ASYMPTOTIC EXPANSION FOR THE $L^1$ NORM OF N-FOLD CONVOLUTIONS

Dissertation

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ABSTRACT

An asymptotic expansion of nonnegative powers of 1/n is obtained which describes the large-n behavior of the $L^1$ norm of the n-fold convolution, \( \| g_n \|_{L^1} = \int_{-\infty}^{\infty} |g_n(x)| \, dx \), of an integrable complex-valued function \( g(x) \), defined on the real line, where, \( g_{n+1}(x) = \int_{-\infty}^{\infty} g(x-y)g_n(y) \, dy \), \( g_1(x) = g(x) \).

Consideration is restricted here to those \( g(x) \) which simultaneously satisfy the following four Assumptions I: \( g(x) \in L^1 \cap L^{s_1} \), for some \( s_1 > 1 \) , II: \( x^j g(x) \in L^1 \), \( (j = 1, 2, 3, ...) \), III: There is only one point, \( t = t_0 \), at which \( |\hat{g}(t)| \) attains its supremum, i.e., \( |\hat{g}(t)| < |\hat{g}(t_0)| = \sup_{s \in \mathbb{R}} |\hat{g}(s)| \), for all \( t \neq t_0 \), IV: \( |\hat{g}(t)(2)|_{t = t_0} < 0 \), where \( \hat{g}(t) \) denotes the Fourier transform of \( g(x) \). We obtain the following Theorem: Let \( g(x) \) satisfy simultaneously Assumptions I,II,III,IV above, and let \( L \) be an arbitrary positive integer, then \( \| g_n \|_{L^1} = |\hat{g}(t_0)|^n \{ \sum_{\ell=0}^{L} c_\ell \left( \frac{1}{n} \right)^\ell + o\left( \left( \frac{1}{n} \right)^L \right) \} \) as \( n \to \infty \), where the coefficients

\[
c_\ell = \frac{1}{\sqrt{2\pi |K_2|}} \int_{-\infty}^{\infty} e^{-\gamma^2 Re(\frac{1}{2\pi K^2})} S_{2\ell}(\gamma) d\gamma, (\ell = 0, 1, 2, 3, ...),
\]

\( S_0(\gamma) = 1 \), and \( S_r(\gamma) = \Sigma_{m=1}^{r} m! \left( \frac{1}{m} \right) \Sigma'_{(m_1, m_2, ..., m_r), m} \Pi_{j=1}^{r} \left[ \Sigma_{j=1}^{n} Q_{j} - j_i(\gamma) Q_{j_1}(\gamma) \right]^{m_j}/m_j! \),

with \( Q_0(\gamma) = 1 \), \( Q_r(\gamma) = \Sigma_{m=1}^{r} H_{2m+r} \left( \frac{-\gamma}{\sqrt{K^2}} \right) \Sigma'_{(m_1, m_2, ..., m_r), m} \Pi_{j=1}^{r} \{ \left( \frac{1}{\sqrt{K^2}} \right)^{2+j} K_{2+j} \}^{m_j}/m_j! \), \( (r = 1, 2, 3, ...) \) and \( K_j = (-i)^j (ln(\hat{g}))^{(j)}(t_0), (j = 2, 3, 4, ...) \), and where the \( H_{m}(u) \) is the monic Hermite polynomial of degree m. Here, \( \Sigma' \) indicates summation over all r-tuples \( (m_1, m_2, ..., m_r) \) where the \( m_j \) run over all nonnegative integers which satisfy
simultaneously the two conditions $\Sigma_{j=1}^{r} m_j = m$ and $\Sigma_{j=1}^{r} j m_j = r$. It is proved that in the special case where the $K_j, j = 2, 3, 4, ..., 2 + p$ are all real, then

$$\lim_{n \to \infty} n^{p+1} \{ \| g_n \|_{L^1} / |\hat{g}(t_0)| \} = c_{p+1} = \frac{(Im(K_{3+p}))^2}{2(3+p)!/(K_2)^{3+p}}$$

As an application of the above Theorem, it is observed that for a $g(x)$ satisfying I,II,III,IV above, the corresponding convolution operator $T_g : L^1 \to L^1$ has

$$\| T_g^n \| = \| g_n \|_{L^1},$$

so that as $n \to \infty$,

$$\| T_g^n \|^{1/n} / |\hat{g}(t_0)| - 1 = b(1/n) + \left( \frac{1}{2} b^2 + c \right)(1/n)^2 + o((1/n)^2).$$

Here, the constants $b = \ln(c_0) = \frac{1}{4} \ln(1 + (Im(K_2)/Re(K_2))^2)$ and $c = c_1/c_0$. Thus, when $ImK_2 \neq 0$ the convergence of $\| T_g^n \|^{1/n}$ to the spectral radius of $T_g$ is less rapid, than when $ImK_2 = 0$
Dedicated to the memory of my parents
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CHAPTER 1
INTRODUCTION

In this work one obtains an asymptotic expansion which describes the large-n behavior of the $L^1$ norm of the n-fold convolution of an integrable complex-valued function, defined on the real line, $g(x)$.

$$\| g_n \|_{L^1} = \int_{-\infty}^{\infty} |g_n(x)| \, dx,$$  \hspace{1cm} (1.1)

where,

$$g_{n+1}(x) = \int_{-\infty}^{\infty} g(x - y) g_n(y) \, dy =$$

$$= \int_{-\infty}^{\infty} dx_n \int_{-\infty}^{\infty} dx_{n-1} \ldots \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_1 g(x - x_n) g(x_n - x_{n-1}) \ldots g(x_2 - x_1) g(x_1)$$  \hspace{1cm} (1.2).

$n = 1, 2, 3, \ldots$ and $g_1(x) = g(x)$. Some asymptotic results are known to the

$$\lim_{n \to \infty} \| g_n \|_{L^1}$$

and some bounds on the rate of approach to the limit value have been reported for the class of those functions, $g(x)$, whose Fourier transform behaves near the origin in a pre-assigned way, Baishanski [2], Humphreys[5]. The idea for the present work
is suggested by the recent publication of Baishanski and Snell [3], who derived an asymptotic expansion,

\[ \| f^n \|_A \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} a_n(j) = \sum_{j=0}^k c_j n^{-j} + o(n^{-k}), \quad n \to \infty \]

\[(1.3),\]

giving thus the limiting value and precise description on rate of approach to this limit, where

\[ a_n(j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-ijt)(f(t))^n dt \]

\[(1.4)\]

with \(f(t)\) defined as the absolutely convergent Fourier series

\[ f(t) = \sum_{j \in \mathbb{Z}} a_1(j) \exp(ijt) \]

\[(1.5)\]

In Snell[6] and Baishanski and Snell[3], attention is focused on the properties of \(f(t)\), what, in probability/statistics language reminds one of the "characteristic function" of the discrete distribution \((a(j))\). Subsequent work [7] has shown that under the conditions on \(f(t)\) assumed by Baishanski and Snell, one gets, for example, that when the \(i^j f^{(j)}(0)\) are all real for \(j = 0, 1, 2, 3, \ldots, 2 + p\) (with \(p\) a nonnegative integer) then,

\[ \lim_{n \to \infty} n^{1+p}(\| f^n \|_A - 1) = \frac{\{Im(i^{3+p} f^{(3+p)}(0))\}^2}{2(3 + p)!(-d/dt)^2|f(t)|_t=0^{3+p}} \]

\[(1.6)\]
In fact, \( \| f^n \|_A = \| a_n(j) \|_{\ell^1} \) the sequence \( (a_n(j))_{j \in Z} \) being in \( \ell^1 \) since its elements are generated by the n-fold convolution of \( (a_1(j))_{j \in Z} \), i.e.,

\[
a_n(j) = \sum_{j \in Z} a_1(k - j)a_{n-1}(j),
\]

\((n = 2, 3, 4, \ldots)\).

In the what follows here assumptions are made and attention is focused on the properties "complex probability density " associated with \( g(x) \) and the methods of Baischkanski and Snell are adapted to obtain asymptotic expansions for \( \| g_n \|_{L^1} \). Our use here of the Faa di Bruno formula has resulted in improvements in some lemmas and in a closed form representation (involving the derivatives of the logarithm of the Fourier transform of \( g(x) \)) for those functions whose integrals give the asymptotic expansion coefficients.
CHAPTER 2
ASSUMPTIONS AND PRELIMINARIES

Hereinafter we denote the j-th order derivative of f(x) by

\[ f^{(j)}(a) = \left. ((d/dx)^j f(x)) \right|_{x=a} \]

We will make frequent use of the Faa di Bruno formula\[1\] for the derivatives of a composite function \( w(x) = (v \circ f)(x) = v(f(x)) \), assuming that all appearing derivatives exist:

\[ w^{(r)}(x) = (v \circ f)^{(r)}(x) = r! \sum_{m=1}^{r} v^{(m)}(u)|_{u=f(x)} \sum_{(m_1,m_2,...,m_r),m} \prod_{j=1}^{r} \frac{(f^{(j)}(x)/j!)^{m_j}}{m_j!} \]

for \( r=1,2,3,... \) \quad (2.0),

where we define the notation \( \sum_{(m_1,m_2,...,m_r),m} \) to indicate that the sum \( \sum' \) is to be taken over all r-tuples \( (m_1,m_2,...,m_r) \) where the \( m_j \) run over all nonnegative integers which satisfy simultaneously the two conditions \( \sum_{j=1}^{r} m_j = m \) and \( \sum_{j=1}^{r} jm_j = r \).

The Fourier transform of \( f(x) \) is denoted by

\[ \hat{f}(t) = \mathcal{F} f(t) = \int_{-\infty}^{\infty} \exp(ixt)f(x)dx \]

Unless otherwise indicated, we restrict attention in all that follows to the elements of \( G \), the class of those complex-valued functions \( g(x) \), which satisfy the following four
Assumptions:

Assumption I.

\[ g(x) \in L^1 \cap L^{s_1} \]

for some \( s_1 > 1 \). This Assumption is seen (with the help of the Hausdorff-Young Inequality: \( \| \hat{f} \|_{L_q} \leq (2\pi)^{1/q} \| f \|_{L_p}, (q = p/(p-1)) \), when \( 1 < p < 2 \)) to ensure that there exists an integer \( n_1 > \max(2, s_1/(s_1-1)) \), such that for all integers \( n \geq n_1 \), the powers \( (\mathcal{F}g(t))^n \in L^1 \). On the other hand \( g(x) \in L^1 \) implies that \( g_n(x) \in L^1 \) and also that \( (\mathcal{F}g_n)(t) = (\mathcal{F}g(t))^n \). So, the Fourier transform inversion theorem is applicable thus:

\[
g_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ixt)(\mathcal{F}g(t))^n \, dt
\]

(2.1),

for all \( n \geq n_1 \). The relation (2.1) will be basic in our examination of the large-n behavior of \( \| g_n \|_{L^1} \).

Assumption II.

\[ x^j g(x) \in L^1, (j = 0, 1, 2, 3, \ldots) \]

This guarantees that \( \hat{g}^{(j)}(t) = \int_{-\infty}^{\infty} (ix)^j \exp(ixt) g(x) \, dx, (j = 0, 1, 2, 3, \ldots) \)

(2.2),

i.e., \( \hat{g}(t) \) is infinitely differentiable, and that \( \hat{g}(t) \) as well as each derivative, \( \hat{g}^{(j)}(t) \), is continuous and bounded, and (recalling the Riemann-Lebesgue Theorem), tends to
zero as \(|t| \to \infty\). Therefore the modulus \(|\hat{g}(t)|\) and each modulus \(|\hat{g}^{(j)}(t)|\) must be functions which attain the value of their respective suprema at one or more points.

Assumption III.

There is only one point, \(t = t_0\), at which \(|\hat{g}(t)|\) attains its supremum. i.e,

\[
|\hat{g}(t)| < |\hat{g}(t_0)| = \sup_{s \in \mathbb{R}} |\hat{g}(s)|
\]

(all \(t \neq t_0\)) (2.3).

Assumption IV: \(|\hat{g}(t)|^{(2)}|_{t=t_0} < 0 \)

(2.4)

By Assumption IV, we limit attention to a common situation where the the graph of \(|\hat{g}(t)|\) vs. \(t\) is ”not too flat” in the vicinity of \(t = t_0\), i.e. the cases where \((d/dt)^2|\hat{g}(t)||_{t=t_0} = 0\) are excluded from consideration here. To proceed, we cast the calculations in the context of ”complex-valued probability density” functions by normalizing the functions and variables here and so, in terms of \(g(x)\), we define \(p(x)\), the complex-valued probability density associated with the real random variable \(X\) and \(h(x)\), the corresponding density function associated with a standardized variable \((X - \alpha)/\sigma\):

\[
p(x) = \exp(it_0 x) g(x)/\hat{g}(t_0) = \frac{1}{\sigma} h((x - \alpha)/\sigma)
\]

(2.4a)

, where \(\sigma > 0\) and \(\alpha\) are reals, to be chosen later. Thus,

\[
\hat{p}(\tau) = \hat{g}(t_0 + \tau)/\hat{g}(t_0)
\]
From the definition of $p(x)$ in (2.4a) it is clear that $p(x)$ satisfies Assumptions I and II. Thus, $\hat{p}(\tau)$ and each derivative $\hat{p}^{(j)}(\tau)$, $(j=1,2,3,...)$ is continuous, bounded, and tends to zero as $|\tau| \to \infty$. Further, $|\hat{p}(\tau)| = |\hat{g}(t_0 + \tau)|/|\hat{g}(t_0)|$ so, by Assumption III and the definition of $t_0$, the point $\tau = 0$ is the only point at which $|\hat{p}(\tau)|$ attains its supremum value, i.e.,

$$|\hat{p}(\tau)| < |\hat{p}(0)| = 1 = \hat{p}(0)$$

(for all $\tau \neq 0$) (2.5).

Associated with the complex-valued probability density $p(x)$ there exist, by Assumption II, the $N_j[p]$, constants defined by

$$N_j[p] = \int_{-\infty}^{\infty} |x|^j |p(x)| dx = \| (.|^j p(.) \|_{L^1}$$

(j=0,1,2,3,...) (2.7a)

and all the complex-valued moments,

$$M_r[p] = (-i)^r \hat{p}^{(r)}(0) = \int_{-\infty}^{\infty} x^r p(x) dx = \alpha_r + i\beta_r$$

(r=0,1,2,3,...) (2.6)

and all the complex-valued cumulants

$$K_r[p] = (-i)^r (ln \hat{p})^{(r)}(0) = \zeta_r + i\eta_r =$$

$$= r! \Sigma_{m=1}^r (-1)^m (m-1)! \Sigma_{(m_1,m_2,...,m_r)_r}^r \Pi_{j=1}^r \frac{(M_j[p]/j!)^{m_j}}{m_j!}$$

(2.4b),
, where this last relation is a direct application of the Faa di Bruno formula to express the $(\ln \hat{p})^{(r)}(0)$ in terms of the $(\hat{p}^{(s)}(0))$. From (2.4) we have also
\[
h(y) = \sigma p(\alpha + \sigma y) = \sigma \exp[i(\alpha + \sigma y)t_0]g(\alpha + \sigma y)/\hat{g}(t_0)
\]
(2.8a)
\[
\hat{h}(\tau) = \exp(-i\alpha \tau/\sigma)\hat{p}(\tau/\sigma) = \exp(-i\alpha \tau/\sigma)\hat{g}(t_0 + \tau/\sigma)/\hat{g}(t_0)
\]
(2.8b).

From the definition of $h$ in (2.4a) it is clear that $h(y)$ satisfies Assumptions I and II. Thus, $\hat{h}(\tau)$ and each derivative $\hat{h}^{(j)}(\tau)$, $(j=1,2,3,...)$, is continuous, bounded, and tends to zero as $|\tau| \to \infty$. Further, $|\hat{h}(\tau)| = |\hat{g}(t_0 + \tau/\sigma)|/|\hat{g}(t_0)|$ so, by Assumption III and the definition of $t_0$, we have that the point $\tau = 0$ is the only point at which $|\hat{h}(\tau)|$ attains its supremum value, i.e.,
\[
|h(\tau)| < |\hat{h}(0)| = 1 = \hat{h}(0)
\]
(for all $\tau \neq 0$) (2.9).

This condition that $|\hat{h}(\tau)|$ (and thus $|\hat{h}(\tau)|^2$) attain its (nonzero) supremum value at $\tau = 0$ requires that $|\hat{h}^{(1)}(0)| = 0$, which implies that the moment $M_1[p]$ be purely real, i.e. $\beta_1 = 0$, as we now see:
\[
0 = \frac{d}{d\tau}|\hat{h}(\tau)(\hat{h}(\tau))^{-1}|_{\tau=0} = \int \int_{-\infty}^{\infty} \frac{(u - v)}{\sigma} e^{i\sigma t_0(u-v)} g(u)g(v) \frac{1}{|\hat{g}(t_0)|^2} \, du \, dv = -2\beta_1/\sigma
\]
where we used Assumption II and the Fubini theorem and recalled that $\beta_1 = \text{Im}(M_1) = \int_{-\infty}^{\infty} x \text{Im}(p(x)) \, dx$. Thus, this gives

$$M_1 = \alpha_1 = \int_{-\infty}^{\infty} x \text{Re}(p(x)) \, dx$$

(2.12)

Further, using $|\hat{h}|(0) = 1$ and $|\hat{h}|^{(1)}(0) = 0$ one has similarly,

$$\frac{|\hat{g}|^{(2)}(t_0)}{|\hat{g}|(t_0)} = |\hat{h}|^{(2)}(0) = \frac{1}{2} (|\hat{h}|^2(0)) = -\frac{1}{2} \int \int_{-\infty}^{\infty} \frac{(u-v)}{\sigma}^2 e^{i\sigma(u-v)} \frac{g(u)g(v)}{|\hat{g}|^2(t_0)} \, du \, dv$$

$$= -(\alpha_2 - \alpha_1^2)/\sigma^2$$

(2.13).

Since Assumption IV requires that the LHS of (2.13) be negative, one must have

$$0 < \alpha_2 - (\alpha_1)^2, \text{thus}, 0 < \alpha_2 - \alpha_1^2 = \int_{-\infty}^{\infty} (u - \int_{-\infty}^{\infty} v \text{Re}(p(v)) \, dv) \, Re(p(u)) \, du$$

(2.14)

which requires that the "variance" corresponding to the density function $\text{Re}(p(x))$ be positive. To see the relations between the convolutions of $g$, $p$ and $h$, we recall (2.4) and (1.2):

$$g_n(x) = e^{-i\alpha x} (\hat{g}(t_0))^n p_n(x) = e^{-i\alpha x} (\hat{g}(t_0))^n \frac{1}{\sigma} h_n \left( \frac{x - n\alpha}{\sigma} \right)$$

(2.15)

,where the rightmost equality can be gotten by straightforward change of variables:

$$x_j = j\alpha + \sigma y_j (j = 1, 2, 3...) \text{ in (1.2).}$$

We recall that $(\hat{g}(t))^n \epsilon L^1$ for all $n > n_1$ and thus by the Riemann Lebesgue theorem applied to (2.1) yields :

$$\lim_{|x| \to \infty} g_n(x) = \lim_{|x| \to \infty} p_n(x) = \lim_{|x| \to \infty} h_n(x) = 0$$
From (2.15) the norms of the n-fold convolutions are related by

$$\| g_n \|_{L^1} = |\hat{g}(t_0)|^n \| p_n \|_{L^1} = |\hat{p}(t_0)|^n \| h_n \|_{L^1}$$

(2.17)

Thus, with this last formula, we can compute the large-n behavior of the norm of $g_n$ by from the large-n behavior of the norm of $h_n$; it is to the elucidation of the behavior of this latter quantity that we direct attention in what follows. The strategy consists of

1. Producing a $(k+1)$-term sum which approximates $h_n(x)$ to $o((1/\sqrt{n})^{k+1})$ uniformly on $(-\infty < x < \infty)$ as $n \to \infty$,
2. Finding a $(k+1)$-term expression which as $n \to \infty$ approximates $|h_n(x)|$ to $o((1/\sqrt{n})^{k+1})$ uniformly on $(-\infty < x < \infty)$ as $n \to \infty$,
3. Evaluating the latter expression on the "central set" $Z_n = \{ x : |x|/\sqrt{n} \leq n^\epsilon \}$ giving asymptotically a result whose integral makes subsequently the "central contributions" to the $L^1$ norm. Here, $0 < \epsilon < 1/2$,
4. Establishing functions which serve as general bounds for $|h_n(x)|$ and integrating these functions over the complement of $Z_n$ to obtain an asymptotic expression for the "noncentral contributions" to the $L^1$ norm as $n \to \infty$. 

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CHAPTER 3

LARGE-N BEHAVIOR OF THE N-FOLD CONVOLUTION

In the Chebyshev-Edgeworth-Esseen Expansion of probability theory the probability density of a sum of $n$ independent identically distributed random variables is expressed as an asymptotic sum in nonnegative powers of $1/\sqrt{n}$ of which the first term leads to the Central Limit Theorem [4]. In this section we produce, without the restrictions of probability theory, a similar expression for the complex-valued density $h_n(x)$ by adapting the methods recently developed by Snell[6] and Baishanski and Snell[3]. To begin, first note that the result of (2.5) and the fact that $g(x)$ satisfies (2.1) imply that

$$2\pi h_n(x) = \int_{-\infty}^{\infty} e^{-ixt} (\hat{h}(t))^n dt$$

(3.1)

for all integers $n \geq n_1$. Second, (2.9) shows that to each $\epsilon > 0$ there corresponds an $M(\epsilon)$ such that for all $t : \epsilon \leq t < \infty$, one has

$$|\hat{h}(t)| \leq sup\{|\hat{h}(s)| : \epsilon \leq s < \infty\} = M(\epsilon) < 1$$

(3.2)
In what follows we investigate the behavior near \( t = 0 \) of \((\hat{h}(t))^n = exp(nH(t))\)

we define

\[
H(t) = ln(\hat{h}(t))
\]

(3.3a)

Recall that \( \hat{h}(0) = 1 \) and that independently of this, we can choose

\[
\alpha = \alpha_1
\]

(3.3ab)

in the definition of \( h(t) \) By this choice,

\[
\alpha = \alpha_1 = M_1[p]
\]

(3.3ac)

which implies that

\[
-i\hat{h}^{(1)}(0) = M_1[h] = K_1[h] = (M_1[p] - \alpha)/\sigma = 0
\]

(3.3b),

\[
-\hat{h}^{(2)}(0) = K_2[h] = (M_2[p] - 2\sigma \alpha M_1[p] + \sigma^2 \alpha^2)/\sigma^2 = (\alpha_2 - \alpha_1^2 + i\beta_2)/\sigma^2
\]

(3.3c)

With these facts and the infinite differentiability of \( \hat{h}(t) \), the Taylor formula gives

\[
\hat{h}(t) = 1 + \frac{1}{2}\hat{h}^{(2)}(\xi)t^2
\]
for a certain $\xi_t$ between 0 and $t$. So,

$$|\hat{h}(t) - 1| \leq \frac{1}{2} |\hat{h}^{(2)}(\xi_t)| t^2 \leq \frac{1}{2} N_2 t^2 \leq \frac{1}{2}$$

for all $|t| \leq T_2$, where by definition the constant $T_2 = 1/\sqrt{N_2}$. This and the triangle inequality yields

$$\frac{1}{|\hat{h}(t)|} \leq \frac{1}{|1 - |\hat{h}(t) - 1||} \leq 2$$

for all $|t| \leq T_2$ (3.3e).

Now since $1/\hat{h}(t)$ is bounded on $[-T_2, T_2]$ and $\hat{h}(t)$ is infinitely differentiable, thus the function $H(t) = ln(\hat{h}(t))$ is infinitely differentiable on $[-T_2, T_2]$ and, by the Faa di Bruno formula:

$$H^{(r)}(t) = \Sigma_{m=1}^{r} (ln(u))^{(m)}|_{u=\hat{h}(t)} \Sigma'_{(m_1,m_2,..,m_r),m} \Pi_{j=1}^{r} \frac{(\hat{h}^{(j)}(t)/j!)^{m_j}}{m_j!} =$$

$$= \Sigma_{m=1}^{r} \frac{(-1)^{m-1}(m-1)!}{(\hat{h}(t))^m} \Sigma'_{(m_1,m_2,..,m_r),m} \Pi_{j=1}^{r} \frac{(\hat{h}^{(j)}(t)/j!)^{m_j}}{m_j!} =$$

for $r = 1, 2, 3, ...$. Also, with this result and (3.3e) we have

$$|H^{(r)}(t)| \leq \Sigma_{m=1}^{r} (m-1)! 2^m \Sigma'_{(m_1,m_2,..,m_r),m} \Pi_{j=1}^{r} \frac{(N_j[h]/j!)^{m_j}}{m_j!} = L_r$$

(3.3g)
for all $|t| \leq T_2$, $r = 1, 2, 3, \ldots$, and where the rightmost equality defines the constant $L_r$.

Applying the Taylor formula to $H(t)$ and its derivative:

$$H(t) - \sum_{j=0}^{2+k} H^{(j)}(0) t^j / j! = H^{(k+3)}(\zeta_t) t^{k+3} / (k + 3)!$$

(3.3h)

for a certain $\zeta_t$ between 0 and $t$.

$$H^{(1)}(t) - \sum_{j=0}^{1+k} H^{(1+j)}(0) t^j / j! = H^{(k+2)}(\chi_t) t^{k+2} / (k + 2)!$$

(3.3i)

for a certain $\chi_t$ between 0 and $t$. But by (3.3g), we have $|H^{(k+3)}(\zeta_t)| \leq L_{k+3}$ and $|H^{(k+2)}(\chi_t)| \leq L_{k+2}$. Using these facts we then get, by (3.3h),

$$H(t) = \sum_{j=0}^{2+k} H^{(j)}(0) t^j / j! + o(t^{k+2}) = At^2 + \sum_{j=1}^{k} B_j t^{2+j} + o(t^{k+2})$$

(as $t \to 0$) (3.3j)

and by (3.3i)

$$H^{(1)}(t) = \sum_{j=0}^{1+k} H^{(1+j)}(0) t^j / j! + o(t^{k+1}) = 2At + \sum_{j=1}^{k} (2 + j) B_j t^{1+j} + o(t^{k+1})$$

(as $t \to 0$) (3.3k)

Here, we have taken into account the facts $H^{(0)}(0) = \ln \hat{h}(0) = 0$ and $H^{(1)}(0) = \hat{h}^{(1)}(0) = iK_1[h] = 0$ and have introduced the constants

$$A = -K_2[h]/2 = -M_2[h]/2 = -(M_2[p] - M_1[p]^2)/(2\sigma^2) = -\alpha_2 - \alpha_1^2 + i\beta_2) / (2\sigma^2)$$

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\[ B_j = i^{2+j} K_{2+j}[h]/((2+j)!)) = i^{2+j} K_{2+j}[p]/((2+j)! \sigma^{2+j}) = \\
= (d/dt_0)^{2+j} \ln(\hat{g}(t_0))/((2+j)! \sigma^{2+j}) \]

\( j = 1, 2, 3, ... \) 

(3.4a)

recalling that \( H^{(r)}(0) = (\ln \hat{h})^{(r)}(0) = i^r K_r[h], r = 1, 2, 3, ... \). So, we obtain

\[ \hat{h}(t) = \exp(H(t)) = \exp[At^2 + \sum_{j=1}^{k} B_j t^{2+j} + o(t^{2+k})] \]

as \( t \to 0 \) 

(3.4b)

**Theorem 1** (Chebyshev-Edgeworth-Esseen Expansion). Let \( h(x) \) satisfy (2.4a) and choose \( \alpha = \alpha_1 = M_1[p] \), then one has, uniformly on \(-\infty < x < \infty\),

\[ h_n(x) = \frac{1}{\sqrt{2n\pi K_2[h]}} \exp\left(\frac{-x^2}{2nK_2[h]}\right) \sum_{j=0}^{k} (\frac{1}{\sqrt{n}})^j Q_j\left(\frac{-x}{\sqrt{n}}\right) + o\left(\frac{1}{\sqrt{n}}\right)^{1+k} \]

as \( n \to \infty \) (3.5a)

where

\[ Q_j(y) = \sum_{s=0}^{j} \sum'_{s_1, s_2, ..., s_j} H e_{2s+j}(-y/\sqrt{K_2[h]}) \Pi_{\ell=1}^{j} ((D_\ell)^{s_\ell} / s_\ell!) \]

(3.5b)
where the sum $\Sigma'$ is over all $j$-tuples $(s_1, s_2, ..., s_j)$ where the $s_\ell$ run over all non-negative integers which satisfy the two conditions $\sum_{\ell=1}^j s_\ell = s$ and $\sum_{\ell=1}^j \ell s_\ell = j$ and where

$$D_\ell = K_{2+\ell}[h]/[(2 + \ell)!((\sqrt{K_2[h]})^{2+\ell})]$$

(3.5c)

($\ell = 1, 2, 3, ...$). Here, the $He_r(x)$ is the monic Hermite polynomial of degree $r$

$$He_r(x) = \exp(x^2/2)(-d/dx)^r\exp(-x^2/2) = H_r(x/\sqrt{2})/(\sqrt{2})^r$$

(3.5d)

and the $H_r(x)$ are the ordinary Hermite polynomials.

Proof. We have by (3.3j) $\hat{h}(t) = \exp(At^2(1 + o(t^2)t^{-2})$ as $t \to 0$ where $Re(A) < 0$, so there exits a $T_0 > 0$ such that

$$|\hat{h}(t)| \leq \exp(-\delta_0 t^2)$$

( for all $|t| \leq T_0$)

(3.6)

, where $\delta_0 = \frac{1}{2}Re(-A)$, which is positive. Also, recall that $|\hat{h}(t)| \to 0$ as $|t| \to \infty$, so there exists an $T_1 > 0$ such that $|\hat{h}(t)| \leq 1/2$ for all $|t| \geq T_1$. In addition, (3.2) gives

$$|\hat{h}(t)| \leq \exp(-\delta_1 t^2)$$

for $T_0 \leq |t| < T_1$
where $\delta_1 = T_1^{-2}\ln(1/M(T_0)) > 0$, since $\exp(-\xi y^2)$ is monotone decreasing in $y$ for positive constant $\xi$. Combining these facts gives bounds for $|\hat{h}(t)|$:

$$|\hat{h}(t)| \leq \exp(-\delta t^2)$$

for $(0 \leq |t| < T_1)$ (3.8a)

and

$$|\hat{h}(t)| \leq \frac{1}{2}$$

for $(T_1 \leq |t| < \infty)$ (3.8b)

, where $\delta = \min(\delta_0, \delta_1)$. Now define $\phi_n = \ln(n)/\sqrt{n}$ and let $n_2$ be the smallest positive integer for which $\phi_n < T_1$, then take $n \geq n_2$ and use the bounds of (3.8) above to obtain the following:

$$\left| \int_{\phi_n \leq |t| < T_1} \exp(-ixt)(\hat{h}(t))^n dt \right| \leq 2T_1 \exp(-\delta(\ln(n))^2) = o(n^{-s})$$

as $n \to \infty$ (3.9a)

and

$$\left| \int_{T_1 \leq |t| < \infty} \exp(-ixt)(\hat{h}(t))^n dt \right| \leq \left( \frac{1}{2} \right)^{n-n_1} \| (\hat{h})^{n_1} \|_{L_1} = o(n^{-s})$$

as $n \to \infty$ (3.9b)

Application of the triangle inequality to (3.1) gives

$$|2\pi h_n(x) - \int_{0 \leq |t| < \phi_n} \exp(-ixt)(\hat{h}(t))^n dt| \leq \left| \int_{\phi_n \leq |t| < T_1} \exp(-ixt)(\hat{h}(t))^n dt \right| +$$
\[ + \int_{T_1 \leq |t| < \infty} \exp(-ixt)(\hat{h}(t))^n dt \]

so the results of (3.9a),(3.9b) yield

\[ 2\pi h_n(x) = \int_{0 \leq |t| < \phi_n} \exp(-ixt)(\hat{h}(t))^n dt + o(n^{-s}) \]

as \( n \to \infty \) (3.10)

for each fixed \( s > 0 \). By the results of (3.9), equation (3.10) holds uniformly on \((-\infty < x < \infty)\) and corresponds to equation (2.11) of Baishanski and Snell, where their \( a_{n\nu} \) is made to correspond formally to the present \( h_n(x) \). To complete the proof here it is sufficient to continue with the unchanged reasonings of these authors. This leads to the conclusion:

\[ h_n(x) = \frac{1}{\sqrt{-4An\pi}} \exp\left(\frac{(\gamma_n(x))^2}{4A}\right) \sum_{j=0}^{k} \left(\frac{1}{\sqrt{n}}\right)^j Q_j(\gamma_n(x)) + O\left(\frac{1}{\sqrt{n}}\right)^{1+k} \]

as \( n \to \infty \) (3.10a)

uniformly on \(-\infty < x < \infty\). Here, \( \gamma_n(x) = -x/\sqrt{n} \), \( Q_0(\gamma) = 1 \), and

\[ Q_r(\gamma) = (i/\sqrt{-4A})^r \sum_{m=1}^{r} H_{2m+r}(\gamma/\sqrt{-4A})(4A)^{-m} \sum_{m_1, m_2, \ldots, m_r, \ldots} B_{m_1, m_2, \ldots, m_r} \]

\[ = \sum_{m=1}^{r} H_{2m+r}(\gamma/\sqrt{-4A}) \sum_{m_1, m_2, \ldots, m_r, \ldots} B_{m_1, m_2, \ldots, m_r} \sum_{j=1}^{r} \left(\frac{i}{\sqrt{-4A}}\right)^{j+2} B_j \frac{m_j}{m!} \]

\[ = \sum_{m=1}^{r} \left(\frac{-1}{\sqrt{2}}\right)^{2m+r} H_{2m+r}(\gamma/\sqrt{-4A}) \sum_{m_1, m_2, \ldots, m_r, \ldots} B_{m_1, m_2, \ldots, m_r} \sum_{j=1}^{r} \left(\frac{1}{\sqrt{K_2[h]}}\right)^{2+j} K_{2+j} \frac{m_j}{m!} \]

\[ = \sum_{m=1}^{r} H_{2m+r}(\gamma/\sqrt{-4A}) \sum_{m_1, m_2, \ldots, m_r, \ldots} B_{m_1, m_2, \ldots, m_r} \sum_{j=1}^{r} \left(\frac{1}{\sqrt{K_2[h]}}\right)^{2+j} K_{2+j} \frac{m_j}{m!} \]

as \( n \to \infty \) (3.10b),

\[ \text{for each fixed } s > 0. \text{ By the results of (3.9), equation (3.10) holds uniformly on } (-\infty < x < \infty) \text{ and corresponds to equation (2.11) of Baishanski and Snell, where their } a_{n\nu} \text{ is made to correspond formally to the present } h_n(x). \text{ To complete the proof here it is sufficient to continue with the unchanged reasonings of these authors. This leads to the conclusion:} \]

\[ h_n(x) = \frac{1}{\sqrt{-4An\pi}} \exp\left(\frac{(\gamma_n(x))^2}{4A}\right) \sum_{j=0}^{k} \left(\frac{1}{\sqrt{n}}\right)^j Q_j(\gamma_n(x)) + O\left(\frac{1}{\sqrt{n}}\right)^{1+k} \]

as \( n \to \infty \) (3.10a)

uniformly on \(-\infty < x < \infty\). Here, \( \gamma_n(x) = -x/\sqrt{n} \), \( Q_0(\gamma) = 1 \), and

\[ Q_r(\gamma) = (i/\sqrt{-4A})^r \sum_{m=1}^{r} H_{2m+r}(\gamma/\sqrt{-4A})(4A)^{-m} \sum_{m_1, m_2, \ldots, m_r, \ldots} B_{m_1, m_2, \ldots, m_r} \sum_{j=1}^{r} \left(\frac{i}{\sqrt{-4A}}\right)^{j+2} B_j \frac{m_j}{m!} \]

\[ = \sum_{m=1}^{r} H_{2m+r}(\gamma/\sqrt{-4A}) \sum_{m_1, m_2, \ldots, m_r, \ldots} B_{m_1, m_2, \ldots, m_r} \sum_{j=1}^{r} \left(\frac{1}{\sqrt{-4A}}\right)^{j+2} B_j \frac{m_j}{m!} \]

\[ = \sum_{m=1}^{r} \left(\frac{-1}{\sqrt{2}}\right)^{2m+r} H_{2m+r}(\gamma/\sqrt{-4A}) \sum_{m_1, m_2, \ldots, m_r, \ldots} B_{m_1, m_2, \ldots, m_r} \sum_{j=1}^{r} \left(\frac{1}{\sqrt{-4A}}\right)^{j+2} B_j \frac{m_j}{m!} \]

\[ = \sum_{m=1}^{r} H_{2m+r}(\gamma/\sqrt{-4A}) \sum_{m_1, m_2, \ldots, m_r, \ldots} B_{m_1, m_2, \ldots, m_r} \sum_{j=1}^{r} \left(\frac{1}{\sqrt{-4A}}\right)^{j+2} B_j \frac{m_j}{m!} \]

as \( n \to \infty \) (3.10a)
(r = 1, 2, 3,...) where we have used the relations between the cumulants $K_{2+j}[h]$ introduced above on the one hand, and the (Baishanski and Snell) constants $A$ and the $B_j, j = 0, 1, 2, ..., k$ on the other. Thus, (3.10a)-(3.10b) give (3.5a),(3.5b),(3.5c), which concludes the proof of Theorem 1.

\[ \square \]
CHAPTER 4

LARGE-N BEHAVIOR OF THE MODULUS OF THE
N-FOLD CONVOLUTION ON THE CENTRAL SET

From the concluding equation of Theorem 1, we use the triangle inequality to obtain

\[ |h_n(x)| = \frac{1}{\sqrt{2n\pi|K_2|}} \exp\left(-\frac{x^2}{2n} \text{Re}\left(\frac{1}{K_2}\right)\right) |\sum_{j=0}^{k} \left(\frac{1}{\sqrt{n}}\right)^j Q_j\left(-\frac{x}{\sqrt{n}}\right)| + \]

\[ + o\left(\left(\frac{1}{\sqrt{n}}\right)^{1+k}\right) \]

uniformly on \(-\infty < x < \infty\)

as \(n \to \infty\) (4.1)

The goal of the present section is to show, following Baishanski and Snell, how the modulus of the linear combination of \(1 + k\) complex-valued polynomials \(Q_j(\gamma)\) on the RHS of (4.1) can be asymptotically represented by an a corresponding linear combination of \(1 + k\) real-valued polynomials \(S_j(\gamma)\):

\[ |\sum_{j=0}^{k} \left(\frac{1}{\sqrt{n}}\right)^j Q_j\left(-\frac{x}{\sqrt{n}}\right)| = \sum_{j=0}^{k} \left(\frac{1}{\sqrt{n}}\right)^j S_j\left(-\frac{x}{\sqrt{n}}\right) + o\left(\left(\frac{1}{\sqrt{n}}\right)^{1+k}\right) \]

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uniformly in $\gamma$ for $\gamma$ on the set $Z_n$, as $n \to \infty$, where

$$Z_n = Z_n(\epsilon_k) = \{x : |x/\sqrt{n}| < n^{\epsilon_k}, \quad 0 < \epsilon_k < 1/(6(k+1))\}$$

as $n \to \infty$ (4.2)

defines the ”central set”, from which the ”central contributions” to the modulus of the LHS of (4.2) arise, and where

$$S_0(\gamma) = 1$$

$$S_1(\gamma) = \frac{1}{2}(Q_1(\gamma) + \bar{Q}_1(\gamma))$$

$$S_r(\gamma) = \frac{1}{2}(Q_r(\gamma) + \bar{Q}_r(\gamma)) + \frac{1}{2} \sum_{j=1}^{r-1} (Q_{r-j}(\gamma) \bar{Q}_j(\gamma) - S_{r-j}(\gamma) S_j(\gamma))$$

($r = 2, 3, 4, ...$) (4.2a)

**Theorem 2.** Let the assumptions of Theorem 1 hold, and define

$$W_n(x) = \frac{1}{\sqrt{4\pi|A|n}} \exp((\gamma_n(x))^2 Re(\frac{1}{4A}))$$

then

$$|h_n(x)| = W_n(x) \Sigma_{j=0}^k (1/\sqrt{n})^j S_j(\gamma_n(x)) + o((1/\sqrt{n})^{k+1})$$

as $n \to \infty$ (4.3)
uniformly for $x \in Z_n(\epsilon_k)$ as $n \to \infty$, where the $S_r(\gamma)$ are polynomials, with $S_0(\gamma) = 1$, and

$$S_r(\gamma) = \sum_{m=1}^{r} m! \left( \frac{1}{2} \right)^{m} \sum'_{(m_1, m_2, \ldots, m_r), m} \prod_{j=1}^{r} [\Sigma^j_{j=1} Q_{j-j_1}(\gamma) \bar{Q}_{j_1}(\gamma)]^{m_j} / m_j!$$

$r=1, 2, 3, \ldots$(4.3a)

The $S_r(\gamma)$ can also be generated by recurrence: $S_1(\gamma) = \frac{1}{2} (Q(\gamma) + \bar{Q}(\gamma))$ and

$$S_r(\gamma) = \frac{1}{2} (Q(\gamma) + \bar{Q}(\gamma)) + \frac{1}{2} \sum_{j=1}^{r-1} (Q_{r-j}(\gamma) \bar{Q}_{j}(\gamma) - S_{r-j}(\gamma) S_{j}(\gamma))$$

$r=2, 3, 4, \ldots$(4.5)

Proof: We use the following

Lemma 2: Let $a_1, a_2, a_3, \ldots$, be a sequence of real-valued constants and let $\rho$ be a positive number such that $|a_j| \leq (\rho)^j$ for $j = 1, 2, 3, \ldots, \ell$. Define

$$G(\lambda) = (1 + a_1 \lambda + a_2 \lambda^2 + \ldots + a_\ell \lambda^\ell)^{\frac{1}{2}}$$

(4.6)

then, for all $|\lambda| \leq 1/(3\rho)$, the function $G(\lambda)$ is infinitely differentiable and one has

$$|G(\lambda) - (1 + \sum_{j=1}^{k} b_j \lambda^j)| \leq E_{k+1}(\rho |\lambda|)^{k+1}$$

(4.7)

where the constant $E_{r+1}$ depends only on $k$, and where now

$$b_r = G^{(r)}(0) / r! = \sum_{m=1}^{r} m! \left( \frac{1}{2} \right)^{m} \sum'_{(m_1, m_2, \ldots, m_r), m} \prod_{j=1}^{r} \frac{a_{m_j}}{m_j}$$

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Proof (Lemma 2): By definition, \( G(\lambda) = \sqrt{1 + \phi(\lambda)} \), where \( \phi(x) = \sum_{j=1}^{k} a_j x^j \).

Now, \( |\phi(\lambda)| \leq \sum_{j=1}^{k} |a_j| |\lambda|^j \leq \rho |\lambda| (1 - |\rho| |\lambda|)/(1 - |\rho|) \leq \frac{1}{2} \) for all \( |\lambda| \leq 1/(3\rho) \).

So, \( \frac{1}{2} \leq 1 - |\phi(x)| \leq |1 + \phi(x)| \), which gives a bounding on the derivatives:

\[
|((\sqrt{1 + u})^{(m)})|_{u=\phi(x)} = m!(1/2)(\sqrt{1 + \phi(x)})^{-m} \leq m!2^{m/2},
\]

(4.9)

\( m = 1, 2, 3... \)

which, together with the fact that

\[
\phi^{(j)}(\lambda) = \sum_{j=r}^{\ell} (j!/(j-r)!) a_j \lambda^{j-r}
\]

(4.10)

allows us to compute the derivatives of \( G(\lambda) \) by the Faa di Bruno formula:

\[
G^{(r)}(\lambda)/r! = \sum_{m=1}^{r} m! \left( \frac{1/2}{m} \right) (\sqrt{1 + \phi(\lambda)})^{-m} \sum_{(m_1, m_2, ..., m_r), m \prod_{j=1}^{r} \frac{(\phi^{(j)}(\lambda)/j!)^{m_j}}{m_j!}}
\]

(4.11)

for all \( |\lambda| \leq 1/(3\rho) \). Since \( G(0) = 1 \) and \( G(\lambda) \) is infintely differentiable in \((-1/(3\rho), 1/(3\rho))\)

thus the Taylor formula gives

\[
G(\lambda) - \sum_{j=0}^{k} G^{(j)}(0) \lambda^j / (j!) = G^{(k+1)}(\xi) \lambda^{k+1} / (k+1)!
\]

(4.12)
for a certain $\xi$ between 0 and $\lambda$. Evaluating the derivatives $G^{(j)}(0)$ by (4.11) and the facts that $\phi(0) = 0, \phi^{(j)}(0) = j!a_j, (j=1,2,3,\ldots,\ell), \phi^{(j)}(0) = 0, (j = \ell + 1, \ell + 2, \ldots)$ and substituting these results in (4.11), (4.12) yields

$$G(\lambda) - (1 + \sum_{j=0}^{k} b_j \lambda^j) = G^{(k+1)}(\xi) \lambda^{k+1} / (k+1)!$$

(4.13)

where,

$$b_r = G^{(r)}(0) / r! = \sum_{m=1}^{r} m! \left( \frac{1}{m} \right) \sum_{(m_1,m_2,\ldots,m_r),m} \prod_{j=1}^{r} \frac{a_{m_j}}{m_j}$$

(4.14)

Next, to obtain the bounding seen in (4.7), we note that the general expression for $G^{(r)}(\lambda)$ given by (4.11), yields, with the triangle inequality and (4.9)

$$|G^{(r)}(\lambda)| / r! \leq \sum_{m=1}^{r} m! \left( \frac{1}{m} \right) |\phi^{(j)}(\lambda)| / j! \leq \rho^j \Delta_j$$

(4.15)

for all $|\lambda| \leq 1/(3\rho)$. But defining $\psi(u) = u + u^2 + u^3 + \ldots + u^\ell$, we have

$$|\phi^{(j)}(\lambda)| \leq \sum_{q=j}^{\ell} (q! / (q - j)!) \rho^q |\lambda|^{q-j} \leq \rho^j |\psi^{(j)}(\rho|\lambda|)| \leq \rho^j \Delta_j$$

(4.16)

$(j = 1, 2, 3, \ldots, \ell, \ell + 1, \ldots)$where we have defined

$\Delta_j = \sup\{|\psi^{(j)}(w)| : 0 \leq w \leq 1/3\}$. These $\Delta_j$ must be finite for each $j$, since $\psi(w)$ and the $\psi^{(j)}(w)$ are continuous functions on the closed interval $0 \leq w \leq 1/3$. Inserting
into (4.12) the bounds on $\phi^{(j)}$ as given by (4.16) and recalling the restrictions on the summation $\Sigma'$ gives

$$
|G^{(r)}(\lambda)|/r! \leq \Sigma_{m=1}^{r} m! \left(\frac{1}{m}\right) 2^{m/2} \Sigma'_{(m_1, m_2, \ldots, m_r), m} \Pi_{j=1}^{r} \frac{(\rho^j \Delta_j)/j!)^{m_j}}{m_j!} = \rho^r E_r
$$

for all $|\lambda| \leq 1/(3\rho)$ where the constant

$$E_r = \Sigma_{m=1}^{r} m! \left(\frac{1}{m}\right) 2^{m/2} \Sigma'_{(m_1, m_2, \ldots, m_r), m} \Pi_{j=1}^{r} \frac{(\Delta_j)/j!)^{m_j}}{m_j!}
$$

(4.17)

(4.18)

$(r = 1, 2, 3, \ldots)$. Note that $E_r$ depends only on $r$ (not on $\lambda$, and not on $\rho$). Thus, application of (4.14) when $|\lambda| \leq 1/(3\rho)$ and $r = k + 1$ gives

$$
|G^{(k+1)}(\xi_\lambda)|/(k + 1)! \leq \sup\{|G^{(k+1)}(t)| : 0 \leq |t| \leq (1/(3\rho))/(k + 1)! \} \leq \rho^{k+1} E_{k+1}
$$

since $|\xi_\lambda| \leq |\lambda| \leq 1/(3\rho)$

(4.19)

. The result of (4.19) is now employed in the Taylor formula ((4.13) to give precisely (4.7). This completes the proof of Lemma 2. To proceed, we now need

Lemma 3: Let $q_0 = 1, q_1, q_2, \ldots, q_k$ be complex-valued coefficients of a polynomial in $\lambda$, real-valued, then

$$
|\frac{\Sigma_{j=0}^{k} q_k \lambda^j}{\Sigma_{j=0}^{k} s_k \lambda^j} - E_{k+1}(\rho|\lambda|)|^{k+1}
$$

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for all \(|\lambda| \leq 1/(3\rho)\), where the real coefficients \(s_j\) are: \(s_0 = 1\) and
\[
s_r = \Sigma_{m=1}^r m! \left(\frac{1/2}{m}\right) \Sigma_{(m_1,m_2,\ldots,m_r),m} \Pi_{j=1}^r (\Sigma_{j_1=0}^j q_{j-j_1} \bar{q}_{j_1})^{m_j}/m_j!
\]
(4.20)

Also, the \(s_j\) can be generated by the recurrence relations: \(s_0 = 1, s_1 = \frac{1}{2}(q_1 + \bar{q}_1)\) and
\[
s_r = \frac{1}{2}(q_r + \bar{q}_r) + \frac{1}{2}\Sigma_{j=1}^{r-1} (q_{r-j} \bar{q}_j - s_{r-j} s_j)
\]
(4.21)

\((r = 2, 3, 4, \ldots)\)

Proof (Lemma 3):
\[
|\Sigma_{j=0}^k q_k \lambda^j|^2 = \Sigma_{j_1,j_2=0}^k q_{j_1} \bar{q}(j_2) \lambda^{j_1+j_2} =
\]
\[
= \Sigma_{m=0}^{k-1} (\Sigma_{j=0}^m \bar{q}_j q_{m-j}) \lambda^m + (\Sigma_{j=0}^k q_j q_{k-j}) \lambda^k + \Sigma_{m_1=0}^{k-1} (\Sigma_{j_1=0}^{m_1} \bar{q}_{k-j_1} q_{k-(m_1-j_1)}) \lambda^{2k-m_1} =
\]
\[
= \Sigma_{m=0}^k (\Sigma_{j=0}^m \bar{q}_j q_{m-j}) \lambda^m + \Sigma_{m_2=k+1}^{2k} (\Sigma_{j_1=0}^{2k-m_2} \bar{q}_{k-j_1} q_{m_2-(k-j_1)}) \lambda^{m_2} =
\]
\[
= 1 + \Sigma_{m=1}^{2k} a_m \lambda^m
\]
(4.23)
, where
\[ a_m = \sum_{j=0}^{m} \bar{q}_j q_{m-j}, \quad (m = 1, 2, 3, \ldots, k) \]
and
\[ a_m = \sum_{j=0}^{2k-m} \bar{q}_{k-j} q_{m-(k-j)}, \quad (m = k + 1, k + 2, 3, \ldots, 2k) \]
(4.24)

Apply now Lemma 2 above with the \( a_j \) of (4.25), \( \ell = 2k \) and any \( \rho \geq \max\{|a_j|^{1/j} : j = 1, 2, 3, ..., 2k\} \) to get
\[ ||\sum_{j=0}^{k} q_j \lambda^j| - \sum_{j=0}^{k} s_k \lambda^j| = |(1 + \sum_{j=1}^{2k} a_j \lambda^j)^{1/2} - (1 + \sum_{j=1}^{k} s_k \lambda^j)| \]
\[ = |G(\lambda) - \sum_{j=0}^{k} s_k \lambda^j| \leq E_{k+1}(\rho|\lambda|)^{k+1} \]
(4.25)

for all \(|\lambda| \leq 1/(3\rho)\), where \( E_{k+1} \) depends only on \( k \). By (4.14) and Lemma 2 now gives, using (4.24)
\[ s_r = b_r = G^{(r)}(0)/r! = \sum_{m=1}^{r} m! \left( \frac{1}{m} \right) \sum_{(m_1, m_2, \ldots, m_r), m} \prod_{j=1}^{r} \frac{(\sum_{j=0}^{m} \bar{q}_{j} q_{j})^{m_j}}{m_j!} \]
\[ (r = 1, 2, 3, \ldots, k) \]
(4.26)

Finally, (4.25) implies that
\[ |\sum_{j=0}^{k} q_j \lambda^j| = \sum_{j=0}^{k} s_j \lambda^j + o(\lambda^{k+1}) \]
as \( \lambda \to 0 \) (4.27)
Thus,

\[ |\sum_{j=0}^{k} q_k \lambda^j|^2 = \sum_{j_1,j_2=0}^{k} q_{j_1} \bar{q}(j_2) \lambda^{j_1+j_2} + o(\lambda^k) \]

\[ = \sum_{m=0}^{k} (\sum_{j=m}^{m} \bar{q}(m-j)) \lambda^m + o(\lambda^k) = \sum_{m=0}^{k} \sum_{j=0}^{s_j s_{m-j}} s_{j_1} \lambda^m + o(\lambda^k) \]

I.e.,

\[ \sum_{m=0}^{k} \lambda^m \sum_{j=0}^{m} (\bar{q}_j q_{m-j} - s_j s_{m-j}) = o(\lambda^k) \]

as \( \lambda \to 0 \) (4.28)

Letting \( \lambda \to 0 \) in (4.28), gives \( 1 = |q_0|^2 = (s_0)^2 \), so, with this result divide both sides of (4.28) by \( \lambda \) and subsequently let \( \lambda \to 0 \) to get \( \Sigma_{j=0}^{1} q_j q_{1-j} = \Sigma_{j=0}^{1} s_j s_{1-j}, \) etc., for \( m = 0, 1, 2, 3, ..., k \) to obtain the basic recurrence relation used in Snell \[ \] and Baishanski and Snell\[ \] :

\[ \Sigma_{j=0}^{m} \bar{q}_j q_{m-j} = \Sigma_{j=0}^{m} s_j s_{m-j} \]

(4.29)

But since \( s_0 = 1 \), (4.22) to be proven is seen to be just a rearrangement of (4.28). This concludes the proof of Lemma 3. With these results we continue the Proof(Theorem 2): Using \( \lambda = 1/\sqrt{n} \) and \( q_j = Q_j(\gamma), j = 0, 1, 2, 3, ... \) and, by (4.27), generating the \( S_j(\gamma) = s_j \), while holding \( \gamma = -x/\sqrt{n} \) fixed, with \( x \in Z_n(\epsilon_k) \), we apply Lemma 3 (4.20) to the polynomial (4.1) and (4.27) gives

\[ |\sum_{j=0}^{k} Q_k \lambda^j| - \sum_{j=0}^{k} S_k \lambda^j | \leq E_{k+1}(\rho(\lambda)) |^{k+1} \]
for all $|\lambda| \leq 1/(3\rho)$. In order to satisfy the assumption of Lemma 2, one can take

$$\rho = \max\{|a_m|^{1/m} : m = 1, 2, 3, \ldots, 2k\}$$

(4.31)

Now since the $a_m$ are determined by the $q_j = Q_j(\gamma)$ which are polynomials of degree $3j$, thus one has

$$|q_j| = |Q_j(\gamma)| \leq r_j \max\{1, |\gamma|^{3j}\} \leq r_j n^{3\epsilon_k j}$$

(4.32)

(j = 0, 1, 2, 3, ..., k)

When we impose the condition that $x \in Z_n(\epsilon_k)$ and that $n = 2, 3, 4, \ldots$. Here, the constants $r_j$ are independent of $n$ and independent of $\gamma$. Now, use of (4.31) and the triangle inequality allows us to bound the $a_m$ given by (4.24):

$$|a_m| \leq n^{3\epsilon_k m} R_m$$

(4.33)

(m = 0, 1, 2, 3, ..., 2k)

where

$$R_m = \sum_{j=0}^{m} r_j r_{m-j}, \quad (m = 1, 2, 3, \ldots, k)$$

and

$$R_m = \sum_{j=0}^{2k-m} r_{k-j} r_{m-(k-j)}, \quad (m = k + 1, k + 2, k + 3, \ldots, 2k)$$

(4.34)
and we now take \( \rho = \max\{|a_m|^{1/m} : m = 1, 2, 3, \ldots 2k\} = R^n_3\epsilon_k \) where we have defined \( R = \max\{|R_m|^{1/m} : m = 1, 2, 3, \ldots 2k\} \), which constant depends only on \( k \). We notice that when

\[
0 < \epsilon_k = \theta/(6(k + 1)), (0 < \theta < 1)
\]

(4.36a)

then the relation \( 1/\sqrt{n} \leq 1/(3\rho) = 1/(3R^n_3\epsilon_k) \) will hold for all sufficiently great \( n \), which verifies that a basic assumption used in Lemma 2 and Lemma 3 is fulfilled. More interestingly, when (4.36a) holds, then the conclusion of Lemma 3, (4.20), gives

\[
W_n(\gamma)||\sum_{j=0}^{k} Q_k \lambda^j| - \sum_{j=0}^{k} S_k \lambda^j|/(1/\sqrt{n})^{k+1} \leq \frac{E_{k+1}}{\sqrt{4\pi|A|}} R^{k+1} n^{3(k+1)\epsilon_k^{1/2}} =
\]

\[
= \frac{E_{k+1}}{\sqrt{4\pi|A|}} R^{k+1} n^{1/2(\theta-1)} \to 0
\]

(4.36)

uniformly for \( x \in Z_n(\epsilon_k) \) as \( n \to \infty \). With \( \lambda = 1/\sqrt{n} \), we therefore have, by (4.1) and the triangle inequality,

\[
||h_n(x)| - W_n(\gamma)(\sum_{j=0}^{k} S_k(\gamma) \lambda^j)| \leq
\]

\[
\leq ||h_n(x)| - W_n(\gamma)|\sum_{j=0}^{k} Q_k(\gamma) \lambda^j| + |W_n(x)(|\sum_{j=0}^{k} Q_k \lambda^j| - \sum_{j=0}^{k} S_k \lambda^j)| \leq
\]

\[
\leq o(\lambda^{k+1}) + o(\lambda^{k+1}) = o(\lambda^{k+1})
\]
This completes the proof of (4.3). The validity of (4.4) and (4.5) follows directly from the (4.27) and (4.28), respectively, wherein \( q_j = Q_j(\gamma) \) and \( s_j = S_j(\gamma) \) are substituted. This concludes the proof of Theorem 2.
CHAPTER 5
UPPER BOUNDS ON THE MODULUS OF THE N-FOLD
CONVOLUTION

The purpose of this section is the production of a function of \( x \) and \( n \) which serve as an upper bound on \( |h_n(x)| \). Integration of this bound over

\[
(Z_n(\epsilon_k))^c = \{ x : n^{\epsilon_k} \leq |x|/\sqrt{n} \}
\]

, the complement to the central set, will then subsequently give a bound on the “noncentral contributions” to the norm \( \| h_n \|_{L^1} \).

**Theorem 3.** Let \( h(x) \) be as defined by (2.8a), with \( \alpha = \alpha_1 \) and satisfying the Assumptions in Chapter 2, then the n-fold convolution, \( h_n(x) \) satisfies,

\[
|h_n(x)| \leq \frac{C(v)}{2\pi} \left( \frac{1}{|x|^v} + \frac{n^v}{x^{2v}} \right) + \frac{D \theta_1^{n/2}}{2\pi \ x^2}
\]

(5.1)

for \( x \neq 0 \), where \( v \) is a positive integer, \( C(v) \) depends on \( v \), but is independent of \( x \) and of \( n \), and where \( D \) and \( \theta_1 \) are independent of \( x \), of \( n \) and of \( v \), with \( 0 < \theta_1 < 1 \).

**Proof (Theorem 3):** We shall need
Lemma 4 (Baishanski-Snell) Let $\rho$ and $\Phi$ be complex valued functions, $v$ and $v + 1$ times continuously differentiable, respectively, on an open interval $I$, and let $\text{supp}(\rho)$ be compact and contained in $I$. If, for every $t \in I$ one has $\text{Re}(\Phi) \leq 0$ and $0 < m < \Phi^{(1)}(t)$ and $|\Phi^{(j)}(t)| \leq M$ for $j = 2, 3, 4, ..., v + 1$, then

$$|\int_I \rho(t) \exp(\Phi(t)) dt| \leq C_v[\rho] \max\left(\frac{1}{m^v}, \left(\frac{M}{m^2}\right)^v\right)$$

(5.2)

Lemma 4 is proved in Baishanski and Snell, who prefaced its use by reasoning that we encapsulate in the following:

Cone Lemma: Let $f(t)$ be a complex-valued function such that $f(t) = at + o(t)$ as $t \to 0$, with $\text{Re}(a) \neq 0$ and let $\lambda_a > |\text{Im}(a)/\text{Re}(a)|$, then there exists an $\eta_a > 0$ such that for all $|t| \leq \eta_a$ one has

$$0 < \frac{1}{\sqrt{1 + \lambda_a^2}} < |\tau f(t) + i|$$

(5.2a)

for an arbitrary real number $\tau$. I.e., there exists an interval centered about $t = 0$, in which $\tau f(t)$ is bounded away from the point $-i$.

Proof (cone lemma): Let $\Gamma_\lambda = \{z : |\text{Im}(z)| \leq \lambda |\text{Re}(z)|\}$ and observe by geometrical considerations that $0 < 1/\sqrt{1 + \lambda^2} < |z + i|$ for all $z \in \Gamma_\lambda$. Moreover, $z \in \Gamma_\lambda$ implies that $\tau z \in \epsilon \Gamma_\lambda$ for arbitrary real $\tau$. Now, since $f(t)/t \to a$ as $t \to 0$ application of the triangle inequality shows that for each $\epsilon > 0$ there exits a $\delta(\epsilon)$ such that $||\text{Im}(f(t))/\text{Re}(f(t))| - |\text{Im}(a)/\text{Re}(a)|| < \epsilon$ for all $|t| \leq \delta(\epsilon)$ So, if $|\text{Im}(a)/\text{Re}(a)| = \lambda_a - \epsilon_a$, where $\epsilon_a > 0$, then $|\text{Im}(f(t))/\text{Re}(f(t))| < \lambda_a - \epsilon_a + \epsilon = \lambda_a - \epsilon_a/2 < \lambda_a$ provided that we choose
\[ \epsilon = \epsilon_a / 2, \text{ as we may. This gives that } f(t)\epsilon \Gamma_{\lambda_a}, \text{ and thus } \tau f(t)\epsilon \Gamma_{\lambda_a}, \text{ with arbitrary real } \tau, \text{ both hold for all } |t| \leq \delta(\epsilon_a / 2) = \delta\left(\frac{1}{2}(\lambda_a - |Im(a)|/Re(a))\right) = \eta_a, \text{ which defines } \eta_a. \text{ Therefore, for arbitrary real } \tau \text{ there holds (5.2a) } 0 < \frac{1}{\sqrt{1+\lambda_a^2}} < |\tau f(t)+i| \text{ everywhere in } -\eta \leq t \leq \eta. \text{(QED(cone lemma)).} \]

To begin, we recall that \( \hat{h}(t) = \exp(H(t)) \), by definition of \( H(t) \), and so

\[ 2\pi h_n(x) = I_n(x) + J_n(x) \]

\[ (5.3) \]

\[ I_n(x) = \int_{-\infty}^{\infty} \exp(nH(t) - ixt)\rho(t)dt \]

\[ (5.4) \]

\[ J_n(x) = \int_{-\infty}^{\infty} \exp(nH(t) - ixt)(1 - \rho(t))dt \]

\[ (5.5) \]

where \( \rho \) is taken to be an infinitely differentiable function with support in \( (-\eta, \eta) \) such that \( 0 \leq \rho(t) \leq 1 \), with \( \rho(t) = 1 \) for all \( |t| \leq \eta_1 \), where \( 0 < \eta_1 < \eta \). For example,

\[ \rho(t) = \beta \left( \frac{t + \eta_1}{\eta - \eta_1} \right), (-\eta < t < \eta_1), \]

\[ \rho(t) = 1, (-\eta_1 \leq t \leq \eta_1), \]

\[ \rho(t) = \beta \left( \frac{\eta - t}{\eta - \eta_1} \right), (\eta_1 < t < \eta) \]

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where the "bridging function" (see Gelbaum, B.R. and Olmsted, J.M.H., "Counterexamples in Analysis" p.40 (reference indicated by B.M. Baishanski))

\[
\beta(u) = 0, \quad (u \leq 0),
\]

\[
\beta(u) = \exp\left(-\frac{1}{u^2} \exp\left(-\frac{-1}{(1-u)^2}\right)\right), \quad (0 < u < 1),
\]

\[
\beta(t) = 1, \quad (1 \leq u).
\]

Now, define

\[
\Phi(t) = nH(t) - ixt,
\]

So, \(\Phi^{(1)}(t) = -x\{\tau H^{(1)}(t) + i\}\), where \(\tau = -n/x\). Recalling eqn() \(H^{(1)} = 2At + o(t)\), as \(t \to 0\), and that \(\text{Re}(A) \neq 0\), so \(|\text{Im}(A)/\text{Re}(A)|\text{is finite} \). Therefore we can apply the cone lemma above, taking therein \(f(t) = H(1)(t)\) and \(a = 2A\). Thus, for any \(\lambda_A > |\text{Im}(A)/\text{Re}(A)|\), the cone lemma guarantees that there exists an \(\eta_A > 0\) such that \(0 < (1 + \lambda_A^2)^{-1/2} < |\Phi^{(1)}(t)|/|x|\) for all \(|t| < \eta_A\). Recalling also that \(|H^{(j)}(t)| \leq L_j, j = 2, 3, 4, ..., v + 1\) and thus \(|\Phi^{(j)}(t)| = n|H^{(j)}(t)| \leq nL(v)\) for all \(|t| \leq T_2 = 1/\sqrt{N_2|h|}\), where \(L(v) = \max\{L_2, L_3, L_4, ..., L_{v+1}\}\) is independent of \(n\). We note also that \(\text{Re}(\Phi(t)) = n\text{Re}(H(t)) = n\text{Re}(A)t^2(1+o(t^2)/t^2) \leq \frac{1}{3}t^2\text{Re}(A) \leq 0\) for all \(|t| \leq T_0\), so we can take \(\eta = \min\{\eta_A, T_2, T_0\}\) in the definition of \(\rho\) to ensure the applicability.
of Lemma 4 to bound $I_n(x)$ of (5.4): we take $m = |x|/\sqrt{1 + \lambda^2}$ and $M = nL(v)$, so Lemma 4 yields (letting $c = 1/\sqrt{1 + \lambda^2}$):

$$|I_n(x)| \leq C \rho(v) \max\{1/(c|x|^v), (nL(v)/(cx^2))^v\} \leq C(v) \left( \frac{1}{|x|^v} + \frac{n^v}{x^{2v}} \right)$$

(5.6)

where $C(v) = C\rho(v) \max\{1/c^v, (L(v)/c^2)^v\}$ is independent of $x$ and of $n$. Next, we use the facts that $\hat{h}(t) \to 0$, and $\hat{h}^{(1)}(t) \to 0$ as $|t| \to \infty$ to evaluate the integral $J_n(x)$ using two successive integration by parts, which gives

$$-x^2 J_n(x) = K_n(x)$$

(5.8a)

where

$$K_n(x) = \int_{-\infty}^{\infty} e^{-ixt} [(d/dt)^2(1 - \rho(t))(\hat{h}(t))^n] dt =$$

$$= K_{n,1}(x) + K_{n,2}(x)$$

(5.8b)

with

$$K_{n,1}(x) = \int_{\eta < |t| \leq \eta} e^{-ixt} [(d/dt)^2(1 - \rho(t))(\hat{h}(t))^n] dt$$

(5.9a)

$$K_{n,2}(x) = \int_{\eta < |t| < \infty} e^{-ixt} [(d/dt)^2(1 - \rho(t))(\hat{h}(t))^n] dt$$

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Noting that
\[ K_{n,1}(x) = \int_{m<|t|\leq \eta} e^{-ixt} (-\rho^{(2)}(t))(\hat{\eta}(t))^n dt + \]
\[ + 2 \int_{m<|t|\leq \eta} e^{-ixt} \rho^{(1)}(t)n(\hat{\eta}(t))n^{-1}\hat{\eta}^{(1)}(t) dt + \]
\[ + \int_{|t|\leq \eta} e^{-ixt} (1 - \rho(t))\{n(n-1)(\hat{\eta}(t))n^{-2}\hat{\eta}^{(1)}(t) + n(\hat{\eta}(t))n^{-1}\hat{\eta}^{(2)}(t)\} dt \]

(5.10)

We recall the facts that \(|\hat{\eta}(t)| \leq M(\epsilon) < 1\) for all \(|t| \geq \epsilon\), \((\epsilon > 0)\) and \(|\hat{\eta}^{(j)}(t)| \leq N_j[h]\), \((j = 1, 2, 3, ...)\) and define
\[ \theta_1 = M(\eta_1) \]

(5.11)

Then, majorization of the integrals of (5.10) by standard methods gives
\[ |K_{n,1}(x)| \leq \theta_1^n \int_{m<|t|<\eta} |\rho^{(2)}(t)| dt + 2n\theta_1^{n-1}N_1[h] \int_{m<|t|<\eta} |\rho^{(1)}(t)| dt + \]
\[ + 2\{n(n-1)\theta_1^{n-2}N_1[h] + n\theta_1^{n-1}N_2[h]\}(\eta - \eta_1) \leq \]
\[ \leq n^2\theta_1^{n-2}C_0 \leq C_1\theta_1^{n/2} \]

(5.12)
(if $0 < \theta_1 < 1$, then $n^2 \theta_1^n \leq (2/\ln(\theta_1))^2 \theta_1^{n/2}, n \geq 0$) where $C_0$ and $C_1$ are independent of $x$ and of $n$. To find an upper bound on $K_{n,2}(x)$, we notice that since $1 - \rho(t) = 1$ for all $|t| \geq \eta$, thus we have

$$K_{n,2}(x) = \int_{\eta < |t| < \infty} e^{-ixt} (d/dt)^2(\hat{h}(t))^n dt =$$

$$= \int_{\eta < |t| < \infty} e^{-ixt} \{ (n(n-1)\hat{h}(t))^{n-2}(\hat{h}(1)(t))^2 + n\hat{h}(t))^{n-1}\hat{h}(2)(t)) \} dt$$

so majorization similar to that giving (5.12) yields here

$$|K_{n,2}(x)| \leq (n(n-1) \int_{\eta < |t| < \infty} |\hat{h}(t)|^{n-2} |\hat{h}(1)(t)| |\hat{h}(2)(t)| N_1[h] dt +$$

$$+ n \int_{\eta < |t| < \infty} |\hat{h}(t)|^{n-1} |\hat{h}(1)(t)| N_2[h] dt \leq$$

$$\leq \{ n(n-1)\theta_1^{n-2-n_1} N_1 + n\theta_1^{n-1-n_1} N_2 \} \| (\hat{h})^n_1 \|_{L^1} \leq$$

$$\leq C_2 \theta_1^{n/2}$$

(5.13),

where $C_2$ is independent of $x$ and of $n$ and we have taken into account that $0 < \eta_1 < \eta$.

Combining (5.12) and (5.13) we get, in view of (5.8)

$$|J_n(x)| = |K_n(x)|/x^2 \leq D \theta_1^{n/2}/x^2,$$

(5.14),
where $D = \max\{C_1, C_2\}$ is independent of $x$ and of $n$. Therefore, (5.6), (5.14) and (5.3) give finally

$$|h_n(x)| \leq \frac{C(v)}{2\pi} \left( \frac{1}{|x|^v} + \frac{n^v}{x^{2v}} \right) + \frac{D \theta_1^{n/2}}{2\pi \frac{x^2}{x^2}}$$

which is (5.1). This concludes the proof of Theorem 3.
CHAPTER 6

LARGE-N BEHAVIOR OF THE $L^1$ NORM OF THE N-FOLD CONVOLUTION

In this section we prove

**Theorem 4.** Let $h(x)$ be be defined by (2.4a), setting $\alpha = \alpha_1 = M_1[p]$ (as in Chapter 3) and let $L$ be a positive integer, then

$$\| h_n \|_{L^1} = \sum_{\ell=0}^{L} e_{2\ell} \left( \frac{1}{n} \right) + o\left( \left( \frac{1}{n} \right)^L \right)$$

(6.1)

as $n \to \infty$, where we have defined the constants

$$e_r = \frac{1}{\sqrt{2\pi |K_2|}} \int_{-\infty}^{\infty} e^{-\frac{\gamma^2}{2} \Re\left( \frac{1}{\sqrt{\pi^2}} \right)} S_r(\gamma) d\gamma$$

(6.2)

Proof(Theorem 4):

We employ the result of Chapter 4 and that of Chapter 5, in evaluating the contributions of the first and second integrals respectively of the following expression:
\[ \| h_n \|_{L^1} = \int_{Z_n(\epsilon_k)} |h_n(x)| \, dx + \int_{\mathbb{R} \setminus Z_n(\epsilon_k)} |h_n(x)| \, dx \]

(6.3)

In what follows we can take \( k = 2L + 1 \) or \( k = 2L + 2 \) accordingly as \( k \) is chosen odd or even. Using the result (4.21) of Chapter 4, we get

\[
\int_{Z_n(\epsilon_k)} |h_n(x)| \, dx =
\int_{|\gamma| \leq n^{s_k}} \left\{ \frac{1}{4\pi |A|} \sum_{j=0}^{k} \frac{\exp(\gamma^2 Re(1/4A)) S_j(\gamma) + \sqrt{n} o((\frac{1}{n})^{(k+1)/2})}{\gamma^{j/2}} \right\} \, d\gamma =
\sum_{j=0}^{k} \left( \frac{1}{n} \right)^{j/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi |A|}} \exp(\gamma^2 Re(1/4A)) S_j(\gamma) \, d\gamma + o((\frac{1}{n})^{s}) + o((\frac{1}{n})^{2k-\epsilon_k})
\]

(6.4)

as \( n \to \infty \), holding for arbitrary \( s > 0 \), since the \( S_j(\gamma) \) is a polynomial and taking into account that \( Re(1/4A) = -a < 0 \), it is straightforward to show that

\[
n^s \left| \int_{|\gamma| \leq n^{s_k}} \exp(-a\gamma^2) S_j(\gamma) \, d\gamma - \int_{-\infty}^{\infty} \exp(-a\gamma^2) S_j(\gamma) \, d\gamma \right| \to 0
\]
as \( n \to \infty \), \( (j = 0, 1, 2, 3, \ldots) \) for all \( s > 0 \).

The second integral of (6.3) is easily bounded by the result of Chapter 5, (5.1):

\[
\int_{\mathbb{R} \setminus Z_n(\epsilon_k)} |h_n(x)| \, dx = \int_{n^{s_k} < |x|/\sqrt{n}} |h_n(x)| \, dx \leq
\leq \int_{n^{s_k} < |x|/\sqrt{n}} \left\{ \frac{C(v)}{2\pi} \left( \frac{1}{|x|^v} + \frac{n^v}{x^{2v}} \right) + \frac{D}{2\pi} \frac{\theta^{n/2}}{x^2} \right\} \, dx =
\]

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\[ = O\left(\frac{1}{n}p_1\right) + O\left(\frac{1}{n}p_2\right) + o\left(\frac{1}{n}\right) = O\left(\frac{1}{n}p_2\right) = o\left(\frac{1}{n}\phi p_2\right) \]

as \( n \to \infty \), for arbitrary \( s > 0 \).

\( (6.5) \).

where \( p_2 = (\epsilon_k + \frac{1}{2})(2v - 1) - v \), and \( p_1 = (\epsilon_k + \frac{1}{2})(v - 1) \). Here, we have used the fact that that \( p_2 - p_1 = v(\epsilon_k - \frac{1}{2}) < 0 \), having recalled that \( 0 < \epsilon_k = \theta/(6k + 6) < \frac{1}{2} \), with arbitrary constant \( \theta : 0 < \theta < 1 \).

Also, the fact that

\[ p_2 = (2v - 1)\epsilon_k - \frac{1}{2} > 0 \]

(for all sufficiently great \( v \))(6.5a).

guarantees the rightmost "equality " in (6.5), for arbitrary constant \( \varphi : 0 < \varphi < 1 \).

We now gather the results of (6.2), (6.4) and (6.5) to get that, for all sufficiently great \( v \),

\[ \| h_n \|_{L^1} = \sum_{j=0}^{k} e_j \left(\frac{1}{n}\right)^{j/2} + o\left(\frac{1}{n}\right)^{\frac{k}{2} - \epsilon_k} + o\left(\frac{1}{n}\phi((2v-1)\epsilon_k - \frac{1}{2})\right) \]

(6.6) as \( n \to \infty \).

Next we see that

\[ e_j = 0, (j = 1, 3, 5, ...) \]

(6.7).
since in the computing of the integral (6.2) we can use the fact that $S_j(-\gamma) = (-1)^j S_j(\gamma), (j = 0, 1, 2, 3, ...)$ which follows from (4.3a) and the property $Q_j(-\gamma) = (-1)^j Q_j(\gamma)$, which in turn follows from (3.5b) and the property of the Hermite polynomials $He_j(-u) = (-1)^j He_j(u)$. Thus, (6.6) and (6.7) give

$$\| h_n \|_{L^1} = \sum_{\ell=0}^{[k/2]} e_{2\ell}(\frac{1}{n})^\ell + o(\frac{1}{n}^{1/2-k-\epsilon_k}) + o(\frac{1}{n}^{\varphi((2v-1)\epsilon_k-1/2)})$$

as $n \to \infty$ (6.8).

Next we can put (6.8) into asymptotic form as $n \to \infty$, i.e., the total of the contributions not included in the explicit sum decay faster than any term included in the explicit sum expression. For convenience in this, we choose herewith in all that follows:

$$\epsilon_k = \theta/(6k + 6) = 1/(12k + 12)$$

and

$$\varphi = \frac{1}{2}$$

(6.9).

According to our application of Theorem 3, and the definition therein of $v$, we have $v = d - 1$, where $d$ is the number of times that $\hat{h}(t)$ is continuously differentiable and we recall at this point that $d$ can be taken arbitrarily great (from Assumption II). With the choices of (6.9), we see that the requirement (6.5a) already becomes satisfied when

$$2 + 3(k + 1) \leq d$$
Consider now

Case 1: let \( k \) be (positive) odd integer. So, \( \frac{1}{2} k = \frac{1}{2}k - \frac{1}{2} \) and this, and the fact that \( \epsilon_k < \frac{1}{2} \), imply that \( \frac{1}{2}k \leq \frac{1}{2}k - \epsilon_k \). Therefore

\[
o((\frac{1}{n})^{\frac{1}{2}k-\epsilon_k}) = o((\frac{1}{n})^{\frac{1}{2}k})
\]

(6.9a).

Also, \( \frac{1}{2}k \leq \frac{1}{2}((2v-1)\epsilon_k - \frac{1}{2}) \) holds already for all \( d \) which satisfy \( 3(k+1)(2k-1)+1 \leq v = d - 1 \), so (6.9a) is certainly satisfied and thus,

\[
o((\frac{1}{n})^{\phi((2v-1)\epsilon_k - \frac{1}{2})}) = o((\frac{1}{n})^{\frac{1}{2}k})
\]

(6.10a).

Therefore, (6.10a) and (6.10b) show that when

\[
k = 1, 3, 5, ... \text{ and } 3(k+1)(2k-1) + 2 \leq d, \text{ then equation } (6.8) \text{ can be expressed as}
\]

\[
\| h_n \|_{L^1} = \sum_{\ell=0}^{[k/2]} e_2\ell\left(\frac{1}{n}\right)^\ell + o(\left(\frac{1}{n}\right)^{\frac{1}{2}k}) \quad \text{as } n \to \infty (6.11).
\]

Case 2: Let \( k \) be a (positive) even integer. So, \( \frac{1}{2}k = \frac{1}{2}k \) and (6.8) becomes

\[
\| h_n \|_{L^1} = \sum_{\ell=0}^{[\frac{k}{2}]-1} e_2\ell\left(\frac{1}{n}\right)^\ell + e_k\left(\frac{1}{n}\right)^{[k/2]} + o(\left(\frac{1}{n}\right)^{\frac{1}{2}k-\epsilon_k}) + o(\left(\frac{1}{n}\right)^{\phi((2v-1)\epsilon_k - \frac{1}{2})})
\]
In this case we have obviously

\[ e_k\left(\frac{1}{n}\right)^{\frac{1}{2k}} = o\left(\left(\frac{1}{n}\right)^{\frac{1}{2k}-1}\right) \]

as \( n \to \infty \) (6.13a).

Also here, \( \left[\frac{1}{2}k\right] - 1 \leq \frac{1}{2}k - \epsilon_k \), since \( 0 < \epsilon_k < 1 \), which gives

\[ o\left(\frac{1}{n}\right)^{\frac{1}{2}k - \epsilon_k} = o\left(\frac{1}{n}\right)^{\frac{1}{2k}-1} \]

as \( n \to \infty \) (6.13b).

and finally both (6.9a) and \( \left[\frac{1}{2}k\right] - 1 \leq \frac{1}{2}((2v - 1)\epsilon_k - \frac{1}{2}) \) will hold for all \( d \) satisfying \( 2 + 3(k + 1)(2k - 3) \leq d \), and this gives

\[ o(\left(\frac{1}{n}\right)^{\varphi((2v-1)\epsilon_k - \frac{1}{2})}) = o\left(\frac{1}{n}\right)^{\frac{1}{2k}-1} \]

as \( n \to \infty \) (6.13c).

So, by (6.13a), (6.13b), (6.13c) we see that

when \( k = 2, 4, 6, \ldots, \) and \( 3(k + 1)(2k - 3) + 2 \leq d \),

then equation (6.12) and thus (6.8), can be expressed as

\[ \| h_n \|_{L^1} = \sum_{\ell=0}^{[k/2]-1} e_{2\ell}\left(\frac{1}{n}\right)^\ell + o\left(\frac{1}{n}\right)^{\frac{1}{2k}-1} \]

as \( n \to \infty \) (6.14).

But since \( \left[\frac{1}{2}k\right] = L \) when \( k \) is the odd positive integer \( 2L + 1 \), and \( \left[\frac{1}{2}k\right] - 1 = L \) when
$k$ is the positive even integer $2L + 2$, we have that when Assumption II is satisfied (and so $d$ is arbitrarily great) both (6.11) and (6.14) can be expressed by the single formula:

$$
\| h_n \|_{L^1} = \sum_{\ell=0}^{L} e^{2\varepsilon (\frac{1}{n})^\ell} + o((\frac{1}{n})^L)
$$

(6.15)

as $n \to \infty$. This concludes the proof of Theorem 4.

**Theorem 4A:** Let $h(x)$ satisfy all of the Assumptions of Chapter 2, but with Assumption II replaced by the following

**Assumption IIA:** $x_j h(x) \in L^1$, ($j = 0, 1, 2, 3, ..., d$),

(6.16),

then the expansion (6.1) holds for all positive integer $k$ such that

$2 + 3(k + 1)(2k - 1) \leq d$.

**Proof (Theorem 4A):** So, $\hat{h}^{(d)}(t)$ exists is bounded and everywhere continuous and tends to zero as $|t| \to \infty$. Recalling the train of reasoning in the proof of Theorem 4 we see now that for all $k$ such that $2 + 3(k + 1)(2k - 1) \leq d$ holds, both (6.11) and (6.14) hold, and so, for all such $k$ (odd or even) the expansion (6.1) also holds. This concludes the proof of Theorem 4A.
We compute $e_0$ from (6.2), recalling that $S_0(\gamma) = 1$ to get

$$e_0 = (1 + (\text{Im}(K_2)/\text{Re}(K_2))^2)^{1/4}$$

(6.17)
CHAPTER 7

SPECIAL CASE WHERE $K_2$ IS REAL

Here, we restrict consideration to the special case where $\hat{h}^{(2)}(0) = M_2[h] = K_2[h] = -2A$ is real. Thus,

$$\sqrt{\text{Re}(\frac{1}{-4A})} = 1/\sqrt{-4A}$$

(7.1)

, which facilitates the calculation of the expansion coefficients, $e_{2j}$ appearing in (6.1).

In this section we prove the following:

Theorem 5: Let $i^j \hat{h}^{(j)}(0)$ be real for $j = 0, 1, 2, 3, ...2 + p$, where $p$ is a nonnegative integer. Then, for all (finite) positive integers $k \geq 2p + 3$ we have

$$\| h_n \|_{L^1} = 1 + \Sigma_{\ell=p+1}^{\lfloor \frac{k-1}{2} \rfloor} e_{2\ell}(\frac{1}{n})^{\ell} + o\left((\frac{1}{n})^{\lfloor \frac{k-1}{2} \rfloor}\right)$$

(7.2)

as $n \to \infty$, where

$$e_{2p+2} = \frac{\{Im(i^{3+p}\hat{h}^{(3+p)}(0))\}^2}{2(3 + p)!(-\hat{h}^{(2)}(0))^{3+p}}$$

$$= \frac{\{Im(K_{3+p}[h])\}^2}{2(3 + p)! (K_2[h])^{3+p}}$$

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\[
\frac{\{Im(M_{3+p}[h])\}^2}{2(3 + p)! (M_2[h])^{3+p}}
\]

(7.3)

Proof (Theorem 5): First, note that since \( M_2[h] = K_2[h] \) is real here. Thus, by (6.17) we have \( e_0 = 1 \). Recalling the fact that \( i^r \hat{h}^{(r)}(0) = (-1)^r M_r[h], (r = 1, 2, 3, ...) \) we see that under the assumptions of theorem 5 here the \( M_r[h], r = 1, 2, 3, ..., (2 + p) \) are all real. But this implies that the \( K_r[h], r = 1, 2, 3, ..., (2 + p) \) are also all real, (recalling Chapter 2), and this implies, in turn, that all the \( Q_j, j = 0, 1, 2, 3, ..., p \) (recalling Chapter 3) are also all real. Recalling the recurrence relation for the \( S_j \), we thus have

\[
S_j = Q_j = \bar{Q}_j, (j = 0, 1, 2, 3, ..., p)
\]

(7.3a).

To proceed, we prove the following

Lemma 5: Let

\[
S_0(\gamma) = 1 = Q_0(\gamma),
\]
\[
S_1(\gamma) = \frac{1}{2}(Q_1(\gamma) + \bar{Q}_1(\gamma)),
\]
\[
S_r(\gamma) = \frac{1}{2}(Q_r(\gamma) + \bar{Q}_r(\gamma)) + \frac{1}{2} \sum_{j=1}^{r-1} (Q_{r-j}(\gamma) \bar{Q}_j(\gamma) - S_{r-j}(\gamma) S_j(\gamma)),
\]

(7.4)

then

\[
S_j = Q_j = \bar{Q}_j, (j = 0, 1, 2, 3, ..., p)
\]

(7.5a)
implies that

\[ S_{p+m} = \frac{1}{2} (Q_{p+m} + Q_{p+m}) = \text{Re}(Q_{p+m}), \quad (m = 0, 1, 2, 3, \ldots, p, p + 1) \]

(7.5b)

Proof (Lemma 5): When \( m = 0 \), (7.5a) obviously gives (7.5b). When \( m = 1 \), set \( r = p + 1 \) in (7.4) and observe that as \( c = 1, 2, 3, \ldots, p \) successively, the \( r - c \) run through \( p, p - 1, p - 2, \ldots, 2, 1 \), so apply (7.5a) to verify that the sum in (7.4) is zero. Thus, (7.5b) does hold for \( m = 1 \), too. Next, take \( 2 \leq m \) and write (7.4) as

\[ S_{p+m} - \text{Re}(Q_{p+m}) = \frac{1}{2} \sum_{c=1}^{m-1} T_c + \frac{1}{2} \sum_{c=m}^{p} T_c + \frac{1}{2} \sum_{c=p+1}^{m+p-1} T_c, \]

(7.6)

where

\[ T_c = Q_{p+m-c} Q_c - S_{p+m-c} S_c. \]

(7.6a)

When \( m \leq p \), then (7.5a) gives \( T_c = 0 \) for each \( c = m, m = 1, \ldots, p \), so the second sum on the RHS of (7.6) is zero. When \( m = p + 1 \) then the second sum on the RHS of (7.6) is empty. Further, observe that as \( c = 1, 2, 3, \ldots, m - 1 \) in the first sum of (7.6), then the index defined as \( j_1 = m - c = m - 1, m - 2, \ldots, 2, 1 \) and, as \( c = p + 1, p + 2, \ldots, p + m - 1 \) in the third sum of (7.6), then the index defined as \( j_2 = c - p = 1, 2, 3, \ldots, m - 1 \), too. Therefore, combining these sums gives

\[
S_{p+m} - \text{Re}(Q_{p+m}) = \frac{1}{2} \sum_{j=1}^{m-1} (Q_{p+j} \bar{Q}_{m-j} + Q_{m-j} \bar{Q}_{p+j} - 2S_{m-j} S_{j+p}) = 50
\]
\[= \sum_{j=1}^{m-1} Q_{m-j} \left\{ \frac{1}{2} (Q_{p+j} + \bar{Q}_{p+j}) - S_{p+j} \right\} = \sum_{j=1}^{m-1} Q_{m-j} (S_{p+j} - \text{Re}(Q_{p+j})) \]

(7.7).

since when \(2 \leq m \leq p + 1\), we have that \(m - j = m - 1, m - 2, \ldots, 2, 1\) implies \(m - j \in \{0, 1, 2, 3, \ldots, p\}\) and so (7.5a) can be applied. From (7.7), it is clear that \(S_{p+j} = \text{Re}(Q_{p+j}), (j = 1, 2, 3, \ldots, m - 1)\) implies that \(S_{p+m} = \text{Re}(Q_{p+m})\), too. But \(S_{p+1} = \text{Re}(Q_{p+1})\) does hold, as seen at the outset. Thus, by induction, \(S_{p+m} = \text{Re}(Q_{p+m})\) holds for \(m = 0, 1, 2, 3, \ldots, p + 1\) This concludes the proof of Lemma 5.

In resume: the \(i^j \hat{h}^{(j)}(0)\) real for \(j = 0, 1, 2, 3, \ldots 2 + p\) implies that \(Q_j(\gamma)\) are real for \(j = 0, 1, 2, 3, \ldots p\) which, in turn implies (by Lemma 5)

\[S_r(\gamma) = \frac{1}{2} (Q_r(\gamma) + \bar{Q}_r(\gamma)), (r = 0, 1, 2, 3, \ldots 2p + 1)\]

(7.7a).

On the other hand, recall that the \(Q_j(\gamma)\) and \(\bar{Q}_j(\gamma)\) defined (Chapter 3) as certain linear combinations of the Hermite polynomials:

\[Q_j(\gamma) = \sum_{m=1}^{j} c_{j,m} H_{2m+j}(\gamma/\sqrt{-4A})\]

(7.7b).

Thus, by (7.7a), the \(S_r(\gamma), (r = 1, 2, 3, \ldots 2p + 1)\), are linear combinations of the \(H_\ell(\gamma/\sqrt{-4A}), with \ell \in \{3, 4, \ldots\}\), each term of which is orthogonal to \(H_0(\gamma/\sqrt{-4A}) =\)
1, with respect to the inner product weighting factor, \(\exp(-\gamma^2(1/(-4A)))\). But the expansion coefficients, \(e_r\), are by definition (Chapter 6) just proportional to the inner product (with respect to precisely this latter weighting factor) of \(S_r(\gamma)\) and \(H_0\). Therefore, inserting these linear combinations (7.7b) into (7.7a) and then substituting the results into (6.2) gives

\[
e_1 = e_2 = e_3 = \ldots = e_{2p+1} = 0
\]

(7.7c).

Next, it will be shown that

\[
S_{2p+2}(\gamma) = \text{Re}(Q_{2p+2}(\gamma)) + \frac{1}{2}(\text{Im}(Q_{p+1}(\gamma)))^2
\]

(7.8).

To prove (7.8), set \(j = 2p + 2\) into (7.4) to get

\[
S_{2p+2} - \text{Re}(Q_{2p+2}) = \frac{1}{2}\sum_{j=1}^{p}(Q_{2p+2-j}\bar{Q}_j - S_{2p+2-j}S_{j+p}) + \\
+ \frac{1}{2}(Q_{p+1}Q_{p+1} - (S_{p+1})^2) + \\
+ \frac{1}{2}\sum_{j=p+2}^{2p+1}(Q_{2p+2-j}\bar{Q}_j - S_{2p+2-j}S_{j+p})
\]

(7.9).

Changing the index in the last sum of (7.9) from \(j\) to \(j_2 = 2p + 2 - j\), which now runs through 1, 2, 3, ..., \(p\), allows this last sum to be combined with the first sum, to yield

\[
S_{2p+2} - \text{Re}(Q_{2p+2}) = \sum_{j=1}^{p}Q_j \left\{ \frac{1}{2}(Q_{2p+2-j} + Q_{2p+2-j}) - S_{2p+2-j} \right\} + \frac{1}{2}(Q_{p+1}Q_{p+1} - (S_{p+1})^2)
\]

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Now in (7.10), the \(2p + 2 - j\) runs through \(2p + 1, 2p, 2p - 1, \ldots, p - 3, p + 2\) so, by Lemma 5 above:

\[
\frac{1}{2}(Q_{2p+1-j} + Q_{2p+2-j}) - S_{2p+2-j} = 0, \quad (j = 1, 2, 3, \ldots, p).
\]

Thus,

\[
S_{2p+2} = \text{Re}(Q_{2p+2}) + \frac{1}{2}(Q_{p+1}Q_{p+1} - (S_{p+1})^2) = \text{Re}(Q_{2p+2}) + \frac{1}{2}(\text{Im}(Q_{p+1}))^2,
\]

having taken into that \(S_{p+1} = \text{Re}(Q_{p+1})\), by Lemma 5, (with \(m = 1\)). Thus, (7.8) is proven. To compute \(\text{Im}(Q_{p+1})\), recall (3.10b) in the form:

\[
Q_{p+1} = \sum_{m=1}^{p+1} H_{2m+p+1} (\gamma/\sqrt{-4A}) \Sigma'_{m_1, m_2, \ldots, m_{p+1}, m} \prod_{j=1}^{p+1} \left( (\frac{i}{\sqrt{-4A}})^{j+2} B_j \right)^m_j \frac{1}{m_j !} =
\]

\[
= \sum_{m=1}^{p+1} \left( \frac{-1}{\sqrt{2}} \right)^{2m+p+1} H_{2m+p+1} (\frac{\gamma}{\sqrt{2} K_2[h]}) \Sigma'_{m_1, m_2, \ldots, m_{p+1}, m} \prod_{j=1}^{p+1} \left( (\frac{1}{\sqrt{K_2[h]}})^{2+j} K_{2+j}(h) \right)^m_j \frac{1}{m_j !} =
\]

(7.11).

and we recall the relation between the cumulants and moments, as in (2.7):

\[
K_r[h] = r! \sum_{m=1}^{r} (-1)^{m} = \prod_{j=1}^{m} \left( \frac{M_j[h][j]}{j!} \right)^{m_j} \frac{1}{m_j !} =
\]

\[
= r! \sum_{m=1}^{r} (-1)^{m} = \prod_{j=1}^{m} \left( \frac{(-i)^j \hat{h}^{(j)}(0)[j]}{j!} \right)^{m_j} \frac{1}{m_j !} =
\]

(\(r=1,2,3,\ldots\)) (7.12).

From (7.12) it is clear that when (as in the assumption in this theorem 5) the \(i^j \hat{h}^{(j)}(0)\) are all real for \(j = 0, 1, 2, 3, \ldots 2 + p\), where \(p\) is a nonnegative integer, then the
$K_1[h], K_2[h], K_3[h], ..., K_{2+p}[h]$ must all be real, and so all of the factors in the product of (7.11) will be real, except possibly, $K_{3+p}[h]$. All parts of the expression of (7.11), other than the product are real, since the the coefficients of the Hermite polynomials and $\frac{\gamma}{\sqrt{2K_2[h]}}$ are real. Observe that all those terms in (7.11) which correspond to $m_{p+1} = 0$ must be real since in these terms the factor of $K_{3+p}[h]$ does not appear. Thus all terms of (7.11) which contribute to $Im(Q_{p+1})$ must be those corresponding to $m_{p+1} = 1$, and this requires, by the sum restrictions, to be those where $m_1 = m_2 = m_3 = ... = m_p = 0$. Therefore,

$$Im(Q_{p+1}) = (-1/\sqrt{2})^{3+p}H_{3+p}(\gamma/\sqrt{-4A})K_{3+p}[h]/(3 + p)!$$

(7.13).

which, when inserted into (7.8) gives $S_{2p+2}(\gamma)$ which, in turn is substituted into (6.2) to evaluate,

$$e_{2p+2} = \frac{1}{\sqrt{4|A|\pi}} \int_{-\infty}^{\infty} e^{(-\gamma^2 Re(\frac{1}{4A}))} S_{2p+2}(\gamma) d\gamma =$$

$$= \frac{1}{\sqrt{4|A|\pi}} \int_{-\infty}^{\infty} e^{(-\gamma^2 Re(\frac{1}{4A}))} \frac{1}{2} (Q_{2p+2}(\gamma) + \bar{Q}_{2p+2}(\gamma)) e^{(-\gamma^2 Re(\frac{1}{4A}))} d\gamma +$$

$$+ \frac{1}{2} C^2 \frac{1}{\sqrt{4|A|\pi}} \int_{-\infty}^{\infty} e^{(-\gamma^2 Re(\frac{1}{4A}))} (H_{p+3}(\gamma/\sqrt{-4A}))^2 d\gamma$$

(7.14),

where $C^2 = (Im(K_{3+p+1})/(3+p)!)/K_2[h]^{3+p}$. The integral above involving $\frac{1}{2} (Q_{2p+2}(\gamma) + \bar{Q}_{2p+2}(\gamma))$ gives zero, since the $Q_{2p+2}(\gamma)$ and its complex conjugate are linear combinations of the Hermite polynomials of degree not less than two, which (with respect to the weight function) are orthogonal to $H_0(u) = 1$. The second integral is well-known, since:

$$\int_{-\infty}^{\infty} (H_m(u))^2 exp(-u^2) du = m!2^m \sqrt{\pi}$$

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Using these facts yields
\[ e_{2p+2} = \frac{(Im(K_{3+p}[h]))^2}{2(3 + p)!(K_2[h])^{3+p}} \]
(7.15),

Also, setting \( r = 3 + p \) in (7.12),
\[
K_{3+p}[h] = (3 + p)!\Sigma_{m=1}^{3+p}(-1)^m(m - 1)!\Sigma'_{(m_1,m_2,...,m_{3+p}),m} \prod_{j=1}^{3+p} \frac{(M_j[h]/j!)^{m_j}}{m_j!} = \\
= (3 + p)!\Sigma_{m=1}^{3+p}(-1)^m(m - 1)!\Sigma'_{(m_1,m_2,...,m_{3+p}),m} \prod_{j=1}^{3+p} \frac{(-i)^j\hat{h}^{(j)}(0)/j!)^{m_j}}{m_j!} \\
\]
(7.16),

Now recall that \( M_j[h] = (-i)^j\hat{h}^{(j)}(0) \) and, that by assumption in this Theorem 5, the \( \hat{h}^{(j)}(0), (j = 0, 1, 2, 3, ..., 2 + p) \) are all real. Thus, all the moments \( M_j[h], (j = 0, 1, 2, 3, ..., 2 + p) \) must be all real, too. Therefore, in each of the above two representations of \( K_{3+p} \), the only factors in the products in which imaginary parts figure must be those factors arising from contributions with \( m_{3+p} \neq 0 \); but in all such contributions, the sum restrictions require that \( m_{3+p} = 1 \), and thus with this, necessarily \( m_j = 0, (j = 1, 2, 3, ..., 2 + p) \) and so \( m = 1 \) is the only possibility. Hence, take the imaginary part of the LHS of (7.16):
\[
Im(K_{3+p}[h]) = (3 + p)!\Sigma_{m=1}^{3+p}(-1)^m(m - 1)!\Sigma'_{(m_1,m_2,...,m_{3+p}),m} \prod_{j=1}^{3+p} \frac{(M_j[h]/j!)^{m_j}}{m_j!} = \\
= (3 + p)!Im(M_{3+p})/(3 + p)! \\
\]
and
\[
Im(K_{3+p}[h]) = (3+p)!\Sigma_{m=1}^{3+p}(-1)^m(m - 1)!\Sigma'_{(m_1,m_2,...,m_{3+p}),m} \prod_{j=1}^{3+p} \frac{(-i)^j\hat{h}^{(j)}(0)/j!)^{m_j}}{m_j!} = \\
= (3 + p)!Im((-i)^{3+p}\hat{h}^{(3+p)}(0))/(3 + p)! \\
\]

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Thus,

$$\text{Im}(K_{3+p}) = \text{Im}(M_{3+p}) = (-1)^{3+p}\text{Im}(i^{3+p}\hat{h}^{(3+p)}(0))$$

(7.18)

which, with the result of (7.15), gives directly (7.3). This concludes the proof of Theorem 5.
CHAPTER 8

REMARKS

The large-n behavior of $\|g_n\|_{L^1}$ of Chapter 2 can be viewed as the large-n behavior of the norm of the n-fold application of the convolution operator $T_g$, where $T_u$ is, for a $u \in L^1$ defined by

$$(T_u(f))(x) = \int_{-\infty}^{\infty} u(x-y) f(y) dy$$

(8.1)

which maps each $f \in L^1$ into $L^1$.

Proposition:

$$\| T_u \| = \sup_{f \in L^1} \left\{ \frac{\| T_u(f) \|_{L^1}}{\| f \|_{L^1}} \right\} = \| u \|_{L^1}$$

(8.2)

Proof(proposition): The rightmost equality in (8.2) can be seen from the following:

on the one hand

$$\| T_u(f) \|_{L^1} \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x-y)| f(y) dy dx / \| f \|_{L^1} \| u \|_{L^1}$$

(8.3)

On the other hand

$$\| u \|_{L^1} - \| u - (u \ast \varphi(., m)) \|_{L^1} \leq \| u - \{ u - (u \ast \varphi(., m)) \} \|_{L^1} \leq \| T_u \|$$

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where \( \varphi(x, m) = m \exp(-(mx)^2)/\sqrt{\pi}, (\| \varphi(., m) \| = 1) \) is the m-th entry in an approximate identity sequence. By a well-known theorem associated with such sequences, \( \| u - (u \ast \varphi(., m)) \|_{L^1} \to 0 \) as \( m \to \infty \). So, letting \( m \to \infty \) in (8.4a), we get \( \| u \|_{L^1} \leq \| T_u \|_{L^1} \), which result, combined with that of (8.3), yields (8.2). This concludes the proof of the Proposition of (8.2).

In the present work, \( g(x) \in L^1 \) and \( g_n(x) \in L^1 \) too, and we have

\[
(T^n g)(f)(x) = \int_{-\infty}^{\infty} g_n(x - y)f(y)dy = (T^n g_n)(f)(x)
\]

(8.5),

for each \( f \in L^1 \). Thus, with (8.5) we can make application of (8.2) to find that

\[
\| T^n g \| = \| T^n g_n \| = \| g_n \|_{L^1}
\]

(8.6),

It is known that

\[
l_{\infty}(\| T^n g \|)^{1/n} = r_{sp}
\]

(8.7),

where \( r_{sp} \) is the spectral radius of the \( T_g \) operator.

Thus, one application of the asymptotic expansion of the present work is a description of approach to the spectral radius of the operator \( T_g \) by the quantity \( (\| g_n \|_{L^1})^{1/n} \)

Taking \( k \geq 3 \) and

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letting $\lambda = 1/n, b = \ln(e_0), c = e_2/e_0$ we have

$$
(|| g_n ||_{L^1})^{1/n} = |\hat{g}(t_0)|\left\{\sum_{\ell=0}^{[\frac{1}{2}(k-1)]} e^{2\ell} (\frac{1}{n})^\ell + o\left(\frac{1}{n}\right)\right\}^{1/n} = 
$$

$$
= |\hat{g}(t_0)| \exp(\lambda \ln(e_0 + e_2 + o(\lambda))) = 
$$

$$
= |\hat{g}(t_0)| \exp(b \lambda + \lambda \ln(1 + c \lambda + o(\lambda))) = 
$$

$$
= |\hat{g}(t_0)| \exp(b \lambda + c \lambda^2 + o(\lambda^2)) = 
$$

$$
= \sup_{t \in \mathbb{R}} |\hat{g}(t)| \left\{1 + b/n + \frac{1}{2} b^2 + c(1/n)^2 + o((1/n)^2)\right\} = 
$$

$$
= \sup_{t \in \mathbb{R}} |\hat{g}(t)| = r_{sp}
$$

(8.8),

as $n \to \infty$. Recalling that $e_0 = (1 + (\text{Im}(K_2)/\text{Re}(K_2))^2)^{1/4}$, we see that when $\text{Im}(K_2) \neq 0$, then $b \neq 0$, so the convergence is slower, like $1/n$, than when $\text{Im}(K_2) = 0$, which gives $e_0 = 1$, so $b = 0$, making the convergence more rapid.

A similar remark can be made in the "discrete case" briefly presented in Chapter 1, where now, given a element of $(a(j))$ of $\ell^1$, one introduces the convolution operator $T_a$ mapping each sequence $(x(j)) \in \ell^1$ into $\ell^1$

$$(T_a(x))(r) = \Sigma_{j \in \mathbb{Z}} a(r - j)x(j),$$

(8.9),

(8.9), (where $r$ runs through $\mathbb{Z}$, the set of all integers). This gives

$$
\| T_a^n \| = \| a_n \|_{\ell^1} = \Sigma_{j \in \mathbb{Z}} |a_n(j)| = \| f^n \|_A
$$
when \( f(t) \), the absolutely convergent Fourier series seen in (1.5), has \( f(0) = 1 \).

A different type of problem presents itself if in the last problem we interchange the roles of \( f(t) \) and that of \( a(j) \). We recall that if 

\[
f(t) = \sum_{j \in \mathbb{Z}} a(j)e^{ijt} \quad \text{and} \quad g(t) = \sum_{j \in \mathbb{Z}} b(j)e^{ijt}
\]

are two absolutely convergent series, then the following convolution

\[
(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t - t_1)g(t_1)dt_1 = \sum_{j \in \mathbb{Z}} a(j)b(j)e^{ijt}
\]

(8.11),

so, for the \( n \)-fold convolution:

\[
f_n(t) = (f * f * ... * f)(t) = \sum_{j \in \mathbb{Z}} a(j)^n e^{ijt}
\]

(8.12),

Problem: to find the \( n \) dependence of the following norm

\[
\| f_n \|_{L^1(T)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_n(t)|dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sum_{j \in \mathbb{Z}} (a(j))^n e^{ijt}|dt
\]

(8.13),

It is straightforward to do the calculation in the following special example where take \( a(j) = z^j \), \( j = 0, 1, 2, 3, ... \) and \( a(j) = 0 \) for \( j = -1, -2, -3, ... \), with \( z = e^{-(\alpha + i\beta)}, 0 < \alpha \). Letting \( \lambda = e^{\alpha} \), we get

\[
\| f_n \|_{L^1(T)} = \frac{2}{\pi} K(\lambda^n) = \sum_{r=0}^{\infty} \left\{ \frac{(2r)!}{(r!2^r)^2} \right\}^2 e^{-2r\alpha}
\]

(8.14),

where \( K(u) \) is the complete elliptic integral of the first kind.
BIBLIOGRAPHY


