OPTIMAL SLIDING MODE CONTROL AND STABILIZATION OF UNDERACTUATED SYSTEMS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

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2007

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Optimal sliding mode control and stabilization of underactuated systems by using sliding mode control are studied in this dissertation.

Sliding mode controls with time-varying sliding surfaces are proposed to solve the optimal control problem for both linear and nonlinear systems. The time-varying sliding surfaces are designed such that the system state is on the sliding surface from the beginning of the motion without reaching phase. Therefore, the behavior of the system is totally determined by these time-varying sliding surfaces. The original optimal control problem is also transformed into finding the optimal sliding surfaces. In some cases, the new problem is easy to solve than the original one. The main advantage of this kind of controls is that they can provide a more robust optimal control to the original problem. The optimal sliding mode control system should work in such a manner. In the region dominated by the system nominal part, the system behavior is mainly governed by optimal control. In the region where perturbation becomes dominant, sliding mode control will take over the main control task. Several approaches are applied to find the optimal solution or an approximation of the optimal solution for linear continuous-time, linear discrete-time and nonlinear optimal control problems.

As a special kind of nonlinear systems, underactuated systems are of great interest in both theoretical research and real applications. For underactuated systems,
which don’t satisfy Brockett’s necessary conditions, non-smooth sliding mode control could be used to robustly stabilize those underactuated systems. Two kinds of underactuated systems, those in the cascaded form and those in the chained form, are studied in the second part of the dissertation. Two sliding mode controllers are presented to stabilize those two kinds of underactuated systems with simulations of three benchmark examples.
This is dedicated to my family . . .
ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my adviser, Dr. Ümit Özgüner, for his guidance and continuous support during my Ph.D. study at The Ohio State University. Especially, I want to thank him for providing me the precious opportunities to participate in the first and second DARPA Grand Challenge. I would also like to thank Dr. Vadmin Utkin and Dr. Andrea Serrani for serving in my dissertation committee.

I want to thank Dr. Keith Redmill, Dr. Qi Chen, Dr. Lu Xu, Dr. Zhijun Tang, Dr. Yiting Liu, Yongjie Zhu, Mingfeng (Alex) Hsieh and everyone else who I was working with during the past five years. It was my pleasure to work with them.

I would like to acknowledge the financial support from the Intelligent Transportation System Fellowship program, the center for automotive research, the department of electrical and computer engineering, and the Ohio State University.

Finally, I would like to thank all my friends and my family for their help and support.
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CHAPTER 1

INTRODUCTION

The concept of sliding mode control (SMC) has received much attention in the past few decades. Sliding mode control is a technique in which an appropriate input is provided so that the system states are confined to a desired sub-manifold of the state space. The concept of sliding mode control was proposed by Utkin [17], who showed that sliding mode could be achieved by changing the controller structure. Therefore, sliding mode control is based on variable structure systems, in which the control input is switched between two control signals. The system state trajectory is forced to move along a chosen manifold in the state space, called the sliding manifold, by using an appropriate variable structure control signal. The sliding manifold itself guarantees the system stability, once the control restrictions are achieved. These restrictions define the existence or not of the sliding mode. The behavior of the closed-loop system is thus governed by the dynamics of the system on the sliding manifold [19], [21]. The main advantage of this technique is that, once the system state reaches the sliding manifold and sliding mode happens, the system dynamics remain insensitive to a class of parameter uncertainties and disturbances. Therefore, sliding mode control is a kind of robust control techniques. In relation to the classical control techniques, sliding mode control is simpler to implement, since only two input control values are
required. Unlike similarly implemented, standard on-off control, system stability is
guaranteed by the sliding manifold. Sliding mode control is considered as a high gain
control, which provides a fast and robust response to the system behavior. The basic
ideas of sliding mode control can be illustrated by the following simple example. For
a second-order continuous system and output error $e$, we have the switching function
$s$ to be defined as

$$s = ce + \frac{d}{dt}e = ce + \dot{e} \tag{1.1}$$

where $c$ is a positive constant number. If the system is on the sliding manifold $s = 0$
(assuming the existence condition), and, consequently, $s = ce + \dot{e} = 0$, which results
in a first-order system. Its solution is $e(t) = e(t_0) \exp[-c(t - t_0)]$, which implies that,
for $c > 0$, the system’s output error $e(t)$ tends exponentially to zero. The condition
for existence of the sliding mode and system stability is defined by

$$s\dot{s} \leq 0 \tag{1.2}$$

The control input $u$ is then defined as

$$u = -M \text{sgn}(s) \tag{1.3}$$

where $\text{sgn}(\cdot)$ is the sign function, which is 1 if $s > 0$, 0 if $s = 0$ and $-1$ otherwise.
The analysis of (1.2) results in the limit values of $M > 0$. With large enough $M,$
the switching control (1.3) can ensure the existence of the sliding mode even in the
presence of matched uncertainties and model errors. After the sliding mode happens,
the system behavior is determined only by the structure of the sliding manifold $s =
ce + \dot{e} = 0$, which is independent of the uncertainties and model errors.
1.1 Literature in sliding mode control

In the works of early years, the plant considered was a linear second-order system modelled in phase variable form. Since then, intensive studies have been carried out on the theory and applications of sliding mode control for various systems, including linear systems [21], nonlinear systems [31], multi-input/multi-output systems [13], discrete-time systems [4], large-scale and infinite-dimensional systems [14], systems with input and/or state delays [11], and stochastic systems [1]. In addition, the objectives of sliding mode control have been greatly extended from stabilization to other control functions. Sliding mode control approaches can also be used to construct state observers [19]. To analyze the sliding motion for systems affine in the control input, the equivalent control methodology has been introduced originally as a regularization technique [17]. It has been shown that the traditional discontinuous control switches in sliding mode so as to imitate the equivalent control in the average sense, which can be broadly defined as the continuous control that would lead to the invariance conditions for the sliding motion. According to the equivalent control methodology, the slow component of the discontinuous control is actually the equivalent control and this component can be extracted by passing the discontinuous control through a low pass filter, whose time constant is sufficiently large to filter out the high frequency switching terms, yet sufficiently small not to eliminate any slow component [19]. The equivalent control idea has also been used in the design of disturbance estimators based on sliding mode state observers [6].

Classical sliding mode control is based on the possibility of making and keeping the switching function $s$ identically zero by means of a discontinuous control acting on the first time derivative of $s$, and switching between high amplitude opposite values with
theoretically infinite frequency. Nevertheless, the implementation of sliding mode control techniques is troublesome in real applications because of the large control effort usually needed to assure robustness properties, and the possibility that the so-called chattering phenomenon can rise due to the limit frequency of the control input devices [19]. In particular the latter may lead to large undesired oscillations that can damage the controlled system. In order to avoid chattering effect, several continuous approximations of the discontinuous sliding mode control have been presented, in which the switching function \( s \) is enforced into a \( \Delta \) boundary layer of the sliding manifold \( s = 0 \). The side effect is the control accuracy is degraded, because the robustness of sliding mode control is lost inside the boundary layer. Moreover, the influence of the uncertainties and disturbance grows with the size \( \Delta \) of the boundary layer. Another way to eliminate the chattering effect is by means of high order sliding mode control approaches. They are characterized by discontinuous controls acting on the higher order time derivative of the switching function \( s \) instead of acting on its first time derivative, as it happens in classical sliding mode. Preserving the main advantages of the original approach with respect to robustness and easiness of implementation, they totally remove the chattering effect, because the discontinuity happens only on the first or higher order time derivative of the control input \( u \).

Due to the use of computers for control purpose, the concept of digital sliding mode control (DSMC) has also been a topic of study in the past two decades. In the case of DSMC design, the control input is applied only at certain sampling instants and the control effort is constant over the entire sampling period. Using an Euler approximation of the time derivative over the sampling period \( T \), Milosavljevic [12]
proposed the concept of quasi-sliding mode by a discrete-time extension of continuous-time sliding mode, \textit{i.e.}, the condition for sliding mode is given as

\[
\lim_{s_k \to 0^+} \Delta s_k \leq 0, \quad \lim_{s_k \to 0^-} \Delta s_k \geq 0
\]  

(1.4)

where \( \Delta s_k = s(t_k) - s(t_{k-1}) \), \( s_k = s(t_k) \), and \( t_k = kT \). To compute the discrete-time equivalent control \( u_k^{eq} \) for DSMC, two concepts have been used. The first one uses \[3\]

\[ s_{k+1} = s(t_{k+1}) = 0, \quad k = 0, 1, \ldots \]  

(1.5)

to obtain \( u_k^{eq} \), while the other employs \[4\]

\[ \Delta_k^s = s_{k+1} - s_k = 0, \quad k = 0, 1, \ldots \]  

(1.6)

It should be noted that (1.5) implies (1.6), however, the reverse is not true. Since (1.6) does not guarantee \( s_k = 0 \), the DSMC based on (1.6) should be more carefully designed to make \( s_k \) converge to the sliding manifold \( s_k = 0 \) \[5\].

Researchers have worked on the idea of robust optimal control in a system using the concept of variable structure control. As mentioned before, the main advantage of this technique is that, once the system state reaches a sliding surface, the system dynamics remain insensitive to a class of parameter uncertainties and disturbances. However, robustness of sliding mode control is guaranteed only after the system state reaches the sliding surface, and therefore robustness is not guaranteed during the reaching phase. Provided a conventional time-invariant sliding surface is considered, the advantage of sliding mode control, namely the desired dynamic behavior of the system, is not obtained for some time after the beginning of motion. As tracking error convergence may merely be asymptotic, the advantage might not even be achieved at all. Furthermore, for given initial conditions, there is trade-off between a short
reaching phase and fast system response in the sliding phase [13]. The more robust the system is owing to a short reaching phase, the slower the response in the sliding phase is. Usually, a conventional SMC is conservatively designed for the worst cases in which the system uncertainties and external disturbances are dominant. Stability or convergence is the main concern of SMC design in such circumstance. However, most real physical systems are not totally unknown to us, and they can be represented by nominal systems with a small portion of uncertainties. When the system nominal part is dominant, stability or convergence is no longer the only concern of control design, and other performance requirements like minimizing input energy should be taken into consideration. On the other hand, it is well known that optimal control provides a systematic design methodology which makes the designed controller “optimal” according to the performance index. The main limitation of optimal or suboptimal control is the requirement of complete system knowledge, or the sensibility to system uncertainties or perturbations. In order to introduce optimal control methods into the sliding mode design, Al-abbass and Özyüner [23] used a time-varying sliding surface in a bilinear form as \( s(x, t) = c(t)x(t) \) instead of conventional sliding surfaces from linear hyper-planes as \( s(x) = cx(t) \), where \( c \) is changed from constant to time-varying. By imposing some special requirements on \( \dot{c}(t) \), stability or other properties can be ensured in the resulting sliding mode dynamics. Young and Özyüner [45], [46] further developed this kind of time-varying sliding surfaces and constructed a sliding surface with the co-states of the plant for both continuous-time and discrete-time systems. In [19], a linear quadratic performance index with time-varying weighing matrices was introduced by Utkin to derive an optimal sliding mode. The resulting sliding mode manifold is also a time-varying manifold. Another kind of time-varying sliding
mode control is introducing extra dynamics into a sliding surface. A technique for designing the sliding surface using the linear quadratic (LQ) approach has been studied by Young and Özgüner [20]. The basic idea is that some states of the system are considered as the control inputs to the subsystem of the other states and LQ methods can be used to find the optimal control, or more precisely the optimal sliding mode. A sliding mode manifold using a linear operator as \( s = \mathcal{L}(x) \) was proposed by Young and Özgüner [42]. Using a state space realization of the linear operator \( \mathcal{L}(\cdot) \), i.e. \( \dot{z} = Fz + Gx \) and \( s = Hz + Kx \), the resulting sliding mode manifold is also defined as the intersection of linear hyper-planes in the extended state space \([x, z]\). Later, Young and Özgüner [44] proposed frequency shaping sliding mode control, which is linked with LQ optimal control and sliding mode control in the frequency domain. They designed compensators using the optimal control and realization methods for linear systems in the sliding mode. The sliding system with a compensator (extra dynamics) is an augmented system which is a higher-order system compared with the original system. However, the designed compensators may not only improve the stability of the sliding system but also yield desired performance and characteristics. Concepts such as model following control [1] and integral sliding mode [2] have been used to design control algorithms that give robust performance. Although most of the recently proposed variable structure control algorithms employ sliding surfaces determined in an \textit{ad hoc} manner, procedures for synthesizing sliding surfaces having optimal properties with respect to various performance indices were proposed in [18]. Unfortunately, research reported there concerned the sliding phase only, and no attention was paid to the system behavior in the reaching phase. Moreover, in the optimal sliding mode design the associated quadratic cost functional only has the
states in its arguments without consideration of the control effort, or uses the equivalent control in the sliding mode as the real control input. In order to overcome those difficulties, Xu and Özgüner [39] presented an optimal sliding mode control approach for linear systems and showed that integrating optimal design methodologies into the sliding surface design can give a robust solution to the optimal control problem with non-vanishing exogenous disturbances. This optimality is involved not only in the sliding phase but also in the reaching phase, and the real control input is used in the performance index instead of the equivalent control input used in the previous results. Later, they [41] extended the optimal sliding mode control for a class of nonlinear systems.

1.2 Related literature in optimal control

There is a huge literature on the topics of optimal control since it emerged in 1950s [61]. The time-optimal control laws (in terms of switch curves and surfaces) were obtained for a variety of second and third-order systems in the early fifties. Then the finite- and infinite-time linear quadratic optimal control problem has been intensively studied for both the unconstrained [58], [49] and constrained [65], [52], [64] cases. The nonlinear optimal control problem has also been the subject of intense research efforts for a long time; however, it is far less mature than the research of the linear optimal control problem. One of the main approaches to solve the nonlinear optimal control problem is the Hamilton-Jacobi-Bellman (HJB) equation. Unfortunately, in most cases this partial differential equation cannot be solved analytically and is computationally intractable due to its high dimensionality. Many researchers tried to extend the existing results, which were developed in the linear optimal control
problem, to solve the nonlinear optimal control problem. Al’Brecht [48] first assumed that the cost functional and system equations are both analytic, and showed that optimal control could be obtained in the form of a power series, whose terms could be sequentially obtained through the solution of a quadratic optimal control problem for the linearized system and subsequent solution of a series of linear differential equations. He also established the convergence form of this power series for single-input systems of the form \( \dot{x} = f(x) + Bu \). Lee and Markus [60] again employed the aforementioned analyticity assumption, and presented a unique and optimal feedback control law \( u = u^* \), which stabilizes the system \( \dot{x} = f(x, u) \), gives rise to a finite valued cost functional, and satisfies the HJB equation near the origin. Lukes [62] later relaxed the analyticity assumption to second-order differentiability. Beard et al. [51] introduced a practical algorithm for computing an optimal feedback control law for nonlinear systems from a sequence of suboptimal control laws that converge to the optimal control law. They solved the HJB equation recursively by using a technique based on Galerkin approximations. The advantage of their approach is that, once an asymptotically stabilizing control law is obtained as the initial solution \( u^{(0)} \), the algorithm will compute a sequence of \( u^{(i)} \) with improved performance. Other recursive algorithms to obtain the optimal control law can also be found in the literature [63]. Freeman and Kokotovic [55] employed the concept of inverse optimality, and showed that control Lyapunov functions are solutions of the HJB equations associated with sensible cost functionals. They used the Sontag’s formula [85] to compute the inverse optimal control law, and pointed out that, if the level curves of the control Lyapunov function fully agreed in shape with the value function, the Sontag’s formula gave the real optimal control. For a special class of nonlinear systems which affine in the
control input \( i.e., \dot{x} = f(x) + g(x)u \) Banks and Mhana [50] introduced the so-called state dependent Riccati equation (SDRE) approach to solve the infinite-time nonlinear optimal control problem. The basic idea is to factorize \( f(x) \) to a linear-like system as \( f(x) = A(x)x \) and solve the Riccati equation at each control instant. Huang and Lu [57] showed that for a given infinite-time nonlinear optimal control problem there exists at least one expansion of \( f(x) \) such that the resulting control law by means of SDRE approach is optimal.

1.3 Literature in stabilization of underactuated systems

In the past decade there has been increasing interest in underactuated systems. These systems are characterized by the fact that they have fewer actuators than the degrees of freedom to be controlled. Underactuated systems have very important applications such as free-flying space robots, underwater robots, surface vessels, manipulators with structural flexibility, etc. [87]. They are used for reducing weight, cost or energy consumption, while still maintaining an adequate degree of dexterity without reducing the reachable configuration space. Some other advantages of underactuated systems include no or less damage while hitting an object, and tolerance for failure of actuators.

For many underactuated systems, it is impossible to use smooth feedback to stabilize the systems around equilibrium states even locally. The key feature of many of these problems is the nonlinear coupling between the directly actuated degrees of freedom and the underactuated degrees of freedom. Brockett [67] has provided the following necessary conditions for the existence of stabilizing smooth feedback laws:
Theorem 1.1. [67] A necessary condition for the existence of a continuously differentiable asymptotically stabilizing feedback law for the system \( \dot{x} = f(x, u) \) is that

1. the linearized system has no uncontrollable modes associated with eigenvalues whose real part is positive;

2. there exists a neighborhood \( N \) of \( x = 0 \) and for each \( \xi \in N \) there is a control \( u_\xi \) steering the system from \( x = \xi \) at \( t = 0 \) to \( x = 0 \) at \( t = \infty \);

3. the map \( (x, u) \to f(x) + ug(x) \) is onto a neighborhood of 0.

For some underactuated systems, these conditions may not be satisfied. If the linearized system has uncontrollable modes associated with eigenvalues whose real part is positive, then the original system cannot be stabilized, even locally, by any smooth state feedback control law. However, Kawski [76] has proved that there exists a Hölder continuous (non-Lipschitz) stabilizing controller for a class of small-time locally controllable affine systems, even if the linearized system has uncontrollable modes associated with eigenvalues whose real part is positive. Qian et al. [82] proposed a non-Lipschitz continuous, globally stabilizing controller for a chain of odd power integrators perturbed by a \( C^1 \) lower-triangular vector field, which may not satisfy Brockett’s necessary conditions. Those results suggest that continuous feedback stabilization may overcome the difficulties encountered by smooth feedback stabilization [85].

Control researchers have given considerable attention to many examples of control problems associated with underactuated mechanical systems, and different control strategies have been proposed, such as back-stepping control [86], energy based or passivity based control [70], [78], adaptive non-smooth control [75], fuzzy control [77]
or intelligent control [68], hybrid control [71], etc. From those underactuated mechanical systems, researchers have generalized two classes of underactuated systems of great interest. One is in the cascaded normal form introduced by Olfati-saber [79] and the other is in the chained form introduced by Murray and Sastry [91]. Olfati-saber [80] also proposed a systematic approach to transform underactuated mechanical systems (e.g. the Acrobot [68], the TORA system [73], the VTOL aircraft [81], and the pendubot [70]) to the cascaded normal form. Although he did not present any general control law to stabilize underactuated systems in the cascaded normal form, the normal form itself is more convenient for control design. On the other hand, the chained form is used to model kinematics of nonholonomic mechanical systems. Murray and Sastry [91] have also given sufficient conditions to convert (via state feedback and coordinates transformation) a generic controllable nonholonomic system with two inputs to the chained form. Many nonholonomic mechanical systems can be described by kinematical models in the chained form or are feedback equivalent to chained form, among which the most interesting examples are a Dubin’s car and a Dubin’s car with multiple trailers [94].

Many researchers have put a great deal of effort on the control of a car-like vehicle in recent years. Most of these works have been concentrated on tracking and posture (including both the position and the orientation of a vehicle with respect to a fixed coordinate frame) stabilization problems. Such posture stabilization problem is more difficult than the tracking problem due to the fact that nonholonomic systems with more degrees of freedom than control inputs cannot be stabilized by any static state feedback control law [67]. For nonholonomic systems, control methods based on linearization techniques are not adequate and a complete nonlinear analysis is required.
Moreover, unlike the conventional differential type mobile robot, a car-like vehicle has a low limit on the turning radius and this constraint makes the problem more difficult. Therefore, many researchers tired to solve the stabilization problem via time-varying or discontinuous control laws or a mixture of both. Open-loop steering with path-planning [90], [92] and feedback stabilizing [88], [89] laws can be found in the literature.

1.4 Dissertation outline

A brief description of the contents of each chapter is presented as follows. In Chapter 2, optimal sliding mode controls (OSMC) with time-varying sliding mode are proposed to solve optimal control problems for linear continuous-time systems. By applying time-varying sliding mode, the reaching phase of SMC disappears or the system state is on the sliding surface from the beginning of the motion. Infinite-time LQR, finite-time LQR and minimum energy control methods are used to obtain the optimal sliding manifold, which makes the closed-loop system’s behavior “optimal” with respect to some performance indices. The concept of Itô Integral is employed to seek a set of optimal parameters for the optimal sliding mode controller. In the remainder of Chapter 2, simulation results will show that the OSMC provides robust solutions to the optimal control problems. In Chapter 3, optimal sliding mode control is extended for a class of nonlinear systems. Three different approaches (i.e. the HJB equation method, the CLF method and the SDRE method) are used to design the optimal sliding manifold. The performance of each method will be compared with each other by means of simulations. In Chapter 4, linear discrete-time systems with rate bounded disturbance and state dependent disturbance are considered. Optimal
discrete-time sliding mode controls are proposed to robustly stabilize such systems with optimal performance. In Chapter 5, sliding mode controls are designed to stabilize two classes of underactuated systems in two general forms, the cascaded normal form and the chained form. Several real application examples validate the effectiveness of sliding mode control of underactuated systems. In the final chapter, the dissertation is concluded with some concluding remarks and some possible directions for future research.
CHAPTER 2

OPTIMAL SLIDING MODE CONTROL OF LINEAR CONTINUOUS-TIME SYSTEMS

2.1 Introduction

Sliding mode control (SMC) has been widely recognized as a powerful control strategy for its ability of making a control system very robust, which yields complete rejection of external disturbances satisfying the matching conditions. The robustness properties are usually achieved by using discontinuous controls. Numerous theoretical studies as well as application researches are reported ([19], [44], [21], etc.). In recent decade, SMC has widely been extended to incorporate new techniques, such as higher-order sliding mode control, dynamic sliding mode control and optimal sliding mode control. These techniques retain the main advantages of SMC and also yield more accuracy and desired performances. A technique for designing the sliding surface using the linear quadratic (LQ) approach has been studied by Young et al. [20]. The basic idea is that some states of the system are considered as the control inputs to the subsystem of the other states and LQ methods can be used to find the optimal control, or more precisely the optimal sliding mode. Young and Özyürek [44] have proposed frequency shaping sliding mode control, which is linked with LQ optimal control and sliding mode control in the frequency domain. They designed compensators using the
optimal control and realization methods for linear systems in the sliding mode. The sliding system with a compensator (extra dynamics) is an augmented system which is a higher-order system compared with the original system. However, the designed compensators may not only improve the stability of the sliding system but also yield desired performance and characteristics. But these methods can only be applied to a linear time-invariant system, and the quadratic cost function only has the states in its arguments without consideration of the control effort.

A typical SMC is conservatively designed for the worst cases in which the system uncertainties and exogenous disturbances are dominant. In such circumstance stability or convergence is the main concern of SMC design. However, most real physical systems are more or less transparent to us with only a small portion of uncertainties. When the system nominal part is dominant, robustness is no longer the only concern of control design and other performance requirements should be taken into consideration, such as minimizing input energy, achieving faster tracking convergence, etc. It is well known that optimal control provides a systematic design methodology which makes the designed controller “optimal” according to the performance index. Numerous researches have shown that the LQ optimal control problem can be solved by using its associated Riccati equation [58]. Other optimal control problems such as minimum-time control problem and minimum-energy control problem for linear systems may be solved by means of Pontryagin Principle [61]. The main limitation of optimal or suboptimal control is the requirement of complete system knowledge, or the sensibility to system uncertainties or perturbations. A possible solution to this problem is to introduce SMC into optimal control whenever uncertainties are present. This can make optimal or suboptimal control more robust and applicable
to systems with uncertainties. Though SMC and optimal control are two extremal control strategies, it is still possible to integrate them together. The integrated control system should work in such a manner. In the region dominated by the system nominal part, the system behavior is mainly governed by optimal control. In the region where perturbation becomes dominant, SMC will take over the main control task. The integrated control method is especially effective when the exogenous disturbance is persistent (non-vanishing) nearby the equilibrium, and the system nominal part is continuous and consists of radically unbounded functions of the system states.

In this chapter, a sliding mode control with dynamic sliding surface is proposed to solve the minimum-energy control, the infinite- and finite-time LQ optimal control problem for linear continuous-time systems. With carefully design of the sliding surface, the reaching phase is eliminated and the robustness of the sliding mode control is ensured from the beginning of the motion. Several simulations results will show that the proposed optimal sliding mode control has better performance than that of the conventional sliding mode control.

2.2 An overview of classical sliding mode control for linear systems

With the aim of outlining the basic ideas of sliding mode control, we consider the following single-input linear system

\[ \dot{x} = Ax + Bu \]  \hspace{1cm} (2.1)

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R} \). A switching surface is then defined as a linear hyperplane

\[ S = \{ x \in \mathbb{R}^n : s(x) = Cx = 0 \} \]  \hspace{1cm} (2.2)
where $C$ is chosen to make $CB \neq 0$ and $A_e = [I - B(CB)^{-1}C]A$ a Hurwitz matrix. The so-called equivalent control $u_{eq}$ can be computed by imposing $\dot{s} = 0$ as

$$\dot{s} = CAx + CBu = 0 \Rightarrow u_{eq} = -(CB)^{-1}CAx$$

(2.3)

And the switching control $u_{sw}$ is designed to force the system converge to the sliding manifold $s = 0$ in finite time

$$u_{sw} = -(CB)^{-1}M \text{sgn}(s)$$

(2.4)

where $M > 0$ is a constant number and $\text{sgn}(s)$ is the signum function, which is defined as

$$\text{sgn}(s) = \begin{cases} 1 & s \geq 0 \\ -1 & s < 0 \end{cases}$$

(2.5)

The existence of sliding mode can be ensured by applying sliding mode control

$$u = u_{eq} + u_{sw}$$

(2.6)

on the linear system (2.1). This can be showed by using a Lyapunov function candidate $V = s^2/2$. The time derivative of $V$ is

$$\dot{V} = s\dot{s} = s[CAx + CBu] = s[CAx + CB(u_{eq} + u_{sw})] = -sM\text{sgn}(s) = -M|s| \leq 0$$

Since $\dot{V} < 0$ when $s \neq 0$, the system will converge to the sliding manifold $s = 0$. Moreover, the convergence time is finite, because $\dot{s} = -M\text{sgn}(s)$. Once the system
reaches \( s = 0 \), it will stay in the sliding manifold forever. In sliding mode, \( s \equiv 0 \) and \( \dot{s} = 0 \), the system dynamics is governed by

\[
\dot{x} = \left[ I - B(CB)^{-1}C \right] Ax = A_e x
\]  

(2.7)

Since \( A_e \) is a Hurwitz matrix, the system state \( x \) will converge to zero asymptotically.

### 2.3 Chattering phenomenon and boundary layer control

Classical sliding mode control is based on the possibility of making and keeping the switching function \( s \) identically zero by means of a discontinuous control acting on the first time derivative of \( s \), and switching between high amplitude opposite values with theoretically infinite frequency. In the presence of switching imperfections, such as switching time delays and small time constants in the actuators, the discontinuity in the feedback control produces particular dynamics in the vicinity of the sliding surface, which is usually called as chattering [19]. This phenomenon may lead to large undesired oscillations that degrade the performance of the system and damage the controlled system. In order to avoid chattering effect, several continuous approximations of the discontinuous sliding mode control have been presented, in which the switching function \( s \) is enforced into a boundary layer of the sliding manifold \( s = 0 \). We also call it boundary layer control, because the control objective now is to make the system approach to and stay in a boundary layer. In the simplest case, the control law merely consists of replacing the signum function by a continuous approximation with a high gain in the boundary layer. For example, the signum function can be replaced by saturation functions shown in Fig. 2.1. The side effect is the control accuracy is degraded, because the robustness of sliding mode control is lost inside the boundary layer. Moreover, the influence of the uncertainties and disturbance grows
Figure 2.1: Saturation function $\sigma_\varepsilon(s)$

with the size of the boundary layer. Another way to eliminate the chattering effect is by means of high order sliding mode control approaches. They are characterized by discontinuous controls acting on the higher order time derivative of the switching function $s$ instead of acting on its first time derivative, as it happens in classical sliding mode. The actual dynamics of the controlled system in the boundary layer can be obtained by standard singular perturbation analysis. Let us consider a simple second-order example

$$
\dot{x}_1 = x_2 \\
\dot{x}_2 = u
$$

(2.8)

The switching function is defined as $s = cx_1 + x_2$ where $c > 0$. Following the procedure discussed in the previous section, we can design a sliding mode control as

$$
u = -cx_2 - M \text{sgn}(s)
$$

(2.9)
If the signum function \( \text{sgn}(\cdot) \) is approximated by a saturation function whose slope is \( 1/\varepsilon \), the system dynamics in the boundary layer \( |s| < \varepsilon \) can be described by

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\varepsilon \dot{x}_2 &= -\varepsilon cx_2 - M(cx_1 + x_2)
\end{align*}
\] (2.10)

The slow motion is defined by setting \( \varepsilon = 0 \)

\[
\dot{x}_{1s} = x_{2s} = -cx_{1s}
\] (2.11)

It is worthwhile to note that the slow motion is identical to the sliding motion in the sliding mode \( s = cx_1 + x_2 = 0 \). In the time scale \( t/\varepsilon \), the fast motion is defined by

\[
\dot{x}_{2f} = -(\varepsilon c + M)x_{2f} - Mcx_{1s}
\] (2.12)

The real dynamics is approximated by superposition of the slow motion and the fast motion

\[
\begin{align*}
x_1 &= x_{1s} = x_{10}e^{-c(t-t_0)} \\
x_2 &= x_{2s} + x_{2f} \\
&= -cx_{10}e^{-c(t-t_0)} + x_{20}e^{-\frac{c+M}{\varepsilon}(t-t_0)} - \frac{M}{\varepsilon c + M}x_{1s} \left[ 1 - e^{-\frac{c+M}{\varepsilon}(t-t_0)} \right]
\end{align*}
\] (2.14)

where \( t_0 \) is the time instant when the system first reaches the switching surface \( s = 0 \), and \( x_{10} \) and \( x_{20} \) are the values of \( x_1 \) and \( x_2 \) at \( t = t_0 \), respectively. Clearly, the continuous approximation of the sliding mode control using saturation functions also stabilizes the system.
2.4 Sliding mode control with time-varying sliding mode manifold

Consider the following single input linear system with a disturbance

\[ \dot{x} = Ax + Bu + B\xi \]  \hspace{1cm} (2.15)

where \( x = [x_1, \cdots, x_n]^T \) is the state vector, \( u \) the single control input, \( \xi \) the bounded extern disturbance, \( |\xi| < \rho \), \( \rho \) a positive constant number. \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times 1} \) and \( (A, B) \) is a controllable pair. Without loss of generality \( A \) and \( B \) are assumed to have the following forms

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 \\
a_1 & a_2 & \cdots & a_{n-1} & a_n
\end{bmatrix} \quad B = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]  \hspace{1cm} (2.16)

because for each controllable pair \((A, B)\) there exists a nonsingular transform matrix \( T \) which makes \((TAT^{-1}, TB)\) have the above forms. To design a sliding mode control for (2.15) a dynamic sliding surface is defined as follow

\[ s(t) = Cx(t) + \phi(t) \]  \hspace{1cm} (2.17)

where \( C = [c_1, \cdots, c_n], \) \( CB \neq 0, \) \( \phi(t) \) is differentiable and to be defined later. It is obvious that \( s(0) = 0 \) if \( \phi(0) = -Cx(0) \). Therefore, the reaching time to the sliding manifold is zero, or the system is on the sliding manifold at the beginning. A sliding mode control is designed to keep the system on the sliding manifold \( s = 0 \), or inside a boundary layer \( \Omega_\varepsilon = \{s||s| < \varepsilon\} \) where \( \varepsilon > 0 \) is a small constant number. The latter one is chosen in order to avoid the chattering effect in the control input. Thus, the sliding mode control is designed as

\[ u = -(CB)^{-1}[CAx + \dot{\phi} + M\sigma_\varepsilon(s)] \]  \hspace{1cm} (2.18)
where $M > \rho |CB|$. $\sigma_\varepsilon(\cdot)$ is a saturation function defined as

\[
\sigma_\varepsilon(s) = \begin{cases} 
1 & \text{if } s > \varepsilon \\
\frac{s}{\varepsilon} & \text{if } |s| \leq \varepsilon \\
-1 & \text{if } s < -\varepsilon
\end{cases}
\] (2.19)

**Theorem 2.1.** The state of (2.15) stays in the boundary layer $\Omega_\varepsilon = \{s||s| < \varepsilon\}$ from the beginning of the motion under the sliding mode control defined in (2.18) if $s(0) = 0$.

**Proof.** A Lyapunov function candidate can be chosen as $V = s^T s/2$. Its time derivative is

\[
\dot{V} = ss = s \left(CAx + CBu + CB\xi + \dot{\phi}\right) = s[-M\sigma_\varepsilon(s) + CB\xi]
\]

On the boundary of $\Omega_\varepsilon$, i.e., $|s| = \varepsilon$,

\[
\dot{V} = s[-M\text{sign}(s) + CB\xi] < 0
\]

because $M > \rho |CB| > |CB\xi|$. Therefore, $\Omega_\varepsilon$ is an invariant set of system (2.15) under the sliding mode control (2.18). If $s(0) = 0$, which locates inside $\Omega_\varepsilon$, the state of (2.15) will stay inside $\Omega_\varepsilon$ forever from the beginning of the motion even in the presence of the disturbance $\xi$. \hfill \square

Thus, the actually applied control is

\[
u = -(CB)^{-1}\left[CAx + \dot{\phi} + \frac{Ms}{\varepsilon}\right]
\] (2.20)

Denote $x_{n+1} = \phi$, $\overline{x} = [x_1, x_2, \cdots, x_n, x_{n+1}]^T$ and $v = \dot{\phi}$. After substituting the new variables and the sliding mode control (2.20) into the original system (2.15), we obtain the closed-loop system as

\[
\dot{\overline{x}} = A\overline{x} + Bv + B\xi
\] (2.21)
where

$$\overline{A} = \begin{bmatrix} A - B(CB)^{-1}(CA + \frac{M}{\varepsilon} C) & -B(CB)^{-1} \frac{M}{\varepsilon} \\ 0 & 0 \end{bmatrix}$$

$$\overline{B} = \begin{bmatrix} -B(CB)^{-1} \\ 1 \end{bmatrix}$$

$$\overline{B}_\xi = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

Since the sliding mode control takes care of the disturbance, we now can focus on designing $v$ for the regulation or tracking problem of the above closed-loop system without consideration of the disturbance $\xi$. In the next sections, several optimal control problems will be discussed in the coordinate of $(\overline{x}, v)$ after applying the sliding mode control (2.20).

### 2.5 Sliding mode control for infinite-time optimal quadratic regulators

#### 2.5.1 Sliding mode manifold design

A special case of the sliding mode manifold design is directly related to linear quadratic optimal regulators. Suppose that we have a quadratic cost functional to be minimized for the original system (2.15)

$$I(x, u) = \frac{1}{2} \int_0^\infty (x^T Q x + Ru^2) dt \quad (2.22)$$

where

$$Q = [q_{ij}] \geq 0, \quad R > 0$$

Similarly, the cost functional $I(x, u)$ can be represented by using the new coordinate $(\overline{x}, v)$ as

$$I(\overline{x}, v) = \frac{1}{2} \int_0^\infty (\overline{x}^T \overline{Q} \overline{x} + 2\overline{x}^T \overline{N} v + \overline{R} v^2) dt \quad (2.23)$$
where

\[
\bar{Q} = \begin{bmatrix}
Q + R(CB)^{-2}(A^T C^T + \frac{M}{\varepsilon}C^T)(CA + \frac{M}{\varepsilon}C) & R(CB)^{-2}(A^T C^T + \frac{M}{\varepsilon}C^T)\frac{M}{\varepsilon} \\
R(CB)^{-2}(CA + \frac{M}{\varepsilon}C)\frac{M}{\varepsilon} & R(CB)^{-2}\frac{M^2}{\varepsilon^2}
\end{bmatrix},
\]

\[
\bar{N} = R(CB)^{-2}\begin{bmatrix} A^T C^T + \frac{M}{\varepsilon}C^T \end{bmatrix}, \quad \bar{R} = R(CB)^{-2}
\]

If the disturbance is not considered, then the original optimal control problem is transformed in the new coordinate as

\[
\min_v I(x, v) = \min_v \frac{1}{2} \int_0^\infty (x^T \bar{Q} x + 2x^T \bar{N} v + \bar{R} v^2) dt \\
\text{s.t.} \quad \dot{x} = \bar{A} x + \bar{B} v
\]

This optimal control problem can be solved by solving the Riccati equation if the pair \((\bar{A}, \bar{B})\) is stabilizable. Therefore, \(C\) is required to make \((\bar{A}, \bar{B})\) stabilizable besides \(CB \neq 0\). This is not a difficult task if \((A, B)\) is controllable. Thus, the optimal solution to the above optimal control problem is given by

\[
\dot{\phi} = v^* = \bar{R}^{-1}(\bar{B}^T P - \bar{N}^T)x
\]

where \(P\) is the solution to the following generalized matrix Riccati equation

\[
-(P\bar{B} - \bar{N})\bar{R}^{-1}(\bar{B}^T P - \bar{N}^T) - P\bar{A} - \bar{A}^T P + \bar{Q} = 0
\]

Denote \(K = [k_1, \cdots, k_{n+1}] = \bar{R}^{-1}(\bar{B}^T P - \bar{N}^T) \in \mathbb{R}^{1 \times (n+1)}\). The optimal choice of \(\dot{\phi}\) can be represented by

\[
\dot{\phi} = \sum_{i=1}^n k_i x_i + k_{n+1} \phi
\]

with initial condition \(\phi(0) = -Cx(0)\). The sliding mode control defined in (2.18) is then implementable.

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2.5.2 Seeking the optimal controller parameters

Now, we take the disturbance $\xi$ into consideration. For simplicity, we can pick up $c_n = 1$ such that $CB = 1$. By using the optimal sliding mode control derived above, the closed-loop system with disturbance is transformed into

$$\dot{x} = \hat{A}x + \hat{B}\xi$$  \hspace{1cm} (2.27)

where

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n & \bar{a}_{n+1} \\ k_1 & k_2 & \cdots & k_n & k_{n+1} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and $\bar{a}_i$ are some constant parameters. Without the disturbance $\xi$, this system will converge to the origin asymptotically. In the presence of the bounded disturbance $\xi$, however, the system states will move randomly but inside a small neighborhood around the origin. Our goal is to find optimal values for $M$ and/or $\varepsilon$ which minimize the size of the neighborhood given sufficient information of the disturbance $\xi$. For the sake of simplicity, we consider the disturbance $\xi(t)$ as a stochastic process and the following assumptions are made:

1. The disturbance is bounded, i.e. $\sup_{0 \leq t \leq \infty} |\xi(t)| \leq \delta$, where $\delta > 0$ is a constant number.

2. $t_1 \neq t_2 \Rightarrow \xi(t_1)$ and $\xi(t_2)$ are independent.

3. $\{\xi(t)\}$ is stationary, i.e. the (joint) distribution of $\{\xi(t_1 + t), \ldots, \xi(t_k + t)\}$ does not depend on $t$. 

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4. The expectation function of $\xi(t)$, $E[\xi(t)] = 0$ for all $t$.

With the above assumptions, the equations of system (2.27) can be represented by stochastic differential equations

\[
\begin{align*}
    \text{dx}_1 &= x_2dt \\
    \vdots \\
    \text{dx}_{n-1} &= x_n dt \\
    \text{dx}_n &= \left[ \sum_{i=1}^{n+1} c_i x_i - \frac{M}{\varepsilon} x_n + \frac{M}{\varepsilon} x_{n+1} \right] dt + dW_t(\omega) \\
    \text{dx}_{n+1} &= \sum_{i=1}^{n+1} c_i x_i dt
\end{align*}
\]

where $W_t(\omega)$ is 1-dimensional Brownian motion starting at the origin. The solution to (2.28) gives us $(n + 1)$ stochastic processes: $x_1(t, \omega), x_2(t, \omega), \ldots, x_{n+1}(t, \omega)$. In order to obtain the solution to (2.28), the Itô integral is needed. In the following, the basic ideas of the Itô integral will be introduced.

**Definition (Itô Integral)** For functions $f(t, \omega) \in [0, \infty) \times \Omega \rightarrow \mathbb{R}$, the Itô integral is

\[
I[f](\omega) = \int_{t_0}^{t_f} f(t, \omega) dW_t(\omega)
\]  

(2.29)

The Itô integral has the following properties:

1. $\int_{t_0}^{t_f} f dW_t = \int_{t_0}^{t_1} f dW_t + \int_{t_1}^{t_f} f dW_t$

2. $\int_{t_0}^{t_f} (cf + g) dW_t = c \int_{t_0}^{t_f} f dW_t + \int_{t_0}^{t_f} g dW_t$ with $c$ constant

3. $E[\int_{t_0}^{t_f} f dW_t] = 0$. Especially, $E[\int_{t_0}^{t_f} W_t dW_t] = 0$

4. $\int_{t_0}^{t_f} f(\tau) dW_\tau = f(t)W_t - \int_{t_0}^{t_f} W_\tau d f_\tau$

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Theorem 2.2. [98] (Existence and uniqueness theorem for stochastic differential equations) Let $t_f > 0$ and $b(\cdot, \cdot) : [0, t_f] \times \mathbb{R}^n \to \mathbb{R}^n$, $\lambda(\cdot, \cdot) : [0, t_f] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ be measurable functions satisfying

$$|b(t, x)| + |\lambda(t, x)| \leq C(1 + |x|); \quad x \in \mathbb{R}^n, \quad t \in [0, t_f]$$

(2.30)

for some constant $C$ and such that

$$|b(t, x) - b(t, y)| + |\lambda(t, x) - \lambda(t, y)| \leq D|x - y|; \quad x, y \in \mathbb{R}^n, \quad t \in [0, t_f]$$

(2.31)

for some constant $D$. Let $Z$ be a random variable such that $E[|Z|^2] < \infty$. Then the stochastic differential equation

$$dX_t = b(t, X_t)dt + \lambda(t, X_t)dW_t, \quad 0 \leq t \leq t_f, \quad X_0 = Z$$

(2.32)

has a unique $t$-continuous solution $X_t(\omega)$ with the property that

$$E \left[ \int_0^{t_f} |X_t|^2 dt \right] < \infty$$

(2.33)

Corollary 2.3. [98] (The Itô isometry)

$$E \left[ \left( \int_{t_0}^{t_f} f(t, \omega)dW_t \right) \left( \int_{t_0}^{t_f} g(t, \omega)dW_t \right) \right] = E \left[ \int_{t_0}^{t_f} f(t, \omega)g(t, \omega)dt \right]$$

(2.34)

for all $f(t, \omega) \in [0, \infty) \times \Omega \to \mathbb{R}$

Especially,

$$E \left[ \left( \int_{t_0}^{t_f} f(t, \omega)dW_t \right)^2 \right] = E \left[ \int_{t_0}^{t_f} f^2(t, \omega)dt \right]$$

(2.35)

Theorem 2.4. [98] (The 1-dimensional Itô formula) Let $X(t)$ be an Itô process given by $dX_t = udt + vdB_t$. Let $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$. Then $Y_t = g(t, X_t)$ is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2$$

(2.36)
where \((dX_t)^2 = (dX_t) \cdot (dX_t)\) is computed according to the rules
\[
dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0, \quad dW_t \cdot dW_t = dt \tag{2.37}
\]

**Definition (n-dimensional Itô process)** Let \(W(t, \omega) = [w_1(t, \omega), ..., w_m(t, \omega)]^T\) denote \(m\)-dimensional Brownian motion. We can form the following \(n\) Itô processes
\[
\begin{aligned}
dx_1 &= u_1 dt + v_{11} dw_1 + \cdots + v_{1m} dw_m \\
\vdots \\
dx_n &= u_n dt + v_{n1} dw_1 + \cdots + v_{nm} dw_m
\end{aligned} \tag{2.38}
\]
Or, in matrix form
\[
dX = ud + vdW \tag{2.39}
\]
where
\[
X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nm} \end{bmatrix}, \quad dW = \begin{bmatrix} dw_1 \\ \vdots \\ dw_m \end{bmatrix} \tag{2.40}
\]

**Theorem 2.5.** [98] (The general Itô formula) Let \(dX(t) = ud + vdW(t)\) be an \(n\)-dimensional Itô process. Let \(g(t, x) = [g_1(t, x), ..., g_p(t, x)]^T\) be a \(C^2\) map from \([0, \infty) \times \mathbb{R}^n\) into \(\mathbb{R}^p\). Then the process \(Y(t, \omega) = g(t, X(t))\) is again an Itô process, whose component number \(k\), \(Y_k\) is given by
\[
dY_k = \frac{\partial g_k}{\partial t}(t, X)dt + \sum_i \frac{\partial g_k}{\partial x_i}(t, X)dx_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t, X)dx_i dx_j \tag{2.41}
\]
where \(dw_i dw_j = \delta_{ij} dt, \quad dw_i dt = dt dw_i = 0\).

**Theorem 2.6.** (The solution to \(n\)-dimensional linear time invariant stochastic differential equation) Consider a matrix stochastic differential equation
\[
dX = AX dt + B dW_t \tag{2.42}
\]
where $dX = [dx_1, ..., dx_n]^T$, $A \in \mathbb{R}^{n \times n}$ is time invariant, $B \in \mathbb{R}^{n \times 1}$ is time invariant and $W_t$ is a 1-dimensional Brownian motion. Then the unique solution to (2.42) is given by

$$X(t) = \exp(At) \left[ X(0) + \int_0^t \exp(-A\tau)BdW_\tau \right]$$

(2.43)

Proof. Multiply $\exp(-At)$ to the both sides of (2.42) and rewrite it

$$\exp(-At)dX - \exp(-At)AXdt = \exp(-At)BdW_t$$

(2.44)

According to the property of integration by parts, the left hand side of (2.44) is $d(\exp(-At)X)$. Therefore, we rewrite (2.44) as

$$d(\exp(-At)X) = \exp(-At)BdW_t$$

(2.45)

Now, consider a function $g : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by

$$g(t, X) = \exp(-At)X$$

(2.46)

Applying the general Itô formula to the function $g(t, X)$ yields

$$d(\exp(-At)X) = (-A)\exp(-At)Xdt + \exp(-At)dX$$

(2.47)

Substituted in (2.45) and after integration, this gives the result

$$X(t) = \exp(At) \left[ X(0) + \int_0^t \exp(-A\tau)BdW_\tau \right]$$

Now, let us go back to the optimization problem. The cost functional is modified to meet the stochastic property of the disturbance $\xi$ by using the expectation of the cost. Suppose that the system states have reached the origin, we want to find optimal
values for $M$ and/or $\varepsilon$ such that the average size of the neighborhood of the variation of the states under the influence of the disturbance $\xi$ is minimized during a long enough time period $T$. Then, the optimization problem is presented as

$$\min_{M, \varepsilon} E[I] = \min_{M, \varepsilon} E \left[ \frac{1}{2} \int_0^T \bar{x}^T \hat{Q} \bar{x} dt \right]$$

s.t.  
$$d\bar{x} = \hat{A}\bar{x} dt + \hat{B}dW_t$$

$$\bar{x}(0) = 0$$

where

$$\hat{Q} = \begin{bmatrix} q_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & q_n & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \quad q_i > 0 \text{ for } i = 1, \cdots, n$$

Applying (2.43) to the original closed-loop system with initial condition $\bar{x}(0) = 0$, we obtain

$$\bar{x}(t) = \int_0^t \exp(\hat{A}(t - \tau)) \hat{B}dW_\tau$$

(2.48)

and by using the Itô isometry

$$E[\bar{x}(t)\bar{x}^T(t)] = E \left[ \left\{ \int_0^t \exp(\hat{A}(t - \tau)) \hat{B}dW_\tau \right\} \left\{ \int_0^t \exp(\hat{A}(t - \tau)) \hat{B}dW_\tau \right\}^T \right]$$

$$= E \left[ \int_0^t \exp(\hat{A}(t - \tau)) \hat{B} \hat{B}^T \{ \exp(\hat{A}(t - \tau)) \}^T d\tau \right]$$

$$= \int_0^t \exp(\hat{A}(t - \tau)) \hat{B} \hat{B}^T \{ \exp(\hat{A}(t - \tau)) \}^T d\tau$$

(2.49)

Denote $\exp(\hat{A}t) = [\hat{a}_{ij}(t)]$, where $\hat{a}_{ij}(t)$ is the $i$th row and $j$th column entry of $\exp(\hat{A}t)$. It should be noted that $\hat{a}_{ij}(t)$ is also a function of $M$ and $\varepsilon$. Recall that

$$\hat{B} \hat{B}^T = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$
We have
\[ E[x_i^2(t)] = \int_0^t \hat{a}_m^2(t - \tau) d\tau = \int_0^t \hat{a}_m^2(\tau) d\tau \text{ for } i = 1, \ldots, n \tag{2.50} \]

Then, the cost functional becomes
\[
E[I] = E \left[ \frac{1}{2} \int_0^T \left( \sum_{i=1}^n q_i x_i^2 \right) dt \right] \\
= \frac{1}{2} \int_0^T \left( \sum_{i=1}^n q_i E[x_i^2] \right) dt \\
= \frac{1}{2} \int_0^T \left( \sum_{i=1}^n q_i \int_0^t \hat{a}_m^2(\tau) d\tau \right) dt \tag{2.51}
\]

Since it is impossible to obtain the analytical expression of $E[J]$ as a function of $M$ and $\varepsilon$, some numerical methods have to be applied to find the optimal values for $M$ and/or $\varepsilon$.

Because the two optimization problems are coupled, we cannot obtain one solution before the other. An iterative optimization procedure is proposed as follows (the superscript $(k)$ is the iteration number)

- **Step 1:** Pick up two suitable $M^{(0)}$ and/or $\varepsilon^{(0)}$ for $M$ and/or $\varepsilon$ respectively;

- **Step 2:** Compute the optimal $c_i^{(k)}$ for $i = 1, \ldots, n + 1$ from (2.24) by using $M^{(k-1)}$ and/or $\varepsilon^{(k-1)}$;

- **Step 3:** Find the numerical minimizers $M^{(k)}$ and/or $\varepsilon^{(k)}$ from (2.51) by using $c_i^{(k)}$;

- **Step 4:** If $(|M^{(k)} - M^{(k-1)}| < \epsilon_1$ and $|\varepsilon^{(k)} - \varepsilon^{(k-1)}| < \epsilon_2)$ or $(|\min E[J]^{(k)} - \min E[J]^{(k-1)}| < \epsilon_3)$, then stop and give out the optimal values $c_i^{(k)}$, $M^{(k-1)}$, $\varepsilon^{(k-1)}$. $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$ are error tolerances. Otherwise, go back to Step 2 and continue the procedure with $k = k + 1$. 

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<table>
<thead>
<tr>
<th>Name</th>
<th>Value</th>
<th>Name</th>
<th>Value</th>
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</thead>
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<tr>
<td>$c_1$</td>
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<td>$c_2$</td>
<td>1</td>
</tr>
<tr>
<td>$M$</td>
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<td>$k_1$</td>
<td>-10</td>
</tr>
<tr>
<td>$k_2$</td>
<td>175.0455</td>
<td>$k_3$</td>
<td>-186</td>
</tr>
</tbody>
</table>

Table 2.1: OSMC parameters for LQR problem

### 2.5.3 Illustrative example

Consider a second order system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + \xi
\end{align*}
\] (2.52)

with a cost functional

\[
I = \frac{1}{2} \int_{0}^{\infty} (x^T Q x + R u^2) dt
\] (2.53)

where

\[
Q = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}, \quad R = 1
\]

We pick up $\varepsilon = 0.1$. The optimal parameters for the optimal sliding mode control (OSMC) $u^{opt, smc}$ are computed from the aforementioned iterative optimization procedure as tabulated in Table (2.1). For the sake of comparison, optimal state feedback control (optimal control) $u^{opt}$ based on $Q$ and $R$ and a conventional sliding mode control (conventional SMC) $u^{smc}$ are also designed. The optimal control $u^{opt}$ can be easily obtained by solving the Riccati equation, and is give by

\[
u^{opt} = -10x_1 - 10.9545x_2
\] (2.54)

and the conventional SMC $u^{smc}$ is given by

\[
u^{smc} = -2x_2 - 15\sigma_{\varepsilon}(2x_1 + x_2)
\] (2.55)
For the disturbance $\xi$, a band-limited white noise is applied, whose maximum magnitude is less than 15. One realization of the white noise is shown in Fig. 2.2. Ten realizations of white noises are generated. Each realization is used in three simulations by using $u^{\text{opt}}$, $u^{\text{smc}}$ and $u^{\text{opt,smc}}$, respectively. The initial conditions for the system equations (2.52) are

$$x_1(0) = 3, \quad x_2(0) = 0$$

The simulation results of the optimal-control, the conventional-SMC and the optimal-SMC under one realization of white noise are shown in Fig. 2.3, 2.4 and 2.5, respectively. For comparison, the responses of $x_1$ of three different controls are put together in Fig. 2.6. From Fig. 2.6, the following observations are made:
1. Before 2 seconds, when $x_1$ is far from the origin, the response of the optimal SMC is very close to that of the optimal control. This means that the optimal SMC is really “optimal”.

2. After 6 seconds, when $x_1$ is around the origin, the variation of the response of the optimal control is much larger than those of the other two sliding mode controls. This is because the optimal control is too weak to offset the influence of the disturbance when the states are close to the origin. While the sliding mode controls can perform more robustly because of their variable structures. This means that the optimal SMC is more robust than the optimal control.

In order to analyze quantitatively the performances of three controls, the values of cost functionals are needed. Since we can not integrate till infinity time, the cost
Figure 2.4: The simulation results of the conventional sliding mode control

Figure 2.5: The simulation results of the optimal sliding mode control

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functional is modified to integrate during a finite time period as

\[ I_1 = \frac{1}{2} \int_0^{20} \left[ 100x_1^2(t) + 100x_2^2(t) + u^2(t) \right] dt \]  

(2.56)

Moreover, the following cost functional is used to measure the robustness of the controls around the origin.

\[ I_2 = \frac{1}{2} \int_0^{20} \left[ 100x_1^2(t) + 100x_2^2(t) \right] dt \]  

(2.57)

The values of cost functionals \( I_1 \) and \( I_2 \) of each simulation are tabulated in Table 2.2 and 2.3, respectively. From Table 2.2 and 2.3, the following observations are made:

1. For \( I_1 \), the optimal control is the best, and the conventional SMC is the worst. However, the performance the optimal SMC is quite close to that of the optimal control, while the conventional SMC is much worse. This means that the optimal sliding mode control is nearly “optimal".

Figure 2.6: Comparison of three controls
<table>
<thead>
<tr>
<th>Noise realization</th>
<th>Optimal control</th>
<th>Sliding mode control</th>
<th>Optimal SMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>482.4</td>
<td>562.5</td>
<td>485.9</td>
</tr>
<tr>
<td>2</td>
<td>479.7</td>
<td>560.4</td>
<td>485.1</td>
</tr>
<tr>
<td>3</td>
<td>478.6</td>
<td>558.0</td>
<td>485.5</td>
</tr>
<tr>
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<td>478.4</td>
<td>557.0</td>
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</tr>
<tr>
<td>5</td>
<td>480.1</td>
<td>560.1</td>
<td>485.7</td>
</tr>
<tr>
<td>6</td>
<td>481.1</td>
<td>561.5</td>
<td>486.2</td>
</tr>
<tr>
<td>7</td>
<td>481.3</td>
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</tr>
<tr>
<td>8</td>
<td>481.1</td>
<td>560.7</td>
<td>485.8</td>
</tr>
<tr>
<td>9</td>
<td>479.0</td>
<td>558.4</td>
<td>484.4</td>
</tr>
<tr>
<td>10</td>
<td>478.8</td>
<td>556.2</td>
<td>485.3</td>
</tr>
<tr>
<td>Average</td>
<td>480.1</td>
<td>559.6</td>
<td>485.4</td>
</tr>
</tbody>
</table>

Table 2.2: $I_1$

2. For $I_2$, the optimal SMC is much better than the optimal control. The average value of $I_2$ of the optimal SMC is only 5.97% of that of the optimal control. This means that the optimal SMC is much more robust than the optimal control.

2.6 Sliding mode control for finite-time optimal quadratic regulators

For a finite-time linear quadratic optimal control problem where the terminal state is free and the terminal time is fixed, the optimal solution is well known, and can be summarized as follows. Over the time interval $[t_0, t_f]$, the optimal control $v(t)$ that minimizes the quadratic cost functional

$$I(v) = \frac{1}{2} \left\{ \mathbf{x}^T(t_f) \mathbf{N}_t \mathbf{x}(t_f) + \int_{t_0}^{t_f} [\mathbf{x}^T(t) \mathbf{Q} \mathbf{x}(t) + 2 \mathbf{x}^T(t) \mathbf{N} \mathbf{v}(t) + \mathbf{v}^T(t) \mathbf{R} \mathbf{v}(t)] dt \right\}$$

(2.58)

where

$$\mathbf{N}_t \geq 0, \; \mathbf{Q} - \mathbf{N} \mathbf{R}^{-1} \mathbf{N}^T \geq 0, \; \mathbf{R} \geq 0$$
Table 2.3: $I_2$

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Noise realization} & \text{Optimal control} & \text{Sliding mode control} & \text{Optimal SMC} \\
\hline
1 & 2.074 & 0.1650 & 0.1584 \\
2 & 2.312 & 0.1538 & 0.1470 \\
3 & 2.909 & 0.1665 & 0.1582 \\
4 & 2.272 & 0.1652 & 0.1588 \\
5 & 2.189 & 0.1621 & 0.1558 \\
6 & 2.683 & 0.1592 & 0.1514 \\
7 & 2.924 & 0.1589 & 0.1509 \\
8 & 2.356 & 0.1627 & 0.1561 \\
9 & 3.000 & 0.1470 & 0.1381 \\
10 & 3.046 & 0.1717 & 0.1640 \\
\hline
\text{Average} & 2.577 & 0.1612 & 0.1539 \\
\hline
\end{array}
\]

with respect to the linear time-invariant system $\dot{x} = \bar{A}x + \bar{B}v$ is given by
\
\[v(t) = -R^{-1}\bar{B}^T P(t) + \bar{N}^T \bar{x}(t)\]  

(2.59)

where $P(t)$ satisfies the differential Riccati equation
\
\[-\dot{P} = P(\bar{A} - \bar{B}R^{-1}\bar{N}^T) + (\bar{A}^T - \bar{N}R^{-1}\bar{B}^T)P - P\bar{B}R^{-1}\bar{B}^T P + \bar{Q} - \bar{N}R^{-1}\bar{N}^T\]  

(2.60)

with
\
\[P(t_f) = \bar{N}_t\]

After we obtain $P(t)$ by backward integrating (2.60) from $P(t_f)$, the optimal solution $v(t)$’s feedback gain is then available.

For the original system (2.15) with a quadratic cost functional
\
\[I(u) = \frac{1}{2} \left\{ x^T(t_f)N_t x(t_f) + \int_{t_0}^{t_f} [x^T(t)Qx(t) + u^T(t)Ru(t)] dt \right\} \]  

(2.61)

the optimal sliding mode control design is similar to the optimal sliding mode control design for the infinite-time LQR problem. By applying the sliding mode control, we
can transform the original finite-time LQR problem to minimize (2.58) with respect to \( \dot{x} = Ax + Bv \). Then, the optimal sliding mode manifold can be computed from (2.59), \( \phi = v \) and \( s =Cx + \phi \).

### 2.7 Sliding mode control for minimum-energy control

#### 2.7.1 Minimum-energy control

In this section, we shall consider the problem of transferring the initial state \( x(0) \) of a linear system to the origin in a fixed time \( T_f = t_f - t_0 \) and keeping the energy consumption of the control input minimum. This optimal control problem is equivalent to minimizing the cost functional

\[
I(u) = \frac{1}{2} \int_{t_0}^{t_f} u^T u dt
\]

with respect to the linear time-invariant system (2.15) with initial condition \( x(0) = x_0 \) and subject to the fixed terminal constraint \( x(t_f) = 0 \).

Applying the sliding mode control (2.20) to the original system (2.15), we obtain the extended system (2.21). The minimum-energy cost functional (2.62) can also be rewritten in the new coordinate as

\[
I(v) = \frac{1}{2} \int_{t_0}^{t_f} [\bar{x}^T \tilde{Q} \bar{x} + 2 \bar{x}^T \tilde{N} v + v^T \tilde{R} v] dt
\]

where

\[
\tilde{Q} = \left[ \begin{array}{cc}
(CB)^{-2}(A^T C + M C^T)(CA + \frac{M}{\varepsilon} C) & (CB)^{-2}(A^T C^T + M C^T) \frac{M}{\varepsilon} \\
(CB)^{-2}(CA + \frac{M}{\varepsilon} C) \frac{M}{\varepsilon} & (CB)^{-2} M^2 \varepsilon
\end{array} \right],
\]

\[
\tilde{N} = \left( CB \right)^{-2} \left[ \begin{array}{c}
\frac{M}{\varepsilon} C^T \\
\frac{M}{\varepsilon} C^T
\end{array} \right], \quad \tilde{R} = (CB)^{-2}
\]

Now let us derive the necessary conditions provided by the Pontryagin Minimum Principle (PMP) for this problem. We first form the Hamiltonian function \( H \) for this
The minimum principle can be used to establish the following theorem:

**Theorem 2.7.** [61] (Pontryagin Minimum Principle) If \( v^*(t) \) is an optimal control and if \( \bar{x}^*(t) \) is the resultant optimal trajectory, then there is a corresponding costate \( p^*(t) \) such that:

1. The state \( \bar{x}^*(t) \) and the costate \( p^*(t) \) satisfy the differential equations

\[
\dot{\bar{x}}^* \quad = \quad \bar{A}\bar{x}^* + \bar{B}v^* \\
\dot{p}^* \quad = \quad -\bar{Q}\bar{x}^* - \bar{N}v^* - \bar{A}^T p^* \tag{2.65}
\]

and the boundary conditions

\[
\bar{x}^*(t_0) = x_0 \quad \bar{x}^*(t_f) = 0 \tag{2.67}
\]

2. The following relation holds for all \( v(t) \in \mathbb{R} \) and \( t \in [t_0, t_f] \):

\[
\frac{1}{2}\bar{x}^T \bar{Q}\bar{x}^* + \bar{x}^T \bar{N}v^* + \frac{1}{2}v^* \bar{R}v^* + \bar{x}^T \bar{A}^T p^* + v^T \bar{B}^T p^* \\
\leq \frac{1}{2}\bar{x}^T \bar{Q}\bar{x}^* + \bar{x}^T \bar{N}v + \frac{1}{2}v^T \bar{R}v + \bar{x}^T \bar{A}^T p + v^T \bar{B}^T p \tag{2.68}
\]

or, equivalently,

\[
\bar{x}^T \bar{N}v^* + \frac{1}{2}v^* \bar{R}v^* + v^* \bar{B}^T p^* \leq \bar{x}^T \bar{N}v + \frac{1}{2}v^T \bar{R}v + v^T \bar{B}^T p \tag{2.69}
\]

The inequality (2.69) implies that the function

\[
\varphi(v) = \bar{x}^T \bar{N}v + \frac{1}{2}v^T \bar{R}v + v^T \bar{B}^T p \tag{2.70}
\]
has an absolute minimum at \( v(t) = v^*(t) \). Since \( v(t) \) is not constrained and since \( \varphi(v) \) is a smooth function of \( v(t) \), we can find the minimum by setting \( \partial \varphi(v)/\partial v = 0 \). But

\[
\frac{\partial \varphi}{\partial v} = \tilde{R}v + \tilde{T}x^* + B^T p^*
\] (2.71)

We thus immediately obtain the optimal solution

\[
v^* = -\tilde{R}^{-1}[\tilde{T}x^* + \tilde{B}^T p^*]
\] (2.72)

Since \( \partial^2 \varphi(v)/\partial v^2 = \tilde{R} = (CB)^{-2} > 0 \), the above optimal solution is truly a minimum. The next step is to substitute the optimal solution (2.72) into the canonical equations consisting of \( x^* \) and \( p^* \) to obtain the system of \( 2n + 2 \) differential equations. For the sake of convenience, we define

\[
\begin{bmatrix}
\tilde{T} \\
p^*
\end{bmatrix}
\quad
\begin{bmatrix}
\tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{T}
\tilde{Q} + \tilde{T}\tilde{R}^{-1}\tilde{H}
\end{bmatrix}
\quad
\begin{bmatrix}
\tilde{B}\tilde{R}^{-1}\tilde{T}
\tilde{A}^T + \tilde{T}\tilde{R}^{-1}\tilde{H}
\end{bmatrix}
\]

After substituting the optimal solution (2.72), we can thus rewrite the system consisting of \( x^* \) and \( p^* \) as

\[
\dot{z} = Wz
\] (2.73)

Let \( \Psi(t, t_0) \) be the \((2n + 2) \times (2n + 2)\) fundamental matrix for the system (2.73), \( i.e., \)

\[
\Psi(t, t_0) = \exp[W(t - t_0)]
\] (2.74)

We partition the \( \Psi(t, t_0) \) matrix into four \((n + 1) \times (n + 1)\) submatrices as follows:

\[
\Psi(t, t_0) = \begin{bmatrix}
\Psi_{11}(t, t_0) & \Psi_{12}(t, t_0) \\
\Psi_{21}(t, t_0) & \Psi_{22}(t, t_0)
\end{bmatrix}
\] (2.75)

Then, the solution of (2.73) can be written as

\[
\begin{bmatrix}
\tilde{T}x^*(t) \\
p^*(t)
\end{bmatrix}
= \begin{bmatrix}
\Psi_{11}(t, t_0) & \Psi_{12}(t, t_0) \\
\Psi_{21}(t, t_0) & \Psi_{22}(t, t_0)
\end{bmatrix}
\begin{bmatrix}
\tilde{T}x^*(t_0) \\
p^*(t_0)
\end{bmatrix}
\] (2.76)
It is necessary that $\bar{x}^*(t_0) = \bar{x}_0$ and $\bar{x}^*(t_f) = 0$. Therefore, from (2.76) we find that

$$\Psi_{12}(t_f, t_0)p^*(t_0) = -\Psi_{11}(t_f, t_0)\bar{x}_0$$

(2.77)

If the matrix $\Psi_{12}(t_f, t_0)$ is nonsingular or invertible, then we can solve for $p^*(t_0)$ in (2.77) and substitute the value

$$p^*(t_0) = -\Psi_{12}^{-1}(t_f, t_0)\Psi_{11}(t_f, t_0)\bar{x}_0$$

(2.78)

into (2.76) to find that

$$\bar{x}^*(t) = [\Psi_{11}(t, t_0) - \Psi_{12}(t, t_0)\Psi_{12}^{-1}(t_f, t_0)\Psi_{11}(t_f, t_0)]\bar{x}_0$$

(2.79)

$$p^*(t) = [\Psi_{21}(t, t_0) - \Psi_{22}(t, t_0)\Psi_{12}^{-1}(t_f, t_0)\Psi_{11}(t_f, t_0)]\bar{x}_0$$

(2.80)

Therefore, the optimal control in (2.72) can be computed as

$$v^*(t) = -\tilde{R}^{-1} \left\{ \tilde{N}[\Psi_{11}(t, t_0) - \Psi_{12}(t, t_0)\Psi_{12}^{-1}(t_f, t_0)\Psi_{11}(t_f, t_0)] ight\} + \tilde{B}^T[\Psi_{21}(t, t_0) - \Psi_{22}(t, t_0)\Psi_{12}^{-1}(t_f, t_0)\Psi_{11}(t_f, t_0)]\bar{x}_0$$

(2.81)

**Remark** The matrix $\Psi_{12}(t_f, t_0)$ is independent of the initial state. It depends on the matrices $\tilde{A}, \tilde{B}, \tilde{Q}, \tilde{N}, \tilde{R}$, on the initial time $t_0$, and on the terminal time $t_f$.

**Theorem 2.8.** If an optimal control exists and if the matrix $\Psi_{12}(t_f, t_0)$ is nonsingular, then the optimal control is unique and is given by (2.81).

**Proof.** If $\Psi_{12}(t_f, t_0)$ is nonsingular, then (2.78) implies that there is a unique initial costate $p^*(t_0)$ corresponding to $\bar{x}_0$, $t_0$ and $t_f$. Therefore, $\bar{x}^*(t)$ and $p^*(t)$ are unique and so is the control $v^*(t)$. This establishes the uniqueness of the extremal control.

If, in addition, an optimal control exists, then it must be $v^*(t)$. \qed
Remark If the matrix $\Psi_{12}(t_f, t_0)$ is singular, then there may exist many initial costates $p^*_i(t_0)$ such that the relation

$$\Psi_{12}(t_f, t_0)p^*_i(t_0) = -\Psi_{11}(t_f, t_0)x_0 \quad i = 1, 2, \cdots \quad (2.82)$$

is satisfied. In this case, the extremal controls are not unique, and so there may be non-unique optimal solutions together with relatively optimal solutions.

2.7.2 Illustrative example

An interesting example is the automatic steering of an autonomous vehicle. The control objective is to steer an automotive vehicle to fulfil a lane change maneuver within a given time interval when it is travelling at a high speed, as depicted in Fig. 2.7. The kinematical model of the vehicle is given by

$$\dot{x}_E = U \cos \theta_r - V \sin \theta_r$$
$$\dot{y}_E = U \sin \theta_r + V \cos \theta_r \quad (2.83)$$
$$\dot{\theta}_r = r$$

where $x_E$ and $y_E$ are the longitudinal and lateral displacements of the vehicle with respect to a fixed inertial reference frame, $\theta_r$ the angular displacement of the vehicle with respect to an inertial axis normal to the plane of motion, $U$ the longitudinal velocity, $V$ the lateral velocity, and $r$ the yaw rate. The vehicle dynamics are governed by the following nonlinear bicycle model [99]

$$\dot{U} = -V r - \frac{1}{m} \left[ A_u U^2 - \left( \frac{T_f}{R_f} + \frac{T_r}{R_r} \right) - \delta F_f(\alpha_f) \right]$$
$$\dot{V} = U r - \frac{1}{m} \left[ F_f(\alpha_f) + F_r(\alpha_r) - \delta \frac{T_f}{R_f} \right] \quad (2.84)$$
$$\dot{r} = \frac{1}{I_z} \left[ b F_f(\alpha_f) - a F_r(\alpha_r) + a \delta \frac{T_f}{R_f} \right]$$
In this model, $T_f$ and $T_r$ are two inputs to the system, which are the front and rear drive/brake torques respectively. $\tau_f$ is the total torque applied through the steering linkage via the power steering system. $\tau_d$ is the disturbance torque. $A_\rho$ is the aerodynamic drag coefficient in the longitudinal direction. $F_f$ and $F_r$ are the front and rear tire forces respectively. $\alpha_f$ and $\alpha_r$ are the slip angles of the front and rear tire respectively. $a$ and $b$ are the distances from the front and rear ends of the vehicle to the center of gravity. $R_f$ and $R_r$ are the effective rolling radii of the front and rear tires respectively. $b_f$ is the viscous damping coefficient. $m$ is the total vehicle mass. $I_\zeta$ and $I_\delta$ are the yaw moment of inertia and the front steering system inertial respectively. $\delta$ is the front steering angle. A schematic diagram of the vehicle is shown in Fig. 2.8. The critical system nonlinearities and uncertainties come from the tire forces, which are determined by the slip angles. These nonlinearities can be
approximated as follows:

\[
\alpha_f = \delta - \arctan\left(\frac{V - ar}{U}\right) \quad (2.85)
\]

\[
\alpha_r = -\arctan\left(\frac{V + br}{U}\right) \quad (2.86)
\]

\[
f(\lambda) = \begin{cases} 
\lambda & 0 \leq \lambda \leq 4 \\
0.125\lambda + 3.5 & 4 < \lambda < 8 \\
4.5 & 8 \leq \lambda 
\end{cases} \quad (2.87)
\]

\[
f_w(w) = \begin{cases} 
0.5w & 0 \leq w \leq 4000 \\
0.05w + 600 & 4000 < w 
\end{cases} \quad (2.88)
\]

\[
F_f(\alpha_f) = -C_r f_w(F_{nf}) f(|\alpha_f|)\text{sgn}(\alpha_f) \quad (2.89)
\]

\[
F_{nf} = mg \frac{b}{a+b} \quad (2.90)
\]

\[
F_r(\alpha_r) = -C_r f_w(F_{nr}) f(|\alpha_r|)\text{sgn}(\alpha_r) \quad (2.91)
\]

\[
F_{nr} = mg \frac{a}{a+b} \quad (2.92)
\]

where \(C_r\) is a constant scaling parameter and \(g\) is the acceleration of gravity.
In order to simplify the problem, we assume that the yaw angle $\theta_r$ is small during the maneuver, i.e., $\sin \theta_r \approx \theta_r$ and $\cos \theta_r \approx 1$. Let $\eta$ be the lateral velocity, and the lateral dynamics can be represented by a second-order system as follows:

$$
\dot{y}_E = \eta \\
\dot{\eta} = \dot{V} + \dot{U} \theta_r + U r - V \theta_r r
$$

After substituting the dynamics of $\dot{U}$ and $\dot{V}$ into the above equations, we can express them as a superposition of nominal linear dynamics and uncertain nonlinear dynamics as follows:

$$
\dot{y}_E = \eta \\
\dot{\eta} = a_y y_E + a_\eta \eta + b_v \delta + \xi(U, V, r, \theta_r, \delta)
$$

where $\xi$ includes all the nonlinearities and uncertainties and can be considered as an external disturbance. The constant nominal parameters $a_y$, $a_\eta$ and $b_v$ are derived from a linearization process with respect to a nominal operation point such that $U = \bar{U}$, $V = 0$, $\bar{\theta}_r = 0$, $r = 0$, $\bar{\delta} = 0$, a front drive torque is applied to overcome the aerodynamic drag, $T_f = R_f A \rho \bar{U}^2$, and no rear braking torque is applied, $T_r = 0$.

The following parameters are obtained:

$$
a_y = 0, \quad a_\eta = -\frac{C_r [f_w(F_{nf}) + f_w(F_{nr})]}{m |\bar{U}|}, \quad b_v = -\frac{T_f/R_f + C_r f_w(F_{nf})}{m}
$$

The terminal constraints are given as

$$
y_E(t_0) = 0, \quad y_E(3) = 5, \quad \eta(t_0) = 0, \quad \eta(3) = 0
$$

The numerical values of the automotive vehicle model are tabulated in Table 2.4.
<table>
<thead>
<tr>
<th>Name</th>
<th>Value</th>
<th>Name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>1300.2 kg</td>
<td>$A_p$</td>
<td>0.36</td>
</tr>
<tr>
<td>$I_z$</td>
<td>1969 kg m$^2$</td>
<td>$I_\delta$</td>
<td>5 kg m$^2$</td>
</tr>
<tr>
<td>$R_f$</td>
<td>0.33 m</td>
<td>$R_r$</td>
<td>0.33 m</td>
</tr>
<tr>
<td>$a$</td>
<td>1.258 m</td>
<td>$b$</td>
<td>1.612 m</td>
</tr>
<tr>
<td>$h$</td>
<td>0.05 m</td>
<td>$b_f$</td>
<td>100 Nm rad s$^{-1}$</td>
</tr>
</tbody>
</table>

Table 2.4: Numerical values of the automotive vehicle model

the nominal and initial longitudinal velocity is $\bar{U} = U(t_0) = 30$ m/s, the nominal linear model parameters are computed as

$$a_y = 0, \quad a_\eta = -3.7471, \quad b_v = -63.389$$

Using this nominal linear model and following the aforementioned minimum-energy sliding mode control design procedure in the previous subsection, the optimal sliding mode controller for the lane change maneuver can be developed. The simulation results are shown in Fig. 2.9–2.11.
Figure 2.9: Time responses of cross range $y_E(t)$ and cross range velocity $\eta(t)$

Figure 2.10: Time response of the longitudinal velocity $U(t)$
Figure 2.11: Trajectory of the automatic steering controlled vehicle: $x_E$ vs. $y_E$
CHAPTER 3

OPTIMAL SLIDING MODE CONTROL OF NONLINEAR SYSTEMS

3.1 Problem statement

In this chapter, the optimal sliding mode control is extended for a class of nonlinear systems, which is described by

\[ \dot{x}_i = x_{i+1}, \quad i = 1, \ldots, n-1 \]
\[ \dot{x}_n = f(x) + u + \xi \]  

(3.1)

where \( x = [x_1, \ldots, x_n]^T \) is the state vector, \( f \) a smooth function with \( f(0) = 0 \) and \( \xi \) a bounded, non-vanishing, exogenous disturbance. A performance index

\[ I = \int_0^\infty [L(x) + u^2]dt \]  

(3.2)

is to be minimized, where \( L(x) \geq 0 \) is a semi-definite positive function of the state \( x \).

3.2 Sliding mode control design

Because of the existence of the exogenous disturbance \( \xi \), the sliding mode control design is applied to obtain robustness against the disturbance around the equilibrium. Instead of the conventional sliding mode control, an integral sliding mode control is
applied to gain one more degree of freedom in control design. This degree of freedom will help to obtain optimality as well as robustness. A switching function is defined as

$$s = c_1 x_1 + c_2 x_2 + \cdots + c_{n-1} x_{n-1} + x_n + \phi$$

(3.3)

where $c_i$’s are constant parameters, and the continuous approximated sliding mode control is designed as

$$u = -c_1 x_2 - \cdots - c_{n-1} x_{n-1} - f - \dot{\phi} - M \sigma_\varepsilon(s)$$

(3.4)

where $M > \sup |\xi|$, $\phi$ is a smooth function and will be defined later, and $\sigma_\varepsilon(\cdot)$ is a saturation function defined as

$$\sigma_\varepsilon(s) = \begin{cases} 
1 & \text{if } s > \varepsilon \\
\frac{s}{\varepsilon} & \text{if } |s| \leq \varepsilon \\
-1 & \text{if } s < -\varepsilon 
\end{cases}$$

(3.5)

where $\varepsilon > 0$ is a small constant.

**Theorem 3.1.** The sliding mode control (3.4) will drive the system (3.1) inside a boundary layer $\Omega = \{|s| < \varepsilon\}$ from the beginning $t = 0$ even in the presence of the disturbance $\xi$, if we enforce $s(0) = 0$ by choosing $\phi(0) = -\sum_{i=1}^{n-1} c_i x_i(0) - x_n(0)$.

**Proof.** Consider a Lyapunov function candidate $V = s^T s / 2$. Its time derivative is

$$\dot{V} = ss$$

$$= s \left( \xi - \frac{M}{\varepsilon} s \right)$$

$$= s \xi - \frac{M}{\varepsilon} s^2$$

On the boundary of $\Omega$, $s = |\varepsilon|$

$$\dot{V} = s \xi - \frac{M}{\varepsilon} \varepsilon^2$$

$$= \varepsilon (|\xi| - M)$$

$$< 0$$
Therefore, the boundary layer $\Omega$ is an invariant set of the closed-loop system of (3.1) using the sliding mode control. Since we enforce $s(0) = 0 \in \Omega$, the closed-loop system of (3.1) will stay inside the boundary layer $\Omega$ from the beginning $t = 0$ even in the presence of the disturbance $\xi$.

Since $s$ always stays inside $\Omega$ or $|s| \leq \varepsilon$, the actually applied control input is

$$u = -c_1x_2 - \cdots - c_{n-1}x_n - f - \dot{\phi} - \frac{M}{\varepsilon} s$$  \hspace{1cm} (3.6)$$

Denote $x_{n+1} = \phi$, $v = \dot{\phi}$ and $\eta = \frac{M}{\varepsilon}$, the dynamics equations of the system (3.1) becomes

$$\begin{align*}
\dot{x}_1 &= x_2 \\
& \vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -\eta c_1 x_1 - \sum_{i=1}^{n-1} (c_i + \eta c_{i+1}) x_{i+1} - \eta x_{n+1} - v \\
\dot{x}_{n+1} &= v
\end{align*}$$ \hspace{1cm} (3.7)$$

or in matrix form

$$\dot{x} = Ax + Bv$$  \hspace{1cm} (3.8)$$

where $x = [x_1, \cdots, x_n, x_{n+1}]^T$, $c_n = 1$ and

$$A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\eta c_1 & -(c_1 + \eta c_2) & \cdots & -\eta & \eta \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
\vdots \\
0 \\
-1
\end{bmatrix}$$

The performance index then becomes

$$I = \int_0^\infty [L + \dot{f}^2 + 2\dot{f}v + v^2] dt$$  \hspace{1cm} (3.9)$$

53
where $\overline{f} = f + \eta c_1 x_1 + \sum_{i=1}^{n-1} (c_i + \eta c_{i+1}) x_{i+1} + \eta x_{n+1}$. It is easy to show that $(A, B)$ is stabilizable. By applying optimal sliding mode control, the equivalent problem becomes a optimal control problem for a linear system with a non-quadratic cost functional. The solution to this optimal control problem then can be used to construct an optimal sliding mode control for the original optimal control problem. To ensure the existence and uniqueness of the optimal control, we should choose $c_i$’s such that $L + \overline{f}^2 > 0$. In the following sections, three different approaches will be proposed to find the optimal $v^*$.

3.3 The HJB equation approach

We assume that the solution to this optimal control problem exists and its value function is $J(\pi) = \min_v I$. The HJB equation for this optimal control problem is

$$
\frac{\partial J}{\partial t} + \min_v \left[ \left( \frac{\partial J}{\partial \pi} \right)^T (A\pi + Bu) + L + \overline{f}^2 + 2\overline{f}v + v^2 \right] = 0 \quad (3.10)
$$

By assuming that $\frac{\partial J}{\partial t} = 0$, we can obtain the optimal control as

$$
v^* = -\overline{f} - \frac{1}{2} B^T \frac{\partial J}{\partial \pi} \quad (3.11)
$$

and the value function $J(\pi)$ satisfies the following HJB equation

$$
\left[ \frac{\partial J}{\partial \pi} \right]^T A\pi - \left[ \frac{\partial J}{\partial \pi} \right]^T B\overline{f} + L - \frac{1}{4} \left[ \frac{\partial J}{\partial \pi} \right]^T BB^T \left[ \frac{\partial J}{\partial \pi} \right] = 0 \quad (3.12)
$$

We further assume that $J(\cdot)$ is analytic, infinite differentiable in $\mathbb{R}^{n+1}$ and can be expanded to a power (Taylor) series around the origin $J(\pi) = \sum_{i=0}^{\infty} J_i(\pi)$ with $J_0(\pi) = J_1(\pi) = 0$. These are justified by the following requirements: The cost-to-go starting from the origin is zero (i.e., $J(0) = 0$) and the control action $v = -\overline{f} - \frac{1}{2} B^T \frac{\partial J}{\partial \pi}$ is zero at the origin (i.e., $\overline{f}(0) = 0$ and $\frac{\partial J}{\partial \pi}(0) = 0$). $J_i(\pi)$ denotes $i$th order polynomials.
For example, if \( n = 2 \), \( J_2(x) = J_{200}x_1^2 + J_{020}x_2^2 + J_{002}x_3^2 + J_{110}x_1x_2 + J_{101}x_1x_3 + J_{011}x_2x_3 \). Similarly, we expand \( f \) and \( L \) as
\[
f(x) = \sum_{i=0}^{\infty} f_i(x)
\]
and
\[
L(x) = \sum_{i=0}^{\infty} L_i(x),
\]
respectively. Since \( J < \infty \) exists and \( L \geq 0 \), \( L_0(x) = L_1(x) = 0 \). Since \( f(0) = 0 \), \( f_0(x) = 0 \).

To make (3.12) hold for any \( x \) it is necessary and sufficient that terms of (3.12) of any order be zero. Since \( L_0(x) = L_1(x) = 0 \) and \( J_0(x) = J_1(x) = 0 \), the 0th-order and 1st-order terms of (3.12) are automatically satisfied. The 2nd order term of (3.12) is
\[
\left[ \frac{\partial J_2}{\partial \bar{x}} \right]^T A \bar{x} - \left[ \frac{\partial J_2}{\partial \bar{x}} \right]^T B \bar{f}_1 + L_2 - \frac{1}{4} \left[ \frac{\partial J_2}{\partial \bar{x}} \right]^T BB^T \left[ \frac{\partial J_2}{\partial \bar{x}} \right] = 0 \tag{3.13}
\]
Substituting \( L_2 \) and \( \bar{f}_1 \) into (3.13), we can obtain \( J_2(\bar{x}) \). Generally, the \( l \)th order term of (3.12) is
\[
\left[ \frac{\partial J_l}{\partial \bar{x}} \right]^T A \bar{x} - \sum_{i=2}^{l} \left[ \frac{\partial J_i}{\partial \bar{x}} \right]^T B \bar{f}_{l-i+1} + L_l - \frac{1}{4} \sum_{i=2}^{l} \left[ \frac{\partial J_i}{\partial \bar{x}} \right]^T BB^T \left[ \frac{\partial J_{l-i+2}}{\partial \bar{x}} \right] = 0 \tag{3.14}
\]
Note that \( \left[ \frac{\partial J_l}{\partial \bar{x}} \right]^T BB^T \left[ \frac{\partial J_{l}}{\partial \bar{x}} \right] = \left[ \frac{\partial J_l}{\partial \bar{x}} \right]^T BB^T \left[ \frac{\partial J_{l}}{\partial \bar{x}} \right] \). Isolating the highest order \( \left[ \frac{\partial J_l}{\partial \bar{x}} \right] \) in (3.14), then yields
\[
\left[ \frac{\partial J_l}{\partial \bar{x}} \right]^T \left( A \bar{x} - B \bar{f}_1 - \frac{1}{2} BB^T \left[ \frac{\partial J_2}{\partial \bar{x}} \right] \right) = \sum_{i=2}^{l-1} \left[ \frac{\partial J_i}{\partial \bar{x}} \right]^T B \bar{f}_{l-i+1} - L_l + \frac{1}{4} \sum_{i=3}^{l-1} \left[ \frac{\partial J_i}{\partial \bar{x}} \right]^T BB^T \left[ \frac{\partial J_{l-i+2}}{\partial \bar{x}} \right] \tag{3.15}
\]
Examining (3.15), we conclude that \( J_i(\bar{x}) \) can be computed from \( J_i(\bar{x}), \ i < l \). After we get \( J_2(\bar{x}) \) by solving (3.13), we can obtain \( J_3(\bar{x}), J_4(\bar{x}), \cdots \) step by step by solving (3.15).

In order to obtain \( J_2(\bar{x}) \), we consider the linearized system
\[
\dot{\bar{x}} = A \bar{x} + Bv \tag{3.16}
\]
with a quadratic cost functional

\[ \tilde{I} = \int_{0}^{\infty} \left[ L_2 + \tilde{f}_1^2 + 2\tilde{f}_1 v + v^2 \right] dt \]  

(3.17)

This is an LQR problem. Since \( L + \tilde{f}^2 > 0 \), \( L_2 + \tilde{f}_1^2 > 0 \) too. Since \((A, B)\) is stabilizable, we conclude that the above LQR problem has a unique optimal solution and its value function \( \tilde{J}_2 \) has only the quadratic terms. The optimal solution is given as

\[ \tilde{v}^* = -\tilde{f}_1 - \frac{1}{2} B^T \frac{\partial \tilde{J}_2}{\partial \tilde{x}} \]  

(3.18)

where \( \tilde{J}_2 \) is the solution to the following HJB equation

\[
\begin{bmatrix}
\frac{\partial \tilde{J}_2}{\partial \tilde{x}} \\
\frac{\partial \tilde{J}_2}{\partial \tilde{x}}
\end{bmatrix}^T A \tilde{x} - \begin{bmatrix}
\frac{\partial \tilde{J}_2}{\partial \tilde{x}} \\
\frac{\partial \tilde{J}_2}{\partial \tilde{x}}
\end{bmatrix}^T B \tilde{f}_1 + L_2 - \frac{1}{4} \begin{bmatrix}
\frac{\partial \tilde{J}_2}{\partial \tilde{x}} \\
\frac{\partial \tilde{J}_2}{\partial \tilde{x}}
\end{bmatrix}^T B B^T \frac{\partial \tilde{J}_2}{\partial \tilde{x}} = 0
\]  

(3.19)

Due to the uniqueness of the optimal solution, \( \tilde{J}_2 \) is identical to the solution to (3.13), i.e., \( \tilde{J}_2 = J_2 \). Then, the linearized closed-loop system is

\[
\begin{align*}
\dot{\tilde{x}} &= A \tilde{x} + B \tilde{v}^* \\
&= \left( A - B \tilde{f}_1 - \frac{1}{2} BB^T \frac{\partial \tilde{J}_2}{\partial \tilde{x}} \right) \tilde{x} \\
&= \hat{A} \tilde{x}
\end{align*}
\]  

(3.20)

where \( \hat{A} = A - B \tilde{f}_1 - \frac{1}{2} BB^T \frac{\partial \tilde{J}_2}{\partial \tilde{x}} \). It should be noted that the linearized closed-loop system \( \dot{\tilde{x}} = \hat{A} \tilde{x} \) is globally asymptotically stable.

**Definition** The \( l \)th order approximation of the value function is defined as

\[ J^l(\tilde{x}) = J_2(\tilde{x}) + \cdots + J_l(\tilde{x}) \]  

(3.21)

**Theorem 3.2.** [62] The \( i \)th \((i \leq l)\) order term of this \( l \)th order approximation, \( J_i(\tilde{x}) \) is the same as the \( i \)th order term of the exact solution of the HJB equation.
A suboptimal control then can be obtained by using the \( l \)th order approximation of the value function

\[
v^* = -\bar{J} - \frac{1}{2} B^T \frac{\partial J^l}{\partial \pi}
\]  

(3.22)

The stability of the closed-loop system can be checked by using \( J^l(\pi) \) as a Lyapunov function. If \( J^l(\pi) > 0 \) and \( \dot{J}^l(\pi) < 0 \), then (3.22) gives a stabilizing control. Otherwise, increase the approximation order until \( J^l(\pi) \) is a Lyapunov function. And the optimal sliding mode control is

\[
\dot{\phi} = v^*, \quad \phi(0) = -\sum_{i=1}^{n-1} c_i x_i(0) - x_n(0) \quad (3.23)
\]

\[
s = \sum_{i=1}^{n-1} c_i x_i + x_n + \phi \quad (3.24)
\]

\[
u = -\sum_{i=1}^{n-1} c_i x_{i+1} - f - \dot{\phi} - \eta s \quad (3.25)
\]

### 3.4 Control Lyapunov function approach

First, let us review the concept of control Lyapunov function and Sontag’s formula. Consider the following optimal control problem in general form

\[
\min_u \int_0^\infty [L(x) + u^2] dt \quad (3.26)
\]

\[
\text{s.t. } \dot{x} = F(x) + G(x)u \quad (3.27)
\]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^1, G(x) \neq 0 \) and \( F(x) \in \mathbb{R}^n \) is a smooth vector field of \( x \). \( L(x) \) is continuously differentiable, positive semi-definite and \((F, L)\) is zero-state detectable.

**Definition** A smooth, positive definite and radially unbounded function \( V(x) \) is called a Control Lyapunov Function (CLF) for system (3.27) if for all \( x \neq 0 \),

\[
\frac{\partial V}{\partial x} G = 0 \Rightarrow \frac{\partial V}{\partial x} F < 0 \quad (3.28)
\]
Freeman and Kokotovic [72] have shown that every CLF solves the HJB equation associated with a sensible cost functional. If we have a CLF \( V(x) \) for system (3.27), then a suboptimal control for system (3.27) with respect to the cost functional (3.26) can be given by Sontag’s formula [84]

\[
    u^* = \begin{cases} 
    - \frac{V_x F + \sqrt{(V_x F)^2 + L(V_x G)^2}}{V_x G} & \text{if } V_x G \neq 0 \\
    0 & \text{if } V_x G = 0 
    \end{cases}
\]

(3.29)

where \( V_x = \frac{\partial V}{\partial x} \). In fact, the Sontag’s formula uses a CLF \( V \) as a substitute for the value function in the HJB approach to the optimal control problem. If the level curves of a CLF \( V \) fully agree in shape with the value function, the Sontag’s formula gives the real optimal control. For an arbitrary CLF \( V \), though in most cases those level curves may not fully agree, this approach based on the Sontag’s formula will result in a suboptimal controller.

To use Sontag’s formula, we need find a CLF for the original system. Again, using the optimal sliding mode control the nonlinear system (3.1) can be transformed to a linear system as in (3.8). Examining \( A \), we find that the sub-matrix consisting of the first \( n \) rows and first \( n \) columns of \( A \) is in a controllable canonical-like form with state feedback and the last row of \( A \) is a zero vector. Therefore, an appropriate set of \( c_i \)'s (\( i = 1, \cdots, n - 1 \)) can make \( A \) satisfy the following requirements:

1. all the eigenvalues of \( A \) in (3.8) are non-positive,

2. in which only one eigenvalue equal to zero,

3. and all eigenvalues of \( A \) are different.

For example, if \( n = 2 \), the eigenvalues of \( A \) are

\[
    \lambda_1 = -\eta, \quad \lambda_2 = -c_1, \quad \lambda_3 = 0
\]

(3.30)
Note that $\eta > 0$. Obviously, if we pick up $c_1 > 0$ and $c_1 \neq \eta$, then the above requirements are all satisfied.

The switching function is actually in the new coordinate

$$s(\bar{x}) = c_1 x_1 + \cdots + c_{n-1} x_{n-1} + x_n + x_{n+1} = [c_1, \cdots, c_{n-1}, 1, 1] \bar{x} = C \bar{x} \quad (3.31)$$

where $C = [c_1, \cdots, c_{n-1}, 1, 1]$.

**Lemma 3.3.** A has all distinct eigenvalues in the left half of the plane, with at most one at the origin, there exists a constant vector $H \in \mathbb{R}^{1 \times (n+1)}$ such that the Lyapunov equation $PA + A^T P = -H^T H$ has a positive semi-definite and symmetric solution $P$.

**Proof.** Since all the eigenvalues of $A$ are non-positive, in which only one eigenvalue equal to zero and all eigenvalues of $A$ are different, there exists an invertible matrix $U \in \mathbb{R}^{(n+1) \times (n+1)}$ such that

$$U^{-1} A U = \begin{bmatrix}
\lambda_1 & \cdots & 0 & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & \lambda_n & 0 \\
0 & \cdots & 0 & 0
\end{bmatrix} \quad (3.32)$$

where $\lambda_i < 0$ ($i = 1, \cdots, n$). Consider $H = [h_1, \cdots, h_n, 0]U^{-1}$, where $h_i(i = 1, \cdots, n)$ are constants, then

$$H e^{tA} = [h_1, \cdots, h_n, 0] \begin{bmatrix}
e^{\lambda_1 t} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & e^{\lambda_n t} \\
0 & \cdots & 1
\end{bmatrix} U^{-1} = [h_1 e^{\lambda_1 t}, \cdots, h_n e^{\lambda_n t}, 0]U^{-1} \quad (3.33)$$

Since $\lambda_i < 0$,

$$H e^{tA}|_{t=0} = -[h_1, \cdots, h_n, 0]U^{-1} = -H \quad (3.34)$$

Therefore,

$$P = \int_0^\infty e^{tA^T} H^T H e^{tA} dt \quad (3.35)$$
exists, and \( P \geq 0 \) because \( e^{tA^T}H^THe^{tA} \geq 0 \) for each \( t \).

\[
PA + A^TP = \int_0^\infty \left( \frac{d}{dt} e^{tA^T}H^THe^{tA} \right) dt = e^{tA^T}H^THe^{tA}\bigg|_0^\infty = -H^TH \quad (3.36)
\]

Thus, \( P \) is a symmetric, positive semi-definite solution to the Lyapunov equation \( PA + A^TP = -H^TH \).

\[\square\]

**Theorem 3.4.** If \( \{B^TP, H, \overline{C}\} \) are linearly independent vectors in \( \mathbb{R}^{1 \times (n+1)} \) and \( P + \overline{C}^T\overline{C}/2 > 0 \), then function \( V = \overline{x}^TP\overline{x} + s^2/2 \) is a CLF for system (3.8), where \( P \) is the symmetric and positive semi-definite solution to the Lyapunov equation \( PA + A^TP = -H^TH \).

**Proof.**

\[
V = \overline{x}^TP\overline{x} + s^2/2 = \overline{x}^T(P + \overline{C}^T\overline{C}/2)\overline{x} \quad (3.37)
\]

If \( P + \overline{C}^T\overline{C}/2 > 0 \), then \( V \) is positive definite. Noting that

\[
\frac{\partial s}{\partial \overline{x}} B = \begin{bmatrix} c_1 & \cdots & c_{n-1} & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 1 \end{bmatrix} = 0 \quad (3.38)
\]

and

\[
\frac{\partial s}{\partial \overline{x}} A\overline{x} = \begin{bmatrix} -\eta c_1 & \cdots & -\eta c_{n-1} & -\eta & -\eta \end{bmatrix} \overline{x} = -\eta s \quad (3.39)
\]

Thus,

\[
\frac{\partial V}{\partial \overline{x}} B = 2\overline{x}^TPB + s\frac{\partial s}{\partial \overline{x}} B = 2\overline{x}^TPB \quad (3.40)
\]

Letting \( \frac{\partial V}{\partial \overline{x}} B = 0 \), we have \( \overline{x}^TPB = B^TP\overline{x} = 0 \). Also,

\[
\frac{\partial V}{\partial \overline{x}} A\overline{x} = \overline{x}^T(PA + A^TP)\overline{x} + s\frac{\partial s}{\partial \overline{x}} A\overline{x} = -\overline{x}^TH^TH\overline{x} - \eta s^2 \leq 0 \quad (3.41)
\]
If we can prove that 
\(- \mathbf{x}^T H \mathbf{x} - \eta \mathbf{s}^2 \neq 0\) for any \(\mathbf{x} \neq 0\) under the condition \(B^T P \mathbf{x} = 0\), then \(V\) is a CLF for system (3.8). We assume that there exists \(x_a \neq 0, x_a \in \mathbb{R}^{(n+1)}\) such that 
\(- x_a^T H x_a - \eta s^2(x_a) = 0\), then 
\(- x_a^T H x_a = 0\) and \(s(x_a) = 0\) because \(H^T H\) is positive semi-definite. Since \(s(\mathbf{x}) = \overline{C} \mathbf{x}\), we have

\[
H x_a = 0 \quad (3.42)
\]
\[
\overline{C} x_a = 0 \quad (3.43)
\]

Using the conditions \(B^T P x_a = 0\) and \(\{B^T P, H, \overline{C}\}\) are linearly independent vectors, we conclude that \(x_a = 0\), which contradicts the assumption that \(x_a \neq 0\). Therefore, if \(B^T P \mathbf{x} = 0\), then \(\frac{\partial V}{\partial \mathbf{x}} A \mathbf{x} < 0\) for any \(\mathbf{x} \neq 0\). By the definition of CLF, \(V = \mathbf{x}^T P \mathbf{x} + s^2/2\) is a CLF for system (3.8).

**Remark** If \(P > 0\), then \(P + \overline{C}^T \overline{C}/2 > 0\) is satisfied for any \(\overline{C}\).

After substituting the CLF \(V\) into the Sontag’s formula, we can obtain a suboptimal sliding mode control. In order to apply the Sontag’s formula, we modify the cost functional by neglecting the cross term between the state and the control in (3.9) as

\[
I = \int_0^\infty [(L + \mathbf{f}^2 + v^2)]dt = \int_0^\infty [\overline{L} + v^2]dt \quad (3.44)
\]

where \(\overline{L} = L + \mathbf{f}^2\). Then the optimal sliding mode control is given as

\[
\dot{\phi} = \begin{cases} 
\frac{- V \mathbf{x} A \mathbf{x} + \sqrt{(V \mathbf{x} A \mathbf{x})^2 + \overline{L}(V \mathbf{x} B)^2}}{V \mathbf{x} B} & \text{if } V \mathbf{x} B \neq 0 \\
0 & \text{if } V \mathbf{x} B = 0 
\end{cases} \quad (3.45)
\]

\[
\phi(0) = - \sum_{i=1}^{n-1} c_i x_i(0) - x_n(0) \quad (3.46)
\]

\[
s = \sum_{i=1}^{n-1} c_i x_i + x_n + \phi \quad (3.47)
\]

\[
u = - \sum_{i=1}^{n-1} c_i x_{i+1} - \mathbf{f} - \dot{\phi} - \eta s \quad (3.48)
\]

where \(V \mathbf{x} = \frac{\partial V}{\partial \mathbf{x}}\).
3.5 State dependent Riccati equation approach

The state dependent Riccati equation (SDRE) approach can be used to solve the following infinite-time nonlinear optimal control problem

\[
\min_u \int_0^\infty \left[ x^T Q(x) x + R(x) u^2 \right] \, dt \\
\text{s.t.} \quad \dot{x} = F(x) + G(x) u, \quad x(0) = x_0
\]

where \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}, Q(x) \) is positive semi-definite, \( R(x) \) positive definite, \( Q(x), R(x), F(x) \) and \( G(x) \) are all sufficiently smooth functions of the state vector \( x(t) \), and \( x_0 \) is the initial conditions of the process. Furthermore, \( F(0) = 0 \), i.e. the origin is the an open loop equilibrium point.

Employing calculus of variations, we let \( H(x, u, \lambda) = x^T Q(x) x + R(x) u^2 + \lambda^T (f(x) + B(x) u) \) be the Hamiltonian function for the problem. Then, the necessary conditions for optimality are

\[
\dot{x} = \frac{\partial H}{\partial \lambda}, \quad \dot{\lambda} = -\frac{\partial H}{\partial x}, \quad 0 = \frac{\partial H}{\partial u}
\]

Solving (3.50) yields the optimal control input \( u^* = -\frac{1}{2} R^{-1}(x) B^T(x) \lambda \). In addition, \( \lambda = \frac{\partial J}{\partial x} \) gives the optimal control input

\[
u^* = -\frac{1}{2} R^{-1}(x) G^T(x) \frac{\partial J}{\partial x}
\]

It should be noted that \( \frac{\partial J}{\partial x} = 0 \) for the infinite-time optimal control problem. Therefore, the HJB equation for the above problem can be written as

\[
x^T Q(x) x + R(x) u^2 + \left[ \frac{\partial J}{\partial x} \right]^T [F(x) + G(x) u] = 0
\]

Substituting the optimal control input \( u^* \) yields The HJB equation for the above problem can be written as

\[
x^T Q(x) x + \left[ \frac{\partial J}{\partial x} \right]^T F(x) - \frac{1}{4} \left[ \frac{\partial J}{\partial x} \right]^T G(x) R^{-1}(x) G^T(x) \frac{\partial J}{\partial x} = 0
\]
where \( J(x) \) is interpreted as the value function of the infinite time nonlinear optimal control problem (3.49). Solving the HJB equation (3.53) will lead to the optimal control policy \( u^* \) by (3.51).

However, it is very difficult to obtain an analytic solution to this equation in most cases. In order to avoid directly solving (3.12), Banks and Mhana [50] have introduced the so-called state dependent Riccati equation (SDRE) approach to solving this optimal control problem. The state dependent Riccati equation (SDRE) approach is derived from the following two expansions. First, the state vector, \( F(x) \) is expanded as the product of a state dependent system matrix, \( A(x) \), and the state vector \( x \)

\[
F(x) = A(x)x
\]  

(3.54)

Substituting in to the original system model yields a linear time-varying system

\[
\dot{x} = A(x)x + G(x)u
\]  

(3.55)

It should be noted that this expansion is generally not unique. Clearly, if \( A_0(x) \) is such that \( f(x) = A_0(x)x \), then any matrix \( \tilde{A}(x) \) such that \( \tilde{A}(x)x = 0 \) gives \( F(x) = [A_0(x) + \tilde{A}(x)]x \). The second expansion is rewriting the infinite-time value function \( J(x) \) as

\[
J(x) = x^TP(x)x
\]  

(3.56)

where \( P(x) \) is a symmetric, matrix valued function of the state \( x \). The quadratic-like form of \( J(x) \) is justified by the fact that the first order and constant terms of \( J(x) \) are zero. The partial derivative of \( J \) with respect to the state \( x \) is

\[
\frac{\partial J}{\partial x} = 2 \left( P(x) + \frac{1}{2}x^T \frac{\partial P(x)}{\partial x} \right)x
\]  

(3.57)
where \( x^T \frac{\partial p_i(x)}{\partial x} \) is a matrix with row \( i \), column \( j \) elements defined by \( x^T \frac{\partial p_{ij}(x)}{\partial x} \) ( \( p_{ij}(x) \) is the \( i \)th row, \( j \)th column of the matrix \( P(x) \)). Let

\[
\Pi(x) = P(x) + \frac{1}{2} x^T \left( \frac{\partial P(x)}{\partial x} \right)
\]

and substitute into (3.12) we find, after some algebraic manipulations

\[
x^T \Pi(x) A(x) + A^T(x) \Pi(x) + Q(x) - \Pi(x) G(x) R^{-1}(x) G^T(x) \Pi(x) | x = 0
\]

For this equation to hold for any \( x \), it is necessary and sufficient that \( \Pi(x) \) satisfies the following SDRE:

\[
\Pi(x) A(x) + A^T(x) \Pi(x) + Q(x) - \Pi(x) G(x) R^{-1}(x) G^T(x) \Pi(x) = 0
\]

(3.60)

If \((A(x), G(x))\) is stabilizable and \((A(x)x, Q(x))\) is zero-detectable, for a given \( x \), the SDRE (3.60) will possess a unique positive definite solution, \( \Pi(x) \). After obtaining the solution to (3.60), we can derive the optimal control policy from

\[
u^* = -R^{-1}(x) G^T(x) \Pi(x) x
\]

(3.61)

If the cost functional has a cross term, i.e., \( I = \int_0^\infty [x^T Q(x)x + 2x^T N(x)u + R(x)u^2] dt \), then the optimal control is given as

\[
u^* = -R^{-1}(x) [G^T(x) \Pi(x) + N(x)] x
\]

(3.62)

where \( \Pi(x) \) is the solution to the generalized SDRE

\[
\Pi(x)[A(x) - G(x) R^{-1}(x) N^T(x)] + [A^T(x) - N(x) R^{-1}(x) G^T(x)] \Pi(x) + Q(x) - \Pi(x) G(x) R^{-1}(x) G^T(x) \Pi(x) - N(x) R^{-1}(x) N^T(x) = 0
\]

(3.63)

Then we call this infinite-time nonlinear optimal control problem as

\[
\text{opt}\{A(x), G(x), Q(x), R(x), N(x)\}
\]
Lemma 3.5. \( \text{opt}\{A(x), G(x), \tilde{Q}(x), R(x), \tilde{N}(x)\} \) and \( \text{opt}\{A(x), G(x), Q(x), R(x), N(x)\} \) have the same optimal solution, if \( \tilde{Q}(x) = Q(x) + Q_0(x) \) and \( \tilde{N}(x) = N(x) + N_0(x) \) where \( x^T Q_0(x) x = 0 \) and \( x^T N_0(x) = 0 \).

Proof. After substituting \( \tilde{Q}(x) \) and \( \tilde{N}(x) \) into the HJB equation, we find that both optimal control problems have the same generalized SDRE as in (3.63). Therefore, they have the same \( \Pi(x) \). Since \( N^T(x) x = \tilde{N}^T(x) x \), their optimal solutions from (3.62) are same too. \( \square \)

Although the computational complexity of the SDRE method is far less than solving directly the HJB equation, Huang and Lu [57] pointed out that the success of this method completely depends on one good choice of the state dependent coefficients \( A(x) \). Not only can the SDRE controller provide a performance which is far off the optimal, it may even fail to yield global stability due to the local nature of this design technique. The right \( A(x) \) is hard to guess from the dynamics of the plant unless the optimal cost or value function \( J(x) \) is known. In light of the above discussion, it would appear that the SDRE approach has only succeeded in converting the nearly impossible problem of solving the HJB equation into an equally difficult task of determining a good \( A(x) \) function. However, for a class of nonlinear systems as defined in (3.1), this factorization can be easily achieved by using the sliding mode control. We assume that the nonlinear function \( f(x) \) in (3.1) can be factorized as

\[
 f(x) = k_1(x)x_1 + k_2(x)x_2 + \cdots + k_n(x)x_n = K(x)x \quad (3.64)
\]

where \( K(x) = [k_1(x), \cdots, k_n(x)] \). We further assume that \( L(x) \) in the cost functional of the original optimal control problem can be written as a quadratic form

\[
 L(x) = x^T Q(x) x = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}(x)x_i x_j, \quad Q(x) > 0 \quad (3.65)
\]
where $q_{ij}(x)$ is the $i$th row, $j$th column element of $Q(x)$. The actually applied control input using the factorization $f(x) = K(x)x$ is

$$u = -\sum_{i=1}^{n-1} c_ix_{i+1} - K(x)x - v - \eta \left( \sum_{i=1}^{n-1} c_ix_i + x_{n+1} \right) = -[\hat{C} + W(x)]\overline{x} - v$$  (3.66)

where $\hat{C} \in \mathbb{R}^{1 \times (n+1)}$ is a constant vector with $\hat{C}x = \sum_{i=1}^{n-1} c_ix_i + \eta \left( \sum_{i=1}^{n-1} c_ix_i + x_{n+1} \right)$, and $W(x) = [K(x), 0]$. By applying this control, the original optimal control problem is changed to

$$\min_v \int_0^\infty [\overline{x}^T \overline{Q}(\overline{x}) + 2\overline{x}^T \overline{N}(\overline{x})v + v^2]dt$$  (3.67)

s.t. $\dot{\overline{x}} = Ax + Bu$

where

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\eta c_1 & -(c_1 + \eta c_2) & \cdots & -\eta \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & \cdots & 0 & -1 & 1 \end{bmatrix}^T$$

$$\overline{Q}(\overline{x}) = \begin{bmatrix} Q(x) & 0 \\ 0 & 0 \end{bmatrix} + \hat{C}^T \hat{C} + W^T(x)W(x) + \hat{C}^T W(x) + W^T(x)\hat{C}$$

$$\overline{N}(\overline{x}) = [\hat{C} + W(x)]^T$$

**Theorem 3.6.** The optimal solution to (3.67) is independent of the factorization $f(x) = K(x)x$.

*Proof.* Suppose that there exists another factorization $f(x) = \tilde{K}(x)x$, where $\tilde{K}(x) = K(x) + K_0(x)$. It is obvious that $K_0(x)x = 0$. From the expressions of $A$ and $B$, we conclude that $A$ and $B$ are both independent of the factorization $f(x) = K(x)x$ or $f(x) = \tilde{K}(x)x$. Suppose that we use the factorization $f(x) = \tilde{K}(x)x$ instead of
\[ f(x) = K(x)x. \] The associated \( \tilde{Q} \) and \( \tilde{N} \) are

\[
\tilde{Q}(x) = \begin{bmatrix} Q(x) & 0 \\ 0 & 0 \end{bmatrix} + \hat{C}^T \hat{C} + \hat{W}^T(x) \hat{W}(x) + \hat{C}^T \hat{W}(x) + \hat{W}^T(x) \hat{C} \\
= Q(x) + W_0^T(x) W_0(x) + W^T(x) W_0(x) + W_0^T(x) W(x) + \hat{C}^T W_0(x) + W_0^T(x) \hat{C} \\
= Q(x) + \tilde{Q}_0(x)
\]

\[
\tilde{N}(x) = [\hat{C} + \hat{W}(x)]^T = N(x) + W_0^T(x)
\]

where \( W_0(x) = [K_0(x), 0] \), \( \hat{W}(x) = W(x) + W_0(x) \) and \( \tilde{Q}_0(x) = W_0^T(x) W_0(x) + W^T(x) W_0(x) + W_0^T(x) W(x) + \hat{C}^T W_0(x) + W_0^T(x) \hat{C} \). Since \( K_0(x)x = 0 \) and \( W_0(x) \bar{x} = 0, \bar{x}^T \tilde{Q}_0(\bar{x}) \bar{x} = 0 \) and \( \bar{x}^T W_0^T(x) = 0 \). By Lemma 4, the optimal solution using the factorization \( f(x) = \tilde{K}(x)x \) is the same as the optimal solution using the factorization \( f(x) = K(x)x \). Therefore the optimal solution is independent of the factorization. \( \Box \)

Theorem 3.6 tells us that any factorization of \( f(x) \) will lead to the same optimal control problem as the ‘right’ factorization of \( f(x) \) does. This fact gives us the freedom to choose any factorization of \( f(x) \).

For second order systems the computation of solving SDRE (3.63) can be done symbolically to obtain a closed form expression for the optimal sliding surface. This is generally not possible for systems of order larger than two. In these cases numerical solution to the Riccati equation is sufficient for our approach. For single input systems these computations may be simplified using spectral factorization results [53].

After we obtain the optimal \( v^* \) from the generalized SDRE (3.63), the optimal sliding mode control is designed as in (3.6) with

\[
\dot{\phi} = v^* \\
\phi(0) = - \sum_{i=1}^{n-1} c_i x_i(0) - x_n(0)
\]
3.6 Simulation Study

Consider an inverted pendulum depicted in Fig. 3.1. The equations of dynamics of the inverted pendulum are modeled as

\begin{align*}
\dot{\theta} &= \omega \\
J \dot{\omega} &= mgl \sin \theta - \beta \omega + \tau + d \tag{3.70}
\end{align*}

where \( \theta \) is the angular displacement, \( \omega \) the angular velocity, \( m \) the mass of the inverted pendulum, \( J \) the inertia of the inverted pendulum, \( \beta \) the coefficient of viscous friction, \( \tau \) the control torque, \( d \) the exogenous disturbance including the wind effect. The exogenous disturbance is assumed to be bounded, \( i.e., |d| < 0.05 \). In simulations, we use a bandlimited white noise as the exogenous disturbance. One realization of the noise is shown in Fig. 3.2. Let \( x_1 = \theta, x_2 = \omega, u = \tau/J \) and \( \xi = d/J \).

\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{mgl}{J} \sin x_1 - \frac{\beta}{J} x_2 + u + \xi \tag{3.71}
\end{align*}

Throughout this section the parameters of the inverted pendulum used to verify the design approaches are tabulated in Table 3.1. Substituting the parameter values into (3.71), we have the inverted pendulum model in the form of (3.1)

\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 10 \sin x_1 - 10x_2 + u + \xi \tag{3.72}
\end{align*}

with \( f(x) = 10 \sin x_1 - 10x_2, n = 2 \) and \( |\xi| < 5 \). A cost functional is chosen by the designer as

\begin{align*}
I &= \int_0^\infty (x_1^2 + 10x_2^2 + u^2) \, dt \tag{3.73}
\end{align*}
Figure 3.1: An inverted pendulum

Table 3.1: Parameter values for the inverted pendulum

<table>
<thead>
<tr>
<th>$m$</th>
<th>0.1 kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>10 m s$^{-2}$</td>
</tr>
<tr>
<td>$l$</td>
<td>0.1 m</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.1 N m s rad$^{-1}$</td>
</tr>
<tr>
<td>$J$</td>
<td>0.01 kg m$^2$</td>
</tr>
</tbody>
</table>
The initial conditions are $x_1(0) = \frac{\pi}{2}$ and $x_2(0) = 0$. The switching function is designed as

$$s = x_1 + x_2 + \phi$$

(3.74)

and the sliding mode control is designed as

$$u = -x_2 - 10 \sin x_1 + 10x_2 - \dot{\phi} - \frac{5}{0.1} s$$

(3.75)

with $c_1 = 1$, $M = 5$, $\varepsilon = 0.1$, $\eta = \frac{M}{\varepsilon} = 50$. Accordingly, other parameters are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -50 & -51 & -50 \\ 0 & 0 & 0 \end{bmatrix}$$

(3.76)

$$B = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T$$

(3.77)

$$\tilde{f} = 10 \sin x_1 + 50x_1 + 41x_2 + 50x_3$$

(3.78)

$$L = x_1 + 10x_2^2$$

(3.79)
3.6.1 The HJB equation approach

First, we compute the second order approximation of the value function, $J_2$.

$$J_1 = 60x_1 + 41x_2 + 50x_3$$  \hspace{1cm} (3.80)

$$A\bar{x} - B\bar{f}_1 = \begin{bmatrix} x_2 \\ 10x_1 - 10x_2 \\ -60x_1 - 41x_2 - 50x_3 \end{bmatrix} = \bar{A}\bar{x}$$  \hspace{1cm} (3.81)

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 10 & -10 & 0 \\ -60 & -41 & -50 \end{bmatrix}$$  \hspace{1cm} (3.82)

Assume that $J_2(\bar{x}) = \frac{1}{2}\bar{x}^T P\bar{x}$, $P^T = P > 0$. $L_2 = \bar{x}^T \bar{Q}\bar{x}$, where

$$\bar{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$  \hspace{1cm} (3.83)

Substituting $J_2$ and $L_2$ into (3.13), we have

$$\bar{x}^T P\bar{A}\bar{x} + \bar{x}^T \bar{Q}\bar{x} - \frac{1}{4}\bar{x}^T PBB^T P\bar{x} = 0$$  \hspace{1cm} (3.84)

$$\bar{x}^T \left( \frac{PA + A^T P}{2} \right) \bar{x} + \bar{x}^T \bar{Q}\bar{x} - \frac{1}{4}\bar{x}^T PBB^T P\bar{x} = 0$$  \hspace{1cm} (3.85)

Denote $\bar{P} = P/2$

$$\bar{x}^T \left( \bar{P}A + \bar{A}^T \bar{P} + \bar{Q} - \bar{P}BB^T \bar{P} \right) \bar{x} = 0$$  \hspace{1cm} (3.86)

The above equation holds for any $\bar{x}$, only if

$$\bar{P}A + \bar{A}^T \bar{P} + \bar{Q} - \bar{P}BB^T \bar{P} = 0$$  \hspace{1cm} (3.87)

The solution to the above Lyapunov equation can be obtained by using the command `care` in MATLAB.

$$P = \begin{bmatrix} 223.1263 & 20.0499 & 0 \\ 20.0499 & 2.2515 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{P} = \begin{bmatrix} 446.2526 & 40.0998 & 0 \\ 40.0998 & 4.5030 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$  \hspace{1cm} (3.88)
\begin{align*}
J_2 &= 223.1263x_1^2 + 40.0988x_1x_2 + 2.2515x_2^2 
\quad \quad \quad \quad (3.89) \\
J_3 &= 0 
\quad \quad \quad \quad (3.90) \\
J_4 &= -22.3320x_1^4 - 6.6501x_1^3x_2 - 0.7642x_1^2x_2^2 - 0.0407x_1x_2^3 - 0.0008x_2^4 
\quad \quad \quad \quad (3.91) \\
J_5 &= 0 
\quad \quad \quad \quad (3.92)
\end{align*}

**Remark** The terms involving \(x_3\) do not appear in \(J_i\)'s. The reason is that \(x_3 = -x_1 - x_2\) on the sliding manifold \(s = 0\), which is not completely independent. This fact can help decreasing the computation complexity by letting the coefficients before the terms involving \(x_3\) in \(J_i\)'s equal to zero.

We choose the 5th order approximation,

\begin{align*}
J^5 &= J_2 + J_3 + J_4 + J_5 \\
&= 223.1263x_1^2 + 40.0988x_1x_2 + 2.2515x_2^2 - 22.3320x_1^4 \\
&\quad - 6.6501x_1^3x_2 - 0.7642x_1^2x_2^2 - 0.0407x_1x_2^3 - 0.0008x_2^4 
\quad \quad \quad \quad (3.93)
\end{align*}

\begin{align*}
v^* &= \mathbf{J} - \frac{1}{2} \mathbf{B}^T \frac{\partial J^5}{\partial \mathbf{x}} \\
&= -10 \sin x_1 - 50x_1 - 41x_2 - 50x_3 + 20.0499x_1 + 2.2515 \mathbf{x}_2 \\
&\quad - 3.3251x_1^3 - 0.7642x_1^2x_2 - 0.0611x_1x_2^2 - 0.0016x_2^3 
\quad \quad \quad \quad (3.94)
\end{align*}

The simulation results using HJB equation approach are shown in Fig. 3.3-3.6.

### 3.6.2 Control Lyapunov Function approach

We choose

\begin{equation}
H = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix} 
\quad \quad \quad \quad (3.95)
\end{equation}
Figure 3.3: Response of $x_1(t)$ using HJB equation approach

Figure 3.4: Response of $x_2(t)$ using HJB equation approach
Figure 3.5: Response of $s(t)$ using HJB equation approach

Figure 3.6: Control input $u(t)$ using HJB equation approach
Denote $P = P^T = [p_{ij}]$. Through $PA + A^T P = -H^T H$, we have the following equations:

$$-100p_{12} = -1$$
$$p_{11} - 51p_{12} - 50p_{22} = -3$$
$$-50p_{12} - 50p_{23} = -1$$
$$2(p_{12} - 51p_{22}) = -9$$
$$-50p_{22} + p_{13} - 51p_{23} = -3$$
$$-100p_{23} = -1$$

$p_{11} = 1.93$, $p_{12} = 0.01$, $p_{13} = 1.93$, $p_{22} = 0.0884$, $p_{23} = 0.01$. We cannot obtain $p_{33}$ from the above equations. Except for the only requirement that $P$ should be positive definite, we can freely choose $p_{33}$. To make $P$ positive definite,

$$p_{33} > 1.9302 \quad (3.96)$$

We pick up $p_{33} = 10$, then

$$P = \begin{bmatrix} 1.93 & 0.01 & 1.93 \\ 0.01 & 0.0884 & 0.01 \\ 1.93 & 0.01 & 10 \end{bmatrix} \quad (3.97)$$

It is easy to check that $C$, $H$ and $B^T P$ are linearly independent. Therefore, by Theorem 3, a control Lyapunov function for the inverted pendulum system is

$$V = \bar{x}^T P \bar{x} + \frac{1}{2} \bar{s}^2$$
$$= \bar{x}^T \left( P + \frac{1}{2} C^T C \right) \bar{x}$$
$$= \bar{x}^T \bar{P} \bar{x}$$
$$= \bar{x}^T \begin{bmatrix} 2.43 & 0.51 & 2.43 \\ 0.51 & 0.5884 & 0.51 \\ 2.43 & 0.51 & 10.50 \end{bmatrix} \bar{x} \quad (3.98)$$

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3.6.3 State dependent Riccati equation approach

The inverted pendulum model (3.71) can be rewritten as

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 10\text{sinc}\left(\frac{x_1}{\pi}\right) x_1 - 10x_2 + u + \xi
\end{align*}
\]  

(3.101)

where \(\text{sinc}(a)\) is defined as

\[
\text{sinc}(a) = \begin{cases} 
\frac{\sin(\pi a)}{\pi a} & \text{if } a \neq 0 \\
1 & \text{if } a = 0
\end{cases}
\]  

(3.102)
Figure 3.8: Response of $x_2(t)$ using CLF approach

Figure 3.9: Response of $s(t)$ using CLF approach
The optimal sliding mode is given as

\[ v^* = -[B^T \Pi(x) + \overline{N}(x)]x \]  \hspace{1cm} (3.103)

where \( \Pi(x) \) is the solution to the generalized SDRE

\[ \Pi[A - BN^T] + [A^T - NB^T]\Pi + \overline{Q} - \Pi BB^T\Pi - \overline{NN}^T = 0 \]  \hspace{1cm} (3.104)

\[ A = \begin{bmatrix} 0 & 1 & 0 \\
-50 & -51 & -50 \\
0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\
-1 \\
1 \end{bmatrix}, \]

\[ \overline{Q}(x) = \begin{bmatrix} 1 + \mu^2 & 41\mu & 50\mu \\
41\mu & 1691 & 2050 \\
50\mu & 2050 & 2500 \end{bmatrix}, \quad \overline{N}(x) = \begin{bmatrix} \mu \\
41 \\
50 \end{bmatrix}, \]

\[ \mu = 50 + 10 \text{sinc} \left( \frac{x_1}{\pi} \right) \]

The simulation results using SDRE approach are shown in Fig. 3.11-3.14.
Figure 3.11: Response of $x_1(t)$ using SDRE approach

Figure 3.12: Response of $x_2(t)$ using SDRE approach
Figure 3.13: Response of $s(t)$ using SDRE approach

Figure 3.14: Control input $u(t)$ using SDRE approach
3.6.4 Comparison of three approaches

The responses of $x_1(t)$ of three approaches are shown in Fig. 3.15. All of them are quite close. Especially, the response using HJB approach and the response using SDRE approach are almost identical. This partly shows that the trajectories are suboptimal. Around the equilibrium, there is variation in the responses caused by the non-vanished disturbance. The variation in the response using CLF approach is significantly smaller than that in the other two responses.

3.7 Conclusion

All three approaches can give a robust optimal control for the inverted pendulum system. The HJB equation approach should be able to get a more accurate solution if we keep increasing the approximation order. Moreover, the optimal sliding switching
function can be designed off-line. But the computation burden of this approach is very heavy if the number of states of the system is larger than 5. For the CLF approach, only a suboptimal solution can be obtained, while the computation complexity of this method is far less than the HJB equation approach. Theoretically, the SDRE approach can obtain the exact optimal solution. However, solving a SDRE off-line for a given system is almost impossible. In that case, numerically solving the SDRE on-line is sufficient for our approach.
CHAPTER 4

OPTIMAL SLIDING MODE CONTROL OF LINEAR DISCRETE-TIME SYSTEMS

4.1 Problem statement

Due to the use of computers for control purpose, discrete-time sliding mode control has been received substantial interest. As pointed out in [5], discrete-time sliding mode control cannot, theoretically, be obtained from its continuous-time counterpart by means of simple equivalence. When the sliding mode control is used in a discrete-time system (e.g., a sampled-data system), the sliding mode motion does not exist because the switching frequency is limited by the sampling rate [12]. Thus, the concept of quasi-sliding mode was proposed by Milosavljevic [12] as a discrete-time extension of continuous-time sliding mode. In this chapter, optimal discrete-time sliding mode control will be presented for linear discrete-time systems with rate bounded disturbance or state dependent disturbance by using the concept of quasi-sliding mode. This kind of optimal discrete-time sliding mode control is featured with the ability to robustly stabilize linear discrete-time systems and the optimality with respect to a quadratic performance index even in the presence of disturbances.
The system model of disturbed linear discrete-time systems is generalized as follows. Using a sampling and hold process on the continuous time systems, the nominal discrete-time systems can be expressed by

\[ x_{k+1} = Fx_k + Gu_k \quad (4.1) \]
\[ x_k = x(kT) \]
\[ u_k = u(kT) \]

where \( x_k = [x_{k1}, x_{k2}, \ldots, x_{kn}]^T \in \mathbb{R}^n \) and \( u_k \in \mathbb{R}^1 \) are the state vector and input, respectively. \( T \) is the sampling period, \( F \) and \( G \) are obtained from integrating the solution of the linear time-invariant system

\[ \dot{x}(t) = Ax(t) + Bu(t) \quad (4.2) \]
\[ u(t) = u_k, \quad t \in [kT, (k+1)T) \quad (4.3) \]
\[ F = \exp(AT) \quad (4.4) \]
\[ \Gamma = \int_0^T \exp(A\tau)d\tau \quad (4.5) \]
\[ G = \Gamma B \quad (4.6) \]

It is assumed that \((F, G)\) is controllable. Note that \( F \) is nonsingular for any nonnegative value of \( T \). The effects of the parametric uncertainties and exogenous disturbances can be modelled by adding an unknown disturbance term into the nominal discrete-time systems as

\[ x_{k+1} = Fx_k + Gu_k + Gd_k \quad (4.7) \]

The disturbance term is expressed as \( Gd_k \) to guarantee the matching condition, where \( d_k \) has the same dimension as \( u_k \). It is assumed that there are two kinds of disturbances. The first kind of disturbance is rate bounded, \( i.e., |d_{k+1} - d_k| < \delta dT \) for any
where $\delta_d > 0$ is a known constant. The second kind of disturbance is state-dependent, i.e., $d_k = [q_1, q_2, \ldots, q_n]x_k$, and $q_i$'s ($i = 1, 2, \ldots, n$) satisfy $|q_i| < \bar{q}$, where $\bar{q} > 0$ is a known constant. Two different optimal discrete-time sliding mode control (ODSMC) approaches will be proposed to overcome those two kinds of disturbances respectively.

4.2 ODSMC of single input systems with rate bounded disturbance

4.2.1 Sliding mode control design

Let the time-varying sliding mode manifold be defined as

$$s_k = Cx_k + \phi_k = 0 \quad (4.8)$$

where $C \in \mathbb{R}^{1 \times n}$ is a constant row vector and $CG \neq 0$ and $\phi_k$ will be defined later. We can choose $\phi_0 = -Cx_0$ such that $s_0 = 0$, i.e., the system state is on the sliding mode manifold $s_k = 0$ at the beginning. The ideal sliding mode should satisfy

$$s_{k+1} = s_k = 0, \quad k = 0, 1, 2, \ldots \quad (4.9)$$

However, because of the disturbance and the limited sampling period, the ideal sliding mode is not obtainable for discrete-time systems, [12]. In this case, the concept of quasi-sliding mode was proposed in [5]. Instead of requiring $s_k$ moving exactly along the sliding mode manifold $s_k = 0$, the quasi-sliding mode only requires that $s_k$ stays inside a boundary layer around the sliding mode manifold.

To obtain an equivalent control for the discrete-time system, we enforce $s_{k+1} = 0$ as

$$s_{k+1} = Cx_{k+1} + \phi_{k+1} = CFx_k + CGu_k + CGd_k + \phi_{k+1} = 0 \quad (4.10)$$
Solving for $u_k$, we obtain the equivalent control

$$u_k^e = -(CG)^{-1}[CFx_k + CGd_k + \phi_{k+1}] \quad (4.11)$$

and the sliding mode control $u_k$ consisting of the equivalent control $u_k^e$ and a switching control $u_k^s$

$$u_k = u_k^e + u_k^s \quad (4.12)$$

where

$$u_k^s = (CG)^{-1}[rs_k - \varepsilon T \text{sign}(s_k)] \quad (4.13)$$

$0 < r < 1$ and $\varepsilon > |CG|\delta_d$.

However, $d_k$ is generally unknown, so the sliding mode control using the information of $d_k$ cannot be implemented. But at time instant $t = kT$, $d_{k-1}$ can be obtained from

$$Gd_{k-1} = x_k - Fx_{k-1} - Gu_{k-1} \quad (4.14)$$

Then $d_{k-1}$ replaces $d_k$ in (4.11) as an approximation and an implementable sliding mode control is designed as

$$u_k = -(CG)^{-1}[CFx_k + CGd_{k-1} + \phi_{k+1} - rs_k + \varepsilon T \text{sign}(s_k)] \quad (4.15)$$

Substituting (4.15) into (4.10)

$$s_{k+1} = rs_k - \varepsilon T \text{sign}(s_k) + CG(d_k - d_{k-1}) = rs_k - \varepsilon T \text{sign}(s_k) + CG\Delta_k \quad (4.16)$$

where $\Delta_k = d_k - d_{k-1}$ and $|\Delta_k| < \delta_d T$.

**Theorem 4.1.** The closed-loop disturbed system (4.7) with the sliding mode control law (4.15) enters the boundary layer

$$\Omega = \{x_k||s_k(x_k)| \leq (\varepsilon + |CG|\delta_d)T\} \quad (4.17)$$
in finite steps and stays inside it thereafter, if $|\Delta_k| < \delta_d T$, $0 < r < 1$ and $\epsilon > |CG|\delta_d$.

**Proof.** First, we consider the case when $s_k$ is outside the boundary layer $\Omega$, i.e. $|s_k| > (\epsilon + |CG|\delta_d)T$. We assume that $s_k > (\epsilon + |CG|\delta_d)T$. From (4.16), we have

$$s_{k+1} = rs_k - \epsilon T + CG\Delta_k = rs_k - \eta$$  \hspace{1cm} (4.18)

where $\eta = \epsilon T - CG\Delta_k$. Since $|\Delta_k| < \delta_d T$ and $\epsilon > |CG|\delta_d$, we have

$$0 < \eta < (\epsilon + |CG|\delta_d)T$$  \hspace{1cm} (4.19)

Therefore,

$$(r - 1)(\epsilon + |CG|\delta_d)T < rs_k - (\epsilon + |CG|\delta_d)T < s_{k+1} < rs_k$$  \hspace{1cm} (4.20)

Since $0 < r < 1$, $s_{k+1}$ will enter or approach to the boundary layer $\Omega$. Thus, $s_k$ will converge to the boundary layer $\Omega$ when $s_k > (\epsilon + |CG|\delta_d)T$. And because $r$ is constant, the maximum number of the steps $N_{\text{max}}$, which the system requires to enter the boundary layer $\Omega$ from any nonzero initial conditions, is bounded by

$$N_{\text{max}} \leq \log_r (\epsilon + |CG|\delta_d)T - \log_r s_0 + 1$$  \hspace{1cm} (4.21)

Similarly, using the property of symmetry we can prove that $s_k$ will enter the boundary layer $\Omega$ in finite steps if $s_k < -(\epsilon + |CG|\delta_d)T$.

Second, we consider the case when $s_k$ is inside the boundary layer $\Omega$, i.e. $|s_k| \leq (\epsilon + |CG|\delta_d)T$. If $s_k > 0$, then

$$(r - 1)(\epsilon + |CG|\delta_d)T < s_{k+1} = rs_k - \epsilon T + CG\Delta_k < rs_k$$  \hspace{1cm} (4.22)

because $|s_k| \leq (\epsilon + |CG|\delta_d)T$. Since $0 < r < 1$, $|s_{k+1}| \leq (\epsilon + |CG|\delta_d)T$ or $s_{k+1}$ is inside the boundary layer $\Omega$. If $s_k = 0$, then

$$-|CG|\delta_d T < s_{k+1} = CG\Delta_k < |CG|\delta_d T$$  \hspace{1cm} (4.23)
which also infers $|s_{k+1}| \leq (\varepsilon + |CG|\delta_d)T$ or $s_{k+1}$ is inside the boundary layer $\Omega$. If $s_k < 0$, then

$$rs_k < s_{k+1} = rs_k + \varepsilon T + CG\Delta_k < (\varepsilon + |CG|\delta_d)T$$

(4.24)

$s_{k+1}$ is inside the boundary layer $\Omega$ too, because $0 < r < 1$ and $|s_k| \leq (\varepsilon + |CG|\delta_d)T$.

Therefore, the system state will stay inside the boundary layer $\Omega$ once it enters $\Omega$.

In all, the sliding mode control law (4.15) makes the state of the closed-loop disturbed system (4.7) enter the boundary layer $\Omega$ in finite steps and stay inside it thereafter.

**Remark** The size of this boundary layer, i.e., $(\varepsilon + |CG|\delta_d)T$, is $O(T)$ and will decrease with smaller sampling periods.

**Remark** For a given sampling period $T$ and a fixed $\delta_d$, the size of this boundary layer is determined by $|CG|$ because the choice of $\varepsilon$ is also limited by $\varepsilon > |CG|$. $G$ is system parameter and not changeable, but $C$ is free to pick up in controller design. A good choice of $C$, which makes $|CG|$ small, will lead to a small size boundary layer and in turn good control performance. However, the side effect is that the control effort will increase, because its inverse $(CG)^{-1}$ appears in the control input $u_k$.

**Remark** Using the proposed sliding mode control, we ensure that the quasi-sliding mode can happen in finite steps. Since we can enforce $s_0 = 0$, the quasi-sliding mode can happen from the beginning.

### 4.2.2 Optimal sliding mode design

Now we will use the optimal control method to design an optimal sliding mode with respect to a quadratic performance index. Considering the nominal system (4.1),
we choose a quadratic performance index as

\[
J = \frac{1}{2} \sum_{k=0}^{\infty} (x_k^T Q x_k + u_k^2)
\]

(4.25)

where \( Q \in \mathbb{R}^{n \times n} \) is a symmetric positive definite matrix. In designing the optimal sliding mode, we assume that the disturbance \( d_k = 0 \). Since the sliding mode can happen from the beginning by enforcing \( s_0 = 0, s_k = 0 \ (k \geq 0) \), and the control input becomes

\[
u_k = -(CG)^{-1}[CF x_k + \phi_{k+1} - r s_k]
\]

(4.26)

and the closed-loop system is

\[
x_{k+1} = [F - G(CG)^{-1}(CF - r C)] x_k - G(CG)^{-1}(\phi_{k+1} - r \phi_k)
\]

(4.27)

Denote \( v_k = \phi_{k+1} - \phi_k \) and \( y_k = [x_k^T, \phi_k]^T \), then

\[
y_{k+1} = F y_k + G v_k
\]

(4.28)

where

\[
F = \begin{bmatrix}
F - G(CG)^{-1}(CF - r C) & G(CG)^{-1}(r - 1) \\
0 & 1
\end{bmatrix}, \quad G = \begin{bmatrix}
-G(CG)^{-1} \\
1
\end{bmatrix}
\]

The quadratic performance index is transformed in the new coordinate as

\[
J = \frac{1}{2} \sum_{k=0}^{\infty} (x_k^T Q x_k + u_k^2)
\]

\[
= \frac{1}{2} \sum_{k=0}^{\infty} [x_k^T Q x_k + (CG)^{-2}(CF x_k + \phi_{k+1} - r s_k)^2]
\]

\[
= \frac{1}{2} \sum_{k=0}^{\infty} [x_k^T Q x_k + (CG)^{-2}((CF - r C)x_k + (1 - r)\phi_k + v_k)^2]
\]

\[
= \frac{1}{2} \sum_{k=0}^{\infty} [y_k^T Q y_k + 2y_k^T H v_k + R v_k^2]
\]

(4.29)
where

\[ Q = \begin{bmatrix} Q + (CG)^{-2}(CF - rC)^T(CF - rC) & (CG)^{-2}(CF - rC)^T(1 - r) \\ (CG)^{-2}(1 - r)(CF - rC) & (CG)^{-2}(1 - r)^2 \end{bmatrix}, \]

\[ H = \begin{bmatrix} (CG)^{-2}(CF - rC)^T \\ (CG)^{-2}(1 - r) \end{bmatrix}, \quad R = (CG)^{-2} \]

If we choose \( C \) and \( r \) such that the pair \((F, G)\) is stabilizable, then the optimal control \( v_k^* \) is given by

\[ v_k^* = -Ky_k \]  \quad (4.30)

where

\[ K = (G^T P G + R)^{-1} (G^T P F + H^T) \]  \quad (4.31)

and \( P \) is the solution to the discrete-time Riccati equation

\[ P = (F - GR^{-1}H^T)^T [P - P G (G^T P G + R)^{-1} G^T P] (F - GR^{-1}H^T) + (Q - H R^{-1} H^T) \]  \quad (4.32)

Therefore, the optimal switching function is given by

\[ s_k = Cx_k + \phi_k \]  \quad (4.33)

\[ \phi_{k+1} = \phi_k - Ky_k, \quad \phi_0 = -Cx_0 \]  \quad (4.34)

4.2.3 Simulation Study

Consider a simple double integrator system

\[ \dot{x}_1 = x_2 \]

\[ \dot{x}_2 = u \]  \quad (4.35)

with initial values \( x_1(0) = 1 \) and \( x_2(0) = 0 \).
When the sampling period $T = 0.05$ second, the discrete-time model is

$$x_{k+1} = Fx_k + Gu_k = \begin{bmatrix} 1 & 0.05 \\ 0 & 1 \end{bmatrix}x_k + \begin{bmatrix} 0.00125 \\ 0.05 \end{bmatrix}u_k$$ (4.36)

When the sampling period $T = 0.01$ second, the discrete-time model is

$$x_{k+1} = Fx_k + Gu_k = \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix}x_k + \begin{bmatrix} 0.00005 \\ 0.01 \end{bmatrix}u_k$$ (4.37)

In the simulations, the disturbance $d_k$ is assumed to be

$$d_k = 0.5 \sin(20kT) + 1$$ (4.38)

It is easy to check that

$$|d_k - d_{k-1}| < 10T, \quad \forall k \geq 1$$ (4.39)

So, if we choose $\varepsilon > 10|CG|$ in the sliding mode control (4.15), the quasi-sliding mode will happen. A performance index is chosen as (4.25) with

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$ (4.40)

Controllers are designed for both $T = 0.05$ and $T = 0.01$, respectively. In order to explore the effect of the choice of $C$, two controllers using different $C$ are designed for $T = 0.05$. The controller parameters are designed by using the aforementioned approach and are tabulated in Table 4.1. The simulation results are shown in Fig. 4.1-4.9. Comparing the results of Controller 1 and Controller 2, we observe that the size of the boundary layer of Controller 2 is smaller than that of Controller 1, so is the deviation of the state caused by the disturbance. However, the control input of Controller 2 is larger than that of Controller 1 at the beginning. It shows that the choice of $C$ does affect the controller’s performance. Comparing the results of Controller 1 and Controller 3, we can see that the size of the boundary layer, in
<table>
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<th>Controller 1</th>
<th>Controller 2</th>
<th>Controller 3</th>
</tr>
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<tr>
<td>$T$</td>
<td>0.05</td>
<td>0.05</td>
<td>0.01</td>
</tr>
<tr>
<td>$r$</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
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<td>0.06</td>
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<tr>
<td>$K$</td>
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<td>[0.0353, 0.0359, 0.5]</td>
<td>[0.4689, 0.4700, 0.5]</td>
</tr>
</tbody>
</table>

Table 4.1: ODSMC Controllers’ Parameters

which the switching function $s_k$ is confined, decreases when the sampling period $T$ decreases. In turn, the control accuracy of the closed-loop system is better when $T$ is smaller.

![Figure 4.1: Response of the system state $x$ using Controller 1 when $T = 0.05$](image)

Figure 4.1: Response of the system state $x$ using Controller 1 when $T = 0.05$
Figure 4.2: Response of the switching function $s$ using Controller 1 when $T = 0.05$

Figure 4.3: Control input of Controller 1 when $T = 0.05$
Figure 4.4: Response of the system state $x$ using Controller 2 when $T = 0.05$

Figure 4.5: Response of the switching function $s$ using Controller 2 when $T = 0.05$
Figure 4.6: Control input of Controller 2 when $T = 0.05$

Figure 4.7: Response of the system state $x$ using Controller 3 when $T = 0.01$
Figure 4.8: Response of the switching function $s$ using Controller 3 when $T = 0.01$

Figure 4.9: Control input of Controller 3 when $T = 0.01$
4.3 ODSMC of single input systems with state-dependent disturbance

4.3.1 Sliding mode control design

Let the switching function be
\[ s_k = C x_k = \sum_{i=1}^{n} c_i x_{ki} \] \hspace{1cm} (4.41)
where \( C = [c_1, c_2, \cdots, c_n] \) is a constant row vector. Denote the incremental change of \( s_k \) as \( \Delta s_{k+1} = s_{k+1} - s_k \). Without consideration of the disturbance, the equivalent control can be obtained by imposing \( \Delta s_{k+1} = 0 \) as
\[ u^e_k = - (CG)^{-1} [CF x_k - C x_k] \] \hspace{1cm} (4.42)
Let the sliding mode control be
\[ u_k = u^e_k + u^s_k \] \hspace{1cm} (4.43)
Substituting this sliding mode control law into \( s_{k+1} \), we have
\[ s_{k+1} = C x_{k+1} \]
\[ = C F x_k + C G u_k \]
\[ = C F x_k + C G (u^e_k + u^s_k) \]
\[ = s_k + C G u^s_k \] \hspace{1cm} (4.44)
The switching part \( u^s_k \) is designed as
\[ u^s_k = (CG)^{-1} (r - 1) s_k + L x_k \] \hspace{1cm} (4.45)
where \(|r| < 1, r \neq 0, L = [l_1, l_2, \cdots, l_n] \). The coefficient vector \( L \) is determined by
\[ l_i = \begin{cases} 0 & |s_k| \leq \delta_k(x_k) \\ -r(CG)^{-1} c_i & |s_k| > \delta_k(x_k) \end{cases} \] \hspace{1cm} (4.46)
where \( M > 0 \). \( \delta_k(x_k) \) is defined as

\[
\delta_k(x_k) = \frac{|CG|}{2\rho} M \sum_{i=1}^{n} |x_{ki}|
\]  

(4.47)

**Theorem 4.2.** For the nominal discrete-time system (4.1), the control law (4.45) makes the switching function \( s_k \) move from any initial condition into the boundary layer defined as \( \Upsilon = \{s_k||s_k| \leq \delta_k(x_k)\} \).

**Proof.** Considering the switching function outside the boundary layer \( \Upsilon \), i.e., \(|s_k| > \delta_k(x_k)\). Substituting the control law (4.45) into (4.44)

\[
s_{k+1} = s_k + CGu_k
\]

\[
= rs_k + CGLx_k
\]

(4.48)

Multiplying both sides of this equation by \( rs_k \) and after a simple manipulation, we have

\[
rs_{k+1} - r^2s_k^2 = rCGs_kLx_k
\]

(4.49)

Since \( l_i = -M \text{sign}(CGs_kx_{ki}) \) and \( M > 0 \),

\[
rs_{k+1}s_k - r^2s_k^2 = rCGs_k \sum_{i=1}^{n} l_ix_{ki}
\]

\[
= -r|CG||s_k|M \sum_{i=1}^{n} |x_{ki}|
\]

\[
< 0
\]

(4.50)

Since \(|s_k| > \delta_k(x_k)\),

\[
rs_{k+1}s_k - r^2s_k^2 = rCGs_k \sum_{i=1}^{n} l_ix_{ki}
\]

\[
= -r|CG||s_k|M \sum_{i=1}^{n} |x_{ki}|
\]

\[
= -2r^2\delta_k(x_k)|s_k|
\]

\[
> -2r^2s_k^2
\]

(4.51)
Therefore,

\[-r^2s_k^2 < rs_k s_{k+1} < r^2s_k^2\]  \hspace{1cm} (4.52)

or

\[|s_{k+1}| < |r||s_k|\]  \hspace{1cm} (4.53)

If a discrete-time Lyapunov function is chosen as

\[V_k = \frac{1}{2}s_k^2\]  \hspace{1cm} (4.54)

then

\[V_{k+1} < r^2V_k \quad (|r| < 1, r \neq 0)\]  \hspace{1cm} (4.55)

which implies that \(s_k\) will move from the outside into the boundary layer \(\Upsilon\).

\[\square\]

\textbf{Lemma 4.3.} If \(s_k \Delta s_{k+1} < -\frac{1}{2}\Delta s_{k+1}^2\), then the sliding mode can happen.

\textit{Proof.} Using the discrete-time Lyapunov function (4.54),

\[V_{k+1} = \frac{1}{2}s_{k+1}^2\]
\[= \frac{1}{2}(s_k + \Delta s_{k+1})^2\]
\[= \frac{1}{2}s_k^2 + s_k \Delta s_{k+1} + \frac{1}{2}s_{k+1}^2\]
\[= V_k + s_k \Delta s_{k+1} + \frac{1}{2}s_{k+1}^2\]  \hspace{1cm} (4.56)

If \(s_k \Delta s_{k+1} < -\frac{1}{2}\Delta s_{k+1}^2\), then

\[V_{k+1} < V_k\]  \hspace{1cm} (4.57)

and \(s_k\) will converge to \(s_k = 0\). Therefore, if \(s_k \Delta s_{k+1} < -\frac{1}{2}\Delta s_{k+1}^2\), the sliding mode can happen.  \[\square\]
Theorem 4.4. For the nominal discrete-time system (4.1), the control law (4.45) makes the switching function $s_k$ converge to the sliding mode manifold $s_k = 0$ when $s_k \in \Upsilon$.

Proof. When the switching function $s_k$ is inside the boundary layer $\Upsilon$, the switching part of control $u^s_k$ becomes

$$u^s_k = -(CG)^{-1}s_k$$

From (4.44)

$$\Delta s_{k+1} = CGu^s_k = -s_k$$

From Lemma 4.3, we conclude that the sliding mode exists and $s_k$ will converge to the sliding mode manifold $s_k = 0$. \qed

Theorem 4.5. For the nominal discrete-time system (4.1), the control law (4.45) makes the switching function $s_k$ converge to the sliding mode manifold $s_k = 0$ from any initial condition.

Proof. Direct result from Theorem 4.2 and 4.4. \qed

4.3.2 Optimal sliding mode design

To design an optimal sliding mode, we assume that $d_k = 0$. Considering the nominal system (4.1), we choose a quadratic performance index as

$$J = \frac{1}{2} \sum_{k=m}^{\infty} x_k^T Q x_k$$

(4.60)

where $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix and $t = mT$ is the time instant when the sliding mode happens.

Since $(F, G)$ is controllable, we take an invertible transformation

$$z_k = Nx_k$$

(4.61)
to transform the system (4.1) into the following controllable canonical form

\[ z_{k+1} = \hat{F}z_k + \hat{G}u_k \]  

(4.62)

where \( z_k = [z_{k1}, z_{k2}, \cdots, z_{kn}]^T \in \mathbb{R}^n \) is the new state vector, \( N \) a \( n \times n \) constant invertible matrix, and

\[
\hat{F} = NFN^{-1} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
f_1 & f_2 & f_3 & \cdots & f_n
\end{bmatrix}, \quad \hat{G} = NG = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

The switching function can be represented in the new coordinate as

\[ s_k = Cx_k = CN^{-1}z_k = \hat{C}z_k \]  

(4.63)

where \( \hat{C} = CN^{-1} = [\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_n]^T \). In the new coordinate, the performance index can be represented as

\[
J = \frac{1}{2} \sum_{k=m}^{\infty} (N^{-1}z_k)^T Q (N^{-1}z_k) \\
= \frac{1}{2} \sum_{k=m}^{\infty} z_k^T (N^{-1})^T Q N^{-1} z_k \\
= \frac{1}{2} \sum_{k=m}^{\infty} z_k^T \hat{Q} z_k 
\]

(4.64)

where \( \hat{Q} = (N^{-1})^T Q N^{-1} \). Note that \( \hat{Q} \) is also a symmetric positive definite matrix, because \( Q \) is positive definite and \( N^{-1} \) is nonsingular. Without loss of generality we suppose \( \hat{c}_n = 1 \). On the sliding mode manifold \( s_k = \hat{C}z_k = 0 \),

\[
z_{kn} = -\hat{c}_1 z_{k1} - \hat{c}_2 z_{k2} - \cdots - \hat{c}_{n-1} z_{k(n-1)}
\]

(4.65)

and the system dynamics is determined by

\[
\bar{z}_{k+1} = W\bar{z}_k + Vz_{kn}
\]

(4.66)
where \( z_k = [z_{k1}, z_{k2}, \cdots, z_{k(n-1)}]^T \in \mathbb{R}^{n-1} \) and

\[
W = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix} \in \mathbb{R}^{(n-1) \times (n-1)}, \quad V = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix} \in \mathbb{R}^{n-1}
\]

\( z_{kn} \) can be considered as a virtual control input to the system (4.66). Correspondingly, the performance index can be rewritten as

\[
J = \frac{1}{2} \sum_{k=m}^{\infty} (z_k^T \hat{Q}_{11} z_k + 2z_k^T \hat{Q}_{12} z_{kn} + \hat{Q}_{22} z_{kn}^2)
\]

(4.67)

where

\[
\hat{Q} = \begin{bmatrix}
\hat{Q}_{11} & \hat{Q}_{12} \\
\hat{Q}_{12}^T & \hat{Q}_{22}
\end{bmatrix}, \quad \hat{Q}_{11} \in \mathbb{R}^{(n-1) \times (n-1)}, \quad \hat{Q}_{12} \in \mathbb{R}^{n-1}, \quad \hat{Q}_{22} \in \mathbb{R}
\]

Note that \( \hat{Q}_{11} \) is positive definite and \( \hat{Q}_{22} \neq 0 \) because \( \hat{Q} \) is positive definite. Since \((W, V)\) is controllable, the optimal solution to minimize this performance index for the system (4.66) exists and has the form

\[
z_{kn} = -Kz_k = -k_1 z_{k1} - k_2 z_{k2} - \cdots - k_{n-1} z_{k(n-1)}
\]

(4.68)

where \( K = [k_1, k_2, \cdots, k_{n-1}] \) is defined by

\[
K = (V^T P V + \hat{Q}_{22})^{-1} (V^T P W + \hat{Q}_{12}^T)
\]

(4.69)

and \( P \) is a symmetric positive definite matrix which satisfies the following discrete-time Riccati equation:

\[
P = (W - V \hat{Q}_{22}^{-1} \hat{Q}_{12}^T)[P - PV(V^T P V + \hat{Q}_{22})^{-1} V^T P](W - V \hat{Q}_{22}^{-1} \hat{Q}_{12}^T) + (\hat{Q}_{11} - \hat{Q}_{12} \hat{Q}_{22}^{-1} \hat{Q}_{12}^T)
\]

(4.70)
After comparing (4.65) and (4.68), we can easily obtain the optimal switching function as
\[ s^*_k = [K, 1]z_k = [K, 1]N x_k = C^* x_k \quad (4.71) \]
where \( C^* = [K, 1]N \). The closed-loop system on this sliding mode manifold is
\[ x_{k+1} = F x_k + G u_k \]
\[ = F x_k - G(C^*G)^{-1}C^* F x_k \]
\[ = [F - G(C^*G)^{-1}C^* F] x_k \quad (4.72) \]

Corollary 4.6. \([F - G(C^*G)^{-1}C^* F]\) is a stable matrix, whose eigenvalues are all inside the unit circle.

### 4.3.3 Robustness analysis

Now we consider the disturbed system (4.7) with state dependent disturbance
\[ d_k = [q_1, q_2, \cdots, q_n] x_k, \quad |q_i| < \bar{q} \ (i = 1, \cdots, n) \quad (4.73) \]
To analyze the robustness of the sliding mode controller, we discuss the case outside and the case inside the boundary layer, respectively.

**Theorem 4.7.** For the disturbed system (4.7), the following sliding mode control law \( u_k^* \) ensures the system state converge to the boundary layer \( \Upsilon^* = |s_k^*| \leq \delta_k^*(x_k) \) from any initial condition.
\[ u_k^* = (C^*G)^{-1}[rs_k^* - C^* F x_k] + L x_k \quad (4.74) \]
\[ l_i = \begin{cases} -r(C^*G)^{-1}c_i^* & \quad |s_k^*| \leq \delta_k^*(x_k) \\ -M \text{sign}(C^*G s_k^* x_{ki}) & \quad |s_k^*| > \delta_k^*(x_k) \end{cases} \quad (4.75) \]
where \( M > \bar{q} \) and \( \delta^*_k(x_k) \) is defined as

\[
\delta^*_k(x_k) = \frac{|C^*G|}{2r} (M + \bar{q}) \sum_{i=1}^{n} |x_{ki}|
\] (4.76)

**Proof.** Considering the switching function outside the boundary layer \( \Upsilon^* \), i.e., \(|s^*_k| > \delta^*_k(x_k)\). Substituting the control law (4.45) into (4.44)

\[
s^*_{k+1} = s^*_k + C^*G u^*_k + C^*G d_k
\]

\[
= rs^*_k + C^*GLx_k + C^*Gd_k
\] (4.77)

Multiplying both sides of this equation by \( rs^*_k \) and after a simple manipulation, we have

\[
rs^*_kk^*_{k+1} - r^2s^2_k = rC^*Gs^*_kLx_k + rC^*Gs^*_kd_k
\] (4.78)

Since \( l_i = -M \text{sign}(C^*G s^*_k x_{ki}) \) and \( M > \bar{q} \),

\[
rs^*_k s^*_{k+1} - r^2 s^2_k = rC^*Gs^*_k \sum_{i=1}^{n} l_i x_{ki} + rC^*Gs^*_k \sum_{i=1}^{n} q_i x_{ki}
\]

\[
< -r|C^*G||s^*_k|(M - \bar{q}) \sum_{i=1}^{n} |x_{ki}|
\]

\[
< 0
\] (4.79)

Since \(|s^*_k| > \delta^*_k(x_k)\),

\[
r s^*_k s^*_{k+1} - r^2 s^2_k = rC^*Gs^*_k \sum_{i=1}^{n} l_i x_{ki} + rC^*Gs^*_k \sum_{i=1}^{n} q_i x_{ki}
\]

\[
= -r|C^*G||s^*_k|M \sum_{i=1}^{n} |x_{ki}| + rC^*Gs^*_k \sum_{i=1}^{n} q_i x_{ki}
\]

\[
> -2r^2 \delta^*_k(x_k)|s^*_k|
\]

\[
> -2r^2 s^2_k
\] (4.80)

Therefore,

\[-r^2 s^2_k < rs^*_k s^*_{k+1} < r^2 s^2_k\] (4.81)
or

\[ |s_{k+1}^*| < |r||s_k^*| \]  \hspace{1cm} (4.82)

Using the discrete-time Lyapunov function (4.54), we have

\[ V_{k+1} < r^2V_k \quad (|r| < 1, r \neq 0) \]  \hspace{1cm} (4.83)

which implies that \( s_k^* \) will move from the outside into the boundary layer \( \Upsilon^* \).

When the system state is inside the boundary layer \( \Upsilon^* \), i.e., \( |s_k^*| \leq \delta_k(x_k) \), the control input becomes

\[ u_k = -(C^*G)^{-1}C^*Fx_k \]  \hspace{1cm} (4.84)

and the closed-loop system is

\[
x_{k+1} = [F - G(C^*G)^{-1}C^*F]x_k + Gd_k \\
= \Psi x_k + Gd_k \]  \hspace{1cm} (4.85)

where \( \Psi = [F - G(C^*G)^{-1}C^*F] \). From Corollary 4.6 we know that \( \Psi \) is a stable matrix. Therefore, there exists a positive definite matrix \( P \), which satisfies \( \Psi^TP\Psi - P = -I \), where \( I \) is a \( n \times n \) identity matrix.

**Theorem 4.8.** If \( \bar{q} < \sqrt{\frac{1}{|G^TPG|}} \), then the sliding mode control law (4.74) makes the disturbed system (4.7) asymptotically stable inside the boundary layer \( \Upsilon^* \).
Proof. Let $V_k = x_k^T P x_k$ be a candidate Lyapunov function.

$$V_{k+1} = x_{k+1}^T P x_{k+1}$$

$$= (\Psi x_k + Gd_k)^T P (\Psi x_k + Gd_k)$$

$$= x_k^T \Psi^T P \Psi x_k + d_k^T G^T P Gd_k$$

$$= x_k^T \Psi^T P \Psi x_k + d_k^T G^T P Gd_k - x_k^T P x_k + V_k$$

$$= x_k^T (\Psi^T P \Psi - P) x_k + (G^T P G)d_k^T d_k + V_k$$

$$\leq -\|x_k\|_2^2 + |G^T P G| q^2 \|x_k\|_2^2 + V_k$$

$$< V_k$$ (4.86)

Therefore, the closed-loop system is asymptotically stable inside the boundary layer $\Upsilon^*$.

\[\square\]

Remark Since $P > 0$ and $G \neq 0$, $G^T P G \neq 0$.

Remark For a given system, if $q < \sqrt{\frac{1}{|G^T P G|}}$ is not satisfied, we need redesign the switching function $s^*_k$ by changing the quadratic performance index.

Theorem 4.9. For the disturbed discrete-time system (4.7), the control law (4.74) with $q < \sqrt{\frac{1}{|G^T P G|}}$ makes the switching function $s^*_k$ converge to the sliding mode manifold $s^*_k = 0$ from any initial condition.

Proof. Direct result from Theorem 4.7 and 4.8. \[\square\]

4.3.4 Simulation study

Consider a second order discrete-time system in the nominal form

$$x_{k+1} = \begin{bmatrix} 1.2 & 0.1 \\ 1 & 0.6 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$ (4.87)
with a state dependent disturbance \( d_k = [q_1, q_2] x_k \), where \( |q_1| < 0.1 \) and \( |q_2| < 0.1 \).

With a transformation \( z_k = N x_k \), where

\[
N = \begin{bmatrix}
10 & 0 \\
12 & 1
\end{bmatrix}
\]

the nominal system (4.87) can be transformed into a controllable canonical form as in (4.62). We choose the quadratic performance index as

\[
Q = \begin{bmatrix}
1 & 0 \\
0 & 10
\end{bmatrix}
\]

Then the optimal sliding mode can be designed by using the aforementioned approach as follow

\[
\hat{Q} = (N^{-1})^T Q N^{-1} = \begin{bmatrix}
14.41 & -12 \\
-12 & 10
\end{bmatrix}
\]

\[
K = -0.8315
\]

\[
C^* = [K, 1] N = [3.6855, 1]
\]

The dynamics of the closed-loop nominal system on the sliding mode manifold is \( x_{k+1} = \Psi x_k \), where

\[
\Psi = F - G (C^* G)^{-1} C^* F = \begin{bmatrix}
1.2 & 0.1 \\
-4.4226 & -0.3685
\end{bmatrix}
\]

(4.88)

The positive definite solution to the Lyapunov equation \( \Psi^T P \Psi - P = -I \) is

\[
P = \begin{bmatrix}
5.6973 & -17.3119 \\
-17.3119 & 64.8024
\end{bmatrix}
\]

(4.89)

From Theorem 6, we can compute the disturbance bound \( \overline{q} \)

\[
0.1 = \overline{q} < \sqrt{\frac{1}{|G^T P G|}} = 0.1242
\]

(4.90)

which implies that the closed-loop disturbed system is asymptotically stable.
In simulations, the disturbance is assumed to be $d_k = [0.09, -0.05]x_k$. The initial conditions are set as $x_1(0) = 1$ and $x_2(0) = 0$. The controller parameters are picked up as $r = 0.5$ and $M = 0.1$. The simulation results are shown in Fig. 4.10-4.12. In Fig. 4.11, we can clearly observe that $s_k$ first converge to the boundary layer $\Upsilon$, and then converge to zero.

4.4 Conclusion

The design of optimal sliding mode control has been proposed to provide robustness to linear discrete-time systems. Two kinds of disturbances, i.e., rate bounded disturbance and state dependent disturbance, are considered respectively. The sliding mode manifold is designed to obtain optimal performance with respect to some
Figure 4.11: $|s_k|$ and $\delta_k(x_k)$

Figure 4.12: Control input $u_k$
quadratic performance indices. For systems disturbed by a rate bounded disturbance, a time-varying sliding mode manifold is used. This time-varying sliding mode provides additional freedom in achieving robustness and optimality. The control accuracy can be greatly improved by picking up suitable controller parameters without decreasing the sampling period. On the other hand, for systems disturbed by a state dependent disturbance, the sliding mode manifold is designed by using Lyapunov theory to ensure the robustness in presence of disturbances. The simulation results illustrate that the optimal sliding mode control provides robust solutions to the optimal control problems.
5.1 Introduction

Two classes of underactuated systems are of great interest among underactuated mechanical systems. One is in the cascaded normal form introduced by Olfati-saber [79] and the other is in the chained form introduced by Murray and Sastry [91]. Olfati-saber [80] proposed a systematic approach to transform underactuated mechanical systems (e.g. the Acrobot [68], the TORA system [73], the VTOL aircraft [81], and the pendubot [70]) to the cascaded normal form. Although he did not present any general control law to stabilize underactuated systems in the cascaded normal form, the normal form itself is more convenient for control design. On the other hand, the chained form is used to model kinematics of nonholonomic mechanical systems. Murray and Sastry [91] have also given sufficient conditions to convert (via state feedback and coordinates transformation) a generic controllable nonholonomic system with two inputs to the chained form. Many nonholonomic mechanical systems can be described by kinematical models in the chained form or are feedback equivalent to chained form, among which the most interesting examples are a Dubin’s car and a Dubin’s car with
multiple trailers [94]. Those two classes of underactuated systems may not satisfy the well known Brockett’s necessary conditions (Theorem 1.1). Therefore, it is impossible to apply a smooth feedback law to, even locally, stabilize them. Thus, many researchers tried to solve the stabilization problem via time-varying or discontinuous control laws or a mixture of both. In this chapter, sliding mode control approaches will be developed, which can globally stabilize all degrees of freedom, including those which are indirectly actuated through the nonlinear coupling, for those two classes of underactuated systems. Since the sliding mode control is not smooth, the proposed sliding mode controllers can stabilize systems, which do not satisfy Brockett’s necessary conditions. The advantage of sliding mode control is its insensitivity to the model errors, parametric uncertainties and other disturbances [19]. Once the system is on the sliding manifold, the system behavior is determined by the structure of the sliding surface. This advantage gives us a little more freedom in controller design, as we can modify the system model by introducing virtual disturbances to satisfy some conditions or requirements.

5.2 Sliding mode control of underactuated systems in the cascaded normal form

5.2.1 System model

The equations of dynamics of an underactuated system can be simplified as

\[
\begin{align*}
m_{11}(q)\ddot{q}_1 + m_{12}(q)\ddot{q}_2 + h_1(q, \dot{q}) &= 0 \\
m_{21}(q)\ddot{q}_1 + m_{22}(q)\ddot{q}_2 + h_2(q, \dot{q}) &= \tau
\end{align*}
\]  

(5.1)

where \( q = [q_1, q_2]^T \), \( q \) and \( \dot{q} \) are the states, \( \tau \) the control input, and \( h_i \)'s contain Coriolis, centrifugal and gravity terms. In [79] and [80] a systematic method is proposed
to transform an underactuated system (5.1) into the following cascaded normal form

\[
\begin{align*}
\dot{x}_1 &= x_2 + d_1 \\
\dot{x}_2 &= f_1(x_1, x_2, x_3, x_4) + d_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2(x_1, x_2, x_3, x_4) + b(x_1, x_2, x_3, x_4)u + d_3
\end{align*}
\]

(5.2)

where \(x_i \in \mathbb{R}^n\) \((i = 1, 2, 3, 4)\) are system states, \(u \in \mathbb{R}^n\) the controls, \(f_1, f_2 : \mathbb{R}^{4n} \rightarrow \mathbb{R}^n\), \(b : \mathbb{R}^{4n} \rightarrow \mathbb{R}^{n \times n}\) nonlinear smooth vector functions, \(b\) is invertible, and \(d_i \in \mathbb{R}^n\) \((i = 1, 2, 3)\) disturbances. The block diagram of (5.2) is depicted in Fig. 5.1. Many underactuated systems can be transformed into the form of (5.2). The inverted pendulum system [78], the translational oscillations with a rotational actuator (TORA) example [73], the vertical takeoff and landing (VTOL) aircraft [81], and the quadrotor helicopter [66] are all examples, to name a few. Our control objective is to stabilize all states of (5.2) to zero. For system (5.2), we have the following assumptions:

Assumption 1. \(f_1(0, 0, 0, 0) = 0\).

Assumption 2. \(\frac{\partial f_1}{\partial x_3}\) is invertible or \(\frac{\partial f_1}{\partial x_4}\) is invertible.

Assumption 3. \(f_1(0, 0, x_3, x_4) = 0\) is an asymptotically stable manifold, i.e., \(x_3\) and \(x_4\) will converge to 0 if \(f_1(0, 0, x_3, x_4) = 0\).

Assumption 1 is a necessary condition for the origin to be an equilibrium of the closed-loop system without consideration of the disturbances. Assumption 2 and 3 are requirements to develop an overall sliding mode controller, and are not necessarily satisfied by some practical systems. However, the existence of \(d_2\) gives us the freedom of choice of \(f_1\), which meets these assumptions. We will show this in TORA example in Section 4.
5.2.2 Sliding mode control design

In this subsection we will present an overall sliding mode control, which considers and controls all states at the same time. First we denote error variables as

\[ e_1 = x_1 \quad e_3 = f_1(x_1, x_2, x_3, x_4) \]
\[ e_2 = x_2 \quad e_4 = \frac{\partial f_1}{\partial x_1} x_2 + \frac{\partial f_1}{\partial x_2} f_1 + \frac{\partial f_1}{\partial x_3} x_4 \]

where \( E_1 = [e_1, e_2, e_3]^T \) \( E_2 = [e_1, e_2]^T \)

To satisfy the matched disturbance condition of the sliding mode control design [19], the following assumptions are also needed.

**Assumption 4.** If \( d_1 \neq 0 \), then the maximum absolute row sum norm of \( \frac{\partial f_1}{\partial x_1} \) is bounded, i.e., \( \| \frac{\partial f_1}{\partial x_1} \|_\infty \leq \beta_1 \) where \( \beta_1 \) is a nonnegative constant.

**Assumption 5.** If \( d_2 \neq 0 \), then the maximum absolute row sum norm of \( \frac{\partial f_1}{\partial x_2} \) is bounded, i.e., \( \| \frac{\partial f_1}{\partial x_2} \|_\infty \leq \beta_2 \) where \( \beta_2 \) is a nonnegative constant.

**Assumption 6.** If \( d_3 \neq 0 \) and if \( \frac{\partial f_1}{\partial x_3} \) is invertible, then the maximum absolute row sum norm of \( \frac{\partial f_1}{\partial x_3} \) is bounded, i.e., \( \| \frac{\partial f_1}{\partial x_3} \|_\infty \leq \beta_3 \) where \( \beta_3 \) is a positive constant.

**Assumption 7.** If \( d_3 \neq 0 \) and if \( \frac{\partial f_1}{\partial x_4} \) is invertible, then the maximum absolute row sum norm of \( \frac{\partial f_1}{\partial x_4} \) is bounded, i.e., \( \| \frac{\partial f_1}{\partial x_4} \|_\infty \leq \beta_4 \) where \( \beta_4 \) is a positive constant.
Assumption 8. The disturbances $d_i$'s are bounded. If $\frac{\partial f_1}{\partial x_4}$ is invertible, then $\|d_1\| < \bar{d}_1\|E_2\|_2$, $\|d_2\| < \bar{d}_2\|E_2\|_2$ and $\|d_3\| < \bar{d}_3 + \bar{d}_4\|\xi(x)\|_2$. If $\frac{\partial f_1}{\partial x_3}$ is invertible, then $\|d_1\| < \bar{d}_1\|E_1\|_2$, $\|d_2\| < \bar{d}_2\|E_1\|_2$ and $\|d_3\| < \bar{d}_3 + \bar{d}_4\|\xi(x)\|_2$. \(d_i \geq 0\) \((i = 1, 2, 3, 4)\) are constant numbers, and $\xi(x)$ is a known vector field of states $x = [x_1, x_2, x_3, x_4]^T$.

If $\frac{\partial f}{\partial x_4} = 0$ and $\frac{\partial f}{\partial x_3}$ is invertible, then a switching surface is defined as $s = c_1 e_1 + c_2 e_2 + c_3 e_3 + e_4$. If $\frac{\partial f}{\partial x_4}$ is invertible, then a switching surface is defined as $s = c_1 e_1 + c_2 e_2 + e_3$, where $c_i$ \((i = 1, 2, 3)\) are positive constant numbers and picked up to make the system dynamics on the sliding manifold $s = 0$ is asymptotically stable. The requirement for the choice of $c_i$ will be discussed later. The reason why we apply two different surfaces is that $s$ needs to be relative degree 1 with respect to the control input $u$.

It is worthwhile to note that the systems, which are described by (5.2) and satisfy Assumption 1-8, do not necessarily satisfy Brockett’s necessary conditions. Consider the following underactuated system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g \sin x_1/l + x_3^2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u
\end{align*}
\]

which is diffeomorphic to an underactuated unstable two degrees of freedom mechanical system studied by Rui et al. [83], where $x_i$ are the states, $u$ the input, $g$ the acceleration of gravity, and $l$ the pendulum length. It is obvious that the linearized system of (5.3) around the origin has an unstable uncontrollable mode associated with
an eigenvalue whose real part is positive. Therefore, this underactuated system cannot be stabilized by any smooth state feedback, because it violates the first Brockett’s necessary condition. However, there exists a continuous state feedback law, which globally stabilizes the system (5.3) [82], [83], e.g.

\[ u = -c \left[ x_4^3 + 18^3 \left( x_3^3 + \frac{3}{2} \left( x_2 + \frac{1}{2} x_1 \right) \right) \right]^{1/3} \quad \text{for a suitable } c > 1 \quad (5.4) \]

If we add \( x_3 \) to and subtract \( x_3 \) from the second equation of (5.3), then system (5.3) can be represented in the general form (5.2) with \( f_1 = g \sin x_1/l + x_3^3 + x_3, f_2 = 0, b = 1, d_1 = 0, d_2 = -x_3 \) and \( d_3 = 0 \). The error variables are defined as before, then

\[ d_2 = -x_3 = g \sin x_1/l + x_3^3 - e_3 \]

\[ \Rightarrow |(1 + x_3^2)x_3| = |e_3 - g \sin x_1/l| \]

\[ \Rightarrow |x_3| \leq |(1 + x_3^2)x_3| = |e_3 - g \sin x_1/l| \leq |e_3| + g|e_1|/l \]

\[ \Rightarrow |d_2| = |x_3| \leq (1 + g/l)\|E_1\|_2 \]

Clearly, Assumption 1-8 are all satisfied.

**Case 1:** \( \partial f_1 / \partial x_4 = 0 \) and \( \partial f_1 / \partial x_3 \) is invertible

The switching surface is defined as \( s = c_1 e_1 + c_2 e_2 + c_3 e_3 + e_4 \). The constants \( c_i \) are picked up as follows. Denote

\[
A_{n1} = \begin{bmatrix}
0 & I_n & 0 \\
0 & 0 & I_n \\
-c_1I_n & -c_2I_n & -c_3I_n
\end{bmatrix}
\]

where \( I_n \) is \( n \times n \) identity matrix, and denote \( \lambda_{\text{left}}(-A_{n1}) \) as the real part of the leftmost eigenvalue of \(-A_{n1}\). \( c_i \) should be picked up to make \( A_{n1} \) Hurwitz and \( \max\{d_1, d_2, \beta_1 d_1 + \beta_2 d_2\} < \lambda_{\text{left}}(-A_{n1}) \). The sliding mode control consists of two
parts: the equivalent control part $u_{eq}$ and the switching control part $u_{sw}$. The equivalent control on the manifold $s = 0$ can be calculated by making $\dot{s} = 0$ as follow

$$u_{eq} = - \left[ \frac{\partial f_1}{\partial x_3} \right]^{-1} \left\{ c_1 x_2 + c_2 f_1 + c_3 \frac{\partial f_1}{\partial x_1} x_2 + c_3 \frac{\partial f_1}{\partial x_2} f_1 + c_3 \frac{\partial f_1}{\partial x_3} x_2 \right\}$$

$$+ \frac{d}{dt} \left[ \frac{\partial f_1}{\partial x_1} x_2 \right] + \frac{d}{dt} \left[ \frac{\partial f_1}{\partial x_2} f_1 \right] + \frac{d}{dt} \left[ \frac{\partial f_1}{\partial x_3} x_4 + \frac{\partial f_1}{\partial x_3} f_2 \right]$$

(5.5)

The switching control part is designed to make $s$ converge to the manifold $s = 0$ as

$$u_{sw} = - \left[ \frac{\partial f_1}{\partial x_3} \right]^{-1} [M\text{sign}(s) + \lambda s]$$

(5.6)

where $M = (c_1 \overline{d}_1 + c_2 \overline{d}_2 + c_3 \beta_1 \overline{d}_1 + c_3 \beta_2 \overline{d}_2)\|E_1\|_2 + \beta_3 (\overline{d}_3 + \overline{d}_4 \|\xi(x)\|_2) + \rho$, and $\rho, \lambda$ are positive constants. The sliding mode control is

$$u = u_{eq} + u_{sw}$$

(5.7)

**Theorem 5.1.** If $\partial f_1 / \partial x_4 = 0$ and $\partial f_1 / \partial x_3$ is invertible, with the control (5.7) all states of (5.2) will asymptotically converge to 0.

**Proof.** First we prove that the sliding mode exists. A Lyapunov function candidate can be selected as $V = s^Ts/2$. Differentiating $V$ and substituting (5.5), (5.6), (5.7) yield

$$\dot{V} = s^T \dot{s} = s^T [c_1 \dot{e}_1 + c_2 \dot{e}_2 + c_3 \dot{e}_3 + \dot{e}_4]$$

$$= s^T \left[ -M\text{sign}(s) - \lambda s + c_1 \overline{d}_1 + c_2 \overline{d}_2 + c_3 \frac{\partial f_1}{\partial x_1} \overline{d}_1 + c_3 \frac{\partial f_1}{\partial x_2} \overline{d}_2 + \frac{\partial f_1}{\partial x_3} \overline{d}_3 \right]$$

$$< -\lambda s^T s - (M - c_1 \overline{d}_1 \|E_1\|_2 - c_2 \overline{d}_2 \|E_1\|_2 - c_3 \beta_1 \overline{d}_1 \|E_1\|_2 - c_3 \beta_2 \overline{d}_2 \|E_1\|_2 - \beta_3 \overline{d}_3$$

$$- \beta_3 \overline{d}_4 \|\xi(x)\|_2) \|s\|_1$$

$$= -\lambda s^T s - \rho \|s\|_1 \leq 0$$

(5.8)

Therefore, under the control (5.7), the system can reach and thereafter stay on the manifold $s = 0$ in finite time. On the manifold $s = 0$ or $e_4 = -c_1 e_1 - c_2 e_2 - c_3 e_3$, the
original system is reduced to
\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2 \\
\dot{e}_3
\end{bmatrix} = \begin{bmatrix}
0 & I_n & 0 \\
0 & 0 & I_n \\
-c_1 I_n & -c_2 I_n & -c_3 I_n
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix} + \begin{bmatrix}
d_1 \\
d_2 \\
\frac{\partial f}{\partial x_1} d_1 + \frac{\partial f}{\partial x_2} d_2
\end{bmatrix}
\] (5.9)
or
\[
\dot{E}_1 = A_{n1} E_1 + D_1
\] (5.10)
where \( D_1 = [d_1, d_2, \frac{\partial f}{\partial x_1} d_1 + \frac{\partial f}{\partial x_2} d_2]^T, \|d_1\| < \bar{d}_1\|E_1\|_2, \|d_2\| < \bar{d}_2\|E_1\|_2, \frac{\partial f}{\partial x_1} d_1 + \frac{\partial f}{\partial x_2} d_2 < (\beta_1 \bar{d}_1 + \beta_2 \bar{d}_2)\|E_1\|_2. \) This is a perturbed system with vanished perturbations \( D_1, \) where \( \|D_1\| < \gamma\|E_1\|_2 \) and \( \gamma = \max\{\bar{d}_1, \bar{d}_2, \beta_1 \bar{d}_1 + \beta_2 \bar{d}_2\}. \) Let \( Q = Q^T > 0, \) then the Lyapunov equation \( PA_{n1} + A_{n1} P^T = -Q \) has a unique solution \( P = P^T > 0, \) because \( A_{n1} \) is Hurwitz. Using \( V_1 = E_1^T P E_1 \) as a Lyapunov function, we have
\[
\dot{V}_1 \leq -\lambda_{\min}(Q)\|E_1\|_2^2 + 2\lambda_{\max}(P)\gamma\|E_1\|_2^2
\] (5.11)
where \( \lambda_{\max}(P) \) and \( \lambda_{\min}(P) \) are the largest and smallest eigenvalues of a positive definite matrix \( P, \) respectively. If \( \gamma < \lambda_{\min}(Q)/2\lambda_{\max}(P), \) then \( \dot{V}_1 < 0 \) and the origin is asymptotically stable. Therefore, the ratio of \( \lambda_{\min}(Q)/2\lambda_{\max}(P) \) is very important and should be as large as possible to allow large perturbations. Actually, this ratio can be maximized with a choice \( Q = I_{3n} \) (pp.342, [97]). When \( Q = I_{3n}, \lambda_{\min}(Q) = 1 \) and \( \lambda_{\max}(P) = 1/(2\lambda_{\left\text{left}}(-A_{n1})). \) If \( \gamma < \lambda_{\left\text{left}}(-A_{n1}), \) the reduced system is asymptotically stable at the origin, and \( e_1, e_2 \) and \( e_3 \) will all converge to zero from any initial conditions. Since \( s = c_1 e_1 + c_2 e_2 + c_3 e_3 + e_4 = 0, e_4 \) will converge to zero too. Recalling \( e_3 = f_1 \) and Assumption 3, we can easily prove that \( x_3 \) and \( x_4 \) will both converge to zero after \( e_1 = x_1, e_2 = x_2, \) and \( e_3 = f_1 \) are stabilized. 

Remark Placing the eigenvalues of \( A_{n1} \) can be done by picking up appropriate \( c_i \) for \( A_{11} \) when \( n = 1 \) or \( I_n = 1, \) because the eigenvalues of \( A_{n1} \) are just multiple \( (n) \) copies of the eigenvalues of \( A_{11}. \)
Case 2: $\partial f_1/\partial x_4$ is invertible

The switching surface is defined as $s = c_1e_1 + c_2e_2 + e_3$. The constants $c_i$ are picked up as follows. Denote

$$A_{n2} = \begin{bmatrix} 0 & I_n \\ -c_1 I_n & -c_2 I_n \end{bmatrix}$$

c_i should be picked up to make $A_{n2}$ Hurwitz and $\max\{d_1, d_2\} < \lambda_{\text{left}}(-A_{n2})$. The equivalent control and the switching control are designed respectively as follows

$$u_{eq} = -\left[\frac{\partial f_1}{\partial x_4}\right]^{-1} \left[ c_1 x_2 + c_2 f_1 + \frac{\partial f_1}{\partial x_1} x_2 + \frac{\partial f_1}{\partial x_2} f_1 + \frac{\partial f_1}{\partial x_3} x_4 + \frac{\partial f_1}{\partial x_4} f_2 \right]$$ (5.12)

$$u_{sw} = -\left[\frac{\partial f_1}{\partial x_4}\right]^{-1} \left[ M \mathrm{sign}(s) + \lambda s \right]$$ (5.13)

where $M = (c_1 d_1 + c_2 d_2 + \beta_1 d_1 + \beta_2 d_2)\|E_2\|_2 + \beta_3 (d_3 + d_4 \|\xi(x)\|_2) + \rho$. Again, the sliding mode control is composed by $u_{eq}$ and $u_{sw}$

$$u = u_{eq} + u_{sw}$$ (5.14)

**Theorem 5.2.** If $\partial f_1/\partial x_4$ is invertible, with the control (5.14) all states of (5.2) will asymptotically converge to 0.

**Proof.** Following the lines of the proof of Theorem 5.1, we can prove that the sliding mode $s = 0$ will happen in finite time. On the sliding manifold $s = 0$ or $e_3 = -c_1 e_1 - c_2 e_2$, the original system is reduced to

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -c_1 I_n & -c_2 I_n \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$ (5.15)

or

$$\dot{E}_2 = A_2 E_2 + D_2$$ (5.16)
where \( D_2 = [d_1, d_2]^T \), and \( \|D_2\|_2 < \max\{\overline{d}_1, \overline{d}_2\} \|E_2\|_2 \). Since \( A_2 \) is Hurwitz and \( \max\{\overline{d}_1, \overline{d}_2\} < \lambda_{\text{left}}(-A_{n2}) \), the above perturbed system of \( E_2 \) is asymptotically stable as shown before. Therefore, \( e_1 = x_1 \) and \( e_2 = x_2 \) will converge to zero. Since \( s = 0 \), \( e_3 = f_1 = 0 \). By applying Assumption 3, we conclude that all states of the original system will converge to zero.

5.2.3 The TORA System

A translational oscillator with rotational actuator (TORA) system is depicted in Fig. 5.2. It has been constructed as a benchmark system to evaluate the performance of various nonlinear controllers ([69], [73], [74]). Let \( z_1 \) be the normalized displacement of the platform from the equilibrium position, \( z_2 = \dot{z}_1 \), \( \theta_1 \) be the angle of the rotor, and \( \theta_2 = \dot{\theta}_1 \). The dynamics of the TORA system are described by

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -z_1 + \varepsilon \theta_2^2 \sin \theta_1 - \frac{\varepsilon \cos \theta_1}{1 - \varepsilon^2 \cos^2 \theta_1} - \frac{\varepsilon^2 \cos \theta_1}{1 - \varepsilon^2 \cos^2 \theta_1} v \\
\dot{\theta}_1 &= \theta_2 \\
\dot{\theta}_2 &= \frac{\varepsilon \cos \theta_1 (z_1 - \varepsilon \theta_2^2 \sin \theta_1)}{1 - \varepsilon^2 \cos^2 \theta_1} + \frac{1}{1 - \varepsilon^2 \cos^2 \theta_1} v
\end{align*}
\]  

(5.17)

where \( v \) is the control input and \( \varepsilon \) a constant parameter which depends on the rotor platform masses and eccentricity. All quantities are normalized in dimensionless units. Employing the following coordinate transformation
\[ x_1 = z_1 + \varepsilon \sin \theta_1 \]
\[ x_2 = z_2 + \varepsilon \theta_2 \cos \theta_1 \]
\[ x_3 = \theta_1 \]
\[ x_4 = \theta_2 \]
\[ v = \frac{\varepsilon \cos x_3 [x_1 - (1 + x_4^2)\varepsilon \sin x_3] + u}{1 - \varepsilon^2 \cos^2 x_3} \]

we can change the TORA system into a cascade form

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -x_1 + \varepsilon \sin x_3 - 11\varepsilon x_3 + 11\varepsilon x_3 \]
\[ \dot{x}_3 = x_4 \]
\[ \dot{x}_4 = u \]  \hspace{1cm} (5.18)

In the second equation we introduce $-11\varepsilon x_3 + 11\varepsilon x_3$, of which $11\varepsilon x_3$ is considered as disturbance, to make Assumption 2 and 3 satisfied. Then (5.18) can be represented by (5.2) with \( f_1 = -x_1 + \varepsilon \sin x_3 - 11\varepsilon x_3, f_2 = 0, b = 1, d_1 = 0, d_2 = 11\varepsilon x_3 \) and
Parameter | Value | Initial condition | Value
---|---|---|---
$\varepsilon$ | 0.1 | $z_1(0)$ | 1 m
| | $z_2(0)$ | 0 m
| | $\theta_1(0)$ | $\pi$
| | $\theta_2(0)$ | 0

Table 5.1: Parameters and initial conditions of the TORA system

\[ d_3 = 0. \] Note that
\[ \frac{\partial f_1}{\partial x_3} = \varepsilon \cos x_3 - 11\varepsilon \neq 0 \] (5.19)
and \( f_1(0,0,x_3,x_4) = 0 \) has only one solution, which is \( x_3 = x_4 = 0 \). Therefore, Assumption 1-3 are all satisfied. Since this system has only one disturbance \( d_2 = 11\varepsilon x_3 \) and \( \frac{\partial f_1}{\partial x_1} = \frac{\partial f_1}{\partial x_2} = 0 \), the matched disturbance condition can be examined only for Assumption 6 and 8. It is obvious that \( \left| \frac{\partial f_1}{\partial x_3} \right| < 12\varepsilon \). Recalling that \( e_1 = x_1 \) and \( e_3 = f_1 = -x_1 + \varepsilon \sin x_3 - 11\varepsilon x_3 \), we have
\[
|d_2| = |11\varepsilon x_3| = |e_3 + x_1 - \varepsilon \sin x_3| \leq |e_3| + |e_1| + |\varepsilon x_3|
\]
\[
\Rightarrow 10\varepsilon |x_3| \leq 2\|e\|_2
\]
\[
\Rightarrow |d_2| = 11\varepsilon |x_3| \leq 2.2\|e\|_2
\]

Thus, Assumption 6 and 8 are satisfied as well, and the aforementioned sliding mode control approach can stabilize the TORA system. A simulation is implemented on the original TORA system equations, and the following parameters and initial conditions are tabulated in Table 5.1. The parameters of the sliding mode controller are tabulated in Table 5.2. The simulation results are shown in Fig. 5.3 and 5.4. We can see that the sliding mode controller successfully stabilizes the TORA system. However,
since we introduce a fictional disturbance $d_2$, the sliding surface is constructed more conservative to ensure stability and in turn the control input required is larger than the result in [73].

### 5.2.4 Flight control of a quadrotor helicopter

A UAV quadrotor is a four rotor helicopter and depicted in Fig. 5.5. More details of its configuration can be found in [66]. The dynamic model of the quadrotor
helicopter can be obtained via a Lagrange approach and a simplified model is given as follow

\[\begin{align*}
\ddot{x} &= u_1 (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi) - K_1 \dot{x}/m \\
\ddot{y} &= u_1 (\sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi) - K_2 \dot{y}/m \\
\ddot{z} &= u_1 (\cos \phi \cos \psi) - g - K_3 \dot{z}/m \\
\ddot{\theta} &= u_2 - lK_4 \dot{\theta}/I_1 \\
\ddot{\psi} &= u_3 - lK_5 \dot{\psi}/I_2 \\
\ddot{\phi} &= u_4 - K_6 \dot{\phi}/I_3
\end{align*}\]

where \((x, y, z)\) are three positions; \((\theta, \psi, \phi)\) three Euler angles, representing pitch, roll and yaw respectively; \(g\) the acceleration of gravity; \(l\) the half length of the helicopter; \(m\) the total mass of the helicopter; \(I_i\)’s the moments of inertia with respect to the
Figure 5.5: A quadrotor configuration

axes; $K_i$’s the drag coefficients; $u_i$’s the virtual control inputs defined as follow

\[ u_1 = \frac{(F_1 + F_2 + F_3 + F_4)}{m} \]
\[ u_2 = l(-F_1 - F_2 + F_3 + F_4)/I_1 \] \hspace{1cm} (5.21)
\[ u_3 = l(-F_1 + F_2 + F_3 - F_4)/I_2 \]
\[ u_4 = C(F_1 - F_2 + F_3 - F_4)/I_3 \]

where $F_i$’s are thrusts generated by four rotors and can be considered as the real control inputs to the system, and $C$ the force to moment scaling factor. The quadrotor model (5.20) can be divided into two subsystems: a fully-actuated subsystem

\[
\begin{bmatrix}
\ddot{z} \\
\ddot{\phi}
\end{bmatrix}
=
\begin{bmatrix}
\frac{u_1 \cos \theta \cos \psi - g}{u_4} \\
\frac{-K_3 \dot{z}}{m} - \frac{K_6 \dot{\phi}}{I_3}
\end{bmatrix}
\] \hspace{1cm} (5.22)

and an underactuated subsystem

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix}
=
\begin{bmatrix}
u_1 \cos \phi & u_1 \sin \phi & \sin \theta \cos \psi & -u_1 \cos \phi \\
u_1 \sin \phi & -u_1 \cos \phi & \sin \phi & \sin \psi \\
-1K_4 \dot{\theta} / I_1 & lK_4 \dot{\psi} / I_1 & -K_2 \dot{y} / m & -K_2 \dot{ym} / I_2
\end{bmatrix}
\] \hspace{1cm} (5.23)
Since drag is very small at low speeds, the drag terms in the above equations can be considered as small disturbances to the systems. For the fully-actuated subsystem (5.22) we can easily construct a rate bounded PID controller and a sliding mode controller to move states $z$ and $\phi$ to their desired values $z_d$ and $\phi_d$, respectively. The desired control input for $z$ is given by

$$u_{1d} = \frac{k_{z1}(z_d - z) + k_{z2} \int (z_d - z)dt - k_{z3} \dot{z} + g}{\cos \theta \cos \psi}$$

(5.24)

A rate bounded control $u_1$ will converge to $u_{1d}$ (pp.575, [97])

$$\dot{u}_1 = k \cdot \text{sat} \left( \frac{k_0 \int (u_{1d} - u_1)dt + k_1 (u_{1d} - u_1)}{\epsilon} \right)$$

(5.25)

where $\epsilon$ is a small positive constant, $k_{z1}, k_{z2}, k_{z3}, k, k_0, k_1$ are all positive, and $\text{sat}(\cdot)$ is the saturation function defined as:

$$\text{sat}(x) = \begin{cases} 
1, & \text{if } x > 1 \\
 x, & \text{if } |x| \leq 1 \\
-1, & \text{if } x < -1 
\end{cases}$$

(5.26)

The initial condition is picked up as $u_1(0) = u_{1d}(0)$ to eliminate the reaching phase. Then, we have

$$|\dot{u}_1| \leq k$$

(5.27)

For $\phi$, a sliding mode control is designed to make $\phi$ converge to its desired value $\phi_d$ quickly:

$$u_4 = -c_\phi \dot{\phi} - M_\phi \text{sgn}(s_\phi) - k_\phi s_\phi$$

(5.28)

where $c_\phi$, $M_\phi$ and $k_\phi$ are controller parameters to be determined and all positive, and $s_\phi = c_\phi(\phi - \phi_d) + \dot{\phi}$ the designed stable sliding surface for $\phi$. The reason why we use a rate bounded control for $u_1$ will be discussed later. Now we will use the aforementioned sliding mode control approach to stabilize the underactuated subsystem (5.23). Since the controllers for the fully-actuated subsystem are independent of
the controller for the underactuated subsystem, we can design a controller with fast
response for \( \phi \), which makes \( \phi \) converge to its desired value very quickly. Thus, it is
reasonable to assume that after some time \( \phi \) becomes almost time invariant. Also, \( u_1 \)
is close to its “steady” state value \( g/(\cos \theta \cos \psi) \) after some time

\[
\left| u_1 - \frac{g}{\cos \theta \cos \psi} \right| \leq \eta \leq \frac{\eta}{|\cos \theta \cos \psi|} \tag{5.29}
\]

where \( \eta > 0 \) is a small constant. Then the lower bound of \( |u_1| \) is given by

\[
|u_1| \geq \left| \frac{g - \eta}{\cos \theta \cos \psi} \right| \tag{5.30}
\]

To develop a controller for the underactuated subsystem the following coordinate
transformation matrix is needed

\[
T = u_1 \begin{bmatrix}
\cos \phi & \sin \phi \\
\sin \phi & -\cos \phi
\end{bmatrix} = u_1 \hat{T}(\phi) \tag{5.31}
\]

Since we assume that \( \phi \) is time invariant, \( \hat{T}(\phi) \) is also time invariant and simply
denoted as \( \hat{T} \). It is worthy noting that the determinant of \( T \) is \(-u_1^2 < 0\) if \( u_1 \neq 0 \).
Therefore, the transformation \( T \) is nonsingular in general operation condition because
\( u_1 \) represents the total thrust on the body in the z-axis and is generally nonzero to
overcome the gravity. Let

\[
\begin{align*}
x_1 &= T^{-1} \begin{bmatrix} x \\ y \end{bmatrix} & x_2 &= T^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} & x_3 &= \begin{bmatrix} \theta \\ \psi \end{bmatrix} & x_4 &= \begin{bmatrix} \dot{\theta} \\ \dot{\psi} \end{bmatrix}
\end{align*}
\]
Check the derivatives of the new states

\[
\dot{x}_1 = T^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \frac{d}{dt} \left( \frac{1}{u_1} \right) \hat{T}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = x_2 - \frac{\dot{u}_1}{u_1} x_1
\]

\[
\dot{x}_2 = T^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \frac{d}{dt} \left( \frac{1}{u_1} \right) \hat{T}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \psi \\ \sin \psi \end{bmatrix} - \hat{T}^{-1} \begin{bmatrix} K_1/m \\ 0 \\ K_2/m \end{bmatrix} \hat{T} x_2 - \frac{\dot{u}_1}{u_1} x_2
\]

\[
\dot{x}_3 = x_4
\]

\[
\dot{x}_4 = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -lK_4/I_1 \\ 0 \\ -lK_5/I_2 \end{bmatrix} x_4
\]

Then the underactuated subsystem (5.23) can be represented by (5.2) with

\[
u = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}, \quad f_1 = \begin{bmatrix} \sin \theta \cos \psi \\ \sin \psi \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad b = I,
\]

\[
d_1 = -\frac{\dot{u}_1}{u_1} x_1, \quad d_3 = \begin{bmatrix} -lK_4/I_1 \\ 0 \\ -lK_5/I_2 \end{bmatrix} x_4,
\]

\[
d_2 = -\frac{\dot{u}_1}{u_1} x_2 - \hat{T}^{-1} \begin{bmatrix} K_1/m \\ 0 \\ K_2/m \end{bmatrix} \hat{T} x_2
\]

Considering (5.27) and (5.30), we have the following bounded functions

\[
\|d_1\| \leq \frac{\|\dot{u}_1 \cos \theta \cos \psi\|}{|g - \eta|} \|x_1\| \leq \frac{\|\dot{u}_1\|}{|g - \eta|} \|x_1\| \leq \frac{k}{|g - \eta|} \|x_1\| = \bar{d}_1 \|x_1\| \leq \bar{d}_1 \|e\|_2 \quad (5.32)
\]

\[
\|d_2\| \leq \frac{k}{|g - \eta|} \|x_2\| + \left\| \begin{bmatrix} K_1/m \\ 0 \\ K_2/m \end{bmatrix} \right\| \|x_2\| \leq \left( \frac{k}{|g - \eta|} + \max \left\{ \frac{K_1}{m}, \frac{K_2}{m} \right\} \right) \|x_2\| = \bar{d}_2 \|x_2\| \leq \bar{d}_2 \|e\|_2 \quad (5.33)
\]

\[
\|d_3\| \leq \max \left\{ \frac{lK_4}{I_1}, \frac{lK_5}{I_2} \right\} \|x_4\| = \bar{d}_4 \|x_4\| \quad (5.34)
\]

Thus, Assumption 8 is satisfied. \( f_1 \) is independent of \( x_1, x_2 \) and \( x_4 \), and

\[
\frac{\partial f_1}{\partial x_3} = \begin{bmatrix} \cos \theta \cos \psi & -\sin \theta \sin \psi \\ 0 & \cos \psi \end{bmatrix}
\]
It should be noted that $\cos \theta \cos \psi > 0$ and the determinant of $\frac{\partial f}{\partial x_3}$ is $\cos \theta \cos^2 \psi > 0$ when $(\theta, \psi) \in (-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$, which is satisfied generally. Therefore, $\frac{\partial f}{\partial x_3}$ is positive definite and invertible. Moreover, the maximum absolute row sum norm of $\frac{\partial f}{\partial x_3}$ is bounded: $\left\| \frac{\partial f}{\partial x_3} \right\|_\infty < 2$. Therefore, Assumption 6 is satisfied as well. The sliding mode control proposed in Section 3 can be used to stabilize the underactuated subsystem (5.23) after the coordinate transformation. The parameters and the initial conditions of the quadrotor for simulation are tabulated in Table 5.3. The parameters of the controllers for the quadrotor are tabulated in Table 5.4. Simulation results for positions, angles and control inputs are shown in Fig. 5.6-5.9. The simulation results
show that the sliding mode controllers with the rate bounded PID controller (5.25) are able to stabilize the quadrotor helicopter. From Fig. 5.7, we find that $\phi$ converges to its desired value faster than other states, so it is safe to consider $\phi$ as time invariant after 5 seconds. This verifies the assumption that $\hat{T}(\phi)$ is time invariant after some time. Fig. 5.9 shows that the rate bounded control $u_1$ successfully tracks the desired PID control $u_{1d}$ with a small time delay.
Figure 5.7: Angle outputs $\theta, \psi, \phi$

Figure 5.8: Control inputs $F_1, F_2, F_3, F_4$
5.3 Sliding mode control of underactuated systems in the chained form

5.3.1 System model

In this section we consider the class of nonholonomic systems described by equations of the form

\[
\begin{align*}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_2u_1 \\
\dot{x}_4 &= x_3u_1 \\
&\vdots \\
\dot{x}_n &= x_{n-1}u_1
\end{align*}
\]  

(5.35)
This class of systems, which is so-called chained form, has been introduced by Murray and Sastry in [91] to model kinematics of nonholonomic mechanical systems. In that paper the authors have also given sufficient conditions to convert (via state feedback and coordinates transformation) a generic controllable nonholonomic system with two inputs to the chained form. Many nonholonomic mechanical systems can be described by kinematic models in chained form or are feedback equivalent to chained form. A special example is the kinematic model of a Dubin’s car as described

\[
\begin{align*}
\dot{x} &= v_1 \cos \theta \\
\dot{y} &= v_1 \sin \theta \\
\dot{\theta} &= \frac{1}{l} v_1 \tan \phi \\
\dot{\phi} &= v_2
\end{align*}
\]

(5.36)

where \(x\) and \(y\) denote the location of the center of the axle between the two rear wheels, \(\theta\) the angle of the car body with respect to the \(x\) axis, \(\phi\) the steering angle with respect to the car body, \(l\) the distance from the front axle to the real axle, and \(v_1\) and \(v_2\) the forward velocity of the rear wheels and the velocity of the steering wheels, respectively. The diagram of a Dubin’s car is shown in Figure 5.10.

As discussed in detail in [91], the Dubin’s car model (5.36) can be put into the chained form (5.35) via the following local state transformation

\[
\begin{align*}
x_1 &= x \\
x_2 &= \frac{1}{l} \sec^3 \theta \tan \phi \\
x_3 &= \tan \theta \\
x_4 &= y
\end{align*}
\]
Figure 5.10: Dubin’s car
and input change

\[ v_1 = \frac{1}{\cos \theta} u_1 \]
\[ v_2 = -\frac{3}{l} \sin^2 \phi \tan \theta \sec \theta u_1 + l \cos^2 \phi \cos^3 \theta u_2 \]

5.3.2 Sliding mode control design

Next, we will present a variable structure controller to stabilize the chained form (5.35). In compatible with the Dubin’s car model, we assume the order of the chained form to be \( n = 4 \). However, the designed controller can be easily extended to higher order (\( n > 4 \)) models.

Two interesting observations of the chain form (5.35) are made:

1. The subsystem consisting of \( x_2, \cdots, x_n \) is a linear time-invariant system if \( u_1 \) is constant.

2. When \( x_2, \cdots, x_n \) are all equal to zero, \( u_1 \) will only influence \( x_1 \).

Based on those observations, the controller design can be divided into two stages. In the first stage, we shall keep \( u_1 \) constant or piecewise constant and converge the other states \( x_2, \cdots, x_n \) to zero in finite time. In the second stage, we shall keep \( u_2 = 0 \) and apply any stabilizing law \( u_1 \) to stabilize \( x_1 \).

The first stage controller

\[ u_1(t) = \begin{cases} 
-\text{sgn}[x_1(0)], & 2m\Delta \leq t < (2m + 1)\Delta \\
+\text{sgn}[x_1(0)], & (2m + 1)\Delta \leq t < (2m + 2)\Delta 
\end{cases} \]
\[ t < T, \ m = 0, 1, \cdots \]

(5.37)

where \( x_1(0) \) is the initial condition of \( x_1 \), \( \Delta \) the switching time interval, \( T \) the time length of the first stage.
For $u_2$, a multi-layer sliding mode controller is applied to ensure the finite-time convergence of the states. The first layer sliding surface function is defined as

$$s_1 = \begin{cases} s_1^+, & \text{if } u_1 = +1 \\ s_1^-, & \text{if } u_1 = -1 \end{cases}$$

(5.38)

where

$$s_1^+ = x_3 + c_1 x_4^{p_1/q_1}$$

(5.39)

$$s_1^- = x_3 - c_1 x_4^{p_1/q_1}$$

(5.40)

c_1 > 0$ is a positive number, $q_1 > p_1 > 0$, $p_1$ and $q_1$ odd numbers, and $p_1$ and $q_1$ are relatively prime. The finite-time convergence can be checked as follows. Assume that $u_1 = 1$. When $t = t_0$, the first layer sliding mode happens, i.e., $s_1^+ = 0$, we have

$$\Rightarrow x_3 + c_1 x_4^{p_1/q_1} = 0$$

$$\Rightarrow x_3 = \dot{x}_4 = -c_1 x_4^{p_1/q_1}$$

$$\Rightarrow x_4(t) = \left[ \frac{q_1 - p_1}{q_1} \left\{ -c_1 t + \frac{q_1}{q_1 - p_1} [x_4(t_0)]^{\frac{q_1 - p_1}{q_1}} \right\} \right]^{\frac{q_1}{q_1 - p_1}} \text{ for } t > t_0$$

Since $q_1 > p_1 > 0$ and $q_1$ and $p_1$ are odd numbers, $x_4$ converges to zero at $t = t_0 + \frac{q_1}{c_1(q_1 - p_1)} [x_4(t_0)]^{\frac{q_1 - p_1}{q_1}}$, so does $x_3$. Similarly, if $u_1 = -1$, the same result happens.

The second layer sliding surface function is defined as

$$s_2 = \begin{cases} s_2^+, & \text{if } u_1 = +1 \\ s_2^-, & \text{if } u_1 = -1 \end{cases}$$

(5.41)

where

$$s_2^+ = \dot{s}_1^+ + c_2 (s_1^+)^{p_2/q_2}$$

(5.42)

$$s_2^- = \dot{s}_1^- + c_2 (s_1^-)^{p_2/q_2}$$

(5.43)
$c_2 > 0$ is a positive number, $q_2 > p_2 > 0$, $p_2$ and $q_2$ odd numbers, and $p_2$ and $q_2$ are relatively prime. Again, on the sliding mode manifold $s_2 = 0$, $s_1$ converges to zero in finite time.

The construction of sliding surface functions continues till the highest layer sliding surface function is relative degree 1 with respect to $u_2$. For $n = 4$, the second layer $s_2$ is enough. Then the sliding mode control is designed to make $s_2$ converge to zero in finite time as

$$u_2 = \begin{cases} u_2^+, & \text{if } u_1 = +1 \\ u_2^-, & \text{if } u_1 = -1 \end{cases} \quad (5.44)$$

where

$$u_2^+ = -\frac{d}{dt} \left[ \frac{p_1}{q_1} c_1 x_{4} \frac{p_1 - q_1}{q_1} x_3 + c_2 \left( x_3 + c_1 x_{4} \right) \frac{p_2}{q_2} \right] - M \text{sgn}(s_2^+) \quad (5.45)$$

$$u_2^- = \frac{d}{dt} \left[ \frac{p_1}{q_1} c_1 x_{4} \frac{p_1 - q_1}{q_1} x_3 + c_2 \left( x_3 - c_1 x_{4} \right) \frac{p_2}{q_2} \right] + M \text{sgn}(s_2^-) \quad (5.46)$$

where $M$ is a positive constant number.

It is worthwhile to note that if $\Delta$ is large enough or there is no constraint on $u_2$ or $\dot{u}_2$, then control inputs in (5.37) and (5.44) can make the states $x_2, \ldots, x_n$ converge to zero in a finite time $\Delta$ by picking up appropriate controller parameters. In that case, $u_1$’s sign doesn’t change during the process of convergence, and $u_2$ doesn’t switch between $u_2^+$ and $u_2^-$. However, in real applications, we don’t want $x_1$ to keep increasing to a huge positive number or decreasing to a huge negative number. Furthermore, there is always a constraint on $u_2$ such as the bound of $u_2$. Therefore, $u_2$ needs to switch when $u_1$ changes its sign. Every time when $u_2$ switches, the converging process of $s_2$ will happen again, so do those of $s_1$ and all the states but $x_1$. 

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The second stage controller

After \( x_2, \cdots, x_n \) converge to zero, the following control inputs can be applied to stabilize \( x_1 \).

\[
\begin{align*}
u_1 &= -k x_1 \\
u_2 &= \begin{cases} u_2^+, & \text{if } u_1 > \varepsilon \\ 0, & \text{if } |u_1| \leq \varepsilon \\ u_2^-, & \text{if } u_1 < -\varepsilon \end{cases}
\end{align*}
\]

where \( k \) is a positive constant number and \( \varepsilon \) is a small positive number.

5.3.3 Stabilization of a Dubin’s car

We now use the variable structure controller proposed in the previous subsection to stabilize a Dubin’s car. Without loss of generality, we assume that \( l = 1 \). The initial conditions are

\[
\begin{bmatrix} x(0) & y(0) & \theta(0) & \phi(0) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 0 \end{bmatrix}
\]

The controller parameters are tabulated in Table 5.5. Figure 5.11 shows the state evolution, Figure 5.12 shows the two velocity controls, Figure 5.13 shows the x-y trajectory of the Dubin’s car, and Figure 5.14 shows the evolution of the two switching functions.

<table>
<thead>
<tr>
<th>Name</th>
<th>Value</th>
<th>Name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta )</td>
<td>2</td>
<td>( T )</td>
<td>4</td>
</tr>
<tr>
<td>( p_1 )</td>
<td>3</td>
<td>( q_1 )</td>
<td>5</td>
</tr>
<tr>
<td>( p_2 )</td>
<td>3</td>
<td>( q_2 )</td>
<td>5</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>1</td>
<td>( c_2 )</td>
<td>1</td>
</tr>
<tr>
<td>( M )</td>
<td>3</td>
<td>( k )</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 5.5: Controller parameters
Figure 5.11: Time evolution of $x(t)$, $y(t)$, $\theta(t)$, $\phi(t)$.

Figure 5.12: Time evolution of the forward velocity control $v_1(t)$ and of the steering velocity control $v_2(t)$.
Figure 5.13: The x-y trajectory of the Dubin’s car.

Figure 5.14: Time evolution of the two switching functions $s_1(t)$ and $s_2(t)$.
5.4 Conclusion

Stabilizing two classes of underactuated systems by using sliding mode control has been studied in this chapter. The non-smooth property of the sliding mode control has been utilized to make it possible to stabilize a nonlinear system, which doesn’t satisfy Brockett’s necessary conditions. The simulation results of there examples show that the proposed sliding mode controllers work effectively to stabilize the underactuated systems.
CHAPTER 6

CONCLUSIONS

6.1 Summary and contribution

The concept of optimal sliding mode control and stabilization of underactuated systems by using sliding mode control are two topics of this dissertation.

In the first part of the dissertation, the concept of optimal sliding mode control design is proposed. Although there is a large volume of literature on sliding mode control, much less research has been carried out on the design of the sliding surface. Most of the existing literature suggests that the sliding surface is designed so that the closed-loop system is stable and has some desirable properties when confined to it. However, not much literature is available for identifying and correlating these desired properties with an appropriate sliding surface. This dissertation is aiming at presenting methods of designing sliding surfaces such that the systems satisfy some optimality. Optimal sliding mode control for linear continuous-time systems is studied in Chapter 2, which integrates the optimal control methodologies into the design of the sliding surface in sliding mode control. Through this way, the property of optimality plus robustness may be obtained for controlling disturbed systems. The proposed optimal sliding mode controls provide robust solutions to the infinite- and finite-time linear quadratic problems and minimum-energy problem. An approach that
utilizes the concept of Itô integral is presented to find a set of optimal parameters of
the designed sliding mode controller for systems disturbed by special signals. Several
simulation results qualitative and quantitatively show the advantages of the proposed
optimal sliding mode control over conventional sliding mode control and conventional
optimal control.

Next, optimal sliding mode control design is extended for a class of nonlinear
systems and linear discrete-time systems, respectively. In Chapter 3, three approaches
(HJB equation approach, CLF approach and SDRE approach) are presented to obtain
an optimal or suboptimal sliding surface for nonlinear systems. The pros and cons
of each approach are discussed via an illustrative example. In Chapter 4, two kinds
of disturbances, rate bounded and state dependent, are considered in the discrete-
time systems. Accordingly, two kinds of optimal discrete sliding mode controllers are
proposed, which are able to robustly and optimally control those disturbed discrete-
time systems.

In the second part, the dissertation is focusing on the stabilization problem of un-
deractuated systems by using sliding mode control. As is well known, underactuated
systems, which do not satisfy Brockett’s necessary conditions, cannot be stabilized
by any smooth feedback law. Because of the discontinuity property of sliding mode
control, it has the potential to stabilize those underactuated systems. Two general
forms of underactuated systems, the cascaded form and the chained form, are of great
interest. Stabilizing sliding mode control laws for underactuated systems in those two
forms are proposed in Chapter 5. Several benchmark real applications including the
TORA system, the quadrotor helicopter and the Dubin’s car are used to test those
sliding mode control laws. The advantages of using sliding mode control to stabilize
an underactuated system are flexibility in designing, easiness of implementation and satisfactory performance.

6.2 Future research directions

Although optimal sliding mode control problem has been solved thoroughly for linear systems, optimal sliding mode control problem for nonlinear systems needs further research effort. The class of nonlinear systems considered in this dissertation only has limited examples. For more general forms of nonlinear systems, it requires more theoretical analysis including analysis of the controllability of nonlinear systems and analysis of the existence of the optimal control. The SDRE approach is more promising than the other two approaches if an appropriate method is found to factorize the system equation.

On the other hand, stabilization of underactuated systems using sliding mode control has wide-open possibilities for the further research directions. There are many underactuated systems in real applications. Using different state feedback and coordinates transformation, they can be transformed into different general forms. We shall explore the possibilities of stabilizing systems using sliding mode control, which are not covered by the two general forms considered in this dissertation.


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