GENERAL NON LINEAR PERTURBATION MODEL OF PHASE NOISE IN LC OSCILLATORS

DISSERTATION

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By

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ABSTRACT

We present a general circuit-based model of LC oscillator phase noise applicable to both white noise and 1/f noise. Using Kurokawa theory, differential equations governing the relationship between amplitude and phase noise at the tank are derived and solved. Closed form equations are obtained for the IEEE oscillator phase noise for both white and 1/f noise. These solutions introduce new parameters which take into account the correlation between the amplitude noise and phase noise and link them to the oscillator circuit operating point. These relations are then used to obtain the final expression for Voltage noise power density across the output oscillator terminals assuming the noise can be modeled by stationary Gaussian processes. For white noise, general conditions under which the phase noise relaxes to closed-form Lorentzian spectra are derived for two practical limiting cases. Further, the buffer noise in oscillators is examined. The forward contribution of the buffer to the white noise floor for large offset is expressed in terms of the buffer noise parameters. The backward contribution of the buffer to the $1/\Delta f^2$ oscillator noise is also quantified. To model flicker noise, the Kurokawa theory is extended by modeling each 1/f noise perturbation in the oscillator as a small-signal dc perturbation of the oscillator operating point. A trap level model of flicker noise is used for the analysis. A rigorous asymptotic analysis is reported to demonstrate how the expressions obtained for a single trap can be applied for a continuum of trap. Again the derivations take into account the correlation existing between amplitude and phase noise at the LC tank. Conditions under which the resulting
flicker noise relaxes to an $1/\Delta f^3$ phase noise distribution are derived. The proposed model is then applied to a practical differential oscillator. A novel method of analysis splitting the noise contribution of the various transistors into modes is introduced to calculate the Kurokawa noise parameters. The modes that contribute the most to white noise and flicker noise are identified. Further the tail noise contribution is analyzed and shown to be mostly up-converted noise. The combined white and flicker noise model exhibits the presence of a number of corner frequencies whose values depend upon the relative strengths of the various noise components. The proposed model is compared with a popular harmonic balance simulator and a reasonable agreement is obtained in the respective range of validity of the simulator and theory. The analytical theory presented which relies on measurable circuit parameters provide valuable insight for oscillator performance optimization as is discussed in the paper. Next we design a practical TSMC 0.18 um differential LC oscillator. The model is then applied on a practical TSMC 0.18 um LC oscillator. In this case also the model matches very well with simulation results. We further analyze the impact of varying the supply voltage of this oscillator using loadline techniques. Finally we compare simulation results with experimental results and see that there is a very close match between simulation and experimental results. From this we conclude that our model matches experimental results as well.
This is dedicated to my parents, Sharbani, Ananta, Basanta and Uma
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Oscillators form a very important part of physical systems. They differ from other electrical circuits in the sense that they are autonomous. Like the ticking on of the universe they have their own rhythm and perhaps a mind of their own. They are found in almost all electronic systems that we can think of, as clocks, frequency synthesizers, dc-ac inverters, cell phones, television, radio and so on.

In RF systems they are used for frequency up-conversion or down-conversion in conjunction with mixers. Basically there are 2 major types of RF oscillators, namely, LC oscillators and ring oscillators. For RF systems, LC oscillators are more popular because of their improved spectral purity. Also recent advances in fabrication technology has enabled chip makers to produce on chip inductors which in turn has further facilitated the development of chip based oscillators for RF frequencies.

Phase Noise is a very important aspect of oscillator design. It is defined as the spectral density of the oscillator voltage (or current) spectrum at an offset from the center frequency of the oscillator relative to the total power of the oscillator. In time domain it is more common to refer to this phenomenon as phase jitter. A perfect oscillator would have localized tones at discrete frequencies (i.e, harmonics) but any corrupting noise spreads these perfect tones, resulting in high power levels at neighboring frequencies. This effect is the
major contributor to undesired phenomena such as inter-channel interference, leading to increased bit error rates (BER) in RF communication systems. Another manifestation of the same phenomenon, jitter, is important in clocked and sampled-data systems. Uncertainties in switching instants caused by noise lead to synchronization problems. With complex modulation schemes like OFDM the requirement of spectral purity becomes ever more stringent. Characterizing how noise affects oscillators is therefore crucial for practical applications. The problem is challenging, since oscillators constitute a special class among noisy physical systems. Their autonomous nature makes them unique in their response to perturbations.

Considerable effort has been laid over the years starting all the way back in 1933 when Van Der Pol first introduced the concept of nonlinearity by expressing the response of an oscillator in terms of a polynomial. There has been a lot of research on the general principles behind the origin of phase noise. Also a significant number of studies have been able to develop equations that adequately represent the phase noise spectrum. However often these equations depend on excessive generalizations or for mathematical ease rely on various convenient functions and/or constants which have to be obtained a priori. To our knowledge no single model depicts the phase noise in terms of simple easy to measure circuit parameters. In this dissertation we shall attempt to develop a model for the phase noise spectrum in terms of easily measurable circuit parameters. We shall then compare our model with a commercial phase noise simulator and see where our models match and whether our model does indeed give a proper representation of the phase noise. Initially we shall consider the origin of phase noise due to white noise only. We then extend our model for 1/f noise sources under some model assumptions. We shall now give a brief review of the work that has so far been accomplished in this field.
1.1 Problem description, History of the problem and review of relevant research in this field

1.1.1 Problem Definition

The goal of our work is to obtain a phase noise expression in terms of simple, easy to measure circuit parameters for both white noise and flicker noise sources. The model should be simple, easy to implement on a standard LC oscillator, should have a circuit focus and should account for the correlation between amplitude deviation and phase deviation. Further the model should help us to characterize the buffer noise in oscillators. The results obtained from this model will then be applied on theoretical (Van Der Pol) oscillator circuit as well as a practical oscillator circuit and then compared with the simulation results obtained from a commercially available phase noise simulator.

1.1.2 Noise in Oscillators

The fundamental cause of phase noise in oscillators is the inherent noise in the oscillator circuit components like resistances, MOSFETS, BJT's etc. The most common noise is additive white Gaussian noise. In addition to white noise in LC tank oscillators, 1/f noise plays an important role in phase noise. While white noise can be easily represented both in time as well as frequency domains, 1/f noise is not so easy to characterize. 1/f noise was first noticed as an excess low-frequency noise in vacuum tubes and then, much later in semiconductors [21]. Models of 1/f noise, based on detailed physical mechanisms, were developed by Bernamont [15] in 1937 for vacuum tubes and by McWhorter [17] in 1955 for semiconductors. 1/f noise has been observed in the resistance of carbon microphones, aqueous ionic solutions, in the frequency of quartz crystal oscillators, in the average seasonal temperature, annual amount of rainfall, rate of traffic flow etc.
The fact that this noise has a $1/f$ frequency dependence implies that at $f = 0$, the noise will blow up. Experimentally people have tried to investigate whether indeed there is a cutoff frequency below which the noise deviates from the $1/f$ dependence and have found none. A detailed analysis of $1/f$ noise was made by Keshner [21] wherein he states that $1/f$ noise is a non stationary random process. Many researchers [1][24] have postulated stationary process models in order to simplify the phase noise derivations. In oscillators this noise has a tendency to get up-converted and get shaped to produce a $-30\text{dB/decade}$ noise spectrum.

1.1.3 White Noise

The origin of phase noise characterization began almost simultaneously with the development of the oscillator during the age of the vacuum tubes. Early models by Van Der Pol and Adler depend on a polynomial like characterization of phase noise. Also most of the early works were focused on white noise sources which is ubiquitous in any system. Further sophisticated works developed later. Well-known references which try to provide a Linear Time Invariant(LTI) model include [10] [7]. These models tend to depict the phase noise in terms of the $Q$ of the LC tank showing a $1/f^2$ dependence, thereby implying infinite noise power near the harmonics [11]. These models greatly simplify the actual noise making process and express the phase noise as a multiple of the device noise. At best these models help us in gaining an insight into noise analysis and oscillator design. These simplified models do not give us a complete understanding of the phase noise spectrum.

Edson [23], introduced the method of obtaining the voltage spectral density of the oscillator output by first computing the autocorrelation function of the same voltage. However the model he proposed suffered from several limitations. It approximated the frequency
dependent part of oscillator with conductances and capacitances. Further the correlation between the phase deviation and amplitude deviation was not considered. However a convenient mathematical method of obtaining the phase noise spectral density was shown.

In order to improve the performance of the models, some approaches incorporated a linear time variant model of noise [2] [12]. These models give us a better explanation of certain portions of the frequency spectrum (like the $1/f^3$ region of an oscillator spectrum arising from $1/f$ noise and the corner frequency between the $1/f^3$ and $1/f^2$ regions). However there is a fundamental assumption of non stationarity in these models which is not valid for phase noise arising due to white noise sources.

A detailed paper on oscillator noise was written by Kaertner [24] wherein he resolves the oscillator response into phase and magnitude components. A differential equation was obtained for the phase error. A similar effort was made by Demir [11] where in he described the oscillator response in terms of a phase deviation and a additive component called orbital deviation. Both Demir and Kaertner obtained the correct Lorentzian spectrum for the PSD due to white noise. However both these models are not circuit focused. Hence in summary though considerable work has been done in the field of oscillator noise analysis, to our knowledge none provide a circuit based approach using simple easy to measure parameters to describe the phase noise.

Hence in summary though considerable work has been done in the field of oscillator noise analysis, to our knowledge none provide a circuit based approach using simple easy to measure parameters to describe the phase noise. In this dissertation we follow the general philosophy of Kaertner and Demir, in that we develop differential equations for the amplitude and phase deviations. We shall rely on the Kurokawa theory [16] which was first developed for analyzing oscillator stability and present a more rigorous application to
phase noise analysis. It uses the concept of linearization about the operating point. The advantage of the Kurokawa approach is that it does have a circuit focus and thus enables us to incorporate the circuit based parameters into the phase noise equations.

In this work we first develop differential equations for the amplitude and phase deviations. The fact that an oscillator can be linearized about its operating point is very much a valid concept as has been demonstrated in numerous papers. We rely on the Kurokawa theory [16] which performs the noise analysis for uncorrelated phase and amplitude deviations. The advantage of this approach is that it gives us a circuit focus and thus enables us to incorporate the circuit based parameters into the phase noise equations. We shall rigorously develop the Phase Noise expression arising due to both phase and amplitude deviations with due consideration for correlations existing between the two. In addition, most models, while giving descriptions of oscillator Phase Noise do not account for the circuit buffer noise or how the noise floor arises. In this paper we show how the buffer noise can be incorporated in the noise model and accounts for the oscillator noise floor.

1.1.4 Flicker Noise

While white noise dominates the phase noise spectrum at large offsets, at smaller offsets, flicker (or $1/f$) noise plays an important role in the LC oscillator. At low offsets it is the primary contributor to the phase noise spectrum of an LC oscillator noise spectrum. The corner frequency between the spectrum due to white and flicker noise should be as low as possible. The ultimate goal of the designer is to obtain a voltage spectrum as sharp as possible around the center frequency, and so flicker noise analysis becomes important.
While white noise can be easily represented both in time as well as frequency domains, 1/f noise is not so easy to characterize. The fact that this noise has a 1/f frequency dependence implies that at $f = 0$, the noise will blow up. Many researchers [1][24] have postulated stationary process models or piecewise stationary models to simplify the phase noise derivations. In oscillators this noise has a tendency to get up-converted and produce a $1/f^3$ noise spectrum.

There are two primary difficulties in analyzing Phase Noise due to flicker noise. First, the difficulty in obtaining suitable expressions for the inverse Fourier transform of flicker noise. Secondly, as we shall see, orthogonally decomposed components of flicker noise may be correlated to each other. This happens due to the presence of memory effects in the flicker noise process. The difficulty lies in obtaining the correlation factor between the two. In this paper we proposes methods to overcome these difficulties. For the first problem, we start our analysis at a single device trap level and then extend it for all traps as shown in [18]. For the second limitation we propose that since the $1/f$ noise process is significant only at very low frequencies, the perturbation it causes to oscillation is equivalent to the perturbation of the dc bias of the devices involved. As we shall see this approximation will greatly aid in our analysis. The proposed method further, enables us to take into consideration, the correlation between the phase and amplitude deviations due to noise. To our knowledge no work so far has obtained an expression for $1/f$ phase noise accounting for the correlation between amplitude and phase deviations.

This dissertation is arranged as follows. In Chapter 2 we give an overview of existing Phase Noise models. In Chapter 3 we describe our model as applied to Phase Noise due to white noise sources. In Chapter 4 we describe the buffer noise concept and how it can be incorporated in our model. In Chapter 5 we extend our model to Flicker noise. In Chapter 6
and 7 we apply the proposed model on a theoretical oscillator (Van Der Pol) and a practical differential oscillator respectively. In Chapter 8 we describe the design of a TSMC 0.18 um differential LC oscillator. In Chapter 9 we describe how our model can be applied on this oscillator as well as study the impact of supply voltage variation on the Phase Noise of this oscillator. Finally in Chapter 10 we conclude with our recommendations and scope of further research.
CHAPTER 2

OVERVIEW OF PHASE NOISE MODELS

The purpose of this chapter is to present a detailed analysis of phase noise in oscillators. Further we will look into the various theories developed for characterizing the phase noise of oscillators in terms of the measurable oscillator parameters.

2.1 Sidebands in Output Voltage Spectrum of Oscillator

Let us analyze a simple oscillator circuit. A ideal oscillator would have an output voltage given by $V_{out}(t) = V_0 \cos[\omega_0 t + \phi_0]$. The output $V_{out}$ of this oscillator is given by

$$V_{out}(t) = A(t) \sin[\omega_0 t + \phi(t)]$$

(2.1)

Here $\phi(t)$ is the phase in the oscillation and may vary with time. $A(t)$ is the amplitude of oscillation. In most systems, the amplitude of oscillation can be sufficiently limited, as a result of which the variation in $A(t)$ can be sufficiently suppressed. Nevertheless the presence of both amplitude and phase fluctuations cause sidebands in the output voltage $V_{out}(t)$ spectrum of the oscillator. Fig 2.2 shows the difference in the output spectrum of an ideal oscillator with that of a real oscillator.
2.2 Phase Noise

Phase noise is a quantity similar to the Signal to Noise Ratio (SNR). However instead of measuring the ratio of the noise to the signal power in the same frequency range, Phase Noise is defined at an offset $\Delta \omega$ from the center frequency $\omega_0$ as follows,

$$\text{Phase Noise } L(\Delta \omega) = 10 \log \left[ \frac{\text{Power Spectral Density at } (\omega_0 + \Delta \omega)}{P_{\text{carrier}}} \right]$$  \hspace{1cm} (2.2)

It can be seen from the above definition that phase noise is a relative quantity since it is measured relative to $P_{\text{carrier}}$ (power at center frequency). The unit of phase noise is therefore dBc/Hz (dB relative to center frequency). A typical phase noise plot is shown in Fig 2.4. We can see four distinct regions are visible, the initial part which has a slope of $\frac{1}{1}$, the second part with a slope of $\frac{1}{7}$. This part then slopes into a $\frac{1}{7}$ region which in
Figure 2.2: The spectrum of an ideal and a practical oscillator

Turn finally leads to a horizontal part corresponding to the noise floor. For example, WLAN 802.11b requires a minimum SNR of 11 dB with a center frequency of 2.412 GHz. Let the phase noise spectrum consist only of a $f^{-1}$ region. Then the total noise power between the offset frequencies $f_{min}$ and $f_{max}$ is given by,

$$P_{noise} = \int_{f_{min}}^{f_{max}} L(\Delta \omega) d(\Delta f)$$  \hspace{1cm} (2.3)

On simplifying the above equation and assuming $L(\Delta f) = \frac{k}{\Delta f^2}$, we get

$$P_{noise} = L \left[ \sqrt{\Delta f_{min}\Delta f_{max}} \right] (\Delta f_{max} - \Delta f_{min})$$  \hspace{1cm} (2.4)
So, the minimum SNR required is given by

\[
10 \log SNR_{\text{min}} = 10 \log P_{\text{sig}} - [10 \log P_{\text{int}} + 10 \log P_{\text{noise}}]
\]  
(2.5)

Hence for the WLAN 802.11b case, suppose we have \( \Delta f_{\text{min}} = 100\text{KHz} \), \( \Delta f_{\text{max}} = 300\text{KHz} \) and the adjacent interferer is 40 dB stronger than the desired signal ie, \( P_{\text{int}} - P_{\text{sig}} = 40 \) dB, then from Eqn 2.5, we get \( 10 \log (P_{\text{noise}}) = -51 \) dB for a minimum SNR of 11 dB. Further, from Eqn 2.4, we get \( L(173 \text{ KHz}) = -104 \text{ dBc/Hz} \). This translates to a phase noise requirement of -99.23 dB at an offset of 100 KHz.
2.3 Phase Noise Models

In this section we shall analyze the various phase noise models that have been developed over the years to try to explain how the phase noise arises and attains its characteristic shape. These models also try to relate the phase noise spectrum to the design parameters of an oscillator like Q factor, bias current, so that the designer using these models can gain some design insight as well.

2.3.1 Leeson's Model

Leeson [10],[4] proposed a phase noise model which had later been modified to account for the $\frac{1}{f^3}$ regions of the phase noise spectrum. The formula is as follows,

$$ L\{\Delta\omega\} = 10 \log \left[ \frac{4FKT}{P_S} \left( 1 + \left( \frac{\omega_0}{2Q_L\Delta\omega} \right)^2 \right) \cdot \left( 1 + \frac{\omega_0}{|\Delta\omega|} \right) \right] $$

(2.6)

Here F is an empirical parameter, k the Boltzmann constant, T the absolute temperature, $P_S$ the average power dissipated in the resistive part of the tank, $\omega_0$ is the oscillation frequency,
$Q_L$ is the loaded Q factor of the tank with all loadings accounted for. As we can see from the equation it is empirical with a goal of accounting for all possible regions of the phase noise spectrum without giving any logic for how such regions might have arrived.

### 2.3.2 Linear Time Invariant Model of Phase Noise

The Time Invariant model of phase noise assumes a linear transformation of device noise to Phase Noise. The device noise is assumed to be completely shaped by the oscillator tank. A simple derivation of the above equation can be made as follows. The impedance of

$$Z(\omega + \Delta \omega) \approx \frac{1}{G_L} \cdot \frac{1}{1 + j2Q_L \frac{\Delta \omega}{\omega_0}}$$

(2.7)
The active device modeled as $-G_m(V_0)$ cancels the tank conductance $G_L$ during steady state. Hence during steady state, the net impedance of the combination is,

$$Z(\Delta \omega) = \frac{v_{\text{out}}(\omega_0 + \Delta \omega)}{i_{\text{in}}(\omega + \Delta \omega)} = -j \frac{1}{G_L} \frac{\omega_0}{2Q_L \Delta \omega} \quad (2.8)$$

Now both the tank as well as the active device will contribute noise. The equivalent noise source arising from $G_L$ can be represented as $\frac{\delta^2}{\Delta f} = 4kT G_L$. If we then combine the equivalent noise contributed by the active device, then the total equivalent current source can be represented as $\frac{\delta^2}{\Delta f} = 4FkT G_L$, where $F$ is fitting factor determined \textit{a posteriori}. Hence the total phase noise can be calculated as,

$$L\{\Delta \omega\} = 10. \log \left( \frac{v_{\text{noise}}^2}{f} \right) = 10. \log \left( \frac{1}{2} |Z(\Delta \omega)|^2 \frac{\delta^2}{\Delta f} \right) = 10. \log \left[ \frac{4FkT}{P_S} \cdot \frac{\omega_0}{2Q \Delta \omega} \right]^2 \quad (2.9)$$

Here $\frac{\delta^2}{\Delta f} = 4FkT$. This phenomenon can be described by Fig 2.6. We can see that the noise spectra gets shaped by the tank impedance thereby giving the characteristic $\frac{1}{f^2}$ shape of the output voltage spectrum. We also note that the expression does not contain any $\frac{1}{f^2}$ region. To account for both of these, the Eqn 2.9 is modified and the $\omega \frac{1}{f^2}$ term included to arrive at the final form given by Eqn 2.6.

In [8] [9] a simple explanation of how this $\frac{1}{f^2}$ noise arises. Since the active device is basically a nonlinear device, its output can be represented as

$$V_{\text{out}} = \alpha_1 V_{\text{in}} + \alpha_2 V_{\text{in}}^2 + \alpha_3 V_{\text{in}}^3$$

Then let us consider an input to the device given by

$$V_{\text{in}}(t) = A_0 \cos \omega_0 t + A_n \cos \omega_n t$$

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The output will exhibit the following important terms

\[ V_{out1}(t) \propto \alpha_2 A_0 A_n \cos(\omega_0 \pm \omega_n) t \]
\[ V_{out2}(t) \propto \alpha_3 A_0^2 A_n^2 \cos(\omega_0 - 2\omega_n) t \]
\[ V_{out3}(t) \propto \alpha_3^2 A_0^2 A_n \cos(2\omega_0 - \omega_n) t \]
Hence if the frequency component $\omega_n$ arises due to the flicker noise in the active device, it will be up converted to $(\omega_0 \pm \omega_n)$ frequencies at the output of the oscillator. This in turn will then be shaped by the tank impedance in the same way the white noise was shaped as described above giving a $\frac{1}{f^2}$ region in the phase noise spectrum. The theory though simple assumes a linear behavior of oscillator which is not true.

### 2.3.3 Linear Time Variant Model of Phase noise

Hajimiri and Lee [2], [5], [4] proposed a linear time variant model of oscillator. They introduce a new quantity known as the Impulse Sensitivity Function (ISF). Let us consider the lossless circuit of Fig. 2.7 where we apply an impulse $\delta(t)$. Suppose initially the oscillation in the lossless circuit was ideal. Now on applying the impulse, the oscillation changes as shown in the same figure 2.8. We also note that applying the impulse at different instants produces different effects. If the impulse is applied while the oscillation amplitude is at its peak, then the oscillation returns to its original state except undergoing a small amplitude change. On the other hand if the impulse is applied while the amplitude is at its lowest point, the oscillation undergoes a noticeable phase change. So the impulse response of this LC circuit may be written as

$$h_\phi(t, \tau) = \frac{\Gamma(\omega_0 \tau)}{q_{\text{max}}} u(t - \tau) \quad (2.10)$$

where $u(t)$ is the unit step function and $\Gamma(x)$ is the impulse sensitivity function. Dividing by $q_{\text{max}}$, the maximum charge displacement across the capacitor, makes the function $\Gamma(x)$ independent of signal amplitude. $\Gamma(x)$ is a dimensionless, frequency and amplitude independent function periodic in $2\pi$. As its name suggests, it encodes information about the sensitivity of the oscillator to an impulse injected at phase $\omega_0 \tau$. In the LC oscillator example, $\Gamma(x)$ has its maximum value near the zero crossings of the oscillation, and a zero value
at maxima of the oscillation waveform. Once the ISF has been determined (by whatever means), we may compute the excess phase through use of the superposition integral. This computation is valid here since superposition is linked to linearity, not time invariance

\[
\hat{\phi}(t) = \int_{-\infty}^{\infty} h(\tau) i(t) d\tau = \frac{1}{q_{\text{max}}} \int_{-\infty}^{t} \Gamma(\omega_{0}\tau) i(t) d\tau
\]

(2.11)

This computation can be visualized with the help of the equivalent block diagram shown in Fig 2.9. To cast this equation in a more practically useful form, we note that the ISF is periodic and therefore expressible as a Fourier series

\[
\Gamma(\omega_{0}\tau) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_{0}\tau + \theta_n)
\]

(2.12)
where the coefficients $c_n$ are real and $\theta_n$ is the phase of the $n$th harmonic of the ISF. We will ignore $\theta_n$ in all that follows because we will be assuming that noise components are uncorrelated, so their relative phase is irrelevant. The value of this decomposition is that, like many functions associated with physical phenomena, the series typically converges rapidly, so that it is often well approximated by just the first few terms of the series. Substituting the Fourier expansion in to Eqn 2.11, and exchanging summation and integration, one obtains

$$\phi(t) = \frac{1}{q_{\max}} \left[ \frac{\Omega_0}{2} \int_{-\infty}^{t} i(\tau) d\tau + \sum_{n=1}^{\infty} c_n \int_{-\infty}^{t} i(\tau) \cos(n\omega_0 \tau) d\tau \right]$$

(2.13)

The corresponding sequence of mathematical operations is shown graphically in Fig 2.10.

Now, suppose, we inject a sinusoidal current whose frequency is near an integer multiple $m$ of the oscillation frequency, so that

$$i(t) = I_m \cos[(m\omega_0 + \Delta\omega)t]$$

(2.14)
Figure 2.9: The equivalent block diagram of the process

where $\Delta \omega \ll \omega_0$. Substituting Eqn 2.13 into Eqn 2.14 and noting that there is a negligible net contribution to the integral by terms other than when $n = m$, one obtains the following approximation:

$$\phi(t) \approx \frac{I_m c_m \sin(\Delta \omega t)}{2q_{max} \Delta \omega} \tag{2.15}$$

The spectrum of $\phi(t)$ therefore consists of two equal sidebands at $\pm \Delta \omega$, even though the injection occurs near some integer multiple of $\omega_0$. Next, we have to perform a phase to voltage conversion, and assuming small amplitude disturbances, we find that the single-tone injection eqn 2.14 results in two equal-power sidebands symmetrically disposed about the carrier

$$P_{SBC}(\Delta \omega) \approx 10 \cdot \log \left( \frac{I_m c_m}{4q_{max} \Delta \omega} \right)^2 \tag{2.16}$$

The foregoing result may be extended to the general case of a white noise source

$$P_{SBC}(\Delta \omega) \approx 10 \cdot \log \left( \frac{\sum_{n=0}^{\infty} c_m^2}{4q_{max}^2 \Delta \omega^2} \right) \tag{2.17}$$

Equation 2.17 implies both upward and downward frequency translations of noise into the noise near the carrier. This equation summarizes what the foregoing equations tell us: components of noise near integer multiples of the carrier frequency all fold into noise near the carrier itself. Noise near dc gets up converted, weighted by coefficient $c_0$, so
1/f device noise ultimately becomes 1/f^3 noise near the carrier; noise near the carrier stays there, weighted by c_1; and white noise near higher integer multiples of the carrier undergoes down conversion, turning into noise in the 1/f^2 region. It is clear from Fig 2.11 that minimizing the various coefficients c_n (by minimizing the ISF) will minimize the phase noise. To underscore this point quantitatively, we may use Parseval’s theorem to write

$$\sum_{n=0}^{\infty} c_n^2 = \frac{1}{\pi} \int_0^{2\pi} |\Gamma(x)|^2 dx = 2\Gamma_{rms}^2$$

(2.18)

so that the spectrum in the 1/f^2 region may be expressed as

$$L(\Delta\omega) = 10 \cdot \log \left( \frac{\xi}{\Delta f \Gamma_{rms}^2} \right)$$

(2.19)

where \( \Gamma_{rms} \) is the rms value of the ISF.

This model can also be used to explain the \( \omega_{1/f^3} \) corner frequency in the phase noise spectrum. Assuming that the current noise behaves as follows in the 1/f region.
where $\omega_{1/f}$ is the $1/f$ corner frequency. Using Eqn 2.19, we obtain the following for the noise in the $1/f^3$ region:

$$L(\Delta \omega) = 10 \cdot \log \left( \frac{\frac{\pi}{2\Delta f} \frac{c_0^2}{\Delta \omega}}{\frac{8q_{max}^2}{\Delta \omega}} \cdot \frac{\omega_{1/f}}{\Delta \omega} \right)$$

(2.21)

which describes the phase noise in the $1/f^3$ region. The $1/f^3$ corner frequency is then

$$\Delta \omega_{1/f^3} = \omega_{1/f} \cdot \frac{c_0^2}{4l_{rms}^2} = \omega_{1/f} \cdot \left( \frac{\Gamma_{dc}}{\Gamma_{rms}} \right)^2$$

(2.22)
We note from the above equation that on lowering the dc value of $\Gamma$ ie, $\Gamma_{dc}$, the corner frequency is reduced and hence the the power spectral density curve has a more thinner central lobe. From equation 2.22 we get a design idea ie, if a differential topology is adopted then due to symmetry, the dc components of the two arms of the differential topology (as shown in Fig. 2.12) cancel each other significantly thereby resulting in a lower value of $\Gamma_{dc}$. Further, we also note that in order to maintain this symmetry, the noise currents from the tail current source has to be kept very low, since the noise currents coming from the tail capacitor are random and hence, can change $\Gamma_x$ which depends on the time instant at
which the noise current is applied. To reduce this noise current from the tail capacitor, a tail capacitance can be used as shown in Fig 2.12.

**Limitations of the LTV Theory**

The LTV theory though providing a good tool for explaining the phase noise spectrum in oscillators especially the $1/f^3$ region, suffers from a number of shortcomings

1. It assumes oscillators are inherently linear time variant, but does not give a concrete reason for this

2. It is based on the parameter $\Gamma(x)$ which is very difficult to determine

3. It does not give an in depth view of oscillator design

**2.3.4 Perturbation Models**

Kaertner [24] and Demir [11] [1] obtain differential equations for describing the frequency and amplitude response of oscillators through perturbation techniques. Due to the involved nature of their work, we shall not go into the details. However, the salient features of both their works is as follows.

1. They obtain differential equations describing the amplitude and phase deviations of the oscillator in terms of Taylor series expansions, assuming the underlying device noise can be completely described stochastically.

2. The stochastic differential equations so obtained are solved to obtain the final expression of phase noise.

3. Since flicker noise is difficult to characterize in time domain, they obtain approximates series solutions.
4. The models depend on complex parameters and have no circuit focus.

5. They require special tools and efficient algorithms to evaluate the model parameters.

So in conclusion, as we stated in the previous chapter, in this work we intend to solve some of the problems with the previous models. As we shall see in the succeeding chapters, though our model follows the general philosophy of Kaertner [24] and Demir [11] and [1], it does have a strong circuit focus. The parameters involved in the model will be easy to measure. Further, our approximation that flicker noise is equivalent to variation of device bias point will lead to a closed form solution for Phase Noise arising due to flicker noise.

2.4 Simulation Tools

The results obtained from our model will be compared with simulation results. The primary simulation tool we use is Agilent ADS. This is a powerful tool and performs a number of simulations like Harmonic Balance(HB), S-Parameter Analysis(SP), DC and Transient. While good for simulation, it has limited layout drawing capabilities.

On the other hand a tool like Cadence Design System allows us to draw the layout of a circuit. The TSMC oscillator described in Chapter 8 was laid out using the Virtuoso tool in Cadence. It has the ability to perform a number of standard analyses like HB, SP, DC. However the simulations are time intensive and inconvenient. So in summary, for the initial simulations of the TSMC oscillator, ADS was used and then Cadence was used for the layout.
Fig 3.1 shows a negative resistance oscillator model. The basic oscillator has been divided into a linear frequency sensitive part (having admittance $Y_L(\omega)$) and a non linear or device part (which is both frequency and amplitude sensitive and has admittance given by $Y_{IN}(A, \omega)$). In a conventional LC oscillator the linear part usually represents the tank. For all derivations henceforth we shall consider the tank to be a parallel $RLC$ circuit. In steady state, in the absence of noise and other perturbing signals, the operating point ($A_0$, $\omega_0$) is given by,

$$Y_L(\omega_0) + Y_{IN}(A_0, \omega_0) = 0$$
$i_N(t)$ represents an equivalent noise source which arises due to noise sources in both the linear and the non-linear parts of the oscillator circuit and which will be extracted in the example shown later. It can be shown (Appendix A) that the phase and amplitude deviations ($\phi$ and $\delta A$) obey the following Langevin equations (3.1) & (3.2), when the oscillator is linearized about its operating point voltage amplitude ($A_0$) and frequency ($\omega_0$)[16].

\[
A_0 \delta A + \frac{d\delta A}{dt} |Y_T'(\omega_0)|^2 = i_{N1} B_T'(\omega_0) + i_{N2} G_T'(\omega_0)
\]

(3.1)

\[
A_0 \left[ \alpha \delta A + |Y_T'(\omega_0)|^2 \frac{d\phi}{dt} \right] = i_{N1} G_T'(\omega_0) - i_{N2} B_T'(\omega_0)
\]

(3.2)

with,

\[
\alpha = G_T'(\omega_0) G_{IN}'(A_0, \omega_0) + B_T'(\omega_0) B_{IN}'(A_0, \omega_0)
\]

\[
\beta = B_T'(\omega_0) G_{IN}'(A_0, \omega_0) - G_T'(\omega_0) B_{IN}'(A_0, \omega_0)
\]

and where $i_{N1}$ and $i_{N2}$ are given by,

\[
i_{N1} = \frac{2}{T} \int_{-T}^{T} i_N \cos(\omega t + \phi(t)) dt
\]

(3.3)

\[
i_{N2} = \frac{2}{T} \int_{-T}^{T} i_N \sin(\omega t + \phi(t)) dt
\]

(3.4)

Note that the processes $\delta A$ and $\phi$ are similar to Ornstein-Uhlenbech processes [14] except that they are correlated. The term $\alpha$ accounts for the correlation between $\phi$ and $\delta A$ (i.e., if $\alpha = 0$ there is no correlation). The terms $Y_T'(\omega_0)$ and $Y_{IN}'(A_0, \omega_0)$ are given by (more general treatment than in [16] which uses a frequency independent $Y_{IN}(A_0)$),

\[
Y_T'(\omega_0) = G_T'(\omega_0) + j B_T'(\omega_0) = \left. \frac{\partial Y_L}{\partial \omega} \right|_{\omega_0} + \left. \frac{\partial Y_{IN}}{\partial \omega} \right|_{A_0, \omega_0}
\]

(3.5)

\[
Y_{IN}'(A_0, \omega_0) = G_{IN}'(A_0, \omega_0) + j B_{IN}'(A_0, \omega_0) = \left. \frac{\partial Y_{IN}}{\partial \delta A} \right|_{A_0, \omega_0}
\]

(3.6)

represent the variation of the admittance with perturbation. If the voltage across the tank is given by

\[
A(t) = A_0 \cos(\omega_0 t + \phi) + \text{small harmonics}
\]

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then the autocorrelation of the voltage for stationary Gaussian noise processes is given by (see derivation in Appendix II),

\[ R_V(\tau) = \frac{1}{2} \left[ A_0^2 + R_{\delta A} \right] \cos(\omega_0 \tau) \exp \left[ -\sigma^2 + R_\phi(\tau) \right] \]  

(3.7)

\( R_{\delta A}(\tau) \) and \( R_\phi(\tau) \) represent the autocorrelation functions of the amplitude deviation \( \delta A \) and the phase deviation \( \phi \) respectively.

The first term in Eqn 3.7 which is proportional to \( A_0^2 \) is the PM noise. The second term involving \( R_{\delta A}(\tau) \) is the AM noise term. Equation 3.7 is derived neglecting a third contribution arising from the correlation between phase and amplitude noises. This is justified as the PM noise will be verified below to be dominant over AM noise: \( A_0 >> c \) so that it can be assumed to be dominant as well over this third contribution.

We shall first consider white noise sources. Flicker noise analysis will be discussed later. As shown in Appendix C,

\[ \text{If } S_{i^\nu}(f) = |e|^2, \ S_{i^\nu_1}(f) = S_{i^\nu_2}(f) = 2|e|^2 \]

As is shown in Appendices D and E the expressions for \( R_{\delta A}(\tau) \) & \( R_\phi(\tau) \) are derived from the Langevin equations (3.1) & (3.2) to be given by:

\[ R_{\delta A}(\tau) = \frac{|e|^2}{A_0 \beta} \exp (-\eta |\tau|) \]  

(3.8)

\[ R_\phi(\tau) = -\frac{|e|^2}{A_0 |Y'_f(\omega_0)|^2} \left( 1 + \frac{\alpha^2}{\beta^2} \right) |\tau| - \frac{\alpha^2 |e|^2}{\beta^3 A_0} \exp (-\eta |\tau|) \]  

(3.9)

where we introduce the frequency \( \eta \):

\[ \eta = \frac{A_0 \beta}{|Y'_f(\omega_0)|^2} \]  

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\[
R_V(\tau) = \frac{1}{2} \left[ A_0^2 + \frac{|e|^2}{A_0^2} \exp\left( -\eta |\tau| \right) \right] \cos(\omega_0 \tau) \exp \left[ -\frac{|e|^2}{A_0^2 |Y_T'(\omega_0)|^2} \left( 1 + \frac{\alpha^2}{\beta^2} \right) |\tau| \left( e^{-\eta |\tau|} - 1 \right) \right] 
\]  
(3.10)

According to the Wiener Khintchin theorem, the power spectral density of the voltage across the tank can now be obtained by taking the inverse Fourier Transform of \( R_V(\tau) \):

\[
S_V(\omega) = \mathcal{F}^{-1}\{ R_V(\tau) \}
\]

Note that \( S_V \) is directly observable on a spectrum analyzer. \( S_\phi(\omega) \) which is derived in Appendix E is the IEEE definition of phase noise [13] requires a phase detector. For \( \alpha = 0 \) (no correlation) a closed form solution under the form of a Lorentzian (defined below) is obtained for \( S_V \). No exact analytic solution is available for the Fourier Transform of \( R_V(\tau) \) when we account for the correlation between the amplitude and phase (\( \alpha \) non zero). However an approximate analytical Lorentzian solution can be obtained for two practical limit conditions:

\[
S_{V,ssb}(\omega) = A_0^2 \left[ \frac{m_1}{m_1^2 + (\omega - \omega_0)^2} \right] + c \left[ \frac{m_1 + \eta}{(m_1 + \eta)^2 + (\omega - \omega_0)^2} \right] 
\]  
(3.11)

with \( c = \frac{|e|^2}{A_0^2 \beta} \) and where we define \( m_1 \) as

\[
m_1 = \begin{cases} 
  m_{01} & \text{for } \Delta \omega >> \eta \text{ or for } \alpha = 0 \text{ and all } \omega \\
  m_{01} \times \left[ 1 + \left( \frac{\eta}{\eta} \right)^2 \right] & \text{for } \eta >> m_{01} \text{ and } \Delta \omega < \eta 
\end{cases}
\]

with \( m_{01} = \frac{|e|^2}{A_0^2 |Y_T'(\omega_0)|^2} \)

The asymptotic result for \( \Delta \omega > \eta \) is derived in Appendix H using the stationary phase approximation. If \( \eta \) is very large the second approximation involving the correlation factor \( (\alpha/\beta)^2 \) is the relevant choice.
For intermediate value of $\eta$ the Fourier transform of $R_V(\tau)$ can readily be calculated numerically. This is illustrated in Figure 3.2 for $\eta/(2\pi) = 3.5$ MHz. As is shown in Figure 3.2 $S_{V,ssb}(\omega)$ is seen to relax for $\Delta\omega >> \eta$ to the limiting Lorentzian with $\alpha = 0$. The voltage noise spectrum is no longer strictly a Lorentzian and an inflexion point is introduced in the PM voltage noise density at the frequency $\eta/(2\pi)$.

The expressions of $S_V$ obtained for $\alpha = 0$ (no correlation) are consistent with other published works [11][24]. A Lorentzian spectrum ensures that the total power of the oscillator remains finite. A $1/\Delta f^2$ spectrum of noise spectral density for all frequencies on the other hand implies infinite oscillator power. Note that for large values of $\Delta\omega = \omega - \omega_0$ compared to $m_1$, the spectrum can be approximated as

$$S_{V,ssb}(\omega) = \frac{A_0^2 m_1}{\Delta\omega^2} + \frac{c(m_1 + \eta)}{\Delta\omega^2}$$

Figure 3.2: Comparison of PM voltage noise spectrum with various Lorentzian approximations.
Integrating the phase noise over frequency it can be shown that,

$$\int_{-\infty}^{\infty} S_V(\omega) \, df = \int_{0}^{\infty} S_{V,\text{ssb}}(\omega) \, df = \frac{A_0^2 + c}{2}$$

which is the same as the power of a noiseless oscillator if AM white noise is neglected ($c = 0$).

Figure 3.3: Comparison of AM and PM white noise spectrum in a differential oscillator

Figure 3.3 compares the AM, PM and (AM+PM) white noise component of $S_{V,\text{ssb}}$ for a differential oscillator (introduced in Section V). On a logarithmic graph the approximate PM/AM Lorentzian spectra have corner frequencies given by $m_1/(2\pi) \approx 0.6 \text{ MHz}$ and $(m_1 + \eta)/(2\pi) \approx 4.2 \text{ MHz}$ respectively.

Since $A_0^2$ (PM noise) exceeds $c$ (AM noise) (by 20dB in Fig 3.3) $m_1/(2\pi)$ is the corner frequency of the total (AM+PM) white noise spectrum.
The inflexion point at $\eta$ observed in the PM noise spectrum is not easily observable in the total (AM+PM) noise spectrum due to the AM noise contribution. Agreement of the theory with the circuit simulator is verified to be within 0.6 dB at high frequency offsets ($\Delta \omega >> m_1$) for the circuit considered.

Both $m_1$ and $m_1 + \eta$ are proportional to $1/|Y'_T(\omega_0)|^2$. For a parallel tank, $|Y'_T(\omega_0)| \simeq 2C$ and is proportional to the tank $Q$. This shows that at large offsets $S_V(\omega)$ is proportional to $1/Q^2$, thus agreeing with Leeson’s model. However the equation derived above presents the voltage noise in terms of easily measurable parameters $Y'_T(\omega_0)$ and $Y'_N(A_0, \omega_0)$ in conventional harmonic balance simulation of oscillators.

The presence of the additional terms ($\alpha/\beta$) in $m_1$ provides greater accuracy in the expression for the Voltage noise density. The ratio $\alpha/\beta$ has to be as low as possible for reducing phase noise.

\[
\frac{\alpha}{\beta} = \frac{[G'_T \ B'_T] \cdot [G'_N \ B'_N]}{[G'_T \ B'_T] \times [G'_N \ B'_N]} = \frac{|Y'_T| \ |Y'_N| \cos \theta}{|Y'_T| \ |Y'_N| \sin \theta} = \cot \theta
\]

where $\theta$ is the angle between the complex vector $Y'_T$ and $Y'_N$. It results that when $\theta$ is $90^\circ$ the noise correlation is minimized: $\alpha/\beta = 0$. When $\theta$ is $0^\circ$ the noise the noise correlation is maximized: $\alpha/\beta = \infty$. This well known result was first inferred by Kurokawa [16] from the inspection of $S_\phi(\omega)$. One of the contribution of the present work for white noise is to introduce the correlation factor $\alpha/\beta = \cot \theta$ and to quantify the impact of the correlation on the overall phase noise spectra $S_V(\omega)$ of the oscillator. In the circuit considered the correlation term is verified in Fig. 3.2 to bring a shift on the order of $\pm 3.8$ dB for frequencies below $\eta/(2\pi)$.

Kurokawa [16] provided a graphical interpretation of the correlation factor. In the limit where $Y'_T = Y'_L$, (active devices contributing minimally to the tank Q factor), $\theta$ is the angle at the oscillator operating point $(\omega_0, A_0)$ between the locus of the device admittance...
line \( Y_{IN}(A, \omega_0) \) as a function of \( A \) and the locus of the circuit admittance line \( Y_L(\omega) \) as a function of \( \omega \). Therefore in the limit where \( Y'_T = Y'_L \) holds the oscillator noise is respectively maximized or minimized when the circuit and device admittance lines are tangent or perpendicular to one another.

For an ideal high Q parallel tank (\( G'_T = G'_L = 0 \)) we have, \( \frac{\eta}{\eta} = \frac{B'_{IN}(A_0, \omega_0)}{G'_{IN}(A_0, \omega_0)} \).

In other words to make the ratio small the variation of device conductance and susceptance with respect to amplitude should be very high and low respectively. This is the case with an active device operating below its \( f_T \) since such a device has a high conductance sensitivity with respect to amplitude. In Chapter 7 we will show how the white noise theory can be applied to a differential oscillator.

### 3.1 Obtaining the circuit parameters \( Y_T(\omega_0) \) and \( Y_{IN}(A_0, \omega_0) \)

Fig 3.4 shows the circuit used for obtaining the values of \( Y_T(\omega_0) \) and \( Y_{IN}(A_0, \omega_0) \).

A signal \( A \sin(\omega_0 t) \) is injected at terminal 3 of the element. The problem associated with measuring \( Y_{IN}(A, \omega) \) is that the value of \( Y_{IN} \) changes with amplitude and frequency. Hence the value of \( Y_{IN} \) at any particular configuration has to be measured by measuring the input voltage and current at that particular configuration. The circuit element in Fig 3.4 does just that. The S parameter matrix of this element is given as follows,

\[
S_{\text{FUNDAMENTAL}} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} \quad S_{\text{OTHER FREQUENCIES}} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

(3.12)

So a signal injected at a particular frequency \( \omega \) and amplitude \( A \) will pass from terminal 3 to 2, then pass through the tank, then pass through the device part. Due to the non linearities of the tank, the signal gets distorted. On the return path only the fundamental signal passes...
from terminal 1 to 4 while other signals are sent to 2. This way by collecting the signal (usually current) at terminal 4 the input and output characteristics of the nonlinear device can be obtained and so $Y_{IN}$ can be easily obtained. By changing $A$ about the operating point, $Y'_{IN}(A_0, \omega_0)$ can be obtained. Similarly by changing $\omega$ about the operating point, $\frac{\partial Y_{IN}}{\partial \omega}$ can be computed, which in turn leads us to $Y'_T(\omega_0)$.

**Figure 3.4:** Schematic of the circuit element used for measuring $Y_T(\omega_0)$ and $Y_{IN}(A_0, \omega_0)$
CHAPTER 4

BUFFER NOISE

Most oscillator circuits have some kind of 2-port buffers to stabilize the load impedance. This can range from a simple pad attenuator to a differential output buffer stage like in the circuit to be considered in Section V. The impact of a 2-port buffer on the oscillator is usually not discussed in details in the literature. Also most simulators neglect the presence of a noise floor introduced by the buffer in the output and shows the noise decreasing infinitely with increasing frequency offset.

Figure 4.1: Mechanism of Buffer Noise Addition

Consider the noisy 2-port equivalent circuit for the buffer circuit shown in Fig 4.1 which features the usual input-referred noise current $i_n$ and noise voltage $v_n$. 
Using the results given in Appendix G the Norton equivalent noise current \( i_{N,\text{buf}} \) injected by the buffer network in the oscillator circuit is given by

\[
i_{N,\text{buf}} = i_n + \frac{v_n}{Z_{in}} + Y_0 v_{n,R_L}
\]

with

\[
Z_{in} = \frac{1}{Y_{in}} = z_{11} - \frac{z_{12} z_{21}}{R_L + z_{22}}
\]

and

\[
Y_0 = \frac{1}{z_{21}} \left( \frac{z_{11}}{Z_{in}} - 1 \right)
\]

It results that the noise current power density \( \overline{i_{N,\text{buf}}^2} \) injected by the buffer in the oscillator is:

\[
\overline{i_{N,\text{buf}}^2} = \overline{i_u^2} + |Y_{in}|^2 \overline{v_n^2} + |Y_0|^2 \overline{v_{n,R_L}^2}
\]

where \( Y_C \) is the correlation admittance \( i_c = Y_C v_n \). In these expressions, \( i_u \) is the uncorrelated component and \( i_c \) the correlated noise component of the total buffer noise current

\[
i_n = i_u + i_c.
\]

The load noise \( \overline{v_{n,R_L}^2} \) and the buffer noises can themselves be expressed as:

\[
\overline{v_n^2} = 4k_B T R_n, \quad \overline{v_{n,R_L}^2} = 4k_B T R_L \quad \text{and} \quad \overline{i_u^2} = 4k_B TG_u.
\]

Thus an additional component \( \overline{i_{N,\text{buf}}^2} \) gets added to \( \overline{i_{N,0}^2} = |\epsilon|^2 \) in the total \( \overline{i_N^2} \) and the definition of \( m_1 \) changes to

\[
m_1 = \frac{(|\epsilon|^2 + |i_{N,\text{buf}}|^2)}{A_0^2 |Y_T(\omega_0)|^2} \left( 1 + \frac{\alpha^2}{\beta^2} \right)
\]

The equivalent voltage noise source \( v_n \) appearing at the input contributes directly to the input-referred noise floor. The total noise at the output of the buffer including the buffer noise is then:

\[
S_{V_{\text{out}}} (\omega) = A_v^2 \left( S_{V_{\text{ssb}}} (\omega) + \overline{v_{n,\text{buf}}^2} \right) = |A_v|^2 \left( S_{V_{\text{ssb}}} (\omega) + \overline{v_n^2} + \overline{v_{n,R_L}^2} \right) \left( \frac{z_{11}}{z_{21}} \right)^2 \quad (4.1)
\]
Figure 4.2: Impact of buffer white noise on the phase noise of a differential oscillator

and where $A_v$ is the voltage gain provided by the buffer:

$$A_v = \frac{z_{21}R_L}{Z_{11}(R_L + z_{22}) - z_{12}z_{21}}$$

The derivation of $v_{N, buf}^2$ is shown in Appendix refappd. $v_n^2$ (i.e., $R_n$) is usually the leading term contributing to the noise floor. The impact of the noise floor is illustrated in Fig. 4.2 for a buffer noise floor of -130 dBV with two different white noise strength of $e^2(1) = 8.5 \times 10^{-18}$ A$^2$/Hz (unusually strong white noise) and $e^2(2) = 2.8 \times 10^{-23}$ A$^2$/Hz (normal weaker white noise). The corner frequency between white noise and noise floor is given by:

$$f_{2F} = \frac{1}{2} \sqrt{\frac{A_0^2 m_1}{v_{N, buf}^2}}$$

Note that $f_{2B}$ for white noise of usual strength $e^2(2)$ is on the order $\eta/(2\pi)$ and therefore $m_1$ should indeed be used instead of $m_0$ in the frequency range ($f < f_{2B}$) where white
noise dominates. Note also that of the 6 dB offset between model (plain line) and simulator
(+) results for $e^2(2)$, 4 dB are due to the new $(1 + \cot^2 \theta)$ correlation factor.
CHAPTER 5

EXTENDING THE MODEL TO FLICKER NOISE

In order to extend the Kurokawa analysis to 1/f noise we can study the effect of a variation of the DC current $i_N = I_0 + \delta i_N$ in any oscillator component, upon the active admittance $Y_{IN}$. Using a linearization scheme similar to the one we employed in the white noise case we get,

$$G_{IN}(A_0, \omega_0, i_N) = G_{IN}(A_0, \omega_0, I_0) + \frac{\partial G_{IN}}{\partial A} \delta A + \frac{\partial G_{IN}}{\partial \omega} \delta \omega + \frac{\partial G_{IN}}{\partial i_N} \delta i_N$$

$$B_{IN}(A_0, \omega_0, i_N) = B_{IN}(A_0, \omega_0, I_0) + \frac{\partial B_{IN}}{\partial A} \delta A + \frac{\partial B_{IN}}{\partial \omega} \delta \omega + \frac{\partial B_{IN}}{\partial i_N} \delta i_N$$

With the additional derivative term at the end, the master equations become,

$$\frac{|Y'_T(\omega_0)|^2 \delta A}{A_0} \frac{d\delta A}{dt} + \beta \delta A = B \delta i_N(t) \quad (5.1)$$

$$\frac{|Y'_T(\omega_0)|^2 \frac{d\phi}{dt}}{A_0} + \alpha \delta A = -A \delta i_N(t) \quad (5.2)$$

where the new constant $A$ and $B$ are given by:

$$A = G'_T(\omega_0)B^{iN'}_{IN}(A_0, \omega_0, I_0) - B'_T(\omega_0)G^{iN'}_{IN}(A_0, \omega_0, I_0)$$

$$B = G'_T(\omega_0)G^{iN'}_{IN}(A_0, \omega_0, I_0) + B'_T(\omega_0)B^{iN'}_{IN}(A_0, \omega_0, I_0)$$

with

$$Y^{iN'}_{IN}(A_0, \omega_0, I_0) = G^{iN'}_{IN}(A_0, \omega_0, I_0) + j B^{iN'}_{IN}(A_0, \omega_0, I_0) = \left. \frac{\partial Y_{IN}}{\partial i_N} \right|_{A_0, \omega_0, I_0}$$
\( \delta i_N(t) \) will be defined by its power density \( S_{\delta i_N,1/f}(\omega) = S/|\omega| \) as is developed in the next section.

### 5.1 Solving the differential equations and obtaining the expression for the voltage noise density

We use the autocorrelation function of the charge trapping model of the flicker noise to find the final noise voltage density as shown in [6]. The goal is to obtain a stationary model of flicker noise. We first assume the autocorrelation function is WSS and equal to (Ornstein-Uhlenbeck assumption [14]),

\[
R_{iN}(\tau) = ke^{-\lambda|\tau|} \quad (5.3)
\]

Taking the Fourier Transform of \( R_{iN}(\tau) \) gives the Noise density as,

\[
S_{iN}(\omega) = \frac{2\lambda k}{\omega^2 + \lambda^2}. \quad (5.4)
\]

This autocorrelation corresponds to a random telegraph noise which has a Lorentzian distribution. A superposition of many of these processes with time constants \( \tau_{\text{trap}}(y) = 1/\lambda = \tau_S \exp(\rho y) \) which are spatially varying with position \( y \) in the oxide (MOS) or wide-bandgap (HFET), will result in a noise process with a \( 1/f \) distribution:

\[
S_{iN,1/f} = \int_0^{d_{\text{max}}} R_{iN}(\tau) dy = -\int_{\lambda_0}^{\lambda_1} R_{iN}(\tau) \frac{d\tau}{\lambda} = \frac{2}{\rho\omega} \left[ \tan^{-1}\left( \frac{\omega}{\lambda_0} \right) - \tan^{-1}\left( \frac{\omega}{\lambda_1} \right) \right] \approx \frac{k\pi}{\rho\omega} = \frac{S}{\omega} \quad \text{for } \lambda_1 < \omega < \lambda_0
\]

where \( \lambda_0 = 1/\tau_S \approx \infty, \lambda_1 = 1/\tau_{\text{trap}}(d_{\text{max}}) \approx 0 \) and \( S = k\pi/\rho \). The superposition is valid since we assume the different traps behave independently.

We take the Fourier transform of Eqns 5.1 & 5.2 and express the RHS of each equation in a simplified form.

\[
\frac{|Y_T^f(\omega)|^2}{A_0} j\omega \delta A(\omega) + \beta \delta A(\omega) = B \delta i_N(\omega) \quad (5.5)
\]
\[ |Y'_f(\omega)|^2 j\omega \phi(\omega) + \alpha \delta A(\omega) = -A \delta i_N(\omega) \] (5.6)

From Eqn (5.5) we obtain,

\[ S_{\delta A}(\omega) = \frac{2B^2\lambda k}{(\lambda^2 + \omega^2) \left( \beta^2 + \frac{\omega^2|Y'_f(\omega)|^4}{A_0^2} \right)} \] (5.7)

Here we have used the expression for \( R_n(\tau) \) from Eqn 5.4 to calculate \( S_{i_N}(\omega) \). From Eqn 5.6 we get,

\[ S_{\phi,\text{trap}}(\omega) = \frac{2\lambda k}{\omega^2|Y'_f(\omega)|^4(\lambda^2 + \omega^2)} \left[ \frac{\omega^2|Y'_f(\omega)|^4A^2}{A_0^2} + (\beta A + \alpha B)^2 \right] \]

The expression above can be simplified as

\[ S_{\phi,\text{trap}}(\omega) = K \left[ \frac{\omega^2 + k_1}{\omega^2(\omega^2 + k_2)(\omega^2 + k_3)} \right] \] (5.8)

with, \( K = 2\lambda k\kappa^2, \kappa = A/|Y'_f(\omega)|^2, k_1 = A_0^2(\beta A + \alpha B)^2/(A^2|Y'_f(\omega)|^4), k_2 = \lambda^2 \) and \( k_3 = A_0^2\beta^2/|Y'_f(\omega)|^4 = \eta^2 \). Equation 5.8 represents the IEEE definition of phase noise for a single trap as shown in [13]. The exact value of \( S_{\phi} \) for all traps is obtained using the following expression.

\[ S_{\phi}(\omega) = -\int_{\lambda_0}^{\lambda_1} S_{\phi,\text{trap}}(\omega) \frac{d\lambda}{\lambda} = \frac{2k\kappa^2(\omega^2 + k_1)}{\omega^2(\omega^2 + k_2)(\omega^2 + k_3)} \left[ \tan^{-1} \left( \frac{\lambda_0}{\omega} \right) - \tan^{-1} \left( \frac{\lambda_1}{\omega} \right) \right] \] (5.9)

Decomposing Eqn (5.8) into partial fractions we obtain,

\[ S_{\phi,\text{trap}}(\omega) = K \left[ \frac{n_1}{\omega^2} - \frac{n_2}{k_2 + \omega^2} - \frac{n_3}{k_3 + \omega^2} \right] \] (5.10)

Where, \( n_1 = k_1/(k_2k_3), n_2 = (k_1 - k_2)/k_2(k_3 - k_2), \) and \( n_3 = (k_1 - k_3)/k_3(k_2 - k_3). \)

From which we obtain,

\[ R_{\phi,\text{trap}}(\tau) = \mathcal{F}^{-1} \left[ \phi^2(\omega) \right] = K \left[ -\frac{n_1|\tau|}{2} - \frac{n_2e^{-\sqrt{k_2}|\tau|}}{2\sqrt{k_2}} - \frac{n_3e^{-\sqrt{k_3}|\tau|}}{2\sqrt{k_3}} \right] \] (5.11)
Now, \( \sigma^2 = R_{\phi,1trap}(0) = K \left[ -\frac{n_2}{2\sqrt{k_2}} - \frac{n_3}{2\sqrt{k_3}} \right] \). Eqn (5.7) can be expanded into partial fractions as,

\[
S_{\delta A,1trap}(\omega) = K' \frac{1}{(\omega^2 + k_3)(\omega^2 + k_2)} = \frac{K'}{k_2 - k_3} \left[ \frac{1}{\omega^2 + k_3} - \frac{1}{\omega^2 + k_2} \right] \tag{5.12}
\]

where \( K' = \frac{2B^2\lambda k A_0^2}{(2\pi^2)} \) and \( k_2 = \lambda^2 \). Taking the Fourier Transform of \( S_{\delta A}(\omega) \), we obtain \( R_{\delta A,1trap}(\tau) \) as,

\[
R_{\delta A,1trap}(\tau) = \frac{K'}{2(k_2 - k_3)} \left[ \frac{e^{-\sqrt{k_3} |\tau|}}{\sqrt{k_3}} - \frac{e^{-\sqrt{k_2} |\tau|}}{\sqrt{k_2}} \right] \tag{5.13}
\]

Hence,

\[
R_{V,1trap}(\tau) = \frac{A_0^2 + R_{\delta A}(\tau)}{2} \cos(\omega_0 \tau) \exp \left[ R_\phi - \sigma^2 \right]
\]

\[
= \frac{A_0^2 + R_{\delta A}(\tau)}{2} \cos(\omega_0 \tau) \exp \left[ -\frac{K}{2} \left( n_1 |\tau| + \frac{n_2}{\sqrt{k_2}} (e^{-\sqrt{k_2} |\tau|} - 1) + \frac{n_3}{\sqrt{k_3}} (e^{-\sqrt{k_3} |\tau|} - 1) \right) \right]
\]

\[
= \frac{A_0^2}{2} \cos(\omega_0 \tau) \exp \left[ -\frac{K}{2} \left( n_1 |\tau| + \frac{n_2}{\sqrt{k_2}} (e^{-\sqrt{k_2} |\tau|} - 1) + \frac{n_3}{\sqrt{k_3}} (e^{-\sqrt{k_3} |\tau|} - 1) \right) \right]
\]

\[
+ \frac{K'}{4\sqrt{k_3}(k_2 - k_3)} \cos(\omega_0 \tau) \exp \left[ -\frac{K}{2} \left( n_1 |\tau| + \frac{n_2}{\sqrt{k_2}} (e^{-\sqrt{k_2} |\tau|} - 1) + \frac{n_3}{\sqrt{k_3}} (e^{-\sqrt{k_3} |\tau|} - 1) \right) \right]
\]

\[
- \frac{K'}{4\sqrt{k_2}(k_2 - k_3)} \cos(\omega_0 \tau) \exp \left[ -\frac{K}{2} \left( n_1 |\tau| + \frac{n_2}{\sqrt{k_2}} (e^{-\sqrt{k_2} |\tau|} - 1) + \frac{n_3}{\sqrt{k_3}} (e^{-\sqrt{k_3} |\tau|} - 1) \right) \right]
\]

\[
\tag{5.14}
\]

As shown in Appendix I using the method of stationary phase the Inverse Fourier Transform of \( R_{\phi,1trap}(\tau) \) for large frequency offset \( \Delta \omega = \omega - \omega_0 \) is given by (using a single side band representation)

\[
S_{V_{1,trap,sub}}(\omega)
\]

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\[ R_{\phi,1/f}(\tau) = -\int_{\lambda_0}^{\lambda_1} R_{\phi,\text{trap}}(\tau) \frac{d\lambda}{\lambda} \]

This expression gives the noise spectral density at large frequency offset for a single trap (i.e. for a particular value of \( \lambda \)). Note that Appendix I can be applied to derive the PM component because the terms \( q_1, q_2, \) and \( q_3 \) which can be identified for a single trap in \( R_{V,\text{trap}} \) verify \( q_1 + q_2 + q_3 = 0 \) thus satisfying the required property \( \sum_k q_k = 0 \).

To obtain the tank voltage 1/f noise spectrum we need first to consider all the traps at once when calculating the phase noise \( R_{\phi} \)

\[ R_{\phi,1/f}(\tau) = -\int_{\lambda_0}^{\lambda_1} R_{\phi,\text{trap}}(\tau) \frac{d\lambda}{\lambda} \]

before we calculate \( R_{V,1/f} \) and \( S_{V,1/f} \). It turns out that a closed form expression for \( R_{\phi,1/f}(\tau) \) can be obtained using among other thing the exponential integral function \( Ei(x) \).

The expression is given below,

\[
R_{\phi,1/f}(\tau) = \frac{K''}{2\rho} \left[ k_1 \left( \frac{1}{\lambda_0} - \frac{1}{\lambda_1} \right) + \frac{k_1}{k_3} \left\{ -\exp(-\lambda_1\tau) - \tau Ei(\lambda_1\tau) + \exp(-\lambda_0\tau) - \tau Ei(\lambda_0\tau) \right\} \right.
\]
\[
\left. + \frac{k_1 - k_3}{2k_3} \left\{ \exp(-\sqrt{k_3\tau}) \left( -Ei(\tau(\lambda_1 + \sqrt{k_3})) + Ei(\tau(\lambda_0 + \sqrt{k_3})) \right) - \exp(\sqrt{k_3\tau}) \left( -Ei(\tau(\lambda_1 - \sqrt{k_3})) + Ei(\tau(\lambda_0 - \sqrt{k_3})) \right) \right\} \right]
\]
\[
+ \frac{k_1 - k_3}{2k_3} \exp(-\sqrt{k_3\tau}) \left\{ \ln \left| \frac{\lambda_1 - \sqrt{k_3}}{\lambda_0 - \sqrt{k_3}} \right| - \ln \left| \frac{\lambda_1 + \sqrt{k_3}}{\lambda_0 + \sqrt{k_3}} \right| \right\} \]
\]

(5.16)

where \( K'' = K/\lambda \). This analytic solution \( R_{\phi,1/f}(\tau) \) is however quite useful for verification purpose as we can then obtain the exact voltage noise density by calculating numerically its inverse Fourier transform. Similarly an exact expression for \( R_{\delta A,1/f}(\tau) \) can be obtained
as,

\[
R_{\delta, A, 1/f}(\tau) = -\frac{K''}{4}\left[\exp\left(-\frac{\sqrt{k_3}|\tau|}{k_3}\right)\left\{\ln\left|\frac{\lambda_1 - \sqrt{k_3}}{\lambda_0 - \sqrt{k_3}}\right| - \ln\left|\frac{\lambda_1 + \sqrt{k_3}}{\lambda_0 + \sqrt{k_3}}\right|\right\} \\
- \frac{1}{k_3}\left\{\exp(-\sqrt{k_3}|\tau|)\left(Ei((\lambda_1 - \sqrt{k_3})|\tau|) - Ei(\lambda_0 - \sqrt{k_3})|\tau|\right) \\
- \exp(\sqrt{k_3}|\tau|)\left(Ei((\lambda_1 + \sqrt{k_3})|\tau|) - Ei(\lambda_0 + \sqrt{k_3})|\tau|\right) \\
- (Ei(\lambda_1|\tau|) - Ei(\lambda_0|\tau|))\right\}\right]
\]

(5.17)

where \(K'' = K'/\lambda\). An analytic expression for \(S_{V, 1/f}\) valid for large frequency offsets can also be obtained using the method of stationary phase. This is due to the fact that the key assumption made in Appendix I namely \(\sum_k q_k = 0\) still holds when summing (integrating) over all the traps. This follows from the dependence of \(R_\phi\) on time. It can be shown that a linear relationship between \(R_\phi\) and time will ensure that the superposition of the \(S_V\) values of a number of traps is the same as that obtained by superposing the noise contributions of the traps first and then obtaining \(S_V\). As seen from Figure 5.1 the relationship is indeed linear. It results by applying the results of Appendix I that for large enough frequency offsets the voltage noise density of flicker noise is simply obtained by averaging over all values of \(\lambda\) the expression obtained for the Voltage Noise Density of a single trap noise:

\[
S_{V, 1/f, \text{stab}}(\omega) = \frac{1}{\rho} \int_{\lambda_0}^{\lambda_1} S_{V, \text{trap}}(\omega) \frac{(-d\lambda)}{\lambda} \\
= \left\{\frac{A_0^2 K'' k_1 + \Delta\omega^2}{2\rho \Delta\omega^2 k_3 + \Delta\omega^2} + \frac{K''}{2\rho k_3 + \Delta\omega^2}\right\} \times \frac{1}{\lambda_2 + \Delta\omega^2} \int_{\lambda_0}^{\lambda_1} \frac{1}{\lambda^2 + \Delta\omega^2} (-d\lambda) \\
= \left\{\frac{A_0^2 K'' k_1 + \Delta\omega^2}{2\rho \Delta\omega^2 k_3 + \Delta\omega^2} + \frac{K''}{2\rho k_3 + \Delta\omega^2}\right\} \times \frac{1}{\Delta\omega} \times \left[\tan^{-1}\left(\frac{\lambda_0}{\Delta\omega}\right) - \tan^{-1}\left(\frac{\lambda_1}{\Delta\omega}\right)\right]
\]

(5.18)
where, $K'' = K/\lambda$ and $K''' = K'/\lambda$. In the limit of $\lambda_0 = \infty$ and $\lambda_1 = 0$ this reduces to

$$S_{V,1/f,ssb}(\omega) = \frac{A_0^2 S K^2}{2} \left( \frac{k_1 + \Delta \omega^2}{\Delta \omega^3(\eta^2 + \Delta \omega^2)} \right) + \frac{A_0^2 S B^2}{2|Y_f(\omega_0)|^4} \frac{1}{\Delta \omega(\eta^2 + \Delta \omega^2)}$$

Figure 5.2 shows the analytic expressions obtained for AM+PM 1/f noise (plain line) and for AM 1/f noise (dotted line). The test circuit used is the differential oscillator to be discussed in Section V. The voltage noise spectra reported are for 1/f noise originating in the tail transistor (later referred at Mode E). Also shown in Figure 5.2 are the results obtained for AM+PM 1/f noise (dashed line) for the uncorrelated case ($\alpha = 0$). Clearly correlation can play an important role for mode E as it leads to a 30 dB decrease in noise in the $1/\Delta f^3$ region. Figure 5.3 and 5.4 show the same results for Mode A and Mode C. Figure

At low offset (here below 1 Hz) the stationary phase approximation fails as indicated by the exact numerical results (circles). A simple estimate of the ceiling voltage noise
Figure 5.2: Comparison of analytic expression for AM+PM and AM with exact numerical solution and simulator results for Mode E.

density $S_{V,1/f} (\text{ceiling})$ and associated corner frequency $\Delta f_{1/f} (\text{ceiling})$ can be obtained by enforcing power conservation and assuming a $1/\Delta f^3$ spectrum up to the corner frequency:

$$S_{V,1/f} (\text{ceiling}) = 0.6 \times \frac{A_0^3}{\sqrt{F}} \quad \text{with} \quad F = \frac{A_0^2 S_i \kappa^2 k_i}{2 \eta^2}$$

$$\Delta f_{1/f} (\text{ceiling}) = \frac{1}{2\pi} \left( \frac{F}{S_{V,1/f} (\text{ceiling})} \right)^{\frac{1}{3}}$$

Agreement with circuit simulation is only of 7 dB for mode E. Better agreement will be obtained for other dominant 1/f modes (A and C) in Section V when correlation brings a weaker correction.

5.2 Obtaining the values of $G_{I,N}^{iN'}(A_0, \omega_0)$ and $B_{I,N}^{iN'}(A_0, \omega_0)$

The test setup described in chapter 3 is used to perturb the oscillator about its oscillating point. Only a single variable is changed at a time (1 dimensional perturbation). The
corresponding values of $Y_{1N}$ are noted. A parabolic least square fit of the derivative is then made to obtain accurately the derivatives. Figure 5.6 shows how the derivative converges using this method.
Figure 5.4: Comparison of analytic expression for AM+PM and AM with exact numerical solution and simulator results for Mode C.

Figure 5.5: Comparison of analytic expression for AM+PM and AM with exact numerical solution and simulator results for all 3 modes summed. In other words this is a comparison of the analytic model with simulation results for flicker noise as a whole.
Figure 5.6: Figure shows how the value of $Y_{IN}$ changes with $i_N$. Also it computes the derivative of $Y_{IN}$ for various $di_N$. 
CHAPTER 6

APPLYING THE MODEL ON A VAN DER POL OSCILLATOR

The Van Der Pol oscillator is shown in Fig 6.1. The non linear current \( I(v) \) is given as

\[
I(v) = -\frac{v}{R} \left( 1 - \frac{a_1}{3}v^2 \right)
\]

The circuit has been divided into 2 parts (a nonlinear part and a linear part ) by the line LL'. \( Y_L(\omega) \) represents the admittance of the linear part , while \( Y_{IN}(A,\omega) \) represents the
admittance of the non linear part. Now let us calculate the values $Y_L(\omega_0)$ and $Y_{IN}(A_0, \omega_0)$

\[
Y_L(\omega) = i\omega C + \frac{1}{j\omega L} = i\omega C \left[1 - \frac{1}{\omega^2 LC}\right] \\
\Rightarrow Y_L(\omega_0) = i\omega_0 C \left[1 - \frac{1}{\omega_0^2 LC}\right] = 0
\]

Hence at the frequency of oscillation, the admittance of the linear part is 0. The Kurokawa coefficient $Y'_L(\omega_0)$ is obtained from the equation above as,

\[
Y'_L(\omega_0) = i2C
\]

From which we get $G'_L(\omega_0) = 0$ and $B'_L(\omega_0) = 2C$.

To compute the admittance $Y_{IN}(A, \omega)$, we proceed as follows. Let us inject a voltage $V = A \sin(\omega_0 t)$ into the nonlinear part. Then

\[
I(v) = -\frac{A}{R} \left(1 - \frac{a_1 A^2}{3} \sin^2(\omega_0 t)\right) \sin(\omega_0 t)
\]

We can assume that since the system is time invariant, the harmonics of $I(v)$ will be periodic with a fundamental frequency $\omega_0$. Taking a Fourier expansion of $I(v)$ we get,

\[
I(v) = I_0 + I_1 \sin(\omega_0 t) + I_2 \sin(2\omega_0 t) + ....
\]

where, $I_0, I_1, I_2 ...$ are the coefficients of the harmonics terms of $I(v)$. Then $Y_{IN}(A, \omega)$ is given by,

\[
Y_{IN}(A, \omega) = \frac{I_1}{A}
\]

The coefficient of the fundamental component $I_1$ is given by

\[
I_1 = \frac{2}{T} \int_0^T I(v) \sin(\omega_0 t) dt = \frac{2}{T} \int_0^T -\frac{A}{R} \left(1 - \frac{a_1^2}{3} \sin^2(\omega_0 t)\right) \sin^2(\omega_0 t) dt
\]

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We can see that $Y_{IN}(A, \omega)$ is independent of $\omega$. Now the Kurokawa coefficient $Y'_{IN}(A_0, \omega_0)$ can be obtained from Eqns 6.1 as,

$$
Y'_{IN}(A, \omega) = \frac{a_1 A}{2R} \Rightarrow Y'_{IN}(A_0, \omega_0) = \frac{a_1 A_0}{2R}
$$

From the result of $A_0$ obtained later we will see that $Y'_{IN}(A_0, \omega_0)$ can be expressed as

$$
Y'_{IN}(A_0, \omega_0) = \frac{\sqrt{a_1}}{R}
$$

Thus $G'_{IN}(A_0, \omega_0) = \frac{\sqrt{a_1}}{R}$ and $B'_{IN}(A_0, \omega_0) = 0$. At the point of oscillation,

$$
Y_{IN}(A_0, \omega_0) = -Y_L(\omega_0) \Rightarrow Y_{IN}(A_0, \omega_0) = 0 \Rightarrow -\frac{1}{R} \left[ 1 - \frac{a_1 A^2}{4} \right] = 0 \Rightarrow A_0 = \frac{2}{\sqrt{a_1}}
$$

Now in our model

\[
\alpha = G'_L(\omega_0)G'_{IN}(A_0, \omega_0) + B'_L(\omega_0)B'_{IN}(A_0, \omega_0)
\]
\[
= 0
\]

\[
\beta = B'_L(\omega_0)G'_{IN}(A_0, \omega_0) - G'_L(\omega_0)B'_{IN}(A_0, \omega_0)
\]
\[
= 2C\frac{\sqrt{a_1}}{R}
\]

\[
|e| = \sqrt{\frac{4kT}{R}}
\]
\[ S_V(\omega) = \frac{A_0^2}{2} \left( \frac{m_1}{m_1^2 + \Delta \omega^2} \right) + \frac{c}{2} \left( \frac{m_1 + m_2}{(m_1 + m_2)^2 + \Delta \omega^2} \right) \]

\[ = \frac{2}{a_1} \left( \frac{4kT a_1 RC^2}{(kT a_1)^2 + (4RC^2) \Delta \omega^2} \right) + \frac{kT}{2C} \left( \frac{4RC^2(kT a_1 + 4C)}{(kT a_1 + 4C)^2 + (4RC^2) \Delta \omega^2} \right) \]

We note that if we consider the deviation of amplitude of oscillation to be negligible, then the S portion of Eqn 6.3 is 0 and for large values of \( \Delta \omega \),

\[ S_V(\omega) \approx \frac{1}{2} \frac{kT}{RC^2 \Delta \omega^2} \quad (6.4) \]

We assume the values of \( L, C, R \) and \( a_1 \) are, \( L = 50pH, C = 141fF, R = 377\Omega \) and \( a_1 = 3 \). We have implemented this circuit in both ADS and Spectre. Fig 6.2 shows the comparison of the results obtained in various models. Fig 6.3 shows the circuit used to obtain the phase noise in Cadence while Fig 6.4 shows the one used in ADS. We see that our model matches well with the simulation results obtained using Spectre (less than 0.035 dB) and ADS (pnfm) (less than 0.15 dB) at high frequencies. At low frequencies (less than 100 Hz), the shape of the PSD in our model is a Lorentzian. By not approximating
to a $1/f^2$ model we ensure that the total power in the oscillator signal is finite. The noise PSD integrated over all frequencies i.e. the total power comes out to be $\frac{A_0^2}{2}$ ($A_0$ is the steady state amplitude of oscillation) which is the same as the power of a noiseless oscillator.
Figure 6.3: Cadence implementation circuit

Figure 6.4: ADS implementation circuit
CHAPTER 7

COMPARISON OF THE PROPOSED MODEL WITH A PRACTICAL DIFFERENTIAL OSCILLATOR

Let us consider the differential oscillator shown in Fig 7.1. The tank is selected to form the linear part of the oscillator while the remaining part of the circuit forms the non-linear part.

![Figure 7.1: A differential oscillator with buffer stage](image)

7.1 Mode Analysis of the oscillator

The Kurokawa analysis is typically used in harmonic balance simulators for calculating the oscillator operating point \((A_0, \omega_0)\) using \(\Gamma_L(\omega_0) \times \Gamma_{IN}(A_0, \omega_0) = 1\). Consequently the
various non-linear device and linear tank admittance are readily available for applying the
circuit based noise model presented above. However a method for calculating the equiva-
lent noise source appearing across the tank is needed to apply the noise theory developed
in the previous sections for the white noise analysis. The methodology introduced for this
purpose will also facilitate the 1/f analysis by reducing the number of independent modes
(noise sources) considered from 5 to 3.

Fig 7.1 shows the schematic of a simple differential oscillator having four 'core' tran-
sistors. $\overline{i_{p1}^2}$ and $\overline{i_{p2}^2}$ represent the noise produced by PMOS transistors while $\overline{i_{n1}^2}$ and $\overline{i_{n2}^2}$
represent the noise produced by the NMOS transistors. Note that for white noise the gate
and drain noise currents (see for example [22])

$$\overline{i_{p/n,d}^2} = 4k_BT\gamma_{MOS}g_m$$

and

$$\overline{i_{p/g}^2} = 4k_BT\delta_{MOS}g_g$$

of opposite NMOS and PMOS transistors (n1 & n2 or p1 & n2) appear in parallel due to
the circuit topology such that we have for example $\overline{i_{p1}^2} = \overline{i_{p1,d}^2} + \overline{i_{p2,g}^2}$. For the rest of the
analysis we define $|\overline{i_{p1}|^2 = |\overline{i_{p2}|^2 = a and }|\overline{i_{n1}|^2 = |\overline{i_{n2}|^2 = b}$. For 1/f noise we similarly
have

$$a = (S_{p1,d} + S_{p2,g})/|\omega|$$

and

$$b = (S_{n1,d} + S_{n2,g})/|\omega|.$$  

The first step for the mode analysis is to split the instantaneous noise currents of the
four transistors into uncorrelated unit mode currents.

Let $N$ be the matrix which converts the transistor noise currents into these uncorrelated
mode currents. i.e.

$$i_{mode} = Ni_{trans},$$

where $i_{mode}$ and $i_{trans}$ are given by,

$$i_{mode} = \begin{bmatrix} i_{m1} \\
            i_{m2} \\
            i_{m3} \\
            i_{m4} \end{bmatrix}$$

and

$$i_{trans} = \begin{bmatrix} i_{p1} \\
            i_{p2} \\
            i_{n1} \\
            i_{n2} \end{bmatrix}$$

(7.1)
Since $i_{\text{mode}}$ consists of uncorrelated mode currents having unit power density, we have,

$$|i_{\text{mode}}|^2 = i_{\text{mode}} \cdot i_{\text{mode}}^T = N i_{\text{trans}} \cdot i_{\text{trans}}^T N^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7.2)$$

It can be shown that the matrix $N$, which leads to the above relation, is given by

$$N = \begin{bmatrix} \frac{1}{a_n} & \frac{1}{a_n} & \frac{1}{b_n} & \frac{1}{b_n} \\ \frac{1}{a_n} & \frac{1}{a_n} & \frac{1}{b_n} & \frac{1}{b_n} \\ c & c & -c & -c \\ c & -c & -c & c \end{bmatrix} \quad (7.3)$$

where,

$$a_n = a \sqrt{\frac{2}{a + b}}, \quad b_n = b \sqrt{\frac{2}{a + b}}, \quad c = \frac{1}{\sqrt{2(a + b)}}$$

Taking the inverse of $N$, we obtain,

$$N^{-1} = \begin{bmatrix} \mu & \mu & \nu & \nu \\ \mu & -\mu & \nu & -\nu \\ \mu & \mu & -\gamma & -\gamma \\ \mu & -\mu & -\gamma & \gamma \end{bmatrix}$$

where, $\mu$, $\nu$ and $\gamma$ are given by,

$$\mu = \sqrt{\frac{ab}{2(a + b)}}, \quad \nu = \frac{a}{\sqrt{2(a + b)}}, \quad \gamma = \frac{b}{\sqrt{2(a + b)}}$$

Now we can express $i_{\text{trans}}$ in terms of $i_{\text{mode}}$ as,

$$i_{\text{trans}} = N^{-1} i_{\text{mode}}$$

We have now expressed $i_{\text{trans}}$ in terms of a certain number of uncorrelated unity strength mode currents. Schematically these modes can be represented as shown in Fig 7.2.

As seen from Tables 7.1 & 7.2 when the noise currents are substituted by a single tone perturbation, each mode contributes differently to the noise current injected across the tank by either up-conversion or direct transfer. For modes A and C the largest contribution to the current $i_N$ across the tank at $f_0 + \Delta f$ is coming from the up-conversion of the input currents.
Figure 7.2: Various modes of a differential oscillator

\[ i_{mA/C} \Delta f \]. For modes B and D the largest contributions to the current \( i_N \) injected across the tank at \( f_0 + \Delta f \) is coming from the transfer of the input currents at \( i_{mB/D} \) at \( f_0 + \Delta f \). It results that modes B and D are most significant for white noise and modes A and C are most significant for \( 1/f \) noise. The tail current leakage of each of the individual mode noises is verified to be usually insignificant in a differential oscillator (the tail current source acts as an open for mode A, B, C and D).
The resultant current across the tank for modes B and D are \(2\mu\) and \((\nu - \gamma)\) respectively. The effect of individual modes can be accounted by adding their relative contributions to \(R_\phi\) and \(R_\delta\) respectively i.e for white noise,

\[
R_\phi = R_{\phi B} + R_{\phi D}
\]

\[
R_\delta = R_{\delta A} + R_{\delta D}
\]

The resulting voltage noise density \(S_V\) obtained will be the same as the one we obtained previously reported in Section 3 if \(|e|^2\) is replaced now by

\[
|e|^2 = |e_B|^2 + |e_D|^2
\]

<table>
<thead>
<tr>
<th>Input Noise offset</th>
<th>Mode A</th>
<th>Mode B</th>
<th>Mode C</th>
<th>Mode D</th>
<th>Mode E</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Delta f)</td>
<td>-96</td>
<td>-316</td>
<td>-84</td>
<td>-299</td>
<td>-89</td>
</tr>
<tr>
<td>(f_0 + \Delta f)</td>
<td>-700</td>
<td>-95</td>
<td>-700</td>
<td>-91</td>
<td>-700</td>
</tr>
<tr>
<td>(2f_0 + \Delta f)</td>
<td>-700</td>
<td>-700</td>
<td>-700</td>
<td>-700</td>
<td>-700</td>
</tr>
<tr>
<td>(3f_0 + \Delta f)</td>
<td>-700</td>
<td>-700</td>
<td>-700</td>
<td>-700</td>
<td>-700</td>
</tr>
</tbody>
</table>

Table 7.1: Response in dBV of the tank voltage at \(H_n\) to a single tone excitation injected in the circuit with 1KHz offset \(\Delta f\) from the various harmonics \(nf_0\).

If we consider the total \(i_N\) flowing across the tank due to modes B and D, we get,

\[
S_{i_N}(\omega) = 4\mu^2 + (\nu - \gamma)^2 = \frac{a + b}{2} = |e|^2
\]

For white noise this results is the same as the noise current density across the tank given in [3]. For 1/f noise we need to calculate \(G_{TN}^{iN}(A_0, \omega_0, I_0)\) and \(B_{TN}^{iN}(A_0, \omega_0, I_0)\) for mode A and C. One can select \(\delta_{iN}\) to be \(2\mu\) and \(\nu\) for mode A and C respectively. For mode
Table 7.2: Response in dBV of the tank voltage at $f_0 + \Delta f$ to a single tone excitation injected in the circuit with 1 MHz offset $\Delta f$ from the various harmonics $n f_0$.

C if we calculate $G_{IN}^{iN}(A_0, \omega_0, I_0)$ and $B_{IN}^{iN}(A_0, \omega_0, I_0)$ using $\delta_{iN} = \nu m_C = 1$ nA for the two top current sources in Figure 7.2 then the two bottom current sources are given by $\gamma m_C = \gamma / \nu \delta_{iN} = a / b \delta_{iN} = a / b \times 1$ nA. Note that for 1/f noise $a / b = S_p / S_n$ is frequency independent.

![Figure 7.3: Impact of the tank voltage at $f_0 + \Delta f$ of a 1 nA perturbation current at various offsets $\Delta f$ for Mode A through E](image-url)
Note that our model neglects the frequency dependence of the oscillator on the pertur-
bating current $\delta_{i_N}$. As can be verified in Figure 7.3, this is a reasonable assumption up to 1 MHz offset for the differential oscillator considered in this paper.

### 7.2 Impact of Tail Noise

For completeness we need to consider the impact of tail noise on the overall voltage noise density. We refer to the tail noise analysis as Mode E. However unlike the previous modes, the noise current of the tail current is directly the mode current. Tables 7.1 and 7.2 show the effect of a single frequency tone with 1KHz and 1MHz offset relative to dc, $\omega_0$ and $2\omega_0$ respectively. The results are similar to those obtained for cases A and C. In other words the up-converted noise dominates over the transfered noise.

### 7.3 Obtaining the circuit parameters $Y_T(\omega_0)$ and $Y_{IN}(A_0, \omega_0)$

The operating point ($f_0, A_0$) ≈ (2.466 GHz, 1.15 V||) for the oscillator was obtained using conventional harmonic balance simulation. An accuracy of 0.1 Hz and 8 $\mu$V was achieved. The convergence accuracy of the voltage and current was set to $1 \times 10^{-15}$ A and V respectively in the harmonic balance simulation. 8 harmonics were used in all oscillator simulations.

By changing $A$ about the operating point, $Y_{IN}'(A_0, \omega_0)$ can be obtained. Similarly by changing $\omega$ about the operating point, $\partial Y_{IN}/\partial \omega$ can be computed, which in turn leads us to $Y_T'(\omega_0)$. A quadratic convergence of the derivatives calculated was observed for shrinking derivative intervals. A least square fit was further used to remove any residual numerical noise. The maximum error bound (i.e., $|\Delta Y_{IN}'(A_0, \omega_0)/Y_{IN}'(A_0, \omega_0)|$) in the derivatives calculated was of $2 \times 10^{-6}$ % in relative magnitude.
7.4 Comparing Simulation results with proposed model for Combined White and 1/f Noise

In this section we consider both white and flicker noise together. As we have already seen when studying white noise and flicker noise to properly evaluate the combined effect of multiple independent noise processes on the oscillator output voltage spectral density we must first sum their respective phase and amplitude autocorrelations

\[
R_{\phi,\text{tot}}(\tau) = R_{\phi,\text{1/f}}(\tau) + R_{\phi,\text{white}}(\tau)
\]

\[
R_{\delta A,\text{tot}}(\tau) = R_{\delta A,\text{1/f}}(\tau) + R_{\delta A,\text{white}}(\tau)
\]

before calculating the voltage spectral density \(S_{V,\text{tot}}\). The commonly used approach which consists of summing the voltage spectral density of white noise and flicker noise

\[
S_{V,\text{tot}}(\tau) = S_{V,\text{1/f}}(\tau) + S_{V,\text{white}}(\tau)
\]
is an approximation. As we shall demonstrate below, the validity of this approximation will depend on the relative strengths of the $1/f$ noise and white noise processes.

Consider the two corner frequencies $\Delta f_{\text{white}}(\text{ceiling}) = \frac{m_1}{(2\pi)}$ and $\Delta f_{1/f}(\text{ceiling})$ at which the voltage spectral density reaches its ceiling value when respectively considering white noise or flicker noise separately.

When the corner frequency $\Delta f_{\text{white}}(\text{ceiling})$ of white noise is larger than the corner frequency $\Delta f_{1/f}(\text{ceiling})$ of flicker noise then no $1/\Delta f^3$ region will be present in the noise spectrum since the ceiling dictated by power conservation has already been reached. This is numerically verified to take place in Figure 7.6 in the presence of a strong white noise $e^2(3) = 8.5 \times 10^{-8}$ A$^2$/Hz when the total voltage density (circles) follows the white noise Lorentzian spectra and not the flicker noise spectra. Clearly in this strong white noise case
Figure 7.6: Comparison of the Voltage noise densities when white and flicker noise summed (circle, square, star) with flicker noise (plain line) and white noise (dashed line, dashed dotted line, dotted line) for three different white noise levels.

the usual summation of the voltage spectral density of white noise and flicker noise would lead to incorrect noise spectra prediction.

On the other hand if the corner frequency $\Delta f_{\text{white}}(\text{ceiling})$ is smaller than the corner frequency $\Delta f_{1/f}(\text{ceiling})$ then the $1/\Delta f^3$ region will be observed in the noise spectrum. This is the case in Figure 7.6 in the presence of the weaker white noise $e^2(1) = e^2(3)/10^7$ and $e^2(2)/(3 \times 10^5)$ when the total (white + flicker) voltage densities (square and star) follow first the $1/\Delta f^3$ flicker noise spectra at low offset frequencies before switching at high offset frequencies to the $1/\Delta f^2$ noise spectra. In such a case the corner frequency between the $1/\Delta f^2$ and $1/\Delta f^3$ regions is given by (assuming mode A dominating):

$$\Delta f_{23} = \frac{1}{2\pi} \frac{k_{1,A} S_A \kappa_A^2}{\eta^2 2m_1}$$
The usual summation of the voltage spectral density of white noise and flicker noise is then an excellent approximation. In summary the simple rules described above for combining noise processes permit us to predict the total noise spectrum from our analytic models without resorting to the time consuming numerical analysis.
CHAPTER 8

DESIGN OF A TSMC 0.18UM LC OSCILLATOR

In this chapter we study the design of a LC oscillator using a TSMC 0.18 um, 6 metal layer 2 poly process. In the next chapter we shall apply our model to this oscillator. This way our model can be validated on a real differential LC oscillator. The general design of the LC oscillator follows from that of the differential oscillator described in the previous chapters using ideal BSIM3 transistors.

8.1 Basic Topology

The circuit diagram of the LC oscillator is shown in Figure 8.1. The circuit exhibits a simple double differential architecture. Though differential oscillators are not good as far as noise performance is considered, having this configuration gives us more flexibility in varying the parameters of the circuit. An optimized circuit (lowest noise) is considered for fabrication. As seen from Figure 8.1, the oscillator has a core consisting of 2 NMOS and 2 PMOS transistors. The current bias of the oscillator is obtained through a current mirror. The center frequency of the oscillator is found to be \( f_0 = 2.549 \text{GHz} \). A buffer stage consisting of a differential topology is connected to the oscillator core. The harmonic balance and the phase noise simulation results are shown in Figures 8.2 and 8.3. Power consumption without parasitics is found to be 91 mW. The layout of the circuit is shown.
in Figure 8.4.

### 8.2 Post Layout Simulation

Usually post layout simulation involves extracting the RC parasitics of the oscillator circuit and then re-simulating with the parasitics included. However, the problem with this approach is that only RC parasitics are included. The inductive parasitics are not included. A micro-strip approximation of the metal and dielectric layers can be used to find the inductive as well as the RC parasitics as shown below.

A multi metal layer process like TSMC 0.18 um consists of 6 metal layers inter-spaced by $SiO_2$ dielectric layers (Figure 8.5). If the thickness of the intermediate layers is neglected (thickness $\approx 0.66 \mu m$), the top-layer metal (thickness $\approx 0.99 \mu m$) can be approximated as the micro-strip line. The bottom layer metal is grounded and acts as the ground.
Once the above approximation has been made, the layout is carried out in such a way so that most of the inter device connections are made through the top layer interconnects. The other metal layers are kept to a minimum. The bottom layer is filled in extensively and is grounded. These features can be seen in the layout Fig 8.4. The circuit for simulating the
circuit post layout is shown in Figure 8.6. ADS is used for the micro-strip like simulation.

8.3 Design Flow

1. Identify the architecture (in this case double differential).
Figure 8.5: The figure shows the cross section of a multi metal layer process. Though only a single metal layer has been shown, there may be more than one. The inter metal layer is filled with $SiO_2$ dielectric.

2. See that the oscillator oscillates. Optimize the design to obtain the lowest noise (optimized design). Design done using ADS.

3. Obtain the layout of the circuit using Virtuoso editor in Cadence.

4. Translate the design back again to ADS, modeling the device interconnects as microstrip lines.

5. Re-simulate and see the performance.

6. Perform final layout optimization.

7. Obtain the gds files for the design.
8.4 Post Layout Simulations Results

The Phase Noise of the optimized circuit is shown in Figure 8.7. The center frequency shifts to 2.397 GHz. The post layout harmonics are shown in Figure 8.8. We then do a supply voltage sweep to see the impact on the Fundamental tone as shown in Fig 8.9. The net power dissipation of the circuit is 104 mW while total wafer area is 867 x 987 sq um.
Figure 8.7: Post Layout Phase Noise simulation of the oscillator

<table>
<thead>
<tr>
<th></th>
<th>Pre Simulation</th>
<th>Post Simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Center Frequency</td>
<td>2.549 GHz</td>
<td>2.397 GHz</td>
</tr>
<tr>
<td>Phase Noise</td>
<td>-100 dBc/Hz at 100 KHz offset</td>
<td>-88 dBc/Hz at 100 KHz offset</td>
</tr>
<tr>
<td>Power of Fundamental</td>
<td>4.976 dBm</td>
<td>-8.262 dBm</td>
</tr>
<tr>
<td>Total Power Consumption</td>
<td>91 mW</td>
<td>104 mW</td>
</tr>
</tbody>
</table>

Table 8.1: Table compares the pre and post simulation results.

Figure 8.8: Post Layout HB simulation of the oscillator output
Figure 8.9: Impact of supply voltage sweep on the oscillator output
CHAPTER 9

APPLYING THE MODEL ON THE TSMC 0.18 UM OSCILLATOR

Figure 9.1 shows the circuit setup for obtaining the parameters necessary for applying the model to the circuit described in chapter 8. As in the case of the ideal differential

Figure 9.1: A TSMC 0.18 um differential oscillator with buffer stage-setup for parameter extraction
oscillator, a least square method was used to obtain the values of the various derivatives used in the model. The oscillation operating point was perturbed by injecting signals with slightly different amplitude keeping the frequency same as the operating frequency and vice versa. Similarly to obtain the derivatives wrt dc bias also, a perturbation of the oscillator operating point was made by injecting dc currents across the transistors according to the schemes for the modes A, C and E. Figure 9.2 shows the result of applying the model for

![Figure 9.2: Comparison of the white noise simulation results and the proposed model. Note that due to the low value of $\frac{a}{b}$, the plot for the two different values of $m_1$ overlap. The match between simulation results and the model is within 3 dB](image-url)
the white noise case. The flicker noise sources of the transistors are turned off by setting the NOIMOD flag in the BSIM3 model of the TSMC 0.18 um transistors to 1 and the KF flag to 0.

The flicker noise parameters are obtained from the BSIM3 model of the transistors. It follows that for the parameter NOIMOD=1, the flicker noise is given by,

\[
\text{Flicker Noise} = \frac{K_f I_d^3}{C_{ox} L_{eff}^2 f_{eff}}
\]  

(9.1)

\(C_{ox}\) is obtained from the TSMC test run parameters available from the MOSIS webpage. \(L_{eff}\) is given by,

\[
L_{eff} = L + \delta L_{eff}
\]

\[
= L + D_{LC} + \frac{L_l}{W L_{wn}} + \frac{L_w}{W_{wn}} + \frac{L_{wl}}{L_{ln} W_{wn}}
\]  

(9.2)

The terms mentioned in the above equations can be obtained from the BSIM3 model specifications for TSMC 0.18 um process. Figure 9.3 compares the simulation results and the proposed model for both white and flicker noise combined.

**9.1 Loadline analysis of the TSMC oscillator**

The only parameter that we can control in TSMC 0.18 um oscillator designed, is the supply voltage. Reducing the supply voltage will reduce the Phase Noise of the oscillator as shown in Figure 9.4. This should either result in reduction of \(|e|^2\) or a reduction of \(\alpha/\beta\). Reduction of \(\alpha/\beta\) in turn means an increase of the angle \(\theta\) as described in Chapter 3. The loadline plots for two different supply voltages are shown in Fig 9.5 and 9.6. As we can see the angle is actually lower for supply voltage equal to 1V. Hence, the main factor in the lower phase noise must be the lower value of \(|e|^2\).
9.2 Experimental Results

Figure 9.7 shows the die micrograph of the fabricated oscillator. The chip area is 1.4 × 1.4 mm². Figure 9.8 shows the output of the oscillator at supply voltage 2V. The Phase Noise can be computed at an offset of 100 KHz from the center frequency to be -87 dBc/Hz. Figure 9.9 shows the same graph with a wider frequency span. The Phase Noise can be computed from this graph to be -108 dBc/Hz at 1MHz offset. These results very nicely match with the simulated results. Comparing the results obtained with high end commercial oscillators like Maxim MAX2753 (which uses a SiGe Bipolar process), which has a Phase Noise of -104 dBc/Hz at 100 KHz offset in the ISM band, we see that the performance of the oscillator needs to be improved. Also the power consumption of
Figure 9.4: Phase Noise at $V_{DD} = 1V$

Table 9.1: Table compares the simulation and experimental results. Note the close agreement between simulation and experiment thereby validating the proposed model since the model results are close to the simulation results.

<table>
<thead>
<tr>
<th></th>
<th>Simulation</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>Center Frequency</td>
<td>2.397 GHz</td>
<td>2.320 GHz</td>
</tr>
<tr>
<td>Phase Noise</td>
<td>-88 dBc/Hz at 100 KHz offset</td>
<td>-87 dBc/Hz at 100 KHz offset</td>
</tr>
<tr>
<td>Power of Fundamental</td>
<td>-8.262 dBm</td>
<td>-3.898 dBm</td>
</tr>
<tr>
<td>Total Power Consumption</td>
<td>104 mW</td>
<td>80 mW</td>
</tr>
</tbody>
</table>

80 mW is a bit too high compared to MAX2753 which has a power consumption of only 10 mW. The best Phase Noise performance obtained in TSMC 0.18 um at the ISM band is -100 dBc/Hz at 100 KHz offset [19].
Figure 9.5: Loadline at $V_{DD} = 2V$

Figure 9.6: Loadline at $V_{DD} = 1V$
Figure 9.7: Die Micrograph of the oscillator having an area of 1.4×1.4mm².

Figure 9.8: Experimental Results: Shows the Phase Noise Spectrum with a span of 1 MHz. The phase noise computed at 100 KHz offset from the center frequency is -87dBc/Hz
Figure 9.9: Experimental Results: Shows the Phase Noise Spectrum with a span of 10 MHz. The phase noise computed at 1 MHz offset from the center frequency is -108dBc/Hz
CHAPTER 10

CONCLUDING REMARKS

10.1 Summary

In this work we have attempted to find simple analytical solutions of the Phase Noise spectrum of an LC oscillator. Analytical models are inherently approximate. The phase noise spectrum of an oscillator is complex. There are minute noisy features, peaks and troughs in the spectrum which are difficult to model. Nevertheless, a model by definition is an idealized representation of a phenomenon. The purpose of a model is to aid the designer in gaining insights and ease in understanding the phenomenon. It is similar to using a map. It is not the real thing but helps us in visualizing the earth. The primary goal of our model is simplicity. Our model incorporates a strong circuit focus. A circuit focus in the model by itself makes the designer aware of the intricacies of the model and how it applies to his design. We have tried to amalgamate the best of the world of models which on one hand despite their mathematical beauty incorporate complex parameters which in turn require their own algorithms to compute; and the world of pure circuit based heuristical models which while simple are not entirely accurate and based on incorrect assumptions.

Our approach follows a non linear perturbation method. Perturbation methods allow us to simplify a complex phenomenon by linearizing it about a reference point. In the case of the oscillator, its operating point provides us the reference. By following the Kurokawa
theory of oscillations, we obtained the linearized equations. The other important aspect of this thesis was the incorporation of correlations existing between the amplitude and phase deviations of the oscillator from the reference. The proposed model takes into account for the first time, correlations existing between the amplitude and phase voltage-noises at the tank (embodied by the $\alpha$ and $\beta$ factors).

As we had stated earlier, while white noise has been studied closely, flicker noise analysis left many holes. The key assumption for the flicker noise part was that the perturbation in oscillation due to flicker noise is equivalent to the perturbation due to device bias current variation. We succeeded in finding asymptotic expression for the voltage noise spectra. These analytic expressions were verified to hold for a wide range of frequencies using numerical analysis relying on the exact solution $R_\phi(\tau)$. Numerical(exact) solutions were obtained which helped show the range of validity of the asymptotic analytic expressions. The results were also compared with HP ADS simulation results and close agreement was obtained in all cases as shown in the previous chapters. A mode theory of noise was developed to facilitate the calculation of the various Kurokawa noise parameters needed. This mode theory also provides valuable insights in the various noise up-conversion/transfer processes. Rules for combining various uncorrelated noise (e.g., white and flicker) were presented and verified with numerical simulation.

While simulation results are a convenient shortcut for designers, they have some limitations. For one thing, simulators are black boxes where the designer has no idea of what is going on. Further, the absence of the correct Lorentzian spectrum for phase noise showing conservation of power tells us that simulator results may not always be correct. A simple circuit focused model comes in handy then.
Another important contribution was the impact of buffer noise. Through a simple two port model we demonstrated how the buffer noise contributes partly to the device and tank noise and also produces the noise floor. This simple but important part of the phase noise spectrum is often ignored. Finally the design experience of the TSMC 0.18 um oscillator gave us a good example of the validity of this model on an actual oscillator. The applicability of the proposed model on this oscillator proves its universality.

The analytic phase noise theory presented here is based on an extension of the Kurokawa oscillator theory and has therefore the advantage of relying on circuit impedances and their derivatives. These terms are easily obtained by harmonic balance simulations and could be obtained experimentally from measurements of the device impedance.

### 10.2 Design Insights

The circuit based-approach used, provides deep insights in the noise processes. Specifically for white noise the analytic model points towards the need to effectively reduce the amplitude and phase noise correlation ($\cot \theta = \alpha / \beta$). According to the analytic models derived in this work, Leeson formula for large offset frequencies should be updated to be (assuming $G_T^* \simeq 0$, $\Delta f_{\text{white}}(\text{ceiling}) < \Delta f_{1/2}(\text{ceiling})$, $\Delta \omega_{23} < \eta$ and $\Delta \omega_{23} < \sqrt{\kappa_1}$):

$$L = \frac{S_{V_{\text{tot}},s,bb}}{\frac{1}{2}A_{0}^{2}} = \frac{1}{P_{S}} \left[ v_{n}^{2}G_{L} + \frac{|e|^{2}(1 + \cot^{2} \theta)}{(2Q)^{3}} \frac{\omega_{0}^{2}}{\Delta \omega^{2}} \right]$$

$$+ \frac{S \left( G_{I_{N}}^{*} - \cot \theta \ B_{I_{N}}^{*} \right)^{2} \omega_{0}^{2}}{(2QG_{L})^{2}} \frac{\omega_{0}^{2}}{\Delta \omega^{3}}$$

for $\Delta f(\text{ceiling}) < \Delta f < \eta/(2\pi)$

with $P_{S} = (1/2)A_{0}^{2}G_{L}$, the resonator $Q = \omega_{0}\text{Im}\{Y_{T}^{*}\}/(2G_{L}) \simeq \omega_{0}C/G_{L}$ and using for example $G_{L} = \text{Re}\{Y_{L}\}$ for the resonator + buffer conductance. Although more accurate formulas are given within this dissertation this simplified expression should be useful to
circuit designers to identify device and circuit parameters which are critical for phase noise optimization.

The above equation can be used to find a lower bound on the Phase Noise wrt transistor width i.e. the best Phase Noise performance that can be achieved by modifying the transistor width. As we can see from the above equation, with increasing bias stability the phase noise due to the flicker noise will reduce. Bias stability depends on transistor width, the higher the width, the lower is the deviation in admittance due to change in bias current. However, increase of width leads to higher white noise current. Hence there has to be a compromise in the width that a transistor can have so as to achieve the lowest possible phase noise. This is basically an optimization problem which we would like to quantify by an example. The flicker noise current of a MOSFET is given by,

\[
\frac{\overline{i_{\text{flicker}}^2 (f)}}{ \overline{v_{\text{flicker}}^2 (f)}} = \frac{K}{WLC_{ox}f}g_m^2
\]  

(10.1)

When the dc bias current of the MOSFET's is constant as is the case in the differential oscillator topology shown,

\[
g_m^2 = 2\mu_nC_{ox}(W/L)I_d
\]

From which the expression for flicker noise current reduces to,

\[
\overline{i_{\text{flicker}}^2 (f)} = \frac{2\mu_nKI_d}{L^2f}
\]

This clearly shows that the flicker noise current is independent of width of the transistor.

On the other hand due to the dependence of the bias derivatives terms \( G_{IN}^d \) and \( B_{IN}^d \) on \( W \), the Phase Noise due to flicker noise current reduces by a factor \( W^p \) (\( p \) ranging from 1 to 2) with change in \( W \) for the same flicker noise current.

Further, the white noise current of a MOSFET is given by,

\[
\overline{i_{\text{white}}^2 (f)} = \frac{8}{3}kTg_m = \frac{8}{3}kT\sqrt{2\mu_nC_{ox}W/L}I_d
\]
which shows that the white noise current increases by a factor $\sqrt{W}$ for increasing $W$. Hence the total Phase Noise can be written as,

$$PN = \frac{M}{\text{constant (noise floor)}} + \frac{N\sqrt{W}}{\text{white noise}} + \frac{R}{W^p}$$

where,

$$N = \frac{a_{\text{white}}G_I^2(1 + \cot^2 \theta)}{8C^2\Delta \omega^2 \sqrt{W_{\text{normal}}}}$$

and

$$R = \frac{a_{\text{flicker}}(G_{IN} - \cot \theta B_{IN}^p)^2W_{\text{normal}}W^p}{8C^2\Delta \omega^2}$$

The minimum value of PN is obtained by taking,

$$\frac{d(PN)}{dW} = 0$$

which gives,

$$W_{\text{min}} = \left(\frac{2pR}{N}\right)^{\frac{1}{2}}$$

(10.3)

It can be shown that $W_{\text{min}}$ is indeed a minima because $\frac{d^2(PN)}{dW^2}(W = W_{\text{min}})$ is always greater than 0. We simplify our calculations by applying noise only across the PMOS transistors and thereby assume that any change in $G_{IN}$ or $B_{IN}$ is due to change in the $W$ of the PMOS transistors only. The Relation between the term $(G_{IN} - \cot \theta B_{IN}^p)^2$ and $W$ can therefore be given by,

$$(G_{IN} - \cot \theta B_{IN}^p)^2 = K\frac{W_{\text{normal}}^p}{W^p}$$

The terms $G_{IN}^p$ and $B_{IN}^p$ only belong to those for Mode C since $\mu = \gamma = 0$ for $b = 0$. Hence plugging values in Equation 10.3 and taking $p = 1$ we get $W_{\text{min}} = 2.16um$. This is indeed what is observed as shown in Fig 10.1 where the lowest phase noise occurs at $W = 2um$. 

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Figure 10.1: Comparison of Phase Noise obtained at various values of $W$. We can see that at $W = 2\mu m$ the lowest Phase Noise is obtained for $\Delta \omega = 100 kHz$.

10.3 Recommendations

Flicker noise involves memory effects. This arises because the strength of the flicker noise depends on the history of trap distribution. Hence a model which incorporates these memory effects will be highly desirable. We applied the white noise part of our theory on the Van Der Pol oscillator. However the flicker noise part could not be applied because of the absence of a connection between the dc bias of the device and the flicker noise produced. In the BSIM3 transistors that we discussed this connection is inherent in the model which is why we were able to get the values of $Y_{IN}$. This turn provides us two avenues of further research. Is it possible to somehow produce this connection in the Van Der Pol oscillator? Conversely, is it possible to obtain a model of the flicker noise part without the dc bias connection?

Our derivation focused principally on PM phase noise which is the dominant term compared to the AM noise which was also derived. There exist however the possibility for a third type of combined AM-PM noise for strong correlated amplitude and phase noises. This noise would be suppressed if the buffer acts an amplitude limiter. [2]. Note that the
expressions obtained for the IEEE phase noise definition $R_\phi(\omega)$ remain themselves unaffected by this approximation.

Yet another work that remains to be done is to obtain a mechanism to translate the model to a simulator. Most of the parameter extraction work described in this dissertation was done manually. HP ADS provides for user defined models which can effectively automate the entire process. Special AEL codes need to be written for this purpose.

Though the model has been developed only for LC oscillators, it may be possible to generalize it for ring oscillators. The ring oscillator circuit may be divided into 2 parts as we had done for the LC oscillator case. However since for the ring oscillator the admittance of both the parts will have amplitude dependence we need to incorporate a derivative term for the admittance of the “tank” part with respect to amplitude and then obtain a new set of equations. This way the model can be generalized for any oscillator.
APPENDIX A

DERIVATIONS OF MASTER EQUATIONS

Let us consider an admittance model of an oscillator as shown in Fig 3.1. The circuit behaves as an open at resonance and a short for the harmonics. The current $i_{IN}(t)$ flowing through the non linear part can be expressed as

$$i_{IN}(t) = Re[A(t)e^{i(\omega t + \phi(t))}Y_{IN}(A_0, w_i)] + \text{large harmonics} \quad (A.1)$$

The voltage $v(t)$ is given by,

$$v(t) = Re[A(t)e^{i(\omega t + \phi(t))}] + \text{small harmonics} \quad (A.2)$$

where, $\omega_i$ is given by

$$\omega_i = \omega + \frac{d\phi}{dt} - j \frac{1}{A(t)} \frac{dA(t)}{dt} = \omega + \delta \omega \quad (A.3)$$

Similarly, the current $i_L$ flowing through the linear part is given by,

$$i_L(t) = Re[A(t)e^{i(\omega t + \phi(t))}Y_L] + \text{large harmonics} \quad (A.4)$$

Hence the current $i_N$ can be given by,

$$i_N = i_L + i_{IN} = Re \left\{ A(t)e^{i(\omega t + \phi(t))}[Y_L(\omega_i) + Y_{IN}(A_0, \omega_i)] \right\} + \text{large harmonics}$$

$$= Re \left\{ A(t)[\cos(\omega t + \phi(t)) + j \sin(\omega t + \phi(t))]\left[G_L(\omega_i) + jB_L(\omega_i)ight] + G_{IN}(A, \omega_i) + jB_{IN}(A, \omega_i) \right\} + \text{large harmonics} \quad (A.5)$$
Now we assume that the quantities $\frac{d\phi}{dt}$ and $\frac{1}{A} \frac{dA}{dt}$ are much smaller than $\omega$. Performing a Taylor series expansion, we obtain the following relations using Equ. (3.5) and (3.6)

$$i_N = \text{Re} \left\{ A(t) \left[ \cos(\omega t + \phi(t)) + j \sin(\omega t + \phi(t)) \right] \left[ (G_L(\omega) + G_{IN}(A, \omega) + G_T^I(\omega) \frac{d\phi}{dt} + B_L(\omega) \right. \right. $$

$$+ \left. \left. B_T^I(\omega) \frac{dA}{dt} \right] + j \left( B_L(\omega) + B_T^I(\omega) \frac{d\phi}{dt} - \frac{G_T^I(\omega) dA}{dt} + B_{IN}(A, \omega) \right) \right\}$$

$$= A(t) \left[ \cos(\omega t + \phi(t)) \left( G_L(\omega) + G_{IN}(A, \omega) + G_T^I(\omega) \frac{d\phi}{dt} + B_L(\omega) \right) \right.$$

$$- \sin(\omega t + \phi(t)) \left( B_L(\omega) + B_T^I(\omega) \frac{d\phi}{dt} - \frac{G_T^I(\omega) dA}{dt} + B_{IN}(A, \omega) \right) \right]\]$$

(A.6)

We note that at steady state when $\frac{dA}{dt} = \frac{d\phi}{dt} = 0, \omega_i = \omega$ and $i_N = 0$

$$G_L(\omega_0) + G_{IN}(A_0, \omega_0) = 0 \quad (A.7)$$

$$B_L(\omega_0) + B_{IN}(A_0, \omega_0) = 0 \quad (A.8)$$

Now multiplying both sides of Eqn A.6 first by $\cos(\omega t + \phi(t))$ and then by $\sin(\omega t + \phi(t))$ and then integrating each of those equations in time over one time period of oscillation we get

$$i_{N1} = \frac{2}{T} \int_{t_0}^{t} i_N \cos(\omega t + \phi(t)) \, dt$$

$$= A(t) \left[ G_L(\omega) + G_{IN}(A, \omega) + G_T^I(\omega) \frac{d\phi}{dt} + B_L(\omega) \right] \quad \text{(A.9)}$$

$$i_{N2} = \frac{2}{T} \int_{t_0}^{t} i_N \sin(\omega t + \phi(t)) \, dt$$

$$= - A(t) \left( B_L(\omega) + B_{IN}(A, \omega) + B_T^I(\omega) \frac{d\phi}{dt} - \frac{G_T^I(\omega) dA}{dt} \right) \quad \text{(A.10)}$$

The equations above can be simplified by noting that

$$G_L(\omega) + G_{IN}(A, \omega) = G_L(\omega) + G_{IN}(A_0 + \delta A, \omega_0) \approx G_L(\omega_0) + G_{IN}(A_0, \omega_0) + \frac{\partial G_{IN}(A_0, \omega_0)}{\partial A} \delta A$$

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Hence Eqns A.9 and A.10 can be simplified as

\[ \frac{i_{N1}}{A_0} = \frac{\partial G_{IN}(A_0, \omega_0)}{\partial A} \delta A + G_T^I(\omega_0) \frac{d\phi}{dt} + \frac{B_T^I(\omega_0)}{A_0} \frac{d\delta A}{dt} \]  
(A.13)

\[ -\frac{i_{N2}}{A_0} = \frac{\partial B_{IN}(A_0, \omega_0)}{\partial A} \delta A + B_T^I(\omega_0) \frac{d\phi}{dt} - \frac{G_T^I(\omega_0)}{A_0} \frac{d\delta A}{dt} \]  
(A.14)

Now eliminating \( \frac{d\phi}{dt} \) from the above two equations we get,

\[ i_{N1}B_T^I(\omega_0) + i_{N2}G_T^I(\omega_0) = A_0 \left[ B_T^I(\omega_0)G_T^{I'}(A_0, \omega_0) - G_T^I(\omega_0)B_T^{I'}(A_0, \omega_0) \right] \delta A + \frac{d\delta A}{dt} |Y_T^I(\omega_0)|^2 \]  
(A.15)

Similarly eliminating \( \frac{d\delta A}{dt} \) from Equ. (A.13) and (A.14),

\[ i_{N1}G_T^I(\omega_0) - i_{N2}B_T^I(\omega_0) = A_0 \left\{ |Y_T^I(\omega_0)|^2 \frac{d\phi}{dt} \right. \\
+ \delta A \left[ G_T^I(\omega_0)G_T^{I'}(A_0, \omega_0) + B_T^I(\omega_0)B_T^{I'}(A_0, \omega_0) \right] \} \]  
(A.16)

The above 2 equations can be simplified to,

\[ \frac{d\delta A}{dt} |Y_T^I(\omega_0)|^2 + A_0 \beta \delta A = i_{N1}B_T^I(\omega_0) + i_{N2}G_T^I(\omega_0) \]  
(A.17)

\[ A_0 \left( |Y_T^I(\omega_0)|^2 \frac{d\phi}{dt} + \alpha \delta A \right) = i_{N1}G_T^I(\omega_0) - i_{N2}B_T^I(\omega_0) \]  
(A.18)

which are the required master equations.
APPENDIX B

DERIVATIONS OF $R_V(T_1, T_2)$

The autocorrelation function of the oscillation for a time interval $\tau = t_2 - t_1$ (as shown in [23])

$$R_V(t_1, t_2) = (A_0^2 + R_{\delta A})\cos\{\omega_0 t + \phi(t)\} \cos\{\omega_0 [t + \tau] + \phi(t + \tau)\}$$

$$= \frac{1}{2} (A_0^2 + R_{\delta A})\cos \phi(t) \cos [\omega_0 \tau + \phi(t + \tau)] - \sin \phi(t) \sin [\omega_0 \tau + \phi(t + \tau)]$$

$$= \frac{1}{2} (A_0^2 + R_{\delta A}) \cos [\omega_0 \tau + \phi(t + \tau) - \phi(t)]$$

$$= \frac{1}{2} (A_0^2 + R_{\delta A}) \cos [\omega_0 \tau \cos [\phi(t + \tau) - \phi(t)] - \sin \omega_0 \tau \sin [\phi(t + \tau) - \phi(t)]]$$

$$= \frac{1}{2} (A_0^2 + R_{\delta A}) \cos [\omega_0 \tau \cos [\phi(t + \tau) - \phi(t)]]$$

(B.1)

This derivation assumes $\delta A$ and $\phi$ are uncorrelated. This assumption is often justified on the basis that the buffer acts as a limiter which suppresses the $\delta A$ fluctuation at its output.

We will keep track of $\delta A$ in our work to monitor that its contribution remains negligible. If $\phi_1 = \phi(t)$ and $\phi_2 = \phi(t + \tau)$ are jointly normal with zero mean and $\sigma_1 = \sigma_2 = \sigma$, then

$$f(\phi(t), \phi(t + \tau); t, t + \tau) = \frac{1}{2\pi \sigma^2 \sqrt{1 - \tau^2}} e^{-\frac{1}{2(1 - \tau^2)} \sigma^2 (\phi_1^2 - 2R_{\phi} \phi_1 \phi_2 + \phi_2^2)}$$

$$R_{\phi}(\tau) = \sigma^2 \tau = E[\phi(t + \tau) \phi(t)]$$

(B.2)  (B.3)

$\tau$ is called the correlation coefficient.
Taking, $\Delta = \phi_1 - \phi_2$ and $S = \phi_1 + \phi_2$, we get

$$d\phi_1 \ d\phi_2 = \frac{d\Delta \ dS}{2}$$

$$\phi_1^2 - 2r\phi_1\phi_2 + \phi_2^2 = \frac{1}{4}\{(S^2 + \Delta^2 + 2\Delta S) - 2r(S + \Delta)(S - \Delta) + S^2 + \Delta^2 - 2\Delta S\}$$

$$= \frac{1}{2}S^2(1 - r) + \frac{1}{2}\Delta^2(1 + r)$$

$$\Rightarrow \frac{\phi_1^2 - 2r\phi_1\phi_2 + \phi_2^2}{2(1 - r^2)\sigma^2} = \frac{S^2}{4(1 + r)\sigma^2} + \frac{\Delta^2}{4(1 - r)\sigma^2} = \frac{S^2}{2\sigma_S^2} + \frac{\Delta^2}{2\sigma_\Delta^2} \quad (B.4)$$

$$\Rightarrow E[\cos(\phi_1 - \phi_2)] = \frac{1}{2\pi\sigma^2\sqrt{1 - r^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(\phi_2 - \phi_1) \exp\left[\frac{\phi_1^2 - 2r\phi_1\phi_2 + \phi_2^2}{2(1 - r^2)\sigma^2}\right] d\phi_1 d\phi_2$$

$$= \frac{1}{4\pi\sigma^2\sqrt{1 - r^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(\Delta) \exp\left[\frac{-S^2}{2\sigma_S^2} - \frac{\Delta^2}{2\sigma_\Delta^2}\right] d\Delta dS$$

$$= \frac{1}{4\pi\sigma^2\sqrt{1 - r^2}} \sqrt{2\pi}\sigma_S^2 \int_{-\infty}^{\infty} \cos(\Delta) \exp\left[-\frac{\Delta^2}{2\sigma_\Delta^2}\right] d\Delta$$

$$\quad (B.5)$$

where,

$$2\sigma_S^2 = 4(1 + r)\sigma^2$$

$$2\sigma_\Delta^2 = 4(1 - r)\sigma^2$$

Using the identity

$$\int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{a^2}\right] \cos x \ dx = a\sqrt{\pi} \exp\left[-\frac{a^2}{4}\right]$$

we get,

$$E[\cos(\phi_1 - \phi_2)] = \exp\left[-\frac{\sigma_\Delta^2}{2}\right] = \exp\left[-\sigma^2 + R_\phi(t_1, t_2)\right] \quad (B.6)$$

Hence from Eqn B.1, we get

$$R_V(t_1, t_2) = \frac{1}{2} \left(A_0^2 + R_\delta\Delta\right) \cos(\omega_0\tau) \exp\left[-\sigma^2 + R_\phi(t_1, t_2)\right] \quad (B.7)$$
APPENDIX C

DERIVATION OF $S_{IN_1}(F)$ AND $S_{IN_2}(F)$

We have,

\[ i_{N_1} = \frac{2}{T} \int_0^T i_N \cos(\omega t + \phi(t)) dt \]
\[ i_{N_2} = \frac{2}{T} \int_0^T i_N \sin(\omega t + \phi(t)) dt \]

The autocorrelation function of $i_{N_1}$ is given by,

\[ R_{i_{N_1}}(t_1, t_2) = \frac{4}{T^2} \int_{t_1-T}^{t_1} \int_{t_2-T}^{t_2} E[i_N(t_1)i_N(t_2)] \cos(\omega t_1 + \phi(t_1)) \cos(\omega t_2 + \phi(t_2)) dt_1 dt_2 \]

From which, we get,

\[ R_{i_{N_1}}(t_1, t_2) = \begin{cases} 0 & \text{for } |t_2 - t_1| > T \\ \frac{4|e|^2}{T^2} \int_0^T \cos^2(\omega t + \phi(t)) dt = \frac{2|e|^2}{T} & \text{for } t_1 - T < t_2 < t_1 \end{cases} \]

The above expression gives us the value of $R_{i_{N_1}}(t_1, t_2)$ as a function of $\tau$. The plot of $R_{i_{N_1}}(t_1, t_2)$ versus $\tau$ is shown in Figure C.1. The total area under the graph is given by,

\[ \int_{-\infty}^{\infty} R_{i_{N_1}}(t_1, t_2) d\tau = 2|e|^2 \]

This is the same area if $R_{i_{N_1}}(t_1, t_2) = 2|e|^2 \delta(\tau)$. Hence in the limit $T \to \infty$, the expression of $R_{i_{N_1}}(t_1, t_2)$ becomes

\[ R_{i_{N_1}}(t_1, t_2) = 2|e|^2 \delta(\tau) \]
Figure C.1: $R_{iN1}(t_1, t_2)$ versus $\tau$
APPENDIX D

DERIVATION OF $R_{\delta A}(T_1, T_2)$

From the Master equation (3.1) we get,

$$i_{N1}B'_T + i_{N2}G'_T = A_0 \beta \delta A + \frac{d\delta A}{dt}|Y'_T|^2$$

Referring to Papoulis’ book [20] we can obtain the following relation from the above, assuming non correlation between the noise components $i_{N1}$ and $i_{N2}$

$$B'_T R_{N1}(t_1, t_2) = A_0 \beta R_{N1 \delta A}(t_1, t_2) + \frac{dR_{N1 \delta A}(t_1, t_2)}{dt_2}|Y'_T|^2$$

$$\Rightarrow 2B'_T |e|^2 \delta(t_2 - t_1) = A_0 \beta R_{N1 \delta A}(t_1, t_2) + \frac{dR_{N1 \delta A}(t_1, t_2)}{dt_2}|Y'_T|^2 \quad (D.1)$$

Solving for $R_{N1 \delta A}$ gives us the following result,

$$R_{N1 \delta A}(t_1, t_2) = 2B'_T |e|^2 \exp \left[ -\frac{A_0 \beta \tau}{|Y'_T|^2} \right] u(\tau) + c_1 \exp \left[ -\frac{A_0 \beta \tau}{|Y'_T|^2} \right]$$

where, $\tau = t_2 - t_1$. Using the boundary condition

$$R_{N1 \delta A}(t_1, -\infty) = 0$$

we get $c_1 = 0$. Hence,

$$R_{N1 \delta A}(t_1, t_2) = 2B'_T |e|^2 \exp \left[ -\frac{A_0 \beta \tau}{|Y'_T|^2} \right] u(t_2 - t_1) \quad (D.2)$$

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Similarly we get the following expression for $R_{N2\delta A}$

$$R_{N2\delta A}(t_1, t_2) = 2 \frac{G_T |e|^2}{|Y_T|^2} \exp \left[ - \frac{A_0 \beta \tau}{|Y_T|^2} \right] u(t_2 - t_1) \quad (D.3)$$

Again from the master equation 3.1 we get,

$$R_{N1\delta A}(t_1, t_2)B_T^l + R_{N2\delta A}(t_1, t_2)G_T^l = A_0 \beta R_{\delta A}(t_1, t_2) + \frac{dR_{\delta A}(t_1, t_2)dt_1 |Y_T|^2}{|Y_T|^2}$$

$$\Rightarrow \frac{dR_{\delta A}}{dt_1} + \frac{A_0 \beta R_{\delta A}}{|Y_T|^2} = 2 \frac{|e|^2}{|Y_T|^2} \exp \left[ - \frac{A_0 \beta (t_2 - t_1)}{|Y_T|^2} \right] u(t_2 - t_1) \quad (D.4)$$

Solving for $R_{\delta A}$ gives us the following result,

$$R_{\delta A}(t_1, t_2) = \frac{|e|^2}{A_0 \beta} \left[ e^{\frac{A_0 \beta (t_1 - t_2)}{|Y_T|^2}} u(t_2 - t_1) + e^{\frac{A_0 \beta (t_2 - t_1)}{|Y_T|^2}} u(t_1 - t_2) \right] + c_2 e^{\frac{A_0 \beta}{|Y_T|^2}}$$

Using the boundary condition $R_{\delta A}(t_1, -\infty) = 0$ we get $c_2 = 0$. Hence,

$$R_{\delta A}(t_1, t_2) = \frac{|e|^2}{A_0 \beta} \left[ e^{\frac{A_0 \beta (t_1 - t_2)}{|Y_T|^2}} u(t_2 - t_1) + e^{\frac{A_0 \beta (t_2 - t_1)}{|Y_T|^2}} u(t_1 - t_2) \right] = \frac{|e|^2}{A_0 \beta} e^{\frac{A_0 \beta |t_2 - t_1|}{|Y_T|^2}} \quad (D.5)$$
APPENDIX E

DERIVATION OF $R_\phi(T_1, T_2)$

Taking the Fourier Transform of the Eqns 3.1 & 3.2, we get

$$j\omega \frac{\delta A}{A_0} |Y'_T(\omega_0)|^2 + \beta \delta A = \frac{1}{A_0} [i_{N1}B'_T(\omega_0) + i_{N2}G'_T(\omega_0)]$$  \hspace{1cm} (E.1)

$$j\omega |Y'_T(\omega_0)|^2 \phi + \alpha \delta A = \frac{1}{A_0} [i_{N1}G'_T(\omega_0) - i_{N2}B'_T(\omega_0)]$$ \hspace{1cm} (E.2)

where,

$$\beta = B'_T(\omega_0)G'_{IN}(A_0, \omega_0) - G'_T(\omega_0)B'_{IN}(A_0, \omega_0)$$ \hspace{1cm} (E.3)

$$\alpha = G'_T(\omega_0)G'_{IN}(A_0, \omega_0) + B'_T(\omega_0)B'_{IN}(A_0, \omega_0)$$ \hspace{1cm} (E.4)

From Eqn E.1 we get the amplitude variation spectral density as,

$$S_{\delta A}(\omega) = \frac{2|Y'_T(\omega_0)|^2|\phi|^2}{A_0^2 \beta^2 + \omega^2|Y'_T(\omega_0)|^4}$$

using $|i_{N1}(\omega)|^2 = |i_{N2}(\omega)|^2 = 2|\phi|^2$. The derivation of this expression is shown in Appendix F.

From the equations E.2, we get the following expression for the $|\phi|^2$

$$S_\phi(\omega) = \frac{|\phi|^2}{\omega^2|Y'_T(\omega_0)|^4} \left[ 2|Y'_T(\omega_0)|^2 \frac{A_0^2}{A_0^2 \beta^2 + \omega^2|Y'_T(\omega_0)|^4} \right]$$

$$= \frac{2|\phi|^2}{A_0^2|Y'_T(\omega_0)|^2} \left( 1 + \frac{\alpha^2}{\beta^2} \right) - \frac{|Y'_T(\omega_0)|^2}{\omega^2|Y'_T(\omega_0)|^4} \times \frac{2|\phi|^2 \alpha^2}{A_0^2 \beta^2} \hspace{1cm} (E.5)$$

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For the uncorrelated $\delta A$ and $\phi$ case, the above equation reduces to

$$ S_{\phi}(\omega) = \frac{2|e|^2}{\omega^2 A_0^2 |Y_T^2(\omega_0)|^2} \quad (E.6) $$

By taking the inverse Fourier transform of the equations (E.5) & (E.6) we obtain for the correlated case,

$$ R_\phi(\tau) = -\frac{|e|^2}{A_0^2 |Y_T^2(\omega_0)|^2} \left( 1 + \frac{\alpha^2}{\beta^2} \right) |\tau| - \frac{|e|^2}{\beta^3 A_0^3} e^{-\beta A_0 |\nu^2(\omega_0)|^2 |\tau|} \quad (E.7) $$

Here we have used the relation $\alpha^2 + \beta^2 = |Y_{1N}(A_0, \omega_0)|^2 |Y_T^2(\omega_0)|^2$. For the uncorrelated case the equation reduces to,

$$ R_\phi(\tau) = -\frac{|e|^2}{A_0^2 |Y_T^2(\omega_0)|^2} |\tau| \quad (E.8) $$

The inverse Fourier Transform of the expressions above have been computed indirectly, since direct computation is difficult. It can be easily shown that,

$$ \mathcal{F} [\zeta |\tau|| = -\frac{2\zeta}{\omega^2} $$

$$ \mathcal{F} [\zeta \exp(-\xi |\tau|)] = \frac{2\zeta \xi}{\xi^2 + \omega^2} $$

Hence, conversely,

$$ \mathcal{F}^{-1} \left[ \frac{1}{\omega^2} \right] = -\frac{1}{2} |\tau| $$

$$ \mathcal{F}^{-1} \left[ \frac{1}{\xi^2 + \omega^2} \right] = \frac{1}{2\xi} \exp(-\xi |\tau|) $$

These results have been used in computing the inverse Fourier Transforms above.
APPENDIX F

DERIVATION OF $S_V(\omega)$ FOR WHITE NOISE CASE

Given the value derived for $R_\phi(\tau)$ and $R_{\delta A}$ and using $\sigma^2 = R_\phi(0)$ we obtain,

$$R_V(\tau) = \frac{1}{2} \left[ A_0^2 + \frac{|e|^2}{A_0 \beta} \exp \left( -\frac{A_0 \beta |\tau|}{|Y_\phi(\omega_0)|^2} \right) \right] \cos(\omega_0 \tau)$$

$$\times \exp \left[ -\frac{|e|^2}{A_0^2 |Y_\phi(\omega_0)|^2} \left( 1 + \frac{\alpha^2}{\beta^2} \right) |\tau| - \frac{\alpha^2 |e|^2}{\beta A_0^2} \left( e^{-\frac{\beta A_0}{|Y_\phi(\omega_0)|^2} |\tau|} - 1 \right) \right]$$

(F.1)

Now taking the Fourier Transform of the above equation and proceeding along the Method of stationary phase (shown in Appendix H to obtain the Fourier Transform of an exponential of an exponential, we obtain,

$$S_{V,\text{PM}}(\omega) = A_0^2 \left[ \frac{m_1}{m_1^2 + (\omega - \omega_0)^2} \right] + c \left[ \frac{m_1 + m_2}{(m_1 + m_2)^2 + (\omega - \omega_0)^2} \right]$$

(F.2)

where, $m_1 = \frac{|e|^2}{A_0^2 |Y_\phi(\omega_0)|^2}$ and $m_2 = \frac{A_0 \beta}{|Y_\phi(\omega_0)|}$, $c = \frac{|e|^2}{A_0 \beta}$. Since $(\omega + \omega_0)^2$ is very high compared to $m_1^2$, those terms containing it in the denominator have been neglected.
APPENDIX G

DERIVATION OF THE PHASE NOISE WITH BUFFER NOISE

Figure G.1: Conventional (a) and modified (b) 2-port noise model equivalent circuits

Consider the two circuits shown in Fig G.1.

Let the Z-parameters of the circuit be $z_{11}$, $z_{12}$, $z_{21}$, and $z_{22}$. For the circuit of Fig G.1(a) the voltages $v_1$ and $v_2$ can be given as

$$v_1 + v_n = (i_1 - i_n)z_{11} + i_2z_{12}$$

$$\Rightarrow v_1 = i_1z_{11} + i_2z_{12} - v_n - i_nz_{11} \quad (G.1)$$

Again

$$v_2 = (i_1 - i_n)z_{21} + i_2z_{22}$$

$$\Rightarrow v_2 = i_1z_{21} + i_2z_{22} - i_nz_{21} \quad (G.2)$$
Similarly for the circuit of Fig G.1 (b) the voltages \( v_1 \) and \( v_2 \) can be given as,

\[
v_1 = (i_1 - i_{n1}) z_{11} + i_2 z_{12} \\
\Rightarrow v_1 = i_1 z_{11} + i_2 z_{12} - i_{n1} z_{11} \tag{G.3}
\]

Again,

\[
v_2 - v_{n2} = (i_1 - i_{n1}) z_{21} + i_2 z_{22} \\
\Rightarrow v_2 = i_1 z_{21} + i_2 z_{22} + v_{n2} - i_{n1} z_{21} \tag{G.4}
\]

Comparing Eqns G.1 and G.2 we get,

\[
i_{n1} = i_n + \frac{v_n}{z_{11}}
\]

and from Eqns G.3 and G.4,

\[
v_{n2} = \frac{z_{21}}{z_{11}} v_n
\]

Alternatively, the voltages \( v_n \) and \( i_n \) can also be given in terms of \( i_{n1} \) and \( v_{n2} \) as follows,

\[
i_n = i_{n1} - \frac{v_{n2}}{z_{21}}
\]

and,

\[
v_n = \frac{z_{11}}{z_{21}} v_{n2}
\]

As seen from Figure G.2, the noise source of the load resistance \( v_{n,R_L} \) can be expressed as two equivalent sources from the equations above as follows,

\[
i_{ni} = -\frac{v_{n,R_L}}{z_{21}}
\]

and

\[
v_{ni} = \frac{z_{11}}{z_{21}} v_{n,R_L}
\]

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The input of the buffer can be represented by the equivalent circuit of Figure G.3. Thus the total Norton equivalent noise source $i_{N, buf}$ will be the superposition of the noise sources due to the buffer and due to the load resistance $R_L$.

\[
i_{N, buf} = i_n + i_{ni} + \frac{v_n + v_{ni}}{Z_{in}}
\]

\[
= i_n - \frac{v_{n,R_L}}{Z_{in}} + \frac{v_n}{Z_{in}} + \frac{v_{n,R_L} \cdot z_{11}}{z_{21} Z_{in}}
\]

\[
= i_n + \frac{v_n}{Z_{in}} + \frac{v_{n,R_L}}{z_{21} Z_{in} - 1}
\]

\[
= i_n + Y_{m}v_n + Y_{o}v_{n,R_L}
\]

\[(G.5)\]
The total noise voltage at the input of the buffer is given by,

\[
v_{N,\text{buf}} = v_n + v_{ni} = v_n + \frac{z_{11}}{z_{21}}v_{n,R_L}
\]  

(G.6)
FOURIER TRANSFORM OF $F(\tau)$ WITH EXPONENTIAL OF EXPONENTIAL

Let us consider $f(\tau)$ having a form as follows,

$$f(\tau) = \exp \left[ -P \left( q_1|\tau| + \frac{q_2}{\eta_2} \left( e^{-\eta_2|\tau|} - 1 \right) \right) \right]$$

where $q_1 \neq q_2$ and $P > 0$. Taking the Fourier Transform of $f(\tau)$ we get

$$\mathcal{F}[f(\tau)] = \int_{-\infty}^{\infty} f(\tau) e^{-j\omega \tau} d\tau$$

$$= \int_{-\infty}^{\infty} \exp \left[ -P \left( q_1|\tau| + \frac{q_2}{\eta_2} \left( e^{-\eta_2|\tau|} - 1 \right) \right) - j\omega \tau \right] d\tau$$

$$= \int_{-\infty}^{0} \frac{d}{d\tau} \exp \left[ -P \left( q_1\tau + \frac{q_2}{\eta_2} (e^{\eta_2 \tau} - 1) \right) - j\omega \tau \right] d\tau$$

$$+ \int_{0}^{\infty} \frac{d}{d\tau} \exp \left[ -P \left( q_1\tau + \frac{q_2}{\eta_2} (e^{-\eta_2 \tau} - 1) \right) - j\omega \tau \right] d\tau$$

$$= \left[ \exp \left( -P \left( q_1\tau + \frac{q_2}{\eta_2} (e^{\eta_2 \tau} - 1) \right) - j\omega \tau \right) \right]_{-\infty}^{0}$$

$$+ \int_{-\infty}^{0} \frac{(-P)q_2\eta_2 e^{\eta_2 \tau} \exp \left[ -P \left( q_1\tau + \frac{q_2}{\eta_2} (e^{\eta_2 \tau} - 1) \right) - j\omega \tau \right]}{\left[ P(q_1 - q_2 e^{\eta_2 \tau}) - j\omega \right]^2} d\tau$$

$$+ \left[ \exp \left( -P \left( q_1\tau + \frac{q_2}{\eta_2} (e^{-\eta_2 \tau} - 1) \right) - j\omega \tau \right) \right]_{0}^{\infty}$$

$$+ \int_{0}^{\infty} \frac{(-P)q_2\eta_2 e^{-\eta_2 \tau} \exp \left[ -P \left( q_1\tau + \frac{q_2}{\eta_2} (e^{-\eta_2 \tau} - 1) \right) - j\omega \tau \right]}{\left[ -P(q_1 - q_2 e^{-\eta_2 \tau}) - j\omega \right]^2} d\tau$$

$$= \text{(H.1)}$$

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The terms within the integration sign can be neglected since they become negligible for higher frequencies (Method of Constant Phase). Hence the final expressions become,

\[ \mathcal{F}[f(\tau)] = \frac{2P(q_1 - q_2)}{P^2(q_1 - q_2)^2 + \omega^2} \quad \text{for } \omega > \eta_2 \quad (H.2) \]

For our purpose, \( q_1 = \frac{|c|^2}{A_0^2|Y_F(\omega_0)|^2} \left( 1 + \frac{\alpha^2}{\beta^2} \right), q_2 = \frac{\alpha^2|c|^2}{\beta^2\beta_0^2|Y_F(\omega_0)|^2}, \eta_2 = \frac{\beta A_0}{|Y_F(\omega_0)|^2}. \) As a result,

\[ \mathcal{F}[f(\tau)] = \frac{2|c|^2}{A_0^2|Y_F(\omega_0)|^2} - \frac{\omega^2}{\omega^2} \quad (H.3) \]
APPENDIX I

DERIVATION OF THE FOURIER TRANSFORM OF THE MORE GENERAL EXPONENTIAL OF AN EPONENTIAL

We will try to obtain an expression for the Fourier transform of expressions of the form

\[ f(\tau) = \exp \left[ -P \left( q_1 |\tau| + \sum_{i=2}^{n} \frac{q_i}{\eta_i} (e^{-\eta_i |\tau|} - 1) \right) \right] \]

where,

\[ \sum_{i=2}^{n} q_i = q_1 \quad \text{and} \quad P > 0 \]

Now taking the Fourier transform of \( f(t) \) we get,

\[ F(\omega) = \mathcal{F}[f(\tau)] \]

\[ = \int_{-\infty}^{\infty} \exp \left[ -P \left( q_1 |\tau| + \sum_{i=2}^{n} \frac{q_i}{\eta_i} (e^{-\eta_i |\tau|} - 1) \right) \right] e^{-i\omega \tau} d\tau \]

\[ = \int_{-\infty}^{0} \exp \left[ -P \left( -q_1 \tau + \sum_{i=2}^{n} \frac{q_i}{\eta_i} (e^{\eta_i \tau} - 1) \right) \right] e^{-i\omega \tau} d\tau \]

\[ + \int_{0}^{\infty} \exp \left[ -P \left( q_1 \tau + \sum_{i=2}^{n} \frac{q_i}{\eta_i} (e^{-\eta_i \tau} - 1) \right) \right] e^{-i\omega \tau} d\tau \]

\[ = \int_{-\infty}^{0} \frac{d}{d\tau} \left\{ \exp \left[ -P \left( -q_1 \tau + \sum_{i=2}^{n} \frac{q_i}{\eta_i} (e^{\eta_i \tau} - 1) \right) - i\omega \tau \right] \right\} d\tau \]

\[ + \int_{0}^{\infty} \frac{d}{d\tau} \left\{ \exp \left[ -P \left( q_1 \tau + \sum_{i=2}^{n} \frac{q_i}{\eta_i} (e^{-\eta_i \tau} - 1) \right) - i\omega \tau \right] \right\} d\tau \]

Integrating by parts we get,

\[ F(\omega) = \left[ \frac{\exp \left[ -P \left( -q_1 \tau + \sum_{i=2}^{n} \frac{q_i}{\eta_i} (\exp^{\eta_i \tau} - 1) \right) - i\omega \tau \right]}{\left[ -P \left( q_1 - \sum_{i=2}^{n} q_i e^{-\eta_i \tau} \right) - i\omega \right]} \right]_{-\infty}^{0} \]

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\[- \int_{-\infty}^{0} - \exp \left[ - P \left( -q_1 \tau + \sum_{i=2}^{n} \frac{q_i}{\eta_i} \left( \exp^{\eta_i \tau} - 1 \right) \right) \right] \left( -P \sum_{i=2}^{n} q_i \eta_i e^{\eta_i \tau} \right) d\tau \]

\[+ \exp \left[ - P \left( q_1 \tau + \sum_{i=2}^{n} \frac{q_i}{\eta_i} \left( \exp^{-\eta_i \tau} - 1 \right) \right) - i\omega \tau \right] \left( -P \sum_{i=2}^{n} q_i \eta_i e^{-\eta_i \tau} \right) d\tau \]

\[= - \frac{1}{i\omega} \int_{-\infty}^{0} \exp \left[ - P \left( -q_1 \tau + \sum_{i=2}^{n} \frac{q_i}{\eta_i} \left( \exp^{\eta_i \tau} - 1 \right) \right) - i\omega \tau \right] \left( \sum_{i=2}^{n} q_i \eta_i e^{\eta_i \tau} \right) d\tau \]

\[- \frac{1}{i\omega} \int_{0}^{\infty} \exp \left[ - P \left( q_1 \tau + \sum_{i=2}^{n} \frac{q_i}{\eta_i} \left( \exp^{-\eta_i \tau} - 1 \right) \right) - i\omega \tau \right] \left( \sum_{i=2}^{n} q_i \eta_i e^{-\eta_i \tau} \right) d\tau \]

\[= - P \sum_{i=2}^{n} q_i \eta_i \int_{-\infty}^{0} \frac{d}{d\tau} \exp \left[ - P \left( -q_1 \tau + \sum_{i=2}^{n} \frac{q_i}{\eta_i} \left( \exp^{\eta_i \tau} - 1 \right) \right) - i\omega \tau + \eta_i \tau \right] \left( -P \left( -q_1 + \sum_{i=2}^{n} q_i e^{\eta_i \tau} \right) - i\omega + \eta_i \right) d\tau \]

\[= - P \sum_{i=2}^{n} q_i \eta_i \int_{0}^{\infty} \frac{d}{d\tau} \exp \left[ - P \left( q_1 \tau + \sum_{i=2}^{n} \frac{q_i}{\eta_i} \left( \exp^{-\eta_i \tau} - 1 \right) \right) - i\omega \tau - \eta_i \tau \right] \left( -P \left( q_1 - \sum_{i=2}^{n} q_i e^{-\eta_i \tau} \right) - i\omega - \eta_i \right) d\tau \]

Integrating by parts once again, we get

\[F(\omega) = - P \sum_{i=2}^{n} q_i \eta_i \left\{ \frac{\exp \left[ - P \left( -q_1 \tau + \sum_{i=2}^{n} \frac{q_i}{\eta_i} \left( \exp^{\eta_i \tau} - 1 \right) \right) - i\omega \tau + \eta_i \tau \right]}{-P \left( -q_1 + \sum_{i=2}^{n} q_i e^{\eta_i \tau} \right) - i\omega + \eta_i} \right\}^{0}_{-\infty} \]

\[- \sum_{i=2}^{n} q_i \eta_i \int_{0}^{\infty} \frac{d}{d\tau} \frac{1}{\left| -P \left( -q_1 + \sum_{i=2}^{n} q_i e^{\eta_i \tau} \right) - i\omega \right|^2} \left( -P \left( -q_1 + \sum_{i=2}^{n} q_i e^{\eta_i \tau} \right) - i\omega + \eta_i \right) d\tau \]

\[= - \sum_{i=2}^{n} q_i \eta_i \int_{-\infty}^{\infty} \frac{d}{d\tau} \frac{1}{\left| -P \left( -q_1 + \sum_{i=2}^{n} q_i e^{\eta_i \tau} \right) - i\omega \right|^2} \left( -P \left( -q_1 + \sum_{i=2}^{n} q_i e^{\eta_i \tau} \right) - i\omega + \eta_i \right) d\tau \]

\[= - \sum_{i=2}^{n} q_i \eta_i \int_{0}^{\infty} \frac{d}{d\tau} \frac{1}{\left| -P \left( q_1 - \sum_{i=2}^{n} q_i e^{-\eta_i \tau} \right) - i\omega \right|^2} \left( -P \left( q_1 - \sum_{i=2}^{n} q_i e^{-\eta_i \tau} \right) - i\omega - \eta_i \right) d\tau \]

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\[ \frac{d}{d\tau} \left\{ \frac{1}{[-P(q_1 - \sum q_i e^{-n\tau}) - i\omega]^2 [-P(q_1 - \sum q_i e^{-n\tau}) - i\omega - \eta_i]} \right\} d\tau \]

(I.3)

The terms within the integration sign will vanish in the limit \( \omega \to \infty \) because they are rapidly varying functions of time. This follows from the Method of Stationary Phase. So \( F(\omega) \) becomes,

\[
F(\omega) = -P \frac{1}{\omega} \sum_{i=2}^{n} \frac{2q_i \eta_i^2}{\eta_i^2 + \omega^2} \]  

(I.4)

### I.1 Obtaining the Fourier Transform of the form \( f(\tau) \cos(\omega_0 \tau) \)

The results obtained in the preceding section can be used to compute the Fourier transform of functions having the form \( f(\tau) \cos(\omega_0 \tau) \)

\[
\mathcal{F}[f(\tau) \cos(\omega_0 \tau)] = \int_{-\infty}^{\infty} f(\tau) \cos(\omega_0 \tau) \exp(-i\omega \tau) d\tau \\
= \frac{1}{2} \left[ \int_{-\infty}^{\infty} f(\tau) \left\{ \frac{\exp(i\omega_0 \tau) + \exp(-i\omega_0 \tau)}{2} \right\} \right] \exp(-i\omega \tau) d\tau \\
= \frac{1}{2} \left[ \int_{-\infty}^{\infty} f(\tau) \exp(-i(\omega - \omega_0) \tau) d\tau + \int_{-\infty}^{\infty} f(\tau) \exp(-i(\omega + \omega_0) \tau) d\tau \right] \\
= \frac{1}{2} \left[ \frac{2P}{(\omega - \omega_0)^2} \sum_{i=2}^{n} \frac{\eta_i^2 q_i}{\eta_i^2 + (\omega - \omega_0)^2} + \frac{2P}{(\omega + \omega_0)^2} \sum_{i=2}^{n} \frac{\eta_i^2 q_i}{\eta_i^2 + (\omega + \omega_0)^2} \right] \\

(I.5)

At RF frequencies \( (\omega + \omega_0)^2 \) becomes very large and hence the second term becomes quite small compared to the first and hence can be neglected. So the final expression becomes,

\[
\mathcal{F}[f(\tau) \cos(\omega_0 \tau)] \approx \frac{P}{(\omega - \omega_0)^2} \sum_{i=2}^{n} \frac{\eta_i^2 q_i}{\eta_i^2 + (\omega - \omega_0)^2} \\

(I.6)
I.2 Fourier Transform of \( f(\tau) \) for a more general case

Let us consider \( f(\tau) \) having a form as follows,

\[
f(\tau) = \exp \left[ -P \left( q_1|\tau| + \frac{q_2}{\eta_2} \left( e^{-\eta_2|\tau|} - 1 \right) \right) \right]
\]

where,

\[ q_1 \neq q_2 \]

Taking the Fourier Transform of \( f(\tau) \) we get

\[
\mathcal{F}\{f(\tau)\} = \int_{-\infty}^{\infty} f(\tau) e^{-j\omega \tau} d\tau
\]

\[
= \int_{-\infty}^{0} \exp \left[ -P \left( q_1|\tau| + \frac{q_2}{\eta_2} \left( e^{-\eta_2|\tau|} - 1 \right) \right) - j \omega \tau \right] d\tau
\]

\[
= \int_{-\infty}^{0} \frac{d}{d\tau} \exp \left[ -P \left( q_1\tau + \frac{q_2}{\eta_2} \left( e^{\eta_2\tau} - 1 \right) \right) - j \omega \tau \right] d\tau
\]

\[
= \left[ \exp \left( -P \left( q_1\tau + \frac{q_2}{\eta_2} \left( e^{\eta_2\tau} - 1 \right) \right) - j \omega \tau \right) \right]_{-\infty}^{0}
\]

\[
= \left[ \exp \left( -P \left( q_1\tau + \frac{q_2}{\eta_2} \left( e^{\eta_2\tau} - 1 \right) \right) - j \omega \tau \right) \right]_{-\infty}^{0}
\]

\[
= \left[ \exp \left( -P \left( q_1\tau + \frac{q_2}{\eta_2} \left( e^{\eta_2\tau} - 1 \right) \right) - j \omega \tau \right) \right]_{0}^{\infty}
\]

\[
\mathcal{F}\{f(\tau)\} = \frac{2P(q_1 - q_2)}{P^2(q_1 - q_2)^2 + \omega^2}
\]

The terms within the integration sign can be neglected since they become negligible for higher frequencies (Method of Constant Phase). Hence the final expressions become,
For our purpose, $q_1 = \frac{|c|^2}{A_0|Y_\tau(\omega_0)|^2} \left(1 + \frac{\omega^2}{\beta^2}\right)$, $q_2 = \frac{\alpha^2|c|^2}{\beta^2 A_0|Y_\tau(\omega_0)|^2}$, $\eta_2 = \frac{\beta A_0}{|Y_\tau(\omega_0)|^2}$ As a result,

$$
\mathcal{F}[f(\tau)] = \frac{2|c|^2}{A_0|Y_\tau(\omega_0)|^2} \frac{|c|^2}{A_0|Y_\tau(\omega_0)|^2} + \omega^2
$$

(1.9)


