NUMERICAL MODELING OF HOMOGENEOUS AND
BIMATERIAL CRACK TIP AND INTERFACIAL COHESIVE
ZONES WITH VARIOUS TRACTION-DISPLACEMENT LAWS

Dissertation

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ABSTRACT

This dissertation investigates problems that involve the cohesive response at a crack tip in a homogeneous material, at a crack tip in a bimaterial interface, and an investigation of the cohesive response of an interface. The first investigation outlines a methodology for computing the non-linear generalized load-displacement response of an edge-cracked beam shaped element with a softening crack-plane cohesive zone. The non-linear load-displacement response is intended for use in various possible techniques of nondestructive characterization of the cohesive material properties using the vibration properties of a cracked cohesive beam. The cracked beam shaped geometry is divided into two bodies which interact across the crack plane through appropriate boundary conditions and each body is loaded by a bending couple away from the crack plane. The mode I cohesive law used here is strictly softening and linear. The direct boundary element method (BEM) is applied to each body and the connecting boundary conditions and an iterative scheme is used to determine the extent of the cohesive zone. The $J$-integral associated with the single crack tip is then calculated from the BEM crack plane tractions and displacements and is used to numerically evaluate the generalized non-linear load-displacement relations. The second investigation outlines a novel approach to numerically model a Dugdale-Barenblatt cohesive zone at an interface crack between two dissimilar materials. The direct BEM approach is once again used here to appropriately apply the constraints in the cohesive zone and obtain a physically meaningful solution for tractions and displacements in the crack plane. The effect of material mismatch and the effect of varying bond
strength on the interface fracture energy as expressed by the individual mode I and mode II contributions to the $J$-integral is also studied. This approach provides a heretofore unavailable convenient method for calculating the local crack tip mode mixity, the total energy release rate and its decomposition into the separate mode contributions as a function of the applied mode mixity and material mismatch. The third investigation concerns the dispersion relations for time-harmonic guided waves in a layer connected to a rigid substrate by a very thin interface layer of material with nonlinear and softening behavior. The interfacial spring stiffness which, is directly included in the dynamic layer boundary conditions, may be interpreted as the local slope of the cohesive law at the static pre-load level. The spring stiffnesses inferred from the dispersion relations of either SH or generalized Rayleigh-Lamb waves from a series of measurements taken at multiple pre-load levels could then be integrated to obtain the cohesive law.
Dedicated, with love, to my FAMILY
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CHAPTER 1

INTRODUCTION

1.1 Material characterization using vibration characteristics

Characterization of material properties and flaws based on vibration characteristics of test samples and structures is a well established quantitative non-destructive evaluation (QNDE) technique. The presence of cracks, damage or plasticity in a structure increases compliance and considerably affects dynamic behavior by reducing the natural frequencies and changing mode shapes, and influencing the forced vibration response as well. The review by Dimarogonas [1] of analyses and experiments on vibrations of cracked structures has many examples of this kind of work. Many techniques have been used to identify the location and size of cracks in structures using frequency measurements. The vibration analysis of cracked beams, almost all of them edge-cracked beams, falls into two categories: open stress free crack surface models (valid if the crack faces are held open throughout a vibration cycles by a static preload or residual stresses) and breathing crack models. The breathing crack model takes into account that, unless otherwise held open, the crack faces are in contact during a part of the cycle and open and stress free during the other part. The crack is open exactly half the time and closed half the time if there is no superposed static loading. This discontinuity in boundary conditions during a vibration cycle is nonlinear and
methods such as harmonic balancing or asymptotic expansions must be used. Gasch [2] identified cracks on a rotor shaft using both open and breathing crack models. Ruotolo, et.al. [3] studied a cracked cantilever beam subjected to harmonic forcing, using the breathing crack model to infer the location and depth of cracks. Gounaris and Papadopoulos [4] considered the open crack model to identify crack depth and location by comparing eigenmodes of cracked and uncracked beams. Liew and Wang [5] have performed a similar analysis using wavelet theory. Bamnios, et.al. [6] suggest that the impedance at the force driving point on a beam substantially changes due to the presence of a crack and depends on the location and size of the crack and on the force location. Techniques, such as first order perturbations [7, 8] and variational formulations [9] have also been employed in analyzing vibrations of cracked structures.

The major kinematic feature of an edge-crack in a beam is the discontinuity the crack allows in the rotation and deflection across the crack plane. The discontinuities are resisted, however, by the bonded ligament in an elastic manner if the crack is in small-scale yielding. This has led several researchers to simplify the vibration problem by using massless rotational and/or shear springs to represent the crack plane [10-14]. The idea originated with Rice and Levy’s line-spring model applied to a static analysis of a through crack in a plate in bending [15]. The model relates the jumps in the rotation and the transverse deflection $w$ across the plane of the crack to the applied loads across the crack plane: the bending moment $M$ and the shear force $Q$, respectively. The spring constants (or their reciprocals the compliances) which relate the loads to the kinematic discontinuities may be found from a two-dimensional elastic analysis of a cracked beam shaped geometry as found in many fracture mechanics handbooks, see the discussion and references in [13]. For an elastic crack tip the resulting compliances are constants, independent of the magnitude of the applied
load or the discontinuity, which depend only on the crack length to beam depth ratio and the elastic modulus.

In virtually all of the beam vibration analyses cited above and the many other analyses cited therein the purpose was to use the vibrational response to locate, number and/or size the crack(s) in the beam. Many of the approaches are meant for application in civil structures with a-priori unknown states of cracking. On the other hand, the present analysis is part of an effort to use laboratory samples with known crack lengths and locations to characterize the cohesive behavior ahead of the crack tip using the vibrational response. The focus of the effort is on using rotational shear springs in the vibration analysis, but in this case, since the extent of the cohesive zone and the plastic stretch in the cohesive zone depend on the load, the stiffness and compliance of the spring depend on the load and the resulting load-displacement relations become nonlinear. This research provides the methodology for computing this nonlinear load-displacement relation given a linear softening cohesive law. Two potential methods for relating the generated nonlinear load-displacement relationship to the nonlinear vibrational response of an edge-cracked beam are presented in Mendelsohn [16] and Mendelsohn, et.al. [17].

A planar cohesive zone formulation allows the volume of the body on either side of the infinitesimal thin cohesive plane to be modeled as linearly elastic, with the nonlinearity included only in the boundary conditions across the cohesive crack plane [18]. In other words the entire fracture process zone (plastic zone) is assumed to be localized in the crack plane, and is characterized in the form of a stress-displacement or traction-displacement (t-δ) law. This makes any numerical solution of a crack boundary value problem easier, but the Boundary Element Method is particularly appropriate for treating cohesive zones [18]. The extensive use of cohesive zones in modeling in recent years is justified by the many materials that do in fact exhibit this kind of localized damage accumulation and failure. Some of the many examples are
ductile metals, in particular when in thin plates or sheets [19-21]. Here a common primary failure mechanism in the thin process zone is void initiation at secondary particles followed by void growth and coalescence [22-28]. A very wide class of metal-matrix, ceramic-matrix and cement-matrix composites also exhibit planar cohesive behavior near macroscopic flaws due to a bridging mechanism involving the way in which the reinforcing phase interacts mechanically with the matrix during progressive matrix cracking. This is a widespread phenomenon which occurs with many different shapes of second phase or reinforcing particles, including the very common fiber reinforced composites [29-40].

From a modeling point of view, cohesive zones were first introduced into the analysis of the crack problem nearly simultaneously by Dugdale [19], Bilby, et.al. [20], and Barenblatt [41]. The first two models, intended for ductile metals, were proposed in the present context as zones of plastic damage ahead of a crack edge, which represents the boundary between completely failed and intact material. Barenblatt on the other hand postulated that in any homogeneous cracked material because the crack face separations are so small near the crack edge, there is a region of the crack surfaces that are held together by atomic cohesive forces of attraction between the two surfaces. The former interpretation has been used far more than the latter, but both ideas lead to the same crack boundary value problem in which the extent of the cohesive zone is found by eliminating the stress singularity at the elastic crack tip. These original analyses assumed that the cohesive stress is constant over the entire cohesive zone. Hillerborg, et.al. [29] applied cohesive modeling to quasibrittle materials like concrete, and, were the first to introduce a softening cohesive law, in which after reaching a peak, the cohesive traction reduces as the plastic stretch increases. They extracted the cohesive law from the macroscopically measured load-elongation curves for the material. The two most important properties of the softening curve are the peak stress or traction $t_0$ and the critical cohesive fracture energy. Linear
softening cohesive zone models have also been used by Geubelle and Rice [42], and Yang and Ravi-Chandar [43] for ductile metals. A bilinear softening model has been extensively used for concrete and other quasibrittle materials [18]. Exponential models have been used by Siegmund and Brocks [44] and Roychowdhury, et.al. [21] to characterize softening in ductile metals. Some other models similar to linear and exponential are also discussed in [45]-[49] in the context of quasi-brittle materials such as concrete and low carbon steel. Linear softening models have been modeled using BEM by Ohtsu and Chahrour [50] and Aliabadi [51] for quasi-brittle materials like concrete to study crack propagation. Ohtsu and Chahrour have used a two domain, plane strain analysis with constant tractions and linear displacements on each element. Aliabadi has used the dual boundary element method. Hanson and Ingraffea [52] and Hanson, et.al. [53] have considered linear and bilinear softening cohesive models for crack growth in concrete. Mendelsohn [16] has used a two-body, iterative, direct boundary element formulation for modelling a Dugdale-Barenblatt cohesive zone in an edge-cracked beam like geometry.

The objective of this research is to develop a methodology for obtaining the nonlinear load-displacement relationship in a two dimensional edge-cracked beam like geometry with a linear softening cohesive crack ahead of the crack tip, and that is subjected to pure bending. The crack and cohesive zone boundary value problem is solved using a two-body, iterative, direct boundary element formulation for a crack under mode-I loading within a weak interface or in a homogeneous material. The two parameters of the linear softening cohesive law are $t_o$, the peak traction and $\delta_o$, the critical crack opening displacement that causes the crack to grow. These parameters are varied over a range of material property values and results for the $J$-integral are obtained for various linear softening laws. For each softening law the $J$-integral must be obtained from the BEM analysis for a range of crack lengths and applied loads. Then, using the relationship between the $J$-integral and generalized
load and displacement (bending moment and jump in slope across the crack plane) for the cracked geometry, the compliance is derived in terms of the $J$-integral. Once the compliance is found at a given load the generalized displacement (jump in slope across the crack) may be calculated. This yields the predicted generalized load vs. displacement relationship.

1.2 Crack at a bimaterial interface

The problem of a crack at the interface of two dissimilar materials has received considerable attention. There are many classical papers concerning evaluation of the elastic stress fields ahead of the crack at a bimaterial interface [57]-[62]. Williams [57] has shown using an eigenfunction analysis that the elastic stresses at the crack tip in both materials share the standard inverse square-root singularity of a crack in a homogeneous material. This is because the index for the power singularity turns out to be complex with real part equal to one half (the same as in the crack in a homogeneous medium problem) and imaginary part which depends on the material mismatch such that it vanishes if the mismatch isn’t present. The imaginary part causes the stresses to rapidly oscillate as a function of position near the crack tip. This finding has been verified independently by Erdogan [58] using the complex variable method, whereby closed form solutions for the stresses are given. Sih and Rice [59] have presented a closed form solution for stresses by extending Williams’ approach and the complex variable method to the problem of bending of plates of dissimilar materials that contain an interface crack. These solutions also show the oscillatory behavior in the stresses. England [60] indicates that the solutions of Williams and Erdogan are not admissible because they predict that the crack faces wrinkle and overlap near the crack tip, i.e. the opposing crack faces interpenetrate each other, which is obviously unphysical. What really happens is that the crack faces are in
contact and the solutions which don’t account for the contact are incorrect. However, if the applied loading is predominately mode I in nature, this oscillatory behaviour is confined to a very small region near the crack tip. Erdogan [61] has confirmed this finding and suggests that the oscillatory phenomenon that is confined to a very small region near the crack tip can be neglected for mode I applied loading. Using the same argument, Rice and Sih [62] have presented a formulation to obtain the complex stress intensity factor, of a crack at a bimetallic interface which also uses the unphysical solution. The real part of the complex stress intensity factor is the mode I stress intensity factor and the imaginary part is the mode II stress intensity factor [59, 62, 71].

Since the oscillatory behaviour indicating interpenetration of material near the crack tip is physically inadmissible, several models have been proposed to account for the crack face contact which is induced by the material mismatch. Comninou [63, 64, 65], the first to provide such a model, introduced frictionless and frictional contact zones of the crack faces near the crack tip. Since the contact extends right up to the crack tip where there is smooth closure the mode I singularity in the normal stress vanishes automatically, while the shear stresses ahead of the crack tip maintain their standard square root singularities and are non-oscillatory. These original results indicated, in agreement with the asymptotic analyses of the non-contact unphysical solution, that when the applied load was pure mode I, the contact zone was very small and numerically difficult to accurately size, but that as the mode II component of the loading increases the contact zone becomes on the order of the crack length for a finite length crack. Technically in any of these scenarios then, the only mechanism for crack propagation along the interface is by pure shear in the plane of the interface, since if there is any crack face contact at all at the crack tip the normal stresses ahead of the crack tip are compressive and only the interface shear stresses can contribute to material failure. Hence, considering the smooth transition
from a crack in a homogeneous medium to an interface crack with a very mild material mismatch, the Comninou contact model predicts an abrupt discontinuity in the mode-mixity at the crack tip from pure mode I to pure mode II. Sinclair [66], recognizing the severity of this prediction, tried to complement the contact model of Comninou by proposing a model that provides a finite crack opening angle prior to loading of the crack tip. This model does permit crack propagation with a tensile component along the interface for a remotely applied tension. Atkinson [67] has proposed two models that introduce an interphase layer between the two materials. In the first model the crack propagates inside the homogeneous interphase layer that has different material moduli than the surrounding materials. This model is not able to truly capture the debonding of the interface since the crack is allowed to grow inside the homogeneous medium. The second model, that is more realistic, allows the crack to propagate at the interface of one of the materials and the interphase layer. There is a gradation in the elastic moduli of the interphase layer between the values of elastic moduli of the two surrounding materials. A modification to Atkinson’s model has been attempted by Boniface and Simha [68] that considers the interphase to be a wedge inclusion with its vertex at the crack tip while giving physically admissible solutions for certain values of wedge angles. Sinclair [69, 70] has discussed most of the crack tip models for a crack at a bimaterial interface in two of his review papers.

We turn now to the effect of plasticity on the state of stress, crack face contact and mode-mixity at an interface crack tip. Rice [71] and Shih and Asaro [72, 73] have addressed the issue of small scale yielding in the bulk ahead of the crack tip for a crack at a bimaterial interface. Shih and Asaro have obtained \( HRR \) type solutions for strain-hardening behaviour in the bulk on either side of the interface ahead of the crack tip and provide conditions for the contact zone to be less than some physical length parameter. The presence of plasticity allowed for a range of applied mode mixities for which there was no crack face contact predicted at all.
In many situations, though failure is confined to the interface, the interface itself exhibits plastic behavior, rather than the bulk materials on either side. In this it seems appropriate to investigate the effect of a cohesive zone formulation in the interface on the state of contact and mode-mixity ahead of the crack tip. There has been very little done along these lines. Tvergaard and Hutchinson [74, 75] used a cohesive law ahead of the crack tip in the interface with a linear rise, a plateau and a linear softening portion. A finite element model of the interface crack tip region is loaded by the displacement field derived from the elastic interface crack solution with complex singularity adjusted for small scale yielding and used to compute crack growth resistance curves for varying cohesive law parameters. The interface plasticity was found to increase the steady state interface toughness, but substantially more so when there is a large mode II component in the applied loading. Also Mohammed and Liechti [76] employed a linear softening cohesive zone in the bimaterial interface at a corner of varying angle (including the zero angle case of an interface crack) in a finite element model to match experimental results for the interfacial displacements which clearly showed the presence of a cohesive zone limited to the interface. While there have been other theoretical and experimental and combined investigations which have given insight into the relationship between applied mode mixity, interface toughness, and in some cases plasticity (Liechti and Knauss [77], Liechti and Hanson [78], Zywicz and Parks [79], Bose and Ponte-Castaneda [80], Ponte-Castaneda and Mataga [81], Bose, et.al. [82], Shih [83], Liechti and Chai [84, 85], Sundararaman and Davidson [86], Ekman, et.al. [87], Zou, et.al. [88]) none of these are concerned directly with the crack face contact issue and none have made use of interfacial cohesive zones. The body of work just discussed all shows that there is still no definitive model of plasticity for the interface crack that can be used to predict experimental results for either overall interfacial toughness or for the actual mode-mixity at the crack tip. However, the work of Mohammed and Liechti [76] seems to indicate strongly that in many situations
cohesive behavior may occur at the interface near the crack tip and play a significant role in the failure process.

In the present research a Dugdale-Barenblatt cohesive zone formulation is considered for an interface crack subjected to a range of applied loading with varying mode mixity from either purely mode I up to the (to be determined) maximum ratio of mode II to mode I loading before crack face contact is induced by the material mismatch at the crack tip. Particular attention is paid to the state of crack face contact and to the mode-mixity at the crack tip. A two-body, direct boundary element formulation is applied to a beam like geometry similar to the four point bend specimen of Mendelsohn et.al. [89] and the interface fracture energy, that is defined by the $J$-integral, and it’s two separate mode I and mode II components are evaluated. A large amount of effort has been invested into obtaining both the mode I and mode II stress intensity factors for homogeneous and interface cracks using contour integral methods: Stern, et.al. [90], Hong and Stern [91], Matos, et.al. [92] and Miyazaki, et.al. [93]. Sun and Jih [94], and Bjerken and Persson [95] have used the direct definition of Griffith’s strain energy release rate $G$ to obtain the stress intensity factors at a bimaterial interface crack. Furthermore, Sun and Jih have given closed form solutions for the stresses and displacements at the crack tips for a combination of loadings. The $M$-integral method, that involves material interaction terms, has been used by Yau and Wang [96] to obtain the stress intensity factors ahead of the bimaterial interface crack tip of an inherently mixed mode problem. However, the $J$-integral is believed to be a true measure of the interface fracture energy and Smelser and Gurtin [97] were the first to indicate that the standard $J$-integral of fracture mechanics extends without change to a bimaterial interface provided the bond line is straight. Khandelwal and Chandra Kishen [98] have presented a complex variable formulation to evaluate a complex $J$-integral around a contour path of small radius within the singularity dominated zone and thereby evaluate accurately the stress intensity factors. Again,
these computations would only be applicable if the loading is such that there is very limited crack face contact.

In the present analysis the contour for integration to evaluate the $J$-integral is shrunk down to be the boundary of the Dugdale-Barenblatt cohesive zone. It is evident by now that the problem of a crack at bimaterial interface is inherently a mixed-mode problem. The problem of a Dugdale cohesive crack in a homogeneous material subjected to mixed-mode loading has been studied by Becker and Gross [99] and Nicholson [100]. Both these analyses apply the von-Mises yield criterion to the cohesive zone and relate the normal and shear cohesive tractions to the remotely applied normal and shear stresses. In the present work, the constraints at the cohesive interface are obtained by relating the normal and shear cohesive tractions to the inherently mixed-mode crack tip stress intensity factors obtained by Sun and Jih [94], and the von-Mises yield criterion applied to the interface. It is also assumed that the interface yields before either of the surrounding materials. The BEM solution with the appropriate constraints is able to give a physically admissible solution for the bimaterial interface.

1.3 Guided wave dispersion at a cohesive interface

The study of wave propagation properties of interface and guided layer wave dispersion relations in the presence of interface damage, contact or roughness and assessment of the constitutive properties across the interface has been extensive. The majority of this work has treated the interface in the extreme limit as freely slipping (totally damaged), or as linearly compliant by employing thin massless springs which relate the interfacial displacement jumps to the interfacial stress, or as a thin elastic or viscoelastic layer. A sampling of this kind of calculation may be found in Jones and Whittier [101], Staecher and Wang [102], Murty [103, 104], Bangar, et.al. [105],
Schoenberg [106], Rokhlin, et.al. [107], Pyrak-Nolte and Cook [108], Mal [109], Nagy and Adler [110], Adler, et.al. [111], Xu, et.al. [112], Huang and Rokhlin [113], Pilarski and Rose [114], Pecorari, et.al. [115, 116], Kundu and Maslov [117], Niklassson, et.al. [118], Guo, et.al. [119], and Leungvichcharoen and Wijeyewickrema [120]. Bulk material nonlinearity in wave guides, contact nonlinearities and other nonlinear stress-displacement laws across an interface all cause higher harmonic generation which affects both guided wave dispersion relations and reflection/transmission behavior. See, e.g. Achenbach and Norris [121], Baik and Thompson [122], Zhou and Shui [123], Shui and Solodov [124], Hirose and Achenbach [125], Scott and Price [126], Price and Scott [127], Hirsekorn [128], Deng [129] and Pecorari [130], Kim, et.al. [131], and the references therein. A lot of this work assumes a weak dynamic nonlinear effect and modifies the linear first-order behavior through a perturbation approach. Most of the interface focused work in this area has been on the reflection/transmission problem.

The present work assumes a lower order approximation in which the cohesive interface’s small amplitude dynamic behavior is assumed to be linearly compliant with the stiffness interpreted as the local slope of the cohesive law at the static pre-load level, and unaffected by the nonlinearity generated higher harmonics. Similar approaches in other settings have been postulated by Achenbach, et.al. [132], Parikh and Achenbach [133], Achenbach and Parikh [134] and Kim, et.al. [131]. The dispersion of SH or generalized Rayleigh-Lamb waves for a layer interacting with a linear spring interface can then be used to identify the stiffness. The spring stiffnesses inferred in this way from a series of measurements taken at multiple pre-load levels could then be integrated to obtain an approximate cohesive law.
CHAPTER 2

DIRECT BEM FORMULATION

On substituting the strain-displacement relations into the constitutive relations for a linear, elastic, isotropic material and then substituting the result into the differential equations of equilibrium, Navier’s equilibrium equations are obtained as,

\[ Gu_{i,jj} + \frac{G}{1-2\nu} u_{j,ji} + b_i = 0, \quad i = 1, 2, 3. \] (2.1)

\( G \) and \( \nu \) are the shear modulus and Poisson’s ratio respectively. \( u_i \) are the displacements and \( b_i \) represents the body force per unit volume. These governing equations for the two dimensional elastic domain \( B \) are transformed into an integral equation over the boundary \( \partial B \) by knowing the fundamental displacement and traction tensors (Green’s functions) due to unit loads in the \( x_1 \) and \( x_2 \) (or \( x \) and \( y \)) directions.

The reciprocity principle is given by [54, 55, 135] as

\[ \int_B \sigma_{ij} \epsilon_{ij}^* dA = \int_B \epsilon_{ij} \sigma_{ij}^* dA. \] (2.2)

where \( * \) denotes an arbitrary equilibrium field in \( B^* \) with boundary \( \partial B^* \), as shown in Figure 2.1, and which contains the domain \( B \) with boundary \( \partial B \), that is also in equilibrium. Applying the divergence theorem to Eqn.(2.2), and knowing that
Figure 2.1: General region \( B^* + \partial B^* \) containing the body \( B + \partial B \) with the same elastic properties

\[
\sigma_{ij,j} + b_i = 0 \quad \text{and} \quad t_i = \sigma_{ij} n_j
\]

\[
\int_B b_i^* u_i dA + \int_{\partial B} t_i^* u_i ds = \int_B b_i u_i^* dA + \int_{\partial B} t_i u_i^* ds.
\] (2.3)

If \( \delta(\xi - x) \) is defined as the Dirac delta function, i.e.,

\[
\int_B \delta(\xi - x) f(x) dA(x) = 0, \xi \notin B
\]

\[
\int_B \delta(\xi - x) f(x) dA(x) = f(\xi), \xi \in B.
\]

Let \( b_1^* = \delta(\xi - x), b_2^* = 0 \) and then let \( b_1^* = 0, b_2^* = \delta(\xi - x) \), i.e., let \( b^* \) be a line load first in the \( x_1 \) direction and then in the \( x_2 \) direction. Substituting these two
body force fields into equation (11) yields

\[ u_j(\xi) = \int_{\partial B} U_{ij}^*(\xi, x)t_j(x)ds(x) - \int_{\partial B} T_{ij}^*(\xi, x)u_j(x)ds(x) + \int_B U_{ij}^*(\xi, x)b_j(x)dA(x), \quad i = 1, 2 \]  

(2.4)

which is a continuous representation of displacements at any point \( \xi \) inside the domain \( B \).

The limit of Eqn. (2.4) when \( \xi \) approaches the boundary \( \partial B \) of \( B \) yields the same equation as Eqn. (2.4) except for a factor of 0.5 on the left hand side \([ \cdot \] \). Therefore

\[ c_{ij}(\xi)u_j(\xi) = \int_{\partial B} U_{ij}^*(\xi, x)t_j(x)ds(x) - \int_{\partial B} T_{ij}^*(\xi, x)u_j(x)ds(x) + \int_B U_{ij}^*(\xi, x)b_j(x)dA(x), \quad i = 1, 2. \]  

(2.5)

\[ c_{ij} = \begin{cases} 
1, & \xi \in B \\
\frac{1}{2}, & \xi \in \partial B \\
0, & \text{otherwise} 
\end{cases} \]

\[ U_{ij}^* = \frac{1}{8\pi G(1-\nu)}[(3-4\nu)\ln \frac{1}{r}\delta_{ij} + \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j}] \]  

(2.6)

\[ T_{ij}^* = \frac{1}{4\pi(1-\nu)r} \left[ \frac{\partial r}{\partial n} ((1-2\nu)\delta_{ij} + 2 \frac{\partial r}{\partial x_j} \frac{\partial r}{\partial x_i}) + (1-2\nu) \left( \frac{\partial r}{\partial x_i} n_j - \frac{\partial r}{\partial x_j} n_i \right) \right] \]  

(2.7)

When \( \xi \in \partial B \), and considering a single body with no body force,

\[ \frac{1}{2}u_1(\xi) = \int_{\partial B} [U_{11}(\xi, x)t_1(x)+U_{12}(\xi, x)t_2(x)]ds(x) - \int_{\partial B} [T_{11}(\xi, x)u_1(x)+T_{12}(\xi, x)u_2(x)]ds(x), \]  

(2.8)
\[ \frac{1}{2} u_2(\xi) = \int_{\partial B} [U_{21}(\xi, x)t_1(x) + U_{22}(\xi, x)t_2(x)]ds(x) - \int_{\partial B} [T_{21}(\xi, x)u_1(x) + T_{22}(\xi, x)u_2(x)]ds(x). \] (2.9)

Now discretize \( \partial B \) into \( M \) straight line elements \( \partial B_k = 1, \ldots, M \). Assume that on \( \partial B_k \) the displacements and tractions are constants.

\[ u_i(x) = u_i^k = \text{constant}; x \in \partial B_k \]
\[ t_i(x) = t_i^k = \text{constant}; x \in \partial B_k \]
\[ i = 1, 2 \]

Applying Equations (2.8) and (2.9) at \( \xi = x_l \) and replacing the exact contour \( \partial B \) with the sum of the lengths of the elements \( \partial B_k \),

\[ \frac{1}{2} u_l^1 = \sum_{k=1}^{M} [U_1^{lk} t_1^k + U_2^{lk} t_2^k - T_1^{lk} u_1^k - T_2^{lk} u_2^k], \] (2.10)

\[ \frac{1}{2} u_l^2 = \sum_{k=1}^{M} [U_1^{lk} t_1^k + U_2^{lk} t_2^k - T_1^{lk} u_1^k - T_2^{lk} u_2^k], \] (2.11)
\[ l = 1, \ldots, M. \]

\[ U_{ij}^{lk} \equiv \int_{\partial B_k} U_{ij}(x_l, x)ds(x) \] (2.12)

\[ \hat{T}_{ij} \equiv \int_{\partial B_k} T_{ij}(x_l, x)ds(x) \] (2.13)

These integrals have been worked out exactly in closed form as a function of \( x_l, x_k \), the orientation of the element, the length of the element, and the material constants, [54, 55].
Reorganizing Equations (2.10) and (2.11), we get;

\[ \sum_{k=1}^{M} [U_{11}^{ik}t_{1}^{k} + U_{12}^{ik}t_{2}^{k}] - (\hat{T}_{11}^{ik} + \frac{1}{2}\delta_{lk})u_{1}^{k} - \hat{T}_{12}^{ik}u_{2}^{k} = 0, \] (2.14)

\[ \sum_{k=1}^{M} [U_{21}^{ik}t_{1}^{k} + U_{22}^{ik}t_{2}^{k}] - \hat{T}_{21}^{ik}u_{1}^{k} - (\hat{T}_{22}^{ik} + \frac{1}{2}\delta_{lk})u_{2}^{k} = 0. \] (2.15)

Combining Equations (2.14) and (2.15) and then writing in matrix form

\[
\begin{bmatrix}
[U_{11}] & [U_{12}] \\
[U_{21}] & [U_{22}]
\end{bmatrix}
\begin{bmatrix}
[t_{1}] \\
[t_{2}]
\end{bmatrix}
+ \begin{bmatrix}
[T_{11}] & [T_{12}] \\
[T_{21}] & [T_{22}]
\end{bmatrix}
\begin{bmatrix}
[u_{1}] \\
[u_{2}]
\end{bmatrix}
= \begin{bmatrix}
[0] \\
[0]
\end{bmatrix}
\] (2.16)

where [U_{ij}] and [T_{ij}] (i, j = 1, 2) have l = 1, ..., M rows and k = 1, ..., M columns and the elements of the Green’s matrices are given by

\[ T_{ij}^{lk} = -(\hat{T}_{ij}^{lk} + \frac{1}{2}\delta_{lk}\delta_{ij}) \] (2.17)

and \( \hat{T}_{ij}^{lk} \) is given by Eqn.(2.13).

This formulation is applied to a beam geometry with an edge-crack as will be discussed in Chapters 3, 4 and 5 for a beam composed of a homogeneous material and in Chapters 6 and 7 for a bimaterial beam with an edge crack at the interface.
CHAPTER 3

BEM ANALYSIS OF LINEAR SOFTENING COHESIVE ZONE

3.1 Vibration problem

Consider a simply supported Euler-Bernoulli beam of rectangular cross section and containing a through-surface edge-crack of length ‘a’, as shown in Figure 3.1(a). If axial forces are neglected and only shear and bending loads are considered, then the crack plane is subjected to a shear force $Q$ and a bending moment $M$ as shown in Figure 3.1(b). The presence of a crack causes a relative jump in displacement and rotation of one flank of the crack relative to the other. The increased compliance due to the presence of the crack can be lumped into a continuous spring of zero width that connects the two faces of the crack [13, 15]. The beam is now divided at the crack plane into two regions and the crack plane replaced by a line-spring of zero-width. The compliance relations for the line-spring are then given by [13]

\[
\Theta = \theta_2 - \theta_1 = \lambda_\theta M, \tag{3.1}
\]

\[
w = w_2 - w_1 = \lambda_w Q, \tag{3.2}
\]
Figure 3.1: (a) Geometry for the simply supported, edge cracked beam. (b) Shear force and bending moment at crack plane. (c) Generic $M - \Theta$ curve.

where $\lambda_\theta$ and $\lambda_w$ are the compliances due to bending moment and shear force respectively. The spring relates the shear force and bending moment to the jump in displacement and rotation at the crack plane. If the spring is linear (i.e. $\lambda_\theta$ does not depend on $M$), the support and line-spring boundary conditions yield a standard eigenvalue problem for the natural frequencies and mode-shapes, in which natural frequencies depend on compliance. If the $M - \Theta$ relationship is non-linear however, as in Figure 3.1(c), the interpretation of the dynamic compliance to be used in a vibration analysis changes. The beam is loaded to a certain static load level along the
non-linear load-displacement curve and small amplitude vibrations are superposed about the static pre-load. Still there are different interpretations of how small amplitude vibrations are superposed about the static load level. One interpretation is described in Mendelsohn [16]. The dynamic compliance is provided by slope of the local tangent to the non-linear load-displacement curve at the static load-level. Another interpretation is described in Mendelsohn, et.al. [17], where a Taylor series expansion is carried out about the static load level on the non-linear load-displacement curve. The non-linear load-displacement curve is therefore the most important ingredient of the vibration analysis.

Under static loading the compliance is related to the $J$-integral [16]. For a crack subject to mode I loading (pure bending), the compliance is given by

$$\lambda_\theta(M,a) = \frac{b}{M} \int_0^a \frac{\partial J_I(M,a)}{\partial M} da,$$

(3.3)

and for a crack subject to mode II loading (pure shear), the compliance is given by

$$\lambda_w(Q,a) = \frac{b}{Q} \int_0^a \frac{\partial J_{II}(Q,a)}{\partial Q} da.$$  (3.4)

$J_I$ and $J_{II}$ denote the $J$-integral in modes I and II respectively. Since the present analysis considers a beam with an edge crack subjected to mode I loading only, $J_I$ will be replaced by $J$ from now on. In the two-dimensional setting $J$ is actually a function of moment per unit thickness out-of-plane, $\bar{M} = M/b$. Also, $J$ can be considered to be a function of the dimensionless crack length $a/W$ and not the actual crack length. Incorporating these changes the generalized relation for compliance $\lambda_\theta$ as a function
of $\bar{M}$ and $a/W$ in bending can then be written as

$$\lambda_\theta(\bar{M}, a/W) = \frac{W}{b} \frac{\bar{M}}{M} \int_0^{a/W} \frac{\partial J(\bar{M}, a/W)}{\partial \bar{M}} d(a/W).$$  

(3.5)

3.2 $J$-integral reduction for a linear softening cohesive model

The $J$-integral which appears in Eqn.(3.5) is computed from a two-dimensional static BEM analysis of an edge-crack with a planar cohesive zone in a beam shaped homogeneous elastic solid subjected to edge-moments. An edge-crack subjected to mode I loading in a homogeneous body in the current beam geometry setting is shown in Figure 3.2(a). The elastic crack tip is at $x_2 = -\frac{W}{2} + a$, while the cohesive zone

![Figure 3.2](image-url)

Figure 3.2: (a) Free body diagram of the cohesive zone ahead of the elastic crack tip. The cohesive zone extends from $x_2 = -\frac{W}{2} + a$ to $x_2 = -\frac{W}{2} + c$. (b) Linear softening traction-displacement ($t - \delta$) law.
extends from \( x_2 = -\frac{W}{2} + a \) to \( x_2 = -\frac{W}{2} + c \). If \( u \) denotes displacement, the cohesive traction \( t \) is a function of the displacement jump \( \delta \) which is defined as

\[
\delta = u_{x_1=0^+} - u_{x_1=0^-}.
\]

The crack tip opening displacement is denoted by \( \delta_t \) and is the value of \( \delta \) at \( x_2 = -\frac{W}{2} + a \). The functional relationship \((t - \delta)\) for a linear softening cohesive model is shown in Figure 3.2(b) and is written as

\[
t(\delta) = t_o [1 - \frac{\delta}{\delta_o}].
\] (3.6)

\( t_o \) is the peak traction (yield stress) and \( \delta_o \) is the critical value of crack opening displacement that causes extension of the elastic crack tip or crack growth. For a crack under mode-II loading (i.e. the crack subject to in-plane shear loading) these would be \( s_o \) and \( \delta_s \). The focus is on obtaining the cohesive response to loadings such that the entire cohesive law is exercised. Cohesive behavior beyond this state (i.e. the cohesive crack growth problem) is not considered here. The contour for evaluation of the \( J \)-integral is chosen to be the right and left faces of the cohesive zone, denoted by \( \Gamma_1 \) and \( \Gamma_2 \) respectively. If the subscript denotes the direction of traction or displacement, the cohesive boundary conditions are written as:

\[
\begin{align*}
\text{Right face (} x_1 = 0^- \text{)} : & \quad t_1 = +t(\delta), \ t_2 = 0, \ u_2 = 0 \quad (3.7) \\
\text{Left face (} x_1 = 0^+ \text{)} : & \quad t_1 = -t(\delta), \ t_2 = 0, \ u_2 = 0. \quad (3.8)
\end{align*}
\]

Since \( \Gamma_1 \) and \( \Gamma_2 \) are chosen to be the contour for integration, and since the singularity at the crack tip is relieved due to the presence of a cohesive zone the \( J \)-integral is
written in the standard form as

\[
J = \int_{\Gamma_1} [W \, dx_1 - t_i \frac{\partial u_i}{\partial x_2}] ds + \int_{\Gamma_2} [W \, dx_1 - t_i \frac{\partial u_i}{\partial x_2}] ds. \tag{3.9}
\]

This becomes

\[
J = - \int_{-\frac{W}{2}+\alpha}^{-\frac{W}{2}+c} t(\delta) \frac{\partial u_{x_1=0+}}{\partial x_2} dx_2 + \int_{-\frac{W}{2}+\alpha}^{-\frac{W}{2}+c} t(\delta) \frac{\partial u_{x_1=0-}}{\partial x_2} dx_2. \tag{3.10}
\]

Realizing that

\[
u_{x_1=0-} = -u_{x_1=0+}, \quad \frac{\partial u_{x_1=0-}}{\partial x_2} = -\frac{\partial u_{x_1=0+}}{\partial x_2}, \text{ and } \delta = 2u_{x_1=0+},
\]

the \( J \)-integral reduces to the area under the softening curve from 0 to \( \delta_t \).

\[
J = \int_0^{\delta_t} t(\delta) d\delta \tag{3.11}
\]

For the linear softening model (Eqn. (3.6)), the \( J \)-integral becomes

\[
J = t_o \delta_t \left[ 1 - \frac{1}{2} \frac{\delta_t}{\delta_o} \right], \tag{3.12}
\]

where \( \delta_t \), the crack tip opening displacement, is always less than the critical value \( \delta_o \), i.e. the crack does not grow. As \( \delta_t \) approaches \( \delta_o \), \( J \) approaches its critical value, \( J_o \),

\[
J_o = \frac{1}{2} t_o \delta_o. \tag{3.13}
\]
\( \delta \) is evaluated using a two-body, iterative, direct boundary element formulation for an edge crack in a beam shaped solid subjected to pure bending, described next.

### 3.3 Formulation with boundary conditions

The beam shaped geometry with an edge crack subjected to pure bending is shown in figure 3.3(a). The total length \( L \) of the beam is taken large enough compared to \( W \) to make the crack tip fields independent of any end effects at the load points, and for this 2-D elasticity model to behave like an Euler-Bernoulli beam. The beam is further divided into two fictitious bodies with the crack plane as the interface between them. The interface is divided into three regions: (i) ligament, (ii) cohesive zone - with a linear softening \( t - \delta \) law, and (iii) open crack as shown in Figure 3.3(b). \( B_1 \) and \( B_2 \) represent the boundaries of the two bodies that are discretized into elements with constant tractions and displacements. The direct boundary element formulation is applied to each of the two bodies which make up the beam under consideration. For each body \((k = 1, 2)\) the reciprocal identity gives the two matrix equations

\[
[kU_{ij}][kt_j] + [kT_{ij}][ku_j] = [0]; \quad k = 1,2.
\] (3.14)

The boundary traction and displacement vectors are \([kt_j]\) and \([ku_j]\), where the leading subscript refers to the body and the trailing subscript refers to the direction of traction or displacement. \([kT_{ij}]\) and \([kU_{ij}]\) \((i,j = 1,2)\) are the infinite space Green’s matrices for tractions and displacements, respectively in body \(k\). Details of these matrices can be found in Brebbia and Dominguez [54] and the doctoral thesis by Young [55]. The boundary conditions are then written as follows in terms of the boundary tractions and displacements.
3.3.1 Interface (ligament and cohesive zone)

In the ligament displacements across the fictitious interface are continuous and tractions are equal and opposite.

\[ 2u_1 = 1u_1, \quad 2u_2 = 1u_2 \]
\[ 2t_1 = -1t_1, \quad 2t_2 = -1t_2 \]
This eliminates four unknowns at each node and gives rise to a linear system of equations for the ligament:

\[
\begin{bmatrix}
[t_1] & [t_2] & [U_1] & [U_2] \\
[t_1] & [t_2] & [U_1] & [U_2] \\
[t_1] & [t_2] & [U_1] & [U_2] \\
[t_1] & [t_2] & [U_1] & [U_2] \\
\end{bmatrix}
\begin{bmatrix}
[u_1] \\
[u_2] \\
[t_1] \\
[t_2] \\
\end{bmatrix}
= \begin{bmatrix}
[0] \\
[0] \\
[0] \\
[0] \\
\end{bmatrix}.
\]

(3.15)

The matrix in Eqn.(3.15) consists of all rows of each of the Green’s submatrices and only those columns corresponding to nodes in the ligament.

In the cohesive zone the tractions are related to their respective displacement jump.

\[
1t_1 = -t_o[1 - \frac{\delta}{\delta_o}] = -t_o[1 - \frac{(1u_1 - 2u_1)}{\delta_o}] \\
2t_1 = +t_o[1 - \frac{\delta}{\delta_o}] = +t_o[1 - \frac{(1u_1 - 2u_1)}{\delta_o}] \\
1t_2 = -s_o[1 - \frac{\delta}{\delta_s}] = -s_o[1 - \frac{(1u_2 - 2u_2)}{\delta_s}] \\
2t_2 = +s_o[1 - \frac{\delta}{\delta_s}] = +s_o[1 - \frac{(1u_2 - 2u_2)}{\delta_s}]
\]

These cohesive zone boundary conditions lead to the linear system of equations in the unknown displacements given by

\[
\begin{bmatrix}
[t_1] + \frac{\delta}{\delta_o}[u_1] & [t_2] + \frac{\delta}{\delta_o}[U_1] & -\frac{\delta}{\delta_o}[U_1] & -\frac{\delta}{\delta_o}[U_1] \\
[t_1] + \frac{\delta}{\delta_o}[U_1] & [t_2] + \frac{\delta}{\delta_o}[U_1] & -\frac{\delta}{\delta_o}[U_1] & -\frac{\delta}{\delta_o}[U_1] \\
-\frac{\delta}{\delta_o}[2U_1] & -\frac{\delta}{\delta_o}[2U_1] & [2T_1] + \frac{\delta}{\delta_o}[2U_1] & [2T_1] + \frac{\delta}{\delta_o}[2U_1] \\
-\frac{\delta}{\delta_o}[2U_1] & -\frac{\delta}{\delta_o}[2U_1] & [2T_1] + \frac{\delta}{\delta_o}[2U_1] & [2T_1] + \frac{\delta}{\delta_o}[2U_1] \\
\end{bmatrix}
\begin{bmatrix}
[u_1] \\
[u_2] \\
[t_1] \\
[t_2] \\
\end{bmatrix}
= \begin{bmatrix}
-t_o[1U_1] - s_o[1U_1] \\
-t_o[1U_1] - s_o[1U_1] \\
t_o[2U_1] + s_o[2U_1] \\
t_o[2U_1] + s_o[2U_1] \\
\end{bmatrix}.
\]

(3.16)
The matrix in Eqn.(3.16) consists of all rows of each of the Green’s submatrices and only those columns corresponding to nodes in the cohesive zone. This formulation is written for general mixed mode I/II loading, however results are given for predominantly mode-I loading. This implies that the value of $s_o$ is taken close to zero.

### 3.3.2 Open crack and boundary $B_1$ and $B_2$ (except load points)

The open crack surfaces and the boundaries $B_1$ and $B_2$ (except the load points) are traction free.

$$i t_1 = i t_2 = 0; \quad i = 1, 2$$

The linear system of equations in the unknown displacements in this case is

$$
\begin{bmatrix}
[i T_{11}] & [i T_{12}] & [0] & [0] \\
[i T_{21}] & [i T_{22}] & [0] & [0] \\
[0] & [0] & [2 T_{11}] & [2 T_{12}] \\
[0] & [0] & [2 T_{21}] & [2 T_{22}]
\end{bmatrix}
\begin{bmatrix}
[1 u_1] \\
[1 u_2] \\
[2 u_1] \\
[2 u_2]
\end{bmatrix}
= 
\begin{bmatrix}
[0] \\
[0] \\
[0] \\
[0]
\end{bmatrix}.
$$

(3.17)

The matrix in Eqn.(3.17) consists of all rows of each of the Green’s submatrices and only those columns corresponding to nodes on the boundary and on the open crack.

### 3.3.3 Load points

$RU$ and $RL$ represent the upper and lower load points on the right body, and $LU$ and $LL$ represent the upper and lower load points on the left body as shown in figure.
3(a).

RL: \( t_1 = P, \ t_2 = 0 \)

RU: \( t_1 = -P, \ t_2 = 0 \)

LU: \( t_1 = P, \ t_2 = 0 \)

LL: \( t_1 = -P, \ t_2 = 0 \)

These boundary conditions lead to the following system of equations in unknown displacements.

\[
\begin{bmatrix}
[T_{11}] & [T_{12}] \\
[T_{21}] & [T_{22}]
\end{bmatrix}
\begin{bmatrix}
[u_1] \\
[u_2]
\end{bmatrix}
= -P
\begin{bmatrix}
[U_{11}] \\
[U_{21}]
\end{bmatrix} \quad (3.18)
\]

\[
\begin{bmatrix}
[T_{11}] & [T_{12}] \\
[T_{21}] & [T_{22}]
\end{bmatrix}
\begin{bmatrix}
[u_1] \\
[u_2]
\end{bmatrix}
= P
\begin{bmatrix}
[U_{11}] \\
[U_{21}]
\end{bmatrix} \quad (3.19)
\]

\[
\begin{bmatrix}
[T_{11}] & [T_{12}] \\
[T_{21}] & [T_{22}]
\end{bmatrix}
\begin{bmatrix}
[u_1] \\
[u_2]
\end{bmatrix}
= P
\begin{bmatrix}
[U_{11}] \\
[U_{21}]
\end{bmatrix} \quad (3.20)
\]

\[
\begin{bmatrix}
[T_{11}] & [T_{12}] \\
[T_{21}] & [T_{22}]
\end{bmatrix}
\begin{bmatrix}
[u_1] \\
[u_2]
\end{bmatrix}
= -P
\begin{bmatrix}
[U_{11}] \\
[U_{21}]
\end{bmatrix} \quad (3.21)
\]

The matrix in each of Equations (3.18 - 3.21) consists of all rows of each of the Green’s submatrices and only those columns corresponding to the respective node of the load point. Equations (3.15 - 3.21) are combined to form a \(4N \times 4N\) linear system where each boundary, \(B_1\) and \(B_2\) is discretized into \(N\) elements. In addition to the \(4N\) unknown tractions and displacements, the length of the cohesive zone is also unknown.
3.4 Solution scheme

The beam is quasi-statically loaded such that significantly observable cohesive zones are formed ahead of the crack tip. The applied moment per unit thickness is calculated as

\[ \bar{M} = P l_e n_e, \]  

(3.22)

where \( P \) is the applied traction at the load points, \( l_e \) is the element length and \( n_e \) is the number of elements between the load points. This moment \( \bar{M} \) is increased incrementally to almost exercise the entire linear softening cohesive law. The crack is not allowed to grow.

The plastic moment per unit thickness, \( M_P \) for a rectangular beam is calculated from basic strength of materials as

\[ M_P = \frac{1}{4} W^2 t_o, \]  

(3.23)

where \( W \) is the height of the cross section of the beam, and \( t_o \) is the yield strength of the material of the beam. The plastic moment is the moment required to cause complete yielding of the cross section of the uncracked beam subjected to pure bending. The dimensionless moment ratio \( M_R \) is then defined as the ratio of the applied moment to the plastic moment.

\[ M_R = \frac{\bar{M}}{M_P} \]  

(3.24)

In all the simulations that follow the value of \( M_R \) is always less than 1. This means that the ligament does not completely yield when the entire cohesive law is exercised.
The height $W$ of the cross section of the beam for all simulations is chosen to be 12.5 mm and the value of the plastic moment can be easily calculated for various values of yield strengths. An automatic iterative solution scheme is employed to obtain the unknown tractions and displacements and the extent of the cohesive zone. The scheme begins with an initial guess of the number of elements that constitute the extent of the cohesive zone. The evaluated normal traction value in an element of the cohesive zone nearest to the ligament is then compared with $t_o$. Iterations are performed on the number of elements in the cohesive zone until the value of the normal traction in the first element of the ligament, nearest to the cohesive zone, is less than $t_o$. The other constraint for the solution is that in the open crack there is no interpenetration of material, i.e. the crack exhibits opening displacements. The solution for displacements is checked for this condition. A representative solution for the normal tractions and displacements at the interface under predominantly mode-I conditions is shown in Figure 4. The interface is composed of 100 elements which provides a sufficient amount of refinement to obtain convergent results. The markers in Figure 3.4(c) show the traction values in the elements in the cohesive zone thus indicating the extent to which the linear cohesive softening law has been exercised and the length of the cohesive zone. In this example the elastic crack is 42 elements long, the cohesive zone is 18 elements long and $\delta_t$ is about 75% of its critical value. The solution shown in figure 3.4 represents a typical aluminum alloy ($G = 28,000 \, MPa$, $\nu = 0.3$, $t_o = 50 \, MPa$ and $\delta_o = 0.0008 \, m$). The total number of elements on the boundaries of both the bodies is 840 making the total size of the linear system $1680 \times 1680$. 

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Figure 3.4: Representative solution for tractions and displacements in the cohesive zone along with the extent of softening. \( E = 72,800 \text{ MPa}, \nu = 0.3, W = 12.5 \text{ mm}, a/W = 0.42, M_R = 0.41, t_o = 50 \text{ MPa} \) and \( \delta_o = 0.0008 \text{ m} \)
CHAPTER 4

PARAMETER VARIATION

4.1 Parameter variation

Several linear softening cohesive laws with three basic kinds of parameter variations are considered. The first kind of parameter variation considers three linear softening cohesive laws in which the total cohesive energy $J_0$, Eqn. (3.13) is maintained constant at the value $20 \ (10^3) \ N/m$. If the crack were in small scale yielding, which is often not the case here, this would correspond to a fracture toughness of $K_{IC} = 40 \ MPa\sqrt{m}$. The peak traction $t_o$ is then varied (from $25 \ MPa$ to $75 \ MPa$). Figure 4.1(a) shows the results for these cohesive laws along with the traction values of the elements (indicated by markers) in the cohesive zone for one value of given crack length $(a/W = 0.42)$ and applied moment $\bar{M}$. However, since $t_o$ is being varied, the moment ratio $M_R$ varies for each case. If the load were increased a little bit from the value used, $\delta_t$ would exceed its critical value for the steepest cohesive law. The trends are that the steeper the softening curve the lower the load is at which the crack propagates, relative to the plastic limit load. Also as the softening curve becomes steeper the cohesive zone becomes markedly shorter.

Figure 4.1(b) shows results for the same parameter variation in which $J_0$ is kept constant at $20 \ (10^3) \ N/m$. The moment ratio $M_R$ is also maintained constant at
0.27, which is the highest moment ratio for the steepest cohesive law prior to crack propagation. As $\delta_t$ for the steepest softening curve approaches its critical value, the size of the cohesive zone for the less steep cohesive laws is significantly less than the size of the cohesive zone for the steepest law.

In the second parameter variation the peak traction $t_o$ is maintained constant (at 50 MPa) and $J_o$ is varied (from $5 \times 10^3$ N/m to $45 \times 10^3$ N/m), thereby generating three linear softening laws. The results for these cohesive laws along with the traction values of the elements in the cohesive zone for one value of crack length ($a/W = 0.42$) and moment ratio ($M_R = 0.2$) are shown in Figure 4.2. Here again a little increase in the load from the value used would cause $\delta_t$ to exceed its critical value for the steepest cohesive law. It is seen that the load required to cause crack propagation is considerably lower than that for less steeper softening curves. Also as $\delta_t$ for the steepest softening curve approaches its critical value, the cohesive zones are just beginning to form for the less steeper softening laws.

In the third parameter variation the value of $\delta_o$ is kept constant (at 0.0018 m) and the peak traction $t_o$ is varied (from 25 MPa to 75 MPa). Representative results for these cohesive laws along with the cohesive traction values for one value of crack length ($a/W = 0.42$) and applied moment $\bar{M}$ are shown in Figure 4.3. Once again since $t_o$ is being varied, the moment ratio $M_R$ varies for each case. The size of the cohesive zone is seen to reduce significantly with increasingly steeper softening curves.

### 4.2 Results for variation of the $J$-integral

As mentioned earlier the value of the crack tip opening displacement $\delta_t$ is sought after from the boundary element calculations. This is the value of the displacement jump at the elastic crack tip and is used in the evaluation of the $J$-integral, Eqn.
(3.12). The $J$-integral has been calculated over a range of crack lengths and moment ratios $M_R$. The variation of the dimensionless $J$-integral ($J/J_o$) with dimensionless crack length ($a/W$) is shown in Figure 4.4, Figure 4.5 and Figure 4.6 at various values of dimensionless moment $M_R$ for all the cohesive laws in the three kinds of parameter variations discussed earlier (Figures 4.1, 4.2, 4.3). It should be noted that $M_R$ is always chosen so that $J/J_o$ is always less than 1, i.e. the crack is not allowed to grow.

Figure 4.4(a) shows results for $J/J_o$ for the least steep cohesive law ($t_o = 25\text{ MPa}, \delta_o = 1.6(10^{-3})\text{ m}$) described in Figure 4.1(a). It is seen that $J/J_o$ increases with crack length $a/W$ and then decreases at higher crack lengths. The size of the cohesive zone also follows a similar trend. This behaviour is more pronounced at higher moment ratios $M_R$. The highest value of applied moment $M$ is 89% of the fully plastic moment. Figure 4.4(b) shows results for $J/J_o$ for the intermediate cohesive law ($t_o = 50\text{ MPa}, \delta_o = 0.8(10^{-3})\text{ m}$) described in Figure 4.1(a). Once again it is seen that $J/J_o$ first increases with increasing $a/W$ and then decreases at higher crack lengths. However, the highest value of the applied moment is 45% of the fully plastic moment. Results for $J/J_o$ for the steepest cohesive law ($t_o = 75\text{ MPa}, \delta_o = 0.553(10^{-3})\text{ m}$) of Figure 4.1(a) are shown in Figure 4.4(c). The highest value of the applied moment is 27% of the fully plastic moment.

Figure 4.5(a) shows results for $J/J_o$ for the steepest cohesive law ($t_o = 50\text{ MPa}, \delta_o = 0.2(10^{-3})\text{ m}$) described in Figure 4.2. The highest value of the applied moment is 23% of the fully plastic moment. Figure 4.5(b) shows results for $J/J_o$ for the intermediate cohesive law ($t_o = 50\text{ MPa}, \delta_o = 0.8(10^{-3})\text{ m}$) described in Figure 4.2. The highest value of the applied moment is 45% of the fully plastic moment. Figure 4.5(c) shows results for $J/J_o$ for the least steep cohesive law ($t_o = 50\text{ MPa}, \delta_o = 1.8(10^{-3})\text{ m}$) described in Figure 4.2. The highest value of the applied moment is 63% of the fully plastic moment.
Figure 4.6(a) shows results for $J/J_o$ for the least steep cohesive law ($t_o = 25 \text{ MPa}, \delta_o = 1.8(10^{-3}) \text{ m}$) described in Figure 4.3. The highest value of the applied moment is 89% of the fully plastic moment. Figure 4.6(b) shows results for $J/J_o$ for the intermediate cohesive law ($t_o = 50 \text{ MPa}, \delta_o = 1.8(10^{-3}) \text{ m}$) described in Figure 4.3. The highest value of the applied moment is 63% of the fully plastic moment. Figure 4.6(c) shows results for $J/J_o$ for the steepest cohesive law ($t_o = 75 \text{ MPa}, \delta_o = 1.8(10^{-3}) \text{ m}$) described in Figure 4.3. The highest value of the applied moment is 51% of the fully plastic moment.

The results shown in Figures 4.4, 4.5 and 4.6 are very sensitive to load increments, especially at the trailing end of the linear softening cohesive law. In other words, a small increment in moment causes a significantly large increase in the value of $\delta_t$ and a large dip in the traction value at the elastic crack tip. A set of values of the $J$-integral can now be used in the numerical scheme that will be discussed in the next chapter to obtain the compliance and non-linear $M-\Theta$ curves.
Figure 4.1: Extent of softening for constant $J_o$ and $a/W$. (a) $\tilde{M}$ is maintained constant. (b) $M_R$ is maintained constant.
Figure 4.2: Extent of softening for constant $t_o$, $a/W$ and applied moment $M$. 

Figure 4.3: Extent of softening for constant $\delta_o$, $a/W$ and applied moment $M$. 

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Figure 4.4: Variation of dimensionless $J$-integral ($J/J_o$) with dimensionless crack length ($a/W$) at various dimensionless moment ratios $M_R$. $J_o$ is maintained constant. (a) $t_o = 25 \text{ MPa}$ and $\delta_o = 0.0016 \text{ m}$. (b) $t_o = 50 \text{ MPa}$ and $\delta_o = 0.0008 \text{ m}$. (c) $t_o = 75 \text{ MPa}$ and $\delta_o = 0.0055 \text{ m}$. 

continued
Figure 4.4 continued

![Graph showing the relationship between J/J₀ and a/W for different values of Mₜ.](image-url)

(c)

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Figure 4.5: Variation of dimensionless $J$-integral ($J/J_o$) with dimensionless crack length ($a/W$) at various dimensionless moment ratios $M_R$. $t_o$ is maintained constant. 

(a) $t_o = 50$ MPa and $\delta_o = 0.0002$ m. (b) $t_o = 50$ MPa and $\delta_o = 0.0008$ m. (c) $t_o = 50$ MPa and $\delta_o = 0.0018$ m.
Figure 4.5 continued
Figure 4.6: Variation of dimensionless $J$-integral ($J/J_o$) with dimensionless crack length ($a/W$) at various dimensionless moment ratios $M_R$. $\delta_o$ is maintained constant. (a) $t_o = 25$ MPa and $\delta_o = 0.0018$ m. (b) $t_o = 50$ MPa and $\delta_o = 0.0018$ m. (c) $t_o = 75$ MPa and $\delta_o = 0.0018$ m.
Figure 4.6 continued
5.1 Numerical analysis for compliance calculations

The methodology for extracting the non-linear $M - \Theta$ curve corresponding to a given linear softening cohesive law, to be used in the vibration problem as discussed in Chapter 3, is now presented. At a given crack length $a/W$ the values of the $J$-integral obtained from the boundary element calculations for various values of applied moment $\bar{M}$ are used to obtain an interpolation polynomial $J_{N-1}(\bar{M})$ using Newton’s method as

$$J_{N-1}(\bar{M}) = \sum_{i=1}^{N} J[\bar{M}_1,..,\bar{M}_i] \prod_{j=1}^{i-1}(\bar{M} - \bar{M}_j).$$

(5.1)

For $N$ points in the original data, $J_{N-1}(\bar{M})$ represents a polynomial of order $(N-1)$. The coefficients of this interpolating polynomial are obtained using divided differences in the standard way and are denoted by $J[\bar{M}_1,..,\bar{M}_i]$ [56]. Using the interpolation polynomial several data points $J_n$ are now created at multiple values of applied moment $\bar{M}_n$ within the range of moments in the original data. $J_n$ denotes the value of $J$-integral at the $n^{th}$ value of applied moment $\bar{M}_n$. The value of the derivative $\frac{dJ}{dM}$ is
obtained numerically at each of the data points using central differences as

$$\frac{dJ}{dM} = \frac{J_{n+1} - J_{n-1}}{2h_M}. \quad (5.2)$$

$h_M$ denotes the increment in moment $\bar{M}$. This procedure is repeated to obtain curves of $\frac{dJ}{dM}$ at several crack lengths $a/W$ that range between 0.06 and 0.5. At a given moment $\bar{M}$ an interpolating polynomial of order $(N - 1)$ for $\frac{dJ}{dM}$ as a function of $(a/W)$, with values of $J$-integral at $N$ crack lengths $a/W$, can be written using Newton’s method as

$$\frac{dJ}{dM}(a/W) = \sum_{i=1}^{N} \frac{dJ}{dM}[(a/W)_1, ..., (a/W)_i] \prod_{j=1}^{i-1} [(a/W) - (a/W)_j]. \quad (5.3)$$

The coefficients of this interpolating polynomial are again obtained using divided differences and are denoted by $\frac{dJ}{dM}[(a/W)_1, ..., (a/W)_i]$. Using the polynomial functions for $\frac{dJ}{dM}(a/W)$ obtained at several values of $\bar{M}$, several data points $\frac{dJ}{dM}$ are now created at multiple values of crack lengths $(a/W)_n$ for crack lengths ranging from 0 to 0.5.

Recall from Eqn. (3.5) that $\lambda_{\theta}$ involves integrating $\frac{dJ}{dM}$ with respect to crack length $(a/W)$. At a given moment $\bar{M}$ this integral is obtained numerically using the trapezoidal rule as

$$\int_0^{0.5} \frac{dJ(a/W)}{dM} d(a/W) = \sum_{i=1}^{n} \frac{(a/W)_i + (a/W)_{i+1}}{2} h_{a/W}. \quad (5.4)$$
\( h_{a/W} \) denotes the increment in crack length \( a/W \). The final expression for \( \lambda_\theta \) can be written as

\[
\lambda_\theta = \frac{W}{b} \frac{1}{M} \sum_{i=1}^{n} \frac{(a/W)_i + (a/W)_{i+1}}{2} h_{a/W}.
\] (5.5)

Figure 5.1 shows the variation of the dimensionless compliance \( \lambda_\theta / \lambda_{\theta e} \) with dimensionless crack length \( a/W \) for various values of moment ratio \( M_R \) obtained using a set of values of the \( J \)-integral corresponding to a single specific \( t - \delta \) cohesive law \((t_o = 50 \, MPa, \, J_o = 45(10^3) \, N/m)\). This law appears as the top law in Figure 4.2 and the middle law in Figure 4.3. The normalization constant \( \lambda_{\theta e} \) represents the value of elastic compliance obtained using a set of values of the \( J \)-integral for the same beam geometry without considering a cohesive zone ahead of the crack tip. The \( J \)-integral for the linear elastic case is obtained in the standard way from the stress intensity factor \( K_I \) that is evaluated from the near tip stress fields using a similar boundary element code [55]. The numerical scheme discussed earlier is used to obtain the elastic compliance \( \lambda_{\theta e} \). Since the factor \( (W/b) \) occurs both in the plastic and elastic compliance relations, the variation of the quantity \((b/W) \lambda_{\theta e}\) with dimensionless crack length \( (a/W) \) is shown in Figure 5.2.

Upon observing the variation of the dimensionless compliance with dimensionless crack length it is seen that at higher crack lengths the increase in compliance due to plastic deformation ahead of the crack tip is less pronounced than at lower crack lengths. Once again, this effect is more dominant at higher values of moment ratio \( M_R \). At very low crack lengths and at low load levels the values of compliance are not reliable due to the inaccuracies involved in numerically obtaining the derivative \( \frac{dJ}{dM} \) near zero. So to get a more meaningful picture results for very low crack lengths are not shown. In order for the compliance to be 10% larger than the elastic the load must be more than 35% of the plastic moment. But, as the load increases the rate
Figure 5.1: Dimensionless compliance ($\lambda_\theta/\lambda_{\theta e}$) curves with dimensionless crack length $a/W$. $t_o = 50$ MPa and $J_o = 45(10^3)$ N/m.

of increase in plastic compliances increases and a 50% increase in plastic compliance occurs for a shorter crack at $M_R$ of only 0.52.

5.2 Extraction of $M - \Theta$ curves

The jump in rotation $\Theta$ at a particular crack length is now evaluated using

$$\Theta = \lambda_\theta M.$$  

This however yields results at $N$ discrete points. To obtain a smooth $M-\Theta$ curve Newton’s method of interpolation is used again to obtain a polynomial $\bar{M}_{N-1}(\theta)$ of
Figure 5.2: Variation of elastic compliance ($\lambda_{\theta e}$) with dimensionless crack length $a/W$.

Normalized $M - \Theta$ curves for the single specific cohesive law ($t_o = 50 \ MPa$, $J_o = 45(10^3) \ N/m$), that appears as the top law in Figure 4.2 and the middle law in Figure 4.3, at several crack lengths are shown in Figure 5.3. $\Theta_e$ represents the elastic value of $\Theta$ at $a/W = 0.5$ obtained from the elastic compliance. These curves are the key input to the analysis presented in [17], where a Taylor series expansion needs to be carried out about a given static load level. The results are refined enough to enable calculations of the first and second derivatives at a given static load level.

$$
\bar{M}_{N-1}(\Theta) = \sum_{i=1}^{N} \bar{M}[\Theta_1, ..., \Theta_i] \prod_{j=1}^{i-1} (\Theta - \Theta_j).
$$

(5.6)
Figure 5.3: Normalized $M - \Theta$ curves at various crack lengths. $t_o = 50 \text{ MPa}$ and $J_o = 45(10^3) \text{ N/m}$.

5.3 Results

Normalized $M - \Theta$ curves for most of the linear softening cohesive laws considered are shown in Figures 5.4, 5.6 and 5.6. Figure 5.4(a) shows the $M - \Theta$ curves at four different crack lengths for the cohesive law defined by $t_o = 25 \text{ MPa}$ and $J_o = 20(10^3) \text{ N/m}$, and considered in an earlier parameter variation. Since no vibration analysis is considered here, it suffices to report the general nature of the curves instead of a more refined set of values. Similar curves at the same four crack lengths are shown in Figure 5.6.
lengths are shown in Figure 5.4(b) for the cohesive law defined by $t_o = 50 \, MPa$ and $J_o = 20(10^3) \, N/m$. For the cohesive law defined by $t_o = 75 \, MPa$ and $J_o = 20(10^3) \, N/m$ the $M - \Theta$ curves could not be obtained due to the fact that the size of the cohesive zone required to exercise the entire cohesive zone is smaller compared to the size of cohesive zone obtained with less steeper cohesive laws. This means that the conditions ahead of the crack tip are small scale yielding or near elastic and do not show significant plastic deformation to enable effective calculation of $M - \Theta$ relations.

Figure 5.5(a) shows the $M - \Theta$ curves at four different crack lengths for the cohesive law defined by $t_o = 50 \, MPa$ and $J_o = 20(10^3) \, N/m$, as part of the parameter variation where the peak cohesive traction is maintained constant at $50 \, MPa$. $M - \Theta$ curves for the cohesive law defined by $t_o = 50 \, MPa$ and $J_o = 5(10^3) \, N/m$ could not be obtained for reasons similar to those discussed for an earlier cohesive law, and are not reported. Figure 5.5(b) shows $M - \Theta$ curves for the cohesive law defined by $t_o = 50 \, MPa$ and $J_o = 45(10^3) \, N/m$.

For the parameter variation that considered the value of the displacement jump to remain constant at $0.0018 \, m$, the corresponding normalized $M - \Theta$ curves are shown in Figure 5.6. Figure 5.6(a) shows the curves for the cohesive law described by $t_o = 25 \, MPa$ and $J_o = 22.5(10^3) \, N/m$. Figure 5.6(b) shows the curves for the cohesive law defined by $t_o = 50 \, MPa$ and $J_o = 45(10^3) \, N/m$ and Figure 5.6(c) shows the $M - \Theta$ curves for the cohesive law defined by $t_o = 75 \, MPa$ and $J_o = 57.7(10^3) \, N/m$. 
Figure 5.4: Normalized $M - \Theta$ curves at various crack lengths. (a) $t_o = 50 \text{ MPa}$ and $J_o = 20(10^3) \text{ N/m}$. (b) $t_o = 50 \text{ MPa}$ and $J_o = 20(10^3) \text{ N/m}$. 
Figure 5.5: Normalized $M - \Theta$ curves at various crack lengths. (a) $t_o = 50\, MPa$ and $J_o = 20(10^3)\, N/m$. (b) $t_o = 50\, MPa$ and $J_o = 45(10^3)\, N/m$. 

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Figure 5.6: Normalized $M - \Theta$ curves at various crack lengths. (a) $t_o = 25 \text{ MPa}$ and $J_o = 22.5 \times 10^3 \text{ N/m}$. (b) $t_o = 50 \text{ MPa}$ and $J_o = 45 \times 10^3 \text{ N/m}$. (c) $t_o = 75 \text{ MPa}$ and $J_o = 57.7 \times 10^3 \text{ N/m}$.
Figure 5.6 continued

\[ M_R \]

- \( a/W = 0.26 \)
- \( a/W = 0.34 \)
- \( a/W = 0.42 \)
- \( a/W = 0.5 \)

- \( t_o = 75 \text{ MPa} \)
- \( J_o = 67.7 \times 10^3 \text{ N/m} \)
CHAPTER 6

DUGDALE COHESIVE ZONE MODEL FOR A

BIMATERIAL INTERFACE CRACK

6.1 Review of crack tip models

Consider a crack at the interface between two materials having elastic moduli $\mu_1, \nu_1$ and $\mu_2, \nu_2$ as shown in Figure 6.1(a). As discussed in the introduction in Chapter 1 several crack tip models have been proposed to account for the crack face contact which is induced by the material mismatch. These models are schematically shown in Figures 6.1(b) to 6.1(f). Figure 6.1(b) shows Comninou’s crack flank contact model [63, 64, 65]. Since the contact extends right up to the crack tip where there is smooth closure, the mode I singularity in the normal stress vanishes automatically, while the shear stresses ahead of the crack tip maintain their standard square root singularities and are non-oscillatory, provided the crack flanks can slide freely relative to each other in the contact. The only mechanism for crack propagation along the interface is then by pure shear in the plane of the interface, since if there is any crack face contact at all at the crack tip the normal stresses ahead of the crack tip are compressive and only the interface shear stresses can contribute to failure of the interface. Sinclair’s [66] finite crack opening angle model, shown in Figure 6.1(c), complements the contact model of Comninou and permits crack propagation with a tensile component.
along the interface for a remotely applied tension. The interphase model proposed by Atkinson [67], shown in Figure 6.1(d), considers the crack to propagate in a finite layer interphase material having properties $\mu_3$, $\nu_3$. This model essentially creates conditions equivalent to a crack tip in a homogeneous material. However, it does not truly capture the physical debonding of the interface. Therefore, the interphase model with continuously varying moduli, as shown in Figure 6.1(e) was introduced. The values of the elastic moduli of the interphase vary continuously between those of the two surrounding materials. The wedge interphase model proposed by Boniface and Simha [68] builds on Atkinson’s interphase model and is shown in Figure 6.1(f). This model considers the interphase to be a wedge inclusion with its vertex at the crack tip. For certain values of wedge angle this model gives a physically admissible solution. In this chapter a Dugdale-Barenblatt cohesive zone model is extended to the bimaterial interface crack. This is schematically shown in Figure 6.1(g).

### 6.2 Dugdale cohesive model and J-integral reduction

Consider a beam like geometry of thickness $b$ that is composed of two dissimilar materials with elastic moduli $\mu_1$, $\nu_1$ and $\mu_2$, $\nu_2$ with an edge crack of length $a$ at their interface as shown in Figure 6.2(a). Also consider a Dugdale-Barenblatt cohesive zone ahead of the crack tip. The elastic crack tip is at $x_2 = -\frac{W}{2} + a$, while the cohesive zone extends to $x_2 = -\frac{W}{2} + c$. The bond line is kept straight so that the standard form of $J$-integral as indicated by Smelser and Gurtin [97] can be applied. The contour for integration is shrunk down to become the right and left faces of the cohesive zone, denoted by $\Gamma_1$ and $\Gamma_2$ respectively. These integration contours are shown on the free body diagram of the cohesive zone in figure 6.2(b). $t_o$ and $s_o$ represent the cohesive tractions in tension and shear respectively and are constant over the length of the cohesive zone.
Figure 6.1: (a) Problem of a crack at the interface of two dissimilar materials. (b) Contact zone model of Comninou. (c) Crack opening angle model of Sinclair. (d) Interphase model of Atkinson. (e) Interphase model with continuously varying moduli-Atkinson. (f) Wedge interphase model of Boniface and Simha. (g) Cohesive zone model.
If the first subscript denotes the body and the second subscript denotes the direction of traction or displacement, the cohesive boundary conditions are written as:

Right face: \( t_{11} = t_o, \ t_{12} = s_o \) \hspace{1cm} (6.1)

Left face: \( t_{21} = -t_o, \ t_{22} = -s_o. \) \hspace{1cm} (6.2)

Since the singularity at the crack tip is relieved due to the presence of a cohesive zone and since the bond line is kept straight [97], the \( J \)-integral can be written in the
standard form as

\[ J = \int_{\Gamma_1} [W dx_1 - t_i \frac{\partial u_i}{\partial x_2}] ds + \int_{\Gamma_2} [W dx_1 - t_i \frac{\partial u_i}{\partial x_2}] ds. \]  \hspace{1cm} (6.3)

This becomes

\[ J = \frac{-W + c}{W + a} \int_{-\frac{W}{2} + a} s_o \frac{\partial u_{22}}{\partial x_2} dx_2 - \int_{-\frac{W}{2} + a} t_o \frac{\partial u_{12}}{\partial x_2} dx_2 + \int_{-\frac{W}{2} + a} s_o \frac{\partial u_{11}}{\partial x_2} dx_2 + \int_{-\frac{W}{2} + a} t_o \frac{\partial u_{21}}{\partial x_2} dx_2. \]  \hspace{1cm} (6.4)

Rearranging terms the final expression for \( J \) becomes

\[ J = \int_{-\frac{W}{2} + a} s_o \frac{\partial u_{22}}{\partial x_2} dx_2 - \frac{-W + c}{W + a} \int_{-\frac{W}{2} + a} t_o \frac{\partial u_{12}}{\partial x_2} dx_2 + \int_{-\frac{W}{2} + a} s_o \frac{\partial u_{11}}{\partial x_2} dx_2 + \frac{-W + c}{W + a} \int_{-\frac{W}{2} + a} t_o \frac{\partial u_{21}}{\partial x_2} dx_2. \]  \hspace{1cm} (6.5)

Therefore the \( J \)-integral that is indicative of the fracture energy of the interface can be obtained if the quantities \( u_{11}, u_{12}, u_{21} \) and \( u_{22} \) are obtained numerically and their gradients over the extent of the cohesive zone are substituted into Eqn. 6.5. This value of the \( J \)-integral consists of the interface fracture energy in mode-I and mode-II, denoted by \( J_I \) and \( J_{II} \) respectively. From Eqn. 6.5 these quantities are written as

\[ J_I = \int_{-\frac{W}{2} + a} t_o \left\{ \frac{\partial u_{11}}{\partial x_2} + \frac{\partial u_{21}}{\partial x_2} \right\} dx_2, \]  \hspace{1cm} (6.6)

\[ J_{II} = \int_{-\frac{W}{2} + a} s_o \left\{ \frac{\partial u_{12}}{\partial x_2} - \frac{\partial u_{22}}{\partial x_2} \right\} dx_2. \]  \hspace{1cm} (6.7)

To apply Eqn. 6.5 estimates of the cohesive tractions \( t_o \) and \( s_o \) are also required. These estimates are believed to depend on the yield strength of the interface bond and
also the local mode mixity at the crack tip and the far field mode mixity in the applied loading. In this chapter the specific formulation for the $J$-integral of the interface crack with a Dugdale-Barenblatt cohesive zone is presented for the beam geometry subject to pure bending with remotely applied moments. A direct, two body, plane stress, elastostatic BEM formulation is used to obtain the solution for tractions and displacements by applying physically admissible constraints in the cohesive zone.

6.3 BEM formulation with boundary conditions

![Boundary element mesh showing the interface between the bodies of the bimetallic beam having boundaries $B_1$ and $B_2$. Couples are applied at the ends ensure pure bending.](image)

![Interface divided into three regions.](image)

Figure 6.3: (a) Boundary element mesh showing the interface between the bodies of the bimetallic beam having boundaries $B_1$ and $B_2$. Couples are applied at the ends ensure pure bending. (b) Interface divided into three regions.
The beam shaped geometry with an edge crack at the interface of two dissimilar materials subjected to pure bending is shown in Figure 6.3(a). This is the same geometry and loading that has been used in Chapter 3. Here again the total length of the beam is taken large enough compared to W to make the crack tip fields independent of any end effects at the load points, and for this 2-D elasticity model to behave like an Euler-Bernoulli beam. The interface is divided into three regions: (i) ligament, (ii) cohesive zone - with constant normal and shear tractions, and (iii) open crack as shown in Figure 6.3(b). $B_1$ and $B_2$ represent the boundaries of the two bodies that are discretized into elements with constant tractions and displacements.

The direct boundary element formulation is applied to each of the two bodies which make up the beam under consideration. For each body ($k = 1, 2$) the reciprocal identity gives a matrix equation:

$$[kU_{ij}] [kt_j] + [kT_{ij}] [ku_j] = [0]; \quad k = 1, 2. \quad (6.8)$$

The boundary traction and displacement vectors are $[kt_j]$ and $[ku_j]$, where the leading subscript refers to the body and the trailing subscript refers to the direction of traction or displacement. $[kT_{ij}]$ and $[kU_{ij}]$ ($i, j = 1, 2$) are the infinite space Green’s matrices for tractions and displacements, respectively in body $k$. These matrices remain unchanged for the bimetallic beam problem. The boundary conditions are then written as follows in terms of the boundary tractions and displacements.
6.3.1 Interface (ligament and cohesive zone)

In the ligament displacements across the bimetallic interface are continuous and tractions are equal and opposite.

\[ 2u_1 = u_1, \quad 2u_2 = u_2 \]

\[ 2t_1 = -t_1, \quad 2t_2 = -t_2 \]

This eliminates four unknowns at each node and gives rise to a linear system of equations for the ligament:

\[
\begin{bmatrix}
1T_{11} & 1T_{12} & 1U_{11} & 1U_{12} \\
1T_{21} & 1T_{22} & 1U_{21} & 1U_{22} \\
2T_{11} & 2T_{12} & -2U_{11} & -2U_{12} \\
2T_{21} & 2T_{22} & -2U_{21} & -2U_{22}
\end{bmatrix}
\begin{bmatrix}
1u_1 \\
1u_2 \\
1t_1 \\
1t_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

(6.9)

The matrix in Eqn. (6.9) consists of all rows of each of the Green’s submatrices and only those columns corresponding to nodes in the ligament.

In the cohesive zone the normal and shear tractions are constant.

\[ 1t_1 = -t_o, \quad 2t_1 = +t_o \]

\[ 1t_2 = -s_o, \quad 2t_2 = +s_o \]
These cohesive zone boundary conditions lead to the linear system of equations in the unknown displacements given by

\[
\begin{bmatrix}
[T_{11}] & [T_{12}] & [0] & [0] & [1u_1] \\
[T_{21}] & [T_{22}] & [0] & [0] & [1u_2] \\
[0] & [0] & [T_{11}] & [T_{12}] & [2u_1] \\
[0] & [0] & [T_{21}] & [T_{22}] & [2u_2]
\end{bmatrix}
= \begin{bmatrix}
-t_o[1U_{11}] - s_o[1U_{12}] \\
-t_o[1U_{21}] - s_o[1U_{22}] \\
t_o[2U_{11}] + s_o[2U_{12}] \\
t_o[2U_{21}] + s_o[2U_{22}]
\end{bmatrix}.
\] (6.10)

The matrix in Eqn.(6.10) consists of all rows of each of the Green’s submatrices and only those columns corresponding to nodes in the cohesive zone.

### 6.3.2 Open crack and boundary \(B_1\) and \(B_2\) (except load points)

The open crack surfaces and the boundaries \(B_1\) and \(B_2\) (except the load points) are traction free.

\[i_{t_1} = i_{t_2} = 0; \ i = 1, 2\]

The linear system of equations in the unknown displacements in this case is

\[
\begin{bmatrix}
[T_{11}] & [T_{12}] & [0] & [0] & [1u_1] \\
[T_{21}] & [T_{22}] & [0] & [0] & [1u_2] \\
[0] & [0] & [T_{11}] & [T_{12}] & [2u_1] \\
[0] & [0] & [T_{21}] & [T_{22}] & [2u_2]
\end{bmatrix}
= \begin{bmatrix}
[0] \\
[0] \\
[0] \\
[0]
\end{bmatrix}.
\] (6.11)

The matrix in Eqn.(6.9) consists of all rows of each of the Green’s submatrices and only those columns corresponding to nodes on the boundary and on the open crack.
6.3.3 Load points

*RU* and *RL* represent the upper and lower load points on the right body, and *LU* and *LL* represent the upper and lower load points on the left body as shown in figure 6.3(a).

- **RL:** \( t_1 = P, \ t_2 = 0 \)
- **RU:** \( t_1 = -P, \ t_2 = 0 \)
- **LU:** \( t_1 = P, \ t_2 = 0 \)
- **LL:** \( t_1 = -P, \ t_2 = 0 \)

These boundary conditions lead to the following system of equations in unknown displacements.

\[
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
1u_1 \\
1u_2
\end{bmatrix}
= -P
\begin{bmatrix}
1U_{11} \\
1U_{21}
\end{bmatrix}
\]

(6.12)

\[
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
1u_1 \\
1u_2
\end{bmatrix}
= P
\begin{bmatrix}
1U_{11} \\
1U_{21}
\end{bmatrix}
\]

(6.13)

\[
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
2u_1 \\
2u_2
\end{bmatrix}
= P
\begin{bmatrix}
2U_{11} \\
2U_{21}
\end{bmatrix}
\]

(6.14)

\[
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
2u_1 \\
2u_2
\end{bmatrix}
= -P
\begin{bmatrix}
2U_{11} \\
2U_{21}
\end{bmatrix}
\]

(6.15)

The matrix in each of Equations (6.12 - 6.15) consists of all rows of each of the Green’s submatrices and only those columns corresponding to the respective node...
of the load point. Equations (6.9 - 6.15) are combined to form a $4N \times 4N$ linear system where each boundary, $B_1$ and $B_2$ is discretized into $N$ elements. In addition to the $4N$ unknown tractions and displacements, the length of the cohesive zone is also unknown. The values of the normal and shear tractions in the cohesive zone that need to be prescribed on the cohesive zone are unknown before the system of equations can be solved. The next section outlines a novel approach to estimate these tractions $t_o$ and $s_o$ before solving the BEM formulation.

### 6.4 Estimates of the normal and shear tractions in the cohesive zone

The approach taken here to estimate the values of the normal and shear cohesive tractions $t_o$ and $s_o$, respectively, for a given far field loading is based on the solutions to the problem of a crack in an infinite homogeneous medium subject to mixed mode loading with Dugdale cohesive zones at the crack tips (Becker and Gross [99] and Nicholson [100]) and the closed form solutions for the mixed mode stress intensity factors at the elastic crack tip for a crack at a bimetallic interface (Sun and Jih [94]).

The solutions of Becker and Gross and Nicholson assume the normal and shear cohesive tractions under mixed mode loading to be proportional to the applied normal and shear tractions to the body. The other relation is obtained by considering the von Mises yield criterion ahead of the crack tip. This idea is made use of for the problem under consideration. If $\sigma_y$ denotes the yield strength of the bond, one of the relations
between \( t_o \) and \( s_o \) can be obtained by applying von Mises criterion as follows.

\[
\sigma_y^2 = \beta t_o^2 + 3s_o^2 \quad \text{Plane strain} \tag{6.16}
\]

\[
\sigma_y^2 = t_o^2 + 3s_o^2 \quad \text{Plane stress} \tag{6.17}
\]

\[
\beta = \begin{cases} 
1 - \nu + \nu^2 & \text{Becker and Gross} \\
(1 - 2\nu)^2 & \text{Nicholson}
\end{cases}
\]

The analysis of Becker and Gross only considers the stress perpendicular to the crack plane at the crack tip, while Nicholson’s approach considers the biaxial nature of the crack tip stress fields by including the stress in the crack plane at the crack tip. Upon applying von Mises criterion using these stress fields the parameter \( \beta \) has a significant constraint effect with increasing Poisson’s ratio in plane strain mixed mode problems considering Nicholson’s approach. This is shown in Figure 5.4.

To obtain another relation between \( t_o \) and \( s_o \) the mixed mode stress intensity factor relations for a crack at a bimetallic interface, given by Sun and Jih [94] are considered. The elastostatic near-tip stress and displacement fields from Sun and Jih [94] for a bimaterial interface crack in an infinite medium are listed in Appendix A. The results required to develop the formulation are listed below.

\[
K_I - iK_{II} = 2\sqrt{2\pi} \ e^{\pi\alpha} (z - a)^{-\frac{1}{2}} \left( \frac{z + a}{z - a} \right)^{i\alpha} \phi_1(z), \tag{6.18}
\]

where \( z = x_1 + ix_2 \), \( \alpha \) is the bi-elastic constant defined as

\[
\alpha = \frac{1}{2\pi} \ln \left[ \frac{\kappa_1}{\mu_1} + \frac{1}{\mu_2} \right] / \left[ \frac{\kappa_2}{\mu_2} + \frac{1}{\mu_1} \right], \tag{6.20}
\]
Figure 6.4: Variation of the parameter $\beta$ with Poisson’s ratio $\nu$ in plane stress and plane strain

$\kappa_j = \begin{cases} 
3 - 4\nu_j, & \text{Plane strain} \\
(3 - \nu_j)/(1 + \nu_j), & \text{Plane stress}
\end{cases}$

and

$\phi_1(z) = \frac{\sigma_{22}^0 - i\sigma_{12}^0}{1 + e^{2\pi\alpha}}(z - 2i\alpha a)(z + a)^{-\frac{1}{2}}(z - a)^{-\frac{1}{2}}(\frac{z + a}{z - a})^{i\alpha}$. \hspace{1cm} (6.21)

Realizing that symmetric and skew symmetric far-field loads are intermixed in expressions for $K_I$ and $K_{II}$,

$K_I = \frac{\sqrt{\pi a} \left[ \sigma_{22}^0 - 2\alpha \sigma_{12}^0 \right]}{\cosh(\alpha\pi)}$, \hspace{1cm} (6.22)
and

\[ K_{II} = \frac{\sqrt{\pi a} \left( \sigma_{12}^0 + 2\alpha \sigma_{22}^0 \right)}{\cosh(\alpha \pi)}. \]  

(6.23)

It should be noted that \( K_I \) and \( K_{II} \) lack a simple physical description as in the homogeneous case. With \( \sigma_{12}^0 = 0 \) and \( \sigma_{22}^0 = \sigma \) (i.e. uniaxial loading), the ratio of \( K_{II} \) to \( K_I \) becomes

\[ \frac{K_{II}}{K_I} = 2\alpha. \]  

(6.24)

Since the mode mixity at cohesive zone initiation should be determined by the elastic crack tip mode-mixity, Eqn.(6.24), we take that as an estimate for the ratio of the shear to the normal cohesive tractions.

\[ \frac{s_o}{t_o} = 2\alpha. \]  

(6.25)

This, combined with one of Equations (6.16) or (6.17) yields the initial estimate of \( t_o \) and \( s_o \) in the BEM formulation, Eqn.(6.10).

### 6.5 Solution scheme

The beam is loaded quasi-statically such that significantly observable cohesive zones are formed ahead of the crack tip. The applied moment per unit thickness is calculated as

\[ \bar{M} = P l e n e, \]  

(6.26)
where $P$ is the applied traction at the load points, $l_e$ is the element length and $n_e$ is the number of elements between the load points.

An automatic iterative solution scheme, similar to the one used in Chapter 3 with additional constraints in the cohesive zone, is employed to obtain the unknown tractions and displacements and the extent of the cohesive zone for a given applied moment and crack length. This scheme also begins with an initial guess of the number of elements that constitute the extent of the cohesive zone. The evaluated normal traction value in an element of the ligament nearest to the cohesive zone is then compared with $t_o$ and the evaluated shear traction value in the same element is compared with $s_o$. Iterations are performed on the number of elements in the cohesive zone until the value of the normal traction in the first element of the ligament, nearest to the cohesive zone, is less than $t_o$ and the value of the shear traction in the same element is less than $s_o$. The other more important constraint for the solution is that in the open crack there is no interpenetration of material. The solution for displacements is checked for this condition.

### 6.6 Sample Results

For a given material mismatch the direction of the shear traction that has been prescribed in the BEM formulation with a Dugdale-Barenblatt cohesive zone is obtained from the linear elastic solution to the same problem. The elastic solution for the same geometry, loading and crack length is obtained from a similar boundary element code [55]. An example of the elastic solution for the tractions and displacements in the two bodies at the bimetallic interface is shown in figure 6.5. The elastic moduli of the two bodies are $\mu_1 = 28,000$ MPa, $\nu_1 = 0.3$, $\mu_2 = 14,000$ MPa and $\nu_2 = 0.3$. Considering the height of the cross section $W$ to be 12.5 mm, the ratio $a/W$ is maintained at 0.28 with 100 elements in the interface. The applied moment $M$
is 732.42 Nm/m. It can be seen from figures 6.5(a) and 6.5(b) that the tractions are singular at the crack tip. From figures 6.5(c) and 6.5(d) it can be seen that the displacements in the open crack do not show any interpenetration. This is attributed to the choice of the mesh, whereby the zone of oscillatory solution as predicted by Williams is less than one element and the current mesh cannot capture this effect.

Figure 6.6 shows the solution for the tractions and displacements at the interface for the Dugdale-Barenblatt cohesive zone for the same material mismatch considered in figure 6.5, the same applied moment $\bar{M}$ and the same crack length $a/W$. The yield strength of the bond is considered to be 50 MPa. Using the two conditions discussed in Section 6.4 the estimates for the cohesive tractions in tension and shear are $t_o = 49.59$ MPa and $s_o = 3.70$ MPa. The automatic iterative scheme checks for all the constraints surrounding the cohesive zone, the most important one being the removal of the physically inadmissible interpenetration of the crack flanks near the crack tip. The solutions for the displacements $u_1$ and $u_2$ in the cohesive zone shows that there is indeed no interpenetration.

For the applied loading discussed the two conditions that give an estimate of the cohesive tractions are the von Mises criterion applied to the interface and the mixed mode elastic stress intensity factor relations which under conditions of plane stress yields

$$\sigma_y^2 = t_o^2 + 3s_o^2 \quad \text{and}$$

$$\frac{s_o}{t_o} = 2\alpha.$$

Admissible solutions without contact of the crack faces were indeed found when $s_o$ and $t_o$ are chosen this way. However, in order to determine if $2\alpha$ is the only possible choice for $\frac{s_o}{t_o}$ that will satisfy the no contact conditions on the crack faces, the ratio $\frac{s_o}{t_o}$ was varied around $2\alpha$. If the ratio is decreased as little as 9.5% then the solutions
show interpenetration of the crack faces which states that the crack faces are indeed in contact for ratios less than $2\alpha$. A new boundary value problem which accounts for contact and includes some sort of friction assumption would have to be posed and solved to find the extent of contact and the actual deformed state of the interface in this case. The present BEM code could easily be extended to handle this. Now, if $\frac{s}{l_0}$ is taken to be anywhere from $2\alpha$ to 12% greater than $2\alpha$, a no contact solution using the present code is obtained. Upon further increase of the mixity ratio the physically inadmissible crack face displacements found using the present BEM code predict again that crack face contact must occur. Over the range of ratios for which a no contact solution was obtained, the value of $J$ varied only by 2%. Also some of this range of mixity ratios could be due to numerical error. While, the crack tip can be exactly located with respect to the mesh at all times, the length of the cohesive zone can only change discretely, one element at a time. For instance if the actual difference in cohesive zone length is, say one quarter of an element length for two different mixity ratios, then the effect of the mixity ratio on this scale cannot be determined accurately. That and the fact that both traction and displacement have constant shape functions indicate that it is quite likely that the actual range of mode mixity ratios $\frac{s}{l_0}$ for which no contact solutions exist is quite narrow and very close to the value $2\alpha$. 
Figure 6.5: Linear elastic solution for the tractions and displacements in the crack plane. $\mu_1 = 28,000$ MPa, $\nu_1 = 0.3$, $\mu_2 = 14,000$ MPa and $\nu_2 = 0.3$. $a/W = 0.28$ and $\bar{M}$ is 732.42 Nm/m
Figure 6.6: An example solution for the tractions and displacements in the crack plane for the Dugdale-Barenblatt cohesive zone model. $\mu_1 = 28,000$ MPa, $\nu_1 = 0.3$, $\mu_2 = 14,000$ MPa and $\nu_2 = 0.3$. $a/W = 0.28$ and $M$ is 732.42 Nm/m. The yield strength of the bond is $\sigma_y = 50$ MPa.
CHAPTER 7

NUMERICAL EVALUATION OF THE \( J \)-INTEGRAL

7.1 Numerical analysis to evaluate the \( J \)-integral

The displacements at every element in the \( N \) elements that make up the cohesive zone in each body \( (u_{11}, u_{12}, u_{21} \text{ and } u_{22}) \) at the bimetallic interface are used to generate interpolation polynomial functions for \( x_2 \) of order \( N-1 \) using Newton’s method as

\[
\begin{align*}
  u_{11,N-1}(x_2) &= \sum_{i=1}^{N} u_{11}[x_{21}, \ldots, x_{2i}] \prod_{j=1}^{i-1} (x_2 - x_{2j}), \\
  u_{12,N-1}(x_2) &= \sum_{i=1}^{N} u_{12}[x_{21}, \ldots, x_{2i}] \prod_{j=1}^{i-1} (x_2 - x_{2j}), \\
  u_{21,N-1}(x_2) &= \sum_{i=1}^{N} u_{21}[x_{21}, \ldots, x_{2i}] \prod_{j=1}^{i-1} (x_2 - x_{2j}), \quad \text{and} \\
  u_{22,N-1}(x_2) &= \sum_{i=1}^{N} u_{22}[x_{21}, \ldots, x_{2i}] \prod_{j=1}^{i-1} (x_2 - x_{2j}).
\end{align*}
\]

The coefficients of these interpolating polynomials are obtained using divided differences in the standard way and are denoted by \( u_{11}[x_{21}, \ldots, x_{2i}], u_{12}[x_{21}, \ldots, x_{2i}], u_{21}[x_{21}, \ldots, x_{2i}] \) and \( u_{22}[x_{21}, \ldots, x_{2i}] \) [56]. Using these interpolation polynomials several data points are now created for each of the displacement components at several values of \( x_{2n} \) within
the range of the coordinate $x_2$ in the cohesive zone. The derivatives of each dis-
placement component with respect to $x_2$ is obtained at each data point using central
differences as

$$\frac{\partial u_{11}}{\partial x_2} = \frac{u_{11,n+1} - u_{11,n-1}}{2h_x}, \quad (7.5)$$

$$\frac{\partial u_{12}}{\partial x_2} = \frac{u_{12,n+1} - u_{12,n-1}}{2h_x}, \quad (7.6)$$

$$\frac{\partial u_{21}}{\partial x_2} = \frac{u_{21,n+1} - u_{21,n-1}}{2h_x}, \quad (7.7)$$

$$\frac{\partial u_{22}}{\partial x_2} = \frac{u_{22,n+1} - u_{22,n-1}}{2h_x}. \quad (7.8)$$

$h_x$ denotes the increment in $x_2$. The integrals of these gradients for substitution in
Eqn. (6.5) are then evaluated using the trapezoidal rule as

$$-\frac{W}{t}^{+c} \int t_o \frac{\partial u_{11}}{\partial x_2} dx_2 = t_o \sum_{i=1}^{n} \frac{\partial u_{11}}{\partial x_2,i} + \frac{\partial u_{11}}{\partial x_2,i+1}{2} h_x, \quad (7.9)$$

$$-\frac{W}{s}^{+c} \int s_o \frac{\partial u_{12}}{\partial x_2} dx_2 = s_o \sum_{i=1}^{n} \frac{\partial u_{12}}{\partial x_2,i} + \frac{\partial u_{12}}{\partial x_2,i+1}{2} h_x, \quad (7.10)$$

$$-\frac{W}{t}^{+c} \int t_o \frac{\partial u_{21}}{\partial x_2} dx_2 = t_o \sum_{i=1}^{n} \frac{\partial u_{21}}{\partial x_2,i} + \frac{\partial u_{21}}{\partial x_2,i+1}{2} h_x, \quad (7.11)$$

$$-\frac{W}{s}^{+c} \int s_o \frac{\partial u_{22}}{\partial x_2} dx_2 = s_o \sum_{i=1}^{n} \frac{\partial u_{22}}{\partial x_2,i} + \frac{\partial u_{22}}{\partial x_2,i+1}{2} h_x. \quad (7.12)$$

Figure 7.1 shows the application of the numerical analysis to the evaluation of
the $J$-integral for the result discussed in Chapter 6, figure 6.6. The elastic moduli
of the two bodies are $\mu_1 = 28,000$ MPa, $\nu_1 = 0.3$, $\mu_2 = 14,000$ MPa and $\nu_2 = 0.3$.
Considering the height of the cross section $W$ to be 12.5 mm, the ratio $a/W$ is
maintained at 0.28 with 100 elements in the interface. The applied moment $\bar{M}$ is $732.42 \, Nm/m$ and the solution yields 10 elements in the cohesive zone. This BEM solution for the displacements in each of the two bodies in the cohesive zone is shown in Figures 7.1(a) and 7.1(b). The interpolated polynomials for the displacements are shown in Figures 7.1(c) and 7.1(d) and the gradients of the displacements are shown in Figures 7.1(e) and 7.1(f).

### 7.2 Effect of elastic mismatch and yield strength of the bond

A parameter variation study exploring the effect of elastic mismatch and the yield strength of the bond is now discussed. Two kinds of variations of the elastic mismatch are considered at a given crack length of $a/W = 0.28$ and applied moment $\bar{M} = 732.42 \, Nm/m$. In the first variation the elastic moduli of the right half of the beam are maintained constant at $\mu_1 = 28,000 \, MPa$ and $\nu_1 = 0.3$. The Poisson’s ratio of the left half of the beam is also maintained constant at $\nu_2 = 0.3$, and $\mu_2$ is varied between 28,000 $MPa$ and 2,800 $MPa$. The yield strength of the interface bond is maintained constant at 50 $MPa$. For every combination of $\mu_1$ and $\mu_2$, the values of $t_o$ and $s_o$ that need to be prescribed in the cohesive zone are calculated using Equations 6.17 and 6.25 for the case of plane stress, and the quantities $J_I$, $J_{II}$ and the absolute value of the $J$-integral are evaluated. The crack tip opening displacement $\delta_t$ and the crack tip shear displacement $\delta_s$ are also evaluated. The variation of the quantities $J_{II}/J_I$ and $\delta_s/\delta_t$ with $\mu_1/\mu_2$ is shown in Figure 7.2(a), while the variation of the absolute value of $J$ with $\mu_1/\mu_2$ is shown in Figure 7.2(b). Since the effective elastic modulus ($\mu^*$) of the bimetallic system is chosen to increase, the interface becomes more compliant and the results show an increasing trend with increase in elastic
mismatch ratio. The effective elastic modulus of the system be written as [95]

$$\frac{1}{\mu^*} = \frac{1}{2} \left[ \frac{1}{\mu_1} + \frac{1}{\mu_2} \right].$$

(7.13)

The size of the cohesive zone however is seen to remain constant. The quantity $n_c/n_{crack}$ that is the ratio of the number of elements in the cohesive zone $n_c$ to the number of elements in the open crack $n_{crack}$ remains 0.36 even though the absolute value of $J$ increases. As seen in Figure 7.2(a) this is due to the increasing contribution of $J_{II}$.

In the second parameter variation the effective elastic modulus of the system $\mu^*$ is maintained constant at 11,200 MPa, and the elastic moduli ratio $\mu_1/\mu_2$ is varied between 2 and 10. This means that the value of $\mu_1$ is not maintained constant. The crack length is maintained at $a/W = 0.28$ and the applied moment is still maintained at $M = 732.42 Nm/m$. In this case the quantities $J_{II}/J_I$ and $\delta_s/\delta_t$ are seen to follow an exactly similar increasing trend with $\mu_1/\mu_2$ as in the case of an increasing effective elastic modulus $\mu^*$, as shown in Figure 7.3(a). The absolute value of $J$ however is seen to almost remain constant around 85 Nm/m, as shown in Figure 7.3(b). The size of the cohesive zone also remains constant with the ratio $n_c/n_{crack} = 0.36$.

The effect of variation of the yield strength of the interface for a given elastic mismatch is now considered. The crack length is maintained constant at $a/W = 0.3$, the applied moment $M = 662.1 Nm/m$ and the values of the elastic moduli are $\mu_1 = 28,000$ MPa, $\nu_1 = 0.3$, $\mu_2 = 14,000$ MPa and $\nu_2 = 0.3$. The values of the cohesive tractions $t_o$ and $s_o$ are once again evaluated using Equations 6.17 and 6.25 under conditions of plane stress. The results showing the variation of $J_{II}/J_I$ and $\delta_s/\delta_t$ with increasing yield strength $\sigma_y$ are shown in Figure 7.4(a). Both these quantities show an increasing trend with $\sigma_y$. Figure 7.4(b) shows the variation of the absolute value of $J$ with $\sigma_y$. $J$ is seen to increase with increasing yield strength and the size of
the cohesive zone is seen to decrease with increasing yield strength. The quantity $n_c/n_{crack}$ is seen to decrease from 0.5 to 0.1 with an increase in yield strength from 35 MPa to 75 MPa as shown in Figure 7.4(c). The length of the cohesive zone is about half the length of the open crack at a yield strength of 35 MPa. This is a significantly large cohesive zone for conditions of small scale yielding to apply. However at a yield strength of 75 MPa the length of the cohesive zone is about one tenth of the length of the crack, which can very well represent conditions of small scale yielding. With the current mesh size and number of elements, the solutions at very low yield strengths could not be obtained as this would lead to the size of the cohesive zone being very large. The solutions at yield strengths greater than 75 MPa the solutions are not reported because the cohesive zone contains only 2-3 elements. This would lead to insufficient input and high numerical errors in the interpolation scheme to evaluate $J$. 


Figure 7.1: Application of the numerical scheme to evaluate the $J$-integral.
Figure 7.2: (a) Variation of $J_{II}/J_I$ and $\delta_s/\delta_t$ with the ratio of elastic moduli for increasing effective elastic modulus. (b) Variation of the absolute value of $J$ with the ratio of elastic moduli for increasing effective elastic modulus.
Figure 7.3: (a) Variation of $J_{II}/J_I$ and $\delta_s/\delta_t$ with the ratio of elastic moduli for constant effective elastic modulus. (b) Variation of the absolute value of $J$ with the ratio of elastic moduli for constant effective elastic modulus.
Figure 7.4: (a) Variation of $J_{II}/J_I$ and $\delta_s/\delta_t$ with the yield strength of the bond. (b) Variation of the absolute value of $J$ with the yield strength of the bond. (c) Variation in the length of the cohesive zone with the yield strength of the bond.
8.1 Cohesive interface between a layer and a rigid substrate

Figure 8.1: An elastic layer of thickness $2h$ connected to a rigid substrate through a thin interface layer

Figure 8.1 shows an elastic layer of thickness $2h$ on a rigid substrate and the $(x_1,x_2)$ coordinate system. The interface between the layer and the substrate is considered to be cohesive and softening. One extreme example of a softening cohesive law is the linear softening law given in Eqn.(8.1) and shown in figure 8.2(a). This assumes that any displacement jump built up during the loading phase of the zone is
negligible compared to the displacement jumps which occur during softening.

\[ \sigma = \sigma_c \left[ 1 - \frac{\delta}{\delta_c} \right] \]  

(8.1)

Here \( \delta_c \) represents the critical value of the displacement jump at rupture. The original Dugdale-Barenblatt assumption of constant stress in the cohesive zone is obtained as a limit of the linear softening law for large \( \delta_c \). Another example of softening behavior, shown in figure 8.2(b), includes a loading phase and is given by the relation

\[ \sigma = e \sigma_c \left[ \frac{\delta}{\delta_c} \right] e^{-\frac{\delta}{\delta_c}} \]  

(8.2)

Here \( \delta_c \) represents the critical displacement jump at which softening begins to occur

\[ e = \exp(1) \]

and \( e = \exp(1) \). Note that theoretically this curve never reaches zero stress, but in practice the tail may be cut off at a suitably small stress and the displacement jump
at which rupture occurs can be approximated. The critical energies required to cause failure are the areas under the curves in Figures 8.2(a) and 8.2(b), denoted by $J_c$ and given by

$$J_c = \frac{1}{2} \sigma_c \delta_c \quad (8.3)$$

and

$$J_c = e \sigma_c \delta_c \quad (8.4)$$

respectively. Both of these cohesive laws may be used in a normal stress setting or in a shear setting. Whatever the cohesive behavior, even if no explicit form is assumed, it is assumed in the linear elastodynamic setting discussed here that the interface is to be pre-loaded statically to a point $(\sigma_o, \delta_o)$ on the cohesive law. This is illustrated in Figure 8.2(b), but applies to any law. The small displacement dynamic disturbance superposed on this pre-loaded state is assumed to invoke linear interface behavior along the tangent to the cohesive law as shown.

Only anti-plane motions are considered for the layer problem in which the only non-zero displacement and stress components are

$$u_3(x_1, x_x, t), \quad \tau_{13}(x_1, x_x, t) = \mu \frac{\partial u_3}{\partial x_1}, \quad \tau_{23}(x_1, x - x, t) = \mu \frac{\partial u_3}{\partial x_2} \quad (8.5)$$

The displacement equation of motion from Achenbach [136] is

$$\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} = \frac{1}{c_T^2} \frac{\partial^2 u_3}{\partial t^2}, \quad (8.6)$$

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where \( c_T = \sqrt{\frac{\mu}{\rho}} \) is the transverse or shear wave speed of the layer. Following the traditional analysis for plane waves travelling down a layer [ ], the out-of-plane displacement \( u_3 \) is taken in the following form.

\[
 u_3(x_1, x_2, t) = \bar{u}_3(x_2)e^{ik(x_1-ct)}, \quad \bar{u}_3(x_2) = [B_1\sin(qx_2) + B_2\cos(qx_2)]
\]  

(8.7)

The stress component \( \tau_{23} \) then becomes

\[
 \tau_{23}(x_1, x_2, t) = \bar{\tau}_{23}(x_2)e^{ik(x_1-ct)}, \quad \bar{\tau}_{23}(x_2) = \mu q[B_1\cos(qx_2) - B_2\sin(qx_2)],
\]  

(8.8)

where \( c \) is the undetermined speed of the plane wave, \( k \) is the wave number and the frequency is \( \omega = kc \). The quantity \( q \) is determined by substituting this displacement form into Eqn.(8.6) as

\[
 q = \sqrt{\frac{\omega^2}{c_T^2} - k^2} = k \sqrt{\frac{c^2}{c_T^2} - 1}.
\]  

(8.9)

A linearization of the cohesive law, \( \sigma = f(\delta) \), at the interface between the layer and the rigid substrate \( (x_2 = -h) \), where \( \sigma \) and \( \delta \) now represent the amplitudes of the interface shear stress and displacement, respectively:

\[
 \sigma = [\bar{\tau}_{23}]_{x_2=-h}, \quad \delta = [\bar{u}_3]_{x_2=-h}.
\]  

(8.10)

In this situation it is assumed that the interface is loaded statically \( (\sigma_o) \) so that the layer is sheared an amount \( (\delta_o) \) as illustrated in figure 8.2b. The small displacement wave, Eqn. (8.7), is assumed to be superposed on the static pre-stress and the dynamic cycling of the interface is assumed to occur such that points \( (\delta, \sigma) \) lie along the tangent to the cohesive law as shown. Letting \( K = \sigma_o'\delta_o \) this gives the dynamic
boundary conditions

\[
\tau_{23} = 0, \quad x_2 = +h, \quad (8.11)
\]

\[
\tau_{23} = K \ddot{u}_3, \quad x_2 = -h, \quad (8.12)
\]

where \( K \) can either be positive (elastic spring behavior) or negative (softening behavior). Equations 8.7, 8.11 and 8.12 yield the frequency equation

\[
2\beta(qh)\sin(qh) - \cos(qh) + \sin(qh)\tan(qh) = 0, \quad (8.13)
\]

where the controlling parameter is

\[
\beta = \frac{\mu}{Kh} \quad (8.14)
\]

which may be thought of as a nondimensional interface compliance. Note that the limiting frequency equations for the well studied free-fixed and free-free layers are obtained by letting \( |\beta| \) go to zero or infinity, respectively. For the free-fixed case (\( \beta = 0 \)) the roots are

\[
\frac{2(qh)_n}{\pi} = \frac{n}{2}, \quad n = 1, 3, 5, \ldots \quad (8.15)
\]

while for the free-free case the roots are

\[
\frac{2(qh)_n}{\pi} = n, \quad n = 0, 1, 2, 3, \ldots \quad (8.16)
\]

For a given \( \beta \), Eqn.(8.13) must be solved numerically to obtain the roots \( (qh)_n \). The non-dimensional dispersion curves are obtained from the definition of \( q \) in Eqn.
(6.5) and using the standard nondimensional frequency and wave-number, \( \Omega \) and \( \zeta \), respectively,

\[
\Omega = \frac{2h\omega}{\pi c_T}, \quad \zeta = \frac{2kh}{\pi} = \frac{2h\omega}{\pi c} \tag{8.17}
\]

which yields

\[
\Omega_n^2 = \zeta_n^2 + \left[ \frac{2(qh)_n}{\pi} \right]^2. \tag{8.18}
\]

### 8.2 Results

For real wave numbers, corresponding to waves which do not attenuate in \( x_1 \), plotting \( \Omega_n \) versus \( \zeta_n \) gives hyperbolas with the smallest value of \( \Omega_n \) always occurring at \( \zeta_n = 0 \). This smallest value, is the nondimensional cutoff frequency

\[
\Omega_{nc}^2 = \left[ \frac{2(qh)_n}{\pi} \right]^2 \tag{8.19}
\]

and it will uniquely depend on interface compliance \( \beta \). The first four branches of the frequency spectrum for \( \beta = 1 \) are shown in Figure 8.3.

Recognizing that the first four free-fixed (\( \beta = 0 \)) branches would be coming from cutoff frequencies (1/2, 3/2, 5/2, 7/2) and the branches shown are seen to be increasingly shifted down by increased flexibility of the interface at \( \beta = 1 \) compared to \( \beta = 0 \). Since the shapes of the branches are always hyperbolas it suffices here to study the dependence of the cutoff frequencies only on the interface parameter. This is shown in Figure 8.4 over a wide range of both positive and negative values of \( \beta \). For positive values of \( \beta \) (Figure 8.4(a)), as the interface stiffness decreases from infinity to zero the \( n^{\text{th}} \) mode cutoff frequency decreases from the \( n^{\text{th}} \) mode free-fixed value to
the \((n-1)\)th mode free-free value. As expected the frequency decreases as the interface stiffness decreases \((\beta \text{ increases})\). The opposite is true for negative softening values of \(\beta\) as shown in Figure 8.4(b) where the cutoff frequencies increase from the \(n\)th mode free-fixed value to the \(n\)th mode free-free value. This clear distinction between an elastic spring and a softening spring should provide an indication of whether softening behavior is being experienced by the small amplitude dynamic displacements and stresses.

As a preliminary study into the effects of positive versus negative stiffnesses in the more complicated problem of in-plane wave motions and how they modify the well-known Rayleigh-Lamb frequency spectrum for the free-free layer [136] has also been carried out. The branches (modes) of the Rayleigh-Lamb frequency spectrum are not simple hyperbolas, but instead are complicated functions of the non-dimensional
wave number. The mapping out of these branches is an exhaustive process, but the cutoff frequencies may be calculated with the same ease as in the SH problem. The in-plane guided wave consists of both longitudinal and vertically polarized shear waves and is divided into symmetric and antisymmetric parts about the midplane of the layer. The stress free boundary conditions at the top yield the same symmetric modes as in the Rayleigh-Lamb spectrum and the spring boundary conditions at the bottom affect only the anti-symmetric modes. Concentrating only on the anti-symmetric modes sensitive to a shear spring the boundary conditions taken at the layer substrate interface are zero normal displacement and a shear spring condition as in Equations (8.11 and 8.12) except between the in-plane shear stress $\tau_{12}$ at the interface and the in-plane tangential displacement $u_1$ at the interface. Referring to Achenbach [ ] for the details, the following anti-symmetric mode frequency equation is obtained.

$$\beta \Omega^2 \frac{\pi}{2} \sqrt{\frac{\Omega^2}{\omega^2} - \zeta^2 + \zeta^2 \tan \left[ \frac{\pi}{2} \sqrt{\frac{\Omega^2}{\omega^2} - \zeta^2} \right]} + \sqrt{\frac{\Omega^2}{\omega^2} - \zeta^2} \sqrt{\Omega^2 - \zeta^2} \tan \left[ \frac{\pi}{2} \sqrt{\Omega^2 - \zeta^2} \right] = 0$$

(8.20)

where all quantities are defined above except for $\bar{\kappa} = \frac{c_L}{c_t}$, where $c_L$ is the longitudinal wave speed given by

$$c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}.$$

$\lambda$ represents the Lame constant and not compliance. The equation for determining the cutoff frequencies $\Omega_c$ is found by setting $\zeta = 0$ in Eqn.(8.20).

$$\beta \frac{\pi}{2} \Omega_c + \tan \left[ \frac{\pi}{2} \Omega_c \right]$$

(8.21)
These frequencies are plotted versus interface compliance $\beta$ in figure 8.5. These results are very similar to the SH results in Figure 8.4. The limiting values are the cutoff frequencies of the free-free and free-fixed modes associated with shear waves only. The free-free and free-fixed modes associated with the longitudinal waves do not appear because of the zero normal displacement boundary condition. A marked difference in the results in figures 8.4 and 8.5 is the appearance of new mode for values of negative greater $\beta$ in absolute value than 1. This mode becomes the first free-free Rayleigh Lamb shear mode at large $|\beta|$.

Either of the results in Figures 8.4 and 8.5 and knowledge of the entire in-plane spectrum show promise for easily identifying the parameter $\beta$. Repeated determinations of this parameter from multiple measurements at different static pre-load levels will reveal the slopes of the nonlinear cohesive law at the different load-levels. These slope and load levels can then be integrated to obtain the measured cohesive law, Figure 8.2(b).
Figure 8.4: Cutoff frequencies of the first three modes of $SH$ layer versus (a) positive non-dimensional interface compliance $\beta$ and (b) absolute value of negative values of interface compliance.
Figure 8.5: Cutoff frequencies of generalized Rayleigh-Lamb waves versus (a) positive non-dimensional interface compliance $\beta$ (1st three modes) and (b) absolute value of negative values of interface compliance (1st four modes).
CHAPTER 9

CONCLUSION

The research described in this dissertation has addressed three kinds of problems that involve cohesive behaviour either at the crack tip or at the interface between two materials. In the first problem a BEM formulation for a linear softening cohesive zone problem in an edge cracked homogeneous beam like geometry which requires an iterative process to determine the length of the cohesive zone to satisfy the non-linear interfacial boundary conditions has been presented. The behaviour of the cohesive zone with respect to the applied load, the peak cohesive traction $t_o$, critical crack opening displacement $\delta_o$ and crack length $a/W$ has been examined for non-propagating cracks. The iterative boundary element solution scheme is robust, as the exact extent of the cohesive zone along with the solution for tractions and displacements are generally obtained within 3-6 iterations with an arbitrary initial guess. For a specific linear softening cohesive law applied to the edge cracked beam shaped geometry the variation of the $J$-integral calculated for various values of applied load and crack lengths $a/W$ is used to develop the algorithm for generating the non-linear $M-\Theta$ curves for the cracked element. These results have potential application in the characterization of the cohesive behaviour ahead of a crack tip in vibration analysis. The non-linear $M - \Theta$ curves are the key input to the vibration analysis used for the characterization. Calculation of frequencies and mode-shapes of vibration of a cracked beam with a linear softening crack tip cohesive zone will be discussed elsewhere.
The second problem involves modeling of a Dugdale-Barenblatt cohesive zone ahead of the crack tip in a bimaterial interface using the BEM approach. A novel approach of applying the cohesive tractions that provides a physically admissible solution for tractions and displacements is presented. The local mode mixity ahead of the crack tip arising due to the material mismatch is explored for a certain far field applied loading. The effect of varying bond strength is also examined. The fracture energy of the interface given by the mixed mode $J$-integral is evaluated numerically. This evaluated value of the $J$-integral is clearly able to provide the separate contributions of the mode-I and mode-II fracture energies.

In the third problem considered the use of linear elastodynamic response of an elastic layer with an interface to infer the nonlinear interface cohesive behavior is presented. The frequency response involves dispersion of guided elastic waves in a layer connected to a substrate by a nonlinear interface. This method is based on the concept that when small amplitude dynamic disturbances interact with a statically pre-loaded interface in the nonlinear range the dynamic interface behavior is approximated by linear spring-like response with the stiffness taken to be the slope of the nonlinear interface stress-displacement law. Based on this concept, calculations show that the frequency spectra of the layer are sensitive to the nonlinear properties of the interface. In particular hardening and softening spring like behaviour are easily distinguished, which could, through a series of measurements at various static pre-loads, allow an entire interface law to be inferred from the frequency spectra at various loads.
APPENDIX A

Elastostatic near-tip stress and displacement fields of a bimaterial interface crack

The explicit form of the asymptotic stress and displacement components can be written as:

\[(\sigma_{11})_j = \frac{K_I}{2\sqrt{2\pi r}} \left[ \omega_j f_{11}^I - \frac{1}{\omega_j} \cos(\theta - \bar{\Theta}) \right] - \frac{K_{II}}{2\sqrt{2\pi r}} \left[ \omega_j f_{11}^{II} + \frac{1}{\omega_j} \sin(\theta - \bar{\Theta}) \right] \] (1)

\[(\sigma_{22})_j = \frac{K_I}{2\sqrt{2\pi r}} \left[ \omega_j f_{22}^I + \frac{1}{\omega_j} \cos(\theta - \bar{\Theta}) \right] - \frac{K_{II}}{2\sqrt{2\pi r}} \left[ \omega_j f_{22}^{II} - \frac{1}{\omega_j} \sin(\theta - \bar{\Theta}) \right] \] (2)
\((\sigma_{12})_j = \frac{K_I}{2\sqrt{2\pi r}} \left[ \omega_j f_{12}^{I} - \frac{1}{\omega_j} \sin(\theta - \bar{\Theta}) \right] - \frac{K_{II}}{2\sqrt{2\pi r}} \left[ \omega_j f_{12}^{II} - \frac{1}{\omega_j} \cos(\theta - \bar{\Theta}) \right] \) (3)

\((u_1)_j = \frac{K_I\sqrt{2\pi r}}{4\pi \mu_j} \left[ \kappa_j \omega_j h_{11} - \frac{1}{\omega_j} h_{12} + \omega_j h_{13} \right] + \frac{K_{II}\sqrt{2\pi r}}{4\pi \mu_j} \left[ \kappa_j \omega_j h_{21} - \frac{1}{\omega_j} h_{22} + \omega_j h_{23} \right] \) (4)

\((u_2)_j = \frac{K_I\sqrt{2\pi r}}{4\pi \mu_j} \left[ \kappa_j \omega_j h_{21} - \frac{1}{\omega_j} h_{22} - \omega_j h_{23} \right] + \frac{K_{II}\sqrt{2\pi r}}{4\pi \mu_j} \left[ -\kappa_j \omega_j h_{11} + \frac{1}{\omega_j} h_{12} + \omega_j h_{13} \right] \) (5)

where

\[ \alpha = \frac{1}{2\pi} \ln \left[ \left( \frac{\kappa_1}{\mu_1} + \frac{1}{\mu_2} \right) / \left( \frac{\kappa_2}{\mu_2} + \frac{1}{\mu_1} \right) \right] \] (6)

\[ \bar{\Theta} = \alpha \ln \left( \frac{r}{2\alpha} \right) + \frac{\theta}{2} \] (7)

\[ \kappa_j = \begin{cases} 3 - 4\nu_j, & \text{Plane strain} \\ (3 - \nu_j)/(1 + \nu_j), & \text{Plane stress} \end{cases} \]

\[ \mu = \text{shear modulus} \]

\[ \nu = \text{Poisson's ratio} \]

\[ \omega_1 = e^{-\alpha(\pi - \theta)} \] (8)

\[ \omega_2 = e^{\alpha(\pi + \theta)} \] (9)

\[ f_{11}^{I} = 3\cos \bar{\Theta} + 2 \alpha \sin \theta \cos (\theta + \bar{\Theta}) - \sin \theta \sin (\theta + \bar{\Theta}) \] (10)
\[ f_{11}^{II} = 3\cos \Theta + 2\alpha \sin \theta \sin (\theta + \Theta) + \sin \theta \cos (\theta + \bar{\theta}) \quad (11) \]

\[ f_{22}^{I} = \cos \Theta - 2\alpha \sin \theta \cos (\theta + \Theta) + \sin \theta \sin (\theta + \bar{\theta}) \quad (12) \]

\[ f_{22}^{II} = \sin \Theta - 2\alpha \sin \theta \sin (\theta + \Theta) - \sin \theta \cos (\theta + \bar{\theta}) \quad (13) \]

\[ f_{12}^{I} = \sin \Theta + 2\alpha \sin \theta \sin (\theta + \Theta) + \sin \theta \cos (\theta + \bar{\theta}) \quad (14) \]

\[ h_{11} = \frac{1}{1 + 4\alpha^2} \left[ \cos (\theta - \bar{\theta}) - 2\alpha \sin (\theta - \bar{\theta}) \right] \quad (15) \]

\[ h_{12} = \frac{1}{1 + 4\alpha^2} \left[ \cos \bar{\theta} + 2\alpha \sin \bar{\theta} \right] \quad (16) \]

\[ h_{13} = \sin \theta \sin \bar{\theta} \quad (17) \]

\[ h_{21} = \frac{1}{1 + 4\alpha^2} \left[ \sin (\theta - \bar{\theta}) + 2\alpha \cos (\theta - \bar{\theta}) \right] \quad (18) \]

\[ h_{22} = \frac{1}{1 + 4\alpha^2} [-\sin \theta + 2\alpha \cos \theta] \quad (19) \]

\[ h_{23} = \sin \theta \cos \bar{\theta} \quad (20) \]

The complex stress functions can readily be obtained by the Hilbert formulation. The first Goursat function \( \Phi_1 \) in the vicinity of the crack tip \( z = a \) can be written as

\[ \phi_1(z) = \frac{\sigma_{22} - i\sigma_{12}^o}{1 + e^{2\pi\alpha}} (z - 2i\alpha a)(z + a)^{-\frac{1}{2}} (z - a)^{-i\alpha} \left( \frac{z + a}{z - a} \right)^{i\alpha}. \quad (21) \]

Removing the singularity \( (z - a)^{-\frac{1}{2}} \) and \( (z - a)^{-i\alpha} \), Rice and Sih \[ \] define the stress intensity factor \( k = k_1 - ik_2 \) as:

\[ k_1 - ik_2 = 2\sqrt{2} e^{\pi\alpha} \text{lim} (z - a)^{\frac{1}{2}} (z - a)^{i\alpha} \Phi_1(z) \quad (22) \]
or

\[
k_1 = \sqrt{\frac{a}{\cosh(\alpha \pi)}} \left( \sigma_{22}^0 \cos(\alpha \ln 2\alpha) + 2\alpha \sin(\alpha \ln 2\alpha) \right) + \left( \sigma_{12}^0 \sin(\alpha \ln 2\alpha) - 2\alpha \cos(\alpha \ln 2\alpha) \right)
\]

(23)

\[
k_2 = \sqrt{\frac{a}{\cosh(\alpha \pi)}} \left( \sigma_{12}^0 \cos(\alpha \ln 2\alpha) + 2\alpha \sin(\alpha \ln 2\alpha) \right) - \left( \sigma_{22}^0 \sin(\alpha \ln 2\alpha) - 2\alpha \cos(\alpha \ln 2\alpha) \right)
\]

(24)

In this definition the ‘ln’ terms have a dimension of length, and consequently, the stress intensity factors become functions of the measuring unit of the crack length.

On the other hand, if the stress intensity factors are defined as

\[
K_I - iK_{II} = 2\sqrt{2\pi} e^{\pi \alpha} \lim_{z \to \pm a} \left( \frac{z - a}{z + a} \right)^{i\alpha} \Phi_1(z)
\]

(25)

then,

\[
K_I = \sqrt{\frac{\pi a}{\cosh(\alpha \pi)}} \left[ \sigma_{22}^0 - 2\alpha \sigma_{12}^0 \right],
\]

(26)

\[
K_{II} = \sqrt{\frac{\pi a}{\cosh(\alpha \pi)}} \left[ \sigma_{12}^0 + 2\alpha \sigma_{22}^0 \right],
\]

(27)

and the ambiguity of dimension is removed.
BIBLIOGRAPHY


