THE ROUSE MODELS IN THE UPPER HALF SPACE

DISSERTATION

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ABSTRACT

The Rouse models are designed to understand the dynamics of a polymer in a dilute solvent. The discrete model is described by a stochastic differential equation while the continuum one is by a stochastic heat equation. When a polymer resides in a polymeric liquid, its motion undergoes obstruction from neighboring polymers. The neighboring polymers build a region where the polymer is able to move. And the polymer makes a reptile-like move. We propose polymer models in a polymeric liquid based on the Rouse models, called by the reflected Rouse models. The dynamics of the reflected Rouse models are described by stochastic equations. We investigate the reflected continuum Rouse model in the upper-half space $\mathbb{R}^3_+$, the dynamics of which is described by a reflected stochastic partial differential equation and prove the existence and uniqueness problems of the reflected stochastic differential equation. And we discuss the curves in $\mathbb{R}^3$ driven by a random curvature and a random torsion and the physical Brownian motion in $\mathbb{R}$. 
To my parents who have great love for their son.
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CHAPTER 1

INTRODUCTION

Polymers have been of great interest in biology, chemistry, engineering and physics. And it has been of great importance to understand the science of polymers in an effort to enhance lives of human being. Human DNA research is one of the most prominent examples which exhibit the importance. Although it is very important and interesting to investigate the chemistry and the biology of polymers, we confine ourselves in the physics of polymers, that is, the physical properties of polymers rather than chemical and biological properties.

Among numerous topics in the physics of polymers, we concentrate on the dynamics of polymers trapped inside a region. In a system of strongly entangled polymers, one polymer is kept from moving freely in the solvent due to polymers nearby. It has been observed that the neighboring polymers form a tube-like region and that the polymer makes a reptile-like move in the region. The motion is called reptation. It has been also noted that the time scale of the change of the region is relatively big compared to that of the polymer due to the fact that the neighboring polymers make the same move.
In this dissertation, we analyze the Rouse model, a polymer model, in a region with nonempty boundary of $\mathbb{R}^3$. The region would be considered as the one formed by the neighboring polymers.

The Rouse model is a theoretical framework to describe dynamics of a polymer in a dilute solvent. In the paper [16] by P. E. Rouse, a polymer in a dilute solvent is considered as a chain of many massless beads connected by harmonic springs while its motion in the solvent is affected by the potential energy along the chain, the friction against the solvent, and surrounding solvent molecules. Each bead represents one monomer molecule of a polymer or a collection of adjacent monomer molecules while each spring represents the bond between adjacent monomer molecules of the polymer or adjacent collections of monomer molecules. In addition, springs are assumed to have no physical presence. We call the model described above the discrete Rouse model. A polymer in a dilute solvent is so small that it seldom touches the solvent container. However, a polymer trapped in a small region would touch the boundary of the region. By a reflected Rouse model, we mean a Rouse model interacting with the boundary of the region. We need an extra ingredient to describe the interaction which is treated in detail later in this chapter.

We treat four Rouse models, discrete, continuum, reflected discrete and reflected continuum Rouse models. The continuum model is the one obtained by increasing the number of beads in the discrete Rouse model. The discrete and continuum models have their own merits in describing the dynamics of the polymer and cast a variety of mathematical challenges. In particular, the reflected continuum Rouse model poses a new set of challenges. One of mathematical challenges treated in the dissertation is the reflected stochastic partial differential equation, in short RSPDE.
In the first section we look into polymer models in general. There are many polymer models that are created to study one aspect or more of polymers. Some models are created to describe the general behavior while the others are created to investigate a specific property in mind. We illustrate the bead-rod models, the bead-spring models, the self-avoiding walk models and the worm-like chain models.

The kinetic theory of polymers and the Rouse models are discussed in the second section. We explain the relevant factors that should be taken into consideration to study dynamics of polymers. The discrete and continuum Rouse models are rigorously constructed and the basic mathematical results related to the discrete and continuum Rouse models are stated. One of the results leads us to continuum Rouse model which is discussed in detail.

In the third section we investigate the reflected discrete and continuum Rouse models and their formulations. We construct the reflected models, discrete and continuum.

Before we go on the background material, we outline the structure of the dissertation. In Chapter 2, we study existence and uniqueness problems of the reflected continuum Rouse models. We prove the new results related the reflected continuum Rouse model in the upper-half space of $\mathbb{R}^3$.

In Chapter 3, we state the results obtained and the problems cast in the course of the study of the Rouse models.
1.1 Polymer Models

A polymer is a large molecule made up of one or more kinds of many monomer molecules. Monomer molecules and geometric structure of monomer molecules determine the biological, chemical and physical characteristics of a polymer they build. To analyze the physics of polymers, scientists started creating models. They noticed that the common structural attribute of many polymers was being linear, which suggests that polymers would be modelled by a chain or a finite curve. And each monomer or a collection of adjacent monomers is regarded as a bead in the chain or ignored in the modelling. In the following we introduce some polymer models.

In the bead-rod chain model, we regard a monomer or a collection of adjacent monomers as a bead and a bond between monomers or collections of them as a rigid rod. The freely jointed chain model is a collection of rigid rods of the same length that are linearly linked without any restriction on bonding. This model is designed to exemplify the flexibility of polymers and realized in probabilistic terms by a finite sum of independent $S_{b_0}^2$-valued random variables, where $b_0$ is the length of the rod and $S_{b_0}^2$ is the sphere in $\mathbb{R}^3$ of radius $b_0$ centered at the origin. However, it has been noted that flexibility of polymers is not due to flexible joints but to their lengths. A more realistic model is the freely rotating chain model. It is the chain that is restricted to have the constant angle between the neighboring bonds. We refer interested readers to Doi and Edward [3], and Grosberg and Khokhlov [8].

A different model, a bead-spring chain model, is built by connecting beads with springs. If a bead-spring chain model does not have restriction on joining bonds, it is called the freely jointed bead-spring chain model. It is easy to work mathematically with the model because it does not have any internal constraint. With Hookean
springs in the model, the model is also called a Gaussian chain because the confor-
mational distribution of the chain is Gaussian. The Gaussian chain has been used for
studying dynamics of polymers and the Rouse model is based on the Gaussian chain.
However, the Gaussian chain has disadvantage that the length of the chain is not
constant and that the chain can be stretched out to any length. We refer interested
readers to Doi and Edward [3], and Grosberg and Khokhlov [8].

The worm-like chain model has been designed for stiff polymers. A polymer is
regarded as a thin, elastic filament which follows the Hooke’s law and has the absolute
minimum energy when it is straight. The double-helix DNA is well approximated by
the worm-like chain model. We refer interested readers to Kratky and Porod[10], and
Saitō, Takahashi and Yunoki [17].

In case that excluded volume effect of polymers is important, the self-avoiding
walk in a lattice is of interest as polymer models. The self-avoiding walk in a lattice
is a random walk such that a walker cannot visit the sites he or she has visited before.
The self-avoiding walk itself is a huge field of study. We refer interested readers to
Vanderzande [20].

1.2 Kinetic Theory of Polymers and Rouse Models

There are several ingredients that govern the motion of a polymer in a dilute
solvent. Depending on the way a model is created, some are included in the modelling
and the others are ignored. In the following we illustrate the important ingredients
for the kinetic theory of polymers.

Polymers are influenced by their own structures. If there is a preferred state,
polymers tend to go to that state. This tendency is described by a potential energy.
For example, the potential energy of the Gaussian chain model is the sum of those of harmonic springs in the model and that of the worm-like chain is the integral of the curvature of the curve.

A polymer in a medium experiences friction against the medium. The friction force is proportional to the velocity of the polymer. And each medium has an intrinsic constant related to friction, the friction constant. The force is the product of the friction constant and the velocity of the polymer.

A polymer in a solvent is also affected by the thermal fluctuation in the solvent, i.e., collisions with the solvent molecules. The thermal fluctuation is treated statistically due to the abundance of collisions and presented by the product of the Brownian force with a constant.

The excluded volume effect results from the interaction between different parts of a polymer that are close in the solvent. It is observed that a polymer creates a region along its trace so that other parts of the polymer cannot come into the region. This is called the excluded volume effect.

The hydrodynamic interaction of a polymer is made through the solvent molecules. Once a polymer moves, it affects the velocity vector field of the solvent and, in turn, the other part of the polymer. This is called the hydrodynamic interaction.

The polymer model of our interest, the discrete Rouse model, is a bead-spring chain model that only assumes a harmonic potential energy, friction and thermal fluctuation, i.e., excluded volume effect and hydrodynamic interaction are ignored in the model. In the model, the beads bear the friction against the solvent and the thermal fluctuation of the solvent, and the springs are the source of the potential energy along the chain. As in the bead-spring chain model, a polymer is modelled
by \( N \) beads and the beads are connected by the springs. We note that increasing \( N \)
does not correspond to adding a monomer molecule to an existing polymer but to
approximating a polymer with more beads.

We construct the model as follows. Suppose that there are \( N \) beads whose posi-
tions in \( \mathbb{R}^3 \) at time \( t \) are denoted by \( R_t^{(N,1)}, \ldots, R_t^{(N,N)} \) and that beads are connected
by harmonic springs whose potential energy is

\[
\frac{3}{2b_N^2} k_B T \sum_{n=2}^{N} \left\| R_t^{(N,n)} - R_t^{(N,n-1)} \right\|^2,
\]

where \( b_N \) is a positive constant, \( k_B \) is the Boltzman constant and \( T \) is the absolute
temperature. In the literatures of polymer dynamics, \( b_N \) is referred to as the \textquote{effective
bond length}. Since the mean length of the chain at equilibrium is

\[
Nb_N,
\]

and the chain approximates a fixed polymer, we argue that \( b_N \) is inversely proportional
to \( N \), i.e., \( b_N = Nb \) for some universal constant \( b > 0 \). Thus the potential energy of
the chain is

\[
\frac{3N^2}{2b^2} k_B T \sum_{n=2}^{N} \left\| R_t^{(N,n)} - R_t^{(N,n-1)} \right\|^2. \tag{1.2.1}
\]

If we assume that

\[
R_t^{(N,0)} = R_t^{(N,1)}, \quad R_t^{(N,N+1)} = R_t^{(N,N)},
\]

then the tensile force at each bead \( R_t^{(N,n)}, n = 1, \ldots, N \), is

\[
\frac{3N^2}{b^2} k_B T \left( R_t^{(N,n-1)} - 2R_t^{(N,n)} + R_t^{(N,n+1)} \right). \tag{1.2.2}
\]

Let \( \zeta \) be the friction constant of the solvent. Then the friction force against the
solvent at each bead \( R_t^{(N,n)}, n = 1, \ldots, N \), is

\[
-\zeta \frac{d}{dt} R_t^{(N,n)}.
\]
We note in the above that the contribution from the thermal fluctuation in the solvent is of the form

$$\sigma_N \dot{B}_t$$

for some constant $\sigma_N \geq 0$, where $\{B_t\}_{t \geq 0}$ is the standard 3-dimensional Brownian motion. In the model we assume that the thermal fluctuation at each bead is independent and identically distributed, and that the average fluctuation produces the motion of a Brownian particle. We let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{B_t^{(N,n)}\}_{t \geq 0}, n = 1, \ldots, N$, the independent 3-dimensional Brownian motions on $(\Omega, \mathcal{F}, P)$. Then the force exerted by the thermal fluctuation at each bead $R_t^{(N,n)}$, $n = 1, \ldots, N$, is

$$\sigma_N \dot{B}_t^{(N,n)}$$

for some constant $\sigma_N$. To determine $\sigma_N$, let $x$ be the particle in the solvent that is driven by the average fluctuation, i.e.,

$$\zeta \dot{x}_t = \frac{1}{N} \sum_{n=1}^{N} \sigma_N \dot{B}_t^{(N,n)}.$$

Then

$$x_t = x_0 + \frac{1}{\zeta N} \sum_{n=1}^{N} \sigma_N B_t^{(N,n)} \text{ and } \mathbb{E} \|x_t - x_0\|^2 = \frac{1}{\zeta^2 N^2} 3\sigma_N^2 N t.$$

Since Einstein equation implies

$$\mathbb{E} \|x_t - x_0\|^2 = 6 \frac{k_B T}{\zeta} t,$$

we have

$$\sigma_N = \sqrt{2\zeta k_B T N}.$$

Thus the force at each bead $R_t^{(N,n)}$, $n = 1, \ldots, N$, is

$$\sqrt{2\zeta k_B T N} \dot{B}_t^{(N,n)}.$$ (1.2.3)
Now we are ready to write the equation for the discrete Rouse model. We suppose that the mass of each bead is $m$. Then, by Newton mechanics, the force at each bead $R_t^{(N,n)}$, $n = 1, \ldots, N$, is

$$m \frac{d^2}{dt^2} R_t^{(N,n)}$$

and equal to the sum of the tensile force, the friction force and the force induced by the thermal fluctuation. Thus the equation of the motion of the discrete Rouse model with mass $m$ is

$$m \frac{d^2}{dt^2} R_t^{(N,n)} = -\zeta \frac{d}{dt} R_t^{(N,n)} + \frac{3N^2}{b^2} k_B T \left( R_t^{(N,n-1)} - 2R_t^{(N,n)} + R_t^{(N,n+1)} \right) + \sqrt{2\zeta k_B T N} \dot{B}_t^{(N,n)}$$

(1.2.4)

for $n = 1, \ldots, N$. Since each bead represents small part of the polymer and the mass of the polymer is minuscule, we ignore the inertial term in (1.2.4). Thus the equation of the motion of the discrete Rouse model is

$$\zeta \frac{d}{dt} R_t^{(N,n)} = \frac{3N^2}{b^2} k_B T \left( R_t^{(N,n-1)} - 2R_t^{(N,n)} + R_t^{(N,n+1)} \right) + \sqrt{2\zeta k_B T N} \dot{B}_t^{(N,n)}$$

(1.2.5)

for $n = 1, \ldots, N$.

(1.2.5) is a $3N$-dimensional stochastic differential equation. In the theory of the stochastic differential equations, (1.2.5) is regarded as an integral equation, i.e., the solution of (1.2.5) satisfies

$$\int_0^t \zeta \, dR_u^{(N,n)} = \int_0^t \frac{3N^2}{b^2} k_B T \left( R_u^{(N,n-1)} - 2R_u^{(N,n)} + R_u^{(N,n+1)} \right) \, du$$

$$+ \int_0^t \sqrt{2\zeta k_B T N} \, dB_u^{(N,n)}$$

or

$$\zeta R_t^{(N,n)} - \zeta R_0^{(N,n)} = \int_0^t \frac{3N^2}{b^2} k_B T \left( R_u^{(N,n-1)} - 2R_u^{(N,n)} + R_u^{(N,n+1)} \right) \, du$$

or
where $R_0^{(N,n)}$ represents the initial position of the $n^{th}$ bead. There are numerous books regarding the stochastic differential equations. We refer interested readers to Karatzas and Shreve [9], and Durrett [5].

The basic mathematical result about the discrete Rouse model is

**Theorem 1.2.1.** Existence of the strong solution and strong uniqueness hold for (1.2.5) with initial condition

$$(R^{(N,1)}, \ldots, R^{(N,N)}) \in \mathbb{R}^{3N}.$$ 

One can find the proof in many books concerning the stochastic differential equations. For example, see the section 5.2 in Karatzas and Shreve [9].

Before we move on to the continuum Rouse model, we rewrite the discrete Rouse model as a map $R^{(N)}$ from $[0, 1] \times [0, \infty)$ into $\mathbb{R}^3$. First let

$$s_n^{(N)} = \frac{n}{N + 1}$$

for $n = 1, \ldots, N$. We define $R^{(N)}$ as follows.

- For $0 \leq s \leq s_1^{(N)}$ and $t \geq 0$,

  $$R^{(N)}(s, t) = R_t^{(N,1)}.$$

- For $s_n^{(N)} \leq s \leq s_{n+1}^{(N)}$ and $t \geq 0$,

  $$R^{(N)}(s, t) = (N + 1) \left( s_{n+1}^{(N)} - s \right) R_t^{(N,n)} + (N + 1) \left( s - s_n^{(N)} \right) R_t^{(N,n+1)}.$$

- For $s_N^{(N)} \leq s \leq 1$ and $t \geq 0$

  $$R^{(N)}(s, t) = R_t^{(N,N)}.$$
And let $W = (W^{(1)}, W^{(2)}, W^{(3)})$ be the triple of independent Brownian sheets on the probability space $(\Omega, \mathcal{F}, P)$. By a Brownian sheet, we mean a random field $B_{s,t}$ with two parameters $(s, t) \in [0, \infty)^2$ such that

- $B_{s,t}$ is Gaussian distributed with mean 0 and variance $s \times t$,
- $\mathbb{E}(B_{s_1, t_1} B_{s_2, t_2}) = (s_1 \wedge s_2) \times (t_1 \wedge t_2)$.

For each $n = 1, \ldots, N$, let

$$B_{t}^{(N)} \left( s_{n}^{(N)} \right) = \sqrt{N+1} \left( W_{s_{n}^{(N)}, t} - W_{s_{n-1}^{(N)}, t} \right).$$

We note that $\{ B_{t}^{(N)} \left( s_{n}^{(N)} \right) \}, n = 1, \ldots, N$, are independent 3-dimensional Brownian motions on $(\Omega, \mathcal{F}, P)$. Then we write (1.2.5) in terms of $R^{(N)}$ as follows.

$$\zeta \frac{d}{dt} R^{(N)} \left( s_{n}^{(N)}, t \right) = \frac{3N^2}{b^2} k_B T \left[ R^{(N)} \left( s_{n-1}^{(N)}, t \right) - 2R^{(N)} \left( s_{n}^{(N)}, t \right) + R^{(N)} \left( s_{n+1}^{(N)}, t \right) \right]$$
$$+ \sqrt{2\zeta k_B T N} B_{t}^{(N)} \left( s_{n}^{(N)} \right). \quad (1.2.6)$$

The continuum Rouse model is supposed to be a limit of the discrete Rouse models as $N$ tends to infinity in a certain sense. We note that for any twice continuously differentiable function $f$ on $[0, 1]$ and any $s \in (0, 1)$,

$$\lim_{N \to \infty} N^2 \left[ f \left( s - \frac{n-1}{N} \right) - 2f(s) + f \left( s + \frac{n+1}{N} \right) \right] = f''(s)$$

and that $f(0) = f \left( s_{1}^{(N)} \right)$ and $f(1) = f \left( s_{N}^{(N)} \right)$ implies

$$f'(0) = 0 = f'(1).$$

Thus (1.2.6) suggests that the equation for the continuum Rouse model $R$ be

$$\zeta \frac{\partial}{\partial t} R(s, t) = \frac{3}{b^2} k_B T \frac{\partial^2}{\partial s^2} R(s, t) + \sqrt{2\zeta k_B T} \dot{W}(s, t) \quad (1.2.7)$$
with \( \frac{\partial}{\partial s} R(0, t) = 0 = \frac{\partial}{\partial s} R(1, t) \) for all \( t > 0 \). (1.2.7) is a stochastic partial differential equation. As were stochastic differential equations, (1.2.7) is regarded as an integral equation. We refer readers to Walsh [21] for the introduction to the topic and cite a result from Walsh [21].

**Theorem 1.2.2.** Let \( r \) be a continuous map on \([0, 1]\) into \( \mathbb{R}^3 \). Then there exists a unique random vector field \( R \) on such that

- \( R(s, 0) = r(s) \) for all \( s \in [0, 1] \) almost surely.
- \( R \) is continuous on \([0, 1] \times [0, \infty)\) almost surely.
- For any \( \psi \in C^{2,1}([0, 1] \times [0, \infty); \mathbb{R}^3) \) with \( \frac{\partial}{\partial s} \psi(0, t) = 0 = \frac{\partial}{\partial s} \psi(1, t) \) and for any \( \tau > 0 \),

\[
\zeta \int_0^1 \Psi(s, \tau) \cdot R(s, \tau) \, ds = \zeta \int_0^1 \Psi(s, 0) \cdot r(s) \, ds \tag{1.2.8}
\]

\[
+ \int_0^\tau \int_0^1 \left( \zeta \frac{\partial \Psi}{\partial t}(s, t) + \frac{3}{\hbar^2} k_B T \frac{\partial^2 \Psi}{\partial s^2}(s, t) \right) \cdot R(s, t) \, ds \, dt
\]

\[
+ \sqrt{2\zeta k_B T} \int_0^\tau \int_0^1 \Psi(s, t) \cdot W(ds, dt)
\]

a.s., where \( \cdot \) denotes the usual inner product in \( \mathbb{R}^3 \).

The proof of the theorem is given in Funaki [6] and Walsh [21].

Let \( r \) be a continuous map on \([0, 1]\) into \( \mathbb{R}^3 \). For each positive integer \( N \), let \( R^{(N)} \) be the unique strong solution of (1.2.6) with the initial condition

\[
\left( r \left( s_1^{(N)} \right), \ldots, r \left( s_N^{(N)} \right) \right)
\]

and \( R \) the solution of (1.2.7) with the initial condition \( r \). And let \( C \) be the set

\[
C \left( [0, 1] \times [0, \infty); \mathbb{R}^3 \right)
\]
of all continuous maps from $[0,1] \times [0,\infty)$ into $\mathbb{R}^3$, equipped with norm
\[
\|f\|_C = \sum_{\tau=1}^{\infty} 2^{-\tau} \min \left\{ 1, \max_{(s,t)\in[0,1]\times[0,\tau]} \|f(s,t)\| \right\}.
\]
Then each $R^{(N)}$ and $R$ induce probability measures $P_{R^{(N)}}$ and $P_R$ on $\mathcal{C}$, respectively, i.e., for any Borel subset $B$ of $\mathcal{C}$
\[
P_{R^{(N)}}(B) = P\left\{ R^{(N)} \in B \right\} \quad \text{and} \quad P_R(B) = P\left\{ R \in B \right\}.
\]

Now we are ready to discuss the convergence.

**Theorem 1.2.3.** $R^{(N)}$ converges in distribution to $R$ as $N \to \infty$, i.e., $P_{R^{(N)}}$ converges weakly to $P_R$.

The proof of the theorem is given in Funaki [6].

### 1.3 The Reflected Rouse Models

The reflected Rouse models are the main topic of the dissertation. We recall that the reflected Rouse models exemplify a polymer in a concentrated solvent that is trapped in a region formed by polymers nearby.

First we illustrate the reflected discrete Rouse model of $N$ beads in an open set $U \subset \mathbb{R}^3$ with nonempty boundary. Let $\hat{L}_{t}^{(N,n)}$ be the force acting on the $n^{th}$ bead $R^{(N,n)}$ which keeps $R^{(N,n)}$ from escaping from the closure of $U$. Then the equation of motion of the discrete Rouse model $\{R^{(N,n)}: n = 1, \ldots, N\}$ of $N$ beads is
\[
\zeta \frac{d}{dt} R^{(N,n)}_{t} = \frac{3}{b^2} k_B T N^2 \left( R^{(N,n-1)}_{t} - 2R^{(N,n)}_{t} + R^{(N,n+1)}_{t} \right) + \sqrt{2\zeta T N} \hat{B}^{(N,n)}_{t} + \hat{L}^{(N,n)}_{t},
\]
and $L^{(N,n)}$, $n = 1, \ldots, N$, satisfy the following:
• For any \( n = 1, \ldots, N \) and any \( \tau > 0 \),

\[
|L^{(N,n)}|_{\tau} = \int_{0}^{\tau} \chi_{\partial U} \left( R^{(N,n)}_{t} \right) d|L^{(N,n)}|_{t},
\]

where \( |L^{(N,n)}| \) denotes the total variation measure of \( L^{(N,n)} \) and \( \chi_{\partial U} \) is the indicator function of the boundary \( \partial U \) of \( U \).

• For any \( n = 1, \ldots, N \),

\[
L^{(N,n)}_{\tau} = \int_{0}^{\tau} n_{\partial U} \left( R^{(N,n)}_{t} \right) d|L^{(N,n)}|_{t},
\]

where \( n_{\partial U}(x) \) is the inward unit normal vector to \( \partial U \) at \( x \) if \( x \in \partial U \) and 0 otherwise.

Two statements above say that \( L^{(N,n)} \) changes only when \( R^{(N,n)} \) touches the boundary \( \partial U \) of \( U \) and that \( R^{(N,n)} \) moves in the direction of the normal vector at the point where \( R^{(N,n)} \) meets \( \partial U \).

The stochastic differential equations like (1.3.1) are called the reflected stochastic differential equations. Skorokhod first studied the reflected stochastic differential equation on the positive real line and since then many have worked on the equations. Important papers regarding the stochastic differential equations are Stroock and Varadhan [18], Tanaka [19] and Lions and Sznitman [11]. Tanaka in [19] proved the strong existence and the pathwise uniqueness for the reflected stochastic equations with Lipschitz coefficients in convex domains. We state the result in the following.

**Theorem 1.3.1.** Suppose that \( U \) is a convex domain. Existence of the strong solution and pathwise uniqueness hold for (1.3.1) with any initial condition

\[
\left( R^{(N,1)}_{0}, \ldots, R^{(N,N)}_{0} \right) \in \mathcal{U}^{N} = \mathcal{U} \times \cdots \times \mathcal{U}.
\]
Just like the Rouse models, we expect the convergence of the reflected discrete
Rouse model in distribution to the reflected continuum Rouse model. It has not been
proven yet. In the following we illustrate the reflected continuum Rouse model in an
open set \( U \subset \mathbb{R}^3 \). As in the reflected discrete Rouse model in an open set \( U \), we need
a term in the equation of the motion of the continuum Rouse model that describes the
reflection against the boundary \( \partial U \) of \( U \), denoted by \( \Lambda \), in the reflected continuum
Rouse models. Then, formally, the equation of motion of the reflected continuum
Rouse model in \( U \) is

\[
\zeta \frac{\partial}{\partial t} R(s, t) = \frac{3}{b^2} k_B T \frac{\partial^2}{\partial s^2} R(s, t) + \sqrt{2\zeta k_B T} \dot{W}(s, t) + \dot{\Lambda}(s, t)
\]

with \( \frac{\partial}{\partial s} R(0, t) = 0 = \frac{\partial}{\partial s} R(1, t) \) for \( t > 0 \) and \( \Lambda \) should satisfy the following:

- For any Borel measurable subset \( A \) of \([0, 1] \times [0, \infty)\),

\[
|\Lambda|(A) = \int_A \chi_{\partial U}(R(s, t)) \ |\Lambda|(ds, dt),
\]

a.s.;

- For any Borel measurable subset \( A \) of \([0, 1] \times [0, \infty)\),

\[
\Lambda(A) = \int_A \mathbf{n}_{\partial U}(R(s, t)) \ |\Lambda|(ds \ dt),
\]

a.s.

Several papers have been published on the reflected stochastic partial differential
the one-dimensional reflected stochastic partial differential equations in the positive
real line with Dirichlet end-point condition, i.e., \( R(0, t) = 0 = R(1, t) \) for all \( t > 0 \).
The results of Nualart and Pardoux [13], and Donati-Martin and Pardoux [4] would
be applied to the reflected continuum Rouse model pinned at the boundary $\partial U$ of $U$. Otobe [14] examines the one-dimensional reflected stochastic partial differential equations in the positive real line with the infinite spatial domain.

In Chapter 2, we discuss (1.3.2) in depth and prove existence and uniqueness problems for (1.3.2) in the upper half space $\mathbb{R}^3_+$. 
In this chapter, we investigate existence and uniqueness of the reflected Rouse model $R$ in an open set $\mathbb{R}_+^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$.

Let $W$ be a triple $(W^1, W^2, W^3)$ of independent Brownian sheets on a probability space $(\Omega, \mathcal{F}, P)$. See Walsh [21] for more about Brownian sheet. For any smooth domain $D \subset \mathbb{R}^3$, let $n_D$ be the inward unit normal vector on the boundary $\partial D$ of $D$ and $n_D(x) = 0$ for $x \notin \partial D$. And we denote by $|\Lambda|$ the total variation measure of a vector-valued measure $\Lambda = (\Lambda^1, \Lambda^2, \Lambda^3)$ on a $\sigma$-algebra $\mathcal{B}$, i.e., for any $A \in \mathcal{B}$

$$|\Lambda|(A) \equiv \sup \sum_{i=1}^{\infty} \left( |\Lambda^1(A_i)| + |\Lambda^2(A_i)| + |\Lambda^3(A_i)| \right),$$

where the supremum is taken over all partitions $\{A_i : i = 1, 2, 3, \ldots\}$ of $A$.

In the previous chapter, we have seen that, given the initial position $R_0$ of the model, the reflected Rouse model $R$ in $\mathbb{R}_+^3$ be described by the reflected stochastic partial differential equation

$$\zeta \frac{\partial}{\partial t} R(s, t) = \frac{3}{b^2} k_B T \frac{\partial^2}{\partial s^2} R(s, t) + \sqrt{2\zeta k_B T} \frac{\partial W}{\partial t} + \frac{\partial^2 \Lambda}{\partial t \partial s},$$

(2.0.1)

with the following :
1. \( R(s, 0) = R_0(s) \) for all \( s \in [0, 1] \) a.s.;

2. \( R \) is continuous on \( [0, 1] \times [0, \infty) \) a.s.;

3. \( R(s, t) \in \mathbb{R}_+^3 \) for all \((s, t) \in [0, 1] \times [0, \infty)\) a.s.;

4. \( \frac{\partial}{\partial s} R(0, t) = \frac{\partial}{\partial s} R(1, t) = 0 \) for all \( t > 0 \);

5. \( \Lambda (A) = \iint_A \chi_{\partial C} (R(s, t)) \Lambda(ds \; dt) \) and \( \Lambda (A) = \iint_A n_{\partial C} (R(s, t)) \; |\Lambda| (ds \; dt) \)
   for any Borel measurable subset \( A \) of \([0, 1] \times [0, \infty)\).

**Definition 2.0.2.** We say that \((R, \Lambda)\) is a weak solution of (2.0.1) with the initial condition \( R_0 \) if

1. \( R(s, 0) = R_0(s) \) for all \( s \in [0, 1] \);

2. \( R \) is continuous on \( [0, 1] \times [0, \infty) \);

3. \( R(s, t) \in \mathbb{R}_+^3 \) for all \((s, t) \in [0, 1] \times [0, \infty)\);

4. \( \Lambda \) is a random Radon \( \mathbb{R}_+^3 \)-valued measure on the Borel subsets of \([0, 1] \times [0, \infty)\)
   and is supported by the set \( \{(s, t) \in [0, 1] : R(s, t) \in \partial C\} \) a.s., i.e., for any Borel measurable subset \( A \) of \([0, 1] \times [0, \infty)\),
   \[
   \Lambda (A) = \iint_A \chi_{\partial \mathbb{R}_+^3} (R(s, t)) \; \Lambda(ds \; dt)
   \]
   a.s.;

5. For any Borel measurable subset \( A \) of \([0, 1] \times [0, \infty)\),
   \[
   \Lambda (A) = \iint_A n_{\partial \mathbb{R}_+^3} (R(s, t)) \; |\Lambda| (ds \; dt)
   \]
   a.s.;
6. For any $\Psi \in C^{2,1}([0,1] \times [0,\infty) ; \mathbb{R}^3)$ with $\frac{\partial}{\partial s} \Psi(0,t) = \frac{\partial}{\partial s} \Psi(1,t) = 0$ for all $t > 0$ and for any $\tau > 0$,

$$
\zeta \int_0^1 R(s,\tau) \cdot \Psi(s,\tau) \, ds - \zeta \int_0^1 R_0(s) \cdot \Psi(s,0) \, ds = \int_0^\tau \int_0^1 \left( \zeta \frac{\partial}{\partial t} \Psi(s,t) + \frac{3}{b^2} k_B T \frac{\partial^2}{\partial s^2} \Psi(s,t) \right) \, ds \, dt \\
+ \int_0^\tau \int_0^1 \Psi(s,t) \cdot \sqrt{2\zeta k_B T W} (ds \, dt) \\
+ \int_0^\tau \int_0^1 \Psi(s,t) \cdot \Lambda (ds \, dt)
$$

a.s., where $\cdot$ denotes the usual scalar product on $\mathbb{R}^3$.

Before we start working on the reflected stochastic partial differential equation (2.0.1), we consider the set $\mathbb{R}_+^3$. We note that the inward unit normal vectors on the boundary of $\mathbb{R}_+^3$ is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, i.e.,

$$
n_{\partial \mathbb{R}_+^3}(x) = \begin{cases} k & \text{if } x \in \mathbb{R}^2 \times \{0\}, \\ 0 & \text{otherwise}, \end{cases}
$$

where $k = (0,0,1)$. With the observation above, we have a proposition which eases the problem.

**Proposition 2.0.3.** $(R, \Lambda) = ((R^1, R^2, R^3), (\Lambda^1, \Lambda^2, \Lambda^3))$ is one and only one weak solution of (2.0.1) with the initial condition $R_0 = (R^1_0, R^2_0, R^3_0)$ if and only if

1. $R^1$ is one and only one weak solution of

$$
\zeta \frac{\partial}{\partial t} R^1 = \frac{3}{b^2} k_B T \frac{\partial^2}{\partial s^2} R^1 + \sqrt{2\zeta k_B T} \frac{\partial}{\partial s} W^1
$$

with the initial condition $R^1_0$ and $\Lambda^1 \equiv 0$.

2. $R^2$ is one and only one weak solution of

$$
\zeta \frac{\partial}{\partial t} R^2 = \frac{3}{b^2} k_B T \frac{\partial^2}{\partial s^2} R^2 + \sqrt{2\zeta k_B T} \frac{\partial}{\partial s} W^2
$$
with the initial condition $R^2_0$ and $\Lambda^2 \equiv 0$.

3. $(R^3, \Lambda^3)$ is a unique pair satisfying the following:

(a) $R^3$ is continuous in $[0, 1] \times [0, \infty)$, $R^3(s, 0) = R^3_0(s)$ for all $s \in [0, 1]$ and $R^3(s, t) \geq 0$ for all $(s, t) \in [0, 1] \times [0, \infty)$, a.s.;

(b) $\Lambda^3$ is a random Radon measure on the Borel subsets of $[0, 1] \times [0, \infty)$ and is supported by the set $\{(s, t) \in [0, 1] : R(s, t) \in \partial R^3_+\}$ a.s., i.e., for any Borel measurable subset $A$ of $[0, 1] \times [0, \infty)$,

$$\Lambda^3(A) = \int \int_A \chi_{\{0\}}(\{R^3(s, t)\} \Lambda^3(ds \, dt),$$

a.s.;

(c) For any $\psi \in C^{2,1}([0, 1] \times [0, \infty))$ with $\frac{\partial}{\partial s}\psi(0, t) = \frac{\partial}{\partial s}\psi(1, t) = 0$ for all $t > 0$ and for any $\tau > 0$,

$$\zeta \int_0^1 R^3(s, \tau)\psi(s, \tau) \, ds - \zeta \int_0^1 R^3_0(s)\psi(s, 0) \, ds$$

$$= \int_0^\tau \int_0^1 R^3(s, t) \left(\zeta \frac{\partial}{\partial t}\psi(s, t) + \frac{3}{b^2} k_B T \frac{\partial^2}{\partial s^2}\psi(s, t)\right) \, ds \, dt$$

$$+ \int_0^\tau \int_0^1 \sqrt{2\zeta k_B T} \psi(s, t) W^3(ds \, dt)$$

$$+ \int_0^\tau \int_0^1 \psi(s, t) \Lambda^3(ds \, dt)$$

a.s.

Proof. Necessity: Let $(R, \Lambda) = ((R^1, R^2, R^3), (\Lambda^1, \Lambda^2, \Lambda^3))$ be one and only one solution of (2.0.1) with the initial condition $R_0 = (R^1_0, R^2_0, R^3_0)$. First we note that, for any Borel set $A \subset [0, 1] \times [0, \infty)$,

$$\Lambda(A) = \int \int_A n_{\partial R^3_+}(R^1(s, t), R^2(s, t), R^3(s, t)) \ |\Lambda| (ds \, dt)$$
\[ \int_A \chi_{\mathbb{R}^2 \times \{0\}} \left( R^1(s, t), R^2(s, t), R^3(s, t) \right) \, k \, |\Lambda| \, (ds \, dt) \]
\[ = \int_A \chi_{\{0\}} (R^3(s, t)) \, k \, |\Lambda| \, (ds \, dt) \]
\[ = \left( 0, 0, \int_A \chi_{\{0\}} (R^3(s, t)) \, |\Lambda| \, (ds \, dt) \right). \]

Thus \( \Lambda^1 \equiv 0 \) and \( \Lambda^2 \equiv 0 \), which imply \( |\Lambda| \equiv |\Lambda^3| \) and so
\[ \Lambda^3(A) = \int_A 1_{\{0\}} (R^3(s, t)) \, |\Lambda^3| \, (ds \, dt). \]

Since \( \Lambda^3 \) is a positive measure, \( \Lambda^3 = |\Lambda^3| \) and
\[ \Lambda^3(A) = \int_A 1_{\{0\}} (R^3(s, t)) \, \Lambda^3(ds \, dt). \]

Hence we have seen the properties for \( \Lambda^i \)'s.

Choose any real-valued \( \psi \in C^{2,1}([0, 1] \times [0, \infty)) \) with \( \frac{\partial}{\partial s} \psi(0, t) = \frac{\partial}{\partial s} \psi(1, t) = 0 \) for all \( t > 0 \) and let \( \Psi^1 = (\psi, 0, 0) \), \( \Psi^2 = (0, \psi, 0) \), \( \Psi^3 = (0, 0, \psi) \). If we apply \( \Psi^i \) to \( R \), where \( i = 1, 2, 3 \), then we see that, for \( i = 1, 2, 3 \),
\[ \zeta \int_0^1 R^i(s, \tau) \psi(s, \tau) \, ds - \zeta \int_0^1 R^i_0(s) \psi(s, 0) \, ds \]
\[ = \int_0^\tau \int_0^1 R^i(s, t) \left( \zeta \frac{\partial}{\partial t} \psi(s, t) + \frac{3}{b^2} k_B T \frac{\partial^2}{\partial s^2} \psi(s, t) \right) \, ds \, dt \]
\[ + \int_0^\tau \int_0^1 \sqrt{2 \zeta k_B T} \psi(s, t) \, W^i(ds \, dt) \]
\[ + \int_0^\tau \int_0^1 \psi(s, t) \, \Lambda^i(ds \, dt), \]
a.s. Since \( R(s, t) \in \overline{C} \) for all \( (s, t) \in [0, 1] \times [0, \infty) \), \( a \leq R^3(s, t) \leq b \) for all \( (s, t) \in [0, 1] \times [0, \infty) \), a.s. Since \( R(s, t) \in \mathbb{R}^3_+ \) for all \( (s, t) \in [0, 1] \times [0, \infty) \) a.s. since \( R^1 \), \( R^2 \) and \( R^3 \) are continuous in \( [0, 1] \times [0, \infty) \), and \( R^3(s, t) \geq 0 \) for all \( (s, t) \in [0, 1] \times [0, \infty) \), a.s.

**Sufficiency**: Let \( (R, \Lambda) = ((R^1, R^2, R^3), (0, 0, \Lambda^3)) \). It is clear that \( R \) is continuous in \( [0, 1] \times [0, \infty) \) and \( R(s, t) \in \mathbb{R}^3_+ \) for all \( (s, t) \in [0, 1] \times [0, \infty) \) a.s. since \( R^1 \), \( R^2 \) and \( R^3 \) are continuous in \( [0, 1] \times [0, \infty) \), and \( R^3(s, t) \geq 0 \) for all \( (s, t) \in [0, 1] \times [0, \infty) \), a.s.
For any Borel measurable subset $A$ of $[0, 1] \times [0, \infty)$,

$$\Lambda(A) = \left( \Lambda^1(A), \Lambda^2(A), \Lambda^3(A) \right)$$

$$= (0, 0, \Lambda^3(A))$$

$$= \left( 0, 0, \int_A \chi_{\{0\}} \left( R^3(s,t) \right) \Lambda^3(ds\,dt) \right)$$

$$= \left( 0, 0, \int_A \chi_{\{0\}} \left( R^3(s,t) \right) \Lambda^3(ds\,dt) \right)$$

$$= \int_A \left( 0, 0, \chi_{\{0\}} \left( R^3(s,t) \right) \right) \Lambda^3(ds\,dt)$$

$$= \int_A \chi_{\partial \mathbb{R}^3} \left( R(s,t) \right) \Lambda(ds\,dt)$$

and

$$\Lambda(A) = \left( \Lambda^1(A), \Lambda^2(A), \Lambda^3(A) \right)$$

$$= (0, 0, \Lambda^3(A))$$

$$= \left( 0, 0, \int_A \chi_{\{0\}} \left( R^3(s,t) \right) \Lambda^3(ds\,dt) \right)$$

$$= \int_A \left( 0, 0, \chi_{\{0\}} \left( R^3(s,t) \right) \right) |\Lambda^3|(ds\,dt)$$

$$= \int_A \left( 0, 0, \chi_{\{0\}} \left( R^3(s,t) \right) \right) |\Lambda|(ds\,dt)$$

$$= \int_A n_{\partial \mathbb{R}^3} \left( R(s,t) \right) |\Lambda|(ds\,dt).$$

Choose $\Psi = (\psi^1, \psi^2, \psi^3) \in C^{2,1}([0, 1] \times [0, \infty); \mathbb{R}^3)$ with $\frac{\partial}{\partial s} \Psi(0,t) = \frac{\partial}{\partial s} \Psi(1,t) = 0$ for all $t > 0$ and let $T > 0$. If you apply $\psi^i$ to $R^i$ for $i = 1, 2, 3$, and add all of the integrals, then we have

$$\zeta \int_0^1 R(s, \tau) \cdot \Psi(s, \tau) \, ds - \zeta \int_0^1 R(s, 0) \cdot \Psi(s, 0) \, ds$$

$$= \zeta \sum_{i=1,2,3} \int_0^1 \left( R_i^i(s, \tau) \psi^i(s, \tau) - R^0_i(s) \psi^i(s, 0) \right) \, ds$$

22
\[
\sum_{i=1,2,3} \int_0^\tau \int_0^1 R^i(s,t) \left( \zeta \frac{\partial}{\partial t} \psi^i(s,t) + \sqrt{2\zeta k_B T} \frac{\partial^2}{\partial s^2} \psi^i(s,t) \right) \, ds \, dt \\
+ \sum_{i=1,2,3} \int_0^\tau \int_0^1 \psi^i(s,t) \sqrt{2\zeta k_B T} \, W^i(ds \, dt) \\
+ \sum_{i=1,2,3} \int_0^\tau \int_0^1 \psi^i(s,t) \Lambda^i(ds \, dt)
\]

= \int_0^\tau \int_0^1 R(s,t) \cdot \left( \zeta \frac{\partial}{\partial t} \Psi(s,t) + \sqrt{2\zeta k_B T} \frac{\partial^2}{\partial s^2} \Psi(s,t) \right) \, ds \, dt \\
+ \int_0^\tau \int_0^1 \Psi(s,t) \cdot \sqrt{2\zeta k_B T} W(ds \, dt) \\
+ \int_0^\tau \int_0^1 \Psi(s,t) \cdot \Lambda(ds \, dt).

Uniqueness property of \( R \) follows easily. This completes the proof.

Proposition 2.0.3 allows us to pursue each component of \( R \) separately. Concerning existence and uniqueness of \( R^1 \) and \( R^2 \), the readers are advised to see Walsh [21].

Thus we are only to prove existence and uniqueness of the pair of \( R^3 \) and \( \Lambda^3 \). We summarize the task in hand in the following theorem about the reflected stochastic partial differential equation.

**Theorem 2.0.4.** Let \( B \) be a Brownian sheet on the probability space \((\Omega, \mathcal{F}, P)\). There exists a unique pair \((u, \eta)\) of a random field \( u \) and a random Radon measure \( \eta \) such that

1. \( u \) is continuous on \([0,1] \times [0,\infty)\), \( u(x,0) = u_0(x) \geq 0 \) for all \( x \in [0,1] \) and \( u(x,t) \geq 0 \) for all \((x,t) \in [0,1] \times [0,\infty)\) a.s.;

2. \( \eta \) is supported by the set \( \{(x,t) \in [0,1] \times [0,\infty) : u(x,t) = 0\} \) a.s., i.e., for any Borel measurable subset \( A \) of \([0,1] \times [0,\infty)\),

\[ \eta(A) = \int \int_A \chi_{\{0\}}(u(x,t)) \, \eta(dx \, dt) \]
3. For any $\psi \in C^{2,1}([0, 1] \times (0, \infty))$ with $\frac{\partial}{\partial x}\psi(0, t) = \frac{\partial}{\partial x}\psi(1, t) = 0 \forall t > 0$ and for any $\tau > 0$,

$$
\int_0^1 \psi(x, \tau) u(x, \tau) \, dx - \int_0^1 \psi(x, 0) u_0(x) \, dx = \int_0^\tau \int_0^1 \left( \frac{\partial}{\partial t}\psi(x, t) + \frac{\partial^2}{\partial x^2}\psi(x, t) \right) u(x, t) \, dx \, dt + \int_0^\tau \int_0^1 \sigma \psi(x, t) B(dx \, dt) + \int_0^\tau \int_0^1 \psi(x, t) \eta(dx \, dt).
$$

In the remainder of the chapter we prove Theorem 2.0.4 and study the properties of the unique solution.

Formally, we would write the stochastic partial differential equation with reflection given the initial condition $u_0$ as follows:

$$
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 B}{\partial x \, \partial t} + \frac{\partial^2 \eta}{\partial x \, \partial t} \quad (2.0.2)
$$

with the following:

1. $u(x, 0) = u_0(x)$ for all $x \in [0, 1]$;

2. $u(x, t) \geq 0$ for all $(x, t) \in [0, 1] \times [0, \infty)$;

3. $\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t) = 0$ for all $t > 0$;

4. For any Borel measurable subset $A$ of $[0, 1] \times [0, \infty)$,

$$
\eta(A) = \iint_A \chi_{\{0\}}(u(x, t)) \, \eta(dx \, dt);
$$

**Definition 2.0.5.** If $(u, \eta)$ satisfies conditions in Theorem 2.0.4, the pair $(u, \eta)$ is called a weak solution of (2.0.2).
Several authors have worked the stochastic partial differential equations with reflection, see Nualart and Pardoux [13], Donati-Martin and Pardoux [4], Otobe [14] and Zambotti [22]. The difference between the work in this dissertation and those in the papers Nualart and Pardoux [13], Donati-Martin [4] lies in the second order differential operator $\frac{\partial^2}{\partial x^2}$. In Nualart and Pardoux [13], Donati-Martin and Pardoux [4], the authors discuss $\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary condition for the spatial variable $x$. In the view of the reflected Rouse model, the work in the papers gives us the solid ground for the model the end points of which are ‘attached’ at one point in a container though we are interested in the model which freely moves in a container. Hence, Neumann condition of the second order differential operator naturally fits the model of our interest.

In the mathematical point, two equations share the same path toward the proof of existence. Regarding uniqueness, however, we have to look elsewhere.

In the following proposition, we reduce (2.0.2) to a partial differential equation with reflection. Before we go into the proposition, we introduce the unique solution $v$ of the stochastic partial differential equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \sigma \frac{\partial^2 B}{\partial x \partial t} \quad (2.0.3)$$

with the initial condition $u_0$ and $\frac{\partial v}{\partial x} (0, t) = \frac{\partial v}{\partial x} (1, t) = 0$ for all $t > 0$. See [21] for detail.

**Proposition 2.0.6.** There exists a unique weak solution $(u, \eta)$ of (2.0.2) if and only if a.s. there exists a unique pair $(z, \eta)$ such that

1. $z$ is continuous on $[0, 1] \times [0, \infty)$, $z(x, 0) = 0$ for all $x \in [0, 1]$ and $z(x, t) + v(x, t) \geq 0$ for all $(x, t) \in [0, 1] \times [0, \infty)$;
2. \( \eta \) is supported by the set \( \{(x,t) \in [0,1] \times [0,\infty) : z(x,t) + v(x,t) = 0\} \), i.e., for any Borel measurable subset \( A \) of \([0,1] \times [0,\infty)\),

\[
\eta(A) = \iint_A \chi_{\{0\}} (z(x,t) + v(x,t)) \eta(\,dx \,dt) ;
\]

3. For any \( \psi \in C^{2,1}([0,1] \times (0,\infty)) \) with \( \frac{\partial}{\partial x}\psi(0,t) = \frac{\partial}{\partial x}\psi(1,t) = 0 \) \( \forall t > 0 \) and for any \( T > 0 \),

\[
\int_0^1 \psi(x,t) z(x,T) \, dx = \int_0^T \int_0^1 \left( \frac{\partial}{\partial t}\psi(x,t) + \frac{\partial^2}{\partial x^2}\psi(x,t) \right) z(x,t) \, dx \, dt \\
+ \int_0^T \int_0^1 \psi(x,t) \, \eta(\,dx \,dt).
\]

Moreover, \((z + v, \eta)\) is one and only one solution of (2.0.2).

**Proof. Sufficiency** : Suppose that there exists a unique pair \((z, \eta)\). Then, for any \( \psi \in C^{2,1}([0,1] \times (0,\infty)) \) with \( \frac{\partial}{\partial x}\psi(0,s) = \frac{\partial}{\partial x}\psi(1,s) = 0 \) \( \forall s > 0 \) and for any \( T > 0 \)

\[
\int_0^1 \psi(x,t) z(x,T) \, dx = \int_0^T \int_0^1 \left( \frac{\partial}{\partial t}\psi(x,t) + \frac{\partial^2}{\partial x^2}\psi(x,t) \right) z(x,t) \, dx \, dt \\
+ \int_0^T \int_0^1 \psi(x,t) \, \eta(\,dx \,dt).
\]

Since \( v \) is the weak solution of (2.0.3),

\[
\int_0^1 \psi(x,t) v(x,T) \, dx - \int_0^1 \psi(x,t) u_0(x) \, dx \\
= \int_0^T \int_0^1 \left( \frac{\partial}{\partial t}\psi(x,t) + \frac{\partial^2}{\partial x^2}\psi(x,t) \right) v(x,t) \, dx \, dt \\
+ \int_0^T \int_0^1 \psi(x,t) \, W(\,dx \,dt).
\]

So we add the above two to get

\[
\int_0^1 \psi(x,t) (z(x,t) + v(x,t)) \, dx - \int_0^1 \psi(x,t) u_0(x) \, dx
\]
\[
= \int_0^t \int_0^1 \frac{\partial \psi}{\partial t}(x, s) (z(x, t) + v(x, t)) \ dx \ ds \\
+ \int_0^t \int_0^1 \frac{\partial^2 \psi}{\partial x^2}(x, s) (z(x, t) + v(x, t)) \ dx \ ds \\
+ \int_0^t \int_0^1 \psi(x, s) \ W(dx \ ds) + \int_0^t \int_0^1 \psi(x, s) \eta(dx \ ds).
\]

The conditions for \( \eta \) easily hold. And uniqueness property \((u, \eta)\) follows easily.

**Necessity**: Let \( z = u - v \). Then it is easy to see that \((z, \eta)\) satisfies the conditions for necessity. And uniqueness property \((z, \eta)\) follows easily. \(\square\)

Proposition 2.0.6 tells us that it is enough to work on the partial differential equation with reflection, given a stochastic field \( v \):

\[
\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 \eta}{\partial x \partial t}
\]

(2.0.4)

with the following properties of \( z \) and \( \eta \):

1. \( z \) is continuous on \([0, 1] \times [0, \infty)\), \( z(x, 0) = 0 \) for all \( x \in [0, 1] \) and, for all \((x, t) \in [0, 1] \times [0, \infty)\),

\[
z(x, t) + v(x, t) \geq 0;
\]

2. \( \eta \) is supported by the set \( \{(x, t) \in [0, 1] \times [0, \infty) : z(x, t) = v(x, t)\} \), i.e., for any Borel measurable subset \( A \) of \([0, 1] \times [0, \infty)\),

\[
\eta(A) = \iint_A \chi_{\{-v(x,t)\}}(z(x,t)) \ \eta(dx \ dt).
\]

**Definition 2.0.7.** If \((z, \eta)\) satisfies conditions in Proposition 2.0.6, then the pair \((z, \eta)\) is called a weak solution of (2.0.4).

The equation (2.0.4) would be interpreted as a stochastic obstacle problem, i.e., \(-v\) is considered as an obstacle for \( z \). For the problem in hand, we employ penalization
method to prove existence of a solution of the stochastic problem. Readers would see Bensoussan and Lions [1], [2] for more discussion about obstacle problems.

2.1 Existence

Regarding the existence of a weak solution of (2.0.4), we work on partial differential equations which approximate (2.0.4) and then look for a limit of solutions of approximate partial differential equations, which turns out to be the desired solution of (2.0.4).

For any $\varepsilon > 0$, we consider a partial differential equation

$$\frac{\partial z^\varepsilon}{\partial t} = \frac{\partial^2 z^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon} (z^\varepsilon + v)^- \quad (2.1.1)$$

with the initial condition $z^\varepsilon(x, 0) = 0$.

**Lemma 2.1.1.** For $\varepsilon > 0$, there exists a unique continuous function $z^\varepsilon$ on $[0, 1] \times [0, \infty)$ such that

$$z^\varepsilon \in \bigcap_{T>0} L^2(0, T; H^1(0, 1)), \quad \frac{\partial z^\varepsilon}{\partial t} \in \bigcap_{T>0} L^2(0, T; H^{-1}(0, 1)),$$

and for any $w \in H^1(0, 1)$,

$$\int_0^1 \frac{\partial z^\varepsilon(x, t)}{\partial t} w(x) \, dx = - \int_0^1 \frac{\partial z^\varepsilon}{\partial x}(x, t) \frac{dw}{dx}(x) \, dx + \int_0^1 \frac{1}{\varepsilon} (z^\varepsilon + v)^- w(x) \, dx$$

a.e. $t > 0$.

Moreover, if $\varepsilon_1 < \varepsilon_2$,

$$z^{\varepsilon_1} \geq z^{\varepsilon_2} \quad \text{for all } (x, t) \in [0, 1] \times [0, \infty).$$
Proof. For existence and uniqueness, see p.106 in Bensoussan and Lions [2].

For \( \varepsilon_1 < \varepsilon_2 \), let \( z^{\varepsilon_1} \) and \( z^{\varepsilon_2} \) be solutions of (2.1.1) for \( \varepsilon = \varepsilon_1 \) and \( \varepsilon = \varepsilon_2 \), respectively. Let \( \psi = z^{\varepsilon_2} - z^{\varepsilon_1} \). For \( T > 0 \), we have

\[
\begin{align*}
\int_0^T \int_0^1 \frac{\partial \psi}{\partial t}(x,t) \psi^+(x,t) \, dx \, dt + \int_0^T \int_0^1 \frac{\partial \psi}{\partial x}(x,t) \frac{\partial \psi^+}{\partial x}(x,t) \, dx \, dt \\
= \int_0^T \int_0^1 \left( \frac{1}{\varepsilon_2} (z^{\varepsilon_2}(x,t) + v(x,t))^+ - \frac{1}{\varepsilon_1} (z^{\varepsilon_1}(x,t) + v(x,t))^+ \right) \psi^+(x,t) \, dx \, dt.
\end{align*}
\]

Since \( a^- - b^- < 0 \) for \( a > b \),

\[
\begin{align*}
\int_0^T \int_0^1 \left( \frac{1}{\varepsilon_2} (z^{\varepsilon_2}(x,t) + v(x,t))^+ - \frac{1}{\varepsilon_1} (z^{\varepsilon_1}(x,t) + v(x,t))^+ \right) \psi^+(x,t) \, dx \, dt \\
= \int_0^T \int_0^1 \left( \frac{1}{\varepsilon_2} (z^{\varepsilon_1}(x,t) + v(x,t))^+ - \frac{1}{\varepsilon_1} (z^{\varepsilon_1}(x,t) + v(x,t))^+ \right) \psi^+(x,t) \, dx \, dt \\
+ \int_0^T \int_0^1 \left( \frac{1}{\varepsilon_2} (z^{\varepsilon_2}(x,t) + v(x,t))^+ - \frac{1}{\varepsilon_1} (z^{\varepsilon_1}(x,t) + v(x,t))^+ \right) \psi^+(x,t) \, dx \, dt \\
= \frac{1}{\varepsilon_2} \int_0^T \int_0^1 \left( (z^{\varepsilon_2}(x,t) + v(x,t))^+ - (z^{\varepsilon_1}(x,t) + v(x,t))^+ \right) \psi^+(x,t) \, dx \, dt \\
+ \left( \frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1} \right) \int_0^T \int_0^1 (z^{\varepsilon_1}(x,t) + v(x,t))^+ \psi^+(x,t) \, dx \, dt \\
\leq 0,
\end{align*}
\]

from which we see that

\[
\int_0^1 (\psi^+(x,T))^2 \, dx = 0,
\]

and so \( z^{\varepsilon_2} \leq z^{\varepsilon_1} \).

In the following lemma, we find an inequality about two different obstacles and solutions of their penalized equations.

**Lemma 2.1.2.** Suppose that \( \tilde{v} \) be continuous on \([0,1] \times [0,\infty)\) and \( \tilde{z}^{\varepsilon} \) be the unique continuous solution of (2.1.1) with \( \tilde{v} \) instead of \( v \). Then, for any \( T > 0 \),

\[
\sup_{(x,t) \in [0,1] \times [0,T]} |z^{\varepsilon}(x,t) - \tilde{z}^{\varepsilon}(x,t)| \leq \sup_{(x,t) \in [0,1] \times [0,T]} |v(x,t) - \tilde{v}(x,t)|.
\]
Proof. Let \( k = \sup_{(x, t) \in [0, 1] \times [0, T]} |v(x, t) - \tilde{v}(x, t)| \) and \( w = z^\varepsilon - \tilde{z}^\varepsilon - k. \) Then

\[
\int_0^T \int_0^1 \left( \frac{\partial z^\varepsilon}{\partial t}(x, t) - \frac{\partial \tilde{z}^\varepsilon}{\partial t}(x, t) \right) w^+(x, t) \, dx \, dt = - \int_0^T \int_0^1 \left( \frac{\partial z^\varepsilon}{\partial x}(x, t) - \frac{\partial \tilde{z}^\varepsilon}{\partial x}(x, t) \right) \frac{\partial w^+}{\partial x}(x, t) \, dx \, dt \\
+ \int_0^T \int_0^1 \frac{1}{\varepsilon} \left( (z^\varepsilon(x, t) + v(x, t))^\varepsilon - (\tilde{z}^\varepsilon(x, t) + v(x, t))\varepsilon \right) w^+(x, t) \, dx \, dt
\]

or

\[
\int_0^T \int_0^1 \frac{\partial w^+}{\partial t}(x, t) \, w^+(x, t) \, dx \, dt + \int_0^T \int_0^1 \left( \frac{\partial w^+}{\partial x}(x, t) \right)^2 \, dx \, dt = \frac{1}{\varepsilon} \int_0^T \int_0^1 \left( (z^\varepsilon(x, t) + v(x, t))^\varepsilon - (\tilde{z}^\varepsilon(x, t) + v(x, t))\varepsilon \right) w^+(x, t) \, dx \, dt.
\]

Note that, for \((x, t) \in [0, 1] \times [0, T]\) with \( w(x, t) \geq 0, \)

\[
z^\varepsilon(x, t) - \tilde{z}^\varepsilon(x, t) \geq k \geq \tilde{v}(x, t) - v(x, t),
\]

and so

\[
z^\varepsilon(x, t) + v(x, t) \geq \tilde{z}^\varepsilon(x, t) + \tilde{v}(x, t).
\]

Then

\[
((z^\varepsilon + v)^\varepsilon - (\tilde{z}^\varepsilon + \tilde{v})^\varepsilon) \, w^+ \leq 0.
\]

Thus we have

\[
\frac{1}{2} \int_0^1 (w^+)^2(x, T) \, dx = \int_0^T \int_0^1 \frac{\partial w^+}{\partial t}(x, t) \, w^+(x, t) \, dx \, dt \leq 0,
\]

and so \( w^+ = 0, \) which implies

\[
z^\varepsilon(x, t) - \tilde{z}^\varepsilon(x, t) \leq \sup_{(x, t) \in [0, 1] \times [0, T]} |v(x, t) - \tilde{v}(x, t)|
\]

for any \((x, t) \in [0, 1] \times [0, T].\)
By interchanging \( \varepsilon \) and \( \tilde{\varepsilon} \), we also have
\[
\tilde{z}^\varepsilon(x, t) - z^\varepsilon(x, t) \leq \sup_{(x, t) \in [0,1] \times [0, T]} |v(x, t) - \tilde{v}(x, t)|
\]
for any \((x, t) \in [0,1] \times [0, T]\). This completes the proof. \( \square \)

The last lemma toward the existence of a weak solution of (2.0.4) is adopted from Bensoussan and Lions [2]. It is the result about existence and uniqueness of a solution of the partial differential equation with a smooth obstacle.

**Lemma 2.1.3.** Let \( \tilde{v} \) be a \( C^\infty \) function on \([0,1] \times [0, \infty)\) such that
\[
\frac{\partial \tilde{v}}{\partial x}(0, t) = \frac{\partial \tilde{v}}{\partial x}(1, t) = 0 \quad \forall t > 0.
\]

Then there exists a unique \((\tilde{z}, \tilde{\eta})\) of a continuous function \( \tilde{z} \) on \([0,1] \times [0, \infty)\) and a positive Radon measure \( \tilde{\eta} \) on Borel subset of \([0,1] \times [0, \infty)\) such that

1. \( \tilde{z}(x, 0) = 0 \) for all \( x \in [0,1] \);

2. \( \tilde{z}(x, t) + \tilde{v}(x, t) \geq 0 \) for all \((x, t) \in [0,1] \times [0, \infty)\);

3. \( \tilde{\eta} \) is supported by the set \( \{ (x, t) \in [0,1] \times [0, \infty) : \tilde{z}(x, t) + \tilde{v}(x, t) = 0 \} \), i.e., for any Borel measurable subset \( A \) of \([0,1] \times [0, \infty)\),
   \[
   \tilde{\eta}(A) = \iint_A \chi_{\{0\}} (\tilde{z}(x, t) + \tilde{v}(x, t)) \, \tilde{\eta}(dx \, dt);
   \]

4. For any \( \psi \in C^{2,1}(\mathbb{R} \times (0, \infty)) \) with \( \frac{\partial}{\partial x} \psi(0, t) = \frac{\partial}{\partial x} \psi(1, t) = 0 \) \( \forall t > 0 \) and for any \( T > 0 \),
   \[
   \int_0^1 \psi(x, t) \tilde{z}(x, T) \, dx = \int_0^T \int_0^1 \left( \frac{\partial}{\partial t} \psi(x, t) + \frac{\partial^2}{\partial x^2} \psi(x, t) \right) \tilde{z}(x, t) \, dx \, dt
   \]
   \[
   + \int_0^T \int_0^1 \psi(x, t) \, \tilde{\eta}(dx \, dt).
   \]
Moreover, $\tilde{z}^\varepsilon$ converges to $\tilde{z}$ uniformly on compact subsets of $[0,1] \times [0,\infty)$ as $\varepsilon \to 0$.

Proof. See p.105 - p.110 in Bensoussan and Lions [2].  

Now we are ready to prove the existence of a weak solution of (2.0.4).

**Theorem 2.1.4.** There exists a weak solution $(z,\eta)$ of (2.0.4), i.e.,

1. $z$ is continuous on $[0,1] \times [0,\infty)$, $z(x,0) = 0$ for all $x \in [0,1]$ and $z(x,t) + v(x,t) \geq 0$ for all $(x,t) \in [0,1] \times [0,\infty)$;

2. $\eta$ is supported by the set $\{(x,t) \in [0,1] \times [0,\infty) : z(x,t) + v(x,t) = 0\}$, i.e., for any Borel measurable subset $A$ of $[0,1] \times [0,\infty)$,

$$
\eta(A) = \int \int_A \chi_{\{0\}}(z(x,t) + v(x,t)) \eta(dx \, dt);
$$

3. For any $\psi \in C^{2,1}([0,1] \times (0,\infty))$ with $\frac{\partial}{\partial x}\psi(0,t) = \frac{\partial}{\partial x}\psi(1,t) = 0 \forall t > 0$ and for any $T > 0$,

$$
\int_0^1 \psi(x,t)z(x,T) \, dx = \int_0^T \int_0^1 \left( \frac{\partial}{\partial t}\psi(x,t) + \frac{\partial^2}{\partial x^2}\psi(x,t) \right) z(x,t) \, dx \, dt + \int_0^T \int_0^1 \psi(x,t) \eta(dx \, dt).
$$

Proof. Let

$$
z(x,t) = \sup_{\varepsilon > 0} z^\varepsilon(x,t)
$$

for $(x,t) \in [0,1] \times [0,\infty)$. And let $\{v_n\}_{n=1}^\infty$ be a sequence of $C^\infty$ functions on $[0,1] \times [0,\infty)$ converging uniformly on compact subsets of $[0,1] \times [0,\infty)$ such that

$$
\frac{\partial v_n}{\partial x}(0,t) = \frac{\partial v_n}{\partial x}(1,t) = 0 \quad \forall t > 0.
$$

Then Lemma 2.1.3 implies that for each $n$, there exists a pair $(z_n,\eta_n)$ such that
1. \( z_n(x, 0) = 0 \) for all \( x \in [0, 1] \);

2. \( z_n(x, t) + v_n(x, t) \geq 0 \) for all \((x, t) \in [0, 1] \times [0, \infty)\);

3. \( \eta_n \) is supported by the set \( \{(x, t) \in [0, 1] \times [0, \infty) : z_n(x, t) + v_n(x, t) = 0\} \), i.e., for any Borel measurable subset \( A \) of \([0, 1] \times [0, \infty)\),

\[
\eta_n(A) = \iint_A \chi_{\{0\}} (z_n(x,t) + v_n(x,t)) \eta_n(\,dx\,dt) ;
\]

4. For any \( \psi \in C^{2,1} (\mathbb{R} \times (0, \infty)) \) with \( \frac{\partial}{\partial x} \psi(0, t) = \frac{\partial}{\partial x} \psi(1, t) = 0 \) \( \forall t > 0 \) and for any \( T > 0 \),

\[
\int_0^1 \psi(x, t) z_n(x, T) \, dx = \int_0^T \int_0^1 \left( \frac{\partial}{\partial t} \psi(x, t) + \frac{\partial^2}{\partial x^2} \psi(x, t) \right) z_n(x, t) \, dx \, dt \\
+ \int_0^T \int_0^1 \psi(x, t) \eta_n(\,dx\,dt) ;
\]

5. \( z_n^\varepsilon \) converges to \( z_n \) uniformly on compact subsets of \([0, 1] \times [0, \infty)\) as \( \varepsilon \to 0 \).

Lemma 2.1.2 implies that for any \( T > 0 \),

\[
\sup_{(x,t) \in [0,1] \times [0,T]} |z^\varepsilon(x,t) - z_n^\varepsilon(x,t)| \leq \sup_{(x,t) \in [0,1] \times [0,T]} |v(x,t) - v_n(x,t)| .
\]

Letting \( \varepsilon \) tend to zero, we have

\[
\sup_{(x,t) \in [0,1] \times [0,T]} |z(x,t) - z_n(x,t)| \leq \sup_{(x,t) \in [0,1] \times [0,T]} |v(x,t) - v_n(x,t)| .
\]

Again letting \( n \) tend to infinity, we conclude, by Dini’s Theorem, that \( z \) is a continuous function to which \( z_n \) converges uniformly on compact subsets of \([0, 1] \times [0, \infty)\) and that \( z(x, t) + v(x, t) \geq 0 \) for all \((x, t) \in [0, 1] \times [0, \infty)\).

With \( z \) in hand, we turn our attention to \( \eta \). Since \( z_n \) converges to \( z \), we can characterize \( \eta \) using \( \eta_n \) as follows. For any \( \psi \in C^{2,1} (\mathbb{R} \times (0, \infty)) \) with \( \frac{\partial}{\partial x} \psi(0, t) = \)
\( \frac{\partial}{\partial x} \psi(1,t) = 0 \) \( \forall t > 0 \) and for any \( T > 0 \),

\[
\int_0^T \int_0^1 \psi(x,t) \eta(dx \, dt) \\
= \int_0^1 \psi(x,t)z(x,T) \, dx - \int_0^T \int_0^1 \left( \frac{\partial}{\partial t} \psi(x,t) + \frac{\partial^2}{\partial x^2} \psi(x,t) \right) z(x,t) \, dx \, dt \\
= \lim_{n \to \infty} \left( \int_0^1 \psi(x,t)z_n(x,T) \, dx - \int_0^T \int_0^1 \left( \frac{\partial}{\partial t} \psi(x,t) + \frac{\partial^2}{\partial x^2} \psi(x,t) \right) z_n(x,t) \, dx \, dt \right) \\
= \lim_{n \to \infty} \int_0^T \int_0^1 \psi(x,t) \, \eta_n(dx \, dt).
\]

Note that, for any compact subset \( K \) of \([0, 1] \times [0, \infty)\),

\[
\iint_K (z(x,t) + v(x,t)) \, \eta(dx \, dt) = \lim_{n \to \infty} \iint_K (z_n(x,t) + v_n(x,t)) \, \eta_n(dx \, dt) \\
= 0,
\]

and so

\[
\iint (z(x,t) + v(x,t)) \, \eta(dx \, dt) = 0,
\]

i.e., \( \eta \) is supported by the set \( \{(x,t) \in [0, 1] \times [0, \infty) : z(x,t) + v(x,t) = 0 \} \). Thus, for any Borel subset \( A \) of \([0, 1] \times [0, \infty)\),

\[
\eta(A) = \iint_A \chi_{\{0\}} (z(x,t) + v(x,t)) \, \eta(dx \, dt).
\]

This completes the proof. \( \square \)

### 2.2 Uniqueness

Uniqueness arguments get clues from the following observation: if \((z_1, \eta_1)\) and \((z_2, \eta_2)\) are two weak solutions of (2.0.4), then, for any \( \psi \in C^{2,1}(\mathbb{R} \times (0, \infty)) \) supported in \( \{z_1 > z_2\} \) with \( \frac{\partial}{\partial x} \psi(0,t) = \frac{\partial}{\partial x} \psi(1,t) = 0 \) \( \forall t > 0 \),

\[
- \int_0^\infty \int_0^1 \left( \frac{\partial}{\partial t} \psi(x,t) + \frac{\partial^2}{\partial x^2} \psi(x,t) \right) (z_1(x,t) - z_2(x,t)) \, dx \, dt \leq 0.
\]
In the following, we connect the observation above with Weak Maximum Principle in PDE and derive a version of the principle that fits our specific purpose. The first lemma is a mere extension of the principle for periodic functions.

**Lemma 2.2.1.** Let \( z(x, t) \) be a \( C^{2,1}(\mathbb{R} \times [0, \infty)) \) function which is periodic in \( x \). Let \( U \) be a non-empty connected open subset of \( \mathbb{R} \times (0, \infty) \). If

\[
\frac{\partial z}{\partial t}(x, t) - \frac{\partial^2 z}{\partial x^2}(x, t) \leq 0 \quad \text{for all } (x, t) \in U,
\]

then

\[
\sup \{ z(x, t) : (x, t) \in \bar{U} \} = \sup \{ z(x, t) : (x, t) \in \partial U \}.
\]

**Proof.** First we assume that

\[
\frac{\partial z}{\partial t}(x, t) - \frac{\partial^2 z}{\partial x^2}(x, t) < 0 \quad \text{for all } (x, t) \in U
\]

Suppose that \( U \) is bounded in the second coordinate, i.e, there is \( T > 0 \) such that \( U \subset \mathbb{R} \times (0, T) \). Since \( z \) is periodic in \( x \), \( z \) is bounded on \( U \). If there is \( (x_0, t_0) \in U \) such that

\[
z(x_0, t_0) = \sup \{ z(x, t) : (x, t) \in \bar{U} \}
\]

then

\[
\frac{\partial z}{\partial t}(x_0, t_0) = 0, \quad \frac{\partial^2 z}{\partial x^2}(x_0, t_0) \leq 0,
\]

and so

\[
\frac{\partial z}{\partial t}(x_0, t_0) - \frac{\partial^2 z}{\partial x^2}(x_0, t_0) \geq 0,
\]

which contradicts the assumption. Thus

\[
\sup \{ z(x, t) : (x, t) \in \bar{U} \} = \sup \{ z(x, t) : (x, t) \in \partial U \}.
\]
Suppose that $U$ is unbounded in the second coordinate. Let $T > 0$ be given, $U_T = U \cap (\mathbb{R} \times (0, T))$ and $\Gamma_T = \partial U \cap (\mathbb{R} \times [0, T])$. By the account we see above, $z$ cannot attain its maximum at any point in $U_T$. If $z$ attains its maximum on $\overline{U_T}$ at $(x_0, t_0) \in \partial U_T - \partial U$, i.e., $t_0 = T$, then $\frac{\partial z}{\partial t}(x_0, t_0) \geq 0$ and $\frac{\partial^2 z}{\partial x^2}(x_0, t_0) \leq 0$, which contradicts the assumption. Thus

$$\sup \{ z(x, t) : (x, t) \in \overline{U_T} \} = \sup \{ z(x, t) : (x, t) \in \Gamma_T \}.$$ 

Since there is a sequence $\{(x_n, t_n)\}$ in $\overline{U}$ such that $z(x_n, t_n) \to \sup \{ z(x, t) : (x, t) \in \overline{U} \}$ as $n$ tends to infinity, each $z(x_n, t_n)$ is bounded above by $\sup \{ z(x, t) : (x, t) \in \overline{U} \}$ and

$$\sup \{ z(x, t) : (x, t) \in \overline{U} \} \leq \sup \{ z(x, t) : (x, t) \in \partial U \},$$

we have

$$\sup \{ z(x, t) : (x, t) \in \overline{U} \} = \lim_{n \to \infty} z(x_n, t_n) \leq \lim_{n \to \infty} \sup \{ z(x, t) : (x, t) \in \Gamma_{t_n} \} \leq \sup \{ z(x, t) : (x, t) \in \partial U \}.$$ 

Thus

$$\sup \{ z(x, t) : (x, t) \in \overline{U} \} = \sup \{ z(x, t) : (x, t) \in \partial U \}.$$ 

Now we assume that

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} \leq 0 \quad \text{on} \ U.$$ 

For $\varepsilon > 0$, let $z_\varepsilon(x, t) = z(x, t) - \varepsilon t$. Then

$$\frac{\partial z_\varepsilon}{\partial t} - \frac{\partial^2 z_\varepsilon}{\partial x^2} < 0 \quad \text{on} \ U.$$
By the same argument above, we see that for any $T > 0$,

$$\sup \{ z_\varepsilon(x, t) : (x, t) \in U_T \} = \sup \{ z_\varepsilon(x, t) : (x, t) \in \Gamma_T \}.$$ 

Since $z_\varepsilon(x, t)$ and $z(x, t)$ are periodic in $x$, $z_\varepsilon$ converges uniformly to $z$ on any closed subset of $\mathbb{R} \times [0, T]$. If

$$\sup \{ z(x, t) : (x, t) \in U_T \} > \sup \{ z(x, t) : (x, t) \in \Gamma_T \},$$

then there is $(x', t') \in \overline{U_T}$ such that

$$z(x', t') > \sup \{ z(x, t) : (x, t) \in \Gamma_T \}.$$ 

However, since $z \geq z_\varepsilon$,

$$\sup \{ z(x, t) : (x, t) \in \Gamma_T \} \geq \lim_{\varepsilon \to 0} \sup \{ z_\varepsilon(x, t) : (x, t) \in \Gamma_T \} \geq \lim_{\varepsilon \to 0} \sup \{ z_\varepsilon(x, t) : (x, t) \in U_T \} \geq \lim_{\varepsilon \to 0} z_\varepsilon(x', t') = z(x', t') > \sup \{ z(x, t) : (x, t) \in \Gamma_T \},$$

which is contradicting. Thus we have

$$\sup \{ z(x, t) : (x, t) \in \overline{U_T} \} = \sup \{ z(x, t) : (x, t) \in \Gamma_T \}.$$ 

Let $\{(x_n, t_n)\}$ be a sequence of points in $\overline{U}$ such that

$$\lim_{n \to \infty} z(x_n, t_n) = \sup \{ z(x, t) : (x, t) \in \overline{U} \}.$$ 

Then

$$\sup \{ z(x, t) : (x, t) \in \overline{U} \} = \lim_{n \to \infty} z(x_n, t_n)$$

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\[ \leq \lim_{n \to \infty} \sup \{ z(x, t) : (x, t) \in \Gamma_n \} \]
\[ \leq \sup \{ z(x, t) : (x, t) \in \partial U \} . \]

Thus we have
\[ \sup \{ z(x, t) : (x, t) \in \overline{U} \} = \sup \{ z(x, t) : (x, t) \in \partial U \} . \]

\[ \square \]

With Lemma (2.2.1) in hand, we prove a weak maximum principle for weak sub-solutions of heat equations.

**Theorem 2.2.2.** Let \( z(x, t) \) be a continuous function on \( \mathbb{R} \times [0, \infty) \) which is periodic in \( x \). Let \( U \) be a non-empty connected open subset of \( \mathbb{R} \times (0, \infty) \). If
\[ - \int_0^\infty \int_{\mathbb{R}} z(x, t) \left( \frac{\partial \phi}{\partial t}(x, t) + \frac{\partial^2 \phi}{\partial x^2}(x, t) \right) \, dx \, dt \leq 0, \]
for any nonnegative \( \phi \in C^{2,1}_c(\mathbb{R} \times [0, \infty)) \) with \( \text{supp} \phi \subset U \), then
\[ \sup \{ z(x, t) : (x, t) \in \overline{U} \} = \sup \{ z(x, t) : (x, t) \in \partial U \} . \]

**Proof.** For any \( \varepsilon > 0 \), we let \( U_\varepsilon = \left\{ (x, t) \in U \mid \inf_{(y, s) \in \partial U} |(x, t) - (y, s)| < \varepsilon \right\} \). Let \( \phi \) be a \( C^\infty(\mathbb{R}^2) \) function defined by
\[ \phi(x, t) = \begin{cases} C \exp\left( \frac{1}{x^2 + t^2 - 1} \right) & \text{if } x^2 + t^2 < 1, \\ 0 & \text{otherwise,} \end{cases} \]
where \( C \) is a constant such that
\[ C^{-1} = \int_{\{x^2 + t^2 < 1\}} \exp\left( \frac{1}{x^2 + t^2 - 1} \right) \, dx \, dt. \]
For any \( \delta > 0 \), we let \( \phi_\delta(x, t) = \frac{1}{\delta^2} \phi\left( \frac{x}{\delta}, \frac{t}{\delta} \right) \) and \( z_\delta \) the convolution of \( z \) with \( \phi_\delta \), i.e.,
\[ z_\delta(x, t) = \int_{(t-\delta)^+}^{t+\delta} \int_{\mathbb{R}} z(y, s) \phi_\delta(x - y, t - s) \, dy \, ds. \]
Note that \( z_\delta(x, t) \) is periodic in \( x \) and \( C^\infty(\mathbb{R} \times (0, \infty)) \) that, for any \((x, t) \in U_\delta\),

\[
\frac{\partial^2 z_\delta}{\partial x^2}(x, t) = \int_0^\infty \int_\mathbb{R} z(y, s) \frac{\partial^2 \phi_\delta}{\partial x^2}(x - y, t - s) \, dx \, ds
\]

\[
= \int_0^\infty \int_\mathbb{R} z(y, s) \frac{\partial^2 \phi_\delta}{\partial x^2}(x - y, t - s) \, dy \, ds,
\]

\[
\frac{\partial z_\delta}{\partial t}(x, t) = \int_0^\infty \int_\mathbb{R} z(y, s) \frac{\partial \phi_\delta}{\partial t}(x - y, t - s) \, dx \, ds
\]

\[
= - \int_0^\infty \int_\mathbb{R} z(y, s) \frac{\partial \phi_\delta}{\partial s}(x - y, t - s) \, dx \, ds.
\]

Then, for \( \delta < \varepsilon \) and for \((x, t) \in U_\varepsilon\),

\[
\frac{\partial z_\delta}{\partial t}(x, t) - \frac{\partial^2 z_\delta}{\partial x^2}(x, t)
\]

\[
= - \int_0^\infty \int_\mathbb{R} z(y, s) \frac{\partial \phi_\delta}{\partial t}(x - y, t - s) \, dy \, ds - \int_0^\infty \int_\mathbb{R} z(y, s) \frac{\partial^2 \phi_\delta}{\partial y^2}(x - y, t - s) \, dy \, ds
\]

\[
\leq 0.
\]

By the lemma above, we have

\[
sup \{ z_\delta(x, t) : (x, t) \in U_\varepsilon \} = sup \{ z_\delta(x, t) : (x, t) \in \partial U_\varepsilon \},
\]

\[
sup \{ z_\delta(x, t) : (x, t) \in U_\varepsilon \cap (\mathbb{R} \times (0, T)) \} = sup \{ z_\delta(x, t) : (x, t) \in \partial U_\varepsilon \cap (\mathbb{R} \times [0, T]) \}
\]

for any \( T > 0 \). Since \( z_\delta \) converges uniformly to \( z \) on \( U_\varepsilon \cap (\mathbb{R} \times (0, T)) \) and \( \partial U_\varepsilon \cap (\mathbb{R} \times [0, T]) \), by letting \( \delta \to 0 \), we have

\[
sup \{ z(x, t) : (x, t) \in \overline{U_\varepsilon} \cap (\mathbb{R} \times (0, T)) \} = sup \{ z(x, t) : (x, t) \in \partial U_\varepsilon \cap \mathbb{R} \times [0, T] \}.
\]

Note that

\[
sup \{ z(x, t) : (x, t) \in \overline{U} \cap (\mathbb{R} \times (0, T)) \} = \lim_{\varepsilon \to 10} sup \{ z(x, t) : (x, t) \in U_\varepsilon \cap (\mathbb{R} \times (0, T)) \}
\]

\[
= \lim_{\varepsilon \to 10} sup \{ z(x, t) : (x, t) \in \partial U_\varepsilon \cap (\mathbb{R} \times [0, T]) \}.
\]

Since \( z(x, t) \) periodic in \( x \), \( z \) is uniformly continuous in \( \overline{U} \cap (\mathbb{R} \times (0, T)) \), i.e., for any \( \gamma > 0 \), there is \( \varepsilon_\gamma > 0 \) such that

\[
\| (x, t) - (x', t') \| < \varepsilon_\gamma \Rightarrow |z(x, t) - z(x', t')| < \gamma.
\]

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Then, for $\varepsilon < \varepsilon_\gamma$ and $(x,t) \in \partial U \cap (\mathbb{R} \times [0,T])$, there is $(x',t') \in \partial U \cap (\mathbb{R} \times [0,T])$ such that

$$z(x,t) < z(x',t') + \gamma,$$

which implies

$$z(x,t) < \sup \left\{ z(\tilde{x},\tilde{t}) : (\tilde{x},\tilde{t}) \in \partial U \cap \mathbb{R} \times [0,T] \right\} + \gamma,$$

and so

$$\lim_{\varepsilon \to 0} \sup \left\{ z(x,t) : (x,t) \in \partial U \cap (\mathbb{R} \times [0,T]) \right\} \leq \sup \left\{ z(\tilde{x},\tilde{t}) : (\tilde{x},\tilde{t}) \in \partial U \cap \mathbb{R} \times [0,T] \right\}.$$

Thus

$$\sup \left\{ z(x,t) : (x,t) \in \overline{U} \cap (\mathbb{R} \times (0,T)) \right\} = \sup \left\{ z(\tilde{x},\tilde{t}) : (\tilde{x},\tilde{t}) \in \partial U \cap \mathbb{R} \times [0,T] \right\}.$$

Since

$$\sup \left\{ z(x,t) : (x,t) \in \overline{U} \right\} = \lim_{T \to \infty} \sup \left\{ z(x,t) : (x,t) \in \overline{U} \cap (\mathbb{R} \times (0,T)) \right\},$$

$$\lim_{T \to \infty} \sup \left\{ z(x,t) : (x,t) \in \partial U \cap (\mathbb{R} \times [0,T]) \right\} \leq \sup \left\{ z(x,t) : (x,t) \in \partial U \right\},$$

we have

$$\sup \left\{ z(x,t) : (x,t) \in \overline{U} \right\} \leq \sup \left\{ z(x,t) : (x,t) \in \partial U \right\},$$

which implies

$$\sup \left\{ z(x,t) : (x,t) \in \overline{U} \right\} = \sup \left\{ z(x,t) : (x,t) \in \partial U \right\}.$$

This completes the proof. \qed

Let $(z,\eta)$ be as in Theorem 2.1.4. We extend the domain of $z$ to $\mathbb{R} \times [0,\infty)$ as follows

$$z(x,t) = \begin{cases} 
    z(x-2n,t) & \text{if } 2n \leq x < 2n+1 \text{ for some integer } n, \\
    z(2n+2-x,t) & \text{if } 2n+1 \leq x < 2n+2 \text{ for some integer } n.
\end{cases}$$
And we also extend $\eta$ to a Radon measure on $\mathbb{R} \times [0, \infty)$ as follows. Let $U$ be any Borel set in $\mathbb{R} \times [0, \infty)$. For each integer $n$, let

$$U_n = \begin{cases} U \cap ([n, n+1] \times [0, \infty)) & \text{if } n \text{ is even}, \\ \{(x, t) \mid (1-x, t) \in U \cap ([n, n+1] \times [0, \infty))\} & \text{if } n \text{ is odd}. \end{cases}$$

Define the extension of $\eta$ by

$$\eta(U) = \sum_{n=-\infty}^{\infty} \eta(U_n).$$

Then $z$ is a continuous function on $\mathbb{R} \times [0, \infty)$ and $\eta$ is a Radon measure on $\mathbb{R} \times [0, \infty)$.

Note that, for any odd integer $n$ and any $\eta$-integrable function $\phi$,

$$\int_0^\infty \int_0^1 \phi(n+1-x, t) \eta(dx \, dt) = \int_0^\infty \int_n^{n+1} \phi(x, t) \eta(dx \, dt).$$

**Proposition 2.2.3.** For any $C^{2,1}_c(\mathbb{R} \times [0, \infty))$ function $\psi$, we have

$$0 = \int_0^\infty \int \psi(x, t) \left( \frac{\partial^2 \psi}{\partial x^2}(x, t) + \frac{\partial \psi}{\partial t}(x, t) \right) dx \, dt + \int \int \psi(x, t) \eta(dx \, dt) + \int \int \psi(x, t) \eta(dx \, dt)$$

**Proof.** Let $\psi \in C^{2,1}_c(\mathbb{R} \times [0, \infty))$ be given. For each integer $n$ and $(x, t) \in [0, 1] \times [0, \infty)$, let

$$\psi_n(x, t) = \begin{cases} \psi(x + n, t) & \text{if } n \text{ is even}, \\ \psi(n + 1 - x, t) & \text{if } n \text{ is odd}. \end{cases}$$

Then, for each $n$,

$$0 = -\int_0^\infty \left( z(1, t) \frac{\partial \psi_n}{\partial x}(1, t) - z(0, t) \frac{\partial \psi_n}{\partial x}(0, t) \right) dt + \int_0^\infty \int_0^1 z(x, t) \left( \frac{\partial^2 \psi_n}{\partial x^2}(x, t) + \frac{\partial \psi_n}{\partial t}(x, t) \right) dx \, dt + \int_0^\infty \int_0^1 \psi_n(x, t) \eta(dx \, dt),$$

where

$$\frac{\partial \psi_n}{\partial x}(1, t) = \lim_{\varepsilon \to 0^+} \frac{\psi_n(1-\varepsilon, t) - \psi_n(1, t)}{-\varepsilon}, \quad \frac{\partial \psi_n}{\partial x}(0, t) = \lim_{\varepsilon \to 0^+} \frac{\psi_n(0+\varepsilon, t) - \psi_n(0, t)}{\varepsilon}.$$
Note that, for an even integer $n$,

\[
\frac{\partial \psi_n}{\partial x}(1, t) = \lim_{\epsilon \to 0^+} \frac{\psi_n(1 - \epsilon, t) - \psi_n(1, t)}{-\epsilon} = \frac{\psi(n + 1 - \epsilon, t) - \psi(n + 1, t)}{-\epsilon} - \frac{\partial \psi}{\partial x}(n + 1, t),
\]

\[
\frac{\partial \psi_{n+1}}{\partial x}(1, t) = \lim_{\epsilon \to 0^+} \frac{\psi_{n+1}(1 - \epsilon, t) - \psi_{n+1}(1, t)}{-\epsilon} = \frac{\psi(n + 1 + \epsilon, t) - \psi(n + 1, t)}{-\epsilon} - \frac{\partial \psi}{\partial x}(n + 1, t),
\]

\[
\frac{\partial \psi_n}{\partial x}(0, t) = \lim_{\epsilon \to 0^+} \frac{\psi_n(0 + \epsilon, t) - \psi_n(0, t)}{-\epsilon} = \frac{\psi(n + \epsilon, t) - \psi(n, t)}{\epsilon} - \frac{\partial \psi}{\partial x}(n, t),
\]

\[
\frac{\partial \psi_{n-1}}{\partial x}(0, t) = \lim_{\epsilon \to 0^+} \frac{\psi_{n-1}(0 + \epsilon, t) - \psi_{n-1}(1, t)}{\epsilon} = \frac{\psi(n - \epsilon, t) - \psi(n, t)}{\epsilon} - \frac{\partial \psi}{\partial x}(n, t)
\]

Now we sum

\[
0 = -\int_0^\infty \left( z(1, t) \frac{\partial \psi_n}{\partial x}(1, t) - z(0, t) \frac{\partial \psi_n}{\partial x}(0, t) \right) dt
\]

\[
+ \int_0^\infty \int_0^1 z(x, t) \left( \frac{\partial^2 \psi_n}{\partial x^2}(x, t) + \frac{\partial \psi_n}{\partial t}(x, t) \right) dx dt + \int_0^\infty \int_0^1 \psi_n(x, t) \eta(dx dt)
\]

over $n$. Notice that all but a finite number of summands are zero for $\psi$ is compactly supported. Then we have

\[
0 = \sum_{n=-\infty}^\infty \int_0^\infty \int_0^1 z(x, t) \left( \frac{\partial^2 \psi_n}{\partial x^2}(x, t) + \frac{\partial \psi_n}{\partial t}(x, t) \right) dx dt
\]

\[
+ \sum_{n=-\infty}^\infty \int_0^\infty \int_0^1 \psi_n(x, t) \eta(dx dt)
\]

since

\[
\frac{\partial}{\partial x} \psi_{2n}(0, t) = -\frac{\partial}{\partial x} \psi_{2n-1}(0, t) \quad \text{and} \quad \frac{\partial}{\partial x} \psi_{2n}(1, t) = -\frac{\partial}{\partial x} \psi_{2n+1}(1, t)
\]

imply

\[
\sum_{n=-\infty}^\infty \int_0^\infty \left( z(1, t) \frac{\partial \psi_n}{\partial x}(1, t) - z(0, t) \frac{\partial \psi_n}{\partial x}(0, t) \right) dt = 0.
\]
We look into each sum. Note that

\[
\sum_{n=-\infty}^{\infty} \int_0^\infty \int_0^1 z(x, t) \left( \frac{\partial^2 \psi_n}{\partial x^2}(x, t) + \frac{\partial \psi_n}{\partial t}(x, t) \right) \, dx \, dt
= \sum_{n: \text{even}} \int_0^\infty \int_0^1 z(x, t) \left( \frac{\partial^2 \psi_n}{\partial x^2}(x, t) + \frac{\partial \psi_n}{\partial t}(x, t) \right) \, dx \, dt
+ \sum_{n: \text{odd}} \int_0^\infty \int_0^1 z(x, t) \left( \frac{\partial^2 \psi_n}{\partial x^2}(x, t) + \frac{\partial \psi_n}{\partial t}(x, t) \right) \, dx \, dt
\]

\[
= \sum_{n: \text{even}} \int_0^\infty \int_0^{n+1} z(x, t) \left( \frac{\partial^2 \psi}{\partial x^2}(x, t) + \frac{\partial \psi}{\partial t}(x, t) \right) \, dx \, dt
+ \sum_{n: \text{odd}} \int_0^\infty \int_0^{n+1} z(x, t) \left( \frac{\partial^2 \psi}{\partial x^2}(x, t) + \frac{\partial \psi}{\partial t}(x, t) \right) \, dx \, dt
\]

\[
= \int_0^\infty \int_{\mathbb{R}} z(x, t) \left( \frac{\partial^2 \psi}{\partial x^2}(x, t) + \frac{\partial \psi}{\partial t}(x, t) \right) \, dx \, dt
\]

and that

\[
\sum_{n=-\infty}^{\infty} \int_0^\infty \int_0^1 \psi_n(x, t) \, \eta(dx \, dt)
= \sum_{n: \text{even}} \int_0^\infty \int_0^1 \psi_n(x, t) \, \eta(dx \, dt) + \sum_{n: \text{odd}} \int_0^\infty \int_0^1 \psi_n(x, t) \, \eta(dx \, dt)
= \sum_{n: \text{even}} \int_0^\infty \int_0^{n+1} \psi(x, t) \, \eta(dx \, dt) + \sum_{n: \text{odd}} \int_0^\infty \int_n^{n+1} \psi(x, t) \, \eta(dx \, dt)
= \sum_{n=-\infty}^{\infty} \int_0^\infty \int_n^{n+1} \psi(x, t) \, \eta(dx \, dt)
= \sum_{n=-\infty}^{\infty} \left( \int_{[n,n+1] \times [0,\infty)} \psi(x, t) \, \eta(dx \, dt) + \int_{\{n\} \times [0,\infty)} \psi(x, t) \, \eta(dx \, dt) \right)
= \int_{\mathbb{R} \times [0,\infty)} \psi(x, t) \, \eta(dx \, dt) + \int_{\mathbb{Z} \times [0,\infty)} \psi(x, t) \, \eta(dx \, dt).
\]
This completes the proof.

Now we are ready to prove uniqueness of a weak solution of (2.0.4).

**Theorem 2.2.4.** There exists a unique pair \((z, \eta)\) such that

1. \(z\) is continuous on \([0, 1] \times [0, \infty)\), \(z(x, 0) = 0\) for all \(x \in [0, 1]\) and \(z(x, t) + v(x, t) \geq 0\) for all \((x, t) \in [0, 1] \times [0, \infty)\);

2. \(\eta\) is supported by the set \(\{(x, t) \in [0, 1] \times [0, \infty) : z(x, t) + v(x, t) = 0\}\), i.e., for any Borel measurable subset \(A\) of \([0, 1] \times [0, \infty)\),
   \[
   \eta(A) = \iint_A \chi_{\{0\}} (z(x, t) + v(x, t)) \eta \, (dx \, dt);
   \]

3. For any \(\psi \in C^{2,1} (\mathbb{R} \times (0, \infty))\) with \(\frac{\partial}{\partial x} \psi(0, t) = \frac{\partial}{\partial x} \psi(1, t) = 0\) \(\forall t > 0\) and for any \(T > 0\),
   \[
   \int_0^1 \int_0^T \psi(x, t) z(x, T) \, dx = \int_0^T \int_0^1 \left( \frac{\partial}{\partial t} \psi(x, t) + \frac{\partial^2}{\partial x^2} \psi(x, t) \right) z(x, t) \, dx \, dt
   + \int_0^T \int_0^1 \psi(x, t) \eta(dx \, dt).
   \]

**Proof.** Suppose that \((z_1, \eta_1)\) and \((z_2, \eta_2)\) are such pairs. Assume that the set \(U = \{(x, t) \in [0, 1] \times [0, \infty) \mid z_1(x, t) > z_2(x, t)\}\) is non-empty and connected. Since

\[
z_1(x, t) + v(x, t) > z_2(x, t) + v(x, t) \geq 0
\]

for any \((x, t) \in U\),

\[
\iint_{\mathbb{R} \times [0, \infty)} \psi(x, t) \eta_1 \, (dx \, dt) + \iint_{\mathbb{R} \times [0, \infty)} \psi(x, t) \eta_1 \, (dx \, dt) = 0
\]

for any nonnegative \(C^{2,1} (\mathbb{R} \times [0, \infty))\) function \(\phi\) with \(\text{supp} \phi \subset U\). Thus, for such \(\phi\),

\[
-\int_0^\infty \int_\mathbb{R} (z_1(x, t) - z_2(x, t)) \left( \frac{\partial \phi}{\partial s} (x, t) + \frac{\partial^2 \phi}{\partial x^2} (x, t) \right) \, dx \, dt
\]
\[
\begin{align*}
&= \int\int_{\mathbb{R} \times [0, \infty)} \phi(x, t) \eta_1 \ (dx \ dt) + \int\int_{\mathbb{Z} \times [0, \infty)} \phi(x, t) \eta_1 \ (dx \ dt) \\
&\quad - \int\int_{\mathbb{R} \times [0, \infty)} \phi(x, t) \eta_2 \ (dx \ dt) - \int\int_{\mathbb{Z} \times [0, \infty)} \phi(x, t) \eta_2 \ (dx \ dt) \\
&= - \int\int_{\mathbb{R} \times [0, \infty)} \phi(x, t) \eta_2 \ (dx \ dt) - \int\int_{\mathbb{Z} \times [0, \infty)} \phi(x, t) \eta_2 \ (dx \ dt) \\
&\leq 0.
\end{align*}
\]

By Theorem 2.2.2,
\[
\sup_U (z_1 - z_2) = \sup_{\partial U} (z_1 - z_2).
\]
Since \(z_1 - z_2 > 0\) on \(U\) and \(\sup_{\partial U} (z_1 - z_2) = 0\), it is contradiction to the fact that \(U\) is non-empty. Thus, \(U\) is an empty set.

By symmetry, we conclude that
\[
z_1 \equiv z_2.
\]

Thus
\[
\eta_1 \equiv \eta_2.
\]

This completes the proof. \(\square\)
CHAPTER 3

RELATED RESULTS AND FUTURE DIRECTIONS

In this chapter we present results obtained cast while we have worked on the Rouse models and discuss the future directions.

3.1 Stochastic Curves

We recall the Frenet-Serret formula for a curve in $\mathbb{R}^3$ from differential geometry. Let $u$ be a smooth curve from $[0, \infty)$ into $\mathbb{R}^3$ parametrized by arc length. And we denote the unit tangent vector, the unit normal vector and the unit binormal vector to $u$ at $t \in [0, \infty)$ by $T(t)$, $N(t)$ and $B(t)$, respectively. Then the curvature $\kappa$ and the torsion $\tau$ are defined by

$$\kappa(t) = \|T'(t)\| \quad \text{and} \quad \tau(t) = \|B'(t)\|.$$  

The Frenet-Serret formula says

$$T'(t) = \kappa(t)N(t)$$
$$N'(t) = -\kappa(t)T(t) + \tau(t)B(t) \quad (3.1.1)$$
$$B'(t) = -\tau(t)N(t)$$
If we are given the data, that is, \( u(0), T(0), N(0), B(0), \kappa \) and \( \tau \), we can reconstruct the curve by solving the differential equation (3.1.1). Thus the curvature and the torsion determine the curve.

Inspired by the observation above, we investigate the situation that the frame field \( \{T, N, B\} \) of a curve would be random. First we rewrite (3.1.1) in the following form

\[
\begin{align*}
    dT(t) &= N(t)\kappa(t)dt \\
    dN(t) &= -T(t)\kappa(t)dt + B(t)\tau(t)dt \\
    dB(t) &= -N(t)\tau(t)dt
\end{align*}
\]

and consider the case

\[
\kappa(t)dt = dX_t^{(1)} \quad \text{and} \quad \tau(t)dt = dX_t^{(2)},
\]

where \( X^{(1)} \) and \( X^{(2)} \) are semimartingales. Then the Frenet-Serret formula for a random curve would be

\[
\begin{align*}
    dT_t &= N_t \circ dX_t^{(1)} \\
    dN_t &= -T_t \circ dX_t^{(1)} + B_t \circ dX_t^{(2)} \\
    dB_t &= -N_t \circ dX_t^{(2)}
\end{align*}
\]

where \( \circ \) denotes the Fisk-Stratonovich stochastic integral. We use the Fisk-Stratonovich stochastic integral rather than the Itô integral since the former behaves more like the ordinary integral in view of change of variable formula. We will see that the Itô integral fails in the Frenet-Serret formula for a random curve.

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space and the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfy the usual condition. Let

\[
X_t^{(1)} = X_0^{(1)} + M_t^{(1)} + A_t^{(1)} \quad \text{and} \quad X_t^{(2)} = X_0^{(2)} + M_t^{(2)} + A_t^{(2)}
\]
be continuous semimartingales, where $M^{(1)}$ and $M^{(2)}$ are continuous local martingales with respect to the filtration $\{F_t\}_{t \geq 0}$, and $A^{(1)}$ and $A^{(2)}$ are the differences of continuous, nondecreasing, adapted processes.

**Proposition 3.1.1.** Let $t, n$ and $b$ be $\mathbb{R}^3$ vectors. Then there exists a unique solution $(T_t, N_t, B_t)$ to (3.1.3) such that $(T_t)_{t \geq 0}$, $(N_t)_{t \geq 0}$ and $(B_t)_{t \geq 0}$ are continuous and adapted $\mathbb{R}^3$-valued processes, and

$$T_0 = t \quad N_0 = n \quad B_0 = b.$$ 

**Proof.** First we rewrite (3.1.3) in terms of Itô integrals:

$$dT_t = N_t \, dX_t^{(1)} + \frac{1}{2} \, d\langle N, M^{(1)} \rangle_t,$$

$$dN_t = -T_t \, dX_t^{(1)} + B_t \, dX_t^{(2)} - \frac{1}{2} \, d\langle T, M^{(1)} \rangle_t + \frac{1}{2} \, d\langle B, M^{(2)} \rangle_t,$$

$$dB_t = -N_t \, dX_t^{(2)} - \frac{1}{2} \, d\langle N, M^{(2)} \rangle_t,$$

or

$$dT_t = N_t \, dX_t^{(1)} - \frac{1}{2} \, T_t \, d\langle M^{(1)} \rangle_t + \frac{1}{2} \, B_t \, d\langle M^{(1)}, M^{(2)} \rangle_t,$$

$$dN_t = -T_t \, dX_t^{(1)} + B_t \, dX_t^{(2)} - \frac{1}{2} \, N_t \, d\langle M^{(1)} \rangle_t - \frac{1}{2} \, N_t \, d\langle M^{(2)} \rangle_t,$$

$$dB_t = -N_t \, dX_t^{(2)} + \frac{1}{2} \, T_t \, d\langle M^{(1)}, M^{(2)} \rangle_t - \frac{1}{2} \, B_t \, d\langle M^{(2)} \rangle_t.$$ 

(3.1.4)

We note that (3.1.3) and (3.1.4) are equivalent in the sense that they both have the same solution given the same initial condition.

Existence of the solution to (3.1.4) is verified by using Picard-Lindelöf iteration scheme. See the proof of (2.1) Theorem on page 375 of Revus and Yor [15] for detail. Similarly, uniqueness holds.

**Theorem 3.1.2.** If the initial value $(t, n, b)$ to (3.1.3) forms an orthonormal frame in $\mathbb{R}^3$, so does the unique solution $(T_t, N_t, B_t)$ to (3.1.3).
Proof. To check the orthonormality, we consider

\[ d \|T_t\|^2 = \sum_{i=1,2,3} 2T_t^{(i)} \circ dT_t^{(i)}, \]
\[ d \|N_t\|^2 = \sum_{i=1,2,3} 2N_t^{(i)} \circ dN_t^{(i)}, \]
\[ d \|B_t\|^2 = \sum_{i=1,2,3} 2B_t^{(i)} \circ dB_t^{(i)}, \]
\[ d(T_t, N_t) = \sum_{i=1,2,3} T_t^{(i)} \circ dN_t^{(i)} + \sum_{i=1,2,3} N_t^{(i)} \circ dT_t^{(i)}, \]
\[ d(N_t, B_t) = \sum_{i=1,2,3} N_t^{(i)} \circ dB_t^{(i)} + \sum_{i=1,2,3} B_t^{(i)} \circ dN_t^{(i)}, \]
\[ d(B_t, T_t) = \sum_{i=1,2,3} B_t^{(i)} \circ dT_t^{(i)} + \sum_{i=1,2,3} T_t^{(i)} \circ dB_t^{(i)}. \]

As before, we rewrite (3.1.5) in terms of Itô integral.

\[ d \|T_t\|^2 = 2 \sum_{i=1,2,3} T_t^{(i)} dT_t^{(i)} + \sum_{i=1,2,3} d\langle T^{(i)} \rangle_t, \]
\[ d \|N_t\|^2 = \sum_{i=1,2,3} 2N_t^{(i)} dN_t^{(i)} + \sum_{i=1,2,3} d\langle N^{(i)} \rangle_t, \]
\[ d \|B_t\|^2 = \sum_{i=1,2,3} 2B_t^{(i)} dB_t^{(i)} + \sum_{i=1,2,3} d\langle B^{(i)} \rangle_t, \]
\[ d(T_t, N_t) = \sum_{i=1,2,3} T_t^{(i)} dN_t^{(i)} + \sum_{i=1,2,3} N_t^{(i)} dT_t^{(i)} + \sum_{i=1,2,3} d\langle T^{(i)}, N^{(i)} \rangle_t, \]
\[ d(N_t, B_t) = \sum_{i=1,2,3} N_t^{(i)} dB_t^{(i)} + \sum_{i=1,2,3} B_t^{(i)} dN_t^{(i)} + \sum_{i=1,2,3} d\langle N^{(i)}, B^{(i)} \rangle_t, \]
\[ d(B_t, T_t) = \sum_{i=1,2,3} B_t^{(i)} dT_t^{(i)} + \sum_{i=1,2,3} T_t^{(i)} dB_t^{(i)} + \sum_{i=1,2,3} d\langle B^{(i)}, T^{(i)} \rangle_t, \]

or

\[ d \|T_t\|^2 = 2 (T_t, N_t) dX_t^{(1)} + (\|N_t\|^2 - \|T_t\|^2) d\langle M^{(1)} \rangle_t \]
\[ + (T_t, B_t) d\langle M^{(1)}, M^{(2)} \rangle_t, \]
\[ d \|N_t\|^2 = -2 (N_t, T_t) dX_t^{(1)} + 2 (N_t, B_t) dX_t^{(2)} \]
\[ + (\|T_t\|^2 - \|N_t\|^2) d\langle M^{(1)} \rangle_t - 2 (T_t, B_t) d\langle M^{(1)}, M^{(2)} \rangle_t \]
\[ d\|B_t\|^2 = -2 (N_t, B_t) \, dX_t^{(2)} + (B_t, T_t) \, d\langle M^{(1)}, M^{(2)} \rangle_t \]

\[ + \left( \|N_t\|^2 - \|B_t\|^2 \right) d\langle M^{(2)} \rangle_t, \]

\[ d\|T_t\|^2 = \left( \|N_t\|^2 - \|T_t\|^2 \right) dX_t^{(1)} + (B_t, T_t) \, dX_t^{(2)} \]

\[ - 2 (T_t, N_t) \, d\langle M^{(1)} \rangle_t + \frac{3}{2} (N_t, B_t) \, d\langle M^{(1)}, M^{(2)} \rangle_t \]

\[ - \frac{1}{2} (T_t, N_t) \, d\langle M^{(2)} \rangle_t, \]

\[ d\|N_t\|^2 = \left( \|N_t\|^2 - \|B_t\|^2 \right) dX_t^{(1)} + (B_t, T_t) \, dX_t^{(2)} \]

\[ - \frac{1}{2} (N_t, B_t) \, d\langle M^{(1)} \rangle_t + \frac{3}{2} (T_t, N_t) \, d\langle M^{(1)}, M^{(2)} \rangle_t \]

\[ - 2 (T_t, B_t) \, d\langle M^{(2)} \rangle_t, \]

\[ d\langle T_t, N_t \rangle = (B_t, T_t) \, dX_t^{(1)} - (T_t, N_t) \, dX_t^{(2)} \]

\[ - \frac{1}{2} (B_t, T_t) \, d\langle M^{(1)} \rangle_t + \left( \frac{1}{2} \|T_t\|^2 - \|N_t\|^2 + \frac{1}{2} \|B_t\|^2 \right) d\langle M^{(1)}, M^{(2)} \rangle_t \]

\[ - \frac{1}{2} (B_t, T_t) \, d\langle M^{(2)} \rangle_t. \]

It is shown that the above stochastic differential equation (3.1.6) regarding \( \|T_t\|^2 \), \( \|N_t\|^2 \), \( \|B_t\|^2 \), \( \langle T_t, N_t \rangle \), \( \langle N_t, B_t \rangle \) and \( \langle B_t, T_t \rangle \) possesses a unique strong solution using the same argument of the proof of Theorem 3.1.2.

However,

\[
\begin{align*}
\|T_t\|^2 &= 1, \\
\|N_t\|^2 &= 1, \\
\|B_t\|^2 &= 1, \\
\langle T_t, N_t \rangle &= 0, \\
\langle N_t, B_t \rangle &= 0, \\
\langle B_t, T_t \rangle &= 0,
\end{align*}
\]

satisfy (3.1.6) and the initial value

\[
\begin{align*}
\|t\|^2 &= 1, \quad \|n\|^2 = 1, \quad \|b\|^2 = 1, \\
\langle t, n \rangle &= 0, \quad \langle n, b \rangle = 0, \quad \langle b, t \rangle = 0.
\end{align*}
\]

Thus

\[
\begin{align*}
\|T_t\|^2 &= 1, \\
\|N_t\|^2 &= 1, \\
\|B_t\|^2 &= 1, \\
\langle T_t, N_t \rangle &= 0, \\
\langle N_t, B_t \rangle &= 0, \\
\langle B_t, T_t \rangle &= 0.
\end{align*}
\]
is the unique solution to (3.1.6). Therefore,

\[(T_t, N_t, B_t)\]

forms an orthonormal frame in \(\mathbb{R}^3\).

Remark 3.1.3. We mention that the usual Itô integral does not work for the Frenet-Serret formula. We give an example which shows the Frenet-Serret formula fail. Let \(W(s) = (W(1)(s), W(2)(s))\) be a 2-dimensional Brownian motion on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Consider the stochastic differential equation

\[
\begin{align*}
    dT_t &= N_t \circ dW_t^{(1)} \\
    dN_t &= -T_t \circ dW_t^{(1)} + B_t \circ dW_t^{(2)} \\
    dB_t &= -N_t \circ dW_t^{(2)}
\end{align*}
\]  

(3.1.7)

It is well-known that there exists a unique strong solution to (3.1.7) with any initial condition. Suppose that

\[
\begin{align*}
    \|T(0)\|^2 &= 1, \\
    \|N(0)\|^2 &= 1, \\
    \|B(0)\|^2 &= 1, \\
    (T(0), N(0)) &= 0, \\
    (T(0), B(0)) &= 0, \\
    (N(0), B(0)) &= 0.
\end{align*}
\]

By use of Itô’s formula, we have

\[
\|T_t\|^2 = \int_0^t 2 \langle T_s, N_s \rangle \ dW_s^{(1)} + \int_0^t \|N_s\|^2 \ ds.
\]

Thus (3.1.7) cannot have the orthonormal solution \((T, N, B)\).

Remark 3.1.4. We consider the case that \(X^{(1)}\) and \(X^{(2)}\) are independent Brownian motions, i.e.,

\[
X_t^{(1)} = B_t^{(1)} \quad \text{and} \quad X_t^{(2)} = B_t^{(2)}
\]
Then the Frenet-Serret formula is
\[
\begin{align*}
    dT_t &= N_t \circ dW_t^{(1)} + B_t \circ dW_t^{(2)} \\
    dN_t &= -T_t \circ dW_t^{(1)} + B_t \circ dW_t^{(2)} \\
    dB_t &= -N_t \circ dW_t^{(2)}
\end{align*}
\] (3.1.8)

In terms of Itô integral, it is
\[
\begin{align*}
    T_t &= T_0 + \int_0^t N_s \; dW_s^{(1)} - \frac{1}{2} \int_0^t T_s \; ds \\
    N_t &= N_0 - \int_0^t T_s \; dW_s^{(1)} + \int_0^t B_s \; dW_s^{(2)} - \int_0^t N_s \; ds \\
    B_t &= B_0 - \int_0^t N_s \; dW_s^{(2)} - \frac{1}{2} \int_0^t B_s \; ds
\end{align*}
\]

Note that
\[
\begin{align*}
    ET_t &= ET_0 + E \int_0^t N_s \; dW_s^{(1)} - \frac{1}{2} E \int_0^t T_s \; ds \\
    &= ET_0 - \frac{1}{2} \int_0^t ET_s \; ds.
\end{align*}
\]

If we let \( f_i(t) = ET_t^{(i)} \) for \( i = 1, 2, 3 \), then
\[
    f_i(t) = f_i(t) - \frac{1}{2} \int_0^t f_i(s) \; ds \quad f_i'(t) = -\frac{1}{2} f_i(t).
\]

Thus we have
\[
    ET_t = e^{-\frac{1}{2}t} ET_0 \quad \text{for all } t \geq 0.
\]

Similarly, we also have
\[
    EN_t = e^{-t} EN_0 \quad EB_t = e^{-\frac{1}{2}t} EB_0 \quad \text{for all } t \geq 0.
\]

We also note from (3.1.6) that
\[
    E \left( T_t^{(i)} \right)^2 = E \left( T_0^{(i)} \right)^2 - \frac{1}{2} \int_0^t E \left( T_s^{(i)} \right)^2 \; ds + \int_0^t E \left( N_s^{(i)} \right)^2 \; ds
\]

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\[
E(N_t^{(i)})^2 = E(T_0^{(i)})^2 + \int_0^t E(T_s^{(i)})^2 \, ds - \int_0^t E(N_s^{(i)})^2 \, ds + \int_0^t E(B_s^{(i)})^2 \, ds
\]
\[
E(B_t^{(i)})^2 = E(T_0^{(i)})^2 + \int_0^t E(N_s^{(i)})^2 \, ds - \frac{1}{2} \int_0^t E(B_s^{(i)})^2 \, ds
\]

Thus
\[
\begin{pmatrix}
E(T_t^{(i)})^2 \\
P(N_t^{(i)})^2 \\
P(B_t^{(i)})^2
\end{pmatrix} = e^{At} \begin{pmatrix}
E(T_0^{(i)})^2 \\
P(N_0^{(i)})^2 \\
P(B_0^{(i)})^2
\end{pmatrix},
\]

where
\[
A = \begin{pmatrix}
-1/2 & 1 & 0 \\
1 & -1 & 1 \\
0 & 1 & -1/2
\end{pmatrix}
\]

From (3.1.6), we see that
\[
E(T_t^{(i)})N_t^{(i)} = E(T_0^{(i)})N_0^{(i)} - \frac{5}{2} \int_0^t E(T_s^{(i)})N_s^{(i)} \, ds.
\]

Thus
\[
E(T_t^{(i)})N_t^{(i)} = e^{-\frac{5}{2}t}E(T_0^{(i)})N_0^{(i)}.
\]

Similarly,
\[
E(N_t^{(i)})B_t^{(i)} = e^{-\frac{5}{2}t}E(N_0^{(i)})B_0^{(i)} \quad E(B_t^{(i)})T_t^{(i)} = e^{-t}E(B_0^{(i)})T_0^{(i)}.
\]

### 3.2 The reflected physical Brownian motion

A reflected Brownian motion \(|B_t|\) in the positive real line \(\mathbb{R}_+\) has been studied for a long time. It is a weak solution to the reflected stochastic differential equation
\[
dY_t = dW_t + dL_t
\]
where \(W_t\) is a Brownian motion, \(Y_t \geq 0\) for all \(t \geq 0\) and \(\int_0^\infty Y_t \, dL_t = 0\). More precisely, by Tanaka formula,
\[
|B_t| = \int_0^t \text{sgn}(B_u) \, dB_u + 2L_t,
\]

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where \( L \) is the local time of the Brownian motion \( B \) at 0. The Skorokhod equation gives a way to analyze \( L \).

The Skorokhod equation is to, given a continuous function \( w \) from \([0, \infty)\) into \( \mathbb{R} \) with \( w(0) \geq 0 \), find a continuous function \( k \) from \([0, \infty)\) into \( \mathbb{R} \) such that

1. \( x(t) \equiv w(t) + k(t) \geq 0 \) for all \( t \geq 0 \),

2. \( k(0) = 0 \) and \( k \) is nondecreasing, and

3. \( k \) changes only at \( t \)'s at which \( x(t) = 0 \).

The equation has a unique solution and the solution provides the precise form of the local time \( L_t \) of the Brownian motion \( B_t \) as

\[
L_t = \max_{0 \leq s \leq t} \left( -\int_0^s \text{sgn} (B_u) \, dB_u \right).
\]

(3.2.2)

Notice from the above that \( |B| \) stays at 0 as long as \( \int_0^t \text{sgn} (B_u) \, dB_u \) reaches new minimum.

If one considers a Brownian motion as a random path of a particle in the real line \( \mathbb{R} \), then the reflected Brownian motion may not be a feasible path of the particle in the positive real line \( \mathbb{R}^+ \) since the impact from the collision against the wall \( x = 0 \) is not shown in (3.2.2).

One learns from elementary physics that, when a ball collides against a wall, its speeds right before and after the collision are equal under the assumption that the ball collides with the wall perfectly elastically. Simple calculation shows that the velocity \( v_+ \) right after the collision is

\[
v_+ = v_- - 2 (v_- \cdot n) n,
\]

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where \( v_- \) is the velocity right before the collision and \( n \) is the inward unit normal vector to the wall at the point of the collision.

Here we investigate the motion of a perfectly elastic particle of mass \( m > 0 \) in the positive real line \( \mathbb{R}_+ = (0, \infty) \). We denote \( Y_t^{(m)} \) by the position of the particle at time \( t \) and assume that the particle undergoes friction. Let \( \zeta > 0 \) be a friction constant and \( W_t \) a one-dimensional Brownian motion on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). And we denote the reflecting force against the wall by \( \dot{\Lambda}_t^{(m)} \). Then the equation of the balance of the forces acting on the particle is written in the following form:

\[
m \frac{d^2 Y_t^{(m)}}{dt^2} = -\zeta \frac{dY_t^{(m)}}{dt} + \sigma \dot{W}_t + \dot{\Lambda}_t^{(m)}
\]

where \( \Lambda^{(m)} \) is nondecreasing only at \( t \) when \( Y_t^{(m)} = 0 \). If we denote the velocity of the particle by \( U_t^{(m)} \), (3.2.5) would be written in the first-order reflected stochastic differential equation

\[
dY_t^{(m)} = U_t^{(m)} dt,
\]

\[
m dU_t^{(m)} = -\zeta U_t^{(m)} dt + \sigma dW_t + d\Lambda_t^{(m)},
\]

and

\[
\int_0^\infty Y_t^{(m)} d\Lambda_t^{(m)} = 0.
\]

We call \( Y_t^{(m)} \) the physically reflected Brownian motion. Due to the assumption of perfectly elastic collision, it is necessary to assume that the speed \( |U_t^{(m)}| \) is continuous in \( t \). This assumption naturally characterizes the reflecting force \( \dot{\Lambda}_t^{(m)} \). If the particle hits the wall with velocity \( V_{t_0-}^{(m)} < 0 \), then the velocity \( U_{t_0+} \) must be \(-U_{t_0-}^{(m)}\) and so \( \dot{\Lambda}_{t_0+}^{(m)} - \dot{\Lambda}_{t_0-}^{(m)} \) is

\[
(-2mU_{t_0-}^{(m)}).
\]

Only when the contact velocity is negative, it is easy to see the reflection of the particle against the wall. However, we should be careful about the case that the
particle lands on the wall with zero velocity, i.e., there is no physical reflection. Then the move of the particle solely depends on the velocity field. In the following section, we consider the case and prove that the particle does not land on the wall with zero velocity.

3.2.1 The Physical Brownian Motion

By the physical Brownian motion we mean the stochastic process \( (X_t^{(m)})_{t \geq 0} \) satisfying the stochastic differential equation

\[
m \frac{d^2}{dt^2} X_t^{(m)} = -\zeta \frac{d}{dt} X_t^{(m)} + \sigma \dot{W}_t. \tag{3.2.5}
\]

If we denote \( \frac{d}{dt} X_t^{(m)} \) by \( V_t^{(m)} \), then (3.2.5) is written in the first-order stochastic differential equation

\[
dX_t^{(m)} = V_t^{(m)} \, dt, \\
m \, dV_t^{(m)} = -\zeta V_t^{(m)} \, dt + \sigma \, dW_t. \tag{3.2.6}
\]

Let the initial position and initial velocity of the physical Brownian motion be \( X_0 \) and \( V_0 \), respectively. Then the unique strong solution \( (X_t^{(m)}, V_t^{(m)}) \) of (3.2.6) is written in the closed form

\[
X_t^{(m)} = X_0^{(m)} + \frac{m}{\zeta} \left( 1 - e^{-\frac{m}{\zeta} t} \right) V_0^{(m)} + \frac{\sigma}{m} W_t - \frac{\sigma}{m} e^{-\frac{\zeta}{m} t} \int_0^t e^{\frac{\zeta}{m} u} \, dW_u \tag{3.2.7}
\]

\[
V_t^{(m)} = e^{-\frac{\zeta}{m} t} V_0^{(m)} + \frac{\sigma}{m} e^{-\frac{\zeta}{m} t} \int_0^t e^{\frac{\zeta}{m} u} \, dW_u. \tag{3.2.8}
\]

Note that \( (X_t^{(m)}, V_t^{(m)}) \) is Gaussian. Then the distribution of \( (X_t^{(m)}, V_t^{(m)}) \) is determined by its mean and covariance matrix, i.e.,

\[
E \left( X_t^{(m)}, V_t^{(m)} \right) = \left( EX_0^{(m)} + \frac{m}{\zeta} \left( 1 - e^{-\frac{m}{\zeta} t} \right) EV_0, \ e^{-\frac{\zeta}{m} t} EV_0 \right), \\
Cov \left( X_t^{(m)}, V_t^{(m)} \right) = \frac{\sigma^2}{\zeta^2} \left( t - 2 \frac{m}{\zeta} \left( 1 - e^{-\frac{\zeta}{m} t} \right) + \frac{m}{2 \zeta} \left( 1 - e^{-\frac{2 \zeta}{m} t} \right) + \frac{1}{2} \left( 1 - e^{-\frac{\zeta}{m} t} \right)^2 \right). \\
\]

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If we denote the covariance matrix $\text{Cov} \left( X_t^{(m)}, V_t^{(m)} \right)$ by $Q(t)$, then the transition probability density function $p^{(m)}(s, x, v; t, y, u)$ of the Markov process $\left( X_t^{(m)}, V_t^{(m)} \right)$ is given by

$$
\frac{1}{2\pi \sqrt{\det Q^{(m)}(t-s)}} \exp \left\{ - \left( y - x - \frac{m}{\zeta} \left( 1 - e^{-\frac{\zeta}{m}(t-s)} \right) v \right) u - e^{-\frac{\zeta}{m}(t-s)} v \right\} \left( Q^{(m)}(t-s) \right)^{-1} \left( y - x - \frac{m}{\zeta} \left( 1 - e^{-\frac{\zeta}{m}(t-s)} \right) v \right).$$

By Itô’s formula, for any $C^2$ function $\phi$, we have

$$
\phi \left( X_t^{(m)}, V_t^{(m)} \right) = \phi \left( X_0^{(m)}, V_0^{(m)} \right) + \int_0^t V_s^{(m)} \frac{\partial \phi}{\partial x} \left( X_s^{(m)}, V_s^{(m)} \right) \, ds - \frac{\zeta}{m} \int_0^t V_s^{(m)} \frac{\partial \phi}{\partial v} \left( X_s^{(m)}, V_s^{(m)} \right) \, ds
$$
$$
+ \frac{\sigma}{m} \int_0^t \frac{\partial \phi}{\partial v} \left( X_s^{(m)}, V_s^{(m)} \right) \, dW_s + \frac{\sigma^2}{2 m^2} \int_0^t \frac{\partial^2 \phi}{\partial v^2} \left( X_s^{(m)}, V_s^{(m)} \right) \, ds
$$
$$
= \phi \left( X_0, V_0 \right) + \frac{\sigma}{m} \int_0^t \frac{\partial \phi}{\partial v} \left( X_s^{(m)}, V_s^{(m)} \right) \, dW_s
$$
$$
+ \int_0^t \left[ \frac{\sigma^2}{2 m^2} \frac{\partial^2 \phi}{\partial v^2} \left( X_s^{(m)}, V_s^{(m)} \right) + V_s^{(m)} \frac{\partial \phi}{\partial x} \left( X_s^{(m)}, V_s^{(m)} \right) - \frac{\zeta}{m} V_s^{(m)} \frac{\partial \phi}{\partial v} \left( X_s^{(m)}, V_s^{(m)} \right) \right] \, ds.
$$

Let $L^{(m)}$ be a differential operator defined by

$$
L^{(m)} \phi(x, v) = \frac{\sigma^2}{2 m^2} \frac{\partial^2 \phi}{\partial v^2}(x, v) + \frac{\partial \phi}{\partial x}(x, v) - \frac{\zeta}{m} v \frac{\partial \phi}{\partial v}(x, v).
$$

We denote the formal adjoint of $L$ by $L^*$. Then the transition probability density function $p(s, x, v; t, y, u)$ satisfies

$$
\frac{\partial p}{\partial s}(s, x, v; t, y, u) = L^{(m)} p(s, x, v; t, y, u), \quad \frac{\partial p}{\partial t}(s, x, v; t, y, u) = (L^{(m)})^* p(s, x, v; t, y, u).
$$

**Lemma 3.2.1.** Let $(X_0, V_0) \neq (0, 0)$. Then, $P_{(X_0, V_0)}$-almost surely,

$$
\left( X_t^{(m)}, V_t^{(m)} \right) \neq (0, 0) \quad \forall t > 0.
$$
Proof. Let
\[
\phi(x, v) = \int_0^\infty e^{-t} p(0, x, v; t, 0, 0) \, dt.
\]
Then \( \phi \) is \( C^2 \) and
\[
L\phi(x, v) = \int_0^\infty e^{-t} L p(0, x, v; t, 0, 0) \, dt
\]
\[
= \int_0^\infty e^{-t} \frac{\partial p}{\partial s}(0, x, v; t, 0, 0) \, dt
\]
\[
= \int_0^\infty e^{-t} \left( -\frac{\partial p}{\partial t}(0, x, v; t, 0, 0) \right) \, dt
\]
\[
= -\int_0^\infty e^{-t} \frac{\partial p}{\partial t}(0, x, v; t, 0, 0) \, dt
\]
\[
= -\int_0^\infty e^{-t} p(0, x, v; t, 0, 0) \, dt,
\]
\[
\text{i.e.,}
\]
\[
L\phi(x, v) = -\phi(x, v).
\]

Then, by Itô’s formula,
\[
\phi\left(X_t^{(m)}, V_t^{(m)}\right) - \phi(X_0, V_0) = \frac{\sigma}{m} \int_0^t \frac{\partial \phi}{\partial v}\left(X_s^{(m)}, V_s^{(m)}\right) \, dW_s + \int_0^t L\phi\left(X_s^{(m)}, V_s^{(m)}\right) \, ds.
\]

Since \( L\phi = -\phi \) and \( \phi \geq 0 \), \( \phi(X_t, V_t) \) is a supermartingale.

For any \( r > 0 \), let \( S_r = \inf \left\{ t : \left(X_t^{(m)}\right)^2 + \left(V_t^{(m)}\right)^2 = r^2 \right\} \). Then, for \( 0 < r < R \) with \( x_0^2 + v_0^2 < R^2 \), by the optional stopping theorem we have
\[
E_{(X_0, V_0)} \phi\left(X_{S_r \wedge S_R}^{(m)}, V_{S_r \wedge S_R}^{(m)}\right) \leq \phi(X_0, V_0).
\]
The above implies
\[
\inf_{x^2 + v^2 = r^2} \phi(x, v) P_{(X_0, V_0)} \left\{ S_r < S_R \right\} \leq E_{(X_0, V_0)} \phi\left(X_{S_r \wedge S_R}, V_{S_r \wedge S_R}\right) \leq \phi(X_0, V_0).
\]
and
\[
P_{(X_0, V_0)} \left\{ S_r < S_R \right\} \leq \frac{\phi(X_0, V_0)}{\inf_{x^2 + v^2 = r^2} \phi(x, v)}
\]
Note that, by Fatou’s lemma,

\[
\lim_{r \to 0} \inf_{x^2 + v^2 = r^2} \phi(x, v) = \lim_{r \to 0} \inf_{x^2 + v^2 = r^2} \int_0^\infty e^{-t} p(0, x; t, 0, 0) \, dt \\
\geq \int_0^\infty \lim_{r \to 0} \inf_{x^2 + v^2 = r^2} e^{-t} p(0, x; t, 0, 0) \, dt \\
= \int_0^\infty \frac{\zeta^2}{2\pi \sigma^2} e^{-t} \sqrt{\frac{2m}{\zeta}} \left(1 - e^{-\frac{2\zeta^2}{m} t}\right) - \left(1 - e^{-\frac{\zeta^2}{m} t}\right)^2 e^{-t} \, dt \\
= +\infty.
\]

Thus

\[
P_{(X_0, V_0)} \{ T_0^{(m)} < S_R^{(m)} \} = 0,
\]

where \( T_0^{(m)} = \inf \{ t : (X_t^{(m)}, V_t^{(m)}) = (0, 0) \} \). Letting \( R \) tend to \( \infty \), we have the desired result, i.e.,

\[
P_{(X_0, V_0)} \{ T_0^{(m)} < \infty \} = 0.
\]

\[\square\]

Suppose that \( u \) is a \( C^2 \) function from \( \mathbb{R} \). Then

\[
u(X_t^{(m)}) = u(X_0) + \int_0^t u'(X_s) \, dX_s \\
= u(X_0) + \int_0^t u'(X_s^{(m)}) V_s^{(m)} \, ds.
\]

Note that

\[
u'(X_t^{(m)}) V_t^{(m)} \\
= u'(X_0) V_0^{(m)} + \int_0^t u'(X_s^{(m)}) \, dV_s^{(m)} + \int_0^t V_s^{(m)} du'(X_s^{(m)}) \\
= u'(X_0^{(m)}) V_0^{(m)} + \int_0^t u'(X_s^{(m)}) \left( -\frac{\zeta}{m} V_s^{(m)} \, ds + \frac{\sigma}{m} \, dW_s \right) + \int_0^t V_s \, du'(X_s^{(m)}) \\
= u'(X_0) V_0^{(m)} - \frac{\zeta}{m} \int_0^t u'(X_s^{(m)}) V_s^{(m)} \, ds + \frac{\sigma}{m} \int_0^t u'(X_s^{(m)}) \, dW_s
\]
\[ + \int_0^t (V_{s}^{(m)})^2 u''(X_{s}^{(m)}) \, ds. \]

Let
\[ f = \begin{cases} \frac{1}{(x-1)^2-1} & 0 < x < 2, \\ 0 & \text{otherwise,} \end{cases} \]

where \( c^{-1} = \int_{-\infty}^{\infty} f(x) \, dx \). For \( n = 1, 2, \ldots \), let \( f_n(x) = n f(nx) \). Let \( u(x) = |x| \) and \( u \) be the convolution of \( u \) and \( f \). Then

\[ u_n(X_t^{(m)}) = u_n(X_0) + \int_0^t u_n'(X_s^{(m)}) V_s^{(m)} \, ds \]

\[ u_n'(X_t^{(m)}) = u_n'(X_0^{(m)}) V_0^{(m)} - \frac{\zeta}{m} \int_0^t u_n'(X_s^{(m)}) V_s^{(m)} \, ds \]

or

\[ m u_n'(X_t^{(m)}) = m u_n'(X_0^{(m)}) V_0^{(m)} - \frac{\zeta}{m} \int_0^t u_n'(X_s^{(m)}) V_s^{(m)} \, ds \]

Then

\[ u_n(X_t^{(m)}) \to |X_t^{(m)}| \text{ a.s.,} \]

\[ u_n'(X_t^{(m)}) \to \text{sgn}(X_t^{(m)}) \text{ a.s.,} \]

\[ \int_0^t u_n'(X_s^{(m)}) \, ds \to \int_0^t \text{sgn}(X_s^{(m)}) \, ds \text{ a.s.,} \]

\[ \int_0^t u_n'(X_s^{(m)}) \, dW_s \to \int_0^t \text{sgn}(X_s^{(m)}) \, dW_s \text{ in probability,} \]

where \( \text{sgn}(x) = 1 \) if \( x > 0 \) and \( \text{sgn}(x) = -1 \) if \( x \leq 0 \). Thus

\[ m \int_0^t (V_s^{(m)})^2 u''(X_s^{(m)}) \, ds \]

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should be convergent in probability and its limit in probability is denoted by $\Lambda^{(m)}_t$.

Note that $\Lambda^{(m)}_t$ is nondecreasing by the convexity of $f_n$. Hence we have

$$d\left|X^{(m)}_t\right| = \text{sgn}\left(X^{(m)}_t\right) V^{(m)}_t \, dt,$$

$$md\left(\text{sgn}\left(X^{(m)}_t\right) V^{(m)}_t\right) = -\zeta \text{sgn}\left(X^{(m)}_t\right) V^{(m)}_t \, dt + \sigma \text{sgn}\left(X^{(m)}_t\right) dW_t + d\Lambda^{(m)}_t.$$

Since

$$E\left(\int_0^t \text{sgn}\left(X^{(m)}_s\right) \, dW_s\right)^2 = t$$

for all $t > 0$, by Lévy’s criterion on Brownian motions,

$$\int_0^t \text{sgn}\left(X^{(m)}_s\right) \, dW_s$$

is a Brownian motion, and we denote it by $B^{(m)}_t$.

We let

$$Y^{(m)}_t = \left|X^{(m)}_t\right| \quad \text{and} \quad U^{(m)}_t = \text{sgn}\left(X^{(m)}_t\right) V^{(m)}_t.$$

Then

$$dY^{(m)}_t = U^{(m)}_t \, dt,$$

$$mdu^{(m)}_t = -\zeta u^{(m)}_t \, dt + \sigma \, dB^{(m)}_t + d\Lambda^{(m)}_t.$$

Lemma 3.2.1 tells us that whenever $X^{(m)}_t = 0$, $X^{(m)}_t$ changes signs, i.e., $V^{(m)}_t \neq 0$.

Thus $U^{(m)}_t$ is discontinuous only at $t$ where $X^{(m)}_t = 0$ and, at such $t$’s,

$$U^{(m)}_{t+} - U^{(m)}_{t-} = \lim_{s \rightarrow t^{+}} \text{sgn}\left(X^{(m)}_s\right) V^{(m)}_s - \lim_{s \rightarrow t^{-}} \text{sgn}\left(X^{(m)}_s\right) V^{(m)}_s$$

$$= \lim_{\epsilon \rightarrow 0} \left[\text{sgn}\left(X^{(m)}_{t+\epsilon}\right) V^{(m)}_{t+\epsilon} - \text{sgn}\left(X^{(m)}_{t-\epsilon}\right) V^{(m)}_{t-\epsilon}\right]$$

$$= 2 \left|V^{(m)}_t\right|$$

$$= -2U^{(m)}_{t-},$$

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i.e.,

\[ U_{t+}^{(m)} = -U_{t-}^{(m)}. \]

And, since

\[ \Lambda_{t+}^{(m)} - \Lambda_{t-}^{(m)} = m\left(U_{t+}^{(m)} - U_{t-}^{(m)}\right) + \zeta \int_{t-}^{t+} U_{s}^{(m)} \, ds - \sigma \left(B_{t+}^{(m)} - B_{t-}^{(m)}\right). \]

\[ \Lambda_{t+}^{(m)} - \Lambda_{t-}^{(m)} = m\left(U_{t+}^{(m)} - U_{t-}^{(m)}\right). \]

For \( t > 0 \) with \( X_{t}^{(m)} \neq 0 \),

\[ U_{t+}^{(m)} = U_{t-}^{(m)}, \]

and

\[ \Lambda_{t+}^{(m)} - \Lambda_{t-}^{(m)} = m\left(U_{t+}^{(m)} - U_{t-}^{(m)}\right) = 0. \]

Since \( \Lambda_{t}^{(m)} \) is nondecreasing in \( t \), there are countably many discontinuities of \( \Lambda \) on a finite interval. If \( T \) is an accumulation point of discontinuities on a finite interval, then there exists a sequence \( \{T_{n}\} \) of discontinuities of \( \Lambda^{(m)} \) such that \( T_{n} \to T \) as \( n \to \infty \). Since \( X_{T_{n}}^{(m)} = 0 \) for each \( n = 1, 2, \ldots \),

\[ V_{T}^{(m)} = \lim_{n \to \infty} \frac{X_{T_{n}}^{(m)} - X_{T}^{(m)}}{T_{n} - T} = 0, \]

i.e., \( \left(X_{T}^{(m)}, V_{T}^{(m)}\right) = (0, 0) \). By Lemma 3.2.1, the set of discontinuities of \( \Lambda \) on a finite interval has no accumulation point a.s., which implies that there must be only a finite number of discontinuities of \( \Lambda^{(m)} \) on a finite interval.

**Proposition 3.2.2.** \( X_{t}^{(m)} \) hits 0 a finite number of times on a finite time interval \( \mathbb{P}_{(X_{0}, V_{0})} \)-almost surely and, if \( V_{0} > 0 \) and if \( \{T_{n}\} \) is an increasing sequence of stopping times at which \( X_{t}^{(m)} = 0 \), then

\[ \int_{0}^{t} \text{sgn} \left(X_{s}^{(m)}\right) \, dW_{s} = \sum_{n=1}^{\infty} (-1)^{n-1} W_{T_{n} \wedge t}. \]
Suppose that $T_1 < T_2$ are two adjacent discontinuities of $\Lambda^{(m)}$. Then, on the interval $(T_1, T_2)$, either

$$Y_t^{(m)} = X_t^{(m)}, U_t^{(m)} = V_t^{(m)} \quad \text{or} \quad Y_t^{(m)} = -X_t^{(m)}, U_t^{(m)} = -V_t^{(m)}.$$

Thus, for $t'$ and $t$ on $(T_1, T_2)$,

$$\Lambda_t^{(m)} - \Lambda_{t'}^{(m)}$$

$$= m \left( U_t^{(m)} - U_{t'}^{(m)} \right) + \zeta \int_{t'}^t U_s^{(m)} \, ds - \sigma \left( B_t^{(m)} - B_{t'}^{(m)} \right)$$

$$= m \left( \text{sgn} \left( X_t^{(m)} \right) V_t^{(m)} - \text{sgn} \left( X_{t'}^{(m)} \right) V_{t'}^{(m)} \right) + \zeta \int_{t'}^t \text{sgn} \left( X_s^{(m)} \right) V_s^{(m)} \, ds$$

$$- \sigma \int_{t'}^t \text{sgn} \left( X_s^{(m)} \right) \, dW_s$$

$$= \text{sgn} \left( X_t^{(m)} \right) \left( m \left( V_t^{(m)} - V_{t'}^{(m)} \right) + \zeta \int_{t'}^t V_s^{(m)} \, ds \right) - \sigma \int_{t'}^t \text{sgn} \left( X_s^{(m)} \right) \, dW_s$$

$$= \text{sgn} \left( X_t^{(m)} \right) \left( m \left( V_t^{(m)} - V_{t'}^{(m)} \right) + \zeta \int_{t'}^t V_s^{(m)} \, ds - \sigma \left( W_t - W_{t'} \right) + \sigma \left( W_t - W_{t'} \right) \right)$$

$$- \sigma \int_{t'}^t \text{sgn} \left( X_s^{(m)} \right) \, dW_s$$

$$= \text{sgn} \left( X_t^{(m)} \right) \left( m \left( V_t^{(m)} - V_{t'}^{(m)} \right) + \zeta \int_{t'}^t V_s^{(m)} \, ds - \sigma \left( W_t - W_{t'} \right) \right)$$

$$+ \sigma \text{sgn} \left( X_t^{(m)} \right) \left( W_t - W_{t'} \right) - \sigma \int_{t'}^t \text{sgn} \left( X_s^{(m)} \right) \, dW_s$$

$$= \text{sgn} \left( X_t^{(m)} \right) \left( mV_t^{(m)} + \zeta \int_0^t V_s^{(m)} \, ds - \sigma W_t \right) - \left( mV_{t'}^{(m)} + \zeta \int_0^{t'} V_s^{(m)} \, ds - \sigma W_{t'} \right)$$

$$+ \sigma \text{sgn} \left( X_t^{(m)} \right) \left( W_t - W_{t'} \right) - \sigma \int_{t'}^t \text{sgn} \left( X_s^{(m)} \right) \, dW_s$$

$$= \text{sgn} \left( X_t^{(m)} \right) \left( mV_t^{(m)} - mV_{t'}^{(m)} \right)$$

$$+ \sigma \text{sgn} \left( X_t^{(m)} \right) \left( W_t - W_{t'} \right) - \sigma \text{sgn} \left( X_t^{(m)} \right) \left( W_t - W_{t'} \right)$$

$$= 0,$
Hence, \( \Lambda^{(m)} \) is a pure jump process.

**Proposition 3.2.3.** Let \( V_0^{(m)} > 0 \). If \( \{T_n\} \) is an increasing sequence of stopping times at which \( X_t^{(m)} = 0 \), then

\[
\Lambda_t^{(m)} = - \sum_{n: \text{odd}} 2 m V_{T_n}^{(m)} + \sum_{n: \text{even}} 2 m V_{T_n}^{(m)}.
\]

**Proof.** We note that \( U^{(m)} \) and \( \Lambda^{(m)} \) share the type of discontinuity since

\[
m U_t^{(m)} = m U_0^{(m)} - \zeta \int_0^t U_s^{(m)} \, ds + \sigma \int_0^t \text{sgn} \left( X_s^{(m)} \right) \, dW_s + \Lambda_t^{(m)},
\]

i.e., if \( U_t^{(m)} \) is only right-continuous at \( t \), then so is \( \Lambda_t^{(m)} \).

Since \( V_0^{(m)} > 0 \), \( \text{sgn} \left( X_t^{(m)} \right) = 1 \) on \([0, T_1)\). And so

\[
m V_t^{(m)} = m V_0^{(m)} - \zeta \int_0^t V_s^{(m)} \, ds + \sigma W_s + \Lambda_t^{(m)}
\]

for \( t \in [0, T_1) \), which implies

\[
\Lambda_t^{(m)} \equiv 0 \quad \text{on} \quad [0, T_1).
\]

Since \( \text{sgn} \left( X_t^{(m)} \right) = -1 \) on \([T_1, T_2] \),

\[
\Lambda_{T_1+}^{(m)} - \Lambda_{T_1-}^{(m)} = m (U_{T_1+}^{(m)} - U_{T_1-}^{(m)}) = m (-V_{T_1+}^{(m)} - V_{T_1-}^{(m)}) = -2m V_{T_1}^{(m)}.
\]

So

\[
\Lambda_{T_1+}^{(m)} = -2m V_{T_1}^{(m)} \quad \text{and} \quad \Lambda_{T_1-}^{(m)} = -2m V_{T_1}^{(m)} \quad \text{on}[T_1, T_2].
\]

The rest is done by induction. \(\square\)

**Theorem 3.2.4.** Let \( x_0 \geq 0 \) and \( v_0 > 0 \). Then there exist \( (Y_t^{(m)}, V_t, \Lambda_t^{(m)}, W_t^{(m)}) \) on a probability space \((\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})\) such that

1. \( Y_t^{(m)} \geq 0 \) for all \( t > 0 \),
2. $U_t^{(m)}$ is right continuous with finite left limit (RCLL) and $|U_t^{(m)}|$ is continuous,

3. $\Lambda_0 = 0$ and $\Lambda_t$ is an increasing process supported by $\{ t \geq 0 : Y_t^{(m)} = 0 \}$, i.e.,

$$\int_0^t Y_s^{(m)} \, d\Lambda_s = 0,$$

4. $\{W_t, \mathcal{F}_t\}$ is a Brownian motion,

5. For all $t \geq 0$, almost surely

$$Y_t^{(m)} = x_0 + \int_0^t U_s^{(m)} \, ds,$$

$$m \, U_t^{(m)} = m \, V_0 - \zeta \int_0^t U_s^{(m)} \, ds + \sigma \, W_t + \Lambda_t.$$  \hfill (3.2.9)

Proof. Let $(X^{(m)}, V^{(m)})$ be the unique strong solution to (3.2.6) with initial value $(x_0, v_0)$, i.e.

$$dX_t^{(m)} = V_t^{(m)} \, dt,$$

$$m \, dV_t^{(m)} = -\zeta \, V_t^{(m)} \, dt + \sigma \, dW_t.$$

As we have seen,

$$\left( |X|, \text{sgn} (X^{(m)}) \, V^{(m)}, - \sum_{n: \text{odd} \ T_n \leq t} 2 \, m \, V_{T_n}^{(m)} + \sum_{n: \text{even} \ T_n \leq t} 2 \, m \, V_{T_n}^{(m)}, \int_0^t \text{sgn} (X_s^{(m)}) \, dW_s \right)$$

satisfy all conditions above except one, i.e., $\text{sgn} (X_t^{(m)}) \, V_t^{(m)}$ is not RCLL.

Since $\text{sgn} (X_t^{(m)}) \, V_t^{(m)}$ is discontinuous only at isolated points of $[0, \infty)$, we simply let $U_t^{(m)}$ agree with $V^{(m)}$ on points of continuity and let $U_t^{(m)}$ be right continuous on $[0, \infty)$. This completes the proof. \hfill \square
3.2.2 Convergence of $\Lambda^{(m)}$ to $2\sigma L$ as $m \to 0$

Let $X_t$ be the solution to

$$\zeta \, dX_t = \sigma \, dW_t \tag{3.2.11}$$

with the initial value $X_0$. Then

$$|X_t| = |X_0| + \frac{\sigma}{\zeta} \int_0^t \text{sgn} (X_s) \, dW_s + \frac{2\sigma}{\zeta} L_t.$$ 

And we let $\left( X_t^{(m)}, V_t^{(m)} \right)$ be the strong solution to (3.2.6) with the initial value $(X_0, V_0)$, i.e.,

$$X_t^{(m)} = X_0 + \int_0^t V_s^{(m)} \, ds,$$

$$m V_t^{(m)} = m V_0 - \zeta \int_0^t V_s^{(m)} \, ds + \sigma W_t.$$

And, if we let $Y_t^{(m)} = \text{sgn} \left( X_t^{(m)} \right)$ and $U_t^{(m)} = \text{sgn} \left( X_t^{(m)} \right) V_t^{(m)}$, then

$$\left| X_t^{(m)} \right| = |X_0| + \int_0^t \text{sgn} \left( X_s^{(m)} \right) V_s^{(m)} \, ds,$$

$$m \, \text{sgn} \left( X_t^{(m)} \right) V_t^{(m)} = m \, \text{sgn} \left( X_0^{(m)} \right) V_0 - \zeta \int_0^t \text{sgn} \left( X_s^{(m)} \right) V_s^{(m)} \, ds$$

$$+ \sigma \int_0^t \text{sgn} \left( X_s^{(m)} \right) \, dW_s + \Lambda_t^{(m)}.$$

It is well-known that $X_t^{(m)}$ converges to $X_t$ almost surely. See Nelson [12]. So $\left| X_t^{(m)} \right|$ converges to $|X_t|$ almost surely.

**Theorem 3.2.5.** $\Lambda_t^{(m)}$ converges to $2\sigma L_t$ in probability.

**Proof.** First we note that

$$\left| X_t^{(m)} \right| - |X_t| = -\frac{m}{\zeta} \, \text{sgn} \left( X_t^{(m)} \right) V_t^{(m)} + \frac{m}{\zeta} \, \text{sgn} \left( X_0^{(m)} \right) V_0$$

$$+ \frac{\sigma}{\zeta} \int_0^t \left( \text{sgn} \left( X_s^{(m)} \right) - \text{sgn} (X_s) \right) \, dW_s + \frac{1}{\zeta} \Lambda_t^{(m)} - \frac{2\sigma}{\zeta} L_t.$$
Note that
\[ \frac{m}{\zeta} \text{sgn} \left( X_t^{(m)} \right) V_t^{(m)} - \frac{m}{\zeta} \text{sgn} \left( X_0^{(m)} \right) V_0 = \frac{m}{\zeta} \text{sgn} \left( X_t^{(m)} \right) V_t^{(m)} - \frac{m}{\zeta} \text{sgn} \left( X_t^{(m)} \right) V_0^{(m)} - \frac{m}{\zeta} \text{sgn} \left( X_0^{(m)} \right) V_0 \]
\[ = \frac{m}{\zeta} \text{sgn} \left( X_t^{(m)} \right) \left( V_t^{(m)} - V_0^{(m)} \right) + \frac{m}{\zeta} \left( \text{sgn} \left( X_t^{(m)} \right) - \text{sgn} \left( X_0^{(m)} \right) \right) V_0 \]

By the argument in Nelson [12],
\[ \frac{m}{\zeta} \text{sgn} \left( X_t^{(m)} \right) V_t^{(m)} - \frac{m}{\zeta} \text{sgn} \left( X_0^{(m)} \right) V_0 \rightarrow 0 \quad \text{as } m \rightarrow 0. \]

And, for each \( t \geq 0 \),
\[ \mathbb{E} \left( \int_0^t \left( \text{sgn} \left( X_s^{(m)} \right) - \text{sgn} \left( X_s \right) \right) \, dW_s \right)^2 = \mathbb{E} \int_0^t \left( \text{sgn} \left( X_s^{(m)} \right) - \text{sgn} \left( X_s \right) \right)^2 \, ds \]
\[ = \int_0^t \mathbb{E} \left( \text{sgn} \left( X_s^{(m)} \right) - \text{sgn} \left( X_s \right) \right)^2 \, ds. \]

Note that
\[ \mathbb{E} \left( \text{sgn} \left( X_s^{(m)} \right) - \text{sgn} \left( X_s \right) \right)^2 = 4 \mathbb{P} \left\{ X_s^{(m)} > 0; X_s \leq 0 \right\} + 4 \mathbb{P} \left\{ X_s^{(m)} \leq 0; X_s > 0 \right\} \]
\[ = 4 \mathbb{P} \left\{ X_s^{(m)} > 0; X_s < 0 \right\} + 4 \mathbb{P} \left\{ X_s^{(m)} > 0; X_s = 0 \right\} \]
\[ + 4 \mathbb{P} \left\{ X_s^{(m)} > 0; X_s < 0 \right\} + 4 \mathbb{P} \left\{ X_s^{(m)} > 0; X_s = 0 \right\} \]
\[ = 4 \mathbb{P} \left\{ X_s^{(m)} > 0; X_s < 0 \right\} + 4 \mathbb{P} \left\{ X_s^{(m)} > 0; X_s < 0 \right\}. \]

Since \( X_s^{(m)} \) converges to \( X_s \) a.s.,
\[ \lim_{m \rightarrow 0} \mathbb{E} \left( \text{sgn} \left( X_s^{(m)} \right) - \text{sgn} \left( X_s \right) \right)^2 = 0, \]
and, by the dominated convergence theorem,
\[ \lim_{m \rightarrow 0} \mathbb{E} \left( \int_0^t \left( \text{sgn} \left( X_s^{(m)} \right) - \text{sgn} \left( X_s \right) \right) \, dW_s \right)^2 = 0. \]
Hence,\
\[
\frac{1}{\zeta} \Lambda_t^{(m)} \rightarrow \frac{2\sigma}{\zeta} L_t \quad \text{as } m \rightarrow 0 \quad \text{in probability.}
\]

3.3 Future Directions

The research presented here opens doors to the rooms of novel mathematical questions. It is believed that some of them have been overlooked.

One question is the convergence of the unique solution of\
\[
\zeta \frac{d}{dt} R_t^{(N,n)} = \frac{3}{b^2} k_B T N^2 \left( R_t^{(N,n-1)} - 2R_t^{(N,n)} + R_t^{(N,n+1)} \right) + \sqrt{2\zeta T N} \dot{B}_t^{(N,n)} + \dot{L}_t^{(N,n)}
\]
to the unique solution of\
\[
\zeta \frac{\partial}{\partial t} R(s,t) = \frac{3}{b^2} k_B T \frac{\partial^2}{\partial s^2} R(s,t) + \sqrt{2\zeta k_B T} \dot{W}(s,t) + \dot{\Lambda}(s,t)
\]
under a condition on the initial values. Funaki [6] answered the question when there is no reflection. Like Funaki’s work, the question relates the discrete models to the continuum model. The mathematical challenge of the question is to relate \( L^{(N,n)} \) to \( \eta \) and to control \( L^{(N,n)} \) as \( N \) increases. Similar questions were investigated by Funaki and Olla [7], and Zambotti [22]. In both papers, the authors proved the convergence of the stationary solutions of reflected stochastic differential equations to the stationary solution of a reflected stochastic partial differential equations. One simple difference is on the boundary condition of the Laplace operator, Dirichlet versus Neumann boundary conditions. The other difference is on the solution, the stationary solutions versus solutions with arbitrary initial values.

The second question is the existence and the uniqueness problems of\
\[
\zeta \frac{\partial}{\partial t} R(s,t) = \frac{3}{b^2} k_B T \frac{\partial^2}{\partial s^2} R(s,t) + \sqrt{2\zeta k_B T} \dot{W}(s,t) + \dot{\Lambda}(s,t)
\]
in more general regions in $\mathbb{R}^3$. Recently, it was known that the existence problem of a one-dimensional reflected stochastic partial differential equation was proved when the spatial variable $s$ ranges over $\mathbb{R}$ and $U = [a, b]$. The method used there can be applied to prove the existence of the reflected Rouse model in $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : a \leq x_3 \leq b\}$. When $s$ ranges over a finite interval, the uniqueness problem is answered in the same frame work presented in the dissertation.

The third question is the existence and uniqueness problems of

\[ m \frac{d^2}{dt^2} X_t^{(m)} = -\zeta \frac{d}{dt} X_t^{(m)} + b \left( X_t^{(m)} \right) + \sigma \left( X_t^{(m)} \right) \dot{B}_t + \dot{\Lambda}_t^{(m)} \tag{3.3.1} \]

in a general domain of $\mathbb{R}^n$ and the convergence of the solutions of (3.3.1) to the solution of

\[ 0 = -\zeta \frac{d}{dt} X_t + b (X_t) + \sigma (X_t) \dot{B}_t + \dot{L}_t \tag{3.3.2} \]

as the mass $m$ tends to zero. In Section 3.2.1, the question was answered with $n = 1$ and $U = (0, \infty)$. It was brought up by the consideration that the beads of the discrete Rouse model bear nontrivial mass. It is understood that the second-order reflected stochastic differential equation (3.3.1) describes the dynamics of a particle with mass $m$ in a region of $\mathbb{R}^n$. Once we have the existence and the uniqueness of the solution of (3.3.1), we can consider the convergence of the solutions as $m$ tends to zero and analyze $L_t$. 

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BIBLIOGRAPHY


