ON A CLASS OF ALGEBRAIC SURFACES WITH 
NUMERICALLY EFFECTIVE COTANGENT BUNDLES

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* * * * *

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ABSTRACT

In this dissertation, we will present two new results in complex geometry. The first one is for a borderline class of surfaces with nef cotangent bundle. The statement of the main theorem on this topic is as follows, suppose $M$ is a general type surface, such that the two Chern numbers satisfy $c_1^2 = c_2$, and $\text{Pic}(M)/\text{Pic}^0(M)$ is generated by $K_M$, then every nontrivial extension of the holomorphic tangent bundle $T_M$ by $K_M$ will be Hermitian flat. As a corollary, we obtain that every surface with the above described properties and with $h^1(T_M \otimes K^*_M) > 0$ will be a generalized theta divisor. The second result is an extended version of Hard Lefschetz theorem for the cohomology with values in Nakano semipositive vector bundles on a compact Kähler manifolds.
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The Geography of Chern Numbers
CHAPTER 1
INTRODUCTION

A major goal in complex geometry is to classify compact complex manifolds in a given dimension up to birational equivalence classes. In early 19th century, Castelnuovo, Enriques and many others had succeeded in creating an impressive geometric theory of birational classification for smooth algebraic surfaces. It was discovered that main birational invariants for an algebraic surface $M$ are the first Betti number $b_1(M) = h^1(M, \mathbb{C})$ and the plurigenera $P_n(M)$ of $M$, which is defined by $P_n(M) = h^0(M, K_M^\otimes n)$.

After sheaf theory had been developed, and applied by Hodge, Serre, Hirzebruch, Grothendieck and others to analytic and algebraic geometry, the classification questions got a decisive progress since 1950’s. In 1960’s and 1970’s Kodaira and others carried out the classification project for projective surfaces, and also extended it to
cover all compact complex surfaces. It is now referred to in the literature as Enrique-Kodaira classification theory. Given a compact complex surface $M$, there are four possibilities for its plurigenera:

1), all $P_n(M)$ vanish;

2), not all $P_n(M)$ vanish, but all are either 0 or 1;

3), $P_n(M) \sim n$;

4), $P_n(M) \sim n^2$.

Nowadays this fact is expressed by saying that the *Kodaira dimension* $\kod(M)$ of $M$ is respectively $-\infty$, 0, 1, or 2. The algebraic surfaces in class 1) are formed by all the *birationally ruled* surfaces, namely, those $M$ that is birational to $\mathbb{P}^1 \times C$, for any compact complex curve $C$. The non-algebraic surfaces in class 1) are so called $VII_0$-class, which are defined as compact complex surfaces with first Betti number $b_1 = 1$ and all $P_n = 0$. For instance, all Hopf surfaces (i.e. compact complex surface with universal cover $\mathbb{C}^2\setminus\{0\}$) belong to this set. There are other examples that belongs to this set as well. However, an exhaustive list is yet to be established. For case 2), there are four types of Kählerian surfaces distinguished by their first Betti numbers and plurigenera, namely the birational classes of tori, hyperelliptic surfaces, $K$-3 surfaces and Enriques surfaces. The non-Kähler ones in case 2) consists of Kodaira surfaces.
which are all affine quotients of $\mathbb{C}^2$. The case 3) are all elliptic surfaces, i.e. there is a smooth curve $B$ and a surjective morphism $p : M \to B$ whose generic fibre is an elliptic curve. Kodaira had studied all possible types of singular fibres, there are 7 of them, and gave a complete classification of this type of surface. Finally, surfaces in class 4), are simply called *surfaces of general type*. This is of course means 99.99% of the surfaces, and not much is known about this class except some very basic results.

It is known that, for any surfaces with non-negative Kodaira dimension, there is a unique surface in each birational equivalent class that is minimal, i.e. can not be blown down any further. This surface is called the minimal model of the birational class.

By Gieseker’s theorem [Gies], p236, we know that there exists a quasi-projective coarse module scheme for minimal surfaces of general type with fixed Chern numbers $c_1^2$ and $c_2$. However, little is known about the structure of the Gieseker scheme in general. Even the very basic questions such as, for which pair $(c_2, c_1^2)$ is the Gieseker scheme non-empty, are not completely answered. The known restrictions on $(c_2, c_1^2)$ are expressed by

1.1. **Theorem** If $M$ is any minimal surface of general type, then

1) $c_1^2(M) + c_2(M) \equiv 0(12)$,
2), \( c_1^2(M) > 0 \) and \( c_2(M) > 0 \),

3), \( c_1^2(M) \leq 3c_2(M) \),

4), \( 5c_1^2(M) - c_2(M) + 36 \geq 0 \) (if \( c_1^2(M) \) is even), \( 5c_1^2(M) - c_2(M) + 30 \geq 0 \) (if \( c_1^2(M) \) is odd).

Let \( c_1^2 = n \) and \( c_2 = m \). This theorem tells us that no integer pair \((m, n)\) with \( n > 3m \) or \( 5n - m + 36 < 0 \), can be the two Chern numbers of any minimal general type surface. In light of this, the question now is following:

If \( D \subset \mathbb{Z}^2 := \{(m,n) | m \in \mathbb{Z}, n \in \mathbb{Z} \} \) is given by the inequalities \( 0 < n \leq 3m \), \( m \leq 5n + 36 \) and \( 12 | (n + m) \), then for which pair \((m, n)\) \in D, does there exist a minimal surface of general type \( M \) with \( c_1^2 = n \) and \( c_2 = m \)?

One can divide \( D \) into two parts \( D_1 := \{(m, n) \in D | n \leq 2m \} \) and \( D_2 := \{(m, n) \in D | n > 2m \} \). \( D_1 \) is the region where \( \tau = \frac{1}{3} p_1 = \frac{1}{3} (c_1^2 - 2c_2) \leq 0 \). Here \( \tau \) is the signature of \( M \) and \( p_1 \) is the first Pontryagin number. Similarly, \( D_2 \) is the region for those \( M \) with positive signature. The simple examples such as, the complete intersections or the theta divisors, which are the smooth ample divisors of the three dimensional abelian varieties, usually lie in the region \( D_1 \). In particular, the theta divisors have \( c_1^2 = c_2 \), so they are on the line \( n = m \). Indeed for a long time only
very few examples with Chern pairs in $D_2$ were known. The geography of the Chern numbers is plot as Figure 1.1.

![Figure 1.1: The Geography of Chern Numbers](image)

Although we can easily find lots of pairs in $D$, but it is difficult to show that all pairs in $D$ can be represented. The combined efforts of Persson ([Pers]) for region $D_1$ and Chen ([Che1] [Che2]) for the regions $D_1$ and $D_2$ gave rise to the following results
1.2. Theorem Given any pair \((m,n) \in D\), there exists a minimal general type surface \(M\) with \(c_1^2 = n\) and \(c_2 = m\), except maybe for points on the finitely many lines \(n - 3m + 4k = 0\) with \(0 \leq k \leq 347\). In fact, for at most finitely many exceptions on these lines all pairs in \(D_1\) occurs as the Chern numbers of minimal general type surfaces.

This theorem does not give surfaces on the borderline \(n = 3m\). By Yau’s deep theorem [Yau1] [Yau2] on Kähler-Einstein metrics

1.3. Theorem (Yau) Let \(M\) be a compact Kähler manifold of dimension \(n\),

(a) if \(c_1 > 0\) (resp. \(c_1 < 0\)), i.e. the \((1,1)\) cohomology class \(c_1\) (resp. \(-c_1\)) is represented by some positive \((1,1)\)-form. Then in any given Kähler class, there exists a Kähler metric with positive (resp. negative) Ricci curvature.

(b) If \(c_1 = 0\) in \(H^2(M, \mathbb{R})\), then in any given Kähler class, there exists a Kähler metric with vanishing Ricci curvature, i.e. Ricci flat.

(c) If \(c_1 < 0\), there exists a unique Kähler Einstein metric with Ricci curvature \(r = -1\).

In particular, we know that any general type surface with \(c_1^2 = 3c_2\) are quotients of the unit ball in \(\mathbb{C}^2\). A natural question would be, which points \((m,n)\) on the line \(n = 3m\)
are represented by the Chern numbers of surfaces, and how can we characterize the other lines.

What we are interested in is a class of general type surfaces on the line $c_1^2 = c_2$ with numerically effective cotangent (nef in short) bundle, which will be defined in chapter 3. The numerical effectiveness is a weaker type of semipositivity than the Griffiths semipositivity, It is widely used in transcendental algebraic geometry because of its flexibility. It is clear that every Griffiths semipositive vector bundle is nef. But the converse is not true, a standard counter example is given in the Chapter 3, due originally to Demailly. The positivity of holomorphic vector bundles has played a very important role in algebraic geometry. It has been used as an essential tool to study and classify compact Kähler manifolds. For example, all the compact Riemann surfaces which are not of general type are exactly those with semipositive curvature. In 1961, T. Frankel [Fran] proved that a compact Kähler surface with positive bisectional curvature is biholomorphic to $\mathbb{P}^2$. He conjectured that the same is true in higher dimensions, which was later is known as the Frankel’s Conjecture. In 1973, Hartshorne raised a more general conjecture, which states that any compact Kähler manifold $M$ of dimension $n$ with ample tangent vector bundle is $\mathbb{P}^n$. Since tangent bundle is ample when the bisectional curvature is positive, Hartshorne’s conjecture is stronger than the Frenkel’s one. But it is still an open question whether
the ample vector bundle is Griffiths positive. In his famous paper [Mori] Mori solved Hartshorne’s conjecture. Around the same time Siu and Yau [SiYa] gave an analytic proof of the Frankel conjecture. Combining the algebraic and analytic tools, Mok [Mok] classified all compact Kähler manifolds with semipositive holomorphic bisectional curvature.

Since nefness is a generalization of the Griffiths semipositivity, it is natural to consider the class of projective manifolds $M$ whose tangent bundles are nef. This has been done by Campana and Peternell [CaPe] for surface case, and Zheng [Zhen1] completed for the case of complex dimension 3. Later Demailly-Peternell-Schneider [DPS1] investigated compact Kähler manifolds with nef tangent bundle, and reduced the problem to the simply connected cases. The conjecture is that such a manifold should be a product of compact simply connected homogenous Kähler manifolds, i.e. the Kähler $\mathbb{C}$-spaces. On the other hand, very little is known for the compact Kähler manifolds with ample or nef cotangent bundles.

In this paper we are going to consider compact complex surfaces $M$ with nef cotangent bundles. When $M$ is not of general type, the nefness and the Enrique-Kodaira classification theory yield that $M$ is either a complex torus, or a elliptic curve bundle over a curve of positive genus. So we should focus on general type surfaces with nef cotangent bundle. By Theorem 4.2 we can know that $c_1^2 \geq c_2$. In
this dissertation, we will focus on the borderline case \( c_1^2 = c_2 \). For this borderline case we have the following conjecture

1.4. **Conjecture** Let \( M \) be a general type surface with nef cotangent bundle and with \( c_1^2 = c_2 \). Then it must be a generalized theta divisor, namely, there exists an equivariant proper holomorphic immersion from the universal cover of \( M \) into \( \mathbb{C}^3 \).

We can show that the conjecture is true under some stronger conditions. The main theorem, which is presented in chapter 5, is the following

1.5. **Theorem** Let \( M \) be a general type surface with \( c_1^2 = c_2 \) such that \( \text{Pic}(M)/\text{Pic}^0(M) \) is generated by the canonical line bundle \( K_M \). Then every nontrivial holomorphic bundle extension

\[
0 \to T_M \to E \to K_M \to 0
\]

of the holomorphic tangent bundle \( T_M \) by \( K_M \) is Hermitian flat.

1.6. **Theorem** Let \( M \) be a general type surface with \( c_1^2 = c_2 \) such that \( \text{Pic}(M)/\text{Pic}^0(M) \) is generated by the canonical line bundle \( K_M \). If \( h^1(T_M \otimes K_M^*) > 0 \), then \( M \) must be an immersed theta divisor.

Theorem 1.6 is a consequence of Theorem 1.5. As a direct corollary of Theorem 1.5, we know that any \( M \) satisfying the conditions of Theorem 1.6 always has nef cotangent bundle. A key lemma used in the proof of the Theorem 1.5 is the following
1.7. Lemma Let $M$ be a general type surface with $c_1^2 = c_2$ and $\text{Pic}(M)/\text{Pic}^0(M)$ is generated by $K_M$. Then the irregularity $q(M) = h^0(M, \Omega^1_M)$ can not be equal to 1 or 2.

In the last chapter we prove an extended version of the Hard Lefschetz Theorem for cohomology with values in Nakano semipositive vector bundles over compact Kähler manifolds. The case of the line bundles was obtained by Mourougane [Mour] in 1999. Demailly-Peternell-Schneider [DPS2] generalized it to pseudo effective line bundles. Our result is the following

1.8. Theorem Let $(E, h)$ be a Nakano semipositive vector bundle of rank $r$ over a compact Kähler manifold $(M, \omega)$ of dimension $n$. Then the wedge multiplication operator $\omega^q \wedge \cdot$ induces a surjective morphism

$$L_h^q = \omega^q \wedge \cdot : H^0(M, \Omega^{n-q}_M \otimes E) \to H^q(M, \Omega^n_M \otimes E)$$

for any $0 \leq q \leq n$. 
CHAPTER 2

STABILITY OF VECTOR BUNDLES

2.1 Coherent sheaf theory

Let $M$ be a compact complex manifold of dimension $n$ and $\mathcal{O}_M$ the structure sheaf of $M$, i.e. the sheaf of germs of holomorphic functions on $M$. We write

$$\mathcal{O}_M^p = \underbrace{\mathcal{O}_M \oplus \cdots \oplus \mathcal{O}_M}_p$$

An analytic sheaf over $M$ is a sheaf of $\mathcal{O}_M$-modules over $M$.

2.1. Definition locally finitely generated sheaf

We say that an analytic sheaf $\mathcal{J}$ over $M$ is locally finitely generated if, given any point $x_0$ of $M$, there exists a neighborhood $U$ of $x_0$ and finitely many sections of $\mathcal{J}_U$ that generate each stalk $\mathcal{J}_x$, $x \in U$, as an $\mathcal{O}_x$-module. This means that we have an exact sequence

$$\mathcal{O}_U^p \rightarrow \mathcal{J}_U \rightarrow 0 \quad (2.1.1)$$

In particular, each stalk $\mathcal{J}_x$ is an $\mathcal{O}_x$-module of finite type.
2.2. Definition coherent sheaf

We say that an analytic sheaf $\mathcal{J}$ over $M$ is coherent if, given any point $x_0$ of $M$, there exists a neighborhood $U$ of $x_0$ and an exact sequence

$$\mathcal{O}_U^q \to \mathcal{O}_U^p \to \mathcal{J}_U \to 0 \quad (2.1.2)$$

This means that the kernel of sequence (2.1.1) is also finitely generated.

We have the following useful lemma, whose proof can be found in [GunR].

2.3. Oka’s Lemma The kernel of any homomorphism $j : \mathcal{O}^q \to \mathcal{O}^p$ is locally finitely generated.

By this lemma we can get the local Syzygy theorem for coherent sheaves.

2.4. Syzygy Theorem Let $\mathcal{J}$ be a coherent sheaf over $M$. Given any $x_0 \in M$, there exists a small neighborhood $U$ of $x_0$, and a finite step free resolution of sheaves:

$$0 \to \mathcal{O}_U^{p_d} \to \cdots \to \mathcal{O}_U^{p_1} \to \mathcal{O}_U^{p_0} \to \mathcal{J}_U \to 0 \quad (2.1.3)$$

with $d \leq n$. 

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2.5. Definition homological dimension

The homological dimension of the stalk \( \mathcal{J}_x \) of coherent sheaf \( \mathcal{J} \) over \( M \), denoted by \( dh(\mathcal{J}_x) \), is defined to be the length \( d \) of a minimal free resolution:

\[
0 \to \mathcal{O}_{x}^{d_1} \to \cdots \to \mathcal{O}_{x}^{p_1} \to \mathcal{O}_{x}^{p_0} \to \mathcal{J}_x \to 0
\]

with \( d \leq n \), which is the stalk version of the Syzygy theorem.

2.6. Definition homological codimension

The homological codimension of the stalk \( \mathcal{J}_x \) of coherent sheaf \( \mathcal{J} \) over \( M \), denoted by \( codh(\mathcal{J}_x) \), is defined to be the maximal length \( p \) of a \( \mathcal{J}_x \)-sequence. The \( p \) is independent of the choice of the sequence. A sequence \( \{a_1, \cdots, a_p\} \) of the element of maximal ideal \( \mathfrak{m} \) of \( \mathcal{J}_x \) is called an \( \mathcal{J}_x \)-sequence if, for each \( i, \ 0 \leq i \leq p - 1 \), \( a_{i+1} \) is not a zero divisor on \( \mathcal{J}_x/(a_1, \cdots, a_i)\mathcal{J}_x \). Then the sequence of the ideals

\[
(a_1) \subset (a_1, a_2) \subset \cdots \subset (a_1, a_2, \cdots, a_i) \subset \cdots
\]

is strictly increasing and there is a maximal \( \mathcal{J}_x \)-sequence.

2.7. Theorem For the stalk \( \mathcal{J}_x \) of coherent sheaf \( \mathcal{J} \) over \( M \)

\[
dh(\mathcal{J}_x) + codh(\mathcal{J}_x) = n
\]

From the definition of the homological dimension we can know that \( \mathcal{J}_x \) is a free \( \mathcal{O}_x \)-module if and only if \( dh(\mathcal{J}_x) = 0 \), or equivalently, \( codh(\mathcal{J}_x) = n \).
2.8. Definition m-th singularity set

For each integer $m$, $0 \leq m \leq n$, the $m$-th singularity set of $\mathcal{J}$ is defined to be

$$S_m(\mathcal{J}) = \{ x \in M; \text{codh}(\mathcal{J}_x) \leq m \} = \{ x \in M; \text{dh}(\mathcal{J}_x) \geq n - m \} \quad (2.1.4)$$

we call $S_{n-1}(\mathcal{J}) = \{ x \in M; \mathcal{J}_x \text{ is not free} \}$ the singularity set of the $\mathcal{J}$.

2.9. Theorem the $m$-th singularity set $S_m(\mathcal{J})$ of the coherent sheaf $\mathcal{J}$ is a closed analytic set of $M$ of dimension $\leq m$.

2.10. Corollary Let $\mathcal{J}$ over $M$ is a coherent sheaf, for any $x \in M$, if there exists $U$, which is a small neighborhood of $x$, together with an exact sequence of sheaves:

$$0 \rightarrow \mathcal{J}_U \rightarrow \mathcal{O}^{P_1}_U \rightarrow \cdots \rightarrow \mathcal{O}^{P_k}_U$$

then $\dim S_m(\mathcal{J}) \leq m - k$.

2.11. Definition rank of coherent sheaf

Let $\mathcal{J}$ be a coherent sheaf over $M$. It is locally free outside of the singularity set $S_{n-1}(\mathcal{J})$. We define the rank of coherent sheaf

$$\text{rank} \mathcal{J} = \text{rank} \mathcal{J}_x \quad x \in M \setminus S_{n-1}(\mathcal{J})$$

2.12. Definition torsion free coherent sheaf

We say that $\mathcal{J}$ over $M$ is a torsion free coherent sheaf, if $\mathcal{J}_x$ is a torsion free $\mathcal{O}_x$-module for each $x \in M$. 
By the above definition we know that every locally free sheaf is obviously torsion free. any coherent subsheaf of a torsion free sheaf is again torsion free. Conversely, we have the following lemma

2.13. Lemma If \( J \) over \( M \) is a torsion free coherent sheaf of rank \( r \), then it is locally a subsheaf of a free sheaf of rank \( r \). i.e. for any \( x \in M \), there exists a neighborhood \( U \) and an injective homomorphism \( j : J_U \to \mathcal{O}_U \).

From this lemma and Corollary (2.10) we get the following

2.14. Corollary Let \( J \) be a torsion free coherent sheaf over \( M \), then \( \dim S_m(J) \leq m - 1 \), for all \( m \). This means that a torsion free coherent sheaf is locally free outside the singularity set of codimension at least 2.

Let us define the dual of a coherent sheaf \( J \) to be \( J^* = \text{Hom}(J, \mathcal{O}) \). There is a natural homomorphism \( \sigma \) from \( J \) to its double dual \( J^{**} \):

\[
\sigma : J \to J^{**}
\]

2.15. Lemma the kernel of \( \sigma \) \( \text{Ker}(\sigma) \) consists exactly of torsion elements of \( J \) and it is a torsion subsheaf of \( J \). So \( \sigma \) is injective if and only if \( J \) is torsion free.

2.16. Definition reflexive coherent sheaf

If \( \sigma : J \to J^{**} \) is bijective, we say that \( J \) is reflexive coherent sheaf.
Every reflexive coherent sheaf is torsion free, and we have the following

**2.17. Lemma** The dual sheaf of any coherent sheaf is reflexive.

**2.18. Lemma** If $\mathcal{J}$ is a reflexive coherent sheaf, then for any $x \in M$, there exists a small neighborhood $U$ of $x_0$ and an exact sequence of sheaves:

$$0 \to \mathcal{J}_U \to \mathcal{O}_U^{p_1} \to \mathcal{O}_U^{p_2}$$

By Corollary (2.10) we know that every reflexive coherent sheaf is locally free outside the singularity set of codimension at least 3.

**2.19. Definition** normal coherent sheaf

We say that coherent sheaf $\mathcal{J}$ over $M$ is normal, if for every open set $U$ and every analytic subset $Z \subset U$ of codimension at least 2, the restriction map

$$r : \Gamma(U, \mathcal{J}) \to \Gamma(U \setminus Z, \mathcal{J})$$

is an isomorphism.

From Lemma (2.13) we know that the restriction map is injective if $\mathcal{J}$ is torsion free. Obviously the torsion free sheaf is not necessarily normal. Conversely, a normal sheaf is not necessarily torsion free, either. However, we do have the following relation between reflexive coherent sheaves and torsion free normal coherent sheaves.
2.20. Lemma  Coherent sheaf $\mathcal{J}$ is reflexive if and only if $\mathcal{J}$ is torsion free and normal.

2.21. Lemma  Let

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0$$

be a short exact sequence of sheaves. If $\mathcal{E}$ is reflexive and $\mathcal{Q}$ is torsion free, then $\mathcal{J}$ is normal.

As a corollary of the Lemma (2.18), we have the following

2.22. Lemma  A reflexive coherent sheaf of rank 1 on a smooth compact complex surface $M$ is a line bundle. A torsion free coherent sheaf of rank 1 on a smooth compact complex curve $M$ is a line bundle.

2.23. Definition  determinant line bundle

For a holomorphic vector bundle $E$ of rank $r$ over a compact complex manifold $M$, its determinant line bundle $\text{det}E$ is defined by $\text{det}E = \wedge^r E$.

For a coherent sheaf $\mathcal{J}$ of rank $r$ over the $M$, its determinant line bundle $\text{det}\mathcal{J}$ is defined by following: for any $x \in M$ and let $U$ is a small neighborhood of $x$ such that there exist a finite step locally free resolution of sheaves:

$$0 \rightarrow \mathcal{E}_n \rightarrow \cdots \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{J}_U \rightarrow 0 \quad (2.1.5)$$
let the $E_i$ denotes the vector bundle corresponding to the sheaf $E_i$, we define

$$\det J_U = \bigotimes_{i=0}^{n} (\det E_i)^{(-1)^i}$$

It can be showed that $\det J_U$ is independent of the choice of the resolution (2.1.5) and that $\det J$ is well defined. We have the following properties of the $\det J$.

**2.24. Proposition** If $J$ is a torsion free coherent sheaf of rank $r$, then there are canonical isomorphisms

$$\det J = (\wedge^r J)^{**}$$

$$(\det J)^* = (\det J^*)$$

Having defined the determinant bundle $\det J$, we can define the first Chern class of $J$ by

$$c_1(J) = c_1(\det J)$$

**2.2 Semistable and stable sheaves**

Let $J$ be a torsion free coherent sheaf over a $n$ dimensional compact Kähler manifold $(M, g)$ with $\omega$ the Kähler form, which is a real positive closed $(1, 1)$-form on $M$. Let $c_1(J)$ be the first Chern class of $J$, it is represented by a real closed $(1, 1)$-form on $M$. 

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2.25. Definition The degree (or \( \omega \)-degree) of \( \mathcal{J} \) is defined to be

\[
\deg(\mathcal{J}) = \int_M c_1(\mathcal{J}) \wedge \omega^{n-1}
\]

2.26. Definition The slope \( \mu(\mathcal{J}) \) is defined to be

\[
\mu(\mathcal{J}) = \frac{\deg(\mathcal{J})}{\text{rank}(\mathcal{J})}
\]

2.27. Definition we say that \( \mathcal{J} \) is semistable (or \( \omega \)-semistable) if for every coherent subsheaf \( \mathcal{J}' \) of \( \mathcal{J} \) with \( 0 < \text{rank}(\mathcal{J}') < \text{rank}(\mathcal{J}) \), we have

\[
\mu(\mathcal{J}') \leq \mu(\mathcal{J})
\]

If moreover for every coherent subsheaf \( \mathcal{J}' \) of \( \mathcal{J} \) with \( 0 < \text{rank}(\mathcal{J}') < \text{rank}(\mathcal{J}) \), the strict inequality

\[
\mu(\mathcal{J}') < \mu(\mathcal{J})
\]

holds, then we say that \( \mathcal{J} \) is stable (or \( \omega \)-stable).

A holomorphic vector bundle \( E \) over \( M \) is said to be semistable (resp. stable) if the sheaf \( \mathcal{E} = \mathcal{O}(E) \) of the \( E \) is semistable (resp. stable). From the definition above, we have to check all subsheaves of \( \mathcal{J} \) to show that \( \mathcal{J} \) is semistable (resp. stable). But the following theorem tell us that we don’t need to consider all subsheaves.

2.28. Theorem Let \( \mathcal{J} \) be a torsion free coherent sheaf over a compact Kähler manifold \( (M, g) \), then
(a) \( J \) is a semistable (resp. stable) if and only if \( \mu(J') \leq \) (resp. <) \( \mu(J) \) for every coherent subsheaf \( J' \) with \( 0 < \text{rank}(J') < \text{rank}(J) \) such that the quotient \( J/J' \) is torsion free.

(b) \( J \) is a semistable (resp. stable) if and only if \( \mu(J) \leq \) (resp. <) \( \mu(J'') \) for every torsion free quotient sheaf \( J'' \) with \( 0 < \text{rank}(J'') < \text{rank}(J) \).

We also have following propositions about semistable or stable sheaves.

2.29. Proposition Let \( J \) be a torsion free coherent sheaf over a compact Kähler manifold \((M, g)\), then

(a) If \( \text{rank}(J) = 1 \), then \( J \) is stable.

(b) Let \( L \) be a line bundle over \( M \), then \( J \otimes L \) is stable (resp. semistable) if and only if \( J \) is stable (resp. semistable).

(c) \( J \) is stable (resp. semistable) if and only if \( J^* \) is stable (resp. semistable).

2.30. Proposition Let \( J_1 \) and \( J_2 \) be two torsion free coherent sheaves over a compact Kähler manifold \((M, g)\), then \( J_1 \oplus J_2 \) is semistable if and only if \( J_1 \) and \( J_2 \) are both semistable and \( \mu(J_1) = \mu(J_2) \).
2.31. **Theorem (Jordan-Hölder)** Given a semistable sheaf \( \mathcal{E} \) over a compact Kähler manifold \((M, g)\), there is a filtration of \( \mathcal{E} \) by subsheaves

\[
0 = \mathcal{E}_{k+1} \subset \mathcal{E}_k \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E}
\]

such that \( \mathcal{E}_i / \mathcal{E}_{i+1} \) are stable, and \( \mu(\mathcal{E}_i / \mathcal{E}_{i+1}) = \mu(\mathcal{E}) \) for \( i = 0, 1, \cdots, k \).
CHAPTER 3

POSITIVITY OF HERMITIAN VECTOR BUNDLES

In this chapter we started with basic definition of curvature for complex vector bundles over smooth differential manifolds. Then we recall the definition and basic properties of Hermitian curvature of Hermitian vector bundles over compact complex manifold, which is the curvature of the unique Hermitian connection. After that we discuss Hermitian flat bundles, positivities for Hermitian vector bundles, and Einstein-Hermitian vector bundles. In the last section we recall the definition for ample vector bundle, which is another type of positivity from the algebraic point of view. We also discuss the relation between ampleness and Griffiths positivity of vector bundle.

3.1 Curvature of vector bundles

Let $E$ be a complex vector bundle of rank $r$ over a smooth differentiable manifold $M$ with $\dim_{\mathbb{R}} M = n$. Denote by $A^p(E) = C^\infty(M, \Lambda^p(T_M^*) \otimes E)$, the space of $C^\infty$ $p$-forms on $M$ with values in $E$. A connection $D$ on $E$ is a linear differential operator
of order 1

\[ D : A^p(E) \to A^{p+1}(E) \]

such that

\[ D(f \wedge u) = df \wedge u + (-1)^{\text{deg } f} f \wedge Du \]

for all forms \( f \in C^\infty(M, \wedge^q(T^*_M)), u \in A^p(E) \). On an open set \( U \subseteq M \) where \( E \) admits a trivialization \( \theta : E_U \cong U \times \mathbb{C}^r \), the connection \( D \) can be written as

\[ Du = \theta du + \Gamma \wedge u \]

where \( \Gamma \in C^\infty(U, \wedge^1(T^*_M) \otimes \text{Hom}(\mathbb{C}^r, \mathbb{C}^r)) \) is a matrix of 1-forms and \( d \) acts componentwisely. Then we have

\[
D^2u = D(du + \Gamma \wedge u) \\
= d(du + \Gamma \wedge u) + \Gamma \wedge (du + \Gamma \wedge u) \\
= d\Gamma \wedge u - \Gamma \wedge du + \Gamma \wedge du + \Gamma \wedge \Gamma \wedge u \\
= (d\Gamma + \Gamma \wedge \Gamma) \wedge u
\]

It is not hard to see that \( D^2 \) is actually linear over \( A^0(E) \). Since \( D^2 \) is globally defined operator, there is a global 2-form, called the curvature form of \( D \),

\[ \Theta(D) \in A^2(\text{Hom}(E, E)) = A^2(E^* \otimes E) \]
such that $D^2u = \Theta(D) \wedge u$ for every form $u \in A^*(E)$.

Next, let us assume that $E$ is endowed with a $C^\infty$ Hermitian metric along the fibres and $\{e_1, \cdots, e_r\}$ is a local $C^\infty$ frame. We have a canonical sesquilinear pairing

$$A^p(E) \otimes A^q(E) \to A^{p+q}(\mathbb{C})$$

$$(u, v) \mapsto \{u, v\} = \sum_{\alpha, \beta} u_\alpha \wedge \overline{v}_\beta (e_\alpha, e_\beta)$$

here $u = \sum_\alpha u_\alpha \otimes e_\alpha$, $v = \sum_\beta u_\beta \otimes e_\beta$, $(e_\alpha, e_\beta)$ is the Hermitian inner product of $E$.

We say that a connection $D$ is \textit{compatible with the metric} if it satisfies the additional property

$$d\{u, v\} = \{Du, v\} + (-1)^{deg u}\{u, Dv\}$$

assuming that $\{e_1, \cdots, e_r\}$ is orthonormal, we will get $\Theta(D)^* = -\Theta(D)$ and

$$\sqrt{-1}\Theta(D) \in A^2(\text{Herm}(E, E))$$

From now on, let us assume that $M$ is the compact complex manifold with $\text{dim}_\mathbb{C} M = n$, and the bundle $E$ is a holomorphic vector bundle of rank $r$. When $E$ is equipped with a Hermitian metric we will call it a \textit{Hermitian vector bundle}. The complexification of the real cotangent bundle splits as $T^*_{\mathbb{C}, z}(M) = T^*_z(M) \oplus T^{*\prime}_z(M)$, into the holomorphic and the antiholomorphic part. $A^{p,q}(E) = C^\infty(M, \wedge^p T^*_M \otimes \wedge^q T^{*\prime}_M \otimes E)$, the space of $C^\infty$ $(p, q)$-forms on $M$ with values in $E$. We can decompose connection
\[ D = D' + D'', \text{ with } D' : A^{p,q}(E) \to A^{p+1,q}(E) \text{ and } D'' : A^{p,q}(E) \to A^{p,q+1}(E). \]

There is a canonical choice of connection for Hermitian vector bundle,

**3.1. Theorem** There is a unique connection \( D \) on a Hermitian vector bundle \( E \), called the Hermitian connection, satisfying the following two conditions

(1) \( D \) is compatible with the complex structure, i.e. \( D'' = \bar{\partial} \).

(2) \( D \) is compatible with the metric

The curvature of the Hermitian connection is called the *Hermitian curvature*. Its curvature form

\[ \sqrt{-1}\Theta(E) := \sqrt{-1}\Theta(D) \in A^{1,1}(Herm(E,E)) \]

Let \( \{z_1, \ldots, z_n\} \) be holomorphic coordinates on open subset \( U \subset M \), \( \{e_1, \ldots, e_r\} \) an orthonormal frame of \( E_U \). On \( U \) one can write

\[ \sqrt{-1}\Theta(E) = \sqrt{-1}\frac{1}{2} \sum_{\substack{1 \leq i,j \leq n \\ 1 \leq \alpha,\beta \leq r}} R_{i\bar{j}\alpha\bar{\beta}} dz_i \wedge d\bar{z}_j \otimes e^\alpha \otimes e_\beta \tag{3.1.1} \]

and identify this curvature form to the following curvature tensor

\[ \tilde{\Theta}(E)(\xi, \bar{\xi}) = \sum_{\substack{1 \leq i,j \leq n \\ 1 \leq \alpha,\beta \leq r}} R_{i\bar{j}\alpha\bar{\beta}} f_{i,\alpha} \bar{f}_{j,\beta} \tag{3.1.2} \]

where \( \xi = \sum_{i,\alpha} f_{i,\alpha} \partial/\partial z_i \otimes e_\alpha \in TM \otimes E \).

If \( \{e_1, \ldots, e_r\} \) is a holomorphic frame of \( E_U \), then in \( U \) we can write

\[ \sqrt{-1}\Theta(E) = \sqrt{-1}\bar{\partial}(\bar{h}^{-1}D'h) \tag{3.1.3} \]
here \( h \) is Hermitian matrix of the Hermitian metric of \( E \). Under the frame, that is \( h = (h_{\alpha\beta}) = (e_\alpha, e_\beta). \)

In general, it is impossible to find a local frame \( \{e_1, \cdots, e_r\} \) of \( E \) that is simultaneously holomorphic and orthonormal. Because under such a frame, we would have the Hermitian curvature \( \Theta(E) = 0 \) on \( U \). On the other hand, if Hermitian curvature \( \Theta(E) = 0 \), we can always find local frames that are holomorphic and orthonormal simultaneously.

3.2. Definition Hermitian flat bundle

We say that a holomorphic vector bundle \( E \) of rank \( r \) over a compact complex manifold \( M \) of dimension \( n \) is Hermitian flat if \( E \) has a flat Hermitian structure, which is given by an open covering \( \{U\} \) and a system of local holomorphic frames \( \{s_U = (e_1, \cdots, e_r)\} \) such that the transition functions \( \{g_{UV}\} \) are all constant unitary matrices in \( U(r):= \{r \times r \text{ unitary matrices}\}. \)

3.3. Theorem Let \( E \) be holomorphic vector bundle of rank \( r \) over a compact complex manifold \( M \) of dimension \( n \), the following conditions are equivalent,

(1) \( E \) is Hermitian flat

(2) \( E \) admits a Hermitian structure \( h \) with flat Hermitian connection, i.e. its curvature vanishes.
(3) $E$ is defined by a representation $\rho : \pi_1(M) \to U(r)$ of $\pi_1(M)$.

We say that a holomorphic local frame $\{e_1, \cdots, e_r\}$ is normal at point $x_0 \in U$ if $h = (\delta_{\alpha\beta})$ at $x_0$, and $\Gamma = 0$ at $x_0$.

The following proposition tell us that Hermitian vector bundles always admit normal local frames and Hermitian curvature tensor (3.1.2) is the obstruction to the existence of orthonormal holomorphic frames.

3.4. Proposition For every point $x_0 \in M$ and every holomorphic coordinate system $\{z_1, \cdots, z_n\}$ at $x_0$, there exists a holomorphic frame $\{e_1, \cdots, e_r\}$ in a neighborhood of $x_0$ such that

$$h_{\alpha\beta}(z) = (e_\alpha(z), e_\beta(z)) = \delta_{\alpha\beta} - \sum_{i,j=1}^n R_{ij\alpha\beta} z_i \bar{z}_j + O(|z|^3)$$

where $R_{ij\alpha\beta}$ are the coefficients of the Hermitian curvature tensor at $x_0 \in M$

3.2 Nakano positivity, Griffiths positivity and Numerically effectiveness

Let $M$ be a compact complex manifold of complex dimension $n$ and $E$ a holomorphic vector bundle of rank $r$ over $M$. We use the same notations as in the last section.

3.5. Definition
a. The Hermitian vector bundle $E$ is said to be Nakano positive (resp. semipositive) if $\tilde{\Theta}(E)(\xi,\bar{\xi}) > 0$ (resp. $\geq 0$) for all non zero tensors $\xi = \sum_{i,\alpha} f_{i,\alpha} \partial/\partial z_i \otimes e_\alpha \in TM \otimes E$.

b. The Hermitian vector bundle $E$ is said to be Griffiths positive (resp. semipositive) if $\tilde{\Theta}(E)(\xi,\bar{\xi}) > 0$ (resp. $\geq 0$) for all non zero tensors $\xi = \varphi \otimes v = \sum_{i,\alpha} f_i \partial/\partial z_i \otimes v_\alpha e_\alpha \in TM \otimes E$.

Nakano positivity (resp. semipositivity) implies Griffiths positivity (resp. semipositivity). We can view that Nakano positivity (resp. semipositivity) as being positive (resp. semipositive) definite on any nonzero $n \times r$ matrix vector, while Griffiths positivity (resp. semipositivity) is just being positive (resp. semipositive) definite on non-zero $n \times r$ matrix vectors of rank 1.

When the rank of the Hermitian vector bundle $E$ is 1, i.e. $E$ is Hermitian line bundle, the Nakano positivity (resp. semipositivity) and Griffiths positivity (resp. semipositivity) are coincide. So we will simply call the Griffiths (or Nakano) positive line bundles positive line bundles.

For positive line bundles we have the following vanishing theorem is a major tool. It is a direct consequence of the Bochner-Kodaira-Nakano inequality which we will discuss in the next section.
3.6. Theorem (Kodaira-Nakano Vanishing)

(a) If $L$ is a positive line bundle over a compact Kähler manifold $M$ of dimension $n$, then

$$H^q(M, \Omega^p(L)) = 0 \quad \text{for} \quad p + q > n$$

(b) If $L$ is a negative line bundle over a compact Kähler manifold $M$ of dimension $n$, then

$$H^q(M, \Omega^p(L)) = 0 \quad \text{for} \quad p + q < n$$

There are several other versions of the vanishing theorem, one of them is the following

3.7. Theorem Let $L$ be a positive line bundle over a compact Kähler manifold $M$ of dimension $n$. Then for any holomorphic vector bundle $E$, there exists a positive integer $k_0$ such that

$$H^q(M, \mathcal{O}(L^k \otimes E)) = 0 \quad \text{for any} \quad q > 0 \quad \text{and any} \quad k \geq k_0$$

A generalized version of the vanishing theorem for numerically effective line bundles is Theorem 6.12 in [Dema]

The *numerically effectiveness* is a weaker notion of positivity than Griffiths positive. It is widely used in transcendental algebraic geometry because it is more flexible. The analytic definition of the numerically effectiveness is the following
3.8. Definition Let \((M, g)\) be compact complex manifold of dimension \(n\) with Hermitian form \(\omega\), \(L\) is a holomorphic line bundle on \(M\). We say that the line bundle \(L\) is \textit{numerically effective} if for every \(\varepsilon > 0\), there exists a smooth Hermitian metric \(h_{\varepsilon}\) on \(L\) such that the Hermitian curvature of \(L\) satisfies

\[
\sqrt{-1} \Theta(L)_{h_{\varepsilon}} \geq -\varepsilon \omega
\]

This means that the curvature of the \(L\) has an arbitrarily small negative part.

3.9. Definition Let \((M, g)\) be compact complex manifold of dimension \(n\) with Hermitian form \(\omega\), \(E\) a Hermitian vector bundle on \(M\). We denote by \(\mathbb{P}(E)\) the projectivized bundle of hyperplanes of \(E\), i.e. \(\mathbb{P}(E) = \bigcup_{x \in M} \mathbb{P}(E^*_x) = \bigcup_{x \in M} (E^*_x - 0)/\mathbb{C}^*\), and denote by \(\mathcal{O}_E(1)\) the associated line bundle on \(\mathbb{P}(E)\), which is the dual of the universal (or tautological) line sub-bundle \(\mathcal{O}_E(-1)\). We say that the vector bundle \(E\) is \textit{numerically effective} if the line bundle \(\mathcal{O}_E(1)\) is numerically effective.

From this definition one can see that Griffiths semipositive vector bundles are always numerically effective, but the converse is not true. A counter example is the following

3.10. Example Let \(C\) be a smooth elliptic curve, \(E\) a rank 2 vector bundle on \(C\) obtained as a non-trivial extension

\[
0 \to \mathcal{O} \to E \to \mathcal{O} \to 0
\]
by the trivial line bundle. Then $E$ is numerically effective, but is not Griffiths semi-positive.

**Proof.** We can see that $E$ is numerically effective by the next proposition. On the other hand, if $E$ is Griffiths semipositive, there will be a Hermitian metric $h$ on $E$ whose curvature tensor $\tilde{\Theta}(E)$ is Griffiths semipositive. Thus $\text{Tr} \tilde{\Theta}(E) = \tilde{\Theta}(\text{det}E) \geq 0$. Since $\text{det}E = O$, the $\text{Tr} \tilde{\Theta}(E) = 0$. We know that the curvature tensor $\tilde{\Theta}(E) = 0$. This implies that the above exact sequence of $E$ will split holomorphically, contradicts our assumption. \qed

The following properties and results about numerically effective vector bundles will be used later.

**3.11. Proposition** Let $X, Y$ be compact complex manifolds, $f : Y \to X$ be a holomorphic map, $E$ be a holomorphic vector bundle over $X$. If $E$ is numerically effective, then $f^* E$ is also numerical effective.

**3.12. Proposition** Let $0 \to F \to E \to Q \to 0$ be an exact sequence of holomorphic vector bundles over a compact complex manifold $(M, g)$ with Hermitian form $\omega$. Then

(a) If $E$ is numerically effective, then $Q$ is numerically effective.

(b) If $F, Q$ are both numerically effective, then $E$ is numerically effective.

(c) If $E, (\text{det}Q)^*$ are both numerically effective, then $F$ will be numerically effective.
3.3 Ample vector bundles

Another important notion on positivities is the ampleness, which captures the algebraic point of view.

3.13. Definition Let $M$ be a compact complex manifold of dimension $n$. A holomorphic line bundle $L$ over $M$ is called an ample line bundle if there is a holomorphic embedding

$$f : M \hookrightarrow \mathbb{P}^N$$

such that $L^k = f^*H$ for some integer $k > 0$, where $H := \mathcal{O}(1)$ is the hyperplane line bundle of $\mathbb{P}^N$.

3.14. Definition Let $M$ be a compact complex manifold of dimension $n$. We say that a holomorphic vector bundle $E$ over $M$ is ample vector bundle, if the associated line bundle $\mathcal{O}_E(1)$ on $\mathbb{P}(E)$ is ample.

By the following Kodaira embedding theorem one can see that for line bundles, the ampleness is equivalent to the positivity notion we discussed before. So we sometimes will call ample line bundles positive line bundles for this reason. But we can not tell from this theorem if ampleness of vector bundles of rank $> 1$ is the same as the Griffiths positivity of vector bundles. It is not hard to show that a Griffiths positive vector bundle is always ample by a straightforward computation of the curvature.
\( \Theta(\mathcal{O}_E(1)) \) of the \( \mathcal{O}_E(1) \), which, at a given point \((p, [v])\) of \( \mathbb{P}(E) \), is the sum of the curvature \( \Theta(E)_{v\bar{v}} \) with the Fubini-Study metric form of the fibre. The converse was conjectured by Griffiths and has not been solved yet.

3.15. Theorem (Kodaira embedding theorem) If \( L \) is a positive line bundle over a compact complex manifold \( M \) of dimension \( n \), then there exists \( k_0 \), such that for any \( k \geq k_0 \),

\[
i_{L^k} : M \to \mathbb{P}^N
\]

is an embedding of the \( M \).

The map \( i_{L^k} \) is defined by the linear system of the line bundle \( L^k \). Blowing up techniques and vanishing theorems were used to prove this theorem. The detail of this theorem can be found from [GrHa].

Since we are only concerned with smooth projective varieties, it is important for us to give a different definition of numerically effectiveness for line bundles, as well as an equivalent definition for the ampleness of line bundle. These definitions only work for projective varieties.

3.16. Definition-Proposition Suppose \( M \) is a projective manifold of dimension \( n \). Then A holomorphic line bundle \( L \) over \( M \) is numerically effective if and only if

\[
L \cdot C = \int_C c_1(L) \geq 0
\]
for any irreducible curve $C \subset M$.

The "only if" part is clear, and in fact, it can be shown that $L^p \cdot Y = \int_Y c_1(L)^p \geq 0$ for any $p$-dimensional subvariety $Y \subset M$. We refer the readers to [Hart] for details. The converse is proved in Prop. 3.18 below. To relate ampleness with this, we have the following

**3.17. Nakai-Moishezon Ampleness Criterion** A holomorphic line bundle $L$ over a projective manifold $M$ of dimension $n$ is ample if and only if

$$L^p \cdot Y = \int_Y c_1(L)^p > 0$$

for any $p$ dimensional subvariety $Y \subset M$.

The following proposition will tell us that this definition of the numerical effective of line bundle on projective manifold is equivalent to the analytic definition 3.8.

**3.18. Proposition** Let $L$ be a holomorphic line bundle over a projective manifold $(M, g)$ with hermitian form $\omega$ of dimension $n$, $H$ an ample line bundle on $M$. Then the following are equivalent,

(a) $L^p \cdot Y = \int_Y c_1(L)^p \geq 0$ for any $p$-dimensional subvariety $Y \subset M$.

(b) For any positive integer $k$, the line bundle $kL \otimes H$ is ample.
(c) For any \( \varepsilon > 0 \), there is a metric \( h_\varepsilon \) on \( L \) whose curvature satisfies \( \Theta(L)_{h_\varepsilon} \geq -\varepsilon \omega \).

Proof. (a) \( \Rightarrow \) (b). If \( L \) satisfies the condition (a) and \( H \) is ample, then clearly \( kL + H \) will satisfy the Nakai Moishezon Ampleness Criterion. Hence \( kL + H \) is ample for any \( k > 0 \).

(b) \( \Rightarrow \) (c). We may select a metric \( h_H \) on \( H \) with positive curvature and set \( \omega = \sqrt{-1}\Theta(H) \). If \( kL + H \) is ample, then this bundle admits a Hermitian metric \( h_{kL+H} \) with positive curvature. Consider the metric \( h_L = (h_{kL+H} \otimes h_H^{-1})^{1/k} \) on \( L \), its curvature is

\[
\sqrt{-1}\Theta(L) = \frac{1}{k}(\sqrt{-1}\Theta(kL + H) - \sqrt{-1}\Theta(H)) \geq -\frac{1}{k}\sqrt{-1}\Theta(H)
\]

So for any given \( \varepsilon > 0 \), by letting \( k \) large enough, we can be sure that \( \frac{1}{k} < \varepsilon \).

(c) \( \Rightarrow \) (a). Under the condition (c), for any irreducible curve \( C \subseteq M \), we have

\[
L \cdot C = \int_C \frac{1}{2\pi} \Theta(L) \geq -\frac{\varepsilon}{2\pi} \int_C \omega
\]

for any \( \varepsilon > 0 \), so we get \( L \cdot C \geq 0 \) by taking \( \varepsilon \to 0 \). Thus (a) is true. \( \square \)

3.19. Definition Einstein-Hermitian vector bundle

Let \((E, h)\) be a Hermitian vector bundle of rank \( r \) over a compact Hermitian manifold \((M, g)\) of dimension \( n \). Define the mean curvature \( K \) of \( E \) by

\[
K = (K^\alpha_{\beta\gamma}) = \sum h^{\alpha\bar{\gamma}} g^{ij} R_{ij\beta\gamma}
\]
here $h^{\alpha\bar{\gamma}} = (h_{\alpha\bar{\gamma}})^{-1}$ and $g^{ij} = (g_{ij})^{-1}$. $R_{ij\bar{\beta}\bar{\gamma}}$ are the coefficients of curvature discussed in (3.1.2). If $K = cI_E$, i.e. $K^\alpha_\beta = c\delta^\alpha_\beta$, $c$ is a constant, then we say that $E$ satisfies the $Einstein$ condition and call $E$ an Einstein-Hermitian vector bundle.

We have the following important theorems about Einstein-Hermitian vector bundles which will be used later.

**3.20. Theorem (Donaldson, Uhlenbeck-Yau)** Let $M$ be a compact complex manifold with ample line bundle $H$ and Kähler form $\omega$ representing the Chern class $c_1(H)$. If a holomorphic vector bundle $E$ over $M$ is $\omega$-stable, then $E$ admits an Einstein-Hermitian structure.

**3.21. Theorem** Let $(E, h)$ be a Hermitian vector bundle over a compact Kähler manifold $(M, g)$ of dimension $n$ with Kähler form $\omega$. If it satisfies the Einstein condition and $c_1(E) = 0$ in $H^2(M, \mathbb{R})$ then

$$\int_M c_2(E) \wedge \omega^{n-2} \geq 0$$

and the equality holds if and only if $(E, h)$ is flat.
CHAPTER 4
INEQUALITIES FOR CHERN CLASSES

In this chapter, we will take a quick tour on the inequalities for Chern classes for ample vector bundles and more generally numerically effective vector bundles.

Let \( P \in \mathbb{Q}[c_1, \ldots, c_r] \) be a weighted homogeneous polynomial of degree \( n \), where each variable \( c_i \) is assigned with degree \( i \). We say \( P \) is called \textit{a numerically positive polynomial for ample vector bundles} if for any projective manifold \( M \) of dimension \( n \), and any ample vector bundle \( E \) over \( M \) with rank \( r \), the following inequality is satisfied

\[
\int_X P(c_1(E), \ldots, c_r(E)) > 0
\]

A theorem of Bloch and Gieseker [BlGi] asserts that the Chern class \( c_n \) is numerically positive for ample bundles provided that \( n \leq r \).

A basis for the \( \mathbb{Q} \) vector space \( \mathbb{Q}[c_1, \ldots, c_r] \) is given by the so-called Schur Polynomials, which is defined as follows.

Denote by \( \Lambda(n, r) \) the set of all partitions of \( n \) by nonnegative integers \( \leq r \). Thus an
element $\lambda \in \Lambda(n,r)$ is specified by a sequence

$$r \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0, \quad \sum \lambda_i = n$$

Each $\lambda \in \Lambda(n,r)$ gives rise to a Schur polynomial $P_\lambda \in \mathbb{Q}[c_1, \ldots, c_r]$ of degree $n$, defined as the $n \times n$ determinant

$$P_\lambda = \left| \begin{array}{cccc} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+n-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\lambda_n-n+1} & c_{\lambda_n-n+2} & \cdots & c_{\lambda_n} \end{array} \right|$$

where by convention $c_0 = 1$ and $c_i = 0$ if $i \notin [0, r]$. Any polynomial $P \in \mathbb{Q}[c_1, \ldots, c_r]$ can be written uniquely as

$$P = \sum_{\lambda \in \Lambda(n,r)} a_\lambda(P) P_\lambda, \quad a_\lambda(P) \in \mathbb{Q}$$

In particular for $\lambda = (1, \cdots, 1)$ we get the $n$’th dual Segre class $s_n(E^*)$. Fulton and Lazrasfeld [FuLa] showed the following

4.1. **Theorem (Fulton and Lazrasfeld)** The polynomial $P$ is a numerically positive polynomial for ample vector bundles if and only if the coefficients

$$a_\lambda(P) \geq 0 \quad \text{for all} \; \lambda \in \Lambda(n,r)$$

and at least one of them is strictly positive.
Demailly-Peternell-Schneider [DePS] generalized the above result to the numerically effective vector bundles, the main theorem is following

4.2. Theorem (Demailly-Peternell-Schneider) Let $E$ be a numerically effective vector bundle over a compact Kähler manifold $M$ equipped with a Kähler form $\omega$. Then for any Schur polynomial $P_\lambda$ of degree $k$ and any analytic subset $Y \subset M$ of dimension $d \geq k$ we have

$$\int_Y P_\lambda(c(E)) \wedge \omega^{d-k} \geq 0$$

For example, if $M$ is a projective surface with numerically effective cotangent bundle, we will have $c_1^2 \geq c_2$. 

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Notation In this chapter a surface means a compact complex manifold of complex dimension 2. All the vector bundles are holomorphic vector bundles. We will use divisor and the line bundle determined by the divisor interchangeably. We will not distinguish between a holomorphic vector bundle and the corresponding sheaf of the holomorphic vector bundle.

Let $M$ be a surface, $E$ a holomorphic vector bundle over $M$. We will fix the following notations:

$\mathcal{O}_M(D)$: the invertible sheaf (or holomorphic line bundle) corresponding to the divisor $D$.

$H^q(M, E)$: the $q$-th cohomology group of the sheaf $\mathcal{O}_M(E)$.

$h^q(M, E)$: the complex dimension of the group $H^q(M, E)$.

$K_M$: the canonical divisor, a divisor such that $\mathcal{O}_M(K_M) = \Lambda^2 \Omega^1_M$.

$\text{Pic}(M)$: the group of all holomorphic line bundles on $M$. 

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$\text{Pic}^0(M)$: the group of holomorphic line bundles with zero first Chern class.

$\chi_{\text{top}}(M) = \sum_i (-1)^i h^i(M, \mathbb{R})$, the Euler number of $M$.

5.1. Main Lemma Let $M$ be a general type surface with $c_1^2 = c_2$. If $\text{Pic}(M)/\text{Pic}^0(M)$ is generated by the canonical line bundle $K_M$, then $q(M) = h^0(M, \Omega^1_M) \neq 1$ or 2.

Proof. If $q(M) = 1$, by Theorem V.13 [Beau] the Albanese map is $p : M \to B$, where $B$ is a smooth elliptic curve. By Theorem V.15 [Beau], $p$ must be a fibration with connected fibres. So the generic fibre $C$ will have self intersection $C^2 = 0$ by Proposition I.8 [Beau], here $C = kK_M$ for some $k > 0$. So $K_M^2 = 0$, which will contradict the original condition $c_1^2 = c_2 \geq 6$ by Noether’s formula $\chi(O_M) = \frac{1}{12}(c_1^2 + c_2)$ and Theorem (1.1).

Now let us assume that $q(M) = 2$. By Theorem V.13 the Albanese map is $\pi : M \to A$, where $A$ is an abelian surface. There are two possibilities for the map $\pi$.

1. the map $\pi$ will contract some curve $C$, possible reducible, to a point.

2. the map $\pi$ is a finite map onto $A$.

For Case 1, By Grauert’s criterion Theorem III.2.1 [BaPV] we know that the intersection matrix $(C_i \cdot C_j)$ is negative definite, here $C = \sum_i C_i$ with each $C_i$ being irreducible. Thus there must be some $i$ such that $C_i^2 \leq 0$. Write $C_i = kK_M$, k must
be positive, so we get $K_M^2 \leq 0$, contradicting to the fact that $c_1^2 = c_2 \geq 6$ by Noether’s formula.

In Case 2, since the map $\pi : M \to A$ is finite, we know that $A$ must be a simple abelian surface, which means that $A$ does not contain any smooth elliptic curve $C$. If it has such a curve $C$, then $C^2 = 0$ by genus formula. So by looking at $\pi^{-1}(C)$ we get a contradiction.

Let $B$ be the branch locus of $\pi$, and $R$ the ramification locus. The Hurwitz formula gives

$$K_M = \pi^* K_A + \sum_i (m_i - 1) R_i,$$

where $R = \sum_i R_i$ is the decomposition into irreducible components and $m_i \geq 2$ is the multiplicity of $\pi$ at a generic point of $R_i$. Since $K_M$ is the generator of the $Pic(M)/Pic^0(M)$, we know that $K_M$ is irreducible. If it is not, let’s say $K_M = D_1 + D_2$, here $D_i = l_i K_M \otimes L_i$, $L_i \in Pic^0(M)$, $l_i > 0$, $i = 1, 2$. Then $K_M \cdot K_M = l_1 K_M^2 + l_2 K_M^2$, contradiction. We also know that $K_A = O_A$, so we know that $R = R_1$ must be irreducible, and $m_1 = 2$. Suppose $d$ is the degree of the map $\pi|_R : R \to B$, then $\pi : M \to A$ has degree $2d$ and $K_M = R$.

Note that we could also assume that $A$ has a principle polarization, i.e. an ample divisor $L$ on $A$ of type of $(1, 1)$, see Definition 4.1 in [BiLa]. This is because if $A$ does not have type $(1, 1)$ polarization, then by Proposition 4.1.2 [BiLa] there will
be an isogeny $p : A \to A'$, onto a principle polarized abelian surface, that is, the $p$
is a surjective homomorphism with finite kernel from the complex torus $A$ onto the
complex torus $A'$, such that $L = p^*L'$, with $L'$ a type $(1, 1)$ line bundle on $A'$. If that
is the case, we could simply replace $\pi : M \to A$ by the composition of $\pi$ and $p$.

From now on, let us assume that $L$ is a type $(1, 1)$ line bundle on $A$. Since we
have $\pi^*B = 2K_M$ and $K_M$ is a generator, we get $B = L$ or $B = 2L$. If $B = 2L$, then
by using Proposition 4.1.2 [BiLa] again we can replace $A$ by another abelian surface
via an isogency, so we could always reduce to the $B = L$ case. Now suppose $B = L$,
by Proposition 4.5.2 of [BiLa] and the vanishing theorem Theorem 3.4.5 of [BiLa] for
complex tori we know that the line bundle $L$ is ample if and only if $h^i(L) = 0$ for
$i = 1, 2$ and $L^2 > 0$. Since $L$ is ample line bundle of type $(1, 1)$, then by Riemann-
Roch Theorem 3.6.3 [BiLa] on complex tori $A$, we get $\chi(L) := \chi(\mathcal{O}_M(L)) = \frac{1}{2}L^2$ (the
more general version of the Riemann-Roch theorem will be given in the proof of the
next lemma). This implies that $h^0(A, L) = \frac{1}{2}L^2 = 1$. So $L^2 = 2$, and the arithmetic
genus of $B$ is

$$p_a(B) = p_a(L) = 1 + \frac{L^2 + K_A.L}{2} = 2$$

and we also have

$$4K_M^2 = \pi^*B.\pi^*B = 2dB^2 = 2dL^2 = 4d, \quad \text{so } K_M^2 = d$$
Since $p_a(B) = 2$, by the Lemma (p505, [GrHa]) we know that the topological genus of $B$ is

$$g(B) \leq p_a(B) - \sum_i \frac{k_i(k_i - 1)}{2}$$

where the $k_i$ are the multiplicities of the singular points. Since $p_a(B) = 2$, we know that $B$ must be one of the following

i. $B$ is a smooth genus 2 curve.

ii. $B$ is a singular rational curve with two multiplicity 2 singular points or one multiplicity 3 singular point.

iii. $B$ is a singular elliptic curve with a multiplicity 2 singular point.

For case i, since $B$ is a smooth genus 2 curve, we will have $\chi_{top}(B) = 2 - 4 = -2$

We also have

$$\chi_{top}(M \setminus R) = 2d\chi_{top}(A \setminus B)$$

$$\chi_{top}(M) = -2d\chi_{top}(B) + \chi_{top}(R) \quad since \quad \chi_{top}(A) = 0$$

$$K^2_M = \chi_{top}(M) = 4d + \chi_{top}(R)$$

$$\geq 4d + 2 - 2(1 + K^2_M) + \sum_i (k_i - 1)^2$$

$$= 2K^2_M + \sum_i (k_i - 1)^2$$
here we used the formula

$$\chi_{top}(R) \geq \chi_{top}(R_0) + \sum_i (k_i - 1)^2$$

where $k_i$ is the multiplicity of the i-th singular point of $R$, and $R_0$ is a smooth curve homologous to $R$, i.e. a smooth curve with the topological genus of $p_a(R)$. The inequality on $K^2_M$ certainly provides a contradiction, so case i can not occur.

For case ii, if there is a singular rational curve in $A$, there will be a holomorphic map $i : \mathbb{P}^1 \to A$. We can lift this map to the universal cover of $A$, $\tilde{i} : \mathbb{P}^1 \to \mathbb{C}^2$, which must be constant by maximum principle. So case ii can not occur either.

For case iii, the singular point of $B$ is either a node or a cusp because it is of multiplicity 2. We divided our discussion into two subcases.

a), If $B$ has a cusp $p$, let $f : \tilde{A} \to A$ be the blowing up of $A$ at $p$, and $\tilde{B}$ the strict transformation of $B$. We have

$$\chi_{top}(B) = \chi_{top}(\tilde{B}) - \sum_i (\# \{f^{-1}(p)\} - 1)$$

so $\chi_{top}(B) = 0$ since $\chi_{top}(\tilde{B}) = 0$. Therefore

$$\chi_{top}(M \setminus R) = 2d \chi_{top}(A \setminus B)$$

$$\chi_{top}(M) = \chi_{top}(R)$$

$$\leq \chi_{top}(\tilde{R}) \leq 0$$

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since $\tilde{R}$ has positive genus, this contradicts the inequality $\chi_{\text{top}}(M) = K_M^2 \geq 6$.

b), if $B$ has a node $p$, similar to the above argument, we will have $\chi_{\text{top}}(B) = \chi_{\text{top}}(\tilde{B}) - 1 = -1$. Therefore

$$\chi_{\text{top}}(M) = -2d\chi_{\text{top}}(B) + \chi_{\text{top}}(R)$$

$$d = 2d + \chi_{\text{top}}(R)$$

$$\chi_{\text{top}}(R) = -d$$

But we also have the following

$$\chi_{\text{top}}(R) = d\chi_{\text{top}}(B) - \sum_{q_i \in \pi^{-1}(p)} (\nu(q_i) - 1)$$

$$-d = -d + \sum_{q_i \in \pi^{-1}(p)} (\nu(q_i) - 1)$$

$$\nu(q_i) = 1$$

So we know that $\pi|_R : R \to B$ is an unbranched cover over $B$. Take a small open set $Z$ of $p$ in $A$ with local holomorphic coordinates $(z_1, z_2)$ such that $Z = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$. Let $U = \pi^{-1}(Z)$, so that $\pi|_U : U \to Z$ is a double covering branched along $z_1 \cdot z_2 = 0$. Write $Z^* = Z \setminus \{z_1z_2 = 0\}$ and $U^* = \pi^{-1}(Z^*)$.

Then $\pi|_U$ and $U$ are determined uniquely by the nonramified covering $U^* \to Z^*$ (see [Stei], Satz1), i.e. by the subgroup $\Gamma = \pi|_{U^*}(\pi_1(U^*)) \subset \pi_1(Z^*)$ of index 2, here $\pi_1(Z^*) = \mathbb{Z} \times \mathbb{Z}$ with generator $w_1 = (1, 0)$ and $w_2 = (0, 1)$, and $w_i$ is the class of a positively oriented little loop around the $z_i$-axis. We can pick generators for $\Gamma$ as follows:
Since $\Gamma \cap (\mathbb{Z}, 0)$ is nontrivial, there is some $(n, 0) \in \Gamma$ generating this intersection, here $n > 0$. Then the quotient $\Gamma/\mathbb{Z} \cdot (n, 0)$ is isomorphic to $\mathbb{Z}$, and there exists some $(q, m) \in \Gamma$, $0 \leq q < n$, $m > 0$, such that $(n, 0)$ and $(q, m)$ generate $\Gamma$. If the $q = 0$, we know that either $n = 1$ and $m = 2$, or $n = 2$ and $m = 1$ since $\Gamma$ is an index 2 subgroup. This will imply that $\pi|_U$ is not ramify over one of the $z$-axis. So we can conclude that $q > 0$. By The proof of Theorem III.5.2 in [BaPV], we know there must exist singularity on $U$. But that contradicts to the smoothness of surface $M$. So the map $\pi : M \to A$ can not be a finite map.

From Case 1 and 2, we conclude that $q(M) \neq 2$. So $q(M) \neq 1$ or 2. \hfill $\Box$

5.2. Theorem Let $M$ be a general type surface with $c_2^2 = c_2$ and suppose that $\text{Pic}(M)/\text{Pic}^0(M)$ is generated by $K_M$. Then any nontrivial holomorphic bundle extension

$$0 \to T_M \to E \to K_M \to 0$$

of the holomorphic tangent bundle $T_M$ by $K_M$ is Hermitian flat.

Proof. First, we will show that $E$ is semistable. It suffices to show that $E \otimes K_M^*$ is semistable by Proposition 2.29.

Let

$$0 \to T_M \to E \to K_M \to 0 \quad (5.0.1)$$
be a nontrivial extension, and twisting the exact sequence (5.0.1) by the line bundle $K_M^*$, we get

$$0 \to T_M \otimes K_M^* \to E \otimes K_M^* \to O_M \to 0$$  \hspace{1cm} (5.0.2)$$

By Whitney sum formula and Chern formula for tensor product, we get

$$c(E \otimes K_M^*) = c(T_M \otimes K_M^*)c(O_M)$$

$$c_1(E \otimes K_M^*) = c_1(T_M) + 2c_1(K_M^*) = 3c_1(K_M^*) = -3c_1(K_M)$$

and from (5.0.1)

$$c(E) = (1 + c_1(T_M) + c_2(T_M))(1 + c_1(K_M))$$

$$= 1 + c_1(T_M) + c_1(K_M) + c_2(T_M) + c_1(T_M)c_1(K_M)$$

The two chern classes of $E$ are

$$c_1(E) = c_1(T_M) + c_1(K_M) = 0$$

$$c_2(E) = c_2(T_M) + c_1(T_M)c_1(K_M) = c_2 - c_1^2 = 0$$

so the slope with respect to $K_M$ is

$$\mu(E \otimes K_M^*) = \frac{\int_M c_1(E \otimes K_M^*) \cup c_1(K_M)}{3} = \frac{-3K_M \cdot K_M}{3} = -K_M^2 = -c_1^2$$

Step1. We want to show that $H^0(M, E \otimes K_M^*) = 0$

First, by the Serre duality and the vanishing theorem, we have

$$H^0(M, T_M \otimes K_M^*) = H^2(M, \Omega^1_M \otimes 2K_M) = 0$$
From (5.0.2) we get

\[ 0 \to H^0(M, T_M \otimes K^*_M) \xrightarrow{i} H^0(M, E \otimes K^*_M) \xrightarrow{\pi} H^0(M, \mathcal{O}_M) \xrightarrow{\delta} H^1(M, T_M \otimes K^*_M) \to \cdots \]

So if there exits \( 0 \neq \sigma \in H^0(M, E \otimes K^*_M) \), then \( \pi(\sigma) = m \mathbf{1} \) with \( 0 \neq m \in \mathbb{C} \), where \( \mathbf{1} \) is the generator of \( H^0(M, \mathcal{O}_M) \). We have

\[ 0 = \delta \pi(\sigma) = \delta(m \mathbf{1}) = m \delta(\mathbf{1}) \]

Thus \( \delta(\mathbf{1}) = 0 \), contradicting our assumption that \( E \) is a non-trivial extension. Thus we must have \( \sigma = 0 \).

Step 2. Let \( \mathcal{F} \subset \mathcal{E} \otimes K^*_M \) be a coherent rank 1 subsheaf with torsion free quotient, here \( \mathcal{E} \otimes K^*_M \) is the sheaf notation of vector bundle \( E \otimes K^*_M \). We want to show that the slope of \( \mathcal{F} \) is no greater than the slope of \( \mathcal{E} \otimes K^*_M \).

By Lemma 2.20, 2.21, 2.22, we know that \( \mathcal{F} \) is a line bundle on \( M \), and we have a short exact sequence of sheaves

\[ 0 \to \mathcal{F} \to \mathcal{E} \otimes K^*_M \to \mathcal{Q} \to 0 \quad (5.0.3) \]

Since \( Pic(M)/Pic^0(M) \) is generated by \( K_M \), we know that \( \mathcal{F} = \mathcal{O}(kK_M \otimes L) \), for some \( L \in Pic^0(M) \) and some \( k \in \mathbb{Z} \). We claim that \( k \leq -1 \).

Twisting the short exact sequence (5.0.3) by the dual of the line bundle \( \mathcal{O}(kK_M \otimes L) \), we get

\[ 0 \to \mathcal{O} \to \mathcal{E} \otimes \mathcal{L}^* \otimes (k + 1)K^*_M \to \mathcal{Q} \otimes \mathcal{L}^* \otimes kK^*_M \to 0 \quad (5.0.4) \]
Also, twisting the short exact sequence (5.0.2) by the dual of the line bundle \( kK_M \otimes L \), we get

\[
0 \to T_M \otimes L^* \otimes (k + 1)K^*_M \to E \otimes L^* \otimes (k + 1)K^*_M \to L^* \otimes kK^*_M \to 0 \quad (5.0.5)
\]

By Theorems 3.18, 3.6 and Serre duality, we have

\[
H^0(M, T_M \otimes L^* \otimes (k + 1)K^*_M) \cong H^2(M, \Omega^1_M \otimes L \otimes (k + 2)K_M) = 0, \quad \text{for} \quad k \geq -1
\]

\[
H^0(M, L^* \otimes kK^*_M) = 0, \quad \text{for} \quad k \geq 1
\]

and \( H^0(M, L^*) = 0 \), if \( k = 0 \) and \( L \) is not the trivial line bundle.

From the short exact sequence (5.0.5) we get that for any \( k \geq 1 \) or \( k = 0 \) with \( L \) nontrivial,

\[
H^0(M, E \otimes L^* \otimes (k + 1)K^*_M) = H^0(M, E \otimes K^*_M) = 0 \quad (5.0.6)
\]

When \( k = 0 \) and \( L \) is trivial, the short exact sequence (5.0.5) becomes

\[
0 \to T_M \otimes K^*_M \to E \otimes K^*_M \to \mathcal{O}_M \to 0
\]

In this case we also know that \( H^0(M, E \otimes K^*_M) = H^0(M, \mathcal{E} \otimes K^*_M) = 0 \) by step 1.

Combining this with (5.0.6), we know that

\[
H^0(M, E \otimes L^* \otimes (k + 1)K^*_M) = H^0(M, \mathcal{E} \otimes \mathcal{L}^* \otimes (k + 1)K^*_M) = 0 \quad \text{for all} \quad k \geq 0
\]
Therefore, the short exact sequence (5.0.4) leads us to the fact that $k \leq -1$.

Since $K_M \cdot L = 0$, we get

$$
\mu(\mathcal{F}) = (kK_M + L) \cdot K_M = kK_M^2 + L \cdot K_M \leq -K_M^2 = \mu(E \otimes K_M^*)
$$

Step 3. It remains to consider the case $\mathcal{F} \subset \mathcal{E} \otimes K_M^*$, where $\mathcal{F}$ is a coherent rank 2 subsheaf with torsion free quotient. We want to show that $\mu(\mathcal{F}) \leq \mu(\mathcal{E} \otimes K_M^*)$.

We have

$$
0 \to \mathcal{F} \to \mathcal{E} \otimes K_M^* \to \mathcal{Q} \to 0 \quad (5.0.7)
$$

where $\mathcal{Q}$ is a torsion free sheaf with rank 1. Taking the dual of this short exact sequence, we have

$$
0 \to \mathcal{Q}^* \to \mathcal{E}^* \otimes K_M \to \mathcal{F}^* \quad (5.0.8)
$$

By Lemma 2.17, we know that $\mathcal{Q}^*$ is local free. So it can be written as $\mathcal{Q}^* = \mathcal{O}(-kK_M \otimes L)$ for some $L \in \text{Pic}^0(M)$ and some $k \in \mathbb{Z}$. Twisting by $\mathcal{O}(2K_M^* \otimes L^*)$ on the short exact sequence (5.0.8), we get

$$
0 \to \mathcal{Q}^* \otimes \mathcal{O}(2K_M^* \otimes L^*) \to \mathcal{E}^* \otimes \mathcal{O}(K_M^* \otimes L^*) \to \mathcal{E}^* \otimes \mathcal{O}(2K_M^* \otimes L^*) \to 0 \quad (5.0.9)
$$

Taking the dual of the exact sequence (5.0.2), we have

$$
0 \to \mathcal{O}_M \to E^* \otimes K_M \to \Omega^1_M \otimes K_M \to 0
$$

By twisting $2K_M^* \otimes L^*$ on the above exact sequence we get

$$
0 \to 2K_M^* \otimes L^* \to E^* \otimes K_M^* \otimes L^* \to \Omega^1_M \otimes K_M^* \otimes L^* \to 0 \quad (5.0.10)
$$
By Theorem 3.6 we get

$$H^0(M, \Omega^1_M \otimes K^*_M \otimes L^*) = 0 \quad \text{and} \quad H^0(M, 2K^*_M \otimes L^*) = 0$$

So the exact sequence (5.0.10) give us

$$H^0(M, E^* \otimes K^*_M \otimes L^*) = H^0(M, E^* \otimes \mathcal{O}(K^*_M \otimes L^*)) = 0$$

We then get, from the exact sequence (5.0.9)

$$H^0(M, \mathcal{Q}^* \otimes \mathcal{O}(2K^*_M \otimes L^*)) = H^0(M, \mathcal{O}(-kK_M \otimes L) \otimes \mathcal{O}(2K^*_M \otimes L^*))$$

$$= H^0(M, \mathcal{O}(-k - 2)K_M) = 0$$

From this we can conclude that $-k - 2 \leq -1$, i.e. $k \geq -1$. This leads to

$$\mu(E \otimes K^*_M) = \mu(E \otimes K^*_M) \leq \mu(\mathcal{Q})$$

By short exact sequence (5.0.7), we have

$$\frac{\int_M [c_1(\mathcal{F}) + c_1(\mathcal{Q})] \cup c_1(K_M)}{3} = \frac{\int_M c_1(E \otimes K^*_M) \cup c_1(K_M)}{3} = \mu(E \otimes K^*_M)$$

$$2\mu(\mathcal{F}) + \mu(\mathcal{Q}) = 3\mu(E \otimes K^*_M)$$

So we get from the above discussion that

$$2\mu(\mathcal{F}) \leq 2\mu(E \otimes K^*_M) \quad \text{i.e.} \quad \mu(\mathcal{F}) \leq \mu(E \otimes K^*_M)$$

Combining steps 1, 2, and 3, we know that $E$ is semistable by Theorem 2.28.
Our next goal is to show that $E$ is Hermitian flat.

If $E$ is stable, then by Theorems 3.20 and 3.21, we know that $E$ must be Hermitian flat since $c_1(E) = 0$, $c_2(E) = 0$ by our construction of $E$. So we may assume that $E$ is semistable but not stable. By our main Lemma (5.1), we may assume that $q(M)$ can not be 1 or 2. We will divide our discussion into the following two cases: $q(M) = 0$ or $q(M) \geq 3$.

Case 1. $q(M) = 0$

By Theorem 2.31, there exists a filtration

$$0 = \mathcal{E}_{k+1} \subset \mathcal{E}_k \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E}$$

where $\mu(\mathcal{E}_i/\mathcal{E}_{i+1}) = \mu(\mathcal{E})$ and $\mathcal{E}_i/\mathcal{E}_{i+1}$ is stable for each $i = 0, 1, \cdots, k$.

Since $\text{rank}(E) = 3$, there are three possibilities for the filtration

i. $0 \subset \mathcal{E}_1 \subset \mathcal{E}$ \hspace{1cm} $\text{rank}(\mathcal{E}_1) = 2$

ii. $0 \subset \mathcal{E}_2 \subset \mathcal{E}$ \hspace{1cm} $\text{rank}(\mathcal{E}_2) = 1$

iii. $0 \subset \mathcal{E}_2 \subset \mathcal{E}_1 \subset \mathcal{E}$ \hspace{1cm} $\text{rank}(\mathcal{E}_2) = 1$, \hspace{1cm} $\text{rank}(\mathcal{E}_1) = 2$

For case i, we will have

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{Q} \to 0$$
where \( \text{rank}(\mathcal{Q}) = 1 \), \( \mathcal{Q} \) is torsion free, and \( \mu(\mathcal{Q}) = 0 \). Taking the dual of the above exact sequence we will have

\[
0 \to \mathcal{Q}^* \to \mathcal{E}^* \to \mathcal{E}_1^* \tag{5.0.11}
\]

Taking the dual of the exact sequence (5.0.1) we have the sequence

\[
0 \to K_M^* \to E^* \to \Omega_M^1 \to 0
\]

So \( H^0(M, E^*) = 0 \) becuase \( H^0(M, K_M^*) = 0 \) and \( H^0(M, \Omega_M^1) = 0 \).

This and the short exact sequence (5.0.11) imply \( H^0(M, \mathcal{Q}^*) = 0 \). On the other hand, we assumed that \( q(M) = 0 \), so \( \text{Pic}^0(M) = 0 \). Thus \( \mu(\mathcal{Q}^*) = 0 \) implies that \( \mathcal{Q}^* = \mathcal{O}_M \), which contradicts the fact that \( H^0(M, \mathcal{Q}^*) = 0 \).

For case ii, we will have

\[
0 \to \mathcal{E}_2 \to \mathcal{E} \to \mathcal{Q} \to 0
\]

where \( \text{rank}(\mathcal{Q}) = 2 \) and \( \mathcal{Q} \) is torsion free. Taking the dual of the above exact sequence we will have

\[
0 \to \mathcal{Q}^* \to \mathcal{E}^* \to \mathcal{E}_2^* \tag{5.0.12}
\]

By Lemmas 2.20, 2.21 and 2.22, we know that \( \mathcal{E}_2 = \mathcal{O}_M \). By Lemma 2.17, \( \mathcal{Q}^* \) is a rank 2 locally free sheaf.
Also by Theorem 2.31 and Proposition 2.29, we know that $Q^*$ is stable. By (5.0.12), we get $c_1(Q^*) = 0$ since $c_1(E) = 0$. We also have

$$0 \to ker(\tau) \to Q^* \xrightarrow{\tau} det(Q^*) \to 0$$

Here $det(Q^*) = \mathcal{O}_M$ by our assumption that $q(M) = 0$, which contradicts to the fact that $Q^*$ is stable by Theorem 2.28.

For case iii, we will have

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to Q \to 0$$

By the same reason as in Case i, we know that this case can not occur. So we conclude that when $q(M) = 0$, the case that $E$ is not stable leads to a contradiction. So $E$ must be stable.

Case 2. $q(M) \geq 3$ Assume that $E$ is not stable, taking the dual of the exact sequence (5.0.1) we get

$$0 \to K^*_M \to E^* \to \Omega^1_M \to 0$$

which leads to the long exact sequence

$$0 \to H^0(M, K^*_M) \to H^0(M, E^*) \to H^0(M, \Omega^1_M) \to H^1(M, K^*_M) \to \cdots$$

From this, we get $H^0(M, E^*) \cong H^0(M, \Omega^1_M)$, since $H^0(M, K^*_M) = 0$ and $H^1(M, K^*_M) \cong H^1(M, 2K_M) = 0$ by Theorem 3.6. So $h^0(M, E^*) \geq 3$. But by Proposition (2.10)
[Take] or Proposition 1.16 [DSP1], we know $h^0(M, E^*) = 3$ and $E$ is Hermitian flat. This concludes the proof that the vector bundle $E$ is Hermitian flat. \hfill \Box

5.3. Definition Let $M$ be a compact complex surface with ample canonical line bundle $K_M$, we say $M$ is an immersed theta divisor if it can be holomorphically immersed into a three dimensional abelian torus.

5.4. Theorem Let $M$ is a general type surface with $c_1^2(T_M) = c_2(T_M)$ and $\text{Pic}(M)/\text{Pic}^0(M)$ is generated by $K_M$. If $h^1(T_M \otimes K_M^*) > 0$, then $M$ will be an immersed theta divisor.

Proof. From $h^1(T_M \otimes K_M^*) > 0$, we will have a nontrivial holomorphic bundle extension

$$0 \to T_M \to E \to K_M \to 0$$

Taking the dual of the above exact sequence, we get

$$0 \to K_M^* \stackrel{i}{\to} E^* \stackrel{\tau}{\to} \Omega^1_M \to 0$$

As a consequence we know that cotangent bundle $\Omega^1_M$ is numerically effective by Proposition 3.12. Let $\pi : \widetilde{M} \to M$ be the universal cover of $M$, then we have the corresponding short exact sequence

$$0 \to K_{\widetilde{M}}^* \stackrel{i}{\to} E_{\widetilde{M}}^* \stackrel{\tau}{\to} \Omega^1_{\widetilde{M}} \to 0$$

Since the vector bundle $E^*$ is Hermitian flat on $M$ by Theorem 5.3, we know that $E_{\widetilde{M}}^*$ is the trivial vector bundle on $\widetilde{M}$. Let $\{e_1, e_2, e_3\}$ be a parallel holomorphic basis for
$E^*_M$, and $\phi_i = \tau(e_i)$, i=1, 2, 3. they are global holomorphic 1-forms on $\widetilde{M}$. For each $1 \leq i \leq 3$, $\phi_i$ is closed since it is parallel. (Or by the following reason: since the $\phi_i$ is holomorphic 1 forms, $d\phi_i = \partial \phi_i$ is holomorphic 2 forms and exact, and we have

$$\int_M d\pi_\ast \phi_i \wedge d\pi_\ast \bar{\phi}_i = \int_M (\pi_\ast \phi_i \wedge d\pi_\ast \bar{\phi}_i) = 0$$

So we will get $d\phi_i = 0$)

Locally, for any point $x \in \widetilde{M}$, there exists a small neighborhood $x \in U_x \subset \widetilde{M}$, such that $\phi_i = dz_i$, where $z_i$ is a holomorphic function on $U_x$. This is guaranteed by the Poincaré lemma, see Chapter 0 [GrHa], for instance. Since the map $\tau : E^*_M \rightarrow \Omega^1_M$ is surjective, we know that at least one of the conditions $\phi_1 \wedge \phi_2 \neq 0$, $\phi_1 \wedge \phi_3 \neq 0$ or $\phi_2 \wedge \phi_3 \neq 0$ should hold. Thus we always have a holomorphic immersion:

$$F_x = (z_1, z_2, z_3) : U_x \rightarrow \mathbb{C}^3$$

It is unique up to $F'_x = F_x + (a_1, a_2, a_3)$, here point $(a_1, a_2, a_3) \in \mathbb{C}^3$. By Theorem VII.7.2 of [KoNo], there is an allowable immersion

$$F = \widetilde{M} \rightarrow \mathbb{C}^3$$

If we fix a point $x_0$, an open set $U_{x_0}$ and a map $F_{x_0}$ as above, then the map $F$ is uniquely determined. Any two such allowable immersions $F$ and $F'$ differ by a translation, i.e. $F = F' + (a_1, a_2, a_3)$. 57
Let $\gamma \in \Gamma$, where $\Gamma \cong \pi_1(M)$ is the group of deck transformations of $\widetilde{M}$, then we have

$$F \circ \gamma = A_\gamma \circ F$$

for some transformation $A_\gamma(z) = z + a_\gamma$ in $\mathbb{C}^3$. Here $a_\gamma$ is a point in $\mathbb{C}^3$. This leads to a homomorphism

$$\tau_\ast : \Gamma \to A(\mathbb{C}^3)$$

$$\gamma \mapsto A_\gamma$$

where $A(\mathbb{C}^3)$ is the group of all translations. Since the group $A(\Gamma) = \text{Im}(\tau_\ast(\Gamma)) \subset \mathbb{C}^3$ is free abelian, we have

$$\pi_1(M) \xrightarrow{\text{onto}} A(\Gamma) \cong \mathbb{Z}^\rho$$

$$\pi_1(M) \xrightarrow{\text{onto}} H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2q} + \text{torsion}$$

So we can conclude that $\rho \leq 2q(M)$.

Next, we are going to show that $\rho = 6$.

We already know that $\mathbb{R} \otimes A(\Gamma) \subseteq \mathbb{R}^6$, if $\mathbb{R} \otimes A(\Gamma) \subseteq \mathbb{R}^5$, then there exists at least one real dimension in $\mathbb{C}^3$ which is invariant under the deck transformation. Without loss of generosity, we may assume that we have non-constant map

$$x_3 : \widetilde{M} \to \mathbb{R}$$

where $x_3 = z_3 + \bar{z}_3$, such that $x_3 \circ \gamma = x_3$ for each $\gamma$ in $\Gamma$. Thus $x_3$ descends to a pluriharmonic function on $M$, as $\partial \bar{\partial} x_3 = \partial \bar{\partial}(z_3 + \bar{z}_3) = 0$. So $x_3$ must be a constant
function, contradicting the fact that $F$ is an immersion.

Thus $\mathbb{R} \otimes A(\Gamma) \cong \mathbb{R}^6$ and $\rho \geq 6$.

By the fact that $2q \geq \rho$, we know that we must be in the case when $q \geq 3$. Since $E$ is Hermitian flat, both $E$ and $E^*$ are nef, thus any holomorphic section of $E^*$ cannot vanish by Proposition 1.16 in [DSP1]. Therefore, $q = h^0(E^*) \leq 3$. Combining this with our previous result, $q \geq 3$, we know that $q = 3$ and $\rho = 6$. So we have a holomorphic immersion $f$ from $M$ to $T^3$, which is a three dimensional abelian torus, as following

$$
\begin{array}{c}
\widetilde{M} \xrightarrow{F} \mathbb{C}^3 \\
\downarrow \pi \quad \downarrow \pi \\
\widetilde{M}/\Gamma \xrightarrow{f} \mathbb{C}^3/A(\Gamma) \\
\parallel \quad \parallel \\
M \xrightarrow{f} T^3
\end{array}
$$
CHAPTER 6

MAIN THEOREMS PART II

**Notation** In this chapter we are going to derive an extended version of Hard Lefschetz Theorem for cohomology with values in a Nakano semipositive vector bundle of rank $r$ on a compact Kähler manifold. To establish this, we need to obtain a Bochner type inequality.

First, let us fix some notations which we will use throughout this chapter.

$(M, g)$: a $n$-dimensional compact Kähler manifold with Kähler metric $g$.

$(E, h)$: a rank $r$ Hermitian vector bundle on $(M, g)$ with Hermitian metric $h$.

$A_{p,q}$: the space of $C^\infty (p, q)$-forms on $M$.

$A_{p,q}(E)$: the space of $C^\infty (p, q)$-forms on $M$ with values in $E$.

$A^r(E)$: the space of $C^\infty r$-forms on $M$ with values in $E$.

$D = D' + \bar\partial$ : the Hermitian connection of $(E, h)$.

$\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ : the Laplacian acting on $A^{p,q}(E)$.

$L = \omega \wedge \cdot : A^{p,q}(E) \to A^{p+1,q+1}(E)$, here $\omega$ is the Kähler form of the Kähler metric $g$. 

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\[ \Lambda = L^*: \text{the adjoint operator of the } L. \]

\[ \Theta := \Theta(D) = D^2: \text{the curvature operator of the } (E, h). \]

**Hard Lefschetz Theorem:** On a \( n \)-dimensional compact Kähler manifold \((M, g)\), the map

\[ L^k : H^{n-k}(M) \to H^{n+k}(M) \]

is an isomorphism for any \( 1 \leq k \leq n \). Also, if we define the primitive cohomology by

\[ P^{n-k} = \text{Ker} L^{k+1} : H^{n-k}(M) \to H^{n+k+2} \]

then we have so-called Lefschetz decomposition,

\[ H^r(M) = \bigoplus_{k \geq 0} L^k P^{r-2k}(M) \]

It is a well-known fact that the following inequality holds,

**Bochner-Kodaira-Nakano inequality** see [Dema] (4.6) for instance. Let \((E, h)\) be a Hermitian vector bundle of rank \( r \) on a compact Kähler manifold \((M, g)\). For any \( u \in A^{p,q}(E) \), it holds that

\[ \| \bar{\partial} u \|^2 + \| \bar{\partial}_h^* u \|^2 \geq (\sqrt{\Theta \Lambda} u, u) \]

This inequality can be used to prove the vanishing theorems. Here we construct a new Bochner type inequality.
6.1. **Lemma (a new Bochner type inequality)** Let \((E, h)\) be a Hermitian vector bundle of rank \(r\) on a compact Kähler manifold \((M, g)\). For any \(v \in A^{n-q,0}(E)\), let \(u = L^q v\), then

\[
0 \leq \|\bar{\partial}u\|^2 + \|\bar{\partial}_h u\|^2 = \|\bar{\partial}v\|^2 + q(\sqrt{-1} \Theta L^{p-1} v, L^q v) \quad q \geq 1
\]

Before proving this lemma, let us recall the definition of the Hodge star operators \(*\) and \(*_E\), and the \(L^2\) norm on the space \(A^{p,q}(E)\) induced from \(g\) and \(h\).

The Hodge star operator \(*\) is defined as a linear (over \(\mathbb{R}\)) isomorphism

\[
* : A^{p,q} \rightarrow A^{n-p,n-q}
\]

by requiring that for any \(\varphi, \psi \in A^{p,q}\),

\[
\varphi \wedge *\psi = g(\varphi, \psi) \frac{\omega^n}{n!}
\]

here \(g(\varphi, \psi)\) is the induced Hermitian metric on \(A^{p,q}\).

In order to write down \(*\) explicitly, let \(\varphi_1, \cdots, \varphi_n\) be a unitary frame of \((1,0)\)-forms. For multi-index \(I = (i_1, \cdots, i_p)\) in \(\{1, \cdots, n\}\), write \(\varphi_I = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p}\) and \(\omega = \frac{\sqrt{-1}}{2} \sum_i \varphi_i \wedge \bar{\varphi_i}\). Also denotes by \(\hat{I}\) the complement of \(I\) in \(\{1, \cdots, n\}\), so that

\[
\varphi_I \wedge \varphi_{\hat{I}} = (-1)^{r(I)} \varphi_1 \wedge \cdots \wedge \varphi_n
\]

Then for any multi-indices \(I, J\) with \(|I| = p\) and \(|J| = q\), \(*\) is given by

\[
*(f \varphi_I \wedge \bar{\varphi}_J) = (\sqrt{-1})^{p+q-n} \sum_{\epsilon I J} f \varphi_I \wedge \bar{\varphi}_J
\]
where \( \varepsilon_{IJ} = (-1)^{\frac{1}{2}n(n-1)+(n-p)q+r(I)+r(J)} \). It is straightforward to check that \( **\psi = (-1)^{p+q}\psi \) for any \((p, q)\)-form \(\psi\), and \(*\left(\frac{\omega^k}{k!}\right) = \frac{\omega^{n-k}}{(n-k)!}\).

Since \( M \) is compact, we can have a Hermitian inner product (the \(L^2\) norm) on each \( A^{p,q} \) defined by

\[
(\varphi, \psi) = \int_M \varphi \wedge *\psi
\]

\( \varphi, \psi \in A^{p,q} \)

The operator \( \bar{\partial}^* = - *\bar{\partial}^* : A^{p,q} \rightarrow A^{p,q-1} \) is adjoint to \( \bar{\partial} \) under this norm.

The star operator \( *_E \) is defined similarly

\[
*_E : A^{p,q}(E) \rightarrow A^{n-p,n-q}(E^*)
\]

by sending \( \sum_\alpha \psi_\alpha \otimes e_\alpha \) to \( \sum_\alpha (*\psi_\alpha) \otimes e_\alpha^* \), where each \( \psi_\alpha \) is a \((p, q)\)-form on \( M \), and \( \{e_1, \cdots, e_r\} \) is a unitary frame of \( E \) with \( \{e_1^*, \cdots, e_r^*\} \) the dual frame.

We can also define \(L^2\)-norm over \( A^{p,q}(E) \) similarly as

\[
(\varphi, \psi) = \int_M \varphi \wedge *_E\psi
\]

\( \varphi, \psi \in A^{p,q}(E) \)

The operator \( \bar{\partial}_h^* = - *_E \bar{\partial}^* : A^{p,q} \rightarrow A^{p,q-1} \) is the adjoint of \( \bar{\partial} \) under the \(L^2\)-norm.

**Proof.** (Proof of the Lemma 6.1)

Since \( M \) is a Kähler manifold and \( E \) is a Hermitian vector bundle, we have the following Kähler identity

\[
[\Lambda, D'] = \sqrt{-1}\bar{\partial}_h^*
\]

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From this we get the following equality from the definition of the Laplacian

$$\sqrt{-1}\Box = \bar{\partial}[\Lambda, D'] + [\Lambda, D']\bar{\partial}$$

(6.0.1)

$$= \bar{\partial}\Lambda D' - \bar{\partial}D'\Lambda + \Lambda D'\bar{\partial} - D'\Lambda\bar{\partial}$$

We also have the following. For any $\eta \in A^r(E)$, since $\omega$ is closed,

$$d(\omega \wedge \eta) = \omega \wedge d\eta$$

so $[L, d] = 0$. Similarly, we have $[L, D] = 0$. Also, we have

$$LD' = D'L, \quad L\bar{\partial} = \bar{\partial}L$$

(6.0.2)

From Equations (6.0.1) and (6.0.2) we have

$$\sqrt{-1}[\Box, L] = \sqrt{-1}\Box L - \sqrt{-1}L\Box$$

$$= (\bar{\partial}\Lambda D' - \bar{\partial}D'\Lambda + \Lambda D'\bar{\partial} - D'\Lambda\bar{\partial})L - L(\bar{\partial}\Lambda D' - \bar{\partial}D'\Lambda + \Lambda D'\bar{\partial} - D'\Lambda\bar{\partial})$$

$$= \bar{\partial}HD' - \bar{\partial}D'H + HD'\bar{\partial} - D'H\bar{\partial}$$

(6.0.3)

where $H = [\Lambda, L] = \sum_{r=0}^{2n}(n - r)P_r$, $P_r$ is the projection map: $A^*(E) \rightarrow A^r(E)$. 

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Thus, for any $\varphi \in A^{p,q}(E)$, we have

$$\sqrt{-1}[\Box, L]\varphi = \sqrt{-1}\Box L\varphi - \sqrt{-1}L\Box \varphi$$

$$= (n - p - q - 1)\bar{\partial}D' - (n - p - q)\bar{\partial}D' +$$

$$(n - p - q - 2)D'\bar{\partial} - (n - p - q - 1)D'\bar{\partial}$$

$$= -(\bar{\partial}D' + D'\bar{\partial})\varphi$$

$$= -\Theta \varphi$$

So we obtain that

$$\Box L\varphi - L\Box \varphi = \sqrt{-1}\Theta \varphi \quad (6.0.4)$$

Since $[L, D] = 0$ and $\Theta = D^2$, we get $[L, \Theta] = 0$.

Now let $v \in A^{n-q,0}(E), q \geq 1$. By the equation (6.0.4), we get

$$\Box L^q v = L\Box L^{q-1}v + \sqrt{-1}\Theta L^{q-1}v$$

$$= L^2\Box L^{q-2}v + 2(\sqrt{-1}\Theta)L^{q-1}v$$

$$= \ldots$$

$$= L^q\Box v + q\sqrt{-1}\Theta L^{q-1}v \quad (6.0.5)$$

Since $v \in A^{n-q,0}(E)$, we know that $\Box v \in A^{n-q,0}(E)$. The inner product

$$(L^q\Box v, L^q v) = \int_M \omega^q \wedge (\Box v) \wedge \ast (\omega^q \wedge v)$$

$$= \int_M (\Box v) \wedge \ast (v) \quad (6.0.6)$$

$$= (\Box v, v)$$

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By Equations (6.0.5) and (6.0.6), we get the following

\[
(\Box L^q v, L^q v) = (L^q \Box v, L^q v) + q(\sqrt{-1} \Theta L^{q-1} v, L^q v)
\]

(6.0.7)

Let \( u = L^q v \in A^{n-q}(E) \), by Equation (6.0.7) we get

\[
\|\bar{\partial} u\|^2 + \|\bar{\partial}_h^* u\|^2 = \|\bar{\partial} v\|^2 + \|\bar{\partial}_h^* v\|^2 + q(\sqrt{-1} \Theta L^{q-1} v, L^q v)
\]

But since \( v \in A^{n-q,0}(E) \) is pure type, so we know that \( \bar{\partial}_h^* v = 0 \). This leads to

\[
\|\bar{\partial} u\|^2 + \|\bar{\partial}_h^* u\|^2 = \|\bar{\partial} v\|^2 + q(\sqrt{-1} \Theta L^{q-1} v, L^q v)
\]

\[\square\]

and the lemma is proved.

**6.2. Theorem** Let \((E, h)\) be a Nakano semipositive vector bundle of rank \(r\) on a compact Kähler manifold \((M, \omega)\). Then, for any \(0 \leq q \leq n\), the wedge multiplication operator \(\omega^q \wedge \cdot\) induces a surjective morphism

\[
L^q_h = \omega^q \wedge \cdot : H^0(M, \Omega_{M}^{n-q} \otimes E) \rightarrow H^q(M, \Omega_{M}^{n} \otimes E)
\]

**Proof.** Let \( \{e_1, \cdots, e_r\} \) be a local orthonormal basis of \( E \), let \( \{\varphi_1, \cdots, \varphi_n\} \) be the unitary coframe of \((1,0)\) forms of \( M \). Write

\[
u = \sum_{|I|=n-q, 1 \leq \alpha \leq r} f_{I, \alpha} \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \overline{\varphi_I} \otimes e_{\alpha} \in H^q(M, \Omega_{M}^{n} \otimes E)
\]
here $\varphi_I = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_q}$, $q \geq 1$. Let $v$ be an element in $A^{n-q,0}(E)$ such that $u = L_h^q(v) = \omega^q \wedge v$, then

$$v = 2^q \sum_{|I|=n-q} f_{I,\alpha} \varphi_I \otimes e_\alpha \quad \varphi_I = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_{n-q}}$$

where $I = (i_1, \ldots, i_p)$ in $\{1, \ldots, n\}$. By the equation (3.1.1) we have

$$\sqrt{-1} \Theta = \sqrt{-1} \sum_{1 \leq i,j \leq n, 1 \leq \alpha, \beta \leq r} R_{ij,\alpha \beta} \varphi_i \wedge \varphi_j \otimes e^*_\alpha \otimes e_\beta$$

The coefficients $R_{ij,\alpha \beta}$ here is differ from the coefficients in (3.1.2) by a positive definite matrix. We have

$$\sqrt{-1} \Theta L_h^q \wedge \ast (L_h^q v) = \sqrt{-1} \Theta v \wedge \ast (Lv)$$

$$= 2^q \sqrt{-1} \sum_{1 \leq i,j \leq n, 1 \leq \alpha, \beta \leq r} R_{ij,\alpha \beta} \varphi_i \wedge \varphi_j \otimes e^*_\alpha \otimes e_\beta \wedge \sum_{|I|=n-q} f_{I,\alpha} \varphi_I \otimes e_\alpha$$

$$\wedge \sum_{|I|=n-q} (-\sqrt{-1})^{n-2q+1} f_{I,\cup \{i\}} \varphi_{I \cup \{i\}} \wedge \varphi_i \otimes e^*_\alpha$$

$$= 2^{n+q+1} \sum_{a,\beta=1}^r \sum_{|K|=n-q+1} f_K v_{i,j} f_K \varphi_{i_{\cup \{j\}}} \varphi_{i_{\cup \{j\}}} \wedge \varphi_i \otimes e^*_\alpha$$

$$= 2^{n+q+1} \sum_{a,\beta=1}^r \sum_{|K|=n-q+1} f_K v_{i,j} f_K \varphi_{i_{\cup \{j\}}} \varphi_{i_{\cup \{j\}}} \wedge \varphi_i \otimes e^*_\alpha$$

$$= 2^{n+q+1} \sum_{a,\beta=1}^r \sum_{|K|=n-q+1} f_K v_{i,j} f_K \varphi_{i_{\cup \{j\}}} \varphi_{i_{\cup \{j\}}} \wedge \varphi_i \otimes e^*_\alpha$$

(6.0.8)
here $I \cup \{ i \} = (i_1, \cdots, i_p, i)$ in $\{ 1, \cdots, n \}$ and $i \notin I$.

$$\left( \sqrt{-1} \Theta L^{q-1} v, L^q v \right) = 2^{n+q+1} \sum_{\alpha, \beta = 1}^{r} \sum_{|K|=n-q+1} \sum_{i,j \in K} f_{K \setminus j, \alpha} \overline{f_{K \setminus i, \beta}} R_{i j \alpha \beta}$$

$$= 2^{n+q+1} \sum_{\alpha, \beta = 1}^{r} \sum_{|P|=n-q-1} \sum_{i,j \notin P} f_{P \cup \{ i \}, \alpha} \overline{f_{P \cup \{ j \}, \beta}} R_{i j \alpha \beta} \quad (6.0.9)$$

$$= 2^{n+q+1} \sum_{|P|=n-q-1} \left( \sum_{\alpha, \beta = 1}^{r} \sum_{i,j=1}^{n} \xi^P_{i, \alpha} \xi^P_{j, \beta} R_{i j \alpha \beta} \right)$$

here we define that

$$\xi^P_{i, \alpha} = \begin{cases} f_{P \cup \{ i \}, \alpha} & \text{if } i \notin P \\ 0 & \text{if } i \in P \end{cases}$$

Let $\xi^P = \{ \xi^P_{i, \alpha} \}_{i, \alpha}$, since that $E$ is Nakano semi-positive, then by the equation (6.0.9) we get

$$\left( \sqrt{-1} \Theta L^{q-1} v, L^q v \right) = 2^{n+q+1} \sum_{|P|=n-q-1} \Theta(\xi^P, \overline{\xi^P}) \geq 0$$

By Lemma 6.1 we have

$$\| \bar{\partial} u \|^2 + \| \bar{\partial}^* u \|^2 \geq \| \bar{\partial} v \|^2$$

For any given cohomology class in $H^q(M, \Omega^* M \otimes E)$, one can pick a harmonic representative $u$ which is an element of $A^{n-q}(E)$ satisfying $\bar{\partial} u = 0$ and $\bar{\partial}^* u = 0$. Let $v \in A^{n-q,0}(E)$ be the element determined by $u = L^q_h(v) = \omega^q \wedge v$. The above Bochner-Kodaira-Nakano type inequality implies that $\| \bar{\partial} v \| \leq 0$, thus $\bar{\partial} v = 0$. So $v \in H^0(M, \Omega^{n-q}_M \otimes E)$, thus the map $L^q_h$ is surjective. This completes the proof of the theorem.  

\[ \square \]
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