (fourth) week of teaching were designed to document their developments and to understand better how both my thinking and that of the students changed.
CHAPTER 5

ANALYSIS AND RESULTS

In this chapter the results of the study on the college students' understanding of rational and negative exponents are analyzed. The results of the pre-interviews of the novice students are presented first, followed by those of the expert students. The interviews are analyzed with particular attention to the understanding the students show of the different forms of exponents and their justification for the interpretations they gave of such exponents.

The purpose of this study is to examine the role of the laws of exponents in the learning and understanding of rational decimal and negative exponents. Special attention was given to the zero exponent because of the persistent problems students had with it: How did the students in this study conceptualize linking and connecting the various definitions of exponents with the Common Definition of Exponents (CDE) and with each other?

The novice students' responses to the interview questions were analyzed and common traits in their concept images of exponents were identified. The interviews of the expert students were analyzed to find out how their knowledge differed from that of the novice students, which aspects were more or less the same, and how the novice students formulated their understanding of various exponents. The results from both the

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novice and the expert students were brought together and an explanation is presented of what this part of the study suggests on the subject of the learning and understanding of rational and negative exponents.

The data from the interviews are presented in two basic forms: matrices with rubrics that provide the relevant data in compact form, organized in such a way as to convey the major areas of analysis (Miles & Hubermann, 1994) and in the form of quotes from the students to provide evidence in the students' own voice. This part of the analysis is concluded by proposing a model of how the novice students in this study organized their knowledge of different forms of exponents and how the various areas are related.

**The Conjecture Driven Teaching Experiment**

After the pre-interviews, and the proposal of a model for the existing knowledge of the novice students, the conjecture driven teaching experiment is analyzed. The results of two groups of students, instructed in two separate sequences of sessions are presented. The responses and activities of the students are displayed in matrix form (Miles & Hubermann, 1994), supported by quotes from the students from the instruction sessions.

From the transcripts of the two groups displayed through the phases, quotes and matrices and the matrices derived from the intermediate interviews of the individual students, a model is constructed for each student to explain how far the student got in her/his quest for understanding of the rational and negative exponents in this teaching experiment. The models are interpreted and explained as part of the description of the effects of the teaching experiment on the knowledge of the five students.

Comparative analysis of the pre-interview (Interview One) and the post-interview (Interview Five) responses provided the basis for describing the differences and
similarities for each student. The intermediate interviews conducted during or right after the teaching experiment, Interviews Two, Three, and Four, provided additional material for acquiring insight in the impact the instruction had on the students' knowledge of exponents.

**The Novice Students' Knowledge of Exponents**

In this next section I discuss the main results of Interview One, the pre-interview, of the novice students on their knowledge of rational and negative exponents. Then Interview One for Group One and Group Two is discussed more indepth.

**Interview One (Pre-Interview)**

Each of the five novice students was interviewed prior to the teaching sessions using a semi-structured format (see Appendix A) and all their verbal explanations were recorded and transcribed. The software program Atlas.ti version 5.0 was used to bring together and organize the transcripts of the teaching experiment. The coding of quotes and the creation of analytical categories was the next step in the process of data reduction. Nine categories were abstracted after close study of the coding for each student. These categories became a rubric for presenting the coding of each student.

*Factoring of integers and knowledge of prime factors.* By inquiring into the students' factoring knowledge their multiplication skills could be studied, their knowledge of exponents with integers, and their familiarity with numbers and with the terminology could be estimated.

*The common definition of exponents (CDE).* The most widely accepted idea of exponents was that they represented the multiplicity of the factors in a repeated multiplication. Careful study of the pronouncements of the students could shed light on
the role of this very common textbook definition of exponents in the learning process of students who were trying to understand what exponents were.

*The zero exponent and its justifications.* The zero exponent together with the exponent 1 represented one of the first occurrences of inconsistent definitions of exponents for the student. The formulation from the CDE did not work very well when the exponent was 0. It also failed for exponent 1. The justifications proposed by textbooks did not seem to address or even be aware of these issues.

*The laws of exponents for integers (LOE).* The justifications for these laws were mostly based on the integer exponents. Students saw only the positive integer justifications and the question arose: how were they going to justify the properties of rational or negative exponents?

*Negative exponents and their justifications.* When exponents were negative, the potential conflict with the common definition of exponents (CDE) became more acute. It made sense to inquire how students reacted to the change from a perfectly meaningful definition to a concept that shut out the CDE.

*Rational exponents, radicals and their connection to integer exponents.* The potential lack of fit between the CDE and the rational exponents increased even more when compared to the lack of fit between the CDE and negative integer exponents. The student had to give up his CDE completely and stick to the definition of rational exponents as presented in the textbook or by the teacher. Knowledge of radical properties and their relations to exponents of the integer kind is usually not required in curricula, but indications were that it is needed. How much awareness do the students have that such
radicals and their properties need to be converted into fractions like entities before he/she can claim that rational exponents exist?

*Decimal exponents and their meaning.* Apart from regular integer exponents, decimal exponents are the most frequent form of exponents that students encounter in their mathematical lifetime. Moreover, decimal exponents have a unique structural connection to the LOE. What were students' notions of decimal exponents and their structure?

*The concept of powers of a number.* One way to trigger the debate about non-integer exponents could be by investigating if students knew what constituted a power of a number. Why are we limiting powers more often than not to numbers that can be written as an integer base with a positive integer exponent?

*The general idea of an exponent applicable to all types of numbers.* Was there a concept of exponents possible that applied to integers, rational numbers, and negative numbers? Were there any notions of such an idea in some students, novice or expert? Did it help the understanding of exponents to have such a concept?

With these categories as rubrics, case-ordered matrices (Miles & Huberman, 1994) were constructed for the novice students, one for Group One (Table 5.1) and one for Group Two (Table 5.2). The tables present in very compressed format the responses of the five students related to the nine categories.

**Summary of the Characteristics for Group One**

Table 5.1 presents the responses of the first two students, Ann and Bob. The comments in each cell of the table reflect the researcher's abstraction of the students' responses to related questions from the pre-interview. The results of the analysis of the
interviews for each student are given through summaries of the knowledge characteristics for the student in the area of exponents.

<table>
<thead>
<tr>
<th></th>
<th>ANN</th>
<th>BOB</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Factoring Prime Numbers; Integer Exponents</strong></td>
<td>Fluent; well defined primes. Formally defined + integer exponents.</td>
<td>Partial mastery of factoring. Uncertain about prime numbers. &quot;Scared by calculations involved.&quot;</td>
</tr>
<tr>
<td><strong>Common Definition of Exponents</strong></td>
<td>Clear notion of CDE. Active use of definition in concepts.</td>
<td>Clear notion of CDE.</td>
</tr>
</tbody>
</table>
5³ = 5*5*5  
5⁰ would be nothing." |
| **Laws of Exponents** | Name unknown. Addition of exponents explained. | OK for positive integers. Avoids positive integers in exponents. Unfamiliar with the name. |
| **Negative Exponents** | Reciprocal of positive exponent form. Justification: school instruction. | Negative exponents are inverses: 1 over a number. Negative exponents are not acceptable: "Can't have negative exponents."  
"They never told me why." Automatically convert to a fraction. |

Table 5.1. Interview One - Group One: Students' knowledge of exponents
Table 5.1 continued

<table>
<thead>
<tr>
<th>Rational Exponents</th>
<th>Connected to radicals. Aware of certain rational analogies.</th>
<th>$5^{1/3} = 3\sqrt[3]{5}$ Unaware of Rational Actions for exponents: $5^{1/3} = 5^{2/6}$ because $1/3 = 2/6$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decimal Exponents</td>
<td>Ignores specific decimal structure: Decimal = fraction.</td>
<td>Unaware of specific properties of decimal exponents. No knowledge about digits of decimal exponents. Attempt to understand $3^{2.1}$ through CDE.</td>
</tr>
<tr>
<td>Powers of Numbers</td>
<td>Numbers are powers of a number if they can be written as positive or negative integer exponent.</td>
<td>Numbers are powers only if the exponent is positive and whole. No negative or rational exponents.</td>
</tr>
<tr>
<td>General Concept of Exponents</td>
<td>No unifying notion of exponents; separate concepts.</td>
<td>Informal notions of exponents, CDE Negative, rational and zero exponents are less clearly articulated.</td>
</tr>
</tbody>
</table>

*Ann.* Ann was unfamiliar with the name "laws of exponents" (LOE) but clearly knew how to calculate with integer exponents. She had a clear idea of the Common Definition of Exponents (CDE) and actively used the definition in concepts like factored forms of numbers. She also tried to use the CDE to explain the negative exponents, and the zero exponents. For Ann, rational exponents were connected to radicals and she was aware of some analogies between radical forms and rational operations.

The knowledge of decimal exponents did not include specific knowledge of the structure of decimals or the system of decimal roots. Ann converted decimals first into
one fraction and then applied the rational exponents' definition to translate the symbol to operational forms with radicals. Negative exponents were associated with fractions or reciprocals but the reason for this definition was not mathematical but was justified as based on her school instruction. The zero exponent had the value 1 but the reasons were unknown, although she proposed an explanation for the value: If $5^1 \times 5^0 = 5^1$ then $5^0 = 1$. Ann does not seem to have a general concept of exponents that can be applied to all the multiple forms of exponents.

**Bob.** Like Ann, Bob was unfamiliar with the term laws of exponents (LOE). He had a clear notion of the CDE, knew how to work with positive integer exponents, but when negative exponents appeared he immediately wanted to replace them with fractions or positive exponents, because "you can't have negative exponents." The CDE was his reason for stating that negative exponents were "forbidden." He did not know why negative exponents were associated with reciprocals or why the zero exponent equaled 1. He tried to explain that $5^0$ should be "nothing." He did this through an interpretation of the CDE, "because 0 is nothing."

The concept of rational exponents did not include knowledge of rational actions as the basis for the equality of forms like $5^{1/3} = 5^{2/6}$, which he simply explained as: there is equality because $1/3 = 2/6$ with no mention of the radical connections. Bob was not able to explain how decimal exponents were structured or what the meaning was of decimal digits in exponents: "I have no idea." He invoked the CDE to explain the form $3^{2.1}$, posing the question how we could take 2.1 times the base 3 to find the value of $3^{2.1}$. There was no sign of a general notion of exponents in his explanations or concepts.
Summary of the Characteristics for Group Two

Table 5.2 presents summaries of the students' characteristics from the pre-interview, Interview One, for Chandra, Dennis and Eddy.

<table>
<thead>
<tr>
<th></th>
<th>CHANDRA</th>
<th>DENNIS</th>
<th>EDDY</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Factoring Prime Numbers; Integer Exponents</strong></td>
<td>Fluent in Factoring; Well-defined primes.</td>
<td>Informal concept of primes.</td>
<td>Basic knowledge of factoring &amp; integer exponents.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(&quot;simplest terms&quot;)</td>
<td>Prime numbers refreshed.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Fluent in factoring.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Positive integer exponents well-defined.</td>
<td></td>
</tr>
<tr>
<td><strong>Common Definition of Exponents</strong></td>
<td>Coherent definition.</td>
<td>Detailed and well-reasoned.</td>
<td>Robust knowledge of CDE.</td>
</tr>
<tr>
<td><strong>Zero Exponent</strong></td>
<td>Value = 1</td>
<td>&quot;Weird&quot;</td>
<td>Value = 1</td>
</tr>
<tr>
<td></td>
<td>Reasons unknown.</td>
<td>Value = 1.</td>
<td>Rule of the game.</td>
</tr>
<tr>
<td></td>
<td>Proposed explanation:</td>
<td>Proposes explanation.</td>
<td>Admits no understanding of zero exponent as being 1.</td>
</tr>
<tr>
<td></td>
<td>According to CDE, value should be zero.</td>
<td>Using CDE and coefficient 1.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Apply &quot;same principle.&quot;</td>
<td>For exp =0, only coefficient 1 remains.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>As up here (CDE).</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Laws of Exponents</strong></td>
<td>Name unfamiliar.</td>
<td>Clear understanding of addition and powers.</td>
<td>Name unknown.</td>
</tr>
<tr>
<td></td>
<td>Strong explanation of law of addition and law of bases.</td>
<td>Familiar with the term &quot;Laws of Exponents.&quot;</td>
<td>Described as &quot;rules.&quot;</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Addition principle of exponents clear.</td>
</tr>
</tbody>
</table>

Table 5.2. Interview One - Group Two: Students' knowledge of exponents
<table>
<thead>
<tr>
<th>Negative Exponents</th>
<th>$5^{-3} = 1/5^3$, but does not know why; school instruction. Negative sign indicates inverses</th>
<th>Concept is &quot;Weird&quot; Neg. exponents indicate &quot;absences&quot; of the base number (CDE). Definition is &quot;logically inconsistent.&quot; &quot;It is an inverse.&quot; Reasons for definition: teachers</th>
<th>Inverses and fractions. Getting rid of the neg. exponents by writing 1 over number. &quot;Whole different concept (CDE and neg. exponents). The concepts don't really fit together.&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rational Exponents</td>
<td>Points to impossibility to apply CDE to understand $5^3$: multiply three times. $5^{1/3}$: multiply $1/3$ times; impossible.</td>
<td>$5^{1/3}$ is not $1/3$ times $5$ (conflict with CDE). $5^{1/3} = 5^{2/6}$ Reason for definition unknown: from teachers!</td>
<td>Just a fraction $5^{1/3} = \sqrt[3]5$ $5^{1/3} = 5^{2/6}$ because $1/3 = 2/6$. No rational action for exponents.</td>
</tr>
<tr>
<td>Decimal Exponents</td>
<td>Convert to fractions. No explanation of digits as related to exponents.</td>
<td>No notion of specific structure of decimal exponents.</td>
<td>No clear notion of the digits in decimal exponents.</td>
</tr>
<tr>
<td>Powers of Numbers</td>
<td>Numbers are powers of a given number if it fits the CDE; negative exponents also make powers: &quot;We need to change the definition.&quot;</td>
<td>A number is a power of the base, if it is divisible by the base number. No explanation for negative or rational powers.</td>
<td>Only positive integer exponents produce powers.</td>
</tr>
<tr>
<td>General Concept of Exponents</td>
<td>Numbers are powers of a given number if it fits the CDE; negative exponents also make powers: &quot;we need to change the definition.&quot;</td>
<td>A number is a power of the base, if it is divisible by the base number. No explanation for negative or rational powers.</td>
<td>Only positive integer exponents produce powers.</td>
</tr>
</tbody>
</table>
**Chandra.** Although the term "laws of exponents" (LOE) was unfamiliar to Chandra, she had a strong mastery of the operations with exponents including rational and negative exponents. Her concept of the CDE was clear. For Chandra the negative sign in an exponent meant "take the inverse." When commenting on rational exponents she pointed to the impossibility of applying the CDE: "5^{1/3}: you can't multiply 1/3 times; it's impossible." Why the negative sign is associated with inverses was unknown to her. The school is where she learned it.

Decimal exponents are converted to fractions and then interpreted with radicals. Chandra was able to explain what $3^{2.1}$ meant by using fractions and radicals but not any structure of decimal roots related to the place value system. The zero exponent had the value 1, but she couldn't remember why. According to her interpretation with the CDE it had to be zero (0)!

Despite her strength in working with exponents and explaining many features of rational and decimal exponents Chandra did not voice a generalized notion of exponents. Her closest answer to a general idea was that for her a number is an exponent as long as we can find an "answer" for the notation and operation.

**Dennis.** Dennis was the first (and only) novice student who actually recalled clearly the term "laws of exponents." He was proficient in his knowledge of the operations on exponents of all three kinds (integer, rational, negative). His concept of the CDE was detailed and clear. The zero exponent had the value 1, and he argued why, although he characterized the concept as "weird" because it went against the CDE interpretation. According to his interpretation with the CDE it should have been 0! His personal explanation used the fact that all numbers have a factor 1 (one) and the zero
exponent tells you that you should not use any factors from the base leaving only the (always present) factor one. Negative exponents were in his words also "weird" and he even stated that the idea of negative exponents was "logically inconsistent": "You can't multiply a number negative three times! " He usually used the calculator to find the values, but knew that they were inverses. Rational exponents were related to the CDE: \(5^{1/3}\); "you're not really having 1/3 at 5 times 1." Rational exponents were equal as soon as the fractions were equal. No mention in his exposition of the radical connection to justify the equality of fractions as the basis for equality of the powers. Decimal exponents were only treated globally: 2.1 was more than 2 so 7\(^{2.1}\) was more than 7\(^2\). No notion of any deeper structure of decimals was found in his exposition. Dennis did not present concepts of exponents other than the fragmented notions that he could recall. No general concept was tried or proposed.

**Eddy.** Eddy showed robust knowledge of the CDE but admitted no familiarity with the term "laws of exponents." (LOE). He characterized the laws as "rules of the game." For Eddy negative exponents could be avoided by writing "1 over the number," and considered the idea of negative exponents as "a whole different concept." His first explanation for negative numbers was fuzzy compared to the CDE. Then he explained such exponents as taking the inverse of a number. Rational exponents were roots but he did not link the fractions of exponents to operations with radicals. Decimal exponents were unclear to him, and he was not able to explain the digits in decimal exponents in any way. The zero exponent had the value one, but he did not know why. "All I know is that it is always one." Eddy was still developing his concepts of exponents and that process
was incomplete. He seemed to be far removed from reflecting on general notions of exponents.

**Knowledge of the Novice Students prior to the Teaching Experiment**

All the students seemed to show attempts to explain their images of exponents of various kinds by trying to fit these new notions of exponents into the definition that they knew and understood fully, the CDE. The zero exponent was conceived as coming from finding a multiplication when zero factors are used or as a special case with an explanation that seemed plausible, but avoided answering the question how the CDE is related to the value of 1. Eddy simply called it a rule of the game.

The negative exponents presented challenges, voiced by students as "weird" or "unacceptable," or "you can't have negative exponents," or they referred to the "schoolteachers" to justify the meaning of negative exponents. Eddy used the description "a whole new concept" for negative exponents to express his perspective on the relation between the CDE and the procedure for negative exponents. He commented that we should "get rid of negative exponents" by switching to 1 over …. Ann did not know why the negative exponents were connected to the inverse of the unit of multiplication. The fact that there was no apparent conceptual connection for the students between the CDE and the traditional definition of negative exponents did not stop them from applying the working definition of negative exponents and solving problems. The authority of the teachers, the school and the existence of "answers" probably overruled their concerns.

With the rational exponents the basic picture was similar in the sense that the CDE seemed to function as the point of departure for interpreting the given definition. Chandra and Dennis voiced this impossible connection to the CDE explicitly by trying to
use the exponent as a counting device. Bob treated the fraction exponent as if only the 
fraction itself was sufficient to find the value of the exponent. No reference was made to 
rational actions to be applied to the underlying properties of roots and powers. This 
notion that all you needed was the value of the exponent seemed to be connected also to 
the unambiguous world of the CDE where one given exponent fixed the multiplicity of 
the whole unit and thus the value of the expression. The notion that the unit itself could 
be counted in "fractions" was not part of the CDE concept. Ann had more awareness of 
the involvement of radicals and powers in the handling of rational exponents.

Decimal exponents were handled as fractions by Ann and Chandra. Bob, Dennis 
and Eddy had no clear notion of any structure relating the digits of the exponent to the 
base or the properties of exponents. Bob made an attempt to use the CDE with the 
exponent 2.1 and thought of using the base "2.1 times."

Each student showed some hesitancy or uncertainty about how one knew if one 
given number is a power of another given number. The common operations with 
exponents seemed to go one way only: given a base and a number, find the value when 
the base has that number as an exponent. The reverse action seemed quite unfamiliar to 
the students. This problem would reappear when the chain of multiplications was 
proposed in the teaching experiment.

The CDE seemed to be the only concept with a clear operational process for and a 
meaningful interpretation of the concept of exponents for novice students. The lack of a 
meaningful extension and development of the CDE into a broader notion that reconciled 
the CDE with the new definitions for the zero exponent, the rational exponent and the 
negative exponent made it unlikely that students like those interviewed would develop on
their own an overarching concept of exponents that was applicable to all types of exponents, from integers to zero, to rational and negative numbers.

**The Expert Students' Knowledge of Exponents**

Two expert doctoral students, one in mathematics (Rabin) and one in architecture (Robby) were interviewed to study their knowledge and understanding of rational and negative exponents. The same questionnaire was used as for the novice students. The results from the interviews are discussed and displayed in Table 5.3 using the same format as used for the novice students. In the next section the results of the expert students are compared with those of the novice students.

There were similarities but also some notable differences between the two expert students in the way they conceptualized exponents and the justifications for the zero exponents, the negative exponents and the rational exponents. The responses (see Table 5.3) to the CDE suggest that both students had a complete and well rounded notion of positive integer exponents as a reflection of the multiplicity of factors, or base numbers, in a repeated multiplication format.

For the zero exponents (see Table 5.3) they used similar steps of reasoning based on two elements: first the LOE to write a division as similar to a subtraction of integer exponents, and second the method of de-encapsulating the concept zero (0) as a number minus itself. Rabin used variables and wrote $X^0 = X^1/X^1 = 1$. This implied that $0 = 1-1$. Robby did not use variables but had the same de-encapsulation of 0 as 1-1: $5^0 = 5^3/5^3 = 1$. This preference for variables on the part of Rabin could have reflected his mathematical background and studies. The use of concrete examples by Robby may also be a reflection of his architectural background.
### Factoring Prime Numbers

**RABIN**
Fluency with factoring and primes.

**ROBBY**
Factoring tree, fluent with process and primes.

### Integer Exponents

<table>
<thead>
<tr>
<th><strong>Common Definition of Exponents</strong></th>
<th>$5^N = 5 \times 5 \times \cdots \times 5$ (N times)</th>
<th>$5^3 = 5 \times 5 \times 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Exponent related to multiplicity of factors.</strong></td>
<td>Exponent related to multiplicity of factors.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Zero Exponent</strong></th>
<th>$X^0 = X^{1-1} = X/X = 1$ for all $X \neq 0$</th>
<th>$5^0 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$5^{(3-3)} = 5^0 \times 5/5 \times 5 \times 5 = 1$</td>
<td></td>
</tr>
</tbody>
</table>

| **Laws of Exponents** | Familiar name; solid knowledge of rules for integer and rational exponents. | Familiar name; well established knowledge of rules for integers and rational exponents. |

<table>
<thead>
<tr>
<th><strong>Negative Exponents</strong></th>
<th>Does not remember how the early definition was presented.</th>
<th>Negative exponent means that you will flip; ...end up flipping the site of the fraction the number is on.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5^{-3} = 1/5^3$</td>
<td>$5^{-3} = (5^{-1})^3 = (1/5)^3$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Rational Exponents</strong></th>
<th>$(64)^{1/3} = 4$</th>
<th>$5^{1/3}$ is the number that must be multiplied by itself three times to equal 5.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(X^{1/n})^m = X^{m/n}$</td>
<td>$5^{1/3} = 5^{2/6}$ because $1/3 = 2/6$ so both are the same power.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$5^{2/6}$ is the number that must be multiplied by itself 6 times to equal $5^2 = 25$.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Decimal Exponents</strong></th>
<th>$7^{2.123787} = \ldots$</th>
<th>Used graphs to illustrate his explanations.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$7^{0.1} = 7^{0.01} = \ldots$</td>
<td>$3^{2.1} = 3<em>3</em>3^{0.1}$</td>
<td></td>
</tr>
<tr>
<td>$7^{1/10} = 7^{1/100} = \ldots$</td>
<td>$9 * 10 \sqrt{3} \approx 9 * 1.11\ldots \approx 10.045$</td>
<td></td>
</tr>
</tbody>
</table>

**Table 5.3.** Interview One: Expert students' knowledge of exponents

---

**Continued**
Table 5.3 continued

| Powers of Numbers | \(F(x) = X^n;\)  
|                  | \(F: \mathbb{R} \rightarrow \mathbb{R}\) for all \(X \neq 0\)  
|                  | \(F(x)\) is called a power of \(X\) if \(n \in \mathbb{Z}\).  
| General Concept of Exponents | \(X^1; 1\) is an exponent.  
|                  | An exponent of a number is the power that the number is raised to.  
|                  | So for positive numbers it is simply the number of times you multiply the base by itself.  
|                  | For negative exponents:  
|                  | flip (\(\rightarrow\) reciprocal) the base and multiply the reciprocal by itself that many times.  
|                  | For fractions: \(4^{\frac{1}{2}} = (2^2)^{\frac{1}{2}} = 2^1\).  
|                  | \(5^{\frac{3}{4}} = 4\sqrt[3]{5}\).  

For both students the LOE seemed to be invariant properties or general patterns of all exponents, ready to be used at any time, so that the equality between the zero exponent of a number and the value 1 followed straight from these LOE. Both students seemed to have no problem thinking of zero as 1-1 or as 3-3. Their concept of zero was clearly much wider and more abstract than the novice students' who may have thought of zero as a mathematical equivalent of "nothing."

Rabin had no clear recollection of how the definition for negative exponents was given in his early years, but he used the inverse relation with no problems at all (Table 5.3). Robby provided a working definition of negative exponents by describing what should be done when simplifying a negative exponent: "A negative exponent means that..."
you will flip; …end up flipping the side of the fraction the number is on." This approach was more "hands-on" than the formal definition.

For rational exponents Rabin had a more formal method with minimal verbal representations or explanations, while the approach of Robby was less formal and with more meaning giving explanations. He explained the symbol $5^{1/3}$ with the words: "$5^{1/3}$ is the number that must be multiplied by itself three times to equal 5." Those words brought together the multiplication involved and the multiplicity 3 that was visible in the fraction $1/3$. What was not explained or not explained enough was the justification for the unexpected exponent of $1/3$. This one third as an exponent needed an intuitive, meaning-giving explanation.

**Summary.** The meaning-giving words were minimal in the justification of Rabin. His (concept) image did not seem to need more than the symbolic forms to make sense for him. The symbols were the concepts; it seemed that they could be exchanged at will. The decimal exponents were explained expertly by Rabin, using the place value system within the exponential world. Robby first used graphs to explain and illustrate his understanding of the decimals and then elaborated his case by using algebra and the decimal forms.

Despite the expert formal knowledge and explanatory skills of both students there was no attempt to formulate a general concept of exponents or a noticeable awareness of the ever changing formulation of what constituted an exponent. The inconsistencies were not really noticed or addressed. It was not clear if the architectural background of Robby created the conditions for a more substantial attempt to give meaning to rational
exponents or describe the negative exponents in a way that gave prominence to the operations involved.

**Comparing Novice with Expert Students**

Upon comparing the results from the novice students and the expert students, we could state that for these two groups of students, the expert students did not show a concept for exponents that covered all the types of exponents from natural numbers to decimal and rational exponents. The segmented approach to exponents that was standard with novice students with their inability to provide explanations for the zero, the negative or the rational exponents was only partially overcome by the expert students in the sense that they actually constructed an explanation for the zero and other exponents, relying on the LOE and formal definitions. But they did not demonstrate an awareness of a generalized idea of exponents. That part of the encapsulation process was set aside or left unfinished.

That the expert students did not develop a generalized concept of exponents despite their strong mathematical background and training in abstract thinking suggests that the fragmented nature of the notions of exponents may not be necessarily overcome with more mathematical training. In terms of the growth of the concepts of novice students this tendency for only partial or formal closure of the conceptual gaps suggests that the disconnections between the CDE and the other exponential concepts for zero, negative, rational and decimal forms, can be overcome with more mathematical training. However, there is no guarantee that this mathematical training will lead to a well rounded notion of exponents that teachers may need to explain this subject meaningfully to other novice students. It also means that especially formal methods need to be supported more
fully with meaningful (verbal) notions and cognitive constructions that support a student's ability to understand the (mathematical) objects involved in her or his thinking. Just having logic applied on formal systems alone doesn't seem to provide all the ingredients for teaching and learning mathematics.

Models of Students' Knowledge of Exponents

The Expert Students. From the analysis of the expert students' knowledge of exponents a model of their exponential knowledge is proposed, as shown in Figure 5.1. The CDE is for both expert students a basic or elementary notion of exponents. With the CDE the expert student could create a clear and meaningful image and operation for positive integer exponents. Negative, rational and the zero exponents could not be accommodated into the schemas for the CDE. Because the expert students had extensive training in mathematics, they were able to mentally transform the laws of exponents into generalized properties of exponents and use those patterns of invariants to justify their notions of rational, negative and the zero exponents. These transformations are represented as arrows in Figure 5.1 with the designation "LOE."

From a formal perspective the connections between various forms of exponents can now be established. There are no mathematical inconsistencies and the system is therefore valid and logical. The need to explain these transformations from an intuitive point of view is secondary. Decimal exponents are examples of rational exponents so there is no motive to study the decimal form extensively. There is no attempt at constructing a concept that explains the wider notion of exponents in a meaningful verbal form.
TheNovice Students. Based on the data and the analysis so far a model of the existing knowledge of the students is proposed as illustrated in Figure 5.2. The basis of the students' understanding is the Common Definition of Exponents (CDE). This CDE concept was meaningful to the students; it was operational and clear in its aspects. The exponent referred to the number of times that a base is multiplied by itself. For an expansion of the notion of exponents to other numbers, like zero, rational and negatives, the students tried to apply the CDE for the interpretation of the new concept, but failed to create a meaningful connection. This failed attempt is represented by the broken arrows.
connecting the CDE box with the three boxes of the zero exponent, the negative exponent and the rational exponent.

![Diagram showing connections between exponent types]

**Figure 5.2: A model of the novice students' knowledge of exponents**

For each of these connections the evidence pointed to either a problematic "I don't know" (Ann, Bob, Chandra.), or an uncomfortable affect like "weird" (Dennis), or an attempt to use the CDE straightforward with counting the multiplicative multiplicity of the base "1/3 times" or" 2.1 times" or "negative-times." Similar failed attempts were reflected in comments on new exponents embodied in statement like "get rid of negative exponents," or "you can't have negative exponents."

The negative exponents were not strongly connected to the rational exponents because all the students had an almost automatic reaction to negative exponents, replacing them by the inverse of the corresponding base. This unexplained or by the
students poorly understood conversion seemed to create the impression that negative exponents were unnatural or undesirable or unmanageable. The areas of knowledge for negative exponents and those for rational exponents didn't seem to be connected through recognition by the students of links with existing knowledge (Hiebert & LeFevre, 1986). Such weakly connected knowledge seemed to be more procedural than conceptual.

The decimal exponents and the rational exponents were connected one-sidedly with the rational numbers, because the handling of decimal numbers occurred exclusively within the framework of fractions. There was no reversal of fraction exponents into decimal form for purposes specifically related to exponents. The decimal structure was replaced by one single fraction and from there the decimal was translated back to the textbook definition with roots. The structure of the digits in decimal exponents was not explained in a systematic way that made these digits clear and meaningful despite the fact that the digits had a powerful connection to almost all the separate parts of the LOE.

The absence of an overarching connecting structure reflected the fact that none of the students interviewed were able to present an argument explaining how all the diverse definitions and calculations could be brought together to integrate and encapsulate the notion of exponent

**The Teaching Experiment**

The teaching experiment was conducted in two groups. The first group, Group One, had two students, Ann and Bob, while the second group, Group Two, had three students, Chandra, Dennis, and Eddy. Group One started one week ahead of Group Two. The separation in time was maintained during the four weeks of instruction. The one week time difference allowed the researcher to adapt the instruction for Group Two if
necessary. In the course of the four weeks of instruction, the two groups became more like interacting entities where experience in one group influenced the instructional emphases in the other.

After the process of data collection for the teaching experiment was complete, the text of both four-week instructional modules and of the three audio-taped open interviews were transcribed. The transcriptions from the video recordings were coded and coded sections reduced to categories. Categories here reflect basic relations between concepts that make up the building blocks for a proposed network of learning leading towards the expanded notion of exponents. These categories were grouped together into a network of phases of learning and instruction, and those phases were used as units of analysis for each student and for the project as a whole.

**The Five Phases of the Teaching Experiment**

In this section the teaching process for the two groups of students is analyzed through a description of the networks of phases and their categories, followed by matrices that display the reduced data per student and per phase. For the actual content of the lessons, the lesson plans, and the worksheets with all the problems, I refer to Appendices B and C.

To reduce the data from the four-week teaching experiment to a manageable format, a process of data reduction was carried out on the transcripts and the video-images of the complete recordings of the work of the groups, the researcher's instruction, the students' reactions and group discussions and the worksheet activity from each student. All the transcripts were coded with the *Atlas ti* software program and brought together into (first-order) learning categories. These categories are the basic groupings of
building blocks of concepts, procedures and their relations as derived from the coded data. The categories were again grouped together into five phases according to their most common mutual relations. A *phase* is thus a higher cluster of connected categories brought together for the teaching experiment to create a foundation for the study of non-integer exponents. The phases are not necessarily time ordered. Each phase corresponds roughly to a section of the teaching experiment and therefore to the lesson plans. Some of the lessons deal with content that reviewed earlier material and such teaching episodes correspond to multiple phases. The relations between categories and phases will be displayed in networks and case-ordered matrices (Miles & Huberman, 1994). Table 5.4 provides a list of the categories within the phases and indicates the lesson containing each category.

**Phase 1.** The first phase is the network of concepts and procedures related to the study of the relation between rates of growth (ROG) and factors of multiplication (FOM) over different periods in a model of population growth. The categories that make up the network for the first phase are *category one:* the elementary relation between the ROG and the FOM. *Category two* of Phase is the multiplicative relationship between the FOM for multiple periods of the unit of multiplication and the corresponding ROG. *Category three* is the relationship between the FOM over a fraction of the unit period of multiplication and the corresponding ROG. The last, *category four* of this first phase is the relationship between the FOM and the ROG when the reverse action of the growth process is studied that calculated the previous population from the existing population.

**Phase 2.** The second phase is the network of concepts and procedures related to the calculation of decimal exponents, the Decimal Exponents Calculation process (DEC).
It brings together a network of three categories. *Category one* studies roots as multiplicative fractions of the unit of multiplication or the FOM. *Category two* brings together the procedures of chains of multiplications, remainders (explained in the section on DEC) and roots. *Category three* is about the relations between decimal exponents and exponential equations.

<table>
<thead>
<tr>
<th>LESSON</th>
<th>CATEGORIES OF TEACHING</th>
<th>PHASE of ANALYSIS</th>
</tr>
</thead>
<tbody>
<tr>
<td>ONE</td>
<td>FOM- ROG</td>
<td>Phase 1</td>
</tr>
<tr>
<td></td>
<td>Multiples – Fractions</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Inverses</td>
<td></td>
</tr>
<tr>
<td>TWO</td>
<td>Decimal Exponents Calculation (DEC)</td>
<td>Phase2</td>
</tr>
<tr>
<td></td>
<td>· Multiplicative Fractions</td>
<td></td>
</tr>
<tr>
<td></td>
<td>· Chain of Multiplication</td>
<td></td>
</tr>
<tr>
<td></td>
<td>· Exponential Equations</td>
<td></td>
</tr>
<tr>
<td>THREE</td>
<td>Rational Actions</td>
<td>Phase 3</td>
</tr>
<tr>
<td></td>
<td>Rational Exponents</td>
<td></td>
</tr>
<tr>
<td>FOUR</td>
<td>Zero Exponents</td>
<td>Phase 4</td>
</tr>
<tr>
<td></td>
<td>Negative Exponents</td>
<td>Phase 5</td>
</tr>
<tr>
<td></td>
<td>· Graphical Symmetry</td>
<td></td>
</tr>
<tr>
<td></td>
<td>· Exponential Equations</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.4. Lessons, categories and phases in the teaching experiment
**Phase 3.** The third phase is the network of rational exponents and rational actions on roots and powers of roots. Rational actions are relations that mimic the operation of ordinary fractions. There are two rational actions on roots and powers of roots referred to in this study. First, if $a$ is any positive number, and $n, m, p$ and $q$ are positive integers, then $\sqrt[n]{a^m} = \sqrt[np]{a^{mp}}$. This relation mimics the operation $mp/np = m/n$ known from ordinary fractions. The second rational action on radicals and powers of radicals is the relation: $\sqrt[n]{(a^p)} = (\sqrt[n]{a})^p$. This relation mimics the rational operation from ordinary fractions that states that $(p/n) = p * (1/n)$. These rational actions form the basis for justifying that $\sqrt[n]{a^p} * \sqrt[m]{a^q} = \sqrt[nm]{a^{nm+qm}}$, which again mimics the operation of ordinary fractions for addition of fractions. Phase 3 has two categories: **category one,** the conversion of roots and powers of roots into rational exponents, and **category two,** analogies between properties of roots and powers of roots on the one hand, and rational actions on multiplicative fractions on the other.

**Phase 4.** The fourth phase is the network on the zero exponent. This phase has two categories. The first category is on connections between the zero exponent and rational exponents. The second category is about the connections between decimal and zero exponents.

**Phase 5.** The fifth phase is the network of negative exponents. It brings together three categories: **category one,** the graphical symmetry between $F(x) = a^x$ and $G(x) = (1/a)^x$, **category two** on exponents and directions and **category three,** solving exponential equations involving negative exponents.
Network of Phases

In Figure 5.3 the network of the five phases of teaching and learning leading up to the (possible) encapsulation of the exponent concept is displayed. This encapsulation is pictured in the last box on the right, with the description of "general concept of exponents." That is not to say that all conceptual development is one directional. It is assumed that higher order development deepens the understanding of lower order concepts. Phase 1 represents the basis with the central notion of factors of multiplication and rates of growth. Phase 2 has the decimal exponents' calculation process and is tied to Phase 3 with rational actions on radicals and roots. The combination of the two phases serves to raise awareness of the nature of fractions used as exponents and the accompanying requirements to make such a connection. Phase 4 represents the zero exponent concepts that causes so much loss of understanding in students. Phase 5 is directly connected to Phase and the general concept because it had specific qualities and requirements. The notions of movement and symmetry had been incorporated in this phase to reinforce intuitive understanding of why negative numbers make sense even with exponents and the CDE. The encapsulation is not guaranteed after the five phases. It depends upon the students' reflective abilities, knowledge and motivation.
Phase 1. Rate of Growth and Factor of Multiplication

In this section I analyze the learning process of Phase. First a diagram displaying Phase and its network is presented. The learning process of Group One is first discussed, illustrated by Table 5.5, with the characteristics, main features and a few short quotes from each student displayed per category in rubrics. The features and quotes are organized to convey in the best possible way the evolution of the researcher's perspective on the thinking process of the individual students and their efforts to understand the concepts under investigation and discussion. After the presentation of the table, also called a case-ordered matrix (Miles & Huberman, 1994), a summary is given of the students' activities and learning steps in this phase. The section concludes with a description of the learning process of Group Two.
Figure 5.4 presents the network of the four categories that constitute Phase. Three categories are related to the study of the relations between the ROG and the FOM in a model of population growth with fixed rates of growth over equal periods. The fourth category is related to the reverse step of division of Factors of Multiplication. The problems the students encountered are from lesson 1 and lesson 2. The worksheet problems are from activity 1.1-1.4 and 2.1-2.2 in Appendix C.

The relations between a fixed rate of growth (in decimals) per unit period and the corresponding factor of multiplication per unit (in decimals) was

$$FOM = 1 + ROG.$$  

The relation between the ROG and the FOM over multiple periods was

$$FOM \text{ (for } n \text{ periods)} = (FOM \text{ for 1 period})^n \text{ and } ROG \text{ (for } n \text{ periods)} = (FOM)^n - 1.$$

---

Figure 5.4: Phase 1. Exploring the connection between the FOM and the ROG

Phase 1 – Group One. In Table 5.5 a summary of responses or actions per student in Group One is displayed for each of the categories in the process of studying the ROG
and the corresponding FOM. The rubrics correspond to the categories for the first phase. It became clear during the teaching that students needed to experience the concept of ROG and FOM under different conditions to understand their interrelationships and in particular the non-linear nature of such connections.

The "Elementary connection" rubric in the table contained not only the students' understanding of the basic connection FOM = 1 + ROG, but also the performance on the graphical application of the ROG on the population applied repeatedly and constructed manually to create discrete points of the exponential graph. The purpose of this step was to link the graph to the non-linear properties of the ROG. This construction also prepared the student for the next inquiry into the relation ROG – FOM for multiple periods.

The rubric in the table for understanding the relation between FOM and ROG for multiple periods was the category on the non-linear relations between the ROG and the FOM. All the students in the teaching experiment had problems understanding that there was no additive or linear connection between the factors and the rates. The non-linearity of the exponential graph also posed problems for every student in the teaching experiment although in different forms.

Partitioning a period of growth and finding the ROG and the FOM was the subject of the next category and rubric in the table. Similar obstacles occurred here for students trying to understand how the relation between ROG and FOM worked under these partitioning conditions. In the rubric for the fourth category the student responses are summarized for the reverse operation of division and how ROG and FOM were then related.
<table>
<thead>
<tr>
<th>Elementary Connection</th>
<th>ANN</th>
<th>BOB</th>
</tr>
</thead>
<tbody>
<tr>
<td>ROG-FOM</td>
<td>Voices meaning of 25% growth. FOM= 1 + ROG Constructs graph of population over 3 periods. Curve not straight.</td>
<td>Unfamiliar with &quot;ROG = 25% year after year.&quot; Accepts FOM = 1+ ROG. Graphs straight line for population over 3 periods. Attempts construction of graph of population over 3 periods with ROG =1/3.</td>
</tr>
</tbody>
</table>

| ROG-FOM for multiple periods | Repeated application of FOM → exponential formula with variable exponent. ROG over 2 periods = 56.25% "Shouldn't it be 25% squared?" | Repeated application of FOM. Transfer to general variable for exponent not obvious. ROG over 2 or more periods constructed manually in graph. (ROG =1/3) Takes two trials and two explanations of the procedure for success. Linear thinking: ROG over 2 periods = 50%. |

| FOM-ROG for fractions of period FOM = 2/3 hrs What is the FOM/hr and the ROG /hr? | FOM=1.333 "The amount of bacteria increases by 33% each hr " "But it does not come out to be 2." Well, I knew it would be 1 point something, something… (1. ---) because we're still doing the current population plus the rate of growth. (1.333)^3 \neq 2 | FOM = 2/3 = 0.6666. Linear model for population growth. After accepting the 25.99% ROG per hr, Bob wonders why the constant number means something different than with his linear model. |

| FOM-ROG in reverse calculations | ROG = 25% then in reverse you need to subtract 1/5 to get back to original number. FOM (for inverse action) = 4/5 = 0.8 Result is a surprise! "Haven't seen it before" | If 2500*1.25 = 3125 then 3125/(1.25) = 2500 (1/1.25) * 3125 = 2500 "Why have 1 over 1.25?" |

Table 5.5. Phase 1 – Group One: Exploring the relation ROG – FOM
The data collected for Ann suggested that she understood the elementary connection between the rate of growth (ROG in decimals) and the factor of multiplication (FOM): \( \text{FOM} = 1 + \text{ROG} \). She was able to represent the relationship in a graph and apply the ROG manually in that graph to obtain successive new populations over 3 periods. The creation of a generalized formula for a population over a variable period of time posed no immediate problems to her. However the concept of the ROG over multiple periods and its actual calculation was more challenging.

For multiple periods in the graph the concept of a ROG over such a multiple period did not seem to be as elementary as the first connection \( \text{FOM} = 1 + \text{ROG} \). Ann needed time to construct the notion that the ROG (in this instance 25% per period) over multiple periods did not follow a linear model. The question: "…shouldn't it be 25% squared?," suggested that the two concepts of FOM and ROG were not yet sufficiently clear, and not fully understood in their functions and properties and how they fitted into the calculation of the population and the subsequent calculations of the rates of growth over time. Ann used an exponential process to find the new population and seemed to assume that the same model also worked for the ROG.

The study of the exponents continued with the question of what the FOM and the ROG per hour would be if there was a doubling in bacteria every 3 hours. The awareness was there to try a form like: \( 1 + 0.333 \). This is partly reminiscent of the elementary connection between ROG and FOM. The 0.333 for the ROG suggested however a linear division of the ROG of 100% (= 1.00) divided equally by the 3 hours necessary for the doubling to occur.
Ann tried various options to see which one worked and which one produced the doubling over 3 periods. Eventually she found the solution by using the cube root of 2. This realization came after discussions with Bob, trying options such as the improbable solution of simple multiplication, the use of numerical examples, and the insight of Ann that exponents were still involved.

The calculation of the population for the previous period, using the concepts of FOM and ROG was new to Ann. First she tried the sequential non-reversed method of going back to figure out the initial population and then use the formula for exponential growth to calculate the population of the previous year.

The intention of the problem was to make the students aware of the possibilities of the FOM in the form of its inverse. The graphical construction of the calculation worked well for Ann and pointed the way to the solution of the FOM and the ROG. The algebraic calculation linking the given FOM to its reciprocal to find the previous population eventually became clear to her.

**Bob.** Bob had no problems connecting the ROG and the FOM in their most elementary form. However the concept of 25% rate of growth for Bob required several attempts before reaching a minimum of understanding. Graphing the 25% growth over multiple periods was also a challenge. Applying the FOM repeatedly did not present a real obstacle for Bob, but developing that relation into a general formula did pose a problem. Bob's steps from repeated multiplication to the general algebraic formula with exponents needed support from the teacher.

Bob had a greater conceptual area to cover than Ann. The more complex relation between the FOM over multiple periods and the corresponding ROG over those multiple
periods needed a substantial shift in Bob's mind from a linear to a multiplicative model. The transcripts suggested that Bob was adding the ROG's when trying to find the ROG over multiple periods. This was not different from the initial thinking of Ann who also tried linear models but with more multiplicative properties associated with the ROG. Bob needed to sort out how to graph with multiple applications of the 25% growth, why the shape of the graph was a non-linear one and how the multiplicative aspects come into play when dealing with both FOM and ROG for such multiple periods. The use of numbers in addition to graphs seemed to trigger better understanding for Bob in developing a basis for the multiplicative nature of the process.

The calculation of the FOM and the ROG per hour, when there was a doubling of a population of bacteria every 3 hours, also went in the direction of proposing linear models first (just like Ann had tried). In contrast to Ann, Bob had more moments of not knowing what to do next than Ann. He tried linear models and rejected them during the discussion, and at some point, after Ann stated that they were on the wrong track with trying to use equal parts (which was essentially the linear model) he drew the conclusion that factors of multiplication did not apply here either. He saw a contradiction between the parts that cannot stay the same, and the ROG that must stay the same. After studying the proposed $3\sqrt[3]{2} \approx 1.2599$ he was able to accept the solution.

**Summary for Ann and Bob in Phase 1.** For both students the elementary connection between the Rate of growth in percentages and the Factor of multiplication as a number was accepted without much effort. However the study of the actual function and properties of both the FOM and the ROG required more mental constructions and more contexts. Both students needed more exposure to problem situations to construct
meaningful notions of what the FOM and the ROG could do and how they fitted into the exponential forms.

There are three areas where the FOM and the ROG function in distinct ways:

First, in the context of multiple periods (in the population problem), that required the insight that powers of the FOM are involved resulting in non-linear ROG's.

Second, in the context of fractions of a unit period and the corresponding FOM and ROG that involved both powers of an unknown FOM and the resulting root taking action again with its new FOM and ROG, producing parts of the initial unit of multiplication.

Third, in the calculation of a FOM for the reverse action of finding a previous population, given the present one. This involved the use of the reciprocal of the Factors of Multiplication and the understanding that division can be conceptualized as multiplication by a factor that is the reciprocal of the regular FOM.

The steps Ann needed to understand the full meaning of the FOM were different from the steps Bob needed. While both started out with the assumption of a linear model for the FOM and ROG, Ann grasped the multiplicative role of the FOM much earlier and clearer than Bob. Ann stated that we need the form FOM = 1 + something as a tool to calculate the FOM per hour when we know that the FOM per 3 hours is 2. That realization of the role of the FOM did not come without recurrences of some remnants of the earlier linear model. The linear mode was visible when Ann used simple division to propose as the FOM/hr the form FOM= 1 + 0.333, using the ROG of 100% and dividing that by 3, to obtain 0.333 and eventually a FOM /hr of 1.333.

Bob did not mention the separation of the FOM into 1+ ROG, and also used linear techniques to propose a solution for the FOM/hr. He used the FOM = 2 per 3 hrs to
divide that into 3 equal parts and tried FOM/hr = 2/3 = 0.666. At one point Bob proposed
the idea of finding a number that raised to the third power would be 2. He used the words
"working backwards" for his proposition. His next observation, however, was that the
number was "less than 1," suggesting a linear concept still at work in his mind.

After our discussions in the sessions leading to a FOM ≈ 1.2599/hr, Bob needed more
explanation to understand that the process was not an additive process.

The third area of reverse application of the FOM had similar differences between
Ann and Bob. Both used an earlier familiar formula to go from P (10) first to P (0) and
then to P (9), and only used the reverse FOM after discussions with the instructor. Ann
understood the reciprocal connection, but Bob felt uncomfortable with the reciprocal idea
and the fractions. More discussions using numerical examples made this part less
challenging for Bob.

Phase 1 - Group Two. In this section the connections between the ROG and the
FOM are studied through various problem situations. The data for the second group were
analyzed as emerging from the video and audio recordings and transcripts of all the
instruction, the worksheets, the three interviews and the test questions. The reduced data
on Chandra, Dennis, and Eddy are summarized in Table 5.6 and relevant quotes are
provided to build our argument for the implications and conclusions, and answering the
research questions. The five phases, used for analyzing the results of the Teaching
Experiment for Ann and Bob are identical for Chandra, Dennis and Eddy. The format of
presentation is similar to the format of the Group One with the exception of the diagrams
of the five phases, which we do not show again.
The study of the FOM under different conditions suggested a pattern of three stages for understanding the relation of the FOM to the ROG. *First* an elementary understanding of the connection FOM = 1 + ROG, followed by a *second* stage in the understanding of the relation FOM and ROG for multiple periods or fraction of a period. When applied to either multiple or fraction periods, linear formats of solving the problems appeared as the first choice with all the students.

While Dennis appeared to be solving correctly the question of the calculation of the FOM over one third of the unit of time (3 weeks), with a cube root of 2 for the FOM over 1 hour (which lead to a 26% ROG over one hour), he still did not have a clear picture of the ROG. Dennis thought that the 26% ROG per one hour should equal 100% after cubing it. He was puzzled by the fact that \((0.26)^3\) does not come even close to 100%! On the other hand, his quest to understand what caused the 100% increase reflected his inquiry into the structure of the problem.

The reverse application of the FOM was the *third* stage in the process of understanding the relation ROG to FOM. The insight how to find the relation when the inverse of the FOM was needed did not seem to be obvious to the students. All the students failed to find the corresponding ROG when going backwards. Several trials with numbers clarified the nature of the relation FOM – ROG for a reverse calculation. Eddy showed problems with finding the FOM over fractions of the period and for multiple periods. He also tried linear relations to solve both the multiple and the inverse problems of the relations FOM to ROG.
<table>
<thead>
<tr>
<th><strong>Elementary Connection. ROG-FOM</strong></th>
<th><strong>CHANDRA</strong></th>
<th><strong>DENNIS</strong></th>
<th><strong>EDDY</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>FOM = 1 + ROG Construction of pop graph over 3 periods.</td>
<td>FOM = 1 + ROG Curve non-linear because of unequal &quot;rise&quot; for equal &quot;run.&quot;</td>
<td>FOM = 1 + ROG</td>
<td></td>
</tr>
<tr>
<td>Curve not straight line. Discussion of why that happens.</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

| **FOM-ROG for multiple. periods**  | **Aware of the role of FOM.** | **FOM solved first with linear model.** | **FOM over multiples is like exponent.** |
| **Multiplicative. connection**     |                              |                                        |

| **FOM-ROG for fractions of period** | **First model for FOM/hr:** | **First model FOM/hr = 1.333. Curve between 3hrs straight line (admits using linear model here). Line only straight between the 3hr period.** | **Expected FOM per 1 hr to be 33.3%. Linear model of process.** |
| **FOM = 2/3hrs**  | =1.333?  |  |  |
| **What is the FOM/hr and the ROG/hr?** | Curve is not a straight line. |  |  |

| **FOM-ROG in reverse calculations** | **To find P(9) given P(10) and the FOM = 1.05,** | **For P(9) out of P(10), and FOM = 1.05. After discussion, use of inverse FOM.** | **Use of formula to find solution. No inverse operation as a first step.** |
|  | "Go to P(0) first then use formula to find P(9)." After discussion: inverse FOM suggested by Dennis. |  |  |

Table 5.6. Phase 1 – Group Two: Exploring the relation ROG-FOM
Phase 2. The Decimal Exponent Calculation Process

In this section, I will discuss first the organization of Figure 5.5 for phase 2 and the content of Table 5.7 that summarizes responses and discussions of Ann and Bob for this part of the teaching experiment. The rubrics of Table 5.7 are the categories of Phase 2. Summaries of the students' work and learning will conclude this part. Because of the central place of the Decimal Exponents Calculation process (DEC) in this teaching experiment, I will discuss that part separately. The worksheets for Table 5.7 and Phase 2 conform to Student activities 3.1-3.6: Solving exponential equations with decimals and applying the DEC process (Appendix B).

In lessons two and three the concepts of decimal exponents' creation and calculation were taught and discussed in conjunction with rational actions on radicals and the process of conversion of such actions into rational exponents. The teachings of lessons two and three are closely related and that is reflected in the close connection between Phase 2 and Phase 3 (see the network of Figure 5.3).

Figure 5.5 represents the network of Phase 2 with three categories of the teaching experiment analysis centered on the concepts and procedures of the decimal exponents' calculation process. The network shows the instruction and learning activity of the students around multiplicative fractions, chains of multiplications, and the process of finding digits of decimal exponents as a special activity to study rational exponents.
The categories of Phase 2 are:

*Category one:* Roots and multiplicative fractions of a unit. One way to build awareness of the structure of rational and decimal exponents was to change the way students think about roots (see left box in Figure 5.5). In this teaching experiment, roots of radical exponent \( n \) were conceptualized as a way to partition numbers multiplicatively into \( n \) equal parts. If \( X^3 = 5 \) then \( X \) was thought of as a multiplicative one-third of 5 and \( X = \sqrt[3]{5} \approx 1.709975947 \). These "fractions" could then be raised to any (positive integer) power. By using numerical examples the concept could be made concrete. However if the student did not understand the regular fractions very well, this part of the teaching could pose problems.

*Category two:* Chains of multiplication. Numbers were "factored" with one specific unit until the remaining quotient was less than the unit but still more than or equal to 1 (see middle box in Figure 5.5). For example 65 = 4 * 4 * 4 * 1.015625 or 65 =
3 * 3 * 3 * 2.407407… the numbers 1.015625 and 2.407407… were called (multiplicative) remainders. Students needed to become acquainted with this form of factoring as a prelude to the decimal exponent calculation process and to become aware that factoring can have more than one form! Bob (and Eddy from the second group) had some problems understanding this part of the teaching, wondering how a number can be equal to the chain of multiplications.

Category three: The decimal system of exponents; exponential equations. What are decimal exponents and how are they calculated? For the success of this part of the teaching experiment (see right box in Figure 5.5), students with well developed concepts of general multiplications, including multiplication with decimals, could do much better than students with outstanding gaps in their multiplication schemas. The application of calculators in this phase was essential.

Phase 2 – Group One. Table 5.7 summarizes the responses and quotes of Ann and Bob. The rubrics are the categories of Phase 2 and they are centered on the decimal calculation process. The main summary and analysis for Ann in this phase is discussed first including the analysis of the DEC format developed by Ann, followed by a similar summary and analysis for Bob.

Ann. Ann worked effortlessly with radicals of higher order (see Table 5.7) and understood quite quickly the concept of thinking of roots as (multiplicative) parts or fractions of the number under the radical sign. When the idea of factoring with chains of multiplication in terms of one unit was discussed, Ann showed no major obstacles in her learning of the method. The concept of multiplicative remainder was understood with ease.
The more complicated transition to the practice of using decimal roots when the remainder was less than the unit was also grasped quite easily. I interpreted these relatively easy transitions to a new approach as suggesting that the new notions did not present cognitive obstacles to Ann and that she found sufficient reasons to link these steps to existing notions in her schemas for multiplication and radicals.

<table>
<thead>
<tr>
<th></th>
<th>ANN</th>
<th>BOB</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Roots as fractions of a unit of multiplication.</strong></td>
<td>Ann has no apparent problems with the concept of multiplicative fractions</td>
<td>Multiplications with fractions and decimals represent challenges to the student.</td>
</tr>
<tr>
<td></td>
<td>√25 = 5 = &quot;1/2 of 5&quot;</td>
<td>The usual regular concept of a fraction goes like this: Suppose you have the number 5, and I ask you to divide this into 4 equal parts, what would you do?</td>
</tr>
<tr>
<td></td>
<td>3√5 = 1.7099 = &quot;1/3 of 5&quot;</td>
<td>Bob: I don't understand the question.</td>
</tr>
<tr>
<td></td>
<td><strong>Additive and multiplicative fractions</strong></td>
<td>Bob: Use the number 25 to describe the concept of half of 25 (multiplicatively).</td>
</tr>
<tr>
<td><strong>Chains of multiplication</strong></td>
<td>No major obstacles during these stages</td>
<td>Chains of multiplications require extra discussions</td>
</tr>
<tr>
<td></td>
<td>I: Suppose you have the number 35 and you divide it by 5. So you are going to factor it in units of 5</td>
<td>35 = 5<em>5</em>1.4;</td>
</tr>
<tr>
<td></td>
<td>Take the number 35 and divide it by 5. (Writes 35/5 at the left hand side of the board) This is equal to?</td>
<td>Bob: Heh? (Bob very surprised by the process).</td>
</tr>
<tr>
<td></td>
<td>Ann: Seven.</td>
<td>If I have the fourth root of 5 (Writes 4√5 ≈ 1.1.4953)</td>
</tr>
<tr>
<td></td>
<td>I: (Writes 35/5 = 7) I call 7 my remainder.</td>
<td>Bob: (Gives all the decimals of this root of five)</td>
</tr>
<tr>
<td><strong>Remainders</strong></td>
<td></td>
<td>I: If I apply this 4 times (Writes 1.4953…x 1.4953…x 1.4953…x 1.4953)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Bob: It's basically like multiplication and division! Cause when you do this, like 5 x 5 = 25 you see what I mean? (Points to the writing on the board that states 25/5 = 5)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Bob: Heh. Division is the opposite of multiplication?</td>
</tr>
</tbody>
</table>

Table 5.7. Phase 2 – Group One: The Decimal Exponents Calculation process
Table 5.7 continued

| Decimal exponent calculation | With calculator, no major obstacles at this stage
| Exponential equations | Ann: Are you saying one-tenth of 5 are equal to 1.1746189…?
| | Chains of multiplications with tenth and hundredth roots as factors
| Focus is more on operational side of the process
| - what kind of operation is involved
| - how multiplication and division are connected
| - How 35 can be equal to the long chain of multiplication

The quotes from Table 5.7 suggest that Ann did not have serious obstacles to using the concept of fractions of a unit for roots. Solving the exponential equation $7^X = 100$, she kept referring to the fractions as the roots, suggesting that at this stage the verbal form of "roots" was easier for her. Ann's format for solving the equation contained all the major aspects of the process: the decimal roots, the remainders, the method to find remainders, and the resulting last remainder that gives an idea how close the process came to the actual value of the unknown number $X$.

**Bob.** Bob had difficulties with percentages, fractions, and the shift from linear thinking to the new context of multiplicative models. The DEC process offered him the possibility to think about fractions, percentages and chains of multiplications in one system. Bob had to cover a considerable "conceptual distance" in terms of procedures he needed for a successful application of the DEC process, but which were not completely encapsulated by him. He still managed to develop a system for DEC and apply it successfully to exponential equations. While Ann had developed a well-rounded system of handling the DEC and her format suggested a structural level of understanding Bob's
format was more operational and less structural. For Ann there were few hurdles about fractions, percentages and chains of multiplication. Bob had to overcome his weaknesses with fractions and decimal multiplications before his DEC process could crystallize into a structural whole.

**Student generated formats for the Decimal Exponents Calculation process (DEC).** Because of the important role of the DEC process in the teaching experiment for understanding rational exponents category three related to the DEC process is discussed in some detail here. One of the core claims of the conjecture in this study was that if students engage in actual construction of rational exponents, they learn such exponents better and with more meaning. The constructive project or context should be part of an intuitively plausible context, with decimal exponent's construction as the major form of rational exponent's learning. The conjecture was that through that process, students learned to connect their early notions of the CDE more meaningfully with the expanded concept of exponents that allowed for rational, decimal, negative and maybe (later) even real numbered exponents.

The general strategy to reach this goal was to let the students calculate step, by step, decimal exponents using the root- calculating facility of the calculator combined with the power function and divisions. This process involved the application of the decimal place value system, interpreted for exponents and the contemplation of the factor called "the remainder." This process of actively calculating decimal exponents and observing the way the "remainder" tended to move closer and closer to 1.000000… could lay a foundation for understanding exponents in all their forms.
The DEC format developed by Ann in the teaching experiment. Ann created a format for the DEC process (Figure 5.6) that was organized to show all the main features of the process. First the Chain of multiplication was visible on the left side of her work in Figure 5.6, then the decimal roots of the unit 7 are displayed with the successive "remainders" of the repeated division process. The emerging decimal exponents are clearly displayed in an orderly style. Finally she included explanations of both the decimal powers and the reasons for shifting from one decimal root to a higher order decimal root, because of the decreasing remainders.

The DEC format developed by Bob. The activity from Bob in Figure 5.7 was from the second week of teaching, recorded as student activity 3.3 from the worksheets. The
question was to solve the equation $3^A = 10$ and show that $A \approx 2.0959$. Bob had to use the repeated division method, combined with the root taking procedure.

![Figure 5.7: The Decimal Exponents Calculation format by Bob](image)

The model used by Bob has two columns: one column (on the left) for the calculations, including the various roots of 3, the decompositions of 10 into (multiplicative) units of 3, divisions and "remainders." The area on the right is for the factored form of 10, with all the decimal exponents explicitly stated.

There were similarities and differences between the formats used by Ann and Bob: Both have the chains of multiplication and the "decimal" roots as smaller units of multiplication. Both document their "remainders" and exhibit the actual calculations based on the "remainders." Both have remainders that show a tendency to get close to
1.000… The differences are in the organization of the components of the process. The "decimal" roots of 3 were not separately listed; the explanation for the remainder as related to the powers of the decimal roots was not present, and the verbal explanation of the decimal exponents was not documented by Bob.

**Phase 2 - Group Two.** Every student in Group Two developed some representation, or model for bringing together the elements that they saw as important to solve the given exponential equation. The three models all produced the decimal exponent 2.366… for the equation $100 = 7^X$. In all models of the students the outcome of the first step with integer exponents set the stage for the next calculation with decimal exponents. All used the calculator because of the high "decimal" roots that have to be produced and are needed for the rest of the calculations. Each student was able to explain her or his method for finding the decimal exponents.

**Differences between DEC formats presented by Chandra, Dennis, and Eddy.**

The method used by Eddy is the most "basic" of the three models. The model presented by Dennis is more elaborate with a compact form of writing the "remainder" as a division between the original number 100 and the intermediate exponential form such as $7^{2.3}$. The method of Chandra was the most structured and complete in the sense of bringing together all the important elements of the calculation and reserving space for all the parts of the method. In the presentation of Eddy the words "division or divided by" occurred more often than with Dennis or Chandra. Eddy's decimal exponents emerged step by step from the calculation process, and the shift of the remainder towards the value 1.000… was not hard to notice.
CHANDRA | DENNIS | EDDY
--- | --- | ---
**Roots as fractions of a unit of multiplication.** | $\sqrt[10]{5} \approx 1.1746…$ as one-tenth of the unit 5. | No noticeable problems with the concept; seems to connect with the notation $5^{\frac{1}{3}}$. | Multiplicative fractions in context accepted. Similarity to roots. |
**Additive and multiplicative fractions** | Comfortable with notion and sequence. Well developed organized system of handling chain of multiplication, remainders and roots. | No signals of confusion or discomfort about the process. | Initial focus on the individual numbers more than on process. Needed assistance to shift attention. |
**Chains of multiplication** | **Roots** | **Remainders** |
**Decimal exponent calculation** | Applied her method correctly in the second week. Develops an integrated system of dealing with both roots and powers of the base unit. | Initial confusion about the method; Unable to apply the method in the second week.(see quote). Developed sequential system of handling calculation of digits. | Unable to fully apply a procedure to calculate the decimal exponents in the second week of instruction. Developed sequential model for finding exponential digits. Similarity with Dennis's model. |
**Exponential Equations** | | |

Table 5.8. Phase 2 – Group Two: The Decimal Exponents Calculation process

In terms of the characteristics of the explanations of the three students, the impression was that Eddy had not internalized the process completely yet. Dennis had more overview, but less concrete information in his model. The remainders were not visibly recorded and the powers of "decimal" roots were indicated by notes on the right side of the form. The decimal exponent form was clearly stated at each stage. Thus it
seemed that Dennis had a more developed, process-oriented concept of the calculation of decimal exponents than Eddy. Chandra seemed to have progressed to a stage that was almost completely structural. The remainders, the roots, the powers, chains of multiplication, all these elements were present in her diagram.

**Phase 3. Rational Exponents and Rational Actions on Exponents**

In this section I analyze the main features of the learning activity of the students in Phase 3 starting with the network and categories of Phase 3, followed by a summary of the learning activities for both students.

Figure 5.8 presents the network. The central issue is the verification by the students of the rational actions on radicals and the conversion of radicals into rational exponents. Phase 3 is closely connected to Phase 2, because they both deal with rational exponents. After the introduction of the DEC process the students verify further what rational exponents are and how they are justified. Phase 3 has two categories discussed below.

Phase 3 has two categories: **Category one: Rational actions on radicals**. The phrase rational actions as used in this study, refers to the following basic operations with roots and powers, needed to build a foundation for the conversion of roots and powers into rational exponents: 

\[ p\sqrt[n]{a^q} = n^p \sqrt[n]{a^{np}} \], for \( a > 0, p, q, n \) positive integers. This corresponded to the rational exponents that claimed that 

\[ a^{q/p} = a^{np/np} \].

The second root relation was 

\[ (p\sqrt[n]{a})^m = p\sqrt[n]{(a^m)^n} \], for \( m \) positive integer; this relation corresponds to the exponential form that defined uniquely the rational exponent \( a^{m/p} \) as being independent of the order of the operations of roots and powers.
Category two: Conversion of the rational actions into rational exponents. The second issue for this phase was the verification by examples that some properties of radicals mimic the regular fractions and that there was a way to use the equalities to create exponents. We had the relation \( p\sqrt[n]{a} \times q\sqrt[m]{a} = pq\sqrt[nq+mp]{a} \), where n, m, p and q were all positive integers. If the roots or radicals were written as parts of a unit whole, and interpreted through the lens of exponents, and for example \( 3\sqrt[3]{5} \approx 1.7099 \ldots \) was viewed as \( 5^{1/3} \) then the statement \( p\sqrt[n]{a} \times q\sqrt[m]{a} = pq\sqrt[nq+mp]{a} \) could be read as \( a^{n/p} \times a^{m/q} = a^{(nq+mp)/pq} \).

Figure 5.8: Phase 3. Rational exponents and rational actions

Phase 3 – Group One. Both students used the calculator extensively to help them calculate higher order roots quickly and for verifying equalities with roots and powers. Ann (see Table 5.9) went further than just verifying equalities with the calculator: she also tried to explain properties of roots that form the basis for the rational actions. While Ann tried to incorporate fractions and roots into one system of calculation and powers,
Bob did not show explicit use of the concept of multiplicative fractions. For both students the use of numerical examples that could be verified on the calculator worked, while the first attempt to use algebra made the teaching less lively (and probably less successful).

<table>
<thead>
<tr>
<th>Verifying basic rational actions on roots and powers</th>
<th>ANN</th>
<th>BOB</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt[p]{a^m} = (\sqrt[p]{a})^m$</td>
<td>Roots converted into factors and grouped</td>
<td>Used the calculator to verify if the equality holds.</td>
</tr>
<tr>
<td>rational exponent:</td>
<td>Moving powers from under the radical sign making it into a power of the root.</td>
<td>Attempts at numerical level.</td>
</tr>
<tr>
<td>def. of $a^{m/p}$</td>
<td>Attempts to find conceptual explanations (see quotes).</td>
<td>No preference voiced for $(\sqrt[p]{a})^m$ compared to $\sqrt[p]{(a^m)}$.</td>
</tr>
<tr>
<td></td>
<td>Clear preference for $(\sqrt[p]{a})^m$ compared to $\sqrt[p]{(a^m)}$.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Verifying basic rational actions on roots and powers</th>
<th>ANN</th>
<th>BOB</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p\sqrt[n]{a^q} = np\sqrt[n]{a^q}$</td>
<td>Regrouping of factors to explain relationship combined with minimal calculator use.</td>
<td>Used the calculator to verify.</td>
</tr>
<tr>
<td>rational exponents:</td>
<td>(see quotes)</td>
<td></td>
</tr>
<tr>
<td>$a^{q/p} = a^{nq/np}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Verifying basic rational actions on roots and powers</th>
<th>ANN</th>
<th>BOB</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p\sqrt[n]{a^m} \times q\sqrt[n]{a^m} = \sqrt[n]{a^{(nq + mp)}}$</td>
<td>Numerical verification with calculators.</td>
<td>Numerical verification.</td>
</tr>
<tr>
<td>rational exponents:</td>
<td>Clear concept of fractions of the unit.</td>
<td></td>
</tr>
<tr>
<td>$a^{n/p} \times a^{m/q} = a^{(nq + mp)/pq}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5.9. Phase 3 – Group One: Rational Actions and Rational Exponents
One quote (from interview three) is included here to illuminate Ann's style of reasoning:

*Ann* (explaining \(16\sqrt{5} = 6\sqrt{5^3} = \sqrt{5};\) the starting point is that \(a\) is a number such that \(a*a*a*a*a*a = 5^3\)):

If you split it up into 2 parts then each one is 2.236 (She writes \(2\sqrt{5}\) and indicates with gestures on the board that 2 small arrows together make 5 and each is then 2.236) So when you have \(5^3\) and you split this up into six parts then \(5^3\) is equal to six of the parts of 2.236. And you can show that, like the first thing that came to my mind and I looked here (Points to the statement \(6\sqrt{5^3} = 2.236\)) and I took the sixth part of both sides.

If we take eight 5's (meaning \(5^8\), and she draws an arrow from 0 to 8) this divides into 16 parts, then we have 16 parts of these small parts, (gestures to indicate the two parts that make 5) which is equal to \(5^8\)!

Ann seemed to transfer her grouping knowledge with little problem into the world of multiplicative fractions. In other words the multiplication was turned into a simple shuffling of factors. Bob did not show similar ease of reasoning. He needed concrete numbers to be convinced of the validity of the steps.

**Phase 3 - Group Two.** The way rational actions on radicals and powers of radicals mimic the rational properties of regular fractions was discussed in Phase 3 of the teaching experiment. There are three distinct approaches in the way the three students of Group Two understand the rational actions involved in exponents (see Table 5.10).

Chandra had a better awareness than Dennis or Eddy of the need to justify the existence of rational exponents and their rational properties by going back to radicals and their properties. Eddy relied on his calculator to verify if radical properties really held, while Dennis relied on fractions to explain other rational properties of exponents and ignored or was not aware of the need to anchor his reasoning in the radicals. Dennis and Eddy had not yet reached the stage of justifying their reasoning on rational exponents in the general
properties of radicals. Eddy used the calculator to verify his notions, while Dennis was not aware of the limitations of his method of using fractions that had not been established fully as exponents. The difference in the type of mathematical habits stemming from the high school environment and the college environment may explain this difference in attitude for Dennis. Eddy was probably still forming his basic concepts of radicals and their properties.

<table>
<thead>
<tr>
<th></th>
<th>CHANDRA</th>
<th>DENNIS</th>
<th>EDDY</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rational actions</strong></td>
<td><strong>with roots and</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>powers</strong></td>
<td>$\sqrt[7]{5^3} = 14 \sqrt[6]{5}$ uses powers of roots to explain relations</td>
<td>$3 \sqrt[5]{5} \times 5 \sqrt[3]{5} = \cdots \sqrt[5]{(5 \cdots)}$? Uses exponential notation of fractions to explain relations $\sqrt[5]{3^2 \times 8 \sqrt[7]{3^7}} = 40 \sqrt[4]{3^{51}}$ Uses fraction notation of exponents to explain relations. No roots or powers. Root and power relations are derived from the fractional exponents.</td>
<td>Uses the same method as Dennis to find the missing radical exponents. His explanations however are carried out with the calculator. He verifies the truth through numerical examples. Verbal explanations: he uses &quot;multiplicative fractions&quot; supported by calculator verifications.</td>
</tr>
<tr>
<td><strong>Corresponding</strong></td>
<td><strong>forms with</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>exponents</strong></td>
<td>$5^{\frac{3}{7}} = 5^{\frac{6}{14}}$</td>
<td>$5 \frac{1}{3} \times 5 \frac{1}{5} = 5 \cdots$ $3 \frac{2}{5} \times 3 \frac{7}{8} = 3 \frac{51}{40}$</td>
<td>$4\sqrt[3]{5} = 12\sqrt[5]{5}$ ....... $(5^{\frac{1}{2}})^{\frac{1}{4}}$ (see quote)</td>
</tr>
</tbody>
</table>

Table 5.10. Phase 3 - Group Two: Rational Actions and Rational Exponents

**Phase 4. The Zero Exponents**

Phase 4 centers on the zero exponent. The zero exponents were studied from the perspective of a (very small) fraction of a unit. The value for exponents zero is derived from the corresponding FOM of the unit with the very small exponents. The zero
exponents were also discussed as a place holders device for the zero as in "no factors of a certain decimal root."

Figure 5.9: Phase 4. The zero exponents

The network of Phase 4 in Figure 5.9 has two categories. **Category one:** The zero exponent and its connections to rational exponents. In the teaching experiment the zero exponent was constructed from the intuitive content of zero. The (intuitive) approach to zero avoided the abstraction that 0 = 1-1 or 0 = 3-3. This abstraction did not always make sense. For example, if we ask the question: How many persons are in this house? Who would think of saying 5-5? **Category two:** zero exponents and the decimal exponents. The decimal exponent structure offered an opportunity to actually have a context where the absence of a factor was one motive for using the zero and for arguing that the value 1 (one) is plausible.
In this section the zero exponent and its connections to rational exponents and the decimal place value system for exponents were examined. Summaries of the students' activities are presented. Because of the special nature of the zero exponent and the idea that abstract approaches may not work for some students the experience from the teaching experiment is discussed here in some detail.

**Phase 4 – Group One.** Ann and Bob reacted in different ways to the teachings on the zero exponent and its connections. Bob had struggled with the idea of zero as a place holder and after some explanations by Ann, the researcher posed some questions to him to observe and record his reactions. Ann followed an explanation linking the zero exponents to higher decimal roots of the unit and smaller fractions of the multiplicative unit with decreasing ROG's that lead to a FOM that got closer and closer to 1.0000000, with an increasing sequence of zero digits. Ann understood that the implication of decreasing intervals was a ROG that would be zero and a FOM that corresponded to multiplication by 1. Bob was still constructing his understanding of various parts of the DEC process, focused mostly on numerical aspects of zero exponents based on higher order roots. This was understandable and could be interpreted as his way of making sense of multiplicative fractions. Table 5.11 displays Bob's numerical insight that the higher decimal roots come closer and closer to one for all the examples that he worked on.

**BOB.**

Q: What is the meaning of this 2 here Bob? (I. points to the digit 2 in the calculations discussed by Ann showing how to solve the equation $3^A = 10$. She finds $3^{2.095} = 9.935$).

Bob: Two three's.

I: Two three's yes. What about the zero here?

Bob: Zero tenths!

I: What about the 5 here?

Bob: Five one-thousandths.
Zero Exponents and Rational Exponents

Smaller fractions of the unit of multiplication = higher roots
Lead to FOM = 1 and ROG = 0
Constant FOM = 1 means a straight line
Conceptual level of understanding.

Numerical understanding of FOM = 1 with higher roots.
No explicit connection to ROG
No explicit graphical connection

Zero Exponents and Decimal Exponents

Zero exponent represents absence of factors;
Used when remainder is less than decimal root
Zero must be a digit in the decimal system

Zero factor; is it necessary?
When do you use it?
Connects to a remainder less than a decimal root.

Table 5.11. Phase 4 – Group One: The Zero Exponent and its connections to Rational Exponents

The Decimal Exponents Calculation process also used the zero digits as an exponent, but now in the strict sense of "absence of certain factors." Bob had some questions as to what this zero meant in the context of calculating decimal digits. This form of the zero is tied to the "remainder" calculations that drop under the decimal root that follows in the place value system. This usage of the zero exponents had connections with the CDE. The place value approach to the zero exponents was an alternative perspective to the zero exponents perspective derived from multiplicative fractions.

Phase 4 - Group Two. In the next paragraphs the students' discussions are investigated individually to understand how they explained or justified that a number raised to the zero exponent was equal to one (1). The discussions refer to a question from
the teacher to use a (positive) fraction (< 1) as a FOM and argue that even then, the rational exponents still point to one (1) as the best value for the zero exponent.

*Eddy.* This student was uncomfortable with a fraction, so he used an integer as the starting point. The concept of the number zero that the students seemed to use was one that came close to the intuitive notion of the absence of quantity ("nothing," something very, very small). With his calculator Eddy seemed to note that the "decimal" fractions (decimal roots) of the number 2 keep getting closer and closer to 1.0000000…. 

*Dennis.* The notion Dennis seemed to have of the reason that a number to zero exponent is one, had graphical elements, a sense of limits and a certainty about the actual value that stems from his high school years. It was unclear how he justified the value 1 with concepts related to the definition or the operation of exponents. Dennis gave an explanation that was unrelated to the zero exponent (see Table 5.12).

*Chandra.* Chandra showed a reasoning that even for fractions the zero exponent must correspond to one: $(0.75)^0 = 1$. She used radicals, the notions of (multiplicative) fractions of a given or chosen FOM, the calculator and graphical arguments for her case. Chandra indicated:

As this exponent gets closer and closer to zero, the numbers get closer and closer to 1, and what this means is that this FOM comes from a period divided into 10 pieces (she works with decimal roots); This FOM (for the 1/100 division) comes from an interval divided into 100 pieces and as we get smaller and smaller intervals (she points to the numbers) the number…the rate of growth between this point and this point," (she indicates on
the board what she talks about, by keeping her fingers close together to indicate a short interval) becomes so small it almost appears to be a straight line from one point to the next. This number here (she points to her final note that reads $0.75^0 = 1$) is the FOM if the rate of growth is zero!

**The zero and the decimal exponent process.** In the procedure for calculating the digits of the decimal exponents, the zero comes up naturally when the remainder is too small to contain even one (1) factor of a certain decimal root fraction. Table 5.12 shows that although all students eventually mastered the main aspects of the DEC process, Dennis and Eddy encountered obstacles when they tried to apply the zero digit to the calculations. For Dennis this was due to his initial tendency to use approximations instead of divisions to find the digits of the DEC process. His method was mostly calculator-based, with no systematic connection to exponents. Any function could have been the subject of analysis, because of the calculator-based procedure that used trial and error to find the digits, with the exception only of the first integer step. When he reduced through trial and error the estimates for the third decimal exponent, he reached the value 1 (trying to solve the equation $6^B = 100$) and still did not find the correct estimate. This caused some confusion. How he finally discovered that he needed to insert a zero is still unclear. Maybe a continuation of the trial and error process explained the last step. However there was no indication that his reasoning was connected to the structure of the exponent. For Eddy, the zero-digit presented difficulties because he was still constructing his notion on the process of the DEC process and admitted that the function of the zero digit was unclear to him.
Table 5.12. Phase 4 – Group Two: The Zero Exponent and its connections to Rational Exponents

**Chandra.** The quotes below from Chandra are on the use of the zero exponent as a place holder. In her system, the zero fits well with the DEC process. She did not elaborate on the interpretation of the role of the zero-place holder as a de facto factor of 1(one). Chandra (solving the equation $6^B = 100$ on the board):

\[
100 \approx 6^2 \times 6^{5/10} \times (1.134023029\ldots) \\
\approx 6^{2.5} \times 6^{7/100} \times (1.00034) \\
\approx 6^{2.570} \times 6^{1/10000} \times (1.000) \\
\approx 6^{2.5701}
\]

Ok! $6^B = 100$; we know that $100/6^2 = 2.7777$ the multiplicative remainder is 2.77777…

Then I found out all the tenth, hundredth and thousandth roots of 6 first. (All the roots are neatly written in a table on the lower left side of the board).

Then 2 whole sixes can come out and then 5 tenths of 6 can come out and it
leaves this (waves with her hand around the remainder, 1.134023). And then you have 7 (copies of) one hundredths can come out and leave that (1.0003484…). (Points to her next line of calculations) "And then you have zero one-thousandths of six (still leaving 1.0003484…).

I: Maybe you should explain the zero over one-thousandth!
Ch: I just did that to keep myself from being confused, to keep the places all together.
I: But why do you have the zero there?
Ch: Because this (she points to her list of roots of 6, the one with $\sqrt[1000]{6} \approx 1.001793366$…) is more than this the one point zero, zero, zero, three (Points now to the remainder in her calculations 1.0003484); you can't divide it out, so you have to go to a smaller unit!

The students' connections from rational and decimal exponents to the zero exponent. Because the zero exponents take a peculiar place in the system of exponents and the understanding of the zero exponent is so diverse for different students, an attempt was made to explain what could have happened in the teaching experiment with the study of the zero as an exponent. A possible explanation why the outcomes for the three students are different could be that Chandra built connections between the radicals, the (multiplicative) fractions, and the shrinking exponents, while supporting her argumentation with graphical images directly related to the "shrinking" exponents and linking the FOM and ROG to the initial value and the new value. These connections gave her reasoning plausibility and force. In Figure 5.10 the possible different routes of concepts and connections taken by Chandra on the one hand and Dennis and Eddy on the other are visualized.
Dennis initially avoided all radicals and proposed a line of reasoning that was not convincingly related to the zero exponent. Eddy studied the radicals numerically and used his calculator to build a case for the zero exponents. It is unclear if he fully understands the connections between the decimal exponents, the properties of radicals, the ROG and the FOM. Eventually all were able to apply the DEC process but the role of the zero exponents in this process may not be entirely clear to Dennis and Eddy.

**Phase 5. Negative Exponents**

In the teaching experiment, the justification for negative exponents was approached from three related directions represented in Figure 5.11 by the three lower
boxes. The first box on the left represents the idea of symmetry that results from graphing exponential functions with base numbers that are inverses of each other: For example $F(x) = (1.25)^x$ and $G(x) = (0.8)^x$. The idea of inverses of Factors of Multiplication, divisions by Factors of Multiplication, and graphs like those of functions $F(x)$ and $G(x)$ that show symmetry are combined with the idea that negative numbers can be thought of as "directed" numbers indicating some reverse operation. These notions are represented by the box in the middle. The method of solving exponential equations involving negative exponents is represented by the box on the right. The categories in Phase 5 associated with these boxes are described below.

![Figure 5.11: Phase 5 - Negative exponents](image)

**Category one:** Graphical symmetry of functions with inverse bases. Special attention was given to units of multiplication and their inverses and their associated functions, $F(x) = a^x$ and $G(x) = (b)^x$ with $b = 1/a$. The inverse relationship produced symmetrical graphs that intersected at $x = 0$. The symmetry was used to clarify the connections between inverses and negative exponents.
Category two: Why exponents have "direction." All students adopted the metaphor that in a graph the division by a unit of multiplication acts like a backward movement along the graph, with the backward movement most conspicuous on the x axis. This was then compared to the directed numbers that students seem to remember from earlier math classes.

Category three: Solving exponential equations with negative exponents. The power of the notion of symmetry between graphs related through the inverse of their units of multiplication or base numbers was presented and applied to the problem of solving exponential equations. An equation like \((0.8)^X = 10\) can be solved by using the symmetry of graphs and first solving \((1.25)^Y = 10\) using the DEC process. 1.25 is the inverse of 0.8 and the DEC process can now be used to find \(Y \approx 10.3\). Based on the symmetry we find \(X = -10.31\).

Phase 5 – Group One. Ann understood how to use the symmetry of the "inverse "graphs to solve the equation \((0.8)^X = 10\) by solving first for \((1.25)^Y = 10\) (Table 5.13). From the examples from the worksheet it seemed that Ann made the association between division and "backward movement" on the graph. There were no signs from Ann that she had problems with this section of the teaching. The notion that division by a number is similar to multiplying by the inverse of that number and that you move in the positive direction with the exponents when multiplying and backwards when you divide, seemed to resonate with Ann.

Bob's responses in Table 5.13 suggested that he did not fully grasp how to use the equation \((1.25)^Y = 10\) to solve \((0.8)^X = 10\). What he seemed to understand and accept as meaningful was the notion of backward movement associated with negative exponents.
He coined the phrase "Factor of Division" for the backward action, using the terminology of "Factors of Multiplication." Bob used concrete examples with few verbal explanations when illustrating his understanding of the relation between division and the inverse of the FOM.

<table>
<thead>
<tr>
<th></th>
<th>ANN</th>
<th>BOB</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Graphical symmetry between curve with a unit of multiplication and the curve with inverse unit of multiplication</strong></td>
<td>For this group the symmetry between the graphs and their inverse counterparts was not extensively discussed.</td>
<td>Symmetry not discussed extensively in this group.</td>
</tr>
<tr>
<td><strong>Why exponents have &quot;direction&quot;</strong></td>
<td>Positive exponents indicate multiplication by the unit; negative exponents indicate division by the unit. Division is similar to multiplication by the inverse of the unit. Graphically you move in the opposite direction.</td>
<td>To move forwards and backwards in the multiplication process. Negative signs should be interpreted as divisions.</td>
</tr>
<tr>
<td><strong>Solving exponential equations with negative exponents</strong></td>
<td>To solve ((0.8)^x = 10), You solve ((1.25)^y = 10) instead. Find (Y) and then take its opposite, (X = -Y) (1/1.25 = 0.8) (inverse FOM)</td>
<td>Divide 10 by 1.25. The next step was unclear.</td>
</tr>
</tbody>
</table>

Table 5.13. Phase 5 – Group One: Negative Exponents

**Summary for Ann and Bob in Phase 5.** The use of the metaphor of movement forward or backward seemed to have made an impression on the students. The notion was used repeatedly when explanations of negative exponents were involved. This could be an instance for arguing that if the (formal) mathematical practice of eliminating all
references to movement in formulating symbolic operations was adopted by mathematical education, this could sever or block meaningful learning options.

**Phase 5 - Group Two.** This part covers the symmetry of graphs with base numbers that were inverses of each other, the connection between division, inverses, the "backward movement" of the exponent through division, and the method of solving exponential equations involving negative exponents by using symmetry. Chandra made the connection between the symmetry and the inverse relation for the base and used that to solve exponential equations with negative exponents. The notion of movement associated with exponents was accepted by her. During most of the teaching Dennis vacillated between adopting new methods and ideas and staying with his knowledge as formed in high school. He accepted the notion of negative exponents as indicating movement because of the reverse counting when making repeated divisions, but his thinking was close to relying strongly on rules and formulas.

The most visual representation of negative exponents explained with movement symbols was created by Eddy (see Table 5.14). He also noticed the opposite relation between the solutions of the equations $(0.8)^x = 10$ and $(1.25)^y = 10$ where $x = -y$ before the other students.
<table>
<thead>
<tr>
<th></th>
<th>CHANDRA</th>
<th>DENNIS</th>
<th>EDDY</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Graphical symmetry between curve with a unit of multiplication</strong>&lt;br&gt;and the curve with inverse unit of multiplication</td>
<td>Noticed the symmetry and tried to use that in calculations.</td>
<td>Did not base his reasoning on the symmetry of the graphs, but tried to use his earlier knowledge. Kept thinking that the connection between the graphs was only the inverse base.</td>
<td>Noticed symmetry early in the teaching.</td>
</tr>
<tr>
<td><strong>Why exponents have &quot;direction&quot;</strong></td>
<td>Associated with inverses and backward movement.</td>
<td>Also backward movement. Kept his high school knowledge intact.</td>
<td>Used the metaphor of movement most consistently with complete diagrams. Representing even zero exponents as &quot;no movement&quot; exponents (see third interview).</td>
</tr>
<tr>
<td><strong>Solving exponential equations with negative exponents</strong></td>
<td>Found the connection between symmetrical graphs and the equation.</td>
<td>After trial and error accepted the symmetry as an easy way to solve equations using inverses.</td>
<td>First to connect the symmetry of the two graphs to a method to solve exponential equations with negative exponents.</td>
</tr>
</tbody>
</table>

Table 5.14. Phase 5 – Group Two: Negative Exponents

**The Interviews During and After the Teaching Experiment**

At the end of the second week of the teaching, Interview Two was conducted with the students. A second interview, Interview Three, was conducted after the third week of teaching, followed by a third interview, Interview Four, after the last sessions were concluded. The interviews (Appendix D) had a semi-structured format with the students.
providing written responses to problems and questions and then giving some verbal comments on their solutions. Most questions were answered with written responses or solutions. Sometimes a student did not respond or did not know how to respond to a question. All the questions were related to the instruction from the teaching experiment. The purpose of the interviews was to obtain additional data on the learning activities of the students and information on the effects of the teaching experiment.

**Triangulation of Data**

Another way of looking at interviews is through the lens of triangulation of data gathered during the qualitative investigation. So far the data collected came from the following sources: First, the interviews with novice and expert students. Then, video recordings made of the teaching experiment sessions and activities. From the recordings transcripts were made and tables were constructed to present reduced data for analysis. The worksheet activities also provided additional data on the learning activities of the students and the apparent evolution of the teaching experiment. Interviews Two, Three and Four provided additional opportunities to increase the trustworthiness of the research and the conclusions derived from the data.

The questions in the interviews were identical or almost identical for all students with some minor variations in the discussions after the written parts. The questions are in Appendix D and the descriptions presented here apply equally to the second group of students (Group Two). Therefore, they will not be listed again in the analysis of the Group Two.

In the next section the three interviews of the teaching experiment are described. The results of these interviews are presented in the following format. First the results of
Ann are discussed in detail with a display in a table. The rubrics for the tables are the questions of the particular interview. A summary of Bob, Chandra, Dennis and Eddy for each of the three interviews is presented after the results for Ann.

**Interview Two with Ann.** Questions 1 and 3 from Interview Two in Table 5.15 cover the problem of the ROG, the FOM and the mechanism for finding the ROG and the FOM over multiple periods. Question 2 tests the student's ability to visualize the application of the ROG and the FOM over multiple periods through graphical construction. Question 4 is related to the need to emphasize the role of the FOM, now working in reverse, or backwards, using division or using the inverse of the FOM. It is also a step to prepare the students for negative exponents. Question 5 continues to build on the need to focus on Factors of Multiplication in the form of radicals that capture another aspect of the FOM, such as the notion of (multiplicative) fractions of a given unit or FOM. Question 6 provides an opportunity for the student to practice her skills in finding decimal exponents and practice the application of the Decimal Exponent Calculation.

The transcripts of the sessions of the teaching experiment suggested that Ann went through various steps to master the relation between the ROG and the corresponding FOM. The method to calculate the ROG for multiple periods was not immediately clear to her. Ann's first attempt (Question 3) to find a ROG over 2 periods, starting with a ROG of 20%, was the idea to use the ROG as a FOM. Initially she tried to calculate the ROG over 2 periods by using 20% x 20% as the new rate of growth over 2 periods, but the responses to the questions 1 to 3 in Table 5.15 suggest that in a second
trial Ann knew how to use the FOM instead of the ROG to solve the multiple period Rate of Growth question.

For the calculation of a ROG over one-fifth of a period of 5 weeks (= a 1-week timeframe) Ann's work suggested, that she knew how to find the ROG using the fifth power of the 1-week unknown FOM and equating that to the given FOM over 5 weeks (see Figure 5.12). Her DEC-calculations were accurate and orderly executed as can be read from her response to question 6.

<table>
<thead>
<tr>
<th>QUESTIONS</th>
<th>ANSWERS</th>
</tr>
</thead>
</table>
| 1. Explain in words and mathematical language what the 20% increase per 5 weeks means. Explain how to calculate the number of infected persons 5 weeks from now if today there are 250 infected persons. | Every 5 wks the number of infected persons increases by 20% of the number of infected persons from the previous 5wks.  
250 + (0.20)(250) = 300 infected persons. |
| 2. Make a graph of the number of infected persons using the 20% increase to construct the number of infected persons step by step. Use a ruler and blank paper. Be very precise. | Diagram shows constructed vertical segments based on a ROG of 20% per 5 wks. New vertical segment = previous segment + 20% of previous vertical segment.  
15 wk-period: (1.2)^3 = 1.728 ⇒ 72.8% |

Table 5.15. Interview Two with Ann
### Table 5.15 continued

<table>
<thead>
<tr>
<th>Question</th>
<th>Calculations and Explanation</th>
</tr>
</thead>
</table>
| 3. What is the rate of increase of the number of infected persons over a 10 week period. | ROG over 10wk period:  
First try: ROG = (0.20)*(0.20)  
Second try: FOM : (1.20)^2 =1.44  
\(1+0.44 \rightarrow \text{rate of growth}\)  
\(250 + x(250) = 300\)  
\(250(1+x) = 300\)  
\(1+x = 300/250 = 1.2\)  
x = 0.20  
- 15 wk period: (1.2)^3 = 1.728  
\(\Rightarrow 72.8\%\)  
- Over 20 wk period:  
  (1.2)^4 = 2.0736  
  \(\Rightarrow 107.36\%\) rate of increase. |
| Over a 15 week period. | Number of infected persons 1 wk earlier was 1075/ (1.02) = 1053.92.  
(Factor of multiplication!) |
| Over a 20 week period. | Calculation of rate of increase per 1wk.  
Use of fifth root of 1.2.  
(see Figure 5.12) |
| 4. If we have 1075 infected persons today, what was the number of infected persons 5 weeks earlier? | Find an approximate number |
| 5. How can we find the rate of increase per 1 week? | Explain the steps and concepts we need to find the answer |
| 6. Suppose we have a unit of multiplication 4.  
We want to know what the measure of 35 is in terms of (multiplicative) units of 4.  
Carry out the calculations and explain what the steps mean. | \(4^4 = 35\)  
\(10\sqrt{4} =1.1487\)  
\(35 = 4*4 *2.1875 (= \text{remainder})\)  
\(2.1875/(1.14875)5 = 1.09375\)  
\(35= 4^2 * 4 ^{5/10} * 1.09375\)  
\(\sqrt[100]{4} = 1.01396\)  
\(1.0937/(1.01396)6 = 1.00645\)  
\(35= 4^2 * 4 ^{5/10} * 4 ^{6/100} * 1.00645\)  
\(35 \approx 4^{2.56}\) |
Interview Three with Ann. The second interview during the teaching experiment, Interview Three, consisted of the nine questions displayed in Table 5.16, with the answers, and inquired about the students' interpretation of rational and decimal exponents.

Ann was able to present a meaningful explanation of the decimal exponent, by invoking the decimal roots. Her explanation of a decimal power of another decimal power used the concept of FOM on the unit $3^{0.7}$, arguing that she has 0.2 of such groups. By "de-encapsulating" the concept back to its radical forms with only integer exponents and decimal roots, she explained the multiplication of the powers. In Question 3 Ann again used radical properties and integer exponents to justify the equality.

She was also able to explain the repeated multiplication from question 4 as the repeated multiplication of $1.25^{1/100}$, repeated 35 times. She seemed to be aware of the
fact that the repetition was either of the "whole unit" or of a part of the unit. Question 5 was equally well explained with the repeated part explicitly stated. Ann's use of the word "parts" suggested that her reasoning with parts in a multiplication context was no hindrance to her.

Question 7 asked: How can we show why this (the multiplication of radical exponents) works?

Ann replied: \[ \left( \frac{1}{5} \right)^{1/3} \] Fractional exponents take parts of the unit of multiplication." This suggests that Ann was actively thinking in terms of fractions of units of multiplication as a way to explain the behavior of radicals. Ann produced a perfect explanation to Question 9 on the DEC, to explain what the decimal exponent meant for the solution of the exponential equation. Her retention of the DEC process seemed to hold so far. In her response to Question 8 she first erred, but quickly corrected herself and produced the right radicals and exponents.

Based on Ann's lucid explanations and justifications for rational actions and decimal exponents the inference could be made that she was reaching understanding of rational and decimal exponents. She was actively using the language of units and parts of units and her verification of exponential laws for decimals or rational numbers came without many confused reasoning steps.
<table>
<thead>
<tr>
<th>QUESTIONS</th>
<th>ANSWERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. What do we mean by the symbol $3^{0.7}$? Describe two ways to approach</td>
<td>$3^{7/10}$</td>
</tr>
<tr>
<td>this symbol.</td>
<td>$10\sqrt{3}$</td>
</tr>
<tr>
<td>(Different forms to interpret the decimal or rational exponent.)</td>
<td>$(10\sqrt{3})^7$</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>2. What is the meaning of the symbol $(3^{0.7})^{0.2}$? Use the language of</td>
<td>$3^{0.7}$ “that’s your unit of multiplication, it’s your group, you could</td>
</tr>
<tr>
<td>factors of multiplication to explain the concept.</td>
<td>say, and you have 0.2 groups...So that’s why you could look at it that</td>
</tr>
<tr>
<td></td>
<td>way. If you want to write it out, you’re taking the $10\sqrt{3}$ of that</td>
</tr>
<tr>
<td></td>
<td>to the 2nd power and you can bring this outside....</td>
</tr>
<tr>
<td></td>
<td>So $3^7$ and then you are taking the $\sqrt{3}$ twice, so you can multiply</td>
</tr>
<tr>
<td></td>
<td>this together. So it’s pretty much gonna look like 14 over 100, which is</td>
</tr>
<tr>
<td></td>
<td>$3^{0.14}$.</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>3. How do we know that $5^{2.4} = 5^2$? Explain why we can’t simply</td>
<td>$10\sqrt{5^{24}} = 5^2 \ast 10\sqrt{5^4}$</td>
</tr>
<tr>
<td>decompose the exponent into 2 + 0.4. Assume that $5^{2.4}$ means $\sqrt[10]{5^4}$.</td>
<td>...if you take the $10\sqrt{5^{10}}$ it is 5, Ok.</td>
</tr>
<tr>
<td></td>
<td>And if you do that twice, you get $5^2$ so you can take $5^{20}$ out,</td>
</tr>
<tr>
<td></td>
<td>because that would be equal to $5^2$ you used in multiplying $5^4$</td>
</tr>
<tr>
<td></td>
<td>And it looks like this side, so it shows why the exponents are added.</td>
</tr>
<tr>
<td></td>
<td>$5^2 \ast 10\sqrt{5^4}$</td>
</tr>
</tbody>
</table>

Table 5.16. Interview Three with Ann
### Table 5.16 continued

<table>
<thead>
<tr>
<th>4. The concept of exponents like $(1.25)^5$ involves Repeated Multiplication.</th>
<th>Repeated multiplication with and without integer exponents: $1.25^5 = 1.25 \times 1.25 \times 1.25 \times 1.25 \times 1.25$ $100\sqrt[100]{(1.25)^{35}} = (1.25)^{1/100} \times \ldots \times (1.25)^{1/100}$ Difference is in the unit: &quot;Whole units&quot; (integers) versus &quot;Multiplying a part of the unit.&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>How can we argue that concepts like $(1.25)^{0.35}$ also involve Repeated Multiplication?</td>
<td>How exactly is Repeated Multiplied? How is the Repeated Multiplication in $(1.25)^{0.35}$ different from the Repeated Multiplication in $(1.25)^5$?</td>
</tr>
<tr>
<td>What exactly is Repeated Multiplied. How is the Repeated Multiplication in $(1.25)^{0.35}$ different from the Repeated Multiplication in $(1.25)^5$?</td>
<td></td>
</tr>
<tr>
<td>5. Take the form $\left(\frac{(1.20)^{1/5}}{5}\right)^5$ and compare it with a form like $\left(\frac{(1.20)^{1/5}}{5}\right)^5$. Describe both in terms of the Repeated Multiplication Model.</td>
<td>$(1.20)^{1/5} \times (1.20)^{1/5} \times (1.20)^{1/5}$ $(1.20)(1.20)(1.20)$</td>
</tr>
<tr>
<td>6. How is the concept of &quot;multiplicative fraction&quot; related to multiplication? Use the unit 5 and the &quot;division of 5 into 10 equal parts&quot; to illustrate your explanations.</td>
<td>5 divided into 10 equal parts. &quot;… if you multiply those ten parts together, it equals 5.&quot;</td>
</tr>
<tr>
<td>7. Radicals behave somewhat like ordinary fractions. For example: $\sqrt[4]{\sqrt[3]{3}} = 12\sqrt[5]{5}$. Which radical exponent is correct? How can we show why this works? How is this related to our concept of rational exponents?</td>
<td>Explaining rational actions on radicals: Why is $\sqrt[4]{\sqrt[3]{3}} = 12\sqrt[5]{5}$? Units of multiplication.</td>
</tr>
<tr>
<td>8. Another fraction-like behavior of radicals is this: $\sqrt[5]{3^2} \times \sqrt[8]{3^7} = 3^{2/5} \times 3^{7/8}$ Explain why this is related to rational exponents.</td>
<td>$\sqrt[5]{3^2} \times \sqrt[8]{3^7} = 3^{2/5} \times 3^{7/8}$ Ann needs more hints to find the right explanation why and how this should be done this way. How to describe and coordinate the rational actions of the radicals?</td>
</tr>
<tr>
<td>9. When we solve equations like $80 = 5^x$ we find $X = 2.722$. We could say that 5 fits into 80 about 2.722 times. Explain this approach to exponents.</td>
<td>Interpretation of the solution for the equation $80 = 5^{2.722}$.</td>
</tr>
</tbody>
</table>
**Interview Four with Ann.** The eight questions in Interview Four (see Table 5.17) dealt with issues from the last week of the teaching experiment and topics covering the whole teaching experiment. They were designed to reveal what the students understood of the subjects as a whole. Question 1 tested understanding of how a fixed rate of growth leads to the repeated multiplication model of growth of a population. Question 2 explored her understanding of the method to solve exponential equations that require negative exponents. Question 3 is similar to question 2 but now with a concrete example to solve. Question 4 invited the student to make an attempt to create a definition of exponents covering integer, fraction and negative exponents in one notion. Question 5 examined the concept of factors of multiplication. Question 6 examined the student's concept of "multiplicative fractions" and their relation to multiplication and division when the exponent was a negative decimal. Question 7 continued the exploration of negative decimal exponents now for negative powers of negative powers and Question 8 was the graphical application of a fixed rate of growth of 20% per time unit, with the stipulation to construct the next value working in two opposite directions: from 0 to + 4 units ahead, then from 0 to - 4 units (backwards). The backward construction required an inverse construction, finding the previous population one time unit earlier.

Ann proposed a definition of the concept of exponents (Question 4 in Table 5.17):

"How many parts of your unit of multiplication you are using in the multiplication and indicating the direction of the number exponent." The inclusion of the phrase: "How many parts" revealed a more refined concept of the Factor of Multiplication or the unit of multiplication. This use of the word "parts" could be interpreted as an indication that Ann had reached a sufficient level of confidence to voice a definition, where the Common
Definition of Exponents had been modified to accommodate the more complex process of rational exponents. The use of "direction" indicated a similar confidence for bringing positive and negative exponents under one roof. All the other questions were answered correctly, with a possible exception of the inverse construction for question 8. In her graph (see Figure 5.13) she used a forward construction only. The reverse construction is unclear. Maybe she limited herself to writing that a division by the factor 1.20 is involved without actually constructing the previous value in the graph.

<table>
<thead>
<tr>
<th>QUESTIONS</th>
<th>ANSWERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Explain why multiplication and division are interconnected in the Exponential function. Show how a rate of growth of say 5% for every year leads to a Repeated Multiplication Model and how multiplication and division are interconnected in this context.</td>
<td>Multiplication makes the unit bigger. &quot;Repeated division is used with negative exponents and makes the unit number smaller and smaller in the opposite direction.&quot; See Figure 5.13.</td>
</tr>
<tr>
<td>2. Suppose we need to solve (0.8^x = 10). Why can’t we solve this with positive exponents? How is this problem related to question 1</td>
<td>&quot;When you repeatedly multiply fractions, the number gets smaller and smaller. In order to get an answer of 10 in the above equation, you need to use the inverse of 0.8. This can be done by using negative exponents.&quot;</td>
</tr>
<tr>
<td>3. Suppose we replace the 0.8 in question 2) by its inverse, 1.25. Explain how the equation (1.25^y = 10) can help us solve the equation from 2)</td>
<td>We can use positive exponents with the equation (1.25^y = 10)</td>
</tr>
<tr>
<td>4. How would you formulate a definition of exponents that covers both the integer case, the rational case and the negative case?</td>
<td>Definition of exponents: How many parts of your unit of multiplication you are using in the multiplication and indicating the direction of the number exponent.</td>
</tr>
</tbody>
</table>

Table 5.17. Interview Four with Ann
Table 5.17 continued

<table>
<thead>
<tr>
<th>5. Why is the factor of multiplication so important in studying rates of growth or decrease in exponential situations?</th>
<th>The factor of multiplication not only includes the rate of growth, but it also accounts for the population of the previous unit of time.</th>
</tr>
</thead>
</table>
| **6. Suppose you have** $(5)^{-0.6}$ **what would that mean in terms of the factor of multiplication and in terms of the definition?** | $(5)^{-0.6} \quad 5^{-0.6/10}$  
$(10\sqrt{1/5})^6$  
factor of division =  
(takes a step back)  
$=1/$factor of multiplication |
| **7. How would we have to interpret a form like** $((5)^{-0.6})^{-2.3}$ **in terms of the factor of multiplication and in terms of the definition?** | $((5)^{-0.6})^{-2.3}$  
$((10\sqrt{1/5})^6)^{-2.3}$  
$((10\sqrt{1/(10\sqrt{1/5})})^6)^{-2.3}$ |
| **8. Graph the exponential model for a 20% increase per unit over 4 such units in both directions** | (see graph in Figure 5.13) |

Figure 5.13: 20% increase over four units (Ann)
I have illustrated the method of analysis in detail for Ann. The content of the questionnaires is similar for all the other students. The organization of the (intermediate) interviews with the other four students follows the same format. From this point on, the results of the interviews will be reported in the form of summaries.

**Bob. Interview Two:** Bob did not fully understand at that time how to construct the repeated application of 20% increase per unit time in a graph. The response to Question 3 also suggested that at the time of the interview Bob did not understand fully how to find the ROG over two periods other than through linear operations of direct additions. From the responses to Questions 1, 2 and 3 there is an indication that the use of concrete numerical examples helped Bob to overcome unsupported conclusions about the ROG over multiple periods. Reverse application of the FOM was not addressed by Bob in this interview. However the DEC process was applied resulting in a two decimal accuracy for the solution.

**Interview Three:** Bob had problems explaining why $5^{2.4} = 5^2 \times 5^{0.4}$ using only definitions and the properties of radicals. He professed not to understand the question. His interpretations of decimal powers of decimal powers were partially correct as he tried to write all his decimal exponents as decimal roots.

The double roots appearing in the radical forms (questions two and five) seemed to confuse him. He did explain the repeated multiplication for decimal powers correctly indicating that it was the decimal roots that were repeatedly multiplied. His explanation of the meaning of decimal exponents was correct, indicating that the DEC process did make sense to him.
Explaining one rational action on radicals was solved numerically correct, but only partially carried out with radicals only. Bob used some mechanical fraction steps to find his final radical form. This form of reasoning I interpret as combining "surface" forms of reasoning with actual mathematical steps. Bob used radicals first to argue that 
\[ \sqrt[5]{3}^2 \times \sqrt[8]{3}^7 = (\sqrt[5]{3})^2 \times (\sqrt[8]{3})^7 \] and then used fractions to finally show that this gave \( 3^{51/40} \).

**Interview Four:** Bob did not reach a point where he was able to propose a definition that could cover both integer and rational exponents. Bob mentioned properties of exponents instead of focusing on the common elements of exponents. He was not able to respond to questions 5, 6 and 7 of the interview. He did propose the use of symmetry of graphs to solve exponential equations that required negative exponents. Bob also explained that the solution was the opposite of the corresponding equation with the inverse as the unit of multiplication.

**Chandra. Interview Two:** Chandra applied the ROG and the FOM correctly, constructed the repeated application of the ROG in a graph and showed proficiency when applying the DEC process. The reverse application of the FOM was not obvious. Chandra used the sequential approach to solve the reverse calculation problem (Finding population of a previous year).

**Interview Three:** Chandra used decimal roots to explain decimal exponents consistently. She also actively used the language and meaning of "parts of the unit" to explain the mechanism of decimal and rational exponents. When addressing the repeated multiplication question she stated not only that parts were repeatedly multiplied but included the refinement that smaller and smaller parts were repeatedly multiplied, accounting for the system of (exponential) number representation beyond integers.
Explaining rational actions with radicals took time but she was able, with some hints, to solve problems correctly. Like all the other students the DEC explanation for decimal exponents did not pose a problem.

*Interview Four:* Chandra explained the symmetry of the inverse graphs precisely and solved the exponential equation with negative exponents. Her interpretation of rational powers of rational powers was flawless. Most importantly she proposed a definition for exponents covering all number types from integer to rational and negative exponents: "*An exponent is the direction, (factor) size, and length of growth of multiplication of a given period.*" The definition is not perfect, but in essence acceptable because it can be interpreted for all rational, decimal and negative numbers.

*Dennis. Interview Two:* Dennis was able to explain the FOM given the ROG. The application of the FOM to calculate populations over multiple periods was no problem for Dennis. His work was well reasoned. He correctly applied reverse thinking to the calculation of the population one year earlier, given the ROG.

*Interview Three:* The decimal exponents were neatly explained with decimal roots, but when he had to show the derivation from question 3 without using fractions and with references to the basic properties of radicals he was unable to separate mechanical fraction manipulations from radical operations. Questions 7 and 8 presented problems for Dennis related to his lack of knowledge how mechanical fraction operations on exponents are related to their origin in radicals.

*Interview Four:* The most important new element is his definition of exponents: "*An exponent indicates repeated multiplication, division or a series of operations involving radical roots.*" This description was correct in a general sense. It reflected
elements treated in the teaching, like repeated multiplication, division, and radicals.

There was no mention of a unit of multiplication or of parts of the unit as important in the multiplication process. There is no reference to an actual exponent in this description. The conclusion could be that Dennis tried to bring various concepts of exponents together but was clearly not ready to tie all elements together in an overall concept. Nevertheless he did not mention a variant of the Common Definition of Exponents as his proposal.

Eddy. Interview Two: Eddy used graphs to explain what a 20% increase means for a population. When calculating the ROG he used numerical examples to understand the ROG and the FOM. His first experiences with the DEC process were difficult but the concrete numerical examples seemed to help him carry out the calculations.

Interview Three: His interpretation of decimal powers of decimal powers and repeated multiplication when there are decimal powers were to the point and correct. He used the language of parts of factors, but verified all statements with the calculator. This behavior suggests that his algebraic knowledge of radicals may be less solid than it needed to be. Eddy translated the meaning of decimal powers as multiples of successive decimal roots without hesitation.

Interview Four: Eddy explained negative exponents in the given equation through symmetry and inverses. His understanding of the construction of the repeated application of the 20% increase in a graph was still vague for him. He preferred to calculate the numbers than construct the graph. The numbers were correct.

Eddy's definition of exponents was: "A number that can be used as in the repeated multiplication step or by using the reciprocal method of a negative number and then complete the problem." Eddy probably had operations in his mind when he tried to
formulate his concept of an exponent. His concept of exponents was close to the CDE in terms of the "number that can be used in repeated multiplication" and because his concept of exponents for negative numbers invoked the reciprocal to "complete the problem.” The partitioning of units for the purpose of using portions for repeated multiplication did appear in his formulation.

In the next section the impact of the teaching experiment on each student is analyzed. The post-interviews are later compared to the pre-interviews in tables and reserved as a tool to triangulate part of the results from the impact evaluation below.

The Impact of the Teaching Experiment on the Students' Knowledge of Exponents

In this section the overall impact of the teaching experiment on the knowledge of the students is summarized for each student. The summaries and conclusions are the foundation for proposing a model of the knowledge for each student after the teaching experiment. The models reflect the basis for the exponential knowledge of the student, the extent of the refinement of the CDE to accommodate the decimal and rational exponents and the way the zero and negative exponents are connected to each other and justified by the student.

Ann. Ann was able to present a unifying notion and formulation of the exponent concept for all the numbers with the exception of the real numbers. The responses from Ann in the teaching experiment and the interviews (two, three and four) suggest that she was able to integrate her existing knowledge with the new notions of multiplicative fractions that form the basis of the counting system for the decimal and rational
exponents. The impact of the teaching experiment on Ann's knowledge of exponents is shown in Figure 5.14.

Figure 5.14: Impact of the Teaching Experiment for Ann

The understanding of the zero exponent involved the calculator, graphical images, and for Ann the idea that ROG = 0 and the FOM = 1 when the exponent is zero. Ann also used the zero exponents successfully in her Decimal Exponents Calculation format as a place holder and she connected the zero exponents to the shrinking interval for finding smaller and smaller fractions of the unit. My hypothesis is that the Common Definition of Exponents is still the most powerful image for Ann, but that she is comfortable with the notion that the unit can be broken up (through radicals) into smaller parts that allow a "finer" more fluent image of exponents that are not necessarily integers. All these points
justify modest connecting arrows in Figure 5.14 from the zero exponent, the rational exponents and the negative exponents to the general concept of exponents as represented in the diagram. The connection is modest because we do not have sufficient empirical data to assume a network that is completely internalized as one chunk of knowledge. The arrows reach up to the top of the diagram indicating an (inferred) encapsulation of the process of constructing exponents for multiple classes of numbers and bringing all the notions under one generalized conceptual scheme.

Bob. Bob was not yet able to present a unifying notion and formulation of the exponent concept for all the numbers. The DEC process, the concept of negative exponents as directed exponents associated with inverses, were both explained by Bob, but the next step of bringing it all together and formulate one notion of exponents that covered all the types of numbers was not observed. The CDE was strong in Bob and that meant a strong connection to repeated multiplication of whole units with integer exponents. Connecting roots and "decimal fractions" was not a real obstacle. Bob's DEC format suggested that that process developed quite well for Bob.

He made progress with the zero exponents, using small decimal exponents to support his claim that the zero exponents could have the value of one (1). This is a change for the better compared to his previous statement that he did not know why the zero exponent was equal to one. The connection between the CDE and the DEC process was less clear from the responses from Bob because the operational aspect was more visible than the abstraction or the establishment of the concept at a level advanced enough for encompassing different exponential cases. In other words, there were no indications that Bob was able to refine his common definition to propose one that allowed
rational numbers and decimal numbers as exponents from one perspective. It seems plausible that the material presented in the teaching experiment required building or re-building of the concepts that Bob had of factoring, percentages, fractions, multiplication, division and some geometric constructions and graphs. Bob struggled with terminology and with making his thinking explicit through precise formulations. It was inevitable that such problems would show up in the four week instructional session. In Figure 5.15 the absence of connections from the zero exponents, the rational exponent and the negative exponents to the top box for a general exponents' concept, indicates the fact that there was no attempt by Bob to propose one single definition for exponents.

![Figure 5.15: Impact of the Teaching Experiment for Bob](image-url)
Chandra. Chandra was able to present a unifying notion and formulation of the exponent concept for all the numbers with the exception of the real numbers (see Figure 5.16). The Common Definition of Exponents was still the basic image for Chandra, but her regular use of the notion of parts of the unit, her reasoning applying fractions of Factors of Multiplication or even fractions of fractions, including decimal fractions, suggested that she was comfortable extending the count for exponents to smaller than whole units. Her schema for exponents was extended, and the CDE could function next to finer grained units. Her explanation for the zero exponents was elaborate and based on a zero notion of "a small, very small quantity." The zero exponent was compared to "no factors," or "nothing" or an "absence" of factors.

Figure 5.16: Impact of the Teaching Experiment for Chandra
When whole multiples of a unit were involved, the connection between CDE and the exponent at hand seemed to be strong. She understood the process of decimal exponents, but linked the radicals and their multiplicity with the CDE and re-formulated her image of exponents to include the rational form. Her definition stated that exponents reflected the size and direction of the amount of the unit involved in the multiplication of such units.

**Dennis.** Dennis was not able to present a unifying notion and formulation of the exponent concept. His format for applying the DEC process was well developed, and his explanations for decimal exponents were very precise, a major change compared to his lack of clear explanations from before the teaching experiment.

His concept of negative exponents did not seem to include statements like "weird" or logically inconsistent but instead used the movement metaphor combined with division and inverses to indicate "direction" for exponents. His zero exponent notion was not different from his earlier statements because he was confusing the zero exponents with the limit of the exponential form when the exponent tends to negative infinity. Figure 5.17 displays the main aspects of the connections Dennis developed or did not develop during the teaching experiment. The connection from the CDE to 'Exponents as multiples of the factors' indicates, that for Dennis the connection between the two notions was strong. The strength of the connection between concepts is reflected in the thickness of the connecting arrows. The inclusion of the DEC process allowed him to expand his ability to respond to questions concerning decimal exponents. But the connection from CDE to rational exponents is not carried through to DEC or to rational actions. The dash arrows indicate very weak connections. Dennis shows no signs of reflection on the
generalized, common nature of exponents. From a calculatory point of view, Dennis does really well, as can be inferred from his self-generated formats for DEC. But he does not present a more generalized notion of exponents that allows for a definition covering the rational cases and CDE simultaneously.

**Figure 5.17: Impact of the Teaching Experiment for Dennis**

**Eddy.** Eddy was not able to present a unifying notion and formulation of the exponent concept for the positive, negative, rational and decimal exponents. He formulated a concept that still revolved more around the operational side of the numbers than around the common properties that make them all exponents. The CDE was not integrated sufficiently into a new image of exponents that could accommodate the rational or decimal case in a consistent way. Eddy seemed to be in the middle of his
efforts to re-connect and re-formulate his schemas for exponents for a more general concept. This is reflected in Figure 5.18 by an arrow connecting the CDE with the exponents and multiplicative fractions' concepts.

The zero exponent was not explained in a way that would indicate strong understanding of how the zero in the exponent leads to a value of 1 (one) for the outcome. However, the fact that Eddy mentioned that the zero exponent was somehow connected to the result of the FOM for smaller "fractions of the unit" with the increasing string of zero's in 1.00000… means that some understanding has probably occurred in Eddy. The arrow connecting the two concepts reflects this conclusion.

His notion of negative exponents included the “movement” metaphor to indicate direction for exponents. Before the teaching experiment he was not able to explain the negative exponent in any way. His format for decimal exponents was effective and allowed him to explain decimal exponents precisely. This ability to explain the meaning of decimal exponents was a major improvement in his knowledge of exponents.

Before the teaching experiment Eddy was unable to explain any of the decimals in the exponent. Figure 5.18 represents those connections with the arrows and the thickness of the lines. The absence of arrows pointing to the general concept indicated that Eddy did not propose a notion for exponents that brought out the common elements in the various definitions of exponents.
In this section the changes in the knowledge of the students after the teaching experiment are analyzed from the perspective of the contrast between the pre-interviews and the post-interviews. The reason for having a separate section for studying the changes based on the pre-interviews and post-interviews are two fold. One is the fact that the instruction introduced new aspects into the teaching of exponents that were not present when the pre-interviews were conducted. The second reason is, that a comparison between the two interviews allowed a partial verification of what the first impact results suggested. This can be interpreted as a form of triangulation on the trustworthiness of the results of the study.
The questions of the pre-interview are identical to the questions in the post-interview. The responses from both interviews are edited and summarized for inclusion in the tables for comparison. To make the presentation more manageable the table for one student was included in detail in this section.

So far Ann was the student with the most detailed tables. Chandra and Ann had similar characteristics in their knowledge, while Dennis, Eddy and Bob, though not identical and with some major differences, shared some common traits on exponents. They all had difficulties with the justification for the zero exponents; they had issues with rational actions on radicals and the conversion of these actions into rational exponents, and all three shared a fragmented approach to exponents, which was only partially resolved. This fragmentation possibly prevented the emergence of a generalized concept of exponents at the end of the teaching experiment. My impression was that Bob who exhibited many gaps in his knowledge could be taken as a good example of the group, so a summary of the results of the changes from the pre-interview to the post-interview for Bob’s knowledge of exponents will be presented first, followed by the summaries of the pre-interview to post-interview changes for the other students.

**Bob.** In Table 5.18 the questions have been reduced to nine content areas and they are organized in the rubrics. Those rubrics are: positive exponents, the zero exponents, negative exponents, the questions "What is an exponent"?, "What is the meaning of $5^{1/3}$"?, Why is $5^{1/3}$ equal to $5^{2/6}$?, and the following questions related to the decimal exponents: Given that the equation $7^X = 61$ has a solution $X = 2.11257$, what is the meaning of the first decimal digit 1 in the solution for X?; what is the meaning of the
second decimal 0.01 in the solution?; how does \(3^2 = 9\) change into 10.0451 as a result of the change from 3 to 2.1?

<table>
<thead>
<tr>
<th>Positive exponents (integers)</th>
<th>PRE-INTERVIEW</th>
<th>POST-INTERVIEW</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5^3) is 5 times itself 3 times.</td>
<td>That’s just 5 with a multiplicity of 3.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Zero exponents</th>
<th>PRE-INTERVIEW</th>
<th>POST-INTERVIEW</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5^0) this would turn into 1</td>
<td>(5^0 = 1). Anything to the zero power =1, because if you would have a (5^{0.001}) you would get a number close to 1, but not quite 1. So the closer (x) times it is to 0 the closer your outcome would be to 1.</td>
<td></td>
</tr>
<tr>
<td>Does not know why.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Using the definition of exponents, (5^0) would mean… nothing!</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Negative exponents (integers)</th>
<th>PRE-INTERVIEW</th>
<th>POST-INTERVIEW</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5^{-3} = 1/ 5^3); Defined this way because you can’t have negative exponents.</td>
<td>(5^{-3}) can be written as 1 over (5^3) which breaks down the number or is called the inverse. The inverse can be used to go backwards using multiplication And that explains (5^{-3})</td>
<td></td>
</tr>
<tr>
<td>You have to convert it into a fraction. Why? Because you can’t leave it that way!</td>
<td></td>
<td></td>
</tr>
<tr>
<td>You can’t write (-5*-5*-5).</td>
<td></td>
<td></td>
</tr>
<tr>
<td>They never told me why it is defined that way.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>What is an exponent?</th>
<th>PRE-INTERVIEW</th>
<th>POST-INTERVIEW</th>
</tr>
</thead>
<tbody>
<tr>
<td>A short way to write a multiple of the base.</td>
<td>I think of an exponent in terms of multiplicity, so you’re either gonna, or a unit, a base or a FOM and you’re gonna to multiply, multiplicatively that number exponent amount of times, or the inverse, you still multiply to go backwards, exponent amount of times.</td>
<td></td>
</tr>
<tr>
<td>It is the way you write it.</td>
<td>(see quote)</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.18. Pre-interview and Post-interview responses for Bob
Table 5.18 continued

<table>
<thead>
<tr>
<th>What is the meaning of $5^{1/3}$?</th>
<th>$\sqrt[3]{5}$</th>
<th>The third root of 5 raised to the first power.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Why is $5^{1/3}$ equal to $5^{2/6}$?</strong></td>
<td>1/3 is half of 2/6; if you multiply 1/3 times 2 you get 2/6. Uses the calculator to verify true relation.</td>
<td>$5^{2/6}$ can be reduced to $5^{1/3}$. $(\sqrt[3]{5})^2$ also gives you the same answer as $(\sqrt[3]{5})^1 \approx 1.709975947…$</td>
</tr>
<tr>
<td>$7^x = 61$ What is the first decimal 0.1 in the solution for X?</td>
<td>Because $7^2 = 49$ and 2.1 is a little higher than 2, that let’s you know that it’s higher. It’s a lone number; once you get… I don’t know.</td>
<td>Two units of 7 fitted twice into 61 and the second decimal means that 7 $^{1/10}$, the tenth root of 7, raised to the first power is the highest number we can get without going over the remainder of 49 divided into 61.</td>
</tr>
<tr>
<td>Meaning of the second decimal 0.01</td>
<td>Checks on the calculator what the decimals show.</td>
<td>Refers to previous response.</td>
</tr>
<tr>
<td>How does $3^2 = 9$ change into 10.0451 as a result of the change from 2 to 2.1?</td>
<td>The decimal causes the change in value. Tries explaining the change with $3 * 3.1$, rejected because that equals 9.3. Tried $3.1 * 3.1 = 9.61$; rejected. How do you know when to multiply something 0.1 times? Do you divide?</td>
<td>The 0.1 is the tenth root of 3 raised to the first, and you multiply that by 9, so you will get a bigger different number than $3^2$. The factor $3^{1/10}$ causes the change.</td>
</tr>
</tbody>
</table>

The notions of Bob about exponents are somewhat vague. The Common Definition of Exponents is still the most vivid image for him, although there are signs that negative and zero exponents became more meaningful for him than they were initially.

Regarding exponents, Bob said: "I think of an exponent in terms of multiplicity, so you're either gonna... or a unit, a base or a FOM and you're gonna to multiply, multiplicatively..."
that number exponent amount of times, or the inverse. You still multiply to go backwards, exponent amount of times."

His CDE was unchanged, but Bob explained the zero exponents by using small decimal values as exponents and calculated the number as being very close to one (1). His concept of zero was "quantity" like and closer to the intuitive notion of "nothing."

There was no sign of the feeling from before the teaching experiment that he did not know why the zero exponents had the value one.

After the teaching negative exponents were associated with inverses and were used to multiply when you go "backwards." The negative sign seemed to indicate the backward movement. The pre-interview notion that "you can't have negative exponents" was not voiced. The image of exponents in his comment brought forth first and foremost the notion of "multiplicity" which I interpret as a left over from the CDE.

The rational exponents were radicals. Although he did not mention multiplicative parts here, the rational actions were in his mind when asked about the two exponents $\frac{1}{3}$ and $\frac{2}{6}$. Through his calculator, the equality of $5^{\frac{1}{3}}$ and $5^{\frac{2}{6}}$ was justified, next to the older notion that $\frac{2}{6}$ can be reduced to $\frac{1}{3}$. Decimal exponents were explained very precisely and he was fully aware of the reasons why $3^{2.1} = 10.045$ was more than $3^2 = 9$, attributing the change to a multiplication of 9 by the factor $3^{\frac{1}{10}}$.

**Ann.** Ann's Common Definition of Exponents was modified in two ways, one was the use of the idea of unit of multiplication, and the other was the use of the phrase "whole units," indicating an awareness of whole units versus smaller parts of units. Her zero exponent notion
included the formulation that with $5^0$, the factor of multiplication is 5, but there are zero fives. She did not mention the reason why the value is one (1). Negative exponents were associated by Ann with divisions and going in the "opposite direction."

Ann proposed a definition of the general exponent that was (imperfect but) basically correct. Her concept for rational exponents was completely formulated in the generalized language of multiplicative fractions or parts; just as her explanation for the equality of exponents 1/3 and 2/6 was also presented in terms of (multiplicative) parts of units. Decimal exponents were fuzzy before the teaching experiment but after the instruction simple explanations were offered.

**Chandra.** Chandra's Common Definition of Exponents included the concept of factor of multiplication and the phrase "whole factors." This was a modification of the CDE that could be considered an improvement. Her new explanation for the zero exponents started with a reference to the CDE but continued with rates of growth and extensions of periods that were going to zero causing the value one.

Negative exponents were conceptualized by Chandra as divisions of the unit of multiplication and the sign was associated with the "reversal in direction of movement" along a graph (of the exponential function). For the concept of exponents she generalized the idea in terms of how much of the base or unit of multiplication was needed to multiply (or divide) to obtain the number. The multiplication was associated to positive exponents and divisions were associated to negative exponents.

For rational exponents radicals were mentioned just like she did before. The decimal exponents were linked to decimal roots, using the structure of the place value system for exponents to show the meaning of the solution of the equation $7^x = 61$. The
difference in value between $3^{2.1} = 10.045$ and $3^2 = 9$ was explained by pointing to the extra factor $3^{0.1} = 1.1161$ that, multiplied by 9, gave the 10.045.

**Dennis.** The Common Definition of Exponents of Dennis did not undergo a strong change, judging from his first response: repeated multiplication of three factors five. For the zero exponents he mentioned that there were no factors multiplied by 1, staying with his first concept of the value of the zero exponent being a kind of left over factor one (1). For the negative exponents repeated division was the operation and the negative sign was the signal for division or for "moving in reverse factor order." His more general exponent concept was repeated multiplication or repeated division without reference to units or (multiplicative) fractions. This change was not a major shift towards more generalization of exponents.

When rational exponents like $5^{1/3}$ were involved, Dennis mentioned the one-third factor of five and the cube root as the meanings of the symbol, a slight shift in language related to the unit of multiplication. Rational actions were not brought into the picture to explain why an exponent $1/3$ was equal to exponent $2/6$. Fraction reductions were the sole reasons presented. Decimal exponents were expertly explained for both the exponential equation $7^x = 61$ and for the change in value from $3^2 = 9$ to $3^{2.1} = 10.045$, through the multiplication of $3^2 = 9$ by the extra factor $10^{\sqrt[3]{3}} = 1.1161$.

**Eddy.** The CDE of Eddy did not change in a noticeable way. However, Eddy designed a representation for positive, negative and zero exponents in his post-interview (Figure 5.19), that extended the movement metaphor into the domain of signals to go "left," "right" or "none of the two."
Figure 5.19: Eddy's explanation of positive, negative and zero exponents

Eddy's concept included the new idea (for him) of describing the multiplication as: how many factors of five (5) would change! When he had negative exponents, the change was in terms of multiplication by the inverse of 5: 0.2. (The backwards movement idea was present in his response to question 4 from the post-interview). When the exponent was zero, no factors of five in either direction would change, effectively leaving the value unchanged and concluding that $5^0 = 1$. The explanation is imperfect but intuitive!

Eddy's general concept for exponents consisted of three parts: one for the symbol (how it should be written), one for the statement that an exponent indicates an incomplete calculation process (answer must be found yet), and a third part that indicates the direction (through the sign). The rational actions were mentioned but carried out through the calculator. His explanations for the decimal exponents were precise, complete and very similar to the others.
Evaluation of the Teaching Experiment

The analysis of the expert students suggests that the development of a concept for exponents for all the numbers from positive integers to rational numbers to negative numbers of all forms, does not emerge spontaneously. The two expert students found their own way for understanding exponents but neither of them presented a concept that covered all numbers under one definition. They were able to work with the concepts, but did not offer an explanation for the inconsistent handling of the nature of the exponent, starting with the CDE and moving to fractions and negative exponents, ignoring the continuously changing concept behind the notion of what constitutes an exponent.

All the novice students had to overcome hurdles to understand the Rates of Growth, the Factors of Multiplication and the way they are interdependent. All students relied for their understanding of the Rates of Growth and the Factors of Multiplication over multiple or partial periods or units first on their existing linear models of thinking. One important part of the teaching experiment was to contrast the multiplicative thinking to linear thinking, and let the students explore the differences with numbers, graphs and manual constructions.

All students seemed to have an intuitive understanding of the notion that inverses, divisions and backward movements along the X-axis in a graph are the basis for negative exponents. Independent of their understanding of other parts of the teaching experiment, the inverse application of Factors of Multiplication was not obvious for any of the students.

**The Decimal Exponents Calculation Process.** All students developed their own format for calculating decimal digits when solving exponential equations. It is
noteworthy that the students who developed the most organized and structured formats were Ann and Chandra, the same students who also voiced generalized concepts for exponents. They were able to voice in their responses a notion of exponents that worked fairly well, by bringing the idea of fractions of the unit in their concept and counting the parts like ordinary rational numbers are counted.

Bob, Dennis and Eddy were very different in their overall knowledge, but they all gave less structured notions of exponents that did not bring all the exponents under one concept. This suggested that the Decimal Exponents Calculation process, combined with the concepts of ROG and FOM, was not sufficient for all students to bring about a conceptual shift toward a full integration of all the exponential definitions and schemas. The deeper cause or basis for this has to be researched further.

**The Conjecture.** There was more understanding for all the students than before in many areas and that could be interpreted as supporting the major organization of the instruction. But not all the students were able to use the information, the new methods, the new tools and the decimal calculation process to gain the knowledge to discover the common aspects of integer, rational, and negative exponents and forge those commonalities into a single idea about exponents. Further study is necessary to bring together more detailed concept building learning strategies to stimulate the construction of networks of concepts, that can bring all the exponential notions together in an organized and convincing way for all students. Three of the five students failed to propose a concept that showed indications of an overall notion of exponents that could explain the nature of integer, rational and negative exponents with a single idea. Eddy and Bob had significant problems with multiplications and fractions and from the
beginning Bob also showed signs of struggling with percentages. Dennis was well aware of the Laws of Exponents, but was struggling to adapt his high school methods and certainties to a way of thinking that placed more emphasis on foundations and less on immediate answers. Were these problems withholding the three students from contemplating more generalized questions such as, for example, what are exponents?
CHAPTER 6

RESULTS AND IMPLICATIONS

In this study the understanding that students exhibit on different forms of exponents was studied. The students participating in the study may not represent the total population of first and second year college students in general. The conclusions from this study should not be generalized. The main purpose of examining a small group of students was to collect detailed information on possible forms of understanding and existing forms of current knowledge among novice students. The testing of the conjecture on the possibility of a different learning path to exponents was another argument to try an teaching experiment with a small group of students from different backgrounds.

In the next section the research questions are discussed, but in a different order than originally posed.

The Research Questions of the Students' Understanding of Exponents

The following questions were posed at the beginning of the study:

1. *What are the novice students' concepts of rational and negative exponents?*

2. *What are the expert mathematics students' concepts of rational and negative exponents, and how do they differ from the novices' concepts?*
Questions 1 and 2 refer to the first part of the investigation to establish what the students know from our perspective. The following questions (3, 4, and 5) refer to the conjecture and the corresponding teaching experiment.

3. *What is the role of the Laws of Exponents in the process of developing rational and negative exponents, from the perspective of the novice students (before the teaching experiment) and as emerging from the teaching experiment?*

4. *How can we model the students' construction and development of the concept of rational and negative exponents as emerging from the teaching experiment and the interviews?*

5. *What is the impact of the teaching experiment on the knowledge of the participating students?*

**Research Question 1: What are the novice students' concepts of rational and negative exponents (prior to the teaching experiment)?**

In this section an analysis of the knowledge of the students prior to the teaching experiment is presented that summarizes the characteristics of what the students know about exponents.

All the students seemed to show attempts to explain their images of exponents of various kinds by trying to fit these new notions of exponents into the definition that they knew and understood fully, the Common Definition of Exponents (CDE). The zero exponents were conceived as finding a multiplication when zero factors are used or as a special case with an explanation that seemed plausible, but avoided answering the
question how the CDE is related to the value of 1. Eddy simply called it "a rule of the game."

The negative exponents also presented challenges, voiced by students as "weird" or "unacceptable," or "you can't have negative exponents," or they referred to the "school teachers" to justify the meaning of negative exponents. Eddy used the description "a whole new concept for negative exponents," to express his perspective on the relation between the CDE and the procedure for negative exponents. He commented that we should "get rid of negative exponents" by switching to 1 over … Ann did not know why the negative exponents were connected to the inverse of the unit of multiplication. The fact that there was no apparent conceptual connection for the students between the CDE and the traditional definition of negative exponents did not stop them from applying the working definition of negative exponents and solving problems. The authority of the teachers, the school and the existence of "answers" probably overruled their concerns.

With the rational exponents the basic picture was similar in the sense that the CDE seemed to function as the point of departure for interpreting the given definition. Chandra and Dennis voiced this impossible connection to the CDE explicitly by trying to use the exponent as a counting device. Bob treated the fraction-exponent as if only the fraction itself was sufficient to find the value of the exponent. No reference was made to rational actions to be applied to the underlying properties of roots and powers. This notion that all you needed was the value of the exponent seemed to be connected also to the unambiguous world of the CDE where one given exponent fixes the value of the expression. Ann had more awareness of the involvement of radicals and powers in the handling of rational exponents.
Decimal exponents were handled as fractions by Ann and Chandra. Bob, Dennis and Eddy had no clear notion of any structure relating the digits of the exponent to the base or the properties of exponents. Bob made an attempt to use the CDE with the exponent 2.1 and thought of using the base "2.1 times."

Each of the five students showed some form of hesitancy or uncertainty about how one knew if one given number is a power of another given number. The common operations with exponents seemed to go one way only: given a base and a number, find the value when the base had that number as an exponent. The reverse action seemed quite unfamiliar to the students. This problem would re-appear when the chain of multiplications was proposed in the teaching experiment.

The CDE seemed to be the only concept that had a clear operational process and a meaningful interpretation for the concept of exponents for novice students. The lack of a meaningful extension and development of the CDE into a broader notion that reconciled the CDE with the new definitions for the zero exponent, the rational exponent and the negative exponent made it unlikely that students like those interviewed would develop on their own, an overarching concept of exponents that was applicable to all types of exponents, from integers, to zero, to rational and negative numbers.

**Research Question 2: What are the expert students' concepts of rational and negative exponents, and how do they differ from the novices' concepts?**

The expert students in this study were able or had learned to turn the properties of exponents into invariants, the Laws of Exponents, or LOE and base their concepts on such mathematical constructions. Formal methods that involved few verbal explanations were part of their mathematical constructions. Rational and decimal exponents were
embedded in algebraic formulas that are mathematically consistent and constructed in such a way that they make sense to mathematicians. The expert students explained decimal exponents very precisely including the place value structure for exponents. The symbol $5^{1/3}$ was described by one student as the number that raised to the third power equals five, or in formula form: $5^{1/3} = a$, such that $a^3 = 5$. Negative numbers as exponents were defined independent of the earlier definitions like the CDE.

For the zero exponents the device was used to transform zero into a difference of integers: $0 = 3-3$ or $0 = 1-1$ and then use the LOE to state that $5^3 / 5^3 = 5^{3-3} = 5^0 = 1$, or that $X^0 = X^1 / X^1 = X^{1-1} = 1$. This device may be perfectly acceptable for many expert or even some novice students, but the responses from all the five novice students in the teaching experiment suggested that the notion $0 = 3-3$ etc. may not impress some students. The zero exponent here was a counting index and the transformation to the difference may cause confusion in students.

The expert students did not seem to experience cognitive conflicts because of such compartmentalized approaches to definitions. The fragmented nature of the various cases was not addressed or even acknowledged. The knowledge of the expert students differed from the novice students basically on the level of the well-founded nature of their explanations from a mathematical point of view. They seemed to use the LOE as an abstraction for the properties of all exponents. The novice students had no such resources available.

Where the novice students were not able to propose explanations (given the many areas where they stated not to know the explanations for the concepts) for rational exponents or for the zero exponents that were rigorous and consistent, the expert students
provided formal explanations to bridge the gaps. The novice students expressed discomfort or conflict when trying to understand rational or negative exponents working with the CDE, the expert students did not show such problems. What they had in common was the same basic foundation for exponents (CDE) and the lack of meaningful explanations for how these exponential forms are linked together. They also had an absence of generalized notions of exponents in common.

**Research Question 3:** What is the role of the Laws of Exponents in the process of developing rational and negative exponents, from the perspective of the novice students (before the teaching experiment) and as emerging from the teaching experiment?

The Laws of Exponents (LOE) operate for all the students as basic properties for computational purposes. Each student in the study was able to work with positive integer exponents using the LOE "spontaneously.” Only one student (Dennis) was aware of the actual name of LOE. All the others just focused on knowing the specific properties that were needed to work with exponents. The first conclusion is that the Laws of Exponents functioned at the computational level for the students.

The notion that the LOE would act as a collection of properties for understanding the invariant patterns or properties for all exponents and thus would provide a strong basis for creating rational and negative exponents, was not confirmed. The strongest indication for this lies in the existing knowledge of the students and their quest for understanding rational or negative exponents. The most powerful concept image of all the students turned out to be the Common Definition of Exponents (CDE). This well established network of ideas states that an exponent represents the number of factors in a repeated multiplication of a certain unit or base number. The concept implies the
existence of a base, or a unit, a stage of multiplication, and the presence of a sufficient number of factors to be multiplied.

The analysis of the interviews and the teaching experiment suggest that this CDE is more powerful than the LOE in forming the notion of students in their efforts to understand what rational exponents or negative exponents are. All the students reported serious conflicts in their learning activity when trying to explain what these rational or negative exponents were. From a learning perspective, the most important activity was the effort of the students for overcoming the notions of the CDE as a basic but conflicting set of ideas for expanding the concept of exponents.

The zero exponent exposed the limitations of the traditional approach to exponents further. All students reported a conflict in trying to apply the CDE to the zero exponent. Dennis tried to find a way around the problem but was only partially successful. All students mentioned the authority of the teacher as a basis for accepting the value of one (1) for zero exponents. Expert students used the method of writing zero (0) as the difference of integers (3-3, or 1-1) and using the LOE to make a convincing argument for 1 as the value for the zero exponents. The question remained: Did students think of zero (0) as 3-3 or 1-1 at this stage in their development or was the stronger image of zero some form of "nothing" or the absence of a quantity? The data in this study suggest that "nothing" was what the novice students had in mind, rather than a form or operation like 3-3 or 1-1.

**Conclusion.** *The Laws of Exponents did not reflect a sufficient body of notions or ideas to function as a base for the formation of expanded concepts of exponents in this teaching experiment in mathematical education.*
**The LOE in the teaching experiment.** In the teaching experiment the additional tools of the rate of growth (ROG) and the factors of multiplication (FOM) were introduced to build foundations for the expanded concepts of exponents. Priority was given to closing the "gap" between integers in the world of exponents by giving more prominence to radicals and to methods for their integration into the world of exponents. The concept of ROG was linked to the FOM by the instructional activities and the relations between these tools and concepts was studied and linked to exponents in general. This part of the learning activity was important because all the students invoked linear methods before finally realizing that other relations are at work, when operating with fixed rates of growth in multiplicative situations. For negative exponents the "movement" metaphor apparently made some impression on the students as all of them adopted this idea in their own expression of what negative exponents were and how they were related to other exponents.

**Conclusion.** In this study new tools (ROG; FOM; Backward movement related to inverses and divisions) were needed for studying the expansion of exponents to rational or negative numbers. Their purpose was to provide more ways to overcome the one-sided limiting image of the CDE in the construction of rational and negative exponents.

**Research Question 4:** How can we model the students' construction and development of the concept of rational and negative exponents as emerging from the teaching experiment and the interviews?

From the evaluation of the role of the LOE and the discussion around this issue the following model (Figure 6.1) is proposed of the students' construction of the concepts of rational and negative exponents as emerging from the teaching experiment:
In this model the CDE is the basis of all the learning activity for the expansion of the concept of exponents. By introducing the notions of Rates of Growth and Factors of Multiplication a foundation was created to overcome the limitations of the CDE. The students with weak knowledge of percentages and fractions struggle with this part of the
learning activity. These weaknesses represented an additional challenge to be overcome but they were not strictly related to the study of exponents.

From the rate of growth and the factor of multiplication the students studied what happened when we had multiple applications of the FOM and how the ROG over multiple steps was affected. This relation between factors and rates of growth was of great importance because it required a new model of thinking for students and a shift away from linear or additive forms of operations to multiplicative properties. In the diagram for the model the box with Exponents as multiples of the unit represent this development.

When smaller rates of growth had to be calculated, depending on the context of the problem, the factor of multiplication was used to find the solution to such ROG calculations. This part was the introduction of fractions of the unit of multiplication through radicals or roots. The next box of exponents on the left of the CDE box represents the development of fractions of the unit.

The students in the study relied heavily on their calculators to produce higher order roots quickly. Having some background knowledge of radicals and roots beyond just the square root helped students Ann and Chandra in particular to find their way with radicals. A metaphor was introduced to make the transition from integers to fractions easier. Roots were called "multiplicative fractions of the unit" or "multiplicative fractions of the initial FOM." The students used both the root terminology and the "fraction" terminology in their explanations. The root idea was used more than the fraction idea.

To give the students a meaningful context to experience how integers and fractions interact as exponents the Decimal Exponents Calculation process was given a
special place in the teaching experiment. Students solved exponential equations with decimals using "decimal" roots. All students developed their own formats for solving and organizing their work on decimal exponents. This activity resulted in a better ability of the students to explain what decimal exponents represented. Placing the DEC and the rational exponents in the center of the diagram convey the important place of this activity.

From the interviews it became clear that initially the students did not differentiate between rational exponent rules from a purely procedural perspective and the conceptual radical basis of rational exponents and their properties. All students initially used both mechanical fraction rules and radical properties to "prove" the LOE for rational exponents. To overcome this confusion, the activity of rational actions for radicals and roots was introduced to argue what rational properties meant for rational exponents. The students used their calculators to verify the properties of radicals. An algebraic approach to rational actions was tried but replaced by an approach with concrete examples.

Ann and Chandra were more successful in separating actions on radicals from mechanical rules transferred from fractions of exponents. The separation between these two conceptual fields (mechanical rules and radical properties) was crossed many times by Dennis and occasionally by Bob and Eddy. This does not mean that Dennis knew less than the other students, since Bob and Eddy used the calculator to verify their results and Dennis tried radical properties.

The box of Negative Exponents represented the notion that, more or less independent of the rational exponents, the directed numbers indicated opposite "movement" for the exponent when inverses or divisions were involved. The connection
from "Exponents as multiples of the unit" to the Negative exponent box represented the relative independence of negative exponents from the discussions about rational exponents.

The top box was connected to the lower boxes, in the following sense. The process of learning exponents reached completion when students developed a general concept of what exponents were. From the Impact section it was inferred that three of the five students did not show convincing signs of having developed a complete understanding of exponents, in the four weeks of the teaching experiment. Complete understanding would be an encompassing, encapsulating construction of a concept of exponents that explained all the various forms that were apparent in positive, negative, and rational numbers as exponents.

The positioning of the LOE in the top box, placed below, not on top of, the general concept of exponents indicated that the study did not find a central role for these laws in the learning of exponents. I would like to emphasize the words "in the learning of" here because these laws are important for understanding exponents, but the learning of exponents required much more than studying the patterns encoded in these laws. Dennis was quite aware of and very knowledgeable about these laws but it did not give him more insight into the common thread that bound all the exponential forms together compared to the two other students who did not show an overarching concept of exponents. This could mean that Dennis needed more time or that he needed more supporting structures to make the step.
Research Question 5: What is the impact of the teaching experiment on the knowledge of the participating students?

All students showed improvements or minimal positive change in exponential concepts over the pre-interview questions. The most significant change was their ability to calculate and explain decimal exponents in a systematic way as their personal formats for finding decimal exponents indicated. Furthermore, the students linked negative exponents to the intuitive action of reverse movement on the axis of a graph, and connected this movement to inverses and divisions.

The changes mentioned so far are both operational and conceptual, but they do not constitute a single complete structural vision of all forms of exponents. Ann and Chandra were able to use the activity from the teaching experiment to bring together their concepts of exponents and build a network that effectively connects all the exponents in this study into one concept. These concepts were not perfect, but they seemed a big step forward compared to the fragmented and inconsistent notions so prevalent in present day textbooks.

Dennis showed improvements in the areas of decimal exponents, with his DEC-format and decimal roots, and negative exponents, that he linked to reverse movement in graphs, related to inverses and divisions. Rational exponents presented a special problem for him. He did not distinguish clearly between mechanical manipulations with fractions and rational actions on radicals and powers of radicals. He had a clear idea of the LOE and thought of all exponents through the lens of the CDE but showed no change in his overall concept of exponents. The fragments were still fragments although probably with fewer inconsistencies.
Eddy and Bob had similar weaknesses with percentages, fractions, multiplications and divisions. They had to overcome some of these while studying the rates of growth and the factors of multiplication. The calculator helped especially with the development of their decimal (DEC) format. The ROG and the FOM seemed to have a positive effect on their concept-image for the zero exponent and for the radicals involved in the rational actions and the DEC process. Bob did manage to link the zero exponents to small exponents in the decimal domain that he could understand with his DEC process. The calculator showed how much the higher "decimal roots" resembled the number 1.000.

**Conclusion**

The main results of the study point to the strong image (Tall & Vinner, 1981) of the CDE in the mind of all the students in the group. Every student tried to reconcile her/his CDE with the textbook definitions of rational, zero, and negative exponents. None of the students mentioned the laws of exponents (LOE) as their basis for accepting the expanded notions of exponents. Only one student had a clear memory of the name of this collection of exponential properties. For this group of students the conclusion could be that the role of the LOE in their learning of the concepts of rational decimal or negative exponents is different than what is suggested by textbooks, that rely exclusively on the expectation that students transfer the properties that are apparent in the LOE of positive integers, to all possible forms of exponents and integrate their understanding of exponents through that "lens."

One important question is still open: what made it possible for two of the five students to formulate a concept of exponents that actually covered - although with some
imperfections - all the three basic number types that were studied: integer exponents, rational exponents and negative exponents. The best definitions given were as follows:

An exponent is the direction, (factor) size, and length of growth of a unit of multiplication of a given period (Chandra).

How many parts of your unit of multiplication you are using in the multiplication and indicating the direction of the number exponent? (Ann).

There are imperfections in these definitions but they do reflect a concept of exponents that incorporate all the types of numbers used for exponents. The difference with earlier notions is that they include the possibility that multiplicative units have "parts" ("size," or "parts of unit") in a natural way and such parts can be counted or measured, including some rudimentary form of direction (sign).

Implications for Teaching and Learning Exponents

The phenomenon of the strength of the CDE and the students' efforts to understand the new exponents with the existing definition was consistent with the notion expressed by Duit & Confrey (1996) that conceptual growth needs to connect with the intuitive concepts of the students. The CDE seemed to function as an intuitive concept for exponents for all the students in the group, including the expert students. The conflicts and feelings of incongruity expressed by all students in the teaching experiment when explaining exponents that were non-positive integers, point to a disconnect between the CDE and the expanded exponents' definitions and concepts as learned by the students. The results so far suggest that it may be important to actively build or create a mental bridge from integer exponents to fractional or decimal exponents as the meaningful transition from integer to non-integer exponents.
The justifications for the zero exponents proposed by novice or expert students, revealed the main weakness for using the LOE as the main vehicle to lay the foundations for rational and negative exponents. While the expert students had no trouble shifting their image of zero from 0 to the more abstract forms of 1-1 or 3-3, the interviews with the students in the teaching experiment group suggest that their notion of zero was primarily of the form 0 as in "nothing" or "nothing left," a quantitative image that was not equivalent to the abstracted form 2-2 or 5-5. This non-equivalent notion begs the question: Is 0 always equal to n-n? For example if you have a 0 % growth of your population, does it mean that it can be represented by 5 % - 5% growth? It may help to shift the explanation for the zero exponent from the drill method or the authority ritual to an understanding that connects with the notions of zero existing in the students' understanding and to connect those notions to decimal and rational roots. Another aspect of the CDE compared to the LOE was that definitions are not always understood by students. The study suggests that when there is a conflict between new definitions and an earlier version of the same concept, it may be helpful to know which concept image (or statement image (Edwards & Ward, 2004; Selden & Selden, 2005; Wilson, 1990)) was more powerful for the student and to adapt the teaching of the new concept accordingly.

The knowledge of the LOE did not mean that the students understood the workings of exponential functions. The study of exponential properties in the context of growth and factors of multiplications revealed that the students approached multiplicative contexts with additive or linear forms of thinking. These problems are consistent with the findings of Confrey & Smith on the problems students had, shifting from rates of growth to factors of multiplication (Confrey, 1991; Confrey & Smith, 1994; 1995). To understand
the connection between rates of growth and factors of multiplication in this teaching experiment, the students had to overcome the natural tendency to *add* the rates of growth when dealing with multiple periods of growth.

When looking for ROG's over parts of a period, the tendency was to *divide* linearly the ROG's over such periods. One student tried to use the ROG as a FOM and raise the ROG to the expected powers to find a new ROG. All the students tried one or more of these methods in some part of the teaching experiment.

When teaching exponents and in particular when the exponential function is studied, it might make sense to insert a section on rates of growth and factors of multiplication. The teaching should pay attention to the three stages of the students growing understanding of the relations between rates of growth and factors of multiplication. A focus of this teaching section could be to create a multiplicative model of growth in the students understanding and connect the model to rational and negative exponents. The difference between additive and multiplicative models of growth should be given sufficient attention.

Prior to the teaching experiment the students were unable to explain in simple terms what the decimals in an exponent represented. The DEC process apparently changed that, because all students were able to provide simple explanations after the teaching experiment. This represented an increase in their skills and at some level in their understanding of decimal exponents. The long term effect of that change in knowledge cannot be stated here, but it would be interesting to know if such knowledge also had a beneficial effect on the students' ability to better understand logarithms.
Future Research

The results of this study cannot be generalized to a larger population. The sample of students was small and there is no reason to consider them representative of the total novice college population. The design of the teaching experiment was such that it was more like a trial teaching experiment for a larger project to find out, how regular classes of students learn the content presented. Conducting such a large-group teaching experiment for exponents would be the next step in my research agenda for exponents.

More specific questions for research could be:

1. What are college students' concepts of zero in such contexts as exponents compared to other contexts like rate of growth, or height, or volume?

2. When is the notion $0 = n - n$ an acceptable notion for zero?

3. How could the concepts of ROG and FOM be better integrated into the pedagogy of exponential functions?

4. What will be needed for a successful encapsulation of the partial notions of exponents into one generalized concept for exponents?

5. What is the best definition for the generalized concept of exponents that also includes the real numbers and is acceptable for novice college students?

6. What would be the impact of a teaching experiment with the same conjecture as proposed in this dissertation on a class of regular freshman college students?
REFERENCES


APPENDIX A

PRE-INTERVIEW AND POST-INTERVIEW QUESTIONNAIRE PROTOCOL
These questions were presented at the beginning and at the end of the teaching experiment.

1. Describe and explain what primes are?
   A. How do you factor the following numbers?
      12 =
      6 =
      4230 =
   B. What are the primes in each number?
   C. What are the exponents of the primes for each number?

2. What is the meaning of $5^3$; $5^0$; $5^{-3}$? Explain your answer as much as you can.

3. Could you explain to me how negative exponents were defined?

4. Suppose N is a positive integer, what is the meaning of $5^N$? How does the notion of exponents change when N becomes a negative integer?
   Why?

5. Why would you say that $5^{-3}$ is a power of 5?
   A. Explain to me what a power of 5 means to you and how $5^{-3}$ fits into that idea?
   B. What is the criterion for deciding if a form is a power of five or not?

6. Let us go back to $5^3$, $5^0$, $5^{-3}$.
   Are all the numbers 3, 0, and -3 exponents?

7. Why do you say that they are or they are not exponents?

8. Could you explain how you see this idea of an exponent?
9. Can you state to me in words what an exponent of a number is?

10. What would be the reason they (the 3 the 0 and the -3) are all called by the same name of exponents?

11. Explain why $7^3 \times 7^{10} = 7^{13}$.

12. Explain why $(\frac{1}{2})^3 \times (1/2)^4 = (1/2) \cdots$

13. Simplify this expression: $(5^3 \times 6^4) = \ldots$. Explain your answer and give a mathematical explanation for your arguments.

14. Two students try to determine how to simplify an expression. The question is to write the expression as a single exponential form: $5^3 \times 6^3 = \ldots$. The options are $(30)^3$ and $(30)^6$.

   What is the right response, and why?

15. What is wrong in the following statement:
   $6^3 \times 3^3 = (18)^6$?

16. If we have $5^3 \times 5^3 = (25)^3$ why is it that we do not add the exponents of the base 5 on the left?

17. Could you explain the laws of exponents? Give some examples of these laws.

18. Do these laws apply to negative exponents? And to an exponent zero? How do you know that they apply when we have negative exponents?

19. What is the meaning of the negative sign in $7^{-4}$?

20. How is that negative sign different from the negative sign in $-7^4$?

21. What do the two negative signs have in common?

22. Do you know any context where exponents are actually used in real life? Explain these uses for me?

23. What does $5^{(1/3)}$ mean to you?

24. How do we know that $5^{(1/3)}$ is the same as $5^{(2/6)}$? Explain how we can justify this equality?

25. Why do you think an equation like $5^x = 3$ cannot have an integer solution?
26. If we solve the equation $7^X = 61$ we find that $X = 2.11257…$

What is the meaning of the first decimal 1 in the solution of this equation?

27. What is the meaning of the second decimal 1 in the solution?

28. How can we know that $X$ cannot be a rational number?

29. Suppose we have a number $3^2$ which is equal to 9 and the number $3^{2.1}$ which is equal to 10.0451…

How would you explain the difference in value between the numbers 9 and 10.0451… as a result of the change in the exponent from 2 to 2.1?

30. What causes the change in the value from 9 to 10.0451…?

31. Explain this difference first without the calculator

32. Can you explain the difference with a calculator?

33. Can you explain to me what the calculator does for you when you explain the difference with the calculator?
APPENDIX B

TEACHING EXPERIMENT
LESSON PLANS
LESSONPLANS

Lesson 1

The development of a formula for the population of a city that has an annual rate of growth of 25 %

General objective:
To show how the stable rate of growth of 25% leads to an exponential function with integer variable.

Sub-objectives:
1: How the 25 % rate of growth can be transformed into the concept of the factor of multiplication.

Comment: The factor of multiplication is the connection to the exponential form. The mechanism that produces this factor and its connection to the exponential form is not obvious to students.

2: To create a graphical image of this process and use the calculator to manually graph points for different values of the initial population.

Lecture:
Every year the population of a city grows by 25%. If this process continues year by year, what will be the population after 10 years? The present population (January 1, 2000) is 2500 persons.
**Student activity 1.1:** (Show all calculations step by step and explain the steps in detail.

- Calculate the population on 1 Jan. 2001.
- Repeat the calculations for all the years until you reach the year 2010.
- Make a list or table with the year, the population for that year and a separate list with the increase of the population as compared with the previous year.
- Make a graph that shows the year and the population for that year. Place the year on the horizontal axis. Plot the population on the vertical axis. Choose your units for the vertical axis wisely. Do not draw a continuous line.

**Question:** Why is it better to plot only dots on your graph?

- Go back to your list or table and calculate \( P(2001)/P(2000) \). We will call this value \( f \). Do the same for \( P(2002)/P(2001) \). What do you notice?
- Continue these calculations for all the years in your list.
- Explain how \( f \) is related to the rate of growth of the city.

**Question:** If the rate of growth of the population for another city is 15 %, what will be the value for \( f \)?

**Student activity 1.2:**

- Try to find a formula for the population of each city after \( n \) years. Explain your method and show why any fixed rate of growth over equal time periods leads to a similar formula.
- Explain how you constructed your formula.

The population of a county grows every year such that the new population one year later could be found by multiplying the population of the previous year by a factor of 1.35.

- Explain how you can find the rate of growth both in decimal form and in percentages for the county.
Student activity 1.3
- What could be a formula for the population of that county after $n$ years?

Student activity 1.4
The number of bacteria in a laboratory grows under optimal conditions and doubles every three hours.
- What is the factor of multiplication per three hours?
- What is the rate of growth per three hours for such a population? (Express your final answer in percentages).
- Why is it important to state the rate of growth per time period?
- Construct a graph for the population growth per three hours over a period of 24 hours. Pick your own initial population. Make sure your choice will allow you to have a reasonable graph.

Important: Use blank non-graph paper to construct your graph and draw the function values exactly to scale.

Lesson 2

The function $F(X) = A \times (1 + a)^X$; $A =$ initial quantity; $a$ is related to the rate of growth.

The exponential function with (positive) integer values for the exponent.

The first steps to create smaller units of multiplication.

Lecture: We have created an exponential function starting from the fixed rate of growth. This fixed rate of growth led to the concept of the factor of multiplication. For 25% rate of growth the exponential function was $F(n) = (1.25)^n$.

How will the function look if we have a 100% rate of growth?
What is the function if the rate of growth is 200%?
When we started the study of the function with fixed rates of growth per unit of time, this led to the exponential function with integer exponents. It is logical to investigate what happens if we assume that over shorter periods of time the rate of growth are still fixed.

Student activity 2.1

- Read and answer the activity on the worksheet.

Lecture: If \( F(X) = (1.25)^X \) the preliminary implication is that \( X \) takes on natural integer numbers and \( X \) represents the number of factors involved in the repeated multiplication with the unit (also called base) of 1.25.

The function \( F(n) = (1.25)^n \) can only take on values determined by the integer variable \( n \).

Use your calculator to create a table of the values from \( n=1 \) to \( n= 10 \)

\[ F(n) = (1.25)^n \] or \( F(X) = (1.25)^X \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(n) )</td>
<td>1.25</td>
<td>1.5625</td>
<td>1.9531</td>
<td>2.4414</td>
<td>3.0518</td>
<td>3.8147</td>
<td>4.768</td>
<td>5.961</td>
<td>7.451</td>
<td>9.313</td>
</tr>
</tbody>
</table>

We will formulate a procedure to let the exponent \( X \) take on rational values by introducing the idea of fractions of the unit 1.25.
Student activity 2.2

- Explain why we use the concept of fractions here under conditions of multiplication.

The justification for trying this is as follows:

It is important to notice that if $X$ takes on the integer values, then $F(X)$ will not be able to have certain values. For example the values between 1.25 and 1.5625 (representing the function values for $n = 1$ and $n = 2$) are out of the reach of $F(X)$!

Let us take on a unit that is easy to handle. Later on we will return to our unit of 1.25. Suppose we have

$F(X) = 3^X$. A table of values for $F(X)$ looks like this:

<table>
<thead>
<tr>
<th>$X$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(X)$</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>81</td>
<td>243</td>
<td>729</td>
<td>2187</td>
<td>6561</td>
<td>19683</td>
<td>59049</td>
</tr>
</tbody>
</table>

We are going to use roots and powers of the unit to construct products that can take on any value between the numbers.

We will show how $N$-roots (like $\sqrt[1]{}, \sqrt[3]{}, \sqrt[4]{}, \sqrt[5]{}, \sqrt[10]{}, \sqrt[10]{\text{etc.}}$) can make it possible to fill up the space between the integer values of the exponents.

If $F(X)$ has as its unit of multiplication the number 3, then we will call $\sqrt[10]{3}$ a tenth multiplicative fraction of 3.

$\sqrt[10]{3} = 1.11612...$

The reason why we use the word fractions is somewhat like this: $(1.11612...)^{10} = 3$.

In other words we have to use the factor $1.11612...$ exactly ten times in a repeated multiplication mode to construct the unit 3 exactly once.
We could think of this number 1.11612… as the tenth part of the factor 3.
To make this more transparent, compare this procedure with classical fractions.
Take the number 3. Instead of multiplication we use addition. If we would split up 3 into
ten equal parts through addition we would come up with the number 3/10, or 0.3.
The additive part comes into play by \( ADDING \ 0.3 + 0.3 + 0.3 + 0.3 + \ldots + 0.3 = 3 \)
(Repeated ten times) to get back to the number 3!
In a multiplicative sense we have 1.11612…* 1.11612…* \ldots = 3 we have the same ten
parts, but now we deal with the operation of multiplication instead of addition!
Obviously the numbers are different, but as we will see, there are some very nice
similarities if we shift our thinking into the direction of measurements and fractions when
we think of exponents, similar to the way we handle additive fractions.

Likewise we call \( \sqrt{3} \) the one-half fraction of the unit 3.
\( 100\sqrt{3} \) is then the hundredth fraction of the unit 3 of multiplication.

We will now study how to use this in a calculation. The fractions that we will use are all
of the powers of ten.
Let us say we have the unit of 3; we need to find out how much of this unit goes into a
number like 15.
In other words what is the solution of the equation \( 3^A = 15 \)?

We know that \( 3^2 = 9 \) and \( 3^3 = 27 \). This means that the first digit of A must be 2.

The next step is to divide 15 by 9 (the highest factor of 3 that can be taken out of 15.
15/9= 1.6666… This division must yield a number less than 3 (Why?).

We cannot use the unit 3 anymore because 1.6666… is less than 3.
The fractions allow us to use a much smaller factor now. Because we try to express our
numbers in a decimal notation, we will use fractions of 10, 100, 1000, etc. of the unit 3.
We know that $10^{\sqrt{3}} = 1.11612…$ (Keep in mind that this means that $(1.11612…)^{10} = 3$)  
The number 1.11612… fits into 1.6666… not more than 4 times, because $(1.11612…)^4 = 1.5518…$ 
What we have found so far is that $15 = 9 \times (1.11612…)^4 \times B$.

How can we find the last number B? If we use our calculator we can see that $15 / (9 \times (1.11612…)^4) = 1.07399…$ so B must be 1.07399… We can do this because all the operations are based on multiplication! 
Again we have to translate this into a fraction of the unit. 
Also we are using the decimal system, and because we have already used the tenth unit 
$(10^{\sqrt{3}} = 1.11612…)$ 
We should use a smaller part of this unit now. 
The next decimal fraction then is the 1/100th part of the fraction, so we find 

$100^{\sqrt{3}} = 1.011046…$ 
In other words, the tenth power of this number is 
$(100^{\sqrt{3}})^{10}$ which is 
equal to 
$10^{\sqrt{3}} = 1.11612…$ 
In formula form: $(1.011046…)^{10} = 1.11612… = \text{tenth part of the unit 3}$ 

Now B = 1.07399… 
We notice that $(1.011046…)^7 = 1.079937…$, 
While $(1.011046…)^6 = 1.0681377…$, 
So the highest power of $100^{\sqrt{3}}$ that fits in the number B is 6. 

Now we have $15 = 9 \times (1.11612…)^4 \times (1.011612)^6 \times C$
Or we have, using the language of exponents:
\[ 15 = 3^2 \cdot (3^{0.1})^4 \cdot (3^{0.01})^6 \cdot C \]

What we have done so far is use the system of fractions of the unit and raise them to an appropriate power to obtain the number 15.

In fact we have shown that
\[ 3^{2.46} \approx 14.918. \]

We have used the following ideas:
1. The unit of our multiplication is 3
2. The roots of 3 are our “fractions of the unit”
3. To create a decimal system in the exponents, we use only tenth roots, like \( 10^{\sqrt{10}(3)} \);
4. Every next factor is determined by dividing the previous factors by the calculated ones.

Try the system by solving step by step the equation \( 3^A = 50 \) with decimal notation.

**Lesson 3**

**To expand procedural and technical knowledge of the decimal exponents**

**Lecture:** We will study decimal exponents further and see how they behave compared to the old laws of exponents.

Take the two equations \( 5^A = 10 \) and \( 5^B = 8 \)
Suppose we multiply \( 5^A \cdot 5^B \)
Then we know that the answer must be \( 10 \cdot 8 = 80 \)
From integer exponents we know that we can add exponents when the base unit is identical and then find the solution to $5^c = 80$
Will this also hold for decimal exponents?
Will $A + B = C$?

**Student activity 3.1**
- Calculate as accurate as possible the values for $A$ and $B$ and $C$. Use the method that we discussed and go to the 4th decimal.
- Is $C \approx A + B$?

**Student activity 3.2**
- If $5^A = 10$. (The same equation as above)  Verify that the equation $5^B = 100$ has the solution $D=2A$. Do this by calculating the value for $D$ and then verifying on paper that it is indeed twice as big as $A$.

Let us repeat how we interpret decimal exponents.
First we interpret notations like $3^4$ as meaning that 4 represents the measure of how much multiplication with the unit 3 is involved in some exponential form.
Also it means that at all times we should have 3 as our unit of measurement.
Moreover, if the change is less than 3 we can still continue our exponent interpretation by using the concept of “fractions of the unit”.
This is done through using the roots of 3. Any root of the unit represents a fraction of that unit. By using the roots, we can introduce finer measurements of the amount of multiplication contained in a number.

**Example**: if we use the unit 3, then the number 9 can be interpreted as a factor (always multiplications!!) containing two units of the factor 3 and the exponent 2 is a measure of how much multiplication involving 3, goes into the number 9.
Now what about the number 10 and the unit 3?
This question can be reformulated as follows:
Solve the equation $3^A = 10$

$A \approx 2.0959…$

**Student activity 3.3**
- Solve the equation $3^A=10$ accurate in 4 decimals.

This means that $10 = (3^1)^2 \times (3^{0.1})^0 \times (3^{0.01})^9 \times (3^{0.001})^9 \times ……$

**Student activity 3.4**
- Formulate what the equation $6^B = 100$ means in terms of exponents and measures of multiplication with units of 6 and decimals.

**Lecture:** Verify that if $1.25^A = 6$ and $1.25^B = 3$ then $1.25^A/1.25^B = 1.25^C$ where $1.25^C = 2$. In particular verify that $A-B=C$

**Student activity 3.5**
- Solve $1.25^A = 6$
- Solve $1.25^B = 3$
- Solve $1.25^C = 2$
- Then carry out the verification.
- Which of the laws of exponents corresponds to this statement?

**Lecture:** The calculations that we have carried out so far suggest that the laws of exponents may also hold for rational and decimal exponents.

**Student activity 3.6**
- Formulate what we have investigated so far concerning rational exponents in the form of decimals.
**Lecture:** We can also combine exponents and investigate what happens when a number is raised to another power and then raised again to another power.

**Example** (In the form of a question)

**Question:** What does it mean if we raise a form like $3^{1.6}$ to the power 1.3?

**Student activity 3.7**
- Formulate your response as you see this.

Before we answer this question let us go back to our original concept of powers of powers.

If we have for example $(3^4)^3$ the meaning of this symbol or notation is the following:

First we explain what is in the parenthesis. That form indicates that we should find a number by using the factor 3 four times in a multiplication.

So we have to find $3*3*3*3$. We know that this number is 81.

Now we have a choice of either expressing the next calculation in terms of 81 or in terms of the original unit 3.

We could write $(3*3*3*3)^4$ or we could write $(81)^4$.

The difference is in the unit you would use for your multiplication.

What this means is that we can have

$$(3*3*3*3) \times (3*3*3*3) \times (3*3*3*3)$$

Or we can have $81*81*81$.

If we use the unit 3, then we have $3^{12}$

If we use the unit 81, then we have $81^3$.

If we use no exponents we have 531441.

It is clear that we *multiply the exponents if we stay with our original unit of 3.*
Student activity 3.8
- How would you write the form \((3^{0.1})^5\) in terms of the unit of multiplication 3?
- Explain your thinking using both the fraction interpretation of exponents and the integer interpretation of exponents.
- Also explain briefly what fraction units mean in this context

Student activity 3.9
- Do the same with \((1.25^{0.01})^6\)
- And with \((0.335^{0.001})^4\)

Student activity 3.10
- Suppose we switch the exponents in our previous activity:
- Explain what \((3^5)^{0.1}\) means.

First calculate the numbers with your calculator:
Find \((3^{0.1})^5 = \ldots\)
Then find \((3^5)^{0.1} = \ldots\)
- What do you notice?

Lecture: If you study the outcome of activity 3.9 carefully you will see that
1. The numbers are identical after the calculations
2. The switching of the exponents from inside out makes no difference in the outcome
3. There is a major difference between the interpretation of \((3^{0.1})^5\) and \((3^5)^{0.1}\)

What is that difference?

Student activity 3.11
- Describe as best as you can what the difference is between the two forms in terms of their calculations
The difference that we discuss here in algebraic form is identical to the following question:

Is \((n\sqrt[3]{3})^M\) identical to \(n\sqrt[3]{3^M}\) ?

*Notice the use of the "√ and the switching of the powers and the roots!*

If we use the language of fractions of factors in a multiplication, and think about exponents as measures of the amount of a certain unit present in a number, then we have:

1- \((n\sqrt[3]{3})\) represents the \(1/n\) fraction of the unit 3 in a multiplication.
2- \((n\sqrt[3]{3})^M\) represents repeated use, \(M\) times of that fraction of the factor 3.
3- \(3^M\) represents the use of the factor 3, \(M\) times.
4- \(n\sqrt[3]{3^M}\) represents the \(1/n\) fraction of the unit \(3^M\).

**Student activity 3.12**

- Analyze the expressions \((p\sqrt[3]{3.55})^A\) and \(p\sqrt[3]{3.55^A}\)
- Explain the meaning of these expressions just like you did for the previous example.

**Student activity 3.13**

- How would the number \((3^{1.6})^{1.3}\) be interpreted?

Hint: Start with \(3^{1.6}\); Read this as \(3^{1*3^{0.6}} = 3^{1*(3^{0.1})^6}\).

Think of \(3^{1.6}\) as a possible new unit of multiplication and then apply the new decimal exponent of 1.3 to that:

Based on our system so far this will represent a multiplication of two forms, each based on the unit.

The first party takes the full, integer measure of \(3^{1.6}\).

So we have our first factor which must be \(3^{1.6}\).

Then we have to take 1/10 of that unit and then repeat the factor three (3) times:

\(10\sqrt[10]{(3^{1.6})}\) repeated 3 times or with mathematical symbols: \((10\sqrt[10]{3^{1.6}})^3\)
If we study this closely we can see that this looks very much like the form we have above.
We are allowed to switch the decimal and the integer and then it shows that we have
$10 \sqrt[3]{3^{1.6}}$ where we take a root of a known quantity.

We have no problem finding the meaning or value of $3^{1.6}$ It just tells us to multiply the
exponents by 3.

It should come as no surprise that taking the 10-th root of that just leads to a division of
the exponent by 10.

In fact what we can see is that a decimal power of a decimal power leads just as naturally
to the product of exponents as happens with integer exponents.

\textbf{Lesson 4}

\textbf{Negative exponents, directed numbers and division.}
\textbf{Connecting division, directed numbers and negative exponents.}

\textbf{Lecture:} If we study the concept of rate of growth, factors of multiplication, and repeated
multiplication, the following picture emerges:
\begin{itemize}
  \item If the rate of growth is a positive number, like 5%, or 10% or 300%, we will
  always have a factor of multiplication that is more than 1.
\end{itemize}

\textbf{Student activity 4.1}
\begin{itemize}
  \item Explain why this is the case.
\end{itemize}
Lecture: If the rate of growth is positive we have a factor of multiplication that is more than 1!!

Suppose we have a factor of multiplication derived from a rate of growth of 10%, or \( f = 1.1 \).

This means that every time we multiply by that factor, we have to count that as a measure of 1 unit.

If the factor is associated with time we could say that for every time period the factor to multiply the previous quantity is that factor 1.1.

Suppose we are looking for a very small fraction of that unit of 1.1, say one-hundredth of it, or \( \frac{1}{100} \sqrt[100]{1.1} \) which according to our calculator is equal to 1.000953556….

Student activity 4.2

- What is the rate of growth associated with 1/100 of the unit 1.1?

Lecture:

It is clear that the factor of multiplication is extremely close to one (1).

If we would take even smaller fractions of the factor we would come even closer to one.

When there is no growth we will say that the rate of growth is equal to zero (0)!

If we think about this we may accept the proposition that the factor of multiplication for a zero rate of growth is just 1(one), because the same quantity is reproduced every time we have zero growth.

Think about a country with a stable population. Every year the rate of growth is zero, and in terms of factors of multiplication, we have \( f = 1 \) every year.

A no change situation corresponds to a multiplication with factors of 1(ONE) and that means that we multiply by one every time.
The number one therefore corresponds to a measurement of zero in terms of units of multiplication.

If I have a unit of multiplication say 5, and someone would ask me how many factors of 5 fit into the factor 1, the only possible answer is zero!!
Any positive value would imply that there is some growth no matter how small!

Student activity 4.3
- Describe your arguments for making exponent 0 correspond to multiplication factor 1.

Lecture:
Let us study division now.

Student activity 4.4
- Graph the Function $F(X) = 2 \times (1.25)^X$
- Create a table first.

Lecture:
We know that if we start at say $X= 4$, then $F (4) = 4.8828125$ (keeping all decimals for the sake of accuracy).
We also know that to get back to $F (3)$ we have to multiply $F (4)$ by the number $1/1.25$ or the decimal 0.8.
$F (3) = F (4) / 1.25$ or $F (3) = F (4) \times 0.8$
In fact if we continue this process of going backwards we will have to use the factor 0.8 repeatedly and we could say that the reverse process is just a form of repeated divisions.
We come to the key notion of this last part of our journey:
Repeated division is the reverse process of repeated multiplications.
Every time we move backwards one unit it is as if the exponents decrease by one. When we arrive at \( X=0 \) we can simply continue with two possible interpretations:

1. Backwards movement corresponds to dividing by the factor of multiplication
2. Backward movement corresponds to the decrease of the exponent.

The two interpretations can be integrated by doing the following;

- Thinking of division as the reverse operation of multiplication
- Use the directed symbol, which is the negative sign, in conjunction with the positive sign, to indicate either multiplication (+) or division (-) when dealing with exponents.
- We can continue our graph for \( F(X) = 2 \times (1.25)^X \) into the negative domain by dividing the value at \( F(0) \) by 1.25, the factor of multiplication.
- We can achieve the same backwards movement by multiplying the given number by 0.8.
- Because of the above we can give meaning to the notion that \( (1.25)^{-1} = 0.8 \)

**Student activity 4.5**

- Complete the table into the negative domain and graph your function \( F(X) \) into the negatives.

**Student activity 4.6**

- Describe how negative exponents can be justified and what the interpretation of the negative sign could be in terms of, or when we deal with exponents

We can also use the laws of exponents to describe the workings of zero and negative exponents.
APPENDIX C

TEACHING EXPERIMENT WORKSHEETS
WORKSHEETS

Student activity 1.1
Every year the population of a city (city # 1) grows by 25%. If this process continues year by year, what will the population be after 10 years? The present population (Jan 1, 2000) is 2500 persons.

a) What does it mean that the population grows by 25% in one year?  
(Use both words and mathematical language to explain your answer in all activities; your verbal explanations are important for the research)

b) What is the increase in population on 1 Jan. 2001 compared to the previous year, on 1 Jan. 2000?

c) What is the population on 1 Jan. 2001?

d) Make a list or table with the year, the population for that year and a separate list with the increase of the population as compared with the previous year.
Table C. 1

Population growth - Assume an initial population of 2500 at the beginning of Jan, 2000

<table>
<thead>
<tr>
<th>Year</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population at beginning of the year</td>
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<tr>
<td>Rate of growth</td>
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<tr>
<td>Population at end of year</td>
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<td></td>
</tr>
</tbody>
</table>

Use blank paper for your graph.

e) Make a graph that shows the year and the population for that year on the last day of the year. Place the year on the horizontal axis. Plot the population on the vertical axis. Choose your units for the vertical axis wisely. Do not connect the plotted points or draw a continuous line.

f) Why is it better to plot only dots on your graph?

g) Go back to your lists or table and calculate \( P(2001)/P(2000) \).

1. What is the value that you find?

2. We will call this value \( f \). Do the same for \( P(2002)/P(2001) \).

3. What do you notice?

4. Continue these calculations for all the years in your list. Use the table below.
Table C.2

Factors of multiplication

|------------------|------------------|------------------|------------------|------------------|

h) 1. Explain how $f$ is related to the rate of growth of the city.

2. If the rate of growth of the population for another city (city # 2) is 15%, what will be the value for $f$?

Student activity 1.2

1. Find a formula for the population of each city after $n$ years (assume 25% growth for city # 1 and 15% growth for city # 2).

2. Explain your method.
   Show that repeated multiplication over the years gives the same result as the exponential function that you have in a 1, with 25% growth. Use the table.
<table>
<thead>
<tr>
<th>Year</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential Formula</td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Repeated Multiplication</td>
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<td></td>
</tr>
</tbody>
</table>

a) Formulate what the connection is between the rate of growth in percentages and the value of $f$.

b) The population of a county grows every year such that the new population one year later could be found by multiplying the population of the previous year by a factor of 1.35. Explain how you can find the rate of growth both in decimal form and in percentages for the county.

**Student activity 1.3**

What could be a formula for the population of that county after $n$ years? (Assume an initial population in 2004 of 6500)
Show that repeated multiplication gives the same results as exponents. Use the calculator and the table
Describe the repeated multiplication method.
Describe the exponential formula that is equivalent with the repeated multiplication procedure

<table>
<thead>
<tr>
<th>Year</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repeated multiplication</td>
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<td></td>
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<tr>
<td>Exponential form</td>
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</tr>
</tbody>
</table>

Notes for activities:

**Student activity 1.4:**
The number of bacteria in a laboratory grows under optimal conditions and doubles every three hours.

a)  1 What is the factor of multiplication per three hours?

2 What is the rate of growth per three hours for such a population? (Express your final answer in percentages).

b)  Why is it important to state the rate of growth per time period?

c)  Construct a graph for the population growth per three hours over a period of 24 hours.
    Pick your own initial population. Make sure your choice will allow you to
have a reasonable graph.

Use the table included in the worksheet (below).

Important: Use blank non-graph paper to construct your graph and draw the function values exactly to scale.

Be aware of the fact that the table has only one-hour steps.

Fill out the 3-hour periods first. Then use the concept of multiplicative fractions (Roots!!) to find the factor of multiplication for the one-hour periods.

After finding the factor, complete the table and create the graph. (Work on separate papers for your graph).

**Table C.3: Growth and factors of multiplication**  (Choose your own initial value at 0 hour)

<table>
<thead>
<tr>
<th>Hour</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td># of bacteria</td>
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<tr>
<td>Factor of multiplication</td>
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</tbody>
</table>

<table>
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<tr>
<th>Hour</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td># of bacteria</td>
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<tr>
<td>Factor of multiplication</td>
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</tbody>
</table>

**Notes on table C.3:**
**Student activity 2.1**

In a city the annual rate of growth of the stray dog population is 5%. For ten years this trend was confirmed and the population of stray dogs in the last year of the investigation was 2925.

a) What was the number of stray dogs one year earlier? Explain your method of calculation.

b) What was the number of dogs two years earlier?

**Student activity 2.1.1**

Depending on how you calculated the early population you may have used the rate of growth all the way to find the earlier population, or you may have used the factor of multiplication per year to find the earlier population. Explain in words which method you used and why.

**Student activity 2.2.1:**

Explain why we use the concept of fractions here under conditions of multiplication.

Use the number 25 to describe the concept of half a factor of 25.

**Student activity 2.2.2**

Explain the concept of half a multiplicative factor if we have an initial factor of 5. What is half that factor?

**Student activity 2.2.3**

What is one third of a factor of 5? Explain the method that you use.

**Student activity 2.2.4**

Describe and explain what one-tenth of a factor of 5 would be.

Compare the *multiplicative* one-tenth of 5 with the *additive* one-tenth of 5.
Student activity 2.2.5

Write the number 155.48 using only factors 5 or fractions of the factor 5, divide your factors of 5 by 10, 100, 1000, etc.

**Your first step could be 155.48 / 5 = .....**

**How do you proceed from there?**

---

Worksheet 3

We will study decimal exponents further and see how they behave compared to the old laws of exponents.

Take the two equations $5^A = 10$ and $5^B = 8$

Suppose we multiply $5^A \times 5^B$

Then we know that the answer must be $10 \times 8 = 80$

From integer exponents we know that we can add exponents when the base unit is identical and then find the solution to $5^C = 80$

**Will this also hold for decimal exponents?**

**Can we find C by adding A and B?**

**Will A + B = C?**

---

Student activity 3.1:

Calculate as accurate as possible the values for A and B and C. Use the method that we discussed and go to the 4th decimal.

Is $C \approx A + B$?
Student activity 3.2:

If $5^A = 10$. (The same equation as above) Verify that the equation $5^D = 100$ has the solution $D=2A$. Do this by calculating the value for $D$ and then verifying on paper that it is indeed twice as big as $A$.

Student activity 3.3:

a) Solve the equation $3^A = 10$. Show that $A \approx 2.0959\ldots$ Use the repeated division method combined with the root taking procedure

b) Explain in words what the digits 2; 0; 9; 5; 9 etc actually mean in this context.

Student activity 3.4:

Evaluate through successive multiplication, roots and powers what the following numbers represent.

a) $3^{2.26185}$, $3^{2.26186}$

b) Explain how the increase from 2.26185 to 2.2186 causes the observed increase in the value of the power.

Student activity 3.5:

Formulate what the equation $6^B = 100$ means in terms of exponents and measures of multiplication with units of 6 and decimals.

Verify that if $1.25^A = 6$ and $1.25^B = 3$ then $1.25^A / 1.25^B = 1.25^C$ where $1.25^C = 2$. In particular verify that $A-B=C$

Student activity 3.6:

1) Solve $1.25^A = 6$ (go to the 4th decimal)
2) Solve $1.25^B = 3$ (4th decimal)
3) Solve $1.25^C = 2$ (4th decimal)

Then carry out the verification.

Which of the laws of exponents corresponds to this statement?

**Student activity 3.7:**
Formulate what we have investigated so far concerning rational exponents in the form of decimals.

**Student activity 3.8:**
How would you write the form $(3^{0.1})^5$ in terms of the unit of multiplication 3?

Explain your thinking using both the fraction interpretation of exponents and the integer interpretation of exponents.
Also explain briefly what fraction units mean in this context

Verify that $(3^{0.1})^5 = 3^{0.5}$ by applying multiplication and assuming that $3^{0.5}$ should correspond to $\sqrt[10]{3^5}$

**Student activity 3.9:**
Do the same with $(1.25^{0.01})^6$
And with $(0.335^{0.001})^4$

**Student activity 3.10:**
Suppose we switch the exponents in our previous activity:
Explain what $(3^{0.5})^{0.1}$ means.
First calculate the numbers with your calculator:

Find \( (3^{0.1})^5 \) =……..

Then find \((3^5)^{0.1}\) = ………..

What do you notice?

If you study the outcome of activity 3.9 carefully you will see that

1. The numbers are identical after the calculations
2. The switching of the exponents from inside out makes no difference in the outcome
3. There is a major difference between the interpretation of \((3^{0.1})^5\) and \((3^5)^{0.1}\)

What is that difference?

**Student activity 3.11:**
Describe as best as you can what the difference is between the two forms in terms of their calculations

If we use the language of fractions of factors in a multiplication, and think about exponents as measures of the amount of a certain unit present in a number, then we have:

1. \((n\sqrt[3]{(3)})\) represents the \(1/n\) fraction of the unit 3 in a multiplication.
2. \((n\sqrt[3]{(3)})^M\) represents repeated use, M times of that fraction of the factor 3.
3. \((3M)\) represents the use of the factor 3, M times.
4. \(n\sqrt[3]{(3^M)}\) represents the \(1/n\) fraction of the unit \(3^M\).

**Student activity 3.12:**
Analyze the expressions: \((p\sqrt[p]{(3.55)})^A\) and \(p\sqrt[p]{(3.55^A)}\)

**Explain the meaning of these expressions just like you did for the previous example.**

**Student activity 3.13:**
How would the number \((3^{1.6})^{1.3}\) be interpreted?
Hint: Start with $3^{1.6}$; Decompose the second, outer exponent into $1.3 = 1 + 0.3$
And decompose the inner exponent as $1.6 = 1 + 0.6$.
Read the first power $3^{1.6}$ as $3^1 \cdot 3^{0.6} = 3^1 \cdot (3^{0.1})^6$

**Question:** Which laws of exponents do you think this action refers to?

Think of $3^{1.6}$ as a possible new unit of multiplication and then apply the new decimal exponent of 1.3 to that:
Based on our system so far this will represent a multiplication of two forms, each based on the unit.
The first party takes the full, integer measure of $3^{1.6}$.
So we have our first factor which must be $3^{1.6}$

Then we have to take $1/10$ of that unit and then repeat the factor three (3) times:
$10\sqrt{(3^{1.6})}$ repeated 3 times or with mathematical symbols: $(10\sqrt{(3^{1.6})})^3$

**Question:** We have here both repeated multiplication and the fraction concept applied under decimal conditions.
Explain this point in your words.

If we study this closely we can see that this looks very much like the form we have above
We are allowed to switch the decimal and the integer and then it shows that we have
$10\sqrt{((3^{1.6})^3)}$ where we take a root of a known quantity.

We have no problem finding the meaning or value of $(3^{1.6})^3$ It just tells us to multiply the exponents by 3.

**Question:**

Why can we characterize this part of the explanation simply on repeated multiplication?

It should come as no surprise that taking the 10-th root of that just leads to a division of the exponent by 10.
In fact what we can see is that a decimal power of a decimal power leads just as naturally to the product of exponents as happens with integer exponents.

**Student activity 3.14:**

*(Use your calculator)*

Find \( X = 3^{1.6} \)

Find \( Y = X^{1.3} \)

Multiply \( 1.6 \times 1.3 = 2.08 = a \)

Show that \( 3^a = 3^2 \times (3^{0.1})^8 = Y \)

**Question: What does this suggests for the laws of exponents in the decimal case?**

- If the rate of growth is a positive number, like 5%, or 10% or 300%, we will always have a factor of multiplication that is more than 1.

**Student activity 4.1:**

Explain why this is the case.

**Student activity 4.2:**

What is the *rate of growth* associated with 1/100 of the unit 1.1?

**Student activity 4.3:**

Describe your arguments for making exponent 0 correspond to factor 1.
Student activity 4.4:
Graph the Function $F(X) = 2^* (1.25)^X$
Create a table first.
Use graph paper (ask the instructor)

Table C.4
*Division and negative exponents*

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>F(X)</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Student activity 4.5:
Complete the table into the negative domain and graph your function $F(X)$ into the negatives.

Student activity 4.6:
Discuss the connection between positive exponents and multiplication, and negative exponents and division.
Student activity 4.6:

Describe how negative exponents can be justified and what the interpretation of the negative sign could be in terms of, or when we deal with exponents.

Explain also the concept of directed numbers when dealing with exponents.

What is the function of the arrows in the table?
APPENDIX D

INTERVIEWS TWO, THREE AND FOUR
QUESTIONNAIRE PROTOCOL
INTERVIEW TWO QUESTIONNAIRE PROTOCOL

1. Explain in words and mathematical language what the 20% increase per 5 weeks means. Explain how to calculate the number of infected persons 5 weeks from now if today there are 250 infected persons.

2. Make a graph of the number of infected persons using the 20% increase to construct the number of infected persons step by step. Use a ruler and blank paper. Be very precise.

3. What is the rate of increase of the number of infected persons over a 10 week period.

4. If we have 1075 infected persons today, what was the number of infected persons 5 weeks earlier?

5. How can we find the rate of increase per 1 week? Explain the steps and concepts we need to find the answer. ROG for 1 wk?

6. Suppose we have a unit of multiplication 4. We want to know what the measure of 35 is in terms of (multiplicative) units of 4. Carry out the calculations and explain what the steps mean.
INTERVIEW THREE QUESTIONNAIRE PROTOCOL

1. What do we mean by the symbol $3^{0.7}$? Describe two ways to approach this symbol.

2. What is the meaning of the symbol $(3^{0.7})^{0.2}$? Use the language of factors of multiplication to explain the concept.

3. How do we know that $5^{2.4} = 52$. Explain why we can’t simply decompose the exponent into $2 + 0.4$. Assume that $5^{2.4}$ means $\sqrt[5]{5^{24}}$

4. The concept of exponents like $(1.25)^5$ involves Repeated Multiplication. How can we argue that concepts like $(1.25)^{0.35}$ also involve Repeated Multiplication? What exactly is Repeated Multiplied.

How is the Repeated Multiplication in $(1.25)^{0.35}$ different from the Repeated Multiplication in $(1.25)^5$?

5. Take the form $(6^{0.20})^5$ and compare it with a form like $(0.20)^3$ Describe both in terms of the Repeated Multiplication Model.

6. What is the concept of “multiplicative fraction”? How is it related to multiplication? Use the unit 5 and the "division of 5 into 10 equal parts" to illustrate your explanations.

7. Radicals behave somewhat like ordinary fractions. For example:

$$4\sqrt[3]{5} = \frac{1.2}{\sqrt[3]{5}}$$

Which radical exponent is correct?
How can we show why this works?
How is this related to our concept of rational exponents?

8. Another fraction-like behavior of radicals is this: \( \sqrt[5]{3^2} \cdot \sqrt[8]{3^7} = \sqrt[40]{3^{31}} \). Explain why this is related to rational exponents.

9. When we solve equations like \( 80 = 5^x \) we find \( x = 2.722 \). We could say that 5 fits into 80 about 2.722 times. Explain this approach to exponents.
INTERVIEW FOUR QUESTIONNAIRE PROTOCOL

1. Explain why multiplication and division are interconnected in the exponential function. Show how a rate of growth of say 5% for every year leads to a Repeated Multiplication Model and how multiplication and division are interconnected in this context.

2. Suppose we need to solve $0.8^x = 10$. Explain why this problem cannot be solved using our positive exponents?
   How is this problem related to Question 1.

3. Suppose we replace the 0.8 in Question 2 by its inverse, 1.25. Explain how the equation $1.25^y = 10$ can help us solve the equation from Question 2.

4. How would you formulate a definition of exponents that covers both the integer case, the rational case and the negative case?

5. Why is the factor of multiplication so important in studying rates of growth or decrease in exponential situations?

6. Suppose you have $(5)^{-0.6}$.
   What would that mean in terms of the factor of multiplication and in terms of the definition?

7. How would we have to interpret a form like $((5)^{-0.6})^{-2.3}$ in terms of the factor of multiplication and in terms of the definition?
8. Graph the exponential model for a 20% increase per unit over 4 such units in both directions.