STATISTICAL MECHANICS OF CELLULAR AUTOMATA

AND RELATED DYNAMICAL SYSTEMS

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To My Parents
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VITAE

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CHAPTER I
INTRODUCTION

Our understanding of many systems in physics, chemistry, and biology depends on our ability to analyze the time evolution of complex structures. Dissipative systems in nature exhibit "self-organization" which may be loosely defined as the spontaneous development of complex structures. Dissipative systems in the physical sciences display complicated spatio-temporal patterns including "chaos".

Simple theoretical approaches have been devised to study "self-organization", "complexity", and "chaos" in dissipative dynamical systems and to develop methods to analyze them. One such approach that will be studied extensively in this thesis involves Cellular Automata (CA), which use discrete variables, introduced by von Neumann and Ulam.\(^1,^2\) Another fruitful area of research has been the study of more traditional mathematical models with continuous variables such as finite numbers of coupled ordinary differential equations and iterative maps.\(^3^-^6\) This thesis also includes an investigation of coupled iterative maps on an infinite lattice.

What is a Cellular Automaton? It consists of a collection of dynamical variables on the sites of a d-dimensional lattice (discrete space), and each of which assumes a discrete set of values. The
system evolves in *discrete* time steps, synchronously according to local rules which determine the variable at a given site at time $t+1$ in terms of the values of the variables in its local neighborhood at time $t$. The rule can be either deterministic or probabilistic. (Detailed definitions are given in Chapter II.)

Cellular Automata, especially of the deterministic kind, have been studied widely in a variety of contexts. Recently, Wolfram has stimulated a great deal of interest by his extensive numerical simulations and classification of (one-dimensional) deterministic CA$^7$ and by highlighting problems in different aspects of their study$^8$. We also note that probabilistic CA have been studied with mathematical rigor under the rubric of locally interacting particle systems$^9,10$ (Refs. 9 and 10 use continuous time evolution); we should also emphasize the deep and beautiful results obtained by the Russian school$^{11,12}$.

We now give a brief survey of some of the applications of CA. Several recent reviews are available$^{13-15}$ and the reader is referred to them for more comprehensive accounts.

Originally, von Neumann used CA to model "self-reproduction"$^{16}$ and to imitate universal computers$^{17}$. Crudely speaking, a system is said to be capable of universal computation if it can simulate the behavior of any other computer; in particular, it can implement any finite algorithm by an appropriate choice of initial conditions. Several systems have been shown to be capable of universal computation including Turing machines$^{17}$ and recently, a two-dimensional, reversible CA based on a discrete approximation to a billiard ball
model. The original model of von Neumann has been considerably simplified and there exist one-dimensional CA equivalent to universal Turing machines. More generally, CA have served as paradigms in computer science.

Cellular Automata have been employed to model chemical and biological systems. To the extent that patterns and growth in these systems are governed by local interactions CA can capture their essential features. In biology, CA have been used in the fields of development, genome structure, and evolution. We mention just a few examples: Three-dimensional CA (or Cellular excitable media) with 3 states (quiescent, excitable, or tired) have been used instead of stiff partial differential equations (PDE's) to model modes of heart muscle. By treating the gene as a binary device and studying randomly-assembled, large but finite automata Kaufmann has modelled genomic regulatory systems. These CA exhibit a variety of ordered dynamical behavior providing a phenomenology that can be compared with the stability, number of cell types, and cellular differentiation of real systems. Automata have been used to model heart fibrillation and tumor growth! One-dimensional and two-dimensional CA have been used to study spatio-temporal oscillations, chaos and pattern formation in reaction-diffusion systems.

Recently Cellular Automata have been studied in the context of physics. Several suggestions have been made that various physically interesting partial differential equations can be approximated by the large-scale behavior of appropriately chosen CA. The
advantage lies in the possibility of constructing massively parallel computing machines to simulate such CA. A very exciting development is the proposal of a hexagonal lattice-gas automaton for Navier-Stokes equation. Pattern formation is another area of interest in many disciplines that can be studied using CA or related systems. Simple models with both discrete and continuous variables have been used to elucidate features of solidification and aggregation. Pictorially, forms similar to those obtained in Hele-Shaw cells and electro-deposition have been obtained. Third, the equivalence of probabilistic CA in d-dimensions to equilibrium statistical mechanical models (See Sec. 2.1) and to directed percolation models in (d+1)-dimensions has been established and exact results for certain three-dimensional Ising and Potts models obtained. A deterministic cellular automaton which simulates the Ising model has been devised and used to study non-equilibrium phenomena, heat flow, mixing etc. The list of possible applications is growing; however, it remains to be seen whether in fact CA or variants thereof will emerge as alternatives to traditional descriptions of nature involving PDE's or even as extremely efficient computational tools.

This thesis is concerned with a study of Cellular Automata and related dynamical systems which have simple constituents and simple rules of evolution but are capable of highly complex behavior. The underlying theme is the application of the well-developed techniques of statistical mechanics, including the Renormalization Group to elucidate various features of these systems. In Chapter II fully probabilistic CA are studied. The occurrence of stationary and
non-stationary states, the nature of certain phase transitions and conditions on rules to possess underlying Hamiltonians are discussed. In Chapter III we\textsuperscript{35} argue that macroscopically-chaotic states do not occur in many-body systems (including PCA) with local interactions and for random initial conditions. The phase diagram for noisy coupled maps on d-dimensional lattices as a function of the control parameter and noise is presented. Chapter IV deals with\textsuperscript{36} deterministic CA with additive rules. Several results for higher-order and d-dimensional additive CA are derived using the properties of circulant matrices on finite fields. Continuous versions of a special class of CA rules\textsuperscript{37} which exhibit the unusual property of non-ergodicity over a finite range of parameter space are investigated in Chapter V using phenomenological time-dependent Ginzburg-Landau equations. They are also shown to exhibit similar behavior. Certain technical details are relegated to the Appendices A-E.
CHAPTER II
STATISTICAL MECHANICS OF PROBABILISTIC
CELLULAR AUTOMATA

2.0 Introduction

Cellular Automata (CA), as we have seen, are simply regular arrays
of variables each of which can assume two or more discrete values and
evolves temporally in discrete steps according to a set of rules. The
extraordinary variety of rules one can invent is reflected in the
complex and diverse behavior observed (typically by numerical means)
in CA. Probabilistic Cellular Automata (PCA), defined in the next
section, bear an obvious intuitive resemblance to dynamical Ising
models \(^{38-40}\) which constitute one special subclass of them. Owing to
their greater diversity progress in categorizing the general types of
behavior possible in CA \(^{41-46}\) lags somewhat behind that for dynamical
Ising systems. In this chapter, we describe work done on a large class
of PCA, a class in which each possible state of the system at an
instant can give rise with non-zero probability to any possible state
at the next step. This work constitutes a first step toward a system-
atic analysis of the wider range of possibilities inherent in PCA.

Given a CA rule one is interested in determining its asymptotic
\((t \to \infty)\) temporal behavior. One would like to understand the dependence
of the nature of the asymptotic state on the rules, and to identify
universality classes of the transitions that occur as the probabilities are varied. Our main results are summarized below:\(^3\text{4}\):

1. We establish the necessary and sufficient condition under which a CA rule is "microscopically reversible", by which one means that the transition probabilities obey detailed balance for some underlying Hamiltonian; hence as \(t \to \infty\) the system is often described by the stationary (equilibrium) Boltzmann distribution governed by that Hamiltonian. The condition depends on whether the rule is applied by updating the spins simultaneously or sequentially. There are rules which correspond to Hamiltonians under simultaneous updating but not under sequential updating and vice versa.

2. Continuous transitions into stationary ferromagnetic states of fully probabilistic CA which do not have associated Hamiltonians (i.e., are "irreversible") but which do have the "up-down" symmetry of Ising models, are argued to fall, both for statics and dynamics, in the same universality classes as do CA with underlying Hamiltonians (i.e., kinetic Ising models). (For dynamics, the CA belong to the universality classes of Ising models with either one\(^3\text{9}\) (viz., the order parameter) or no\(^3\text{8}\) conserved variables, according as the rules conserve or do not conserve that variable.) Thus, at ferromagnetic critical points, fully PCA coarse grained to sufficiently large length scales possess underlying Hamiltonians, even if they do not on microscopic scales.

3. We show that there exists, as in equilibrium statistical mechanics\(^4\text{7}\), a systematic expansion for probabilistic CA in inverse powers of \(d\), the dimension. In the \(d=\infty\) limit one obtains mean field
theory (MFT): the evolution of the CA is described by an iterative map \(^3\) with one variable (the "magnetization") which, for appropriate rules exhibits time-dependent asymptotic behavior, including limit cycles, chaos, etc. \(^4\) Using the insight obtained from the analytic results available at \(d=\infty\), we construct rules in \(d=2\) which, produce, under numerical simulation, non-stationary states, viz., two, three, and four cycles, as \(t\to\infty\). We have also found it difficult to obtain compelling evidence for chaotic states. This suggests the surprising fact that fluctuations can stabilize stationary states.

2.1 Notation and Formalism

We consider CA on \(d\)-dimensional hypercubical lattices with \(N\) sites. At each site \(i\), there is a dynamical variable denoted by \(S_i\) which can assume \(q\) values. We will refer to \(S_i\) as a spin. The state of the system is specified by the values of the \(N\) spins \(\{S_i\}\). The system evolves in discrete time according to a set of rules which, for probabilistic cellular automata, are defined by the transition probabilities for the system to go from one state to another. The transition probability \(T(\{S_i'\}|\{S_i\})\) to go from state \(\{S_i\}\) at time \(t\) to state \(\{S_i'\}\) at time \(t+1\) can be written as a product of local transition probabilities:

\[
T(\{S_i'\}|\{S_i\}) = \prod_{i} Q(S_i'|S_i)_{v_i} .
\]  

(2.1.1)

We denote by \(Q(S_i'|S_j)_{v_i}\) the probability that the \(i^{th}\) spin assumes the value \(S_i'\) at time \(t+1\) given that the spins in its neighborhood \(v_i\) have the values \(\{S_j\}_{v_i}\) at time \(t\). A PCA rule is defined by specifying the neighborhood \(v_i\) and the transition probabilities \(Q\).
If $Q$ is independent of $S_i$, i.e., the future of the spin $S_i$ does not depend on its present value but only on those of its neighbors, the rule is called a peripheral rule. Formally, for a peripheral rule, $Q(S_i^+|\{S_j\}_{\nu_i}) = Q(S_i^+|\{S_j\}_{\nu_i})$, where $\nu_i$ is the neighborhood of $i$ excluding itself. If $Q(S_i^+|\{S_j\}_{\nu_i}) = Q(S_i^+|\sum_{j \in \nu_i} S_j)$ the rule is said to be totalistic, i.e., the future of $S_i$ depends on the sum of the present values of its neighborhood spins. If all the $Q$'s are either 0 or 1 the rule is said to be deterministic.

In general, the neighborhood of $i$ $\nu_i$ can contain the site $i$ itself and $z$ other sites. There are clearly $q^{z+2}$ transition probabilities. However, in general, only $(q-1)q^{z+1}$ $Q$'s are independent since $Q$'s satisfy the normalization condition

$$\sum_{S_i^+} Q(S_i^+|\{S_j\}_{\nu_i}) = 1$$  \hspace{1cm} (2.1.2)

for each of the $q^{z+1}$ configurations of the neighborhood.

As an illustrative example, consider a one-dimensional PCA, where $S_i$ is an Ising variable ($S_i = \pm 1$). Let the neighborhood $\nu_i$ be \{i-1, i, i+1\}. Since $q=2$, $z=2$, there are $2^3$ independent $Q$'s which we exhibit below:

$$Q(1|1, 1, 1) = \alpha_1 \hspace{1cm} Q(1|1, 1, -1) = \alpha_2$$

$$Q(1|1, -1, 1) = \alpha_3 \hspace{1cm} Q(1|1, -1, 1) = \alpha_4$$

$$Q(1|1, -1, -1) = \alpha_5 \hspace{1cm} Q(1|1, -1, -1) = \alpha_6$$

$$Q(1|-1, -1, -1) = \alpha_7 \hspace{1cm} Q(1|-1, -1, -1) = \alpha_8$$

$Q(1|1, 1, -1) = \alpha_2$ means that if spins at site $i-1$, $i$, and $i+1$ assume values 1, 1, and -1 respectively at time $t$, $S_i$ will take on value 1 with probability $\alpha_2$ at time $t+1$. Observe that the normalization
condition \( \text{Eq. (2.1.2)} \) implies that \( Q(-1|1,1,1) = 1-Q(1|1,1,1) = 1-v_1 \), etc. By imposing up-down symmetry in spin space, and left-right symmetry in real space the number of independent \( Q \)'s can be reduced. Also, if \( v_2 = v_3 = v_4 \), and \( v_5 = v_6 = v_7 \) the rule is totalistic.

The time evolution of the system is described by a discrete master equation \(^{48}\) for \( P([S_1], t) \), the probability that the system is in the state \( [S_1] \) at time \( t \):\(^{b} \)

\[
P([S_1], t+1) = \sum_{[S_k]} T([S_1]|[S_k])P([S_k], t) \tag{2.1.4}
\]

\[
= \sum_{[S_k]} \left[ \prod_{i} Q([S_1]|[S_j]_i) \right] P([S_k], t) \tag{2.1.5}
\]

Or in words, the probability for a particular state \( [S_1] \) to obtain at time \( t+1 \) is given by the probability for a state \( [S_k] \) to occur at time \( t \), multiplied by the transition probability to go from \( [S_k] \) to \( [S_1] \) summed over all possible states \( [S_k] \) at time \( t \). Equation (2.1.5) clearly describes simultaneous (synchronous) updating of the spins; any rule \( Q \) can, alternatively, be applied spin by spin, i.e., sequentially. CA, as conventionally defined, are always updated simultaneously. We broaden the definition here to include sequential updating. In what follows we are going to restrict ourselves to fully PCA, i.e., all the microscopic transition probabilities are strictly positive \( (0 < Q < 1) \); i.e., every state can be reached from every other state. We also concentrate on simultaneously-updated 2-state (i.e., \( S_1 = \mp 1 \) is an Ising variable) PCA unless otherwise specified.

The behavior of the system can be described in terms of correlation functions defined as ensemble averages of products of spin operators. The \( n \)-point correlation function is defined by
\[ \langle a_n(l_1, \ldots, l_n) \rangle_t \equiv \langle \prod_{j=1}^{n} S_{l_j} \rangle_t \equiv \prod_{j=1}^{n} S_{l_j} P([S_i], t) \]. A complete understanding of the system involves the calculation of all such correlation functions, a hopeless task.

We now describe a powerful approximate method, mean-field theory (MFT) and corrections to it. Without loss of generality, we can write \( Q = a(1 + S_1^1 F) \). From the normalization condition for \( Q \), we obtain \( a = \frac{1}{2} \). So the general form for \( Q \) is

\[ Q(S_1^1 | [S_j]_{\nu_1}) = \frac{1}{2} [1 + S_1^1 F([S_j]_{\nu_1})] \] \hspace{1cm} (2.1.6)

Substituting the form of \( Q \) given above in the master equation (2.1.5) we obtain

\[ \langle a_n(l_1, \ldots, l_n) \rangle_{t+1} = \langle \prod_{j=1}^{n} F([S_j]_{\nu_{1j}}) \rangle_t \] \hspace{1cm} (2.1.7)

Since in general \( F \) is a polynomial of degree \( z+1 \), the right-hand side (RHS) of Eq. (2.1.7) contains correlation functions of much higher order than \( n \). Thus one is led to an intractable, infinite hierarchy of coupled equations involving correlation functions of progressively higher order. This is reminiscent of the BBKGY hierarchy in classical statistical mechanics\(^{49}\). For large \( z \), however, this hierarchy can be systematically decoupled in an expansion in powers of \( 1/z \). This notion is familiar from equilibrium statistical mechanics, where "infinite-ranged" (\( z = \infty \)) models are well known to yield the mean-field approximation and has provided considerable insight into the behavior of finite \( z \)-systems. It is therefore reasonable to explore this limit in CA, which need not even approach a time-independent distribution as \( t \rightarrow \infty \). The simplicity of large \( z \) can be crudely understood as follows: The values \( S_1^1 \) and \( S_j^1 \) of the spins at sites \( i \) and \( j \) at time

\[ \text{time} \]
t+1 are determined by their neighbors at time t. The influence an individual spin exerts in determining $S_i^t$ ($S_j^t$) is of $O(1/z)$. If $i$ and $j$ share $k$ neighbors, they induce a correlation between $S_i^t$ and $S_j^t$ of $O(k/z^2)$; when $z$ is large the two-point correlation is of $O(1/z)$.

In the infinite-range limit ($z \to \infty$), therefore, all correlations vanish and the only quantity required to characterize the time evolution of the system is the uniform magnetization $M(t) \equiv \langle S_i^t \rangle_t$. In this limit $M(t)$ obeys the equation:

$$M(t+1) = F(M(t))$$  \hspace{1cm} (2.1.8)

Thus MFT reduces to a one-variable iterative map $^3$.

Before proceeding to the next section, we will discuss an interesting connection: The time-evolution of any simultaneously-updated d-dimensional PCA is equivalent to the equilibrium statistical mechanics of a (d+1)-dimensional model with a corresponding spin Hamiltonian $^3 H_{d+1}$. The temporal direction in the PCA corresponds to the $(d+1)^{st}$ dimension in the statistical mechanical model. Note that one needs a fully PCA to obtain a $H_{d+1}$ with all couplings finite.

This correspondence can be understood as follows: Let $V_t = \{S_i^t\}$ denote the state of the PCA at time $t$; this corresponds to a d-dimensional hyperplane of a configuration of the (d+1)-dimensional model. Thus each configuration of the (d+1)-dimensional model can be labelled by $\{V_m\}$ which is precisely a specific space-time evolution of the PCA. The probability for a configuration $\{V_t\}$ to occur is

$$P(\{V_m\}) = \ldots P(V_t|V_{t-1}) P(V_{t-1}|V_{t-2}) \ldots P(V_1|V_0)$$  \hspace{1cm} (2.1.9)

where $P(V_t|V_{t-1})$ is the conditional probability to find the CA in state $V_t = \{S_i^t\}$ given it was in state $V_{t-1} = \{S_i^{t-1}\}$ at time $t-1$. Using
the master equation [Eq. (2.1.4)], we observe that

\[ P(V_t|V_{t-1}) = T([S_j^t]|[S_j^{t-1}]) \]

(2.1.10)

Hence, in terms of the local probabilities

\[ P((V_m)) = \prod_t \prod_{i,j} Q(S_i^{t+1}|[S_j^t]_{V_i}) \]

(2.1.11)

Clearly \( P((V_m)) \) can be written in the form \( e^{-H_{d+1}((V_m))} \) with

\[ H_{d+1} = -\sum_t \sum_i \ln Q(S_i^{t+1}|[S_j^t]_{V_i}) \]

(2.1.12)

Thus given a PCA rule, the form of \( H_{d+1} \) can be identified. It will involve all possible couplings allowed by symmetry within the set \([S_i^{t+1},[S_j^t]_{V_i}]\), and usually contains multispin couplings. The coupling constants are functions of the parameters in \( Q \) and hence, are not all independent. An important feature of the \((d+1)\)-dimensional model is that its partition function is unity. This is obvious because the conditional probabilities in Eq. (2.1.9) are normalized.

We illustrate the identification of \( H_{d+1} \) with a simple example. Consider the up-down symmetric, totalistic version of the nearest-neighbor 1d PCA defined in Eq. (2.1.3),

\[ Q(1|3) = \alpha_1 \quad Q(1|-3) = 1 - \alpha_1 \]

\[ Q(1|1) = \alpha_2 \quad Q(1|-1) = 1 - \alpha_2 \]

(2.1.13)

This corresponds to a two-dimensional statistical mechanical model with spins labelled by \( S_{i,t} \) and the Hamiltonian is easily obtained:

\[ -H_{d+1} = \sum_t \sum_{i} \ln Q(S_{i,t+1}|S_{i-1,t},S_{i,t},S_{i+1,t}) \]

\[ = \sum_t \sum_{i} [A + B(S_{i-1,t}S_{i,t} + S_{i,t}S_{i+1,t} + S_{i-1,t}S_{i+1,t}) \]

\[ + C S_{i,t+1} (S_{i-1,t} + S_{i,t} + S_{i+1,t}) + D S_{i,t+1} S_{i-1,t} S_{i,t} S_{i+1,t}] \]

(2.1.14)
Substituting the form of $Q$ we find

$$A = \frac{1}{6} \ln[\alpha_1(1-\alpha_1) \alpha_2^2 (1-\alpha_2)^3]$$

$$B = \frac{1}{8} \ln \frac{\alpha_1(1-\alpha_1)}{\alpha_2(1-\alpha_2)}$$

$$C = \frac{1}{8} \ln \frac{\alpha_1 \alpha_2}{(1-\alpha_1)(1-\alpha_2)}$$

$$D = \frac{1}{8} \ln \left[ \frac{\alpha_1}{1-\alpha_1} \left( \frac{1-\alpha_2}{\alpha_2} \right)^3 \right].$$

(2.1.15)

Note that $A$ is an additive constant and the other three parameters are not independent but satisfy the relation

$$e^{-4B} = \cosh(3C+D)/\cosh(C-D).$$

(2.1.16)

Such correspondences have been employed to deduce various results for directed percolation\(^{32}\) and multicritical phenomena\(^{50}\) but these are not our present concern.

### 2.2 Reversible PCA Rules

One class of PCA whose behavior is well understood consists of kinetic Ising models, i.e., CA rules which satisfy the detailed balance conditions for some associated Hamiltonian. In general, the detailed balance condition requires that there exists a Hamiltonian $H$ such that for any two states $\{S_1\}$ and $\{S'_1\}$

$$\frac{T(\{S'_1\}|\{S_1\})}{T(\{S_1\}|\{S'_1\})} = e^{-[H(\{S'_1\}) - H(\{S_1\})]}.$$

(2.2.1)

A PCA which satisfies Eq. (2.2.1) is said to be reversible. (Note that a factor of $1/k_B T$ has been absorbed in $H$.) If a system is reversible, it has a stationary solution to the master equation
[Eq. (2.1.4)] given by the Boltzmann (Gibbs) distribution
\[ P_0([S_j]) = e^{-H([S_j])}. \]
Of course, the existence of this stationary solution does not always guarantee that the system reaches the equilibrium state described by \( P_0 \) for all (or even almost all) initial conditions. This will be discussed at the end of the section. In most cases, however, the system can be expected to go to equilibrium with the distribution \( P_0 \). In this case all averaged quantities will be independent of time. Since this is an especially simple possibility in determining the asymptotic behavior of PCA it is useful to know whether a given rule satisfies the detailed balance condition or not. In the following we demonstrate that this can indeed be done and give the general form of the transition probability \( Q(S_1'|[S_j]_{v_1}) \) compatible with reversibility. It has been brought to our attention that Vasilyev has derived some of the results in this section using more mathematical techniques. We remind the reader that the asymptotic state can be (and, in fact, in many cases is) stationary for rules which do not satisfy detailed balance.

A. Simultaneously-updated 2-state PCA

For simultaneously-updated PCA the detailed balance condition can be written as follows:

\[
\prod_1^{\Pi} \frac{Q(S_1'|[S_j]_{v_1})}{Q(S_1|[S_j']_{v_1})} = e^{-[H([S_1'])-H([S_1])]} \equiv e^{-\Delta H} \quad (2.2.2)
\]

or equivalently,

\[
\sum_1 \left[ \ln Q(S_1'|[S_j]_{v_1}) - \ln Q(S_1|[S_j']_{v_1}) \right] = -\Delta H \quad (2.2.3)
\]
We now determine what form \( Q \) has to take in order to satisfy Eqs. (2.2.2) or (2.2.3). We write, without any loss of generality,

\[
\ln Q(S_i'|[S_j]_{v_1}) = A([S_j]_{v_1}) + S_i'B([S_j]_{v_1}) . \tag{2.2.4}
\]

From the normalization condition on \( Q \), it is easy to see that,

\[
A([S_j]_{v_1}) = -\ln 2 \cosh B([S_j]_{v_1}) \tag{2.2.5}
\]

So we only need determine \( B \). In general, \( B \) can be written as

\[
B \equiv S_1C_1([S_j]_{\mu_1}) + C_2([S_j]_{\mu_1}) . \tag{2.2.6}
\]

(We remind the reader that \( \mu_1 \) refers to the neighborhood of \( i \) excluding itself.) This yields for \( \ln Q \),

\[
\ln Q(S_i'|[S_j]_{v_1}) = A([S_j]_{v_1}) + S_i'S_1C_1([S_j]_{\mu_1}) + S_i'C_2([S_j]_{\mu_1}) . \tag{2.2.7}
\]

We will show that in order for Eq. (2.2.2) to be satisfied for arbitrary pairs of states \( [S_i] \) and \( [S_i'] \), \( C_1 \) has to be a constant and \( C_2 \) contains terms at most linear in \( [S_j] \). First we notice from Eq. (2.2.3) that in general \( H \) can contain any interaction (singlet, pair, triplet, etc.) whose range is not longer than the farthest distance in the neighborhood of the rule. By considering a term in the Hamiltonian of the form \( \lambda S_1S_2 \) (for some constant \( \lambda \)) where sites 1 and 2 belong to the neighborhood of site 0, we will establish that \( C_1 \) cannot contain terms quadratic in \( [S_j]_{\mu_1} \). Let the system evolve from a state \( [S_i] \) to a state \( [S_i'] \) chosen such that \( S_i'S_2 = -S_1S_2 \), i.e., \( S_1S_2 \) changes sign. This implies that there is a contribution to the RHS of Eq. (2.2.3) of the form \( 2\lambda S_1S_2 \). Let us investigate contributions to the left-hand side (LHS) of the same form. If \( C_1([S_j]_{\mu_0}) \) contains a quadratic term, since \( S_1, S_2 \in [S_j]_{\mu_0} \) there will be a term proportional to \( S_1S_0S_1S_2 - S_0S_0S_1S_2 \).
(Note that there are other terms independent of \( S_0 \) which contribute to \( \lambda S_1 S_2 \) from \( A \) and from \( C_2 \).) The term from the quadratic part of \( C_1 \) has a different sign \((\pm S_1 S_2)\) depending on whether \( S_0' \pm S_0 \). Since Eq. (2.2.3) must be true for all pairs of states for a given \( \lambda \), the only consistent solution is that the quadratic terms in \( C_1 \) must be zero. Similar considerations can be used to demonstrate that only a constant term independent \([S_j]_{\mu_1} \) survives.

Now we consider the term \( C_2([S_j]_{\mu_1}) \) in Eq. (2.2.7) and show that it can only contain constant and linear terms. Let us suppose that there is a quadratic term in \( C_2 \) which yields terms of the form \( S_1^j S_j S_k \) in \( \ln Q \) when \( j, k \in \mu_1 \). Consider a specific term in \( H \), say, \( \lambda S_0 S_1 S_2 \) which changes sign when the system evolves from \([S_j]\) at time \( t \) to \([S_j']\) at \( t+1 \), yielding a contribution to the RHS of Eq. (2.2.3) \( 2\lambda S_0 S_1 S_2 \).

(For illustrative purposes we will assume that both the sites 1 and 2 belong to \( \mu_0 \) the neighborhood of 0; however, \( \mu_1 \) does not contain both 0 and 2, and \( \mu_2 \) does not contain both 0 and 1. The following argument is easily modified to incorporate other possibilities.) On the LHS of Eq. (2.2.3) terms of the form \( S_0 S_1 S_2 \) can only arise from \( A \) and \( C_2 \) if the latter contains quadratic terms: \( A \) gives rise to terms of the form \( S_0 S_1 S_2 - S_0' S_1 S_2' \) while \( C_2 \) yields \( S_0 S_1 S_2 - S_0' S_1 S_2' \). Note that the contribution of \( A \) can come from \( i=0, 1, 2 \) and \( 0, 1, 2 \in \mu_1 \) and \( i=0, \) and \( 1, 2 \in \mu_0 \).

Now Eq. (2.2.3) must be satisfied for all pairs of configurations such that \( S_0 S_1 S_2 = -S_0' S_1 S_2' \). If this change in sign is because \( S_0 = -S_0' \) (and \( S_1 S_2 = S_1' S_2' \)) the contribution due to \( C_2 \) is proportional to \(-2S_0 S_1 S_2 \).

However, if \( S_1 = S_1' \) (and \( S_0 = -S_0' S_2 = S_2' \)) the contribution is \(+2S_0 S_1 S_2 \). The terms coming from \( A \) remain unaltered in both cases. Thus we
conclude that $C_2$ cannot contain quadratic terms if detailed balance is to be obeyed. A similar argument (involving higher-order couplings in $H$) can be carried out to show that $C_2$ does not contain terms with three or more powers of \{$S_j$\).

We now study the form of the linear term in $C_2(\{S_j\}_{\mu_1})$, i.e.,

$$\sum_{\alpha} c_\alpha S_{1+\alpha}$$

where the sum is over $d$-dimensional Bravais lattice vectors $\alpha$ such that $1+\alpha \in \mu_1$. Assume that $AH$ contains a term $2CS_1S_{1+\alpha}$. The LHS of Eq. (2.2.3) has the terms (in addition to contributions from $A$)

$$(S_1'S_{1+\alpha} - S_1S_{1+\alpha}) (c_\alpha - c_{-\alpha}).$$

Clearly, depending on whether $S_1$ or $S_{1+\alpha}$ changes sign we obtain different contributions. Thus the only way for Eq. (2.2.3) to be satisfied is to demand that $c_\alpha = c_{-\alpha}$. This means that the neighborhoods are symmetric, i.e., if $i$ is a neighbor of $j$, $j$ is a neighbor of $i$ and the couplings in $Q$ have to be symmetric, i.e., $S_i'$ depends on $S_{1+\alpha}$ in exactly the same way as $S_1'$ depends on $S_1$.

So we finally obtain,

$$B(\{S_j\}_{\mu_1}) = a + b S_1 + \sum_{\alpha} c_\alpha S_{1+\alpha} \quad (2.2.8)$$

where $a$, $b$, $c_\alpha$ are constants with $c_\alpha = c_{-\alpha}$. Therefore, a rule satisfies the detailed balance condition if and only if it has the following form:

$$Q(S_1'|\{S_j\}_{\mu_1}) = \frac{BS_1'}{e^{1/2\cosh B}}$$

$$= \frac{1}{2} \left[ 1 + S_1' \tanh(a+b S_1 + \sum_{i+\alpha \in \mu_1} c_\alpha S_{1+\alpha}) \right] \quad (2.2.9)$$

for arbitrary constants $a$, $b$ and \{$c_\alpha$\} subject only to the condition $c_\alpha = c_{-\alpha}$. This is a striking result, among the multitude of rules that can be concocted only those rules which have form specified by
Eq. (2.2.9) can satisfy detailed balance with an underlying Hamiltonian. The corresponding Hamiltonian for a rule of the form in Eq. (2.2.9) is easy to obtain. Substituting Eq. (2.2.9) for \( Q \) into Eq. (2.2.2)

\[
\cosh(B([S_j]_{\nu_1})) e^{aS_1} \frac{S_1^c S_{1+a}}{|\prod_{i} \cosh(B([S_j]_{\nu_1}) + aS_i)|} \frac{S_1^c S_{1+a}}{e^{i\alpha}} = e^{-\Delta H} . \tag{2.2.10}
\]

It is easy to see that the quantity in the second bracket on the LHS is identically equal to one since \( c = -\alpha \). So the Hamiltonian is given by \( \mathcal{H} / \)

\[
\mathcal{H}(S_1) = -\sum_{i} [\ln \cosh(a+bS_1 + \sum_{c \alpha} S_{1+a}) + aS_i] \tag{2.2.11}
\]

with \( i + a \in \nu_1 \).

8. Simultaneously-updated q-state PCA

Following the lines of the proof outlined in the previous section, we can determine necessary and sufficient conditions for q-state (\( q > 2 \)) PCA to satisfy detailed balance and to possess an underlying translationally invariant q-state Potts Hamiltonian of the form \( \mathcal{H}' \):

\[
\mathcal{H}_p = \sum_{i} J_i \delta(S_1,0) + \sum_{(ij)} J_{ij} \delta(S_i, S_j) + \sum_{(ijk)} J_{ijk} \delta(S_i, S_j, S_k) + \ldots \tag{2.2.12}
\]

where \( \delta(S_1, S_j, S_k, \ldots) = 1 \) if \( S_1 = S_j = S_k = \ldots \), and vanishes otherwise. \( J \) is the field acting along the direction of the state 0 and \( (ij) \), \( (ijk) \) \ldots represent all possible pairs, triplets \ldots of sites on the lattice. We will label the q-states of a spin \( S_i \) by \( 0, 1, \ldots, q-1 \).

Once more we write

\[
\ln Q(S_1^i | [S_j]_{\nu_1}) = A([S_j]_{\nu_1}) + B(S_1^i, [S_j]_{\nu_1}) \tag{2.2.13}
\]
where \( A \) contains all the terms independent of \( S_1 \). From the normalization condition on \( Q \), we deduce that
\[
A(\{S_j\}_{v_1}) = -\ln(\sum_{S_1} B(\{S_i\}, \{S_j\}_{v_1})) .
\]  

(2.2.14)

The form of the Hamiltonian \( H_P \) implies that \( B \) has the following general form:
\[
B = a \delta(S_1', 0) + \sum_j a_{ij} \delta(S_1', S_j) + \sum_{j,k} a_{ijk} \delta(S_1', S_j, S_k) + \ldots
\]
\[
(2.2.15)
\]

We can show that all the terms except the first two have to be zero and
\[
B(S_1', \{S_j\}_{v_1}) = a_0 \delta(S_1', 0) + a_1 \delta(S_1', S_1)
\]
\[
+ \sum_{\alpha} a_{\alpha} \delta(S_1', S_1 + \alpha)
\]  

(2.2.16)

for arbitrary constants \( a_0, a_1, a_{\alpha} \) where \( \alpha \) is a vector on the lattice such that \( 1+i+\alpha \in v_1 \), and \( a_{-\alpha} = a_{\alpha} \) (symmetric coupling).

So, for a \( q \)-state PCA to be reversible and have the corresponding Potts Hamiltonian [Eq. (2.2.12)], the transition probability \( Q \) has to have the following form:
\[
Q(S_1' | \{S_j\}_{v_1}) = \frac{e^{a_0 \delta(S_1', 0) + a_1 \delta(S_1', S_1) + \sum_{\alpha} a_{\alpha} \delta(S_1', S_1 + \alpha)}}{\sum_{S_1} e^{a_0 \delta(S_1', 0) + a_1 \delta(S_1', S_1) + \sum_{\alpha} a_{\alpha} \delta(S_1', S_1 + \alpha)}} .
\]  

(2.2.17)

The associated Potts Hamiltonian is given by
\[
H_P(\{S_i\}) = -\sum_i [\ln(\sum_{S_i} e^{a_0 \delta(S_1', 0) + a_1 \delta(S_1', S_1) + \sum_{\alpha} a_{\alpha} \delta(S_1', S_1 + \alpha)} + a_0 \delta(S_1', 0))]
\]
\[
+ a_0 \delta(S_1', 0))
\]  

(2.2.17)
C. **Sequentially-updated 2-state PCA**

We will now establish conditions under which a rule specified by a given $Q$ obeys detailed balance when applied sequentially. By sequential updating we refer to updating single spins chosen at random. Since only one spin can change at a time, Eq. (2.2.2) is simple and becomes,

$$
\frac{Q(S'_1 | [S_j]_{\nu_1}, S_1)}{Q(S_1 | [S_j]_{\nu_1}, S'_1)} = e^{H([S_j]) - H([S'_j])}
$$

(2.2.18)

Using the general form of $Q$ displayed in Eq. (2.1.6), we separate the $S_1$ dependence of $F$ and write $F = f_1([S_j]_{\mu_1}) + S_1 f_2([S_j]_{\mu_1})$. Therefore, Eq. (2.2.18) can be written as,

$$
\frac{1 + S'_1 (S'_1 f_2 + f_1)}{1 + S_1 (S_1 f_2 + f_1)} = e^{-\Delta H}
$$

(2.2.19)

Clearly, if $S'_1 = S_1$, $\Delta H = 0$ and Eq. (2.2.19) is satisfied. If, on the other hand, $S'_1 = -S_1$, we obtain the condition

$$
f_1 = \tilde{f}_2 S_1 \tanh(\frac{1}{2} \Delta H)
$$

(2.2.20)

where we have defined $\tilde{f}_2 = 1 - f_2$. Thus, $f_1$ and $\tilde{f}_2$ cannot be independent functions and $Q$ must satisfy:

$$
Q(S'_1 | [S_j]_{\nu_1}) = \frac{1}{2} [1 + S'_1 \tilde{f}_2 (1 - \tanh(\frac{1}{2} \Delta H))] 
$$

(2.2.21)

where $\tilde{f}_2$ is an arbitrary function of $[S_j]_{\mu_1}$, subject to the restriction that $0 < \tilde{Q} < 1$. In contrast to simultaneously-updated PCA (see Eq. (2.2.9)) the criterion for reversibility depends significantly on the neighborhood $\nu_1$. For example, for $d$-dimensional near-neighbor rules (i.e.,
\( \mu_1 \) contains only near-neighbor sites), the associated Hamiltonian can have at most near-neighbor interactions, i.e.,

\[
H = a_0 \sum_{j=1}^N S_j + \frac{1}{2} \sum_{j=1}^N a_\alpha S_j S_{j+\alpha} \tag{2.2.22}
\]

where the sum on \( i \) runs over all lattice sites and \( \alpha \) runs over the 2d unit vectors such that \( i+\alpha \in \mu_1 \). Obviously, \( a_\alpha = a_{-\alpha} \) by translational invariance. In this case, \( Q \) takes the form

\[
Q(S_j^i | S_j^1_{\mu_1}) = \frac{1}{2} \left[ 1 + S_j^i S_j^1 \{ 1 - \tilde{f}_2 (1 + \tanh(a_0 S_j^1 + \sum_\alpha a_\alpha S_{j+\alpha})) \} \right] . \tag{2.2.23}
\]

Again, \( a_0, a_\alpha \) are arbitrary constants with the restriction \( a_\alpha = a_{-\alpha} \).

Note that if \( a_0 = 0 \), Eq. (2.2.22) describes the up-down symmetric, Ising Hamiltonian. The choice \( \tilde{f}_2 = 1 \) (or equivalently \( f_2 = 1 - \tilde{f}_2 = 0 \) which implies that \( Q = 1 + S_j^i f_j^1 (\{ S_j^1_{\mu_1} \}) \) is peripheral) leads to

\[ Q = \frac{1}{2} (1 - S_j^1 \tanh(\sum_\alpha a_\alpha S_{j+\alpha})), \]

the well-known Glauber\(^{38}\) form. Rules with further neighbor couplings in \( Q \) correspond to multi-spin interaction terms in \( H \) and therefore, more terms are allowed in the argument of the hyperbolic tangent that appears in \( Q \) (see Eq. (2.2.21)). Thus, as the range of the rule increases the restrictions on \( Q \) required to ensure the existence of an associated Hamiltonian become progressively less stringent. In infinite-ranged PCA, \( H \) can have infinite-ranged, arbitrary multi-spin interactions and there is no restriction on the argument of the \( \tanh \) in Eq. (2.2.21) (except that the couplings have to be symmetric).

In the case of infinite-ranged, totalistic rules, the Hamiltonian can be obtained as follows. From Eq. (2.2.20) we have,

\[
\frac{\Delta H}{2S_1^j} = -\frac{\Delta H}{\Delta S_1} = \tanh^{-1} \frac{f_1}{f_2} \tag{2.2.24}
\]
Since for infinite-ranged, totalistic rules, $\mathcal{H}$ depends on the sum of all the spins $\mathcal{H}([S_i]) = \mathcal{H}(M)$ where $M = \frac{1}{N} \sum_{i} S_i$. We obviously have $\frac{\Delta \mathcal{H}}{\Delta M} = N \frac{\Delta \mathcal{H}}{\Delta S_i}$. Direct integration leads to the following form for the Hamiltonian:

$$\mathcal{H}(M) = -N \int_{0}^{M} \frac{f_1(m')}{f_2(m')} \, \text{tanh}^{-1} \left[ \frac{f_1(m')}{f_2(m')} \right] \, \text{dm}' . \tag{2.2.25}$$

There is an amusing connection worth noticing. For sequentially-applied infinite-ranged PCA with the Hamiltonian (2.2.25) we can write down a Gibbs free energy for the equilibrium system with a fixed magnetization. It can be shown (see Appendix A) that the extrema of this free energy are in one-to-one correspondence with the fixed points of the map (Eq. (2.1.8)) which describes the evolution of the simultaneously-updated CA with the same transition probability $Q$. Moreover, every stable fixed point of Eq. (2.1.8) corresponds to a minimum of the free energy, although not all minima correspond to stable fixed points.

The above results [Eqs. (2.2.9) and (2.2.21)] clearly show that whether a given CA rule satisfies detailed balance condition or not depends on whether it is applied simultaneously or sequentially. The same local specification for a rule, i.e., the same neighborhood and the same form of $Q$ can correspond to a Hamiltonian under simultaneous updating but not under sequential updating and vice versa. For example, the near-neighbor rule with $Q$ of the form given in Eq. (2.2.9) and $b=0$ does not satisfy the detailed balance condition when applied sequentially although it does when applied simultaneously. Only when $b=0$ is the rule reversible under both types of updating, though the
respective underlying Hamiltonians differ.

Before closing this section, we would like to emphasize again that detailed balance does not always guarantee that the system reaches equilibrium state with the corresponding \( P_0([S_j]) \propto e^{-H([S_j])} \). For example, in a ferromagnet below the phase transition temperature the system is nonergodic. Depending on the initial condition, it can go to a state with either positive or negative magnetization. The distribution is no longer described by \( P_0 \) given above. The same situation can manifest itself in PCA. For example\(^\text{54} \), consider a near-neighbor simultaneously-updated reversible PCA. In this case \( Q \) has the form shown in Eq. (2.2.9) with \( a=b=0 \), i.e., it corresponds to an up-down symmetric and peripheral rule. The system decouples into two sublattices \( A \) and \( B \), where the \( [S'_A] \) and \( [S'_B] \) depend only on \( [S_A] \) and \( [S_B] \) respectively. Now, imagine starting out the system with one sublattice (say \( A \)) mostly up and the other mostly down; if we choose \( c_a \)'s to be sufficiently large and positive, at the next time step both sublattices will flip most of their spins. The staggered magnetization \( M^x(t) = \langle S_A \rangle_t - \langle S_B \rangle_t \) will change sign at every time step and therefore, execute a two-cycle. This is not surprising if we consider the \((d+1)\)-dimensional statistical model to which the CA corresponds. There we have two three-dimensional interpenetrating and decoupled sublattices with spins of one sublattice mostly up and the other mostly down. The large \( c_a \) values correspond to a temperature below the critical value. The two-cycle state in the two-dimensional reversible PCA corresponds to phase coexistence in the 3d Ising system.
The above example shows that reversible PCA may not always go to a time-independent state. However, in general, the equilibrium state can be expected asymptotically.

2.3 Ferromagnetic Transitions in PCA

Consider a PCA with a given neighborhood \( v_1 \) and the most general possible form of \( Q \). As the parameters that occur in the definition of \( Q \) (e.g., \( a_1, a_2, \) etc. in Eq. (2.1.3)) are varied the asymptotic state of the PCA can change, from stationary to non-stationary, or from a stationary state without long-range spatial order to an ordered state etc. We refer to different asymptotic states as different phases. It is of interest to understand the nature of transitions between these phases and to classify them as has been done for equilibrium statistical mechanical models. In this section, we will consider the nature of continuous transitions which can occur between stationary states of either reversible or irreversible PCA which are up-down symmetric, i.e.,

\[
Q(S_i' | \{S_j\}_{v_1}) = Q(-S_i' | \{-S_j\}_{v_1}).
\]  

(2.3.1)

We add a technical comment. For irreversible PCA a stable phase corresponds to a stationary measure that cannot be expressed in the Boltzmann form in the thermodynamic limit and which is extremal in that it is not expressible as a linear combination of other stationary measures\(^{55}\). In particular, we investigate ferromagnetic transitions where \( \langle S_i \rangle \) goes from zero to a non-zero value continuously. Transitions from stationary to non-stationary states will be considered in the next chapter. We will first discuss mean field theory (MFT) and
compare our results with those from the theory of equilibrium phase transitions \(^5\text{6}\). Then we study the effects of fluctuations on the nature of these transitions.

A. Mean-field Theory

In Sec. 2.1 we deduced that for an \(\infty\)-ranged PCA, all correlations between different sites vanish and the time evolution is completely characterized by the magnetization \(M(t)\equiv \langle S_i \rangle_t\). \(M(t)\) obeys a one-variable recursion relation

\[
M(t+1) = F(M(t)) \ . \tag{2.3.2}
\]

If up-down symmetry is imposed \(F\) becomes an odd function of \(M\). In general, \(F(M)=AM+BM^3+O(M^5)\) with \(A\neq 0\). The asymptotic state of the PCA is determined by the asymptotic behavior of the iterative map given by Eq. (2.3.2). The map has an obvious fixed point at \(M^*=0\) corresponding to the "paramagnetic" state. Other fixed points with \(M^* \neq 0\) ("ferromagnetic" state) may occur depending on the form of \(F(M)\). The stability criterion for a fixed point is \(|F'(M^*)|<1\). Rules for which at some critical value of the parameters a stable paramagnetic fixed point loses its stability to a pair of ferromagnetic fixed points \((\pm M^*)\) (symmetric pitchfork bifurcation) lead to continuous ferromagnetic transitions. One can calculate the exponents \(B\) and \(\gamma\) for such transitions within the \(\infty\)-ranged model using the general form of \(F\). However, an alternate formulation at finite \(d\) is needed for computing \(\nu\) and \(\eta\). For pedagogical reasons, we will use a specific example to work out such a MFT in finite \(d\) and use it to evaluate the critical exponents which hold in general.
We study a 2-dimensional nearest-neighbor, peripheral, totalistic rule. Let \( X_1 = \sum_{j \in U_1} S_j \) be the sum of the values of the neighbors of 1. There are only 2 independent parameters in specifying \( Q(S_1'|X_1) \):

\[
Q(1|4) = \alpha_1 \quad \text{and} \quad Q(1|2) = \alpha_2.
\] (2.3.3)

Up-down symmetry requires \( Q(1|0) = Q(-1|0) = \frac{1}{2} \). The general form of \( Q \) is given by

\[
Q(S_1'|X_1) = \frac{1}{2} [1 + S_1'(aX_1 + bX_1^3)].
\] (2.3.4)

We use Eq. (2.3.3) and make the following identifications:

\[
a = \frac{1}{3} (32\alpha_2 - 10\alpha_1 - 11),
\] (2.3.5)

\[
b = \frac{8}{3} (\frac{1}{2} + \alpha_1 - 2\alpha_2).
\]

Thus we obtain

\[
M(t+1) = a(X_1)_t + b(X_1^3)_t.
\] (2.3.6)

In the mean-field approximation we neglect all correlations between spins i.e., \( \langle S_1 S_j \rangle = \langle S_1 \rangle \langle S_j \rangle \). Hence, Eq. (2.3.6) becomes

\[
M(t+1) = A M(t) + B M^3(t)
\] (2.3.7)

with \( A = \alpha_1 + 2\alpha_2 - 3/2 \) and \( B = \frac{1}{2} + \alpha_1 - 2\alpha_2 \). (2.3.8)

Equivalently, mean-field theory corresponds to factoring the probability distribution for the state of the system \( P(S_1,t) \) into a product of single site probabilities at all \( t \). Thus we can also arrive at Eq. (2.3.7) by considering the probability \( p \) for site 1 to be up at time \( t \) and deriving an expression for \( p' \) the probability for
\[ p' = \alpha_1 p^4 + 4p^3(1-p)\alpha_2 + 3p^2(1-p)^2 + 4p(1-p)^3(1-\alpha_2) + (1-p)^4(1-\alpha_1) . \] (2.3.9)

Substituting \( M(t+1)=2p'-1 \) and \( M(t)=2p-1 \) we get back Eq. (2.3.7).

For \( A < 1 \), the only fixed point of Eq. (2.3.7) is \( M^* = 0 \). For \( A > 1 \), \( M^* = 0 \) becomes unstable and a pair of ferromagnetic fixed points \( M^* = \pm \left( \frac{1-A}{B} \right)^{1/2} \)

occur arbitrarily close to 0. Thus \( A_c = \frac{(\alpha_1 + 2\alpha_2)}{2} = 5/2 \) corresponds to a critical point. For \( A < A_c \), the system is paramagnetic while for \( A > A_c \), it is ferromagnetic. When \( A > A_c^+ \), we have

\[ M^* \sim (A - A_c)^{1/2} , \] (2.3.10)

yielding \( B = 1/2 \). Since this result depends only on the form \( M(t+1) = AM(t) + BM^2(t) + \ldots \) it is valid in general. Along the time direction to linear order in \( M \), one has

\[ M(t) = A^t M(0) + O(M^3) , \]
i.e., when \( A \rightarrow A_c^- \), \( 1 \)  \hspace{1cm} (2.3.11)

\[ M(t) \sim M(0) e^{-t/\xi_1} \sim M(0) e^{t \ln(1-(A_c-A))} . \] (2.3.12)

Therefore we obtain

\[ \xi_1 = (A_c - A)^{-1} . \] (2.3.13)

Hence, along the time direction the correlation length \( \xi_1 \) diverges \( (\xi_1 \sim (A_c - A)^{-\nu_1}) \) with \( \nu_1 = 1 \).

To deduce the spatial correlation length exponent \( \nu_1 \) in the plane, we fix the spin at the origin to be up and calculate the expectation value of the spin at site \((i,j)\) for \( A < A_c \). Let \( M_{1,j} = \langle S(i,j) \rangle \) denote
the asymptotic stationary value. Fixing the spin at the origin breaks translational invariance and we have instead of Eq. (2.3.7)

\[ M_{1,j} = \frac{1}{4} A(M_{1}+1,j+M_{1},j+1+M_{1}-1,j+M_{1}-j-1) \left( 1 - \delta_{1,0} \delta_{j,0} + O(\delta^{3}) \right). \] (2.3.14)

The Fourier transform of Eq. (2.3.14) yields

\[ G(q) \sim \frac{1}{1-A(1-q^{2}/4)} \sim \frac{1}{(A_{c}-A)+cq^{2}}. \] (2.3.15)

(We refer to this as \( G(q) \) since \( M(1,j) \) is the expectation value of the spin at \((1,j)\) given that the spin at \((0,0)\) is fixed to be up.) This is simply the familiar Ornstein-Zernike form and we immediately obtain \( v_{\parallel}=1/2, \ n=0 \) and \( \gamma=1 \). The standard dynamical critical exponent \( z \) is defined by \( z=v_{\perp}/v_{\parallel} \) and therefore we have \( z=2 \).

So, the MF values of the critical exponents \( \beta, \gamma, v, n, z \) are all the conventional results of equilibrium critical phenomena. In equilibrium critical phenomena, the upper critical dimensionality \( d_{c} \) can be identified as that \( d \) at which the hyperscaling relation \( s=v/2(d-z+n) \) is satisfied by the MF exponents. Assuming the validity of this identification for CA we find, from the MF exponents, as in the static equilibrium case \( d_{c}=4 \). For \( d>4 \), then, all continuous transitions into ferromagnetic states in PCA are characterized by the standard MF exponents.

B. Fluctuations

To study the effect of fluctuations on continuous ferromagnetic transitions in PCA for \( d<d_{c}=4 \), consider the Langevin equation:

\[ \frac{\partial \Psi_{1}}{\partial t} = V_{1}[\Psi_{j}] + \zeta_{1}(t) \] (2.3.16)
where $\psi_i$, a classical field at site $i$, assumes any value between $-\infty$ and $+\infty$, $V_1$ is an analytic function of $\{\psi_i\}$, and $\xi$ is a Gaussian random noise variable of zero mean. Critical behavior of kinetic Ising models (i.e., PCA with underlying Hamiltonians) is described by time-dependent Ginzburg-Landau equations of the form (2.3.16) with $V_1(\{\psi_j\})=-\Gamma \partial H/\partial \psi_i$ and $(\xi_i(\tau) \xi_j(\tau'))=2\Gamma \delta_{ij} \delta(\tau-\tau')$. Here $H(\{\psi_j\})$ is the Ginzburg-Landau representation of the underlying Hamiltonian, $\Gamma$ is the dissipation constant, and this specific choice for $\langle \xi_i \xi_j \rangle$ ensures that in the $t \to \infty$ limit the system is described by the Boltzmann distribution $e^{-H(\{\psi_j\})}$, corresponding to $\frac{1}{H(\{\psi_j\})}$. Critical phenomena in kinetic models have been intensively studied through the application of the $\epsilon$-expansion to this special case of Eq. (2.3.16).$^40$

It is reasonable to assume that the critical behavior of CA which do not admit underlying Hamiltonians can likewise be described by Eq. (2.3.16), with $V_1$ not expressible as $-\Gamma \partial H/\partial \psi_i$ for any $H$ (i.e., $\frac{\partial V_1}{\partial \psi_i} \neq \frac{\partial V_1}{\partial \psi_j}$). The approximation of the discrete PCA dynamics by a Langevin equation is obviously better justified for sequentially-updated CA than for simultaneously-updated CA; while it is not clear that the continuous-time Langevin equation correctly describes simultaneously-applied irreversible rules, we hypothesize that irreversible PCA are like reversible ones (Ising models) in that universality classes for continuous ferromagnetic transitions are independent of how the rules are applied. This hypothesis certainly holds in MFT.

For short-ranged CA with up-down symmetry $V_1$ must then be an odd function of $\{\psi_j\}$. For example,$^40$

$$V_1 = A \sum_{j \in \mathcal{V}_1} \psi_j + B \sum_{j,k,l} \psi_j \psi_k \psi_l + O(\psi^5) \quad (2.3.17)$$
represents, for arbitrary coefficients $A$ and $B$, a totalistic rule.

Note that for $B \neq 0$, $\frac{\partial V}{\partial \psi_j} \neq \frac{\partial V}{\partial \psi_1}$ and hence, there is no underlying Hamiltonian. This eliminates the fluctuation-dissipation theorem and relaxes the constraint on the noise-noise correlation. We can have multiplicative noise in Eq. (2.3.16), or loosely speaking, $(\zeta_i(t) \zeta_j(t'))$ can be an arbitrary even (by up-down symmetry) function of $[\psi_j]$, e.g.,

$$
(\zeta_i(t) \zeta_j(t')) = 2 \delta_1 \delta(t-t') \Gamma_0 [1 + \Gamma_1 \psi_1^2 + \ldots ] \equiv 2 \delta \delta(t-t') \Gamma(\psi)
$$

for constants $\Gamma_0$ and $\Gamma_1$. One way inferring a form of $(\zeta_i(t) \zeta_j(t'))$ appropriate to a given system is by using a master equation for discrete-valued variables and the Kramers-Moyal expansion.48

We will now verify that within $\epsilon$-expansion the extra terms which prevent the CA from having an underlying Hamiltonian are irrelevant in the renormalization group (RG)59 sense. Therefore, we will conclude that the critical behavior is the same as that of reversible PCA (i.e. kinetic Ising models). It is convenient to employ the functional integral scheme60,61 (the Martin-Siggia-Rose formalism) to do this calculation.

The idea is to write the average over the random variables $\{\psi(1,t)\}$ as a functional integral over $\{\psi(1,t)\}$ and a so-called response field $\{\tilde{\psi}(1,t)\}$, with a weight given by a Boltzmann factor with an effective "Hamiltonian", $e^{-\mathcal{H}_{\text{eff}}}$. For the Langevin equation given in Eq. (2.3.16) $\mathcal{H}_{\text{eff}}$ is given by (see Appendix B for a derivation),

$$
\mathcal{H}_{\text{eff}} = \int dt \int d^d x [i \tilde{\psi}(\psi - V(\psi)) - \mathcal{L}^2].
$$

(2.3.19)

For the irreversible PCA given in Eq. (2.3.17), it is easy to see
that $H_{\text{eff}}$, in Fourier space, is

$$-H_{\text{eff}} = \int_{q, \omega} \tilde{\psi}(-q, -\omega) [-i \omega + \Gamma_0 (r + q^2)] \psi(q, \omega)$$

$$+ \Gamma_0 \int_{q_1, \omega_1} \int_{q_2, \omega_2} \int_{q_3, \omega_3} 1 \tilde{\psi}(-q_1 - q_2 - q_3, -\omega_1 - \omega_2 - \omega_3) \psi(q_1, \omega_1) \psi(q_2, \omega_2) \psi(q_3, \omega_3)$$

$$\times \psi(q_2, \omega_2) \psi(q_3, \omega_3) \left( u_1 u_2 q^2 + O(q^4) \right)$$

$$- \Gamma_0 \int_{q, \omega} \tilde{\psi}(-q, -\omega) \tilde{\psi}(q, \omega)$$

$$- \Gamma_0 \Gamma_1 \int_{q_1, \omega_1} \int_{q_2, \omega_2} \int_{q_3, \omega_3} \tilde{\psi}(-q_1 - q_2 - q_3, -\omega_1 - \omega_2 - \omega_3) \psi(q_1, \omega_1) \psi(q_2, \omega_2) \psi(q_3, \omega_3)$$

$$+ \ldots \quad (2.3.20)$$

where $\int_{q, \omega} \equiv \frac{1}{(2\pi)^{d+1}} \int \! dq \int \! d\omega$, and the ellipses refer to higher-order terms. We have used $\Gamma_0$ for the coefficient of the $q^2$ term so that $H_{\text{eff}}$ reduces to its reversible form when the extra terms $u_2$, $\Gamma_1$, etc. are set equal to zero.

Now, imagine doing the RG transformation, by integrating out the high-momentum components in the range $[A/b, A]$ and rescaling as follows

$$q \to q' = b q, \quad \omega \to \omega' = b^2 \omega$$

$$\psi_q \to \psi'_{q'} = \xi^{-1} \psi_q, \quad \tilde{\psi}_q \to \tilde{\psi}'_{q'} = \xi^{-1} \tilde{\psi}_q . \quad (2.3.21)$$

By fixing the coefficient of $\omega \tilde{\psi}$ to be one and requiring that the $\tilde{\psi}^2$ term scale the same way as the $q^{2\omega}$ term, we get,

$$\xi = b^{2 + z - 1}, \quad \tilde{\xi} = b^{2 + z + 1}$$

$$\Gamma_0' = \Gamma_0 b^{z-2} . \quad (2.3.22)$$

We can now verify that the extra terms responsible for the irreversibility of PCA are all irrelevant near $d_c = 4$. For example, the $\Gamma_1$
term scales like

$$\Gamma'_1 = \xi^{-2} b^{-3(d+z)} \Gamma_1 \frac{\Gamma_0}{\Gamma'_0}$$

$$= b^{2d+4z-3d-3z+2-2} \Gamma_1 = b^{-(d-2)} \Gamma_1 . \quad (2.3.23)$$

and $q^2 u_2 \psi^3$ term scales like

$$u'_2 = \xi^3 b^{-3(d+z)-2} \frac{\Gamma_0}{\Gamma'_0} u_2 = b^{-(d-2)} u_2 . \quad (2.3.24)$$

Similarly, one can show that all the higher-order terms are irrelevant near $d=4$. In other words, in $d=4-c$ the dynamical fixed point of the standard kinetic Ising model with no conserved variables is stable with respect to all additional terms introduced by eliminating the underlying Hamiltonian. We conclude that fully probabilistic CA with up-down symmetry and a non-conserved order parameter fall, both for statics and dynamics, in the universality class of the ordinary Ising model with no conservation laws. Similar arguments show that CA rules which conserve the order parameter give rise to ferromagnetic transitions in the universality class of the kinetic Ising model wherein the order parameter is the only conserved quantity. Thus one reaches the striking conclusion that near a second order phase transition irreversibility on microscopic length scales renormalizes away, producing, on very large length scales, a reversible system!

We note that numerical results $^{62}$ on a simultaneously applied irreversible rule (see Chapter V) in $d=2$ give a magnetization exponent $\beta$ fully consistent with Ising model value of $1/8$. In the case of CA which are not fully probabilistic but have one absorbing state $^{45,63}$, $\Gamma(\psi=0)=0$. This places such models in a different universality class.
2.4 Non-stationary States

In this section we explore the occurrence of non-stationary states in PCA. In Sec. 2.2 we derived criteria for a fully probabilistic CA to obey detailed balance. Since these are enormously restrictive most CA are in fact irreversible. This does not necessarily imply that they will exhibit time-dependent behavior as $t \to \infty$. It is easy, however, to concoct a rule which displays two-cycle behavior. Consider the totalistic, peripheral rule, studied in the previous section. If $a_1$ is made arbitrarily small then an initial state with $M_0 = 1$ will execute a two-cycle flip-flopping between positive and negative values of $M$. More interesting is the possibility of time-dependent states which are more complicated e.g., chaotic. We restrict our attention to spatially homogeneous states for simplicity. We first show that diverse non-stationary behaviors occur within mean-field theory ($d = \infty$) for appropriately chosen rules. Then we discuss MFT in finite dimensions and $1/d$ corrections to MFT. Finally, we report on numerical simulations we have performed on simultaneously updated PCA in two dimensions. The parameter space is large and an exhaustive search is prohibitively time-consuming. However, a noteworthy feature of our simulations is the strong stabilizing effect of fluctuations on stationary states. For small numbers of neighbors, $z = 4, 8, 12, \ldots$, no non-stationary states except two cycles were found. A fuller understanding of this surprising result, and in a more general context, is the theme of Chapter III.
A. MFT and $1/d$ Expansion

Recall that the time evolution of a simultaneously-updated $\omega$-ranged model is described by the one-variable iterative map

$$M(t+1) = F(M(t))$$  \hspace{1cm} (2.4.1)

Different CA rules yield different functions $F$ [see Eq. (2.1.6)]. One can choose the CA rule such that $F$ is a non-monotonic function and has a fixed point $M^*$; by varying the parameters in the rule so as to achieve $F'(M^*)<1$ we can make the fixed point unstable and bifurcate into a 2-cycle. In this case, the magnetization alternates in time between two distinct values. Indeed, for appropriate choices of $F$, it is possible to find all the diverse features of single-variable maps, notably bifurcation sequences accumulating to chaotic states (see Appendix C for a review). That is, in MFT, magnetization as a function of time can be stationary, periodic, or chaotic.

Note that when simultaneously-updated $\omega$-ranged PCA are reversible $F$ assumes the form given in Eq. (2.2.9). This leads to a monotonic function and the criterion $F'(M^*)<1$ necessary for fixed points to bifurcate is not satisfied. Hence, the asymptotic state is stationary. This is in contrast to claims by Choi and Huberman in Refs. 41 and 42. Note that for sequentially applied 2-state PCA rules at $d=\infty$, all rules with symmetric couplings are reversible, and therefore, we do not expect complex temporal behavior.

Now we would like to investigate if these temporally-varying states that occur in MF limit can actually survive the inclusion of fluctuations. For simplicity, we consider nearest-neighbor totalistic rules in $d$-dimensions (i.e., the rule depends symmetrically on the
near-neighbor spins). Since the number of neighbors $z$ is $2d$ and the
MF limit corresponds to $d \rightarrow \infty$, fluctuation corrections are conveniently
studied in a systematic $1/d$ (or $1/z$) expansion. To $O(1/d)$ both $M(t)$
and the near-neighbor correlation function $G(t) \equiv \langle S_i S_j \rangle_t - M^2(t)$, for $j \neq 1$,
are required for a complete characterization of the time evolution of
the system; all other two- and higher- point correlation functions are
at most $O(1/d^2)$. The equations obeyed by $M(t)$ and $G(t)$ are easily
derived and we do so in Appendix D:

$$M(t+1) = F(M(t)) + \frac{1}{4d} \left( 1 - M^2(t) \right) \left[ f_1^2(M(t)) + M(t)f_2^2(M(t)) \right]$$

$$+ f_2^2(M(t)) G(t)$$

$$G(t+1) = f_2^2(M(t)) G(t) + \frac{1}{d} \left( 1 - M^2(t) \right) f_2(M(t)) \left[ f_1^2(M(t)) + M(t)f_2^2(M(t)) \right]$$

$$(2.4.2)$$

where $F = f_1 + S_1 f_2$. Note that $f_2^2 < 1$ and therefore, as $d \rightarrow \infty$ $G$ decays to
zero and a one-variable description is sufficient. Equation (2.4.2)
is a 2-variable iterative map. Similarly, to any finite order in $1/d$
the hierarchy of Eq. (2.1.7) reduces to an $n$-variable iterative map
for some finite $n$. Two-variable iterative maps are known to exhibit
the Feigenbaum period-doubling route to chaos and more complex
behaviors. So one might be tempted to draw the conclusion that all
the qualitative features of MFPT ought to occur to $O(1/d)$ for appro-
priate rules. However, one has to be careful in interpreting
Eq. (2.4.2). First, it is a perturbation expansion and non-pertur-
bative term of the form $e^{-d}$ which are not included might play a
crucial role. More importantly, even given the form of Eq. (2.4.2) it
may not have chaotic solutions due to the range of variation of the
parameters in the rule in the region of validity of the approximation. For example, consider the 2d nearest-neighbor, totalistic, up-down symmetric rule studied in Sec. 2.3. MFT predicts

\[ M(t+1) = AM(t) + BM^3(t) \]  

(2.4.3)

One might naively conclude that by varying the parameters \( a_1 \) and \( a_2 \) in the rule, and therefore, \( A \) and \( B \) in Eq. (2.4.3), period-doubling bifurcations and chaos should occur. However, it is trivial to establish that the model cannot produce a single bifurcation around the ferromagnetic fixed point let alone the infinite sequence of bifurcations and chaos. Equation (2.4.3) has fixed points 0, \( \pm \sqrt{(1-A)/B} \). Recall that \( a_1 \) and \( a_2 \) are probabilities (0 < \( a_1 \), \( a_2 < 1 \)). Hence, \( |A|, |B| < 3/2, |A+B| < 1 \), and \( |A-B| < 2 \). In order for the ferromagnetic fixed point to bifurcate we need \( f'(M^*) < 1 \) but \( f'(M^*) = 3BM^2 + A |M^*|^3 = 3-2A \geq 0 \). Hence, the ferromagnetic fixed point can become unstable but does not bifurcate. Also \( f'(M^* = 0) = A \). Since \( -3/2 < A < 3/2 \), the paramagnetic fixed point can bifurcate into a 2-cycle but it can be easily checked that no further bifurcations occur. For completeness we display the MF phase diagram for this model in Fig. 1. The central unshaded region is paramagnetic, i.e., the system converges to the \( M = 0 \) fixed point. The upper-right hatched triangle is a ferromagnetic region where the asymptotic state of the system is stationary and the average magnetization \( M \) is non-zero. The lower-left hatched triangle is a 2-cycle region where the magnetization goes between +\( M_0 \) and -\( M_0 \) on alternate time steps. No further bifurcation occurs. The bold curved line given by

\[ a_1 = \frac{1}{2} \left[ 1 + \frac{2(2a_2 - 1)}{1 + (2a_2 - 1)^2} \right] \]

represents reversible PCA, i.e., for these values of \( a_1 \) and \( a_2 \) the model has an underlying Hamiltonian. We see from
Figure 1  Mean field phase diagram for the PCA defined in Eq. (2.3.3). The central portion, the upper triangle, and the lower triangle represent paramagnetic, ferromagnetic, and two-cycle regions respectively. The bold curve represents reversible PCA.
this MF phase diagram that reversible PCA can go to a 2-cycle state which is related to two-phase coexistence in higher dimension (see corresponding discussion at the end of Sec. 2.2).

Therefore we conclude that it is dangerous to infer the existence of non-stationary states merely from the form of the MF equations! However, one might naively expect the \( d \to \infty \) predictions to continue to hold qualitatively at finite \( d \). To check this expectation reliably we must resort to numerical simulations which include the effect of fluctuations neglected in MFT.

B. Numerical Results

To test whether the MF predictions actually occur in low dimensions where fluctuations are strong, we have searched numerically for period-doubling sequences and chaos in simultaneously-updated PCA in \( d=2 \). We have computed \( M(t) \), the average magnetization and studied the power spectrum of \( M \). The space of parameters characterizing the possible probabilities is, of course, vast; selecting simple rules in a relatively arbitrary way never led us to time-dependent states (other than to trivial two-cycles, e.g., the one in which the system flips between essentially all up and essentially all down at alternate time steps). We systematized the search by considering totalistic CA with neighborhoods consisting of \( (2n+1) \times (2n+1) \) squares of sites for \( n=1,2,...,8 \). Using the known criterion for the occurrence of non-stationary state in MFT as a guide, e.g., choosing \( F \) defined in Eq. (2.1.6) to be a quadratic or a sine function and \( |F'(M)| \) large, we have succeeded in constructing rules which undergo limit cycles with
periodicities 2 and 4 on samples of size up to 200x200 and times up to \(10^5\) steps. We have also obtained a 3-cycle by choosing the transition probabilities close to the deterministic value. The three- and four-cycles occurred, however, only for \(z \geq 49\) and \(z \geq 288\), respectively. We have never found a periodic state with period >4, much less a complete bifurcation sequence, which MFT predicts. The numerical results certainly suggest that it is extremely difficult to obtain chaotic and higher cycle states. Note that in Ref. 43 Kinzel reports finding no chaotic states in searches in PCA with one absorbing state. In the next chapter we will argue that the absence of chaotic states when fluctuations are included is actually a general result.

2.5 Concluding Remarks

The work described in this chapter constitutes a modest step towards a systematic study of the rich and complex behaviors exhibited by PCA using statistical mechanical methods. In concluding the chapter we will highlight some of our results and draw attention to potential avenues for further work.

We have focused on spatially homogeneous properties, i.e., zero-wavevector modes \(\langle S_{i,t} \rangle\) independent of the site \(i\). We have argued that continuous ferromagnetic transitions in PCA belong in the universality class of kinetic Ising models in spite of their irreversibility. We employed the \(\epsilon\)-expansion and as was emphasized these are restricted to showing local stability. Hence, it would be useful to pursue non-perturbative methods like the Monte Carlo Renormalization Group\(^{64}\) to evaluate critical exponents for PCA in \(d=2\) (e.g., the
model in Eq. (2.3.3)).

An aspect of PCA that will in fact be discussed in greater detail in Chapter III is the uniqueness of the asymptotic state, i.e., does the asymptotic state depend on the initial condition? In the absence of fluctuations, i.e., in MFT, it is easy to construct models where depending on $M(t=0)$ the state at $t=\infty$ is a temporal 2-, 3-, or 6-cycle. However, numerical simulations, which include fluctuations, fail to show such behavior.

Another interesting area that remains to be explored is that of spatially-modulated phases. In spin systems it is well-known that competing interactions give rise to commensurate and incommensurate phases. While within MFT in $d=2$ we have succeeded in obtaining simple modulations, we have not studied the effect of fluctuations. The lack of reversibility allows the existence of temporal cycles and this permits an additional mechanism for relieving frustration and competition.

The numerical results of Sec. 2.4 emphasize the stabilizing effect of fluctuations on stationary states, in particular, the difficulty in obtaining chaotic behavior in $M(t)$. The next chapter addresses this question in depth and argues that it is not possible for PCA to exhibit chaotic behavior for any averaged quantity!

We have studied rules which are homogeneous in space and time. Random PCA, i.e., CA in which the rules are different in space and/or in time remain to be explored in any detail.
FOOTNOTES

a In two dimensions, if the neighborhood of site \( i \) contains \( i \) and its four nearest-neighbor sites it is known as a von-Neumann neighborhood. If \( i \) contains \( i \) and its four nearest- and four next-nearest neighbor sites, it is referred to as a Moore neighborhood.

b We point out that the master equation employed in Ref. 41, [Eq. (2.3)] is not correct. Their equation reads

\[
P([S_i], t+\tau) = [1-\sum_k W_k(S_k)] P([S_i'], t) \\
+ \sum_k W_k(-S_k) P(S_i, \ldots -S_k, \ldots S_N, t)
\]

where \( W_k(S_k) \) is the probability that \( S_k \rightarrow -S_k \) at time \( t \). The first term presumably represents the probability for the system to remain in the state \([S_i] \). The term \([1-\sum_k W_k(S_k)]\) is a probability but it is unbounded! Consider infinite temperature when \( W_k(S_k)=W_k(-S_k)=1/2 \). Then the first term is \([1-\frac{N}{2}]P\) where \( N \) is the number of sites.

Clearly, the equation is incorrect and cannot be used for a systematic analysis in general.

c It is easy to see from Eq. (2.2.11) that there is no simultaneously-updated PCA with an underlying Hamiltonian of a nearest-neighbor, ferromagnetic Ising model. This was pointed out earlier by Vichniac.17 See, however, Ref. 33.

d It is straightforward to include more general interactions in the Hamiltonian, e.g., different fields along the \( q \)-directions and more complicated multispin terms,

\[
J_{ij}^a \delta(s_i^a, s_j^a) \quad \text{where} \quad a, b \in [0,1, \ldots q-1] .
\]

e For the kinetic Ising model, which is reversible, by the fluctuation-dissipation theorem the coefficient of the \( q^2\psi^2 \) term is the constant \( \Gamma_0 \) that appears in the noise-noise correlation function. Since we are considering the fixed point which describes such a model (in order to examine irreversible deviations from it) we demand that the \( q^2\psi^2 \) term and the \( \psi^2 \) term (which arises from the integration of the Gaussian noise and hence is initially equal to \( \Gamma_0 \)) scale the same way.
CHAPTER III
COHERENCE, CHAOS, AND BROKEN SYMMETRY
IN CLASSICAL, NOISY, MANY-BODY SYSTEMS

3.0 Introduction

Though non-stationary (periodic and chaotic) behavior of dynamical systems with few degrees of freedom has been studied extensively during the last decade, understanding of analogous phenomena in many-body systems is less advanced. Most of the theoretical effort to date has centered on systems without external noise, e.g., noiseless partial differential equations (PDEs) or deterministic cellular automata. Treatments of noisy many-body systems have been largely confined to approximations analogous to mean field theory (MFT) in equilibrium statistical mechanics. Our work is motivated in part by the fact that in equilibrium models fluctuations ignored in MFT can cause dramatic changes in the phase diagrams and transitions predicted in their absence. It is not infrequent that entire phases turn out to be artifacts of MFT; i.e., they are rendered unstable by fluctuations and hence, do not occur.

One of our motivations is the following: Many deterministic model dynamical systems (either maps or small sets of coupled ordinary differential equations) originally derived as approximations to many-body systems (e.g., PDEs, PCA, etc.) have been shown to exhibit non-stationary behavior. A famous example is the Lorenz model,\textsuperscript{67} three
coupled, first-order non-linear differential equations, obtained by truncation of the Navier-Stokes equation for a convecting fluid. It has not been carefully established that the non-stationary states of these zero-dimensional (0d) approximants (i.e., consisting of a finite number of modes) actually occur in the full many-body system which has complex spatial fluctuations, and in realistic cases, external noise. It is, however, commonly believed that the non-stationary behavior of many-body systems is faithfully represented by their 0d approximants in that the finite number of modes correspond to spatially-coherent modes of the former and describe their time-dependence, at least qualitatively. These modes are sometimes localized, but are often argued to be spatially extended. For example, (a) in PCA, Huberman et al. have predicted that noise-averaged quantities (k=0 mode) can vary chaotically in time and exhibit all other phenomena that occur in one-variable maps. However, we recall that in our numerical simulations we were unable to see compelling evidence for chaotic trajectories and never saw the full bifurcation sequence that occurs in 1-variable maps. (b) In PDE numerical evidence for chaotic behavior in averaged quantities has been reported by Bishop et al.68

It is thus clearly of interest to study the effect of fluctuations on the non-stationary states obtained within finite-mode approximations or within MFT. In this chapter, we discuss our analysis which constitutes the beginnings of a systematic investigation. It is based on physical arguments and numerical simulations and is by no stretch of the imagination rigorous. Our analysis is for model systems and we have not made any effort to connect to specific experiments.
3.1 The Model and Summary of Results

The analysis and conclusions of this chapter will be presented in the context of noisy\textsuperscript{69,70} coupled maps on regular d-dimensional lattices. We emphasize that our results apply equally well to other dynamical systems. Specifically, we consider a model whose evolution in discrete time is described by\textsuperscript{3/}

\[
\psi_{n+1}(i) = f\left(\frac{1}{z} \sum_{j \in v_1} \psi_n(j)\right) + \zeta_n(i)
\]

\[= f(\bar{\psi}_n(i)) + \zeta_n(i).
\] (3.1.1)

Here \(\psi_n(i)\) is the dynamical variable at the nth time step on the ith site of a d-dimensional hypercubic lattice with N sites. The function \(f\) which defines the map depends on a the control parameter \(r\). As \(r\) is increased, \(f\) is assumed to undergo a bifurcation sequence leading to chaos at a critical value \(r=r_c\). For simplicity we will use the quadratic\textsuperscript{71} map, i.e., \(f(x) = rx(1-x)\). \(\zeta_n(i)\) represents the noise, which is assumed to be Gaussian distributed with zero mean and mean square deviation given by \((\zeta_n(i)\zeta_n'(j)) = \sigma^2 \delta_{ij} \delta_{nn'}\). The coupling between different sites appears in the argument of \(f\), where \(j\) runs over the neighborhood of site \(i\), \(v_1\), which contains \(z\) (<\infty) sites. In the following, we will choose for simplicity the neighborhood of \(i\) to consist of the site \(i\) itself and its 2d nearest neighbors. Other choices for \(v_1\) will not alter our main results as long as the range remains finite.

The question we will address is how the noisy many-body system behaves as the control parameter \(r\) and the variance of the noise \(\sigma\) are varied. In particular, we will consider spatial averages, say
\[ \psi_n = \frac{1}{N} \sum \psi_n(i) \ (N^{+\infty}), \text{ or equivalently, in a homogeneous system, noise-} \]

averaged quantities, e.g., \( (\psi_n(1)) = \psi_n \) and investigate how they

behave as a function of time \( n \). This is a very complicated many-body

problem, since we are interested in the thermodynamic limit, where the

number of degrees of freedom goes to infinity. However, in the

following we will appeal to physical arguments and employ numerical

simulations to obtain a qualitative (sometimes semi-quantitative)

description of the macroscopic behavior of the system. In contrast,

previous work \cite{72-74} has been largely restricted to 1d and has focussed

on different questions.

For future convenience we will use standard techniques and cast

this problem in the language of equilibrium statistical mechanics.

This is precisely analogous to the mapping in Chapter II where we saw

that every \( d \)-dimensional CA corresponds to a \( (d+1) \)-dimensional statisti-

cal mechanical model. First note that a general noise-averaged

correlation function \( \langle 0 \rangle \) where \( 0 \) is any function of \( \{ \psi_n(i) \} \) can be

written as

\[ \langle 0 \rangle \equiv \int \prod_{i,n} [d\zeta_n(i) \ e^{-\zeta_n^2(1)/2\alpha^2}] \ 0/Z \quad (3.1.2) \]

with the "partition function"

\[ Z \equiv \int \prod_{i,n} d\zeta_n(i) \ e^{-\zeta_n^2(1)/2\alpha^2}. \quad (3.1.3) \]

Using the equation of motion (3.1.1) to change variables from \( \{ \zeta_n(i) \} \)

to \( \{ \psi_n(i) \} \) one obtains

\[ Z = \int \prod_{i,n} d\psi_n(i) \ e^{-H/\alpha^2} \quad (3.1.4) \]
where the Hamiltonian \( H \) is given by

\[
H = \frac{1}{2} \sum_{n,1} (\psi_{n+1}(1) - f(\psi_n(1)))^2.
\]  

(3.1.5)

This is the Hamiltonian for the effective \((d+1)\)-dimensional system (\( n \) labels the time direction and \( i \) the \( d \) spatial dimensions). Note that \( \sigma^2 \) plays the role of temperature. Because we are dealing with discrete space-time variables it can be shown that the Jacobian of the transformation from \( \{\zeta_n(1)\} \) to \( \{\psi_n(1)\} \) is unity.\(^7^0\) This representation makes clear that the noiseless limit, \( \sigma=0 \), corresponds to the zero-temperature limit of the \((d+1)\)-dimensional model. Alternatively, it can be viewed as the Mean-Field (MF) limit of statistical mechanics where we simply minimize the Hamiltonian with respect to \( \{\psi_n(1)\} \). In either case, we recover, as we must, Eq. (3.1.1) with \( \zeta_n(1) \) set equal to zero.

For the benefit of the reader we now briefly summarize the central features of our results in Fig. 2. First recall that with increasing \( r \) the single variable \((0d)\) quadratic map with \( \sigma=0 \) undergoes\(^3\) an infinite sequence of period-doubling bifurcations leading, at \( r=r_c=3.569... \), to chaos. As \( r \) increases beyond \( r_c \), an inverse bifurcation sequence of bands (interrupted by periodic windows) occurs.\(^7^5\) We remind the reader that a relevant summary of the behavior of the quadratic map can be found in Appendix C. For \( \sigma=0 \) and \( r<r_c \) the coupled map system has stable, spatially-uniform \( 2^m \)-cycles which correspond precisely to those of \( 0d \), exhibit the same period-doubling cascade, and persist for sufficiently small noise. For \( \sigma>0 \) and \( N \rightarrow \infty \) they are, however, spatially-uniform only in that the
Figure 2 Schematic phase diagram as a function of the control parameter \( r \) and noise \( \sigma \) for coupled lattice maps in \( d>1 \). The dotted line represents the zeros of the Liapunov exponent for an individual site.
noise-averaged value \( \langle \psi_n(i) \rangle \) is independent of \( i \), and periodic only in that \( \langle \psi_n(i) \rangle \) (or, equivalently, the spatial average, \( \sum_i \psi_n(i)/N \), for a given realization of the noise variables) executes a \( 2^m \)-cycle in \( n \). These states are connected through period-doubling transitions, argued below to belong, for \( \sigma > 0 \), in the universality class of the \( d \)-dimensional kinetic Ising model; for \( \sigma = 0 \) they are mean-field-like. The successive period-doubling transitions break the discrete time-translation symmetry of Eq. (3.1.1) progressively; (i.e., the \( 2^m \)-cycle is invariant only under time translations of \( 2^m \) time steps.) For any fixed \( \sigma \), however, the period-doubling sequence terminates at some maximum period which increases with decreasing \( \sigma \), diverging as \( \sigma \to 0 \); thus states of arbitrarily long period occur for small enough noise. (This is closely analogous to the "bifurcation gap" predicted\(^76\) for noisy \( O_d \) maps.)

As \( r \) increases beyond the point where this maximum occurs, the system undergoes an inverse sequence of period-doubling transitions, becoming stationary (a 1-cycle) as \( r \) approaches 4. There is, strikingly, no distinct chaotic phase, i.e., no phase, even for \( \sigma = 0 \), in which the time translation symmetry is completely broken so that \( \langle \psi_n(i) \rangle \) varies chaotically in time. The system has a "chaotic" regime, but the chaos is purely local, not collective; i.e., the evolution of any individual variable is chaotic (as measured by a Lyapunov exponent \( \lambda \)),\(^77\) but the system remains periodic at the macroscopic level. A schematic \( \lambda = 0 \) line is shown (dashed) in Fig. 2. To its right \( \lambda > 0 \), and the system is locally chaotic. Unlike period-doubling boundaries this line does not represent a broken symmetry or
phase transition, so $\lambda$ varies analytically through it for any $\sigma > 0$.

For $\sigma = 0$ the system is often multistable\(^72-74\) (i.e., different initial conditions produce distinct stable states, only the most robust of which is shown in Fig. 2). As argued below, however, such multistability cannot persist at generic points of Fig. 2 for $\sigma > 0$.

3.2 Mean-Field (Noiseless) Limit

We begin our analysis by considering the noiseless (i.e., $\sigma = 0$) limit of Eq. (3.1.1). In this limit we have

$$\psi_{n+1}(1) = f\left[\frac{1}{z} \sum_{j \notin v_i} \psi_n(j)\right], \quad (3.2.1)$$

an infinite set of coupled deterministic equations. As we saw in Sec. 3.1, Eq. (3.2.1) corresponds to the zero-temperature limit or to the mean-field theory of an anisotropic $(d+1)$-dimensional statistical mechanical model. In general, it is difficult to determine the complete set of solutions to the MF equations in $d$ spatial dimensions for arbitrary values of the control parameter $r$. However, an obvious solution is the one that is spatially uniform. If the system is spatially uniform initially, i.e., $\psi_n(1) = \psi_n$ independent of $i$ for $n = 0$, we see from Eq. (3.2.1) that it will remain so at all subsequent times and the problem reduces to that of a single variable map $\psi_{n+1} = f(\psi_n)$ for all dimensions $d$. Thus, in the noiseless limit, the coupled $d$-dimensional map does have spatially uniform, temporally cyclic or chaotic states which correspond precisely to those of the $0d$ map. That is MF predicts that the $d$-dimensional coupled lattice map can display the period-doubling route to chaos. We note parenthetically
that in statistical mechanical language, we have a MF solution which describes a state that varies periodically or chaotically in one (time-like) direction and is uniform in the remaining d-directions.

Eq. (3.2.1) can have many other solutions: in particular, solutions that are spatially non-uniform in d-dimensions and whose spatial average \( \psi_n \equiv \frac{1}{N} \sum_1^N \psi_i(1) \) exhibits cycles or chaos temporally. For example, consider a value of the control parameter \( r \) such that the Od map exhibits a stable 2-cycle with values \( \psi_1^* \) and \( \psi_2^* \). Now imagine starting a two-dimensional system with alternating strips (domains) of length \( L_1 \) and \( L_2 \) with the \( \psi \) in each strip assuming values \( \psi_1^* \) and \( \psi_2^* \) respectively. After many iterations the system can be expected to converge to a solution where the basic strip structure is maintained with a transition region connecting the different domains such that the system as a whole executes a 2-cycle. Thus we expect a MF solution that is non-uniform in space, and whose average value executes a 2-cycle in time (the values will be different from \( \psi_1^* \) and \( \psi_2^* \) and depend on \( L_1 \) and \( L_2 \)). These considerations have been verified numerically on a 50x50 lattice. The profile of the domain wall depends, not surprisingly, on \( L_1 \) and \( L_2 \). Such periodic arrays of solitons and antisolitons (domain walls/kinks) and other non-periodic sequence of solitons have been observed and studied in detail in \( d=1 \) by Kaneko.72

In contrast to \( d=1 \), for \( d>2 \), random initial conditions always lead to the uniform MF solution. This has been checked for several randomly chosen initial values. Thus for \( r < r_c \) the uniform solution appears to be the most stable one empirically, although we have been unable to argue global stability analytically.
For $r>r_c$, it is also possible to construct MF solutions that are non-uniform in space, and whose average values are chaotic in time. Consider an initial condition which is a superposition of a uniform value and a long but finite wavelength modulation that is commensurate with the lattice. For all values of $r$, the system will obviously remain spatially periodic in the absence of noise. For $r>r_c$, it is easy to see that the average value can be chaotic in time for certain wavelength modulations. This has again been verified numerically. However, for $r>r_c$, in contrast to $r<r_c$, the uniform solution is not the most stable one.

In summary, MFT predicts that d-dimensional coupled lattice maps can exhibit cycles and chaos on the average and exhibit the period doubling route to chaos. However, for random initial conditions, the MF solution is uniform for $r<r_c$ but non-uniform for $r>r_c$. While multistability does occur for $\sigma=0$, in Fig. 2, we have displayed the most robust of these states.

3.3 Fluctuations for $r<r_c$

In real systems both spatial fluctuations and noise are present; the question then is whether the periodic and chaotic states found within MFT in the previous section survive the inclusion of fluctuations. In this section we address this problem for the range of control parameter values $r<r_c$. Since for $r<r_c$, random initial conditions always lead to spatially uniform states we will concentrate on the stability of such states and ignore those states that result from special initial conditions. We will demonstrate that spatially uniform solutions remain stable both within linear stability
analysis and with respect to the most disruptive of the non-linear modes, the domain-wall excitations. Of course, in the presence of noise the solutions are only uniform in that the noise-averaged value \( \langle \psi_1^n \rangle \) is independent of \( n \) and the fluctuations from this value are small (proportional to \( \sigma \)).

A. Linear Stability Analysis

One can easily see that the uniform \( 2^m \)-cycle states of MFT are linearly stable. Let the uniform MF solution \( \psi_0^\ast \) be perturbed by a small amount \( \delta \psi_0(1) \) at time \( n=0 \) where \( \delta \psi_0(1) = \psi_0^\ast(1) - \psi_0^\ast \). We will compute \( \delta \psi_n(1) = \psi_n(1) - \psi_n^\ast \) the deviation of \( \psi_n(1) \) at the \( n \)th time step from the uniform solution \( \psi_n^\ast \). From linear stability analysis (or equivalently, by calculating Gaussian fluctuations in the language of statistical mechanics), we have, in \( q \)-space:

\[
\delta \psi_{n+1}(q) = f'(\psi_n^\ast) a(q) \delta \psi_n(q) + \zeta_n(q)
\]  

(3.3.1)

where \( a(q) \) depends on the neighborhood one chooses. In the non-peripheral near-neighbor case under consideration,

\[
a(q) = \frac{1}{2} \left( 1 + 2 \sum_{1=1}^{d} \cos q_1 \right)
\]

with \( d = 2d+1 \). Averaging (3.3.1) over noise, we obtain,

\[
\langle \delta \psi_{n+1}(q) \rangle = f'(\psi_n^\ast) a(q) \langle \delta \psi_n(q) \rangle
\]

(3.3.2)

Upon iterating (3.3.2), we see immediately that since \( \prod_{1}^{n} f'(\psi_1^\ast) \sim e^{\lambda n} \) for large \( n \) with the Liapunov exponent \( \lambda < 0 \) for a stable \( 2^m \)-cycle state, perturbations around the MF uniform solution \( \{ \psi_n^\ast \} \) of any wave vector \( \vec{q} \) will decay exponentially in time on the average. The mean square deviation is given by
\[ \langle |\delta \psi_{n+1}(\hat{q})|^2 \rangle = \sigma^2 + a^2(q)f \cdot \hat{2}(\psi^n) \langle |\delta \psi_n(q)|^2 \rangle. \] (3.3.3)

Again, upon iteration and integrating over all \( \hat{q} \), we get
\[ \langle \delta \psi_{n+1}^2(\hat{q}) \rangle \approx \sigma^2. \] This means that, not only does any perturbation around the uniform MF solution decay on the average, but also that the system remains basically uniform as long as \( \sigma \) is small. So, we arrive at the conclusion that MF uniform cycle states are linearly stable with respect to fluctuations. This, of course, does not establish the global stability of the cycle states. To do that one must consider the effect of nonlinear terms in (3.1.1), or equivalently, the terms of higher order than quadratic in \( \delta \psi \) of the Hamiltonian (3.1.5) and this we now proceed to do.

B. Beyond Linear Stability Analysis

The correspondence between coupled lattice maps (3.1.1) and the statistical mechanical model (3.1.5) enables us to apply the physical and intuitive considerations developed in statistical mechanics to the problem of coupled lattice maps. It is well known from equilibrium statistical mechanics that of the nonlinear fluctuations in Ising models, the potentially most disruptive are "domain wall" excitations.

Let us first review domain wall arguments in Ising models.\(^78\) Domain walls are boundaries which separate a region of up spins from one of down spins. In zero field and at low temperatures an infinite flat interface between two equally stable phases does not move. Note that by up-down symmetry the probability for those boundary spins which are up to flip down is the same as that for the down spins to flip up. This implies that the domain wall has no preferred
direction to move and has zero mean-propagation velocity, i.e., it is stable. For example, in d=1, domain walls are always flat, i.e., have zero curvature, and have "areas" independent of the size of the domains they enclose. So in d=1, the domains are always stable in the sense that they need not shrink with time. Thermal fluctuations cause the number of domains to proliferate and ultimately destroy any long-range order in one dimension.

In d>1 domain walls can have finite curvature: any wall separating a finite domain of one phase in a sea of the other has finite curvature. Consider a square domain of size $R$ of up spins in a background of down spins in two dimensions for simplicity. Down spins on the boundary are surrounded by three down spins and one up spin. Thus while it is possible for such a spin to flip up it is prohibitively expensive energetically, at low T. The same is the case for up spins on the boundary except those at the corners; here the up spin is surrounded by 2 up spins and 2 down spins and it is equally favorable for it to be down. Thus the domain gets eroded systematically from the corners. The motion of the corners is diffusive and is like that of a particle executing random walk on a lattice with equal probability for hopping to the right or to the left. So the time $\tau$ required for the top layer to erode is the mean time for two random walkers starting $R$ sites apart to meet which is roughly $O(R^2): \tau \sim R^2$. The total time required for the domain to disappear is not $R\tau$ since subsequent rows of the domain begin to erode long before the first layer has been eliminated. Thus the time required to strip the top layer is the rate-limiting step and dominates the decay time of the
domain and hence we conclude that the domain disappears in a time $\tau \sim R^2$.

For coupled lattice maps, similar "domain wall" excitations occur for all $2^m$-cycles. In this case, domain walls are boundaries which separate two regions of appropriate $2^m$-cycle states that are out of phase with each other. To understand the effect of such excitations consider first the case $m=1$ for $d>1$ in the noiseless limit and denote the MF 2-cycle values by $\psi_1'$ and $\psi_2'$. Since in one time step the two domains $\psi_1'$ and $\psi_2'$ exchange identities, by symmetry, the domain wall cannot have a preferred direction to move but is stable. A finite domain of $\psi_1'$, linear size $R$, immersed in a sea of $\psi_2'$ will, under the once-iterated map $f^2$, shrink with time for all $d>1$ because of its finite curvature. It will disappear, as in the equilibrium Ising models, in a time proportional to $R^2$. Again, as in Ising models, sufficiently small noise alters only the proportionality constant (and the two values $\psi_1'$ and $\psi_2'$), not the qualitative phenomenology. This has been checked by numerical experiments.

The above argument tacitly assumes a simple structure for the domains of attraction of the 2 two-cycles. We explain this below: Consider the once-iterated map $f^2$ which has two stable fixed points $\psi_1'$ and $\psi_2'$ separated by an unstable fixed point $\psi'$. Since $\psi_1'$ ($i=1,2$) is stable, there is a finite interval around it which is part of its domain of attraction. Denoting the intervals by $I_1$ and $I_2$ we observe that $I_1$ and $I_2$ are contiguous (separated by a single point, the unstable fixed point $\psi'$). Now imagine a d-dimensional system ($d>1$) at time $n=0$ with a perfectly flat interface separating a domain with $\psi=\psi_1'$ from one with $\psi=\psi_2'$. Now the future of a spin $\psi_{n+1}(1)$ is
determined by the mean of the neighboring spins $\bar{\psi}_n(1)$; it is evident that $\bar{\psi}_n(1)$ belongs to the union of the intervals $I_1$ and $I_2$.

Therefore, the system evolves in such a way that the effect of the initial interface is confined to its vicinity: far from it the spins assume values $\psi_1$ and $\psi_2$ asymptotically while near it a stable profile is formed. We belabor this point for the following reason.

In the case of the 3-cycle (and other cycles for $r>r_c$) as we shall discuss in the next section, there are no contiguous intervals of attraction. Furthermore, the gaps between the intervals have a complex interleaving of the basins of attraction so that $\bar{\psi}_n(1)$ is in the gaps, hence, $\bar{\psi}_n(1)$ executes transient but long-lived chaotic motion. This propagates the influence of the wall deep inside the domain and can destroy the domain wall structure! However, for the $2^m$-cycles for $r<r_c$ the basins of attraction possess the structure that ensures that our naive domain wall arguments are valid.

We conclude the for $d>1$ the two-cycles are stable with respect to domain formation for "sufficiently small $\sigma$", i.e., the noise averaged quantity $\langle \psi_n(1) \rangle$, or equivalently the spatial averaged quantity

$$\frac{1}{N} \sum_{n=1}^{N} \psi_n(1)$$

undergoes, in the thermodynamic limit, a 2-cycle for large $n$.

Crudely speaking when $\sigma$ becomes the order of the spacing between the two-cycle $\sigma \approx |\psi_1 - \psi_2|$, the distinction between $\psi_1$ and $\psi_2$ blurs, and the two-cycle disorders, i.e., undergoes a transition to a temporally uniform state (1-cycle). Recall that any cycle higher than 1-cycle is a state of broken discrete time translation symmetry. The 1-cycle is the most symmetric and therefore the most disordered state.
For $m > 1$, we can again argue that $2^m$-cycle is stable with respect to domain formation. For example, when $m=2$ we consider $f^4$. $f^4$ has four stable fixed points (four phases). One can argue in the same way as in the two-cycle case that any finite domain of one phase immersed in the sea of the other phase will shrink to zero, i.e., uniform MF cycle states are stable with respect to domain formation. So we conclude for any given $m$, the many-body system exhibits a $2^m$-cycle for $d > 1$ for sufficiently small $\sigma$. Again, when $\sigma$ becomes roughly comparable to the smallest spacing between cycle values, the $2^m$-cycle undergoes a transition to a $2^{m-1}$-cycle. For a given $r(<r_c)$ value which corresponds to a stable $2^m$-cycle in the 0d map, as we increase $\sigma$ from zero, the many-body system will change its state from $2^m$-cycle to $2^{m-1}$-cycle, then to $2^{m-2}$-cycle, etc. Eventually, when the noise is so large that distinction between elements of any cycle is impossible, the system becomes stationary (1-cycle) on the average. For any given $\sigma > 0$, however, there is a $\overline{m}(\sigma)$, such that cycle lengths greater than $2\overline{m}(\sigma)$ cannot be observed. When we increase $r$, the period-doubling sequence does not go to completion; it terminates at the maximum period $2\overline{m}(\sigma)$ which the system can achieve for $r < r_c$, and $\overline{m}(\sigma)$ increases with decreasing $\sigma$, diverging as $\sigma \to 0$. States of arbitrary high period occur in the vicinity of the point $r = r_c$, $\sigma = 0$. This structure is clearly embodied in the phase diagram of Fig. 2.

It is usually assumed that the form of the noise distribution does not alter the qualitative behavior of the system. While this is generally true in $d > 1$, it does not hold for $d \leq 1$. 
Consider a single-variable iterative map \(d=0\) in the 2-cycle regime. The once-iterated map \(r^2\) has 2 stable fixed points \(\psi_1^*\) and \(\psi_2^*\) with an unstable fixed point \(\psi^*\) in between. In the noiseless limit \(\sigma=0\) the system converges to one of the stable fixed points, say, \(\psi_1^*\) (depending on the initial condition), i.e., it executes a 2-cycle \(\psi_1^*, \psi_2^*, \psi_1^*\). Now add unbounded noise (e.g., Gaussian noise) to the system. Since in this case, the noise variable can assume any value with non-zero probability, eventually the noise kicks the system from the neighborhood of \(\psi_1^*\) to that of \(\psi_2^*\). Now the system continues to execute a 2-cycle, but out of phase with the previous one. The noise can kick the system back and forth and if we look at the noise ensemble of systems all of which start with the same initial condition, we find that the noise averaged value executes a 1-cycle. So for \(d=0\) and Gaussian noise, noise-averaged quantities can only be stationary. This result can be interpreted using the \((0+1)\)-dimensional statistical mechanical correspondence: the absence of long-range order (broken symmetry) in 1d at finite temperature implies that no non-stationary states are possible in noisy 0d maps. Note that these results contradict a statement by Shraiman et al.\(^7\) that, even for the 0d noisy map \(\langle \psi_n \rangle\) need not approach a constant as \(n \to \infty\) for Gaussian noise\(^b/\).

Now consider the system with bounded noise. We can choose the range (width) to be sufficiently small that it is never enough to move the system from the vicinity of one fixed point of \(r^2\) to the other. In this case the noise-averaged value indeed executes a 2-cycle! Periodic states can occur for zero-dimensional systems for bounded noise in contrast to Gaussian noise.
For $d=1$, as in equilibrium Ising systems, domain wall excitations play a more important role. Since in $d=1$ a domain wall enclosing a droplet of size $R$ has no curvature, i.e., has an area which does not increase with $R$, droplets need not shrink with time; any non-zero Gaussian noise will proliferate the number of droplets, eventually destabilizing any cycle with $m \geq 1$. So, in $d=1$, as in $d=0$, only bounded noise can give rise to cycle-states. For Gaussian noise only stationary (1-cycle) states are possible, although metastable cycle states can occur and can sometimes be very long-lived.

The difference between bounded and unbounded noise also manifests itself in statistical mechanics. Even in a 0d statistical mechanical system bounded noise need not destroy long-range order. The situation for 0d maps has an analog in the motion of a classical particle in a double-well potential (We are considering a particle in continuous space and time). The two potential minima correspond to the two stable fixed points of $x^2$ and the local maximum of the potential, to the unstable fixed point. Unbounded noise can overcome the potential barrier and knock the particle from one potential well to the other. If the wells are symmetric, the average position of the particle will be zero. However, if the noise is bounded and the width is small (e.g. smaller than the magnitude of largest gradient in the potential well) the particle cannot escape from one well and go to the other. In this case, we have broken symmetry even in zero dimension! This is, of course, not in contradiction with the usual statement that 'there is no symmetry breaking or no non-ergodicity in $d=1$ at finite temperature', since, unlike statistical mechanical systems,
the bounded noise systems are not ergodic to begin with (i.e., not every state can reach every other state with non-zero probability). So, broken symmetry has been built into the bounded noise system.

3.4 Fluctuations for \( r > r_c \)

Now we analyze the parameter range \( r > r_c \) which is the chaotic regime of the 0d map.

A. Linear Stability Analysis

We investigate spatially uniform solutions and demonstrate that the uniform chaotic state obtained in MFT is linearly unstable with respect to spatial fluctuations. Consider a MF chaotic trajectory \( \{ \psi_n \} \) and let \( \delta \psi_n(j) = \psi_n(j) - \psi_n \) be the deviation of \( \psi_n(j) \) from the uniform chaotic solution \( \psi_n \). Given a small perturbation \( \delta \psi_0(q) \) with wavevector \( q \) around \( \psi_0 \) at time \( n = 0 \), the deviation in q-space after \( n+1 \) time steps can be calculated by iterating Eq. (3.3.2). To linear order we have

\[
\langle \delta \psi_{n+1}(q) \rangle = \sum_{l=0}^{n} f'(\psi_1^n) a_l(q) \delta \psi_0(q). \tag{3.4.1}
\]

When \( n \) is large

\[
\langle \delta \psi_{n+1}(q) \rangle \sim e^{n(\lambda-|z_n|a(q))} \delta \psi_0(q). \tag{3.4.2}
\]

For any given uniform chaotic state characterized by a Liapunov exponent \( \lambda > 0 \), there exists a \( q_0 \) such that for all perturbations of sufficiently long wavelength \( 0 < q < q_0 \), \( \langle \delta \psi(q) \rangle \) will grow exponentially with time destroying the spatial uniformity of the chaotic state. Random initial conditions contain all Fourier components
and therefore, the uniform chaotic state is unstable, even with \( \sigma = 0 \) for all except special initial conditions.

For \( \sigma > 0 \), the mean square deviation for large \( n \) is given by

\[
\langle |\delta \psi_{n+1}(q)|^2 \rangle \sim \sigma^2 e^{2\lambda n} a^{2n}(q) \quad (3.4.3)
\]

Upon integrating over all \( q \), we have,

\[
\langle \delta \psi_{n+1}^2(1) \rangle \sim \sigma^2 e^{2\lambda n} \int \frac{dq}{(2\pi)^d} a^{2n}(q) \quad (3.4.4)
\]

For finite \( d \), as \( n \to \infty \), the integral over \( q \) can be performed by saddle-point integration and shown to decrease no faster than a power of \( n \). So even for infinitesimal \( \sigma \), \( \langle \delta \psi_n^2(1) \rangle \) will grow exponentially for large \( n \), due to the positive Liapunov exponent. Thus even though the noise-averaged value \( \langle \psi_n(i) \rangle \) is independent of \( i \), the mean square deviation from the MF solution at each site can become arbitrarily large. For any member of the noise ensemble the spread of \( \psi_n(i) \) is not \( O(\sigma) \) as in the case \( r < r_c \) but is much larger. We conclude therefore that the uniform chaotic state is unstable with respect to noise and spatial fluctuations.

The physical interpretation of this instability is clear. The extreme sensitivity of the chaotic regime means that even tiny, noise-induced spatial non-uniformities will grow exponentially with time, completely destroying, at long times, the spatial coherence.

B. General Argument for Non-existence of Coherent Chaos

We have just seen that spatially uniform chaotic states cannot survive the inclusion of fluctuations because of their sensitivity to initial conditions. Now we will argue heuristically that this
sensitivity also precludes any state, including those that are spatially non-uniform, that exhibits coherent chaos. By coherent chaos we have in mind chaotic behavior of any Fourier mode that is macroscopically occupied, i.e., if \( \psi_n(q) = \sum \omega_n(1) e^{ig \cdot r} \) then \( \psi_n(q) \) is \( O(N) \) (not \( O(\sqrt{N}) \)) and is a chaotic function of \( n \) with excursions also of \( O(N) \). Of course any extended mode not necessarily a Fourier mode is equally satisfactory as a candidate for coherent chaos.

As a prelude to our argument we make two observations: First notice that in order to have coherent chaos the system must have long-range spatial coherence. If the correlations among \( \psi_n(i) \)'s at different sites extend only over some finite distance, \( \xi \), the spatial average over large number (\( \rightarrow \infty \)) of uncorrelated regions of size \( \xi^d \) cannot give rise to a temporally chaotic state. (This will become transparent later). Second if the system exhibits coherent chaos, any local variable such as \( \psi_n(i) \) must also exhibit chaotic motion. Now, we will argue that the long-ranged spatial coherence and local chaotic motion are incompatible with each other and therefore, coherent chaos cannot exist.

Suppose the system exhibits local chaotic motion. When \( \sigma = 0 \), imagine at time \( n = 0 \) the variable at site \( i \) is perturbed by a very small amount \( \delta \psi_0(i) \). Since the system is locally chaotic, \( \delta \psi_n(i) \) grows like \( e^{\lambda n} \delta \psi_0(i) \) where \( \lambda > 0 \) is the appropriate Liapunov exponent. When, roughly speaking, \( \delta \psi_n(i) \sim 1 \) (or the bandwidth, see end of the section), i.e., when \( n \geq n^* \equiv -\frac{1}{\lambda} \ln|\delta \psi_0(i)| \), \( \psi_n(i) \) has lost all memory of the value it would have achieved in the absence of the perturbation. Now consider any other site \( j \). Since for near-neighbor
coupled maps information cannot be transmitted from site to site with 
speed greater than unity, \(j\) evolves in ignorance of the fact that \(i\) 
was perturbed until time \(n=|i-j|\), when it first learns of the pertur-
bation. If \(|i-j|>n^*\), then by the time \(j\) receives this information, \(i\) 
is completely decorrelated from the value it would have had in the 
absence of the perturbation. Thus \(i\) and \(j\) have been decorrelated by 
the perturbation.

When \(\sigma>0\), similar perturbations induced by the noise occur at 
every instant, constantly destroying the assumed correlation between \(i\) 
and \(j\). Roughly speaking, the maximum distance over which the variables 
can maintain their spatial coherence is \(n^*\), with \(\delta \psi_0(1)\sim \sigma\) and 
\(\xi \leq -\ln \sigma / \lambda\). Even for \(\sigma=0\), however, random initial conditions are a source of 
decorrelating perturbations which prevent the system from establish-
ing correlations over lengths exceeding \(-\ln \delta \psi_0(1) / \lambda\), where \(\delta \psi_0(1)\) is a 
measure of the randomness in the initial condition. Thus long-range 
coherence cannot exist in a locally chaotic system except for some 
special initial conditions and \(\sigma=0\). Therefore, we have argued the 
incompatibility of local chaotic behavior and spatial coherence and 
hence, coherent chaotic states cannot occur.

What then becomes of the MF chaotic state in the presence of 
fluctuations? One's first guess is that the chaotic meanderings of 
the variables in the various uncorrelated regions averages the 
temporal structure away, producing, at a macroscopic level, a 
stationary state (1-cycle). In general, however, the situation can be 
more complicated. The chaotic attractor can consist of many bands. 
Starting from random initial conditions, the many-body system will,
therefore, for d>1, organize itself so that the bands are visited in synchrony everywhere on the lattice. Within each band, however, the previous arguments imply that the many-body system 'self-averages', producing an n-independent value of \( \frac{1}{N} \sum \psi_n(i) \) (or \( \langle \psi_n(i) \rangle \)). Thus for \( r>r_c \) the many-body system with d=1 does not become macroscopically chaotic but remains periodic, undergoing an inverse bifurcation sequence which produces a stationary state at large \( r \). As in \( r<r_c \) for a fixed \( c \), there is a maximum period \( 2^m(\sigma) \) the system can achieve. Also, for fixed \( r \), as \( \sigma \) is increased, when \( \sigma \) becomes comparable to the smallest gap between bands the system makes a transition from a \( 2^m \)-cycle to a \( 2^{m-1} \) cycle. This continues, until eventually the system goes to a stationary state (1-cycle) for very large \( \sigma \). Now it is easy to explain the observation regarding the non-occurrence of chaotic behavior for a state without spatial coherence. If the mean value is to execute chaotic motion, one needs an infinity of bands which are visited aperiodically. Since, the smallest gap between bands has to be greater than \( O(\sigma) \) and the range of \( \psi \) is bounded this is clearly impossible. The most one can have are a finite number of bands leading to periodic behavior for the spatially averaged value. We note in passing that even for \( \sigma=0 \) the intersite coupling changes the values of \( r \) at which the inverse bifurcations occur from the values where the bands merge in the Od map! Also, as in \( r<r_c \), in d=1 only stationary state (1-cycle) can occur for Gaussian noise.

What happens to those periodic windows of stable states which occur in the chaotic regime of the Od map? Equations (3.3.2) and (3.3.3) show that all the MF stable cycles are linearly stable in the
noisy coupled map system. However, for \( r > r_c \) the primitive cycles arise from tangent bifurcation. In contrast to pitchfork bifurcations, after a tangent bifurcation the intervals around different fixed points which form a subset of the respective basins of attractions, are not adjacent to each other. More importantly, the gaps between the intervals consist of the different domains of attraction intricately intermingled so that points very close to each other belong to different basins. For example, consider an \( r \) such that the Od map has a stable 3-cycle. Let the three stable fixed points of \( f^3 \) be denoted by \( \psi_1, \psi_2, \) and \( \psi_3 \) and the corresponding intervals around them by \( I_1, I_2, \) and \( I_3. \) These intervals are separated by gaps which form a "strange repeller" in that the Od system executes "transient chaos" before settling to one of the stable fixed points.\(^7\) This is related to the existence of an unstable 3-cycle in the gaps. Now consider the coupled lattice map system for such an \( r \) and \( \sigma = 0. \) Consider a domain of \( \psi_1 \) inside a background of \( \psi_2 \) at time \( n. \) The future of a boundary spin \( \psi_{n+1}(i) \) is determined by \( \psi_{n}(i) = \frac{1}{Z} \sum_{j \in \psi_{n}} \psi_{n}(j) \) which falls in the gap between \( I_1 \) and \( I_2 \) and thus \( \psi_{n+1}(i) \) wanders randomly for a long time. This influences its neighbors which in turn execute transient chaotic motion. For the domain-wall to be stable one has to have coherent chaotic motion which we have argued is impossible. Thus the system loses spatial coherence and the 3-cycle state is destroyed. What occurs instead is that the spatial average executes a 2-cycle - this has been observed numerically. This phenomenon is more general and the window cycles are all replaced by spatially non-uniform \( 2^m \)-cycles for \( \sigma > 0. \) For \( \sigma = 0 \) there is multistability and for special
initial conditions as we saw earlier window cycles can occur. These results permit us to complete the phase diagram in Fig. 2.

3.5 **Numerical Results**

We have verified the physical arguments outlined in Section 3.3 and 3.4 by extensive simulations in 1d and 2d for both \( r < r_c \) and \( r > r_c \). The numerical results are in good agreement with our physical picture. Here we present some of the results for 2d coupled quadratic maps. For comparison, we also include corresponding 0d (i.e., MF with uniform initial condition) result. Since for \( d > 1 \) the form of the noise distribution is not important, the noise used in the simulations is uniformly distributed in the interval \( [-\frac{\sigma}{2}, \frac{\sigma}{2}] \). Simulations with Gaussian noise show similar results. For \( \sigma > 0 \) one must modify \( f(x) \) for large \( x \) to keep \( \psi_n(i) \) roughly confined to the unit interval. Any \( f \) with \( f(\langle 0,1 \rangle) \) for all \( x \) outside \( (0,1) \) accomplishes this. We have arbitrarily mapped \( x > 1 \) onto \( 0.9995 \) and \( x < 0 \) onto \( 0.0005 \). Our results are insensitive to this choice. All the simulations shown here were performed with random initial conditions and all the histograms include 2048 data points.

\( r < r_c \). Fig 3 shows the histogram of a 0d map at \( r = 3.46 \) and \( \sigma = 0 \). It exhibits sharp 4-cycles. Fig. 4 is the histogram of the average value \( \frac{1}{N} \sum_{i} \psi_n(i) \) for 2d coupled quadratic maps (see Eq. (3.2.1)) at the same \( r \) and \( \sigma = 0.01 \) for a 100x100 lattice. Again the graph shows sharp 4-cycles. The width of the spatial distribution at a fixed time is not zero but is of the order of \( \sigma \). This means that the uniform MF 4-cycles is indeed stable for small noise as was demonstrated in the
Figure 3  Histogram for a 0d map at $r=3.46$ and $\sigma=0$. It exhibits a sharp 4-cycle.
Figure 4  Histogram for a 100 x 100 lattice at r=3.46 and \( \sigma=0.01 \). The sharp 4-cycle persists for small noise.
previous sections. Fig. 5 shows the histogram for the same size lattice and the same r but with larger noise levels, $\sigma=0.12$. Now $\sigma$ is comparable to the smallest spacing between the cycles, hence, the distinction between these cycles blurs, and we expect a transition a from 4-cycle to a 2-cycle. We indeed observe 2 instead of 4-cycles in this figure!

$R > R_c$. Fig. 6 shows the histogram of a 0d map at $r=3.572$ and $\sigma=0$. It consists of 8 chaotic bands. Fig. 7 and Fig. 8 are corresponding histograms for $\sigma=0.01$ and lattices of size 20x20 and 200x200 respectively. We see 8 noisy cycles in Fig. 7. As the lattice size increases N the width of the peaks decreases like $N^{-1/2}$, and we get 8 sharp spikes in Fig. 8. The corresponding power spectra are shown in Fig. 9 and Fig. 10. Comparing these two figures, we see that as the size of the system increases the level of the background noise in the power spectrum decreases. So, the numerical evidence supports our claim that in the $N \to \infty$ limit, the coupled lattice map system does not execute chaotic motion on the average but remains periodic. Again, when we increase the noise, the system makes a transition from $2^m$-cycle to $2^{m-1}$-cycle. Fig. 11 and Fig. 12 show the histograms at the same $r=3.572$ value with $\sigma=0.06$ and $\sigma=0.12$ respectively for a 100x100 lattice. We see as we increase $\sigma$, the system goes from an 8-cycle to a 4-cycle and then to a 2-cycle.

Even in the noiseless case, spatial-average does not display chaotic motion as long as we start with random initial conditions. For example, Fig. 14 shows the histogram for a 100x100 lattice with random initial condition and $r=3.8$, $\sigma=0$, where the 0d map exhibits
Figure 5  Histogram for a 100 x 100 lattice at $r=3.46$ and $\sigma=0.12$. The system exhibits a 2-cycle. (cf. Fig. 4)
Figure 6  Histogram for a 0d map at $r=3.572$ and $s=0$. It shows 8 chaotic bands.
Figure 7  Histogram for a 20 x 20 lattice at $r=3.572$ and $\sigma=0.01$. It shows a noisy 8-cycle.
Figure 8  Histogram for a 200 x 200 lattice at $r=3.572$ and $\sigma=0.01$.  It shows 8 sharp spikes in contrast to the noisy 8-cycle in Fig. 7.
Figure 9  Power spectrum for a 20 x 20 lattice at $r=3.572$ and $\sigma=0.01$. It shows a noisy 8-cycle.
Figure 10  Power spectrum for a 200 x 200 lattice at r=3.572 and 
σ=0.01. The background noise decreases as the system 
size increases. (cf. 20 x 20 lattice results in Fig. 9)
Figure 11 Histogram for a 100 x 100 lattice at $r=3.572$ and $\sigma=0.06$. It shows 4 sharp spikes.
Figure 12  Histogram for a 100 x 100 lattice at r=3.572 and σ=0.12. It shows 2 sharp spikes. (cf. Fig. 11)
Figure 13  Histogram for a 0d map at r=3.8 and σ=0. It contains one chaotic band.
Figure 14  Histogram for a 100 x 100 lattice with a random initial condition at $r=3.8$ and $\sigma=0$. It shows 2 spikes.
a chaotic band (Fig. 13). Again, the widths of the peaks in Fig. 14 diminish like $N^{-1/2}$ as we go to larger and larger $N$. Observe that, the correspondence between $2^M$-chaotic bands in Oa and $2^M$-cycles in the many-body system is not always true; this is presumably because the intersite couplings and the noise change the effective value of the control parameter $r$. Fig. 15 and Fig. 16 show the spatial distributions of the system studied in Fig. 14 for two consecutive times. We see that the distributions are not well separated. This is in contrast to the situation for 2- and 4- cycles etc. for $r<r_c$. At $r=3.83187$ the Oa system has a 3-cycle (Fig. 17), however this 3-cycle is unstable. Fig. 18 shows that a $100 \times 100$ lattice at the same $r$ value and with $\epsilon=0.01$ displays a 2-cycle. This 2-cycle has a very broad spatial distribution.

Before we conclude this section, we consider DCA which are known to exhibit chaotic patterns and discuss how our arguments for the non-existence of macroscopic chaos apply in that context.

Based on extensive numerical simulations Wolfram has classified all one-dimensional DCA into four distinct universality classes:

1) Evolution leads to a homogeneous state,
2) Evolution leads to periodic structures,
3) Evolution leads to a chaotic pattern,
4) Evolution leads to complex structures, sometimes long-lived.

The distinction between Class 3 and Class 4 is conjectured to be as follows: while random initial conditions lead to chaotic behavior, the value of a given site in a Class 3 CA may be determined by a
Figure 15 Histogram of the spatial distribution at time $n$ for a 100 x 100 lattice at $r=3.8$ and $\sigma=0$. 
$r = 3.8$
$\sigma = 0$
$N = 10^4$

Figure 16 Histogram of the spatial distribution at time $n+1$ for a $100 \times 100$ lattice at $r=3.8$ and $\sigma=0$. (cf. Fig. 15)
Figure 17  Histogram for a Od map at $r=3.83187$ and $\sigma=0$. It exhibits a 3-cycle.
Figure 18. Histogram for a 100 x 100 lattice at \( r = 3.83187 \) and \( \sigma = 0.01 \). It shows 2 spikes.
simple algorithm. In contrast, a particular site value for a Class 4 CA may only be determined by an algorithm equivalent in complexity to explicit simulation. In particular, Class 4 CA are thought to be capable of universal computation.

On the basis of our general arguments we do not expect Class 3 or Class 4 DCA to exhibit macroscopic chaos. We have tested this expectation by numerical simulations on 1d DCA. The rules considered by Wolfram are as follows. Let \( \sigma_i(t) \) denote the value (0 or 1) of the 'spin' at site \( i \) at time \( t \). The rule is totalistic, i.e.,

\[
\sigma_i(t+1) = f(\sum_{j \in \mathcal{N}_i} \sigma_j(t)) \equiv f(\overline{\sigma}_i(t))
\]

(3.5.1)

where the neighborhood \( \mathcal{N}_i = \{i-2,i-1,i,1,i+1,i+2\} \), and the range of \( f \) is obviously \( \{0,1\} \). Since \( \overline{\sigma}_i \) can only take on values 0,1,...,5 the rule is specified by fixing the 6 values \( f(0), f(1), \ldots, f(5) \).

Wolfram labels a rule by the following numerical code. The number \( \sum_{j=0}^{5} 2^j f(j) \) is associated with a given rule. Rules 20 and 52 belong to Class 4 while Rules 2, 10, 12, 14, 18, 22, 26, 28, 30, 34, 38, 42, 44, 46, 50 are expected to belong in Class 3. In Fig. 19 we display the histogram of the spatial average \( \frac{1}{N} \sum_{i=1}^{N} \sigma_i(t) \) for \( N=2000 \) sites, for rule 26 for a run consisting of 2048 time steps with random initial conditions. Note that we see a narrow band. Upon increasing the size \( N \), one finds that the width of the 1-cycle peak decreases, consistent with the expected \( N^{-\frac{1}{2}} \) behavior. Fig. 20 shows the corresponding histogram with \( N=20,000 \). Similar results hold for other rules both in Class 3 and 4. Thus, on the average, in the thermodynamic limit, these DCA do not appear to exhibit chaotic behavior.
Figure 19  Histogram for the average value of a 1d DCA (Rule 26) with 2 x 10^3 sites.
Figure 20  Histogram for the average value of a 1d DCA (Rule 26) with $2 \times 10^4$ sites. (cf. Fig. 19)
3.6 **The Phase Diagram and the Nature of Phase Transitions**

We have assembled the phase diagram from the qualitative analysis of Secs. 3.2–3.4 and verified it by numerical simulations whose results were presented in the previous section. In this section we discuss multistability, the occurrence of local chaos, and establish the nature of the phase transitions between $2^{m-1}$- to $2^m$-cycle states. Our first remark concerns multistability: For $\sigma=0$ the many-body system is often multistable i.e., different initial conditions produce distinct stable states. (Only the most robust of these are shown in Fig. 2.) For example, let the value of $r$ be such that the 0-dimensional map has a stable 3-cycle. If we start with nearly uniform initial conditions, the system will actually go to a uniform 3-cycle which corresponds to that of the 0d map. However, starting from more random initial conditions, we always found one of the spatially non-uniform 2-cycles. In general $2^m$-cycles are found to be more stable than the "window" cycles; i.e., a flat domain wall separating these two states moves preferentially to eat up the "window" cycle. Thus for any $\sigma>0$ the "window" cycles become metastable and only the $2^m$-cycles are stable.\(^{11}\) This absence of multistability for $\sigma>0$ is general: One expects\(^ {12}\) that for local, spatially-symmetric interactions, two states can be equally stable only by accident or symmetry. Since any Gaussian noise eventually moves the system into the most stable state, multistability is prohibited at generic points of parameter space. This does not of course mean that multistability will not be seen numerically or experimentally for small noise. Note that we have not addressed the question of the time required to go from a metastable state to the stable state!
Our second remark concerns the occurrence of local chaos. Though there is no macroscopic chaotic state, there is, just as in 0d, a sharp (dashed) line in Fig. 2 separating chaotic from periodic variation of the individual variables. This line can be defined as the place where a Liapunov exponent, $\lambda$, computed, e.g., from the sensitivity to initial conditions of $\psi_n(1)$ for a particular $i$, changes sign.

For $\sigma > 0$, $\lambda$ can presumably be defined in terms of the noise-averaged quantity $\langle |\delta \psi_n(1)|^2 \rangle$ where $\delta \psi_n(i)$ is the change in $\psi_n(i)$ resulting from a tiny perturbation, $\delta \psi_0(i)$, at $n=0$. It cannot be computed from $\langle \delta \psi_n(1) \rangle$, as was suggested by Shraiman et al. for 0d, since this will always produce a $\lambda < 0$. There is no collective effect, however, so, for any $d$, $\lambda$ should go through zero in a way governed by an effective 0d map. For $\sigma > 0$, therefore, $\lambda$ should change sign analytically as a function of $r$, given that no 0d map can have singularities for $\sigma > 0$; (this would be tantamount to a 1d Ising model having a phase transition at finite temperature).

We now discuss the nature of the phase transitions in Fig. 2. Recall that successive period-doubling bifurcations break the discrete time-translation symmetry of Eq. (3.1.1) progressively. Consider the transition between 1- and 2-cycles. Note that the 2-cycle is only invariant under time translation of 2 time steps.

First consider the noiseless ($\sigma=0$) case. Here MFT obtains and we have the spatially homogeneous solutions corresponding to the 0d map for $r < r_C$. We expand $f^2(\psi_n)$ around the 1-cycle fixed point value $\psi^*$ in powers of the deviation $\delta \psi_n \equiv \psi_n - \psi^*$ and obtain

$$\delta \psi_{n+2} = a_1 \delta \psi_n + a_2 \delta \psi_n^2 + a_3 \delta \psi_n^3 + \ldots \tag{3.6.1}$$
where for quadratic $f$

$$
a_1 = [f'(\psi^*)]^2
$$

$$
a_2 = \frac{1}{2} f''(\psi^*) f'(\psi^*) [1 + f'(\psi^*)]
$$

$$
a_3 = \frac{1}{2} [f''(\psi^*)]^2 f'(\psi^*) .
$$

The appearance of non-zero fixed points of Eq. (3.6.1) signifies new fixed points of $f^2$, i.e., a transition from one-cycle to two-cycles. It is clear that the critical point occurs at $f'(\psi^*) = 1$ and precisely at this point both $a_1$ and $a_2$ vanish. Note that (3.6.1) is simply a discretized version of the MF time-dependent Ginzburg-Landau (TDGL) equation. The simultaneous vanishing of $a_1$ and $a_2$ is precisely what occurs at the liquid-gas critical point which is in the Ising-universality class. Hence, the exponents assume their corresponding mean-field values.

For non-zero noise($\sigma \neq 0$) we expand $f^2$ around $\psi^*$ where $\psi^* = \lim_{n \to \infty} \psi_n(i)$ is the uniform, noise-averaged 1-cycle value. Note that $\psi^*$ will not be equal to $f(\psi^*)$, in general. Let $\phi_n(i) = \psi_n(i) - \psi^*$. The expansion yields

$$
\phi_{n+2}(1) = a + b \phi_n(1) + c \nabla^2 \phi_n(1) + d\phi_n^2(1)
$$

$$
+ e\phi_n^3(1) + g[\phi_n(1)] \zeta_n(1) + \tilde{g} \zeta_n^2(1)
$$

$$
+ \ldots
$$

(3.6.3)

where $\nabla$ is a lattice gradient and the ellipses indicate higher-order terms in either powers of $\phi$ or additional lattice derivatives. The form $g$ is given by

$$
g[\phi_n(1)] = g_0 + g_1 \phi_n(1) + g_2 \phi_n^2(1) + \ldots$$

with $g_0 \neq 0$. 

[58]
The constant term $a$ (given by $a = (1 + f') \Delta + \frac{1}{\xi} f'' \Delta^2 + \ldots$ where $\Delta = f(\psi^n) - \psi^n$) can be ignored since order by order in perturbation theory $\psi^n$ is chosen so that $\phi_n \to 0$ for all $n$.

The application of standard renormalization group techniques to determine the nature of the phase transition is facilitated by taking the continuum limit in space and time and performing a Fourier transform to $(q, \omega)$ variables. The corresponding time-dependent Ginzburg–Landau (TDGL) equation assumes the following form in more conventional notation:

$$-i\omega \Phi(q, \omega) = -\Gamma[(r + q^2)\Phi(q, \omega) + v\Phi^2 + u\Phi^3] + \left[1 + \Gamma_1 \Phi + \Gamma_2 \Phi^2\right] \tilde{\zeta} + \tilde{\zeta}^2 + \ldots \quad (3.6.4)$$

Here we have written the terms schematically, e.g.

$$\Phi = \int \frac{dq_1}{(2\pi)^d} \int \frac{d\omega_1}{2\pi} \Phi(q_1, \omega_1) \zeta(q - q_1, \omega - \omega_1)$$

The noise $\zeta$ has the usual correlation function $\langle \zeta(q_1, \omega_1) \zeta(q_2, \omega_2) \rangle = 2\Gamma_1(q_1 + q_2)\delta(\omega_1 + \omega_2)$. As usual, we have changed to a spherical Brillouin zone with unit radius for convenience. Equation (3.6.4) has the form of a TDGL equation for the liquid-gas phase transition except for the additional terms $\Gamma_1, \Gamma_2, \tilde{\zeta}$ etc. In order to establish the irrelevance of these terms around the liquid-gas critical point we follow the usual procedure: we eliminate the short wavelength modes $\Phi(q, \omega)$ with $b^{-1} |q| < 1$ where $b > 1$ is the scale factor. We then rescale the variables as follows:
\[ q \rightarrow q' = bq \quad \omega \rightarrow \omega' = b^2 \omega \]

\[ \phi(q,\omega) \rightarrow \phi(q',\omega') = \xi^{-1} \phi(q,\omega) \quad (3.6.5) \]

\[ \zeta(q,\omega) \rightarrow \zeta'(q',\omega') = Z^{-1} \zeta(q,\omega) . \]

We shall need to adjust \( Z \) and \( \xi \) so that the RG transformation will have a fixed point. The primed equation for the long-wavelength modes reads

\[ -i\omega' \phi'(q',\omega') = -\Gamma b^{2-2}[(b^2 r^2 + q'^2)\phi'(q',\omega') + b^2 \phi'^2 \]

\[ + \xi b^2 \phi^3] + \xi^{-1} b^2 Z[1+\Gamma_1 \xi \phi' + \xi^2 \Gamma_2 \phi'^2] \zeta' \]

\[ + \Gamma_2 b^2 \xi^{-1} b^2 \zeta'^2 . \quad (3.6.6) \]

Now the renormalized parameters can be identified e.g.,

\[ \Gamma' = b^{2-2} \Gamma . \]

If we demand that the form of the equation remains the same (coefficient of \( \xi' \) is 1), we see that \( Z = \xi b^{-2} \). Requiring that

\[ \langle \zeta'(q',\omega') \zeta'(q',\omega') \rangle = 2 \Gamma' \delta(q' + q') \delta(\omega' + \omega') \]

leads to

\[ \xi = b^{d/2 + 2} \quad \text{and} \quad Z = b^{d/2 + 1} . \quad (3.6.7) \]

It is trivial to check the naive scaling dimensions of the extra terms:

\[ u' = u \xi^{2b-2(d+z)+2} = b^{4-d} u , \]

\[ \Gamma'_2 = \Gamma_2 \xi^{2b-2d-2z} = \Gamma_2 b^{-d+2} , \]

\[ \Gamma'_1 = \Gamma_1 \xi^{b-d-z} = \Gamma_1 b^{-(d/2-1)} . \quad (3.6.8) \]

Hence, \( \Gamma_2 \) and \( \Gamma_2 \) are irrelevant for \( d>2 \). Similarly, we can show that \( \tilde{\Gamma} \) is irrelevant for all \( d \).
Therefore, we conclude that the period doubling transitions (so long as they remain continuous) fall into the same universality class as the Ising model. 81

3.7 Continuous-Time Systems

Our arguments for the non-existence of macroscopic chaotic states in the previous sections are based on general physical considerations and therefore, should hold for systems which evolve in continuous time. In this section, we show explicitly that the uniform chaotic state is unstable with respect to fluctuations. We also establish that sufficiently strong long-ranged couplings can stabilize the uniform chaotic state.

Consider an $n$-component vector $\tilde{\psi}(i,t)$ at each site $i$ of a $d$-dimensional lattice. The components may be viewed as the concentrations of $n$ species participating in a chemical reaction. 82 In the absence of noise, the time evolution of these dynamical variables is described by a coupled set of deterministic differential equations:

$$ \frac{d\tilde{\psi}(i,t)}{dt} = \tilde{\dot{\psi}}(i,t) = \tilde{\mathbf{f}}(\{\tilde{\psi}(j,t)\}) \quad (3.7.1) $$

where $\tilde{\mathbf{f}}$ is a vector-valued function. If $\tilde{\psi}(i,t)$ is independent of $i$, $\tilde{\psi}(i,t) = \tilde{\psi}^{(0)}(t)$ (e.g., the chemical concentrations are uniform in space) and the infinite set of equations in (3.7.1) reduces to $n$ coupled ordinary differential equations:

$$ \tilde{\dot{\psi}}^{(0)}(t) = \tilde{\mathbf{f}}(\tilde{\psi}^{(0)}(t)) \quad (3.7.2) $$

Since these are non-linear differential equations as the control
parameters in $\dot{f}$ (e.g., the flow rate in the chemical reaction) are varied one or more of the components $\psi_0^{(0)}$ can display chaotic behavior. More precisely, a small perturbation in the initial condition $\delta \psi^{(0)}(0)$ causes increasingly large deviations $\delta \psi^{(0)}(t) = \dot{\psi}(t) - \psi^{(0)}(t)$ for large $t$. The deviation obeys the linear equation

$$\dot{\delta \psi^{(0)}}(t) = L(t) \delta \psi^{(0)}(t),$$

(3.7.3)

where $L(t)$ is a matrix whose elements are given by

$$L_{ab}(t) = \frac{\partial f_a}{\partial \psi_b} \bigg|_{\psi^{(0)}(t)}.$$ 

The formal solution to (3.7.3) can be written in terms of the time-ordered product:

$$\delta \psi^{(0)}(t) = T \exp \left\{ \int_0^t L(t') dt' \right\} \delta \psi^{(0)}(0).$$

(3.7.4)

If the system is chaotic, for large $t$, $\delta \psi^{(0)}(t)$ will diverge exponentially with an exponent $\lambda$ defined below: 77

$$\delta \psi^{(0)}(t) \sim e^{\lambda t} \delta \psi^{(0)}(0)$$

(3.7.5)

Let $\Lambda = \lim_{t \to \infty} \frac{1}{t} \log \| L(t) L(t) \|$. The logarithms of the eigenvalues of $\Lambda$ are the Liapunov exponents $\lambda > \lambda' > \lambda'' > \ldots$.

In the presence of noise, the system is no longer uniform. Linearizing Eq. (3.7.1) around the uniform solution $\dot{\psi}^{(0)}(t)$ yields for $\delta \psi^{(1,t)} = \psi^{(1,t)} - \psi^{(0)}(t)$,

$$\dot{\delta \psi}^{(1,t)} = L(t) \frac{1}{2} \sum_{j \in v_1} \delta \psi^{(j,t)} + \delta \psi^{(1,t)}$$

(3.7.6)

where for simplicity we have assumed that $f$ depends on the sum of the neighbors ($v_1$ is the neighborhood of $i$ as before). $\delta \psi^{(1,t)}$ is a noise variable with $\langle \delta \psi^{(1,t)} \rangle = 0$ and $\langle \delta \psi^{(1,t)} \delta \psi^{(1,t')} \rangle = \sigma^2 \delta(t-t')$. In Fourier space, (3.7.6) becomes
\[ \dot{\delta\psi}(q,t) = L(t)\dot{a}(q)\delta\psi(q,t) + \dot{\zeta}(q,t) \]  

and for near-neighbor couplings

\[ a(q) = [1 + 2 \sum_{l=1}^{d} \cos q_l ]/(1+2d) \]  

(3.7.8)

The solution of Eq. (3.7.7) can be formally represented as before by

\[ \delta\psi(q,t) = T e^{0} \dot{\delta\psi}(q,0) \]

\[ + \int_{0}^{t} dt' T e^{t-t'} \int_{0}^{t} dt'' L(t') a(q) dt'' \zeta(t'') \]  

(3.7.9)

Upon taking noise averages and using (3.7.5) we expect, heuristically,

\[ \langle \delta\psi(q,t) \rangle \sim e^{a(q)\lambda t} \langle \delta\psi(q,0) \rangle \]  

(3.7.10)

Since for q sufficiently small a(q)>0 long-wavelength perturbations will grow exponentially destroying the spatial coherence and therefore, render the uniform chaotic state unstable. Note that for any short-ranged interaction, a(q) ->1 as q->0 and hence, the above conclusion remains valid.

We now argue that if we suppress spatial fluctuations by adding long-ranged forces the uniform chaotic state can be made stable.

Consider adding the term \(-c(\psi(1,t)-\bar{\psi}(t))\) to Eq. (3.7.1) where \(\bar{\psi}(t) = \frac{1}{N} \sum_{i=1}^{N} \psi(i,t)\) is the spatial average value. This is an infinite-ranged interaction which forces the local value of \(\psi(i)\) to be equal to the spatial average. Linear stability analysis can be performed as before obtaining an additional term on the RHS of Eq. (3.7.7) of the form \(-c(1-\dot{a}_{q,0})\dot{\delta\psi}(q,t)\). It is easy to deduce that for any \(\dot{q} \neq 0\)
\( (\delta \psi(q,t)) = e^{(\Lambda a(q) - \epsilon)t} (\delta \psi(q,0)). \)  \hspace{1cm} (3.7.11)

Clearly there is a critical \( \epsilon_c \), \( \epsilon_c > \lambda \) such that for \( \epsilon > \epsilon_c \) all fluctuations with \( \delta \neq 0 \) decay in time and the uniform chaotic state is stable. Our arguments are far from rigorous but we believe they are convincing.

3.8 Concluding Remarks

We first comment on the relation between the results described in this chapter for coupled lattice maps and those for related systems, viz., PCA, PDE's which can exhibit, in the MF (noiseless) limit, period-doubling bifurcations and transitions to chaos. PCA differ from the system studied in this chapter primarily in that the variables are discrete rather than continuous. Just as for equilibrium Ising models where the continuous Ginzburg-Landau-Wilson representation is believed to produce topologies and critical properties of phase diagrams identical to those of the discrete spin Ising model, it seems reasonable that all our results apply to non-stationary states of PCA. In particular, period-doubling transitions, which can occur for any \( d > 1 \) belong in the Ising universality class. States wherein the average value varies chaotically in time do not survive beyond the MF approximation for any \( d < \omega \). This explains our failure to observe chaotic states. We also saw only periods 2, 3, and \( 4^k \). It is clearly easier to produce long period states in coupled maps than in PCA, since in the former the noise and the control parameter are manifestly independent, so it is possible to add tiny amounts of noise to any regime of the noiseless system. In contrast, for PCA, changing the probabilities alters both (noise and the control parameter)
simultaneously so that it is difficult to control. Also the discrete-
valuedness takes the system far away from MF considerations! As the
number of states increases it is easier to produce long-period states.

We also expect that PDE's (except for \( \sigma = 0 \) and special initial
conditions) cannot support macroscopically-chaotic states, i.e.,
states in which spatially extended modes are macroscopically occupied
and vary chaotically in time. Also for fixed noise, there is a maximum
allowed period. We recall that Bishop and co-workers\(^{68}\) have studied
numerically the ac-driven, linearly-damped sine-Gordon system in \( d=1 \):

\[
\phi_{tt} - \phi_{xx} + \sin \phi = \Gamma \sin(\omega t) - c \phi_t .
\]  

(3.8.1)

For special initial conditions (e.g. the sine-Gordon breather profile
for \( \phi(x,0) \)) they observe chaotic behavior for the average value of \( \phi \).
While this by itself does not contradict our results, we predict that
these chaotic variations will diminish like \( N^{-\frac{1}{2}} \) with increasing
sample size, \( N \), for random initial conditions.

It is important to emphasize that the non-occurrence of chaotic
states is not specific to systems which display the period-doubling
route to chaos. The intrinsic instability of macroscopic chaotic
states should hold equally well for other routes to chaos.\(^{83,84}\)

We recall that there is no true (thermodynamic) phase transition
into a chaotic state, local variables can go from periodic to chaotic
behavior. So experiments which clearly distinguish between micro-
scopic and macroscopic measurements are needed. Theoretically, it
appears that extending our work to make contact with chemical reaction
experiments holds the greatest promise.
FOOTNOTES

a One can consider other equations of motion. For example,
\[ \psi_{n+1} = f(\psi_n) + \epsilon \sum_{\alpha} [g(\psi_{n+\alpha}) - g(\psi_{n})] + \zeta_n(1) \]
where \( \alpha \) are lattice vectors such that \( i+j \in \mathbb{Z} \) and \( g(x) \) is some smooth function e.g., \( x \) or \( f \) itself. The function \( f \) must be modified for large values of the argument so that the \( \psi_n \)'s are confined to \([0,1]\). We believe our general results should hold for such diffusively coupled maps. However details such as which states are the most robust for \( \sigma = 0 \) may be different.

b In Ref. 70 Shraiman, Wayne, and Martin (SWM) studied the effect of Gaussian noise on zero-dimensional systems which undergo period-doubling transitions to chaos. \((x_{n+1} = x_n(1-x_n) + \zeta_n)\). With the aid of functional integral techniques they derived an expression for the Liapunov exponent \( \lambda \) and by an elegant application of a "coarse-graining" procedure they derived a scaling form for \( \lambda = \lambda(r, \sigma) = (r_{so} - r)^{\phi}((r_{so} - r)^{t_{so}}) \). However, we disagree with their interpretation of this result. By our arguments, \( x_\sigma(y) = \langle x_n \rangle \) the noise-averaged value of \( x_n \) given the initial condition \( x_0 = y \) should approach a constant value independent of \( y \). Otherwise, we would have broken symmetry at finite temperatures in the corresponding one-dimensional equilibrium statistical mechanical model. Hence, the Liapunov exponent as defined by SWM should always be negative in the presence of noise. This contradicts the discussion before Eq. 19 of Ref. 70, which asserts that \( \lambda > 0 \) which in turn would imply symmetry-breaking. (Once more with emphasis: we believe that not only is chaotic behavior prohibited but so is any periodic behavior.) It appears to us that their scaling function \( \phi \) should not have a zero for \( \omega = 0 \). However, \( \lambda \) can be defined in terms of \( (\Delta x_n)^2 \) where \( \Delta x_n \) is the change in \( x_n \) resulting from a tiny perturbation \( \Delta x_0 \) at \( n = 0 \). Note that the results of Ref. 69, however, do not contradict our results. Their definition of the Liapunov exponent is different and it is rather obvious that it can be positive. Also these authors find finite cycles which is consistent even at \( d = 0 \) since their noise is bounded not Gaussian (cf. Sec. 3.3)
We point out that Walgraef et al. in Ref. 81 argued that the transition from the temporally uniform to the periodic state in the 'Brusselator' belongs in the universality class of XY, rather than the Ising, model. Note that they are dealing with a continuous time system and hence, a periodic state ($\psi(t) \sim \cos(\omega(t-t_0))$) for some $\omega$ has a broken continuous symmetry corresponding to the arbitrariness in the choice of $t_0$. Consequently, the order parameter, the Fourier transform $\psi(\omega)$, is complex (magnitude and phase) and has XY-like symmetry. The corresponding transition is absent in discrete time. We believe that period-doubling transitions in continuous time belong in the Ising universality class as do their discrete time counterparts.
CHAPTER IV
EXACT RESULTS FOR DETERMINISTIC CELLULAR AUTOMATA WITH ADDITIVE RULES

4.0 Introduction

Deterministic CA (DCA) are CA that evolve according to deterministic rules. DCA exhibit very rich and complex behavior, as observed, typically, in numerical simulations. In general, the exact determination of the temporal behavior of DCA is difficult, if not impossible. However, for a particular class of DCA, called additive\textsuperscript{25} (or linear) CA, on finite lattices the time evolution can be studied analytically. This chapter describes an algebraic analysis of such rules. However, before specializing our discussion to additive DCA, we introduce some general concepts.

The time evolution of finite DCA can be conveniently represented by state transition diagrams. A state transition diagram is a directed graph with the vertices of the graph corresponding to the states of the CA. The vertices are connected by directed edges which represent the transition between the CA states at each time step. (Note that the graph may be a union of disjoint subgraphs). Observe that since the CA rule is deterministic, each vertex has a unique successor. However, there may be zero, one, or more predecessors for each CA state. Since the number of CA states is finite the CA evolving from any initial state must eventually enter a cycle. Thus the transition diagram
consists of cycles (which may be points!) with trees which consist of transient states rooted on the vertices in the cycle. The state at the top of a tree which does not have a predecessor will be referred to as a leaf. For example, Fig. 21 (a) represents a 5-cycle with balanced trees of height 2, while Fig. 21 (b) is the state transition diagram for the 1d 2-state near-neighbor mod 2 rule with 15 sites. (See Appendix E.)

The state transition diagram of a finite CA completely describes its temporal behavior. We can directly read off a host of properties, the cycle lengths, the fraction of states on each cycle of a given length, the maximum transient time (time before a state settles into a cycle which is given by the length of the highest tree), the fraction of states which do not belong to a cycle, etc. So understanding the time evolution of a DCA is equivalent to determining the topology of the state transition diagram.

In this chapter we describe a new algorithm to analyze additive CA based on the properties of circulant matrices. 86 Several results about the structure of the state transition diagrams of multidimensional and higher-order CA are established. In particular, we show how the behavior of a $p^q$-state ($p$ is a prime, $q$ any integer) additive CA in $d$-dimensions on a finite lattice with periodic boundary conditions can be determined explicitly. Conditions on rules which determine whether they are reversible or not are derived. CA with irreversible rules exhibit transient behavior before settling into periodic attracting sets (cycles). Cycle lengths, lengths of transients, multiplicities of cycles, and other properties are shown to be calculable for finite additive CA.
Figure 21  

(a) A 5-cycle with balanced trees of height 2.

(b) State transition diagram for the CA (with 15 sites) discussed in Appendix E. Multiplicities of the 1-, 3-, and 15-cycles are 4, 20, and 1088 respectively.
An additive CA is a CA whose rules are such that the value of the variable at site \( i \) at time \( t+1 \), depends on some linear superposition of the values of its neighboring sites at time \( t \), or some \( s \) time steps earlier (\( s \)-order CA). Throughout this chapter, we represent the variable on site \( i \) at time \( t \) by \( \sigma_i(t) \). The \( p^q \) values assumed by \( \sigma_i(t) \) can be thought of as belonging to a field \( K \). \(^{87}\)

The state of a CA with \( N \) sites at time \( t \), \( \{\sigma_i(t)\} \), can be represented as an \( N \)-component vector. The states of the CA form a linear vector space (on the field \( K \) to which \( \sigma_i \) belongs). The linearity of the rules implies that the time evolution of the CA can be represented by a linear operator, a matrix, acting on the vector space of states. This matrix, denoted by \( A \), will be referred to as the transition matrix, and \( \sigma(t+1) = A \sigma(t) \). The elements of \( A \) are also required to belong to the field \( K \) and all operations are defined on this field.

The state transition diagram for a given CA is determined by the properties of its transition matrix \( A \), especially by its eigenvalues. In Sec. 4.1 we provide a crude and intuitive outline of how to determine the state transition diagram from the properties of \( A \). The rigorous mathematical proofs constitute the bulk of the chapter, Secs. 4.2-4.4.

4.1 Basic Ideas

In order to outline the basic ideas we discuss the simplest possible case. More complicated cases can be handled by generalizing these ideas. As we will explain, the topology of the state
transition diagrams is determined by the eigenvalues of the transition matrix $A$ and the corresponding Jordan form. When the dimension of $A$ is large, both its eigenvalues and the Jordan form are difficult to obtain. However, since the CA rules are homogeneous and we impose periodic boundary conditions the matrix $A$ is actually a circulant matrix, i.e.,

$$
A = \begin{pmatrix}
    a_0 & a_1 & \ldots & a_{N-1} \\
    a_{N-1} & a_0 & \ldots & a_{N-2} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_1 & a_2 & \ldots & a_0 \\
\end{pmatrix} \equiv \text{circ} (a_0, a_1, \ldots, a_{N-1}). \ (4.1.1)
$$

This special form allows us to determine the eigenvalues and the Jordan form easily. In Sec. 4.2 we explain the procedure for doing so but here we assume that the eigenvalues and Jordan form of $A$ can be found and discuss the calculation of the state transition diagram.

We distinguish the following two cases:

(a) none of the eigenvalues of $A$ are zero and
(b) some of the eigenvalues are zero.

Consider case (a) first. There exists a time $L$ such that $A^L=I$, the identity matrix. This can be most easily seen if $A$ can be diagonalized. (If not, in the basis in which $A$ assumes the Jordan form one can deduce the result trivially). The eigenvalues of $A$, $\{\lambda_i\}$, also belong to a field, and hence, there exists a finite number, $L$, called the order such that $\lambda_1^L=1$. Now time evolution by $t$ steps is
accomplished by using $A^t$. In the diagonal basis the elements of $A^t$ are given by $\lambda_1^t$. Even though different $\lambda_1$ have different orders $l_1$, when $t=L$, the least common multiple of all the $l_1$, we get $A^L=I$. Since we $\vec{\sigma}(t+L)=A^L\vec{\sigma}(t)=\vec{\sigma}(t)$, the state of the system after a time $L$ returns to its original state for any initial state. In other words, every state is on a cycle, and the state transition diagram consists exclusively of cycles; no trees (transient states which do not recur) are present. Clearly, every state has a unique predecessor and such a CA is said to be reversible. So, given an additive CA we can determine whether it is reversible or not by simply checking if the transition matrix has zero eigenvalues.

Now, what are the cycle lengths? Clearly $L$ is a cycle length, in fact, it is the maximum cycle length. There can be smaller cycle lengths. For example, if the orders of eigenvalues are $l_1 < l_2 < l_3$ ... obviously $A^{l_1} \neq I$. But if the initial state vector is such that it has non-zero components only in the subspace corresponding to eigenvalues having order $l_1$, then after time $l_1$ the state will come back to its initial state even though $A^{l_1} \neq I$. So we can have cycles with length $l_1$. Similarly there are cycles with length $l_2$, $l_3$ ... Now, imagine that the initial state vector has non-zero projection only onto the subspace corresponding to eigenvalues whose orders are $l_1$ and $l_2$. The state returns to its initial state after a time which is the least common multiple ($\text{lcm}$) of $l_1$ and $l_2$, so, we can have cycles of length $\text{lcm}$ ($l_1, l_2$), etc. Therefore, in the case where $A$ can be diagonalized the cycle lengths are given by the orders of the eigenvalues or the least common multiples of any combinations of them. Obviously, the largest
cycle length L is divisible by the lengths of all the other smaller cycles. The multiplicities of each cycle length can then be calculated.

In case (b), A has zero eigenvalues. Again consider the problem in a basis in which A is diagonal. If the initial state vector has non-zero projection on to the subspace corresponding to zero eigenvalues, it is obvious that the repeated application of A, will make the projection vanish. Hence, the states which eventually survive in the time evolution have vanishing projection on the zero-eigenvalue subspace. Any state with non-zero components in the zero-eigenvalue subspace can only be a transient state. Such states correspond to the trees rooted on cycles. These DCA are irreversible since, given a state, its predecessor cannot be determined uniquely. The cycle lengths and multiplicities can be determined in the same way as in the reversible case (case (a)). The structure of the trees depends on the dimension of the Jordan blocks which correspond to zero eigenvalues. We merely call the reader's attention to one property here: Trees rooted at all vertices on all cycles of the state transition diagram for a given additive CA are identical! (See Sec. 4.3 for the proof.)

The rest of the chapter is devoted to a complete analysis with rigorous proofs.

4.2 Notation and Preliminaries

We begin by establishing notation and proving some basic theorems about circulant matrices employed frequently in the rest of the
chapter. For notational compactness, this section is restricted to one-dimensional CA.

Consider a finite one-dimensional lattice with $N$ sites and periodic boundary conditions. The variable at each of the $N$ sites can assume one of $p^q$ values belonging to a field $K$. We characterize the state (configuration) of the CA at time $t$ by an $N$-component column vector $\vec{\sigma}(t)$ whose $i$th entry $\sigma_{i-1}(t)$ is the value the $i-1$ site takes at time $t$. The definition of the CA is completed by specifying the rule according to which the state evolves in time. In this paper we consider additive rules which have the following general form

$$
\sigma_i(t+1) = \sum_{j=0}^{N-1} a_j \sigma_{i+j}(t) \tag{4.2.1}
$$

where the coefficients $\{a_j\}$ belong to the same field $K$ as do the $\sigma_i$'s. [Periodic boundary conditions imply $\sigma_{i+N}(t) = \sigma_i(t)$]. Different choices for $\{a_j\}$ yield different rules. Additive CA are amenable to exact analysis because of the linearity of (4.2.1). It is useful to rewrite (4.2.1) as $\vec{\sigma}(t+1) = A \vec{\sigma}(t)$ where the transition matrix $A$ is an $N\times N$ circulant matrix,

$$
A = \text{circ}(a_0, a_1, \ldots, a_{N-1}) \equiv \sum_{i=0}^{N-1} a_i \Pi_N^i \tag{4.2.2}
$$

where $\Pi_N$ itself is an $N\times N$ circulant matrix defined as

$$
\Pi_N \equiv \text{circ}(0, 1, 0, \ldots, 0) \tag{4.2.3}
$$

Theorem 2.1. Let $K$ be a finite field of character $p$. $|K| = p^q$. If $(p,N)=1$, $\Pi_N$ can be diagonalized by $F_N$ (with elements in the extension field of $K$),
\[ P_N^{-1} \Pi_N P_N = \text{diag}(\eta_0, \eta_1, \ldots, \eta_{N-1}) \]  

(4.2.4)

where \( \eta_j = \eta_j^1 \), \( (P_N)_{ij} = \frac{1}{N} (\eta_{i-j})^{j-1} \) and \( \eta \) is an \( N \)th primitive root of unity over \( K \).

Theorem 2.2. Let \( K \) and \( \Pi_N \) be as before. Suppose \( N = p^l \) with \( (p, r) = 1 \). Then \( \Pi_N \) is similar to a direct sum of \( r \) Jordan blocks each of size \( p^l \times p^l \). The eigenvalues of \( \Pi_N \), each of which has multiplicity \( p^l \), are \( \xi_0, \xi_1, \ldots, \xi_{r-1} \) where \( \xi_1 = \xi^1 \) and \( \xi \) is an \( r \)th primitive root of unity.

Proof: The minimum polynomial of \( \Pi_N \) is

\[ m(\Pi_N) = x^N - 1 = (x^r - 1)^l = \det (xI - \Pi_N) \]  

(4.2.5)

Since \( (p, r) = 1 \), \( x^r - 1 \) is separable over \( K \). Hence, the Jordan blocks are given by Theorem 2.2.

Observe that Theorem 2.1 is a special case of Theorem 2.2 when \( l = 0 \).

Theorem 2.3. Let \( N, p, l, r, \) and \( \{\xi_i\} \) be as above. All \( N \times N \) circulant matrices \( A = \text{circ}(a_0, a_1, \ldots, a_{N-1}) \) are similar to a direct sum of \( r \) \( p^l \times p^l \) blocks:

\[ A = \bigoplus_{k=0}^{r-1} A(k), \]

where the \( A(k) \)'s are \( p^l \times p^l \) matrices given by,

\[ A(k) = \sum_{i=0}^{N-1} a_i (\xi_k I + C)^i = \sum_{i=0}^{p^l-1} b(k)_i C^i. \]  

(4.2.6)

Here \( I \) is a \( p^l \times p^l \) unit matrix and \( C \) is a \( p^l \times p^l \) matrix given by
$$C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ & \vdots & 0 \\ & \vdots & \ddots & \vdots \\ & \vdots \\ & \vdots & \ddots & 1 \\ 0 & & & \end{pmatrix}$$

(4.2.7)

$$b_0^{(k)} = \sum_{i=0}^{N-1} a_i \xi_k^i \equiv \lambda_k$$  \hspace{1cm} (4.2.8)

and $b_j^{(k)}$ can be written formally as

$$b_j^{(k)} = \frac{1}{j!} \frac{d_j}{d(\xi_k)^j} b_0^{(k)}$$  \hspace{1cm} (4.2.9)

The proof follows directly from theorem 2.2 and Eq. (4.2.2).

In particular, when $l=0$, i.e., $(p,N) = 1$, all circulant matrices $A$ can be diagonalized.

Now, consider $(A^{(k)})_p^l$. From (4.2.6), using the binomial expansion, we get

$$(A^{(k)})_p^l = \left( \sum_{i=0}^{p^l-1} (b_i^{(k)} C_i^1) p_1^l \right) = \sum_{i=0}^{p^l-1} (b_i^{(k)})_p^l (C_i^1) p_1^l$$

$$= \lambda_k^p I,$$  \hspace{1cm} (4.2.10)

where we have used $C_p^l = 0$ and $pB = 0$ for all matrices $B$ over the field of character $p$. Hence, $A_p^l$ can be diagonalized as

$$A_p^l \sim \bigoplus_{i=0}^{r-1} \lambda_i^p I$$  \hspace{1cm} (4.2.11)

Remark. From (4.2.8), we see that for each $k$, $\lambda_k$ is an element of the extension field of $K$, $K(\xi)$. If $\lambda_k \neq 0$, $\lambda_k$ has finite order.
\[ l_k = \text{ord}(\lambda_k), \text{ and } l_k | p^n - 1, \text{ where } n = \text{lcm} (q, \text{ord}_r p). \text{ Ord}_r p \text{ is the least positive integer } j \text{ such that } p^j - 1 \text{ mod}(r). \text{ In particular, when } q = 1, n = \text{ord}_r p. \]

Definition. The generating polynomial of \( A = \text{circ}(a_0, \ldots, a_{N-1}) \) which defines an additive CA through (4.2.1), is

\[ a(x) = \sum_{i=0}^{N-1} a_i x^i \]

\[ (4.2.12) \]

Theorem 2.4. An additive CA (4.2.1) has zero eigenvalue if and only if \( g(x) = \text{gcd}(a(x), x^r - 1) \neq 1 \). The dimension of the eigenspace of \( A \) with eigenvalue zero is \( p^\deg(g(x)) \).

Proof. Let \( g(x) \neq 1 \). Since \( g(x) | x^r - 1 \) and \( x^r - 1 \) is separable, all the roots of \( g(x) \), \( \xi_{k_i} (i = 1, 2, \ldots, \deg(g(x))) \), are distinct and are roots of \( x_r - 1 \). Since \( g(x) | a(x) \), \( g(\xi_{k_i}) = 0 \) implies \( a(\xi_{k_i}) = 0 \). Recall that the eigenvalue \( \lambda_k = a(\xi_{k_i}) \) by (4.2.8). Therefore, there are \( \deg(g(x)) \) zero eigenvalues. The second part of Theorem 2.4 is a direct corollary of Theorem 2.3.

4.3. Higher-Order CA

In this section we consider \( N \)-site, \( p^q \)-state additive CA (with periodic boundary conditions) whose time-evolution is non-Markovian, i.e., the state of the CA at time \( t \) depends on the states at the \( s(\geq 2) \) preceding time steps, \( t-1, t-2, \ldots, t-s \). For simplicity, we consider \( s \)-order CA in one dimension; the higher dimensional case is similar and can be treated analogously (see next section). We provide an algorithm for computing the state transition diagram explicitly. The conditions under which the class of CA rules are reversible, i.e., all states belong to cycles, are established: it depends on the influence
of the state at time \( t-s \) only. A general Theorem (3.2) regarding the topology of transition diagrams is proved. The fraction of the states on cycles is computed and explicit expressions for cycle lengths and multiplicities in terms of the orders of the non-zero eigenvalues of the transition matrix are given.

We discuss the CA in terms of the state vector \( \mathbf{b}(t) \), defined by

\[
\mathbf{b}(t) = (\mathbf{c}^{tr}(t-s+1), \mathbf{c}^{tr}(t-s+2), ..., \mathbf{c}^{tr}(t))
\]

(4.3.1)

where \( \mathbf{c} \) is defined as before. The time evolution of \( \mathbf{b}(t) \) is given by

\[
\mathbf{b}(t+1) = \mathbf{A}\mathbf{b}(t).
\]

The transition matrix \( \mathbf{A} \) has dimension \( sN \) and is given by

\[
\mathbf{A} = \begin{bmatrix}
0 & I & & \\
0 & I & & \\
& & \ddots & 0 \\
& & & \ddots & \\
& & & & \ddots & \\
A_0 & A_1 & \cdots & \cdots & \cdots & A_{s-1}
\end{bmatrix}
\]

(4.3.2)

where \( I \) is an \( N \times N \) unit matrix and \( A_1 \)'s are \( N \times N \) circulant matrices. \( A_1 = \text{circ}(a_0^{(1)}, a_1^{(1)}, \ldots, a_{N-1}^{(1)}) \) describes the influence of \( \mathbf{c}(t-s+1) \) on \( \mathbf{c}(t) \).

We now explore the topology of the state transition diagram; one property of interest is clearly the occurrence of cycles. Since the state vector \( \mathbf{b}(t) \) at \( t+1 \) is \( \mathbf{A}\mathbf{b}(t) \), \( \mathbf{b}(t) \) will be on a cycle if the projection of \( \mathbf{b}(t) \) on any invariant subspace of \( \mathbf{A} \) with eigenvalue zero vanishes. We now proceed to determine the conditions under which \( \mathbf{A} \) has no zero eigenvalues and thus all states are on cycles.

By Theorem 2.3, \( A_1 \) is similar to a direct sum of upper triangular
matrices $A_1^{(k)} = \sum_{k=0}^{r-1} A_1^{(k)}$, where $N = r p^q$ as before and $A_1^{(k)}$ is a $p^q \times p^q$ matrix given by Eqs. (4.2.6) - (4.2.9). Moreover, the matrix $T$ which transforms $A_1$ to $A_1$ is the same for all $i$ (it is the matrix that transforms $\Pi_\lambda$ to its Jordan form). Therefore, we have

$$
\Lambda = \begin{pmatrix}
T & -1 & T & & 0 & I \\
T & & T & \ddots & & I \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & & & & & \ddots \\
& & & & & & T
\end{pmatrix} = \begin{pmatrix}
A_0 & A_1 & \cdots & A_{s-1}
\end{pmatrix}
$$

(4.3.3)

The characteristic polynomial of $A$ is obviously the same as that of $A$. Performing elementary operations on $xI - A \otimes B$, we obtain

$$
B \sim \begin{pmatrix}
I & & & 0 \\
& I & & \\
& & \ddots & \\
& & & 0 & I
\end{pmatrix}
$$

(4.3.4)

where $B_0$ is an $N \times N$ upper triangular matrix given by

$$
B_0 = x^s I - \sum_{i=0}^{s-1} x^i A_1 = x^s I - \sum_{i=0}^{r-1} \sum_{k=0}^{s-1} x^i A_1^{(k)}
$$

(4.3.5)

Since $B$ is upper triangular, the characteristic polynomial of $A$ is the product of the diagonal elements of $B_0$. The diagonal elements of the $k$th block of $B_0$ are all identical and they are simply,

$$
f_k(x) = x^s - \sum_{i=0}^{s-1} x^i A_1^{(k)}
$$

(4.3.6)
where } \lambda_1^{(k)} \text{, the } k\text{th eigenvalue of } A_1 \text{, is given by}
\begin{equation}
\lambda_1^{(k)} = \sum_{j=0}^{N-1} a_j^{(1)} \xi_j^{k}
\end{equation}

as before.

If } \lambda_0^{(k)} \neq 0 \text{ for every } k \text{ none of the eigenvalues of } A \text{ are zero. Hence, all possible states of CA will be on cycles. If, on the other hand, } \lambda_0^{(k)} = 0 \text{ for some } k \text{, } A \text{ will have zero eigenvalues. In this case, some states will not be on cycles and will appear only as vertices on trees.}

Combining (4.3.6) and Theorem 2.4 we have,

Theorem 3.1. For additive CA of order } s \text{, each state } \mathbf{\hat{z}} \text{ is on a cycle (i.e. the rule is reversible) if and only if } \text{gcd}(a_0(x), x^r - 1) = 1 \text{ where } a_0(x) \text{ is the generating polynomial of } A_0.

When } \text{gcd}(a_0(x), x^r - 1) \equiv \text{g}(x) \neq 1 \text{, there will be states } \mathbf{\hat{z}} \text{ which belong to trees rooted on cycles. We now establish a general property of trees.}

Theorem 3.2. The trees rooted at all vertices on all cycles of the state transition diagram of the CA defined by (4.3.2) are identical.

Proof. There exists a transformation matrix } Q \text{, such that } Q^{-1}AQ = J \text{, where } J \text{ is a Jordan matrix. A state vector } \mathbf{\hat{z}} \text{ is on a cycle if } \mathbf{\hat{z}} \text{ belongs to an invariant subspace of } A \text{ with non-zero eigenvalue. Consider the largest invariant space } V^1 \text{ of } A \text{ with eigenvalue zero. Within } V^1 \text{ there can still be invariant subspaces } J_1, J_2, \ldots, J_k \text{, each one of which corresponds to a block in } J \text{ with zero diagonal elements. For each subspace } J \text{, there exists a generator } \mathbf{\hat{u}}_1 \text{, such that, if the}
dimension of $J_1$ is $d_1$, then $u_1, A^2u_1, \ldots, A^{d_1-1}u_1 \neq 0$ form a basis basis of the subspace $J_1$ and $A^{d_1}u_1 = 0$. Thus all the elements in $V_1$ form one tree with the root $\hat{z} = 0$ and leaves

$$\sum_{i=1}^{k} \sum_{j=0}^{d_1-1} a_{j1} A^{d_1-j} u_1, \text{ where } a_{j1} \in K(\xi) \text{ and } a_{01} \neq 0 \text{ for some } 1.$$

Now, in the Jordanized basis, the transition matrix can be written as $A = A_0 \oplus A_1$, where $A_1$ is a direct sum of all the Jordan blocks which nonzero diagonal elements. Correspondingly, $\hat{x} = \hat{z}_0 \oplus \hat{z}_1$, and hence, the time evolution is given by $\hat{x}(t+1) = (A_0 \oplus A_1) \hat{x}(t)$. For each $\hat{x}$ on a cycle, i.e., $\hat{x}_0 = 0$, construct the set,

$$S_1 = \{ \hat{x}_0 \oplus (A_1^{d_1})^{d_1-1} + \sum_{i=1}^{k} a_{d_1-1,1} A^{d_1-1}u_1 | a_{d_1-1,1} \in K(\xi), a_{d_1-1,1} \neq 0 \}$$

(4.3.8)

for some 1},

,$$
\vdots
\vdots
\vdots
$$

$$S_d = \{ \hat{x}_0 \oplus (A_1^{d_1})^{d_1-1} + \sum_{i=1}^{k} a_{d_1-1,d_1} A^{d_1-1}u_1 | a_{d_1-1,d_1} \in K(\xi), a_{d_1-1,d_1} \neq 0 \}$$

(4.3.9)

for some 1},

,$$
\vdots
\vdots
\vdots
$$

max \{d_1\}

Then, $U_{S_1}$ is a tree rooted on $\hat{z}$ and is isomorphic to the

tree rooted on $\hat{z}_1 = 0$. QED.
Note that the above proof does not depend on the $A_{1}$'s being circulant matrices. In fact, Theorem 3.2 is valid for arbitrary (including inhomogenous) additive CA irrespective of dimensionality, order, or nature of boundary conditions.

Corollary. Let $k$ be the number of Jordan blocks with zero diagonal elements in $A$, and $d_{m}$ be the dimension of the largest of those $k$ blocks. The height of the tree rooted at each vertex of a cycle is $d_{m}$. In particular, if all the $k$ blocks have the same size $d_{m}$, the tree is balanced and has in-degree $|K|^{k}$. [A tree is said to be balanced if the number of predecessors (in-degree) is the same for all vertices on the tree and the distance from each leaf to the root is also the same. The height of a tree is defined to be the largest distance from the leaf to the root.]

Let us define

$$g_{1}(x) = \gcd(a_{1}(x), g(x)),$$  \hspace{1cm} (4.3.10)

$$g_{i}(x) = \gcd(a_{i}(x), g_{i-1}(x)) \text{ for } i=2, 3, \ldots, s-1,$$  \hspace{1cm} (4.3.11)

$$n_{1}^{'} = \deg(g_{1}(x)), \quad n_{5}^{'} = 0 \text{ and } n_{0}^{'} = \deg(g(x)).$$  \hspace{1cm} (4.3.12)

Clearly, $n_{1} = n_{1}^{'} + n_{s}^{'}$ is the number of $f_{j}$'s which have zero roots of multiplicity $i$. So the total number of zeros on the diagonal of the Jordan form of $A$ is

$$\lambda = p^{s} \sum_{i=1}^{s} n_{i}^{'}.$$  \hspace{1cm} (4.3.13)

From the proof of Theorem 3.2, we have,

Theorem 3.3. The fraction of the configurations of a CA defined by (4.3.2) that are on cycles is $|K|^{-\lambda}$. 
Remark. All the states of the CA are in $K^N$ while the Jordan forms and generating vectors we have been discussing belong to $K(\xi)^N$. Since $K^N$ is a full lattice in $K(\xi)^N$, the intersection of $K^N$ with any subspace $V$ of $K(\xi)^N$, which has dimension $n$, is also a subspace of $K^N$ with dimension $n$.

We now obtain the multiplicities and the lengths of the cycles in terms of the order of the non-zero eigenvalues of the transition matrix $A$. By solving (4.3.6) for each $k$, we get all eigenvalues $\zeta_1$ of $A$.

Then $A = J = \Theta(\zeta_1 I + C_{d_1})$, where $C_{d_1}$ is a $d_1 \times d_1$ matrix of the form in (2.7).

Let $I$ be the smallest integer such that $p \not\equiv d$, where $d$ is the dimension of the largest Jordan block of $J$; we then have,

$$A^{p^I} \sim \Theta(\zeta_1 I + C_{d_1})^{p^I} = \Theta \zeta_1^{p^I} I. \quad (4.3.14)$$

That is, $A^{p^I}$ is similar to a diagonal matrix. Denote the non-zero eigenvalues of $A$ by $\zeta_1$, $(i = 0, 1, \ldots, r')$. We have thus shown

Theorem 3.4. The possible cycle lengths of a CA defined by

(4.3.2) are ord $(\zeta_1)$, $p$ ord $(\zeta_1)$, $\ldots$, $p^{\ell}$ ord $(\zeta_1)$ and their least common multiples, $p^{\ell} \text{lcm} \text{ (ord } (\zeta_1_1), \text{ ord } (\zeta_1_2), \ldots, \text{ ord } (\zeta_1_m))$

$(i = 0, 1, 2, \ldots, r', m = 2, 3, \ldots, r' \text{ and } i \not\equiv \bar{I})$, where $\ell$ is the smallest integer such that $p^\ell \equiv d_1$ ($d_1$ is the dimension of the Jordan block corresponding to the non-zero eigenvalue $\zeta_1$).

The maximum cycle length $L$ is

$$L = p^\bar{I} \times \text{lcm} \text{ (ord } (\zeta_0), \text{ ord } (\zeta_1), \ldots, \text{ ord } (\zeta_{r'})). \quad (4.3.15)$$

So we see that all possible cycle lengths are divisors of the maximum cycle length $L$, and in turn, $L$ is a divisor of $p^\bar{I}(p^{N-1})$. (See
the remark in Sec. 4.2).

Knowing the cycle length \( L(i) \), its multiplicity \( m(L(i)) \) can be computed. Let \( K^{N_s} = V_o \oplus V \) where \( V_o \) and \( V \) are subspaces corresponding to zero and non-zero eigenvalues respectively. Let \( D_1 \) be the dimension of the largest subspace \( S_1 \) of \( V \) such that the maximum cycle length in \( S_1 \) is \( L(i) \). Two cases must be distinguished. If no other cycles with length different from \( L(i) \) are present in \( S_1 \), then \( m(L(i)) \) is given by,

\[
m(L(i)) = \frac{((p^q)^{D_1} - 1)}{L(i)}
\]

where the numerator is the total number of states (excluding the zero state) in the \( D_1 \)-dimensional subspace. If, on the other hand, \( S_1 \) contains several distinct cycles with length denoted by \( L(j_1), L(j_2), \ldots, L(j_n) \) then, the multiplicity is,

\[
m(L(i)) = \frac{((p^q)^{D_1} - \sum_{k=1}^{n} L(j_k)m(L(j_k)-1))}{L(i)}
\]

4.4. CA in Higher Dimensions

The methods described in previous sections can be applied to arbitrary integer dimensions. As an illustrative example, we present the results for two-dimensional CA when the state at time \( t \) depends only on the state at time \( t-1 \). Results for one-dimensional CA can be recovered as a special case by setting the number of rows equal to one while the higher-dimensional cases can be treated inductively. The extension to higher-order CA in higher dimensions is tedious but straightforward.

Consider a \( p^q \)-state CA on a two-dimensional lattice of \( M \) rows and \( N \) columns with periodic boundary conditions. We can represent the
state of CA by an MN-component column vector \( \mathbf{\varphi}(t) \) with the elements representing the state of N sites in each of the M rows successively. Because of the periodicity in both directions, the transition matrix \( \mathbf{A} \) is a circulant of circulant matrices:

\[
\mathbf{A} = \text{circ} (A_0, A_1, \ldots, A_{M-1}) \tag{4.4.1}
\]

Here, each \( A_i \) is an \( N \times N \) circulant matrix which describes the influence of the \( i \)th row on the future.

\[
A_i = \text{circ} (a_0^{(1)}, a_1^{(1)}, \ldots, a_{N-1}^{(1)}) \tag{4.4.2}
\]

\( \mathbf{A} \) can be written in terms of the \( M \times M \) matrix \( \Pi_M \) (defined before) as,

\[
\mathbf{A} = \sum_{i=0}^{M-1} \Pi_M^i \odot A_i \tag{4.4.3}
\]

First, we discuss the simple case, \( (M, p) = 1 \) and \( (N, p) = 1 \). Recall that \( \Pi_M \) and \( A_1 \) can both be diagonalized by \( F_M \) and \( F_N \) respectively (Theorems 2.1 and 2.3),

\[
\Pi_M \sim \Omega = \text{diag} (\xi_0, \xi_1, \ldots, \xi_{M-1})
\]

\[
A_1 \sim A_1 = \text{diag} (\lambda_0^{(1)}, \lambda_1^{(1)}, \ldots, \lambda_1^{(N-1)}) \tag{4.4.4}
\]

where \( \xi_i = \xi_1 \), \( \xi \) is an \( M \)-th primitive root of unity and

\[
\lambda_j^{(1)} = \sum_{k=0}^{N-1} a_k^{(1)} \eta_j^k \eta_j = \eta_j, \quad \eta_j = \eta_j, \quad \eta \text{ is an } N \text{th primitive root of unity.}
\]

Therefore,
\[ \Pi_M^1 A_1 = (F_M \Omega^1 F_M^{-1}) \otimes (F_N A_1 F_N^{-1}) \]

\[ = (F_M \otimes F_N) (\Omega^1 \otimes A_1) (F_M \otimes F_N)^{-1} \]  \hspace{1cm} (4.4.5)

and \[ A = (F_M \otimes F_N) \sum_{l=0}^{M-1} (\Omega^1 \otimes A_1^l) (F_M \otimes F_N)^{-1} \]  \hspace{1cm} (4.4.6)

Hence, the eigenvalues of \( A \) are given by,

\[ \rho_{kj} = \sum_{l=0}^{M-1} \xi_k^l \lambda_j^l = \sum_{l=0}^{N-1} (\sum_{l=0}^{M-1} a_k^{(l)} \xi_k^l \eta_j^l) \]

\[ \equiv \sum_{l=0}^{N-1} b_{lk} \eta_j^l \quad \text{for} \quad k=0,1, \ldots, M-1 \]

\[ \text{and} \quad j=0,1, \ldots, N-1 \]  \hspace{1cm} (4.4.7)

where

\[ b_{lk} = \sum_{l=0}^{M-1} a_k^{(l)} \xi_k^l \]  \hspace{1cm} (4.4.8)

Theorem 4.1. Let

\[ b_k(x) = \sum_{l=0}^{N-1} b_{lk} x^l \]  \hspace{1cm} (4.4.9)

All configurations of the CA evolving under the rule specified by (4.4.1) for \((p, M) = 1\), and \((p, N) = 1\) are on cycles if and only if for \(k=0,1, \ldots, M-1\),

\[ \gcd(b_k(x), x^{N-1}) \equiv g_k(x) = 1 \]  \hspace{1cm} (4.4.10)

Proof. By (4.4.7), \( \rho_{k,j} = b_k(\eta_j) \). If (4.4.10) holds, then \( b_k(\eta_j) \neq 0 \) for all \( j \). Therefore, \( \rho_{k,j} \neq 0 \). Since \( \rho_{k,j} \) is in a finite extension of a finite field, \( A^h = I \) for \( h = \lcm(\text{ord}(\rho_{k,j})) \). Therefore, all configurations are on cycles.
Corollary 4.1 The fraction of configurations that are on cycles for the CA evolving under the rules specified by (4.4.1) for \((p,M)=1\),

\[
\sum_{i=0}^{M-1} \text{deg} \left( g_{1}(x) \right) \quad \text{and} \quad (p, N) = 1 \\text{i.e.} \quad |K|.
\]

Now, consider the case \((p, N)\neq 1, (p, M)\neq 1\). Let \(M=s^k p^l, N=r^k p^l\) and \((s,p)=(r,p)=1\). By Theorem 2.3, all the \(A_1^{(j)}\) can be simultaneously transformed to a direct sum of \(r\) blocks \(A_1^{(j)} \) each of size \(p^k \times p^l\) by a non-singular matrix \(T\), and \(N_1^i\) can be transformed to a direct sum of \(s\) blocks \(P_j^1\) each of size \(p^k \times p^l\) by \(S\), where \(A_1^{(j)}\) and \(P_j^1\) are given by

\[
\begin{align*}
A_1^{(j)} &= \sum_{m=0}^{N-1} a_m^{(i)} (\eta_j I + C)^m, \\
P_j^1 &= (I \xi_j + C)^{-1},
\end{align*}
\]

\((\eta_1 = \xi_1, \eta_2 = \xi_2\) and \(\eta\) and \(\xi\) are the \(r\)th and \(s\)th primitive roots of unity respectively). Hence,

\[
A = \sum_{i=0}^{M-1} (S P^1 S^{-1}) \otimes (T A_1^{(j)} T^{-1})
\]

\[
= (S \otimes T) \sum_{i=0}^{M-1} \sum_{j=0}^{s-1} \sum_{m=0}^{r-1} [(\otimes P_j^1) \otimes (\otimes A_1^{(m)})] (S \otimes T)^{-1}
\]

The diagonal elements have the same form as (4.4.7), and so we have, Corollary 4.2. All configurations of the CA evolving under the rule specified by (4.4.1) are on cycles if and only if for \(k=0,1,\ldots, s-1\)

\[
\text{gcd} \left( b_k(x), x^{r-1} \right) \equiv g_k(x) = 1.
\]
Obviously, the number of \( \rho_{ki} \), which are zero is given by \( \sum_{i=0}^{s-1} \deg(g_i(x)) \) and each zero has degeneracy \( p_k^k \). So we have,

Corollary 4.3. The fraction of the states that are on cycles for the CA evolving under the rule specified by (4.4.1) is \(|K|^{-\lambda} \), where

\[
\lambda = p_k^{k+1} \sum_{i=0}^{s-1} \deg(g_i(x)) \quad (4.4.15)
\]

Theorem 4.2. The trees rooted at all vertices on all cycles of the state transition diagram of the CA evolving under the rule specified by (4.4.1) are identical.

The proof is exactly the same as for Theorem 3.2.

The structure of the trees is given exactly as in the Corollary in Sec. 4.3.

We now determine the lengths of cycles. Note that

\[
\Lambda \sim \Lambda = \sum_{i=0}^{M-1} \sum_{j=0}^{s-1} \sum_{m=0}^{r-1} (\otimes \ (p_1^i \otimes (\otimes \ \Lambda_{1}^{(m)}))) \quad (4.4.16)
\]

Since \( p_1^i \) and \( \Lambda_{1}^{(m)} \) are upper triangular blocks, \( \Lambda \) is also an upper upper triangular matrix. Therefore,

\[
\Lambda \sim \Lambda = \sum_{i=0}^{s-1} \sum_{j=0}^{M-1} \sum_{m=0}^{r-1} (\otimes \ (p_1^i \otimes (\otimes \ \Lambda_{1}^{(m)}))) \quad (4.4.17)
\]

The diagonal element of \( \Lambda \) are eigenvalues \( \rho_{jm} \) of \( A \)

\[
\rho_{jm} = \sum_{i=0}^{N-1} \sum_{t=0}^{M-1} \sum_{n=0}^{r-1} a_1^{(t)} \xi_j^{n} \frac{i}{m} \quad (4.4.18)
\]

where \( \tilde{\lambda} \) is the smallest integer such that \( p_{\tilde{\lambda}} \leq \tilde{d} \) where \( \tilde{d} \) is the dimension of the largest Jordan block of \( A \), \( AP^{\tilde{\lambda}} \) is similar to a diagonal matrix.
We therefore, have,

Theorem 4.3. The possible cycle lengths of a CA evolving under
the rule specified by (4.4.1) are $p'^1$ times the orders of the non-zero
eigenvalues of $A$, $p'^1 \text{ord}
(\rho_{jm}) (1 \leq j_m)$. $l_{jm}$ is the smallest integer
such that $p^l_{jm} d_i$ where $d_i$ is the dimension of the Jordan block
(corresponding to $\rho_{jm}$) and their least common multiples

\[ p'^i \times \text{lcm} (\text{ord}(\rho_{11}^{i1}), \text{ord}(\rho_{12}^{i2}), \ldots, \text{ord}(\rho_{m'm}^{j1})) \]

$s'$ is the total number of distinct non-zero eigenvalues and $1 \leq s'$. The maximum cycle length $L$ is

\[ L = \frac{e}{p'^i \times \text{lcm} (\text{ord}(\rho_{00}), \text{ord}(\rho_{01}), \ldots, \text{ord}(\rho_{s'-1 r'-1}))}. \]

Again, the possible cycle lengths are divisors of $L$, and $L$ is a divisor of

\[ p^{i1} (p^{j1} \alpha), \]

where $\alpha = \text{lcm} (q, \text{ord}_p, \text{ord}_q)$. The multiplicities of cycles can be worked out exactly as in

Sec. 4.3.

As an illustrative example we have worked out the multiplicities
and lengths of cycles of a CA on a two-dimensional square lattice of
size $N \times N$ (with $N$ up to 22) with the following rule:

\[ \sigma_{i,j}^{(t+1)} = \sigma_{i+1,j}^{(t)} + \sigma_{i-1,j}^{(t)} + \sigma_{i,j+1}^{(t)} + \sigma_{i,j-1}^{(t)} \mod 2 \]

(4.4.18)

The results are displayed in Table 1. The requisite factorization of
polynomials was carried out using MACSYMA.

In summary, we have shown that in order to compute the state
transition diagram one simply finds the Jordan form $J$ of the transition
matrix $A$. Then the order of the non-zero diagonal elements of $J$
(eigenvalues of $A$) yields the cycle lengths. To decide whether a
<table>
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<td>3</td>
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<tr>
<td>4</td>
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</tr>
<tr>
<td>5</td>
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<td></td>
<td>[14]: (248,-(120,(8,-1)))(7,0)/(3,-1)</td>
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<td></td>
<td>[4]: (64,-1)(62,0); [6]: (96,-(33,-1))(31,0)/(1,1);</td>
</tr>
<tr>
<td></td>
<td>[12]: (192,-(65,-1))(62,0)/(1,1)</td>
</tr>
<tr>
<td>21</td>
<td>[1]: (4,0); [7]: (84,-1)(4,0)/(3,-1); [9]: (48,-1)(4,0)/(3,1);</td>
</tr>
<tr>
<td></td>
<td>[21]: (88,0)(24,1)(12,1)(6,1)(1,1); [63]: (396,-(132,(48,-1)))(4,0)/(6,-1)</td>
</tr>
<tr>
<td>22</td>
<td>[1]: (0,0); [31]: (200,-1)/(5,-1); [62]: (200,-1)(199,0)/(5,-1)</td>
</tr>
</tbody>
</table>
state \( \hat{\sigma} \) is on a cycle or how far it is away from a cycle, one needs to know the matrix which transforms \( A \) to \( J \). The state \( \sigma \) is on a tree if it has non-zero components in the zero-eigenvalue subspace. This can be determined easily by examining \( \hat{\sigma} \) in a Jordanized basis. Furthermore, the distance from the root can be obtained by considering the projection of \( \hat{\sigma} \) onto an appropriate basis for the zero-eigenvalue zero-eigenvalue subspace. All the computations are standard and the number of steps needed is \( O(N^3) \) where \( N \) is the dimension of the transition matrix. Therefore, all additive CA discussed in this chapter are computationally reducible and their time evolution can be determined explicitly.
FOOTNOTES

a Henceforth, root of unity refers to a root of unity over K.

b From now on in this section, $\xi$ rather than $\sigma$ is referred to as a state of the CA.

c Note A itself is not necessarily an $NM \times NM$ circulant matrix. In three dimensions, the transition matrix will be a circulant of circulant matrix and so on.

d The cases (a) \((p,N)=(p,M)=1\), (b) \((p,M) \neq (p,N)=1\), and (c) \((p,N) \neq (p,M)=1\) correspond to (a) $k=0$, (b) $k \neq k=0$, and (c) $k \neq k=0$ respectively.

e Since $\rho_{jm}^D$ is a Frobenius automorphism of $K(\xi, \eta)$, the order of $\rho_{jm}^D$ is the same as that of $\rho_{jm}$.
CHAPTER V
NON-ERGODIC BEHAVIOR IN
LOCALLY INTERACTING CONTINUOUS SYSTEMS

5.0 Introduction

The ferromagnetic kinetic Ising model in d>1 displays phase coexistence along the line h=0, T<Tc i.e., it is non-ergodic in a one-dimensional region in the 2-dimensional (T,h) parameter space. Hence, at a generic point, for an arbitrary initial condition the system reaches the stationary Boltzmann distribution asymptotically and is in an unique, stable phase. In a series of pioneering papers, the Russian mathematician, A. L. Toom constructed a class of irreversible PCA which he proved rigorously are non-ergodic over a finite volume of parameter space. There have been investigated further. More recently Bennett and Grinstein have demonstrated phase coexistence by an appealing physical argument based on domain growth kinetics. They also resolved the apparent paradox of having non-ergodicity in a region of co-dimension 0 by noting the correspondence of Toom's models to (d+1)-dimensional equilibrium statistical mechanical models (see Sec. 2.1). As discussed earlier, the parameters of the (d+1)-dimensional model are not all independent and are so constrained as to make the free energy zero. Hence, in the (d+1)-dimensional model, non-ergodicity only occurs in a lower-dimensional region of parameter space; however, it is possible for

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irreversible systems to have two (or more) stable phases in a finite region of parameter space.

The work we have described above was done for systems in discrete space-time with discrete variables. In this chapter we explore the role played by discreteness through the study of appropriately generalized time-dependent Ginzburg-Landau (TDGL) models.\textsuperscript{58} We define systems with continuous dynamical variables, evolving in continuous time on a lattice and demonstrate that they can also display stability of two phases (stationary states) over a finite region of parameter space. The crucial feature responsible for the striking behavior in Toom's rule is thus identified with the irreversibility which occurs due to the asymmetry of the neighborhood and is not due to the discreteness of the model. Our TDGL models contain extra terms which incorporate the asymmetry of the neighborhood; these terms though irrelevant in the RG sense for the purposes of critical phenomena profoundly alter the phase diagram. We have also extended our analysis to models in continuous space.

5.1 Review of Toom's NEC Rule

In this section we discuss a specific, simple example of the class of PCA rules introduced by Toom, the so-called North-East-Center (NEC) rule. The model is a 2-state, simultaneously-updated PCA in two dimensions and consists of a square lattice of spins \( S_i(t) \) where \( i \) labels the sites and \( t \) denotes the discrete time variable. The dynamical variable (spin) \( S_i \) assumes values \( \pm 1 \). The value of the spin \( S_i(t) \) at time \( t \) depends probabilistically on its neighbors at time \( t-1 \). In the NEC rule, the neighborhood of site \( i \), is asymmetric, and consists of
itself, its northern and eastern neighbors, i.e., $S_i$, $S_{i+1}$, and $S_{i+N}$. The updating is visualized most easily in two steps. The first step is deterministic: The value of the spin at site $i$ and time $t+1$ is the majority value of its neighbors at time $t$. The second step introduces noise (probabilistic perturbation) into the system. After the first step each spin which is up is flipped down with probability $p$ (i.e., remains up with probability $1-p$) and any spin that is down is flipped up with probability $q$. The above two steps together define Toom's NEC rule which can be succinctly described by the following transition probability,

$$Q(S_i'|S_i, S_{i+1}, S_{i+N}, S_{i+E}) = \frac{1}{2} \left[ 1 + S_i'(h+ (1-r)\Delta_1) \right]$$

(5.1.1)

where $S_i'$ is the value of the spins at time $t+1$,

$$h \equiv q-p, \quad r \equiv p+q, \quad \text{and} \quad \Delta_1 \equiv \frac{1}{2} \left[ S_i + S_{i+1} + S_{i+N} + S_{i+E} - S_i S_{i+1} S_{i+N} S_{i+E} \right]$$

(5.1.2)

Note that the transition probability given in Eq. (5.1.1) clearly cannot be written in the hyperbolic tangent form we derived earlier (see Sec. 2.2) and hence, Toom's NEC rule is irreversible. The noise is characterized by an amplitude $r$, and a bias $b=h/r$. When the bias is non-zero, i.e., $p \neq q$, the rule is not up-down symmetric.

For $p>0$, $q>0$, the rule is fully probabilistic, and a finite lattice is consequently ergodic. Toom has shown rigorously that for sufficiently small noise, biased or not, the system is non-ergodic. In other words, when $p$ and $q$ are small, there exist two stable phases and the asymptotic behavior of the system depends on the initial conditions. The simultaneous existence of two stable phases, i.e., phase coexistence, is, of course, not unusual. An Ising model in
zero magnetic field below its critical temperature does the same. One can think of the amplitude $r$ of the noise as the temperature $T$ of the Ising model and $h=q-p$ which breaks up-down symmetry as the magnetic field. What makes the NEC rule unusual is its non-ergodic behavior in the presence of biased noise, i.e., over a finite region of the $(r,h)$ plane, in contrast to the Ising model.

A simple and appealing argument explaining this unusual behavior on the basis of domain wall kinetics has been put forth by Bennett and Grinstein.\textsuperscript{62} We recapitulate their arguments. The stability of phases is best investigated in terms of the ability of the majority phase to eliminate islands of the minority phase. The NEC rule (with $h=q-p\neq 0$) behaves very differently from the Ising model (with $h=0$) as we shall see next.

Let us review the relevant features of domain wall motion\textsuperscript{78,92} in a ferromagnetic Ising model at $h=0$ and $T<T_c$. Up-down symmetry makes the two coexisting phases equally stable. As described in Sec. 3.3, a flat interface separating a region of up spins from one of down spins has zero mean velocity, while a domain of linear size $R$, which has walls of finite curvature, will shrink and disappear in a time $\tau\sim R^2$. Here we demonstrate this in the framework of the phenomenological TDGL equation for an Ising system:

$$
\frac{\partial \psi(\vec{r},t)}{\partial t} = -\Gamma \frac{\delta F}{\delta \psi} + \zeta(\vec{r},t) = -\Gamma [r\psi + u\psi^3 - \psi^2] + \zeta(\vec{r},t) \quad (5.1.3)
$$

where $\psi(\vec{r},t)$ is a coarse-grained spin variable and $F$ is the Ginzburg-Landau (GL) free-energy functional. $\zeta$ is a Gaussian noise with zero mean and $\langle \zeta(\vec{r},t)\zeta(\vec{r}',t') \rangle = 2\Gamma \delta(\vec{r}-\vec{r}')\delta(t-t')$. The uniform equilibrium
value of \( \psi \) is given by \( \frac{\delta \mathcal{P}}{\delta \psi} |_{\psi^*} = 0 \), \( \psi^* = \sqrt{-\Gamma / U} \). For a flat interface \( \nabla^2 \psi = \frac{\partial^2 \psi}{\partial z^2} \) (\( z \) is the interface normal). If we demand that far away from the interface \( \psi(\pm \infty) = \pm \psi^* \), in the noiseless limit we can indeed find a stationary solution for the interface:

\[
\psi(z,t) = \psi^* \tanh(\sqrt{\Gamma / 2} \ z) \equiv \psi^* \tanh \frac{z}{\xi} \tag{5.1.4}
\]

On the other hand, if we have a gently curved interface the gradient term in the TDGL equation \( \nabla^2 \psi \) is equal to \( \frac{\partial^2 \psi}{\partial z^2} - \kappa \frac{\partial \psi}{\partial z} \) where \( \kappa \) is the curvature. Now the "stationary" solution is given by

\[
\psi(z,t) = \psi^* \tanh[\sqrt{\Gamma / 2} (z - \Gamma \xi t)] \tag{5.1.5}
\]

This describes an interface moving along the \( z \)-direction with velocity \( v = \Gamma \kappa \). Thus a curved interface moves with a velocity proportional to its curvature. A circular domain of radius \( R \) will shrink with a velocity \( \dot{R} \propto -\kappa = -R^{-1} \) and hence, \( R \) decreases with time as \( t^{\frac{1}{2}} \). This is the famous Lifshitz (Cahn-Allen) result. Hence, when \( h=0 \) and \( T \ll J \), if the spins are initially predominantly up the minority islands of down spins will shrink and disappear due to its finite curvature. The addition of noise does not alter the results qualitatively.

When a small symmetry-breaking field \( h \) is turned on, the velocity of the domain walls surrounding a region of the favored phase, is increased by a term proportional to \( h \). Thus islands larger than a critical radius proportional to \( h^{-1} \) will grow. Hence, for \( h>0 \), even if the initial state is mostly down, a sufficiently large island of up spins will be nucleated by thermal fluctuations and will grow and ultimately the magnetization reaches a positive value. So the presence of the magnetic field destroys phase coexistence.
However, for PCA which evolve according to Toom's NEC rule the dynamics of domain walls is quite different. The quantities analogous to $T$ and $h$ in this case are $r=q+p$ and $h=q-p$ respectively. First, consider the motion of an interface in the noiseless limit. Consider the vertical ($\theta=0^\circ$) and horizontal ($\theta=90^\circ$) interfaces. (The interface normal makes an angle $\theta$ with the $x$-axis.) They are stationary because the majority value of the neighborhood of a boundary spin is the same as the spin itself (see Fig. 22 (a) and (b)). From Fig. 22 (c), it is evident that a $45^\circ$-interface will drift southwestward with constant speed irrespective of which phase is on which side of the interface because a boundary spin below the interface is always in a minority! This is different from the Ising model where all interfaces are stationary. How does this alter what happens to minority islands?

First consider the fate of an island of up spins in the shape of an isosceles right triangle with the hypotenuse forming a $45^\circ$-interface. For the noiseless NEC rule it is easy to see that since the southern and western boundaries are stationary while the northeastern boundary moves with unit velocity the island will be eliminated in a time proportional to its initial size.

Consider an arbitrary-shaped island (See Fig. 22 (d)) of up spins in a sea of down spins. The boundary clearly consists of pieces of $0^\circ$, $90^\circ$, $45^\circ$-interfaces. The upper-right hand part of the island necessarily contains a $45^\circ$-interface. This will move southwestward, overtaking the stationary $0^\circ$- and $90^\circ$-pieces, causing the upper-right
Figure 22 Domain walls and domains (a) 0°-wall (b) 90°-wall (c) 45°-wall and (d) an arbitrary domain.
hand part to become flat and contain increasing portions of 45°-interface, in turn causing the island to shrink. Any piece of 45°-interface on the lower-left hand part drifts and causes the island to grow initially. However, its motion, in contrast, is restricted by the 0°- and 90°-pieces on the sides and it will eventually disappear. The 45°-interface from the upper right will sweep the island and eliminate it in a time proportional to its size. Note that unlike the Ising model, the interface velocities are independent of the domain size.

We emphasize that the above argument applies to minority islands of either sign. Since the velocity of the 45°-interface is a non-zero constant, and the addition of small amount of noise (biased or otherwise) can only alter the velocity by an amount proportional to the noise (p and q) the instability of minority islands persists in the presence of sufficiently small noise. Hence, a small symmetry-breaking field does not destroy phase coexistence!

5.2 Mean-Field Theory for the NEC Rule

We give a pedagogical account of Mean-Field Theory (MFT) for the NEC rule in this section. Consider the magnetization \( m(t) = \langle S_1(t) \rangle \) as a function of time. Using the methods of Sec. 2.1, and the transition probability \( \mathcal{Q} \) given by Eq. (5.1.1) we obtain

\[
m(t+1) = -\frac{1}{2} (1-r)m^3(t) + \frac{3}{2} (1-r)m(t) + h \equiv f(m(t)). \tag{5.2.1}
\]

Since the asymptotic state is obtained by repeated iteration of (Eq. 5.2.1), its stable (attracting), fixed points correspond to stationary asymptotic values of the magnetization. Two or more stable fixed points imply phase coexistence.
First let us consider the case when h=0 (up-down symmetric). When r>1/3 Eq. (5.2.1) has a unique stable fixed point at m"=0 and hence, asymptotically, the system reaches a stationary state with zero magnetization, i.e., the paramagnetic phase. When r is decreased below 1/3, the m"=0 fixed point becomes unstable and two new, stable fixed points \( m_{\pm} = \pm \left( \frac{1-3r}{1-r} \right)^{\frac{1}{2}} \) are generated. The unstable fixed point m"=0 separates the domains of attraction of the two stable fixed points. Depending on whether the initial value of the magnetization, m_0, is greater or less than zero the system reaches an asymptotic stationary state (ferromagnetic phase) with magnetization equal to m_+ or m_- respectively. This is precisely the phenomenon of phase coexistence. At r=1/3 all three fixed points merge continuously into one at m"=0. So we have a continuous phase transition from a paramagnetic phase to a ferromagnetic phase at the critical point r=1/3 with conventional MF exponents.

When h≠0, the analysis is similar. Note that Eq. (5.2.1) is invariant under \( h \to -h \) and \( m \to -m \). So we need only discuss the case h>0. When h is small we can perform linear stability analysis. When r is large, the stable fixed point at m"=0 in the absence of h is shifted to m">0 (by an amount of \( O(h) \)) and remains stable (since small h can only change the stability to \( O(h) \)). Hence, the asymptotic magnetization is no longer zero but proportional to h. When r is sufficiently small, the original unstable fixed point at m"=0 is shifted to \( m_c < 0 \) and the two stable fixed points m_± are shifted toward larger magnetization (by terms \( O(h) \)) but remain stable. Hence, in this problem two-phase coexistence persists. Note that this is in contrast
to the Ising model where even though the two fixed points of
Eq. (5.2.1) at $h=0$ remain locally stable for small $h$ (apart from being
shifted) there is a free energy function whose global minimum is the
equilibrium state and hence, one obtains a unique phase corresponding
to the lowest free energy. Phase coexistence, in fact, is determined
by equality of the free energies. In the case of the irreversible
NEC rule, however, there is no such function and hence, both phases
can coexist. The domains of attraction of the two stable fixed points
$m_{\pm}$ (phases) separated by the unstable fixed point $m_c$ are no longer
symmetric about $m_c$ and $m_{\pm}$ are no longer equal in magnitude as was to
be expected. The system favors the up phase, in that both the magneti-
ization ($m_+$) and the size of its domain of attraction are greater than
the corresponding values for the down phase. Note that in order to
get the down phase asymptotically the initial magnetization has to be
more negative than $m_c(<0)$.

The boundary which separates the phase coexistence region and
the "paramagnetic" region occurs when the RHS of Eq. (5.2.1) changes
from having three fixed points to having just one. The phase
boundary is given by

$$h^2 = \frac{(1-3r)^3}{27(1-r)}.$$  \hspace{1cm} (5.2.2)

Fig. 23 shows the mean-field phase diagram for the NEC rule. The
hatched portion corresponds to the region of phase coexistence and its
boundary is given above. The contrast with the well-known phase
diagram of the Ising model in the ($T,h$) plane is obvious.

We now show how to compute the critical exponents $\beta$ and $\delta$. From
Eq. (5.2.1) with $h=0$ we have $|m_{\pm}| = \left( \frac{3(r_c-r)}{1-r} \right)^{1/2}$
yielding $\beta=1/2$. 
Figure 23 Mean field phase diagram for Toom's NEC rule. $r=p+q$, $h=q-p$. The shaded portion corresponds to two-phase coexistence.
Setting $r = r_c = 1/3$ yields $m = h^{3/2}$ and hence, $\delta = 3$. Note that the shape of the boundary is described by the cross-over exponent $3/2$.

5.3 Continuous Versions of Toom's Rule

In the previous section we saw that the occurrence of non-ergodicity over a finite region of parameter space, which we will refer to as Toom-like behavior, in the irreversible NEC rule was due to the ability of both phases to eliminate islands of the other even in the presence of a symmetry-breaking field. It is of interest to determine whether such unusual interface dynamics can occur in the continuum. In this section we present our analysis of different versions of the NEC rule: (A) discrete variables on a lattice evolving in continuous time (B) continuous variables on a lattice evolving in continuous time and finally (C) continuous variables in continuous space-time. We will demonstrate that Toom-like behavior can persist in these models as long as we incorporate the asymmetry of the neighborhood in the time-evolution equation.

A. Discrete Variables on a Lattice in Continuous Time

This case is implemented by considering the sequentially-updated version of the PCA with the NEC rule. We have performed numerical simulations on lattices up to $100 \times 100$ with the spins being chosen at random and updated with the local transition probability $Q$ given by Eq. (5.1.1). We have found that for random initial conditions the system reaches a unique "paramagnetic phase" with $m = \langle S_4 \rangle$ non-zero when the parameters $p$ and $q (p=q)$ are chosen deep in the expected one-phase
region. In contrast, in the expected two-phase coexistence region depending on the magnetization of the initial configuration \( m_0 \) two different asymptotic stationary states with positive and negative values of \( m \) were obtained. In addition, we have verified numerically that in the 2-phase region both phases eliminate islands of the other phase. Thus, in this case we have empirically established that Toom-like behavior obtains.

B. Continuous Variable on a Lattice with Time Continuous

Consider continuous dynamical variables \( \{\psi_i(t)\} \) on a square lattice whose sites are denoted by \( i \) (we retain the discreteness of space) and which evolve in continuous time \( t \). We write down a phenomenological TDGL equation which incorporates the asymmetry of the neighborhood:

\[
\frac{d\psi_i}{dt} = -c[\psi_i + u\psi_i^3 - c\nabla_L^2\psi_i - h - \alpha(\psi_{i+1} - \psi_{i-1} + \psi_{i+N} + \psi_{i-N}) + \zeta_i]
\]  

(5.3.1)

where \( \nabla_L \) is the lattice gradient and \( h \) is the symmetry-breaking field.

The choice of the term proportional to \( \alpha(\alpha>0) \) is motivated by the form of the transition probability for the discrete model (see Eqs. (5.1.1) and (5.1.2)). The asymmetry of the neighborhood appears in this term (Toom-like term). When \( \alpha=0 \), the ordinary TDGL equation for a lattice Ising model is recovered. One expects the system to exhibit Toom-like behavior when \( \alpha \) is increased and we establish this below.

Consider the noiseless limit with \( h=0 \). The uniform stationary solution is given by \( \psi^* = 0 \) for \( r \geq 3\alpha \); there are 3 solutions
\[ \psi^* = 0, \psi^* = \pm \sqrt{3\alpha - r} \] for \( r < 3\alpha \). (We have set \( u = 1 \) for convenience.) Let an infinite, flat, interface be located along the \( x \)-axis at \( t = 0 \) separating an uniform "up-phase" with value \( \psi^+ \) from a uniform "down-phase with value \( \psi^- \). We study the time evolution of the interface and determine its asymptotic \((t \to \infty)\) behavior.

The case \( c = 0 \) can be studied analytically. Since \( \psi \) varies only along the \( y \)-direction Eq. (5.3.1) becomes a 1d equation in space. Denoting the \( y \)-coordinate by \( \xi \) we have

\[
\frac{d\psi^*}{dt} = \gamma [(\tau - 2\alpha)\psi^* + \psi_{n=0}^3 + \alpha\psi^*_{n=1}]
\]

\[ \equiv L[\psi^*, \psi_{n=1}] \quad (5.3.2) \]

The initial conditions are \( \psi^*_k = \psi^+ \) for \( k \geq 0 \) and \( \psi^*_k = \psi^- \) for \( k < 0 \). Clearly \( \psi^*_k(t) = \psi^+ \) for \( k \geq 0 \) and all \( t \). Next we discuss the asymptotic behavior of \( \psi^*_k(t \to \infty) \) for \( k < 0 \). Consider \( \psi^* \) for \( k = -1 \) whose time evolution is described by

\[
\frac{d\psi_{-1}}{dt} = L[\psi_{-1}, \psi^+] \quad (5.3.3)
\]

with \( \psi_{-1}(t = 0) = \psi^- \). \( \psi_{-1}(t \to \infty) \) is given by the zero of \( L \) whose domain of attraction contains the initial value of \( \psi_{-1} \), i.e., \( \psi^- \). The zeros of \( L[\psi, \psi^+] \) are \( \psi^+ \), and \( \psi_{a,b}^- = \frac{1}{2}[\psi^+ \mp \sqrt{-(\alpha + 3\tau + \alpha^2)}] \). Two cases must be distinguished.

a) \( \alpha < -r \): In this case, \( \psi^- < \psi_{a,b}^- \psi_b^- < 0 \). It is easy to verify that \( L[\psi_{-1}, \psi^+] > 0 \) for \( \psi_{-1} < \psi_b^- \). Consequently \( d\psi_{-1}/dt > 0 \) for \( \psi_{-1} < \psi_b^- \) and \( \psi_{-1}(t \to \infty) = \psi^+ \). (In other words, the initial value \( \psi^- \) belongs to the domain of attraction of \( \psi^+ \)). Having determined \( \psi_{-1}(t \to \infty) \), we can deduce \( \psi_{-2}(t \to \infty) \) and so on. \( \psi_{-2}(t \to \infty) \) is in turn determined by the zeros
of $L[\psi_2, \psi^+_{b}]$. Since, $L[x,y]$ is a monotone increasing function of $y$, $L[\psi, \psi_{-1}] < L[\psi^+, \psi^+]$ and therefore, $\psi_{-2}(t) < \psi_{-1}(t)$. For $t=\infty$ this implies $\psi_{-2}(t=\infty) < \psi_{b}$. Since, $L[\psi^+, \psi_{-1}] > 0$ for $t>0$, we conclude that the zero of $L[\psi_{-2}, \psi_{b}] > 0$ occurs between $\psi_{-}$ and $\psi_{b}$ and $\psi_{-2}(t)$ converges asymptotically to this value. Similarly, $\psi_{-3}(t)$ converges asymptotically to a value between $\psi_{-}^+$ and $\psi_{-2}(t=\infty)$. These values $\psi_{-1}(t=\infty), \psi_{-2}(t=\infty), \ldots$ approach $\psi_{-}^+$ rapidly. In fact, a few lattice spacings from $l=0$ (of the order of the coherence length $(3\alpha-r)^{-\frac{1}{2}}$) the asymptotic, stationary value is practically $\psi_{-}^+$. The rather well-localized interface is thus stable and has zero velocity.

b) $\alpha > -r$: $\psi_{a,b}^+$ are not real and the only real zero of $L[\psi, \psi^+]$ is $\psi_{-}^+$. Hence, $\psi_{-1}(t=\infty) = \psi_{+}^+$. Similarly, we can argue that $\psi_{k}(t=\infty) = \psi_{+}^+$ for all $k$, or in other words the interface moves down and is unstable. The system eventually becomes a homogeneous, one-phase system. One can understand this as follows. As $\alpha$ increases, the term $\Gamma \psi_{k+1}$ becomes more important in Eq. (5.3.2) and when $\alpha > |r|$ this term dominates, forcing $\psi_{k}^+$ to have the same value as $\psi_{k+1}^+$.

The vertical ($0^\circ$-) boundary behaves exactly as the horizontal one by symmetry, i.e., for $\alpha < -r$ it has zero velocity while for $\alpha > -r$ it has finite velocity.

Now consider the $45^\circ$-interface with all the spins "above" it being $\psi_{+}^+$ and spins below it being $\psi_{-}^+$ at $t=0$. Let $l$ label the diagonal rows. The corresponding equation is

$$\frac{\partial \psi_{l}^+}{\partial t} = -\Gamma[(r-\alpha)\psi_{l}^+ + \psi_{l}^3 - 2\alpha \psi_{l+1}]$$  (5.3.4)

Following exactly the analysis outlined for Eq. (5.3.2) we obtain the
result that for \( \alpha > -r/5 \) the 45°-interface moves with finite velocity.

We conclude then that in the range \( \frac{1}{5} \alpha < |r| \), with \( r < 0 \), the horizontal and vertical interfaces are stationary while the 45°-interface moves with constant velocity. This is clearly sufficient to prove that "Toom-like" behavior persists since all minority domains will be wiped out (see argument in Sec. 5.1). When \( \alpha = -r \), one can evaluate the velocities numerically and verify that \( v(45°) > \sqrt{2v(0°)} \) which is sufficient to eliminate minority islands of either sign.

When \( h \neq 0 \), the uniform stationary solution is given by

\[
\psi^3 + (r-3\alpha)\psi^2 - h = 0 \tag{5.3.5}
\]

Phase coexistence can occur only if Eq. (5.3.5) has more than one real root. This happens if

\[
\left( \frac{r-3\alpha}{3} \right)^3 + \left( \frac{h}{2} \right)^2 \leq 0 . \tag{5.3.6}
\]

Condition (5.3.6) can always be satisfied by making \( h \) sufficiently small and \( r \) large and negative. The solution of (5.3.5) will be of the form \( \psi = ah + bh^2 + \ldots \). Following the same analysis as in the \( h = 0 \) case, we deduce the conditions for the 0°-interface to be stationary and the 45°-interface to drift in a southwesterly direction irrespective of which phase is on which side of the boundary. The conditions are (to linear order in \( h \)).

\[
|r| - \alpha > \frac{3h}{\sqrt{3\alpha - r}} , \quad \text{and} \quad 5\alpha - |r| > \frac{3h}{\sqrt{3\alpha - r}} \tag{5.3.7}
\]

So, we see that for small but finite \( h \), we can choose \( r \) and \( \alpha \) such that the conditions (5.3.6) and (5.3.7) are satisfied and Toom-like behavior persists.
When $c>0$, we had to resort to numerical solution of the set of differential equations given by (5.3.1) with $\zeta=0$. We have verified that the scenario outlined above continues to be valid. Again when $\zeta<0$, we only expect the velocities to be changed by terms $O(\sigma)$ and for sufficiently small noise our results will continue to hold.

C. Continuous Variables in Continuous Space-time

We report preliminary results for the occurrence of non-ergodic behavior in a finite region of parameter space in systems in which the variables and space-time are continuous.

We begin by postulating a phenomenological TDGL equation which incorporates the asymmetry of the neighborhood. A naive first attempt is to add terms which are not invariant under parity ($\vec{r} \rightarrow -\vec{r}$), say the continuum limit of the Toom-like term in Eq. (5.3.1), i.e., $\vec{\nabla} \psi$.

Consider the model described by

$$\frac{\partial \psi}{\partial t} = -\Gamma [-r\psi - \psi^2 \psi_{xy} - \alpha_0(\psi_x + \psi_y)] + \zeta \quad (5.3.8)$$

where $r>0$ and we have suppressed the $\vec{r}$ and $t$ dependence of $\psi$. Our discussion will ignore the noise term which for small values of the variance leaves the qualitative phenomenology unaltered.

Now imagine a flat interface with normal along an arbitrary direction $\hat{z}$ which makes an angle $\theta$ with the $x$-axis. It is easy to see that the solution of (5.3.8) with $\zeta=0$ is

$$\psi(z,t) = \sqrt{r/u} \tanh \left( \sqrt{r/2} (z+c_0(s+c)t) \right) \quad (5.3.9)$$

where $s=\sin\theta$, $c=\cos\theta$ and $z$ is the co-ordinate along $\hat{z}$. Equation (5.3.9) describes an interface moving with a velocity
\[ V(\theta) = -2\alpha_0 (\sin \theta + \cos \theta) \]. So in this model, a flat interface of arbitrary orientation will move (except for \( \theta = 135^\circ \) where \( s + c = 0 \)) and the 45°-interface moves with the maximum velocity \(-\sqrt{2} \alpha_0\). Note, however, that \( v(45^\circ) = \sqrt{2} v(0^\circ) \) and hence, it is clear that an isosceles right triangle will drift with a speed \( \alpha_0 \) and will not be eliminated. (This ignores curvature effects, i.e., treats the boundaries of the finite island as infinite lines). Of course, the inclusion of curvature eliminates the islands but this is no different from the Ising model and a small symmetry-breaking field renders one phase unstable. So the model described by Eq. (5.3.8) fails to exhibit Toom-like behavior. This is in contrast to the system described by Eq. (5.3.2) which contains only linear lattice gradients. However, in the continuum these correspond to the inclusion of higher derivatives.

So next we try adding an asymmetric term of the form \( c_0(\psi_{xxx} + \psi_{yyy}) \) to Eq. (5.3.8). The motion of a flat interface with normal along \( \hat{z} \) \((\hat{z} \cdot \hat{x} = \cos \theta)\) is described by

\[
\frac{\partial \psi}{\partial t} = -\Gamma [-r\psi - \psi_{zz} + \psi^3 - \alpha_0(\theta)\psi_z - \epsilon_0(\theta)\psi_{zzz}] \tag{5.3.10}
\]

where \( \alpha_0(\theta) = \alpha_0(s + c) \) and \( \epsilon_0(\theta) = \epsilon_0(s^3 + c^3) \). We rescale \( \psi, z, \) and \( t \) such that Eq. (5.3.10) has the following canonical form

\[
\frac{\partial \psi}{\partial t} = \psi + \frac{1}{2} \psi_{zz} - \psi^3 + \alpha \psi_z + \epsilon \psi_{zzz} \tag{5.3.11}
\]

Again, the co-ordinate transformation, \( z + z + at \) allows us to transform the \( \psi_z \) term away and we obtain

\[
\psi_t = \psi + \frac{1}{2} \psi_{zz} - \psi^3 + \epsilon \psi_{zzz} \tag{5.3.12}
\]
When \( c=0 \), a stationary solution to Eq. (5.3.12) obeying the appropriate spatial boundary conditions is the soliton solution \( \psi_0(z)=\tanh z \). For small \( c \) we do a perturbation expansion \(^93\) in \( c \) and attempt to determine the velocity of the interface to lowest order in \( c \). We seek a solution of a form which separates the two effects of the perturbation, the change in the velocity and the change in shape, i.e.,

\[
\psi(z,t) = \psi_0(y) + \sum_{j=1}^{\infty} c^j \psi_j(y,t')
\]

(5.3.13)

where \( y=z+c(c) t \) and \( t'=t \). We choose \( c(c) \) order by order in \( c \) such that it is the velocity of the travelling wave. Then the second term merely describes a change in shape of the soliton. Let \( c(c) = \sum_{k=1}^{\infty} c_k c^k \); substituting (5.3.13) into (5.3.12) and by comparing terms we obtain to \( O(c) \):

\[
C_1(\psi_0)_{yy} + (\psi_1)_t = \psi_1 - 3\psi_0\psi_1 + \frac{1}{2} (\psi_1)_{yy} + (\psi_0)_{yyy}
\]

\[
\equiv L\psi_1 + (\psi_0)_{yyy}
\]

(5.3.14)

where we have defined the operator \( L \) by

\[
L \equiv 1 - 3\psi_0^2 + \frac{1}{2} \frac{\partial^2}{\partial y^2}
\]

(5.3.15)

We now proceed to solve Eq. (5.3.14) for \( \psi_1 \) and determine \( C_1 \) self-consistently. Let \( \{u_n\} \) be the eigenfunctions of the linear, self-adjoint operator \( L \) with eigenvalues \( \{\lambda_n\} \). These can be evaluated explicitly \(^94\) and one finds that \( \lambda_0=0 \) and \( \lambda_j<0 \) for \( j\geq1 \). We also note that the eigenfunction, \( u_0 \), corresponding to the zero-mode is given by \( u_0=\sqrt{\frac{3}{2}}(\psi_0)_{yy} \). We expand \( \psi_1 \) and \( (\psi_0)_{yyy} \) in terms of \( \{u_n(y)\} \):

\[
\psi_1 = \sum_{n=0}^{\infty} a_n(t)u_n(y) \; ; \; (\psi_0)_{yyy} = \sum_{n=0}^{\infty} b_n u_n(y)
\]

(5.3.16)
Since \( \psi_0(y) \) and \( \{u_n(y)\} \) are known, \( b_n \) can be evaluated explicitly. Substituting Eq. (5.3.16) in Eq. (5.3.14) and using the orthonormality of \( \{u_n\} \) we find

\[
\dot{a}_n(t) = \lambda_n a_n(t) + \left( b_n - C_1 \frac{2}{\sqrt{3}} a_n, 0 \right)
\]  

(5.3.17)

We choose \( C_1 \) such that \( a_0 \) vanishes. This enforces the separation in Eq. (5.3.13) whereby the second term only describes a change of shape. If \( \dot{a} \neq 0 \), this leads to a term proportional to \( \epsilon t \psi_0' \) in \( \psi_1 \) which for small \( t \) corresponds to translating \( \psi_0 \) by \( O(\epsilon t) \), i.e., a Galilean boost. Mathematically, the term linear in \( t \) corresponds to a resonance and must be forced to vanish. It is easily verified that this can be done to all orders in \( \epsilon \). To \( O(\epsilon) \) this leads to \( C_1 = \frac{\sqrt{3}}{2} b_0 \). Since \( \lambda_n < 0 \) for \( n \geq 1 \), \( a_n(t) \) reaches a constant \( f_n = -b_n/\lambda_n \) asymptotically. Hence, we obtain the asymptotic form of \( \psi \) to \( O(\epsilon) \):

\[
\psi(z, t) = \psi_0(y) + \epsilon \psi_1(y, t)
\]

\[
= \tanh(z + \frac{\sqrt{3}}{2} b_0 \epsilon t) + \epsilon \sum_{j=1}^{\infty} f_j u_j(z + \frac{\sqrt{3}}{2} b_0 \epsilon t) + O(\epsilon^2)
\]  

(5.3.19)

We have then a travelling wave solution with velocity \( -\frac{\sqrt{3}}{2} b_0 \epsilon \). An explicit evaluation of \( b_0 \) from Eq. (5.3.16) yields \( -8\sqrt{3}/15 \) and hence, the velocity \( v \) is given by \( \frac{3}{5} \) up to terms \( O(\epsilon^2) \). Transforming back to the original variables yields the interface velocity to \( O(\epsilon) \):

\[
v(\theta) = \Gamma \left[ -a_0(s+c) + \frac{2}{5} \epsilon_0 (s^3+c^3)r \right]
\]  

(5.3.20)

Now consider an right triangular domain with the hypotenuse along \( \theta \) and neglect curvature effects. Even though all three edges
move the hypotenuse moves sufficiently fast to eliminate the island.

To see this let \( L_0 \) be the height of the triangle initially. After a
time \( t \) the height of the triangle can be shown to be

\[
L(t) = \frac{c}{s} |v(0)| t + L_0 - \frac{|v(0)| t}{s} + t |v(0)| .
\] (5.3.21)

Therefore,

\[
L(t) - L_0 = \frac{t}{s} [(s+c)|v(0)| - |v(0)|] < 0 .
\] (5.3.22)

So the triangular island will disappear in a time proportional to
\( L_0/\epsilon \). Hence, the addition of small symmetry-breaking field and noise
will not alter our results.

Finite curvature effects will modify the interface velocities
calculated above and our analysis has to be generalized. A careful
analysis including the effect of curvature is currently underway.
Intuitively, it should enhance the shrinking of the domain and we
expect Toom-like behavior to persist.

In conclusion, in this chapter we have analyzed several
continuous versions of the NEC rule and demonstrated that the
occurrence of non-ergodic behavior in a finite region of parameter
space depends on incorporating the asymmetry of the neighborhood and
the discreteness of the variables and/or space-time is unimportant.
APPENDIX A

A CONNECTION BETWEEN SEQUENTIALLY-UPDATED AND SIMULTANEOUSLY-UPDATED INFINITE-RANGED PCA
In this Appendix we show that the extrema of the free-energy of sequentially-updated, $\omega$-ranged PCA are in one-to-one correspondence with the fixed points of the iterative map describing the time evolution of simultaneously-updated CA with the same transition probability $Q$. We have shown that every sequentially-updated, $\omega$-ranged rule with the transition probability

$$Q(S_1'|[S_j]) = \frac{1}{2} (1+S_1'|F) = \frac{1}{2} [1+S_1'(f_1+S_1'f_2)]$$  (A.1)

corresponds to a Hamiltonian. In particular, when the rule is totalistic the Hamiltonian is given by [cf. Eq. (2.2.25)]

$$H(M) = -N \int_0^M \tanh^{-1} \left( \frac{f_1(m')}{f_2(m')} \right) \, dm'$$  (A.2)

where $\tilde{f}_2 = 1-f_2$. The Gibbs free energy of the equilibrium system described by (A.2) in units of $k_B T$ is,

$$G = H + N[(\frac{1+M}{2})\ln(\frac{1+M}{2})+(\frac{1-M}{2})\ln(\frac{1-M}{2})]$$  (A.3)

where the second term is the entropy of the system (in units of $k_B$).

The extrema of this free energy, $m_0$, are given by $\frac{\partial G}{\partial M}|_{m_0} = 0$. Upon differentiating (A.3) we have,

$$m_0 = f_1(m_0)/\tilde{f}_2(m_0) \ .$$  (A.4)

The minima of $G$, $\tilde{m}$, satisfy $\partial^2 G/\partial M^2|_{\tilde{m}} > 0$. After some straightforward manipulations, we get

$$[f_1(\tilde{m})/\tilde{f}_2(\tilde{m})]' - 1 < 0 \ .$$  (A.5)

Now, consider the same rule (A.1) applied simultaneously. From Eq. (2.16)-(2.18), the magnetization obeys the iterative equation
\[ M(t+1) = f_1(M(t)) + M(t) f_2(M(t)). \]  
(A.6)

The fixed point of (A.6) is given by

\[ m^* = f_1(m^*)/\tilde{f}_2(m^*). \]  
(A.7)

We see from (A.4) and (A.7) that \( m_0 \) and \( m^* \) are in one-to-one correspondence as claimed. Moreover, stable fixed points satisfy,

\[ |F'(M)|_{m^*} < 1, \text{ i.e.,} \]

\[-2 < f_1' - \tilde{f}_2 - m^* \tilde{f}_2' < 0. \]  
(A.8)

We see immediately that (A.8) implies (A.6) but not vice versa.
APPENDIX B

DERIVATION OF $H_{\text{eff}}$ IN EQUATION (2.3.19)
Let $\psi_1(t)$ be a classical random variable (where $\mathbf{i}$ may be taken to represent the sites of a discrete lattice or continuous space) which satisfies a non-linear equation of motion:

$$\dot{\psi}_1(t) = V_1([\psi_j(t)]) + \zeta_1(t) \quad (B.1)$$

where $\zeta_1(t)$ represents a Gaussian noise variable with zero mean and $\langle \zeta_1(t)\zeta_j(t') \rangle = 2\Gamma_{ij}\delta(t-t')$. The study of the statistical dynamics of such systems, in particular, the calculation of correlation functions and response functions poses formidable problems. In Ref. 60 Martin, Siggia, and Rose (MSR) constructed a formal Heisenberg operator theory by introducing a fictitious "adjoint" operator $\tilde{\psi}_1(t)$ for each classical variable $\psi_1(t)$ and appropriate equal-time commutation relations to generate the dynamics. The calculation of response functions is done by computing expectation values of time-ordered products of operators. A more transparent reformulation of MSR's results is provided by the functional integral approach elucidated in Refs. 61. We give a brief derivation of $H_{\text{eff}}$ (Eq. (2.3.19)) using this approach.

We wish to compute the noise average of $F([\psi_j(t)])$ where $F$ is any arbitrary functional of $[\psi_j(t)]$. This will be accomplished by re-expressing the average over $[\zeta_j(t)]$ as an average over $[\psi_1(t)]$. To this end we write the noise average as

$$\langle F([\psi_j(t)]) \rangle = \langle \Pi \int dx_k(t) \delta(x_k - \psi_k) \rangle F([x_j(t)]) \quad (B.2)$$

(Note that the integrals over $x_k(t)$ constitute functional integrals.)

Now we change variables in the argument of the $\delta$-functionals as follows. Since $\psi_j(t)$ are solutions of Eq. (B.1) we have

$$\Pi \delta(x_j(t) - \psi_j(t)) = \Pi \delta(\dot{x}_j - V_j - \zeta_j) \quad (B.3)$$
where \( J \) is a functional Jacobian. These formal manipulations can be assigned meaning as usual by discretizing time and adopting a limiting process. The Jacobian can be shown to depend on the limiting procedure; these different representations are presumably equivalent. We will not explicitly display the Jacobian henceforth. Using the standard representation of the \( \delta \)-functional

\[
\delta(\dot{\psi}_j - \nu_j - \zeta_j) \sim \int \mathcal{D} \tilde{\psi}_j \ e^{i \int dt \tilde{\psi}_j (t)(\dot{\psi}_j - \nu_j - \zeta_j)},
\]

we can re-write Eq. (B.2) as

\[
\langle F[\psi_j] \rangle = Z^{-1} \int \mathcal{D} \psi_j \int \mathcal{D} \tilde{\psi}_j \ F[\psi_j] \ .
\]

\[
e^{\frac{i}{2} \int dt \tilde{\psi}_j (t)[\psi_j - \nu_j] \ - \frac{i}{2} \int dt \tilde{\psi}_j \zeta_j},
\]

where the partition function \( Z = \langle 1 \rangle \). The noise average can be performed easily since these are Gaussian integrals and we obtain,

\[
\langle F[\psi_j] \rangle = Z^{-1} \int \mathcal{D} \psi_j \mathcal{D} \tilde{\psi}_j \ F[\psi_j] \ .
\]

\[
e^{\int dt [1(\dot{\psi}_j - \nu_j)\tilde{\psi}_j - \Gamma \tilde{\psi}_j^2]}.
\]

We can now read off the effective Hamiltonian from the exponential:

\[
-\mathcal{H}_{\text{eff}} = \sum_j \int dt \ [1\tilde{\psi}_j (\dot{\psi}_j - \nu_j) - \Gamma \tilde{\psi}_j^2]
\]

(B.7)

For continuous space, we have,

\[
-\mathcal{H}_{\text{eff}} = \int dt \int d^d x \ [i\tilde{\psi}(x)[\dot{\psi}(x) - V(\psi(x))] - \Gamma \tilde{\psi}^2(x)]
\]

(B.8)

which is Eq. (2.3.19).
APPENDIX C

REVIEW OF SINGLE-VARIABLE QUADRATIC MAP
Non-linear, one-variable, iterative maps have been studied for a long time. In particular, the period-doubling route to chaos and the associated scaling behavior have been investigated intensively.\textsuperscript{3,5,6} We provide a brief review of relevant results for the quadratic map in this Appendix.

Consider the iterative map

\[ x_{n+1} = r x_n (1-x_n) \equiv f(x_n) \]  \hfill (C.1)

where \( x_n \in [0,1] \subseteq I \) is the dynamical variable at the \( n \)th time step. The control parameter \( r \) is restricted to \([0,4]\) so that if \( x_n \in I, x_{n+1} \in I \). The iterates \( \{x_j\} \) form a sequence which is referred to as the trajectory. The problem is to determine the asymptotic \((n \to \infty)\) behavior of \( \{x_j\} \) for any initial \( x_0 \) for different \( r \). In general, the sequence will settle into a subset of \( I \), called the attractor, for almost all \( x_0 \), for a given \( r \). The class of maps we focus on have a unique, stable attracting set. We would like to see how the attractor changes as \( r \) is varied.

When \( 1 < r < 3 \) the attractor contains one point, \( x^* \), the fixed point of the map \((x^* = f(x^*) = (r-1)/r)\). The trajectory becomes stationary (executes a 1-cycle) asymptotically. When \( r = 4 \), the substitution \( x_n = \sin^2 \theta_n \) yields

\[ \theta_{n+1} = 2 \theta_n \quad (\theta_n \in [0, \pi/4]) \]

\[ = \pi - 2 \theta_n \quad (\theta_n \in [\pi/4, \pi/2]) \]  \hfill (C.2)

It is clear that \( \theta_n \) (and \( x_n \)) behave chaotically. The attractor in \( x \)-space is almost all of \( I \). In fact, an invariant, ergodic measure \( \rho \)
exists which crudely represents a histogram of how often each point is visited. For $r=4$, $\rho = (x(1-x))^{-\frac{1}{2}}$. The attractor has a stretching property, or equivalently, the system has sensitive dependence on initial conditions. There is an exponential separation of infinitesimally close points characterized by the Liapunov exponent $\lambda$. For two sequences $\{x_n\}$ and $\{y_n\}$ where $y_0 = x_0 + \epsilon$, one expects $|y_n - x_n| \sim e^{\lambda n}$. Note that since the interval $I$ is bounded, for a given $\epsilon, n$ cannot be too large! We define $\lambda$ for the one-variable map by

$$\lambda = \lim_{n \to \infty} \lim_{\epsilon \to 0} \frac{1}{n} \ln \left| \frac{y_n - x_n}{\epsilon} \right|$$

(C.3)

For $r=4$, using the transformation to $\theta$ variables it can be shown that $\lambda = 2\ln 2$. It is also easy to demonstrate that (C.3) is equivalent to

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \ln \left( \prod_{j=1}^{n} |f'(x_j)| \right)$$

(C.4)

where $\{x_j\}$ is the trajectory and for ergodic measures $\lambda$ is independent of $x_0$. For the fixed point $x^*$ using (C.4), $\lambda$ is negative, $\lambda = 2\ln |2-r|$, for $r \in (1, 3)$.

What happens between $r=3$ and $r=4$? What route does the system traverse in going from fixed-point behavior to chaotic behavior? We summarize the answer briefly.

A fixed point is linearly stable when $|f'(x^*)| < 1$ for this condition guarantees that points close to $x^*$ converge to it. For the quadratic map $f'(x^*) = r(1-2x^*) = 2-r$ so that for $r \in (1, 3)$ $|f'(x^*)| < 1$ and we have a stable fixed point. When $r$ is increased past 3, $x^*$ becomes unstable. By considering the once-iterated map $f(f(x)) = f^2(x)$, the
asymptotic behavior can be determined. For \( r > 3 \), two new stable fixed points of \( f^2, x_1^n, x_2^n \) on either side of \( x^n \) are generated. This is the so-called period-doubling (pitchfork) bifurcation. Since \( f(x_1^n) = x_2^n \) and \( f(x_2^n) = x_1^n \), the attractor consists of two points and the system executes a 2-cycle \( x_1^n, x_2^n, x_1^n, x_2^n, \ldots \). When \( r \) is increased further, at \( r_2 \), \( |f^2'(x_1^n)| = |f^2'(x_2^n)| = |f'(x_1^n)f'(x_2^n)| \) exceeds 1, the 2-cycle becomes unstable and a stable 4-cycle appears. This scenario recurs and an infinite sequence of period doubling bifurcations occur and accumulate geometrically \(^7\) at \( r = r_c \approx 3.569945672 \ldots \). [The condition for a \( 2^m \)-cycle to be stable is \( \prod_{j=1}^{2^m} f'(x_j^n) < 1 \) where \( \{x_j^n\} \) are the \( 2^m \) points on the cycle and this corresponds to \( \lambda < 0 \) from (C.4).]

For \( r > r_c \) the system displays chaotic behavior interrupted by periodic windows. In detail, the structure is intricate and complex. For the quadratic map, the chaotic attractor contains \( 2^m \) disjoint intervals (bands), and the trajectory moves periodically from band to band, but the precise location within each band is random. The parameter values for which periodic attractors occur form intervals in \( (r_c, 4) \) while the set of values of \( r \) for which chaotic behavior occurs contains no intervals but still has non-zero measure! This means that if \( r \) is chosen randomly there is a non-zero probability of obtaining chaotic behavior. When \( r \) is increased we have the phenomenon of band merging (inverse bifurcation) i.e., the chaotic attractor changes from containing \( 2^m \) bands to \( 2^{m-1} \) bands until for sufficiently large \( r \) one has a single band.

We explore the 2-band structure in some detail. \(^7\) Consider the iterates of the critical point \( x_c = 1/2 \). Clearly \( f(x_c) \) gives the upper bound for the attractor and \( f^2(x_c) \) the lower bound. A little thought
shows that \( f^3(x_c) \) is the lower bound of the second band and \( f^4(x_c) \) the upper bound of the first band. So the intervals \([f^2(x_c), f^4(x_c)]\) and \([f^3(x_c), f(x_c)]\) form the two bands and clearly \( f \) maps each interval onto the other. Band merging occurs when \( f^3(x_c) = f^4(x_c) \) and the gap between the bands vanishes.

For \( r > r_c \) all the \( 2^m \)-cycles that occur for \( r < r_c \) are unstable. However, new, stable, primitive cycles are created by tangent bifurcation. Each of the primitive cycles subsequently undergoes, as \( r \) is increased, the usual cascade of pitchfork bifurcations to chaos. Hence, each cycle and its period-doubled successors occupy an interval in parameter space which is referred to as a window. The windows are shorter for longer-period primitive cycles. The last odd cycle to appear is the 3-cycle which occurs at \( r = 1 + \sqrt{3} \).
APPENDIX D

DERIVATION OF RECURSION RELATIONS TO O(1/d)
Consider a simultaneously-updated near-neighbor totalistic rule in $d(\geq 1)$ dimensions described by the local transition probability,

$$ Q(S_i'|[S_j]_{v_i}) = \frac{1}{2} [1+S_i'(f_1+S_i f_2)], \quad (D.1) $$

where $f_1$ and $f_2$ depend on the sum of the $2d=2$ neighbors of $S_i, x_i=\frac{1}{2}\sum_{j \in v_i} S_j$. We want to calculate the magnetization $M(t) = \langle S_i \rangle_t$ which obeys the equation

$$ M(t+1) = \langle f_1 \rangle_t + \langle S_i f_2 \rangle_t, \quad (D.2) $$

and the near-neighbor correlation function $0(t) = \langle S_i S_j \rangle_t - \langle S_i \rangle_t \langle S_j \rangle_t$ which clearly satisfies

$$ G(t+1) = \langle [f_1(x_i)+S_i f_2(x_i)][f_1(x_j)+S_j f_2(x_j)] \rangle_t - M^2(t+1) \quad (D.3) $$

where $i$ and $j$ are near neighbors. To $O(1/d)$ only these two quantities need to be considered. Note that the influence of $S_j$ on $S_i$ when $j \in v_i$ is $O(1/d)$. Consider next-near-neighbor spins $S_i$ and $S_k$. Correlations are sustained via the two spins which belong to both $v_i$ and $v_k$. Clearly, the correlations are $O(1/d^2)$ for large $d$. This argument can be generalized to spins farther apart and higher-spin correlations since the number of common neighbors is $O(1)$.

We now evaluate Eq. (D.2). The first term is handled by a formal power series expansions of $f(x_i)$ around $M$:

$$ \langle f_1(x_i) \rangle = f_1(M) + f_1'(M) \langle (x_i-M) \rangle + $$

$$ \frac{1}{2} f_1''(M) \langle (x_i-M)^2 \rangle + \ldots \quad (D.4) $$

By definition, $\langle (x_i-M) \rangle$ vanishes. It is easy to compute $\langle (x_i-M)^2 \rangle$:
\[(x_1-M)^2 = \frac{1}{z^2} \sum_{j,k \in \nu_1} (S_jS_k - 2MS_j + M^2)\]

\[= \frac{1}{z} (1-M^2) + O(1/z^2) . \quad (D.5)\]

We have used the fact that \(S_jS_k - (S_j)(S_k)\) is \(O(1/z^2)\) when \(j, k\) are not near neighbors. Higher-order terms in Eq. (D.4) can be shown to be at most \(O(1/z^2)\) since higher powers of \(x_1-M\) involve higher-order correlation functions. Therefore, to \(O(1/d)\) we have

\[f_1(x_1) = f_1(M) + \frac{1}{2} f_1''(M) \frac{1}{2d} (1-M^2) . \quad (D.6)\]

Similarly, we compute the second term on the right hand side of Eq. (D.2):

\[S_1f_2 = Mf_2(M) + f_2'(M) \frac{1}{z} \sum_{j} (S_j(M))\]

\[+ \frac{1}{2} f_2''(M) \frac{1}{z^2} \sum_{j} (S_j(M))^2 + \]

\[\ldots \quad (D.7)\]

We have,

\[\sum_{j \in \nu_1} (S_j-M) = zG \quad (D.8)\]

\[\sum_{j \in \nu_1} (S_j-M)^2 = \sum_{j,k} (S_jS_k - 2MS_j + M^2)\]

\[= z(M(1-M^2) + O(1) . \quad (D.9)\]

Again, we can argue that the terms we have neglected are at most \(O(1/d^2)\). Gathering all the terms together we obtain, to \(O(1/d)\):
\[ M(t+1) = F(M(t)) + \frac{1}{4d} (1-M^2(t)) \left[ f_1''(M(t)) \right. \\
\left. + M(t) f_2''(M(t)) \right] + f_2'(M(t)) G(t). \] (D.10)

This is precisely the first equation in Eq. (2.4.2).

Proceeding analogously we can derive the equation for \( G \) to \( O(1/d) \).

Note

\[ G(t+1) = (F(x_1)F(x_j))_t - M^2(t) \]
\[ = (f_1(x_1)f_1(x_j)) + (f_2(x_1)f_1(x_j)s_1) \]
\[ + (f_1(x_1)f_2(x_j)s_j) + (f_2(x_1)f_2(x_j)s_1s_j) \]
\[ - M^2(t). \] (D.11)

Each of the terms can be evaluated as before to \( O(1/d) \) and we arrive at

\[ G(t+1) = f_2^2(M(t))G(t) + \frac{(1-M^2(t))}{d} f_2'(M(t))[f_1'(M(t)) + (M(t)f_2'(M(t))]. \] (D.12)
APPENDIX E

ILLUSTRATION OF THE FORMALISM IN CHAPTER IV USING RULE 90
In this appendix, we illustrate the use of the general formalism developed in Chapter IV by determining the state transition diagram for a simple rule. We have chosen the one-dimensional nearest-neighbor mod 2 rule (rule 90 in Ref. 7), i.e., \( \sigma_1(t+1) = \sigma_{i-1}(t) + \sigma_{i+1}(t) \mod 2 \) as a pedagogical example. We proceed as follows to obtain the state transition diagram for a lattice with \( N = 2^r \) sites.

1. Write down the transition matrix and its eigenvalues. The transition matrix \( A \) (see (4.2.2)) for this rule is given in terms of \( a_1 = a_{N-1} = 1 \) with the rest of \( a_i \)'s being zero. The eigenvalues of \( A \) are determined from (4.2.8):

\[
\lambda_k = \xi^k + \xi^{N-1-k} \quad (k = 0, 1, \ldots, N-1)
\]

where \( \xi \) is an \( r \)th primitive root of unity.

2. Determine the order of the eigenvalues. This is done as follows:

Given \( N \) and hence \( r \), one factorizes \( x^r - 1 \) on \( GF(2) \). \( x^r - 1 = f_1(x) \cdots f_n(x) \).

Here \( f_n(x) \) is an \( n \)th degree polynomial whose roots are \( r \)th primitive roots of unity, i.e. \( f_n(\xi) = 0 \). Then, one finds the smallest \( m \) such that \( (\lambda_k)^m = 1 \mod f_n(\xi) \) (Since \( m | 2^N - 1 \), one only needs to try those integers which are factors of \( 2^N - 1 \)). The order of \( \lambda \) is given by \( m \).

For example, let \( N = 15 \). Factorizing \( X^{15} - 1 \) gives \( n = 4 \) and \( f_4(x) = x^4 + x + 1 \).

Since \( 2^N - 1 = 15 \), the order of \( \lambda_k \) can only be 1, 3, 5 or 15. For \( \lambda_1 = \xi + \xi^{-1} \), it turns out \( \lambda_1^{15} = 1 \mod f_4(\xi) \). Therefore, \( \text{ord}(\lambda_1) = 15 \).

Similarly, one can find the orders of the rest of the eigenvalues.

The results are: \( \lambda_0 = 0 \), two eigenvalues have order one (i.e., \( n_1 \), the number of order one eigenvalues, is 2), four have order three \( (n_3 = 4) \) and eight have order fifteen \( (n_{15} = 8) \).

3. Determine the Jordan form of \( A \), cycle lengths and dimension of subspace corresponding to various cycles. For \( N = 15 \), \( A \) can be
diagonalized (Theorem 2.1), so the cycle lengths are simply given by
the order of \( \lambda \)'s, i.e., 1, 3, and 15 (Theorem 4.3). The dimension \( D_1 \)
of the subspace which corresponds to cycle of length 1 is calculated
as follows (see the end of Sec. 4.3). \( D_1 = n_1 = 2 \), \( D_3 = n_1 + n_3 = 6 \) and
\( D_{15} = n_1 + n_3 + n_{15} = 14 \). The state transition diagram follows from the
above. Since there is one zero eigenvalue (\( d_m = 1 \)), the rule is
irreversible. Thus the height of the tree rooted on each vertex of
cycles is one (corollary in Sec. 4.3). The fraction of states which
are on cycles is \( p^{-d_m} = 2^{-1} \) (corollary 4.3). The multiplicity of
of cycles are given by (4.3.16) and (4.3.17): \( m(1) = 2^2 - 1 = 3 \).
\( m(3) = [2^6 - m(1) - 1]/3 = 20 \) and \( m(15) = [2^{14} - 3m(3) - m(1) - 1]/15 = 1088 \).
Including the trivial one-cycle corresponding to the zero state, the
multiplicity of cycles of length one is \( m(1) + 1 = 4 \). Similarly, one can
work out the even \( N \) case. If \( N = 2^k \cdot r \), we get the same eigenvalues as
in \( N = r \) except that in this case the degeneracy of eigenvalues are
increased by a factor \( 2^k \). For example, for \( N = 30 = 2 \cdot 15 \), the cycle
lengths are: 1, 3, 15, 2x1, 2x3, 2x15 while for \( N = 60 = 2^2 \cdot 15 \), the lengths
are: 1, 3, 15, 2x1, 2x3, 2x15, 4x1, 4x3, 4x15 (Theorem 4.3). For \( N = 30 \), the
dimension of zero-eigenspace \( d_m \) is 2. Therefore, the height of height
of each tree is 2 and the fraction of states which are on cycles is
\( p^{-d_m} = 2^{-2} \). The multiplicity of each cycle length is given by (4.3.16)
and (4.3.17) with \( D_1 = 2 \), \( D_2 = 4 \), \( D_3 = 6 \), \( D_6 = 12 \), \( D_{15} = 14 \) and \( D_{30} = 28 \).
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