Model-Based Feedback Control of Subsonic Cavity Flows -
Control Design

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

By

Xin Yuan, B.E., M.S.

* * * * *

The Ohio State University

2006

Dissertation Committee:
Andrea Serrani, Co-Adviser
Hitay Özbay, Co-Adviser
Mohammad Samimy

Approved by

Co-Adviser

Co-Adviser
Graduate Programm of
Electrical Engineering
In this dissertation, we present and discuss development, implementation, and experimental results of reduced-order model based feedback control of subsonic cavity flows. Model based feedback control of subsonic flows have been studied and implemented by the flow control group at the Collaborative Center of Control Science (CCCS) at the Ohio State University (OSU). The team, composed of researchers from the departments of Electrical Engineering and Mechanical Engineering at the Ohio State, the Air Force Research Laboratory, and NASA Glenn Research Center, possesses synergistic capabilities in all of the required multidisciplinary areas of experimental data acquisition, computational flow simulation, low dimensional modeling, controller design, and experimental validation.

The goal of the CCCS effort is to develop tools and methodologies for the use of closed-loop aerodynamic flow control to manipulate the flow over maneuvering air vehicles. The problem chosen for the initial study by the CCCS flow team is control of the resonant noise generated by subsonic flow past an open cavity. This phenomenon is characterized by a strong coupling between the flow dynamics and the flow-induced acoustic field that can lead to self-sustained resonance.

Two approaches towards model development have been studied in this dissertation. One aims at representing the physical properties of the system by dynamical models in transfer function forms, referred to as the physics-based linear model in
this dissertation. The other approach we have followed is based on proper orthogonal
decomposition (POD) and Galerkin projection methods involving the flow governing
equations, which is referred to as the nonlinear model or Galerkin model in the dis-
sertation. Each model mentioned above can be further divided into two types: model
derived from numerical simulation data and model derived from real time experimen-
tal data. Different types of feedback controllers have been designed for corresponding
flow models. Closed-loop system performance has been evaluated by both numeri-
cal simulation and experimental implementation as well. These results confirm that
model based feedback control represents a promising approach to flow control even
in its current infancy state.
This is dedicated to my mother and my father.
ACKNOWLEDGMENTS

I would like to express my sincere thanks to all of those who helped me throughout my doctoral program. First and foremost, I would like to thank my two advisors, Prof. Hitay Özbay and Prof. Andrea Serrani, for their intellectual guidance, constant encouragement, incredible support, and being models for me all the time.

I would like to thank everyone in the flow control group at OSU. I am especially grateful to Prof. Mo Samimy for all the opportunities and support he has provided me in the past five years, probably in more ways than I am aware. Thanks to Dr. Önder Efe for his support and friendship. Thanks to Drs. Marco DeBiasi, Peng Yan, Kihwan Kim, and Jesse Little for all their support in the lab and valuable discussions. I particular thank Edgar Caraballo for his support side by side.

I would like to acknowledge the financial support from the Collaborative Center of Control Science (CCCS) at the Ohio State University (OSU) and from Air Force Research Laboratory (AFRL). I especially thank Dr. James H. Myatt from AFRL.

I also want to thank all my friends, especially to my two best friends Yiheng Du and Long Pan for the words of encouragement. They have made my graduate study abroad not lonely.

I am most grateful to my wonderful family for teaching me the value of hard work and perseverance and encouraging me to always do my best. Thanks Mom, Dad, and my two sisters for all love, care, and motivation throughout the course of my
life. I sincerely thank my mother-in-law for her tremendous help with taking care of my daughter. Thank you especially to my husband, Jiachao, for supporting and enlightening me during every step of my life. I am indebted to my most loved daughter, Sirui, for the sacrifice of time I should be staying with you. You are the wonder of my world.
VITA

1999  . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . B.E. Harbin Institute of Technology

2001 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . M.S. Harbin Institute of Technology

2001-present . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . Graduate Fellow,
               The Ohio State University.

PUBLICATIONS

Research Publications


Mo Samimy, Marco Debsaisi, Edgar Caraballo, Hitay Özbay, Mehmet Önder Efe, Xin Yuan, Jim DeBonis, James H. Myatt “Development of Closed-Loop Control for

FIELDS OF STUDY

Major Field: Electrical Engineering
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapters</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>iv</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>v</td>
</tr>
<tr>
<td>Vita</td>
<td>vii</td>
</tr>
<tr>
<td>List of Tables</td>
<td>xiii</td>
</tr>
<tr>
<td>List of Figures</td>
<td>xiv</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Motivation</td>
<td>1</td>
</tr>
<tr>
<td>1.2 A Multidisciplinary Project</td>
<td>2</td>
</tr>
<tr>
<td>1.3 Experimental Apparatus</td>
<td>3</td>
</tr>
<tr>
<td>1.4 Literature Review</td>
<td>8</td>
</tr>
<tr>
<td>1.4.1 Passive Control vs. Active Control</td>
<td>11</td>
</tr>
<tr>
<td>1.4.2 Linear Model vs. Nonlinear Model</td>
<td>13</td>
</tr>
<tr>
<td>1.4.3 Numerical Simulation Data vs. Experimental Data</td>
<td>15</td>
</tr>
<tr>
<td>1.5 Organization of this Dissertation</td>
<td>15</td>
</tr>
<tr>
<td>2. Physics Based Linear Model and $H_\infty$ Controller Design</td>
<td>17</td>
</tr>
<tr>
<td>2.1 Delay-Based Linear Model</td>
<td>18</td>
</tr>
<tr>
<td>2.1.1 Shear layer</td>
<td>19</td>
</tr>
<tr>
<td>2.1.2 Acoustics</td>
<td>20</td>
</tr>
<tr>
<td>2.1.3 Scattering and Receptivity</td>
<td>21</td>
</tr>
<tr>
<td>Chapter 2: Parameter Tuning for the Flow Dynamics</td>
<td></td>
</tr>
<tr>
<td>------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>2.2 Parameter Tuning for the Flow Dynamics</td>
<td></td>
</tr>
<tr>
<td>2.2.1 Numerical Simulations Based Model</td>
<td></td>
</tr>
<tr>
<td>2.2.2 Experimental Data Based Model</td>
<td></td>
</tr>
<tr>
<td>2.3 H∞ Controller Design for Linear Model Based on Numerical Simulation Data</td>
<td></td>
</tr>
<tr>
<td>2.3.1 Plant Factorization</td>
<td></td>
</tr>
<tr>
<td>2.3.2 Weighting Functions</td>
<td></td>
</tr>
<tr>
<td>2.3.3 Equivalent Design</td>
<td></td>
</tr>
<tr>
<td>2.3.4 Toker-Özbay Design Results</td>
<td></td>
</tr>
<tr>
<td>2.3.5 Simulation Results</td>
<td></td>
</tr>
<tr>
<td>2.4 Linear Controller Design for Model Based on Experimental Data</td>
<td></td>
</tr>
<tr>
<td>2.4.1 Weighting Functions</td>
<td></td>
</tr>
<tr>
<td>2.4.2 Implementation Results</td>
<td></td>
</tr>
<tr>
<td>2.4.3 Smith Predictor</td>
<td></td>
</tr>
<tr>
<td>2.4.4 PID Controller Design</td>
<td></td>
</tr>
<tr>
<td>2.5 Parallel Proportional Control with Delay</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 3: POD/Galerkin Projection Method Based Nonlinear Model and LQ Controller Design</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. POD/Galerkin Projection Method Based Nonlinear Model and LQ Controller Design</td>
</tr>
<tr>
<td>3.1 Reduced-Order Modeling</td>
</tr>
<tr>
<td>3.1.1 POD and Snapshot Method</td>
</tr>
<tr>
<td>3.1.2 Galerkin Projection</td>
</tr>
<tr>
<td>3.1.3 Control Separation</td>
</tr>
<tr>
<td>3.1.4 Stochastic Estimation</td>
</tr>
<tr>
<td>3.2 Simulation Data Based Model and Feedback Controller Design</td>
</tr>
<tr>
<td>3.2.1 Equilibrium Analysis</td>
</tr>
<tr>
<td>3.2.2 Static Output Feedback Control</td>
</tr>
<tr>
<td>3.2.3 Optimal State Feedback Control</td>
</tr>
<tr>
<td>3.2.4 Simulation Results</td>
</tr>
<tr>
<td>3.3 Experimental Data Based Model and Feedback Controller Design</td>
</tr>
<tr>
<td>3.3.1 Equilibrium Analysis and Coordinate Transformation</td>
</tr>
<tr>
<td>3.3.2 LQ Controller Design</td>
</tr>
<tr>
<td>3.3.3 Mathematical Analysis of the Performance of LQ Control</td>
</tr>
<tr>
<td>3.3.4 Experimental Results</td>
</tr>
<tr>
<td>3.3.5 Dynamic Observer Design</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Chapter 4: Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>4. Conclusion</td>
</tr>
</tbody>
</table>

Appendices:
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Parameters of the linear model based on numerical simulations of the Navier-Stokes equations.</td>
<td>24</td>
</tr>
<tr>
<td>2.2</td>
<td>Parameters of the linear model based on experimental data.</td>
<td>25</td>
</tr>
<tr>
<td>2.3</td>
<td>Parameters of the weighting functions for controller design for the simulation data based model.</td>
<td>29</td>
</tr>
<tr>
<td>2.4</td>
<td>Parameters of the weighting functions for controller design for experimental data based model.</td>
<td>39</td>
</tr>
<tr>
<td>3.1</td>
<td>Robustness of the optimal LQ controllers.</td>
<td>72</td>
</tr>
<tr>
<td>3.2</td>
<td>Region of attraction of the optimal LQR state-feedback controllers.</td>
<td>75</td>
</tr>
<tr>
<td>3.3</td>
<td>Flow cases from which PIV data are acquired for POD modeling at Mach 0.3.</td>
<td>78</td>
</tr>
<tr>
<td>3.4</td>
<td>Galerkin flow models derived from PIV data described in Table 3.3.</td>
<td>78</td>
</tr>
<tr>
<td>3.5</td>
<td>Eigenvalues of the open-loop system matrices of each flow model.</td>
<td>80</td>
</tr>
<tr>
<td>3.6</td>
<td>LQ feedback control gains and values of the scaling factor $\alpha$ for each model.</td>
<td>81</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>2</td>
</tr>
<tr>
<td>1.2</td>
<td>4</td>
</tr>
<tr>
<td>1.3</td>
<td>6</td>
</tr>
<tr>
<td>1.4</td>
<td>8</td>
</tr>
<tr>
<td>1.5</td>
<td>9</td>
</tr>
<tr>
<td>1.6</td>
<td>12</td>
</tr>
<tr>
<td>1.7</td>
<td>13</td>
</tr>
<tr>
<td>2.1</td>
<td>19</td>
</tr>
<tr>
<td>2.2</td>
<td>20</td>
</tr>
<tr>
<td>2.3</td>
<td>24</td>
</tr>
<tr>
<td>2.4</td>
<td>25</td>
</tr>
</tbody>
</table>
2.5 Bode plot of the weighting functions and the plant based on numerical simulation data. ........................................... 29

2.6 Impulse response of $H_{FIR}(s)$. ............................................. 34

2.7 Magnitude of $H(s)$, $H_{FIR}(s)$ and $H_{IIR}(s)$. ......................... 35

2.8 Bode plot of the optimal controller $C$ (2.27). ........................... 36

2.9 Magnitude of the sensitivity and complimentary sensitivity function $S$ and $T$. ......................................................... 37

2.10 Magnitude of the closed loop transfer function compared with that of the open loop system. .......................................... 37

2.11 Comparison of the power spectrum of the time domain simulation results of the the open-loop system and closed-loop system - linear model matching numerical simulations data. ......................... 38

2.12 Comparison of the power spectrum of the time domain simulation results of the the open-loop system and closed-loop system - linear model matching experimental data. Thin line is the open-loop output spectrum without control, thick line is the closed-loop output spectrum with the $H_{\infty}$ controller. ......................................................... 40

2.13 Effect of $H_{\infty}$ controller on Mach 0.30 flow. Thin line is the open-loop (baseline) flow SPL, thick line is closed-loop flow SPL with the $H_{\infty}$ controller. ......................................................... 41

2.14 Simulink diagram of the $H_{\infty}$ controller implementation. .................. 41

2.15 Block representation of the cavity flow and the control loop with extra block introduced. ......................................................... 43

2.16 Structure of the Smith predictor controller designed for $G(s)$. .......... 44

2.17 Effect of Smith predictor based control on Mach 0.3 flow. Thin line is the open-loop (baseline) flow SPL, thick line is closed-loop flow SPL with Smith predictor control. ......................................................... 45
2.18 Effect of PD control on Mach 0.3 flow. Thin line is the baseline flow SPL spectrum and thick line is the spectrum with PD control.

2.19 Spectra for Mach 0.30 cavity flow system excited by the PD-like (effectively a proportional) controller. The P, D, and control signals were processed as the pressure transducer signals. Reference dB levels are applicable only for the SPL spectra of the pressure transducer signals.

2.20 Diagram of the parallel-proportional (PP) controller with time delay.

2.21 Spectra for Mach 0.30 cavity flow system excited by the PP with delay controller. The P₁, P₂ and control signals were processed as the pressure transducer signals. Reference dB levels are applicable only for the SPL spectra of the pressure transducer signals.

2.22 Spectra for Mach 0.30 cavity flow system excited by the \( H_\infty \) controller with delay. The \( H_1 \), \( H_2 \) and control signals were processed as the pressure transducer signals. Reference dB levels are applicable only for the SPL spectra of the pressure transducer signals.

3.1 Flow domain separation, \( \Omega_1 \) is marked by the solid dots, \( \Omega_2 \) is marked by the hollow dots.

3.2 Runge-Kutta simulation of the Galerkin model of cavity flow at Mach 0.38: baseline.

3.3 Equilibrium points of the closed-loop system vs. static output feedback control gain \( K \in [-\infty, 1058] \cup [2417, \infty] \).

3.4 Zoom of Figure 3.3 where \( K \in [100, 100] \).

3.5 Root locus for \( K \in [-\infty, 1058] \cup [2417, \infty] \).

3.6 Root locus for the modified model with equilibrium at the origin.

3.7 Closed-loop simulation results with LQ controller and LQ observer applied to the nonlinear Galerkin model – output pressure \( \tilde{p}(t) \).

3.8 Closed-loop simulation results with LQ controller and LQ observer applied to the nonlinear Galerkin model – first mode magnitude \( \tilde{a}_1(t) \).
3.9 Simulink diagram of linear LQ controller and LQ observer applied to the original nonlinear system. ........................................ 77

3.10 Simulation of states $a(t)$ of the open loop system. ............... 79

3.11 Simulation results of the closed loop system with LQ control for M0 flow model, (a) Open loop and closed-loop eigenvalues, (b) Closed-loop response of states. .......................................................... 82

3.12 Diagram of the closed-loop system with LQ state feedback control. 83

3.13 Simulation results of the closed loop system with different scaling factor $\alpha$ for M3 flow model. (a) Root Locus, (b) Closed loop responses at location of transducer 3. ................................................................. 85

3.14 Sound pressure level in closed-loop experiments at Mach 0.3 with (a) M0-based LQ controller measured at sensor 5, (b) M0-based LQ controller measured at sensor 6, (c) M4-based LQ controller measured at sensor 5, (d) M4-based LQ controller measured at sensor 6. ........................................ 90

3.15 Sound pressure level in closed-loop experiments at Mach 0.27 with (a) M0-based LQ controller measured at sensor 5, (b) M0-based LQ controller measured at sensor 6, (c) M4-based LQ controller measured at sensor 5, (d) M4-based LQ controller measured at sensor 6. 91

3.16 Sound pressure level in closed-loop experiments at Mach 0.32 with (a) M0-based LQ controller measured at sensor 5, (b) M0-based LQ controller measured at sensor 6, (c) M4-based LQ controller measured at sensor 5, (d) M4-based LQ controller measured at sensor 6. ........................................ 92

3.17 Sound pressure level of open-loop forced flow with forcing frequency at 3920 Hz at (a) Mach 0.27 measured at sensor 5, (b) Mach 0.27 measured at sensor 6, (c) Mach 0.30 measured at sensor 5, (d) Mach 0.30 measured at sensor 6, (e) Mach 0.32 measured at sensor 5, (f) Mach 0.32 measured at sensor 6. .................................................. 93

3.18 Comparison of open loop SPL and closed-loop SPL with LQ controller with real-time states provided by LQ observer and stochastic estimation method at (a) Mach 0.27, (b) Mach 0.30. ................................. 97
CHAPTER 1

INTRODUCTION

1.1 Motivation

The problem addressed in this dissertation is to reduce the resonant tones generated by subsonic flow past an open cavity by using model-based feedback controllers. Cavity flow is characterized by a strong coupling between the flow dynamics and the flow-induced acoustic field that can lead to self-sustained resonance. The basic mechanism behind the generation of cavity flow resonance is shown in Figure 1.1, and can be explained as follows: Small disturbances are amplified by the shear layer, and produce acoustic waves when they impinge on the downstream corner; these acoustic waves then propagate upstream, and excite further disturbances in the shear layer, creating a feedback loop, which often leads to self-sustained oscillations at discrete resonant frequencies.

Suppressing the resonance in cavity flows has a wide range of real-world applications, such as landing-gear and weapons bays in aircraft, flow in gas transport systems, sunroofs and windows in automobiles. In aircrafts, flow-induced cavity resonance is known to cause, among many other undesired effects, store damage and
1.1 Basic mechanism of a cavity flow resonance [36].

Figure 1.1: Basic mechanism of a cavity flow resonance [36].

Airframe structure fatigue in weapons bays, unsafe departure and inaccurate delivery of munitions stored in the weapons bays.

1.2 A Multidisciplinary Project

Cavity resonance suppression is chosen as a benchmark problem by the flow control group at the Collaborative Center of Control Science at the Ohio State University toward the development of a systematic approach to closed-loop aerodynamic flow control. The team, composed of researchers from the departments of Electrical Engineering and Mechanical Engineering at the Ohio State University, the Air Force Research Laboratory, and NASA Glenn Research Center, possesses synergistic capabilities in all of the required multidisciplinary areas of experimental data acquisition, computational flow simulation, low dimensional modeling, controller design, and experimental validation. Specifically, experimental set up has been built by researchers from the the Gas Dynamics and Turbulence Laboratory (GDTL) at OSU. They have also performed open-loop experiments and collected velocity and pressure data for model development. Computational flow simulation has been carried out by
researchers from the NASA Glenn Research Center. Flow models have been developed by a collective effort of researchers from the Mechanical Engineering and Electrical Engineering Department of OSU. Feedback control system has been designed by researchers from the Electrical Engineering Department of OSU. Experimental integration has been achieved by collaboration of the whole group.

1.3 Experimental Apparatus

The experimental facility used in this study is a small blow-down wind tunnel located at the Gas Dynamics and Turbulence Laboratory (GDTL) of OSU. For current subsonic studies, flow is directed to the 50.8 mm (2 in) by 50.8 mm (2 in) test section, Figure 1.2, through a smoothly contoured converging nozzle before exhausting to the atmosphere. A length of $L = 50.8$ mm (2 in), variable depth cavity is recessed in the floor of the wind tunnel. This cavity spans the entire width of the test section. In this study the cavity depth was held at $D = 12.7$ mm (0.5 in) corresponding to an aspect ratio, $L/D$, of 4. Optical quality windows surround the test section and provide an entrance and viewing area for laser diagnostic studies. Additional details on the experimental facility and methods can be found in [16, 30].

A 2-dimensional particle image velocimetry (PIV) system is used for flow visualization and quantitative measurements of the flow velocity field, which is called snapshots used later for modeling purpose. PIV is a non-invasive measurement technique which allows the flow remain undisturbed. This technique involves the use of microscopically small, highly reflective particles, which are added to the flow to aid in analysis. A laser illuminates a thin layer of the flow so that only the particles in that light sheet reflect the laser’s light. One camera is mounted orthogonally to the
Figure 1.2: Experimental set up showing the test section with the cavity and the actuator, (a) photo, (b) scaled drawing.
light sheet and captures images. Two successive images and an algorithm based on statistical probability are used to determine the speed and the direction of the moving particles. The velocity of the flow can be determined knowing that the particles move with the same velocity as the flow. In the current work, a velocity field grid of 128 by 128 over the approximate measurement domain of 50.8 mm (2 in) by 50.8 mm (2 in) is employed. This translates to each velocity vector being separated by approximately 0.4 mm, which is fine enough for spatial derivative computation. The snapshots of the flow velocity are then used to extract dominant coherent structures of the flow by proper orthogonal decomposition (POD) method.

Sensors used in this study are Kulite dynamic pressure transducers (with a flat frequency response up to 50 kHz) placed in various locations in the test section arranged as shown in Figure 1.3 and at the actuator exit to measure pressure fluctuations. The Kulite signals are band-pass filtered between 100 and 10,000 Hz to remove spurious frequency components. For state estimation, dynamic pressure measurements are recorded simultaneously with the PIV measurements. In the current study, 1000 PIV snapshots are recorded for each flow/actuation condition explored. For each PIV snapshot, 128 samples from each of the transducers 1-6 in Figure 1.3 are acquired at 50 kHz rate. The temporal location of the laser pulse with respect to the pressure traces is resolved by recording the laser Q-switch TTL simultaneously with the pressure signals. The experiment is designed in such a way that the Q-switch TTL falls near the middle of a pressure data sequence. This simultaneous sampling of the laser Q-switch signal and the pressure signals allows, for each snapshot, the identification of the section of pressure time traces corresponding to the instantaneous velocity field.
For closed-loop control of the flow, a dSPACE 1103 DSP board connected to a Dell Precision Workstation 650 is used. This system utilizes four independent, 16-bit A/D converters each with 4 multiplexed input channels that allow simultaneous acquisition and control processing of 4 signals. It also allows almost simultaneous acquisition and processing of additional signals at a rate up to 50 kHz per channel to produce at the same rate a control signal from a 14-bit output channel. As before, the pressure signals are band-pass filtered between 100 and 10,000Hz to remove unwanted frequency components.

A 2-D synthetic-jet type speaker is used as actuator issuing from a high-aspect ratio converging nozzle embedded in the cavity leading edge and exhausting through a small slot at an angle of 30° with respect to the main flow, as shown in Figure 1.2. A Selenium D3300Ti compression driver provides the mechanical oscillations necessary to create a zero net mass, non-zero net momentum flow for actuation. The actuator
signals are produced by either a BK Precision 3011A function generator for open-
loop forcing or by a dSPACE 1103 DSP control board in closed-loop studies and are
amplified by a Crown D-150A amplifier.

The characteristics of the actuator has been investigated in this study. White
noise signals band-limited up to 10,000 Hz are applied to the compression driver as
an input voltage $V_a$. The pressures inside and outside of the exit slot, $p_i$ and $p_o$
respectively, are measured by Kulite sensors and the jet velocity, $v_j$, exiting across
the slot is acquired by a hotwire setup. A detailed measurement setup is shown in
Figure 1.4. Given the various root mean square (RMS) values of the input voltage, the
corresponding RMS values of each measurement are compared in Figure 1.5. All the
measured variables show output linearity with respect to the input in the absence of
free stream, while the results with free stream at Mach 0.3 present that linearities are
significantly affected by free stream condition. The outside pressure $p_o$ loses linearity
as shown in Figure 1.5 (b), since its location corresponds to the leading edge of the
cavity shear layer such that the cavity oscillation dominates its response. The jet
velocity $v_j$ shown in Figure 1.5 (c) also increases considerably with the presence of
nonzero free stream velocity. In contrast, the inside pressure $p_i$ preserves linearity
fairly well, see Figure 1.5 (a), since its location is isolated from free stream compared
with other two variables. Therefore, the relationship between the inside pressure and
the input voltage will facilitate to obtain a linear dynamic model for the actuator.
System identification for the actuator dynamics and its feedback compensation are
under investigation.
1.4 Literature Review

In this dissertation, we present and discuss development, implementation, and experimental results of reduced-order model based feedback control of subsonic cavity flows. Model based feedback control of subsonic flows have been studied and implemented by the flow control group at the Collaborative Center of Control Science (CCCS) at the Ohio State University (OSU). The team, composed of researchers from the departments of Electrical Engineering and Mechanical Engineering at the Ohio State, the Air Force Research Laboratory, and NASA Glenn Research Center, possesses synergistic capabilities in all of the required multidisciplinary areas of experimental data acquisition, computational flow simulation, low dimensional modeling, controller design, and experimental validation.

The goal of the CCCS effort is to develop tools and methodologies for the use of closed-loop aerodynamic flow control to manipulate the flow over maneuvering air
Figure 1.5: Comparison of the RMS values of (a) inside pressure $p_i$, (b) outside pressure $p_o$, (c) jet velocity $v_j$, with respect of white noise input of different RMS values.
vehicles. The problem chosen for the initial study by the CCCS flow team is control of the resonant noise generated by subsonic flow past an open cavity. This phenomenon is characterized by a strong coupling between the flow dynamics and the flow-induced acoustic field that can lead to self-sustained resonance.

Two approaches towards model development have been studied in this dissertation. One aims at representing the physical properties of the system by dynamical models in transfer function forms, referred to as the physics-based linear model in this dissertation. The other approach we have followed is based on proper orthogonal decomposition (POD) and Galerkin projection methods involving the flow governing equations, which is referred to as the nonlinear model or Galerkin model in the dissertation. Each model mentioned above can be further divided into two types: model derived from numerical simulation data and model derived from real time experimental data. Different types of feedback controllers have been designed for corresponding flow models. Closed-loop system performance has been evaluated by both numerical simulation and experimental implementation as well. These results confirm that model based feedback control represents a promising approach to flow control even in its current infancy state. Research on cavity-flow resonance trace back to the early fifties. Rossiter [35] first developed a semi-empirical formula, which was later modified and improved by Heller and Bliss [27], as given by (1.1), for predicting the frequencies of cavity-flow resonance, today referred to as Rossiter frequencies or modes.

\[
Sr_n = \frac{f_n L}{U_\infty} = \frac{n - \varepsilon}{M_\infty \left\{ 1 + [(\gamma - 1)/2]M_\infty^2 \right\}^{1/2} + 1/\beta}
\]

(1.1)

where \( n \) is an integer mode number corresponding to the number of vortices spanning the cavity length \( L \); \( U_\infty \) and \( M_\infty \) are the freestream velocity and Mach number; \( \varepsilon \) is the phase lag (in fraction of a wave length) between the interaction of a large
scale structure in the shear layer with the cavity trailing edge and the formation of a corresponding upstream traveling disturbance (phase shift of the acoustic scattering process); and $\beta = \frac{U_c}{U_\infty}$ is the ratio of the convective speed of the large-scale structures to the freestream velocity. The predicted Rossiter frequencies of our setup ($L = 50.8\text{mm}$) with $\varepsilon = 0.25$ and $\beta = 0.66$ are presented in Figure 1.6 as a function of the flow Mach number. Circles in the figure represent the frequency of the resonant peaks measured in our experimental setup. More specifically, closed circles represent dominant peaks, and open circles represent other peaks appearing in multimode resonance. In different flow conditions, either a strong single-mode resonance (Mach number 0.25-0.31 and 0.39-0.5) occurs or multiple modes (Mach number 0.32-0.38) exist in the flow. Rapid switching between modes has also been observed in multimode conditions [12, 15, 57]. The random switching between multiple modes on a rapid time scale places large bandwidth and fast time response requirements on the actuation scheme and feedback control algorithm.

1.4.1 Passive Control vs. Active Control

A comprehensive review of the state of the art in cavity flow control can be found in [13] with emphasis on experimental implementation of open-loop and closed-loop control approaches. Flow control can be usually divided into two general categories: passive and active control, as shown in Figure 1.7. In passive control, which is easy to implement and has wide-spread applications, control is accomplished by geometrical modifications to the flow system, e.g., employing rigid fixed fences, spoilers, ramps [27, 45, 54] and cylinder or rods placed in the boundary layer near the leading edge of the cavity [32, 50]. These devices are simple, inexpensive, and reliable, but may not
work well at off-design conditions since they have little or no capability for adjustment to changing flow requirements. In contrast, mass and/or momentum is added to the flow in active control, which can be further divided into open-loop and closed-loop control. Different open-loop control strategies have been explored in recent years with varying degrees of success [16, 26, 46]. However, open-loop control lacks the robustness and flexibility needed for application in dynamic flight environments as well. To this end, closed-loop flow control is potentially well-suited to the successful management of flows since it allows adaptability to varying flow conditions. In addition, closed-loop control shows the potential to significantly reduce power requirements in comparison to open-loop control strategies [11]. However, it is only in recent years that feedback control has been intensively studied to control aerodynamic flow [11, 22, 23, 25, 29,
The results are encouraging, but also indicate that many issues remain to be understood and solved.

1.4.2 Linear Model vs. Nonlinear Model

The most apparent difficulty in dealing with flow control problem is that the tools of classical control systems theory are not directly applicable, since such systems display spatial continuity and strong nonlinear behavior while also posing formidable modeling challenges due to their infinite dimensionality, a complexity introduced by the governing Navier-Stokes equations [3]. In order to design and successfully implement a closed-loop control strategy, it is necessary to obtain a reduced-order model.
of the flow, which can capture the important dynamics of the flow and the effect of actuation while remaining sufficiently simple to allow its use in model-based feedback control design.

Two approaches towards model development have been carefully studied in this dissertation. One aims at representing the physical properties of the system by dynamical models in transfer function forms [37, 41, 58]. In the cited works, it is shown that the shear layer, scattering, cavity acoustics and receptivity can be represented dynamically as transfer functions. Due to the reflections from the upstream wall of the cavity, after some propagation delay time, the reflections interact with the oncoming flow and a delay-based coupled dynamics arises. It must be noted that the devised form of the transfer function matches the frequency content of the open-loop baseline data obtained from numerical simulation of Navier-Stokes equations or from real-time measured experimental data.

The other approach we have followed in the development of a reduced-order model is based on POD and Galerkin projection methods. This technique relies on the energy-containing eddies in the flow that can be extracted from either numerical simulation data or experimental data by using the spatial correlation tensor of the velocity field in the form of spatial eigenmodes called POD modes [4, 17, 28, 31, 49]. These structures are the most dominant features in the flow and arguably are the only entities that can effectively be controlled. The dynamics of the flow are obtained as the evolution of mode amplitudes are obtained by projecting the Navier-Stokes equations onto the POD basis. This results in a set of nonlinear ordinary differential equations (ODEs), which can be used for controller design [36]. Unfortunately, projecting directly the Navier-Stokes equations onto the POD modes results in an autonomous
system, which is not useful for controller design purpose since the effect of actuation is implicitly included in the system. Consequently, the equations must be recast in a form expressing the effect of actuation explicitly by incorporating a boundary control separation method [18, 19, 20] so that the resulting ODEs can be used as a model for feedback controller design using the tools of control theory.

1.4.3 Numerical Simulation Data vs. Experimental Data

The two modeling approaches described above require using either numerical simulation data or experimental data for identification. Both numerical simulation and experimental measurement have played important roles in the modeling process for providing either frequency contents of the cavity resonance for linear model parameter identification or detailed flow field information for POD expansion. In the early stage of this research work before the PIV system became available for the experimental setup, two-dimensional direct numerical simulations (DNS) were carried out to obtain detailed flow field information for use in the reduced-order model development [6, 7, 43]. In the following chapters of this dissertation, cavity flow models obtained from numerical simulation data and experimental data are both investigated.

1.5 Organization of this Dissertation

The rest of this dissertation is organized as follows.

In chapter 2, we introduce the physics-based linear model developed in [37, 41, 58] and H∞ controller design based on it. First, the underlining mechanism of the model are examined and the parameters of the linear model are optimized to match the magnitudes and frequencies of the resonant peaks obtained from either numerical
simulation data or real-time experimental data. Then, $H_\infty$ controllers are synthesized based on the models. Other linear controllers, e.g., Smith predictor based controller and PID controller are explored and implementation results of all linear controllers have been compared. Experimental results show that suppression of the dominant resonant Rossiter mode has always been accompanied with activating of another resonant mode when linear controllers are applied in the closed-loop system. A significant improvement can be achieved by adding a zero to the controller transfer function at the potentially activated Rossiter mode.

In chapter 3, we explore the nonlinear cavity model obtained from combination of the reduced-order modeling tools: proper orthogonal decomposition(POD), Galerkin projection along with control separation, and stochastic estimation method. Optimal LQ controller and LQ observer are designed based on the flow model in state space form. Numerical simulations have been performed to evaluate the closed-loop system performance. Robustness and region of attraction of the controllers are investigated numerically. When experimental implementations are being conducted, a scaling factor is introduced to the closed-loop system due to actuation limitation. The performance of the scaled LQ controller is analyzed from a mathematical perspective. Experimental results are presented and compared.

In chapter 4, we summarize the results of the dissertation, offer some conclusions, and suggest future directions.

Appendix A gives an example of Galerkin projection and control separation method on one-dimensional heat equation.
CHAPTER 2

PHYSICS BASED LINEAR MODEL AND $H_\infty$ CONTROLLER DESIGN

As mentioned in the introduction, the most apparent difficulty in dealing with flow control problem is that the tools of classical control systems theory are not directly applicable, since the flow dynamics are governed by the Navier-Stokes equations, which describe an infinite-dimensional and highly nonlinear system. These equations cannot be solved sufficiently fast for any practical models, and they cannot be used in any “internal model control” scheme. Recently, a physics-based linear model was proposed and used in control of cavity oscillations [41, 58]. This model represents the basic physical properties of the cavity flow on the basis of separate components such as scattering, acoustics, shear layer and receptivity, representing each physical process with a linear transfer function. In this chapter, we use the transfer function representation and demonstrate that the parameters of this transfer function model can be optimized to match either the numerical simulations results or the experimental results. Utilizing the devised models, $H_\infty$ controllers based on the Toker-Özbay formula [52] are synthesized. Closed-loop simulation results are presented, which show that the undesired peaks in the frequency response can be successfully eliminated.
Controller designed for experimental based model is implemented in real-time experiments and the results show that the $H_\infty$ controller reduces the dominant resonant tone for which it is designed, but introduces resonant tones at other Rossiter frequencies. Similar observations can be obtained for other linear control methods (e.g. Smith predictor control and PID control). The real time implementation shows that a significant improvement can be achieved by adding a tunable time delay between two proportional feedback loops. This chapter is based on the work published in [59, 63].

2.1 Delay-Based Linear Model

The model for the cavity dynamics is based on the Rossiter mechanism described in the introduction: small disturbances are amplified by the shear layer, and produce acoustic waves when they impinge on the downstream corner; these acoustic waves then propagate upstream, and excite further disturbances in the shear layer, creating a feedback loop, which often leads to self-sustained oscillations at discrete resonant frequencies. This process is depicted in the block diagram shown in Figure 2.1, where each component of the mechanism described above is represented by a separate transfer function.

As shown in Figure 2.1, the process of the shear layer is represented by the block $G$. Blocks $S_c$, $A$, $R$ represent the processes of scattering, acoustics and receptivity, respectively. The influences of the actuator and sensor are represented by $V$, and $S_m$. The controller $C$ is going to be designed later in this chapter. The loop inside the dotted rectangle represents the overall process of the cavity flow characterized by self-sustained oscillations. The overall transfer function for the cavity flow is given by:
Detailed descriptions of each physical process will be given as follows.

### 2.1.1 Shear layer

The shear layer is modeled as a second-order system with a time delay

\[ G(s) = G_0(s)e^{-s\tau_s}, \]  

(2.2)

where

\[ G_0(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}. \]  

(2.3)

In the second-order system \( G_0(s) \), \( \omega_0 \) and \( \zeta \) are the natural frequency and the damping ratio, respectively. The time delay \( \tau_s \) is the convection time for a disturbance to travel.
the length of the cavity. It can be computed by

\[ \tau_s = \frac{L}{c_p}, \]  

(2.4)

where \( L \) is the length of the cavity, and \( c_p \) is the mean phase speed. While it has been pointed out that the model (2.2) of the shear layer has less physical justification than a model that can be obtained from the linear stability theory [41], it nevertheless provides the same general features, and is easier to analyze. Moreover, its adjustable parameters make it possible to be tuned to better match any simulation data or experimental data.

\subsection{Acoustics}

The model we use in our research for acoustic propagation in the cavity is shown in Figure 2.2, in which \( p_0 \) and \( p_L \) denote pressure disturbances at the leading and trailing edges, respectively. The overall transfer function of the acoustic propagation can be written in the form

\[ A(s) = \frac{e^{-s\tau_a}}{1 - r(s)e^{-2s\tau_a}}, \]  

(2.5)
where the time delay $\tau_a = L/a$, represents the acoustic lag between the trailing edge and the leading edge, being $L$ the cavity length and $a$ the sound speed inside the cavity. The frequency response of $A(s)$ shows that harmonic peaks with different magnitudes can be obtained by making the attenuation factor of the reflection $r(s)$ frequency dependent as

$$r(s) = \frac{r}{1 + s/\omega_r}. \quad (2.6)$$

The coefficient $r$ measures the total efficiency of the reflection process, e.g. if both reflections are perfect, then $r = 1$; if each reflection reduces the amplitude by 0.5, then $r = 0.25$.

### 2.1.3 Scattering and Receptivity

In Rossiter’s empirical formula for predicting cavity frequencies, the scattering and receptivity effects are treated together as a simple phase lag, independent of frequency. In the physics-based linear modeling method [41], Rossiter’s approach is followed and the processes of scattering and receptivity are modeled as constant gains, here denoted as $K_S$ and $K_R$ respectively in this chapter.

Furthermore, dynamics of the actuator ($V$) and sensors ($S_m$) are neglected and considered as unitary transfer functions.

This model is admittedly crude, but for the purposes of control, it provides a reasonable starting point before a more accurate and suitable model can be obtained.

### 2.2 Parameter Tuning for the Flow Dynamics

Our studies have demonstrated that the physics based linear model developed by Rowley et al. [36, 41] exhibits certain degrees of flexibility to match the open
loop frequency response obtained either from numerical simulation of the Navier-
Stokes equations or from experimental measurements. In order to analyze this, we
have performed several tests to see which parameter is responsible for introducing
what sort of modification into the frequency contents (Bode plot). Following is a list
summarizing our conclusions in this aspect:

• As $\omega_0$ increases, the dominant peak of $G(j\omega)$ moves towards higher frequencies.
The frequency domain picture is stretched to the right.

• As $\zeta$ increases, the values of the peaks in $G(j\omega)$ get lowered, and the frequency
content becomes more flattened.

• An increase in $K_S$ lifts up the entire frequency domain picture while magnifying
the peak values slightly.

• If $r$ is increased, more harmonic peaks appear particularly in the higher fre-
quencies in $G(j\omega)$.

• As $\tau_s$ increases, the frequency response acquires more harmonic fluctuations
(peaks) in the low frequencies in $G(j\omega)$. Further increments lead to more wavy
low frequency behavior.

• Changes in $\tau_a$ cause small translations with some small changes in the peak
magnitudes.

• As $K_R$ increases, the peak magnitudes get larger.

Apparently, the above information constitutes a knowledge base to direct how to
tune the parameters to achieve desired frequency response.
In the early stage of this research work, before experimental data were available, two-dimensional direct numerical simulations (DNS) of the Navier-Stokes equations in the cavity were carried out and real-time solutions of the system output were obtained. Providing the numerical simulation and the linear model with the same white noise input, we used trial and error method to tune the model parameters to match the frequency contents of both outputs. In a later stage, the experimental data became available and the same strategy was carried out to tune the parameters to match the linear model to the experimental data. Therefore, two linear models with different parameters are presented later in this chapter, one based on numerical simulations data and one based on experimental data. $\mathcal{H}_\infty$ controllers have been designed for each one of them.

2.2.1 Numerical Simulations Based Model

The result of the model matching based on numerical simulation data is illustrated in Figure 2.3, from which it is seen that outputs of the linear model match that of the numerical simulation of the Navier-Stokes equations very well. The parameters of the linear system are given in Table 2.1, for which a robust controller is designed.

2.2.2 Experimental Data Based Model

The sound pressure level (SPL) of the pressure measured at the central cavity floor at Mach 0.3 shows a strong single resonant peak at the third Rossiter mode as given in Figure 2.4 (thin line). Parameters of the linear model are tuned to match this experimental result as given in Table 2.2. The result of the model matching based on experimental data is illustrated by Figure 2.4.
Figure 2.3: Power spectral density comparison of pressure data obtained from full order numerical simulation of the Navier-Stokes equations (solid line) and simulation of model (2.1) (dash line).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_0$</td>
<td>200 rad/sec</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>0.95</td>
</tr>
<tr>
<td>$\tau_s$</td>
<td>0.0195 sec</td>
</tr>
<tr>
<td>$\tau_a$</td>
<td>0.001 sec</td>
</tr>
<tr>
<td>$r$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\omega_r$</td>
<td>100 rad/sec</td>
</tr>
<tr>
<td>$K_R$</td>
<td>0.408</td>
</tr>
<tr>
<td>$K_S$</td>
<td>1.65</td>
</tr>
</tbody>
</table>

Table 2.1: Parameters of the linear model based on numerical simulations of the Navier-Stokes equations.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_0$</td>
<td>18000 rad/sec</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>0.03</td>
</tr>
<tr>
<td>$\tau_s$</td>
<td>0.0006 sec</td>
</tr>
<tr>
<td>$\tau_a$</td>
<td>0.0001 sec</td>
</tr>
<tr>
<td>$r$</td>
<td>0.01</td>
</tr>
<tr>
<td>$\omega_r$</td>
<td>20000 rad/sec</td>
</tr>
<tr>
<td>$K_R$</td>
<td>0.01</td>
</tr>
<tr>
<td>$K_S$</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 2.2: Parameters of the linear model based on experimental data.

Figure 2.4: Power spectral density comparison of pressure data obtained from experimental measurement (thin line) and from simulation of model (2.1) (thick line).
2.3 $H_\infty$ Controller Design for Linear Model Based on Numerical Simulation Data

Based on the model given in section 2.2.1, a $H_\infty$ controller can be designed to reduce the effect of the external noise on the system response. In our particular case, for the cavity flow control, we can define a sensitivity minimization problem to compare the open loop and closed loop responses of the system shown in Figure 2.1.

The open loop system response to the noise is

$$|P(j\omega)| = \left| \frac{K_SA(j\omega)G(j\omega)}{1 - K_RK_SA(j\omega)G(j\omega)} \right|,$$

and the closed loop response is

$$\frac{|P(j\omega)|}{|1 + P(j\omega)C(j\omega)|},$$

where dynamics of actuator and sensor are neglected and $C(j\omega)$ is the control law to be designed. The goal is to minimize the weighted sensitivity over all stabilizing controllers, the weight being the plant itself. Note that the function $(1 + P(j\omega)C(j\omega))^{-1}$ is the sensitivity function of the feedback system shown in Figure 2.1. Similar techniques from linear robust control theory have been used for cavity flow control in [39]. In this work, the authors discussed the effects of actuator saturation as well, using describing function analysis.

2.3.1 Plant Factorization

Before illustrating the $H_\infty$ design procedure, the plant transfer function is factorized for a further simplification of the design. Inserting the transfer functions of the
shear layer and the acoustics into the plant transfer function, we get

\[
P(s) = \frac{G(s)K_SA(s)}{1 - K_RG(s)K_SA(s)},
\]

\[
= \frac{G_0(s)e^{-s\tau_A}K_S \frac{e^{-s\tau_A}}{1 - r(s)e^{-s\tau_A}}}{1 - K_RG_0(s)e^{-s\tau_A}K_S \frac{e^{-s\tau_A}}{1 - r(s)e^{-s\tau_A}}},
\]

\[
= \frac{K_SG_0(s)e^{-s(\tau_s + \tau_a)}}{1 - r(s)e^{-2s\tau_A} - K_RG_SG_0(s)e^{-s(\tau_s + \tau_a)}}. 
\]

(2.9)

Define

\[
M_n(s) = e^{-h_1s}, \text{ where } h_1 = \tau_s + \tau_a, 
\]

(2.10)

\[
M_2(s) = e^{-2s\tau_A}, 
\]

(2.11)

\[
N_{o2}(s) = K_SG_0(s) = \frac{K_S}{1 + 2\zeta s/\omega_0 + s^2/\omega_0^2}, 
\]

(2.12)

\[
N_{o1}(s) = (1 - K_RG_{o2}(s)M_n(s) - r(s)M_2(s))^{-1}. 
\]

(2.13)

Then the plant transfer function \( P(s) \) can be factorized into the form

\[
P(s) = N_{o1}(s)N_{o2}(s)M_n(s). 
\]

(2.14)

The plants models are stable for the numerical values listed in Table 2.1 and Table 2.2. For different numerical values it is possible to have an unstable plant, then finitely many unstable modes may appear from the roots of the equation \( 1/N_{o1}(s) = 0. \) This situation can be handled as well. In our study, we propose to use Toker-Özbay formula [52] to design the controller.
### 2.3.2 Weighting Functions

The optimal $H_\infty$ cost is defined by

$$\gamma_{\text{opt}} := \inf_{C \in \mathcal{C}} \left\| \begin{pmatrix} W_1S \\ W_2T \end{pmatrix} \right\|_\infty$$

$$= \inf_{C \in \mathcal{C}} \sup_{\omega} \left( |W_1(j\omega)S(j\omega)|^2 + |W_2(j\omega)T(j\omega)|^2 \right)^{1/2}, \quad (2.15)$$

where $\mathcal{C}$ is the set of all compensators stabilizing $P$, $S = (1 + PC)^{-1}$ and $T = 1 - S$ are the sensitivity and complementary sensitivity functions and $W_1$ and $W_2$ are the performance and stability weighting functions. Since the goal of the controller is to suppress the peak value of $|P(j\omega)S(j\omega)|$, we choose the performance weighting function $W_1(s)$ capturing all the peaks of $P(s)$ as given in (2.17), so that the oscillation magnitudes at the dominating modes can be suppressed. To take into account the uncertainties in high frequency, we set the stability weighting function $W_2(s)$ as given by (2.19). Parameters of the weighting functions are given in Table 2.3, while Bode plots of these weighting functions and the plant are depicted in Figure 2.5.

\[
W_{1o}(s) = k_1 \frac{(1 + s/\omega_{1n})}{(1 + s/\omega_{1d})}, \quad (2.16)
\]

\[
W_1(s) = W_{1o}(s)(1 - r(s)M_2(s))N_0(s), \quad (2.17)
\]

\[
W_{2o}(s) = \epsilon_2 s(1 + s/\omega_{1n}), \quad (2.18)
\]

\[
W_2(s) = W_{2o}(s)(1 - r(s)M_2(s)). \quad (2.19)
\]

Since the plant and the weighting functions are both infinite dimensional, there is no direct and easy solution to this weighted sensitivity minimization problem. For the case where the plant, or the weights, is finite dimensional, the problem can be solved using certain procedures from operator theory [21]. In this particular case, we solve the problem by exploiting the special structure of the plant and the weights.

28
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_1$</td>
<td>2</td>
</tr>
<tr>
<td>$\omega_{1d}$</td>
<td>500</td>
</tr>
<tr>
<td>$\omega_{1n}$</td>
<td>1000</td>
</tr>
<tr>
<td>$\epsilon_2$</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table 2.3: Parameters of the weighting functions for controller design for the simulation data based model.

Figure 2.5: Bode plot of the weighting functions and the plant based on numerical simulation data.
The controller design problem can be simplified by solving an equivalent design problem. Recall that the transfer function of the plant can be written in the form

\[
P = \frac{K_S G_0(s) e^{-s\tau_a}}{1 - K_R K_S G_0(s) e^{-s\tau_a}} \frac{e^{-s\tau_a}}{1 - \tau(s) e^{-s\tau_a}},
\]

\[
= \frac{N_{02}(s) M_n(s)(1 - r(s) M_2(s))^{-1}}{1 - K_R N_{02}(s) M_n(s)(1 - r(s) M_2(s))^{-1}}.
\]  

(2.20)

Let

\[
P_1(s) = N_{02}(s) M_n(s)(1 - r(s) M_2(s))^{-1},
\]

then

\[
P(s) = \frac{P_1(s)}{1 - K_R P_1(s)}.
\]  

(2.22)

It’s easy to see that any controller \(C(s)\) in the form

\[
C(s) = C_1(s) + K_R
\]

(2.23)

can stabilize the plant \(P\) if \(C_1\) can stabilize \(P_1\) because the poles of the feedback system \((P_1, C_1)\) are equal to those of the feedback system \((P, C)\). This can be shown:

\[
\frac{P}{1 + PC} = \frac{P_1}{1 - K_R P_1} \frac{1}{1 + \frac{P_1}{1 - K_R P_1}(C_1 + K_R)},
\]

\[
= \frac{P_1}{1 - K_R P_1 + P_1 C_1 + K_R P_1},
\]

\[
= \frac{P_1}{1 + P_1 C_1}.
\]  

(2.24)

Define a fictitious system

\[
P_2(s) = N_{02}(s) M_n(s),
\]  

(2.25)

30
then any controller $C_1$ of the form

$$C_1(s) = C_2(s)(1 - r(s)M_2(s))$$  \hspace{1cm} (2.26)

can stabilize $P_1$ if $C_2$ can stabilize system $P_2$. The optimal controller can be obtained in the form

$$C(s) = C_2(s)(1 - r(s)M_2(s)) + K_R$$  \hspace{1cm} (2.27)

by designing $C_2$ for a mixed sensitivity minimization problem with $\gamma < \gamma_{opt}$ satisfying

$$\left\| \begin{pmatrix} W_{1o}S_2 \\ W_{2o}T_2 \end{pmatrix} \right\|_{\infty} \leq \gamma,$$  \hspace{1cm} (2.28)

where the sensitivity and complimentary sensitivity functions $S_2$, $T_2$ with respect to the fictitious system $P_2$ are assumed to be

$$S_2 = (1 + P_2C_2)^{-1},$$

$$T_2 = 1 - S_2.$$  \hspace{1cm} (2.29)

By definition, it can be seen that the performance specifications of both systems are equivalent, which is shown as follows

$$|W_1S| = \left| W_{1o}(1 - rM_2)N_{o1} \frac{1}{1 + PC} \right|,$$

$$= \left| W_{1o}(1 - rM_2)N_{o1} \frac{1}{1 + \frac{P_2(1-rM_2)^{-1}}{1 - K_R P_2(1-rM_2)^{-1}}(C_2(1 - rM_2) + K_R)} \right|,$$

$$= \left| W_{1o}(1 - rM_2)N_{o1} \frac{1 - K_R P_2(1 - rM_2)^{-1}}{1 + P_2C_2} \right|,$$

$$= \left| W_{1o} \frac{1}{1 + P_2C_2} \left( (1 - rM_2)N_{o1} - K_R P_2 N_{o1} \right) \right|,$$

$$= \left| W_{1o}S_2 \left( \frac{1}{1 - K_R P_2(1-rM_2)^{-1}} - \frac{K_R P_2(1 - rM_2)^{-1}}{1 - K_R P_2(1-rM_2)^{-1}} \right) \right|,$$

$$= \left| W_{1o}S_2 \right|.$$
For the robustness of the system, it can be shown that

\[ |W_2T| = |W_{2o}(1 - rM_2) \frac{PC}{1 + PC}|, \]

\[ = |W_{2o}(1 - rM_2) \frac{N_{o1}N_{o2}M_n(C_2(1 - rM_2) + K_R)}{1 + N_{o1}N_{o2}M_n(C_2(1 - rM_2) + K_R)}|, \]

\[ = |W_{2o}(1 - rM_2) \frac{1}{1 - rM_2 - K_RP_2} \frac{P_2(C_2(1 - rM_2) + K_R)}{1 + \frac{1}{1 - rM_2 - K_RP_2} P_2(C_2(1 - rM_2) + K_R)}|, \]

\[ = |W_{2o}(1 - rM_2) \frac{P_2C_2(1 - rM_2) + K_RP_2}{1 - rM_2 - K_RP_2 + P_2C_2(1 - rM_2) + K_RP_2}|, \]

\[ = |W_{2o}(1 - rM_2) \frac{P_2C_2(1 - rM_2) + K_RP_2}{(1 - rM_2) + P_2C_2(1 - rM_2)}|, \]

\[ \leq |W_{2o}T_2(1 - rM_2)| + |W_{2o}K_RP_2S_2|. \]

Then the system associated with plant \( P \) and controller \( C \) is robustly stable, when the uncertainties of this system are sufficiently bounded. The equivalence between designing \( C_2 \) for \( P_2 \) and designing \( C \) for \( P \) has been established by skillfully choosing the corresponding weighting functions as described by (2.16)-(2.19).

### 2.3.4 Toker-Özbay Design Results

By an equivalent design procedure, the problem is significantly simplified in such a way that \( C_2 \) can be computed explicitly [52, 33]. The optimal controller \( C_2 \) for \( P_2 \) is in the form

\[ C_2(s) = \left( \frac{\gamma}{\gamma_{\text{min}}} - \frac{\gamma_{\text{min}}}{\gamma} \right) \frac{N_{o2}^{-1}(s)}{(1 + as + bs^2)} \left( \frac{1}{1 + H(s)} \right). \]  

(2.30)
To compute the optimal performance level $\gamma$, define

$$
\gamma_{\text{min}} := k_1 \frac{\omega_{1d}}{\omega_{1n}},
$$

$$
\gamma_{\text{max}} := k_1,
$$

and

$$
x := \sqrt{\frac{k_1^2 - \gamma^2}{\gamma^2 - \gamma_{\text{min}}^2}},
$$

$$
b := \frac{\epsilon_2 \sqrt{1 - (\gamma_{\text{min}}/\gamma)^2}}{k_1 \omega_{1d}},
$$

$$
a := \sqrt{2b + \epsilon_2^2 (k_1^{-2} - \gamma^{-2})}.
$$

Taking the largest value of $\gamma$ satisfying the equality (2.31) below in the allowable range $\gamma_{\text{min}} < \gamma < \gamma_{\text{max}}$ gives the optimal $\gamma$ needed in the optimal controller formula (2.30). For the current problem, the optimal $\gamma$ is obtained as $\gamma = 1.9484$ from the relation

$$
\pi = h_1 \omega_{1d} x + \tan^{-1} x + \tan^{-1} \frac{\omega_{1d} x}{\omega_{1n}} + \tan^{-1} \frac{a \omega_{1d} x}{1 - b \omega_{1d} x^2}.
$$

(2.31)

### 2.3.5 Simulation Results

To implement the controller in discrete time, $H(s)$ is expanded as a finite impulse response (FIR) filter and an exponential decay term, i.e. $H(s) = H_{\text{FIR}}(s) + H_{\text{IIR}}(s)$, which are derived as follows

$$
H_{\text{FIR}}(s) = \frac{(\omega_{1d} + \omega_{1n}) s + \omega_{1n} \omega_{1d} - \omega_x^2 + \frac{\gamma}{\gamma_{\text{min}}} \omega_{1d}^2 (1 + x^2)(d_1 s + d_0) e^{-h_1 s}}{s^2 + \omega_x^2},
$$

(2.32)

$$
H_{\text{IIR}}(s) = \frac{\gamma}{\gamma_{\text{min}}} \left( \omega_{1d}^2 (1 + x^2)(c_1 s + c_0) - 1 \right) e^{-h_1 s},
$$

(2.33)
where $c_0, c_1, d_0, d_1$ and $\omega_x$ are defined as

\[
  c_0 = \frac{(b(1 - b\omega_x^2) - a^2)}{a} d_1, \\
  c_1 = -bd_1, \\
  d_0 = d_1 \frac{(b\omega_x^2 - 1)}{a}, \\
  d_1 = \frac{-a}{\omega_x^2 a^2 + (1 - b\omega_x^2)^2}, \\
  \omega_x = \omega_1 d x.
\]

The impulse response of $H_{FIR}$ given in Figure 2.6 has finite support $[0, h_1]$. The magnitudes of $H_{FIR}(s)$, $H_{IIR}(s)$ and $H(s)$ are shown in Figure 2.7.

Since it has been demonstrated that the impulse response of $H_{FIR}(s)$ is restricted to the time interval $[0, h_1]$, $H_{FIR}(s)$ can be easily implemented as a FIR filter of duration $h_1$. The discrete-time realization of $H_{FIR}(s)$ requires only $h_1/T_s$ states,
where $T_s$ is the sampling period. Theoretically, the infinite dimensional controller can be implemented through a finite impulse response (FIR) filter approximation.

The Bode plot of the controller is shown in Figure 2.8. The optimal sensitivity function $S(j\omega)$ and complimentary sensitivity function $T(j\omega)$ obtained are given in Figure 2.9. Figure 2.10 illustrates the Bode plot of the closed loop control system compared to that of the open loop plant. As the figure suggests, the controller modifies the frequency content of the open loop system significantly at the dominant modes. The resonant peaks in the frequency response of the open loop system are suppressed and the improvement is obvious. Figure 2.11 compares the power spectrum of the time domain simulation results of the the open-loop system and closed-loop system. The numerical simulation results in Figure 2.11 show that when the plant is taken
to be the linear model defined above, then the $H_\infty$ controller is able to suppress the strong oscillation peaks seen at the output.

2.4 Linear Controller Design for Model Based on Experimental Data

In the following section, an $H_\infty$ controller designed for the experimental based model given in section 2.2.2 is presented. The same design strategy used in section 2.3 is followed, as a result, we focus on the real-time implementation results and analysis in this part. Closed-loop experimental results of the $H_\infty$ controller are compared with those obtained using other types of linear feedback controllers, e.g. Smith predictor and PID controller. Similar observations can be obtained for other linear control methods.
Figure 2.9: Magnitude of the sensitivity and complimentary sensitivity function $S$ and $T$.

Figure 2.10: Magnitude of the closed loop transfer function compared with that of the open loop system.
2.4.1 Weighting Functions

The weighting functions in this design are chosen as follows, with parameters given in Table 2.4:

\[
W_{1o}(s) = k_1 \frac{\omega_1^2}{(s^2 + 2\zeta_1 \omega_1 s + \omega_1^2)}, \quad (2.34)
\]

\[
W_1(s) = W_{1o}(s)(1 - r(s)M_2(s))N_{o1}(s), \quad (2.35)
\]

\[
W_{2o}(s) = \epsilon_2 s^2, \quad (2.36)
\]

\[
W_2(s) = W_{2o}(s)(1 - r(s)M_2(s)). \quad (2.37)
\]
Notice that since the uncontrolled flow at Mach 0.30 exhibit a strong single Rossiter mode, the performance weighting function $W_1(s)$ is chosen to capture the single dominant resonant peak of $P(s)$ so that the sensitivity function $S(s)$ at that frequency can be suppressed.

By following the same method given in the previous section, an $H_{\infty}$ controller is designed and validated in numerical simulations. The numerical simulation result shown in Figure 2.12 illustrates that when the plant is taken to be the linear model defined above matching the single Rossiter mode observed in experiments, then the $H_{\infty}$ controller is able to suppress the strong sinusoidal oscillations seen at the output. Experimental implementation of this controller is discussed in the next section.

### 2.4.2 Implementation Results

We explore real time implementations of linear feedback controllers in Simulink and dSPACE systems as described in Chapter 1. The $H_{\infty}$ controller was designed for Mach 0.3, a flow dominated by strong single-mode resonance at about 2800 Hz, see the thin line in Figure 2.13. The controller diagram for the real time implementations is depicted in Figure 2.14. As seen in Figure 2.13, the $H_{\infty}$ controller suppresses the

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_1$</td>
<td>4</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>18000</td>
</tr>
<tr>
<td>$\zeta_1$</td>
<td>0.03</td>
</tr>
<tr>
<td>$\epsilon_2$</td>
<td>$2 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 2.4: Parameters of the weighting functions for controller design for experimental data based model.
Figure 2.12: Comparison of the power spectrum of the time domain simulation results of the open-loop system and closed-loop system - linear model matching experimental data. Thin line is the open-loop output spectrum without control, thick line is the closed-loop output spectrum with the $H_\infty$ controller.

main frequency of oscillation (third Rossiter mode), but it leads to strong oscillation with frequency about 1900 Hz, which corresponds to the second Rossiter mode for this flow. The linear controller is designed for the unforced, single-mode resonant flow and thus the potential excitement of the other Rossiter modes is not taken into account. Similar results were obtained for other types of linear feedback controllers, e.g. Smith predictor and PID controller, which are discussed in the following sections.

We could address this issue by selecting the uncertainty weight of the $H_\infty$ control to include sharp peaks at these “dormant” Rossiter modes (which do not appear in the open loop response), but this will complicate the $H_\infty$ design which is already complex due to infinite dimensionality of the plant and weights. Based on the observation that
Figure 2.13: Effect of $H_\infty$ controller on Mach 0.30 flow. Thin line is the open-loop (baseline) flow SPL, thick line is closed-loop flow SPL with the $H_\infty$ controller.

Figure 2.14: Simulink diagram of the $H_\infty$ controller implementation.
new Rossiter modes are excited when the linear controller is applied in the closed-loop system, we can add a new block into the closed-loop system to represent this dynamics, as shown in Figure 2.15. In the feedback loop where the controller is applied to the system, the block $Q(s)$ is chosen in the form

$$Q(s) = \frac{1}{1 + q(s)e^{-hs}}$$

(2.38)

to take into account the nonlinear dynamics of excitement of new Rossiter mode. $q(s)$ and $h$ can be chosen so that dynamics of $Q(s)$ includes peaks at those “dormant” Rossiter frequencies. Then controller in practise should be modified as

$$C_{\text{new}}(s) = C(s)(1 + q(s)e^{-hs})$$

(2.39)

where $C(s)$ is the $H_{\infty}$ controller originally designed based on the linear system. This scheme will be later proved to be effective in experiments.

### 2.4.3 Smith Predictor

Smith predictors are simple controllers typically used for solving time delay system, like $P_2(s)$ defined above. However, Smith predictors are very sensitive to plant parameter uncertainties. In order to match the phase of the linear model to that of the cavity output, the delay term in the linear model described by (2.14) was tuned again as

$$h_1 = \tau_s + \tau_a + \tau_0,$$

(2.40)

where $\tau_0 = 1.4$ and $\tau_s, \tau_a$ are the same as before. $P_2(s)$ is defined the same as before

$$P_2(s) = K_sG_0(s)e^{-h_1s} := P_0(s)e^{-h_1s}.$$  

(2.41)

A Smith predictor controller can be obtained as

$$C_2 = \frac{C_0}{1 + P_0C_0(1 - e^{-h_1s})}.$$  

(2.42)
To stabilize the system \( P_2(s) \) and the overall controller for the cavity flow model is the same as before, that is,

\[
C(s) = C_2(s)(1 - r(s)M_2(s)) + K_R. \tag{2.43}
\]

The Smith predictor structure is shown in Figure 2.16, where \( C_0 \) is the stabilizing controller for the delay-free part \( P_0 \), chosen such that the delay-free closed-loop system \( \frac{P_0(s)}{1+P_0(s)C_0(s)} \) is stable. Then, the closed loop feedback response from noise to output becomes the delayed version of the above closed-loop transfer function, which is designed from the delay-free part of the plant. This idea is used for the delayed second-order transfer function representation of the cavity flow system. For \( C_0 \) we took a simple first order controller placing the poles as far left in the complex plane as possible.
Figure 2.16: Structure of the Smith predictor controller designed for $G(s)$.

The closed-loop experimental results with the Smith predictor controller shown in Figure 2.17 are very similar to the closed-loop results with the $H_\infty$ controller. The controller is successful in eliminating the main frequency of oscillation (third Rossiter mode), but it leads to strong oscillations with frequency about 1900Hz, which coincides with the second Rossiter mode. Similar results are obtained with a PID controller discussed as follows.

### 2.4.4 PID Controller Design

The classical PID controller has three terms, proportional, integral and the derivative:

$$C(s) = K_p + \frac{K_i}{s} + \frac{K_d s}{\epsilon s + 1}, \quad (2.44)$$

while a small time constant $\epsilon$ is introduced for causality of the controller. There are several tuning techniques for the PID gains for linear plants. In the actual experimental set-up we first used a proportional (P) controller only, and searched for its best gain by trial and error. Then we have noticed that adding an integral term does not change the system behavior due to the actuator dynamics. The integral term
Figure 2.17: Effect of Smith predictor based control on Mach 0.3 flow. Thin line is the open-loop (baseline) flow SPL, thick line is closed-loop flow SPL with Smith predictor control.

does not have any significant effect on the system response since the loudspeaker cannot produce very low frequency signals, where integral action is most significant. Similarly the derivative term did not have a significant impact on this system, so we replaced it with a first order filter with an adjustable cut-off frequency to obtain a PD like controller in the form

\[
C(s) = K_p + K_d \frac{\tau_d s}{\tau_d s + 1}.
\]  

(2.45)

Note that when \( \tau_d \) is small we have a derivative action, with appropriate scaling of \( K_d \). Otherwise, the system behaves like a P-P controller, two parallel proportional terms. While this is equivalent to a single proportional term, we keep them separated, as in the sequel we will add time delays to each term to introduce zeros in the controller transfer function.
Effect of PD control has been tested on Mach 0.3 flow (see Figure 2.18) and the results are very similar to the other linear controllers, i.e., $H_\infty$ controller and the Smith predictor controller. The third Rossiter mode is successfully suppressed, while the second Rossiter resonance is excited. This is expected, since the linear controllers were designed for the baseline case and thus excitation of the other Rossiter modes was not taken into consideration. Such behavior has also been observed by other researchers using linear feedback control [5, 56, 58]. Based on these experimental observations, the controllers should be re-designed in such a way that the second Rossiter mode does not get excited.

Figure 2.18: Effect of PD control on Mach 0.3 flow. Thin line is the baseline flow SPL spectrum and thick line is the spectrum with PD control.
2.5 Parallel Proportional Control with Delay

The optimal parameters of the PD controller (2.45) for elimination of the main frequency of oscillation (third Rossiter mode) of the Mach 0.30 flow were found from a brute force search method, as

$$K_p = 8, \ K_d = 0.04, \ \tau_d = 200.$$  \hspace{1cm} (2.46)

However, the PD controller with these parameters led to strong oscillation in the neighborhood of the second Rossiter mode, as shown in Figure 2.19. Note that the P, D, and the control spectra correspond to voltage signals produced by the controller and not to sound pressure levels. Nevertheless, for the sake of comparison, they are presented alongside the SPL spectra from the Kulite transducers placed at the actuator exit and in the middle of the cavity floor. Therefore, the dB levels in this figure refer only to the spectra of the Kulite signals.

The closed-loop experimental results of the linear designs ($H_\infty$, Smith predictor and PID controllers) confirm that the plant has a nonlinear behavior and linear designs lead to results which cannot be predicted from the linear analysis. Observations of the control signals in Figure 2.19 motivated us to add a tunable time delay, $h$, to the second term in (2.45), which introduced a phase shift for signals operating in the neighborhood of the second Rossiter frequency:

$$C_{PD}(s) = K_p + K_d \frac{\tau_d s}{\tau_d s + 1} e^{-hs}. \hspace{1cm} (2.47)$$

Furthermore, having observed that, due to the relatively large value of $\tau_d$, this controller acts like a parallel-proportional (PP) control with individual delay terms, we simplified the controller as

$$C_{PP}(s) = K_p(1 + e^{-hs}). \hspace{1cm} (2.48)$$
Figure 2.19: Spectra for Mach 0.30 cavity flow system excited by the PD-like (effectively a proportional) controller. The P, D, and control signals were processed as the pressure transducer signals. Reference dB levels are applicable only for the SPL spectra of the pressure transducer signals.

A schematic diagram of such a controller as implemented in our experimental setup is given in Figure 2.20. It has been verified that the performance of this controller is very much like the performance of the PD-like controller with the same time delay.

A value of $h = 260 \mu s$ was used to introduce a $180^\circ$ phase shift for signals operating in the neighborhood of the second Rossiter mode thus effectively placing a “zero” at the corresponding frequency. In fact, it is straightforward to see that

$$1 + e^{-hj\omega} \bigg|_{\omega=1932 Hz} = 1 + e^{-2.6 \times 10^{-4} \times 2\pi \times 1932j} = 0.$$  \hspace{1cm} (2.49)

As a result, the performance of this controller is greatly improved as shown in Figure 2.21 where, similarly to Figure 2.19, one can observe the spectra of the signals in different points of the closed loop. Combining the $260 \mu s$ phase shifted P signals produces a control signal whose spectrum is characterized by frequency cancelation
at 1932 Hz and its odd harmonics, whereas a modest reinforcement is produced at the even harmonics. This effect of phase shift introduced by the delay term has also been applied to the $H\infty$ controller as

$$C_{\text{delay}}(s) = C(s)(1 + e^{-hs}),$$

(2.50)

where $C$ is the $H\infty$ controller derived as given by (2.27). The results given in Figure 2.22 show that the performance of the $H\infty$ controller is substantially improved by introducing a zero to the controller at the hidden Rossiter frequencies. This, at the same time, verified the model modification scheme (see Figure 2.15) proposed in the previous section.

It should also be noted that the highest peak of the forcing signal at the actuator exit occurs at a frequency of about 3800 Hz, a value close to 3920 Hz, one of the optimal forcing frequencies for reducing the resonance of the Mach 0.30 cavity flow [16].
Figure 2.21: Spectra for Mach 0.30 cavity flow system excited by the PP with delay controller. The $P_1$, $P_2$ and control signals were processed as the pressure transducer signals. Reference dB levels are applicable only for the SPL spectra of the pressure transducer signals.

Figure 2.22: Spectra for Mach 0.30 cavity flow system excited by the $H_\infty$ controller with delay. The $H_1$, $H_2$ and control signals were processed as the pressure transducer signals. Reference dB levels are applicable only for the SPL spectra of the pressure transducer signals.
In this chapter, the approach we have followed in the development of a reduced-order model is based on proper orthogonal decomposition (POD) and Galerkin projection methods. POD method relies on the energy-containing vortices in the flow that can be extracted using the spatial correlation tensor of the velocity field in the form of spatial eigenmodes called POD modes. These structures are the most dominant features in the flow and arguably are the only entities that can effectively be controlled. The dynamics of the flow, as the evolution of mode amplitudes, are obtained by projecting the Navier-Stokes equations onto the POD basis. This results in a set of nonlinear ordinary differential equations (ODEs), which can be used for model-based control. Unfortunately, projecting directly the Navier-Stokes equations onto the POD modes results in an autonomous system, which is not useful for controller design since the effect of actuation is implicitly included in the system. Consequently, the equations must be recast in a form expressing the effect of actuation explicitly by incorporating a boundary control separation method [18, 19, 20], so that a feedback controller can be designed using the tools of control theory. The resulting ODEs including the effect of actuation are used as a model for a state feedback controller.
design. Implementation of this controller is allowed by using a stochastic estimation method to estimate flow variables from real-time surface pressure measurements. As before, reduced-order models are obtained based on two different types of data: numerical simulation data and experimental data. Numerical simulation data, though later revealed to be inconsistent with experimental measurements, serves for a good starting point for development of reduced order models of subsonic cavity flows. Robustness and region of attraction of the controllers designed for numerical data based nonlinear model are explored in simulation. Controller designed for experimental data based nonlinear models have been validated in real-time experiments. In implementation, the control signal must be limited to a given range to prevent the actuator from damages. Therefore, an additional scaling factor has to be introduced into the control law to avoid constant saturation of the actuator. The performance of the scaled feedback control is both analyzed mathematically and assessed experimentally, and agreement is achieved between both methods. Results presented in this chapter have been presented in [60, 61, 62].

### 3.1 Reduced-Order Modeling

The motion of the fluid within the cavity is governed by the Navier-Stokes equations [3]. In this study, we investigate the compressible Navier-Stokes equations, which can be written in a nondimensional form [36] as

\begin{align}
    u_t + uu_x + v u_y + h_x &= \frac{1}{Re_L}(u_{xx} + u_{yy}), \\
    v_t + uu_x + vv_y + h_y &= \frac{1}{Re_L}(v_{xx} + v_{yy}), \\
    h_t + uh_x + vh_y + (r - 1)h(u_x + v_y) &= 0,
\end{align}

where \( u(x, t) \), \( v(x, t) \) is the flow velocity in the streamwise and vertical direction, respectively, \( h(x, t) \) is the enthalpy, \( x \in \mathbb{R}^2 \) denotes the coordinate vector \((x, y)\),
Re$_L$ is the Reynolds number, and $r$ is the ratio of specific heats ($r = 1.4$ for air). Subscripts denote partial derivatives. Such a set of partial differential equations (PDEs), even though precisely describe the dynamics of the flow, can hardly serve for a model for controller design. To this end, it is essential to obtain a reduced-order model with a explicit input-output relation. The modeling technique we adopted to obtain a low dimensional model combines detailed flow velocity and pressure data with the governing Navier-Stokes equations, and consists of the following four steps:

1. The POD method is used to extract the “most important” features in a turbulent flow, POD modes, from the detailed temporal or spatial correlation data in the flow, which can be obtained using either numerical simulations or laser based planar flow measurements.

2. The Navier-Stokes equations are projected onto the POD modes using Galerkin projection to obtain a reduced-order flow model, which consists of a set of nonlinear ODEs. These equations govern the time evolution of the magnitudes of the POD modes.

3. A control separation method is incorporated in the Galerkin projection procedure to make the effect of external actuation explicitly appear in the ODEs.

4. Stochastic estimation is used to correlate the flow velocity field to surface pressure data to provide both state-to-output and output-to-state relations.

3.1.1 POD and Snapshot Method

The POD method was introduced to the fluid dynamics community by Lumley [31] as an objective tool to extract large scale structures in a turbulent flow. It
uses detailed temporal or spatial correlation data in the flow to determine a finite-dimensional subspace which contains the “most important” features of the flow in the sense of energy.

The goal of POD can be stated as follows [38]: Let $H$ be a Hilbert space, with inner product $\langle \cdot, \cdot \rangle$. Given an ensemble of data $\{ u^k \in H \mid k = 1, \ldots, m \}$, the task is to find a subspace $S$ of fixed dimension $N < m$, such that the error $E(\| u - P_S u \|)$ is minimized. Here, $\| \cdot \|$ is the induced norm on $H$, $u$ represents the entire ensemble $\{ u^k \}$, $P_S$ is the orthogonal projection onto the subspace, and $E(\cdot)$ denotes an average over $k$. This problem can be cast into the solution of the following integral eigenvalue problem [38]

$$\int_\Omega R(x, z)\phi(z)dz = \lambda\phi(x),$$ (3.2)

where $R$ is the averaged two point velocity correlation tensor which will be described later, eigenfunctions $\phi(x)$ give the optimal basis of subspace $S$ in terms of energy, $x$ and $z$ represent the vector coordinates of any two points in the domain of interest $\Omega$, which in our case is two dimensional. This infinite dimensional problem can be approximated by finite summation of discretized data over the entire domain $\Omega$, which leads to a standard eigenvalue problem.

In what follows, we briefly illustrate the POD procedures by using the $u$ component (streamwise velocity) in the Navier-Stokes equations as an example. Two different POD methods are examined, one is the conventional POD method and the other is the method of snapshots. The direct POD method [4, 17, 28] favors time-resolved data over a long time period at a few spatial locations, like hot-wire type data [24]. However, this method is computational intensive as the spatial resolution increases. The method of snapshots favors spatially-resolved, but time-uncorrelated snapshots.
of the flow field. Such data can be easily obtained by advanced laser-based planar flow
diagnostics such as particle image velocimetry (PIV) or planar Doppler velocimetry
(PDV), or by numerical simulations. In the current study, the method of snapshots
is adopted.

Assume an ensemble of the streamwise velocity \( \{u(x_i, t_j)\} \) has been obtained at
position \( x_i = x_1, x_2, ..., x_m \), and at time \( t_j = t_1, t_2, ..., t_k \). The mean flow velocity can
be calculated as

\[
\bar{u}(x_i) = \frac{1}{k} \sum_{j=1}^{k} u(x_i, t_j),
\]
and the velocity fluctuation can be obtained as

\[
\tilde{u}(x_i, t_j) = u(x_i, t_j) - \bar{u}(x_i).
\]

The POD procedures are summarized as follows.

**Conventional POD method**

When there is a large time resolved data set, \( k >> m \), a direct POD method
can be conveniently applied. Let the matrix \( \tilde{U} \in \mathbb{R}^{k \times m} \) collect the ensemble of the
velocity fluctuation as follows:

\[
\tilde{U} = \begin{pmatrix}
\tilde{u}(x_1, t_1) & \tilde{u}(x_2, t_1) & \cdots & \tilde{u}(x_m, t_1) \\
\tilde{u}(x_1, t_2) & \tilde{u}(x_2, t_2) & \cdots & \tilde{u}(x_m, t_2) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{u}(x_1, t_k) & \tilde{u}(x_2, t_k) & \cdots & \tilde{u}(x_m, t_k)
\end{pmatrix}.
\]

The two point correlation tensor over time is calculated by:

\[
R_{ij} = \frac{1}{k} \sum_{n=1}^{k} \tilde{u}(x_i, t_n)\tilde{u}(x_j, t_n) \in \mathbb{R}, \quad i, j = 1, 2, ..., m.
\]

The POD modes \( \phi_i \) can be obtained by directly solving the eigenvalue problem

\[
R \phi_n = \lambda_n \phi_n,
\]
where \( \phi_n, n = 1, ..., m \), constitute a basis of the optimal subspace, in the sense that the error in the projection onto the subspace is minimized. It can be shown that the POD eigenmodes are orthonormal

\[
< \phi_i, \phi_j > = \delta_{ij}, \tag{3.8}
\]

with \( \delta_{ij} \) being the Kronecker delta function

\[
\delta_{ij} = \begin{cases} 
1, & i = j, \\
0, & i \neq j.
\end{cases} \tag{3.9}
\]

The coefficients \( a_n(t) \) of the POD expansions can be computed by projecting the velocity field onto the subspaces spanned by \( \phi_n \). This yields

\[
a_n(t_j) = < \tilde{U}(x, t_j), \phi_n >, \quad n = 1, ..., m, \quad j = 1, ..., k, \tag{3.10}
\]

where \( \tilde{U}(x, t_j) \) represent the entire velocity field at a given time instant \( t_j \), and is given by the \( j \)-th row of \( \tilde{U} \). The truncated POD model with \( N < m \) modes is obtained as

\[
u(x_i, t_j) \approx \tilde{u}(x_i) + \sum_{n=1}^{N} a_n(t_j) \phi_n(x_i). \tag{3.11}
\]

Usually most of the flow kinetic energy can be captured by a small number of POD modes, which enables the reconstruction of the large coherent structures of flow. More details of this method can be found in [17, 28].

**Method of snapshots**

The snapshot method proposed by Sirovich [49] is an alternative way of computing the POD modes when there is a large spatial resolved data set and sufficiently large number of time realizations for the instantaneous field \( (k \ll m) \). The two point correlation tensor of independent snapshots is calculated over the spatial domain as

\[
C_{ij} = \frac{1}{m} \sum_{n=1}^{m} \tilde{u}(x_n, t_i) \tilde{u}(x_n, t_j), \quad i, j = 1, 2, ..., k. \tag{3.12}
\]
The eigenfunctions $z_n$ are obtained by solving the eigenvalue problem:

$$C z_n = \lambda_n z_n. \tag{3.13}$$

Any POD modes $\phi_n$ is a linear combination of snapshots computed as

$$\phi_n = \sum_{j=1}^{k} z_n(t_j) \tilde{U}(x, t_j). \tag{3.14}$$

Similarly, the POD eigenmodes $\phi_n$ can be shown to be orthonormal seen in (3.8). Projecting the snapshots onto the POD basis yields again the time evolution of each spatial basis as given in (3.10). The truncated POD model with $N << k$ modes is given by (3.11). The POD expansion (3.11) represents, as an example, the streamwise velocity in terms of the modes that contain the major portion of the kinetic energy in the flow. Similar expressions can be written for other flow quantities as well.

\[
\begin{align*}
    u(x, t) &\approx \bar{u}(x) + \sum_{n=1}^{N} a_n^u(t) \phi_n^u(x), \\
v(x, t) &\approx \bar{v}(x) + \sum_{n=1}^{N} a_n^v(t) \phi_n^v(x), \\
h(x, t) &\approx \bar{h}(x) + \sum_{n=1}^{N} a_n^h(t) \phi_n^h(x),
\end{align*}
\tag{3.15}
\]

where the superscripts $u$, $v$, $h$ denote the flow variable corresponding to the given POD expansion. In our study, following [36], a vector-valued POD expansion was obtained by combining the three flow variables into a vector $\mathbf{q} = (u \ v \ h)^T$. By properly defining the corresponding inner product for the vector-valued flow variable $\mathbf{q}(x, t)$ (see [36]), a vector-valued POD expansion can be computed as

$$\mathbf{q}(x, t) \approx \bar{\mathbf{q}}(x) + \sum_{n=1}^{N} a_n^q(t) \phi_n^q(x). \tag{3.16}$$

Using vector-valued POD expansion allows an even lower-order model with $N$ variables $a_n^q$, instead of $3N$ variables $a_n^u$, $a_n^v$, $a_n^h$ obtained by the scalar-valued POD expansion. Details of the snapshots method and its application in the present context are given in [6, 42].
3.1.2 Galerkin Projection

The second step in the process of deriving a reduced-order model is the projection of the Navier-Stokes equations onto the POD modes $\phi_n(x)$. The result of this procedure is a set of nonlinear ODEs for the modes magnitudes $a_n(t)$. For the sake of simplicity, here we still use the single $u$ component with scalar valued POD expansion to describe the Galerkin projection method.

Substituting the scale-valued POD expansion $u(x, t)$ into the Navier-Stokes equations gives

$$
\sum \dot{a}_n^u \phi_n^u + \left( \bar{u} + \sum a_n^u \phi_n^u \right) \left( \frac{\partial \bar{u}}{\partial x} + \sum a_n^u \frac{\partial \phi_n^u}{\partial x} \right) + \left( \bar{v} + \sum a_n^v \phi_n^v \right) \left( \frac{\partial \bar{u}}{\partial y} + \sum a_n^v \frac{\partial \phi_n^v}{\partial y} \right) + \left( \frac{\partial \bar{h}}{\partial x} + \sum a_n^h \frac{\partial \phi_n^h}{\partial x} \right) = \frac{1}{Re} \left( \frac{\partial^2 \bar{u}}{\partial x^2} + \sum a_n^u \frac{\partial^2 \phi_n^u}{\partial x^2} \right),
$$

(3.17)

where summations are over $n = 1$ to $N$.

Second, the expanded Navier-Stokes equation (3.17) are projected onto each POD modes $\phi_k^u$, $k = 1, 2, ..., N$, that is, the inner product of each term in (3.17) with each POD modes is computed. Recall that the POD modes are orthonormal (3.8), so only the $k$-th mode amplitude derivative $\dot{a}_k(t)$ remains when projecting the Navier-Stokes equations onto the $k$-th POD mode $\phi_k^u$. The standard Galerkin projection results in a system of ODEs which is called the *Galerkin model* hereafter:

$$
\dot{a}_k^u = b_k^u + \sum_{i=1}^N (c_{ik} a_i^u + d_{ik} a_i^v + e_{ik} a_i^h) + \sum_{i,j=1}^N (p_{ijk} a_i^u a_j^u + q_{ijk} a_i^u a_j^v), \quad k = 1, 2, ..., N. \tag{3.18}
$$

A detailed descriptions of the coefficients in the above equation can be found in [36].

The Galerkin model can be written in matrix form as

$$
\dot{a} = F + Ga + \left( \begin{array}{c} a^T H^1 a \\ \vdots \\ a^T H^N a \end{array} \right), \tag{3.19}
$$

58
where $F \in \mathbb{R}^{N \times 1}$, $G \in \mathbb{R}^{N \times N}$ and $H^i \in \mathbb{R}^{N \times N}$, $i = 1, 2, ..., N$. At this stage, the effect of actuation which is performed at the level of the boundary condition of (3.1) is not explicit in the model.

### 3.1.3 Control Separation

The standard Galerkin projection fails to exhibit the effect of actuation, resulting a model which is not useful for controller design. Several different methods are currently being investigated to derive a model where the effect of actuation appears explicitly in the equations. The method used in the present work is based on a spatial sub-domain separation idea [20], which divides the whole flow domain $\Omega$, into two parts: the actuation domain $\Omega_1$ and the remaining domain $\Omega_2$ as shown in Figure 3.1.

The idea is to calculate the spatial inner product separately over the actuation domain $\Omega_1$ and the remaining flow domain $\Omega_2$ as

$$< \cdot, \cdot >_\Omega = < \cdot, \cdot >_{\Omega_1} + < \cdot, \cdot >_{\Omega_2}. \quad (3.20)$$
The flow dynamics is regarded as being dominated by the actuator in the domain \( \Omega_1 \), and governed by the PDEs (3.1) over the domain \( \Omega_2 \). The starting point is to notice that the control input \( \Gamma(t) \) must satisfy the POD expansion

\[
u(x, t)|_{\Omega_1} = \Gamma(t)\cos(\theta) = \bar{u}(x)|_{\Omega_1} + \sum_{i=1}^{N} a_i(t)\phi_i(x)|_{\Omega_1}, \tag{3.21}\]

where \( \theta \) is the angle at which the control enters into the main flow, in our case \( \theta = 30^o \), and \( \Omega_1 \) denotes the domain where the control input enters the system (solid dots in Fig. 3.1). This in turn gives the following equation

\[
a_k(t)\phi_k(x)|_{\Omega_1} = \Gamma(t)\cos(\theta) - \bar{u}(x)|_{\Omega_1} - \sum_{i=1}^{N} (1 - \delta_{ik})a_i(t)\phi_i(x)|_{\Omega_1}. \tag{3.22}\]

Computing the inner products separately over the two domains and repeatedly using equation (3.22) in the procedure of Galerkin projection yield a new system of ODEs as follows:

\[
\dot{a} = F + Ga + \begin{pmatrix} a^T H^1 a \\ \vdots \\ a^T H^N a \end{pmatrix} + B\Gamma + \begin{pmatrix} (\bar{B}^1 \Gamma)^T a \\ \vdots \\ (\bar{B}^N \Gamma)^T a \end{pmatrix}. \tag{3.23}\]

Notice that the resulting matrices \( F, G, H^i, i = 1, 2, \ldots, N \) have the same dimensions but different values from the ones derived without control separation in (3.19), and \( B \in \mathbb{R}^{N \times 1}, \bar{B}^i \in \mathbb{R}^{N \times 1}, i = 1, 2, \ldots, N \). An example of control separation method applied to one dimensional heat equation is given in the appendix.

### 3.1.4 Stochastic Estimation

The Galerkin system (3.23) provides a state space model of the cavity flow dynamics, suitable for controller design. However, the state \( a(t) \) of the system is not directly available, and must be estimated from flow variables that can be measured in real-time. In any realistic setting, real-time experimental data could only be obtained via
surface measurements, e.g. surface pressure or surface shear stress measurements. In the current work, a stochastic estimation method is employed to correlate the states of the flow model $a(t)$ to surface pressure $p(t)$. Stochastic estimation was originally proposed and used by Adrian [1] to extract coherent structures from a turbulent flow field. The technique can be used to estimate flow variables at any point by using statistical information about the flow at a limited number of fixed locations. This method has been used to study flow structures in various flows [2, 14], and as a tool in POD modeling technique to estimate the modes magnitudes from experimental measurements in subsonic jets [34] and in cavity flows [42, 53]. In recent years, it has been used to estimate the modes magnitudes of POD models for feedback flow control [8, 9, 25, 48].

In the present work, linear and quadratic stochastic estimation were both explored to correlate the states of the flow model (3.23) and real-time measurements of surface pressures. Taking quadratic stochastic estimation as an example and using 6 locations as indicated in Figure 1.3, the estimates of the states can be written in the following form:

$$\hat{a}_n(t) = \sum_{i=1}^{6} C_{ni}p_i(t) + \sum_{i=1}^{6} \sum_{j=1}^{6} D_{nij}p_i(t)p_j(t), \quad n = 1, ..., N. \quad (3.24)$$

The matrices of the estimation coefficients $C, D$ are computed by minimizing the average mean square error between the values of $a_n$ obtained from the snapshots (3.10) and the estimated ones $\hat{a}_n$ (3.24) at the same time

$$e_{\hat{a}_n} = \frac{1}{k} \sum_{j=1}^{k} (\hat{a}_n(t_j) - a_n(t_j))^2 / k. \quad (3.25)$$
where \( k \) is the number of total time instants of interest. By requiring the partial derivative of \( e_{an} \) with respect to \( C \) and \( D \) to vanish, that is
\[
\frac{\partial e_{an}}{\partial C_{ni}} = 0, \quad \frac{\partial e_{an}}{\partial D_{nij}} = 0, \quad i, j = 1, ..., 6, \quad n = 1, ..., N, \quad (3.26)
\]
the corresponding values of \( C \) and \( D \) which minimize \( e_{an} \) in (3.25) can be directly obtained.

In a similar fashion, a state-output model can be also obtained by linear stochastic estimation method in the form of
\[
p_n(t) = \sum_{i=1}^{N} \bar{C}_{ni} a_i(t), \quad n = 1, ..., 6, \quad (3.27)
\]
which can be used for dynamic observer design.

### 3.2 Simulation Data Based Model and Feedback Controller Design

Finite-element simulations of the cavity flows have been conducted in the following three cases:

1. In absence of external input \( \Gamma(t) = 0 \), i.e., the baseline case at Mach 0.38;
2. In presence of an external sinusoidal excitation of the form \( \Gamma(t) = A \sin(2\pi f_c t) \), at Mach 0.38 with \( A = 40 \), \( f_c = 500 \text{ Hz} \);
3. In presence of an external sinusoidal excitation of the form \( \Gamma(t) = A \sin(2\pi f_c t) \), at Mach 0.38 with \( A = 40 \), \( f_c = 900 \text{ Hz} \).

Using the reduced order modeling tools described before, cavity flow models in the form
\[
\dot{a} = F + Ga + \begin{pmatrix} a^T H^1 a \\ \vdots \\ a^T H^N a \end{pmatrix} + B\Gamma + \begin{pmatrix} (\bar{B}^1 \Gamma)^T a \\ \vdots \\ (\bar{B}^N \Gamma)^T a \end{pmatrix}.
\]
have been derived from CFD simulation data with $N = 5$. The output equation is obtained by linear stochastic estimation as

$$p(t) = \bar{C}a(t). \quad (3.28)$$

Here, $p(t)$ is the output pressure at the central cavity floor and $\bar{C} \in \mathbb{R}^{1 \times N}$. For the sake of brevity, only the results for the baseline case are presented in the sequel.

### 3.2.1 Equilibrium Analysis

In order to gain insight into the properties of the nonlinear Galerkin model, we started with determining the location of the equilibrium points of the Galerkin model. Since this task involves solving a nonlinear algebraic matrix equation derived from (3.23) with $\Gamma = 0$, an analytic solution is intractable, and numerical methods must be sought. In our case, a Newton iterative method was implemented to calculate the equilibrium points of (3.23), that is, the roots of the nonlinear equation

$$f(a) := F + Ga + \begin{pmatrix} a^T H^1 a \\ \vdots \\ a^T H^N a \end{pmatrix} = 0. \quad (3.29)$$

The method implements the steepest descent iteration

$$a_{k+1} = a_k - J^{-1}(a_k) f(a_k), \quad (3.30)$$

where $J(a_k)$ is the Jacobian matrix

$$J = \frac{\partial f(a)}{\partial a}. \quad (3.31)$$

It should be noticed that the solution of the Newton method of (3.30) depends on the initial conditions and it is usually not unique. Evaluation of the modes magnitudes obtained by the numerical simulations shows that there exists a feasible region for the modes magnitudes given by
Figure 3.2: Runge-Kutta simulation of the Galerkin model of cavity flow at Mach 0.38: baseline.

\begin{equation}
|a_1| < 1.72,
|a_2| < 2.11,
|a_3| < 1.42,
|a_4| < 1.23,
|a_5| < 0.65,
\end{equation}

and so solutions obtained by means of the Newton algorithm that lie outside the feasible region need to be discarded. An example of the time history of the modes magnitudes $a_i(t)$ of the Galerkin model (3.23), obtained by means of computer simulations, is shown in Figure 3.2. With this in mind, the unique feasible equilibrium solution $a_0$ relative to the baseline cavity flow at Mach 0.38 is computed by the
Newton iterative method as

\[ a_0 = \begin{bmatrix} 0.1457 \\ 0.1541 \\ 0.0064 \\ 0.0026 \\ -0.1370 \end{bmatrix}. \] (3.33)

The reduced order flow model (3.23) is linearized around the equilibrium \( a_0 \) to obtain

\[ \dot{a} = \left. \frac{\partial f(a)}{\partial a} \right|_{a=a_0} a := J(a_0)a, \] (3.34)

where the Jacobian matrix \( J(a_0) \) is given by

\[ J(a_0) = \begin{bmatrix} 0.0739 & -0.5392 & -0.1266 & -0.0188 & 0.0627 \\ 0.4714 & 0.0525 & -0.3956 & 0.0808 & 0.0597 \\ 0.0486 & 0.0545 & -0.1943 & 1.0659 & 0.2010 \\ 0.0617 & 0.0221 & -0.7684 & -0.1731 & 0.0075 \\ -0.0451 & -0.0555 & -0.0730 & 0.0022 & -0.2352 \end{bmatrix}. \] (3.35)

The set of eigenvalues of Jacobian matrix is

\[ \lambda(J(a_0)) = \begin{bmatrix} -0.1626 + 0.9363i \\ -0.1626 - 0.9363i \\ 0.0401 + 0.4834i \\ 0.0401 - 0.4834i \\ -0.2312 \end{bmatrix}. \] (3.36)

Similar results have been obtained for the forced cases with forcing frequencies at 500 Hz and 900 Hz. It is necessary to point out that the eigenvalues presented in this section are different from the correct values by an order of magnitude. This is caused by a negligence of the fact that time is dimensionless during calculation. However, qualitative analysis remains correct.

### 3.2.2 Static Output Feedback Control

Static output feedback control, i.e, a control of the form

\[ \Gamma = -Kp(t) \] (3.37)
is the simplest feedback law that can be implemented. By substituting (3.28) into (3.37), the control law can be obtained as

\[ \Gamma = -K \tilde{C}a(t). \]  

(3.38)

By substituting the controller (3.38) into the nonlinear state space flow model (3.23), we can investigate the closed-loop system performance that can be obtained by static output feedback for different values of the gain \( K \). We provide the plot of \( K \) vs. the closed-loop equilibrium point and the corresponding root locus, which shows how the locations of the closed-loop eigenvalues change with respect to \( K \). The first plot is reported in Figure 3.3, while the root locus is shown in Figure 3.5. Note that for \( K \in [-\infty, 1058] \cup [2417, \infty] \), the equilibria of the closed-loop system change continuously and the eigenvalues change continuously as well. On the contrary, the Newton algorithm fails to yield a converging solution within the feasible region for \( K \in [1059, 2416] \).

Since very high gain solutions are of no practical use due to unavoidable limitations in the control authority, we restrict the analysis to a reasonable range of the feedback gain \( K \). As shown in Figure 3.4, the sensitivity of the equilibrium solution is very small for \( K \in [-100, 100] \). Moreover, it is clear from Figure 3.5 that stabilization cannot be achieved using proportional feedback since the unstable eigenvalues remain in the right half plane with respect to \( K \in [-\infty, 1058] \cup [2417, \infty] \). The forced cases show no significant difference from the baseline case, and similar conclusions can be drawn.

Since the presence of feedback control does not alter the location of the equilibrium, it makes sense to shift the coordinates of the state space to the unique equilibrium point \( a_0 \). Shifting the equilibrium of (3.23) to the origin corresponds to
Figure 3.3: Equilibrium points of the closed-loop system vs. static output feedback control gain $K \in [-\infty, 1058] \cup [2417, \infty]$.

Figure 3.4: Zoom of Figure 3.3 where $K \in [100, 100]$. 
Figure 3.5: Root locus for $K \in [-\infty, 1058] \cup [2417, \infty]$.

Singling out the effect of the mean flow from the low order model, and considering the local behavior of the system around the mean flow. Letting $\tilde{a} = a - a_0$, we obtain

\[
\dot{\tilde{a}} = \dot{a} = F + G(\tilde{a} + a_0) + \begin{pmatrix}
(\tilde{a} + a_0)^T H^1(\tilde{a} + a_0) \\
\vdots \\
(\tilde{a} + a_0)^T H^N(\tilde{a} + a_0)
\end{pmatrix} + B\Gamma + \begin{pmatrix}
(\tilde{B}^1\Gamma)^T(\tilde{a} + a_0) \\
\vdots \\
(\tilde{B}^N\Gamma)^T(\tilde{a} + a_0)
\end{pmatrix}
\]

\[
= F + Ga_0 + \begin{pmatrix}
a_0^T H^1 a_0 \\
\vdots \\
a_0^T H^N a_0
\end{pmatrix} + G\tilde{a} + \begin{pmatrix}
\tilde{a}^T H^1 a_0 + a_0^T H^1 \tilde{a} \\
\vdots \\
\tilde{a}^T H^N a_0 + a_0^T H^N \tilde{a}
\end{pmatrix} + \begin{pmatrix}
\tilde{a}^T H^1 \tilde{a} \\
\vdots \\
\tilde{a}^T H^N \tilde{a}
\end{pmatrix}
\]

\[
+ B\Gamma + \begin{pmatrix}
(\tilde{B}^1\Gamma)^T a_0 \\
\vdots \\
(\tilde{B}^N\Gamma)^T a_0
\end{pmatrix} + \begin{pmatrix}
(\tilde{B}^1\Gamma)^T \tilde{a} \\
\vdots \\
(\tilde{B}^N\Gamma)^T \tilde{a}
\end{pmatrix}.
\]

(3.39)
Recall that $a_0$ is the equilibrium solution of the unforced system, that is,

$$F + Ga_0 + \begin{pmatrix} a_0^T H^1 a_0 \\ \vdots \\ a_0^T H^N a_0 \end{pmatrix} = 0. \quad (3.40)$$

Since $\Gamma$ and $a_i^T H^i a_0, \ i = 1, \ldots, N$ are all scalars, the terms can be reorganized and the model in the new set of coordinates $\tilde{a}$ can be written as

$$\dot{\tilde{a}} = \tilde{G} \tilde{a} + \begin{pmatrix} \tilde{a}^T H^1 \tilde{a} \\ \vdots \\ \tilde{a}^T H^N \tilde{a} \end{pmatrix} + \tilde{B} \Gamma + \begin{pmatrix} (\tilde{B}^1 \Gamma)^T \tilde{a} \\ \vdots \\ (\tilde{B}^N \Gamma)^T \tilde{a} \end{pmatrix}, \quad (3.41)$$

where

$$\tilde{G} = G + \begin{pmatrix} \tilde{a}_0^T (H^1 + H^1^T) \\ \vdots \\ \tilde{a}_0^T (H^N + H^N^T) \end{pmatrix},$$

$$\tilde{B} = B + \begin{pmatrix} \tilde{B}^1 \Gamma \\ \vdots \\ \tilde{B}^N \Gamma \end{pmatrix} a_0. \quad (3.42)$$

Similarly, letting $\tilde{p} = p - p_0$, the output equation (3.28) can be rewritten as

$$\tilde{p} = p - p_0 = \tilde{C} \tilde{a} - \tilde{C} a_0 = \tilde{C} \tilde{a}. \quad (3.43)$$

The new modified model has the same structure as before but with the equilibrium transformed to the origin, which is more convenient for controller design and stability analysis.

The static output feedback $\Gamma = -K \tilde{p}$, which is the most appropriate for the model (3.41)–(3.43), does not alter the equilibrium at $a_0$ for the original POD model (3.23) even in presence of high-gain feedback, as opposed to the feedback law $\Gamma = -K p$ considered previously. The analysis reveals that the origin of (3.41) cannot be stabilized by the static output feedback $\Gamma = -K \tilde{p}$, as the root locus of the compensated system (shown in Figure 3.6) possesses two branches which remain inside the open right-half
complex plane for all values of the gain $K$. It is quite obvious that static output feedback controllers are inadequate for the POD model (3.41)–(3.43), and that the use of state feedback or dynamic output feedback controllers is required.

### 3.2.3 Optimal State Feedback Control

The Jacobian linearization of (3.41) and (3.43) at the origin is readily obtained as

$$
\begin{align*}
\dot{\tilde{a}} &= \tilde{G}\tilde{a} + \tilde{B}\Gamma, \\
\dot{\tilde{p}} &= \tilde{C}\tilde{a}.
\end{align*}
$$

It has been verified that the system (3.44) is controllable and observable for all considered three cases. This allows standard state-space methodologies to be implemented to develop a controller (either by full state feedback or dynamic output feedback) to stabilize the origin of the linear system. This, in turn, yields a controller that *locally*
stabilizes the origin of the nonlinear system (3.41). A convenient and well-established methodology for the controller design is offered by linear-quadratic optimal control. Let the cost function $J_c$ be

$$J_c = \int_0^\infty (\tilde{a}^TW\tilde{a} + W_\Gamma \Gamma^2)dt,$$  \hfill (3.45)

where $W_\tilde{a} > 0$ and $W_\Gamma > 0$ are the positive definite state weighting matrix and the scalar control weight, respectively. In our design, the weights have been chosen as

$$W_\tilde{a} = I_{5 \times 5},$$

$$W_\Gamma = 1.$$  \hfill (3.46)

The solutions of optimal state feedback controllers

$$\Gamma = -K_i \tilde{a},$$ \hfill (3.47)

where the subscript $i = 0, 1, 2$ stands for the baseline case, forced case with $f_c = 500$ Hz and $f_c = 900$ Hz respectively, have been obtained from the solution of the associated Riccati equations, and read as

$$K_0 = \begin{bmatrix} -1688.8 & -491.7 & -31.4 & 1050.9 & 210.1 \end{bmatrix}^T \quad \text{(baseline),}$$

$$K_1 = \begin{bmatrix} 16.8 & -188.3 & 85.6 & 83.8 & 50.8 \end{bmatrix}^T \quad \text{(}f_c = 500\text{Hz),}$$

$$K_2 = \begin{bmatrix} 12.9 & -141.2 & 64.2 & 56.9 & 43.5 \end{bmatrix}^T \quad \text{(}f_c = 900\text{Hz).}$$ \hfill (3.48)

Similarly we design a full state observer

$$\dot{\hat{\tilde{a}}} = \hat{G}\hat{\tilde{a}} + \hat{B}\Gamma + L(\hat{p} - \tilde{p}),$$ \hfill (3.49)

$$\hat{\tilde{p}} = \hat{C}\hat{\tilde{a}},$$ \hfill (3.50)

where $\hat{\tilde{a}}, \hat{\tilde{p}}$ are respectively the estimated state and estimated output, and $L$ is the optimal output-injection gain obtained by minimizing the cost function of the same
<table>
<thead>
<tr>
<th></th>
<th>(C_0)</th>
<th>(C_1)</th>
<th>(C_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_0)</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>(P_1)</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>(P_2)</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>

Table 3.1: Robustness of the optimal LQ controllers.

form as (3.45) for the dual system. Choosing again the weights as identity matrices, the optimal \(L_i\) for each case are given as follows,

\[
L_0 = \begin{bmatrix} -1.3 & 1.3 & -0.8 & -0.2 & -0.6 \end{bmatrix}^T \quad \text{(baseline)},
\]

\[
L_1 = \begin{bmatrix} -1.3 & 1.6 & -0.8 & -0.4 & -0.9 \end{bmatrix}^T \quad (f_c = 500\text{Hz}), \quad (3.51)
\]

\[
L_2 = \begin{bmatrix} -1.2 & 1.3 & -0.7 & -0.2 & -0.7 \end{bmatrix}^T \quad (f_c = 900\text{Hz}).
\]

3.2.4 Simulation Results

In section 3.2, we show simulation results for the application of the optimal state feedback controllers and observer designed in the previous section to the nonlinear model described by (3.41). For all the three cases the robustness of the controllers to parameter variations has been evaluated. Since the controllers have been designed on the basis of the linearized models, their validity for the nonlinear model may be restricted to a possibly small neighborhood of the equilibrium. For this reason, estimates of the regions of attraction of each case have been obtained numerically.

Controller Robustness

Each controller given by (3.48) was designed for a fixed model, while it may be required to work under uncertain conditions, under the presence of disturbances, or even for a model different from the nominal one that the controller was designed for.
Therefore, it is important to investigate the robustness provided by the controllers. A simple test is to check if a controller designed on a specific model works for the other two models as well. Denote the reduced order flow models obtained for the baseline case, the forced case with $f_c = 500$ Hz and $f_c = 900$ Hz by $P_0$, $P_1$, $P_2$, and the controller designed for each model by $C_0$, $C_1$, $C_2$, respectively. The results are given in Table 3.1, where the entry ‘yes’ means that the controller can stabilize the corresponding nonlinear model and the entry ‘no’ means that the controller can not stabilize that model.

**Region of Attraction**

An important aspect of the design is to evaluate the region of attraction of the origin in closed loop provided by each controller. As a matter of fact, since the design is based on linear control theory for the linearized model, attractivity is guaranteed only for initial conditions within a (possibly small) neighborhood of the origin of the nonlinear model. For the nonlinear system (3.41), a theoretical analysis of the region of attraction is difficult and beyond the scope of the present work. Instead, the region of attraction obtained by each controller $C_i$ can be estimated numerically. For the sake of simplicity, we will discuss the estimation of the region of attraction for the system in closed-loop with the state-feedback controller $\Gamma = -K_i \tilde{a}$ only. Since the region of attraction is the largest open invariant set which has the property that each point of the set is asymptotically attracted to the origin, an estimate of the domain of attraction is obtained looking for compact sets which contain the origin in their interior, are invariant under the flow of the closed-loop system, and such that all the point at its boundary are attracted to the origin.
The obvious choice for the compact sets in question is given by the level sets of the Lyapunov functions for each linearized closed-loop system, that is

\[ \mathcal{V}_i(\tilde{a}) = \tilde{a}^T S_i \tilde{a}, \quad 0 \leq i \leq 2, \]  

(3.52)

where \( S_i \) is the solution of the Lyapunov equation

\[ S_i (\tilde{G} - \tilde{B} K_i) + (\tilde{G} - \tilde{B} K_i)^T S_i = -I \]

(3.53)

corresponding to the given choice for \( K_i, 0 \leq i \leq 2 \). The level sets \( \Omega^i_c \) are defined as

\[ \Omega^i_c = \{ \tilde{a} \in \mathbb{R}^N \text{ such that } \mathcal{V}_i(\tilde{a}) \leq c \}, \]

(3.54)

where \( c \) is a positive constant. The boundary of \( \Omega^i_c \), denoted by \( \partial \Omega^i_c \) is a closed \((N - 1)\)-dimensional surface in \( \mathbb{R}^N \). Fix \( c > 0 \), and consider, for the system (3.41) in closed-loop with the state feedback controller \( \Gamma = -K_i \tilde{a} \), arbitrary initial conditions \( \tilde{a}(0) \) such that \( \tilde{a}(0) \in \partial \Omega^i_c \). Integrating numerically the differential equations of the closed-loop system allows us to determine whether initial conditions in \( \partial \Omega^i_c \) are attracted to the origin or not. The search is initialized with \( c > 0 \) small enough so that the state trajectory with the corresponding initial condition converges to the origin. We enlarge \( c \) until we find \( c_{\text{max}} \) such that the state trajectory with initial condition in \( \partial \Omega^i_c \) diverges for any \( c > c_{\text{max}} \). Then, an estimate of the region of attraction is given by the corresponding level set \( \Omega^i_{c_{\text{max}}} \). Similarly to the analysis of the controller robustness, we apply a linearly designed controller to the nonlinear model described by (3.41) for all three cases. The estimated region of attraction represented by the value of \( c_{\text{max}} \) is given in Table 3.2.4, where a value \( c_{\text{max}} = 0 \) corresponds to an unstable equilibrium at the origin in closed-loop, consistently with the results given in Table 3.1. Finally, an example of simulation results is given.
Table 3.2: Region of attraction of the optimal LQR state-feedback controllers.

<table>
<thead>
<tr>
<th></th>
<th>$C_0$</th>
<th>$C_1$</th>
<th>$C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_0$</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$P_1$</td>
<td>0</td>
<td>0.9</td>
<td>1.1</td>
</tr>
<tr>
<td>$P_2$</td>
<td>0</td>
<td>7</td>
<td>11</td>
</tr>
</tbody>
</table>

in Figure 3.7 and Figure 3.8, which show respectively the output $\tilde{p}(t)$ and the mode magnitude corresponding to the first POD mode $\tilde{a}_1(t)$ for each the closed-loop systems $(P_0, C_0)$, $(P_1, C_1)$, and $(P_2, C_2)$ corresponding to the main diagonal in Table 3.1 and Table 3.2. The Simulink diagram of the linear LQ controller and LQ observer implementation is given in Figure 3.9. For each case, the initial condition has been selected within the estimated region of attraction, and the corresponding trajectory decays asymptotically to the origin. The simulation results show that the closed-loop pressures and states of all three cases are stabilized and are driven to the desired conditions.

3.3 Experimental Data Based Model and Feedback Controller Design

In the previous section, flow models derived from numerical simulation data have been discussed, and controllers designed for the models have been tested in simulations. These controllers have not been validated through experiments because the model itself derived from the simulation data shows a remarkably different open-loop response from the experiments. So it is necessary to obtain a flow model directly from the experimental data, for which experimental validation of the controller is
Figure 3.7: Closed-loop simulation results with LQ controller and LQ observer applied to the nonlinear Galerkin model – output pressure $\tilde{p}(t)$.

Figure 3.8: Closed-loop simulation results with LQ controller and LQ observer applied to the nonlinear Galerkin model – first mode magnitude $\tilde{a}_1(t)$.
meaningful. The PIV system provides a detail estimation of the flow velocity field which can be used by POD method for modeling purpose.

In the following section, we study flow models derived from experimental data, design controllers for them, present the real-time implementation results, and interpret the results from a mathematical perspective. A single-mode flow at Mach number $M = 0.3$ has been selected as our reference baseline case because it has been shown that the actuator has sufficient authority to significantly alter the flow and real-time feedback control can be implemented at this Mach number [16]. Several flow conditions have been studied, including the baseline flow and four open-loop forced flow at different forcing frequencies as shown in Table 3.3. Flow models based on PIV data of different flow conditions are then obtained for control design, as summarized in Table 3.4. Here, we mainly present the results obtained for the baseline flow model M0, while all other cases are discussed briefly. The order of the flow model (3.23) has been chosen to be $N = 4$ to achieve a tradeoff between accuracy and applicability.

Figure 3.9: Simulink diagram of linear LQ controller and LQ observer applied to the original nonlinear system.
<table>
<thead>
<tr>
<th>Flow cases</th>
<th>Open loop forcing frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>F0</td>
<td>no forcing</td>
</tr>
<tr>
<td>F1</td>
<td>1610 Hz</td>
</tr>
<tr>
<td>F2</td>
<td>1830 Hz</td>
</tr>
<tr>
<td>F3</td>
<td>3250 Hz</td>
</tr>
<tr>
<td>F4</td>
<td>3920 Hz</td>
</tr>
</tbody>
</table>

Table 3.3: Flow cases from which PIV data are acquired for POD modeling at Mach 0.3.

<table>
<thead>
<tr>
<th>Model</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>M0</td>
<td>Galerkin model based on F0 snapshots</td>
</tr>
<tr>
<td>M1</td>
<td>Galerkin model based on F0 and F1 snapshots</td>
</tr>
<tr>
<td>M2</td>
<td>Galerkin model based on F0 and F2 snapshots</td>
</tr>
<tr>
<td>M3</td>
<td>Galerkin model based on F0 and F3 snapshots</td>
</tr>
<tr>
<td>M4</td>
<td>Galerkin model based on F0 and F4 snapshots</td>
</tr>
</tbody>
</table>

Table 3.4: Galerkin flow models derived from PIV data described in Table 3.3.

of the model. It has been shown that 4 POD modes are sufficient to reconstruct the dominant coherent structures of the flow [44] and simple enough for design and analysis of feedback control.

### 3.3.1 Equilibrium Analysis and Coordinate Transformation

The fourth-order nonlinear system (3.23) is still quite complicated for classical control theory to apply directly. For a further simplification of the model, we look for a linearization around the equilibrium point, which corresponds to the mean flow.
The equilibrium has been calculated by the Newton iterative method (3.30). Runge-Kutta simulation of the ODE system (3.23) with $\Gamma = 0$ is shown in Figure 3.10, which suggests the bounded range for the feasible solution in this case.

In the same way as presented in Section 3.2, the origin of the coordinates is shifted to the equilibrium point, which results in a new state space model in the form of (3.41).

### 3.3.2 LQ Controller Design

Recall that the linear approximation of (3.41) at the origin is given by

$$\dot{\hat{a}} = \widetilde{G}\hat{a} + \widetilde{B}\Gamma.$$  \hfill (3.55)

The eigenvalues of the system matrix $\widetilde{G}$ are given in Table 3.5.
Table 3.5: Eigenvalues of the open-loop system matrices of each flow model.

<table>
<thead>
<tr>
<th>Model</th>
<th>Open loop eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>M0</td>
<td>$\lambda_{1,2} = 1597 \pm 7023i, \lambda_3 = -3652, \lambda_4 = -880$</td>
</tr>
<tr>
<td>M1</td>
<td>$\lambda_{1,2} = 270 \pm 6491i, \lambda_3 = -1847, \lambda_4 = -623$</td>
</tr>
<tr>
<td>M2</td>
<td>$\lambda_{1,2} = 433 \pm 12458i, \lambda_3 = -397 \pm 1779i$</td>
</tr>
<tr>
<td>M3</td>
<td>$\lambda_{1,2} = 1493 \pm 7532i, \lambda_3 = -2742, \lambda_4 = -674$</td>
</tr>
<tr>
<td>M4</td>
<td>$\lambda_{1,2} = 1397 \pm 7062i, \lambda_3 = -2871, \lambda_4 = -697$</td>
</tr>
</tbody>
</table>

It can be seen the open-loop matrix of model M0 possesses two unstable complex conjugate eigenvalues and two stable real eigenvalues. Other flow cases studied in this work show the same qualitative features, that is, 2 unstable complex conjugate eigenvalues plus 2 stable eigenvalues. Since the pairs $(\tilde{G}, \tilde{B})$ are controllable, linear state-feedback design based on the linearized model (3.55) offers a simple approach to the design of a controller for the nonlinear Galerkin model. Recall that the stochastic estimation method provides a way to estimate the states of the Galerkin model from real-time surface pressure measurements with (3.24). The availability of real-time estimates of the state of the Galerkin model allows the use of linear state-feedback control to stabilize the origin of the linearized model. This, in turn, yields a controller that locally stabilizes the origin of the nonlinear Galerkin system. Linear-quadratic (LQ) optimal controllers have been designed with the positive definite weighting functions for the state vector and the control signal are chosen as

$$W_a = I_{4 \times 4}, \quad W_T = 1.$$  

LQ design results in asymptotic stabilization of the origin, while keeping the control energy small. The controller gains are computed as given in Table 3.6. As it is well
Table 3.6: LQ feedback control gains and values of the scaling factor $\alpha$ for each model.

<table>
<thead>
<tr>
<th>Model</th>
<th>Controller gains</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>M0</td>
<td>$(-56, 8.8, -417, -12.8)$</td>
<td>0.265</td>
</tr>
<tr>
<td>M1</td>
<td>$(2.6, -19.9, -27, -16.1)$</td>
<td>1</td>
</tr>
<tr>
<td>M2</td>
<td>$(1.7, 69.5, 36.3, -33)$</td>
<td>0.62</td>
</tr>
<tr>
<td>M3</td>
<td>$(-51.9, 50.8, -320, -261)$</td>
<td>0.25</td>
</tr>
<tr>
<td>M4</td>
<td>$(17.6, 209, 11.6, -147)$</td>
<td>0.5</td>
</tr>
</tbody>
</table>

known, applying the LQ state feedback control to the linearized system results in mirroring all the right-half plane eigenvalues of the matrix to the left half plane, while the left-half plane eigenvalues are left practically unchanged, as shown in Figure 3.11. Results of nonlinear simulations of the closed-loop system (3.23 - 3.47) show that the trajectory $a(t)$ converges to the corresponding equilibrium point $a_0$. This indicates that, at least in principle, the LQ controller designed for the linear approximation (3.55) succeeds in stabilizing the equilibrium of the 4-dimensional nonlinear Galerkin model (3.23).

The diagram of the closed-loop implementation of the LQ optimal controller is given in Figure 3.12. It is important to point out that, to prevent damaging the actuator, the control input signal must be limited to the range $\pm 10V$. Since the LQ control results in large control gains as shown in Table 3.6. As a matter of fact, constant saturation of the actuator was observed during closed-loop experiments. In order to keep the actuator below the saturation limits, a constant scaling factor $\alpha > 0$ was introduced into the feedback loop

$$\Gamma_\alpha(t) = -\alpha K\hat{a}(t). \quad (3.56)$$
Figure 3.11: Simulation results of the closed loop system with LQ control for M0 flow model, (a) Open loop and closed-loop eigenvalues, (b) Closed-loop response of states.
The largest feasible scaling factor for each controller investigated in this research has been found as given in Table 3.6, and the actual control law was scaled by $\alpha$.

The performance of the closed-loop systems obtained with the scaled LQ control (3.56) replacing (3.47) has been fully evaluated in simulation, and the results obtained for model M3 are shown in Figure 3.13. The results for the other models show a similar behavior, and are therefore omitted. From Figure 3.13 (a), it can be noted that the closed-loop eigenvalues are moved to the left-half plane only when $\alpha > 0.5$. However, Figure 3.13 (b) shows that, though the scaled LQ control is not capable of asymptotically stabilizing the origin of the nonlinear Galerkin model, as shown for the scaling factor of $\alpha = 0.25$, it nevertheless provides a significant reduction of the amplitude of the stable limit cycle. These results are in agreement with a
mathematical analysis given in the next section, which predicts a reduction of the amplitude of limit cycle corresponding to the fundamental cavity tone as the gain $\alpha$ increases from 0 to 0.5, with complete suppression of the oscillation only possible for $\alpha > 0.5$.

3.3.3 Mathematical Analysis of the Performance of LQ Control

In what follows, we present a simple analysis, carried out on the basis of the nonlinear Galerkin model with $N = 4$, to explain the closed-loop results obtained with the LQ feedback control. Recall that, after the preliminary transformation $\tilde{a} = a - a_0$, the Galerkin system can be written in the form (3.41). Let $T \in \mathbb{R}^{4 \times 4}$ be a nonsingular transformation that puts $\tilde{G}$ in modal form

$$ T\tilde{G}T^{-1} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \tag{3.57} $$

where

$$ L_1 = \begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix}, \quad L_2 = \begin{pmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, $$

$\sigma > 0$, $\omega > 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$. Partitioning the state vector according to the above decomposition, the Galerkin system is written in the new coordinates as

$$ \dot{\eta} = L_1 \eta + M_1 \Gamma_\alpha + \varphi_1(\eta, \zeta) + \gamma_1(\eta, \zeta) \Gamma_\alpha, $$

$$ \dot{\zeta} = L_2 \zeta + M_2 \Gamma_\alpha + \varphi_2(\eta, \zeta) + \gamma_2(\eta, \zeta) \Gamma_\alpha, $$

where

$$ T\tilde{a} = \begin{pmatrix} \eta \\ \zeta \end{pmatrix}, \quad T\tilde{B} = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, $$

and

$$ \varphi_i(\eta, \zeta) = O(\|\eta\|^2, \|\zeta\|^2), \quad \gamma_i(\eta, \zeta) = O(\|\eta\|, \|\zeta\|), \quad i = 1, 2. $$
Figure 3.13: Simulation results of the closed loop system with different scaling factor $\alpha$ for M3 flow model. (a) Root Locus, (b) Closed loop responses at location of transducer 3.
The control law $\Gamma_{\alpha} = -\alpha K \tilde{a}$ can be expressed in the new coordinates as

$$\Gamma_{\alpha} = -\alpha K_1 \eta - \alpha K_2 \zeta,$$

for some matrices $K_1$ and $K_2$. Since it has been verified that the control input does not affect the location of the stable eigenvalues of the open-loop matrix $\tilde{G}$, as it is typically the case for LQ-based design, necessarily $K_2 = (0 \ 0)$. Therefore, the closed-loop system can be written in the form

$$\begin{align*}
\dot{\eta} &= (L_1 - \alpha M_1 K_1) \eta + \varphi_1 (\eta, \zeta) - \alpha \gamma_1 (\eta, \zeta) K_1 \eta, \\
\dot{\zeta} &= -\alpha M_2 K_2 \eta + L_2 \zeta + \varphi_2 (\eta, \zeta) - \alpha \gamma_2 (\eta, \zeta) K_1 \eta.
\end{align*}$$

An easy computation shows that the eigenvalues of the matrix $(L_1 - \alpha M_1 K_1)$ are given by

$$\lambda(L_1 - \alpha M_1 K_1) = (1 - 2\alpha)\sigma \pm j\sqrt{\omega^2 + 4\alpha \sigma^2 (1 - \alpha)},$$

and thus

$$\lambda(L_1 - \alpha M_1 K_1) \approx (1 - 2\alpha)\sigma \pm j\omega,$$

since $\omega \gg \sigma$, and $\alpha \in [0, 1]$. Letting $\mu = 1 - 2\alpha$, one obtains (modulo a unitary transformation)

$$L_1 - \alpha M_1 K_1 = \begin{pmatrix} \mu \sigma & -\omega \\
\omega & \mu \sigma \end{pmatrix},$$

and thus the spectrum of the closed-loop matrix

$$\tilde{L}(\mu) = \begin{pmatrix} L_1 + (\mu/2 - 1/2) M_1 K_1 & 0 \\
(\mu/2 - 1/2) M_2 K_2 & L_2 \end{pmatrix}$$

splits into a pair of purely imaginary eigenvalues and a pair of negative real eigenvalues when $\mu = 0$. This implies the existence of a **center manifold** for the trajectories of the Galerkin system. Specifically, denote for the sake of simplicity

$$\tilde{L}(\mu) = \begin{pmatrix} \tilde{L}_{11}(\mu) & 0 \\
\tilde{L}_{21}(\mu) & L_2 \end{pmatrix},$$
\[ \Phi_i(\eta, \zeta, \mu) = \varphi_i(\eta, \zeta) - \alpha \gamma_i(\eta, \zeta)K_i\eta, \quad i = 1, 2 \]

and write the closed-loop Galerkin system as

\[
\begin{align*}
\dot{\mu} &= 0 \\
\dot{\eta} &= \bar{L}_{11}(\mu)\eta + \Phi_1(\eta, \zeta, \mu) \\
\dot{\zeta} &= \bar{L}_{21}(\mu)\eta + L_2\zeta + \Phi_2(\eta, \zeta, \mu)
\end{align*}
\]  

(3.60)

where

\[ \Phi_i(\eta, \zeta, \mu) = O(\|\eta\|^2, \|\zeta\|^2) \quad \text{for all} \quad i = 1, 2 \]

and we have added a trivial dynamics for the bifurcation parameter \( \mu \). The Center Manifold Theorem [10] establishes the existence of an exponentially attracting submanifold of the state space, which is described by the graph of a smooth mapping \( \zeta = \pi(\eta, \mu) \) satisfying \( \pi(0, \mu) = 0 \), \( (\partial\pi/\partial\eta)(0, \mu) = 0 \), and

\[
\frac{\partial\pi}{\partial\eta} [\bar{L}_{11}(\mu)\eta + \Phi_1(\eta, \pi(\eta, \mu), \mu)] = \bar{L}_{21}(\mu)\eta + L_2\pi(\eta, \mu) + \Phi_2(\eta, \pi(\eta, \mu), \mu)
\]

for all \((\eta, \mu)\) in a neighborhood of \((0, 0)\). This allows to reduce the analysis of the dynamics of the Galerkin system to its restriction onto the center manifold, which in the given set of coordinates reads as

\[
\begin{pmatrix}
\dot{\eta}_1 \\
\dot{\eta}_2
\end{pmatrix} = \begin{pmatrix}
\mu \sigma & -\omega \\
\omega & \mu \sigma
\end{pmatrix} \begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix} + \begin{pmatrix}
\Phi_{11}(\eta, \pi(\eta, \mu), \mu) \\
\Phi_{12}(\eta, \pi(\eta, \mu), \mu)
\end{pmatrix} .
\]

(3.61)

A near-identity transformation into Poincarè normal form [55] yields

\[
\begin{align*}
\Phi_{11}(\eta, \pi(\eta, \mu), \mu) &= (-a(\mu)\eta_1 - b(\mu)\eta_2)(\eta_1^2 + \eta_2^2) + O(||\eta||^5), \\
\Phi_{12}(\eta, \pi(\eta, \mu), \mu) &= (b(\mu)\eta_1 - a(\mu)\eta_2)(\eta_1^2 + \eta_2^2) + O(||\eta||^5),
\end{align*}
\]

from which, using polar coordinates \((\rho, \theta) = (\sqrt{\eta_1^2 + \eta_2^2}, \tan^{-1}(\eta_2/\eta_1))\), one obtains the system

\[
\begin{align*}
\dot{\rho} &= \mu \sigma \rho - a(\mu)\rho^3 + O(\rho^5), \\
\dot{\theta} &= \omega + b(\mu)\rho^2 + O(\rho^4).
\end{align*}
\]  

(3.62)
The structure of the reduced system reveals that the original Galerkin system has a locally exponentially stable equilibrium at the origin for $\mu < 0$, and undergoes a Hopf-Poincaré-Andropov bifurcation at $\mu = 0$, with a stable limit cycle for $\mu > 0$. The amplitude and frequency of the limit cycle are given respectively by

$$\rho^* = \sqrt{\frac{\mu \sigma}{a(\mu)}}, \quad \omega^* = \omega + b(\mu) \frac{\mu \sigma}{a(\mu)},$$

from which, since $a(\mu) = O(1)$, it is readily seen that the amplitude of the oscillation decreases as $\mu \to 0^+$. Recalling that $\mu = (1 - 2\alpha)$, the results of the analysis can be summarized as follows:

1. If it is required to set $\alpha < 0.5$ to avoid saturating the actuator, the origin of the Galerkin system can not be stabilized at all.

2. If this is the case, the application of linear feedback can still lower the amplitude of the limit cycle, but only up to the critical value imposed by the actuator limits.

The experimental results given in the next section seem to support the results of the analysis, as linear feedback is capable to attenuate the resonance in the cavity to a certain extent, while complete suppression seems to be unattainable with the given actuation. However, it may still be possible to reduce the amplitude of the cavity tone beyond the limit achievable using linear feedback, resorting to different control strategies (nonlinear feedback or time-varying feedback). In particular, a viable strategy to be pursued is to increase $a(\mu)$, shaping the center manifold $\zeta = \pi(\eta, \mu)$ by means of nonlinear feedback.
3.3.4 Experimental Results

The performance of the scaled control law has been tested experimentally, for different flow conditions. Hereby, we present the results obtained in controlling different flows with controllers designed based on model M0 and M4. Specifically, we present the sound pressure level (SPL) spectra acquired by transducers 5 and 6 on the cavity wall (see Figure. 1.3). In addition, a comparison is made with the results obtained using open-loop sinusoidal excitation at 3290 Hz. In Figure 3.14-3.17, the thin lines represent the open-loop pressure spectrum, and the thick lines represent either the pressure spectrum of the closed-loop system with the scaled LQ controller applied or the pressure spectrum of the open-loop forced flow.

The results for closed-loop LQ control at the reference flow condition Mach 0.3 are presented in Figure 3.14. The M0-based LQ controller produces a considerable attenuation of the resonance peak with a redistribution of the energy into various modes, especially lower frequency modes, with much lower energy level (Figure 3.14 (a-b)). When the model incorporating the forcing (Figure 3.14 (c-d)) is considered in place of the M0 model, the attenuation of the resonance peak is comparable to the M0-based case. It is worth noting that the results show many similarities at the same transducer location with different model-based LQ controllers. For example, transducer 5 detects a wide peak around the second Rossiter mode in both M0-based and M4-based LQ control cases, and transducer 6 detects a new peak around 1500 Hz in both cases too. The results indicates that the dynamics of the flow have been well captured by the linear flow models and this information has been utilized by the closed-loop LQ controllers.
Figure 3.14: Sound pressure level in closed-loop experiments at Mach 0.3 with (a) M0-based LQ controller measured at sensor 5, (b) M0-based LQ controller measured at sensor 6, (c) M4-based LQ controller measured at sensor 5, (d) M4-based LQ controller measured at sensor 6.
Figure 3.15: Sound pressure level in closed-loop experiments at Mach 0.27 with (a) M0-based LQ controller measured at sensor 5, (b) M0-based LQ controller measured at sensor 6, (c) M4-based LQ controller measured at sensor 5, (d) M4-based LQ controller measured at sensor 6.
Figure 3.16: Sound pressure level in closed-loop experiments at Mach 0.32 with (a) M0-based LQ controller measured at sensor 5, (b) M0-based LQ controller measured at sensor 6, (c) M4-based LQ controller measured at sensor 5, (d) M4-based LQ controller measured at sensor 6.
Figure 3.17: Sound pressure level of open-loop forced flow with forcing frequency at 3920 Hz at (a) Mach 0.27 measured at sensor 5, (b) Mach 0.27 measured at sensor 6, (c) Mach 0.30 measured at sensor 5, (d) Mach 0.30 measured at sensor 6, (e) Mach 0.32 measured at sensor 5, (f) Mach 0.32 measured at sensor 6.
Notwithstanding the successful suppression at the designed condition, the model-based LQ controllers have also been tested at several off-design flow conditions, e.g., Mach 0.27 and Mach 0.32 as given in Figure 3.15 and Figure 3.16. In these off-design flow conditions, similar benefits and characteristics of the closed-loop system are maintained. At Mach 0.27, suppression of the resonance peak is accompanied with introduction of a new peak around the third Rossiter mode with a lower energy level (Figure 3.15). Similar closed-loop characteristics are observed at Mach 0.32, new peak is introduced around 2100 Hz (Figure 3.16 (a-c)) while the original resonance peak is suppressed. The peak around 1500 Hz is more obvious at transducer 6 of the M4-based case (Figure 3.16 (d)). It is speculated that feedback control induces in the system a state similar to multi-mode resonance where switching occurs between the resonant and the forcing modes. In commenting on similar peak reductions obtained with open-loop forcing, Cattafesta et al. [11] suggested that a competition exists between the fundamental mode and the forced mode for the available energy that can be extracted from the mean flow. At different Mach numbers, different forcing modes are activated and the competition results vary. To summarize, the model-based feedback control considerably achieved its goal by greatly suppressing the target resonance peak and exhibits good robustness for the off-design conditions.

The results of open-loop forcing control are tested for comparison [16], as shown in Figure 3.17. A sinusoidal input with fixed amplitude at frequency 3920 Hz is applied as a feed-forward control. It can been seen that attenuation of the resonance peak is generally accompanied by the introduction of one or two new significant peaks with open-loop forcing control. In the worst case, the new resonant peak has even larger magnitude than the original baseline resonance peak as the Mach number increases to
0.32, as shown in Figure 3.17 (e), (f). It can be concluded from the figures that open-loop forcing control has inferior performance on suppressing the resonant peak and less robustness of the closed-loop control. Even though the original acoustic oscillation can be broken down by the sinusoidal forcing input, open-loop forcing control has no ability to anticipate where the energy goes. Therefore, it cannot further attenuate the new resonances.

The SPL spectrum of the LQ control resembles what previously obtained using a parallel-proportional with time delay control. Furthermore, these two controllers present similar robustness for some off-design conditions. The similarities between the scaled LQ control and the parallel-proportional with time delay control suggest that, although through different processes, similar physical mechanisms are activated at the receptivity region of the cavity shear layer by both these real-time feedback controls.

Analogous results (not shown here) were obtained analyzing the spectra from the other flow models as discussed in Table 3.4 and other flow conditions.

### 3.3.5 Dynamic Observer Design

In alteration to stochastic estimation method, a LQ observer has been designed in the form

\[
\dot{\hat{a}} = \tilde{G}\hat{a} + \tilde{B}\Gamma + L(\bar{p} - \bar{C}\hat{a}),
\]
\[
\tilde{p} = \bar{C}\hat{a},
\] (3.64)

based on the linearized flow model

\[
\dot{\hat{a}} = \tilde{G}\hat{a} + \tilde{B}\Gamma,
\]
\[
\tilde{p} = \bar{C}\hat{a}.
\] (3.65)

This observer has been implemented together with the LQ controller designed above in experiments. Instead of using stochastic estimation for real time stats estimates,
here the observer is used to estimate the real time states for the LQ controller from a single pressure measurement. Comparison of these two methods has been made as shown in Figure 3.18. It can be seen that the feedback control results with the LQ observer are worse than the feedback control results with stochastic estimation method, which suggests that the LQ observer provides an incorrect states estimation for the nonlinear system.

This can be explained by the previous analysis. In section 3.3.3, it has been shown that if $0 < \alpha < 0.5$, there exits a limit cycle with amplitude and frequency

$$\rho^* = \sqrt{\frac{\mu \sigma}{a(\mu)}}, \quad \omega^* = \omega + b(\mu)\rho^*^2.$$  

Here $\omega$ is the frequency of the linear model, and $\omega^*$ is the frequency in steady state. Since the LQ observer is designed based on the linearized model, the observer assumes a frequency of the oscillation $\omega \approx 1800$ Hz, while the frequency of the oscillation for the nonlinear model is $\omega^* \approx 3000$ Hz.
Figure 3.18: Comparison of open loop SPL and closed-loop SPL with LQ controller with real-time states provided by LQ observer and stochastic estimation method at (a) Mach 0.27, (b) Mach 0.30.
CHAPTER 4

CONCLUSION

This dissertation presents and discusses a comprehensive feedback control research work aiming at reducing the acoustic resonance induced in cavity flows. The methodology includes (1) experimental setup and data acquisition; (2) model identification and reduced order modeling; (3) feedback control design and analysis; (4) closed-loop simulation and experimental implementation.

Two different approaches to reduced-order modeling have been studied in this thesis. One exploits the strength of representing the physical properties of the flow by linear dynamics models, referred to as the physics-based modeling. The other approach we have followed is based on proper orthogonal decomposition (POD) and Galerkin projection methods involving the flow governing equations, which is referred to as the nonlinear model or Galerkin model. For each approach, models have been identified using both numerical simulation data and experimental data.

In chapter 2, the physics based linear model proposed by Rowley [41] has been carefully examined, and $H_\infty$ controller has been designed. First, the parameters of the linear model have been identified based on numerical simulation data and experimental data. The structure of the linear model has been investigated and an equivalent $H_\infty$ control problem has been proposed by suitably choosing the weighting functions
to simplify the problem. Closed-loop simulations and experimental implementation have been carried out. Closed-loop experimental results show that the $H_{\infty}$ controller reduces the dominant resonance for which it is designed, but introduces resonances at other frequencies. Similar observations can be obtained by other linear control methods (e.g. Smith predictor control and PID control). A further study shows that a significant improvement can be achieved by introducing a phase shift into the feedback control signals.

Chapter 3 presents and discusses nonlinear reduced-order model development and corresponding controller design, analysis, and experimental implementation results. Both large eddy numerical simulation data and particle image velocimetry (PIV) data have been used for proper orthogonal decomposition (POD) techniques to extract the most energetic flow features. Galerkin projection of the Navier-Stokes equations onto the POD modes is used to derive a set of nonlinear ordinary differential equations, which governs the time evolution of the modes. A control separation technique has been incorporated in the Galerkin projection process to separate the effect of actuation at the boundary. Stochastic estimation has been used to correlate surface pressure data with flow field data and dynamic surface pressure measurements have been used to estimate the state of the Galerkin model in real-time.

It has been shown that the equilibrium of the nonlinear Galerkin model corresponding to the mean flow is exponentially unstable and cannot be stabilized using a static output feedback controller. A modified model with the equilibrium transformed to the origin has been derived and further analyzed. A Linear Quadratic (LQ) optimal state feedback controller and an LQ optimal observer have been developed for
the linearized models, and its effectiveness tested on the nonlinear POD model. Numerical analysis has been used to test the controller robustness and give an estimate of the region of attraction.

Linear-quadratic optimal controllers designed on the basis of experimental data have been validated experimentally. It was necessary to introduce a scaling factor into the feedback loop to account for the limitation caused by actuator saturation. The performance of the scaled feedback control has been analyzed from a mathematical perspective, and evaluated in real time implementation. Experimental results are in qualitative agreement with the analysis. The results show a significant attenuation of the resonant tone with a redistribution of the energy into lower frequency modes and robustness for some off-design conditions. These results confirm that reduced-order model based feedback control represents a promising approach to flow control, although much remains to be done to understand the effect of closed-loop actuation.
APPENDIX A

CONTROL SEPARATION EXAMPLE: ONE DIMENSIONAL HEAT EQUATION

Here we use one dimensional heat equation as an example to illustrate the Galerkin projection and control separation method. The equation of motion is

\[ u_t = c^2 u_{xx}. \]  \hspace{1cm} (A.1)

The boundary condition and initial condition are defined as

\[ \begin{cases} u(0, t) = 0, & u(1, t) = \Gamma(t). \\ u(x, 0) = 0, & x \in [0, 1]. \end{cases} \]  \hspace{1cm} (A.2)

The POD expansion is

\[ u(x, t) = \sum_{j=1}^{N} a_j(t) \phi_j(x). \]  \hspace{1cm} (A.3)

Insert the POD expansion into the dynamic equation (A.1):

\[ \sum_{j=1}^{N} \dot{a}_j(t) \phi_j(x) = c^2 \sum_{j=1}^{N} a_j(t) \xi_j(x), \]  \hspace{1cm} (A.4)

where

\[ \xi_j(x) = \frac{\partial \phi_j(x)}{\partial x^2}. \]  \hspace{1cm} (A.5)

The POD modes are orthonormal:

\[ < \phi_j, \phi_k > = \begin{cases} 0 & j \neq k, \\ 1 & j = k. \end{cases} \]  \hspace{1cm} (A.6)
Calculate inner products to both sides of equation (A.4) with POD mode $\phi_k(x)$ gives

$$
\dot{a}_k(t) = c^2 \sum_{j=1}^{N} a_j(t) \phi_k^T(x) \xi_j(x) > \Omega 
$$

$$
= c^2 \sum_{j=1}^{N} a_j(t) \phi_k^T(x) \xi_j(x) 
$$

$$
= c^2 \sum_{j=1}^{N} a_j(t) [\phi_k^T(\vec{x}^o) \xi_j(\vec{x}^o) + \phi_k(1) \xi_j(1)] 
$$

$$
= c^2 \sum_{j=1}^{N} a_j(t) \phi_k^T(\vec{x}^o) \xi_j(\vec{x}^o) + c^2 \sum_{j=1}^{N} a_j(t) \phi_k(1) \xi_j(1) 
$$

$$
= c^2 \sum_{j=1}^{N} a_j(t) \phi_k^T(\vec{x}^o) \xi_j(\vec{x}^o) + c^2 \phi_k(1) \sum_{j=1}^{N} a_j(t) \xi_j(1) 
$$

$$
+ c^2 \phi_k(1) \sum_{j=1}^{N} (1 - \delta_{jk}) a_j(t) \xi_j(1) + c^2 \phi_k(1) a_k(t) \xi_k(1), 
$$

(A.7)

here $\vec{x}^o = [0, 1)$, $\Omega = [0, 1] = \vec{x}^o \cup \{1\}$. The boundary condition should satisfy equation (A.3)

$$
\Gamma(t) = u(1, t) = \sum_{j=1}^{N} a_j(t) \phi_j(1), 
$$

(A.8)

$$
= a_k(t) \phi_k(1) + \sum_{j=1}^{N} (1 - \delta_{jk}) a_j(t) \phi_j(1), 
$$

so

$$
a_k(t) \phi_k(1) = \Gamma(t) - \sum_{j=1}^{N} (1 - \delta_{jk}) a_j(t) \phi_j(1). 
$$

(A.9)

Multiply $c^2 \xi_k(1)$ on both sides gives

$$
c^2 a_k(t) \phi_k(1) \xi_k(1) = c^2 \Gamma(t) \xi_k(1) - c^2 \sum_{j=1}^{N} (1 - \delta_{jk}) a_j(t) \phi_j(1) \xi_k(1). 
$$

(A.10)
Insert equation (A.10) into equation (A.7) gives

\[ \dot{a}_k(t) = c^2 \sum_{j=1}^{N} a_j(t) \phi_k^T(\vec{x}^o) \xi_j(\vec{x}^o) + c^2 \phi_k(1) \sum_{j=1}^{N} (1 - \delta_{jk}) a_j(t) \xi_j(1) \]

\[ + c^2 \Gamma(t) \xi_k(1) + c^2 \sum_{j=1}^{N} (1 - \delta_{jk}) a_j(t) \left( \phi_k(1) \xi_j(1) - \phi_j(1) \xi_k(1) \right) \]

\[ = c^2 \sum_{j=1}^{N} a_j(t) \phi_k^T(\vec{x}^o) \xi_j(\vec{x}^o) + c^2 \Gamma(t) \xi_k(1) \]

\[ + c^2 \sum_{j=1}^{N} (1 - \delta_{jk}) a_j(t) \left( \phi_k(1) \xi_j(1) - \phi_j(1) \xi_k(1) \right) \]

\[ = c^2 \sum_{j=1}^{N} a_j(t) \phi_k^T(\vec{x}^o) \xi_j(\vec{x}^o) + c^2 \Gamma(t) \xi_k(1) \]

\[ + c^2 \sum_{j=1}^{N} a_j(t) \left( \phi_k(1) \xi_j(1) - \phi_j(1) \xi_k(1) \right) \]

\[ = \left( c^2 \sum_{j=1}^{N} a_j(t) \phi_k^T(\vec{x}^o) \xi_j(\vec{x}^o) + c^2 \sum_{j=1}^{N} a_j(t) \phi_k(1) \xi_j(1) \right) \]

\[ + c^2 \Gamma(t) \xi_k(1) + c^2 \sum_{j=1}^{N} a_j(t) \left( - \phi_j(1) \xi_k(1) \right) \]

\[ = c^2 \sum_{j=1}^{N} a_j(t) \phi_k^T(\vec{x}) \xi_j(\vec{x}) + c^2 \Gamma(t) \xi_k(1) \]

\[ + c^2 \Gamma(t) \xi_k(1) + c^2 \sum_{j=1}^{N} a_j(t) \left( - \phi_j(1) \xi_k(1) \right) \]

\[ = c^2 \sum_{j=1}^{N} a_j(t) \left( \phi_k^T(\vec{x}) \xi_j(\vec{x}) - \phi_j(1) \xi_k(1) \right) + c^2 \xi_k(1) \Gamma(t). \]

The final model is

\[ \dot{a} = Aa + B\Gamma, \quad \text{(A.12)} \] where

\[ \begin{align*}
A_{ki} &= c^2 \left( \phi_k^T(\vec{x}) \xi_j(\vec{x}) - \phi_j(1) \xi_k(1) \right), \\
B_k &= c^2 \xi_k(1).
\end{align*} \quad \text{(A.13)} \]
BIBLIOGRAPHY


