PRICING IN THE ACTUARIAL MARKET

DISSERTATION

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* * * * *

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ABSTRACT

In contrast to the price of derivative securities in a complete financial market, which can be uniquely determined by constructing a portfolio that hedges the payoff, the price of insurance risk typically is not unique. This is because the price of insurance risk is also determined by supply and demand for insurance products in the market. No mathematical model to date has fully captured the interactions between insurance markets and financial markets that characterize modern corporate practice.

Young and Zariphopoulou (2001) introduced an expected utility approach to price dynamic insurance risk. Their valuation is based on comparing maximal expected utility under two scenarios: with and without incorporating the sale of a given insurance product.

Their work gives reserve prices for the company and for the customer; that is, upper and lower bounds on the price of the insurance product so that the transaction can occur. However, Young and Zariphopoulou did not specify which particular value in the interval between reserve prices should be adopted by the insurer. In this paper, we will solve this problem and generalize their work by incorporating a demand curve which addresses the market demand of a certain insurance product at a given price. In this way, insurance policies are treated en masse and their price can be calculated using the Dynamic Programming Principle.
Dedicated to my family
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CHAPTER 1
INTRODUCTION

Nowadays insurance companies are much more involved in financial markets than in the recent past. This is quite different from the traditional view of corporate practice where insurers deposit the premiums of their customers in a bank account and make a profit if the accumulated value of the account exceeds the total claim payment. Now insurance companies invest the premium stream in various kinds of financial securities and get a much higher return than fixed interest rate while, of course, exposing themselves to a certain extra degree of risk. This activity greatly lowers the price of insurance products and makes the insurer more competitive. In this way, insurers trade off risks in the financial markets and liabilities in their portfolio of insurance policies.

In the history of actuarial science, various models have been proposed to describe actuarial liabilities for property and casualty insurance. One important mathematical and philosophical advance came in 1662 from a surprising source, a London draper named John Graunt. His great achievement was to show the regularities of the patterns of life and death in a group of people, despite uncertainty about the future lifetime of any given person. He had the original idea of making a statistical analysis of the London Bills of Mortality. Inspired by his work, actuarial scientists developed
mortality functions to model the frequency of deaths at a given age in a sufficiently large population.

One of the most popular models for property insurance is the compound Poisson process. We assume the times at which claims are made form a Poisson process and individual claims are independent, identically, distributed (iid) random variables. This model is a result of the following natural assumptions: (1) the probability of one claim occurring in a short time interval is proportional to $h$, the length of the interval, (2) the probability of more than two or more claims in the same interval is of order $o(h)$, (3) the portfolio is so large that distinct claim amounts are statistically independent, and (4) the claim amount can be regarded as a sample drawn from a fixed probability distribution.

On the financial market side, Harry Markowitz’s 1952 Ph.D. thesis *Portfolio Selection* laid the ground work for mathematical theory of finance. His work was elaborated by Robert Merton through the introduction of stochastic calculus into the study of finance in 1969. At the same time as Merton’s work, and with Merton’s assistance, Black and Scholes developed their celebrated option pricing formula which provided a satisfying solution to an important practical problem, that of finding a fair price for a European call option. This work won the 1997 Nobel Prize in Economics.

For these classical works the most simple model of a stock price is the geometric Brownian motion. The idea of using continuous time random walks to model stock price fluctuations can be traced back to 1900 when Bachelier used Brownian motion as a model of stock prices on the Paris Bourse. It is interesting to note this work predates the first mathematically rigorous construction of Brownian motion by Wiener in 1920’s.
Because of the mixing together of financial and actuarial industry practice, it is imperative to combine the two types of mathematical models so that the activities of insurers can be properly analyzed and insurance products be fairly priced. A substantial body of work has already been produced, for example, see [5] by Bowers and [10] by Gerber & Pafumi. We are particularly interested in an approach that is based on expected utility arguments and produces so-called reservation prices. This methodology was introduced by Hodges and Nerburger in [11] for the valuation of European call options in the presence of transaction costs. In 2002, Young and Zariphopoulou extended this work in [23] and calculated indifference prices for both buyer and seller of an insurance contract, which can be regarded as upper and lower bounds on the price at which transaction can occur in the market.

Is this the end of the story? It seems not. The insurance company needs to announce a particular price between the upper and lower bound to customers instead of offering them a price range. In this work, we take the point of view of the insurer and construct a realistic mathematical model that describes how the prices of insurance products are determined. We assume the following:

(1) we are clear about the demand and supply relation in the industry, that is, the demand curve is known to us; (2) our utility function is given; and (3) the goal is to maximize the final expected utility for some given time $T$ in the future. Given these assumptions, we can finally determine the optimal price at which expected utility is maximized.

Let us describe this in general terms. The actuarial market is an incomplete market, which means that there is no unique price for derivative products. According to Young and Zariphopoulou, given the utility function of the insurer and the writer, one can calculate reservation prices, which give lowest price such that the insurance
company is willing to offer the business and highest price the customers are willing to take. In our work, we add one more ingredient, namely the demand curve, this will finally pin down optimal price that the insurer should offer.

The rest of our work is organized as follows: In Chapter Two we review the elements of discrete financial mathematics and carefully study the binomial model. Mathematically they are very straightforward, however, we obtain an understanding of some deep concepts of financial market such as arbitrage, completeness and hedging. In Chapter Three, we look at mathematical finance in a continuous time setting, and the classic Black-Scholes work is presented. Chapter Four focuses on the celebrated dynamic programming principle, which is a powerful tool to solve stochastic control problems. The core of our work is Chapter Five, where we formulate the optimal insurance price problem, discuss the existence and uniqueness of solutions, and also give some concrete examples where the optimal price can be found analytically. Combining our theory with the idea of Monte-Carlo simulation, in Chapter Six we perform numerical analysis and present the algorithm which can solve more complex and practical problems. In Chapter Seven we summarize our findings and some further research directions are suggested.
CHAPTER 2
PRICING IN BINOMIAL MODELS

2.1 The One-step Binomial Model

First let us quickly review the one-step binomial model. In this model we have two
times, which we call $t = 0$ and $t = 1$ for convenience. The time $t = 0$ denotes the
present time and $t = 1$ denotes the future time. Viewed from $t = 0$, there are two
states at $t = 1$. They will be called upstate and downstate.

There are two tradable assets:

1. A risky asset $S$, such as a stock. For the risky asset, $S(1) = S_u = S(0) \times u$ in
   upstate with probability $p_u$ and $S(1) = S_d = S(0) \times d$ in downstate with probability
   $p_d = 1 - p_u$, where $u$ and $d$ are some real numbers;

2. A riskless asset $B$, such as a bond. For the riskless asset, we have $B(1) = B(0) \times (1 + r)$ in both states. We can think $r$ as the interest rate.

An essential feature of an efficient market is that any trading strategy can not turn
nothing into something without running the risk of loss. The meaning of this principle
will become clearer in the next section. One important assumption we have to make
in order for our model to be efficient is that the rate of return of the bond is between
those of the two states of the stock:

$$d < 1 + r < u$$
Now let us suppose $X$ is any claim that will be paid at time $t = 1$, where $X(1) = f(S(1))$ for some given function $f$. As $X$ is an asset whose value depends on $S$, it is called a derivative asset written on $S$. Two important examples are:

$$X(1) = [S(1) - K]^+: \text{X is called an European call option on } S \text{ with strike price } K$$

$$X(1) = [K - S(1)]^+: \text{X is called an European put option on } S \text{ with strike price } K$$

### 2.2 The Risk Neutral Probability

A natural question we want to ask is: how to price derivative assets? Intuitively, we can find the present value at $t=0$ of the expected $X$ payoff at $t=1$:

$$X(0) = \frac{E_p(X(1))}{1 + r} = \frac{f(S_{1u}^u) \times p_u + f(S_{1d}^d) \times p_d}{1 + r}$$

This turns out not to be a good model since this price gives free lunch: people can make money from the market with zero cost and no risk, which is the result of Law of One Price. For more details, see [12], page 2. To make our argument more strict, we will need the following definitions:

**Definition 2.1 (Arbitrage opportunity)** An arbitrage opportunity is an asset (or a portfolio of assets) whose value at $t = 0$ is zero and whose value at $t = 1$ is never negative, but has a strictly positive value at $t = 1$ with a strictly positive probability.

**Definition 2.2 (Risk Neutral Probability)** We call $\tilde{p}$ and $1 - \tilde{p}$ the risk neutral probability which is defined as follows:

$$\tilde{p} = \frac{(1 + r) - d}{u - d}$$

We claim that the reasonable price of $X$ in our model is the present value at $t = 0$ of the expected $X$ payoff at $t = 1$, but the expectation is taken under the risk neutral
probability defined above. That is, regardless the real probability \( p_u \) and \( p_d \), as long as they are strictly positive, we will assign a subjective probability \( \tilde{p} \) to the event \( S(1) = S(0) \times u \), and \( 1 - \tilde{p} \) to the event \( S(1) = S(0) \times d \), then we can find the price of \( X \) from the intuitive way: compute the expectation of the payoff discounted by the interest rate:

\[
X(0) = \frac{\tilde{E}(X(1))}{1 + r} = \frac{f(S_1^u) \times \tilde{p} + f(S_1^d) \times (1 - \tilde{p})}{1 + r}
\]

This formula looks mysterious since it is not even related to \( p_u \) and \( p_d \)! Actually we obtain this formula by using the following ideas:

Step 1: find a portfolio whose payoff is the same as \( X \) at time \( t = 1 \) in each state, which is called the replicating portfolio.

Step 2: find the price of the replicating portfolio at \( t = 0 \), then we assign this price to \( X \) at \( t = 0 \). This is the unique price which excludes arbitrages.

**Example:** Suppose \( S(0) = B(0) = 1, S_1^u = 1.5, S_1^d = 0.5, B(1) = 1.1 \). Given these information, suppose we have a financial product and whose payoffs are \( X(1) = 5.6 \) in the upstate and \( X(1) = 2.6 \) in the down state, we want to find the price of \( X \) at \( t=0 \).

Solution: Following the ideas given above, by solving a linear equation system, we can easily find a replicating portfolio: if we have 3 shares of stock and 1 share of bond, then the value of the portfolio coincides with \( X \) in both states at \( t = 1 \), so a fair price for \( X \) at \( t = 0 \) is \( X(0) = 3 \times S(0) + B(0) = 4 \).

**Remark:** In the above example, if we find the risk neutral probability \( \tilde{p} = \frac{1.1 - 0.5}{1.0} = 0.6 \) first and then calculate the discounted value of the expected \( X \) payoff with respect to the risk neutral probability, we will get the same answer. Here the information of real probability \( p_u \) and \( p_d \) is not even given.
2.3 Multiperiod Binomial model

We now extend the model in section 1.1 to multiple periods. Now suppose that we have an \( N \)-period model, that is, we consider \( t = 0, 1, 2, ... N \). For each period, our riskless bond \( B \) will earn an interest at the given rate \( r \). The price of the stock may either move up by the factor \( u \) with probability \( p_u \), or move down by the factor \( d \) with probability \( p_d = 1 - p_u \). Note that these probabilities could be different at different times. But since we assume that the parameters \( u, d \) and \( r \) are constants for each period, the risk neutral probability remains the same for all time intervals.

To describe this in the probability sense, we imagine that we toss a coin repeatedly, say \( N \) times. Whenever we get a head, the stock price moves up; if we get a tail, the stock price moves down. We consider a probability space \( (P, \Omega, \mathcal{F}_N) \): \( \Omega = \{\omega_1 \omega_2 ... \omega_N; \omega_i = H \text{ or } T\} \), \( \mathcal{F}_N \) is the power set of \( \Omega \), and \( P(\omega_1 \omega_2 ... \omega_N) = P_1(\omega_1) * P_2(\omega_2) * ... * P_N(\omega_N) \).

As we have discussed before, the probability \( P \) is irrelevant to our pricing: we have to find the risk-neutral probability \( \tilde{P} \). In fact, if we define \( \tilde{P}(\omega_1 \omega_2 ... \omega_N) = \prod_{i=1}^{N} \tilde{p}(\omega_i) \), where \( \tilde{p}(H) = \frac{1+r-d}{u-d} \), \( \tilde{p}(T) = \frac{u-1-r}{u-d} \), then \( \tilde{P} \) is the risk neutral probability in the multiperiod model.

\( \mathcal{F}_N \) can be viewed as information accumulated up to time \( N \). Similarly we can define \( \mathcal{F}_n \), which contains the information up to time \( n \). Let \( \Pi_n \) be the \( n \)-th coordinate map of \( \omega \in \Omega \), that is \( \Pi_n(\omega) = \omega_n \), then \( \mathcal{F}_n \) is the \( \sigma \)-field generated by \( \Pi_1, \Pi_2, ... \Pi_n \).

Suppose \( V_N \) is an \( \mathcal{F}_N \)-measurable random variable which is the payoff of a derivative security, we want to find the non-arbitrage price of \( V \) at time \( t \) before \( N \). We have the following theorem:
Theorem 2.3 (Hedging in the multiperiod binomial model). Consider the $N$-period binomial asset-pricing model, with $0 < d < 1 + r < u$, and with

$$\tilde{p} = \frac{1 + r - d}{u - d}, \quad \tilde{q} = \frac{u - 1 - r}{u - d}$$

(2.1)

Let $V_N$ be a random variable depending on the first $N$ coin tosses $\omega_1 \omega_2 ... \omega_N$. Define recursively backward in time the sequence of random variables $V_{N-1}, V_{N-2}, ... V_0$ by

$$V_n(\omega_1 \omega_2 ... \omega_n) = \frac{1}{1 + r} [\tilde{p} V_{n+1}(\omega_1 \omega_2 ... \omega_n H) + \frac{1}{1 + r} [\tilde{q} V_{n+1}(\omega_1 \omega_2 ... \omega_n T)]$$

(2.2)

so that each $V_n$ depends on the first $n$ coin tosses $\omega_1 \omega_2 ... \omega_n$, where $n$ ranges between $N - 1$ and 0. Next define

$$\Delta_n(\omega_1 \omega_2 ... \omega_n) = \frac{V_{n+1}(\omega_1 \omega_2 ... \omega_n H) - V_{n+1}(\omega_1 \omega_2 ... \omega_n T)}{S_{n+1}(\omega_1 \omega_2 ... \omega_n H) - S_{n+1}(\omega_1 \omega_2 ... \omega_n T)}$$

(2.3)

where again $n$ ranges between 0 and $N-1$. If we set $X_0 = V_0$ and define recursively forward in time the portfolio values $X_1, X_2..X_N$ by

$$X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n)$$

(2.4)

then we will have

$$X_N(\omega_1 \omega_2 ... \omega_N) = V_N(\omega_1 \omega_2 ... \omega_N)$$

for all $\omega_1 \omega_2 ... \omega_N$. (2.5)

For a proof, please see [21], page 12.

Remark: The previous theorem gives us the algorithm to compute the non-arbitrage price of derivative assets. Also, it tells us how to construct a portfolio to hedge the risk. From (2.2), we know that the discounted value of our portfolio, $\frac{V_n}{(1+r)^n}$, as well as the stock price, is a martingale under the risk neutral probability. Also, at each time $n$, the number of shares of stock in the replicating portfolio $\Delta_n$ is a random variable whose value depends on the result of all coin tosses up to time $n$, so it is an adapted dynamic strategy.
2.4 A More General One-period Model

In this chapter we consider a financial market with \( N \) different financial assets. The discussion can provide us some insight into some important concepts such as hedging and completeness. All results are standard, the reader may refer to [2], Chapter Three for more details. We consider the following model: the market only exists at two points in time \( t = 0 \) and \( t = 1 \). The price of the \( i \)th asset at time \( t \) will be denoted by \( S^i_t \). Thus we have a price vector process \( \{S_t\}_{t=0,1} \), and we view each price vector as a column vector, i.e.

\[
S_t = \begin{pmatrix}
S^1_t \\
S^2_t \\
\vdots \\
S^N_t
\end{pmatrix}
\]

The randomness in the system is modeled by assuming that we have a finite sample space \( \Omega = \{\omega_1, \ldots, \omega_M\} \), and the probability \( P(\omega_i) \) is strictly positive for each \( i = 1, 2, \ldots N \). The price vector \( S_0 \) is assumed to be deterministic and known to us, but the price vector at time \( t = 1 \) depends on the state \( \omega \in \Omega \).

Let us define the matrix \( D \) by

\[
D = \begin{pmatrix}
S_1^1(\omega_1) & \ldots & S_1^1(\omega_M) \\
S_1^2(\omega_1) & \ldots & S_1^2(\omega_M) \\
\vdots & \ddots & \vdots \\
S_1^N(\omega_1) & \ldots & S_1^N(\omega_M)
\end{pmatrix} = \begin{bmatrix} d_1, d_2, \ldots, d_M \end{bmatrix}
\]

where \( d_1, d_2, \ldots, d_M \) denote the column vectors of \( D \). We now define a portfolio as an
N-dim row vector \( h = [h^1, h^2 \ldots h^N] \) with the interpretation that \( h^i \) is the units of the \( i \)-th asset we buy at time \( t = 0 \) and keep until time \( t = 1 \). The value process of our portfolio will be a stochastic process \( V_t^h \) defined by

\[
V_t^h = \sum_{i=1}^{N} h^i S_t^i = h \cdot S_t
\]

**Definition 2.4** The portfolio \( h \) is an arbitrage portfolio if it satisfies the following conditions

\[
\begin{align*}
V_0^h &< 0 \\
V_1^h &\geq 0 \text{ with probability 1}
\end{align*}
\]

An arbitrage portfolio is thus basically a deterministic money making machine, and we interpret the existence of an arbitrage portfolio as equivalent to a serious case of mispricing on the market. In other words, we want our model to be arbitrage free.

**Proposition 2.5** The market is arbitrage free if and only if there exist nonnegative numbers \( z_1, z_2, \ldots z_M \) such that the following vector equality holds

\[
S_0 = \sum_{j=1}^{M} z_j S_1(\omega_j)
\]

On component form this reads as

\[
S_0^i = \sum_{j=1}^{M} z_j S_1^i(\omega_j)
\]

For proof, please refer to page 28, [2].

We want to give an economic interpretation of this result. Motivated by discussion in section (1.2) we define the real nonnegative numbers \( q_1, q_2, \ldots, q_M \) by

\[
q_j = \frac{z_j}{\beta}, \text{ where } \beta = \sum_{j=1}^{M} z_j
\]
then we may interpret \( q_1, q_2, ..., q_M \) as a probability distribution \( Q \) on \( \Omega \) by setting \( Q(\omega_i) = q_i \), and we can reformulate our previous result.

**Proposition 2.6** The market is arbitrage free if and only if there exists a probability distribution \( Q \) on \( \Omega \) and a real constant \( \beta \) such that

\[
S_0 = \beta E^Q[S_1]
\]

Such a measure, or probability distribution \( Q \) is called a *martingale measure*, or a *risk neutral distribution*.

We may want to know whether there exists a natural economic interpretation of the factor \( \beta \) above. Actually if we suppose that there exists a risk free investment alternative, say, a bond, and the rate of return is given by \( r \), then \( \beta \) is the discount ratio under this rate, i.e.

\[
\beta = \frac{1}{1+r}
\]

From the above arguments, we have the following result, which in its far reaching generalizations is known as “the first fundamental theorem of mathematical finance”

**Proposition 2.7 (First Fundamental Theorem)** Assume that there exits a risk free asset, and denote the corresponding risk free interest by \( r \). Then the market is arbitrage free if and only if there exits a measure \( Q \) such that

\[
S_0 = \frac{1}{1+r} E^Q[S_1]
\]

(2.7)

For a proof, see [22], page 231.

Our main problem is how to price financial derivatives or, in other words, contingent claims; for which the definition is given as the following:
Definition 2.8 A contingent claim is any random variable $X$, defined on $\Omega$.

We would like to price the claim $X$ consistently with the underlying a priori given assets $S^1, S^2, .. S^N$, or in other words, we would like to price the claim $X$ in such a way that there are no arbitrage opportunities on the extended market consisting of $X, S^1, S^2, ..., S^N$. If we want to exclude any arbitrage opportunities, we must price $X$ with the same martingale measure for the underlying market. So we have the following corollary:

Corollary 2.9 Consider a given claim $X_1$ at $t = 1$. In order to avoid arbitrage, $X_0$, the price at time $t = 0$, must be priced according to the formula:

$$X_0 = \frac{1}{1 + r} E^Q[X]$$

where $Q$ is a martingale measure for the underlying market.

2.5 Completeness of the Binomial Model

At this point, we will discuss an important concept, which describes whether it is possible to generate the payoff of a claim at $t = 1$ by forming portfolios in the underlying assets. We still assume that we have $N$ risky assets $S^1, S^2, ..., S^N$, and there also exits a risk free asset with rate of return given by $r$.

Definition 2.10 Consider a contingent claim $X$. If there exists a portfolio $h$, based on the underlying assets, such that

$$V^h_1 = X_1$$

with probability 1

then we say that $X$ is replicated, or hedged by $h$. Such a portfolio is called a replicating, or hedging portfolio. If every contingent claim can be replicated, we say that the market is complete.
It is easy to characterize completeness in our market from an algebraic point of view, and we have the following result.

**Proposition 2.11** The market is complete if and only if the rows of the matrix $D$ spans $\mathbb{R}^M$, i.e. if and only if $D$ has rank $M$.

The proof is quite straightforward, and it is a direct result from the linear algebra. We can use the same tool to prove the following important results.

**Proposition 2.12 (Second Fundamental Theorem)** Assume that the model is arbitrage free. Then the market is complete if and only if the martingale measure is unique.

The reader can refer to [2], page 33.

Finally, we want to conclude this chapter with a proposition which summarize our findings.

**Proposition 2.13** The following hold:

1. The market is arbitrage free if and only if there exists a martingale measure $Q$.
2. The market is complete if and only if the martingale measure is unique.
3. For any claim $X$, the only prices which are consistent with absence of arbitrage are of the form:

   $X_0 = \frac{1}{1 + r} E^Q[X]$

   where $Q$ is a martingale measure for the underlying market.
4. If the market is incomplete, then the different choices of martingale measures $Q$ will generally give rise to different prices.
(5) If $X$ is replicable, even in an incomplete market, the price will not depend on the particular choice of martingale measure $Q$. If $X$ is replicable, then

$$V_0^h = \frac{1}{1+r} E^Q[X]$$

for all martingale measures $Q$ and for all replicating portfolios $h$. 
CHAPTER 3
CONTINUOUS FINANCIAL MATHEMATICS

3.1 Probability Space and Brownian Motion

In this chapter, we consider the continuous model in classic financial mathematics. First, we give out the general setting:

**Definition 3.1:** Let $\Omega$ be a nonempty set, and let $\mathcal{F}$ be a collection of subsets of $\Omega$. We say that $\mathcal{F}$ is a $\sigma$-algebra provided that:

(i) the empty set $\emptyset$ belongs to $\mathcal{F}$,

(ii) whenever a set $A$ belongs to $\mathcal{F}$, its complement $A^c$ also belongs to $\mathcal{F}$, and

(iii) whenever a sequence of sets $A_1, A_2, \ldots$ belongs to $\mathcal{F}$, their union $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{F}$.

**Definition 3.2:** Let $\Omega$ be a nonempty set, and let $\mathcal{F}$ be a $\sigma$-algebra of subsets of $\Omega$. A probability measure $P$ is a function that, to every set $A \in \mathcal{F}$, assigns a number in $[0, 1]$, called the probability of $A$ and written $P(A)$. We require:

(i) $P(\Omega) = 1$, and

(ii) (countable additivity) whenever $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$, then

\[ P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n) \]

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The triple \((\Omega, \mathcal{F}, P)\) is called a \textit{probability space}.

Let \((\Omega, \mathcal{F})\) and \((\Omega', \mathcal{F}')\) be two measurable spaces. We say that \(X : \Omega \to \Omega'\) is an \(\mathcal{F}/\mathcal{F}'\)-measurable if for each \(B \in \mathcal{F}'\), \(X^{-1}(B) \in \mathcal{F}\). We call \(X\) an \(\mathcal{F}/\mathcal{F}'\)-random variable, or simply random variable if there would be no confusion. It is called an \(\mathcal{F}\)-random variable when \((\Omega', \mathcal{F}') = (\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))\). We also need to define the stochastic process on a probability space:

**Definition 3.3** Let \(I\) be a nonempty index set and \((\Omega, \mathcal{F}, P)\) be a probability space. A family \(\{X_t, t \in I\}\) of random variables from \((\Omega, \mathcal{F}, P)\) to \(\mathbb{R}^m\) is called a stochastic process. For any \(\omega \in \Omega\), the map \(t \to X_t(\omega)\) is called a sample path.

Next, for a given measurable space \((\Omega, \mathcal{F}, P)\), we introduce a monotone family of sub-\(\sigma\)-fields \(\mathcal{F}_t \subseteq \mathcal{F}\), \(t \in [0, T]\). By monotonicity we mean

\[
\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \quad \text{for any } 0 \leq t_1 \leq t_2 \leq T
\]

Such a family is called a filtration. Set \(\mathcal{F}_{t+} = \cap_{s>t} \mathcal{F}_s\) for any \(t \in [0, T]\) and \(\mathcal{F}_{t-} = \cup_{s<t} \mathcal{F}_s\) for any \(t \in (0, T]\). If \(\mathcal{F}_{t+} = \mathcal{F}_t\) (resp. \(\mathcal{F}_{t-} = \mathcal{F}_t\)), we say that \(\{\mathcal{F}_t\}_{t \geq 0}\) is right (resp. left) continuous. The four-tuple \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) is called a \textit{filtered probability space}.

**Definition 3.4** We say that \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) satisfies the usual condition if \((\Omega, \mathcal{F}, P)\) is complete, \(\mathcal{F}_0\) contains all the \(P\)-null sets in \(\mathcal{F}\), and \(\{\mathcal{F}_t\}_{t \geq 0}\) is right continuous.

The following definitions describe that \(\{\mathcal{F}_t\}_{t \geq 0}\) contains the information of the stochastic process \(\{X_s, t \geq s \geq 0\}\), from weak to strong:
Definition 3.5 Let \((Ω, ℱ, \{ℱ_t\}_{t≥0}, P)\) be a filtered probability space and \(X_t\) be a process taking values in \((ℝ^m, ℬ(ℝ^m))\).

The process \(X_t\) is said to be \textit{measurable} if the map \((t, ω) → X_t(ω)\) is \((ℬ[0, T] × ℱ)/ℬ(ℝ^m))\) measurable.

The process \(X_t\) is said to be \textit{adapted} if for each \(t\), the map \(ω → X_t(ω)\) is \(ℱ_t/ℬ(ℝ^m))\) measurable.

The process \(X_t\) is \{\(ℱ_t\)\}_{t≥0}-\textit{progressively measurable} if for all \(t ∈ [0, t]\), the map \((s, ω) → X_s(ω)\) is \(ℬ[0, T] × ℱ_t/ℬ(ℝ^m))\) measurable.

Let us now introduce an extremely important example of stochastic process, called Brownian motion, which plays a unique role in probability theory. It is a canonical example of both a Markov process and a martingale.

Definition 3.6 Let \((Ω, ℱ, \{ℱ_t\}_{t≥0}, P)\) be a probability space. An \{\(ℱ_t\)\}_{t≥0}-adapted \(ℝ^m\)-valued process \(W(.)\) is called an \(m\)-dimensional \{\(ℱ_t\)\}_{t≥0}-\textit{Brownian motion} over \([0, ∞)\) if for all \(0 ≤ s < t\), \(W(t) − W(s)\) is independent of \(ℱ_s\) and is normally distributed with mean 0 and covariance \((t − s)I\). Namely, for any \(0 ≤ s < t\)

\[
E[W(t) − W(s)|ℱ_s] = 0 \\
E[(W(t) − W(s))(W(t) − W(s))^T|ℱ_s] = (t − s)I
\]

In addition, if \(P(W(0) = 0) = 1\), then \(W(.)\) is called an \(m\)-dimensional \textit{standard} \{\(ℱ_t\)\}_{t≥0}-\textit{Brownian motion} over \([0, ∞)\)

There are several different ways to construct a Brownian motion. We refer interested readers to [14].
3.2 Log-Normal Distribution of Stock Price under Risk Neutral Probability

In this section we investigate the binomial model and show that the limit of a properly scaled multiperiod binomial asset-pricing model leads to a stock price with a log-normal distribution. We build a model for a bond price and stock price on the time interval from 0 to \( t \) by choosing an integer \( n \) and constructing a binomial model for the prices that take \( n \) steps per unit time. We assume that \( nt \) is an integer. Within a single time period of length \( \frac{1}{n} \), we set \( r_n = \frac{r}{n} \), the up factor to be \( u_n = 1 + \frac{\sigma}{\sqrt{n}} \) and the down factor to be \( d_n = 1 - \frac{\sigma}{\sqrt{n}} \). The risk neutral probability is given by:

\[
\tilde{p} = \frac{1 + r_n - d_n}{u_n - d_n} = \frac{\sigma + r/\sqrt{n}}{2\sigma}, \quad \tilde{q} = \frac{u_n - 1 - r_n}{u_n - d_n} = \frac{\sigma - r/\sqrt{n}}{2\sigma}
\]

We determine the price at time \( t \) from \( nt \) coin tosses, the probability of each toss giving a head or tail is \( \tilde{p} \) and \( \tilde{q} \). Let us assume that the number of heads and tails are \( H_{nt} \) and \( T_{nt} \) separately, and the biased random walk \( M_{nt} \) is the number of heads minus the number of tails in these \( nt \) coin tosses. According to our assumptions in Chapter One, the stock price at time \( t \) is

\[
S_n(t) = S(0)u_n^{H_{nt}}d_n^{T_{nt}} = S(0)(1 + \sigma\sqrt{n})^{\frac{1}{2}(nt + M_{nt})}(1 - \sigma\sqrt{n})^{\frac{1}{2}(nt - M_{nt})} \tag{3.1}
\]

**Theorem 3.7** As \( n \to \infty \), the distribution of \( S_n(t) \) in (3.1) converges to \( S(t) \) in distribution, where \( S(t) \) is defined by

\[
S(t) = S(0)\exp\{\sigma W(t) + (r - \frac{1}{2}\sigma^2)t\} \tag{3.2}
\]

Proof of Theorem 3.7: It suffices to show that the distribution of

\[
\ln S_n(t) = \ln S(0) + \frac{1}{2}(nt + M_{nt})\ln(1 + \frac{\sigma}{\sqrt{n}}) + \frac{1}{2}(nt - M_{nt})\ln(1 - \frac{\sigma}{\sqrt{n}}) \tag{3.3}
\]
converges to the distribution of

\[ \ln S(t) = \ln S(0) + \sigma W(t) + (r - \frac{1}{2} \sigma^2) t \]

where \( W(t) \) is a standard Brownian motion. To do this, we need to use the following Taylor expansion

\[ f(x) = \ln(1 + x) = x - \frac{1}{2} x^2 + O(x^3) \]

We apply this to (3.3) with \( x = \frac{\sigma}{\sqrt{n}} \) and \( x = -\frac{\sigma}{\sqrt{n}} \), then we have

\[ \ln S_n(t) = \ln S(0) + \frac{1}{2}(nt + M_{nt})\left(\frac{\sigma}{\sqrt{n}} - \left(\frac{\sigma^2}{2n} + o\left(\frac{1}{n}\right)\right)\right) \]
\[ + \frac{1}{2}(nt - M_{nt})\left(-\frac{\sigma}{\sqrt{n}} - \left(\frac{\sigma^2}{2n} + o\left(\frac{1}{n}\right)\right)\right) \]
\[ = \ln S(0) + nt\left(-\frac{\sigma^2}{2n} + o\left(\frac{1}{n}\right)\right) + M_{nt}\left(\frac{\sigma}{\sqrt{n}} + o\left(\frac{1}{n}\right)\right) \]
\[ = \ln S(0) - \frac{1}{2} \sigma^2 t + \sigma X^{(n)}(t) + h.o.t. \]

The term \( X^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} \) that appears in the last line is a scaled biased random walk. To find its limit process, let us consider its generator. Let \( f \in C^2[\mathbb{R}] \), then the generator \( L_n \) is

\[ L_n f(x) = \lim_{h \to 0} \frac{E[f(X^{(n)}(h))] - f(x)}{h} \]

Let us suppose \( X^{(n)}(t) = x = \frac{k}{\sqrt{n}}, h = \frac{1}{n}, \) then \( X^{(n)}(t + h) = \frac{k + 1}{\sqrt{n}} \) with probability \( \tilde{p} \) and \( X^{(n)}(t + h) = \frac{k - 1}{\sqrt{n}} \) with probability \( \tilde{q} \), therefore:

\[ L_n f(x) = \lim_{h \to 0} \frac{E[f(X^{(n)}(h))] - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{f(x + \frac{1}{\sqrt{n}}) \times \tilde{p} + f(x - \frac{1}{\sqrt{n}}) \times \tilde{q} - f(x)}{1/n} \]

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If we plug in the value of \( \tilde{p}, \tilde{q} \), use Taylor expansion on the function \( f \) at point \( x \) and let \( n \) approach \( \infty \), we get:

\[
\lim_{n \to \infty} L_n f(x) = \frac{r}{\sigma} f'(x) + \frac{1}{2} f''(x) + o(n)
\]

From classical theory of stochastic process, we know that \( X^{(n)}(t) \) approaches to the following process in distribution.

\[
dX_t = \frac{r}{\sigma} dt + dW_t
\]

where \( W_t \) is a standard Brownian motion. Substitute this into \( \ln S_n(t) \) and let \( n \to \infty \) we get the desired result. This completes the proof of theorem 3.7.

**Remark:** In Theorem 3.7, we assume that the asset price \( S(t) \) has constant instantaneous rate of return and volatility for simplicity. Both parameters are allowed to be time-varying and random. This example includes all possible models of an asset price process that is always positive, has no jumps, and is driven by a single Brownian motion.

### 3.3 Pricing and hedging in the Black-Scholes model

In this section we use the classical example of European option to explain the pricing method of derivative assets developed by Black and Scholes in 1970’s. The idea is the same as that in the discrete model. First we try to construct a portfolio that replicates the payoff of the derivative asset at the expiration date, then the price of this replicating portfolio at the initial time will be the unique fair price for our derivative asset. Also, as in the discrete model, we can find the hedging strategy from our calculation.
To carry out this process, we will need the following well-known Itô-Doeblin formulas for Brownian motion and Itô process.

**Theorem 3.8 (Itô-Doeblin formula for Brownian motion)** Let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x), f_x(t, x)$ and $f_{xx}(t, x)$ are defined and continuous, and let $W(t)$ be a Brownian motion. Then, for every $T \geq 0$

$$f(T, W(T)) = f(0, W(0)) + \int_0^T f_t(t, W(t)) dt + \int_0^T f_x(t, W(t)) dW(t) + \frac{1}{2} \int_0^T f_{xx}(t, W(t)) dt$$

We can also write this in the following differential form:

$$df(t, W(t)) = f_t(t, W(t)) dt + f_x(t, W(t)) dW(t) + \frac{1}{2} f_{xx}(t, W(t)) dt$$

**Theorem 3.9 (Itô-Doeblin formula for Itô process)** Let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x), f_x(t, x)$ and $f_{xx}(t, x)$ are defined and continuous, and let $X(t)$ be an Itô process of the form

$$dX(t) = \Delta(t) dW(t) + \Theta(t) dt$$

then, for every $T \geq 0$

$$f(T, X(T)) = f(0, X(0)) + \int_0^T f_t(t, X(t)) dt + \int_0^T f_x(t, X(t)) dX(t) + \frac{1}{2} \int_0^T f_{xx}(t, X(t)) dX(t) dX(t)$$

and the term $dX(t) dX(t)$ can be computed using the following rules:

$$dtdt = 0, dt dW(t) = 0, dW(t) dW(t) = dt$$

For a proof, we refer the readers to [19], page 44.

Let us consider an European call option that pays $[S(T) - K]^+$ at time $T$. The value of this call at any time depends on the time $t$ (more precisely, on time to expiration
\[ T - t \) and the current stock price \( S(t) \). As usual we assume the price of the stock follows the geometric Brownian motion with mean rate of return \( \mu \) and volatility \( \sigma \).

\[
dS(t) = \mu S(t)dt + \sigma S(t)dW(t)
\]

Notice that here we assume the rate of return is \( \mu \) instead of \( r \) in Theorem 3.7. This is because in Theorem 3.7 we are trying to determine the price process under risk neutral probability but here we are looking at the same process under physical (real) probability. As we discussed above, the price of the option is a function of \( t \) and \( S(t) \).

Using \( c(t, s(t)) \) to denote the price of the call option, let us use Itô formula to find the differential of the price:

\[
d(c(t, S(t))) = c_t(t, S(t))dt + c_x(t, S(t))dS(t) + \frac{1}{2}c_{xx}(t, S(t))dS(t)dS(t)
= [c_t(t, S(t)) + \mu S(t)c_x(t, S(t)) + \frac{1}{2}c_{xx}(t, S(t))\sigma^2 S(t)]dt
+ \sigma S(t)c_x(t, S(t))dW(t)
\]

Using this result, by applying Itô formula to \( f(t, x) = e^{-rt}x \), we can find the differential of the discounted price:

\[
d(e^{-rt}c(t, S(t))) = e^{-rt}[-rc(t, S(t)) + c_t(t, S(t)) + \mu S(t)c_x(t, S(t))]
+ \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t))]dt + e^{-rt}\sigma S(t)c_x(t, S(t))dW(t)
\]

As in the discrete model, we are trying to find a replicating portfolio valued \( X(t) \) for \( c(t, S(t)) \). Suppose that at each time \( t \), the investor holds \( \Delta(t) \) shares of stock. The remainder of the portfolio value, \( X(t) - \Delta(t)S(t) \), is invested in the money market account. We can find the differential of the investor’s portfolio value:

\[
dx_t = \Delta dS(t) + r(X(t) - \Delta(t)S(t))dt
= rX(t)dt + \Delta(t)(\mu - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)
\]

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Using Itô formula, we can find the differential of the discounted value:

\[ d(e^{-rt}X(t)) = \Delta(t)(\mu - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t) \]

Since we assume that \( X(t) \) is the replicating portfolio of \( c(t, S(t)) \), then they should have the same differentials. Comparing their differentials, if we equate the \( dt \) and \( dW(t) \) terms and simplify, considering the final payoff as terminal condition, we get the Black-Scholes-Merton partial differential equation:

\[
\begin{cases}
rc(t, x) = c_t(t, x) + rxc_x(t, x) + \frac{1}{2}\sigma^2x^2c_{xx}(t, x) \\
c(T, x) = (x - K)^+
\end{cases}
\tag{3.4}
\]

To find out the hedging strategy, we can equate the \( dW(t) \) term

\[ \Delta(t) = c_x(t, S(t)) \]

### 3.4 Risk Neutral Pricing

In Chapter Two we have discussed how to price financial derivative products using risk neutral probability. The basic idea is that we assign a new probability which is different from the real one to the states of the stock so that the expected rate of return becomes the same as the riskless asset, even this is not the case in reality: usually people are expecting higher returns for more risky securities. Here we will follow the same approach, a very important tool we are going to use is Girsanov’s Theorem. All results are standard, for example, see [22].

We begin with a probability space \((\Omega, \mathcal{F}, P)\) and a nonnegative random variable \(Z\) satisfying \(EZ = 1\). We then define a new probability \(\tilde{P}\) by the formula

\[ \tilde{P}(A) = \int_A Z(\omega)dP(\omega) \text{ for all } A \in \mathcal{F} \]
\tag{3.5}
If $Z > 0$ almost surely, then $P$ and $\tilde{P}$ are equivalent, that is, they agree on sets which have probability zero.

**Theorem 3.10 (Girsanov, one dimension).** Let $W(t)$, $0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, and $\mathcal{F}(t)$, $0 \leq t \leq T$, be a filtration for this Brownian motion. Let $\Theta(t)$, $0 \leq t \leq T$, be an adapted process. Define

$$Z(t) = \exp\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2} \int_0^t \Theta^2(u)du\} \quad (3.6)$$

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)du \quad (3.7)$$

and assume that

$$E\int_0^T \Theta^2(u)Z^2(u)du < \infty \quad (3.8)$$

Set $Z = Z(T)$. Then $EZ = 1$ and under the probability measure $\tilde{P}$ given by (3.5), the process $\{\tilde{W}(t)\}_{0 \leq t \leq T}$ is a Brownian motion.

Our next goal is to use Theorem (3.10) to construct a risk neutral probability so that the mean rate of return is the same as interest rate $r$. We know that under the physical probability, the price of the stock follows geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW(t)$$

If we let $\Theta = \frac{\mu - r}{\sigma}$ and apply Itô’s formula to the function $f(t, x) = e^{-rt}x$, we obtain:

$$d(e^{-rt}S_t) = \sigma e^{-rt}S_t[\Theta dt + dW(t)]$$

$$= \sigma e^{-rt}S_t d\tilde{W}(t)$$

Where $\tilde{W}(t)$ is as defined in Theorem (3.10). From above we can see that the stock price, discounted by the riskless interest rate $r$, is a martingale. Therefore, the original price has rate of return $r$ under the risk neutral probability $\tilde{P}$, as defined in Theorem (3.10).
Now let us find a formula to compute the price of a portfolio $X(t)$ whose payoff $X(T)$ is known and assume that $X(T)$ if $\mathcal{F}_T$ measurable. The portfolio is a combination of stocks and bonds. Since the discounted price of both are martingales under the risk neutral probability, the discounted price of the portfolio, $e^{-rt}X(t)$, is also a martingale, given that the investing strategy is adapted. According to the martingale property of the discounted price:

$$e^{-rt}X(t) = \tilde{E}[e^{-rT}X(T)|\mathcal{F}_t]$$

we get:

$$X(t) = \tilde{E}[e^{-r(T-t)}X(T)|\mathcal{F}_t]$$

(3.9)

Now let us use the above idea to find the price of an European option, in other words, we are trying to find an explicit solution to equation (3.4). Since we know that the payoff of the option at time $t = T$ is $X(T) = (S(T) - K)^+$, (3.9) becomes

$$X(t) = \tilde{E}[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}_t]$$

Since $S(t)$ follows the geometric Brownian motion, we can explicitly write $S(T)$ out:

$$S(T) = S(t) \exp\{\sigma(\tilde{W}(T) - \tilde{W}(t)) + (r - \frac{1}{2}\sigma^2)(T - t)\}$$

$$= S(t) \exp\{-\sigma\sqrt{T-t}Y + (r - \frac{1}{2}\sigma^2)(T - t)\}$$

where $Y$ is a standard normal variable independent of $\mathcal{F}_t$. Note that $S(t)$ is $\mathcal{F}_t$-measurable, we obtain:

$$c(t, S(t)) = \tilde{E}[e^{r(T-t)}(S(t) \exp\{-\sigma\sqrt{T-t}Y + (r - \frac{1}{2}\sigma^2)(T - t)\} - K)^+]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{r(T-t)}(S(t) \exp\{-\sigma\sqrt{T-t}Y + (r - \frac{1}{2}\sigma^2)(T - t)\} - K)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

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After a long but straightforward calculation, we can get the well-known Black-Scholes-Merton formula for European call option:

$$BSM(t, T, x, K, r, \sigma) = xN(d_+(T - t, x)) - e^{-r(T-t)}K N(d_-(T - t, x))$$

where

$$d_+ = d_+ + \sigma \sqrt{T - t} = \frac{1}{\sigma \sqrt{T - t}} \left[ \log \frac{x}{K} + (r + \frac{1}{2} \sigma^2)(T - t) \right]$$

and $N(.)$ is the accumulative probability function of stand normal variables

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} exp\left(-\frac{z^2}{2}\right)dz$$

**Remark:** From Feynman-Kac formula, we also see that (3.9) is the solution of (3.4), where the dynamics of $X(t)$ is given by:

$$dX(t) = rX(t)dt + \sigma X(t)dW(t)$$

under the risk neutral probability.

### 3.5 Portfolio Dynamics and Completeness

In this section let us consider a financial market consisting of different assets such as stocks, bonds with different maturities, or various kinds of financial derivatives. We take the price dynamics of the various assets as given, and the objective is to derive the dynamics of (the value of) a so-called self-financing portfolio.

**Definition 3.11** Let the $N$-dimensional price process $\{S(t); t \geq 0\}$ be given:

1. A portfolio is any $\mathcal{F}^S_t$-adapted $N$-dimensional process $\{h(t); t \geq 0\}$
2. The portfolio $h$ is said to be *Markovian* if it is of the form

$$h(t) = h(t, S(t))$$
for some function measurable function $h$.

3. The value process $V^h$ corresponding to the portfolio $h$ is given by

$$V^h(t) = \sum_{i=1}^{N} h_i(t)S_i(t)$$

4. A consumption process is any $\mathcal{F}^S_t$-adapted one-dimensional process \{c(t); t \geq 0\}

5. A portfolio-consumption pair $(h, c)$ is called self-financing if the value process $V^h$ satisfies the condition

$$dV^h(t) = \sum_{i=1}^{N} h_i(t)dS_i(t) - c(t)dt$$

In the previous section, we have discussed how to specifically compute the price European call option in the Black-Scholes model. The replicating strategy is also given as a byproduct. Actually if using the same idea, we can prove that within the same model, the price of any derivative asset whose payoff is $\mathcal{F}_T$ measurable can be uniquely determined by creating a hedging portfolio. Thus the Black-Scholes model, which consists one stock and one bond, is complete in the following sense:

**Definition 3.12** We say that a $T$-claim $X$ can be replicated, alternatively that it is reachable of hedgeable, if there exists a self-financing portfolio $h$ such that

$$V^h(T) = X(T) \text{ P-a.e}$$

In this case we say that $h$ is a hedge against $X$. Alternatively, $h$ is called a replicating or hedging portfolio. If every contingent claim is reachable we say that the market is complete.

Let us consider a model with $M$ traded underlying assets plus one risk free asset(i.e. totally $M+1$ assets). We assume that the price process of the underlying assets are driven by $N$ independent Wiener processes.
While discussing completeness and absence of arbitrage it is important to realize that these concepts work in opposite directions. Let the number of random sources \( N \) be fixed, then every new underlying asset added to the model (without increasing \( N \)) may give us a potential opportunity of creating an arbitrage portfolio, so in order to have an arbitrage free market the number \( M \) of underlying assets must be small in comparison to the number of random sources \( N \).

On the other hand, we see that every new underlying asset added to the model gives us new possibilities of replicating a given contingent claim, so the completeness requires \( M \) to be large in comparison to \( N \).

We do not formulate and prove a precise result here, but the following “Meta Theore”, is nevertheless extremely useful. We can use it as a general criterion to judge whether a model is complete and whether it is arbitrage free.

**Theorem 3.13 (Meta Theorem)** Let \( M \) denote the number of the underlying traded assets in the model excluding the risk free asset, and let \( N \) denote the number of random sources. Generically we then have the following relations:

1. The model is arbitrage free if and only if \( M \leq N \)
2. The model is complete if and only if \( M \geq N \)
3. The model is complete and arbitrage free if and only if \( M = N \).

**Remarks**

1. Here we can see that the results of above discussion is very similar to (essentially, it is) linear algebra. We can think \( M \) to be the number of linear equations, \( N \) is
the number of unknowns, completeness is the same as existence of solutions, free of arbitrage is essentially the uniqueness of solutions.

2. A source of randomness may not necessarily be a Wiener process. Another example of a random source would be a counting process such as a Poisson process. In this context it is important to note that if the prices are driven by a Poisson process with different jump sizes then the appropriate number of random sources equals the number of different jump sizes.

3. In the above arguments we assume that the pay-off function and underlying assets are determined by the same random sources. If this is not the case, for example, if the pay-off is also driven by some random sources that are outside those which drive the price of underlying assets, then there is no way to hedge the pay-off and the market is therefore incomplete.
CHAPTER 4
STOCHASTIC CONTROLS

4.1 Formulation of Stochastic Optimal Control Problems

To solve many problems we are faced with the following task: there is a diffusion system, which is described by an Itô stochastic differential equation; there are many alternative decisions that can affect the dynamics of the system; there are some constraints that the decisions and/or the state are subject to; and there is a criterion that measures the performance of the decisions. The goal is to optimize the criterion by selecting a non-anticipative decision among the ones satisfying all the constraints. Such problems are called stochastic optimal control problems. We present two, called “strong” and “weak”, formulations of the stochastic optimal control problems in this section.

Given a filtered probability space \((Ω, ℱ, \{ℱ_τ\}_τ, P)\) satisfying the usual condition on which an \(m\)-dimensional standard Brownian motion \(W(.)\) is defined. Considering the following controlled stochastic differential equation:

\[
\begin{aligned}
    dx(t) &= b(t, x(t), u(t))dt + σ(t, x(t), u(t))dW(t) \\
    x(0) &= x_0 \in \mathbb{R}^n
\end{aligned}
\]  

(4.1)

where \(b : [0, T] × \mathbb{R}^n × U → \mathbb{R}^n\), \(σ : [0, T] × \mathbb{R}^n × U → \mathbb{R}^{n \times m}\), \(U\) is a given separable metric space and \(T\) is fixed. The function \(u(.)\) is called the control representing the
action, or decision of the controllers. At any time instant the controller is knowledgeable of all the information up to time \( t \), modeled by the filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \), but not able to foretell what is going to happen afterwards. Namely, the control \( u(.) \) is taken from the set of all \( \{ \mathcal{F}_t \}_{t \geq 0} \)-adapted functions, which is called a feasible control. Sometimes we also have some state constraint given by:

\[
x(t) \in S(t), \text{ for any } t \in [0, T], \text{ P-a.e.} \tag{4.2}
\]

Next, we introduce a cost functional as follows:

\[
J(u(.)) = E \left\{ \int_0^T f(t, x(t), u(t)) dt + h(x(T)) \right\} \tag{4.3}
\]

**Definition 4.1** Let \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, P) \) be given satisfying the usual conditions and \( W(t) \) be a given \( m \)-dimensional standard \( \{ \mathcal{F}_t \}_{t \geq 0} \)-Brownian motion. A control \( u(.) \) is an \( s \)-admissible control, and \( (x(.), u(.)) \) is an \( s \)-admissible pair, if

1. \( u(.) \) is feasible;
2. \( x(.) \) is the unique solution of equation (4.1);
3. some prescribed state constraints are satisfied;
4. \( f(., x(.), u(.)) \) and \( h(x(T)) \) are integrable so that (4.3) is well-defined.

The set of all \( s \)-admissible control is denoted by \( U_{ad}^s[0, T] \). Our stochastic optimal control problem under strong formulation can be stated as follows:

**Strong Formulation.** Minimize (4.3) over \( U_{ad}^s[0, T] \). That is, to find \( \bar{u}(.) \in U_{ad}^s[0, T] \) (if it ever exists), such that

\[
J(\bar{u}(.)) = \inf_{u(.) \in U_{ad}^s[0, T]} J(u(.)) \tag{4.4}
\]

We say that the problem is *finite* if the right-hand side of (4.3) is finite. It is *solvable* if there exists a solution. Any \( \bar{u}(.) \in U_{ad}^s[0, T] \) satisfying (4.4) is called an *s-optimal*
control. The corresponding \( \bar{x}(.) \) and the state-control pair \( (\bar{x}(.), \bar{u}(.)) \) are called an \textit{s-optimal state process} and an \textit{s-optimal pair}, respectively.

We see that in the strong formulation, the underlying probability space with filtration \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P) \) along with the Brownian motion \( W(.) \) are given before we actually start to solve the problem. However, sometimes it is reasonable to consider these elements as parts of the control. This is the case especially when one applies the dynamic programming principle to solve a stochastic optimal control problem originally under the strong formulation, see [26], for example. Therefore, we need to have another formulation of the problem.

**Definition 4.2** A 6-tuple \( \pi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P, W(.), u(.)) \) is called a \textit{w-admissible} pair control, and \( (x(.), u(.)) \) a \textit{w-admissible pair}, if

1. \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P) \) is given satisfying the usual conditions;
2. \( W(.) \) is an \( m \)-dimensional standard Brownian motion defined on \( (\Omega, \mathcal{F}, P) \);
3. \( u(.) \) is an \( \{\mathcal{F}_t\}_{t\geq 0} \)-adapted process on \( (\Omega, \mathcal{F}, P) \) taking values in \( U \);
4. \( x(.) \) is the unique solution of the equation (4.1);
5. some prescribed state constraints are satisfied;
6. \( f(. , x(.), u(.)) \) and \( h(x(T)) \) are integrable so that (4.3) is well-defined.

The set of all \( w \)-admissible control is denoted by \( U_{ad}^w[0, T] \). Sometimes we might write \( u(.) \in U_{ad}^w[0, T] \) instead of \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P, W(.), u(.)) \in U_{ad}^w[0, T] \) when there is no confusion. Our stochastic optimal control problem under strong formulation can be stated as follows:

**Weak Formulation.** Minimize (3.3) over \( U_{ad}^w[0, T] \). That is, to find \( \bar{\pi}(.) \in U_{ad}^w[0, T] \) (if it ever exists), such that

\[
J(\bar{\pi}(.)) = \inf_{\pi(.) \in U_{ad}^w[0, T]} J(\pi(.))
\]
We say that the problem is \textit{finite} if the right-hand side of (3.3) is finite. It is \textit{solvable} if there exists a solution. Similarly, we can define \textit{w-optimal control}, the \textit{w-optimal state process}, and the \textit{w-optimal pair} accordingly.

4.2 Existence of Optimal Control

We are going to discuss the existence of optimal controls under the strong and weak formulations. The essential idea is the following: a convex function defined on some compact metric space attains its minimum. Here we will only present the results and skip the proofs here.

\textbf{Existence under strong formulation}

Let us first look at the existence of an optimal control under the strong formulation. We the following assumptions:

\textbf{S1}: The set $U \subseteq \mathbb{R}^k$ is convex and closed, and the functions $f$ and $h$ are convex and for some $\delta, K > 0$, we have

$$f(x, u) \geq \delta|u|^2 - K, h \geq -K, \text{ for any } (x, u) \in \mathbb{R}^n \times U$$

\textbf{S2}: The set $U \subseteq \mathbb{R}^k$ is convex and compact, the functions $f$ and $h$ are convex.

\textbf{Theorem 4.3} Suppose the coefficients $b(x, u)$ and $\sigma(x, u)$ in (4.1) are linear combinations of $x$ and $u$. Under either (S1) or (S2), if the strong formulation in Definition 4.1 is finite, then it admits an optimal control.

For a proof, see [26], page 67.

\textbf{Remark} We have noticed that Theorem 3.3 is valid only when the coefficients are of linear form, which is unpleasant because we may need the existence in more general
settings. We need the linearity assumption to prove the convergence of the minimizing sequence in an infinite dimensional space where the local compactness is missing. However, the linearity assumption may be relaxed, interested readers may refer to specialist literatures such as [9].

**Existence under weak formulation**

Now we want to examine the existence of an optimal control under the weak formulation. Let us make the following assumptions:

**W1**: \((U,d)\) is a compact metric space and \(T > 0\)

**W2**: The maps \(b, \sigma, f\) and \(h\) are all continuous, and there exists a constant \(L > 0\) such that for \(\phi(t,x,u) = b(t,x,u), \sigma(t,x,u), f(t,x,u), h(x)\), we have

\[
|\phi(t,x_1,u) - \phi(t,x_2,u)| \leq L|x_1 - x_2| \text{ for any } t \in [0,T], x_1, x_2 \in \mathbb{R}^n, u \in U \quad (4.7)
\]

\[
\phi(t,0,u) \leq L, \text{ for any } (t,u) \in [0,T] \times U \quad (4.8)
\]

**W3**: For any \((t,x) \in [0,T] \times \mathbb{R}^n\), the set

\[
\{(b_i(t,x,u), (\sigma\sigma^T)^{ij}(t,x,u), f(t,x,u))| u \in U, i = 1, \ldots, n, j = 1, \ldots, m\}
\]

is convex in \(\mathbb{R}^{n+nm+1}\).

**Theorem 4.4** Under (W1)-(W3), if the weak formulation in Definition 4.2 is finite, then it admits an optimal control.

The proof of Theorem 4.4 is much harder and technical than the proof of Theorem 4.3, the basic idea behind it is the so-called relaxed control, which is needed in order to provide some compact structure. see [26], page 69.
4.3 Dynamic Programming Principle and HJB Equations

One of the principal approaches in solving optimization problems is the derivation of a set of necessary conditions that must be satisfied by an optimal solution. For example, in obtaining an optimum of a finite-dimensional function, one relies on the zero-derivative condition for the unconstrained case or the Kuhn-Tucker condition for the constrained case, which are necessary conditions for optimality. These necessary conditions become sufficient under certain convex conditions. Optimal control problems are essentially optimization problems in infinite-dimensional spaces. The maximum principle, formulated and derived by Pontryagin in the 1950’s, is a powerful tool to solve this kind of problem. It states that any optimal control along with the optimal trajectory must solve the so-called Hamiltonian system, which is a two-point boundary value problem (and also be called forward-backward differential equation), plus a maximum condition of the Hamiltonian function.

Another powerful approach to solving optimal control problems is called the dynamic programming principle. It is a mathematical technique for making a sequence of interrelated decisions, which can be applied to many optimization problems. To apply this method to optimal control, we consider a family of optimal control problems with different initial times and states. We establish relationships among these problems via the Hamilton-Jacobi-Bellman Equation, which is a nonlinear partial differential equation of first-order in the deterministic case and second-order in the stochastic case. If the HJB equation is solvable, either analytically or numerically, then we can obtain an optimal feedback control by taking the maximizer/minimnizer of the Hamiltonian involved in the HJB equation. This is the verification technique. The classical dynamic programming approach requires that the HJB equation admits classical solutions, meaning that the solutions be smooth enough. Since this may not be
the case in general, and to overcome this difficulty, Crandall and Lions introduced the so-called \textit{viscosity solutions} in [7] in early 1980s.

In this paper we solve our stochastic control problems mainly using the dynamic programming principle, so we discuss this approach in more details.

First we set up the framework. Let $T > 0$ be given and assume that the control space $U$ is a metric space. For any $(s, y) \in [0, T) \times \mathbb{R}^n$, consider the state equation:

$$
\begin{cases}
    dx(t) = b(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dW(t) \\
    x(s) = y
\end{cases}
$$

along with the cost functional

$$J(s, y; u(.)) = E\{ \int_s^T f(t, x(t), u(t))dt + h(x(T)) \}$$

(4.10)

For each given $(s, y) \in [0, T) \times \mathbb{R}^n$, we need to find $\bar{u}(.)$ in some certain space $U^*[s, T]$ such that

$$J(s, y; \bar{u}(.)) = \inf_{u(.) \in U^*[s, T]} J(s, y; u(.))$$

where $U^*[s, T]$ is defined as all adapted strategies such that (4.9) admits a unique solution and the resultant integral is well defined, see the previous section for more details.

Under appropriate continuity and integrability assumption, (4.9) admits a unique solution $x(.) = x(.; s, y, u(.))$ and the cost function (4.10) is well-defined. Thus we can define the following function $V : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ by

$$
V(s, y) = \inf_{u(.) \in U^*[s, T]} J(s, y; u(.)), (s, y) \in [0, T) \times \mathbb{R}^n
$$

(4.11)

$$V(T, y) = h(y), y \in \mathbb{R}^n
$$

(4.12)

$V(s, y)$ is called the \textit{value function} with initial data $(s, y)$. 37
The following is the stochastic version of Bellman's principle of optimality

**Theorem 4.5** Let \( V(s, y) \) be defined as in (4.11) and (4.12), then

\[
V(s, y) = \inf_{u(\cdot) \in U^*} \mathbb{E} \left\{ \int_s^\tau f(t, x(t; s, y, u(\cdot)), u(t)) dt + V(\tau, x(\tau; s, y, u(\cdot))) \right\}
\]

Proof: see [26], page 180.

We refer to Theorem 4.5 as the *dynamic programming principle*. This equation gives a relationship among the family of problems with different initial condition via the value function. The essence of the above theorem can be roughly interpreted as the following: global optimality implies local optimality; in other words, this is to say that if \( \bar{u}(\cdot) \) is optimal on \([s, T]\), then the restriction of \( \bar{u}(\cdot) \) on \([\tau, T]\) must also be optimal for any \( \tau \in (s, T) \). This is not hard to understand, since if \( \bar{u}(\cdot) \) is not optimal on \([\tau, T]\), and we can find a better control say \( \tilde{u} \), then we can form a better control by piecing together \( \bar{u} \) on \([s, \tau]\) with \( \tilde{u} \) on \([\tau, T]\), which will result a better control on the whole interval \([s, T]\). This contradicts with the global optimality of \( \bar{u}(\cdot) \).

The dynamic programming principle is very complicated, and it seems impossible to solve such an equation directly. However we can derive a formal partial differential equation that the value function \( V(\ldots) \) should satisfy, based on Theorem (4.5).

**Proposition 4.6** Suppose the value function is defined and \( V \in C^{1,2}([0, T] \times \mathbb{R}^n) \).

Then \( V \) is a solution of the following second order partial differential equation:

\[
\begin{aligned}
-\frac{\partial v}{\partial t} + \sup_{u \in U} G(t, x, u, -v_x, -v_{xx}) &= 0 \\
V|_{t=T} &= h(x)
\end{aligned}
\]

where

\[
G(t, x, u, p, P) = -\frac{1}{2} tr(P \sigma(t, x, u) \sigma(t, x, u)^T) + <p, b(t, x, u)> - f(t, x, u)
\]
for any \((t, x, u, p, P) \in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^n\)

We call (4.13) the *Hamilton-Jacobi-Bellman* equation (HJB equation, for short). The function \(G(t, x, u, p, P)\) is called the *generalized Hamiltonian*. Proposition 4.6 provides us with a powerful tool which can be directly applied. However, it is restrictive because it requires differentiability of order one in time variable and order two in wealth. To overcome this deficiency, we examine the concept of *viscosity solution*, which only requires the continuity of the value function.

**Definition 4.7** A function \(v \in C([0, T] \times \mathbb{R}^n)\) is called a *viscosity subsolution* of (4.13) if

\[
V(T, x) \leq h(x) \text{ for any } x \in \mathbb{R}^n \tag{4.15}
\]

and for any \(\phi \in C^{1,2}([0, T] \times \mathbb{R}^n)\), where \(v - \phi\) attains a local maximum at \((t, x) \in [0, T] \times \mathbb{R}^n\), we have

\[
-\phi_t(t, x) + \sup_{u \in U} G(t, x, u, -\phi_x(t, x), \phi_{xx}(t, x)) \leq 0 \tag{4.16}
\]

A function \(v \in C([0, T] \times \mathbb{R}^n)\) is called a *viscosity supersolution* of (4.13) if the inequalities "\(\leq\)" are switched into "\(\geq\)" and the "local maximum" are changed into "local minimum" in above definition. Further, if \(v \in C([0, T] \times \mathbb{R}^n)\) is both a viscosity subsolution and a viscosity supersolution of (4.15) and (4.16), then it is called a *viscosity solution*.

Before we state our main theorem, let us first introduce some assumptions:

- **(V1)** \((U, d)\) is a Polish space and \(T > 0\).
- **(V2)** The maps \(b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times m}, f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}, \text{ and } h : \mathbb{R}^n \to \mathbb{R}\) are uniformly continuous, and there exists a constant \(L > 0\) such that for \(\phi(t, x, u) = b(t, x, u), \sigma(t, x, u), f(t, x, u), h(x), \)

\[
|\phi(t, x, u) - \phi(t, y, u)| \leq L|x - y|
\]
|φ(t, x, u)| ≤ L

for any \( t \in [0, T], \ x, y \in \mathbb{R}^n, \ u \in U \).

The following two theorems are useful to solve problems using dynamic programming principle when classical solution is not available. Detailed proof can be find in Chapter 4 of [26].

**Theorem 4.8.** Let (V1) and (V2) hold. Then the value function \( V \) is a viscosity solution of (4.13).

**Theorem 4.9.** Let (V1) and (V2) hold. Then the HJB equation (4.13) admits at most one viscosity solution \( v(\cdot, \cdot) \) in the class of functions which satisfy the following:

\[
|V(s, y)| \leq K(1 + |y|) \text{ for any } (s, y) \in [0, T] \times \mathbb{R}^n \tag{4.17}
\]

\[
|V(s, y) - V(t, x)| \leq \{|y-x| + (1 + |y|\vee|x|)|t-s|^{1/2}\}, \forall s, t \in [0, T], x, y \in \mathbb{R}^n \tag{4.18}
\]
CHAPTER 5
PRICING OF INSURANCE PRODUCTS

5.1 Introduction

The purpose of this chapter is to introduce a coherent method for the valuation of insurance risk in a dynamic market setting. The valuation of dynamic risks has been a fundamental issue in financial markets, primarily in the area of derivative securities. As we discussed in Chapter Two, one successful pricing theory is based on a strategy in which one creates a portfolio that accurately replicates the payoff of the product: the risk associated with the financial product is completely hedged, thus one can argue that the valuation of the product must be the cost of setting up the hedging portfolio. This is the key ingredient of the celebrated Black-Scholes method that has been a landmark in derivative asset pricing (Black & Scholes, 1973). Despite its success, which resulted in the great growth of derivative markets, the Black-Scholes approach breaks down entirely once the fundamental assumptions of the characteristics of the market are removed. These assumptions include completeness of the market, absence of transaction costs and constant volatility, to name a few.

In the case of incomplete markets, there is no universal theory that successfully addresses all aspects of pricing, for example, numeraire properties, specification of hedging strategies and robustness of prices. Various alternative pricing mechanisms
have been developed that are strongly oriented towards the specific nature of each market friction. We are particularly interested in one that is based on the expected utility arguments, which is built around the insurance company’s preference towards the risks that cannot be eliminated. This methodology was first introduced in [11] by Hodges & Neuberger (1989) for the valuation of European calls in the presence of transaction costs and later extended by Davis(1993), see [8]. Since then, a substantial body of work has been introduced by using either stochastic control methods (for example, Zariphopoulou, [27]), or by using martingale theory arguments (Karatzas & Kou, [13]).

In an incomplete market, the risk neutral probability is not unique. Different martingale measures will generally generate different prices. The actuarial market is an important example of an incomplete market, and there may not be “fair price” for an insurance product: if one purchases the same product from different insurers, the price could be considerably different. The market price is determined by insurer’s preference towards the risk as well as the relationship of supply and demand. Many mathematicians and actuarial scientists have pointed out that among the different martingale measures which exclude arbitrage opportunities, the final choice is made by the market; for example see [2]. However, no mathematical model to date has fully captured this mechanism. In this chapter, we will construct a realistic model to accurately describe this situation. We are trying to answer the following question: given the demand curve and financial market, what are the firm’s optimal strategies (investing strategy and pricing strategy) so that the expected final utility is maximized? Generally, it is very hard to find the optimal price analytically since it entails solving a partial differential equation which may not have closed form solutions. But when we choose an exponential utility function and a linear demand curve, we can explicitly write out the price. For the more general situation, we suggest that one
use Monte-Carlo simulation. In our model, it is very clear that the market demand
and supply relation plays an important role in determining of the final market price.
The parameters of the demand curve, such as elasticity, can be explicitly found in
the price formula.

5.2 Background

Financial Market

Except otherwise stated, we take the view of the insurance company in our study.
Suppose that we have access to a bond market with deterministic rate of return $r$. We
can also invest money in a stock market. For simplicity, assume that there is only one
stock whose price process follows the geometric Brownian motion with mean rate of
return $\mu$ and volatility $\sigma$. In general, we are investigating our strategy in the setting
of the standard Black-Scholes model. We have the following:

$$
\begin{align*}
    dB_t &= rB_t dt \\
    dS_t &= \mu S_t dt + \sigma S_t dW_t
\end{align*}
$$

where $B_t$ and $S_t$ are the price processes of the bond and the stock respectively.

To avoid arbitrage opportunities, we suppose $r < \mu$. This is reasonable because the
stock investors are expecting a higher rate of return from the stock, as the price of
taking the risk.

Utility Function

The goal of a company is to maximize the profit. But for actuarial markets and fi-
nancial markets, where the profit is random because of financial and insurance risks,
we have to compromise and try to maximize the expected final utility function. The
utility function is used to describe the extent to which a particular person or company is willing to assume risks to achieve a profit. Our evaluation of risk is based on two premises. The first one is that the investors prefer more to less; the second one is that people are more sensitive to loss than the same amount of profit. These two assumptions are summarized in an increasing concave function called the utility function. Many utility functions have been proposed: for example, if we assume constant absolute risk aversion (CARA), \( U(X) = -e^{-\alpha X} \) would meet our requirements; for constant relative risk aversion (CRRA), \( U_{\gamma}(X) = \frac{X^{1-\gamma}}{1-\gamma} \) would be appropriate; where \( \alpha \) and \( \gamma \) are coefficient of absolute risk aversion and coefficient of relative risk aversion, respectively. For a more complete introduction, please refer to [6], Chapter Four.

As we will see in subsequent sections, the specification of the optimal price might be a formidable task. However, it turns out that using exponential utility facilitates the computation and, thus, the specification of the price. Because our purpose is primarily to introduce the principle of utility maximization in a dynamic setting, our analytic examples use only this class of utility functions. Using this idea of utility maximization, we can solve this type of question with much more complex utility functions and price evolution processes through computer simulation.

In our analytical examples, we use the CARA model and the utility function is given by

\[
U = u(X) = -e^{-\alpha X}
\]  

(5.3)

where \( X \) is the endowment of the company.

**Demand Curve**

As we mentioned before, there is no “fair price” for an insurance product because
the actuarial market is incomplete, but this does not mean a company could charge any price they want for a certain insurance product. One important constraint a firm needs to consider in a market is the demand curve.

In a perfect competitive market, we assume that the market is so large that each participant, either a supplier or a customer, is a price taker. In other words, a single company has to accept the price set by the whole market. If a company is trying to sell a particular product at a higher price than the market price, then it will lose all its customers. This is the case in the stock market, for example, you can not sell a stock at a higher price than the spot market price. But the actuarial market is a different story, since it is close to an oligopoly market. This means that each insurance company has a certain autonomy to set the price for its product. Some customers may have a preference to a certain insurance company because of past experience, customer service level, or some other reasons. Even if a particular insurance company charges more for the same product than its competitors, it may still have may of its customers.

Young and Zaiphopoulou have investigated this in details: in [23] they came up with the theory of reserved price in dynamic setting. According to their work, we can calculate reservation prices for each insurer and buyer. To establish the writer’s reservation price, for example, one examines the maximal expected utility with and without writing the claim. The compensation at which the writer is indifferent between the two alternative investment opportunities yields her reservation price. It is the lowest price the writer would take, which we denote as $P^-$. Similarly, we can calculate the reservation price for the buyer, which is the highest price acceptable to the customer, denoted as $P^+$. In order for a transaction to take place, we need to
have \( P^- \leq P^+ \). Based on these facts, we assume that our insurance company has a given demand curve:

\[
Q = q(P), P \in [P^-, P^+] \tag{5.4}
\]

where \( q(.) \) is a decreasing function. A more detailed discussion will be carried out in section 5.4.

**Loss Process**

For simplicity we suppose that all our customers are the same. Let us consider the property insurance, to be specific, we consider a certain car insurance contract. We assume that the loss process of each our customers is compound poisson. That is, without considering the time value of money, the loss process for each customer is

\[
l_t = \sum_{0}^{N_t} Y_i \tag{5.5}
\]

where \( Y_i' \)s are i.i.d. and \( N_t \) is a poisson process with specific intensity \( \lambda \). It is clear that the severity of loss is determined by the density function of \( Y_i' \)s.

Observation: If we have \( N \) independent customers at the same time, then the total loss process is compound poisson with the same severity, but now the intensity parameter is given by \( N\lambda \).

In case of life insurance, we will investigate a group of people who are at the same age and health condition, thus we can use the same mortality function for each single customer. We consider the typical life-term insurance: we pay 1 at the time of death, if the customer dies before the expiration date of the contract and 0 otherwise.
We have two stochastic processes in our model, namely, the Brownian motion in the financial market and the compound poisson process in of the actuarial market. The first step of our mission is to combine them. This method follows [18].

Let \((\Omega^f, \mathcal{F}^f, P^f)\) be a probability space on which is defined a Brownian motion \(W_t, t \geq 0\), and assume the stock price follows the geometric Brownian, which is given by (5.1) and (5.2). We equip this space with the filtration \(\mathcal{F}^f_t\) which is the augmented one generated by \(W_t\). A purely financial derivative is a random variable \(H^f \in L^2(P^f, \mathcal{F}^f)\).

Let us consider now another filtered probability space \((\Omega^a, \mathcal{F}^a, P^a)\) with a Poisson process \(N_t\) defined. The filtration \(\mathcal{F}^a_t\) is the augmented one generated by the Poisson process. This space carries a pure insurance risk process which describe the development of insurance claims. An insurance claim is a random variable on \((\Omega^a, \mathcal{F}^a, P^a)\) that is in \(L^2(P^a, \mathcal{F}^a)\).

Next we are looking at the adapted space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\), which is defined as the product space of the two individual ones. The construction of the combined model is defined as follows: we let \(\Omega = \Omega^f \times \Omega^a\) and \(P = P^f \otimes P^a\). Let us define the \(\sigma\)-algebra \(\mathcal{N}\) generated by all subsets of the null-sets from \(\mathcal{F}^f \otimes \mathcal{F}^a\), that is:

\[
\mathcal{N} = \sigma\{F \subset \Omega^f \times \Omega^a | \exists G \in \mathcal{F}^f \otimes \mathcal{F}^a : F \subset G, (P = P^f \otimes P^a)G = 0\}
\]

Next we define \(\mathcal{F} = (\mathcal{F}^f \otimes \mathcal{F}^a) \vee \mathcal{N}\) and also the following filtration on the product space (extension of the original filtrations):

\[
\mathcal{F}^1_t = (\mathcal{F}^f_t \otimes \{\emptyset, \Omega^a\}) \vee \mathcal{N}, \mathcal{F}^2_t = (\{\emptyset, \Omega^f\} \otimes \mathcal{F}^a_t) \vee \mathcal{N}
\]

The following lemma shows that we can construct a probability space which models the combination the financial and actuarial risk:

**Lemma 5.1** The filtrations defined above:
1. Satisfy the usual conditions;
2. They are independent;
3. The filtration \((\mathcal{F}_t)_{t \geq 0}\) defined by \(\mathcal{F}_t = \mathcal{F}_1^t \vee \mathcal{F}_2^t\) satisfies the usual conditions, moreover, \(\mathcal{F}_t = (\mathcal{F}_t^1 \otimes \mathcal{F}_t^a) \vee \mathcal{N}\)

For a proof, please refer to [1], page 19.

### 5.3 Background Results on Stochastic Optimization and Expected Utility

We first go through a quick review of Merton’s model which examines the optimal investment strategies of an individual who, endowed with initial wealth, seeks to maximize his or her expected utility of terminal wealth. We are not considering the insurance market at this moment. Suppose we have a stock and a bond available whose prices are given by (5.1) and (5.2). Under strategy \(\{\Delta(t)\}\), at time \(t\), we have \(\Delta(t)\) invested in the stock and the rest of the asset is in bond, then we can easily derive the differential equation satisfied by the wealth:

\[
dX_t = \frac{\Delta(t)}{S_t} dS_t + \frac{X_t - \Delta(t)}{B_t} dB_t
\]

using (5.1) and (5.2), this yields

\[
dX_t = (rX_t + (\mu - r)\Delta(t)) dt + \Delta(t)\sigma dW_t
\]  \hspace{1cm} (5.6)

Let us define the value function as the following:

\[
U(x, t) = \sup_{\{\Delta(t)\} \in \mathcal{A}} \mathbb{E}[u(X_T)|X(t) = x] \hspace{1cm} (5.7)
\]

The set \(\mathcal{A}\) is the set of admissible policies that are \(\mathcal{F}_s\)-progressively measurable, (in which \(\mathcal{F}_s\) is the augmentation of \(\sigma(W_u, t \leq u \leq s)\)) and that satisfy the integrability
condition $E \int_t^T \Delta(s)^2 ds < \infty$. By Theorem 4.5, using Itô formula, the classical principle of dynamic programming yields:

$$U(x,t) \geq E[U(X_{t+h}, t + h)|X_t = x]$$

$$= U(x,t) + E\left[ \int_t^{t+h} [U_t(X_s, s) + U_x(X_s, s)(rX_s + (\mu - r)\Delta(s)) + \frac{1}{2}U_{xx}(X_s, s)\Delta(s)^2] ds | X_t = x \right]$$

Notice that in the setting of Theorem 4.5, $f(t, x, u) = 0$ because we are only concerned with the final utility. By subtracting both sides by $U(x,t)$ and then dividing both sides by $h$, when $h$ approaches zero, we obtain:

$$0 \geq U_t + U_x(x,t)(rx + (\mu - r)\Delta(t)) + \frac{1}{2}U_{xx}(x,t)\Delta(t)^2\sigma^2$$

This yields the HJB equation for the value function:

$$U_t + \max_{\Delta(t)} \left\{ \frac{1}{2}U_{xx}(x,t)\Delta(t)^2\sigma^2 + U_x(x,t)(\mu - r)\Delta(t) \right\} + rxU_x(x,t) = 0 \quad (5.8)$$

where $\Delta(t)$ is admissible. One can compare this with (4.13), but since we are maximizing our objective function here, some alteration is made. Since the utility function $u$ is concave, using the linearity of the state equation (5.6), one can show that the value function itself is concave. If it can be shown that the value function is smooth ($V(x,t) \in C^{2,1}([0, T])$), then the result, which is well known by now (see [28]), yields that the value function equals the unique smooth solution of the HJB equation, and the maximum in (5.8) is well-defined and achieved at

$$\Delta^*(t) = \frac{(\mu - r)U_x}{\sigma^2 U_{xx}} \quad (5.9)$$

Substitue (5.9) into (5.8) and considering the terminal condition, we get the HJB equation:

$$\begin{cases} 
U_t - \frac{U_x^2(\mu - r)^2}{2U_{xx}(x,t)\sigma^2} + rxU_x = 0 \\
U(x,T) = u(x)
\end{cases} \quad (5.10)$$
**Remark** If the HJB equation does not have smooth solutions, one needs to work with a relaxed class of solutions. It turns out that a rich class of weak solutions, which are appropriate for this model, are the so-called *viscosity solutions*. We have discussed this in general stochastic control model in Chapter Four. In context of expected utility methods, Zariphopoulou first introduced viscosity solutions in [27], which have by now become a standard tool for the analysis of stochastic optimization models of market with frictions. For an overview, see the review article [28].

**EXAMPLE.** As discussed before, the exponential utility function will facilitate the computation of the value function, so we use CARA model here. Suppose \( u(x) = -e^{-\alpha x} \), in which \( \alpha > 0 \) is the risk aversion coefficient. Let us try to find a solution satisfying (5.10). We are looking for a solution in the form of \( V(x, t) = -e^{-\alpha x A(t)} + B(t) \). By substituting this into (5.10), we can find the differential equations satisfied by \( A(t) \) and \( B(t) \):

\[
\begin{align*}
A'(t) + rA(t) &= 0 \\
A(T) &= 1 \\
B'(t) &= \frac{(\mu - r)^2}{2\sigma^2} \\
B(T) &= 0
\end{align*}
\]

We can solve them quite easily and thus get a solution of (5.10)

\[
V(x, t) = -e^{-\alpha x e^{r(T-t)}} - \frac{(\mu - r)^2}{2\sigma^2} (T-t)
\]

(5.11)

From (5.9), we can find the optimal strategy:

\[
\Delta(t) = \frac{(\mu - r)}{\sigma^2} e^{-r(T-t)} \frac{e^{-\alpha x}}{\alpha}
\]

(5.12)
Remark: Under the exponential utility function, the money invested in the stock market at time $t$ is equal to \( \frac{(\mu - r) e^{-(r(T-t))}}{\sigma^2 \alpha} \). Notice that this amount is deterministic and not related to our initial wealth $X_t$. This is true only when we use the CARA model. If we have the CRRA utility function, we will always invest a deterministic portion of our wealth to the stock market, according to [16].

5.4 Reservation Prices

In this section we review an important pricing technique, the principle of equivalent utility, which was first introduced by Hodges & Neuberger in [11] for the valuation of European calls in the presence of transaction costs. This principle has been extended to price the risks of uncertainties that do not correspond to fluctuations of a tradable asset; for example, the actuarial risk. The main ingredient of the pricing methodology is the use of individual risk preferences toward the risks that cannot be hedged. The risk preferences are introduced via utility functions for the buyer and the seller of the insurance contract. To establish the writer’s reservation prices, for example, one examines his or her maximal expected utility with and without writing the claim. The compensation at which the writer is indifferent between the two alternatives yields the reservation price.

Here we suppose both parties of the insurance contract have the opportunity to invest in a riskless asset and a risky one with the goal of maximizing their expected utility of terminal wealth. In the absence of any additional insurance liabilities, the investor seeks to maximize the expected utility of the terminal wealth, the value function is defined as:

\[
U^f(w, t) = \sup_{\{\Delta(t)\} \in A} E[u(W_T)|W_t = w]
\]

The set $A$ is the set of admissible policies, that are $\mathcal{F}_s$-progressively measurable in
which $\mathcal{F}$ is the augmented filtration generated by the Brownian motion, and also satisfy the appropriate integrability condition. The superscript $f$ indicates that the investor is free of insurance liabilities or benefits.

If the insurer insures the risk, then we need to define a value function similar to $U^I(w, t)$. We assume that the insurance liability $L_T$ is payable at time $T$ and that the liability cannot be traded after its transfer from the buyer to the insurer and before its expiration. In this time horizon, only trading between the two available market assets is allowed. Then the value function of the agent is defined to be

$$U^I(w, t) = \sup_{\{\Delta(t)\} \in \mathcal{A}'} E[u(W_T - L_T) | W_t = w]$$

For the insurance company, $U^I$ is the value function if the company insures the risk $L_T$, while $U^f$ is the value function if the company does not accept the risk. Note that the set of admissible strategies $\mathcal{A}'$ in $U^I$ is different from the one in $U^f$ because the former also contains the information of the actuarial randomness.

**Definition 5.2** The reservation price of the insurer, $P^I(w, t)$, is defined as the compensation such that

$$U^f(w, t) = U^I(w + P^I, t)$$

Similarly, the reservation price of the buyer, $P^B(w, t)$, is defined as the obligation such that

$$U^I(w - P^B, t) = U^I(w, t)$$

In the definition of $P^I$, the left hand side is the maximal expected utility of the insurance company when it does not write the insurance. The right hand side is the one where the company writes the insurance: the initial wealth is increased by the premium $P^I$, but it also has a liability $L_T$. Therefore, $P^I$ defines the lowest price
that is acceptable for the insurer to take the insurance risk, and of course, the agent is glad to sell the risk at a higher price. We have the similar interpretation for $P^B$ which is the highest price that is acceptable to the buyer, otherwise the buyer would rather take the risk by itself. It can be shown that if we use CARA model and use the same risk aversion coefficient $\alpha$ for both parties, then $P^I = P^B$. However, in reality, one expects that the insurer’s risk aversion will be less than the buyer’s, from which it will follow that $P^I < P^B$.

According to the above definition, two agents must trade the liability at prices within above spread, i.e. at some $P \in [P^I, P^B]$. Thus the demand curve is supported in $[P^I, P^B]$: the demand curve does not exist for $P < P^I$ because the insurer is not willing to offer such a business; for $P^I \leq P \leq P^B$, the demand curve is continuously decreasing to 0; when $P > P^B$, the demand curve vanishes.

5.5 Optimal Property Insurance Price Under Lump Sum Premium Payment

In this section let us first investigate the optimal property insurance problem where all the premiums are paid at the beginning of the contract. We have the following assumptions: all the insurance policies will cover the period between 0 and $T$. We do not take any more customers after $t = 0$. Also we assume that all the premiums are paid at $t = 0$. This model is static from the point of view that the price is determined at the beginning and is not allowed to change. On the other hand it is dynamic in the following sense: we can trade in the financial market continuously at any time between 0 and $T$. We will assume that we have a complete financial market, i.e. we assume there is no market frictions such as transaction cost and market constraint. For a review of financial market with frictions, see [25].
Now our wealth process is governed by the following SDE:

$$\begin{align*}
\begin{cases}
   dX_t &= (rX_t + (\mu - r)\Delta(t))dt + \Delta(t)\sigma dW_t - dL_t \\
   X(0-) &= x
\end{cases}
\end{align*}$$

(5.13)

where $L_t$ is the loss process which follows:

$$\begin{align*}
\begin{cases}
   dL_t &= Y_{N_t}dN_s \\
   L_0 &= 0
\end{cases}
\end{align*}$$

(5.14)

where $Y_i$'s are i.i.d. with certain distribution, which describes the severity of the total loss. $N_s$ is a poisson process with parameter $\tilde{\lambda}$. Here we can not derive our HJB equation directly from Theorem 3.6, as we did in the previous section. The difficulty lies in the fact that our wealth has jumps at the occurrences of loss.

**Existence and Uniqueness**

We start with the existence and uniqueness of the solution to the problem: given the assumption of the model, can we say that there exists an optimal price which maximize the final expected utility? Also, if we can assure the existence, is the solution unique? It turns out that we may not guarantee uniqueness of the solution, the reason is that we think $P$ as a control parameter; when we adjust $P$, there are two consequences. First, our initial endowment is changed because our premium income will be different as we change $P$. If our initial endowment is $X(0-) = x$, after we collect the premium, our wealth immediately after collecting the premium at time 0 is given by $X(0) = x + Pq(P)$. Second, since we have different number of customers, the intensity of the poisson process $N_t$ in (5.14) is different, and it is given by $\tilde{\lambda} = q(P)\lambda$. Given the concavity of the utility function, it can be shown that the value function is increasing with respect to the initial endowment. It is also clear that the value function is a decreasing function of the intensity parameter of
Poisson process. However, it is hard to predict what is happening when both factors are changing. Fortunately, the existence is clear.

Let the demand curve \( q = q(P) \), and the utility function \( u = u(X) \) be given. For each customer, suppose the loss process is compound Poisson with intensity parameter \( \lambda \) and the claims \( \{Y_1, Y_2, Y_3, \ldots\} \) are i.i.d., defined on \( (\Omega, \mathcal{F}, P) \).

**Definition 5.3** We say \( \Delta(t) \) is an admissible strategy if it satisfies the following:

1. \( \Delta(t) \) is \( \{\mathcal{F}_t\} \) adapted where \( \{\mathcal{F}_t\} \) is the augmented filtration generated by \( W_t \) and \( N_t \).

2. \( \Delta(t) \) satisfies the integrability condition \( \int_0^T \Delta^2(t) < \infty \).

3. The following stochastic differential equation admits a unique strong solution:

\[
\begin{cases}
    dX_t = (rX_t + (\mu - r)\Delta(t))dt + \Delta(t)\sigma dW_t \\
    X_0 = x
\end{cases}
\]  

(5.15)

It is clear that given price \( P \), the wealth process satisfies the following stochastic differential equation:

\[
\begin{cases}
    dX_t = (rX_t + (\mu - r)\Delta(t))dt + \Delta(t)\sigma dW_t - Y_{N_t} dN_t^P \\
    X_0 = x + Pq(P)
\end{cases}
\]  

(5.16)

where \( \Delta(t) \) is an admissible strategy and \( N_t^P \) is a Poisson process with parameter \( q(P)\lambda \). If we use \( X^P(.) \) to denote the solution of (5.16), our objective is:

**Optimal Property Insurance Price Problem (Lump Sum Payment)**: Find \( P^* \) such that

\[
\sup_{\Delta(t) \in A} E[u(X^{P^*}(T))] = \max_{P \in [P^-, P^+]} \sup_{\Delta(t) \in A} E[u(X^P(T))]
\]
Lemma 5.4: For any admissible strategy $\Delta(t)$, (5.16) admits a unique strong solution.

Proof: By definition, for any admissible strategy $\Delta(t)$ and initial condition, we know that (5.15) admits a unique strong solution. For any $T, N_T^P < \infty$ almost surely. Conditioning on $N_T^P = k$ and $Y_1 = y_1, Y_2 = y_2, \ldots Y_k = ye$, suppose the jumps take place at $0 < t_1 < t_2 < \ldots < t_k < T$, we can construct a solution as following:

For $t \in [0, t_1)$, let $X^1_t$ be the unique strong solution of the following
$$
\begin{cases}
  dX_t = (rX_t + (\mu - r)\Delta(t))dt + \Delta(t)\sigma dW_t \\
  X_0 = x + Pq(P)
\end{cases}
$$

For $t \in [t_1, t_2)$, let $X^2_t$ be the unique strong solution of the following
$$
\begin{cases}
  dX_t = (rX_t + (\mu - r)\Delta(t))dt + \Delta(t)\sigma dW_t \\
  X(t_1) = X^1_{t_1} - y_1
\end{cases}
$$

For $t \in [t_2, t_3)$, let $X^3_t$ be the unique strong solution of the following
$$
\begin{cases}
  dX_t = (rX_t + (\mu - r)\Delta(t))dt + \Delta(t)\sigma dW_t \\
  X(t_2) = X^2_{t_2} - y_2
\end{cases}
$$

Continuing this process, we can find a solution for each subinterval. Let $X(t)$ be defined such that $X(t) = X^i(t)$ when $t \in [t_{i-1}, t_i)$, it is clear that $X(t)$ is a solution of (5.16). The uniqueness follows from the uniqueness of solution of (5.15).

Now let us prove the existence result:

Theorem 5.5 Suppose the utility function is convex and Lipschitz continuous, and $Y_i$’s are in the class of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, then the Optimal Insurance Price Problem (Lump Sum Payment) has a unique solution.

To prove Theorem (5.5), we need the following regularity result:
Proposition 5.6 Suppose \( X_t \) is the strong solution of

\[
\begin{align*}
  dX_t &= (rX_t + (\mu - r)\Delta(t))dt + \Delta(t)\sigma dW_t - Y_N dN_t \\
  X_0 &= x
\end{align*}
\] (5.17)

Let \( \lambda \) be the intensity parameter of the poisson process and \( \Delta(t) \) be a Lipschitz continuous function of \( X(t) \). In addition, suppose \( u \) is Lipschitz continuous and \( Y_i \in L^2(\Omega, \mathcal{F}, P) \), then \( E[u(X_T)] \) is a continuous function: (a) with respect to \( x \), (b) with respect to \( \lambda \).

Proof of proposition (5.6): (a) Suppose \( X_t \) is the solution of the following stochastic differential equation of

\[
\begin{align*}
  dX_t &= (rX_t + (\mu - r)\Delta(t))dt + \Delta(t)\sigma dW_t \\
  X_0 &= x
\end{align*}
\]

If we follow the standard argument using recursive approximation and Gronwall’s inequality, which can be found in many literatures, see [15], section 4.5 for example, we can show that \( X_t \) is continuous with respect to initial conditions:

\[
\lim_{y \to x} \sup_{t \in [0,T]} E(|X^y_t - X^x_t|^2) = 0
\]

where \( X^x_t \) and \( X^y_t \) are solutions with initial conditions \( x \) and \( y \). Furthermore we have the following estimate:

\[
E(|X^y_t - X^x_t|^2) \leq K(L,T)(y - x)^2, \forall t \in [0, T]
\] (5.18)

In this expression, \( K(L,T) \) is a constant depends on expiration date \( T \) and Lipschitz constant \( L \). Now we consider equation (5.17), that is, we take the jumps into consideration. Consider a fixed path of the compound poisson process; that is, condition on \( N_T = k \) and \( Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k \), if the jumps take place at \( 0 < t_1 < t_2 < \ldots < t_k < T \), both \( X^y_t \) and \( X^x_t \) jump at the same time instant with the
same size, and their difference is not changed. Therefore, estimation (5.18) is still valid. Using Jensen’s inequality and the Lipschitz condition, we obtain

\[
|E[u(X^x_T)] - E[u(X^y_T)]|^2 \leq (E|u(X^x_T) - u(X^y_T)|)^2 \\
\leq L^2(E|X^x_T - X^y_T|)^2 \\
\leq L^2E|X^x_T - X^y_T|^2 \\
\leq L^2K(T)(y - x)^2
\]

This proves part (a)

(b) Suppose for \(i = 1, 2\), \(X^{(i)}_t\) is the solution of the following:

\[
\begin{cases}
    dX_t = (rX_t + (\mu - r)\Delta(t))dt + \Delta(t)\sigma dW_t - Y_{N_t^i}dN_t^i \\
    X_0 = x
\end{cases}
\]

where \(N_t^i\) is a poisson process with \(\lambda_i\), we need to show

\[
\lim_{\lambda_2 \to \lambda_1} |EX^{(2)}(T) - EX^{(1)}(T)| = 0
\]

Suppose \(\lambda_2 > \lambda_1\) and let \(h = \lambda_2 - \lambda_1\). Now we suppose \(\{Z_1, Z_2, \ldots\}\) is another i.i.d. sequence on \((\Omega, F, P)\) with the same distribution function as \(Y_t^i's\), and \(M_t\) be an independent poisson process with parameter \(h\). (Now we need to modify our definition of the filtration \(\{F_t\}_{t \geq 0}\) so that the information of \(M_t\) is included)

Consider \(X^{(3)}_t\) which is the solution of the following:

\[
\begin{cases}
    dX_t = (rX_t + (\mu - r)\Delta(t))dt + \Delta(t)\sigma dW_t - Y_{N^1_t}dN^1_t - Z_{M_t}dM_t \\
    X_0 = x
\end{cases}
\]

Because \(M_t + N^1_t\) and \(N^2_t\) are both poisson processes with parameter \(\lambda_2\), also \(Z_t^i's\) and \(Y_t^i's\) have the same distribution, by the uniqueness of solution of (5.19), the stochastic
processes \(X_t^{(2)}\) and \(X_t^{(3)}\) have the same finite dimensional distribution. So we only need to show:

\[
\lim_{h \to 0} |E u(X_T^{(3)}) - E u(X_T^{(1)})| = 0
\]

On event \(A_0 = \{M_T = 0\}\), it is clear that \(X_T^{(3)} = X_T^{(1)}\)

On event \(A_j = \{M_T = j\}\), for \(j = 1, 2, 3, \ldots\), we can follow the argument in (a) and show that

\[
|E[u(X_T^{(1)})] - E[u(X_T^{(3)})]| \leq LC(T)j E(Y^2)^{1/2}
\]

Therefore, on \(\Omega\) we have

\[
|E[u(X_T^{(1)})] - E[u(X_T^{(3)})]| \leq LC(T)E(Y^2)^{1/2} \sum_{i=0}^{\infty} jP(A_i)
\]

\[
= LC(T)E(Y^2)^{1/2}E(M_T)
\]

\[
= LC(T)E(Y^2)^{1/2}hT
\]

This proves that \(|E[u(X_T^{(1)})] - E[u(X_T^{(3)})]|\) approaches zero as \(h \to 0\).

Now the proof of Theorem (5.5) is straightforward.

**Proof of Theorem (5.5):** Our \(P\) takes value in a bounded closed interval \([P^-, P^+]\).

If we adjust \(P\), the two parameters \(x\) and \(\lambda\) change continuously. By proposition(5.6), \(E[u(X_T^P)]\) is a continuous function of \(P\). Therefore, the maximizer exists.

### HJB Equation

Let us use the dynamic programming principle to derive the HJB equation for our value function under the presence of an insurance contract. We define the value function as before, but we need to consider our payments for the claims.

\[
U(x, t) = \sup_{\Delta(t) \in \mathcal{A}} E[u(X_T)|X_t = x]
\] (5.21)
where $X(t)$ is the solution of (5.16). Let $\{L_t\}$ be the total loss process of our customers defined in section 5.2. In this model we assume that the number of customers remains constant and the loss of each customer is compound poisson, so that $\{L_t\}$ is also compound poisson with a different intensity parameter, say, $\tilde{\lambda}$.

Now we derive the HJB equation by analyzing the infinitesimal behavior of the loss process. In general, the dynamic programming principle gives:

$$U(x,t) \geq E[U(X_{t+h}, t+h)|X_t = x]$$ \hspace{1cm} (5.22)

Let us assume that $\{\Delta^*(s) : t \leq s \leq t+h\}$ is the optimal admissible strategy that the insurer follows. Denote by $X^*_s$ the wealth under $\{\Delta^*(s)\}$. Suppose our customers do not have an accident during $[t, t+h]$. According to the asymptotic behavior of the poisson process in an infinitesimal small time interval, the probability of no accident occurring is $1 - \tilde{\lambda}h + O(h^2)$. In this case, we do not need to pay anything and the insurer’s value function equals

$$E[U(X^*_t, t+h)|X_t = x].$$

On the other hand, if our customers encounter one accident during $[t, t+h]$, we need pay out an amount of money $Y$, where $Y$ is a random variable which has a certain distribution function, and this happens with a probability $\tilde{\lambda}h + O(h^2)$. In this case, the value function becomes

$$E[U(X^*_t - Y, t+h)|X_t = x].$$

Since the probability of more than one accident happening is of order $h^2$, (5.22) becomes

$$U(x,t) = (1 - \tilde{\lambda}h)E[U(X^*_t, t+h)|X_t = x]$$

$$+ (\tilde{\lambda}h)E[U(X^*_t - Y, t+h)|X_t = x] + O(h^2).$$

60
We use equality instead of inequality since we assume that our strategy is optimal and the maximum is achieved at \( \{ \Delta^*(s) \} \). By assuming enough regularity conditions and appropriate integrability on the value functions and their derivatives, we can apply the Itô formula to \( U(X_{t+h}^*, t+h) \):

\[
E[U(X_{t+h}^*, t+h)] = U(x, t) + E\left[ \int_t^{t+h} \{ U_t(X_s^*, s) + (rX_s^* + (\mu - r)\Delta^*(s))U_x(X_s^*, s) + \frac{1}{2}\sigma^2\Delta^2(s)U_{xx}(X_s^*, s) \} ds | X_t = x \right].
\]

We can obtain a similar expression for \( E[U(X_{t+h}^* - Y, t+h) | X_t = x] \):

\[
E[U(X_{t+h}^* - Y, t+h)|X_t = x] = E[U(x - Y, t)] + E\left[ \int_t^{t+h} \{ U_t(X_s^* - Y, s) + (rX_s^* + (\mu - r)\Delta^*(s))U_x(X_s^* - Y, s) \} ds | X_t = x \right] + E\left[ \int_t^{t+h} \frac{1}{2}\sigma^2\Delta^2(s)U_{xx}(X_s^* - Y, s) ds | X_t = x \right].
\]

By substituting the above equations into (5.22), we get the following equality

\[
U(x, t) = U(x, t)(1 - \lambda h)
\]

\[
+ \lambda h E[U(x - Y, t)] + (1 - \lambda h) E\left[ \int_t^{t+h} \{ U_t(X_s^*, s) + (rX_s^* + (\mu - r)\Delta^*(s))U_x(X_s^*, s) \} ds | X_t = x \right] + (1 - \lambda h) E\left[ \int_t^{t+h} \frac{1}{2}\sigma^2\Delta^2(s)U_{xx}(X_s^*, s) ds | X_t = x \right] + \lambda h E\left[ \int_t^{t+h} \{ U_t(X_s^* - Y, s) + (rX_s^* + (\mu - r)\Delta^*(s))(X_s^* - Y, s) \} ds | X_t = x \right] + \lambda h E\left[ \int_t^{t+h} \frac{1}{2}\sigma^2\Delta^2(s)U_{xx}(X_s^* - Y, s) ds | X_t = x \right] + O(h^2).
\]

If we subtract \( U(x, t) \) from both sides and divide by \( h \), we obtain:

\[
0 = \lambda(E[U(x - Y, t)] - U(x, t))
\]

\[
\frac{1 - \lambda h}{h} \left[ \frac{1}{E\left[ \int_t^{t+h} \{ U_t(X_s^*, s) + (rX_s^* + (\mu - r)\Delta^*(s))U_x(X_s^*, s) \} ds | X_t = x \right]} - \frac{1}{E\left[ \int_t^{t+h} \frac{1}{2}\sigma^2\Delta^2(s)U_{xx}(X_s^*, s) ds | X_t = x \right]} \right] + O(h).
\]
By taking the limit as \( h \) goes to 0 from the right, we get
\[
0 = \tilde{\lambda}(E[U(x - Y, t)] - U(x, t)) + U_t + (r x + (\mu - r) \Delta^*(t)) U_x \\
+ \frac{1}{2} \sigma^2 \Delta^2(t) U_{xx}.
\]
Here \( \Delta^*(t) \) is the optimal strategy, in terms of stochastic control, the above equation can be written as
\[
0 = U_t + \tilde{\lambda}(E^a[U(x - Y, t)] - U(x, t)) + r x U_x \\
+ \max_{\{\Delta(t)\} \in A} \{(\mu - r) \Delta(t) U_x + \frac{1}{2} \sigma^2 \Delta^2(t) U_{xx}\}.
\]
Notice that our the expected value operator is only related to the randomness of \( Y \), which is a purely actuarial risk and independent of the financial one, so the expected value is taken on \((\Omega^a, \mathcal{F}^a, P^a)\). We use a superscript \( a \) to denote this. If it can be shown that the value function is smooth \((U(x, t) \in C^{2,1}(\mathbb{R} \times [0, T]))\) and convex, then the value function equals the unique smooth solution of the HJB equation, and the maximum is achieved at \( \Delta^*(t) \):
\[
\Delta^*(t) = -\frac{(\mu - r) U_x}{\sigma^2 U_{xx}}, \tag{5.23}
\]
and the resultant HJB equation, together with the terminal condition is given by:
\[
\begin{cases}
U_t + r x U_x = \frac{[(\mu - r) U_x]^2}{2 \sigma^2 U_{xx}} + \tilde{\lambda}(E^a[U(x - Y, t)] - U(x, t)) = 0 \\
U(x, T) = u(x)
\end{cases}
\tag{5.24}
\]
\textbf{Remark:} Since (5.9) is the same as (5.23), it appears that the optimal investment strategies of the insurance company are the same with and without an insurance contract liability. However, this may not be true because the value function in (5.23) involves the insurance information, and is different from that in (5.9).

\textbf{Solution to the Optimal Insurance Price Problem (Lump Sum Payment)}
Consider the time period \([0, T]\). If the initial wealth \(x\), utility function \(U = u(X)\) and demand curve \(q = q(P)\) are given, what is the best strategy for the insurance company? Here we have two decision variables \(\Delta(t)\) and \(P\), where \(\Delta(t)\) is the amount of our asset which is invested in the stock market at time \(t\). The remainder, \(X(t) - \Delta(t)\) is invested in the bond. \(P\) is the price we charge for each customer. This price is determined at time \(t = 0\) and is not subject to change until the end of contract \(t = T\).

For fixed \(P\), we are trying to maximize the expected utility of terminal wealth:

\[
\bar{U}(x, 0; P) = \sup_{\{\Delta(t)\}} E[u(X_T^P)|X(0-) = x] = U(x + Pq(P), 0),
\]

(5.25)

where \(\{\Delta(t)\}\) are the admissible policies that are \(\mathcal{F}_s\)-progressively measurable (\(\mathcal{F}_s\) is as defined in Lemma 5.1), and that satisfy integrability condition.

Given the demand curve \(Q = q(P)\), if the insurer sets the price to be \(P\), then there will be \(q(P)\) customers. If we have initial wealth \(x\) before time 0, after the insurer collects the premium at \(t = 0\), the wealth is given by \(X(0) = x + P \star q(P)\). Also, since we assume that the number of customers remains constant during the time period \([0, T]\), the total loss process is compound poisson with parameter \(\bar{\lambda} = q(P) \star \lambda\).

Our approach will be the following:

Step 1. Fix any price \(P\), we change the intensity parameter to \(\bar{\lambda} = q(P) \star \lambda\) and the initial wealth to \(X(0) = x + P \star q(P)\).

Step 2. Solve the HJB equation, considering \(P\) as a parameter, and denote the solution by \(\bar{U}(x, t; P)\).

Step 3. Vary \(P\) within \([P^-, P^+]\), then find the \(P\) which maximizes the utility function \(\bar{U}(x, 0; P)\). Denoted this price by \(P^*\). It is clear that \(P^*\) is the optimal price we should select to maximize the final expected utility.

**Remark:** In step 3, if the solution is smooth with respect to \(P\), we can find the
optimal price using the classic techniques, for example, the first and second derivative tests. When the solution is not differentiable, numerical methods are necessary.

**EXAMPLE**

Now let us study a concrete example to implement the ideas discussed above. An exponential utility function facilitates the calculation of the value function, so in this example we assume that our utility is given by $u(x) = -e^{-\alpha x}$. We will see later that even with this simple utility function, the result is still rather complex. Next, we need a demand curve. Since our purpose is to introduce the idea of pricing insurance using utility maximization, we choose the simplest one, the linear demand curve

$$q(P) = H - kP, \forall P \in [P^-, P^+].$$

We want to give an economic interpretation of the parameters $H$ and $k$. It is clear that $k$ is the elasticity of the demand curve. It describes how sensitive the customers are to the price change of the goods. One may think $H$ is the number of customers when $P = 0$. However, this is not correct because we have mentioned that the demand curve is supported in $[P^-,P^+]$ and does not exist when $P < P^-$. We can give another interpretation: we know that the number of customer will become 0 at the price of $P^+$, so we have $H = kP^+$. In general, if the demand curve is given, it is easy to find $P^+$.

Let us start with the following HJB equation:

$$
\begin{align*}
U_t + rxU_x - \frac{[(\mu - r)U_x]^2}{2\sigma^2U_{xx}} + \tilde{\lambda}(E^a[U(x - Y, t)] - U(x, t)) &= 0 \\
U(x, T) &= -e^{-\alpha x}
\end{align*}
\tag{5.26}
$$

From section (5.3), we know that

$$V(x, t) = -e^{-\alpha xe^{(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T-t)}$$
solves the following partial differential equation:

\[
\begin{cases}
U_t - \frac{U_x^2(\mu - r)^2}{2U_{xx}(x,t)\sigma^2} + rxU_x = 0 \\
U(x,T) = -e^{-\alpha x}
\end{cases}
\] (5.27)

Now we are looking for a solution to (5.26) in the form 

\[U(x,t) = V(x,t)\Psi(t)\]. By substituting this expression into (5.26), we get:

\[V_t\Psi + V\Psi' + rxV_x\Psi - \frac{(\mu - r)^2V_x^2}{2\sigma^2\Psi V_x} + \lambda[E^a(V(x-Y,t)\Psi(t) - V(x,t)\Psi(t)) = 0
\]

Because \(V\) solves the HJB equation (5.27), the first, third and fourth term cancel and we have:

\[V\Psi' + \lambda[E^a(V(x-Y,t) - V(x,t)]\Psi = 0.
\] (5.28)

Let \(f(x)\) be the distribution function of a single claim, then

\[E^aV(x-Y,t) - V(x,t) = \int V(x,t)\exp\{-\alpha ye^{r(T-t)}\}f(y)dy - V(x,t) = V(x,t)[E^ae^{\alpha Ye^{r(T-t)}} - 1]
\]

\[= V(x,t)[M_Y(e^{\alpha e^{r(T-t)}}) - 1],
\]

where \(M_Y(.)\) is the moment generate function of \(Y\). Considering the boundary condition of (5.26), we obtain the following ordinary differential equation which can be easily solved

\[
\begin{cases}
\Psi'(t) + \lambda[M_Y(e^{\alpha e^{r(T-t)}}) - 1]\Psi(t) = 0 \\
\Psi(T) = 1
\end{cases}
\]

The solution is given by

\[\Psi(t) = \exp\{\int_t^T \lambda[M_Y(e^{\alpha e^{r(T-s)}}) - 1]ds\}.
\]

Therefore we have solved (5.26):

\[U(x,t) = -\exp\{-\alpha xe^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T-t) + \int_t^T \lambda[M_Y(e^{\alpha e^{r(T-s)}}) - 1]ds\}(5.29)
\]
Now if the insurer, given the wealth \( x \), has set the price to be \( P \), then there will be \( q = H - kP \) customers entering contracts with the company. There are two consequences. First, we have more initial wealth because of the premium, and it is given by

\[
X(0) = x + qP = x + PH - kP^2
\]

Second, we would have the liability of paying \( H - kP \) customers’ loss between 0 and \( T \); so we should substitute \( \tilde{\lambda} = (H - kP)\lambda \).

By substituting these values into (5.29), at time \( t = 0 \), the value function of the insurer is:

\[
\bar{U}(x, 0; P) = U(x + Pq(P), 0) = -\exp\{-\alpha(x + PH - kP^2)e^{rT} - \frac{(\mu - r)^2}{2\sigma^2}T
+ \int_0^T (H - kP)\lambda[M_Y(e^{\alpha e^{r(T-s)}}) - 1]ds\}
\]

here \( P \) is the only decision variable. Differentiating with respect to \( P \), we get

\[
\frac{\partial U}{\partial P}(x, 0; P) = U\{-\alpha(H - 2kP)e^{rT} - k \int_0^T \lambda[M_Y(e^{\alpha e^{r(T-s)}}) - 1]ds\}.
\]

If we set the partial derivative equal to zero, a critical point is found:

\[
P^* = \frac{H}{2k} + \frac{e^{-rT}}{2\alpha} \int_0^T \lambda[M_Y(e^{\alpha e^{r(T-s)}}) - 1]ds.
\]

(5.30)

To verify \( P^* \) maximize \( \bar{U}(x, 0, P) \), we take the second derivative, and plug in \( P^* \):

\[
\frac{\partial^2 U}{\partial P^2}(x, 0; P) = 2k\alpha e^{rT}U(x, 0; P^*) < 0.
\]

Let us investigate (5.30) further. Note that the \( \alpha \) in (5.30) is the risk aversion coefficient of the insurer, which can be used to determine the reservation price \( P^- \), the least price at which we are willing to accept the risk. According to page 271,[23], it is given by

\[
P^- = \frac{1}{\alpha}e^{-rT} \int_0^T \lambda[M_Y(e^{\alpha e^{r(T-s)}}) - 1]ds.
\]
On the other hand, after we choose a demand curve, we can determine $P^+$ by setting $q(P) = 0$ and solving for $P$. In this case, $P^+ = \frac{H}{k}$. Comparing these results with (5.30), we see that $$P^* = \frac{P^- + P^+}{2}$$

In this example, it is clear that $P^*$ falls into the interval $[P^-, P^+]$. However, this may not be true in general. The reader will find a numerical example in Chapter Six. We have the following discussion:

1. If $P^* \leq P^-$, then we will set the price to be $P^-$, which means we will take as many customers as we can, as long as the price is no less than the indifference price.
2. If $P^- < P^* < P^+$, our optimal price is given by (5.30).
3. If $P^* \geq P^+$, then we will set the price to be $P^+$. The result is $q=0$, that is, our best strategy is not to take any customers.

Now let us summarize our findings above. To be honest, it should not be called a theorem since it is just a summary of a lengthy calculation. However, for the convenience of the reader, we present our results in the following theorem:

**Theorem 5.7** Under the settings of Optimal Property Insurance Problem (Lump Sum Payment), if the utility function is CARA with parameter $\alpha$ and the demand curve is linear with elasticity $k$ and intercept $H$, then to maximize the expected final utility, the only critical point of the price is given by:

$$P^* = \frac{H}{2k} + \frac{e^{-rt}T}{2\alpha} \int_0^T \lambda[M_Y(e^{\alpha e^{(T-s)}}) - 1]ds$$

Furthermore, $P^*$ is the middle point of the interval $[P^-, P^+]$, and it is the optimal price.

**Remark:** When we choose a linear demand curve $q(P) = H - kP$, the total premium collected is $q(P)P = HP - kP^2$, which is a quadratic function with maximum achieved
at $P = \frac{H}{2k} = \frac{P^+}{2}$. When $0 < P < \frac{H}{2k}$, if we increase $P$, we will collect more premium. On the other hand, the number of customers will decrease; thus we will have a larger initial wealth and less future liability. Therefore, the optimal price must be greater than $\frac{H}{2k}$. $P^*$ in (5.30) is obviously greater than this value.

We will continue our discussion of this example (with a quadratic demand curve) by performing some numerical analysis in the next chapter.

### 5.6 Optimal Property Insurance Price Under Continuous Premium Payment

In this section, instead of assuming that all the premiums are collected at $t = 0$, we suppose that our customers make the payment continuously. That is, we announce our price, which is a premium rate, say $P$ dollars per unit time, at $t = 0$, and our customers will make their payment continuously according to this rate. If we make this change, we will see later that our HJB equation will change considerably.

We still consider the time period $[0, T]$. During the time period we have access to a riskless bond and a risky stock whose price processes $B_t$ and $S_t$ is given by (5.1) and (5.2) respectively. Our utility function, demand curve and the loss process of a single customer are given by (5.3), (5.4) and (5.5) respectively. The only difference is that the meaning of $P$ in this section is different from that in the previous section: in previous sections we denote the price by $P$, but here $P$ means the rate of payment in which customers make their payment to the insurer.

Suppose the initial wealth of the insurer is $x$. The stochastic partial differential
equation which describes the evolution of the wealth is given by the following:

\[
\begin{align*}
\begin{cases}
dX_t &= rX_t dt + (\mu - r)\Delta_t + \sigma \Delta_t dB_t + Pq(P) dt - YdN_t^P \\
X_0 &= x
\end{cases}
\end{align*}
\] (5.31)

where \(\Delta(t)\) is our investment strategy, which is the amount of our wealth invested in the stock market. We also assume that \(\{\Delta(t)\}\) is admissible. In (5.31), the first three terms describe the change of wealth resulting from our investment in the financial market, the fourth term is due to premium collection, and the last one is the claim payments. \(N_t^P\) is poisson with parameter \(q(P)\lambda\).

Our objective is:

**Optimal Property Insurance Price Problem (Continuous Payment):** Find the premium rate \(P^*\) such that

\[
\sup_{\Delta(t)\in \mathcal{A}} E[u(X_t^P(T))] = \max_{P \in [P^-, P^+]} \sup_{\Delta(t)\in \mathcal{A}} E[u(X_t^P(T))]
\]

where \(X_t^P(t)\) is the solution of (5.31). We define the value function \(U(x, t)\) the same as before:

\[
U(x, t) = \sup_{\{\Delta(t)\} \in \mathcal{A}} E[u(X_t)] | X_t = x
\]

where \(\mathcal{A}\) is the admissible set defined as before. Suppose that at time \(t\), our wealth is \(X_t = x\). If we follow the same reasoning as last section, we obtain the following inequality:

\[
U(x, t) \geq E[U(X_{t+h}, t + h)|X_t = x]
= (1 - \tilde{\lambda}h)E[U(X_{t+h}, t + h)|X_t = x] + (\tilde{\lambda}h)E[U(X_{t+h} - Y, t + h)|X_t = x]
\]

Here we have taken the claim payments into account, so we consider the wealth process between the claim payments, which is given by (5.31), except that the last term being removed.
By assuming proper integrability and smoothness of the coefficients, $U(x, t)$ satisfies:

\[
E[U(X_{t+h}^*, t+h)|X_t = x] = U(x, t) + E\left[\int_t^{t+h} \{U_t(X_{s}^*, s) + (rX_s^* + (\mu - r)\Delta^*(s))U_x(X_{s}^*, s)\}ds \big| X_t = x\right]
\]

where

\[
\Delta^*(s) = \frac{1}{2}\sigma^2\Delta^2(s)U_{xx}(X_{s}^*, s)
\]

We can obtain a similar expression for $E[U(X_{t+h}^* - Y, t+h)|X_t = x]$:

\[
E[U(X_{t+h}^* - Y, t+h)|X_t = x] = E[U(x - Y, t)] + E\left[\int_t^{t+h} \{U_t(X_{s}^* - Y, s) + (rX_s^* + (\mu - r)\Delta^*(s))U_x(X_{s}^* - Y, s)\}ds \big| X_t = x\right]
\]

where $X_t^*$ is the wealth process under optimal strategy.

By substituting the above into the original inequality, we obtain the following

\[
U(x, t) = U(x, t)(1 - \bar{\lambda}h) + \bar{\lambda}hE[U(x - Y, t)] + (1 - \bar{\lambda}h)E\left[\int_t^{t+h} \{U_t(X_{s}^*, s) + (rX_s^* + (\mu - r)\Delta^*(s))U_x(X_{s}^*, s)\}ds \big| X_t = x\right]
\]

\[
+ (1 - \bar{\lambda}h)E\left[\int_t^{t+h} Pq(P)U_x(X_{s}^*, s) + \frac{1}{2}\sigma^2\Delta^2(s)U_{xx}(X_{s}^*, s)ds \big| X_t = x\right]
\]

\[
+ \bar{\lambda}hE\left[\int_t^{t+h} \{U_t(X_{s}^* - Y, s) + (rX_s^* + (\mu - r)\Delta^*(s))U_x(X_{s}^* - Y, s)\}ds \big| X_t = x\right]
\]

\[
+ \bar{\lambda}hE\left[\int_t^{t+h} Pq(P)U_x(X_{s}^* - Y, s) + \frac{1}{2}\sigma^2\Delta^2(s)U_{xx}(X_{s}^* - Y, s)ds \big| X_t = x\right] + O(h^2)
\]

The equality is used because we assume the optimal strategy $\Delta^*$ is adopted. By subtracting $U(x, t)$ from both sides and let $h$ goes to zero, we obtain:

\[
0 = \bar{\lambda}(E[U(x - Y, t)] - U(x, t)) + U_t + (rX + \mu - r)\Delta^*(t) + Pq(P)U_x + \frac{1}{2}\sigma^2\Delta^2(t)U_{xx}
\]

It is easy to see that $\Delta^*(t) = -\frac{\mu - r}{U_{xx}\sigma^2}$ is the optimal strategy, substitute this into above equation, we get:

\[
0 = U_t + \bar{\lambda}(E[U(x - Y, t)] - U(x, t)) + rU_x + Pq(P)U_x - \frac{(\mu - r)^2U_x^2}{2U_{xx}\sigma^2}
\]
Therefore we have derived the HJB equation satisfied by $U(x,t)$.

**EXAMPLE:**

Now let us look at an example, where we still use our favorite exponential utility function. The HJB equation becomes:

\[
\begin{aligned}
U_t + \lambda(E[U(x - Y, t)] - U(x, t)) + r x U_x + P q(P) U_x - \frac{(\mu - r)^2 U_x^2}{2 U_{xx} \sigma^2} &= 0 \\
U(x, T) &= -e^{-\alpha x}
\end{aligned}
\]  

Motivated by the methodology we used to solve the last example, let us first solve the following equation

\[
\begin{aligned}
U_t + r x U_x + P q(P) U_x - \frac{(\mu - r)^2 U_x^2}{2 U_{xx} \sigma^2} &= 0 \\
U(x, T) &= -e^{-\alpha x}
\end{aligned}
\]  

i.e., we ignore the term generated by the claim payment at this time. We are trying to find a solution of (5.33) in the following form:

\[V(x, t) = -e^{-\alpha x A(t) + B(t)}\]

If we plug the above into (5.33), after comparing the $x$ coefficient and constant term and considering the initial condition, we obtain two separate ordinary differential equations

\[
\begin{aligned}
-\alpha A'(t) - r \alpha A(t) &= 0 \\
A(T) &= 1
\end{aligned}
\]  

\[
\begin{aligned}
B'(t) - \frac{(\mu - r)^2}{2 \sigma^2} - \alpha P q(P) A(t) &= 0 \\
B(T) &= 0
\end{aligned}
\]
We first solve (5.34) and get

\[ A(t) = e^{r(T-t)} \]

By plugging this into (5.35), we can easily get a solution for \( B(t) \)

\[ B(t) = \frac{(\mu - r)^2}{2\sigma^2} (t - T) + \frac{\alpha P q(P)}{r} (1 - e^{r(T-t)}) \]

Thus we have found a solution for (5.33):

\[ V(x, t) = -\exp\{-\alpha x e^{r(T-t)} + \frac{(\mu - r)^2}{2\sigma^2} (t - T) + \frac{\alpha P q(P)}{r} (1 - e^{r(T-t)})\} \]  
(5.36)

Next, let us try to find a solution to (5.32) in form of

\[ U(x, t) = V(x, t)\psi(t) \]  
(5.37)

By plugging (5.37) into (5.32), notice that in this case \( \tilde{\lambda} = \lambda * q(P) \), we obtain the following:

\[ 0 = V_t\psi + V\psi' - \frac{(\mu - r)^2 V_x^2 \psi^2}{2V_{xx}\sigma^2} + rV_x\psi + P q(P) V_x\psi + \lambda q(P) [E V(x - Y, t) - V(x, t)]\psi \]

Since \( V(x, t) \) solves (5.33), the first, third, fourth and fifth terms cancel, we end up with a simple ordinary differential equation for \( \psi(t) \)

\[ V(x, t)\psi' + \lambda q(P) [E V(x - Y, t) - V(x, t)]\psi = 0 \]

following the same argument as in the previous example, we get

\[ V\psi' + \lambda q(P) [V E^a(\alpha e^{r(T-t)}) - V] \psi = 0 \]

where the superscript \( a \) means that the expected value is taken with respect to the actuarial randomness. The solution is given by

\[ \psi(t) = \exp\{-\int_t^T \lambda q(P) [M_Y(\alpha e^{r(T-s)}) - 1] ds\} \]  
(5.38)
By (5.36), (5.37) and (5.38), we have solved (5.32)

\[ U(x, t) = -\exp\{-\alpha xe^{r(T-t)} + \frac{(\mu - r)^2}{2\sigma^2}(t - T) + \frac{\alpha Pq(P)}{r}(1 - e^{r(T-t)}) \}
- \int_t^T \lambda q(P)[M_Y(\alpha e^{r(T-s)} - 1)]ds \]

Now we can start to find the optimal price. The idea is as the following: at time \( t = 0 \), we are trying to maximize our final expected utility which is give by \( \bar{U}(x, 0; P) \), where \( P \) is a parameter. In case the value function is smooth and convex, we can just use first derivative to find the maximizer.

Now let us choose a demand curve for our example. For simplicity, let us just use the linear demand curve \( q(P) = H - kP \) where \( H \) and \( k \) are positive. In this case, the value function at \( t = 0 \) is given by:

\[ \bar{U}(x, 0; P) = -\exp\{-\alpha xe^T + \frac{(\mu - r)^2}{2\sigma^2}(-T) + \frac{\alpha HP - k\alpha P^2}{r}(1 - e^{rT}) \}
+ \int_0^T \lambda(H - kP)[M_Y(\alpha e^{r(T-s)} - 1)]ds \]

by differentiating with respect to \( P \), we get

\[ \frac{\partial \bar{U}}{\partial P}(x, 0; P) = \bar{U}(x, 0; P)\left\{ \frac{1}{r}(1 - e^{-\alpha T})(H\alpha - 2k\alpha P) - k \right\} \int_0^T \lambda[M_Y(\alpha e^{r(T-s)} - 1)]ds \]

If we set the derivative equal to zero, we have found a critical point

\[ P^* = \frac{H}{2k} + \frac{r\lambda \int_0^T [M_Y(\alpha e^{r(T-s)} - 1)]ds}{2(1 - e^{-\alpha T})} \]

(5.39)

We follow the same discussion as the previous example:

1. If \( P^* \leq P^- \), then we will set the price to be \( P^- \), which means we will take as many customers as we can, as long as the price is no less than the indifference price.

2. If \( P^- < P^* \leq P^+ \), our optimal price is given by (5.39).
3. If $P \geq P^*$, then we will set the price to be $P^+$. This will result the number of customers to be zero, that is, our best strategy is not to take any customers.

Again, we summarize our findings in the following theorem

**Theorem 5.8** Under the settings of Optimal Property Insurance Problem (Continuous Payment), if the utility function is CARA with parameter $\alpha$ and the demand curve is linear with elasticity $k$ and intercept $H$, then:

To maximize the expected final utility, the only critical point of the price is given by:

$$P^* = \frac{H}{2k} + \frac{r \lambda \int_0^T [My(\alpha e^{r(T-s)} - 1)] ds}{2(1 - e^{-\alpha T})}$$

Furthermore, if $P^*$ is in the interval $[P^-, P^+]$, then it is the optimal price; otherwise, the optimal price is given by one of the two endpoints.

### 5.7 Optimal Casualty Insurance Price

In this section we consider the problem of optimal life insurance price which leads to the maximum expected utility. Our model here is pretty much the same as the one we studied in section 5.5: we have access to the same financial market consisting of the stock and the bond whose price processes are given by (5.1) and (5.2); the insurer’s risk preference is described by a utility function which is generally an increasing concave function and we use the exponential utility function (5.3); finally we are facing a demand curve (5.4). However, the product we are interested now is the life insurance. Suppose we have a group of people aged $x$ at time 0. To be specific, we consider an individual aged $x$, who is seeking to buy term life insurance that will pay 1 unit at time of death if the individual dies before time $T$, and 0 otherwise. For simplicity we write $(x)$ to refer to a typical individual. We formulate our problem as following:
It is clear that given price $P$, the wealth process satisfies the following stochastic differential equation:

$$
\begin{align*}
\frac{dX_t}{X_t} &= (r X_t + (\mu - r) \Delta(t)) dt + \Delta(t) \sigma dW_t - dM_t^P \\
X_0 &= x + P q(P)
\end{align*}
$$

(5.40)

where $\Delta(t)$ is an admissible strategy and $M_t^P$ is the mortality process. If we use $X^P(.)$ to denote solution of (5.40), our objective is:

**Optimal Casualty Insurance Price Problem (Lump Sum Payment) :** Find $P^*$ such that

$$
\sup_{\Delta(t) \in A} E[u(X^{P^*}(T))] = \max_{P \in [P^-, P^+]} \sup_{\Delta(t) \in A} E[u(X^P(T))]
$$

Let $F_x(t)$ be the cumulative distribution function of the time until the death of $(x)$, that is, $F_x(t)$ is the probability of $(x)$ dies before $t$. We use $h q_{x+t}$ to denote the probability that $(x)$ will die before time $t + h$ given that he or she is alive at time $t$, then,

$$
h q_{x+t} = \frac{F_x(t + h) - F_x(t)}{1 - F_x(t)}
$$

By assuming enough smoothness of the distribution function, we can define the hazard function, otherwise known as the force of mortality, $\lambda_x(t)$, given by $\lambda_x(t) = f_x(t)/(1 - F_x(t))$, in which $f_x$ is the probability density function of the time until the death of $(x)$.

**Observation:** If the distribution function is smooth, then $h q_{x+t} = \lambda_x(t) h + O(h^2)$ as $h \to 0$.

Suppose we have $n$ customers at time $t$, we want to find the distribution of the number of people that dies between time $t$ and $t + h$. By assumption, each person
has a probability of death given by $hq_{x+t}$, the total number of death is distributed according to the $Binomial(n, hq_{x+t})$. Let $m$ be the number of death in $[t, t+h]$, then according to the above discussion, we have

$$P[m = k] = C^n_k h^k q^k_{x+t} (1 - hq_{x+t})^{n-k}$$

Since $hq_{x+t}$ is of order $h$, only the probability of $k = 0$ and $k = 1$ is significant. It is easy to see

$$P(k = 0) = 1 - n\lambda_x(t)h + O(h^2), \quad P(k = 1) = n\lambda_x(t)h + O(h^2)$$

Now let us derive our HJB equation. In this case, the death of each individual can happen only once, so if we have paid the claim to a customer, we do not have him or her in our business any longer. This is quite different from the example of car insurance. As a result, besides the wealth $x$ and time $t$, we also need to include the number of customers $n$ as an independent variable. Let the value function be

$$U(x, t, n) = \sup_{\{\Delta(t)\} \in A} E[u(X_T)|X_t = x, N_t = n]$$

We proceed similarly as in section 4.5, the classic dynamic programming principle gives:

$$U(x, t, n) \geq (1 - n\lambda_x(t))E[U(X_{t+h}, t+h, n)|X_t = x, N_t = n]$$

$$+n\lambda_x(t)E[U(X_{t+h} - 1, t+h, n-1)|X_t = x, N_t = n]$$

After a straightforward but tedious calculation, considering the boundary condition and terminal condition, we end up with the following HJB equation:

$$\begin{cases}
U_t + rxU_x - \frac{(\mu - r)^2 U_{xx}^2}{2U_x^2 \sigma^2} + n\lambda_x(t)[U(x-1, t, n-1) - U(x, t, n)] = 0 \\
u(x, T, n) = -e^{-\alpha x} \\
u(x, t, 0) = V(x, t)
\end{cases}$$
where $V(x, t)$ is the solution of (5.27). The above equation is easy to solve when $n = 1$, and the solution is given by

$$u(x, t, 1) = V(x, t)\psi(t)$$

and $\psi(t)$ is the solution of the following ODE

$$\begin{align*}
\psi'(t) + \lambda_x(t)[\exp(\alpha e^{r(T-t)}) - \psi(t)] &= 0 \\
\psi(T) &= 1
\end{align*}$$

(5.41)

For general $n$, we can show by induction that

$$u(x, t, n) = V(x, t)\psi(t)^n$$

Now let us try to determine the optimal price for the life insurance described at the beginning of this section. Suppose we set the price to be $P$, for simplicity, we still adopt linear demand curve here, then the number of customers is given by

$$q(P) = H - kP$$

Considering the premium we have collected, the value function at $t = 0$ is

$$U(x, 0, q(P)) = -e^{-\alpha(x+HP-kP^2)e^{rT}} - \frac{(\mu - r)^2}{2\sigma^2} T \psi(0)^{H-kP}$$

We set the partial derivative with respect to $P$ equal to zero and find the critical $P$ value:

$$P^* = \frac{\alpha H + k \ln \psi(0)}{2\alpha k}$$

(5.42)

Notice that this optimal price also depends on the distribution of the mortality function, this is because that $\psi(t)$ is the solution of an ODE whose coefficient depends on $\lambda_x(t)$. 

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Having found the unique critical point, we can perform the same discussion as before: if $P^*$ falls into the interval $[P^-, P^+]$, then $P^*$ is the optimal price we should set; otherwise, we should set our price to be one of the end points. The result is summarized below:

**Theorem 5.9** Under the settings of Optimal Casualty Insurance Problem (Lump Sum Payment), if the utility function is CARA with parameter $\alpha$ and the demand curve is linear with elasticity $k$ and intercept $H$, then:

To maximize the expected final utility, the only critical point of the price is given by:

$$P^* = \frac{\alpha H + k \ln \psi(0)}{2\alpha k}$$

where $\psi(t)$ is the solution of (5.41). Furthermore, if $P^*$ is in the interval $[P^-, P^+]$, then it is the optimal price; otherwise, the optimal price is given by one of the two endpoints.
CHAPTER 6
NUMERICAL ANALYSIS

6.1 Introduction

In Chapter Five we have solved the insurance price problem analytically and presented several examples where the solutions are explicitly given. However it is important for the reader to be aware that it is generally very hard to achieve such a solution. We have made a lot of assumptions which facilitate our calculation: we use the geometric Brownian model for stock price where the mean rate of return and volatility are held constant; we use utility functions in exponential form; we assume the loss process is compound poisson and each claim size is independent of the state variable. These assumptions may not be realistic, especially in the long range. The purpose of this chapter is, using the ideas of utility maximization, to provide the reader a handy tool to solve optimal price problems in absence of those “nice but unrealistic” assumptions mentioned above. Also, we would like to perform some numerical analysis of the Optimal Property Insurance Price Problem under the lump sum payment assumption. This may provide the reader some insight about how the utility, optimal price and other parameters are related to each other.

One proposal would be to solve the HJB equations numerically. This turns out not to be a good method because the HJB equations are quite different from those elliptic or
hyperbolic differential equations which can be properly handled by classic numerical methods. Let us use equation (5.10) as an example. For this equation, $U_{xx}$ is a factor of the denominator for the second term, we can not solve this equation using linear methods. Even if we linearize this problem and use the finite difference method to approximate the equation, the stability of the resultant matrix is questionable. Also, it is very possible that $U_{xx}$ is close to zero at certain points, where we may have round-off error problems.

Another possible way to work out this problem is to use Monte-Carlo simulation, which is a very popular method for asset pricing in finance and economics. This method was introduced in 1949 in [17] by Metropolis and Ulam in their article entitled “The Monte Carlo method”. The American mathematicians John Von Neumann and Stanislaw Ulam are considered its main originators. The theoretical foundation had been known long before the birth of the method. However, because of simulation of random variables by hand is a laborious process, use of the Monte-Carlo method as a universal method became practical only with the advent of computers. As for the name “Monte-Carlo”, it is derived from the city in the Principality of Monaco famous for its casinos. The point is that one of the simplest mechanical devices for generating random numbers, which is a crucial part of the method, is the Roulette wheel.

6.2 Monte-Carlo Method

**The General Scheme of the Monte-Carlo Method** Suppose that we need to calculate some unknown quantity $\mu$ which is very hard to find analytically. Let us find a random variable $\xi$ with $E[\xi] = \mu$ and $Var[\xi] = \sigma^2 < \infty$. If we take $N$ independent samples from this random variable, say $\xi_1, \xi_2, \xi_3, ... \xi_N$, it follows from
the central limit theorem that the distribution of the average \( \bar{\xi} = \frac{\xi_1 + \xi_2 + \ldots + \xi_N}{N} \)
will be approximately normal with mean \( \mu \) and variance \( \frac{\sigma^2}{N} \). Roughly speaking, according to the rule of “three sigmas”, we have:

\[
P\{|\bar{\xi} - \mu| < \frac{3\sigma}{\sqrt{N}}\} \approx 0.997 \quad (6.1)
\]

This approximation tells us that if we have \( N \) large enough, the probability that our estimate is within the acceptable range is more than 99% . Equation (6.1) gives us both the method for calculating \( \mu \) and the error estimate.

**Example:** Suppose we are trying to find the integral \( \int_{0}^{1} x^3 dx \), which is easy to calculate and the value is 0.25. Let us use the Monte-Carlo method to solve this problem, the basic idea is the following: to calculate the integral value is equivalent to finding the area \( A \) which is bounded by \( x \)-axis, \( x = 1 \) and \( y = x^3 \).

Obviously this area is within the unit square \([0, 1] \times [0, 1]\), which has area 1. Suppose we take a random point uniformly from the unit square, then the probability that the point falls into \( A \) is exact the same as the integral value, which is what we want to calculate. Let \( P_1, P_2, \ldots P_N \) be random points uniformly distributed in the square, and let \( \xi_1, \xi_2, \ldots \xi_N \) be defined by \( \xi_i = 1 \) if \( P_i \) falls into \( A \) and 0 otherwise, then we have

\[
E[\xi] = \int_{0}^{1} x^3 dx = \frac{1}{4}, \text{ and } Var[\xi] = \left(\frac{1}{4}\right)\left(1 - \frac{1}{4}\right) = \frac{3}{16}
\]

As a result, the average value \( \bar{\xi} = \frac{\xi_1 + \xi_2 + \ldots + \xi_N}{N} \) will converge to the value of the integral.

Based on our discussion above, we give the following algorithm:

1. Set \( n = 0 \)
2. Take a point \((x, y)\) uniformly from \([0, 1] \times [0, 1] \)
Figure 6.1: Graph of $y = x^3$
3. Increase the counter variable \( n \) by 1 if \( y \leq x^3 \)

4. Repeat 2 and 3 \( N \) times, then \( \frac{n}{N} \) should converge to the value of the integral as \( N \) get large.

The results of the numerical experiment are given in the table 6.1:

<table>
<thead>
<tr>
<th>Number of Points (N)</th>
<th>Numerical Result (n/N)</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.3</td>
<td>0.05</td>
</tr>
<tr>
<td>100</td>
<td>0.29</td>
<td>0.04</td>
</tr>
<tr>
<td>1,000</td>
<td>0.252</td>
<td>0.002</td>
</tr>
<tr>
<td>10,000</td>
<td>0.2487</td>
<td>0.0013</td>
</tr>
<tr>
<td>100,000</td>
<td>0.25237</td>
<td>0.00237</td>
</tr>
<tr>
<td>200,000</td>
<td>0.250975</td>
<td>0.000975</td>
</tr>
<tr>
<td>300,000</td>
<td>0.2496067</td>
<td>0.0003933</td>
</tr>
</tbody>
</table>

Table 6.1: An Example Using Monte-Carlo Method
**Remark:** We have the following comments about the Monte-Carlo Method:

1. From Table (6.1), we see that the error of Monte-Carlo method does not necessarily decrease as we increase the number of experiments $N$. It is a probabilistic process. It is very possible that if we use only 4 points and you can get the real value 0.25, and we can not do better than that by adding more points. Of course, we may also get some number which is far away from the true value for small $N$. However, as we increase $N$, the probability that we get a value that is close enough to the real value will increase, according to the rule of thumb (6.1).

2. One key step for Monte-Carlo method is random number generation. Various methods to generate random numbers have been proposed, however, none of them generates real “random” numbers. Actually they only seem to be random numbers with certain distribution, and are in fact referred to as *pseudorandom*. However, the user needs not to worry about this too much because the results of good random number generators always pass the set of statistics tests and may be treated as “truly” random. See Chapter One of [20] for a more complete discussion.

### 6.3 An Algorithm for Optimal Price Problem

We present a specific algorithm which can be used to solve the problems discussed in Chapter Five. It is based on the following idea: First we choose a price which is in $[P^-, P^+]$, for this given price, we can determine our initial wealth, the number of customers, and thus the intensity parameter of the compound poisson process. Second, for each initial condition, we use the computer to simulate the stochastic process, say 500 times; for each simulation, we can get a final wealth $X_T$ and calculate the final utility function, and then find the average value of the 500 simulations. Finally, we find out the price whose resultant average final utility reaches the maximum.
A natural question readers may ask is the reliability of this method. Since the purpose of this section is to give out a practical method instead of to carry out the strict mathematical discussion, we roughly discuss the following simplified case where there are only two prices available to choose. That is, suppose $P_0$ is the optimal price for our problem, and $P_1$ is another price with $U(0, x, P_1) < U(0, x, P_0)$, what is the probability of our program giving out $P_1$ as the answer? Under some mild assumptions, we can show that the second moment of $u(X_T)$ is finite for any given $P$, say bounded by $M$. If we simulate the process $N$ times, when $N$ is large enough, the average final utilities $\{\bar{u}_i\}_{i=0,1}$ will be approximately normal with mean $U(0, x, P_i)$ and variance less than $M/N$. Their difference $\bar{u}_0 - \bar{u}_1$ would also be normal with a positive mean and variance bounded by $2M/N$. From this brief discussion we easily see that when $N$ is large enough, our machine could give us the correct answer with a probability almost equal to 1.

The code can be written according to the following algorithm:

1. Set $P=P^-$, find out number of customers, initial wealth, and intensity parameter of the loss process.
2. Simulate the wealth process according to equation (5.13), find $u(X_T)$.
3. Repeat step 2 $N$ times.
4. Find the average value of $u(X_T)$.
5. Increase the price $P$ by 1, repeat step 1,2,3,4.
6. Repeat 5, until $P = P^+$.
7. Find the $P$ where the average value of $u(X_T)$ reaches the maximum.

Some typical paths of the wealth generated by MATLAB are given in figures 6.2-6.5, with $r = 0.1, \sigma = 0.4, \mu = 0.3, \tilde{\lambda} = 1$ and various initial wealth values. From the plots one may notice that the paths are smoother when $t$ is small and becoming
more oscillating as $t$ is large. The explanation for this interesting phenomenon is that if we look at (5.12), when we use CARA model, $\Delta(t)$ is an increasing function of $t$. Therefore, at the beginning, a large portion of money is in the deterministic bond, and the wealth evolution is more deterministic; as time increases, the investor is putting more money in the stock, which results a larger volatility.

![Wealth Path 1: $X(t)$ vs. $t$](image)

Figure 6.2: Wealth Path 1: $X(t)$ vs. $t$

A graph of utility versus price is shown in figure 6.6, with $x = 15000, r = 0.1, \mu = 0.3, \sigma = 0.4 and \lambda = 0.01$. From our graph, we can see that the utility is roughly a convex function of price $P$. The reader can also verify this by calculating the second derivative of the value function in Chapter Five. One may also notice that the zigzag
Figure 6.3: Wealth Path 2: $X(t)$ vs. $t$

Figure 6.4: Wealth Path 3: $X(t)$ vs. $t$
Figure 6.5: Wealth Path 4: $X(t)$ vs. $t$

shape of the graph in spite of the fact that the value function is smooth; this is a result of lack of sufficient number of simulations due to the limit of time and equipment.

We want to point out that the algorithm may only be restricted to problems with CRRA and CARA utility functions because the investment strategies $\Delta(t)$ under other utility function may not have a explicit form.
6.4 More Discussion on Optimal Property Insurance Problem (Lump Sum Payment)

Now let us go back to investigate the analytical example given in section 5.5. We have found the following analytical solution of (5.26):

\[ U(x, t) = -\exp\left\{-\alpha xe^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T - t) + \int_t^T \lambda[Y(e^{\alpha e^{r(T-s)}} - 1)]ds\right\} \]

Now we consider the following quadratic demand curve \( q = H - kP^2 \). Following the same argument, if we set the price to be \( P \), then the value function at the initial time is given by:

\[ \bar{U}(x, 0; P) = U(x + Pq(P), 0) = -\exp\left\{-\alpha(x + PH - kP^3)e^{rT} - \frac{(\mu - r)^2}{2\sigma^2}T + \int_0^T (H - kP^2)\lambda[Y(e^{\alpha e^{r(T-s)}} - 1)]ds\right\} \]
here $P$ is the only decision variable. Taking the derivative with respect to $P$, we get

$$\frac{\partial U}{\partial P}(x, 0; P) = U \{-\alpha(H - 3kP^2)e^rT - 2kP \int_0^T \lambda[M_Y(e^{\alpha e^r(T-s)}) - 1] ds\}$$

If we set partial derivative equal to zero, it is equivalent to solve a quadratic equation. We obtain two solutions, one is positive and another one negative. We are only interested in the positive solution, which is a critical point.

It is important to realize that we can determine $P^*$ as well as reservation prices from these information. Basically $P^*$ is the unique positive solution of the quadratic solution of

$$3k\alpha e^rTP^2 - 2k \int_0^T \lambda[M_Y(e^{\alpha e^r(T-s)}) - 1] ds P - \alpha e^rTH = 0 \quad (6.2)$$

$P^+$ is simply the root of the demand curve, $\sqrt{\frac{H}{k}}$; and according to [23], $P^-$ is given by:

$$P^- = \frac{1}{\alpha} e^{-rT} \int_0^T \lambda[M_Y(e^{\alpha e^r(T-s)}) - 1] ds \quad (6.3)$$

To write out the solution analytically, we have to choose a specific distribution for a single claim size. For simplicity, we choose the exponential distribution with parameter $c$, that is:

$$f(y) = \begin{cases} 
ce^{-cy}, & y \geq 0 \\
0, & y < 0 
\end{cases}$$

For this choice, the moment generating function is:

$$M_Y(t) = \frac{1}{1 - t/c}$$
Figure 6.7: $P^-$ vs $\alpha$
Figure 6.8: $P^*$ vs $\alpha$
For certain choice of parameters, a graph of $P^-$ versus $\alpha$ is given in figure 6.7. From the graph we can see that $P^-$ is U-shaped and has two vertical asymptotes. The first one at $\alpha = 0$ is caused by the diminishing of the denominator. The second one at $\alpha \approx .26$ is because of the nature of the moment generating function of distribution: $M_Y(t)$ approaches infinity when $t$ is close to $c$. We also plot the graph of $P^*$ in figure 6.8, from the graph we can see that $P^*$ is also U-shaped and has similar vertical asymptotes.

In figure 6.9, we plot $P^-, P^*, P^+$ in the same graph. We are only interested in the interval when $P^- < P^+$ so that transactions can take place. We will choose our optimal price from $P^-, P^*$ and $P^+$ according to the relative position of $P^*$ compared with the reservation prices. Sometimes it is more straightforward to look that the following quantity:

$$\rho(\alpha) = \frac{P^* - P^-}{P^+ - P^-}$$

We set optimal price to be $P^*$ when $0 < \rho < 1$; $P^-$ if $\rho < 1$ and $P^+$ if $\rho > 1$. A figure of $\rho$ corresponding to the values in figure 6.9 is given in figure 6.10.
Figure 6.9: $P^-, P^*$ and $P^+$ vs $\alpha$
Figure 6.10: $\rho$ vs $\alpha$
CHAPTER 7
CONCLUSION AND FUTURE RESEARCH DIRECTIONS

In this paper we have described how the market demand-supply relationship can affect prices in an incomplete market. For an individual insurance company with a given risk preference, one can derive the corresponding Hamilton-Jacobi-Bellman equations from the analysis of infinitesimal time behavior of the poisson process, and the prices can be calculated by finding the maximum of the value function. We formulate the optimal insurance problem and prove existence of a solution in Theorem 5.5. We have compared the examples of optimal property insurance price where the premium is paid in a lump sum, as opposed to the situation where the premium is paid continuously. We also investigated the case of casualty insurance price. The results are summarized in Theorem 5.7-5.9.

Since our purpose is to introduce the idea of pricing insurance risks using Dynamic Programming Principle, in each of our examples, we have used the exponential utility function and Poisson process to facilitate our calculation. We suggest that one use Monte-Carlo simulation to deal with more complex models. We can also incorporate consumption that represents the money that is used for daily operation and management cost. For simplicity, we have used the geometric Brownian motion to model the price evolution of our stock market; using the same method, we can easily extend our work to time dependent mean rate of return and diffusion coefficient. We
have assumed constant interest rate, which is realistic in the short-term range. For long-term model, one can also use some stochastic process to simulate the interest rate such as CIR model. Also, the behavior of investor under other utility functions than CRRA and CARA is still unknown, which greatly restrict the applicability of numerical methods.
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