ANALYSIS OF ELECTROMAGNETIC WELL-LOGGING TOOLS.

DISSERTATION

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By

Yik-Kiong Hue, B.S., M.S.

* * * * *

The Ohio State University

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Dissertation Committee:

Professor Fernando Teixeira, Adviser
Professor Prabhakar Pathak
Professor Roberto Rojas-Teran
Professor Paul Evans

Approved by

Adviser
Department of Electrical Engineering
This dissertation presents numerical and pseudoanalytical modeling techniques for
the analysis of electromagnetic well logging tools in complex geophysical formations. The well-logging tools modeled here include measurement-while-drilling/logging-while-
drilling (MWD/LWD) tools employing horizontal coil and for tilted-coil antenna ar-
rays. The electromagnetic response of such tools are studied in cylindrical layered
formations, including eccentric borehole, dipping bed layers, and/or anisotropic con-
ductivities.

The three main techniques used for modeling are the finite-difference time-domain
(FDTD), a pseudoanalytical method, and numerical mode-matching. The FDTD
method is implemented three-dimensional (3-D) cylindrical coordinates to handle
arbitrary 3-D complex formations. The FDTD method developed here incorporates
the following extensions: (i) A perfectly matched layer (PML) absorbing boundary
condition in cylindrical coordinates to mimic open region problems, (ii) an efficient
frequency domain data extraction technique to yield frequency domain data from
early time domain data, (iii) two locally-conformal FDTD (LC-FDTD) techniques to
model eccentric boreholes, and (iv) a scaling permittivity technique to overcome the
Courant stability limit at the low frequency range.

To provide validation results and a faster solution method in simpler formations,
including multiple asymmetric cylindrical layered formations with eccentric boreholes,
a pseudoanalytical method is also developed. This method is based on a decomposition of the solution into spectral components and on the use of the addition theorem for cylindrical waves to tackle eccentric borehole problems.

Finally, a NMM method is developed and employed to model tilt-coil antenna array in multilayered anisotropic formations. The present NMM is based on an expansion of the field components in terms of vertical eigenmodes, and incorporates a PML in the vertical direction. B-splines are used to expand the fields in the vertical direction and cylindrical Bessel/Hankel functions are used for the radial dependency. The results from each method are validated against each other, showing very good agreement.
To my families . . .
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September 24th, 1977 .......................Born - Kuala Lumpur, Malaysia

2000 .................................B.S. Electrical Engineering,
The Ohio State University

2003 .................................M.S. Electrical Engineering,
The Ohio State University.

2003-present ............................Graduate Research Associate,
The Ohio State University.

PUBLICATIONS

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Journal Publications

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Table of Contents</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>iv</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>v</td>
</tr>
<tr>
<td>Vita</td>
<td>vi</td>
</tr>
<tr>
<td>List of Tables</td>
<td>xii</td>
</tr>
<tr>
<td>List of Figures</td>
<td>xiii</td>
</tr>
<tr>
<td>Chapters:</td>
<td></td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Organization of the Dissertation</td>
<td>3</td>
</tr>
<tr>
<td>2. Well-Logging Tools and the Well-Logging Borehole Environment</td>
<td>6</td>
</tr>
<tr>
<td>2.1 Formations</td>
<td>6</td>
</tr>
<tr>
<td>2.2 Well-Logging Tools</td>
<td>8</td>
</tr>
<tr>
<td>3. Three-Dimensional Finite-Difference Time-Domain Simulation of MWD/LWD Tool Responses</td>
<td>11</td>
</tr>
<tr>
<td>3.1 Introduction</td>
<td>11</td>
</tr>
<tr>
<td>3.2 Cylindrical FDTD Formulations</td>
<td>13</td>
</tr>
<tr>
<td>3.2.1 PML Absorbing Boundary Condition in 3-D Cylindrical FDTD Grids</td>
<td>14</td>
</tr>
<tr>
<td>3.2.2 Locally Conformal FDTD for Eccentric Borehole Problems</td>
<td>15</td>
</tr>
</tbody>
</table>

viii
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2.3 Frequency Data Extraction for Treated FDTD Excitations</td>
<td>19</td>
</tr>
<tr>
<td>3.2.4 Permittivity Scaling for Low Frequency Problems</td>
<td>21</td>
</tr>
<tr>
<td>3.3 FDTD Modeling of LWD Tools</td>
<td>23</td>
</tr>
<tr>
<td>3.4 Numerical Results and Validation</td>
<td>24</td>
</tr>
<tr>
<td>3.4.1 Homogeneous Formations</td>
<td>24</td>
</tr>
<tr>
<td>3.4.2 Eccentric Tools</td>
<td>25</td>
</tr>
<tr>
<td>3.4.3 Eccentric Tools in Dipping Beds</td>
<td>27</td>
</tr>
<tr>
<td>3.4.4 LWD Tools at Lower Frequencies</td>
<td>28</td>
</tr>
<tr>
<td>3.5 Conclusions</td>
<td>29</td>
</tr>
<tr>
<td>4. Analysis of Tilted Coil Antennas in Cylindrical Multilayered and Eccentric Borehole Formations</td>
<td>46</td>
</tr>
<tr>
<td>4.1 Introduction</td>
<td>46</td>
</tr>
<tr>
<td>4.2 Tilted-Coil Well-Logging Tools</td>
<td>47</td>
</tr>
<tr>
<td>4.3 Pseudoanalytic Formulation</td>
<td>49</td>
</tr>
<tr>
<td>4.3.1 Multicylindrically layered media</td>
<td>51</td>
</tr>
<tr>
<td>4.3.2 Eccentric borehole</td>
<td>52</td>
</tr>
<tr>
<td>4.4 Pseudoanalytical Formulation: Numerical Considerations</td>
<td>55</td>
</tr>
<tr>
<td>4.4.1 Numerical Integration Issues for Multicylindrically Layered Media</td>
<td>55</td>
</tr>
<tr>
<td>4.4.2 Improving Convergence for Eccentric Boreholes</td>
<td>57</td>
</tr>
<tr>
<td>4.5 Validation against FDTD</td>
<td>59</td>
</tr>
<tr>
<td>4.6 Results</td>
<td>60</td>
</tr>
<tr>
<td>4.6.1 Homogeneous Formations</td>
<td>60</td>
</tr>
<tr>
<td>4.6.2 Homogeneous Formations with Mud layer</td>
<td>60</td>
</tr>
<tr>
<td>4.6.3 Eccentric Boreholes</td>
<td>64</td>
</tr>
<tr>
<td>4.7 Conclusions</td>
<td>65</td>
</tr>
<tr>
<td>5. Vertical Eigenstates NMM with Uniaxial PML For Simulation of Tilted-Coil Antenna Arrays in Anisotropic Formations</td>
<td>73</td>
</tr>
<tr>
<td>5.1 NMM Formulation</td>
<td>76</td>
</tr>
<tr>
<td>5.2 Incorporating Multicylindrical-Vertical layers</td>
<td>82</td>
</tr>
<tr>
<td>5.3 Transimpedance</td>
<td>85</td>
</tr>
<tr>
<td>5.4 Numerical Issues</td>
<td>86</td>
</tr>
<tr>
<td>5.4.1 Matrix Inversion and Ill-Conditioning Issues</td>
<td>86</td>
</tr>
<tr>
<td>5.4.2 B-Spline Basis Functions</td>
<td>87</td>
</tr>
<tr>
<td>5.4.3 Convergence at Interfaces along the z-direction</td>
<td>89</td>
</tr>
<tr>
<td>5.4.4 PML Performance in NMM</td>
<td>94</td>
</tr>
<tr>
<td>5.5 Simulation Results</td>
<td>97</td>
</tr>
<tr>
<td>5.5.1 FDTD Reference Results</td>
<td>97</td>
</tr>
</tbody>
</table>
5.5.2 NMM Simulation Setup ........................................... 97
5.5.3 Homogeneous Formation with Invasion Zone .................. 97
5.5.4 Two-layer Isotropic Formation .................................. 98
5.5.5 Two-layer Anisotropic Formation ............................... 99
5.5.6 Comparison of Tilted-Coil and Horizontal-Coil Results ...... 99
5.5.7 Tool response in dipping bed using dipole formulation ...... 100
5.5.8 Apparent resistivity in dipping anisotropic formations ...... 101

5.6 Conclusion ......................................................... 102

6. Conclusions and Suggestions For Future Work .................... 123

Appendices:

A. Further Details on 3-D Cylindrical FDTD Formulation .......... 125

B. Fields Due to An Arbitrarily Oriented Ring Source in Cylindrical Multi-
    Layered Media ..................................................... 132
    B.1 Tilt-Coil Antenna Electric Current Density .................. 132
    B.2 Homogeneous Medium Solution ................................ 134
    B.3 Transimpedance ................................................. 137
    B.4 Radiation Integral ............................................. 139
    B.5 Solution Incorporating Multicylindrical Layers .............. 140
    B.6 Plane Wave Reflection from a PEC Cylinder .................. 145
        B.6.1 Reflection and Transmission coefficient matrix ...... 145
        B.6.2 Solution Incorporating Steel Mandrel modelled as PEC 147
    B.7 Graf’s Addition Theorem ....................................... 148
    B.8 Spectral Components of the Source, c_{TM} and c_{TE} ...... 150
    B.9 Source Condition .............................................. 152

C. Details on the Numerical Mode Matching Implementation .......... 154
    C.1 Transverse Field Component .................................. 154
    C.2 Matrix Representation of the Transverse Components ......... 156
    C.3 Finding J_{\nu q} and J_{\mu q} ................................... 159
    C.4 NMM: Source condition ....................................... 161
    C.5 Local reflection/transmission coefficients for two cylindrical-vertical
        layers .......................................................... 162
    C.6 NMM: Transimpedance ......................................... 164
    C.7 NMM: magnetic dipole formulation ............................. 165
Bibliography
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Phase velocity, wavelength, timestep, and normalized CPU time with</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>scaled permittivities.</td>
<td></td>
</tr>
<tr>
<td>5.1</td>
<td>The below table shows the inverse of the condition number of the matrix</td>
<td>87</td>
</tr>
<tr>
<td></td>
<td>within the round bracket of Eq. 5.45 and Eq. 5.61. The Ill-posed means</td>
<td></td>
</tr>
<tr>
<td></td>
<td>the matrix in Eq. 5.45 and the well-posed means the matrix in Eq. 5.61.</td>
<td></td>
</tr>
<tr>
<td>5.2</td>
<td>( e_{z\nu} ) computed by pseudoanalytical method, NMM homogeneous</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>formulation.</td>
<td></td>
</tr>
<tr>
<td>5.3</td>
<td>( e_{\phi\nu} ) due to TE(_z) mode computed by pseudoanalytical</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>method, NMM homogeneous formulation.</td>
<td></td>
</tr>
<tr>
<td>5.4</td>
<td>( e_{\phi\nu} ) due to TM(_z) mode computed by pseudoanalytical</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>method, NMM homogeneous formulation.</td>
<td></td>
</tr>
<tr>
<td>5.5</td>
<td>The error of the numerical method is given here. The analytical referenced</td>
<td>96</td>
</tr>
<tr>
<td></td>
<td>value for a low-loss (( \sigma=0.0005 )) homogeneous medium is ( V_{\text{ref}}=0.000032288142 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>- 0.006039728286i. The ring source is tilted with angle ( \theta_T = 45^\circ ),</td>
<td></td>
</tr>
<tr>
<td></td>
<td>and the receiver is a ring source with tilted angle ( \theta_R = 45^\circ ).</td>
<td></td>
</tr>
<tr>
<td>5.6</td>
<td>Voltage measured at the receiver with different formation profiles. The</td>
<td>98</td>
</tr>
<tr>
<td></td>
<td>distance between transmitter and receiver is 30 in. Transmitter and receiver</td>
<td></td>
</tr>
<tr>
<td></td>
<td>are tilted by 45(^\circ) in the same direction. The antennas are mounted</td>
<td></td>
</tr>
<tr>
<td></td>
<td>on the mandrel with a diameter of 8 in.</td>
<td></td>
</tr>
</tbody>
</table>
LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>The above figure illustrates the <em>vertical</em> and <em>directional</em> well-logging. The wireline tool is employed to estimate the log. The MWD/LWD tool is geo-steered to the estimated oil-zone.</td>
</tr>
<tr>
<td>2.2</td>
<td>The two tools above represent the conventional (left) and novel directional (right) MWD tool. The conventional tool uses horizontal-coil antennas and the novel directional tool employ tilted-coil antennas and horizontal-coil antennas to provide directional sensitivity.</td>
</tr>
<tr>
<td>3.1</td>
<td>Averaging procedure used in the edge-based conformal FDTD scheme showing, on the top, the intersection between the regular FDTD mesh and curved interface between two conductive media, at the left, the original problem with different conductivities along a single edge, and at the right, the equivalent problem with effective conductivities along each edge.</td>
</tr>
<tr>
<td>3.2</td>
<td>Linear weighted averaging procedure used in the face-based conformal dielectric FDTD scheme. The top figure shows the intersection between a curved interface and the FDTD dual grid mesh. The bottom figure depicts the equivalent problem using effective conductivities over each dual grid face.</td>
</tr>
<tr>
<td>3.3</td>
<td>Example of the convergence behavior of the phase difference between LWD tool receiver voltages using the 2E2U approach.</td>
</tr>
<tr>
<td>3.4</td>
<td>The basic configuration of the LWD tool inside a 5 [in] radius borehole.</td>
</tr>
<tr>
<td>3.5</td>
<td>Illustration of the cross-section of the LWD tool discretization in a layered formation.</td>
</tr>
</tbody>
</table>
3.6 FDTD and NMM results for phase difference (PD) and amplitude ratio (AR) of a LWD tool in a uniform formation (infinitely thick bed) for various values of formation conductivities. The mud conductivity is fixed at 0.0005 [mho/m].

3.7 Horizontal cross-section of an eccentric borehole problem.

3.8 Conductivity distribution along the horizontal cross-section of the cylindrical FDTD for an eccentric LWD tool scenario.

3.9 FDTD simulation results versus pseudoanalytical results of eccentric LWD tool responses. Phase difference and amplitude ratio are plotted against the mandrel offset $\Delta x$ (see Fig. 3.7). The borehole has 12 [in] radius and is filled with oil-based mud $\sigma = 0.0005$ [mho/m]. The surrounding homogeneous formation has $\sigma = 10$ [mho/m]. Very good agreement is observed between the FDTD and pseudoanalytical results.

3.10 FDTD simulation results versus pseudoanalytical results of eccentric LWD tool responses. Phase difference and amplitude ratio are plotted against the mandrel offset $\Delta x$ (see Fig. 3.7). The borehole has 12 [in] radius and is filled with oil-based mud $\sigma = 10.0$ [mho/m]. The surrounding homogeneous formation has $\sigma = 0.1$ [mho/m]. Very good agreement is again observed between the FDTD and pseudoanalytical results.

3.11 Influence of eccentricity and borehole size on LWD tool response. (a) Case 1: Borehole size is varied while keeping mandrel centered. (b) Case 2: Borehole size is varies while keeping the offset distance $\Delta x$ fixed.

3.12 FDTD simulation of the response of eccentric LWD tools in a homogeneous formation. A high contrast case is considered with mud and formation conductivities equal to 10 and 0.0001 [mho/m], respectively. Case 1 curve is the LWD tool responses versus borehole radius while the tool is kept at the center of the borehole (see Fig. 3.11(a)). Case 2 curve is the LWD tool response versus borehole size while keeping a fixed distance $d = 4.75$ [in] from the mandrel axis to closest borehole wall (see Fig. 3.11(b)).
3.13 Electric field ($\phi$-component) distribution on the horizontal cross-section of the cylindrical FDTD grid for an eccentric LWD tool. The top figure depicts an expanded view of the field distribution on the formation. The bottom figure is a zoomed view of the field distribution close to the borehole. The mud conductivity is $\sigma_{mud} = 0.01$ [mho/m], and the formation conductivity is $\sigma_{for} = 0.1$ [mho/m].

3.14 Electric field ($\phi$-component) distribution on the horizontal cross-section of the cylindrical FDTD grid for an eccentric LWD tool. The top figure depicts an expanded view of the field distribution on the formation. The bottom figure is a zoomed view of the field distribution close to the borehole. The mud conductivity is again $\sigma_{mud} = 1.0$ [mho/m], and the formation conductivity is now $\sigma_{for} = 0.1$ [mho/m] in this case.

3.15 FDTD simulations of eccentric LWD tools in formations with dipping beds. The upper and lower layer conductivities is $\sigma = 1$ [mho/m], while the 60 in thick dipping bed has $\sigma = 0.01$. The borehole has radius 9 [in] and is filled with oil-based mud with $\sigma = 0.0005$. The label ECCE in this plot represents the offset distance $\Delta x$ in inches (see Fig. 3.7).

3.16 Comparison of FDTD and NMM results of a 500kHz LWD tool response in a three-layer formation with lower, middle and upper layer conductivities equal to 1, 0.01 and 1 [mho/m], respectively. The conductivity of the mud is 2 [mho/m]. The middle bed is 60 [in] thick.

3.17 Comparison of FDTD and NMM results of a 500kHz LWD tool response in a three-layer formation with lower, middle and upper layer conductivities again equal to 1, 0.01 and 1 [mho/m], respectively. The mud conductivity is now 0.0005 [mho/m]. The middle bed is 60 [in] thick.

4.1 Tilted-coil LWD tool. The tilting of the transmitter and receivers provides directionality to the tool response.

4.2 Tilted coil antenna in a multicylindrical layered medium. The formation is assumed axially symmetric. The source layer is denoted by $i = 0$. Outer layers have positive index, while inner layers have negative index. $\vec{n}_T$ is the normal unit vector to the coil antenna plane.
4.3 Illustration of eccentric borehole geometry. The tool (mandrel) axis is located \( d \) away from the borehole axis. Nonprimed coordinates are centered on the borehole axis, while primed coordinates are centered on the mandrel axis. The transmitter coil antenna is located inside the borehole (mud filled) region. \( \tilde{\Gamma}_\nu \) is the generalized reflection coefficient due to the mandrel. \( \tilde{N}_\nu \) is the generalized reflection coefficient due to outer formation layers. \( \phi'_E \) is enforced to be zero, and the primed coordinate system is placed at the corresponding location with respect to the unprimed coordinate system.

4.4 The logarithm of \(|w(k_z)|\) given in Eq. 4.14 and \(|w(k_z) - w^h(k_z)|\), where \( w^h(k_z) \) is the homogeneous component of the transimpedance.

4.5 Comparison of 3-D FDTD and pseudoanalytical results. The tool is located in a homogeneous formation with either \( \sigma = 0.1 \text{ mho/m or 1.0 mho/m} \). Three different tilt angles for the transmitter are considered for each conductivity choice: 0°, 20° and 45°. Pseudoanalytic results are represent by symbols and FDTD results by solid curves.

4.6 Comparison of 3-D FDTD and pseudoanalytical results for a tool located in a cylindrical borehole surrounded by a homogeneous formation. Two pairs of borehole mud and formation conductivities are considered: \( \sigma_{mud/for} = 10/0.1 \text{ mho/m and } \sigma_{mud/for} = 0.0005/1.0 \text{ mho/m} \). Three different transmitter tilt angles are considered for each pair of conductivities: 0°, 20° and 45°. Pseudoanalytic results represent by symbols and FDTD results by solid lines.

4.7 Phase and amplitude of the voltage measured at the receivers in homogeneous medium with conductivity indicated in the legend. The transmitter coil source is tilted by 45°. The \( x \)-axis is the receivers tilted angle. Fig. 4.7(a) and Fig. 4.7(b) depicts the phase of the voltage measured at first and second receivers; Fig. 4.7(c) and Fig. 4.7(d) depicts the amplitude of the voltage measured at first and second receivers.

4.8 Tool response for a tool setup with transmitter tilt angle equal to 45°, and two receivers with the same tilt angle. In the above, the phase difference and amplitude ratio are plotted against the receiver tilt angle.
4.9 Tool response for a tool setup with transmitter tilt angle equal to 45°, and second receiver with the tilt angle fixed at 0°. The phase difference and the amplitude ratio are plotted against the first receiver tilt angle. 68

4.10 Phase and amplitude of the voltage measured at the receivers in homogeneous medium with conductivity as indicated in the legend. The transmitter coil source is tilted with angle 0°. The x-axis represents the receivers (both) tilt angle. Fig. 4.10(a) and Fig. 4.10(b) depicts the phase of the voltage measured at first and second receivers; Fig. 4.10(c) and Fig. 4.10(d) depicts the amplitude of the voltage measured at first and second receivers. 69

4.11 Tool response for a tool setup with transmitter tilt angle equal to 0°, and the two receivers having the same tilt angle. In the above, phase difference and amplitude ratio are plotted against the receiver tilt angle. 70

4.12 Tool response for a tool setup with transmitter tilt angle equal to 0° and the second receiver having tilt angle fixed at 0°. The phase difference and amplitude ratio are plotted against the first receiver tilt angle. 71

4.13 Comparison of 3-D FDTD and pseudoanalytical results for a tool located in an eccentric cylindrical borehole surrounded by a homogeneous formation. Two pairs of borehole mud and formation conductivities are considered: \( \sigma_{mud/for} = 10/0.1 \) and \( \sigma_{mud/for} = 0.1/10.0 \) mho/m. The parameter \( d \) represents the eccentricity of the tool in inches (see Fig. 4.3). Both transmitter and receivers have tilt angle equal to 45°. Pseudoanalytical results are represented by symbols and FDTD results by solid curves. See more details in the text. 72

5.1 Illustration of a tilted-coil well-logging tool within a layered formation with horizontal invaded beds. 75

5.2 The above illustrates the direction relates to the reflection and transmission coefficient. The arrow pointing to the left is associated with “-” and pointing to the right is associated with “+”. 84

5.3 The formation used to study the conditioning of the ill-posed and well-posed formulation of the generalized reflection matrix. 88

5.4 Second-order B-spline function. The four knot points of the second-order B-spline functions are \( (0,1/3,2/3,1) \). 89
5.5 The basis function formulated from the second-order b-spline function in an element. ................................................................. 91

5.6 Formation used to study the gridding effect on the accuracy of the results close to the interfaces. The domain size along \( z \) is 5 m. with the interface located at 2.5 m. .................................................. 92

5.7 Nodal positions for the grids used to test the convergence of NMM at bed boundaries (interfaces). A lower slope for the traces above means the grid is denser. The interface is located at 2.5 m. ................. 93

5.8 NMM results for the amplitude ratios of a tool with spacing of the first and second receiver are 30 in and 24 in crossing a bed boundary. In this case, the transmitter tilt angle is 45\(^\circ\). The \( x \)-axis is the distance of transmitter from the boundary interface. ................................. 94

5.9 The example of the setup grid to study the accuracy of the method. .......................................................... 95

5.10 PML performance for the NMM formulation in a homogeneous medium. The error on the amplitude (a) and phase (b) of the voltage are shown. The values in the legend represent the real and imaginary parts of the complex stretching variables of the PML \((a_z, f_z)\). The \( x \)-axis represents the source position. The ring source is tilted with angle \( \theta_T = 45^\circ \), and the receiver is a ring source with angle \( \theta_R = 45^\circ \). ........................ 104

5.11 Example of a mesh used for FDTD simulations for well-logging tools employing tilted-coil antenna. Non-uniform grid is adapted in both the \( z \) and \( \rho \)-direction. The horizontal lines at the \( z \)-axis indicate the position of the transmitter and receivers. ................................. 105

5.12 FDTD and NMM results for the phase of the measured voltage at the first and second receivers. The problem includes a borehole filled with oil-based mud \((\sigma_{\text{mud}}=0.0005 \text{ mho/m})\) and an isotropic formation with two layers. The top layer has conductivity of 1 mho/m and the bottom layer has conductivity of 5 mho/m. The transmitter tilt angle is 45\(^\circ\). Four receiver angles are considered: 10\(^\circ\), 20\(^\circ\), 30\(^\circ\), and 40\(^\circ\). The \( x \)-axis represents distance of the transmitter to the interface. ................................. 106

xviii
5.13 FDTD and NMM results for the amplitude of the measured voltage at the first and second receivers. The problem includes a borehole filled with oil-based mud ($\sigma_{\text{mud}}=0.0005 \, \text{mho/m}$) and two-layer isotropic formation. The top layer has conductivity of 1 mho/m and the bottom layer has conductivity of 5 mho/m. The transmitter tilt angle is 45°. Four receiver angles are considered: 10°, 20°, 30°, and 40°. The abscissa represents the distance of the transmitter to the interface.

5.14 FDTD and NMM results for the phase of the measured voltage at the first and second receivers. The problem includes a borehole filled with oil-based mud ($\sigma_{\text{mud}}=0.0005 \, \text{mho/m}$) and two-layer uniaxial-anisotropic formation. The top layer has conductivity $\sigma_{h/v} = 1/5 \, \text{mho/m}$ and bottom layer has $\sigma_{h/v} = 5/1 \, \text{mho/m}$. The transmitter tilt angle is 45°. Four receiver angles are considered: 15°, 25°, 35°, and 45°. The abscissa represents the distance of the transmitter to the interface.

5.15 FDTD and NMM results for the amplitude of the measured voltage at the first and second receivers. The problem includes a borehole filled with oil-based mud ($\sigma_{\text{mud}}=0.0005 \, \text{mho/m}$) and two-layer uniaxial-anisotropic formation. The top layer has conductivity $\sigma_{h/v} = 1/5 \, \text{mho/m}$ and bottom layer has conductivity $\sigma_{h/v} = 5/1 \, \text{mho/m}$. The transmitter tilt angle is 45°. Four receiver angles are considered: 15°, 25°, 35°, and 45°. The log is given with the x-axis defined as the distance of the transmitter to the interface.

5.16 Borehole problem with a formation including invasion, isotropic layer and anisotropic layers. This formation can be misread by the conventional well-logging tool since a horizontal-coil antenna excites only TE mode.

5.17 Phase and amplitude of the computed voltage measured. Conventional and tool results are compared. The abscissa represents the distance of the transmitter the mid-point (0 in) of the formation depicted in Fig. 5.16.

5.18 Responses of the conventional tool and the tilted-coil tool in the formation of the Fig. 5.16. The conventional tool has misread the vertical component formation conductivity, and its response is flatter than the tilted-coil tool.
5.19 Borehole problem with a formation including invasion zone and anisotropic layers. In this example, the horizontal component for the conductivity of all layers is equal to 0.1 mho/m. This formation is misread by the conventional well-logging tool since a horizontal-coil antenna excites only TE modes. ............................................. 113

5.20 Phase and amplitude of the computed voltage at the receivers. Conventional and tilted-coil tool results are compared. The abcissa represents the distance of the transmitter to the mid-point (0 in) of the formation depicted in Fig. 5.19. ............................................. 114

5.21 Responses of the conventional tool and the tilted-coil tool in the formation depicted in the Fig. 5.19. The conventional tool is not sensitive to the vertical component formation conductivity, and its response is flat. The tilted-coil tool response on the other hand, changes according to the surrounding $\sigma_v$ values. ............................................. 115

5.22 The above Figure illustrates the model used to obtain the tool response in dipping formations using NMM dipole formation. The coil antenna is replaced by a magnetic dipole point in the normal direction of the coil plane. The tool response is modeled by taking the phase difference and amplitude ratio of the magnetic field component at the center of the receiver coil positions in the normal direction of the receiver coil plane. In the above, the tool is logging into the formation with $\theta$ degree dip with respect to the normal to the beds. ............................................. 116

5.23 Phase difference and amplitude ratio of the magnetic field measured 30 in and 24 in away from the source. The tool logs inside a three-layer formation with isotropic conductivities of 1.0/0.01/1.0 for upper/middle/lower layer, respectively. The direction of the magnetic dipole source and the measured magnetic fields are aligned with the log axis. The five dip angles are $0^\circ$, $15^\circ$, $30^\circ$, $45^\circ$ and $60^\circ$. The $x$-axis represent the distance of the source point to the boundary of the middle and the bottom layer (D in Fig. 5.22). ............................................. 117
5.24 Phase difference and amplitude ratio of the conventional tool (employing horizontal coil antenna). The tool logs inside a three-layer formation with isotropic conductivities of 1.0/0.01/1.0. In the FDTD model, both mandrel and borehole are modeled. The dipping bed is modeled using the staircase approximation [1]. The $x$-axis represent the distance of the transmitter to the interface of middle and bottom layers.

5.25 Phase difference and amplitude ratio of the magnetic field measured 30 in and 24 in away from the source. The tool logs inside a three-layer formation with anisotropic conductivities with $\sigma_h=1.0/0.01/1.0$ and $\sigma_v=1.0/0.1/1.0$ for upper/middle/lower layer, respectively. The direction of the magnetic dipole source and the measured magnetic fields are aligned with the log axis. This resembles the conventional tool employing horizontal-coil antennas. The five dip angles are 0°, 15°, 30°, 45° and 60°. The $x$-axis represent the distance of the source point to the interface of middle and bottom layers.

5.26 Phase difference and amplitude ratio of the magnetic field measured 30 in and 24 in away from the source. The tool logs inside a three-layer formation with anisotropic conductivities with $\sigma_h=1.0/0.01/1.0$ and $\sigma_v=1.0/0.1/1.0$ for upper/middle/lower layer, respectively. The direction of the magnetic dipole source and the second measured magnetic fields are tilted 45° away from the log axis. This is similar to a tilted-coil tool. The five dip angles are 0°, 15°, 30°, 45° and 60°. The $x$-axis represent the distance of the source point to the interface of middle and bottom layers.

5.27 Apparent resistivity using phase difference and amplitude ratio. In this case, $\sigma_h=0.1$ mho/m.

5.28 Apparent resistivity using phase difference and amplitude ratio. In this case, $\sigma_h=0.5$ mho/m.

A.1 A unit cell of the staggered FDTD grid scheme for spatial discretization of electromagnetic fields on the cylindrical grid.

B.1 Tilted transmitting ring (coil) antenna with current density $\vec{J}_T$. 

xxi
B.2 Illustration of a tilted coil antenna on a multicylindrical layered medium. The electrical properties are axially symmetric. The source layer is denoted as layer $i=0$. Layers outside the source layer have $i > 0$. Layers inside the source layer have $i < 0$. ........................ 140

B.3 A plane wave incident upon the PEC cylinder. ......................... 145

B.4 Steel mandrel wounded by a tilted coil antenna in a homogenous formation. ................................................................. 147

B.5 The above triangle denotes the relation between the origin ($0$), source ($\rho'$) and observation point ($\rho$) for the Graf’s Addition Theorem .... 148
CHAPTER 1

INTRODUCTION

Electromagnetic (EM) well-logging tools are one of the many classes of well-logging tools (others include Gamma Ray (GR) tools and Nuclear Magnetic Resonance Imaging (NMRI) tools) used for applications in oil exploration. EM logging tools have been subject of interest for many decades due to their capability to measure the conductivity (or resistivity) of an earth formation. The basic concept can be explained by a simple tool which consist of single transmitter coil and single receiver coil inside a borehole. The transmitter coil excites an alternating current (ac) inducing an alternating EM field that propagates/diffuses through the earth formation. This EM field induces an electric current on the receiver coil. The receiver current is proportional to the conductivity of the formation. Hence, the resistivity of the formation can be estimated. If the porosity of the earth is also known, well-logging analysts can use Archie formula [2] as a guideline to determine the saturation of oil within the formation surrounding the logging tool.

In practice, EM tools can operate thousands feet underneath the earth surface in a complex environment where its detailed properties are unknown. The measured conductivity represents an average effective conductivity of the earth formation in the vicinity of the EM tool. The formation itself can exhibit inhomogeneities such
as layered beds, dipping beds, anisotropic response, and invaded zones. Moreover, the logging tool may be misaligned with the borehole (eccentric tool) due to gravitational pull or mechanical vibration. In order to obtain accurate formation evaluation, modeling and analysis of the response of EM tools in complex formations is essential.

Before the advent of high performance computing, costly experiment or approximate analytical techniques were the only option to study EM tool response. With the advance of computing capabilities in the decade, well-logging analysts have turned to computer modelling as the most cost-effective way to understand the tool response in complex formations.

This dissertation is devoted to introduce and develop new extensions of both numerical and pseudoanalytical methods to study the response of EM tools in complex geophysical environments. Within this general objective, this dissertation can be divided into three main parts. The first part is the development of a brute-force three-dimensional (3D) simulation approach using the finite-difference time-domain (FDTD) technique. This technique uses finite-differences to discretize Maxwell curl equation in the time domain. FDTD is capable of computing EM tool response in complex 3D environments, including arbitrarily inhomogeneous and anisotropic earth formations. The second part deals with the extension of a pseudoanalytical approach to analyze well-logging tools with tilted-coil antennas (TCAs) in cylindrical multilayered formations and eccentric boreholes. The method is less flexible than FDTD but much faster and efficient for preliminary studies to estimate the response of well-logging tools employing TCAs in $z$-invariant formations. The third part of this dissertation is devoted to the development of a hybrid approach based on the numerical mode matching (NMM) method that combines both numerical and
analytical techniques. NMM solves the Maxwell equation by computing numerical
eigenmodes that satisfy an inhomogeneous in one direction (transverse/longitudinal
after an spectral decomposition), and by using mode propagators to describe them
in the remaining one-dimensional (1D) ordinary differential equation (ODE). This
method is used to analyze well-logging tools employing TCAs in layered anisotropic
formations.

1.1 Organization of the Dissertation

This dissertation is organized as follows. In Chapter 2, we give a brief description
of the typical geophysical environments under study and the geometry and mode of
operation of the measurement-while-drilling (MWD)/logging-while-drilling (LWD)
tools use for formation evaluation.

In Chapter 3 [3], the three-dimensional (3D) simulation and analysis of the prob-
lem using FDTD is discussed. We simulate the response of logging-while-drilling tools
in complex 3D borehole environments using FDTD scheme in cylindrical coordinates
in order to better conform to the (cylindrical) tool geometry. Several improvements
are incorporated to the FDTD algorithm to improve its computational efficiency and
its accuracy for geometries/media present in well-logging problems: (i) a 3-D FDTD
cylindrical discretization is employed to avoid staircasing discretization errors in the
transmitter, receiver, and mandrel geometries; (ii) an anisotropic-medium (unsplit)
perfectly matched layer (PML) absorbing boundary condition in cylindrical coordi-
nates is incorporated in the FDTD algorithm, leading to more compact grids and
reduced memory requirements; (iii) a simple and efficient algorithm is employed to
extract frequency-domain data (phase and amplitude) from early-time FDTD data;
(iv) a permittivity scaling is applied to overcome the Courant limit of FDTD and allow faster simulations in the low-frequency regime; And, finally, (v) two locally conformal FDTD (LC-FDTD) techniques are applied to better model the geometry of eccentric boreholes. The FDTD results are validated against NMM results for problems where the latter is applicable, and against pseudoanalytical results for eccentric borehole problems.

In Chapter 4 [4], we study the well-logging tools employing the TCAs. This is a relatively novel class of tools that provide directional (azimuth) information and improved estimates of anisotropy when compared to conventional tools that use horizontal coils. We analyze the response of well-logging tools employing TCAs in eccentric and cylindrical multilayered borehole formations by two approaches. The first approach is based on pseudoanalytical approach, previously applied for concentric borehole problems and augmented here to include eccentric borehole geometries. The second approach is based on 3D numerical simulations using FDTD algorithm discussed in Chapter 3.

In Chapter 5 [5], we analyze the TCAs in anisotropic earth formations using the NMM approach. We extend and employ the NMM approach to analyze the electromagnetic response of well-logging tools employing TCAs in anisotropic (uniaxial) layered earth formations. In this method, the field components are expanded in terms of longitudinal (vertical) eigenmodes to facilitate the analysis of TE and TM fields that coexist in this case. The perfectly matched layer (PML) is incorporated into the NMM formulation to simulate the radiation condition in the longitudinal (vertical) direction. The NMM results are compared against 3-D simulations using FDTD discussed in Chapter 3.
Finally, Chapter 6 summarizes the most important results of the dissertation and suggests possible directions of future research.
CHAPTER 2

WELL-LOGGING TOOLS AND THE WELL-LOGGING BOREHOLE ENVIRONMENT

Well-logging or formation evaluation refers to the process of acquiring information of the earth formation properties (mainly resistivity and porosity) versus the depth, or time around the well. The currently available technology for well logging can be divided into two classes. The first and conventional class is the wireline logging. In this case, the log is carried out after the drilling operation is completed. Wireline logging employs a electrical cable to carry the tool down to the borehole and to transmit data. The second class is the measurement-while-drilling (MWD) or logging-while-drilling (LWD), where the logging and drilling are performed simultaneously. In this dissertation, we will focus on the later due to its increased importance for the oil exploration industry. This chapter gives a brief introduction of the typical formation scenario present in well-logging, and on typical MWD tools.

2.1 Formations

The term formation in the formation evaluation context means the earth media surrounding the borehole. Earth formations usually consist of horizontal layers each having different properties and varying thickness. Earth media can be categorized
by their electrical properties (permittivity, conductivity), which can often (but not always) be assumed uniform within a layer. For a fixed tool geometry, the current (field) from the transmitter(s) may propagate through several layers before it reaches the receiver(s). The variation on the tool response as the tool crosses layer interfaces is known as shoulder-bed effects.

The formation conductivity is often also exhibits anisotropy [6]. This could be due to fractured formations that when filled with salted water (during off-shore drilling, for example) display a higher conductivity in the direction parallel to the fracture plans than in the perpendicular direction [7]. Or, it can be due to sand/clays laminae with differential electrical properties that can be treated as macro-anisotropy [8].

In order to minimize the operational cost and environmental impact, directional drilling has become increasingly prevalent in recent years. By using directional drilling, only movement of the oil platform in the surface is drastically minimized. An estimation of the log is pre-computed by a vertical wireline log, so that the MWD/LWD could efficiently geo-steer to the (horizontal) oil zone. During directional drilling, the layer becomes dipped with respect to the well-logging tool axis. Fig. 2.1 illustrates a well-logging tool as it geo-steers to the oil zone and then remains inside that zone. Moreover, during directional drilling, due to the gravitational pull on the tool (and the always present mechanical vibration), the tool axis can be displaced inside the borehole. The degree of off-center between the tool axis and the borehole axis is denoted as tool eccentricity. The eccentric effect can become significant if the borehole is large with respect to the diameter of the tool itself.

In the process of drilling, the drilling fluid that fills the borehole is called mud. The mud can be either water-based (high conductivity) or oil-based (low conductivity) and
Figure 2.1: The above figure illustrates the *vertical* and *directional* well-logging. The wireline tool is employed to estimate the log. The MWD/LWD tool is geo-steered to the estimated oil-zone.

can infiltrate into formation depending on the porosity and the differential pressure. In this case, an *invasion zone* is formed as a transition region between the mud and the formation. The invasion zone exhibits electric properties intermediate to the borehole mud and the actual formation. The cross-section profile of the invasion zone can be either circular (in vertical-drilling) or oval/elliptic (in directional-drilling).

### 2.2 Well-Logging Tools

A large variety of electromagnetic well-logging tools are employed in practice, with different geometries, number of receivers/transmitter, and frequency of operation. The class of well-logging tools we will discuss here is mostly restricted to
MWD/LWD tools. These tools are more cost efficient for applications in challenging environments (such as deep sea exploration) when compared to wireline tools because they can perform real-time formation evaluation, which facilitate geo-steering to minimize platform movements. MWD/LWD tools are placed very close to the drill bit and are built to withstand extreme mechanical vibration.

In terms of electromagnetic properties, MWD/LWD tools can measure the resistivity of the formations. In this case, the operating frequency for this tool ranges from 100kHz to about 10GHz, and is perhaps most commonly used at 2MHz. A typical geometry uses one transmitter and two receivers coil antennas wrapped around the metallic tool mandrel, as illustrated in Fig. 2.2. The parameter of interest in this case are the phase difference and amplitude ratio of the voltages, (induced by the wave propagated through the formations), at the two receivers. In practice, multiple transmitter and receiver antennas with designated spacings and frequencies can be employed to provide different depths (perpendicular to the tool axis) of investigation.

Throughout this dissertation whenever we refer to MWD or LWD well-logging tools, it means electrical resistivity ones.
Figure 2.2: The two tools above represent the conventional (left) and novel directional (right) MWD tool. The conventional tool uses horizontal-coil antennas and the novel directional tool employ tilted-coil antennas and horizontal-coil antennas to provide directional sensitivity.
CHAPTER 3

THREE-DIMENSIONAL FINITE-DIFFERENCE TIME-DOMAIN SIMULATION OF MWD/LWD TOOL RESPONSES

3.1 Introduction

As mentioned in Chapter 1, numerical simulation of electromagnetic well-logging tools in complex borehole environments is of great importance for the interpretation of measurement data and characterization of oil reservoirs [9],[10],[11].

Numerical methods such as finite element [12] (FEM), finite-difference frequency-domain (FDFD) [13], and transmission line matrix (TLM) methods [11], as well as pseudoanalytical methods such as numerical mode matching [14] (NMM) and conjugate gradient fast Fourier-Hankel transform (CG-FFHT) type methods [15] have been employed in the past to study the electromagnetic response of well-logging tools. Numerical methods such as FEM and FDFD discretize Maxwell’s equations directly, and hence can be applied to study general tool geometries and responses in arbitrary three-dimensional (3-D) formations. However, such (frequency-domain) methods require the solution of a large sparse linear system. This can become impractical in current machines for large 3-D well-logging problems, especially when the number of unknowns grows beyond $10^6 - 10^7$. Moreover, the presence of large resistivity
contrasts, which is common in many geophysical formations, poses an additional challenge for both FEM and FDFD methods because the associated linear system can become ill-posed. The NMM and CG-FFHT, on the other hand, are much faster and efficient methods but cannot be easily applied to arbitrary tool geometries and 3-D formations.

In this Chapter, we discuss the implementation and use of finite-difference time-domain (FDTD) methods for the modeling of well-logging tools. Similarly to FEM and FDFD, the FDTD method can solve for arbitrary 3-D formations since it discretizes Maxwell’s equations directly. However, FDTD involves an explicit time-domain update method. As a result, FDTD does not require the solution of a linear system (matrix-free) and larger problems can be solved. Indeed, FDTD can be classified as an optimal method in the sense that it exhibits $O(N)$ computational complexity per time step and requires $O(N)$ memory, where $N$ is the number of unknowns. The drawback is that, since it requires a time stepping process, the FDTD solution can be slower for small problems than FEM and FD solutions in the frequency domain. Here, we discuss the application of several techniques to the standard FDTD algorithm to solve the well-logging problem more efficiently and accurately. (i) We implement the FDTD in a 3-D cylindrical grid to avoid staircasing discretization errors in the transmitter, receiver, and mandrel geometries. (ii) An anisotropic-medium (unsplit) perfectly matched layer (PML) absorbing boundary condition in cylindrical coordinates is applied to the FDTD algorithm, leading to more compact grids and reduced memory requirements for a problem of given (physical) size. (iii) A simple and efficient algorithm is applied to extract frequency domain data from early-time FDTD data, allowing phase and amplitude data to be obtained in shorter simulation
times. (iv) For low frequency problems, a scaling on permittivity values is applied to overcome the Courant limit and allow for larger time step sizes (and hence shorter simulation times). (v) In order to examine the effect of eccentricity on the tool response, two locally-conformal FDTD (LC-FDTD) schemes are applied to model non-conformal borehole/formation interfaces.

The Chapter is organized as follows. In Section 3.2, we discuss aspects of the FDTD formulation for the well-logging problem. In Section 3.3, we describe in more detail the particular logging-while-drilling (LWD) tool geometry under consideration and its discretization using a cylindrical FDTD grid. In Section 3.4, we validate the FDTD results against NMM results and illustrate various applications of the algorithm to borehole problems involving eccentric tools, dipping bed formations, and/or high contrast between mud/formation conductivities. Finally, in Section 3.5, we present some final remarks and draw the main conclusions.

3.2 Cylindrical FDTD Formulations

The use of a cylindrical FDTD algorithm instead of traditional Cartesian FDTD is advantageous for well-logging tools because it automatically conforms to the (cylindrical) tool geometry (mandrel, drill collar, and antennas) thus avoiding staircasing errors caused by discretization. The use of a fully 3-D algorithm is important to study arbitrary formations, not restricted to axisymmetric geometries.

The spatial discretization adopted for the cylindrical FDTD here utilizes a staggered cylindrical grid [16],[17] (refers to Appendix A). The cell size is uniform in
the longitudinal, $z$, and azimuthal, $\phi$, directions. In the radial, $\rho$, direction, non-uniform discretization is utilized to reduce overall memory requirements. Details on the spatial discretization are given in Section 3.3.

### 3.2.1 PML Absorbing Boundary Condition in 3-D Cylindrical FDTD Grids

In order to implement the FDTD algorithm for open-domain problems, an absorbing boundary condition is needed to truncate the computational domain and avoid spurious reflections from the computational boundaries. The 3-D anisotropic-medium cylindrical PML introduced in [17] is applied here for this purpose. Unlike the split-field PML formulation [16],[18],[19], the anisotropic-medium PML does not require modifications on Maxwell’s equations. Instead, only the constitutive relations inside the PML region are modified. In cylindrical coordinates, the PML constitutive parameters (permittivity and permeability) are characterized by diagonal tensors $\bar{\epsilon}, \bar{\mu}$ given by [17]

\[
\bar{\epsilon}_{\text{PML}} = \epsilon \bar{\Lambda}_{[\rho,\phi,z]}(\rho, z; \omega) \\
\bar{\mu}_{\text{PML}} = \mu \bar{\Lambda}_{[\rho,\phi,z]}(\rho, z; \omega)
\]

with

\[
\bar{\Lambda}_{[\rho,\phi,z]}(\rho, z; \omega) = \hat{\rho} \hat{\rho} \rho s_{\rho} + \hat{\phi} \hat{\phi} \rho s_{\phi} + \hat{z} \hat{z} \rho s_{z} 
\]

Here, $\hat{\rho}$ is the analytic continuation of coordinate $\rho$ to a complex variable domain, and $s_{\rho}, s_{z}$ are frequency-dependent complex stretching variables [19], defined as:

\[
s_{\rho}(\rho) = a_{\rho}(\rho) + i \frac{\Omega_{\rho}(\rho)}{\omega} \\
s_{z}(z) = a_{z}(z) + i \frac{\Omega_{z}(z)}{\omega}
\]
\[ \hat{\rho} = \int_0^\rho s_\rho(\rho')d\rho' = \int_0^\rho \left( a_\rho(\rho') + i\frac{\Omega_\rho(\rho')}{\omega} \right) d\rho' = b_\rho(\rho) + i\frac{\Delta_\rho(\rho)}{\omega} \]  

(3.6)

where \( a_\rho(\rho), a_z(z) \) (real stretching parameter) and \( \Omega_\rho(\rho), \Omega_z(z) \) (PML conductivity) are functions of position only. The purpose of these variables is to change the eigenfunctions of the problem inside the PML [17] so as to produce reflectionless absorption. By choosing the \( \Omega_{\rho,z} > 0 \), propagating eigenfunctions of Maxwell’s equations are mapped into exponentially decaying eigenfunctions inside the PML. Moreover, by letting \( a_{\rho,z} > 1 \), faster decay of evanescent modes are induced inside the PML. Both of these variables are chosen to gradually increase along the normal coordinate following a polynomial profile inside the PML. Inside the physical domain, \( \Omega_{\rho,z} = 0 \), and \( a_{\rho,z} = 1 \), so that the original equations are recovered. The optimal values and polynomial profile for \( \Omega_{\rho,z} \) and \( a_{\rho,z} \) are problem-dependent. In general, the best choice depends on the spatial discretization size, the number of cell used for the PML, the frequency spectrum of the excitation, and both the conductivity and permeability of the background medium. It is not our objective here to perform a detailed study on the optimal choices for these parameters.

### 3.2.2 Locally Conformal FDTD for Eccentric Borehole Problems

Because of mechanical vibrations and/or gravitational pull effects, electromagnetic well-logging tools are not always perfectly aligned with the borehole axis. This leads to so-called *eccentricity* effects [20]. These are particularly prevalent in highly deviated drilling where the gravitational pull is more intense.
Figure 3.1: Averaging procedure used in the edge-based conformal FDTD scheme showing, on the top, the intersection between the regular FDTD mesh and curved interface between two conductive media, at the left, the original problem with different conductivities along a single edge, and at the right, the equivalent problem with effective conductivities along each edge.

In the cylindrical FDTD, we choose the mandrel and antenna geometries to be conformal to the grid. As a result, eccentric tools produce borehole/formation interfaces that are not conformal to the cylindrical grid. Here, we model these interfaces by locally-conformal (LC) FDTD algorithms. These algorithms have been previously explored to model curved dielectric interfaces in Cartesian grids [21]. In this section, we describe the application of the LC-FDTD approaches to the conductive borehole wall interfaces in the 3-D cylindrical grid.

(a) Edge-based LC-FDTD

The edge-based LC-FDTD approach utilizes an arithmetic mean over edges to derive effective permittivities, $\varepsilon_x^{eff}(i,j,k)$ and $\varepsilon_y^{eff}(i,j,k)$. For conductive interfaces, effective conductivities can be obtained in an analogous manner.
Fig. 3.1 illustrates this approach. The curved interface crosses the edges $E_x(i, j, k)$ and $E_y(i, j, k)$. The conductivity used to update the associated fields is replaced by the effective one given by

$$\sigma_{x}^{\text{eff}}(i, j, k) = (\Delta x_2(i, j, k)\sigma_2 + (\Delta x - \Delta x_2(i, j, k))\sigma_1)/\Delta x(i, j, k) \quad (3.7)$$

$$\sigma_{y}^{\text{eff}}(i, j, k) = (\Delta y_2(i, j, k)\sigma_2 + (\Delta y - \Delta y_2(i, j, k))\sigma_1)/\Delta y(i, j, k) \quad (3.8)$$

respectively (linear weighting). Since the curved interface does not penetrate the remaining edges $E_x(i, j + 1, k)$ and $E_y(i + 1, j, k)$, the conductivity used to update the remaining field components is simply the corresponding conductivity of the region where the associated edge is situated. This modification can be incorporated with a minimal amount of extra bookkeeping in the cylindrical FDTD algorithm by identifying the edges intersected by the curved borehole/formation interface, and changing the corresponding conductivities by effective ones. In the next section we discuss the application of face-based LC-FDTD for this same problem. The face-based LC-FDTD is more intuitively more appealing and indeed leads to more accurate results than edge-based LC-FDTD. We nevertheless include some edge-based results in the examples that follow since this approach is still used in the FDTD literature.

**b) Face-based LC-FDTD**

The second LC-FDTD approach utilizes face-based averaging. In this approach, a weighted mean is again used to obtain effective conductivities but the averaging process is now applied to the fields normal to the dual grid face where the curved interface exists. Both the permittivity and conductivity are associated with a particular electric field, so when a material interface crosses the surface of a dual grid cell, the permittivity and conductivity is weighted accordingly.
Figure 3.2: Linear weighted averaging procedure used in the face-based conformal
dielectric FDTD scheme. The top figure shows the intersection between a curved
interface and the FDTD dual grid mesh. The bottom figure depicts the equivalent
problem using effective conductivities over each dual grid face.

Fig. 3.2 illustrates this second approach. From Maxwell’s equations in the integral
form applied at a given dual grid cell face, we have

\[
\int \int_S \left( \epsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} \right) \cdot d\vec{S} = \int \int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint l \vec{H} \cdot d\vec{l} \tag{3.9}
\]

Assuming constant electric field over each dual grid cell face of the FDTD grid,
the above equation simplifies to

\[
(S_1 + S_2) \epsilon \frac{\partial \bar{E}}{\partial t} + (\sigma_1 S_1 + \sigma_2 S_2) \bar{E} = \oint l \vec{H} \cdot d\vec{l} \tag{3.10}
\]

here, \(S_1\) and \(S_2\) represent the partial areas of each dual grid cell face corresponding to
conductivities \(\sigma_1\) and \(\sigma_2\), respectively (gray and white areas in Fig. 3.2). The effective
conductivity \(\sigma_{\text{eff}}\) then becomes the weighted average below

\[
\sigma_{\text{eff}} = \frac{\sigma_1 S_1 + \sigma_2 S_2}{S_1 + S_2}. \tag{3.11}
\]
Note that this approach applies to all three electric field components. The effective conductivities used in the FDTD update equations for each electric field component use the $S_1$ and $S_2$ calculated on the corresponding normal (dual grid) face.

### 3.2.3 Frequency Data Extraction for Treated FDTD Excitations

Well-logging problems usually involves CW (single frequency) operation (CW tools). For problems where the steady-state, CW behavior of the fields is of interest, it may be very costly to run the time-domain simulation for long periods of time to extract frequency-domain data, particularly for explicit time-domain methods, such as FDTD, in large 3-D domains. As a result, it is important to develop techniques to extract accurate frequency data at early times during the simulation.

To achieve this, we employ here a treatment on the time-domain source excitation and use the so-called two-equation two-unknown (2E2U) algorithm proposed in [22] for time-to-frequency conversion. The 2E2U is a very simple approach based on the solution of two linear equations in single-frequency analysis. Assuming a sinusoidal excitation at the source, one can approximate the field components at any location in the computational domain in the generic form

$$
A \sin(\omega t_1 + \theta) = q_1 \\
A \sin(\omega t_2 + \theta) = q_2
$$

(3.12)

where $A$ and $\theta$ are the amplitude and the phase, and $\omega = 2\pi f$ is the angular frequency. These equations are an approximation because, in practice, one cannot have a pure sinusoidal excitation but only a truncated one due to the turn-on effect of the sources at $t = 0$. The turn-on effect produces high-frequency contamination and a DC offset as discussed later on. Given any two time instants, $t_1$ and $t_2$ and corresponding field
values \( q_1 \) and \( q_2 \) from the FDTD simulation, the unknowns \( A \) and \( \theta \) can be obtained as

\[
\theta = \arctan \left[ \frac{q_2 \sin(\omega t_1) - q_1 \sin(\omega t_2)}{q_1 \cos(\omega t_2) - q_2 \cos(\omega t_1)} \right]
\]

\[
A = \left| \frac{q_1 \sin(\omega t_1 + \theta)}{\sin(\omega t_1 + \theta)} \right| \tag{3.13}
\]

Successive \( t_1 \) and \( t_2 \) are used during the FDTD update until convergence is reached for \( A \) and \( \theta \). In practice, \( t_1 \) and \( t_2 \) should be chosen sufficiently apart to avoid ill-conditioning when solving for \( A \) and \( \theta \). In practice, a satisfactory rule of thumb is to choose the absolute time difference between \( t_1 \) and \( t_2 \) to be around one tenth of the period \( T = 1/f \).

Because of the turn-on effect at \( t = 0 \), the early excitation of the truncated sinusoidal source needs to be modified; otherwise, high frequency contamination and DC-offset [23] (i.e., a \( 1/t \) slowly decaying time-average signal) will occur, leading to a slower convergence in the frequency data extraction. A smooth turn-on of the (pseudo-)sinusoidal source can be used to avoid such DC-offset. In addition, a smooth turn-on avoids high-frequency contamination from the discontinuity on the derivative at \( t=0 \). In our case, we choose a time-domain excitation of the form

\[
v_s(t) = r(t) \sin(\omega t), \quad \text{where } r(t) \text{ is a raised cosine (RC) ramp function given by}
\]

\[
r(t) = \begin{cases} 
0 & t < 0 \\
0.5[1 - \cos(\omega t/2\alpha)] & 0 \leq t \leq \alpha T \\
1 & t > \alpha T
\end{cases} \tag{3.14}
\]

where \( T = 2\pi/\omega \) is the period of the sine function, and \( \alpha \) is the number of sine wave cycles during the ramp duration \( \alpha T \).

This particular excitation has the desirable properties that both the function and its first derivative are continuous for all values of \( \alpha \) at \( t = 0 \). However, the DC-offset is zero only for values of \( \alpha \) multiples of 0.5.
Figure 3.3: Example of the convergence behavior of the phase difference between LWD tool receiver voltages using the 2E2U approach.

Fig. 3.3 illustrates a typical convergence result for the phase difference between the LWD receivers using a ramp-sinusoidal function with \( \alpha = 0.5 \), and the 2E2U approach. The phase difference in this particular example converges after a time period approximately equals to 1.1\( T \).

3.2.4 Permittivity Scaling for Low Frequency Problems

In the FDTD algorithm, the Courant stability condition in a medium with \( \mu \), \( \epsilon \), and \( \sigma \) establishes an upper bound on the time step, \( \Delta t_c \), given by [24]

\[
\Delta t_c = v^{-1} \left[ (\Delta \rho_{min})^{-2} + (\rho_{min} \Delta \phi)^{-2} + (\Delta z_{min})^{-2} \right]^{-1/2}
\]  

(3.15)
where \( v \) is the phase velocity given by

\[
v = (\epsilon \mu)^{-1/2}
\]  

(3.16)

In well-logging problems, the spatial discretization resolution is often dictated by the skin depth inside the conductive formation and not by the wavelength (since the former can be smaller than the latter). As a result, the maximum time step \( \Delta t_c \) can become very small compared to the period of the wave. A small \( \Delta t_c \) implies that the total number of time steps necessary to achieve convergence can become very large, especially for low frequency tools in high conductivity formations.

Under these conditions, however, the displacement current is much smaller than the conduction current. When this occurs (diffusion dominated regime), one can simply scale up the permittivity \( \epsilon \) of the medium to increase the maximum allowed time step [25], as can be seen by writing Maxwell’s equations in frequency domain:

\[
\nabla \times \vec{E} - i\omega \mu \vec{H} = 0,
\]

\[
\nabla \times \vec{H} - (\sigma - i\omega \epsilon)\vec{E} = \vec{J}_s,
\]

(3.17)

Here, all field quantities are now complex phasors, and \( \vec{J}_s \) is the current excitation.

If \( \sigma \gg \omega \epsilon \), the above simplifies to

\[
\nabla \times \vec{E} - i\omega \mu \vec{H} = 0,
\]

\[
\nabla \times \vec{H} - \sigma \vec{E} = \vec{J}_s,
\]

(3.18)

which does not depend on \( \epsilon \) anymore. Hence, by scaling up \( \epsilon \) (while still satisfying \( \sigma \gg \omega \epsilon \)), the Courant condition is relaxed without altering the physics of the problem. Permittivity scaling was also used in [26] to solve a diffusive time-domain problem. In the present case, the only necessary post-processing in the final results is the scaling back of field amplitudes to incorporate the change on the intrinsic impedance. This
scaling will depend on the way the excitation is introduced in the problem. If an
electric source is used, then a scaling up is effected on the magnetic field. For a
magnetic source, a scaling down is effected on the electric field.

3.3 FDTD Modeling of LWD Tools

The LWD tool configuration used here is illustrated in Fig. 3.4. The frequencies
of operation in the examples considered range from 100kHz to 2MHz. The tool has
a 8 [in] diameter steel mandrel at its center. The transmitter and receiver antennas
consist of 4.5 [in] radius circular wire loops wrapped around the mandrel. To simulate
these loop antennas in the FDTD grid, a spatial discretization employing either a
gap source excitation or a continuous source excitation can be utilized. In the gap
source excitation, a voltage source is impressed between two points (gap) of the wire
transmitter antenna by enforcing the electric field to assume predetermined values
along FDTD edges on a straight line between these two points. In the continuous
source excitation, an electric current source is impressed throughout the FDTD nodes
comprising the wire transmitter antennas by enforcing the magnetic field to assume
predetermined values over small loops (Ampere’s law) enclosing each FDTD node
along the wire antenna. In the simulations that follow, we utilize continuous source
excitations. The two single-loop receiver antennas are situated at 30 and 24 [in]
away from a single-loop transmitter antenna. The quantities of interest are the phase
difference (PD) and the amplitude ratio (AR) between the voltages measured at the
two receiver antennas:

\[ PD = \theta_r2 - \theta_r1 \]

\[ AR = \frac{A_r2}{A_r1} \]  

(3.19)
where $\theta$ and $A$ is the phase and amplitude of the receiver voltages, and the subscripts $r1$ and $r2$ denote receiver 1 and 2, respectively.

A $\rho - z$ cross-section plane of the LWD tool discretization is depicted in Fig. 3.5. The discretization is non-uniform in the $\rho$ direction, and includes two cells between the coils. Outside the coils, $\Delta \rho$ is gradually increased to minimize memory requirements. The maximum cell size is chosen as $(\Delta \rho)_{max} = \delta/5$ where $\delta$ is the skin depth corresponding to the largest conductivity value in the formation. The increase in $\Delta \rho$ is gradual to avoid spurious reflections due to abrupt changes in the discrete impedance of the grid (which depends on the local cell size). The skin depth is given as

$$
\delta = \frac{1}{\omega \sqrt{\mu \epsilon}} \left\{ \frac{1}{2} \left[ \sqrt{1 + \left( \frac{\sigma}{\omega \epsilon} \right)^2} - 1 \right] \right\}^{-1/2} [m]
$$

(3.20)

For $\frac{\sigma}{\omega \epsilon} \gg 1$, this reduces to:

$$
\delta = \frac{2}{\omega \mu \sigma}
$$

(3.21)

A uniform discretization is employed along both the longitudinal, $z$, and azimuthal, $\phi$, directions.

3.4 Numerical Results and Validation

3.4.1 Homogeneous Formations

Fig. 3.6 depicts the phase difference response of the LWD tool in a borehole surrounded by homogeneous formations (infinitely thick bed) with different conductivities. The tool operates at 2MHz and the borehole has 10 [in] diameter in this case. The conductivity of the mud (borehole fluid) is equal to $5 \times 10^{-4}$ [mho/m] (oil-based mud). A two-dimensional (2-D) version of the algorithm can be employed in this case since the problem is axisymmetric. The domain is discretized using a $(N_\rho, N_z)$
= (50,180) grid. The PML is set up using 10 cells both in $\rho$ and $z$ directions and a cubic taper profile for both the real and imaginary parts of the stretching variables. The cell discretization size in the longitudinal direction in $\Delta z = 3.81$ [cm], while in the radial direction $\Delta \rho$ varies from 0.635 [cm] close to the mandrel to 3.18 [cm] at the outer edges. This latter value is determined according to the skin depth of the formation with largest conductivity. The results are compared against NMM results in Fig. 3.6, showing very good agreement for all conductivity values.

### 3.4.2 Eccentric Tools

We next simulate an eccentric borehole problem, where the LWD tool is not aligned with the borehole axis, as illustrated in Fig. 3.7. The conductivity distribution over an horizontal cross-section ($xy$-plane) of the 3-D FDTD grid is depicted in Fig. 3.8, where face-based averaging of the borehole and formation conductivities are visible at the borehole/formation interface.

Fig. 3.9 shows results from the simulation of an LWD tool in a mud with $\sigma = 0.0005$ [mho/m]) and 12 [in] radius borehole. The abscissa in this figure represents the mandrel offset, i.e., the distance between the mandrel axis and the borehole axis, as indicated in Fig. 3.7. The tool operates at 2 MHz. The formation is homogenous with $\sigma = 10$ [mho/m], and different eccentricities are considered. The discretization of the domain in this case utilizes a $(N_{\rho}, N_{\phi}, N_z) = (50,76,430)$ grid. The PML has 10 cells with cubic taper profiles in both the $\rho$ and $z$ directions. The discretization is non-uniform in $\rho$ direction, with $\Delta \rho$ varying from 0.635 [cm] to 18.77 [cm], and uniform in $z$ direction, with $\Delta z = 1.524$ [cm]. Fig. 3.9 compares the face-based and edge-based LC-FDTD against pseudoanalytical results, showing very good agreement.
Face-based LC-FDTD consistently perform slightly better than the edge-based LC-FDTD.

In general, the effect of eccentricity on the tool response is larger when the conductivity contrast between the mud and the surrounding formation is high. To illustrate this point, Fig. 3.10 shows results for the same 2MHz LWD in a 12 [in] borehole now with water-based mud with $\sigma = 10$ [mho/m]) in homogeneous formation with $\sigma = 0.1$ [mho/m]. The discretization utilizes the same parameters as the previous example. The variation on the (absolute) phase difference and amplitude ratio as a function of the eccentricity is clearly less pronounced in this case. Face-based and edge-based LC-FDTD again show very good agreement against pseudoanalytical results.

The maximum degree of eccentricity depends on the ratio between the borehole and mandrel radii. In other words, for a given tool geometry (mandrel radius), the maximum variation on the tool response due to eccentricity effects can be established as a function of the borehole radius. To illustrate the latter point, we consider two borehole scenarios. In the first scenario, we let the borehole size vary while keeping the logging tool centered (Case 1), as illustrated in Fig. 3.11(a). In the second scenario, we let the borehole size vary while keeping the distance from the logging tool to the borehole wall fixed (Case 2) and equal to $\Delta x = 4.75$ [in], as illustrated in Fig. 3.11(b). We consider a high contrast case where the conductivity of the mud is $\sigma = 10$ [mho/m] and the conductivity of the formation is $\sigma = 0.0001$ [mho/m]. The phase difference and amplitude ratio between the receiver voltages is plotted against the borehole radius in Fig. 3.12. The difference between Case 1 and Case 2 is produced solely by the eccentricity effect. These two curves provides us with a range of values for phase difference and amplitude ratio in a borehole of given size when the degree of
eccentricity is not exactly known (as often occurs in practice). From this figure, we observe that the variation of the tool response due to eccentricity effects is more limited for smaller size boreholes, as expected.

To illustrate qualitatively the effect of the eccentricity on the field distribution, we plot in Fig. 3.13 and Fig. 3.14 the $\phi$-component electric field distribution along the horizontal cross section of the borehole. In this case, the mandrel axis is situated 3.5 [in] away from the borehole axis in a borehole with 9 [in] radius. In Fig. 3.13, the conductivity of the mud and formation are $\sigma_{mud}=0.01$ and $\sigma_{for}=0.1$, respectively. In Fig. 3.14, the conductivity of the mud and formation are $\sigma_{mud}=1.0$ and $\sigma_{for}=0.1$, respectively. In these figures, we observe that the field penetrates asymmetrically into the formation due to eccentricity effect. The field penetration profile not only depends on the degree of eccentricity but also on the relative conductivities of the mud and the surrounding formation.

3.4.3 Eccentric Tools in Dipping Beds

The present FDTD method allows the simulation of LWD tool problems in eccentric and dipping bed environments simultaneously. Fig. 3.15 shows the simulation results for an LWD tool operating at 2MHz on a three-layer formation with dipping bed for various eccentricities and dipping angles. The middle layer (dipping bed) is 60 [in] thick. The upper and lower layers have conductivity $\sigma = 1$ [mho/m], whereas the middle layer has conductivity $\sigma = 0.01$ [mho/m]. The borehole has 9 [in] radius. We consider an oil-based mud with $\sigma = 0.0005$ [mho/m].

The discretization employs a $(N_\rho,N_\phi,N_z) = (50,60,175)$ grid. The PML has 10 cells with cubic taper profiles in both the $\rho$ and $z$ directions. The discretization is
non-uniform in \( \rho \) direction, with \( \Delta \rho \) varying from 0.635 [cm] to 7.11 [cm], and uniform in \( z \) direction, with \( \Delta z = 5.08 \) [cm]. We use the face-based LC-FDTD to model the eccentric geometry and a staircasing approximation to model the dipping bed. The ECCE label in Fig. 3.15 represents the offset distance \( \Delta x \) (in inches) of the eccentric tool as indicated in Fig. 3.7. Compared to [27], [28], the noneccentric results in these figures show very similar behavior according to the dip angle.

From these plots, we can observe two main effects caused by the eccentricity. First, as mentioned before, the change on the apparent resistivity due to the eccentricity effect is larger when the contrast between the mud and the resistivity (conductivity) of the surrounding layer is larger. Note that, in this example, the larger contrast occurs in the upper and lower layers (leftmost and rightmost regions of the figure, respectively). In addition, the eccentricity produces an increase in the horn effect in the highly deviated bed case for this example.

### 3.4.4 LWD Tools at Lower Frequencies

Fig. 3.16 shows the simulation results for a LWD tool operating at 500kHz, using a water-based mud with \( \sigma = 2 \) [mho/m] in a three-layer formation where the center bed is again 60 [in] thick. The FDTD results are validated against NMM results showing very good agreement. This problem is axisymmetric and the discretization of the 2-D domain in this case uses a \( (N_\rho, N_z) = (50,130) \) grid. Since the skin depth is larger at this lower frequency, the cell size in the \( \rho \) direction varies from 0.635 [cm] to 14.24 [cm]. Note that, for lower frequencies, large variations on the cell size have less impact on the discrete impedance because the number of grid points per wavelength is much larger. The \( z \) direction cell size is uniformly discretized with \( \Delta z = 7.62 \) [cm].
The PML is inserted over the ten outermost cells in along the radial direction and over the bottom ten and top ten cells along the longitudinal, z, direction. The PML employs a cubic tapered profile on both the real and imaginary part of the stretching variables.

Next, we simulate the same tool operating at 100kHz. The discretization in this case employs a \((N_\rho, N_z) = (40, 200)\) grid. The discretization cell is non-uniform in the \(\rho\) direction varying from 0.635 [cm] to 31.83 [cm], and uniform in the \(z\) direction with \(\Delta z = 7.62\) [cm]. The ten-layer PML employs a cubic tapered profile only in the real part of the stretching variables. No imaginary stretching is used since this is now an essentially diffusive problem. Permittivity scaling is desirable at this frequency because the ratio of the displacement current to the conduction current is very small, \(\frac{\omega \varepsilon}{\sigma} \approx 5.56 \times 10^{-4}\). A time step much larger than established by the Courant condition 3.15 can then be used, as illustrated in Table 3.1. Fig. 3.17 shows a comparison between FDTD and NMM results for the phase difference and amplitude ratio in this case. FDTD results use \(\varepsilon_r = 1, 17.975,\) and 179.75. This figure shows that the scaled permittivity results still provide very accurate results. The phase velocity, wavelength, time step, and normalized CPU time for \(\varepsilon_r = 1, 17.975\) and 179.75 are given in Table 3.1.

### 3.5 Conclusions

We have described the application of several techniques to model eccentric borehole problems in dipping formations using cylindrical FDTD. Results from FDTD
simulations were validated against pseudoanalytical and NMM results (when the latter are applicable) in a number of scenarios, including eccentric tools with high conductivity contrast between mud and formation and for varying degrees of eccentricity.

For the examples considered, the effect of eccentricity on the tool response is observed to be more important when the conductivity contrast between the mud and the surrounding formation is high. The specific behavior of the amplitude ratio and phase difference versus the degree of eccentricity depends on the particular mud and formation conductivities involved, as illustrated in Fig. 3.9 and Fig. 3.10. When considered together with deviated formations, it was observed that eccentric geometries can produce an increase in the horn effect for large dipping angles, as shown in Fig. 3.15.

The maximum degree of eccentricity in a borehole problem is a function of the ratio between the mandrel and borehole radii. For a given ratio, curves indicating the predicted variations on the tools response caused to eccentricity effects can be obtained by simulating the responses of both centered (axisymmetric) and maximally eccentric cases, such as provided in Fig. 3.12.

The use of PML as absorbing boundary condition for the cylindrical FDTD algorithms allows the use of more compact grids around the logging tool, and the use of locally conformal FDTD techniques reduces the staircasing error present in the representation of the borehole wall for eccentric problems. Finally, for tools operating at lower frequencies, the use of permittivity scaling has shown to be useful in reducing the computation time for diffusive problems. For example, the CPU time necessary to run a 100KHz LWD tool after scaling is about 13% of the original CPU time (with the actual permittivity) for the case considered in Table 3.1.
<table>
<thead>
<tr>
<th>$\epsilon_r$</th>
<th>$\sigma/\omega \epsilon$</th>
<th>$v(\times 10^6)$ [m/s]</th>
<th>$\lambda$ [m]</th>
<th>$\Delta t$ [ps]</th>
<th>normalized CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1797.5</td>
<td>299.79</td>
<td>2997.9</td>
<td>13.03</td>
<td>1.000</td>
</tr>
<tr>
<td>17.975</td>
<td>100</td>
<td>70.71</td>
<td>707.10</td>
<td>55.24</td>
<td>0.336</td>
</tr>
<tr>
<td>179.75</td>
<td>10</td>
<td>22.36</td>
<td>233.60</td>
<td>174.69</td>
<td>0.139</td>
</tr>
</tbody>
</table>

Table 3.1: Phase velocity, wavelength, timestep, and normalized CPU time with scaled permittivities.
Figure 3.4: The basic configuration of the LWD tool inside a 5 [in] radius borehole.
Figure 3.5: Illustration of the cross-section of the LWD tool discretization in a layered formation
Figure 3.6: FDTD and NMM results for phase difference (PD) and amplitude ratio (AR) of a LWD tool in a uniform formation (infinitely thick bed) for various values of formation conductivities. The mud conductivity is fixed at 0.0005 [mho/m].
Figure 3.7: Horizontal cross-section of an eccentric borehole problem.
Figure 3.8: Conductivity distribution along the horizontal cross-section of the cylindrical FDTD for an eccentric LWD tool scenario.
Figure 3.9: FDTD simulation results versus pseudoanalytical results of eccentric LWD tool responses. Phase difference and amplitude ratio are plotted against the mandrel offset $\Delta x$ (see Fig. 3.7). The borehole has 12 [in] radius and is filled with oil-based mud $\sigma = 0.0005$ [mho/m]. The surrounding homogeneous formation has $\sigma = 10$ [mho/m]. Very good agreement is observed between the FDTD and pseudoanalytical results.
Figure 3.10: FDTD simulation results versus pseudoanalytical results of eccentric LWD tool responses. Phase difference and amplitude ratio are plotted against the mandrel offset $\Delta x$ (see Fig. 3.7). The borehole has 12 [in] radius and is filled with oil-based mud $\sigma = 10.0$ [mho/m]. The surrounding homogeneous formation has $\sigma = 0.1$ [mho/m]. Very good agreement is again observed between the FDTD and pseudoanalytical results.
Figure 3.11: Influence of eccentricity and borehole size on LWD tool response. (a) Case 1: Borehole size is varied while keeping mandrel centered. (b) Case 2: Borehole size is varied while keeping the offset distance $\Delta x$ fixed.
Figure 3.12: FDTD simulation of the response of eccentric LWD tools in a homogeneous formation. A high contrast case is considered with mud and formation conductivities equal to 10 and 0.0001 [mho/m], respectively. Case 1 curve is the LWD tool response versus borehole radius while the tool is kept at the center of the borehole (see Fig. 3.11(a)). Case 2 curve is the LWD tool response versus borehole size while keeping a fixed distance $d = 4.75$ [in] from the mandrel axis to closest borehole wall (see Fig. 3.11(b)).
Figure 3.13: Electric field ($\phi$-component) distribution on the horizontal cross-section of the cylindrical FDTD grid for an eccentric LWD tool. The top figure depicts an expanded view of the field distribution on the formation. The bottom figure is a zoomed view of the field distribution close to the borehole. The mud conductivity is $\sigma_{\text{mud}} = 0.01 \text{ [mho/m]}$, and the formation conductivity is $\sigma_{\text{for}} = 0.1 \text{ [mho/m]}$. 
Figure 3.14: Electric field ($\phi$-component) distribution on the horizontal cross-section of the cylindrical FDTD grid for an eccentric LWD tool. The top figure depicts an expanded view of the field distribution on the formation. The bottom figure is a zoomed view of the field distribution close to the borehole. The mud conductivity is again $\sigma_{\text{mud}} = 1.0$ [mho/m], and the formation conductivity is now $\sigma_{\text{for}} = 0.1$ [mho/m] in this case.
Figure 3.15: FDTD simulations of eccentric LWD tools in formations with dipping beds. The upper and lower layer conductivities is $\sigma = 1$ [mho/m], while the 60 in thick dipping bed has $\sigma = 0.01$. The borehole has radius 9 [in] and is filled with oil-based mud with $\sigma = 0.0005$. The label ECCE in this plots represents the offset distance $\Delta x$ in inches (see Fig. 3.7)
Figure 3.16: Comparison of FDTD and NMM results of a 500kHz LWD tool response in a three-layer formation with lower, middle and upper layer conductivities equal to 1, 0.01 and 1 [mho/m], respectively. The conductivity of the mud is 2 [mho/m]. The middle bed is 60 [in] thick.
Figure 3.17: Comparison of FDTD and NMM results of a 500kHz LWD tool response in a three-layer formation with lower, middle and upper layer conductivities again equal to 1, 0.01 and 1 [mho/m], respectively. The mud conductivity is now 0.0005 [mho/m]. The middle bed is 60 [in] thick.
4.1 Introduction

The behavior of electromagnetic logging tools incorporating transmitter and/or receiver coils tilted with respect to the borehole axis has been the subject of interest recently [29], [30]. In applications related to oilfield exploration, tilted-coil antennas not only provide directional (azimuthal) sensitive data [31], but can also provide improved estimates of anisotropy. In logging-while-drilling/measurement-while-drilling (LWD/MWD) applications for example, sensitivity to azimuthal orientation around the borehole axis can be employed for improved geo-steering in directional drilling.

Mandrel, borehole, and (piecewise homogeneous) invasion effects in tilted-coil logging tools designs can be characterized using pseudoanalytical formulations. Previous studies have considered cylindrically symmetric borehole problems [32]. Such results show that some configurations are well approximated by (tilted) point dipole antennas, with only minor corrections necessary, but other configurations can depart significantly from the dipole response, especially when invasion effects are considered [32].
Because of its azimuthal sensitivity, logging tools with tilted coil antennas are attractive for use in deviated or horizontal drilling. However, because of mechanical vibrations and the directional gravitational pull, the tool axis is often eccentric to the borehole axis. Therefore, it is important to study the effect of eccentricity [20] on the electrical response (and azimuthal sensitivity) of these tools.

In this Chapter, we analyze the electrical response of well-logging tools using tilted-coil antenna designs in both cylindrical multilayered and eccentric borehole problems using (1) a pseudoanalytic formulation for tilted coils [32] augmented here to include eccentric cases [20] and (2) the three-dimensional (3-D) cylindrical finite-difference time-domain (FDTD) algorithm from chapter 3 extended to tilted coils. Both approaches can incorporate mandrel effects.

4.2 Tilted-Coil Well-Logging Tools

Conventional well-logging tools [14], [15], [17], [33] usually employ horizontal loop antennas. Conventional tools can extract information about the surrounding formation only along the axial (vertical) direction. A tilted-coil LWD/MWD tool consists of tilted antennas that provide directional sensitivity (azimuthal information) as well. Unlike a standard tool, a tilted-coil tool not only excites TE\(_z\) fields, but also TM\(_z\) fields, and can receive both modes. Therefore, it is advantageous over a standard tool when mode coupling occurs, such as in eccentric boreholes. A typical logging tool consists of one tilted transmitter and two tilted receivers around a cylindrical steel mandrel, as illustrated in Fig. 4.1. Here, we consider coil antennas with radius as measured along the radial direction fixed at 4.5 in. As a result, the larger the tilt angle, the larger is the actual radius of the loops (by a cosine factor). Moreover, the
Figure 4.1: Tilted-coil LWD tool. The tilting of the transmitter and receivers provides directionality to the tool response.
loop antennas assume an elliptic shape, not a circular one, when tilted. The antennas are wrapped around a cylindrical metallic mandrel with 4 in radius. Both angles (elevation and azimuth) of the antennas can be changed, as illustrated in Fig. B.2. The center of the transmitter antenna is located at $z = 0$ in. The centers of the first and second receiver antennas are located at $z = 30$ and $z = 24$ in, respectively. The operating frequency is 2MHz. The parameters of interest are the phase difference (PD) and amplitude ratio (AR) between the voltages at the two receiver antennas.

$$\begin{align*}
PD &= \theta_{r2} - \theta_{r1} \\
AR &= \frac{A_{r2}}{A_{r1}}
\end{align*}$$

(4.1)

where $\theta$ and $A$ denote phase and magnitude of the voltage at the receivers, and the subscripts $r1$ and $r2$ refer to receivers 1 and 2, respectively.

### 4.3 Pseudoanalytic Formulation

An analysis of tilted-coil LWD tools for circularly symmetric formations has been presented in [32] based on the methodology developed in [33]. Here, we extend this analysis to tilted coil antennas multicylindrically layered and eccentric formations using the methodology introduced for dipole sources in [20]. We assume a time harmonic excitation of the form $e^{-i\omega t}$.

To solve this problem, the electric and magnetic field are first decomposed into spectral components as

$$\begin{bmatrix} E_z \\ H_z \end{bmatrix} = \frac{1}{2\pi} \sum_{\nu = -\infty}^{+\infty} e^{i\nu\phi} \int_{-\infty}^{+\infty} dk_z e^{ik_zz} \begin{bmatrix} e_{z\nu} \\ h_{z\nu} \end{bmatrix}$$

(4.2)

In terms of the transformed fields, Maxwell equations reduce to two ordinary differential equations (ODEs) in terms of the radial variable $\rho$ [33], [34], [35], [36].
The general solution for these ODEs are

\[
\begin{bmatrix}
  e^{z\nu} \\
  h^{z\nu}
\end{bmatrix} = H_{\nu}^{(1)}(k_\rho \rho) \bar{a}_\nu + J_{\nu}(k_\rho \rho) \bar{b}_\nu \tag{4.3}
\]

where \(H_{\nu}^{(1)}\) is the Hankel function of first-kind [36], \(J_{\nu}\) is the Bessel function, \(k_\rho = \sqrt{k^2 - k_z^2}\), \(\text{Im}(k_\rho) > 0\) and \(k^2 = i\omega\mu(\sigma - i\omega\epsilon)\). The amplitude vectors \(\bar{a}_\nu\) and \(\bar{b}_\nu\) are \(2\times1\) column vectors to be determined by enforcing appropriate boundary conditions.

From Eq. 4.3, the \(\phi\) components of the transformed fields can be obtained as [36]

\[
\begin{bmatrix}
  e^{\phi\nu} \\
  h^{\phi\nu}
\end{bmatrix} = \bar{H}_{\nu}^{(1)}(k_\rho \rho) \bar{a}_\nu + \bar{J}_{\nu}(k_\rho \rho) \bar{b}_\nu \tag{4.4}
\]

where

\[
\bar{J}_{\nu} = \frac{1}{k_\rho^2} \begin{bmatrix}
  -\nu k_z J_{\nu}(k_\rho \rho) & -i\omega\mu k_\rho J'_{\nu}(k_\rho \rho) \\
  -(\sigma - i\omega\epsilon) k_\rho J_{\nu}(k_\rho \rho) & -\nu k_z J'_{\nu}(k_\rho \rho)
\end{bmatrix} \tag{4.5}
\]

and similarly for \(\bar{H}_{\nu}^{(1)}\).

To model tilted coil antennas, a ring source is used (see Fig. B.2). The source is given by

\[
J = I_T \delta(\rho - \rho_T) \delta(z - z_T)(\dot{\phi} + \dot{z}\xi_T), \tag{4.6}
\]

with

\[
\zeta_T = z_T + \rho_T \tan \theta_T \cos(\phi - \phi_T), \tag{4.7}
\]

\[
\xi_T = \tan \theta_T \sin(\phi - \phi_T), \tag{4.8}
\]

where \(\theta_T, \phi_T\) are the axial and azimuthal tilt angle of the ring source, and \(\rho_T\) and \(z_T\) are the \(\rho\) and \(z\) position of the ring source. By enforcing boundary conditions along the source location, the following solution for a tilted ring source excitation in a homogeneous medium is obtained [32]

\[
\begin{bmatrix}
  e^{h z\nu} \\
  h^{h z\nu}
\end{bmatrix} = \begin{cases}
  J_{\nu}(k_\rho \rho) \bar{C}^-_\nu & \rho \leq \rho_T \\
  H_{\nu}^{(1)}(k_\rho \rho) \bar{C}^+_\nu & \rho > \rho_T
\end{cases} \tag{4.9}
\]
The wave amplitudes \( C_{\nu}^{\pm} \) are defined as 
\[
C_{\nu}^{\pm} = \frac{-i\pi}{2c_{TE}} G_{\nu}^{\mp}(k_{\rho} \rho T) \left[ \frac{\omega_{\nu} \mu_{\nu}}{k_{\nu}} G_{\nu}^{\pm}(k_{\rho} \rho T) \right]
\]
where \( c_{TE} = i J_{\nu}(k_{\rho} \rho T \tan \theta T) e^{-i(k_{z} \rho T + \nu \phi T)} \) (4.10)

\( C_{\nu}^{\pm} \) are wave amplitudes that satisfy the boundary conditions at the ring source. In the above, \( G_{\nu}^{-} \) stands for \( J_{\nu} \) and \( G_{\nu}^{+} \) stands for \( H_{\nu} \), respectively. The voltage at the receivers can be determined by a line integral of the electric field along the (loop) receivers positions.

### 4.3.1 Multicylindrically layered media

In this case, we assume a multicylindrically layered formation along the \( \rho \) direction, as illustrated in Fig. B.2. We still consider a \( \phi \)-invariant formation. The \( z \) component of the field in the layer 0 (source layer) can then be expressed as
\[
\begin{bmatrix}
e_{z \nu} \\
h_{z \nu}
\end{bmatrix} = \begin{cases}
[H_{\nu}^{(1)}(k_{\rho i} \rho) \bar{M}_{0 \nu} + J_{\nu}(k_{\rho i} \rho) \bar{I}] \bar{b}_{0 \nu} & \rho < \rho T \\
[H_{\nu}^{(1)}(k_{\rho i} \rho) \bar{I} + J_{\nu}(k_{\rho i} \rho) \bar{N}_{0 \nu}] \bar{a}_{0 \nu} & \rho > \rho T
\end{cases}
\]
(4.11)

where \( k_{\rho i} = \sqrt{k_{i}^2 - k_{z}^2} \), \( k_{i} \) is the wave number in layer \( i \). \( \bar{M}_{0 \nu} \) and \( \bar{N}_{0 \nu} \) are 2 × 2 generalized reflection matrices [33]. These two matrices can be derived recursively to incorporate (multiple) reflections from both inner \( (i < 0) \) and outer \( (i > 0) \) layers.

The coefficients \( \bar{a}_{0 \nu} \), the \( \bar{b}_{0 \nu} \) for a tilted-coil antenna in a multicylindrically layered medium can be obtained by comparing Eq. B.26 and Eq. 4.11. This leads to the following linear system
\[
\begin{pmatrix}
\bar{N}_{0 \nu} & -\bar{I} \\
\bar{I} & -\bar{M}_{0 \nu}
\end{pmatrix} \begin{bmatrix}
\bar{a}_{0 \nu} \\
\bar{b}_{0 \nu}
\end{bmatrix} = \begin{bmatrix}
-\bar{C}_{\nu}^{-} \\
\bar{C}_{\nu}^{+}
\end{bmatrix}
\]
(4.12)

After obtaining \( \bar{a}_{0 \nu} \), the \( \bar{b}_{0 \nu} \), the induced voltage at the receiver due to a unit current source at the transmitter (transimpedance) can be calculated as
\[
V = \int_{0}^{\infty} dk_{z} \cos\{k_{z}(z_{R} - z_{T})\} w(k_{z})
\]
(4.13)
with
\[ w(k_z) = -\rho_R \sum_{\nu=0}^{\infty} \gamma \cos(\nu(\phi_R - \phi_T)) \cdot [d_{TE}(\nu, k_z)e_{\phi \nu}(\rho_R, k_z) + d_{TM}(\nu, k_z)e_{z\nu}(\rho_R, k_z)] \]

(4.14)

where
\[ d_{TE} = i^{-\nu} J_{\nu}(\kappa_R), \quad d_{TM} = -\frac{\nu i^{-\nu}}{k_z \rho_R} J_{\nu}(\kappa_R), \quad (4.15) \]
\[ \kappa_R = k_z \rho_R \tan \theta_R \]

and \( \gamma = 1 \) if \( \nu = 0 \) or \( \gamma = 2 \) otherwise.

### 4.3.2 Eccentric borehole

In practice, due to gravitational pull effects and/or mechanical vibrations, the logging tool axis is often misaligned with the borehole axis. Therefore, analysis of eccentricity effects is important. In this Section, we apply the methodology developed in [20] to the case of tilted-coil antennas tools. Fig. 4.3 illustrates an eccentric tool located \( d \) in away from the borehole axis. The segment joining the borehole and mandrel axes makes an angle \( \phi_E \) with the positive \( x \) axis. The prime coordinate system is aligned with the mandrel axis, while the unprimed system is aligned with the borehole axis. The \( z \) components in the primed system write as

\[
\begin{bmatrix} E_z \\ H_z \end{bmatrix} = \frac{1}{2\pi} \sum_{\nu'}^{+\infty} \int_{-\infty}^{+\infty} e^{ik_z z} \cdot \left[ H_{\nu'}^{(1)}(k_{\rho'}) \bar{\Gamma}_{\nu'} + J_{\nu'}(k_{\rho'}) \right] \bar{b}_{\nu'} \quad \rho' \leq \rho_T
\]

(4.16)

where \( \bar{\Gamma}_{\nu'} \) is the reflection matrix at the mandrel given by

\[
\bar{\Gamma}_{\nu'} = \begin{bmatrix} -\frac{J_{\nu'}(k_{\rho a})}{H_{\nu'}^{(1)}(k_{\rho a})} & 0 \\ 0 & -\frac{J'_{\nu'}(k_{\rho a})}{H_{\nu'}^{(1)}(k_{\rho a})} \end{bmatrix}
\]

(4.17)
Figure 4.2: Tilted coil antenna in a multicylindrical layered medium. The formation is assumed axially symmetric. The source layer is denoted by $i = 0$. Outer layers have positive index, while inner layers have negative index. $\vec{n}_T$ is the normal unit vector to the coil antenna plane.
In the region $\rho' > \rho_T$, the solution is given by

$$
\begin{align*}
\left[ \frac{E_z}{H_z} \right] = \frac{1}{2\pi} \sum_{\nu=\pm\infty}^{+\infty} e^{i\nu' \phi'} \int_{-\infty}^{+\infty} dk_z e^{ik_z z} \cdot \left\{ H_{\nu'}^{(1)}(k_{\nu' \rho'}) \tilde{\Gamma}_{\nu'} + J_{\nu'}(k_{\nu' \rho'}) \bar{b}_{0\nu'} + \right. \\
\left. \quad \left[ H_{\nu'}^{(1)}(k_{\nu' \rho'}) \bar{C}_{\nu'}^{+} - J_{\nu'}(k_{\nu' \rho'}) \bar{C}_{\nu'}^{-} \right] \right\} \\
\end{align*}
$$

(4.18)

where $\bar{C}_{\nu'}^{+}$ and $\bar{C}_{\nu'}^{-}$ are determined from the boundary conditions. This expression can be compared with the solution in terms of the nonprimed system.

$$
\begin{align*}
\left[ \frac{E_z}{H_z} \right] = \frac{1}{2\pi} \sum_{\nu=\pm\infty}^{+\infty} e^{i\nu \phi} \int_{-\infty}^{+\infty} dk_z e^{ik_z z} \cdot \left[ H_{\nu}^{(1)}(k_{\nu \rho}) + J_{\nu}(k_{\nu \rho}) \bar{N}_{0\nu} \right] \bar{a}_{0\nu}
\end{align*}
$$

(4.19)

where $\bar{N}_{0\nu}$ is the generalized reflection matrix at the borehole wall. Eq. 4.18 and 4.19 can be matched at the fictitious boundary condition shown by the dashed circle in Fig. 4.13. To match the two equations, Graf’s addition theorem for Bessel functions [37] is used.

$$
\Psi_{\nu}(\nu') e^{i\nu' \phi'} = \sum_{\nu=\pm\infty}^{\infty} \Psi_{\nu}(\nu) J_{\nu-\nu'}(d) e^{-i(\nu'-\nu) \phi_E}
$$

(4.20)

where $\Psi$ represents a solution of Bessel’s equation. Using the above identity, the outgoing wave component of Eq. 4.18 can be written as

$$
\sum_{\nu'=\pm\infty}^{+\infty} e^{i\nu' \phi'} H_{\nu'}^{(1)}(k_{\nu' \rho'}) \left[ \tilde{\Gamma}_{\nu'} \bar{b}_{0\nu'} + \bar{C}_{\nu'}^{+} \right] = \\
\sum_{\nu=\pm\infty}^{+\infty} e^{i\nu \phi} H_{\nu}^{(1)}(k_{\nu \rho}) \sum_{\nu'=\pm\infty}^{+\infty} J_{\nu'-\nu'}(k_{\nu' \rho}) e^{-i(\nu'-\nu) \phi_E} \left[ \tilde{\Gamma}_{\nu'} \bar{b}_{0\nu'} + \bar{C}_{\nu'}^{+} \right]
$$

(4.21)

Similarly, the standing wave component of Eq. 4.18 can be written as

$$
\sum_{\nu'=\pm\infty}^{+\infty} e^{i\nu' \phi'} J_{\nu'}(k_{\nu' \rho'}) \left[ \bar{b}_{0\nu'} - \bar{C}_{\nu'}^{-} \right] = \\
\sum_{\nu=\pm\infty}^{+\infty} e^{i\nu \phi} J_{\nu}(k_{\nu \rho}) \cdot \sum_{\nu'=\pm\infty}^{+\infty} J_{\nu'-\nu'}(k_{\nu' \rho}) e^{-i(\nu'-\nu) \phi_E} \left[ \bar{b}_{0\nu'} - \bar{C}_{\nu'}^{-} \right]
$$

(4.22)

By comparing these two equations with Eq. 4.19, we have

$$
\bar{a}_{0\nu} = \sum_{\nu'=\pm\infty}^{+\infty} J_{\nu'-\nu'}(k_{\nu' \rho}) e^{-i(\nu'-\nu) \phi_E} \left[ \tilde{\Gamma}_{\nu'} \bar{b}_{0\nu'} + \bar{C}_{\nu'}^{+} \right]
$$

(4.23)
\[\mathbf{N}_0 \mathbf{a}_0 = \sum_{\nu'=-\infty}^{+\infty} J_{\nu'-\nu}(k_0 d) e^{-i(\nu'-\nu)\phi_E} \left[ \mathbf{b}_{0\nu'} - \mathbf{C}_{\nu'} \right] \quad (4.24)\]

Multiplying Eq. 4.23 with \( \mathbf{N}_0 \) and subtracting from Eq. 4.24 gives the following linear system, for all \( \nu \).

\[\sum_{\nu'=-\infty}^{+\infty} J_{\nu'-\nu}(k_0 d) e^{-i(\nu'-\nu)\phi_E} \left[ \mathbf{I} - \mathbf{N}_0 \mathbf{I}_{\nu'} \right] \mathbf{b}_{0\nu'} = \sum_{\nu'=-\infty}^{+\infty} J_{\nu'-\nu}(k_0 d) e^{-i(\nu'-\nu)\phi_E} \left[ \mathbf{N}_0 \mathbf{C}_{\nu'}^+ + \mathbf{C}_{\nu'}^- \right] \quad (4.25)\]

By solving the above linear system, \( \mathbf{b}_{0\nu'} \) can be determined. This gives both \( e_{z\nu'} \) and \( e_{\phi'\nu'} \). Note that, in numerical calculations, the above infinite sum on the azimuthal indexes \( \nu \) and \( \nu' \) needs to be truncated. For a given accuracy, the number of modes depends on the borehole size and on the degree of eccentricity. In the results that follow, we use 30 azimuthal modes after ensuring from convergence tests that inclusion of additional modes has negligible effect in the results. The voltage of the receivers can then be computed using Eq. 4.13 with the kernel \( w(k_z) \) now given by

\[w(k_z) = -\rho'_R \sum_{\nu'=-\infty}^{+\infty} e^{i\nu'\phi'_R} \cdot [d_{TE}(\nu', k_z)e_{\phi'\nu'}(\rho'_R, k_z) + d_{TM}(\nu', k_z)e_{z\nu'}(\rho'_R, k_z)] \quad (4.26)\]

Note that since the eccentric problem is not invariant along \( \phi' \) anymore, \( \mathbf{b}_{0\nu'} \) has none of the symmetry properties present in the circularly symmetric case. For example, the summation over \( \nu' \) above cannot be folded as done in Eq. 4.14.

4.4 Pseudoanalytical Formulation: Numerical Considerations

4.4.1 Numerical Integration Issues for Multicylindrically Layered Media

The integral in Eq. 4.13 has a slow (algebraic) convergence. However, the difference \( \tilde{V} = V - V^h \), where \( V^h \) is the homogeneous case integral, converges exponentially. \( \tilde{V} \) can be considered as the voltage due to the formation. The field corresponding to
Figure 4.3: Illustration of eccentric borehole geometry. The tool (mandrel) axis is located $d$ away from the borehole axis. Nonprimed coordinates are centered on the borehole axis, while primed coordinates are centered on the mandrel axis. The transmitter coil antenna is located inside the borehole (mud filled) region. $\tilde{\Gamma}_E$ is the generalized reflection coefficient due to the mandrel. $\tilde{N}_\nu$ is the generalized reflection coefficient due to outer formation layers. $\phi'_E$ is enforced to be zero, and the primed coordinate system is placed at the corresponding location with respect to the unprimed coordinate system.
larger eigenvalue $k_z$ does not propagate far away from the source. Thus, $V^h$ converges very fast. Fig. 4.4 denotes the logarithmic of the $|w(k_z)|$ and $|w(k_z) - w^h(k_z)|$ for the case where the $\sigma_{\text{mud}} = 0.0005$ and $\sigma_{\text{formation}} = 1.0$.

$V^h$ can be computed analytically using the radiation integral [38] [39]

$$V^h = -i\omega \frac{\mu \rho_T \rho_R}{4\pi} \int_{-\pi}^{\pi} d\phi' \int_{-\pi}^{\pi} d\phi \bar{u}_T \cdot \bar{u}_R \frac{e^{i k_R}}{R} \tag{4.27}$$

where

$$\bar{u}_T \cdot \bar{u}_R = \cos(\phi - \phi') + \tan \theta_T \tan \theta_R \sin(\phi - \phi_T) \sin(\phi - \phi_R) \tag{4.28}$$

$$R = \sqrt{[\rho_T^2 + \rho_R^2 - 2 \rho_T \rho_R \cos(\phi - \phi') + ((z_R - \zeta_R(\phi)) - (z_T - \zeta_T(\phi)))^2]} \tag{4.29}$$

and $z - \zeta(\phi)$ is the position of the antenna in the $z$-direction. After numerical integration of $\tilde{V}$, $V^h$ can be added to it to obtain $V$.

### 4.4.2 Improving Convergence for Eccentric Boreholes

For faster convergence, the voltage due to the eccentric borehole formation is extracted as in Eq. 4.30,

$$\tilde{b}_{\nu \nu'} = \tilde{C}_{\nu} + \tilde{X} \tilde{\Omega}_{\nu'}, \tag{4.30}$$

where first term above corresponds to the homogeneous formation contribution (known). The second term is the eccentric borehole formation contribution, with $\tilde{X}$ being a free parameter to be determined. With this decomposition, the linear system in Eq. 4.25 becomes

$$\sum_{\nu' = -\infty}^{+\infty} J_{\nu - \nu'}(k_p d) e^{-i \nu' \phi_E} \left[ I - \tilde{N}_{\nu} \tilde{\Gamma}_{\nu'} \right] \tilde{X} \tilde{\Omega}_{\nu'} = \sum_{\nu' = -\infty}^{+\infty} J_{\nu - \nu'}(k_p d) e^{-i \nu' \phi_E} \left\{ \tilde{N}_{\nu} \left[ \tilde{C}_{\nu'} + \tilde{\Gamma}_{\nu'} \tilde{C}_{\nu'}^{-} \right] \right\} \tag{4.31}$$
Figure 4.4: The logarithm of $|w(k_z)|$ given in Eq. 4.14 and $|w(k_z) - w^h(k_z)|$, where $w^h(k_z)$ is the homogeneous component of the transimpedance.

for all $\nu$. The condition number of this system can be modified by choosing an appropriate $\bar{X}$. The $\phi'$ and $z$ components of the field contributed by the eccentric term are given by

$$
\bar{e}_{z\nu} = H^{(1)}(k_{\rho'\rho_R}) \Gamma_{\nu}^{11} C_{\nu'}^{TM} + \left[J_{\nu'}(k_{\rho'\rho_R}) + H^{(1)}(k_{\rho'\rho_R}) \Gamma_{\nu}^{11}\right] x_1 \Omega_{\nu'}^{TM}
$$

$$
\bar{e}_{\phi'\nu} = \left[H_{\nu'}^{11} \Gamma_{\nu'}^{11} C_{\nu'}^{TM} + H_{\nu'}^{12} \Gamma_{\nu'}^{22} C_{\nu'}^{TE}\right] + \left[J_{\nu'}^{11} + H_{\nu'}^{12} \Gamma_{\nu'}^{22}\right] x_1 \Omega_{\nu'}^{TM}
$$

+ $\left[J_{\nu'}^{12} + H_{\nu'}^{12} \Gamma_{\nu'}^{22}\right] x_2 \Omega_{\nu'}^{TE}$

(4.32)

where

$$
\bar{J}_{\nu'} = \begin{bmatrix} J_{\nu'}^{11} & J_{\nu'}^{12} \\ J_{\nu'}^{21} & J_{\nu'}^{22} \end{bmatrix}, \quad \bar{H}_{\nu'} = \begin{bmatrix} H_{\nu'}^{11} & H_{\nu'}^{12} \\ H_{\nu'}^{21} & H_{\nu'}^{22} \end{bmatrix},
$$

$$
\bar{\Gamma}_{\nu'} = \begin{bmatrix} \Gamma_{\nu'}^{11} & 0 \\ 0 & \Gamma_{\nu'}^{22} \end{bmatrix}, \quad \bar{X} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix},
$$

(4.33)

are given in Eqs. 4.5 and B.69.
Large borehole with large argument $k_z$

When the borehole size is large, for large argument $k_z$, the amplitude of $\vec{N}_\nu$ is several order smaller than the $\vec{\Gamma}_\nu'$. The linear system matrix is reduced to two independent equation given in vector form as

$$\sum_{\nu'=-\infty}^{+\infty} J_{\nu'-\nu'}(k_{\rho}d)e^{-iw'\phi_E} \vec{b}_{0\nu'} = \sum_{\nu'=-\infty}^{+\infty} J_{\nu'-\nu'}(k_{\rho}d)e^{-iw'\phi_E} \vec{C}_{\nu'},$$  \hspace{1cm} (4.34)$$

so that

$$\vec{b}_{0\nu'} = \vec{C}_{\nu'},$$ \hspace{1cm} (4.35)$$

In other word, the $\vec{\Omega}'_{\nu}$ given in Eq. 4.30 is just a zero vector in this case. The argument used in Section 4.4.1 can also be applied here.

4.5 Validation against FDTD

The FDTD method described in Chapter 3 can be extended to analyze this problem.

Since the antennas can tilt in both elevation and azimuth angles, the FDTD grid in this case is not conformal to the antenna geometry along the $z$-direction. A staircasing approximation along the $z$-direction becomes necessary to model the tilted antennas. To correctly capture the curvature of the tilt coil, the grid size in the $z$-direction should be sufficiently fine. Moreover, a continuous current source is assumed, instead of gap source, in order to resemble the analysis of the pseudoanalytical formulation.
4.6 Results

4.6.1 Homogeneous Formations

We first simulate tilted-coil tools in a homogeneous formation. We chose two different formation conductivities, $\sigma = 0.1 \text{ mho/m}$ and $1.0 \text{ mho/m}$. For each conductivity value, we simulate three different transmitter tilt angles ($\theta_T = 0^\circ, 20^\circ$ and $45^\circ$). The receivers are tilted along the same direction as the transmitter.

Fig. 4.5 compares 3-D FDTD and pseudoanalytic results. The FDTD solution uses a grid with $(N_p, N_\phi, N_z) = (40,160,350)$ cells. As mentioned before, the 3-D grid employs non-uniform grid cells in the $\rho$ direction. For $\sigma = 0.1 \text{ mho/m}$, the radial cell sizes vary from $\Delta \rho = 0.635 \text{ cm}$ close to the mandrel to 18.767 cm at the outer end. For $\sigma = 1.0 \text{ mho/m}$, the cell sizes vary from $\Delta \rho = 0.635 \text{ cm}$ to 7.11 cm to capture the smaller skin depth. Very good agreement is observed between FDTD and pseudoanalytical results.

4.6.2 Homogeneous Formations with Mud layer

We next consider a cylindrical borehole in a homogeneous formation. The conductivity of the borehole mud is assumed uniform. The radius of the borehole is 5 in. Two different cases are considered, one with low mud conductivity and high formation conductivity, and the other with the reverse choice. For each scenario, we simulate three transmitter tilt angles, equal to $0^\circ$, $20^\circ$ and $45^\circ$. The receivers are tilted in the same direction as the transmitter.

Fig. 4.6 compares 3-D FDTD and pseudoanalytical results. For the FDTD simulation, we use a grid with $(N_p, N_\phi, N_z) = (40,160,350)$ cells. For the case $\sigma_{\text{mud/for}} = 10.0/0.1 \text{ mho/m}$, $\Delta \rho$ varies from 0.635 cm to 22.52 cm. For the case $\sigma_{\text{mud/for}} =$
Figure 4.5: Comparison of 3-D FDTD and pseudoanalytical results. The tool is located in a homogeneous formation with either $\sigma=0.1$ mho/m or 1.0 mho/m. Three different tilt angles for the transmitter are considered for each conductivity choice: $0^\circ$, $20^\circ$ and $45^\circ$. Pseudoanalytic results are represent by symbols and FDTD results by solid curves.
0.0005/1.0 mho/m, $\Delta \rho$ varies from 0.635 cm to 7.11 cm. The comparison between the FDTD results and the pseudoanalytical results shows again very good agreement.

In order to perform adequate formation evaluation, a logging tool needs to ideally provide unique tool responses for different homogeneous medium. The pseudoanalytical formulation can gives us preliminary set of design rules for the directional resistivity tool. Fig. 4.7 shows as the phase and amplitude (dB) of the first and second receivers vary with the tilt angle, in different homogeneous media. The transmitter in this case is tilted by 45°. The mud conductivity equals to 0.0005 mho/m. We investigate in more detail the tool response with setup of transmitter angle of 45° and the two receivers with the same tilt angle. Fig. 4.8 depicts the tool response of the tool. The phase difference and amplitude ratio (dB) are plotted against the receiver tilt angle. From this figure, there is more than one apparent conductivity mapped to the same amplitude ratio when the receiver is tilted by an angle greater than 20° (non-uniqueness). The same phenomenon happens to the phase difference when the receiver tilt angle is greater than 40°. Hence, this is not an optimal design for a directional tool.

Another tool design is depicted in Fig. 4.9, where tilt angle of the second receiver is fixed at 0°. The tool response is plotted versus the first receiver tilt angle. The phase difference for this tool provides a bad estimate of the formation conductivity when the first receiver is tilted angle around 20°. On the other hand, the amplitude ratio provides a good estimate if the first receiver tilt angle is chosen in the negative range.

Fig. 4.10 shows the phase and amplitude of the voltage at first and second receiver with a horizontal-coil transmitter. In this case, the response of the antenna does not
Figure 4.6: Comparison of 3-D FDTD and pseudoanalytical results for a tool located in a cylindrical borehole surrounded by a homogeneous formation. Two pairs of borehole mud and formation conductivities are considered: $\sigma_{mud/for} = 10/0.1$ mho/m and $\sigma_{mud/for} = 0.0005/1.0$ mho/m. Three different transmitter tilt angles are considered for each pair of conductivities: $0^\circ$, $20^\circ$ and $45^\circ$. Pseudoanalytic results represent by symbols and FDTD results by solid lines.
change significantly with respect to the tilt angle. Fig. 4.11 shows the tool response of a tool design with horizontal-coil transmitter and two receivers having same tilt angle. For any receiver tilted angle, this gives very good design since the tool response in different medium exhibits different response. Fig. 4.12 shows the tool response of a tool design having a horizontal-coil transmitter and a horizontal-coil second receiver. The tool response is plotted against the tilted angle of the first receivers. Again, this configuration gives very good design. Note that in an axissymmetric formation, the tool response of this tool is similar to a conventional (horizontal-coil) tool. In contrast, if one or both receiver employ tilted-coils, the tool response is different when the formation is asymmetric (due to dipping beds or eccentric borehole).

4.6.3 Eccentric Boreholes

To better illustrate eccentricity effects, a larger borehole size is chosen in this case, together with a larger conductivity contrast between mud and formation. The radius of the borehole is chosen equal to 12 in. Two formation conductivity cases are considered, as shown in Fig. 4.3. The transmitter tilted angle equal to 45°. The eccentricity $\phi_E$ angle is chosen to be $-90^\circ$. The receivers are again tilted in the same direction as the transmitter.

Fig. 4.13 compares 3-D FDTD and pseudoanalytic results. For $\sigma_{\text{mud/for}} = 10./0.1$ mho/m, we use a FDTD grid with $(N_{\rho},N_{\phi},N_z) = (40,160,350)$ cells and $\Delta \rho$ varying from 0.635 cm to 2.25 cm. For $\sigma_{\text{mud/for}},0.1/10.0$ mho/m, we use a grid with $(N_{\rho},N_{\phi},N_z)$ (40,200,350) cells and $\Delta \rho$ varying from 0.635 cm to 18.76 cm. Very good agreement is again observed between the two approaches.
4.7 Conclusions

We have discussed the analysis of tilted-coil logging tools in multilayered and eccentric borehole problems using a pseudoanalytical approach scheme and compare the results with 3-D FDTD scheme. Comparison between the two approaches for well-logging tools in different borehole and formation scenarios has shown very good agreement. The pseudoanalytical approach is considerably less costly than FDTD modeling. As such, it can be used as a tool for preliminary tool evaluation and testing of design principles. On the other hand, despite being computationally more costly, FDTD is flexible enough to account for highly complex formations features, such as arbitrary invasion zone profiles and dipping beds. As such, FDTD can be used to evaluate in more detail the impact of each of these features on the response of a particular tool. In terms of the main limitations of each approach, the cylindrical FDTD method applied to tilted antennas is restricted by the grid geometry and staircasing approximation. In order to have a good approximation, the cell size along the $z$ direction need to be fine enough to capture the tilted antenna geometry. For the pseudoanalytical formulation, if the mandrel eccentricity is very large, the linear system given in Eq. 4.31 below can become quite ill-conditioned. In practice, however, this does not happen for usual well-logging tool geometries because borehole and tool radii are comparable.

The work presented in this chapter has considered isotropic media. One of the applications of tilted coil tools is to obtain improved estimates of anisotropy. We will consider the response of tilted coil tools in eccentric borehole through anisotropic formations in the next chapter using numerical mode matching.
Figure 4.7: Phase and amplitude of the voltage measured at the receivers in homogeneous medium with conductivity indicated in the legend. The transmitter coil source is tilted by 45°. The x-axis is the receivers tilted angle. Fig. 4.7(a) and Fig. 4.7(b) depicts the phase of the voltage measured at first and second receivers; Fig. 4.7(c) and Fig. 4.7(d) depicts the amplitude of the voltage measured at first and second receivers.
Figure 4.8: Tool response for a tool setup with transmitter tilt angle equal to 45°, and two receivers with the same tilt angle. In the above, the phase difference and amplitude ratio are plotted against the receiver tilt angle.
Figure 4.9: Tool response for a tool setup with transmitter tilt angle equal to 45°, and second receiver with the tilt angle fixed at 0°. The phase difference and the amplitude ratio are plotted against the first receiver tilt angle.
Figure 4.10: Phase and amplitude of the voltage measured at the receivers in homogeneous medium with conductivity as indicated in the legend. The transmitter coil source is tilted with angle $0^\circ$. The $x$-axis represents the receivers (both) tilt angle. Fig. 4.10(a) and Fig. 4.10(b) depicts the phase of the voltage measured at first and second receivers; Fig. 4.10(c) and Fig. 4.10(d) depicts the amplitude of the voltage measured at first and second receivers.
Figure 4.11: Tool response for a tool setup with transmitter tilt angle equal to 0°, and the two receivers having the same tilt angle. In the above, phase difference and amplitude ratio are plotted against the receiver tilt angle.
Figure 4.12: Tool response for a tool setup with transmitter tilt angle equal to 0° and the second receiver having tilt angle fixed at 0°. The phase difference and amplitude ratio are plotted against the first receiver tilt angle.
Figure 4.13: Comparison of 3-D FDTD and pseudoanalytical results for a tool located in an eccentric cylindrical borehole surrounded by a homogeneous formation. Two pairs of borehole mud and formation conductivities are considered: $\sigma_{\text{mud/for}} = 10/0.1$ and $\sigma_{\text{mud/for}} = 0.1/10.0$ mho/m. The parameter $d$ represents the eccentricity of the tool in inches (see Fig. 4.3). Both transmitter and receivers have tilt angle equal to 45°. Pseudoanalytical results are represented by symbols and FDTD results by solid curves. See more details in the text.
CHAPTER 5

VERTICAL EIGENSTATES NMM WITH UNIAXIAL PML FOR SIMULATION OF TILTED-COIL ANTENNA ARRAYS IN ANISOTROPIC FORMATIONS

The response of the well-logging tools in anisotropic formations has been subject of continual interest [40], [41], [42], [43], [44], because the earth conductivity in the vertical direction $\sigma_v$ and horizontal direction $\sigma_h$ are often different. Conventional well-logging tools employing horizontal-coil antennas excite only TE modes and hence are sensitivity only to the horizontal component (i.e. perpendicular to the tool axis) of the formation conductivity. This may lead to inaccurate formation evaluation, not only in anisotropic earth formations but also in formations having dipping beds [7], [45] that occur, for example, during deviated drilling. Logging tools with tilted-coil antenna arrays (TCAs) have been proposed as an alternative to provide better formation evaluation in these scenarios [29].

To model the electromagnetic field in the presence of a mandrel and a multilayered formation including invasions, it is convenient to use a numerical mode matching (NMM) approach. Using separation of variables, the problem can be solved using numerical method in one direction and analytical method in another direction. This leads to a computationally less intensive and accurate method than the FDTD
method discussed in Chapter 3 [46]. In [14], [47], NMM is formulated by expanding horizontal eigenmodes in terms of triangular basis functions. These eigenmodes are then matched in the vertical direction. In contrast, [48] uses sinusoidal basis function to represent vertical eigenmodes. These eigenmodes are then matched in the radial direction. NMM modeling of a three-dimensional (3-D) formation uses attempted in [49] by using a two-dimensional (2-D) eigenmodes expansion and matching in the vertical direction. For low conductivity formations, a perfectly matched layer (PML) was also used in [50], [51] to mimic the radiation condition.

In this chapter, we derive and implement a new NMM approach to model well-logging tool employing TCAs in a multilayered-uniaxial-anisotropic formation, as illustrated in Fig. 5.1. Unlike NMM modeling of conventional logging tools with horizontal-coil antenna, which involves only TE\_z modes, the new NMM formulation fully incorporates the cross-coupling of TM\_z and TE\_z modes. The present NMM utilizes vertical (longitudinal) eigenmodes instead of transversal eigenmodes. Furthermore, it integrates a perfectly matched layer (PML) in the vertical direction to allow for accurate solution in low conductivity formations and/or (borehole) mud.

The spectral decomposition used the new NMM approach is similar to the one presented for the pseudoanalytical method in Chapter 4. The main differences are that eigenmodes in the vertical direction are now solved using numerical method instead expanding in terms of spectral component (e^{ikz}) to account for the layered formation properties.
Figure 5.1: Illustration of a tilted-coil well-logging tool within a layered formation with horizontal invaded beds.
5.1 NMM Formulation

Due to the 3-D geometry of tilted-coil antenna source, a vertical (parallel to the axis of the tool) numerical eigenmodes expansion is employed. Vertical eigenmodes of each cylindrical-vertical layer (in the domain \([z_{\text{min}}, z_{\text{max}}]\)) are obtained and formal solutions for the fields within each layer are expanded in terms of these eigenmodes. Within each layer, the eigenmodes are propagated in the transverse direction using mode propagators (in case Bessel or Hankel function). By enforcing the appropriate boundary conditions across the interface of any two cylindrical-vertical layers, reflection and transmission coefficient terms can be found. Here, we first describe the source expansion into these numerical eigenmodes. We then derive generalized reflection coefficients, that include mandrel and borehole effects.

Since the earth formations considered here are uniaxial anisotropic, the permittivity and permeability tensors, which incorporate the PML complex stretching coordinate, are given as

\[
\bar{\epsilon}^*(z) = \begin{pmatrix} \epsilon^*_h \\ \epsilon^*_v \end{pmatrix}, \quad \bar{\mu}^*(z) = \begin{pmatrix} \mu^*_h \\ \mu^*_v \end{pmatrix}
\]

(5.1)

where \(\epsilon^*_h = (\epsilon_h - \sigma_h j\omega) s_z\); \(\epsilon^*_v = (\epsilon_v - \sigma_v j\omega)/s_z\); \(\mu^*_h = \mu_h s_z\) and \(\mu^*_v = \mu_v/s_z\). \(s_z\) is the PML complex stretching variable along the \(z\)-direction. The superscript * indicates media tensors that incorporate the PML complex stretching variable. We assume the formation to be piecewise constant in \(z\)-direction. The vector wave equations governing the electric and magnetic fields can be written as

\[
\mu_h^* \nabla \times (\bar{\mu}^{*-1} \nabla \times \mathbf{E}) - \nabla (\epsilon^*_h \nabla \cdot \bar{\epsilon}^* \mathbf{E}) - \omega^2 \epsilon^*_h \bar{\epsilon}^* \mathbf{E} = i\omega \mu^*_h \mathbf{J} - \nabla \frac{q}{\epsilon^*_h},
\]

(5.2)
or
\[
\epsilon_h^* \nabla \times (\tilde{\epsilon}^{-1} \nabla \times H) - \nabla (\mu_h^*^{-1} \nabla \cdot \tilde{\mu}^* H) - \omega^2 \epsilon_h^* \tilde{\mu}^* H = \epsilon_h^* (\nabla \times \tilde{\epsilon}^{-1} J) - \nabla \frac{\sigma_m}{\mu_h^*}. \tag{5.3}
\]
respectively, where \( J \) is given in Eq. 4.6. In the above equations, the second term on l.h.s. and r.h.s equal to each other and are added to the above equation in order to simplify the later development of the extraction of the \( z \)-component. In general, Eq. 5.2 represents TM mode solutions and Eq. 5.3 represents TE mode solutions, respectively, which are coupled in this case. We first decompose the field components using the following spectral decomposition \[52]\]
\[
F(\rho, \phi, z) = \frac{1}{\sqrt{2\pi}} \sum_{\nu=-\infty}^{\infty} e^{i\nu\phi} f_{\nu}(\rho, z), \tag{5.4}
\]
and
\[
f_{\nu}(\rho, z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \tilde{f}_{\nu}(k_{\rho}, z) J_{\nu}(k_{\rho} \rho) k_{\rho} dk_{\rho}, \tag{5.5}
\]
where \( J_{\nu}(k_{\rho}) \) is the Bessel function of order \( \nu \) and it can also be replaced by the Hankel function of the first kind of order \( \nu \). In the above \( \tilde{f} \), \( f \), and \( F \) stand for any field or source component. We extract the \( z \)-component of the electric field of Eq. 5.2 in a source-free region, which leads to the following equation for the spectral components
\[
(- \frac{1}{\epsilon_{\nu}} k_{\rho}^2 + \frac{\partial}{\partial z} \frac{1}{\epsilon_h^*} \frac{\partial}{\partial z} + \omega^2 \mu_h^*) \tilde{d}_{z\nu}(k_{\rho}, z) = 0, \tag{5.6}
\]
where \( \tilde{d}_z = \epsilon_{\nu}^* \tilde{e}_z \). Below the subscript \( \nu \) and dependency on \( k_{\rho} \) of the spectral components of the fields are omitted, for simplicity. By expanding the electric flux density with appropriate set of basis functions \( S(z) \),
\[
\tilde{d}_z(k_{\rho}, z) = \sum_{n=1}^{\infty} a_n S_n(z), \tag{5.7}
\]
we arrive to the following generalized eigenfunction problem

$$\sum_{n=1}^{\infty} a_n \left[ \frac{d}{dz} \epsilon_h^* \frac{d}{dz} + \omega^2 \mu_h^* \right] S_n(z) = k^2 \rho \sum_{n=1}^{\infty} a_n \frac{1}{\epsilon_v^*} S_n(z). \quad (5.8)$$

By projecting the above into the space spanned by \( S_n(z) \) with inner product \(< f, g >\) defined as \( \int_{z_{\min}}^{z_{\max}} f g^* dz \) and truncate the sum at \( N \) terms, the above equation reduces to a linear system (generalized eigenvalue problem) as [36]

$$\bar{L}_\epsilon \cdot \bar{a} = k^2 \rho \bar{p}_\epsilon \cdot \bar{a} \quad (5.9)$$

where \( \bar{L}_\epsilon \) and \( \bar{p}_\epsilon \) are \( N \times N \) matrices with elements given by

$$(\bar{L}_\epsilon)_{mn} = -\langle S'_m(z), \epsilon^{-1}_h S'_n(z) \rangle + \langle S_m(z), \omega^2 \mu^*_h S_n(z) \rangle \quad (5.10)$$

and

$$(\bar{p}_\epsilon)_{mn} = \langle S_m(z), \epsilon^{-1}_v S_n(z) \rangle. \quad (5.11)$$

In the above subscript \( \epsilon \) represents TM modes. TE modes are obtained via duality by replacing \( \epsilon \) with \( \mu \). We use double overhead bar to represent matrices and single overhead bar to represent column vectors. We use an index \( q \) to specify the \( q=1,...,N \) eigenvalues \( k_{pq} \) and corresponding eigenvector \( \bar{a}_q \) of Eq. 5.9. Since the matrices \( \bar{L}_\epsilon \) and \( \bar{p}_\epsilon \) are symmetric (in reciprocal media), the eigenvectors are orthogonal, i.e.,

$$\bar{a}_p^t \cdot \bar{p} \cdot \bar{a}_q = \delta_{pq} C_q \quad (5.12)$$

where \( C_q \) is a constant. By letting \( \bar{a}_q \rightarrow \frac{1}{\sqrt{C_q}} \bar{a}_q \), we have an orthonormal set of eigenmodes. Once the eigenmodes have been found, we can expand the source in term of them. In the source region by taking, the product of the \( z \)-component of Eq. 5.6 with \( \epsilon^* v \) before the transformation of Eq. 5.5 we have

$$(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} - \frac{\nu^2}{\rho^2} + \epsilon^*_v \frac{\partial}{\partial z} \epsilon^{-1}_h \frac{\partial}{\partial z} + \omega^2 \mu^*_h \epsilon^*_v) dz_2(\rho, z) = -i \omega \mu^*_h \epsilon^*_v j_z(\rho, z) \quad (5.13)$$
where the second term in the r.h.s. of Eq. 5.2 is zero because $\nabla \cdot \mathbf{J} = 0$. We express the electric flux density using the eigenmodes found in Eq. 5.9 as

$$
\psi_{\epsilon q}(z) = \sum_{n=1}^{N} (\bar{a}_q)_n S_n(z) \quad (5.14)
$$

and

$$
d_z(\rho, z) = \sum_{q=1}^{N} b_{\epsilon q}(\rho) \psi_{\epsilon q}(z). \quad (5.15)
$$

where $b_{\epsilon q}(\rho)$ is the propagation term for eigenmode $\psi_{\epsilon q}(z)$. By substituting the above into Eq. 5.13, and testing the resulting equation with $\psi_{\epsilon p}(z)\epsilon^{*-1}$, we have

$$
\begin{align*}
&\frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} b_{\epsilon q}(\rho) - \frac{\nu^2}{\rho^2} b_{\epsilon q}(\rho) \\
&+ \sum_{q=1}^{N} \left( \psi_{\epsilon q}(z), \left( \frac{d}{dz} \epsilon^{*-1} \frac{d}{dz} + \omega^2 \mu_h \right) \psi_{\epsilon p}(z) \right) b_{\epsilon p}(\rho) \\
&= -i\omega \mu_h^* j_{\epsilon vq} \delta(\rho - \rho_T).
\end{align*}
$$

where

$$
\begin{align*}
&j_{\epsilon vq} = \sum_{n=1}^{N} a_{\epsilon q n} \frac{I_T}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \xi_T S_n(\zeta_T) e^{-i\nu \phi}.
\end{align*}
$$

(5.17)

The meaning of $\xi_T$ and $\zeta_T$ is given in Appendix B.1 and the derivation of $j_{\epsilon v m}$ (and $j_{\mu v m}$ below) is given in Appendix C.3. Because the eigenmodes are orthonormal,

$$
\begin{align*}
&\left\langle \psi_{\epsilon q}(z), \left( \frac{d}{dz} \epsilon^{*-1} \frac{d}{dz} + \omega^2 \mu_h \right) \psi_{\epsilon p}(z) \right\rangle \\
&= \bar{a}_q^T \cdot \bar{F}_\epsilon \cdot \bar{a}_p \\
&= \bar{a}_q^T (k_{\rho p} \bar{P}_\epsilon \cdot \bar{a}_p) \\
&= k_{\rho q}^2 \delta_{qp}.
\end{align*}
$$

(5.18)

Eq. 5.16 can be written as Bessel’s equation as

$$
\begin{align*}
&\left( \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} - \frac{\nu^2}{\rho^2} + k_{\rho q}^2 \right) b_{\epsilon q}(\rho) = -i\omega \mu_h^* j_{\epsilon vq} \delta(\rho - \rho_T)
\end{align*}
$$

(5.19)
A similar expansion can be derived for TE modes leading to

\[
\left( \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} - \frac{\nu^2}{\rho^2} + k_{pq}^2 \right) b_{pq}(\rho) = -j_{\mu\nu} \frac{1}{\rho} \frac{d}{d\rho} (\rho \delta(\rho - \rho_T)) \tag{5.20}
\]

where

\[
j_{\mu\nu} = \sum_{n=1}^{N} a_{\mu\nu n} \frac{I_T}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi S_n(\zeta_T) e^{-i \nu \phi} \tag{5.21}
\]

To determine the mode propagators \(b_{eq}\) and \(b_{\mu q}\), we need to satisfy the source condition. The formal solution is written in terms of Bessel function or Hankel function of first kind of order \(\nu\). In a homogeneous medium (along the radial direction), they are

\[
b_{eq}(\rho) = \begin{cases} 
J_{\nu} (k_{pq} \rho) c_{eq}^- \rho \leq \rho_T ; \\
H_{\nu} (k_{pq} \rho) c_{eq}^+ \rho \geq \rho_T ;
\end{cases} \tag{5.22}
\]

\[
b_{\mu q}(\rho) = \begin{cases} 
J_{\nu} (k_{pq} \rho) c_{\mu q}^- \rho \leq \rho_T ; \\
H_{\nu} (k_{pq} \rho) c_{\mu q}^+ \rho \geq \rho_T ;
\end{cases} \tag{5.23}
\]

with \(c^\pm\) is the source related amplitude coefficient, (details can be found in Appendix C.4)

\[
c_{eq}^\pm = -\frac{\pi}{2} j_{\nu \omega} \mu_p \rho_T C_\nu^\pm (k_{pq} \rho_T) \tag{5.24}
\]

\[
c_{\mu q}^\pm = -\frac{i\pi}{2} j_{\nu \omega} k_{pq} \rho_T C_\nu^\pm (k_{pq} \rho_T) \tag{5.25}
\]

where, \(C^-\) is a Bessel function of order \(\nu\) and \(C^+\) is a Hankel function of first kind and order \(\nu\).

Now, \(d_z\) and \(b_z\) can be written as

\[
\begin{bmatrix} d_z(\rho, z) \\
b_z(\rho, z) \end{bmatrix} = \sum_{q=1}^{N} \begin{bmatrix} J_{\nu} (k_{pq} \rho) c_{eq}^- \psi_{eq}(z) \\
J_{\nu} (k_{pq} \rho) c_{\mu q}^- \psi_{\mu q}(z) \\
H_{\nu} (k_{pq} \rho) c_{eq}^+ \psi_{eq}(z) \\
H_{\nu} (k_{pq} \rho) c_{\mu q}^+ \psi_{\mu q}(z) \end{bmatrix} \phi_T ; \rho \geq \rho_T ; \tag{5.26}
\]

or, put into a more compact matrix form, as

\[
\mathbf{\overline{A}}_z = \begin{bmatrix} d_z \\
b_z \end{bmatrix} = \overline{\mathbf{S}}(z) \cdot \overline{\mathbf{a}} \cdot \overline{\mathbf{C}}^\pm (\rho) \cdot \overline{\mathbf{c}}^\pm \tag{5.27}
\]
\[ \tilde{S}(z) = \begin{bmatrix} \bar{S}(z) & 0 \\ 0 & \bar{S}(z) \end{bmatrix}_{(2N \times 2)}, \quad (5.28) \]

\[ \tilde{a} = \begin{bmatrix} \bar{a}_e & 0 \\ 0 & \bar{a}_\mu \end{bmatrix}_{(2N \times 2N)} , \quad (5.29) \]

\[ \bar{C}^\pm(\rho) = \begin{bmatrix} \bar{C}_e^\pm(\rho) & 0 \\ 0 & \bar{C}_\mu^\pm(\rho) \end{bmatrix}_{(2N \times 2N)} , \quad (5.30) \]

and

\[ \tilde{c}^\pm = \begin{bmatrix} \tilde{c}_{\mu v}^\pm \\ \tilde{c}_{\mu \mu}^\pm \end{bmatrix}_{(2N \times 1)} . \quad (5.31) \]

The matrix \( \bar{C}_{e/\mu}^\pm \) is diagonal matrix with elements \( C_{e/\mu}^\pm(k_{\rho(\epsilon/\mu)q}\rho) \).

In order to incorporate extra cylindrical-vertical layers, we satisfy the boundary conditions across the interfaces by enforcing continuity of tangential field components. From Maxwell equations, the transverse \( \phi \)-components are related to \( z \)-components by,

\[
\left( \frac{\partial}{\partial z} + \frac{1}{\mu_h} \frac{\partial}{\partial \rho} \right) e_\phi = \frac{i\nu}{\rho} \frac{\partial}{\partial z} \frac{1}{\mu_h^* \epsilon_v^*} d_z - i\omega \frac{1}{\mu_h^* \epsilon_v^*} d\rho \,,
\]

\[
\left( \frac{\partial}{\partial z} + \frac{1}{\epsilon_h} \frac{\partial}{\partial \rho} \right) h_\phi = \frac{i\nu}{\epsilon_v^* \epsilon_v^*} \frac{\partial}{\partial \rho} d_z + \frac{i\nu}{\epsilon_v^* \epsilon_v^*} \frac{\partial}{\partial \rho} d\rho \,. \quad (5.32)
\]

Hence, the transverse components can be expressed in matrix form as

\[
\tilde{A}_\phi = \begin{bmatrix} e_\phi \\ h_\phi \end{bmatrix} = \begin{bmatrix} L^{-1}_\mu \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{i\nu}{\epsilon_v^* \epsilon_v^*} \frac{\partial}{\partial \rho} & \frac{i\nu}{\epsilon_v^* \epsilon_v^*} \frac{\partial}{\partial \rho} \\ \frac{i\nu}{\epsilon_v^* \epsilon_v^*} \frac{\partial}{\partial \rho} & \frac{i\nu}{\epsilon_v^* \epsilon_v^*} \frac{\partial}{\partial \rho} \end{bmatrix} \cdot \bar{S}(z) \cdot \tilde{a} \cdot \bar{C}^\pm(\rho) \cdot \tilde{c}^\pm \quad (5.34)
\]

where \( L_\mu = \left( \frac{d}{\mu_h \epsilon_h^* \epsilon_h^*} \frac{d}{dz} + \frac{d}{d\rho} \right) \) and \( L_\epsilon = \left( \frac{d}{\epsilon_h \epsilon_h^* \epsilon_h^*} \frac{d}{dz} + \frac{d}{d\rho} \right) \). By using an operator projection of the form [53], \( \bar{L}\bar{S}(z) \approx \bar{S}(z) \langle \bar{S}(z), \bar{L}\bar{S}(z) \rangle \), Eq. 5.34 becomes

\[
\tilde{A}_\phi = \bar{S}(z) \cdot \bar{S}^{-1}_\mu \cdot \tilde{M}^\pm(\rho) \cdot \tilde{c}^\pm \quad (5.35)
\]

where

\[
\bar{L} = \begin{bmatrix} \bar{L}_\mu & 0 \\ 0 & \bar{L}_\epsilon \end{bmatrix} , \quad (5.36)
\]
5.2 Incorporating Multicylindrical-Vertical layers

We perform this extra step to formulate in terms of \( K \) which is a diagonal matrix containing the square root of the eigenvalues. The superscript ' above means derivatives with respect to the argument.

Using the Eq. 5.9, the matrix product \( \bar{L}^{-1} \cdot \bar{M}^\pm \) can be rearranged into

\[
\bar{L}^{-1} \cdot \bar{M}^\pm = \begin{bmatrix}
\frac{i\omega}{p} \bar{L}_\mu \cdot \bar{a}_\mu & \bar{C}_\epsilon \cdot \bar{C}_\epsilon^\prime & -i\omega \bar{p}_\mu \cdot \bar{a}_\mu & \bar{C}_\mu^\pm & \bar{K}_\mu^\pm \\
\frac{i\omega}{p} \bar{p}_\mu \cdot \bar{a}_\mu & \bar{C}_\epsilon^\prime & -i\omega \bar{C}_\mu \cdot \bar{C}_\mu^\prime & \bar{C}_\mu^\prime & \bar{K}_\mu^\prime \\
\frac{\bar{L}_\epsilon \cdot \bar{a}_\mu}{\bar{C}_\mu} & \bar{C}_\epsilon & \bar{C}_\epsilon^\prime & \bar{C}_\mu & \bar{K}_\mu^\pm \\
\end{bmatrix} \cdot \begin{bmatrix}
\bar{C}_\epsilon^\pm \\
\bar{C}_\mu^\pm \\
\end{bmatrix}
\]

(5.39)

and \( \bar{K} \) is a diagonal matrix containing the square root of the eigenvalues. The superscript ' above means derivatives with respect to the argument.

We perform this extra step to formulate in terms of \( \bar{Y}^\pm \) because \( \bar{Y}^\pm \) is better conditioned than \( \bar{M}^\pm \).

In summary, we can express the formal solution of the field within a cylindrical-vertical layer as

\[
\bar{A}_z(\rho, z) = \bar{S}(z) \cdot \bar{a} \cdot \bar{C}^+(\rho) \cdot \bar{c}^+
\]

(5.40)

\[
\bar{A}_\phi(\rho, z) = \bar{S}(z) \cdot \bar{Y}^\pm (\rho) \cdot \bar{C}^+(\rho) \cdot \bar{c}^+
\]

(5.41)

5.2 Incorporating Multicylindrical-Vertical layers

We can express the field within the source layer as

\[
\bar{A}_{0z}(\rho, z) = \bar{S}(z) \cdot \bar{a} \cdot \begin{cases}
\bar{H}_0(\rho) \cdot \bar{R}_{0,-1}^+ + \bar{J}_0(\rho) \cdot \bar{d}_0^- & \rho_{-1} < \rho \leq \rho_T; \\
\bar{H}_0(\rho) + \bar{J}_0(\rho) \cdot \bar{R}_{0,1}^- \cdot \bar{d}_0^+ & \rho_T \leq \rho \leq \rho_0.
\end{cases}
\]

(5.42)

\[
\bar{A}_{0\phi}(\rho, z) = \bar{S}(z) \cdot \begin{cases}
\bar{Y}_0^+(\rho) \cdot \bar{H}_0(\rho) \cdot \bar{R}_{0,-1}^+ + \bar{Y}_0^-(\rho) \cdot \bar{J}_1(\rho) \cdot \bar{d}_0^- & \rho_{-1} < \rho \leq \rho_T; \\
\bar{Y}_0^+(\rho) \cdot \bar{H}_0(\rho) + \bar{Y}_0^-(\rho) \cdot \bar{J}_1(\rho) \cdot \bar{R}_{0,1}^- \cdot \bar{d}_0^+ & \rho_T \leq \rho \leq \rho_0.
\end{cases}
\]

(5.43)
where the subscript 0 represents the source layer. In the above, $\tilde{R}^{+}_{0,-1}$ and $\tilde{R}^{-}_{0,1}$ are generalized reflection coefficients from the inner and outer layers, which can be computed using the recursive formula [33] [36],

$$
\tilde{R}^{+}_{n,n-1} = \tilde{R}^{+}_{n,n-1} + T^{-}_{n-1,n} \tilde{R}^{+}_{n-1,n-2} \cdot (I - \tilde{R}^{-}_{n-1,n} \cdot \tilde{R}^{+}_{n-1,n-2})^{-1} \cdot \tilde{T}^{-}_{n,n-1},
$$

and

$$
\tilde{R}^{-}_{n,n+1} = \tilde{R}^{-}_{n,n+1} + T^{-}_{n+1,n} \tilde{R}^{-}_{n+1,n+2} \cdot (I - \tilde{R}^{+}_{n+1,n} \cdot \tilde{R}^{-}_{n+1,n+2})^{-1} \cdot \tilde{T}^{+}_{n,n+1}.
$$

The local reflection and transmission coefficient are given as

$$
\tilde{R}^{-}_{n,n+1} = \tilde{J}^{-}_{n} \cdot (\tilde{I} - \tilde{\Psi}_{n+1} \cdot (\tilde{Y}^{+}_{n+1})^{-1} \cdot \tilde{Y}^{-}_{n})^{-1} \cdot (\tilde{\Psi}_{n+1} \cdot (\tilde{Y}^{+}_{n+1})^{-1} \cdot \tilde{Y}^{+}_{n} - \tilde{I}) \cdot \tilde{H}_{n} \tag{5.46}
$$

$$
\tilde{T}^{+}_{n,n+1} = (\tilde{H}^{-}_{n+1})^{-1} \cdot (\tilde{Y}^{+}_{n+1})^{-1} \cdot [\tilde{Y}^{+}_{n} \cdot \tilde{H}_{n} + \tilde{Y}^{-}_{n} \cdot \tilde{J}_{n} \cdot \tilde{R}^{-}_{n,n+1}] \tag{5.47}
$$

$$
\tilde{R}^{+}_{n+1,n} = \tilde{H}^{-1}_{n+1} \cdot (\tilde{\Psi}_{n+1} - (\tilde{Y}^{-}_{n+1})^{-1} \cdot \tilde{Y}^{+}_{n+1})^{-1} \cdot ((\tilde{Y}^{-}_{n+1})^{-1} \cdot \tilde{Y}^{-}_{n+1} - \tilde{\Psi}_{n+1}) \cdot \tilde{J}_{n+1} \tag{5.48}
$$

$$
\tilde{T}^{-}_{n+1,n} = (\tilde{J}^{-}_{n})^{-1} \cdot (\tilde{Y}^{-}_{n})^{-1} \cdot [\tilde{Y}^{+}_{n+1} \cdot \tilde{H}_{n+1} + \tilde{Y}^{-}_{n+1} \cdot \tilde{J}_{n+1}] \tag{5.49}
$$

where $\tilde{\Psi}_{n+1} = \tilde{a}_{0}^{i} \cdot \tilde{p}_{n+1} \cdot \tilde{a}_{n+1}$ (see Appendix C.5). In the above, the superscript $+/-$ refers to outgoing/incoming wave as illustrated in the diagram of Fig. 5.2.

The amplitude coefficient of the source layer in Eq. 5.42 is solved by enforcing the source condition at $\rho_{T}$ as

$$
\tilde{d}^{-}_{0} = \left(\tilde{I} - \tilde{R}^{-}_{0,1} \cdot \tilde{R}^{+}_{0,1} \cdot \tilde{R}^{-}_{0,1}\right)^{-1} \cdot \left(\tilde{c}^{-} + \tilde{R}^{-}_{0,1} \cdot \tilde{c}^{+}\right) \tag{5.50}
$$

$$
\tilde{d}^{+}_{0} = \tilde{R}^{+}_{0,-1} \cdot \tilde{d}^{-}_{0} + \tilde{c}^{+}
$$

The amplitude coefficient of other layers can be derived from $\tilde{d}^{+}_{0}$ using a recursive algorithm such as describe in [33].
Figure 5.2: The above illustrates the direction relates to the reflection and transmission coefficient. The arrow pointing to the left is associated with “−” and pointing to the right is associated with “+”.
5.3 Transimpedance

Once the fields in the receiver layer (denoted by subscript $r$) are known, the induced voltage on the receiver due to a unit current source at the transmitter can be determined. This transimpedance is given in terms of volume integral of the electric field along the receiver as

$$ Z_{RT} = \frac{V_R}{I_T} = \frac{-1}{I_T} \iiint_V \delta(\rho - \rho_R) \delta(z - \zeta_R(\phi))(\hat{\phi} + \hat{\xi}_R(\phi)) \cdot \vec{E} d\nu $$

(5.51)

If the field is given as

$$ \vec{A}_{rz}(\rho, z) = \bar{\vec{S}} (z) \cdot \bar{\vec{A}}_r \cdot \bar{\Pi}_{rz}(\rho), $$

(5.52)

$$ \vec{A}_{r\phi}(\rho, z) = \bar{\vec{S}} (z) \cdot \bar{\vec{A}}_r \cdot \bar{\Pi}_{r\phi}(\rho), $$

(5.53)

where $\bar{\Pi}_{rz}$ and $\bar{\Pi}_{r\phi}$ are given by (from Eq. 5.42 and Eq. 5.43)

$$ \bar{\Pi}_{rz} = \left\{ \begin{array}{ll}
\bar{H}_r(\rho) \cdot \bar{R}_{r,r-1} + \bar{J}_r(\rho) \cdot \bar{d}_r^- & \rho \leq \rho_T; \\
\bar{H}_r(\rho) + \bar{J}_r(\rho) \cdot \bar{R}_{r,r+1} & \rho_T \leq \rho,
\end{array} \right. $$

(5.54)

$$ \bar{\Pi}_{r\phi} = \left\{ \begin{array}{ll}
\bar{Y}_r^+(\rho) \cdot \bar{H}_r(\rho) \cdot \bar{R}_{r,r-1} + \bar{Y}_r^-(\rho) \cdot \bar{J}_r(\rho) \cdot \bar{d}_r^- & \rho \leq \rho_T; \\
\bar{Y}_r^+(\rho) \cdot \bar{H}_r(\rho) + \bar{Y}_r^-(\rho) \cdot \bar{J}_r(\rho) \cdot \bar{R}_{r,r+1} & \rho_T \leq \rho,
\end{array} \right. $$

(5.55)

$Z_{RT_z}$ can be expressed as

$$ Z_{RT_z} = -\frac{\rho_R}{I_T} \sum_{\nu} \bar{d}_{\nu}^f \ast \bar{\vec{a}}_r \cdot \bar{\Pi}_{rz}, $$

(5.56)

$$ \bar{d}_{\nu} = \int_{-\pi}^{\pi} d\phi \frac{1}{\sqrt{2\pi}} e^{i\nu\phi} \zeta_R S(\zeta_R) $$

(5.57)

and

$$ Z_{RT_{\phi}} = -\frac{\rho_R}{I_T} \sum_{\nu} \bar{d}_{\mu}^f \ast \bar{\Pi}_{r\phi}, $$

(5.58)
\[ \tilde{d}_{\mu\nu} = \int_{-\pi}^{\pi} d\phi \frac{1}{\sqrt{2\pi}} e^{i\nu\phi} \tilde{S}(\zeta_R) \] (5.59)

where the operator * above is defined as a dot product on the first half of the right column vector. Appendix C.6 presents a more the detailed derivation.

5.4 Numerical Issues

5.4.1 Matrix Inversion and Ill-Conditioning Issues

The accuracy of the proposed approach depends on the number of modes utilized. Unlike the NMM approach in [47], [54], the number of transversal eigenmodes required to handle the problem is usually not larger than 45 modes. For problems requiring merely many modes, the ratio of the maximum and minimum eigenvalues can reach $10^6 \sim 7$, leading to ill-condition of the matrices involved. For the local reflection/transmission coefficient in Eq. 5.46-5.49, we factor out the diagonal matrices $\tilde{J}$ and $\tilde{H}$, (which can be inverted easily) from the inverse matrices. This leads to a more well-conditioned formulation. Moreover, Eq. 5.44, 5.45 and 5.50 need to be reformulated as

\begin{align*}
\tilde{R}_{n+1}^- &= \tilde{R}_{n+1}^- + \tilde{T}_{n+1,n}^- \cdot \tilde{R}_{n+1,n+2}^- \cdot \tilde{H}_{n+1}^{-1}(\rho_{n+1}) \cdot \tilde{H}_{n+1}(\rho_{n+1}) - \tilde{H}_{n+1}(\rho_{n+1}) \cdot \tilde{R}_{n+1,n}^- \cdot \tilde{R}_{n+1,n+2}^- \cdot \tilde{H}_{n+1}(\rho_{n+2})^{-1} \\
&= \tilde{H}_{n+1}(\rho_{n+1}) \cdot \tilde{T}_{n,n+1}^-,
\end{align*}
(5.60)

\begin{align*}
\tilde{R}_{n,n-1}^+ &= \tilde{R}_{n,n-1}^+ + \tilde{T}_{n-1,n}^+ \cdot \tilde{R}_{n-1,n-2}^+ \cdot \tilde{H}_{n-1}^{-1}(\rho_{n-2}) \cdot \tilde{H}_{n-1}(\rho_{n-2}) - \tilde{H}_{n-1}(\rho_{n-2}) \cdot \tilde{R}_{n-1,n}^+ \cdot \tilde{R}_{n-1,n-2}^+ \cdot \tilde{H}_{n-1}(\rho_{n-2})^{-1} \\
&= \tilde{H}_{n-1}(\rho_{n-1}) \cdot \tilde{T}_{n,n-1}^+.
\end{align*}
(5.61)
Table 5.1: The below table shows the inverse of the condition number of the matrix within the round bracket of Eq. 5.45 and Eq. 5.61. The Ill-posed means the matrix in Eq. 5.45 and the well-posed means the matrix in Eq. 5.61.

<table>
<thead>
<tr>
<th>ν</th>
<th>n=0</th>
<th>n=1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ill-Posed</td>
<td>Well-Posed</td>
</tr>
<tr>
<td>0</td>
<td>1.06e-12</td>
<td>3.23e-4</td>
</tr>
<tr>
<td>1</td>
<td>3.83e-20</td>
<td>2.86e-4</td>
</tr>
<tr>
<td>2</td>
<td>4.71e-22</td>
<td>3.72e-4</td>
</tr>
</tbody>
</table>

\[
\dd_0^- = \tilde{J}_0^{-1}(\rho_{-1}) \cdot \left( \tilde{J}_0(\rho_0) \cdot \tilde{J}_0^{-1}(\rho_{-1}) - \tilde{J}_0(\rho_0) \cdot \tilde{R}_{0,1}^- \cdot \tilde{R}_{0,-1}^- \cdot \tilde{J}_0^{-1}(\rho_{-1}) \right)^{-1} \cdot \tilde{J}_0(\rho_0) \cdot \left( \tilde{c}^- + \tilde{R}_{0,1}^- \cdot \tilde{c}^+ \right).
\]

In the above, the matrices to be inverted within the round brackets are better conditioned. To illustrate the improvement, we consider an extreme case with 250 modes within a 2.5 m domain. The formation considered is shown in Fig. 5.3. Table 5.4.1 shows the reciprocal of the condition number of the matrices in the round bracket in Eq. 5.45 and Eq. 5.61, respectively. This results show that a dramatic improvement can be obtained in the condition number.

### 5.4.2 B-Spline Basis Functions

The cross-coupling of TM and TE modes in the cylindrical formation forces the use of curl-conformal basis functions. We use the second-order B-spline basis function [55], as illustrated in Fig. 5.4. A second order B-spline function has four knot points. When these points are equally spaced, a uniform B-spline interpolation is obtained. Within every element of the computational grid, three basis functions are used to approximate
Mandrel

$\rho_0 = 5 \text{ in.}$
$\rho_1 = 6 \text{ in.}$
$\rho_2 = 7 \text{ in.}$

$\sigma_0 = 0.0005 \text{ mho/m}$
$\sigma_1 = 0.005 \text{ mho/m}$
$\sigma_2 = 0.05 \text{ mho/m}$
$\sigma_3 = 0.5 \text{ mho/m}$

Figure 5.3: The formation used to study the conditioning of the ill-posed and well-posed formulation of the generalized reflection matrix.

a function, as shown in Fig. 5.5. These three functions are given as

\begin{align*}
B_1(z) &= \frac{(z_2 - z)^2}{(l_2 + l_1)l_2} \\
B_2(z) &= -\frac{(z - z_1)(z - z_2)}{(l_1 + l_2)l_2} - \frac{(z - z_2)(z - z_1)}{(l_2 + l_3)l_3} - \frac{l_1(z - z_2)}{(l_1 + l_2)l_2} + \frac{l_3(z - z_1)}{(l_2 + l_3)l_2} \\
B_3(z) &= \frac{(z - z_1)^2}{(l_2 + l_3)l_3}
\end{align*}

(5.63) (5.64) (5.65)

where

\begin{equation}
l_1 = z_1 - z_0, \quad l_2 = z_2 - z_1, \quad l_3 = z_3 - z_2.
\end{equation}

(5.66)

and $z_i, i=1,...,4$ are the knot points.

We also use Lagrangian basis functions [56], [57]. However, these functions are not curl-conforming and hence, the result is not as accurate as the B-spline case.
Figure 5.4: Second-order B-spline function. The four knot points of the second-order B-spline functions are (0,1/3,2/3,1).

The results due to the two type of basis functions are compared in Table. 5.2, 5.3 and 5.4. The results generated by the B-spline basis function agree very well to the pseudoanalytical results. The formation used in this comparison includes a borehole with 10 in. diameter filled with mud having conductivity of $\sigma_{\text{mud}} = 1.0$ mho/m. The homogeneous formation itself has conductivity $\sigma = 5.0$ mho/m.

5.4.3 Convergence at Interfaces along the $z$-direction

We examine the efficiency of the B-splines at interface between two horizontal layers. The cross-coupling effect of TM and TE modes is effected by diagonal terms of $\bar{M}$ in Eq 5.37. These cross-coupling terms are singular because the derivative of the normal field (distributions) across the interface is not continuous, leading to Dirac
<table>
<thead>
<tr>
<th>$\nu$</th>
<th>Pseudoanalytical</th>
<th>Lagrangian 3rd order</th>
<th>B-spline 2nd order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1.987e-3+j3.106e-4</td>
<td>-1.986e-3+j3.099e-4</td>
<td>-1.987e-3+j3.106e-4</td>
</tr>
<tr>
<td>2</td>
<td>2.111e-6-j1.028e-5</td>
<td>2.339e-6-j1.108e-5</td>
<td>2.111e-6-j1.028e-5</td>
</tr>
<tr>
<td>3</td>
<td>1.812e-8+j2.906e-8</td>
<td>-2.761e-7+j4.171e-6</td>
<td>1.819e-8+j2.904e-8</td>
</tr>
<tr>
<td>4</td>
<td>-1.335e-10-j3.685e-11</td>
<td>1.046e-7-j6.163e-6</td>
<td>-1.331e-10-j4.041e-11</td>
</tr>
</tbody>
</table>

Table 5.2: $e_{z\nu}$ computed by pseudoanalytical method, NMM homogeneous formulation.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>Pseudoanalytical</th>
<th>Lagrangian 3rd order</th>
<th>B-spline 2nd order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-4.649e-3+j7.930e-4</td>
<td>-4.649e-3+j7.930e-4</td>
<td>-4.648e-3+j7.930e-4</td>
</tr>
<tr>
<td>2</td>
<td>3.102e-4+j1.191e-3</td>
<td>3.102e-4+j1.191e-3</td>
<td>3.103e-4+j1.191e-3</td>
</tr>
<tr>
<td>3</td>
<td>-8.026e-6+j2.098e-6</td>
<td>-7.964e-6+j2.098e-6</td>
<td>-8.026e-6+j2.008e-6</td>
</tr>
<tr>
<td>4</td>
<td>-6.759e-9-j2.636e-8</td>
<td>-5.881e-8-2.20e-8</td>
<td>-6.764e-9-2.714e-8</td>
</tr>
<tr>
<td>5</td>
<td>5.598e-11-j7.564e-10</td>
<td>9.231e-10+j3.971e-8</td>
<td>5.616e-11-j1.359e-11</td>
</tr>
</tbody>
</table>

Table 5.3: $e_{\varphi\nu}$ due to TE$_z$ mode computed by pseudoanalytical method, NMM homogeneous formulation.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>Pseudoanalytical</th>
<th>Lagrangian 3rd order</th>
<th>B-spline 2nd order</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-3.109e-4-j1.168e-3</td>
<td>-3.111e-4-j1.165e-3</td>
<td>-3.109e-4-j1.168e-3</td>
</tr>
<tr>
<td>3</td>
<td>8.003e-6-j2.070e-6</td>
<td>8.391e-6-j6.755e-6</td>
<td>8.004e-6-j2.070e-6</td>
</tr>
<tr>
<td>4</td>
<td>6.897e-9+j2.732e-8</td>
<td>-1.365e-7-j6.163e-6</td>
<td>6.900e-9+j2.732e-8</td>
</tr>
<tr>
<td>5</td>
<td>-5.647e-11+j2.296e-11</td>
<td>1.059e-7-j5.378e-6</td>
<td>-5.672e-11+j1.416e-11</td>
</tr>
</tbody>
</table>

Table 5.4: $e_{\varphi\nu}$ due to TM$_z$ mode computed by pseudoanalytical method, NMM homogeneous formulation.
Figure 5.5: The basis function formulated from the second-order b-spline function in an element.

delta functions. If multiple junctions ($z'_n$) are present, matrix $\mathbf{\tilde{D}}$ in Eq. 5.38 can be expressed as

$$
\mathbf{\tilde{D}} = \left\langle \mathbf{\tilde{S}}(z), \frac{d}{dz}p(z)\mathbf{\tilde{S}}^t(z) \right\rangle 
$$

$$
= \left\langle \mathbf{\tilde{S}}(z), p(z) \frac{d}{dz}\mathbf{\tilde{S}}^t(z) \right\rangle + \left\langle \mathbf{\tilde{S}}(z), \mathbf{\tilde{S}}^t(z) \sum_n (p(z'_{n+}) - p(z'_{n-}))\delta(z - z'_n) \right\rangle 
$$

(5.67)

where

$$
p(z) = \frac{1}{\mu(z)\epsilon(z)}. \tag{5.68}
$$

Because the Dirac delta is inside the inner product, we can write an element in the second term of Eq. 5.67 as

$$
(\mathbf{\tilde{D}})_{ij} = \int_{z_{\text{min}}}^{z_{\text{max}}} S_i(z)S_j(z)(p(z'_{n+}) - p(z'_{n-}))\delta(z - z'_n)dz = (p(z'_{n+}) - p(z'_{n-}))S_i(z'_n)S_j(z'_n) 
$$

(5.69)
In order to capture the correct behavior, five smaller grid or more is needed close to the interfaces.

We employ different grids to study the convergence at the interfaces. The formation under test is shown in Fig. 5.6.

![Diagram of formation](image)

Figure 5.6: Formation used to study the gridding effect on the accuracy of the results close to the interfaces. The domain size along z is 5 m. with the interface located at 2.5 m.

The domain size is 5 m and the interface is located 2.5 m. Different grids are used to discretize the domain. Four uniform grids with 200, 300, 400, and 600 elements, and three non-uniform grids with 200, 240 and 280 elements are used. The position of the index is given in Fig. 5.7. For non-uniform grids, the grid elements in the vicinity of the interface are more closely spaced.
Figure 5.7: Nodal positions for the grids used to test the convergence of NMM at bed boundaries (interfaces). A lower slope for the traces above means the grid is denser. The interfaces is located at 2.5 m.

Fig. 5.8 depicts the amplitude ratio of the tilt-coil well-logging tool. The spacing between the transmitter and the first and second receivers are 30 in and 24 in, respectively. Fig. 5.8 shows that the grids with element sizes of 0.025 m (200 elements), 0.0167 m (300 elements), and 0.0125 m (400 elements) are not enough to correctly capture the field behavior at the vicinity of the interface. In terms of the well-logging tool response, it will be affected when the receiver is close to the interface (24/30 in). For the non-uniform grids, we have elements of size 0.005 m close to the interface. All three non-uniform grids cases and the highly refined grid case with 600 elements converge to the same results.
Figure 5.8: NMM results for the amplitude ratios of a tool with spacing of the first and second receiver are 30 in and 24 in crossing a bed boundary. In this case, the transmitter tilt angle is 45°. The x-axis is the distance of transmitter from the boundary interface.

5.4.4 PML Performance in NMM

We next investigate the performance of the PML along the z-direction. In this study, we consider a low loss ($\sigma = 0.0005$ mho/m) homogeneous medium, which is a worst case scenarios.

The accuracy is first studied versus the number of modes. Without incorporating the PML, we first stretched the end points to increasing distances. A ring source is located at 1.25 m. in the middle of a 0 - 2.5 m. domain. The ring receiver is located 30 in. away from the source point (equal to the distance between the first receiver to the transmitter of the well-logging tool considered). The end points are
stretched to -283.72 m. and 286.22 m, with in a region with 20 elements (bottommost and topmost elements). The results are compared against the results generated from radiation integral in homogeneous space in Appendix B.4. Fig. 5.9 depicts the grid use in this case, where there is a total of 100 elements within the domain. Table 5.5 shows the numerical error for the NMM. With more than 100 elements modes, the NMM results show excellent agreement with the analytical solution (radiation integral).

Based on the reference point given in Table. 5.5, we evaluate the performance of the NMM incorporating PML. The grid used here is no longer stretched at the two ends. Instead, We use a total of 100 elements within the 2.5 m. domain, with 20 elements at the top and bottom regions having same size as the inner elements but incorporating the PML (non-zero PML conductivities). Fig. 5.10 shows the performance of the
<table>
<thead>
<tr>
<th># of modes</th>
<th>V</th>
<th>Amplitude error (dB)</th>
<th>Phase error (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0.000032287943 - 0.006038415512i</td>
<td>-36.63</td>
<td>-61.42</td>
</tr>
<tr>
<td>75</td>
<td>0.000032288221 - 0.006039698029i</td>
<td>-53.00</td>
<td>-75.95</td>
</tr>
<tr>
<td>100</td>
<td>0.000032288145 - 0.006039723219i</td>
<td>-60.76</td>
<td>-84.97</td>
</tr>
<tr>
<td>200</td>
<td>0.000032288151 - 0.006039723992i</td>
<td>-61.48</td>
<td>-84.59</td>
</tr>
</tbody>
</table>

Table 5.5: The error of the numerical method is given here. The analytical referenced value for a low-loss ($\sigma=0.0005$) homogeneous medium is $V_{ref}=0.000032288142 - 0.006039728286i$. The ring source is tilted with angle $\theta_T = 45^\circ$, and the receiver is a ring source with tilted angle $\theta_R = 45^\circ$.

PML. The errors are defined as

\[
\text{Amplitude Error} = 10 \log_{10} \frac{||V|| - ||V_{ref}||}{||V_{ref}||},
\]  
(5.70)

and

\[
\text{Phase Error} = 10 \log_{10} \frac{|\theta(V) - \theta(V_{ref})|}{\pi}.
\]  
(5.71)

Different profiles for the real $a_z$ and imaginary $f_z$ stretching coordinate variable $s_z=a_z+\frac{f_z}{\omega}$ of the PML are studied. Interestingly, the phase error for the case with no PML is very low. However, the voltage result is good only if both the error in the amplitude and phase are small. Note that, we can further stretch the grids outside the domain of interest to increase the PML layer and improve even more the approximation to the radiation condition.
5.5 Simulation Results

5.5.1 FDTD Reference Results

Using FDTD to model the log of tilted-coil antenna array tools is quite expensive since very fine meshing is required in the vicinity of the antenna to minimize staircasing due to the tilted geometry. We use such FDTD results here as a reference to compare against NMM results. The grid size is spaced 1/8 in. uniformly around the tool, and is gradually increased with a rate of 1.05 between adjacent cells away from the tool. To avoid variations due to different meshing, the log is modelled by placing the tool at the center of the mesh, and moving the bed interfaces relatively to the tool. Fig. 5.11 illustrates the non-uniform mesh used for the FDTD simulation.

5.5.2 NMM Simulation Setup

For the NMM simulations, we use the following general guidelines. For every 2.5 m along z direction, we use 100 elements (of regular size). In the vicinity of the interfaces, we use 10 smaller elements with reduced element size of 0.005 m. At the top and bottom ends of the domain, we use 20 elements with size ten times larger the regular element size for the PML layer. The PML use a fourth order profile for the complex stretching variables, with 5.0 and 9e-4 as the maximum values real and imaginary parts, respectively. The above guidelines are followed for all NMM simulation, unless indicated otherwise.

5.5.3 Homogeneous Formation with Invasion Zone

We first compare NMM results against the pseudoanalytical formulation described in Chapter 4. Table 5.6 illustrates the voltage measured at the receiver that is 30
Table 5.6: Voltage measured at the receiver with different formation profiles. The distance between transmitter and receiver is 30 in. Transmitter and receiver are tilted by 45° in the same direction. The antennas are mounted on the mandrel with a diameter of 8 in.

in away from the transmitter. Four different formation profiles are studied. The overall domain size of the NMM is 5 m. The transmitter is placed in the center of the domain. The transmitter and receiver are both tilted 45° in the same direction. The NMM results and the pseudoanalytical results show excellent agreement.

5.5.4 Two-layer Isotropic Formation

Next, we compare the FDTD and NMM results in a two-layer isotropic formation. Fig. 5.12 and Fig. 5.13 show the phase and amplitude computed at the two receivers using the two methods. The problem includes a 10 in diameter borehole filled with oil-based mud (σ_mud=0.0005 mho/m), and a two-layer formation. The conductivity of the top layer is 1.0 mho/m and the conductivity of the bottom layer is 5.0 mho/m. The transmitter tilt angle is fixed at 45°. Four receiver tilt angles are considered:

\[\sigma_0=5.0 \times 10^{-4} (\rho_0=5\text{in}), \sigma_1=0.05\]
\[\sigma_0=5.0 \times 10^{-4} (\rho_0=5\text{in}), \sigma_1=0.1 (\rho_1=7\text{in}), \sigma_2=0.5\]
\[\sigma_0=2.0 (\rho_0=5\text{in}), \sigma_1=1.5 (\rho_1=7\text{in}), \sigma_2=1.0\]
\[\sigma_0=2.0 (\rho_0=5\text{in}), \sigma_1=1.0 (\rho_1=7\text{in}), \sigma_2=0.5 (\rho_2=10\text{in}), \sigma_3=0.1\]
10°, 20°, 30°, and 40°. The abcissa represents the distance from the transmitter to the interface. The FDTD and NMM results show excellent agreement.

### 5.5.5 Two-layer Anisotropic Formation

We compared the FDTD and NMM results in a two-layer anisotropic formation. Fig. 5.14 and Fig. 5.15 illustrate the phase and amplitude computed at the two receivers using the two methods. In this case, we consider a borehole with 5 in diameter filled with oil-based mud surrounded by a two-layer anisotropic formation. The horizontal and vertical component of the conductivity of the bottom layer are 5.0 and 1.0 mho/m. In the top layer, they are 1.0 and 5.0 mho/m, respectively. The transmitter tilt angle is fixed at 45°. Four receiver tilted angles are considered: 15°, 25°, 35°, and 45°. The results generated from the two methods again show excellent agreement.

### 5.5.6 Comparison of Tilted-Coil and Horizontal-Coil Results

We contrast the log of a conventional (horizontal-coil) well-logging tool and a tilted-coil tool in the formation depicted in Fig. 5.16. This formation includes an invasion zone, and both isotropic and anisotropic layers. The conventional well-logging tool has the geometry shown in Fig. 3.4. The tilted-coil well-logging tool has similar geometry but with transmitter and the second (close to the transmitter) receiver coils tilted by 45 degree in the same direction. The phase and amplitude of the voltage pair calculated at the two receivers are shown in Fig. 5.17. In this figure, ‘ref’ denotes conventional tool. The response (phase difference and amplitude ratio) of the two tools in the formation are given in Fig. 5.18. From this Figures, it can be seen that the conventional tool has a much flatter response, since it excites only TE modes and
is not sensitive to $\sigma_v$ values. More specifically, this response resembles this response of
the conventional tool in a 3-layer isotropic formation with conductivity 1.0/0.1/1.0 for
the upper, middle (60 in thick) and bottom layer, respectively. The tilted coil tool, on
the other hand, shows a much better sensitivity to the true formation conductivities.

We also show another extreme case using the formation given in Fig 5.19. In this
formation, the horizontal components of the conductivity in all layers are the same.
Hence, the conventional tool responds to these layers as if they were homogeneous.
Fig. 5.20 shows the voltage pair calculated measured at the two receivers for the two
tools. Fig. 5.21 shows the actual tool responses.

5.5.7 Tool response in dipping bed using dipole formulation

Here we illustrate the use of NMM formulation to model well-logging tools in
dipping bed layers using a dipole approximation [44], [58]. In this model, the trans-
mitter coil antenna is replaced by a tilted magnetic dipole in the direction normal
to the coil plane. This approximation works well because, at the 2MHz frequency
of operation, the wavelength is much larger than the coil diameter. The resulting
magnetic field at 24 in and 30 in away from the source and pointing along the normal
direction to the receiver coil plane is calculated by the NMM. The phase difference
and amplitude ratio of the tool is then approximated by the phase difference and
amplitude ratio of the magnetic field components at those two points. In this case,
the mandrel and borehole effects are neglected. The dipole model is illustrated in
Fig. 5.22. Directional logging can be modeled by aligning the magnetic dipole and
the two observation points along a line at a $\theta$ angle from the longitudinal axis (see
Fig. 5.22). Mathematical details on this formulation are given in Appendix C.7.
Fig. 5.23 shows the tool response in a three-layer formation with isotropic conductivities $\sigma = 1/0.01/1$ for upper/middle/lower layer, respectively. In this case, the orientation of the magnetic source point and the observation point are along the same direction, which resembles the conventional tool. The dip angles considered are 0°, 15°, 30°, 45°, and 60°. By comparing this result with the result shown in Fig. 5.24, generated using FDTD [1], we note that presence of the borehole and mandrel minimize the horn effect.

Fig. 5.25 shows the tool response in a three-layer anisotropic formation with horizontal conductivities $\sigma_h = 1/0.01/1$ and vertical conductivities $\sigma_v = 1/0.1/1$ for upper/middle/lower layer, respectively. The dip angles considered are 0°, 15°, 30°, 45°, and 60°. Fig. 5.26 shows the tool response of a different tool logging in the same formation. In this case, the magnetic source dipole and the second magnetic field component are tilted by 45° (with $\phi = 0$) away from the log axis. This mimics the response of a tilted coil tool discussed in the previous section. The comparison between these two results again show that the response of the tilted-coil tool is significantly different than the conventional (horizontal-coil) tool.

### 5.5.8 Apparent resistivity in dipping anisotropic formations

We also illustrate the use of dipole formulation for computing the apparent resistivity of anisotropic homogeneous formation for different dip angles. First, the phase difference and amplitude ratio of isotropic formation with different conductivities are generated as a lookup table $[\phi_0, \sigma_h]$ and $[A_0, \sigma_h]$. The parameters $\phi_0$ and $A_0$ are defined as

\[
\phi_0 = \theta(V_0)
\]

\[
A_0 = 20 \log_{10} |V_0|
\]

(5.72)
In the above, $V_0$ are the voltage ratio measured at the two receivers in a homogeneous medium with conductivity $\sigma_h$ and adjusted for air-hang correction described in [44].

Then, the apparent phase ($R_{aph}$) and amplitude ($R_{aam}$) resistivity are determined using interpolation

\[
R_{aph} = \frac{1}{\text{interp}(\phi_0, \sigma_h, \phi)}
\]
\[
R_{aam} = \frac{1}{\text{interp}(A_0, \sigma_h, A)}
\]

where the function interp, given a lookup table $[x, y]$, computes an interpolated ordinate $y_i$, given an index value $x_i$, i.e.,

\[
y_i = \text{interp}(x, y, x_i).
\]

Fig. 5.27 and Fig 5.28 depict the apparent resistivity in anisotropic homogeneous formations with $\sigma_h$ equal to 0.1 and 0.5 [mho/m] for different dipping angles. They are plotted against the anisotropic ratio given by

\[
k = \sqrt{\frac{\sigma_h}{\sigma_v}}
\]

This NMM results show very good agreement against the results shown in [44].

5.6 Conclusion

In this chapter, we have derived a novel NMM approach employing vertical eigenvector and including PML to simulate the titled-coil antenna tools in anisotropic formations. The vertical eigenmodes are obtained via a generalized eigenfunction problem. Vertical modes are propagated in the transverse direction using mode propagators. This NMM approach is more efficient compared to conventional NMM approaches because reflection and transmission of the multiple cylindrical-vertical layers
need to be computed only once. In addition, the voltage component due to the forma-
tion can be easily extracted, providing more insight of the underlying physics. In term
of limitations, the method has a restriction on the source location in the transverse
direction. Moreover, the convergence of the method is grid dependent. If there exists
a large ratio between the maximum and minimum eigenvalues (ill-conditioning), the
method may converge slowly, or not converge at all.

The present NMM formulation has been validated against both pseudoanalytical
results and FDTD results, showing very good agreement. Using NMM, we have also
compared the tool response of conventional tool and the novel tool in anisotropic
formation, showing that the conventional tool misread the actual formation conduc-
tivity. We have also shown how to model the tool response of the horizontal-coil tool
and tilted-coil tool in the dipping bed formation using this NMM dipole formulation.
Figure 5.10: PML performance for the NMM formulation in a homogeneous medium. The error on the amplitude (a) and phase (b) of the voltage are shown. The values in the legend, represent the real and imaginary parts of the complex stretching variables of the PML \((a_z, f_z)\). The \(x\)-axis represents the source position. The ring source is tilted with angle \(\theta_T = 45^\circ\), and the receiver is a ring source with angle \(\theta_R = 45^\circ\).
Figure 5.11: Example of a mesh used for FDTD simulations for well-logging tools employing tilted-coil antenna. Non-uniform grid is adapted in both the $z$ and $\rho$-direction. The horizontal lines at the $z$-axis indicate the position of the transmitter and receivers.
Figure 5.12: FDTD and NMM results for the phase of the measured voltage at the first and second receivers. The problem includes a borehole filled with oil-based mud ($\sigma_{mud}=0.0005$ mho/m) and an isotropic formation with two layers. The top layer has conductivity of 1 mho/m and the bottom layer has conductivity of 5 mho/m. The transmitter tilt angle is $45^\circ$. Four receiver angles are considered: $10^\circ$, $20^\circ$, $30^\circ$, and $40^\circ$. The abscissa represent distance of the transmitter to the interface.
Figure 5.13: FDTD and NMM results for the amplitude of the measured voltage at the first and second receivers. The problem includes a borehole filled with oil-based mud ($\sigma_{mud}=0.0005$ mho/m) and two-layer isotropic formation. The top layer has conductivity of 1 mho/m and the bottom layer has conductivity of 5 mho/m. The transmitter tilt angle is 45°. Four receiver angles are considered: 10°, 20°, 30°, and 40°. The abcissa represents the distance of the transmitter to the interface.
Figure 5.14: FDTD and NMM results for the phase of the measured voltage at the first and second receivers. The problem includes a borehole filled with oil-based mud ($\sigma_{\text{mud}}=0.0005$ mho/m) and two-layer uniaxial-anisotropic formation. The top layer has $\sigma_{h/v}=1/5$ mho/m and bottom layer has $\sigma_{h/v}=5/1$ mho/m. The transmitter tilt angle is $45^o$. Four receiver angles are considered: 15°, 25°, 35°, and 45°. The abcissa represents the distance of the transmitter to the interface.
Figure 5.15: FDTD and NMM results for the amplitude of the measured voltage at the first and second receivers. The problem includes a borehole filled with oil-based mud ($\sigma_{\text{mud}}=0.0005$ mho/m) and two-layer uniaxial-anisotropic formation. The top layer has conductivity $\sigma_{h/v}=1/5$ mho/m and bottom layer has conductivity $\sigma_{h/v}=5/1$ mho/m. The transmitter tilt angle is 45°. Four receiver angles are considered: 15°, 25°, 35°, and 45°. The log is given with the $x$-axis defined as the distance of the transmitter to the interface.
Figure 5.16: Borehole problem with a formation including invasion, isotropic layer and anisotropic layers. This formation can be misread by the conventional well-loging tool since a horizontal-coil antenna excites only TE mode.
Figure 5.17: Phase and amplitude of the computed voltage measured. Conventional and tool results are compared. The abcissa represents the distance of the transmitter the mid-point (0 in) of the formation depicted in Fig. 5.16.
Figure 5.18: Responses of the conventional tool and the tilted-coil tool in the formation of the Fig. 5.16. The conventional tool has misread the vertical component formation conductivity, and its response is flatter than the tilted-coil tool.
Figure 5.19: Borehole problem with a formation including invasion zone and anisotropic layers. In this example, the horizontal component for the conductivity of all layers is equal to 0.1 mho/m. This formation is misread by the conventional well-logging tool since a horizontal-coil antenna excites only TE modes.
Figure 5.20: Phase and amplitude of the computed voltage at the receivers. Conventional and tilted-coil tool results are compared. The abcissa represents the distance of the transmitter to the mid-point (0 in) of the formation depicted in Fig. 5.19.
Figure 5.21: Responses of the conventional tool and the tilted-coil tool in the formation depicted in the Fig. 5.19. The conventional tool is not sensitive to the vertical component formation conductivity, and its response is flat. The tilted-coil tool response on the other hand, changes according to the surrounding $\sigma_v$ values.
Figure 5.22: The above Figure illustrates the model used to obtain the tool response in dipping formations using NMM dipole formation. The coil antenna is replaced by a magnetic dipole point in the normal direction of the coil plane. The tool response is modeled by taking the phase difference and amplitude ratio of the magnetic field component at the center of the receiver coil positions in the normal direction of the receiver coil plane. In the above, the tool is logging into the formation with $\theta$ degree dip with respect to the normal to the beds.
Figure 5.23: Phase difference and amplitude ratio of the magnetic field measured 30 in and 24 in away from the source. The tool logs inside a three-layer formation with isotropic conductivities of 1.0/0.01/1.0 for upper/middle/lower layer, respectively. The direction of the magnetic dipole source and the measured magnetic fields are aligned with the log axis. The five dip angles are 0°, 15°, 30°, 45° and 60°. The x-axis represent the distance of the source point to the boundary of the middle and the bottom layer (D in Fig. 5.22).
Figure 5.24: Phase difference and amplitude ratio of the conventional tool (employing horizontal coil antenna). The tool logs inside a three-layer formation with isotropic conductivities of 1.0/0.01/1.0. In the FDTD model, both mandrel and borehole are modeled. The dipping bed is modeled using the staircase approximation [1]. The x-axis represent the distance of the transmitter to the interface of middle and bottom layers.
Figure 5.25: Phase difference and amplitude ratio of the magnetic field measured 30 in and 24 in away from the source. The tool logs inside a three-layer formation with anisotropic conductivities with $\sigma_h=1.0/0.01/1.0$ and $\sigma_v=1.0/0.1/1.0$ for upper/middle/lower layer, respectively. The direction of the magnetic dipole source and the measured magnetic fields are aligned with the log axis. This resembles the conventional tool employing horizontal-coil antennas. The five dip angles are 0°, 15°, 30°, 45° and 60°. The $x$-axis represent the distance of the source point to the interface of middle and bottom layers.
Figure 5.26: Phase difference and amplitude ratio of the magnetic field measured 30 in and 24 in away from the source. The tool logs inside a three-layer formation with anisotropic conductivities with $\sigma_h=1.0/0.01/1.0$ and $\sigma_v=1.0/0.1/1.0$ for upper/middle/lower layer, respectively. The direction of the magnetic dipole source and the second measured magnetic fields are tilted 45° away from the log axis. This is similar to a tilted-coil tool. The five dip angles are 0°, 15°, 30°, 45° and 60°. The x-axis represent the distance of the source point to the interface of middle and bottom layers.
Figure 5.27: Apparent resistivity using phase difference and amplitude ratio. In this case, $\sigma_h=0.1$ mho/m.
Figure 5.28: Apparent resistivity using phase difference and amplitude ratio. In this case, $\sigma_h = 0.5$ mho/m.
CHAPTER 6

CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

Accurate modeling of electromagnetic well-logging tools provide an alternative to costly experimental tests. It allows the analysis of the response of newly designed tools in various formations before any prototype is built. In this thesis, we have proposed and discussed three approaches for the modeling of the well-logging tools.

In Chapter 3, we have discussed a brute-force numerical approach using three-dimensional (3-D) finite-difference time-domain method in cylindrical grids. We have considered several extensions on the FDTD technique for this problem: (i) use of perfectly matched layer (PML) extended to cylindrically coordinates to truncate the finite-difference grid; (ii) use of locally-conformal (LC) FDTD to model eccentric borehole; (iii) employing a frequency data extraction method with proper FDTD excitation; (iv) scaling the permittivity in the lower frequency regime to overcome the Courant stability limit. The resulting FDTD method is flexible enough to model any 3D formation including layered formations, eccentric boreholes, dipping beds and arbitrary invasion zones. The accuracy of the method has been validated against NMM and pseudoanalytical results, where the latter is applicable, showing very good agreement.
Novel well-logging tool employing tilted-coil antennas (TCAs) were modelled in Chapter 4 and Chapter 5. In Chapter 4, we have discussed an extension of a pseudoanalytical approach to model such tilted-coil antenna arrays in cylindrical multilayered formations, including eccentric boreholes. In Chapter 5, we have discussed the development of a vertical eigenmode numerical mode matching (NMM) algorithm incorporating PML to model tilted-coil antennas in multilayered anisotropic formations. The NMM algorithm employs numerical matching in the vertical direction and analytical expansion in the transverse (radial) direction. Numerical issues to improve the convergence and reduce ill-conditioning for the NMM have also been discussed. The novel NMM approach has been validated against FDTD results and used for the analysis of the tilted-coil well-logging in multilayered anisotropic formation.

In terms of future work, we plan to extend our studies of novel tilted-coil well-logging tools in dipping bed formations. This would allow better insight to the exploration of such tools for geosteering application.

In terms of algorithm improvements for FDTD, a possible avenue of research is to develop alternating-direction-implicit schemes for the well-logging cylindrical FDTD problem. The ADI scheme splits curl operator in two half-time steps and has been proven to be unconditional stable regardless of the time step size. Hence, the time step is only bounded the accuracy and shorter simulation times can be achieved.
APPENDIX A

FURTHER DETAILS ON 3-D CYLINDRICAL FDTD FORMULATION

In this appendix, we consider some details of the cylindrical FDTD formulation discussed in Chapter 3. A cylindrical coordinate system is chosen to discretize Maxwell’s equation to better conform with the geometry of well-logging tools.

Consider the differential form of the Maxwell’s equations given as

\[
\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} - \vec{M}, \quad (A.1)
\]

\[
\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} + \vec{J} \quad (A.2)
\]

where \(\vec{E} [V/m]\) is the electric field intensity, \(\vec{H} [A/m]\) is the magnetic field intensity, \(\sigma [mho/m]\) is the electrical conductivity, \(\epsilon [F/m]\) is the dielectric permittivity, \(\mu [H/m]\) is the permeability of the medium, \(\vec{J} [A/m^2]\) is the impressed (source) electric current density and \(\vec{M} [V/m^2]\) is the impressed (source) magnetic current density.

Equations A.1-A.2 can be represented by six scalar equations in the three dimensional
cylindrical coordinate system as

$$
\begin{align*}
\frac{\mu}{\partial t} \frac{\partial H_\rho}{\partial t} &= \frac{1}{\rho} \frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} - M_\rho \quad (A.3) \\
\frac{\partial H_\phi}{\partial t} &= \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} - M_\phi \quad (A.4) \\
\frac{\mu}{\partial t} \frac{\partial H_z}{\partial t} &= \frac{\partial E_\phi}{\partial \rho} + \frac{1}{\rho} E_\phi - \frac{1}{\rho} \frac{\partial E_\rho}{\partial \phi} - M_z \quad (A.5) \\
\frac{\epsilon}{\partial t} \frac{\partial E_\rho}{\partial t} &= \frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} - \sigma E_\rho - J_\rho \quad (A.6) \\
\frac{\epsilon}{\partial t} \frac{\partial E_\phi}{\partial t} &= \frac{\partial H_\rho}{\partial z} - \frac{\partial E_\phi}{\partial \rho} - \sigma E_\phi - J_\phi \quad (A.7) \\
\frac{\epsilon}{\partial t} \frac{\partial E_z}{\partial t} &= \frac{\partial H_\phi}{\partial \rho} + \frac{1}{\rho} H_\phi - \frac{1}{\rho} \frac{\partial H_\rho}{\partial \phi} - \sigma E_z - J_z. \quad (A.8)
\end{align*}
$$

The notation to identify the electric or magnetic field at any time and spatial location in the cylindrical FDTD lattice is given as

$$(i, j, k) = (i \Delta \rho, j \Delta \phi, k \Delta z), \quad (A.9)$$

where, $\Delta \rho, \Delta \phi,$ and $\Delta z$ are the cylindrical lattice space increments (assumed uniform presently) in the $\rho, \phi, z$ cylindrical coordinate directions, respectively. Figure A.1 depicts the location of the electromagnetic field components associated with a given space point. To represent any function $u$ of space and time evaluated at a discrete point in the cylindrical grid and at a discrete point in time, we employ the following index notation

$$u(i \Delta \rho, j \Delta \phi, k \Delta z, n \Delta t) = u_{i,j,k} \quad (A.10)$$

Using Taylor’s series expansion, the temporal derivative of the function $u$ at $(i \Delta \rho, j \Delta \phi, k \Delta z)$ and time $t = n \Delta t$ can numerical approximate as

$$\left. \frac{\partial u}{\partial t} \right|_{t=n\Delta t} \approx \frac{u_{i,j,k}^{n+1/2} - u_{i,j,k}^{n-1/2}}{\Delta t}. \quad (A.11)$$

126
Figure A.1: A unit cell of the staggered FDTD grid scheme for spatial discretization of electromagnetic fields on the cylindrical grid.
Note that the approximation using a central difference in time produces a second order accuracy. Similar to the temporal derivative, the spatial derivative of the function along the \( \rho \)-direction can be approximated as

\[
\frac{\partial u}{\partial \rho} \bigg|_{\rho = i \Delta \rho} \approx \frac{u_{i+1/2,j,k}^n - u_{i-1/2,j,k}^n}{\Delta \rho}.
\] (A.12)

Applying finite difference numerical approximations of Equations A.11-A.12 on Equations A.3-A.8, we can write the final finite difference update equations for the six field components in a cylindrical cell \( (i, j, k) \) as

\[
\mu_{i+1/2,j,k} \frac{H_{i+1/2,j,k}^{n+1/2} - H_{i+1/2,j,k}^{n-1/2}}{\Delta t} - \left( \frac{E_{\phi(i+1/2,j,k+1/2)}^n - E_{\phi(i+1/2,j,k-1/2)}^n}{\Delta z} \right) - M_{\rho(i+1/2,j,k+1/2)}^n
\] (A.13)

\[
\mu_{i+1/2,j,k} \frac{H_{i+1/2,j,k}^{n+1/2} - H_{i+1/2,j,k}^{n-1/2}}{\Delta t} - \left( \frac{E_{z(i+1/2,j,k+1/2)}^n - E_{z(i+1/2,j,k-1/2)}^n}{\Delta z} \right) - M_{\phi(i+1/2,j,k+1/2)}^n
\] (A.14)

\[
\rho_{i+1/2,j,k} \frac{H_{i+1/2,j,k}^{n+1/2} - H_{i+1/2,j,k}^{n-1/2}}{\Delta t} - \left( \frac{E_{\phi(i+1/2,j,k+1/2)}^n - E_{\phi(i+1/2,j,k-1/2)}^n}{\Delta \rho} \right) - M_{\rho(i+1/2,j,k+1/2)}^n
\] (A.15)

\[
\epsilon_{i+1/2,j,k} \frac{E_{\rho(i+1/2,j,k)}^n - E_{\rho(i+1/2,j,k)}^{n-1/2}}{\Delta t} - \left( \frac{H_{i+1/2,j,k}^{n+1/2} - H_{i+1/2,j,k}^{n-1/2}}{\Delta z} \right) + \sigma_{i+1/2,j,k} E_{\rho(i+1/2,j,k)}^{n+1/2} - J_{\rho(i+1/2,j,k)}^{n+1/2}
\] (A.16)

\[
\epsilon_{i+1/2,j,k} \frac{E_{\phi(i+1/2,j,k)}^n - E_{\phi(i+1/2,j,k)}^{n-1/2}}{\Delta t} - \left( \frac{H_{i+1/2,j,k}^{n+1/2} - H_{i+1/2,j,k}^{n-1/2}}{\Delta \rho} \right) + \sigma_{i+1/2,j,k} E_{\phi(i+1/2,j,k)}^{n+1/2} - J_{\phi(i+1/2,j,k)}^{n+1/2}
\] (A.17)
\[ \epsilon_{i,j,k+\frac{1}{2}} \frac{E_{z(i,j,k+\frac{1}{2})}^{n+1} - E_{z(i,j,k+\frac{1}{2})}^{n}}{\Delta t} - \frac{H_{\phi(i+\frac{1}{2},j,k+\frac{1}{2})}^{n+\frac{1}{2}} - H_{\phi(i+\frac{1}{2},j,k+\frac{1}{2})}^{n}}{\Delta \rho} \]

\[ + \frac{1}{i \Delta \rho} \left[ H_{\phi(i+\frac{1}{2},j+\frac{1}{2},k)}^{n+\frac{1}{2}} - H_{\phi(i+\frac{1}{2},j+\frac{1}{2},k)}^{n+\frac{1}{2}} \rho(i,j+\frac{1}{2},k+\frac{1}{2}) \right] + \frac{\sigma_{i,j,k+\frac{1}{2}}}{z(i,j,k+\frac{1}{2})} E_{z(i,j,k+\frac{1}{2})}^{n+\frac{1}{2}} \]  

(A.18)

Note that all the quantities on the right hand side are half time step earlier which has already been computed. The electrical properties \((\epsilon, \mu, \sigma)\) are defined in the every grid point. The \(\rho\) term in the scalar equations is defined in the same space point as the point of interest. In the Equation A.16-A.18, the \(E\) terms on the right hand side which are on time step \(n+1/2\) is approximated as follow

\[ E_{\rho(i+\frac{1}{2},j,k)}^{n+\frac{1}{2}} = \frac{E_{\rho(i+\frac{1}{2},j,k)}^{n+1} + E_{\rho(i+\frac{1}{2},j,k)}^{n}}{2} \]  

(A.19)

\[ E_{\phi(i,j+\frac{1}{2},k)}^{n+\frac{1}{2}} = \frac{E_{\phi(i,j+\frac{1}{2},k)}^{n+1} + E_{\phi(i,j+\frac{1}{2},k)}^{n}}{2} \]  

(A.20)

\[ E_{z(i,j,k+\frac{1}{2})}^{n+\frac{1}{2}} = \frac{E_{z(i,j,k+\frac{1}{2})}^{n+1} + E_{z(i,j,k+\frac{1}{2})}^{n}}{2} \]  

(A.21)

A similar approximation is applied to the \(E_{\phi}\) and \(H_{\phi}\) terms on the right hand side of the Equation A.15,A.18 where the field is not defined (refer to Figure A.1). The approximation is given as

\[ E_{\phi(i+\frac{1}{2},j+\frac{1}{2},k)}^{n+\frac{1}{2}} = \frac{E_{\phi(i+\frac{1}{2},j+\frac{1}{2},k)}^{n} + E_{\phi(i+\frac{1}{2},j+\frac{1}{2},k)}^{n+1}}{2} \]  

(A.22)

\[ H_{\phi(i+j+\frac{1}{2},k+\frac{1}{2})}^{n+\frac{1}{2}} = \frac{H_{\phi(i+j+\frac{1}{2},k+\frac{1}{2})}^{n+1} + H_{\phi(i+\frac{1}{2},j,k+\frac{1}{2})}^{n}}{2} \]  

(A.23)

Hence, rearranging the Equation A.13-A.18 by using the above approximation, we obtain the finite difference time-domain update equations as

\[ H_{\rho(i+j+\frac{1}{2},k+\frac{1}{2})}^{n+\frac{1}{2}} = H_{\rho(i,j+\frac{1}{2},k+\frac{1}{2})}^{n-\frac{1}{2}} + \frac{\Delta t}{\mu_{i,j+\frac{1}{2},k+\frac{1}{2}}} \left[ 1 - \frac{E_{z(i,j+\frac{1}{2},k+\frac{1}{2})}^{n+1} - E_{z(i,j+\frac{1}{2},k+\frac{1}{2})}^{n}}{\Delta \phi} \right] \]

\[ - \left( \frac{E_{\phi(i,j+\frac{1}{2},k+\frac{1}{2})}^{n+1} - E_{\phi(i,j+\frac{1}{2},k+\frac{1}{2})}^{n}}{\Delta z} \right) - M_{\rho(i+\frac{1}{2},j,k+\frac{1}{2})}^{n} \]  

(A.24)

129
\[ H^{n+\frac{1}{2}}_{\phi(i+\frac{1}{2},j,k+\frac{1}{2})} = H^{n-\frac{1}{2}}_{\phi(i+\frac{1}{2},j,k+\frac{1}{2})} + \frac{\Delta t}{\mu_{i+\frac{1}{2},j,k+\frac{1}{2}}} \left[ \frac{E^n_{\rho(i+\frac{1}{2},j,k+1)} - E^n_{\rho(i+\frac{1}{2},j,k)}}{\Delta \rho} \right] + \frac{\Delta t}{\rho_{i+\frac{1}{2},j,k+\frac{1}{2}}} \left[ \frac{E^n_{z(i+1,j,k+\frac{1}{2})} - E^n_{z(i,j,k+\frac{1}{2})}}{\Delta z} - M^n_{\phi(i+\frac{1}{2},j,k+\frac{1}{2})} \right] \quad (A.25) \]

\[ H^{n+\frac{1}{2}}_{z(i+\frac{1}{2},j,k+\frac{1}{2})} = H^{n-\frac{1}{2}}_{z(i+\frac{1}{2},j,k+\frac{1}{2})} + \frac{\Delta t}{\mu_{i+\frac{1}{2},j,k+\frac{1}{2}}} \left[ \frac{E^n_{\phi(i+1,j,k+\frac{1}{2})} - E^n_{\phi(i,j,k+\frac{1}{2})}}{\Delta \rho} \right] + \frac{1}{(i + \frac{1}{2})\Delta \rho} \left[ \frac{E^n_{\phi(i+1,j+\frac{1}{2},k)}}{2} - \frac{E^n_{\phi(i+1,j+1,k)} - E^n_{\rho(i+\frac{1}{2},j,k+\frac{1}{2})}}{\Delta \phi} \right] - M^n_{z(i+\frac{1}{2},j,k+\frac{1}{2})} \quad (A.26) \]

\[ E^{n+1}_{\rho(i+\frac{1}{2},j,k)} = \frac{2\epsilon_{i+\frac{1}{2},j,k} - \sigma_{i+\frac{1}{2},j,k} \Delta t}{2\epsilon_{i+\frac{1}{2},j,k} + \sigma_{i+\frac{1}{2},j,k} \Delta t} E^n_{\rho(i+\frac{1}{2},j,k)} + \frac{2\epsilon_{i+\frac{1}{2},j,k}}{2\epsilon_{i+\frac{1}{2},j,k} + \sigma_{i+\frac{1}{2},j,k} \Delta t} \left[ \frac{1}{(i + \frac{1}{2})\Delta \rho} \left( \frac{H^{n+\frac{1}{2}}_{z(i+\frac{1}{2},j+\frac{1}{2},k)} - H^{n+\frac{1}{2}}_{z(i+\frac{1}{2},j-\frac{1}{2},k)}}{\Delta z} \right) - \frac{H^{n+\frac{1}{2}}_{\phi(i+\frac{1}{2},j,k+\frac{1}{2})} - H^{n+\frac{1}{2}}_{\phi(i+\frac{1}{2},j,k-\frac{1}{2})}}{\Delta \phi} \right] - J^{n+\frac{1}{2}}_{\rho(i+\frac{1}{2},j,k)} \quad (A.27) \]

\[ E^{n+1}_{\phi(i,j+\frac{1}{2},k)} = \frac{2\epsilon_{i,j+\frac{1}{2},k} - \sigma_{i,j+\frac{1}{2},k} \Delta t}{2\epsilon_{i,j+\frac{1}{2},k} + \sigma_{i,j+\frac{1}{2},k} \Delta t} E^n_{\phi(i,j+\frac{1}{2},k)} + \frac{2\epsilon_{i,j+\frac{1}{2},k}}{2\epsilon_{i,j+\frac{1}{2},k} + \sigma_{i,j+\frac{1}{2},k} \Delta t} \left[ \frac{1}{(i + \frac{1}{2})\Delta \rho} \left( \frac{H^{n+\frac{1}{2}}_{\rho(i+\frac{1}{2},j+\frac{1}{2},k)} - H^{n+\frac{1}{2}}_{\rho(i+\frac{1}{2},j-\frac{1}{2},k)}}{\Delta z} \right) - \frac{H^{n+\frac{1}{2}}_{\phi(i+\frac{1}{2},j,k+\frac{1}{2})} - H^{n+\frac{1}{2}}_{\phi(i+\frac{1}{2},j,k-\frac{1}{2})}}{\Delta \phi} \right] - J^{n+\frac{1}{2}}_{\phi(i,j+\frac{1}{2},k)} \quad (A.28) \]

\[ E^{n+1}_{z(i,j,k+\frac{1}{2})} = \frac{2\epsilon_{i,j,k+\frac{1}{2}} - \sigma_{i,j,k+\frac{1}{2}} \Delta t}{2\epsilon_{i,j,k+\frac{1}{2}} + \sigma_{i,j,k+\frac{1}{2}} \Delta t} E^n_{z(i,j,k+\frac{1}{2})} + \frac{2\epsilon_{i,j,k+\frac{1}{2}}}{2\epsilon_{i,j,k+\frac{1}{2}} + \sigma_{i,j,k+\frac{1}{2}} \Delta t} \left[ \frac{1}{i\Delta \rho} \left( \frac{H^{n+\frac{1}{2}}_{\rho(i+1/2,j,k+\frac{1}{2})} - H^{n+\frac{1}{2}}_{\rho(i+1/2,j,k+\frac{1}{2})}}{\Delta \phi} \right) + \frac{H^{n+\frac{1}{2}}_{\phi(i+\frac{1}{2},j,k+\frac{1}{2})} + H^{n+\frac{1}{2}}_{\phi(i-\frac{1}{2},j,k+\frac{1}{2})}}{2} \right) - \left( \frac{H^{n+\frac{1}{2}}_{\phi(i+\frac{1}{2},j,k+\frac{1}{2})} - H^{n+\frac{1}{2}}_{\phi(i-\frac{1}{2},j,k+\frac{1}{2})}}{\Delta \phi} \right) - J^{n+\frac{1}{2}}_{z(i,j,k+\frac{1}{2})} \right] \quad (A.29) \]
Now, we have the explicit FDTD updating equation where the field components only depend on its previous value and the surrounding field components, that also previously stored in the computer memory. Note also that unlike the Cartesian coordinate systems, the \( \phi \)-coordinate in the cylindrical coordinate system has to incorporate periodic boundary conditions. The electric field components at \( j = N_\phi \) (\( N_\phi \) is the number of grid points in the \( \phi \)-direction) are equal to the electric field components at \( j = 0 \), and the magnetic field components at \( j = -1/2 \) are equal to the magnetic field components at \( j = N_\phi - 1/2 \).
APPENDIX B

FIELDS DUE TO AN ARBITRARILY ORIENTED RING SOURCE IN CYLINDRICAL MULTILAYERED MEDIA

This appendix provides more details on the derivation of the pseudo-analytic formulation described in Chapter 4.

B.1 Tilt-Coil Antenna Electric Current Density

We derive the current density for the tilt-coil antenna in this section. Compared to the wavelength, the thickness of the wire antenna is negligible. Hence, we replace the antenna by a filamentlike current density.

Consider the tilt-coil antenna in Fig. B.1. The normal vector to the tilted coil antenna plane is given by

\[ \vec{n}_T = \hat{x}_T \sin \theta_T \cos \phi_T + \hat{y}_T \sin \theta_T \sin \phi_T + \hat{z}_T \cos \theta_T \quad \text{(B.1)} \]

The coordinates of the wire are given by

\[ \vec{r}_T = \rho_T - \hat{z}_T \zeta_T(\phi) \quad \text{(B.2)} \]

where \( \rho_T \) is the radial coordinate of the coil (fixed) and

\[ \zeta_T(\phi) = \hat{z}_T \cdot \vec{r}_T = (z_T + \rho_T \tan \theta_T \cos(\phi - \phi_T)) \].

\[ (B.3) \]

132
Figure B.1: Tilted transmitting ring (coil) antenna with current density $\vec{J}_T$.

The differential of the above expression gives an infinitesimal vector along the direction of the antenna

$$d\vec{l}_T = \frac{d\vec{r}_T}{d\phi}d\phi = (\hat{\phi} + \hat{z}\xi_T(\phi))\rho_T d\phi,$$  \hfill (B.4)

where

$$\xi_T(\phi) = \tan \theta_T \sin(\phi - \phi_T).$$  \hfill (B.5)

The unit vector on the direction of $d\vec{l}_T$ gives the direction of the tilted-coil at $\vec{r}_T$. Thus, the current density along the coil can be written as

$$\vec{J}_T = I_T \delta(\rho - \rho_T)\delta(z - \zeta_T(\phi))\vec{u}_T$$  \hfill (B.6)

where $\vec{u}_T = \frac{1}{\rho_T} \frac{d\vec{l}_T}{d\phi}$, and $\delta(\cdot)$ is the Dirac delta function.
B.2 Homogeneous Medium Solution

We describe here the detailed derivation of the pseudoanalytical formulation in the homogeneous medium case. The time-harmonic Maxwell equations $e^{-i\omega t}$ convention are written as follow

\[ \nabla \times \vec{E} = i\omega \mu \vec{H} \]  
(B.7)

\[ \nabla \times \vec{H} = (\sigma - i\omega \epsilon) \vec{E} + \vec{J}_T \]  
(B.8)

\[ \nabla \cdot \mu \vec{H} = 0 \]  
(B.9)

\[ \nabla \cdot \epsilon \vec{E} = \rho \]  
(B.10)

By eliminating $\vec{E}$ or $\vec{H}$ in the above, and by extracting the $z$-components, we have

\[ \nabla^2 E_z + k^2 E_z = -i\omega \mu \vec{J}_T \cdot \hat{z} \]  
(B.11)

\[ \nabla^2 H_z + k^2 H_z = - (\nabla \times \vec{J}_T) \cdot \hat{z} \]  
(B.12)

The $z$ components is considered here because the $\phi$ and $\rho$ field components can be derived from the them [36]

A Fourier transformation in the $z$-direction and a Fourier series expansion in the $\phi$-direction are performed to express the field components in terms of its spectral components as follows

\[ f_{j\nu}(\rho, k_z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} dz \vec{F}(\rho, \phi, z) \cdot \hat{j} e^{-i(k_z z + \nu \phi)}. \]  
(B.13)

where $j=\phi, \rho, z$. The inverse transformation is given as

\[ F_j(\rho, \phi, z) = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu \phi} \int_{-\infty}^{\infty} dk_z f_{j\nu}(\rho, k_z) e^{ik_z z}. \]  
(B.14)
In terms of the spectral components, Eq. B.11 and Eq. B.12 reduce to two ordinary differential equations in terms of the radial variable

$$\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d e_{\nu \rho}}{d\rho} \right) + \left( k^2_{\rho} - \frac{\nu^2}{\rho^2} \right) e_{\nu \rho} = i \omega \mu c_{TM} \delta(\rho - \rho_T)$$

(B.15)

$$\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d h_{\nu \rho}}{d\rho} \right) + \left( k^2_{\rho} - \frac{\nu^2}{\rho^2} \right) h_{\nu \rho} = -c_{TE} \frac{d}{d\rho} \left( \delta(\rho - \rho_T) \right)$$

(B.16)

where $k_{\rho} = \sqrt{k^2 - k_z^2}$. The coefficients $c_{TM}$ and $c_{TE}$ are spectral amplitudes of the source given in B.8. Note that a minus sign on the l.h.s of Eq. B.15 is included in $c_{TM}$. To guarantee the radiation condition is satisfied, we choose $\text{Im}(k_{\rho}) > 0$.

The azimuthal and radial components of the fields can be derived from the axial components as follows

$$\vec{E}_s = \frac{1}{k_{\rho}^2} \left[ ik_z \nabla_s E_z - i \omega \mu \hat{z} \times \nabla_s H_z \right]$$

(B.17)

$$\vec{H}_s = \frac{1}{k_{\rho}^2} \left[ ik_z \nabla_s H_z + i \omega \epsilon \hat{z} \times \nabla_s E_z \right]$$

(B.18)

where $\nabla_s = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi}$. In matrix form, we have

$$\begin{bmatrix} e_{\rho \nu} \\ h_{\rho \nu} \end{bmatrix} = \frac{1}{k_{\rho}^2} \begin{bmatrix} -\frac{vk_z}{\rho} & -i \omega \mu \frac{d}{d\rho} \\ -(\sigma - i \omega \epsilon) \frac{1}{\rho} & -\frac{vk_z}{\rho} \end{bmatrix} \begin{bmatrix} e_{\nu \rho} \\ h_{\nu \rho} \end{bmatrix}$$

(B.19)

$$\begin{bmatrix} e_{\phi \nu} \\ h_{\phi \nu} \end{bmatrix} = \frac{1}{k_{\rho}^2} \begin{bmatrix} -ik_z \frac{d}{d\rho} & \frac{\omega \mu}{\rho} \\ (\sigma - i \omega \epsilon) \frac{\mu}{\rho} & ik_z \frac{d}{d\rho} \end{bmatrix} \begin{bmatrix} e_{\nu \rho} \\ h_{\nu \rho} \end{bmatrix}$$

(B.20)

The solutions to Eq. B.15 and Eq. B.16 are

$$\begin{bmatrix} e_{\nu \rho} \\ h_{\nu \rho} \end{bmatrix} = H^{(1)}_{\nu}(k_{\rho} \rho) \tilde{a}_\nu + J_{\nu}(k_{\rho} \rho) \tilde{b}_\nu$$

(B.21)

where $J_{\nu}$ and $H^{(1)}_{\nu}$ are Bessel and Hankel functions of the first kind of order $\nu$. The unknown $\tilde{a}_\nu$ and $\tilde{b}_\nu$ coefficients comprise the amplitude vector and will be determined by applying appropriate source conditions.

The azimuthal component from Eq. B.19 is given by
\[
\begin{bmatrix}
\varphi
\end{bmatrix} = \frac{1}{\sigma} \begin{bmatrix}
-\frac{\nu_k \varphi}{\rho} - \varphi
\end{bmatrix}
\]

\[
\begin{bmatrix}
\psi
\end{bmatrix} = \frac{1}{\sigma} \begin{bmatrix}
-\nu_k \psi - \psi
\end{bmatrix}
\]

\[
\begin{bmatrix}
\varphi
\end{bmatrix} = \frac{1}{\nu_k} \begin{bmatrix}
three \varphi - \varphi
\end{bmatrix}
\]

\[
\begin{bmatrix}
\psi
\end{bmatrix} = \frac{1}{\nu_k} \begin{bmatrix}
three \psi - \psi
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
\varphi
\end{bmatrix} = \frac{1}{\nu_k} \begin{bmatrix}
-\nu_k \varphi - \varphi
\end{bmatrix}
\]

\[
\begin{bmatrix}
\psi
\end{bmatrix} = \frac{1}{\nu_k} \begin{bmatrix}
-\nu_k \psi - \psi
\end{bmatrix}
\]

and \( G_{\nu} \) stands for \( J_{\nu} \) or \( H_{\nu}^{(1)} \). The prime denotes derivative with respect to the argument. In order to find \( \bar{a} \) and \( \bar{b} \), we need to solve the source condition (see Sec. B.9) given by

\[
\begin{bmatrix}
\varphi
\end{bmatrix} = \frac{1}{\nu_k} \begin{bmatrix}
-\nu_k \varphi - \varphi
\end{bmatrix}
\]

\[
\begin{bmatrix}
\psi
\end{bmatrix} = \frac{1}{\nu_k} \begin{bmatrix}
-\nu_k \psi - \psi
\end{bmatrix}
\]

Thus, we can write the solution for the homogeneous medium as

\[
\begin{bmatrix}
\varphi
\end{bmatrix} = \begin{bmatrix}
J_{\nu}(k_{\psi} \rho) \bar{C}_{\nu}^{-} & \rho \leq \rho_T
H_{\nu}^{(1)}(k_{\psi} \rho) \bar{C}_{\nu}^{+} & \rho > \rho_T
\end{bmatrix}
\]

\[
\begin{bmatrix}
\psi
\end{bmatrix} = \begin{bmatrix}
J_{\nu}(k_{\psi} \rho) \bar{C}_{\nu}^{-} & \rho \leq \rho_T
H_{\nu}^{(1)}(k_{\psi} \rho) \bar{C}_{\nu}^{+} & \rho > \rho_T
\end{bmatrix}
\]

where \( \bar{C}_{\nu}^{-} \) and \( \bar{C}_{\nu}^{+} \) are 2x1 column vectors. As a result of the boundary condition at the source given by Eq. B.24 and Eq. B.25, we have

\[
\bar{C}_{\nu}^{+} = \frac{-i\pi}{2} c_{TE} \begin{bmatrix}
\frac{i \omega \mu}{k_{\psi}} J_{\nu}(k_{\psi} \rho_T)
\end{bmatrix}
\]

\[
\bar{C}_{\nu}^{-} = \frac{-i\pi}{2} c_{TE} \begin{bmatrix}
\frac{i \omega \mu}{k_{\psi}} H_{\nu}^{(1)}(k_{\psi} \rho_T)
\end{bmatrix}
\]
B.3 Transimpedance

Here, we compute the voltage induced on the receiver due to the field excited by a unit current source from the transmitter. This is called transimpedance. To compute the transimpedance, we integrate the electric field along the receiver position.

$$Z_{RT} = \frac{V_R}{I_T} = -\frac{1}{I_T} \oint \vec{E} \cdot d\vec{l}_R$$ (B.29)

The $d\vec{l}_R$ is similar to Eq. B.4, however, $d\vec{l}_R$, in this case it refers to the receiver coil antenna(s). $\vec{E}$ is found by inverse transforming Eq. B.26 and Eq. B.27, i.e.,

$$\vec{E} = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu\phi} \int_{-\infty}^{\infty} dk_z e^{ik_z z} \left[ \hat{\phi}e_{\phi\nu} + \hat{z}e_{z\nu} \right]$$ (B.30)

Note that the radial component is ignore, since $d\vec{l}_R$ only involves azimuthal and longitudinal components. The azimuthal component of Eq. B.29 is found by integrating the receiver coil path

$$Z_{RT\phi} = -\frac{1}{I_T} \int_0^{\infty} d\rho \int_{-\pi}^{\pi} \rho d\phi \int_{-\infty}^{\infty} dz \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu\phi} \int_{-\infty}^{\infty} dk_z e^{izk_z} \cdot e_{\phi\nu}(\rho, k_z) \cdot \delta(\rho - \rho_R) \delta(z + \rho_R \tan \theta_R \cos \phi)$$

$$= -\frac{\rho_R}{I_T} \int_{-\infty}^{\infty} dk_z \sum_{\nu=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi e^{i\nu\phi} e^{-ik_z \rho_R \tan \theta_R \cos \phi} e_{\phi\nu}(\rho_R, k_z)$$

using the identity $J_n(x) = \frac{i^{-n}}{2\pi} \int_{-\pi}^{\pi} e^{ix} \cos \phi e^{in\phi} d\phi$ and $J_n(-x) = (-1)^n J_n(x)$, the above can be simplified to

$$Z_{RT\phi} = -\frac{\rho_R}{I_T} \int_{-\infty}^{\infty} dk_z \sum_{\nu=-\infty}^{\infty} i^{-\nu} J_\nu(k_z \rho_R \tan \theta_R) e_{\phi\nu}(\rho_R, k_z)$$ (B.31)

The axial component is written as

$$Z_{RTz} = -\frac{1}{I_T} \int_0^{\infty} d\rho \int_{-\pi}^{\pi} \rho d\phi \int_{-\infty}^{\infty} dz \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu\phi} \int_{-\infty}^{\infty} dk_z e^{ik_z z} \cdot e_{z\nu}(\rho, k_z) \tan \theta_R \sin \phi \cdot \delta(\rho - \rho_R) \delta(z + \rho_R \tan \theta_R \cos \phi)$$

$$= -\frac{\rho_R}{I_T} \int_{-\infty}^{\infty} dk_z \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e_{z\nu}(\rho_R, k_z) \int_{-\pi}^{\pi} d\phi e^{i\nu\phi} e^{-ik_z \rho_R \tan \theta_R \cos \phi} \tan \theta_R \sin \phi.$$
Using integration by parts for the inner integral \( \Omega = \int_{-\pi}^{\pi} d\phi e^{-ik_z \rho_R \tan \theta_R \cos \phi} \tan \theta_R \sin \phi e^{i\nu \phi} \)

with \( u = \frac{1}{ik_z \rho_R} e^{-ik_z \rho_R \tan \theta_R \cos \phi} \) and \( v = e^{i\nu \phi} \), we have

\[
\Omega = \frac{e^{i\nu \phi}}{ik_z \rho_R} e^{-ik_z \rho_R \tan \theta_R \cos \phi} \bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} d\phi \frac{\nu}{k_z \rho_R} e^{i\nu \phi} e^{-ik_z \rho_R \tan \theta_R \cos \phi}. \tag{B.32}
\]

Since the first term of the above is zero, the axial component reduces to

\[
Z_{RT} = -\frac{\rho_R}{I_T} \int_{-\infty}^{\infty} dk_z \frac{1}{2\pi} \sum_{\nu = -\infty}^{\infty} e_{z\nu}(\rho_R, k_z) \int_{-\pi}^{\pi} d\phi \frac{-\nu}{k_z \rho_R} e^{i\nu \phi} e^{-ik_z \rho_R \tan \theta_R \cos \phi} \tag{B.33}
\]

This can also be written as

\[
Z_{RT} = \int_{-\infty}^{\infty} dk_z w(k_z) e^{ik_z (z_R - z_T)} e^{i\nu (\phi_R - \phi_T)} \tag{B.34}
\]

where

\[
w(k_z) = -\frac{\rho_R}{I_T} \sum_{\nu = -\infty}^{\infty} (d_{TE}(\nu, k_z) e_{\nu R}(\rho_R, k_z) + d_{TM}(\nu, k_z) e_{z\nu}(\rho_R, k_z)) \tag{B.35}
\]

and

\[
d_{TE} = i^{-\nu} J_\nu(\kappa_R),
\]

\[
d_{TM} = -\frac{\nu i^{-\nu}}{k_z \rho_R} J_\nu(\kappa_R), \tag{B.36}
\]

\[
\kappa_R = k_z \rho_R \tan \theta_R.
\]
B.4 Radiation Integral

The field radiated by a loop antenna in a homogeneous medium can also be easily found using the radiation integral [38], [39], [59]. In this case, we have

\[
\vec{E}(\vec{r}) = -i\omega \vec{A} = -i\omega \mu \frac{\int\int\int_{V'} J(\vec{r}') e^{ikR} dV'}{R} \tag{B.37}
\]

where \(\vec{r}'\) denotes the source location and \(R = |\vec{r} - \vec{r}'|\). The voltage induced on the receiver coil is given by

\[
V = -\oint_{L_R} \vec{E} \cdot d\vec{l}_R = -i\omega \oint_{L_R} \vec{A} \cdot d\vec{l}_R. \tag{B.38}
\]

The source location is given by Eq. 4.6, and the receiver location can be expressed in the similar fashion. As a result, the transimpedance is given by

\[
Z_{RT}^h = -i\omega \frac{\mu \rho_T \rho_R}{4\pi} \int_{-\pi}^{\pi} d\phi' \int_{-\pi}^{\pi} \frac{d\phi}{2} \bar{u}_T \cdot \bar{u}_R e^{ikR} \tag{B.39}
\]

where

\[
\bar{u}_T \cdot \bar{u}_R = \cos(\phi - \phi') + \tan \theta_T \tan \theta_R \sin(\phi - \phi_T) \sin(\phi' - \phi_R), \tag{B.40}
\]

and

\[
R = \sqrt{\rho_T^2 + \rho_R^2 - 2\rho_T \rho_R \cos(\phi - \phi') + ((z_R + \zeta_R(\phi)) - (z_T + \zeta_T(\phi')))^2}. \tag{B.41}
\]
B.5 Solution Incorporating Multicylindrical Layers

A recursive algorithm to incorporate generalized reflection and transmission coefficients from multicylindrical layers is presented in [33]. For completeness, we will review it here. The solution for the $z$-components of the fields in layer $i$ is of the form

\[
\begin{bmatrix}
  e_z \\
  h_{z} \\
\end{bmatrix} = \begin{cases} 
  [H^{(1)}(k_p, \rho) \bar{M}_i + J(k_p, \rho) \bar{b}_i] & \rho < \rho_T \\
  [H^{(1)}(k_p, \rho) + J(k_p, \rho) \bar{N}_i] \bar{a}_i & \rho > \rho_T 
\end{cases} \quad (B.42)
\]
where the $\bar{M}_i$ and $\bar{N}_i$ are reflection matrices of the inward and outward interfaces, respectively. In this section, we omit subscript $\nu$, for simplicity, but keep in mind that all matrices and vectors here have $\nu$ dependence. The reflection and transmission matrices can be derived recursively to incorporate multiple reflections due to the multiple layers at the inner regions and the outer regions.

The coefficients $\bar{a}_i$ and $\bar{b}_i$ for each layer can be derived from $\bar{M}_i$ and $\bar{N}_i$ and by incorporating the source condition at the source point.

The local reflection and transmission matrix coefficients are by considering the following four types of reflection/transmission coefficients [36] as:

1. $\bar{\Gamma}_i^+$ is the matrix coefficient of the reflection at interface $R_i$ of a standing wave in medium $i + 1$ into an outgoing wave in medium $i + 1$.

2. $\bar{\Gamma}_i^-$ is the matrix coefficient of the reflection at interface $R_i$ of an outgoing wave in medium $i$ into a standing wave in medium $i$.

3. $\bar{T}_i^+$ is the matrix coefficient of the transmission at interface $R_i$ of an outgoing wave in medium $i$ into an outgoing wave in medium $i + 1$.

4. $\bar{T}_i^-$ is the matrix coefficient of the transmission at interface $R_i$ of a standing wave in medium $i + 1$ into a standing wave in medium $i$.

The $\bar{T}$ and $\bar{\Gamma}$ matrices only depend on the media and the geometry, and not on the location or type of source. They describe locally the physics of reflection and transmission of the waves at a specific boundary. These interactions can be divided in two cases:

Case (1): A standing wave of order $\nu$, of arbitrary “amplitude” $\bar{a}$ is incident from medium $i + 1$ onto boundary at $R_i$. An outgoing wave is reflected and a standing
wave is transmitted. Continuity of the \( z \)-component of the field gives

\[
\left[ J(k_{\rho i+1} R_i) \bar{\mathbf{I}} + H(k_{\rho i+1}) \bar{\Gamma}_i^z \right] \bar{a} = J(k_{\rho i} R_i) \bar{T}_i^z \bar{a}, \tag{B.43}
\]

while continuity of the \( \phi \)-component gives

\[
\left[ \bar{J}(k_{\rho i+1} R_i) \bar{\mathbf{I}} + \bar{H}(k_{\rho i+1}) \bar{\Gamma}_i^\phi \right] \bar{a} = \bar{J}(k_{\rho i} R_i) \bar{T}_i^\phi \bar{a}. \tag{B.44}
\]

Solving the above two equations, we have

\[
\bar{\Gamma}_i^- = A_i^{-1} \left[ J(k_{\rho i} R_i) \bar{J}(k_{\rho i+1} R_i) - \bar{J}(k_{\rho i} R_i) J(k_{\rho i+1} R_i) \right], \tag{B.45}
\]

and

\[
\bar{T}_i^- = \frac{2\omega}{\pi k^2_{\rho i+1} R_i} A_i^{-1} \begin{bmatrix} 0 & -\mu \\ \epsilon_{i+1} & 0 \end{bmatrix}, \tag{B.46}
\]

where

\[
\bar{A}_i = \left[ H(k_{\rho i+1} R_i) \bar{J}(k_{\rho i} R_i) - J(k_{\rho i} R_i) \bar{H}(k_{\rho i+1} R_i) \right]. \tag{B.47}
\]

Case (2): An outgoing wave of order \( \nu \) is incident from medium \( i \) onto the boundary at \( R_i \), a standing wave is reflected, and an outgoing wave is transmitted. The continuity of tangential fields gives

\[
\left[ H(k_{\rho i} R_i) \bar{\mathbf{I}} + J(k_{\rho i} R_i) \bar{\Gamma}_i^t \right] \bar{a} = H(k_{\rho i+1} R_i) \bar{T}_i^t \bar{a}, \tag{B.48}
\]

and

\[
\left[ \bar{H}(k_{\rho i} R_i) \bar{\mathbf{I}} + \bar{J}(k_{\rho i} R_i) \bar{\Gamma}_i^t \right] \bar{a} = \bar{H}(k_{\rho i+1} R_i) \bar{T}_i^t \bar{a} \tag{B.49}
\]

Solving the above two equations, we have

\[
\bar{\Gamma}_i^t = \bar{A}_i^{-1} \left[ H(k_{\rho i} R_i) \bar{H}(k_{\rho i+1} R_i) - \bar{H}(k_{\rho i} R_i) H(k_{\rho i+1} R_i) \right], \tag{B.50}
\]

and

\[
\bar{T}_i^t = \frac{2\omega}{\pi k^2_{\rho i} R_i} \bar{A}_i^{-1} \begin{bmatrix} 0 & -\mu \\ \epsilon_i & 0 \end{bmatrix}. \tag{B.51}
\]
Using these matrix coefficients, we can determine matrices $\tilde{M}_i$ and $\tilde{N}_i$. When $\rho < R$, a standing wave in region $l$, $l \leq 0$, is in general due to reflection of an outgoing wave in region $l$ and the transmission of a standing wave in region $l + 1$, i.e.,

$$\tilde{b}_l = \tilde{\Gamma}_l^+ \tilde{M}_l \tilde{b}_l + \tilde{T}_l^+ \tilde{b}_{l+1}.$$  \hspace{1cm} (B.52)

On the other hand, an outgoing wave in region $l + 1$ is in general due to the transmission of an outgoing wave from region $l$ and the reflection of a standing wave from region $l + 1$, i.e.,

$$\tilde{M}_{l+1} \tilde{b}_l = \tilde{T}_l^+ \tilde{M}_l \tilde{b}_l + \tilde{\Gamma}_l^- \tilde{b}_{l+1},$$  \hspace{1cm} (B.53)

so that

$$\tilde{M}_{l+1} = \tilde{\Gamma}_l^- + \tilde{T}_l^+ \tilde{M}_l \left[ \tilde{I} - \tilde{T}_l^+ \tilde{M}_l \right]^{-1} \tilde{T}_l^- .$$  \hspace{1cm} (B.54)

This gives a recursive algorithm to determine all $\tilde{M}_i$, for $l \leq 0$, the recursion beginning with the regularity condition $\tilde{M}_{-m} = \tilde{0}$ for the innermost layer $m$, unless the innermost layer is a PEC cylinder where in this case, $\tilde{M}_{-m}$ is given by Eq. B.69.

A similar condition exist for the $\tilde{N}_i$. An outgoing wave in region $k + 1$ is in general due to the reflection of the standing wave in region $i + 1$ and the transmission of the outgoing wave in region $k$, i.e.,

$$\tilde{a}_{k+1} = \tilde{\Gamma}_k^- \tilde{N}_{k+1} \tilde{a}_{k+1} + \tilde{T}_k^+ \tilde{a}_k.$$  \hspace{1cm} (B.55)

Moreover, a standing wave in region $k$ is in general due to a transmission of a standing wave from region $k + 1$ and a reflection of the outgoing wave at region $k$, i.e.,

$$\tilde{N}_k \tilde{a}_k = \tilde{T}_k^- \tilde{N}_{k+1} \tilde{a}_{k+1} + \tilde{\Gamma}_k^+ \tilde{a}_k,$$  \hspace{1cm} (B.56)

and we have the recursive relation as

$$\tilde{N}_k = \tilde{\Gamma}_k^+ + \tilde{T}_k^- \tilde{N}_{k+1} \left[ \tilde{I} - \tilde{T}_k^- \tilde{N}_{k+1} \right]^{-1} \tilde{T}_k^+ .$$  \hspace{1cm} (B.57)
This allow us to determine $\bar{N}_k$ for all $k \leq 0$ recursively, since we know that $\bar{N}_n = 0$.

After obtaining the matrix coefficients recursively, the amplitude of $\bar{a}_0$ and $\bar{b}_0$ can be determined by matching the source condition at the ring source. Once $\bar{a}_0$, $\bar{b}_0$, $\bar{M}_0$ and $\bar{N}_0$, are determined the fields everywhere can be determined. The $z$-component of the fields is given by

$$
\begin{bmatrix}
  e_z \\
  h_z
\end{bmatrix} = \begin{cases} 
  [H^{(1)}(k_\rho \rho) \bar{M}_0 + J(k_\rho \rho)] \bar{b}_0 & \rho < \rho_T \\
  [H^{(1)}(k_\rho \rho) + J(k_\rho \rho) \bar{N}_0] \bar{a}_0 & \rho > \rho_T.
\end{cases}
$$

(B.58)

The electric current ring source located at $\rho = \rho_T$ in a homogeneous medium is expressed as Eq. B.26. Matching the fields at $\rho = \rho_T^+$ and $\rho = \rho_T^-$, we have the following relation.

$$
\begin{bmatrix}
  \bar{N}_0 \\
  \bar{I}_0
\end{bmatrix} = \begin{bmatrix}
  \bar{a}_0 \\
  \bar{b}_0
\end{bmatrix} = \begin{bmatrix}
  -\bar{C}^- \\
  \bar{C}^+
\end{bmatrix}
$$

(B.59)

The value of $\bar{a}_i$ and $\bar{b}_i$ can be determined using Eq. B.55 and Eq. B.52. The $\phi$-component of the fields can be derived accordingly.
B.6 Plane Wave Reflection from a PEC Cylinder

Plane wave reflection from a PEC cylinder is a classic problem can be found in [34], [60]. We include a short derivation here for completeness.

![Diagram of a plane wave incident upon the PEC cylinder.](image)

Figure B.3: A plane wave incident upon the PEC cylinder.

B.6.1 Reflection and Transmission coefficient matrix

To derive the TM\textsubscript{z} mode plane wave reflection coefficient for the cylinder as in Fig. B.3, we assume an incident plane wave of the form,

\[ E_{z}^{\text{inc}} = E_{o}e^{-ikx} = E_{o} \sum_{n=-\infty}^{\infty} i^{-n} J_{n}(k \rho)e^{in\phi}. \]  

(B.60)

The total field everywhere outside the cylinder is

\[ E_{z}^{\text{tot}} = E_{z}^{\text{inc}} + E_{z}^{\text{scat}}, \]  

(B.61)
where the scattered electric field has the following form

\[ E_{scat}^z = E_o \sum_{n=-\infty}^{\infty} i^{-n} a_n H_n^{(1)}(k\rho)e^{in\phi}. \]  

(B.62)

Hence the total field at the surface of the cylinder \( \rho = a \) is given by

\[ E_{tot}^z = E_o \sum_{n=-\infty}^{\infty} i^{-n} [J_n(ka) + a_n H_n^{(1)}(ka)] e^{in\phi}. \]  

(B.63)

Since the cylinder is a PEC, enforcing the boundary condition of \( E_{tot}^z = 0 \) gives

\[ a_n = -\frac{J_n(ka)}{H_n^{(1)}(ka)}. \]  

(B.64)

For the TE\(_z\) mode, the incident magnetic field is given by

\[ H_{inc}^z = H_o e^{-ikx} = H_o \sum_{n=-\infty}^{\infty} i^{-n} J_n(k\rho)e^{in\phi}. \]  

(B.65)

and the total magnetic field outside the cylinder is

\[ H_{tot}^z = H_{inc}^z + H_{scat}^z \]

\[ = H_o \sum_{n=-\infty}^{\infty} i^{-n} [J_n(k\rho) + b_n H_n^{(1)}(k\rho)] e^{in\phi}. \]  

(B.66)

The boundary condition on the PEC cylinder, in this case, requires \( E_{tot}^\phi = 0 \) at \( \rho = a \).

Thus, we derive the \( E_{tot}^\phi \) from \( H_{tot}^z \),

\[ E_{tot}^\phi = \frac{1}{i\omega\epsilon} \left( \nabla \times H_{tot}^z \right)_\phi 
\]

\[ = -\frac{1}{i\omega\epsilon} \frac{\partial H_z}{\partial \rho} \]  

(B.67)

\[ = -\frac{k}{i\omega\epsilon} H_o \sum_{n=-\infty}^{\infty} i^{-n} \left[ J'_n(k\rho) + b_n H_n^{(1)}(k\rho) \right] e^{in\phi}. \]

So that, \( E_{tot}^\phi = 0 \) at \( \rho = a \) implies

\[ b_n = -\frac{J'_n(ka)}{H_n^{(1)}(ka)}. \]  

(B.68)

Now we can put them in a matrix form consistent with the formulation in Appendix B.5,

\[ \tilde{R}_{0,-1}^+ = \begin{bmatrix} -\frac{J_n(k_0a)}{H_n^{(1)}(k_0a)} & 0 \\ 0 & -\frac{J'_n(k_0a)}{H_n^{(1)}(k_0a)} \end{bmatrix} \]  

(B.69)
B.6.2 Solution Incorporating Steel Mandrel modelled as PEC

Well-logging tools include a long steel mandrel. The tilted-coil antenna is mount around the mandrel as illustrated in Fig. B.4. The solution in the homogeneous formation is given by

\[
\begin{bmatrix}
  e_{z \nu} \\
  h_{z \nu}
\end{bmatrix} = \begin{cases}
  \left( J_{\nu}(k_{\rho}\rho)I + H_{\nu}^{(1)}(k_{\rho}\rho)\bar{R}_{21} \right) \bar{C}^{-}_{\nu} & a < \rho \leq \rho_T \\
  H_{\nu}^{(1)}(k_{\rho}\rho) \left( \bar{C}^{+}_{\nu} + \bar{R}_{21} \bar{C}^{-}_{\nu} \right) & \rho > \rho_T
\end{cases} \tag{B.70}
\]

\[
\begin{bmatrix}
  e_{\phi \nu} \\
  h_{\phi \nu}
\end{bmatrix} = \begin{cases}
  \left( J_{\nu}(k_{\rho}\rho)\bar{I} + \bar{H}_{\nu}^{(1)}(k_{\rho}\rho)\bar{R}_{21} \right) \bar{C}^{-}_{\nu} & a < \rho \leq \rho_T \\
  \bar{H}_{\nu}^{(1)}(k_{\rho}\rho) \left( \bar{C}^{+}_{\nu} + \bar{R}_{21} \bar{C}^{-}_{\nu} \right) & \rho > \rho_T
\end{cases} \tag{B.71}
\]
B.7 Graf’s Addition Theorem

Here, we review the Graf’s addition theorem used in eccentric borehole problem.

\[ \Psi_{\nu}(w)e^{i\nu\chi} = \sum_{k=-\infty}^{\infty} \Psi_{\nu+k}(u)J_k(v)e^{ika} \]  

(B.72)

where \( \Psi \) represent either a Bessel function or a Hankel function and the other parameters are as indicated in Fig. B.5. The above equation assumes \( \phi_E=0 \) as shown.

Figure B.5: The above triangle denotes the relation between the origin (0), source \((\rho')\) and observation point \((\rho)\) for the Graf’s Addition Theorem.
in Fig. B.5. Letting $l = \nu + k$, we have

$$
\Psi_\nu(w)e^{i\nu(\beta + \phi_E)} = \sum_{l=-\infty}^{\infty} \Psi_l(u)J_{l-\nu}(v)e^{il(\alpha + \phi_E)} \tag{B.73}
$$

or

$$
\Psi_{\nu'}(\rho')e^{i\nu'\phi'} = \sum_{\nu=-\infty}^{\infty} \Psi_{\nu}(\rho)e^{i\nu\phi} J_{\nu-\nu'}(d)e^{-i(\nu'-\nu)\phi_E} \tag{B.74}
$$
B.8 Spectral Components of the Source, \( c_{TM} \) and \( c_{TE} \)

We derive the spectral component of the source. For simplicity, we assume both \( z_T \) and \( \phi_T \) are zero, without loss of generality. We define

\[
c_{TM} = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} dz \hat{J}_T \cdot \hat{z} e^{-i(k_z z + \nu \phi)}
\]

\[
= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} dz \delta(z + \rho_T \tan \theta_T \cos \phi) u_T \cdot \hat{z} e^{-i(k_z z + \nu \phi)}
\]

\[
= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi}{\sin \phi} \sin \phi e^{i(k_z \rho_T \tan \theta_T \cos \phi - \nu \phi)}
\]

\[
= \frac{i}{2\pi} \int_{-\pi}^{\pi} d\phi \tan \theta_T \sin \phi \sin(\nu \phi) e^{i k_z \rho_T \tan \theta_T \cos \phi}
\]

Using integration by parts \( \int vdu = uv - \int u dv \) with \( du = \tan \theta_T \sin \phi e^{i k_z \rho_T \tan \theta_T \cos \phi}, u = -\frac{1}{ik_z \rho_T} e^{i k_z \rho_T \tan \theta_T \cos \phi}, v = \sin(\nu \phi) \) and \( dv = i \nu \cos(\nu \phi) \), we have

\[
c_{TM} = -\frac{1}{2\pi k_z \rho_T} \sin(\nu \phi) e^{i k_z \rho_T \tan \theta_T \cos \phi} \bigg|_{-\pi}^{\pi} + \frac{\nu}{2\pi k_z \rho_T} \int_{-\pi}^{\pi} d\phi \cos(\nu \phi) e^{i k_z \rho_T \tan \theta_T \cos \phi}
\]

\[
= \frac{\nu}{\pi k_z \rho_T} \int_{0}^{\pi} d\phi \cos(\nu \phi) e^{i k_z \rho_T \tan \theta_T \cos \phi}.
\]

Using the identity \( J_n(x) = \frac{i^{-n}}{\pi} \int_{0}^{\pi} \cos(n \phi) e^{ix \cos \phi} d\phi \), this becomes

\[
c_{TM} = \frac{\nu}{k_z \rho_T} i^{\nu} J_{\nu}(k_z \rho_T \tan \theta_T) \quad \text{(B.75)}
\]

Similarly, we can define

\[
c_{TE} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} dz \hat{J}_T \cdot \hat{\phi} e^{-i(k_z z + \nu \phi)}
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} dz \delta(z + \rho_T \tan \theta_T \cos(\phi)) e^{-i(k_z z + \nu \phi)}
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi}{\sin \phi} e^{i(k_z \rho_T \tan \theta_T \cos \phi - \nu \phi)}
\]

\[
= \frac{1}{\pi} \int_{0}^{\pi} d\phi e^{i k_z \rho_T \tan \theta_T \cos \phi} \cos(\nu \phi)
\]

\[
c_{TE} = i^{\nu} J_{\nu}(k_z \rho_T \tan \theta_T) \quad \text{(B.76)}
\]
If, $z_T$ and $\phi_T$ are not zero, we can simply make the substitution $c_{TM/TE} \rightarrow c_{TM/TE} e^{i(k_z z_T + \nu \phi_T)}$. 
B.9 Source Condition

We derive the source condition to determine the jump conditions [62] for the fields at the (ring) source location. The source condition can be derived from Eq. B.15 and Eq. B.16. From convenience, they are rewritten here

\[
\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d e_{zv}}{d\rho} \right) + \left( k^2 - \frac{\nu^2}{\rho^2} \right) e_{zv} = i \omega \mu \rho c_{TM} \delta (\rho - \rho_T), \quad (B.77)
\]

\[
\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d h_{zv}}{d\rho} \right) + \left( k^2 - \frac{\nu^2}{\rho^2} \right) h_{zv} = -c_{TE} \frac{d}{d\rho} (\delta (\rho - \rho_T)). \quad (B.78)
\]

By integrating Eq. B.77 from \( \rho_T^- \) to \( \rho_T^+ \), we have

\[
\int_{\rho_T^-}^{\rho_T^+} \frac{d}{d\rho} e_{zv} \bigg|_{\rho_T^-}^{\rho_T^+} + \int_{\rho_T^-}^{\rho_T^+} \rho \left( k^2 - \frac{\nu^2}{\rho^2} \right) e_{zv} = i \omega \mu \rho c_{TM},
\]

where the second term would be zero because the integrand is bounded. Thus,

\[
\rho \frac{d}{d\rho} e_{zv} \bigg|_{\rho_T^-}^{\rho_T^+} = i \omega \mu \rho c_{TM}
\]

(B.79)

We can also integrate Eq. B.77 one more, leading to

\[
\Rightarrow \int_{\rho_T^-}^{\rho_T^+} \frac{d}{d\rho} e_{zv} + \int_{\rho_T^-}^{\rho_T^+} C = i \omega \mu \int_{\rho_T^-}^{\rho_T^+} \rho c_{TM}
\]

\[
\Rightarrow e_{zv} \bigg|_{\rho_T^-}^{\rho_T^+} = 0
\]

(B.80)

Using a similar procedure for Eq. B.78

\[
\int_{\rho_T^-}^{\rho_T^+} \frac{d}{d\rho} h_{zv} + \int_{\rho_T^-}^{\rho_T^+} \rho \left( k^2 - \frac{\nu^2}{\rho^2} \right) h_{zv} = -c_{TE} - c_{TE} \int_{\rho_T^-}^{\rho_T^+} \frac{d}{d\rho} \delta (\rho - \rho_T)
\]

where, in the sense of distributions,

\[
\int_{\rho_T^-}^{\rho_T^+} \frac{d}{d\rho} \delta (\rho - \rho_T) d\rho = \rho \delta (\rho - \rho_T) - \int_{\rho_T^-}^{\rho_T^+} \delta (\rho - \rho_T) d\rho
\]
we have
\[
\rho \frac{d}{d\rho} h_{z\nu} \bigg|_{\rho_T^+} - c_{TE} - c_{TE} \rho \delta(\rho - \rho_T) \bigg|_{\rho_T^+} + c_{TE}
\]
(B.81)
\[
\frac{d}{d\rho} h_{z\nu} \bigg|_{\rho_T^-} = 0.
\]
Similarly, by integrating it twice we obtain
\[
h_{z\nu} \bigg|_{\rho_T^+} = -c_{TE} \quad (B.82)
\]
From the above four conditions, we can derive transverse component conditions using Eq. B.23. In this case,
\[
e_{\phi\nu} \bigg|_{\rho_T^+} \neq - \frac{\nu k_z}{k^2 \rho} e_{z\nu} \bigg|_{\rho_T^-} - \frac{i \omega \mu}{k^2 \rho} h_{z\nu} \bigg|_{\rho_T^-} + c_{TM}
\]
(B.83)
and,
\[
h_{\phi\nu} \bigg|_{\rho_T^+} = \frac{-i (\sigma - i \omega)}{k^2 \rho} \frac{d}{d\rho} e_{z\nu} \bigg|_{\rho_T^-} - \frac{\nu k_z}{k^2 \rho} h_{z\nu} \bigg|_{\rho_T^-} + c_{TM}
\]
(B.84)
So that, we arrive to the source condition given in matrix form as
\[
\begin{bmatrix} e_{z\nu} \\ h_{z\nu} \end{bmatrix} \bigg|_{\rho_T^+} - \begin{bmatrix} e_{z\nu} \\ h_{z\nu} \end{bmatrix} \bigg|_{\rho_T^-} = -c_{TE} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (B.85)
\]
\[
\begin{bmatrix} e_{\phi\nu} \\ h_{\phi\nu} \end{bmatrix} \bigg|_{\rho_T^+} - \begin{bmatrix} e_{\phi\nu} \\ h_{\phi\nu} \end{bmatrix} \bigg|_{\rho_T^-} = -c_{TM} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (B.86)
\]
APPENDIX C

DETAILS ON THE NUMERICAL MODE MATCHING IMPLEMENTATION

C.1 Transverse Field Component

Here, we find the expression of the transverse component in terms of longitudinal component. In the uniaxial-anisotropic medium, we decompose Maxwell equation as

\[
(\nabla + \hat{z} \frac{\partial}{\partial z}) \times (\vec{E}_s + \vec{E}_z) = i\omega (\mu_h \vec{H}_s + \mu_v \vec{H}_z) \quad \text{(C.1)}
\]

\[
(\nabla + \hat{z} \frac{\partial}{\partial z}) \times (\vec{H}_s + \vec{H}_z) = -i\omega (\epsilon_h \vec{E}_s + \epsilon_v \vec{E}_z) \quad \text{(C.2)}
\]

where subscript \( s/h \) and \( z/v \) for the field/constitutive properties stands for transverse and longitudinal components, respectively. Here, the constitutive properties are function of \( z \). Matching each component of the above equations, we have

\[
i\omega \mu_h \vec{H}_s = \nabla \times \vec{E}_z + \hat{z} \times \frac{\partial}{\partial z} \vec{E}_s, \quad \text{(C.3)}
\]

\[
-i\omega \epsilon_h \vec{E}_s = \nabla \times \vec{H}_z + \hat{z} \times \frac{\partial}{\partial z} \vec{H}_s, \quad \text{(C.4)}
\]

\[
i\omega \mu_v \vec{H}_z = \nabla \times \vec{H}_s, \quad \text{(C.5)}
\]

\[
-i\omega \epsilon_v \vec{E}_z = \nabla \times \vec{H}_s, \quad \text{(C.6)}
\]

and using the identities \( \hat{z} \times (\nabla \times \vec{E}_z) = \nabla \times \vec{E}_z \) and \( \hat{z} \times (\hat{z} \times \vec{E}_s) = -\vec{E}_s \), we have

\[
i\omega \mu_h (\hat{z} \times \vec{H}_s) = \nabla \times \vec{E}_z - \hat{z} \frac{\partial}{\partial z} \vec{E}_s \quad \text{(C.7)}
\]
\[-i \omega_{eh} (\hat{z} \times \vec{E}_s) = \nabla_s \vec{H}_z - \frac{\partial}{\partial z} \vec{H}_s \]  \hspace{1cm} (C.8)

If we substitute Eq. C.7 into Eq. C.8,

\[-i \omega_{eh} (-\vec{E}_s) = \hat{z} \times (\nabla_s \vec{H}_s) - \frac{\partial}{\partial z} \left( \frac{1}{i \omega \mu_h} \nabla_s \vec{E}_z - \frac{1}{i \omega \mu_h} \frac{\partial}{\partial z} \vec{E}_s \right)\]

\[\frac{\partial}{\partial z} \frac{1}{\mu_h} \frac{\partial}{\partial z} \vec{E}_s + \omega^2 \epsilon_h \vec{E}_s = \frac{\partial}{\partial z} \frac{1}{\mu_h} \nabla_s \vec{E}_z - i \omega \hat{z} \times \nabla_s \vec{H}_z \]  \hspace{1cm} (C.9)

is obtained. Otherwise, if we substitute, Eq. C.8 into Eq. C.7, we have

\[i \omega_{eh} (-\vec{H}_s) = \hat{z} \times (\nabla_s \vec{E}_z) - \frac{\partial}{\partial z} \left( \frac{1}{-i \omega \epsilon_h} \nabla_s \vec{H}_z - \frac{1}{-i \omega \epsilon_h} \frac{\partial}{\partial z} \vec{H}_s \right)\]

\[\frac{\partial}{\partial z} \frac{1}{\epsilon_h} \frac{\partial}{\partial z} \vec{H}_s + \omega^2 \mu_h \vec{H}_s = \frac{\partial}{\partial z} \frac{1}{\epsilon_h} \nabla_s \vec{H}_z + i \omega \hat{z} \times \nabla_s \vec{E}_z. \]  \hspace{1cm} (C.10)

In cylindrical coordinates, \( \nabla_s = \hat{\rho} \frac{\partial}{\partial \rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial}{\partial \phi} \), and the \( \phi \)-component of the field is given as (we assume a field with \( e^{i \nu \phi} \) dependency)

\[\left( \frac{\partial}{\partial z} \frac{1}{\mu_h} \frac{\partial}{\partial z} + \omega^2 \epsilon_h \right) E_{\phi} = \frac{i \nu}{\rho} \frac{\partial}{\partial z} \frac{1}{\mu_h \epsilon_v} D_z - \frac{i \omega}{\mu_v} \frac{1}{\rho} \frac{\partial}{\partial \rho} B_z \]  \hspace{1cm} (C.11)

\[\left( \frac{\partial}{\partial z} \frac{1}{\epsilon_h} \frac{\partial}{\partial z} + \omega^2 \mu_h \right) H_{\phi} = \frac{i \omega}{\epsilon_v} \frac{1}{\rho} \frac{\partial}{\partial \rho} D_z + \frac{i \nu}{\rho} \frac{1}{\partial z} B_z \]  \hspace{1cm} (C.12)
C.2 Matrix Representation of the Transverse Components

To derive the matrix representation for the transverse field components, we rewrite Eq. 5.34,

\[
\begin{bmatrix}
  e^\phi \\
  h^\phi 
\end{bmatrix}
= \begin{bmatrix}
  L_\mu^{-1} \\
  0 
\end{bmatrix}
\cdot
\begin{bmatrix}
  \frac{i\omega \frac{\partial}{\partial \rho} \frac{1}{\mu^*_v}}{i\omega} & -i\omega \frac{1}{\mu^*_v} \\
  \frac{i\omega \frac{\partial}{\partial \rho} \frac{1}{\mu^*_v}}{i\omega} & 0 
\end{bmatrix}
\cdot
\bar{S}'(z) \cdot \bar{a} \cdot \bar{C}'^\pm(\rho) \cdot \bar{c}^\pm
\]  
(C.13)

where \( L_\mu = \left( \frac{d}{dz} \mu^*_h \frac{d}{dz} + \omega^2 \epsilon^*_h \right) \) and \( L_\epsilon = \left( \frac{d}{dz} \epsilon^*_h \frac{d}{dz} + \omega^2 \mu^*_h \right) \). For the off diagonal elements \((M_{12}, M_{12})\) the operator \( \frac{\partial}{\partial \rho} \) can be incorporated into propagation term \( \bar{C}'(\rho) \), and expressed as

\[
M_{12} = -i\omega L_\mu^{-1} \cdot \frac{1}{\mu^*_v} \bar{S}'(z) \cdot \bar{a}_\mu \cdot \bar{C}'^\pm(\rho) \cdot \bar{K}_\mu \cdot \bar{c}_\mu^\pm, \quad (C.14)
\]

and

\[
M_{21} = i\omega L_\epsilon^{-1} \cdot \frac{1}{\epsilon^*_v} \bar{S}'(z) \cdot \bar{a}_\epsilon \cdot \bar{C}'^\pm(\rho) \cdot \bar{K}_\epsilon \cdot \bar{c}_\epsilon^\pm, \quad (C.15)
\]

where \( \bar{K} \) is a diagonal matrix containing the square root of the eigenvalues. The superscript ‘ above means derivatives with respect to the argument. Now, by projecting the operators as \( \mathcal{L}\bar{S}'(z) \approx \bar{S}'(z) \langle \bar{S}(z), \mathcal{L}\bar{S}'(z) \rangle \), the off diagonal elements (operators) can be rewritten as

\[
M_{12} = -i\omega \bar{L}_\mu^{-1} \cdot \bar{p}_\mu \cdot \bar{a}_\mu \cdot \bar{C}'^\pm(\rho) \cdot \bar{K}_\mu \cdot \bar{c}_\mu^\pm, \quad (C.16)
\]

\[
M_{21} = i\omega \bar{L}_\epsilon^{-1} \cdot \bar{p}_\epsilon \cdot \bar{a}_\epsilon \cdot \bar{C}'^\pm(\rho) \cdot \bar{K}_\epsilon \cdot \bar{c}_\epsilon^\pm, \quad (C.17)
\]

where

\[
\bar{L}_\epsilon = \langle \bar{S}_\epsilon, \left( \frac{d}{dz} \epsilon^*_h \frac{d}{dz} + \omega^2 \mu^*_h \right) \bar{S}' \rangle, \quad \bar{p}_\epsilon = \langle \bar{S}, \frac{1}{\epsilon^*_v} \bar{S}' \rangle,
\]

\[
\bar{L}_\mu = \langle \bar{S}_\mu, \left( \frac{d}{dz} \mu^*_h \frac{d}{dz} + \omega^2 \epsilon^*_h \right) \bar{S}' \rangle, \quad \bar{p}_\mu = \langle \bar{S}, \frac{1}{\mu^*_v} \bar{S}' \rangle.
\]
Similarly, the diagonal terms can be written as

\begin{align}
M_{11} &= \frac{i\nu}{\rho} \bar{S}^t \cdot \bar{L}_\mu \cdot \bar{a}_\epsilon \cdot \bar{C}_\epsilon^\pm (\rho) \cdot \bar{c}_\epsilon^\pm \\
M_{22} &= \frac{i\nu}{\rho} \bar{S}^t \cdot \bar{L}_\epsilon \cdot \bar{a}_\mu \cdot \bar{C}_\mu^\pm (\rho) \cdot \bar{c}_\mu^\pm
\end{align}

where

\begin{align}
\bar{D}_\mu &= \left\langle \bar{S}, \frac{1}{dz} \mu^*_\epsilon \nu^* \right\rangle, \\
\bar{D}_\epsilon &= \left\langle \bar{S}, \frac{1}{dz} \epsilon^*_\mu \nu^* \right\rangle.
\end{align}

Now, Eq. C.13 can be decomposed as

\begin{equation}
\begin{bmatrix}
\epsilon \\
\epsilon
\end{bmatrix} = \bar{S}^t (z) \cdot \bar{L}^{-1} \cdot \bar{M}^\pm (\rho) \cdot \bar{c}^\pm
\end{equation}

where

\begin{equation}
\bar{L} = \begin{bmatrix}
\bar{L}_\mu & 0 \\
0 & \bar{L}_\epsilon
\end{bmatrix},
\end{equation}

\begin{equation}
\bar{M}^\pm (\rho) = \begin{bmatrix}
\frac{\nu}{\rho} \bar{D}_\mu \cdot \bar{a}_\epsilon \cdot \bar{C}_\epsilon^\pm & -i\omega \bar{p}_\mu \cdot \bar{a}_\epsilon \cdot \bar{C}_\mu^\pm \cdot \bar{K}_\epsilon \\
i\omega \bar{p}_\epsilon \cdot \bar{a}_\mu \cdot \bar{C}_\epsilon^{\pm'} \cdot \bar{K}_\epsilon & \frac{\nu}{\rho} \bar{D}_\epsilon \cdot \bar{a}_\mu \cdot \bar{C}_\mu^{\pm'}
\end{bmatrix}.
\end{equation}

Since from Eq.5.9, we have

\begin{equation}
\bar{L} \cdot \bar{a} = \bar{p} \cdot \bar{a} \cdot \bar{K}^2
\end{equation}

\[\Rightarrow \bar{a} \cdot \bar{K}^{-2} = \bar{L}^{-1} \cdot \bar{p} \cdot \bar{a},\]

we can rewrite

\begin{align}
\bar{L}^{-1} \cdot \bar{M}^\pm &= \begin{bmatrix}
\frac{\nu}{\rho} \bar{L}_\mu^{-1} \cdot \bar{D}_\mu \cdot \bar{a}_\epsilon & -i\omega \bar{a}_\mu \cdot \bar{C}_\mu^\pm \cdot \bar{K}_\epsilon^{-1} \cdot \bar{C}_\epsilon^\pm^{-1} \\
i\omega \bar{a}_\epsilon \cdot \bar{C}_\epsilon^{\pm'} \cdot \bar{K}_\epsilon^{-1} \cdot \bar{C}_\epsilon^\pm^{-1} & \frac{\nu}{\rho} \bar{L}_\epsilon^{-1} \cdot \bar{D}_\epsilon \cdot \bar{a}_\mu
\end{bmatrix} \cdot \begin{bmatrix}
\bar{C}_\epsilon^\pm & 0 \\
0 & \bar{C}_\mu^\pm
\end{bmatrix}
\end{align}

\[= \bar{Y}^\pm (\rho) \cdot \bar{C}^\pm (\rho).
\]

This extra step is done because matrix \( \bar{Y}^\pm \) is better conditioned than matrix \( \bar{M}^\pm \).
The $\rho$-component is given as

\[
\begin{bmatrix}
  e_\rho \\
h_\rho
\end{bmatrix} =
\begin{bmatrix}
  L^{-1}_\mu & 0 \\
  0 & L^{-1}_\epsilon
\end{bmatrix}
\cdot
\begin{bmatrix}
  \frac{\partial}{\partial z} \frac{1}{\mu} \frac{\partial}{\partial \rho} \\
  -i \omega \frac{1}{\epsilon} \frac{\partial}{\partial \rho}
\end{bmatrix}
\cdot
\begin{bmatrix}
  i \omega \frac{1}{\mu} \frac{i \omega}{\rho} \\
  -i \omega \frac{1}{\epsilon} \frac{i \omega}{\rho}
\end{bmatrix}
\cdot
\begin{bmatrix}
d_z \\
b_z
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  L^{-1}_\mu & 0 \\
  0 & L^{-1}_\epsilon
\end{bmatrix}
\cdot
\begin{bmatrix}
  \frac{\partial}{\partial z} \frac{1}{\mu} \frac{\partial}{\partial \rho} \\
  -i \omega \frac{1}{\epsilon} \frac{\partial}{\partial \rho}
\end{bmatrix}
\cdot
\begin{bmatrix}
  \bar{s}^t(z) \cdot \bar{a} \cdot \bar{C}^\pm(\rho) \cdot \bar{c}^\pm
\end{bmatrix}
= \bar{s}(z) \cdot \bar{L}^{-1}_\mu \cdot \bar{D}_\mu \cdot \bar{a} \cdot \bar{C}^\pm(\rho) \cdot \bar{K}_e \cdot \bar{D}_e \cdot \bar{a} \cdot \bar{C}^\pm(\rho) \cdot \bar{K}_e
\]

\[
\bar{s}(z) \cdot \bar{L}^{-1}_\mu \cdot \bar{D}_\mu \cdot \bar{a} \cdot \bar{C}^\pm(\rho) \cdot \bar{K}_e \cdot \bar{D}_e \cdot \bar{a} \cdot \bar{C}^\pm(\rho) \cdot \bar{K}_e
\]

\[
= \bar{s}(z) \cdot \bar{X}^\pm(\rho) \cdot \bar{c}^\pm
\]

158
C.3 Finding $j_{\nu q}$ and $j_{\mu q}$

Here, we show the derivation of the source spectral component used in the NMM formulation. If $\psi_q(z)$ is the eigenmodes for the TM mode, then, the source term of the TM mode can be derived as

$$j_{\nu q} = I_T \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \int_{-z_{\text{min}}}^{z_{\text{max}}} dz \delta(z - \zeta_T(\phi)) \xi_T(\phi) \psi_q(z) e^{-i\nu\phi}$$

$$= I_T \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \xi_T(\phi) \psi_q(\zeta_T(\phi)) e^{-i\nu\phi}$$

$$= I_T \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \xi_T(\phi) \sum_{n=1}^{N} a_{\nu q n} S_n(\zeta_T(\phi)) e^{-i\nu\phi}$$

$$(C.27)$$

The integral ($\tilde{j}_{\nu q}$, term other than the coefficients $a_{\nu q n}$) of the above equation can be evaluated numerically from $-\pi$ to $\pi$ for $n$ basis functions. The coefficients, $a_{\nu q n}$, are determined from solving Eq. 5.9. Thus,

$$j_{\nu q} = \sum_{n=1}^{N} a_{\nu q n} \tilde{j}_{\nu q n}(\rho_T, \theta_T, \phi_T)$$

$$(C.28)$$

where

$$\tilde{j}_{\nu q n} = I_T \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \xi_T(\phi) S_n(\zeta_T(\phi)) e^{-i\nu\phi}.$$  

$$(C.29)$$

The matrix representation of Eq. C.28 would be

$$\bar{\mathbf{j}}_{\nu} = \bar{\mathbf{a}}_x \cdot \bar{\mathbf{j}}_{\nu}$$

$$(C.30)$$
The same procedure can be applied to TE modes, and \( j_{\mu \nu q} \) is going to be

\[
\begin{align*}
  j_{\mu \nu q} &= \frac{I_T}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \int_{z_{\min}}^{z_{\max}} dz \delta(z - \zeta_T(\phi)) \psi_{\mu q}(z) e^{-i\nu \phi} \\
  &= \frac{I_T}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \psi_{\mu q}(\zeta_T(\phi)) e^{-i\nu \phi} \\
  &= \frac{I_T}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \sum_{n=1}^{N} a_{\mu q n} S_n(\zeta_T(\phi)) e^{-i\nu \phi} \\
  j_{\mu \nu q} &= \sum_{n=1}^{N} a_{\mu q n} \frac{I_T}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi S_n(\zeta_T(\phi)) e^{-i\nu \phi}.
\end{align*}
\] (C.31)

It can be simplified as,

\[
\begin{align*}
  j_{\mu \nu q} &= \sum_{n=1}^{N} a_{\mu q n} \tilde{j}_{\mu \nu n}(\rho_T, \theta_T, \phi_T) \\
  \text{where} \\
  \tilde{j}_{\mu \nu n}(\rho_T, \theta_T, \phi_T) &= \frac{I_T}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi S_n(\zeta_T(\phi)) e^{-i\nu \phi}. \quad (C.33)
\end{align*}
\]

The matrix representation of Eq. C.32 is

\[
\tilde{J}_{\mu \nu} = \tilde{a}_\mu \cdot \tilde{J}_{\mu \nu}
\] (C.34)
C.4 NMM: Source condition

Here we derive the source condition used in the NMM formulation. We have

\[
\frac{1}{\rho} \frac{d}{d\rho} \left( \frac{d}{d\rho} - \frac{\nu^2}{\rho^2} + k_{\rho q}^2 \right) b_{\epsilon q}(\rho) = -i\omega \mu_{h,\epsilon \nu q} \delta(\rho - \rho_T) \tag{C.35}
\]

\[
\frac{1}{\rho} \frac{d}{d\rho} \left( \frac{d}{d\rho} - \frac{\nu^2}{\rho^2} + k_{\mu q}^2 \right) b_{\mu q}(\rho) = -j\mu_{q} \frac{1}{\rho} \frac{d}{d\rho} (\rho \delta(\rho - \rho_T)), \tag{C.36}
\]

for \( q = 1 \) to \( N \). The formal solution would be either Bessel function or Hankel function of order \( \nu \). In the source layer,

\[
b_{\epsilon q}(\rho) = \begin{cases} J_\nu(k_{\rho q} \rho) c_{\epsilon q}^-, & \rho \leq \rho_T; \\ H_\nu(k_{\rho q} \rho) c_{\epsilon q}^+, & \rho > \rho_T. \end{cases} \tag{C.37}
\]

\[
b_{\mu q}(\rho) = \begin{cases} J_\nu(k_{\mu q} \rho) c_{\mu q}^-, & \rho \leq \rho_T; \\ H_\nu(k_{\mu q} \rho) c_{\mu q}^+, & \rho > \rho_T. \end{cases} \tag{C.38}
\]

The source condition can be derived from Appendix B.9

\[
\frac{d}{d\rho} b_{\epsilon q}(\rho_T^+) - \frac{d}{d\rho} b_{\epsilon q}(\rho_T^-) = -i\omega \mu_{h,\epsilon \nu q} j_{\epsilon \nu q}, \quad b_{\epsilon q}(\rho_T^+) - b_{\epsilon q}(\rho_T^-) = 0. \tag{C.39}
\]

\[
\frac{d}{d\rho} b_{\mu q}(\rho_T^+) - \frac{d}{d\rho} b_{\mu q}(\rho_T^-) = 0, \quad b_{\mu q}(\rho_T^+) - b_{\mu q}(\rho_T^-) = -j_{\mu \nu q}. \tag{C.40}
\]

The solution are

\[
c_{\epsilon q}^- = -\frac{\pi}{2} j_{\epsilon \nu q} \omega \rho_T H_\nu(k_{\rho q} \rho_T), \quad c_{\epsilon q}^+ = -\frac{\pi}{2} j_{\epsilon \nu q} \omega \rho_T J_\nu(k_{\rho q} \rho_T), \tag{C.41}
\]

\[
c_{\mu q}^- = -\frac{i\pi}{2} j_{\mu \nu q} k_{\rho q} \rho_T H'_\nu(k_{\rho q} \rho_T), \quad c_{\mu q}^+ = -\frac{i\pi}{2} j_{\mu \nu q} k_{\rho q} \rho_T J'_\nu(k_{\rho q} \rho_T). \tag{C.42}
\]
C.5 Local reflection/transmission coefficients for two cylindrical-vertical layers

Here, we show how to find local reflection/transmission coefficients using a two cylindrical-vertical layer example. The source layer is denoted by a subscript 0, where the formal solution writes as

$$\bar{A}_{0z}(\rho, z) = \bar{S}^t(z) \cdot \bar{a}_0 \cdot \begin{cases} \[\bar{H}_0(\rho) \cdot \bar{R}^+_{0,-1} + \bar{J}_0(\rho)\] \cdot \bar{d}^-, \quad \rho_{-1} < \rho \leq \rho_T; \vspace{1em} \\
\[\bar{H}_0(\rho) + \bar{J}_0(\rho) \cdot \bar{R}_{0,1}^-\] \cdot \bar{d}^+, \quad \rho_T \leq \rho \leq \rho_0. \end{cases} \quad \text{(C.43)}$$

$$\bar{A}_{0\phi}(\rho, z) = \bar{S}^t(z) \cdot \begin{cases} \[\bar{Y}^+_0(\rho) \cdot \bar{H}_0(\rho) \cdot \bar{R}^+_{0,-1} + \bar{Y}^-_1(\rho) \cdot \bar{J}_0(\rho)\] \cdot \bar{d}^-, \quad \rho_{-1} < \rho \leq \rho_T; \vspace{1em} \\
\[\bar{Y}^+_0(\rho) \cdot \bar{H}_0(\rho) + \bar{Y}^-_1(\rho) \cdot \bar{J}_1(\rho) \cdot \bar{R}_{0,1}^-\] \cdot \bar{d}^+, \quad \rho_T \leq \rho \leq \rho_0. \end{cases} \quad \text{(C.44)}$$

The next (outward) layer is denoted by subscript and has 1 formal solution

$$\bar{A}_{1z}(\rho, z) = \bar{S}^t(z) \cdot \bar{a}_1 \cdot \bar{H}_1(\rho) \cdot \bar{T}^+_{0,1} \cdot \bar{d}^+, \quad \rho > \rho_0; \quad \text{(C.45)}$$

$$\bar{A}_{1\phi}(\rho, z) = \bar{S}^t(z) \cdot \bar{Y}^+_1(\rho) \cdot \bar{H}_1(\rho) \cdot \bar{T}^+_{0,1} \cdot \bar{d}^+, \quad \rho > \rho_0. \quad \text{(C.46)}$$

Enforcing the boundary conditions on the tangential components at the interface, we arrive at the following equation in terms of the longitudinal components,

$$\bar{E}^{-1}_0 \cdot \bar{S}^t(z) \cdot \bar{a}_0 \cdot \left[\bar{H}_0(\rho_0) + \bar{J}_0(\rho_0) \cdot \bar{R}_{0,1}^-ight] = \bar{E}^{-1}_1 \cdot \bar{S}^t(z) \cdot \bar{a}_1 \cdot \bar{H}_1(\rho_0) \cdot \bar{T}^+_{0,1}, \quad \text{(C.47)}$$

where

$$\bar{E} = \begin{bmatrix} \epsilon(z) & 0 \\
0 & \mu(z) \end{bmatrix}. \quad \text{(C.48)}$$

Projecting the above equation on the basis function set and multiply the resulting equation with $\bar{a}^t$, we have

$$\left[\bar{H}_0 + \bar{J}_0 \cdot \bar{R}_{0,1}\right] = \bar{\Psi}_{0,1} \cdot \bar{H}_1 \cdot \bar{T}^+_{0,1} \quad \text{(C.49)}$$
where $\tilde{\Psi}_{n,n+1} = \tilde{a}^t_n \cdot \tilde{b}_{n+1} \cdot \tilde{a}_{n+1}$. This matrix matches eigenmodes of layer $n$ to layer $n+1$. Similarly, the transverse component gives

$$\tilde{Y}^+_0 \cdot \tilde{H}_0 + \tilde{Y}^-_0 \cdot \tilde{J}_0 \cdot \tilde{R}_{0,1} = \tilde{Y}^+_1 \cdot \tilde{H}_1 \cdot \tilde{T}^+_{0,1}$$ (C.50)

With the above two boundary conditions, we can express the reflection and transmission coefficient as

$$\tilde{R}_{-0,1} = \tilde{J}_0^{-1} \cdot (\tilde{I} - \tilde{\Psi}_{0,1} \cdot (\tilde{Y}^+_1)^{-1} \cdot \tilde{Y}^-_0)^{-1} \cdot (\tilde{\Psi}_{0,1} \cdot (\tilde{Y}^+_1)^{-1} \cdot \tilde{Y}^-_0 - \tilde{I}) \cdot \tilde{H}_0, \quad (C.51)$$

and

$$\tilde{T}^+_{0,1} = (\tilde{H}_1)^{-1} \cdot (\tilde{Y}^+_1)^{-1} \cdot \left[ \tilde{Y}^+_0 \cdot \tilde{H}_0 + \tilde{Y}^-_0 \cdot \tilde{J}_0 \cdot \tilde{R}_{0,1} \right]. \quad (C.52)$$

The other two other local reflection/transmission (inward) coefficients can be determined using the same approach.

The general form of the four local coefficients are

$$\tilde{R}_{n,n+1}^{-} = \tilde{J}_n^{-1} \cdot (\tilde{I} - \tilde{\Psi}_{n,n+1} \cdot (\tilde{Y}^+_n)^{-1} \cdot \tilde{Y}^-_n)^{-1} \cdot (\tilde{\Psi}_{n,n+1} \cdot (\tilde{Y}^+_n)^{-1} \cdot \tilde{Y}^-_n - \tilde{I}) \cdot \tilde{H}_n \quad (C.53)$$

$$\tilde{T}^+_{n,n+1} = (\tilde{H}_{n+1})^{-1} \cdot (\tilde{Y}^+_n)^{-1} \cdot \left[ \tilde{Y}^+_n \cdot \tilde{H}_n + \tilde{Y}^-_n \cdot \tilde{J}_n \cdot \tilde{R}^-_{n,n+1} \right] \quad (C.54)$$

$$\tilde{R}_{n+1,n}^{+} = \tilde{H}_{n+1}^{-1} \cdot (\tilde{\Psi}_{n,n+1} - (\tilde{Y}^-_n)^{-1} \cdot \tilde{Y}^+_n)^{-1} \cdot ((\tilde{Y}^-_n)^{-1} \cdot \tilde{Y}^-_n \cdot \tilde{R}^-_{n,n+1}) \cdot \tilde{J}_{n+1} \quad (C.55)$$

$$\tilde{T}^{-}_{n+1,n} = (\tilde{J}_n)^{-1} \cdot (\tilde{Y}^-_n)^{-1} \cdot \left[ \tilde{Y}^+_n \cdot \tilde{H}_{n+1} \cdot \tilde{R}^{+}_{n+1,n} + \tilde{Y}^-_n \cdot \tilde{J}_{n+1} \right] \quad (C.56)$$

Multiple cylindrical-vertical layers can also be included in recursive fashion, as described in Appendix B.5.
C.6 NMM: Transimpedance

Here we show the derivation of the transimpedance used in the NMM formulation. This transimpedance is given in terms of volume integral of the electric field along the receiver position (see Appendix B.1). In the homogeneous formation, the longitudinal component of the transimpedance is given as

$$Z_{RT_z} = -\frac{1}{I_T} \int_{-\infty}^{\infty} dz \int_{0}^{\pi} d\rho \int_{-\pi}^{\pi} \rho d\phi \delta(\rho - \rho_R)\delta(z - \zeta_R(\phi))E_z$$

$$= -\frac{\rho_R}{I_T} \sum_{\nu} \int_{-\pi}^{\pi} d\phi \frac{1}{\sqrt{2\pi}} e^{i\nu\phi} \xi_R(\phi) \bar{S}^l(\zeta_R(\phi)) * \bar{\mathbf{p}} \cdot \bar{\mathbf{a}} \cdot \bar{\mathbf{C}}(\rho_R) \cdot \bar{c}$$

where

$$\bar{d}_{\nu} = \int_{-\pi}^{\pi} d\phi \frac{1}{\sqrt{2\pi}} e^{i\nu\phi} \xi_R(\phi) \bar{S}(\zeta_R(\phi))$$

and the transverse component of the transimpedance is given as

$$Z_{RT_\phi} = -\frac{1}{I_T} \int_{-\infty}^{\infty} dz \int_{0}^{\pi} d\rho \int_{-\pi}^{\pi} \rho d\phi \delta(\rho - \rho_R)\delta(z - \zeta_R(\phi))E_\phi$$

$$= -\frac{\rho_R}{I_T} \sum_{\nu} \int_{-\infty}^{\infty} dz \int_{-\pi}^{\pi} d\phi \frac{1}{\sqrt{2\pi}} e^{i\nu\phi} \delta(z - \zeta_R(\phi)) \bar{S}^l(z) * \bar{\mathbf{Y}} \cdot \bar{\mathbf{C}}(\rho_R) \cdot \bar{c}$$

$$= -\frac{\rho_R}{I_T} \sum_{\nu} \int_{-\pi}^{\pi} d\phi \frac{1}{\sqrt{2\pi}} e^{i\nu\phi} \bar{S}^l(\zeta_R(\phi)) \cdot \bar{\mathbf{Y}} \cdot \bar{\mathbf{C}}(\rho_R) \cdot \bar{c}$$

where

$$\bar{d}_{\mu\nu} = \int_{-\pi}^{\pi} d\phi \frac{1}{\sqrt{2\pi}} e^{i\nu\phi} \bar{S}(\zeta_R(\phi))$$
C.7 NMM: magnetic dipole formulation

Here, we derive NMM source excitation coefficients of a magnetic point source. The magnetic point source is given by

\[
\vec{M} = m_s l \delta(\rho - \rho') \delta(\phi - \phi') \delta(z - z') (\cos \theta \hat{z} + \sin \theta \hat{\rho})
\]  

(C.61)

where \( m_s \) is the magnetic source current and \( l \) is the dipole length. In the above, the coordinate with prime subscript denotes the source location. The magnetic dipole points at an angle \( \theta \) from the longitudinal \( z \)-direction. Since

\[
\delta(\phi - \phi') = \frac{1}{\sqrt{2\pi}} \sum_{\nu = -\infty}^{\infty} e^{i\nu(\phi - \phi')},
\]

(C.62)

we let \( \phi' = 0 \), so that the \( \nu \) harmonic of \( \left( \nabla \frac{1}{\mu_h} \nabla \cdot \vec{M} \right) / \omega \), \( (\mu_h \nabla \times \vec{\mu}^{-1} \vec{M})_z \) and \( (i\omega \epsilon_h \vec{M})_z \) is written as

\[
\left( \nabla \frac{1}{\mu_h} \nabla \cdot \vec{M} \right)_{z\nu} = \frac{1}{\sqrt{2\pi}} \frac{m_s l}{i\omega} \left( \frac{\sin \theta}{\rho} \frac{d}{d\rho} \delta(\rho - \rho') \frac{d}{dz} \frac{\delta(z - z')}{\mu_h} \right.
\]

\[
+ \cos \theta \frac{\delta(\rho - \rho')}{\rho} \frac{d}{dz} \frac{1}{\mu_h} \frac{d}{dz} \delta(z - z'),
\]

(C.63)

\[
(\mu_h \nabla \times \vec{\mu}^{-1} \vec{M})_{z\nu} = -\frac{i\nu}{\sqrt{2\pi}} m_s l \sin \theta \frac{1}{\rho^2} \delta(\rho - \rho') \delta(z - z'),
\]

(C.64)

and

\[
(i\omega \epsilon_h \vec{M})_{z\nu} = \frac{1}{\sqrt{2\pi}} i\omega \epsilon_h m_s l \cos \theta \frac{\delta(\rho - \rho')}{\rho} \delta(z - z'),
\]

(C.65)

Let \( \psi_{q}(z) \) be the eigenmodes along the longitudinal direction. In this case,

\[
\left( \nabla \frac{1}{\mu_h} \nabla \cdot \vec{M} \right)_{z\nu q} = \frac{1}{\sqrt{2\pi}} \frac{m_s l}{i\omega} \left( \frac{\sin \theta}{\rho} \frac{1}{\mu_h(z')} \frac{d}{d\rho} \delta(\rho - \rho') \psi_{q}(z') \right.
\]

\[
+ \cos \theta \frac{\delta(\rho - \rho')}{\rho} \left( \frac{k_{p\mu m}^2}{\mu_h(z')} - \omega^2 \epsilon_h(z') \right) \psi_{q}(z'),
\]

(C.66)

\[
(\mu_h \nabla \times \vec{\mu}^{-1} \vec{M})_{z\nu q} = -\frac{i\nu}{\sqrt{2\pi}} m_s l \sin \theta \frac{1}{\rho^2} \delta(\rho - \rho') \psi_{q}(z'),
\]

(C.67)
The Helmholtz equations for the electric and magnetic fields with magnetic source excitation are given by

\begin{equation}
\mu_h \nabla \times \left( (\vec{\mu} - 1) \nabla \times \vec{E} \right) - \nabla \left( \left( \epsilon_h \right) ^{-1} \nabla \cdot \vec{E} \right) - \omega^2 \mu_h \vec{E} = -k_h \nabla \times ((\vec{\mu} - 1) \vec{M}), \tag{C.69}
\end{equation}

and

\begin{equation}
\epsilon_h \nabla \times \left( (\vec{\epsilon} - 1) \nabla \times \vec{H} \right) - \nabla \left( \left( \mu_h \right) ^{-1} \nabla \cdot \vec{H} \right) - \omega^2 \epsilon_h \vec{H} = -\nabla \frac{\rho_m}{\mu_h} + i \omega \epsilon_h \vec{M}. \tag{C.70}
\end{equation}

Thus, the mode propagator satisfies the following equations,

\begin{equation}
\frac{1}{\rho} \frac{d}{d\rho} \frac{d}{d\rho} - \nu^2 \rho^2 + k^2 \rho \rightarrow \frac{1}{\rho} \frac{d}{d\rho} \delta(\rho - \rho'), \tag{C.71}
\end{equation}

\begin{equation}
\frac{1}{\rho} \frac{d}{d\rho} \frac{d}{d\rho} - \nu^2 \rho^2 + k^2 \rho \rightarrow \frac{1}{\rho} \frac{d}{d\rho} \frac{1}{\rho} \delta(\rho - \rho'), \tag{C.72}
\end{equation}

with solution given by

\begin{equation}
b_{\nu} = \frac{\nu \pi}{2} \sqrt{\frac{2}{2\pi}} m_s l \sin \theta \psi_{\nu} (z') \frac{1}{\rho} \left\{ \begin{array}{ll}
H_{\nu} (k_{\nu} \rho \rho') & , \rho < \rho' ; \\
J_{\nu} (k_{\nu} \rho \rho') & , \rho > \rho'.
\end{array} \right\} \tag{C.73}
\end{equation}

and

\begin{equation}
b_{\rho} = b_{\rho 1} (\rho) + b_{\rho 2} (\rho) \tag{C.74}
\end{equation}

where

\begin{equation}
b_{\rho 1} (\rho) = \frac{i \pi}{2} \sqrt{\frac{2}{2\pi}} m_s l \sin \theta \psi_{\nu} (z') \frac{1}{\rho} \left\{ \begin{array}{ll}
H_{\nu} (k_{\nu} \rho \rho') & , \rho < \rho' ; \\
J_{\nu} (k_{\nu} \rho \rho') & , \rho > \rho'.
\end{array} \right\} \tag{C.75}
\end{equation}

\begin{equation}
b_{\rho 2} (\rho) = \frac{i \pi}{2} \sqrt{\frac{2}{2\pi}} m_s l \cos \theta \psi_{\nu} (z') \quad \frac{1}{\rho} \left\{ \begin{array}{ll}
H_{\nu} (k_{\nu} \rho \rho') & , \rho < \rho' ; \\
J_{\nu} (k_{\nu} \rho \rho') & , \rho > \rho'.
\end{array} \right\} \tag{C.76}
\end{equation}
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170


