A STUDY OF THE DYNAMIC BEHAVIOR OF PIECEWISE
NONLINEAR OSCILLATORS WITH TIME-VARYING STIFFNESS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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ABSTRACT

The dynamic behavior of a piecewise-nonlinear mechanical oscillator with parametric and external excitations is investigated. The viscously damped oscillator is subjected to a periodically time-varying, piecewise nonlinear restoring function. Typical applications represented by this oscillator are highlighted. A multi-term harmonic balance formulation is used in conjunction with discrete Fourier transforms and a parametric continuation scheme to determine steady-state motions of the system due to both parametric and external excitations. The accuracy of the analytical solutions is demonstrated by comparing them to direct numerical integration solutions and available experimental data for a special case. Floquet theory is applied to determine the stability of the steady-state harmonic balance solutions.

This solution method is first applied on a single-degree-of-freedom piecewise nonlinear time-varying system to find steady state period-1 and period-$\eta$ ($\eta > 1$) motions. The system is characterized by a symmetric restoring function, which consists of three segments: a clearance (dead-zone) segment and two continuously nonlinear segments defined by a linear component, a quadratic term and a cubic term. Detailed parametric studies are presented to quantify the combined influence of clearance, quadratic and cubic nonlinearities within reasonable ranges of all other system parameters.
comparison between time-varying and time-invariant systems is also provided to
demonstrate the influence of the parametric and external excitations on a piecewise
nonlinear system. As a specific application, an elastic sphere-plane interface is studied
by using this solution method. The dynamic model of the sphere-plane system includes
both a continuous nonlinearity associated with the Hertzian contact and a clearance-type
nonlinearity due to contact loss. The accuracy of the dynamic model and solution
method is demonstrated through comparisons with experimental data and numerical
solutions. A single-term harmonic balance approximation is used to derive a criterion for
contact loss to occur. The influence of harmonic external excitation $f(\tau)$ and damping
ratio $\zeta$ on the steady state response is also demonstrated.

Finally, the solution method is extended to a generalized multi-degree-of-freedom
dynamical system with multiple clearances, time-varying coefficients, and piecewise
nonlinear characteristics. This generalized formulation is applied to a three-degree-of-
freedom gear-bearing system to demonstrate its applicability.
To my parents
Ma Jiping and Lin Pin
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NOMENCLATURE

A  the harmonic amplitude of the HBM solutions
C  damping matrix
CN  continuously nonlinear
d  damping function
DFT  discrete Fourier transforms
DFM  describing function method
DSI  double-side impacts
E  modulus of elasticity
f  external force
g  restoring function
H  periodic state matrix
P  total number of rolling elements in contact of bearings
h  sampling intervals of discrete Fourier transforms
H  \( H = \Lambda/\eta \)
HBM  harmonic balance method
i  tooth index of a spline
J  Jacobian matrix
\( \mathbf{K} \) stiffness matrix

\( M \) total number of Fourier harmonics of \( f(\tau) \)

\( \mathbf{M} \) mass matrix

MDOF multiple degree-of-freedom

\( N \) number of time samples of discrete Fourier transforms

\( n \) a positive integer

PL piecewise linear

PN piecewise nonlinear

\( R \) total number of Fourier harmonics of \( u(\tau) \)

\( \mathbf{S} \) vector of harmonic balance equation

SDOF single-degree-of-freedom

SSI single-side impacts

\( T \) period

TI time-invariant

TV time-varying

\( \mathbf{u} \) vector of Fourier coefficients of \( u(\tau) \)

\( u \) displacement

\( v \) Fourier coefficients of \( g[u(\tau)] \)

\( w \) stiffness

\( \mathbf{Y} \) variation matrix

\( \alpha_1 \) coefficient of linear component of \( g[u(\tau)] \)

\( \alpha_2 \) coefficient of quadratic component of \( g[u(\tau)] \)
\( \alpha_3 \) coefficient of cubic component of \( g[u(\tau)] \)

\( \Gamma \) phase angle between \( w(\tau) \) and \( f(\tau) \)

\( \Delta \) small variation

\( \delta \) circumferential spline tooth position (spacing) errors

\( \zeta \) damping ratio

\( \eta \) subharmonic index

\( \theta \) \( \theta = \Lambda \tau / \eta \)

\( \Lambda \) total number of Fourier harmonics of \( w(\tau) \)

\( \Lambda \) dimensionless fundamental excitation frequency

\( \tau \) dimensionless time

\( \nu \) Poisson ratio

\( F \) monodromy matrix

\( \psi \) angular position of the \( r \)-th rolling element in contact of bearings

**Subscripts**

\( i, j, n, r \) positive integers

r.m.s. root-mean-square

\( \kappa, \mu \) positive integers

**Superscripts**

\( (.) \) differentiation with respect to \( \tau \)

\( m \) a positive integer

\( T \) matrix transpose

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CHAPTER 1

INTRODUCTION

1.1 Introduction

In this study, the dynamic behavior of piecewise nonlinear (PN) oscillators having periodically time-varying (TV) parameters is investigated. As a representative of such systems, a single-degree-of-freedom (SDOF) mechanical oscillator is shown in Figure 1.1, which consists of a unit mass subjected to PN spring and damping elements. The system is excited by a periodically TV stiffness in the form of parametric excitation as well as a periodic external force acting on the mass. The equation of motion of this oscillator is given in dimensionless form as

\[
\ddot{u}(\tau) + 2\zeta \{d[u(\tau)]\}^m \dot{u}(\tau) + w(\tau)g[u(\tau)] = f(\tau),
\]  

(1.1)

where \(\tau\) is dimensionless time, an overdot denotes differentiation with respect to \(\tau\), \(u(\tau)\) is the displacement of the unit mass, \(d[u(\tau)]\) represents a PN damping function, and \(\zeta\) is the damping ratio. Periodic functions \(w(\tau)\) and \(f(\tau)\) act as an internal and external
Figure 1.1. A physical model of a mechanical oscillator with unit mass and backlash.
excitations, respectively. The restoring function \( g[u(\tau)] \) may take different forms depending on the application investigated. In this study, two special cases are considered. The first one is defined as

\[
g[u(\tau)] = \begin{cases} 
\sum_{i=1}^{3} \alpha_i [u(\tau) - 1]^i, & u(\tau) > 1, \\
0, & |u(\tau)| \leq 1, \\
\sum_{i=1}^{3} (-1)^{i-1} \alpha_i [u(\tau) + 1]^i, & u(\tau) < -1.
\end{cases} \tag{1.2a}
\]

As Figure 1.2(a) shows for different values of \( \alpha_2 \) and \( \alpha_3 \), this form of \( g[u(\tau)] \) consists of three segments: a clearance (dead-zone) segment for \( |u(\tau)| \leq 1 \), and two continuously nonlinear segments for \( u(\tau) > 1 \) and \( u(\tau) < -1 \). The nonlinear segments are defined by a linear stiffness component of slope \( \alpha_1 \), a quadratic nonlinearity term with coefficient \( \alpha_2 \), and a cubic nonlinearity term with coefficient \( \alpha_3 \). The second form of \( g[u(\tau)] \) is defined as

\[
g[u(\tau)] = \begin{cases} 
(1 + \rho u(\tau))^{\frac{1}{\rho}}, & u(\tau) > -\frac{1}{\rho}, \\
0, & u(\tau) \leq -\frac{1}{\rho}.
\end{cases} \tag{1.2b}
\]

This equation that is illustrated in Figure 1.2(b) represents a Hertzian contact allowing separations. Here, \( \rho \) is a constant that is \( \rho = \frac{2}{3} \) for a sphere-plane contact of bodies...
Figure 1.2. Different forms of $g[u(\tau)]$ and $d[u(\tau)]$: (a) equation (1.2a) given $\alpha_1 = 1$, and: (- - -) $\alpha_2 = 0.2$, $\alpha_3 = -0.1$, (-----) $\alpha_2 = \alpha_3 = 0$, and (- ?- ?) $\alpha_2 = 0.2$, $\alpha_3 = 0$; (b) equation (1.2b) given $\rho = \frac{2}{3}$.
made of the same material. When \( u(\tau) > -\frac{1}{\rho} \), the contact is maintained all the time, and the model represents a typical continuously nonlinear system. However, if \( u(\tau) \leq -\frac{1}{\rho} \) for certain \( \tau \), the contact is lost, representing a PN system.

In equation (1.1), \( d[u(\tau)] \) is a damping function subject to a power index \( m \). Here, if \( m = 0 \), the damping force \( f_d(\tau) \) is proportional to \( u \) linearly. When \( m = 1 \), \( d[u(\tau)] \) represents a PN function as in the application of a sphere-plane problem with contact loss, and is given as

\[
d[u(\tau)] = \begin{cases} 
[1 + \rho u(\tau)]^p, & u(\tau) > -\frac{1}{\rho}, \\
0, & u(\tau) \leq -\frac{1}{\rho}.
\end{cases}
\] (1.3a)

When \( p = 0 \), \( d[u(\tau)] \) reduces to a piecewise linear (PL) function

\[
d[u(\tau)] = \begin{cases} 
1, & u(\tau) > -\frac{1}{\rho}, \\
0, & u(\tau) \leq -\frac{1}{\rho},
\end{cases}
\] (1.3b)

such that the damping is a linear viscous one when the sphere and plane are in contact, and zero when separations happen. For \( p \neq 0 \), \( f_d(\tau) \) is a PN function. Specifically, \( p = \frac{1}{2} \) represents a damping force that is in proportion to the contact radius, while for
\( p = 1 \) and \( p = \frac{3}{2} \), \( f_d(\tau) \) is proportional to the contact area and the elastic restoring force, respectively.

Another critical feature of equation (1.1) is the periodically TV stiffness \( w(\tau) \). In the case of a rolling element bearing, the total number of rolling elements (balls or cylindrical rollers) in contact fluctuates between two integers \( m \) and \( m + 1 \) as the roller cage rotates relative to inner and outer races of the bearing. Similarly, in a spur gear pair, the total number of loaded tooth pairs typically alternates between 1 and 2. In both applications, such rotation (hence time) dependent changes cause the overall stiffness at the interface to vary periodically. This is represented by \( w(\tau) \) in equation (1.1).

1.2 Relevant Applications

Equation (1.1) governs a number of common mechanical components or systems, which have discontinuous and non-differentiable nonlinearities. Various examples of such systems include spline-shaft interfaces [1-5], Hertzian contact problems [6-13], rolling element bearings [14-15], and gear pairs [17-18]. In this section, brief formulation for each of these typical applications will be presented to demonstrate how these systems are governed by equation (1.1).

1.2.1 Spline Interfaces

Splined structures such as splined shafts [1], spline couplings [2-4] and spline joints [5] are used commonly in rotating machinery for carrying torque. A spline interface can carry large loads, allow quick assembly and disassembly, provide relative axial motion as
needed, and is very cost effective and compact [2]. Most studies have been focused on stress distribution on contacting surfaces of spline interfaces, which may result in fretting damage, including wear and fatigue [1-4]. A very limited number of investigations exist for dynamic modeling of spline interfaces. Kahraman [5] proposed a torsional dynamic model to represent a spline joint of a gear-shaft pair, as illustrated in Figure 1.3. In this study, deflections were assumed to take place at the mating spline tooth pairs of the shaft and the mating gear. An equation of motion was proposed in dimensionless form as [5]

\[ \ddot{u}(\tau) + 2\zeta \dot{u}(\tau) + g[u(\tau)] = f(\tau). \]  

(1.4)

This is a special case of equation (1.1) with the restoring function \( g[u(\tau)] \) expressed as

\[ g[u(\tau)] = \begin{cases} 
    n[u(\tau) - 1] - \sum_{i=1}^{n} \delta_i, & (\delta_n + 1) \leq x(\tau), \\
    r[u(\tau) - 1] - \sum_{i=1}^{r} \delta_i, & 1 \leq u(\tau) < (\delta_r + 1), \\
    0, & -1 < u(\tau) < 1, \\
    -r[u(\tau) - 1] + \sum_{i=1}^{r} \delta'_{n+1-i}, & (\delta'_{n-r} + 1) < u(\tau) \leq -1, \\
    -n[u(\tau) - 1] + \sum_{i=1}^{n} \delta'_i, & u(\tau) \leq -(\delta'_1 + 1), 
\end{cases} \]  

(1.5a)

where \( \delta_i = (b_i - a)/a \), \( \delta'_i = (b'_i - a)/a \), \( i = 1 \) to \( n \), \( b_i \) and \( b'_i \) are the clearances at front (loaded) and back flanks, and \( a \) is the minimum clearance that is same for both sides. If the tooth spacing error is neglected, every tooth carries the same amount of load.
Figure 1.3. A schematic model of a shaft-gear pair with a spline joint [5].
Therefore, $g[u(\tau)]$ can be reduced to a simple PL or PN function with three segments similar to Figure 1.2(a). Kahraman [5] showed that if the tooth spacing error is assumed to vary linearly at the spline interface as $\delta_i = (i-1)\delta, \ (i \in [1,n])$, $g[u(\tau)]$ can be approximated to the following PN form

$$G[u(\tau)] = \begin{cases} 
\frac{[u(\tau) - 1]}{2} + \frac{[u(\tau) - 1]^2}{2\delta}, & u(\tau) > 1, \\
0, & |u(\tau)| < 1, \\
\frac{[u(\tau) + 1]}{2} - \frac{[u(\tau) + 1]^2}{2\delta}, & u(\tau) \leq -1,
\end{cases}$$

(1.5b)

which is clearly in the form of equation (1.2a) with $\alpha_1 = 1/2$, $\alpha_2 = 1/(2\delta)$, and $\alpha_3 = 0$. For other types of tooth spacing error distribution, $g[u(\tau)]$ reduces to different forms within the family of curves defined by equation (1.2a). In summary, the torsional dynamic model of a spline interface having unavoidable tooth spacing errors can be described by a TI version of equation (1.1) with $w(\tau) = 1$.

1.2.2 A Sphere-Plane Hertzian Contact

The vibration between contact interfaces of elastic-plastic solids may result in fatigue and wear of contacting parts, or generate excessive high-level noise in machines. Hertzian theory was utilized extensively to model the flexibility of the contact interfaces [6-13]. In reference [12], the double sphere-plane Hertzian contact was simplified to a
SDOF system shown in Figure 1.4. The dimensionless equation of motion for this system is given as

\[
\ddot{u} + 2\zeta \dot{u} + g[u(\tau)] = 1 + f(\tau)
\]  

(1.6a)

where \( u(\tau) \) is the displacement of the unit mass, and \( f(\tau) \) is the dynamic component of external force. In equation (1.6a), \( g[u(\tau)] \) is a PN restoring function defined by equation (1.2b) representing a Hertzian contact allowing contact lose. Using Taylor series expansion, equation (1.2b) can be approximated by Taylor series with the first three terms as

\[
g[u(\tau)] = \begin{cases} 
1 + \frac{3}{2} \rho u(\tau) + \frac{3}{8} [\rho u(\tau)]^2 - \frac{1}{16} [\rho u(\tau)]^3, & \text{if } u(\tau) > -\frac{1}{\rho}, \\
0, & \text{if } u(\tau) \leq -\frac{1}{\rho}.
\end{cases}
\]

(1.6b)

This indicates that for \( u(\tau) > -\frac{1}{\rho} \), \( g[u(\tau)] \) can be described by linear, quadratic and cubic segments approximately. Therefore, the forms of \( g[u(\tau)] \) defined by equations (1.2b) and (1.6b) are similar to each other qualitatively. It is also clear that a sphere-plane contact problem can be treated as a special case of equation (1.1) with TI stiffness \( w(\tau) = 1 \).
Figure 1.4. The dynamic model of a SDOF sphere-plane contact oscillator [12].
1.2.3 A Rolling Element Bearing Support

In order to model rotor systems supported by rolling element bearings, the bearing compliance and radical clearances must be included in the dynamic model. This can be done by expressing the bearing flexibility as a PN function, like the one defined by Kahraman and Singh [14]

\[
g(t) = \begin{cases} 
  k_t \sum_{r=1}^{H} \left[ y(t) \cos \psi_r - b \right]^q \cos \psi_r, & |y(t)| > b, \\
  0, & |y(t)| \leq b, \\
  -k_t \sum_{r=1}^{H} \left[ y(t) \cos \psi_r - b \right]^q \cos \psi_r, & |y(t)| < b,
\end{cases}
\]

(1.7)

where \( y(t) \) is the radial displacement of the rotor, \( k_t \) is the contact stiffness of an individual rolling element, \( \psi_r \) is the angular position of the \( r \)-th rolling element in contact measured from the direction of preload, \( q \) is the power of the nonlinear force-displacement relationship \( (q = \frac{3}{2} \) for ball bearings and \( q = \frac{10}{9} \) for roller bearings), \( 2b \) is the diametral clearance of the bearing, and \( H \) is the total number of roller elements within load zone during rotation. Defining a dimensionless displacement as \( u(t) = y(t)/b \) and a dimensionless time \( \tau = \omega_n t \) where \( \omega_n \) is the natural frequency of the linear model for the gear mesh, \( g[y(t)] \) can be transformed in dimensionless form as
Once the bearing parameters are known, \( g[u(\tau)] \) can be reduced to the form as in equation (1.2a). One complication here is that the number of rolling elements in contact \( H \) may be a variable number, which is dependent on the preload applied and the radial clearance \( b \) [15, 16]. When the cage holds rolling elements from rotating with the shaft, and the same amount of radial bearing force is maintained, then \( H \) remains constant yielding \( w(\tau) = 1 \) in equation (1.1). For most typical cases, when the roller cage rotates, the number of rolling elements within the load zone fluctuates between two integers \( m \) and \( m+1 \). If the static preload on bearing is high, and \( b \) is relatively small, especially for ball-type rolling elements, the variation of \( H \) in load zone has an insignificant effect on the nonlinear stiffness characteristics of the bearing [16]. In such cases, \( w(\tau) = 1 \), and otherwise, a TV stiffness \( w(\tau) \) is needed in equation (1.1) for more general cases.

1.2.4 A Gear Pair

A spur gear pair can be modeled as a semi-definite two degree-of-freedom (DOF) purely torsional dynamic system by assuming that shafts and bearings are rigid, and bearings have no radial clearances. Such a dynamic model of a gear pair is illustrated in
Figure 1.5 [17]. After eliminating the rigid body mode of the system, the model is transformed to a SDOF system governed by following equation

$$\ddot{y}(t) + \frac{c}{m_{eq}} \dot{y}(t) + \frac{k(t)}{m_{eq}} g[y(t)] = \frac{r_1 T_1(t)}{I_1} + \frac{r_2 T_2(t)}{I_2} + \dot{e}(t). \tag{1.9a}$$

Here, $y(t) = r_1 \theta_1(t) + r_2 \theta_2(t) - e(t)$, which defines the difference between dynamic and static transmission errors along the line of action, $r_i$ and $\theta_i$ ($i = 1, 2$) represent the base circle radius and rotational displacement of the gear and pinion, respectively. $c$ is the viscous damping coefficient, $k(t)$ is the TV mesh stiffness, $e(t)$ is the static transmission error, $m_{eq} = I_1 I_2 / (r_1^2 I_2 + r_2^2 I_1)$ is the equivalent mass of gear pair, and $I_i$ and $T_i$ ($i = 1, 2$) denote the rotary inertias and torques of gear $i$. The nonlinear restoring function $g[y(t)]$ has often been expressed approximately in a PL form as

$$g[y(t)] = \begin{cases} 
  y(t) - b, & y(t) > b, \\
  0, & |y(t)| \leq b, \\
  y(t) + b, & y(t) < -b. 
\end{cases} \tag{1.9b}$$

By defining various new dimensionless variables, such as time $\tau = \omega_n t$, TV stiffness $w(\tau) = k(t)/k_m = 1 + k_m(t)$, where $k_m$ is the mean component of $k(t)$, natural frequency of the corresponding undamped linear system $\omega_n = \sqrt{k_m/m_{eq}}$, damping ratio
Figure 1.5 A torsional dynamic model of a gear pair.
\[ \zeta = c / (2\sqrt{k_m/m_{eq}}), \text{ displacement } u(\tau) = y(t)/b, \text{ and external excitation} \]

\[ f(\tau) = \frac{m_{eq}}{bk_m} \left[ \frac{r_1}{I_1} T_1(t) + \frac{r_2}{I_2} T_2(t) \right] + \ddot{\epsilon}(t) = f_m + f_a(\tau), \text{ a dimensionless equation is obtained as} \]

as [17]

\[
\dddot{u}(\tau) + 2\zeta \ddot{u}(\tau) + \omega^2 u(\tau) = f(\tau), \quad (1.10a)
\]

\[
g[u(\tau)] = \begin{cases} 
  u(\tau) - 1, & u(\tau) > 1, \\
  0, & |u(\tau)| \leq 1, \\
  u(\tau) + 1, & u(\tau) < -1.
\end{cases} \quad (1.10b)
\]

In equation (1.10b), the mesh stiffness is assumed as linear in the regions where contact is maintained. However, the overall average stiffness of a gear mesh interface increases with load mainly due to the Hertzian contact deformations [18]. Therefore, a PN form of \( g[u(\tau)] \) as defined in equation (1.2a) might be required, as discussed in reference [19].

By including quadratic and cubic terms in equation (1.10b), the variation of average mesh stiffness of a gear pair with load can be included in the dynamic model. Therefore, one can conclude that equation (1.1) governs a gear pair as well.
1.3 Literature Review

There are a vast number of published studies on dynamic modeling of the components mentioned in previous section. These studies are mostly limiting cases of equation (1.1). These models differ in terms of the type of excitation considered, configurations analyzed, numbers of DOF included, and analysis methodologies employed [20, 21]. Furthermore, the classification can be based on whether nonlinearities and TV coefficients are considered or not.

The dynamic behavior of systems with linear restoring functions is well understood. Modal summation and perturbation methods have been used to study TI [22-29] and TV [30-43] versions of these systems. Dynamic behaviors of oscillators having continuous nonlinearities and TI [44-63] or periodically TV [62-76] coefficients have also been studied extensively, and rich nonlinear phenomena have been exhibited. Since the discontinuous characteristic of \( g[u(\tau)] \) is the critical point of equation (1.1), studies about systems with continuous \( g[u(\tau)] \) will not be reviewed in section.

To review the studies of other cases of equation (1.1) with PL or PN restoring functions subjected to TI or TV stiffness, equation (1.2a) is utilized as the example form of \( g[u(\tau)] \). Four different limiting cases of equation (1.1) can be obtained as listed in Table 1.1: (i) PL systems with TI coefficients \( (\alpha_2 = \alpha_3 = 0 \text{ and } w(\tau) = 1) \), (ii) PL systems with periodically TV coefficients \( (\alpha_2 = \alpha_3 = 0 \text{ and periodic } w(\tau)) \), (iii) PN systems with TI coefficients \( (\alpha_2 \neq 0, \alpha_3 \neq 0 \text{ and } w(\tau) = 1) \), and (iv) PN systems with
<table>
<thead>
<tr>
<th>Model Name</th>
<th>$w(\tau)$</th>
<th>$g[u(\tau)]$</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>Piecewise linear TI (PLTI)</td>
<td>constant</td>
<td>$\begin{cases} u(\tau) - 1, &amp; u(\tau) &gt; 1, \ 0, &amp; -1 \leq u(\tau) \leq 1, \ u(\tau) + 1, &amp; u(\tau) &lt; -1. \end{cases}$</td>
<td>14, 17, 78-102</td>
</tr>
<tr>
<td>Piecewise linear TV (PLTV)</td>
<td>periodic</td>
<td>$\begin{cases} \sum_{i=1}^{n} \alpha_i [u(\tau) - 1], &amp; u(\tau) &gt; 1, \ 0, &amp; -1 \leq u(\tau) \leq 1, \ \sum_{i=1}^{n} (-1)^{i-1} \alpha_i [u(\tau) + 1], &amp; u(\tau) &lt; -1. \end{cases}$</td>
<td>103-116</td>
</tr>
<tr>
<td>Piecewise nonlinear TI (PNTI)</td>
<td>constant</td>
<td>$\begin{cases} \sum_{i=1}^{n} \alpha_i [u(\tau) - 1], &amp; u(\tau) &gt; 1, \ 0, &amp; -1 \leq u(\tau) \leq 1, \ \sum_{i=1}^{n} (-1)^{i-1} \alpha_i [u(\tau) + 1], &amp; u(\tau) &lt; -1. \end{cases}$</td>
<td>8-9, 11-13, 118-121, 126-134</td>
</tr>
<tr>
<td>Piecewise nonlinear TV (PNTV)</td>
<td>periodic</td>
<td>$\begin{cases} \sum_{i=1}^{n} \alpha_i [u(\tau) - 1], &amp; u(\tau) &gt; 1, \ 0, &amp; -1 \leq u(\tau) \leq 1, \ \sum_{i=1}^{n} (-1)^{i-1} \alpha_i [u(\tau) + 1], &amp; u(\tau) &lt; -1. \end{cases}$</td>
<td>18, 125</td>
</tr>
</tbody>
</table>

Table 1.1 Special cases of equation (1.1)
periodically TV coefficients ($\alpha_2 \neq 0$, $\alpha_3 \neq 0$ and periodic $w(\tau)$). Following literature review focuses on these discontinuous systems.

1.3.1 *Piecewise-Linear Time-Invariant (PLTI) Dynamic Models*

PL models have been used extensively to describe dynamics of mechanical and electro-mechanical systems subjected to discontinuous boundary conditions [77]. The models in this group have TI coefficients only, and consider $g[u(\tau)]$ formed by multiple linear segments like the one defined by equation (1.10).

Since PLTI systems have been studied thoroughly, only representative studies are reviewed here. Den Hartog [78-79] is perhaps one of the first investigators studying forced response of PL systems by analytical methods. In reference [78], dynamics of a SDOF system excited harmonically and subjected to dry friction were studied by expressing discontinuous friction force in Fourier series. Later, this work was extended to other types of PL restoring functions [79], and the piecewise linear method was implemented to find solutions of trilinear systems crossing one discontinuous point. More results on the same system were presented later [80].

Maezawa [81] proposed a procedure to predict the steady state periodic response of an asymmetric PL system. The nonlinear restoring and damping forces were treated as external excitations that were approximated by Fourier series. The governing equation was solved analytically, and the predictions were compared with analog simulation results [82]. An application of this method on an ultrasonic carving machine was
presented [83]. Later, the approach was extended to obtain sub-harmonic [84] and super-harmonic responses [85] of a harmonically excited symmetric trilinear system, and a periodically excited asymmetric PL system [86], both of which are undamped.

Shaw and Holmes [87] studied the dynamic behavior of a periodically forced asymmetric PL oscillator (including zero gap and impact) by using piecewise linear method. A wide array of nonlinear dynamic phenomena was predicted by applying bifurcation theory. Moon and Shaw [88] also set up an experiment formed by an elastic beam with nonlinear boundary conditions to demonstrate chaotic phenomenon from a simple dynamical system. More detailed treatments on similar systems followed [89], and a symmetric PL model with two rigid constrains was analyzed by the same procedure for local and global dynamic behavior [90, 91]. As an application of theoretical studies, dynamics of an impact print hammer were investigated based on an asymmetric PL model [92], and criteria to improve the performance of impact printers were proposed [93].

The idea of the piecewise linear method that was proposed earlier [79, 87] to study PL systems was generalized later by Natsiavas [94] to obtain double-crossing periodic solutions with some restrictions on initial conditions. This procedure was evolved to carry on the stability and bifurcation analysis of PL systems [95]. Later, Natsiavas and Gonzalez [96] extended previous work to asymmetric trilinear oscillators and dry friction problems [97]. All these studies [94-97] for obtaining steady state forced periodic response by applying the piecewise linear method excluded mean load term of external excitation.
Another group of investigators used different types of HBM to analyze PLTI systems in frequency domain. Comparin and Singh [98] applied describing function method to investigate the frequency response of a SDOF impact pair having harmonic external excitation. This procedure was extended to a coupled multi-degree-of-freedom (MDOF) semi-definite system with multiple clearances [99]. Later, Kahraman and Singh [17] included internal excitation in the model, and evolved to an asymmetric geared rotor-bearing system with gear backlash and bearing clearances [14]. Padmanabhan and Singh [100] examined the resonance coupling of a 2-DOF PLTI system by describing function method. Later, a semi-analytical method was proposed, which combines shooting method and parametric continuation techniques, and was applied to study a SDOF impacting pair model [101]. Recently, Kim et al. [102] revisited the impact pair problem by providing complete HBM solutions.

1.3.2 Piecewise-Linear Time-Varying (PLTV) Dynamic Models

Periodically TV coefficients or parametric (internal) excitations are common in many systems as described earlier. While continuous nonlinear systems having periodically TV parameters have been attracted significant attention [30-43, 62-76], the same cannot be said for PLTV systems.

One of the early investigations into steady state response of PLTV systems is by Kahraman and Singh [103]. In this study, the numerical integration technique was applied to predict the dynamic response of a spur gear pair. A generalized 3-DOF geared rotor-bearing PLTV model was proposed, and a simplified SDOF gear pair model and the
The interactions between backlash, and internal and external excitations were investigated, and compared with published experimental data. Sato, Yamamoto and Kawakami [104] introduced a similar SDOF PLTV model of a gear pair with harmonic external excitation. This study concentrated on bifurcation analysis of system response, and the existence of chaotic phenomena was predicted.

Later, Kahraman and Blankenship [105-107] used multi-term HBM in conjunction with DFT to obtain the steady state response of a SDOF PLTV oscillator that is excited parametrically and externally. Period-1 response was predicted and compared to experimental data measured from a spur gear pair [105]. The period-1 sub-harmonic motions of this system were determined by using the same analysis method [106]. Comprehensive experimental results were explained, and the existence of many dynamical phenomena, such as long-period subharmonic and chaotic motions, was demonstrated [107]. In a recent study, the analysis was extended to a 2-DOF PLTV model representing a multi-mesh gear train [108].

Natsiavas and Theodossiades [109] proposed a methodology which combines the piecewise linear method with a perturbation method to analyze a SDOF PLTV system with weak TV coefficients. A detailed explanation of this approach was given by analyzing a PLTV gear-pair model with weak external harmonic excitation [110], and a similar model with mean load only [111]. Later on, Theodossiades and Natsiavas investigated a 3-DOF [112] and a 5-DOF [113] PLTV geared rotor-bearing models by direct numerical integration method.
Padmanabhan and Singh [114] examined the forced system response of a SDOF and a 3-DOF PLTV geared models by shooting method. Raghothama and Narayanan [115] proposed a procedure to study a 3-DOF geared rotor-bearing system by incremental HBM. In this PLTV model, only gear mesh stiffness was assumed TV, and the stiffness of bearings was treated as linear. Limited periodic solutions were obtained by this method, and bifurcation analysis of the model on the route to chaos was carried out by numerical method. Belovodsky, Tsyfansky and Beresnevich [116] introduced an asymmetric SDOF PLTV vibro-machine model with a PLTV damping force and PLTI restoring force. The governing equation was solved by numerical methods, and a design procedure for parametric vibro-machines was obtained by parametric studies.

1.3.3 Piecewise Nonlinear Dynamic Models

A limited number of studies on PN systems is available, and only a very few of them considered TV coefficient. Also, the term “piecewise-nonlinear” was occasionally used to describe a PL system [117]. A brief review about these studies of PN systems is provided in this section.

Even though many real-life systems have PN type properties [119, 120], most investigators choose PL models to describe these systems mainly because of the lack of proper analysis methods [14]. As one of the few exceptions, Kahraman and Singh [121] investigated a geared system with both clearance and continuous nonlinearities by numerical integration. Three kinds of nonlinearities in a SDOF oscillator, quadratic, cubic and non-integer, were considered separately for different levels of excitations.
Szabelski and Litak [18] studied the forced chaotic response of a SDOF PNTV gear pair model with cubic nonlinearity by numerical approach, and demonstrated the effect of the cubic nonlinearity on chaotic motions.

PN models can also describe impact damping when clearances exist between impacting bodies [122], such as clutch-disc component in drive shaft systems [123, 124]. Azar and Crossley [125] proposed a MDOF PNTV model to describe a geared torsional system that contains a gear pair, and drive and load shafts. The impacting force at the contacting zone was expressed by PNTV stiffness, and the model was analyzed by numerical method for both unloaded and loaded cases. Later, Padmanabhan and Singh [126] investigated a 2-DOF automotive transmission model with piecewise impact damping and periodic excitation by shooting method. The effect of linear stiffness, mean load and alternating force on system response for different damping models with single and dual external excitations was demonstrated by an extensive parametric study. Kim et al. [127] extended this work to a more general impact damping model using various analysis methods.

In order to study the dynamics of offshore structures such as articulated loading platform, Choi and Lou [128] proposed an asymmetric two-piece nonlinear model, which includes linear damping, a TI stiffness, and harmonic external excitation without mean load. Harmonic and subharmonic motions were predicted by applying the multi-term HBM. Kim and Noah [129] studied this model with the same HBM procedure, and used Newton-Raphson method to obtain the solutions. Raghothama and Narayanan [130] investigated this PNTI model by incremental HBM, considering a generalized nonlinear
restoring function with linear, quadratic and cubic components. Wang [131] studied the
same model with periodic external excitation by finite element in time method, a
numerical integration method that allows conventional shape functions to be used within
finite time elements.

Cveticanin [132] studied dynamic behaviors of a preloaded SDOF system with PN
restoring function having linear and quadratic terms assuming solutions in the form of
Jacobi elliptic functions. Gottlieb [133] considered free vibrations of an undamped
SDOF oscillator with a preload type PN restoring function having positive powers
(integer or inverse integer). Exact solutions were obtained for some special cases by
using Beta and Gamma functions. Ji and Hansen [134] examined a harmonically excited
SDOF system, which has a restoring function that is formed by three segments. The
upper and lower segments are saturation type PL functions, and the middle piece has
impact damping and cubic nonlinearity component. The equation was analyzed by using
piecewise linear method, and perturbation methods were applied on the middle nonlinear
section with the assumption of weak nonlinearity. Another group of investigations on
dynamics of Hertzian contacts [8-9, 11-13] using PNTI models will be reviewed in
chapter 4 in the context of a sphere-plane contact problem.

1.4 Dissertation Objectives

While a significant number of studies were performed on continuously nonlinear and
piecewise linear systems with or without TV parameters, a limited number of published
studies exist for PN systems with constant parameters [12, 119-121, 126-134], as reviewed in previous section. Since most of these investigations were numerical in nature [18, 121, 125, 132-133], and application oriented, dynamics of PNTI systems are still not fully understood. While semi-analytical and analytical methods, such as shooting method [12, 126] and HBM [127-130] were applied occasionally, a systematic analytical study on the dynamic behavior of systems governed by equation (1.1) is yet to be performed. In addition, dynamics of PN systems with TV parameters have not been studied in detail. Accordingly, the main objectives of this study are as follows:

- Propose a general analytical procedure to obtain steady state response of PNTV systems by using the multi-term HBM in conjunction with DFT and a parametric continuation method.

- Examine the accuracy of multi-term HBM solutions through a comparison to corresponding numerical integration results.

- Analyze the stability of predicted steady state motions using Floquet theory.

- Demonstrate the overall nonlinear dynamic behavior of the examined systems in terms of both period-1 and period-η (η >1, subharmonic) motions, and investigate
the dynamic response in the frequency ranges of the parametric instabilities of corresponding linear systems.

- Describe the influence of key system parameters including quadratic and cubic nonlinearities of restoring function, periodic internal and external excitations, and damping on the steady state response.

- Identify significant differences between PLTV and PNTV systems as well as between PNTI and PNTV systems.

- Apply the analytical procedure to a generic elastic sphere-plane contact problem with contact loss. Validate the theoretical model and the solution method by comparing the predictions to numerical and experimental data. Derive conditions for contact losses to occur.

- Extend the methodology to demonstrate its applicability to study the dynamic behavior of MDOF, multiple clearance PNTV systems.
1.5 Dissertation Outline

This dissertation consists of six chapters. In chapter 2, period-1 motions of a SDOF PN mechanical oscillator with parametric and external excitations are investigated. In chapter 3, the subharmonic (period-$\eta$, $\eta > 1$) motions of the same PN mechanical oscillator are investigated within the range of resonances. In chapter 4, the solution methodology presented in previous chapters is applied to study the dynamic behavior of an elastic sphere-plane contact interface analytically. In chapter 5, the solution method is extended to analyze a generalized MDOF system with multiple clearances and PNTV components, and is demonstrated on examining a 3-DOF gear-bearing system. Finally, chapter 6 provides a summary and major contributions of this study as well as a list of recommendations for future work.
2.1 Introduction

In this chapter, the period-1 response of a SDOF system governed by a version of equation (1.1) is studied. For this case, a viscously damping model is used, corresponding to \( m = 0 \) in equation (1.1). The equation of motion for this case is

\[
\ddot{u}(\tau) + 2\zeta \dot{u}(\tau) + w(\tau) g[u(\tau)] = f(\tau).
\]  

Here, PN restoring function \( g[u(\tau)] \) is defined by equation (1.2a). As mentioned in chapter 1, a significant number of studies were performed on CN and PL systems with or without TV parameters. A limited number of published studies exist for PN systems, which combine both continuous and clearance nonlinearities. Most of these studies fall into the PNTI category [5, 6, 8-10, 12, 121, 125, 126, 128, 130, 131] with a constant stiffness coefficient, and many used numerical techniques with the exception of references [126, 130]. Meanwhile, the dynamic behavior of PNTV systems, as defined
by equations (2.1) and (1.2a), is yet to be studied in detail. Accordingly, this chapter focuses on the dynamic response of a SDOF PNTV oscillator governed by equation (2.1).

Specific objectives of this chapter are to:

• obtain the steady state response analytically by using a multi-term HBM in conjunction with DFT,

• demonstrate the accuracy of harmonic balance solutions by comparison to numerical integration results,

• formulate the single-term steady state response by using the describing function method,

• describe the impact of continuous nonlinearities of different types and magnitudes on the steady state response of the PNTV systems,

• quantify the influence of TV stiffness on the response through the comparison between the solutions of PNTI and PNTV systems, and

• describe the effect of key system parameters such as damping ratio $\zeta$, alternating stiffness amplitude, mean load $f_1$, and alternating external force amplitude on the dynamic response.

A parametric continuation scheme is utilized to find solutions while passing through turning points. Floquet theory is employed to examine the stability of the steady state motions.
2.2. Multi-term Period-1 Response to Periodic Excitations

2.2.1 Analytical Procedure

A general method for obtaining steady state period-1 solutions of equation (2.1) is presented in this section. This method combines a multi-term HBM formulation with DFT, which was applied to PL systems earlier successfully [105, 106, 135-137]. Here, \( w(\tau) \) and \( f(\tau) \) are written in the form of truncated Fourier series as

\[
w(\tau) = 1 + \sum_{\kappa=1}^{K} \left[ w_{2\kappa} \cos(\kappa \Lambda \tau) + w_{2\kappa+1} \sin(\kappa \Lambda \tau) \right], \tag{2.2}
\]

\[
f(\tau) = f_1 + \sum_{\mu=1}^{M} \left[ f_{2\mu} \cos(\mu \Lambda \tau) + f_{2\mu+1} \sin(\mu \Lambda \tau) \right], \tag{2.3}
\]

where \( \Lambda \) is the dimensionless fundamental excitation frequency, \( w_1 = 1 \) is the mean component of the stiffness function, and \( w_{2\kappa} \) and \( w_{2\kappa+1} \) are the \( \kappa \) -th harmonic amplitudes of \( w(\tau) \). \( f_1 \) is the mean load applied to the unit mass, and \( f_{2\mu} \) and \( f_{2\mu+1} \) are the \( \mu \) -th harmonic amplitudes of \( f(\tau) \). \( \kappa \) and \( \mu \) are integer valued harmonic indices. By defining \( \theta = \Lambda \tau \), equation (2.1) becomes

\[
\Lambda^2 \frac{d^2 u(\theta)}{d\theta^2} + 2\zeta \Lambda \frac{du(\theta)}{d\theta} + w(\theta) g[u(\theta)] = f(\theta). \tag{2.4}
\]
The unknown steady state period-1 response $u(\theta)$ and the PN restoring function $g[u(\theta)]$ can also be expressed in Fourier series form as

$$u(\theta) = u_1 + \sum_{r=1}^{R} [u_{2r} \cos(r\theta) + u_{2r+1} \sin(r\theta)],$$  \hspace{1cm} (2.5)$$

$$g[u(\theta)] = v_1 + \sum_{r=1}^{R} [v_{2r} \cos(r\theta) + v_{2r+1} \sin(r\theta)].$$  \hspace{1cm} (2.6)$$

While representing $g[u(\theta)]$ in Fourier series form seems unreasonable, the use of discrete Fourier transforms in determining its coefficients captures the piecewise properties properly [135-137] as it will be done later. By substituting equations (2.2), (2.3), (2.5) and (2.6) into equation (2.4) and enforcing harmonic balance, a vector equation $S = 0$ is obtained, and the elements of $S$ are defined as

$$S_1 = v_1 - f_1 + \frac{1}{2} \sum_{k=1}^{K} [w_{2k} v_{2k} + w_{2k+1} v_{2k+1}],$$  \hspace{1cm} (2.7a)$$

$$S_{2r} = -\Lambda^2 r^2 u_{2r} + 2\zeta \Lambda r u_{2r+1} + v_{2r} + v_{1} w_{2r} - f_2 r$$

$$+ \frac{1}{2} \sum_{k=1}^{K} w_{2k} [v_{2(k-r)} + v_{2(k+r)} + v_{2(r-k)}]$$  \hspace{1cm} (2.7b)$$

$$+ \frac{1}{2} \sum_{k=1}^{K} w_{2k+1} [v_{2(k-r)+1} + v_{2(k+r)+1} - v_{2(r-k)+1}], \quad r \in [1, R].$$

$$32$$
\[ S_{2r+1} = -\Lambda^2 r^2 u_{2r+1} - 2\zeta \Lambda r u_{2r} + v_{2r+1} + v_{1} w_{2r+1} - f_{2r+1} \]

\[ + \frac{1}{2} \sum_{k=1}^{K} w_{2k} \left[ -v_{2(k-r)+1} + v_{2(k+r)+1} + v_{2(r-k)+1} \right] \]

\[ + \frac{1}{2} \sum_{k=1}^{K} w_{2k+1} \left[ v_{2(k-r)} - v_{2(k+r)} + v_{2(r-k)} \right], \quad r \in [1, R] \]  

(2.7c)

The coefficients \( v_i \) of \( g[u(\theta)] \) can be expressed in terms of unknown Fourier coefficients of the response \( \mathbf{u} = [u_1 \ u_2 \ u_3 \ \ldots \ u_{2R} \ u_{2R+1}]^T \) by utilizing the DFT \([105, 106, 135, 136]\).

The values of \( u(\theta) \) at discrete values of \( \theta = nh \) are

\[ u^{(n)} = u_1 + \sum_{r=1}^{R} \left[ u_{2r} \cos\left(\frac{2\pi r n}{N}\right) + u_{2r+1} \sin\left(\frac{2\pi r n}{N}\right) \right], \]  

(2.8a)

where \( h = 2\pi/(N\Lambda) \), \( N \geq 2R \), and \( n \in [0, N-1] \). Using equation (1.2a), the \( n \)-th discrete value of \( g[u(\theta)] \) is given as

\[ g^{(n)} = \begin{cases} 
\sum_{i=1}^{3} \alpha_i [u^{(n)} - 1]^i, & u^{(n)} > 1, \\
0, & |u^{(n)}| \leq 1, \\
\sum_{i=1}^{3} (-1)^{i-1} \alpha_i [u^{(n)} + 1]^i, & u^{(n)} < -1,
\end{cases} \]  

(2.8b)
and the Fourier coefficients of \( g[u(\theta)] \) are calculated by taking the inverse DFT of equation (2.8) as

\[
v_1 = \frac{1}{N} \sum_{n=0}^{N-1} g^{(n)}, \quad (2.9a)
\]

\[
v_{2r} = \frac{2}{N} \sum_{n=0}^{N-1} g^{(n)} \cos \frac{2\pi rn}{N}, \quad (2.9b)
\]

\[
v_{2r+1} = \frac{2}{N} \sum_{n=0}^{N-1} g^{(n)} \sin \frac{2\pi rn}{N}. \quad (2.9c)
\]

Having \( v_j \), the vector equation \( S = 0 \) can be solved for \( u \) by employing the Newton-Raphson method as

\[
\mathbf{u}^{(m)} = \mathbf{u}^{(m-1)} - [\mathbf{J}^{-1}]^{(m-1)} \mathbf{S}^{(m-1)}, \quad (2.10)
\]

where the value of \( \mathbf{u}^{(m)} \) at the \( m \)-th iteration is obtained from the values of \( \mathbf{S}^{(m-1)} \) and \( \mathbf{u}^{(m-1)} \), and \( \mathbf{J} \) is the Jacobian matrix whose elements are given as

\[
\frac{\partial S_1}{\partial u_j} = \frac{\partial v_1}{\partial u_j} + \frac{1}{2} \sum_{\kappa=1}^{K} \left( w_{2\kappa} \frac{\partial v_{2\k\eta}}{\partial u_j} + w_{2\k\eta+1} \frac{\partial v_{2\k\eta+1}}{\partial u_j} \right), \quad (2.11a)
\]
\[ \frac{\partial S_{2i}}{\partial u_j} = -H^2 i^2 \delta_{2i} + 2\zeta H i \delta_{2i+1} + \frac{\partial v_{2i}}{\partial u_j} + w_{\kappa} \frac{\partial v_1}{\partial u_j} 
+ \frac{1}{2} \sum_{k=1}^{K} w_{2k} \left[ \frac{\partial v_{2(k\kappa-i)}}{\partial u_j} + \frac{\partial v_{2(k\kappa+i)}}{\partial u_j} + \frac{\partial v_{2(i-\kappa\kappa)}}{\partial u_j} \right] 
+ \frac{1}{2} \sum_{k=1}^{K} w_{2k+1} \left[ \frac{\partial v_{2(k\kappa-i)+1}}{\partial u_j} + \frac{\partial v_{2(k\kappa+i)+1}}{\partial u_j} - \frac{\partial v_{2(i-\kappa\kappa)+1}}{\partial u_j} \right] \tag{2.11b} \]

\[ \frac{\partial S_{2i+1}}{\partial u_j} = -H^2 i^2 \delta_{2i+1} - 2\zeta H i \delta_{2i+1} + \frac{\partial v_{2i+1}}{\partial u_j} + w_{\kappa+1} \frac{\partial v_1}{\partial u_j} 
+ \frac{1}{2} \sum_{k=1}^{K} w_{2k} \left[ -\frac{\partial v_{2(k\kappa-i)+1}}{\partial u_j} + \frac{\partial v_{2(k\kappa+i)+1}}{\partial u_j} + \frac{\partial v_{2(i-\kappa\kappa)+1}}{\partial u_j} \right] \n+ \frac{1}{2} \sum_{k=1}^{K} w_{2k+1} \left[ -\frac{\partial v_{2(k\kappa-i)}}{\partial u_j} + \frac{\partial v_{2(k\kappa+i)}}{\partial u_j} + \frac{\partial v_{2(i-\kappa\kappa)}}{\partial u_j} \right] \tag{2.11c} \]

where \( i \in [1, R] \), \( j \in [1, 2R + 1] \), \( \delta_i \) is the Kronecker delta, and

\[ \delta_i = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \tag{2.12} \]

The partial derivatives \( \frac{\partial v_i}{\partial u_j} \) in equation (2.11) can be express as

\[ \frac{\partial v_1}{\partial u_1} = \frac{1}{N} \sum_{n=0}^{N-1} |\Phi_n| E_n, \tag{2.13a} \]
\[ \frac{\partial v_1}{\partial u_{2j}} = \frac{1}{N} \sum_{n=0}^{N-1} |\Phi_n| E_n \cos \frac{2\pi jn}{N}, \quad (2.13b) \]

\[ \frac{\partial v_1}{\partial u_{2j+1}} = \frac{1}{N} \sum_{n=0}^{N-1} |\Phi_n| E_n \sin \frac{2\pi jn}{N}, \quad (2.13c) \]

\[ \frac{\partial v_{2i}}{\partial u_1} = \frac{2}{N} \sum_{n=0}^{N-1} |\Phi_n| E_n \cos \frac{2\pi in}{N}, \quad (2.13d) \]

\[ \frac{\partial v_{2i}}{\partial u_{2j}} = \frac{2}{N} \sum_{n=0}^{N-1} |\Phi_n| E_n \cos \frac{2\pi in}{N} \cos \frac{2\pi jn}{N}, \quad (2.13e) \]

\[ \frac{\partial v_{2i}}{\partial u_{2j+1}} = \frac{2}{N} \sum_{n=0}^{N-1} |\Phi_n| E_n \cos \frac{2\pi in}{N} \sin \frac{2\pi jn}{N}, \quad (2.13f) \]

\[ \frac{\partial v_{2i+1}}{\partial u_1} = \frac{2}{N} \sum_{n=0}^{N-1} |\Phi_n| E_n \sin \frac{2\pi in}{N}, \quad (2.13g) \]

\[ \frac{\partial v_{2i+1}}{\partial u_{2j}} = \frac{2}{N} \sum_{n=0}^{N-1} |\Phi_n| E_n \sin \frac{2\pi in}{N} \cos \frac{2\pi jn}{N}, \quad (2.13h) \]

\[ \frac{\partial v_{2i+1}}{\partial u_{2j+1}} = \frac{2}{N} \sum_{n=0}^{N-1} |\Phi_n| E_n \sin \frac{2\pi in}{N} \sin \frac{2\pi jn}{N}, \quad (2.13i) \]

where

\[ E_n = \alpha_1 - 2\alpha_2 + 3\alpha_3 - 2\Phi_n (\alpha_2 - 3\alpha_3) u_n + 3\alpha_3 u_n^2, \quad (2.14a) \]
The Newton-Raphson iteration starts with an initial guess \( u^{(0)} \) and a control parameter \( \Lambda \). The process is repeated until the steady state solution \( u^{(m)} \) converges within a predefined error limit. Then \( \Lambda \) is set to the next value of interest by increasing or decreasing until a turning point impedes continuation. In order to find the location of the turning point, the artificial-parameter generic homotopy method is utilized [138, 139]. The iteration process is continued by using the value of \( u^{(m)} \) at the turning point as the new initial guess, and change \( \Lambda \) in the opposite incremental direction.

### 2.2.2 Stability Analysis

The stability of the steady state response is determined by using Floquet theory [140]. Introducing a small variation \( \Delta u(\tau) \) to the periodic solution \( u_0(\tau) = u_0(\tau + T) \) where \( T \) is the least period of \( u(\tau) \), the following variational equation is obtained:

\[
\Delta \ddot{u} + 2 \zeta \Delta \dot{u} + \Psi(\tau) \delta [u_0(\tau)] \Delta u = 0,
\]  

(2.15a)

where
\[
\tilde{g}[u(\tau)] = \begin{cases} 
\sum_{i=1}^{3} i\alpha_i [u-1]^{i-1}, & u > 1, \\
0, & |u| \leq 1, \\
\sum_{i=1}^{3} [-1]^{i-1} i\alpha_i [u+1]^{i-1}, & u < -1.
\end{cases}
\] (2.15b)

Defining \( y = [\Delta u(\tau) \ \Delta \dot{u}(\tau)]^T \) where \( y \) is a state vector, equation (2.15a) can be transformed into a system of first-order equations on the state plane in matrix form as

\[
y'(\tau) = H(\tau) y(\tau),
\] (2.15c)

where \( H(\tau) = H(\tau + T) \) is the periodic state matrix that can be expressed as

\[
H(\tau) = \begin{bmatrix} 0 & 1 \\ -w\tilde{g} & -2\zeta \end{bmatrix}.
\] (2.15d)

The local stability of \( u(\tau) \) is dependent on \( \Delta u(\tau) \). If \( \Delta u(\tau) \) decays or stays bounded when \( \tau \to T \), then \( u(\tau) \) is locally stable; otherwise it is unstable. Moreover, the variation of \( \Delta u(\tau) \) during one minimal period can be measured from examining the eigenvalues of a monodromy matrix \( F \) that is dependent on the variational equation (2.15a).
There are different ways to obtain \( F \). From reference [2.31], \( F = Y(T) \), where \( Y = [y_1(\tau) \ y_2(\tau)] \) is the fundamental matrix solution of equation (2.15c) with the initial condition \( Y(0) = I_2 \), and \( I_2 \) is a \( 2 \times 2 \) identity matrix. Thus, one way to obtain \( F \) is through direct numerical integration of equation \( F = Y(T) \) with \( u_0(\tau) \), \( w(\tau) \) and \( \tilde{g}[u_0(\tau)] \) constructed from previously solved Fourier coefficients \( u_i \) and \( v_i \).

Hsu and Cheng [38] proposed an efficient numerical method to calculate \( F \) approximately. The state matrix \( H \) is described as a series of step functions \( H_m \) at \( M \) discrete time intervals \( T = m\Delta \) which is written as

\[
H_m = \frac{1}{\Delta} \int_{(m-1)\Delta}^{m\Delta} H(\tau)d\tau, \quad m \in [1, M]. \tag{2.16}
\]

Hence, \( F \) can be estimated approximately as

\[
F \cong \prod_{m=1}^{M} \left[ I_2 + \sum_{n=1}^{N} \left( \frac{\Delta H_m^n}{n!} \right) \right] \tag{2.17}
\]

where \( N \) is the number of terms in the series expansion of exponential matrix \( e^{H_n} \). Thus, \( F \) can be constructed from discrete Fourier transform method.
The local stability of $u_0(\tau)$ is determined by the eigenvalues $\lambda_1$ and $\lambda_2$ of $F$, which are called as Floquet multipliers. If both $\lambda_1$ and $\lambda_2$ are less than unity, the solution is stable. Otherwise, it is an unstable solution.

2.2.3 Comparison to Numerical Integration Results

A typical forced response is illustrated in Figure 2.1 for an oscillator having a periodically TV stiffness function with $w_3 = 0.3, w_5 = 0.15, w_7 = 0.1$ ($K = 3$), all other $w_i = 0$, and $\zeta = 0.05$. Here, the root-mean-square (RMS) amplitude of the steady state response is defined as $u_{rms} = \left[ \sum_{r=1}^{R} \left( u_{2r}^2 + u_{2r+1}^2 \right) \right]^{1/2}$. The PN restoring function $g[u(\tau)]$ contains both quadratic and cubic coefficients $\alpha_2 = 0.1$ and $\alpha_3 = 0.2$, in addition to a linear term $\alpha_1 = 1$. The oscillator is subjected to a mean load $f_1 = 0.5$ with no alternating external force ($f_i = 0$, $i = 2, 3, \ldots$). Figure 2.1(a) shows the $u_{rms}$ values of both stable and unstable HBM solutions as a function of dimensionless frequency $\Lambda$. These solutions are obtained by using six harmonic terms ($R = 6$) in equation (2.6). In addition, $u_{rms}$ obtained by direct numerical integration of equation (2.1) using backward differentiation formulas is also shown in Figure 2.1(a). Within each period, 2048 points are considered in the numerical integration to represent the transition times between the piecewise solution regimes reasonably well. A very good agreement is observed between the solutions from both methods. Similarly, the values of $u_1$ predicted by HBM and the numerical integration method compare well in Figure 2.1(b) for the same case, further suggesting that the HBM solutions are indeed correct.
Figure 2.1(a) $u_{rms}$ and (b) $u_1$ values of forced response ($R = 6$) as a function of dimensionless frequency $\Lambda$ given $\alpha_2 = 0.1$, $\alpha_3 = 0.2$, $f_1 = 0.5$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$, $w_5 = 0.15$, $w_7 = 0.1$ ($K = 3$), all other $w_i = 0$, and $\zeta = 0.05$. (---) stable HBM solutions, (-----) unstable HBM solutions, and (□) numerical integration solutions.
It is noted from Figure 2.1(a) that, at $\Lambda \approx 0.9$, the stable (lower branch) solution loses its stability, causing a jump-up to another stable (central branch) solution. Further examining these solutions in time domain indicates that the lower branch solution is such that $u(\tau) > 1$ for all $\tau$. This suggests that the motion is contained at the right piecewise segment of $g[u(\tau)]$, and the clearance nonlinearity has no influence on this motion. Using the same terminology introduced by Comparin and Singh [98], such motions are called no-impact (NI) motions. NI motions will always take place in the right piecewise segment since the external force has a mean component forcing a contact in that segment. Similarly in Figure 2.1(a), for the solutions on central branch, $|u(\tau)|<1$ when $\Lambda \in [0.75, 1.25]$, indicating that separation takes place. Here, these motions are named as single-sided-impact (SSI) motions that demonstrate a typical softening type behavior due to contact loss. Finally, reducing $\Lambda$ on the central branch, SSI motions lose the stability at $\Lambda \approx 0.75$. Either a jump-down to the lower branch NI motion or a jump-up to another stable motion is possible. These higher amplitude motions are such that $u(\tau) < -1$ for certain $\tau$, which indicates that these motions have back contacts following separations. The mass travels through the entire clearance region to initiate contact at the left piecewise segment. These motions are called double-sided-impact (DSI) motions.

2.3. Single-term Response to Harmonic Excitations

Figure 2.2 illustrates the first three harmonic amplitudes of the multi-term HBM solution given in Figure 2.1. Here, the $r$-th harmonic amplitude is defined as
Figure 2.2. Harmonic content of the steady state HBM solution shown in Figure 2: (a) $A_1$, (b) $A_2$, and (c) $A_3$. 
\[ A_r = [u_{2r}^2 + u_{2r+1}^2]^{1/2} \text{ where } u_{rms} = \left[ \sum_{r=1}^{R} A_r^2 \right]^{1/2}. \]

Comparing the \( u_{rms} \) values in Figure 2.1(a) to those in Figure 2.2 shows that the resonance peak near \( \Lambda = \frac{1}{r} \) is dominated by the \( A_r \) component of the response. Therefore, if the response near any particular resonance frequency of \( \Lambda = \frac{1}{r} \) is sought, the corresponding single-term solution could provide a reasonably accurate approximation of overall response. In this section, the feasibility of such solutions is explored by using the describing function method [77].

First, \( w(\tau) \) and \( f(\tau) \) in equation (2.1) are expressed as purely harmonic excitations:

\[
\begin{align*}
\text{(2.18a)}
\quad w(\tau) &= 1 + w_{2\kappa+1} \sin(\kappa \Lambda \tau), \\
\text{(2.18b)}
\quad f(\tau) &= f_1 + f_{2\mu+1} \sin(\mu \Lambda \tau + \Gamma_{\mu}),
\end{align*}
\]

where \( \kappa \Lambda \) and \( \mu \Lambda \) are the excitation frequencies of \( w(\tau) \) and \( f(\tau) \), respectively. Harmonic indices \( \kappa \) and \( \mu \) are commensurate positive integers, i.e. \( \mu = m\kappa \), or \( \kappa = n\mu \) with \( m \) and \( n \) being positive integers. \( w_{2\kappa+1} \) and \( f_{2\mu+1} \) are harmonic amplitudes of \( w(\tau) \) and \( f(\tau) \), and \( f_1 \) is the mean component of \( f(\tau) \). \( \Gamma_{\mu} \) is the phase angle between two excitations. The unknown steady state solution \( u(\tau) \) is assumed as

\[
\begin{align*}
\text{(2.19)}
\quad u(\tau) &= u_1 + u_{2\kappa+1} \sin(\kappa \Lambda \tau + \beta_{\kappa}) + u_{2\mu+1} \sin(\mu \Lambda \tau + \beta_{\mu}),
\end{align*}
\]
where \( u_1 \) is a mean component, \( u_{2\kappa+1} \) and \( u_{2\mu+1} \) are amplitudes of the alternating components at frequencies \( \kappa \Lambda \) and \( \mu \Lambda \) respectively, and \( \beta_\kappa \) and \( \beta_\mu \) are the corresponding phase angles. Defining \( \varphi_\kappa = \kappa \Lambda \tau + \beta_\kappa \), \( \varphi_\mu = \mu \Lambda \tau + \beta_\mu \), with \( \kappa = m \mu \), then

\[
\varphi_\kappa = m \varphi_\mu + A_\mu ,
\]  

(2.20)

where \( A_\mu = \beta_{m \mu} - m \beta_\mu \). With this, equation (2.19) and \( g[u(\tau)] \) are given as:

\[
u(\varphi_\mu ) = u_1 + u_{2\mu+1} \sin \varphi_\mu + u_{2m\mu+1} \sin( m \varphi_\mu + A_\mu ),
\]  

(2.21a)

\[
g[u(\varphi_\mu )] = v_1 + v_{2\mu+1} \sin \varphi_\mu + v_{2m\mu+1} \sin( m \varphi_\mu + A_\mu ).
\]  

(2.21b)

Here, \( v_1 \), \( v_{2\mu+1} \) and \( v_{2m\mu+1} = v_{2\kappa+1} \) are so-called describing functions [77]. The following two cases are of special interest.

2.3.1 \( \kappa = \mu \).

In this case, the parametric and forcing excitation frequencies are same. Therefore, the frequencies of primary resonance peak due to two excitations \( f(\tau) \) and \( w(\tau) \) are both at \( \Lambda = l/\kappa = l/\mu \). For \( \kappa = \mu \), \( u(\varphi ) = u_1 + \hat{u}_{2\kappa+1} \sin \varphi \) and \( g[u(\varphi )] = v_1 + \hat{v}_{2\kappa+1} \sin \varphi \), where \( \varphi_\kappa = \varphi_\mu = \varphi \), \( \hat{u}_{2\kappa+1} = 2u_{2\kappa+1} \) and \( \hat{v}_{2\kappa+1} = 2v_{2\kappa+1} \). By substituting the harmonic
forms of \( w(\tau), f(\tau), u(\tau) \) and \( g[u(\tau)] \) into equation (2.1), following coupled algebraic equations are obtained

\[
(\hat{v}_{2k+1} - \kappa^2 \Lambda^2 \hat{u}_{2k+1}) \cos \beta_k - 2\zeta \kappa \Lambda \hat{u}_{2k+1} \sin \beta_k - f_{2k+1} \cos \Gamma_k + v_1 w_{2k+1} = 0, \tag{2.22a}
\]

\[
(\hat{v}_{2k+1} - \kappa^2 \Lambda^2 \hat{u}_{2k+1}) \sin \beta_k + 2\zeta \kappa \Lambda \hat{u}_{2k+1} \cos \beta_k - f_{2k+1} \sin \Gamma_k = 0, \tag{2.22b}
\]

\[
v_1 - f_1 + \frac{1}{2} \hat{v}_{2k+1} w_{2k+1} \cos \beta_k = 0. \tag{2.22c}
\]

Since \( g[u(\tau)] \) has a single-term harmonic, \( v_1 \) and \( \hat{v}_{2k+1} \) are defined as [77]

\[
v_1 = \frac{1}{2\pi} \int_0^{2\pi} g[u(\phi)] d\phi, \tag{2.23a}
\]

\[
\hat{v}_{2k+1} = \frac{1}{\pi} \int_0^{2\pi} g[u(\phi)] \sin \phi d\phi. \tag{2.23b}
\]

Here, three distinct impact regimes are possible, namely NI, SSI, and DSI. Accordingly, \( g[u(\phi)] \) can take one of three shapes illustrated in Figure 2.3 (for \( \alpha_1 = 1, \alpha_2 = 0.1 \) and \( \alpha_3 = 0.2 \)). By setting \( u(\phi) = u_1 + \hat{u}_{2k+1} \sin \phi = \pm 1 \), the transition angles are found to be \( \varphi_1^\pm = \pi \mp \sin^{-1} \gamma_- \) and \( \varphi_2^\pm = 2\pi \pm \sin^{-1} \gamma_+ \) where \( \gamma_\pm = (u_1 \pm 1)/\hat{u}_{2k+1} \). Thus, \( g[u(\phi)] \) in equation (2.3) is written as
Figure 2.3. Displacement function $g(\varphi)$ for $\kappa = \mu$ and (a) no-impact, (b) single sided impact and (c) double sided impact regimes, given $\alpha_1 = 1$, $\alpha_2 = 0.1$ and $\alpha_3 = 0.2$. $G_{\text{max}} = \sum_{i=1}^{3} \alpha_i (u_1 - \hat{u}_k)^i$ and $G_0 = \sum_{i=1}^{3} \alpha_i (u_1 - 1)^i$. 

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With the following definitions

\[
A_0^\pm = \sum_{n=1}^{3} n (u_1, \pm 1)^n \left[ (u_1, \pm 1)^{n-2} \hat{u}_{2k+1} \right], \quad (2.24a)
\]

\[
A_1^\pm = \sum_{n=1}^{3} n (u_1, \pm 1)^{n-1} \hat{u}_{2k+1} + \frac{1}{4} n(n-1)(u_1, \pm 1)^{n-3} \hat{u}_{2k+1}^2, \quad (2.24b)
\]

\[
A_2^\pm = -\sum_{n=1}^{3} n(n-1)(u_1, \pm 1)^{n-2} \hat{u}_{2k+1}^2, \quad (2.24c)
\]

\[
A_3 = -\frac{1}{4} \alpha_3 \hat{u}_{2k+1}^3, \quad (2.24d)
\]

equation (2.23c) can be expressed as

\[
g[u(\phi)] = \begin{cases} 
A_0^- + A_1^- \sin \phi + A_2^- \cos 2\phi + A_3 \sin 3\phi, & \phi \in [0, \phi_1^-) \cup \phi_1^+ \cup 2\pi ] \\
0, & \phi \in [\phi_1^-, \phi_2^-] \cup [\phi_2^+, \phi_1^+], \\
A_0^+ + A_1^+ \sin \phi + A_2^+ \cos 2\phi + A_3 \sin 3\phi, & \phi \in (\phi_2^-, \phi_2^+).
\end{cases} \quad (2.25)
\]
If \( u_1 - \tilde{u}_{2\kappa+1} > 1 \), there are no separations (NI regime) as in Figure 2.3(a), and equations (2.23a,b) are reduced to

\[
v_1 = A_0^-, \quad \hat{v}_{2\kappa+1} = A_f^-.
\]

(2.26a,b)

Meanwhile, if \( |u_1 - \tilde{u}_{2\kappa+1}| \leq 1 \), SSI solutions exist, and describing functions take the form

\[
v_1 = \frac{1}{2\pi} \left[ A_0^- (\pi + 2 \sin^{-1} \gamma_-) + 2A_1^- (1 - \gamma_-^2)^{1/2} + 2A_2^- \gamma_- (1 - \gamma_-^2)^{1/2} - \frac{2}{3} A_3 \cos(3 \sin^{-1} \gamma_-) \right].
\]

(2.27a)

\[
\hat{v}_{2\kappa+1} = \frac{1}{\pi} \left[ \frac{1}{2} A_1^- (\pi + 2 \sin^{-1} \gamma_-) + (2A_0^- - A_1^- \gamma_- - A_2^- + A_3 \gamma_-)(1 - \gamma_-^2)^{1/2} \right.
\]

\[
- \frac{1}{3} A_2^- \cos(3 \sin^{-1} \gamma_-) - \frac{1}{4} A_3 \sin(4 \sin^{-1} \gamma_-) \right].
\]

(2.27b)

Finally, the mathematical condition \( u_1 - \tilde{u}_{2\kappa+1} < -1 \) leads to DSI solutions as in Figure 2.3(c), and the describing functions are as follows:

\[
v_1 = \frac{1}{2\pi} \left\{ A_0^- (\pi + 2 \sin^{-1} \gamma_-) + A_0^+ (\pi - 2 \sin^{-1} \gamma_+) \right.
\]

\[
+ 2(A_1^- + A_2^- \gamma_-)(1 - \gamma_-^2)^{1/2} - 2(A_1^+ + A_2^+ \gamma_+)(1 - \gamma_+^2)^{1/2}
\]

\[
+ \frac{2}{3} A_3 [\cos(3 \sin^{-1} \gamma_+ - \cos(3 \sin^{-1} \gamma_-))].
\]

(2.28a)
Equations (2.22) can be solved numerically for each impact regime separately to obtain the steady state response. Results of this single-term describing function formulation were found to be the same as to the multi-term HBM for $R = 1$ and harmonic $f(\tau)$ and $w(\tau)$.

2.3.2 $\kappa = 2\mu$

In this case, the frequency of the primary resonance due to $f(\tau)$ coincides with the frequency of the fundamental parametric resonance of $w(\tau)$ at $\Lambda = 2/\kappa = 1/\mu$. Letting $\phi = \phi_\mu$ and $A_\mu = \beta_2 - 2\beta_\mu$, $u(\tau)$ and $g[u(\tau)]$ are written as

$$u(\tau) = u_1 + u_{2\mu+1} \sin \phi + u_{4\mu+1} \sin(2\phi + A_\mu),$$

$$g[u(\tau)] = v_1 + v_{2\mu+1} \sin \phi + v_{4\mu+1} \sin(2\phi + A_\mu).$$

In this case, the harmonic balance yields the following set of algebraic equations

$$\hat{v}_{2\kappa+1} = \frac{1}{\pi} \left\{ 2A_0^- (1 - \gamma_-^2)^{1/2} - 2A_0^+ (1 - \gamma_+^2)^{1/2} + A_1 \left[ \frac{1}{2} (-\pi + 2 \sin^{-1} \gamma_-) \right. \right.$$}

$$- \gamma_- (1 - \gamma_-^2)^{1/2} + \pi \right] + A_1^+ \left[ \frac{1}{2} (\pi - 2 \sin^{-1} \gamma_+) + \gamma_+ (1 - \gamma_+^2)^{1/2} \right]$$

$$+ A_2 \left[ - (1 - \gamma_-^2)^{1/2} - \frac{1}{3} \cos(3 \sin^{-1} \gamma_-) \right]$$

$$+ A_2^+ \left[ (1 - \gamma_+^2)^{1/2} + \frac{1}{3} \cos(3 \sin^{-1} \gamma_+) \right]$$

$$+ A_3 \left[ \gamma_+ (1 - \gamma_+^2)^{1/2} - \gamma_- (1 - \gamma_-^2)^{1/2} \right.$$

$$\left. - \frac{1}{4} \sin \left( 4 \sin^{-1} \gamma_- \right) + \frac{1}{4} \sin \left( 4 \sin^{-1} \gamma_+ \right) \right\} \left(2.28b\right)$$
\[ (v_{2\mu+1} - \mu^2 \Lambda^2 u_{2\mu+1}) \cos \beta_\mu + (\frac{1}{2} v_{2\mu+1} w_{4\mu+1} - 2\zeta \mu \Lambda u_{2\mu+1}) \sin \beta_\mu - f_{2\mu+1} \cos \Gamma_\mu = 0 \] (2.30a)

\[ (v_{2\mu+1} - \mu^2 \Lambda^2 u_{2\mu+1}) \sin \beta_\mu + (\frac{1}{2} v_{2\mu+1} w_{4\mu+1} + 2\zeta \mu \Lambda u_{2\mu+1}) \cos \beta_\mu, \]

\[ - f_{2\mu+1} \sin \Gamma_\mu = 0 \] (2.30b)

\[ (v_{4\mu+1} - 4\mu^2 \Lambda^2 u_{4\mu+1}) \cos \beta_{2\mu} - 4\zeta \mu \Lambda u_{4\mu+1} \sin \beta_{2\mu} + v_1 w_{4\mu+1} = 0, \] (2.30c)

\[ (v_{4\mu+1} - 4\mu^2 \Lambda^2 u_{4\mu+1}) \sin \beta_{2\mu} - 4\zeta \mu \Lambda u_{4\mu+1} \cos \beta_{2\mu} = 0, \] (2.30d)

\[ v_1 - f_1 + v_{4\mu+1} w_{4\mu+1} \cos \beta_{2\mu} = 0. \] (2.30e)

In this case, describing functions for the dual-input, nonlinear system are defined as [77]

\[ v_1 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} g[u(\varphi, \varphi_1)] d\varphi d\varphi_1, \] (2.31a)

\[ v_{2\mu+1} = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} g[u(\varphi, \varphi_1)] \sin \varphi d\varphi d\varphi_1 \] (2.31b)

\[ v_{4\mu+1} = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} g[u(\varphi, \varphi_1)] \sin \varphi_1 d\varphi d\varphi_1, \] (2.31c)

where \( \varphi_1 = 2\varphi + A_\mu \). The transition angles \( \varphi_1^\pm \) and \( \varphi_2^\pm \) are calculated by solving equations \( u(\tau) = \pm 1 \) numerically which yields
\[ \gamma_+ + \sin \varphi + \gamma_1 \sin 2\varphi + \gamma_2 \cos 2\varphi = 0, \quad (2.32a) \]

\[ \gamma_- + \sin \varphi + \gamma_1 \sin 2\varphi + \gamma_2 \cos 2\varphi = 0, \quad (2.32b) \]

where \( \gamma_\pm = (u_\pm 1)/u_\mu \), \( \gamma_1 = (u_2 \mu \cos A_\mu)/u_\mu \), and \( \gamma_2 = (u_2 \mu \sin A_\mu)/u_\mu \). With these, \( g[u(\tau)] \) is written as

\[
g[u(\varphi)] = \begin{cases} 
\sum_{i=1}^{3} \alpha_i [u_1 - 1 + u_{2\mu + 1} \sin \varphi] \\
+ u_{4\mu + 1} \sin (2\varphi + A_\mu) \bigg], & \varphi \in [0, \varphi_1^-] or (\varphi_1^+, 2\pi], \quad \varphi \in [\varphi_1^- , \varphi_2^-] or [\varphi_2^+ , \varphi_1^+], \\
0, & \varphi \in (\varphi_2^- , \varphi_2^+). 
\end{cases} 
\]

This equation can be written as

\[
g[u(\varphi)] = \begin{cases} 
\sum_{i=0}^{6} (B_i^- \sin i\varphi + C_i^- \cos i\varphi), & \varphi \in [0, \varphi_1^-] or (\varphi_1^+, 2\pi] \\
0, & \varphi \in [\varphi_1^- , \varphi_2^-] or [\varphi_2^+ , \varphi_1^+], \\
\sum_{i=0}^{6} (B_i^+ \sin i\varphi + C_i^+ \cos i\varphi), & \varphi \in (\varphi_2^- , \varphi_2^+). 
\end{cases} 
\]

with following definitions:
\[ C^\pm_0 = \alpha_1 (u_1 \pm 1) \mp \alpha_2 [(u_1 \pm 1)^2 + \frac{1}{2} u_\mu^2 + \frac{1}{2} u_{2\mu}^2 ] \]
\[ + \alpha_3 [(u_1 \pm 1)^3 + \frac{3}{2} (u_1 \pm 1)(u_\mu^2 + u_{2\mu}^2 ) - \frac{5}{4} u_\mu^2 u_{2\mu} \sin A_\mu ] , \]  
(2.34b)

\[ B^\pm_1 = \alpha_1 u_\mu \mp \alpha_2 [2(u_1 \pm 1)u_\mu - u_\mu u_{2\mu} \sin A_\mu ] \]
\[ + 3\alpha_3 [(u_1 \pm 1)^2 u_\mu + \frac{1}{2} u_\mu^3 - (u_1 \pm 1)u_\mu u_{2\mu} \sin A_\mu + \frac{1}{2} u_\mu^2 u_{2\mu} ] , \]
(2.34c)

\[ C^\pm_1 = \mp \alpha_2 u_\mu u_{2\mu} \cos A_\mu + 3\alpha_3 (u_1 \pm 1)u_\mu u_{2\mu} \cos A_\mu , \]
(2.34d)

\[ B^\pm_2 = \alpha_1 u_{2\mu} \cos A_\mu \pm 2\alpha_2 (u_1 \pm 1)u_{2\mu} \cos A_\mu \]
\[ + 3\alpha_3 [(u_1 \pm 1)^2 u_{2\mu} \cos A_\mu + \frac{1}{2} u_\mu^2 u_{2\mu} \cos A_\mu + \frac{1}{4} u_{2\mu}^2 \cos A_\mu ] , \]
(2.34e)

\[ C^\pm_2 = \alpha_1 u_{2\mu} \sin A_\mu \pm \alpha_2 [-\frac{1}{2} u_\mu^2 + 2(u_1 \pm 1)u_{2\mu} \sin A_\mu ] \]
\[ + 3\alpha_3 [\frac{1}{2} (u_1 \pm 1)u_\mu^2 + (u_1 \pm 1)^2 u_{2\mu} \sin A_\mu + u_\mu^2 u_{2\mu} \sin A_\mu ] , \]
(2.34f)

\[ B^\pm_3 = \mp \alpha_2 u_\mu u_{2\mu} \sin A_\mu \]
\[ + \alpha_3 [-\frac{1}{4} u_\mu^3 + 3(u_1 \pm 1)u_\mu u_{2\mu} \sin A_\mu + \frac{3}{2} u_\mu^2 u_{2\mu} \cos 2A_\mu ] , \]
(2.34g)

\[ C^\pm_3 = \mp \alpha_2 u_\mu u_{2\mu} \cos A_\mu \]
\[ + 3\alpha_3 [\frac{1}{4} u_{2\mu}^2 \sin A_\mu - (u_1 \pm 1)u_\mu u_{2\mu} \cos A_\mu + \frac{1}{2} u_\mu^2 u_{2\mu} \sin 2A_\mu ] , \]
(2.34h)

\[ B^\pm_4 = \mp \frac{1}{2} \alpha_2 u_{2\mu}^2 \sin 2A_\mu + 3\alpha_3 [-\frac{1}{4} u_\mu^2 u_{2\mu} \cos A_\mu + \frac{1}{2} (u_1 \pm 1)u_{2\mu}^2 \sin 2A_\mu ] , \]
(2.34i)

\[ C^\pm_4 = \mp \frac{1}{2} \alpha_2 u_{2\mu}^2 \cos 2A_\mu + 3\alpha_3 [-\frac{1}{4} u_\mu^2 u_{2\mu} \cos A_\mu - \frac{1}{4} (u_1 \pm 1)u_{2\mu}^2 \cos 2A_\mu ] , \]
(2.34j)

\[ B^+_5 = B^-_5 = -\frac{3}{2} \alpha_3 u_\mu u_{2\mu}^2 \cos 2A_\mu , \]
(2.34k)

\[ C^+_5 = C^-_5 = -\frac{3}{2} \alpha_3 u_\mu u_{2\mu}^2 \sin 2A_\mu , \]
(2.34m)

\[ B^+_6 = B^-_6 = -\frac{1}{4} u_{2\mu}^2 \cos 3A_\mu , \]
(2.34n)

\[ C^+_6 = C^-_6 = -\frac{1}{4} u_{2\mu}^2 \sin 3A_\mu . \]
(2.34o)
Given equation (2.34a), equations (2.31) can be evaluated numerically to obtain the Fourier coefficients \( v_1, v_{2\mu + 1} \) and \( v_{4\mu + 1} \) for all three types of impact regimes, and equations (2.30) can be used to compute \( u_1, u_{2\mu + 1} \), and \( u_{4\mu + 1} \). Implementation of this solution, however, required more effort than the previous case of \( \kappa = \mu \).

Based on solutions presented above, it can be stated that the single-term formulation using describing function method is laborious to implement since one would need a new formulation for every value of \( m \) where \( \kappa = m\mu \), and would be required to obtain solutions for each impact regime separately. These formulations are no longer straightforward in the case of dual-input describing functions \( (m > 1) \), and solutions are dependent heavily on numerical techniques.

### 2.4. Parametric Studies

The parameter set for equation (2.1) is given as \( P \in \{ \alpha_1, \alpha_2, \alpha_3, f_1, f_i, w_i, \zeta, \Lambda \} \). Since the focus of the study is on the influence of clearance, quadratic (\( \alpha_2 \)), and cubic (\( \alpha_3 \)) nonlinearities, three cases are considered: (i) oscillators having clearance and cubic nonlinearities: \( \alpha_1 = 1, \alpha_2 = 0 \), and variable \( \alpha_3 \); (ii) oscillators having clearance and quadratic nonlinearities: \( \alpha_1 = 1, \alpha_3 = 0 \), and variable \( \alpha_2 \); and (iii) oscillators having all three types of nonlinearities: \( \alpha_1 = 1 \), and variable \( \alpha_2 \) and \( \alpha_3 \). Moreover, to limit the parametric study within a reasonable size, the influences of system parameters \( f_1, f_i, w_i \)
and $\zeta$ on the steady state response are demonstrated only for case (i) in which $\alpha_1=1$, $\alpha_2=0$, and $\alpha_3$ has different values. The range of dimensionless frequency is defined as $\Lambda \in [0, 1.5]$. This range covers all primary and superharmonic resonance peaks of interest. Here, the primary resonance frequency represents the undamped natural frequency of the corresponding linear system at $\Lambda =1$ while the superharmonic resonances at $\Lambda = \frac{1}{\kappa}$ ($\kappa \geq 2$) are due to the nonlinear effects [139]. In equation (2.6), the maximum index of Fourier series is chosen as $R=6$, which has sufficient accuracy for the analysis, as shown in Figure 2.1. Only $u_1$ and $u_{rms}$ values are used to represent $u(\tau)$ since the presentation of all harmonic amplitudes is not feasible. In all figures of following sections, solid and dashed lines represent the stable and unstable period-1 motions, respectively.

2.4.1 $\alpha_1 =1$, $\alpha_2 =0$ and Variable $\alpha_3$

In this case, PN function $g[u(\tau)]$ is formed by two nonlinear segments that are defined by a linear term $\alpha_1 =1$ and a cubic nonlinear $\alpha_3$ term. The value of $\alpha_3$ is varied to obtain softening ($\alpha_3 <0$) and hardening ($\alpha_3 >0$) types of cubic nonlinearities. The case of $\alpha_3 =0$ is also included to represent the corresponding PL system.

The effect of $f_1$ on the steady state forced response for $\alpha_3 \neq 0$ is illustrated in Figures 2.4 and 2.5 for $f_i = 0$ ($i \geq 2$), $\omega_3 =0.3$, $\zeta =0.05$, $\alpha_1 =1$, $\alpha_2 =0$. The values of $\alpha_3$ and $f_1$ are varied, and the resultant changes on $u_{rms}$ and $u_1$ of the system response are observed. In Figures 2.4(a) and 2.5(a) for $f_1 =0.25$, regardless of the value of $\alpha_3$,
Figure 2.4. $u_{rms}$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -0.1$, 0.0, 0.2 and 0.4, given $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$, $\zeta = 0.05$ for (a) $f_1 = 0.25$, (b) $f_1 = 0.5$, (c) $f_1 = 0.75$, and (d) $f_1 = 1.0$. (—) stable and (−) unstable HBM solutions.
Figure 2.5. \( u_1 \) of the same oscillator as that in Figure 3; (a) \( f_1 = 0.25 \), (b) \( f_1 = 0.5 \), (c) \( f_1 = 0.75 \), and (d) \( f_1 = 1.0 \). (---) stable and (−−) unstable HBM solutions.
the motion is nearly linear for most $\Lambda$ values except near $\Lambda = 1$, where a softening-type nonlinear response is formed by SSI solutions corresponding to the primary resonance due to internal harmonic excitation $w_3$. Additionally, the responses for $\alpha_3 \neq 0$ and $\alpha_3 = 0$ are quite close because of the fact that the vibrations take place in a portion of $g[u(\tau)]$ where the cubic nonlinearity is not significant enough. For instance, at $\Lambda = 1$ where $u_1 \approx 1.15$ and $u_{rms} = 0.35$, $u(\tau)$ varies roughly between 0.8 and 1.5, and $g[u(\tau)]$ is nearly linear within this range. However, with the increasing of $f_1$, the value of $u_1$ is enlarged as well, which moves the range of $u(\tau)$ to the right in Figure 1.2(a), where cubic nonlinearity becomes more important. Thus, the behavior of the oscillator is changed significantly as shown in Figures 2.4(b) and 2.5(b), which have the same parameters as those of Figures 2.4(a) and 2.5(a), except $f_1 = 0.50$.

The effect of $\alpha_3$ in Figures 2.4 and 2.5 is such that a positive $\alpha_3$ (hardening) shifts the response to the right, and reduces the vibration amplitudes slightly while the overall shape of the motion remains similar to the case of $\alpha_3 = 0$. Meanwhile, a negative $\alpha_3$ (softening) causes the primary resonance peak to shift to the left, and the vibration amplitudes may increase significantly on the upper branch before a jump-down takes place. In Figures 2.4(c, d) and 2.5(c, d) for $f_1 = 0.75$ and 1.0, respectively, the same effect of a hardening $\alpha_3$ is observed except the changes in resonance frequencies, and vibration amplitudes become more severe. The response for $\alpha_3 = -0.1$ differs drastically among these four cases. In Figure 2.4(c), the primary resonance peak is at nearly $\Lambda = 0.8$, and the upper branch solution loses its stability at about $\Lambda = 0.7$. There is no
stable period-1 motion for $\Lambda \in [0.63, 0.7]$, in which stable subharmonic motions exist [2.48]. The superharmonic resonance peak (at half the primary resonance frequency) becomes larger, and exhibits an SSI solution. Finally, when $f_1 = 1.0$ in Figures 2.4(d) and 2.5(d), the PL system for $\alpha_3 = 0$ displays DSI motions at the end of the SSI branch. These DSI motions are eliminated for $\alpha_3 > 0$. For $\alpha_3 = -0.1$, there is no stable period-1 motion for $\Lambda < 0.82$. In summary, the behavior displayed in Figures 2.4 and 2.5 indicates that the impact of cubic nonlinearity depends heavily on the value of the mean load. A larger $f_1$ makes the influence of $\alpha_3$ more significant. On the contrary, a PL approximation for $g[u(\tau)]$ might be sufficient when $f_1$ is small regardless of $\Lambda$.

The effect of damping ratio $\zeta$ is illustrated in Figure 2.6 for the same parameters as in Figure 2.4 except $\omega_3 = 0.3$, and $\zeta = 0.025, 0.05, 0.075$, and 0.1. As expected, a lower $\zeta$ value results in larger $u_{rms}$ amplitudes as well as several types of nonlinear behaviors including SSI and DSI motions and superharmonic resonances. For instance, in Figure 2.6(a) for $\zeta = 0.025$, all response curves for $\alpha_3 \geq 0$ show hardening type DSI motions at the end of large-amplitude softening type SSI motions. For $\alpha_3 = -0.1$, on the other hand, DSI motions do not exist while the SSI motions have even higher amplitudes. Further increasing the value of $\zeta$, first DSI motions are eliminated, and then the SSI amplitudes are reduced significantly. For instance, motions are mostly NI type for $\zeta = 0.1$ as shown in Figure 2.6(d).

Figure 2.7 demonstrates the combined influence of $\alpha_3$ and $w(\tau)$ on $u_{rms}$, where $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -0.1, 0, 0.2$ and 0.4, $\zeta = 0.05$, $f_1 = 0.5$, and all other $f_i = 0$. 

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Figure 2.6. $u_{rms}$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -0.1$, 0.0, 0.2 and 0.4, given $f_i = 0$ ($i \geq 2$), $f_1 = 0.5$, $w_3 = 0.3$ for (a) $\zeta = 0.025$, (b) $\zeta = 0.05$, (c) $\zeta = 0.075$, and (d) $\zeta = 0.1$. (--) stable and (--) unstable HBM solutions.
Figure 2.7. $u_{rms}$ of an oscillator with $\alpha_1 = 1, \alpha_2 = 0$ and $\alpha_3 = -0.1, 0.0, 0.2$ and $0.4$, given $f_i = 0$ ($i \geq 2$), $f_1 = 0.5$, $\zeta = 0.05$ for (a) $w_3 = 0.1$, (b) $w_3 = 0.2$, (c) $w_3 = 0.3$, and (d) $w_3 = 0.4$. (−−) stable and (− −) unstable HBM solutions.
The stiffness function \( w(\tau) \) is considered to be harmonic with amplitude \( w_3 = 0.1, 0.2, 0.3 \) and 0.4. Figure 2.7 indicates that the magnitude of \( u_{rms} \) gets larger when \( w_3 \) is increased. When \( w_3 = 0.1 \), all response curves are approximately linear with no impacts as shown in Figure 2.7(a). This agrees with previous studies on PL systems, which stated that separations could not occur if \( w_3 \leq 2\zeta \) \([105, 106]\). When \( w_3 = 0.2 \), softening type nonlinear curves from SSI motions are introduced. Increasing \( \alpha_3 \) reduces the amplitude of vibrations, and moves the resonance peaks to higher frequencies. This is also true with larger \( w_3 \) values as in Figures 2.7(c) and (d) for \( w_3 = 0.3 \) and 0.4, respectively. When \( w_3 = 0.4 \), the PL system (\( \alpha_3 = 0 \)) exhibits DSI motions, which are eliminated when \( \alpha_3 = 0.2 \) and 0.4. DSI motions do not exist for \( \alpha_3 = -0.1 \) either, and the amplitude of the superharmonic resonance (at half the frequency of the primary resonance) peak is quite large with a slight separation.

In Figure 2.8, a harmonic external excitation \( f(\tau) \) is considered without any stiffness fluctuations (\( w(\tau) = 1 \)). The system parameters are kept the same as in Figure 2.7 except \( \zeta = 0.05 \) and \( f_3 = 0.05, 0.1, 0.2 \) and 0.3. For \( \alpha_3 = 0 \), a good agreement with previous studies on PLTI system \([17]\) is obtained, in the sense that increasing \( f_3 \) enlarges the amplitudes in the vicinity of the primary resonance. In Figure 2.8(b), both SSI and DSI motions are presented for \( f_3 = 0.1 \). When the ratio of \( f_3/f_1 \) is small, say \( f_3/f_1 < 0.5 \), no significant superharmonic resonances are evident in Figure 2.8, especially for the PL system, unlike Figure 2.7 for \( w(\tau) \neq 0 \). However, when \( f_3/f_1 \) is large enough (\( f_3/f_1 = 0.6 \) in Figure 2.8(d)), super-harmonic resonance amplitudes are
Figure 2.8. $u_{rms}$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -0.1, 0.0, 0.2$ and $0.4$, given $w_i = 0 \ (i \geq 2)$, $f_1 = 0.5$, $\zeta = 0.05$ for (a) $f_3 = 0.05$, (b) $f_3 = 0.1$, (c) $f_3 = 0.2$, and (d) $f_3 = 0.3$. (−−) stable and (−−) unstable HBM solutions.
increased significantly. This suggests that not only the individual values of $f_1$ and $f_3$, but also the ratio $f_3/f_1$, are critical. For different values of $f_1$ and $f_3$, having the same ratio $f_3/f_1$, the response curves maintain the same qualitative shape and primary resonances. The influence of $\alpha_3$ is the same as in previous cases.

In Figure 2.9, harmonic internal and external excitations, $w(\tau)$ and $f(\tau)$, are applied simultaneously. Here $\kappa = \mu = 1$, and $w(\tau)$ and $f(\tau)$ are in-phase. This is accomplished by considering $f_3$ and $w_3$ only in equation (2.3). With $w_3 = 0.3$ kept constant, $f_3$ is varied from 0.01 to 0.2 in Figure 2.9. For a very small $f_3$, the nonlinear response in Figure 2.9(a) is primarily due to $w_3$, which can be confirmed by a comparison with Figure 2.7(c). As the value of $f_3$ is increased, the amplitude of response decreases first as shown in Figures 2.9(b) and (c) to a point that there are only NI motions in Figure 2.9(c) for $f_3 = 0.1$. However, $u_{rms}$ increases and softening-type SSI response reappears in Figure 2.9(d) for $f_3 = 0.2$. This suggests that $w(\tau)$ and $f(\tau)$ tend to cancel each other when they are in-phase. It can also be shown for in-phase harmonic excitations that $u_{rms}=0$ and $u_1$ is a constant when $w_i = f_i/f_1$ ($i \geq 2$) regardless of the value of $\alpha_i$.

An 180 degrees out-of-phase condition is accomplished in Figure 2.10 by simply setting $w_3 = 0.3$ and $f_3 = -0.01$, -0.05, -0.1 and -0.2. Here, two excitations act in such a way that their effects on system response add to each other. In contrast to Figure 2.9, response amplitudes are enlarged significantly with increasing $f_3$. Very large DSI
Figure 2.9. $u_{\text{rms}}$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -0.1$, 0.0, 0.2 and 0.4, given $w_3 = 0.3$, $f_1 = 0.5$, $\zeta = 0.05$ for (a) $f_3 = 0.01$, (b) $f_3 = 0.05$, (c) $f_3 = 0.1$, and (d) $f_3 = 0.2$. (--) stable and (--) unstable HBM solutions.
Figure 2.10. $u_{\text{rms}}$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -0.1$, 0.0, 0.2 and 0.4, given $\omega_3 = 0.3$, $f_1 = 0.5$, $\zeta = 0.05$ for (a) $f_3 = -0.01$, (b) $f_3 = -0.05$, (c) $f_3 = -0.1$, and (d) $f_3 = -0.2$. (--) stable and (--) unstable HBM solutions.
motions are observed in Figure 2.10(d) for \( w_3 = 0.3 \) and \( f_3 = -0.2 \) when \( \alpha_3 \geq 0 \), and large superharmonic resonance peaks are obtained when \( \alpha_3 = -0.1 \). A similar dependence of the response on the phasing relationship between \( w(\tau) \) and \( f(\tau) \) was reported earlier for PL systems as well [105, 106].

Next, consider the case when \( 2\kappa = \mu \) as shown in Figure 2.11. Here, all parameters except \( w_i \) and \( f_i \ (i \geq 2) \) are the same as those in Figure 2.10. The value of \( w_3 \) is kept constant at 0.3, and \( f_5 \) is varied from 0.01 to 0.2. In Figure 2.11(a), the nonlinear response is mainly due to \( w_3 \) since the value of \( f_5 \) is relatively small. \( w_{2\kappa+1} = w_3 \) results in a primary resonance near \( \Lambda = \frac{1}{\kappa} = 1 \) and a superharmonic resonance near \( \Lambda = \frac{1}{2\kappa} = \frac{1}{2} \). Examining Figures 2.11(a-d), one concludes that the response near \( \Lambda = 1 \) remains almost the same for different \( \alpha_3 \) since \( w_3 \) is kept constant, confirming that \( f_5 \) has a negligible effect near \( \Lambda = 1 \). Increasing the value of \( f_{2\mu+1} = f_5 \), the response near \( \Lambda = \frac{1}{\mu} = \frac{1}{2} \) is amplified. SSI motions appear in Figure 2.11(b) for \( f_5 = 0.05 \), and DSI motions appear in Figure 2.11(d) for \( f_5 = 0.2 \). The effects of \( w(\tau) \) and \( f(\tau) \) are superimposed in terms of primary resonances, and superharmonic activity at \( \Lambda = \frac{1}{3\kappa} = \frac{1}{3} \) is increased significantly. The effect of increasing \( \alpha_3 \) is the same as before, in terms of decreased \( u_{rms} \) and increased primary resonance frequencies.
Figure 2.11. $u_{rms}$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -0.1, 0.0, 0.2$ and 0.4, given $w_3 = 0.3$, $f_1 = 0.5$, $\zeta = 0.05$ for (a) $f_5 = 0.01$, (b) $f_5 = 0.05$, (c) $f_5 = 0.1$, and (d) $f_5 = 0.2$. (---) stable and (−−) unstable HBM solutions.
Figure 2.12. $u_{rms}$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -0.1$, 0.0, 0.2 and 0.4, given $w_3 = 0.3$, $f_1 = 0.5$, $f_3 = 0.05$, $\zeta = 0.05$ for (a) $w_5 = 0.1$, (b) $w_5 = 0.2$, (c) $w_5 = 0.3$, and (d) $w_5 = 0.4$. (--) stable and (--) unstable HBM solutions.
Figure 2.12 represents the response when $\kappa = 2\mu$ with all parameters the same as in Figure 2.11. In this figure, $f_3=0.05$ and $w_5=0.1, 0.2, 0.3$ and $0.4$. In line with Figures 2.9(a) and 2.10(a), Figure 2.12(a) has a resonance peak near $\Lambda = \frac{1}{\mu} = 1$ due to $f_{2\mu+1} = f_3$, and another peak at $\Lambda = \frac{1}{\kappa} = \frac{1}{2}$ due to $w_{2\kappa+1} = w_5$. The value of $w_5 = 0.1$ is not large enough to either cause SSI motions near $\Lambda = \frac{1}{\kappa} = \frac{1}{2}$ or a parametric resonance near $\Lambda = \frac{2}{\kappa} = 1$. As a result, the activity near $\Lambda = 1$ can mostly be attributed to $f_3$. For the rest of the cases in Figures 2.12(b-d), a larger $w_5$ has a greater influence on the vibration amplitudes around $\Lambda = 1$. DSI motions appear when $w_5 \geq 0.2$, and the $u_{rms}$ values increase drastically in Figures 2.12(c, d) near $\Lambda = 1$. This is because the parametric resonance due to $w_5$ and the primary resonance due to $f_3$ act near the same frequency of $\Lambda = \frac{2}{\kappa} = \frac{1}{\mu} = 1$. Meanwhile, in Figure 2.12(d) for $w_5=0.4$ superharmonic resonances are created at $\Lambda = \frac{1}{2\kappa} = \frac{1}{4}$ and $\Lambda = \frac{1}{3\kappa} = \frac{1}{6}$ as well.

2.4.2 $\alpha_1 = 1$, $\alpha_3 = 0$, and Variable $\alpha_2$

In this case, $g[u(\tau)]$ is defined by a constant linear term $\alpha_1 = 1$ and a quadratic nonlinear term $\alpha_2$ with no cubic term ($\alpha_3 = 0$). The value of $\alpha_2$ is varied between $-0.1$ and $0.4$ including $\alpha_2 = 0$. All of the parametric studies presented in Section 2.4.1 were carried out for the case of quadratic nonlinearities as well. As the influence of $\alpha_2$ is quite similar to $\alpha_3$, at least qualitatively, only a representative example is shown here in Figure 2.13. This figure presents the combined effect of $f_1$ and $\alpha_2$ on the response.
Figure 2.13. $u_{rms}$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = -0.1, 0.0, 0.2$ and $0.4$ and $\alpha_3 = 0$, given $f_1 = 0 \ (i \geq 2)$, $w_3 = 0.3$, $\zeta = 0.05$ for (a) $f_1 = 0.25$, (b) $f_1 = 0.5$, (c) $f_1 = 0.75$, and (d) $f_1 = 1.0$. (—) stable and (− −) unstable HBM solutions.
By comparing Figures 2.4 and 2.13, several differences can be noted. (i) For softening cases ($\alpha_2 = -0.1$ versus $\alpha_3 = -0.1$), the responses are quite different. The response curves for $\alpha_2 = -0.1$ are much like extensions of those of the PL system in contrast to the erratic and mostly unstable behavior for $\alpha_3 = -0.1$. (ii) For the hardening cases, the results for $\alpha_2 = 0.2$ and 0.4 are quite similar to corresponding cases in Figure 2.5 for $\alpha_3 = 0.2$ and 0.4 except the amplitudes increase slightly. The changes near $\Lambda = 1$ with $\alpha_2$ are not as significant as the corresponding changes with $\alpha_3$ in Figure 2.4. This suggests that the effect of $\alpha_2$ on system response is not as significant as that of $\alpha_3$ having the same values. (iii) DSI motions are predicted for $\alpha_2 = -0.1$ in Figures 2.13(c, d), while no DSI motions are evident in Figure 2.4 for $\alpha_3 = -0.1$.

2.4.3 $\alpha_1 = 1$ and Variable $\alpha_2$ and $\alpha_3$

As the final case, both quadratic and cubic nonlinearities are considered simultaneously. Four sets of parameters are used: (i) $\alpha_2 = \alpha_3 = -0.1$, (ii) $\alpha_2 = -0.1$, $\alpha_3 = 0.2$, (iii) $\alpha_2 = 0.2$, $\alpha_3 = -0.1$, and (iv) $\alpha_2 = \alpha_3 = 0.2$. In Figure 2.14, the response curves for these four cases are compared for different $f_1$ values. The corresponding restoring functions $g[u(\tau)]$ are plotted schematically in the upper right corner for each case to demonstrate the shape of $g[u(\tau)]$. In Figure 2.14(a), both nonlinearities are of the softening type, which makes the system response very dramatic and quite unstable. No stable period-1 solutions could be found for $f_1 = 1.0$ within this frequency range for this case. However, if the signs of $\alpha_2$ and $\alpha_3$ are opposite, they tend to cancel each
Figure 2.14. $u_{rms}$ of an oscillator with $\alpha_1 = 1$, variable $\alpha_2$ and $\alpha_3$, given $f_1 = 0$ ($i \geq 2$), $w_3 = 0.3$, $\zeta = 0.05$, $f_1 = 0.25$, 0.5, 0.75 and 1.0 for (a) $\alpha_2 = \alpha_3 = -0.1$, (b) $\alpha_2 = -0.1$, $\alpha_3 = 0.2$, (c) $\alpha_2 = 0.2$, $\alpha_3 = -0.1$, and (d) $\alpha_2 = \alpha_3 = 0.2$. (---) stable and (–--) unstable HBM solutions.
other. In Figure 2.14(c), the response curves are very similar to the results of the corresponding PL system, and the amplitudes near primary and superharmonic resonances are similar to those of the PL system for each $f_1$ value considered. In addition, $\alpha_3$ has a more dominant effect on system response than $\alpha_2$. This can be confirmed by comparing the results in Figures 2.14(b) and (d).

2.5. Conclusions

In this chapter, the dynamic behavior of a PN oscillator subjected to a mean load and combined parametric and external excitations is considered. The oscillator has a periodically TV stiffness as well as a restoring function formed by clearance and continuous nonlinearities. Analytical solutions are obtained by employing the multi-term HBM in conjunction with the Newton-Raphson method, DFT, and the generic homotopy method. The stability of steady state response is determined by applying Floquet theory. The HBM solutions are verified by comparing them to the direct numerical integration results. In addition, DFM is used to derive single-term harmonic balance solutions for the same system.

Several key characteristics of the system are demonstrated through a detailed parametric study:

- The effect of continuous nonlinearities is very sensitive to the value of mean load $f_1$. Both quadratic and cubic nonlinearities are more influential for larger $f_1$ values.
• The amplitudes of $u_{rms}$ near the primary resonance peaks are decreased, and the resonance peaks are shifted to high frequencies with increases of the values of $\alpha_2$ and $\alpha_3$.

• In the presence of a parametric excitation of harmonic order $k$, a primary resonance at $\Lambda = \frac{1}{k}$ is obtained as well as a sizable superharmonic resonance at $\Lambda = \frac{1}{2k}$.

• For a case of external excitation $f(\tau)$ only, superharmonic resonances appear only when the ratio of alternating component to mean load $f_i/f_1$ is quite large.

• When both internal and external excitations act simultaneously, they appear to cancel each other when two excitations are in phase and of the same fundamental frequency, and add to each other when they are 180 degrees out of phase. Moreover, when $w_i = f_i/f_1$, $u_{rms} = 0$, and $u_1$ remains constant regardless of the values of $\alpha_2$, $\alpha_3$ and $\Lambda$.

• While both SSI and DSI motions are predicted for $\alpha_2 \geq 0$ and $\alpha_3 \geq 0$, DSI motions are not common for $\alpha_3 < 0$.

• The overall impact of quadratic nonlinearity on the system response seems to be less significant than that of a cubic nonlinearity.
CHAPTER 3

SUBHARMONIC RESONANCES OF A MECHANICAL OSCILLATOR WITH PERIODICALLY TIME-VARYING PIECEWISE NONLINEAR STIFFNESS

3.1. Introduction

In chapter 2, the period-1 response of equation (2.1) was studied by using multi-term HBM. Meanwhile, mechanical systems having \( g[\tau(t)] \) as in equation (1.2a) are expected to exhibit subharmonic and chaotic motions as well. Choi and Noah [135] examined the subharmonic response of a SDOF bilinear system with time invariant coefficients using HBM. Later, Choi and Lou [128] determined the forced steady state response of a system with asymmetric PNTI stiffness by an improved HBM algorithm that incorporates fast Fourier transforms. Kahraman and Blankenship [106, 107] demonstrated the existence of period-2, subharmonic resonances through measurements of a spur gear pair, and predicted subharmonic response (period-2 and period-3) analytically by using a PL version of \( g[\tau(t)] \) with \( \alpha_2=\alpha_3=0 \). Natsiavas and Theodossiades [111] applied a piecewise perturbation method to show that period-1 and period-\( \eta \) motions may coexist for a PL oscillator with TV stiffness. Recently, Al-shyyab and Kahraman [108] predicted period-\( \eta \) and chaotic motions using a PL dynamic model
of a multi-mesh gear system. However, these studies were limited to either a PN version of \( g[u(\tau)] \) with constant coefficients [128], or a PL version with or without parametric excitation [106-108, 111, 135].

In this chapter, the formulation presented in chapter 2 is extended to predict period-\( \eta \) subharmonic motions exhibited by equation (2.1). Multi-term HBM is used in conjunction with DFT to obtain steady state period-\( \eta \) response. The accuracy of HBM solutions is demonstrated through a comparison with numerical integration solutions. The results of a parametric study are presented to describe the combined influence of continuous nonlinearities and the clearance on the steady state response of the PN system defined by equation (2.1). In addition, the effect of system parameters, such as \( \zeta \), \( w(\tau) \), mean load \( f_1 \), and \( f(\tau) \) on period-\( \eta \) motions is investigated.

3.2. Multi-term Period-\( \eta \) Response to Periodic Excitations

The multi-term HBM formulation that was used to study period-1 motions is expanded to predict period-\( \eta \) (\( \eta \geq 1 \)) motions of the same system. The excitations \( w(\tau) \) and \( f(\tau) \) are still defined by equations (2.2) and (2.3). By defining \( \theta = \Lambda \tau/\eta \) and \( H = \Lambda/\eta \), where \( \eta \) is the subharmonic index, equation (2.1) becomes

\[
H^2 \frac{d^2 u(\theta)}{d\theta^2} + 2\zeta H \frac{du(\theta)}{d\theta} + w(\theta)g[u(\theta)] = f(\theta).
\]  

(3.1)
The unknown steady state, period-\(\eta\) response \(u(\theta)\) is expressed in Fourier series form as

\[
u(\theta) = u_1 + \sum_{r=1}^{R} \left[ u_{2r} \cos(r\theta) + u_{2r+1} \sin(r\theta) \right], \tag{3.2}
\]

where \(R\) is the number of harmonics that would be sufficient to describe the steady state period-1 response. For subharmonic motions, \(\eta R\) harmonic terms are considered in equation (3.2). Similarly, the restoring function \(g[u(\theta)]\) is written in Fourier series as

\[
g[u(\theta)] = v_1 + \sum_{r=1}^{R} \left[ v_{2r} \cos(r\theta) + v_{2r+1} \sin(r\theta) \right]. \tag{3.3}
\]

Substituting equations (2.2), (2.3), (3.2) and (3.3) into equation (3.1) and enforcing harmonic balance as in Chapter 2 result in a vector equation \(S = 0\), where \(S = [S_1, S_2, S_3, \ldots, S_{2\eta R}, S_{2\eta R+1}]^T\) is given as

\[
S_1 = v_1 - f_1 + \frac{1}{2} \sum_{k=1}^{K} \left[ w_{2k} v_{2k\eta} + w_{2k+1} v_{2k\eta+1} \right], \tag{3.4a}
\]

\[
S_{2r} = -H^2 r^2 u_{2r} + 2\zeta H r u_{2r+1} + v_{2r} + v_1 w_{2r\eta} - f_{2r\eta}
+ \frac{1}{2} \sum_{k=1}^{K} \left[ w_{2k} \left( v_{2(k\eta-r)} + v_{2(k\eta+r)} + v_{2(r-k\eta)} \right) \right]
+ \frac{1}{2} \sum_{k=1}^{K} \left[ w_{2k+1} \left( v_{2(k\eta-r)+1} + v_{2(k\eta+r)+1} - v_{2(r-k\eta)+1} \right) \right], \quad r \in [1, \eta R]. \tag{3.4b}
\]
\[ S_{2r+1} = -H^2 r^2 u_{2r+1} - 2\zeta_0 H r u_{2r} + v_{2r+1} + v_1 w_{(2r+1)+1} + f_{(2r+1)+1} \]
\[ + \frac{1}{2} \sum_{k=1}^{K} w_{2k} \left[-v_{2(k\eta-r)+1} + v_{2(k\eta+r)+1} + v_{2(r-k\eta)+1}\right] \]
\[ + \frac{1}{2} \sum_{k=1}^{K} w_{2k+1} \left[v_{2(k\eta-r)} - v_{2(k\eta+r)} + v_{2(r-k\eta)}\right], \quad r \in \left[1, \eta R\right] \] (3.4c)

The coefficients \( v_i \) of \( g[u(\theta)] \) can be expressed in terms of unknown Fourier coefficients of the response \( u = [u_1 \ u_2 \ u_3 \ ... \ u_{2R} \ u_{2R+1}]^T \) by utilizing the DFT [105, 106, 135, 136]. The value of \( u(\theta) \) at discrete \( \theta_n = nh \) is given as

\[ u^{(n)} = u_1 + \sum_{r=1}^{\eta R} \left[u_{2r} \cos\left(\frac{2\pi rn}{N}\right) + u_{2r+1} \sin\left(\frac{2\pi rn}{N}\right)\right], \] (3.5)

where \( h = \frac{2\pi}{(NA)} \), \( N \geq 2R \), and \( n \in \left[0, N - 1\right] \). Likewise, the discrete value of \( g[u(\theta)] \) at \( \theta_n \) is obtained from equation (1.2a), which is same as defined in equation (2.8). The Fourier coefficients \( v_{2r} \) and \( v_{2r+1} \) of \( g[u(\theta)] \) in equation (3.3) are calculated by taking inverse DFT of equation (2.8), which are defined in equation (2.9). With \( v = [v_1 \ v_2 \ v_3 \ ... \ v_{2\eta R} \ v_{2\eta R+1}]^T \) given in equation (2.9), the above \( 2\eta R + 1 \) algebraic equations \( S = 0 \) can be solved for \( 2\eta R + 1 \) unknowns \( u_i \) by using Newton-Raphson method as equation (2.10).
3.3. Comparison to Direct Numerical Integration Results

In order to evaluate the accuracy of HBM solutions from equations (3.5) and (2.10), equation (2.1) is integrated numerically utilizing a variable-order backward differentiation formula. An example case of a periodic $w(\tau)$ ($w_2 = 0.4$, $w_4 = 0.15$, $w_6 = 0.05$, $(K = 3)$, and all other $w_i = 0$) is considered here. This approximates the mesh stiffness of an unmodified spur gear pair with an involute contact ratio of 1.37 [141]. The external force is assumed to be constant with $f_1 = 0.5$ and $f_i = 0$, $(i > 1)$. Moreover, $\zeta = 0.01$, and $g[u(\tau)]$ contains linear, quadratic and cubic terms as $\alpha_1 = 1$, $\alpha_2 = 0.1$, and $\alpha_3 = 0.2$, respectively.

In Figure 3.1(a), the RMS amplitudes of both stable and unstable period-$\eta$ $(\eta = 1, 2, 3)$ HBM solutions are shown as a function of $\Lambda$. These solutions are obtained by solving equation (2.1) for different $\eta$ separately. In order to find period-$\eta$ solutions, $6\eta$ harmonic terms ($R = 6$) are assumed in equation (3.2). The RMS value of the response is defined as $u_{\text{rms}}^\eta = \left[ \sum_{r=1}^{\eta R} (u_{2r}^2 + u_{2r+1}^2) \right]^{\frac{1}{2}}$. A good agreement is observed between HBM and numerical solutions. Period-2 motions emerge when period-1 motions become unstable near the first parametric resonance frequency of $\Lambda = 2$. Stable period-1 and period-2 motions coexist for $\Lambda \in (1.38, 1.93)$ while only period-2 motions are present when $\Lambda \in (1.93, 2.45)$. The shape of period-2 resonance peak is similar to that of a typical primary resonance of period-1 motions near $\Lambda = 1$ with a pair of softening branches (one stable and one unstable) bent to the left due to contact loss (single-sided
Figure 3.1. (a) $u_{rms}$ and (b) $u_1$ values of period-$\eta$ ($\eta = 1, 2, \text{ and } 3$) motions ($R = 6$) as a function of $\Lambda$ given $\alpha_2 = 0.1$, $\alpha_3 = 0.2$, $f_1 = 0.5$, $f_j = 0$ ($i \geq 2$), $w_2 = 0.4$, $w_4 = 0.15$, $w_6 = 0.05$ ($K = 3$), all other $w_i = 0$, and $\zeta = 0.01$. (---) stable HBM solutions, (-----) unstable HBM solutions, and (o) numerical integration solutions.
impacts (SSI)) followed by another pair of hardening branches owing to back contact (double-sided impacts (DSI)).

A period-3 subharmonic resonance exists in Figure 3.1(a) near $\Lambda = 3$. Similar to PL systems as in reference [107], a “boomerang-shaped” island arises, which is entirely isolated from the stable period-1 motions below. Similarly, a comparison of the mean component of system response $u_1$ from HBM and numerical integration is shown in Figure 3.1(b), which further illustrates the agreement between two methods. While the results from HBM and the numerical integration method agree satisfactorily in Figure 3.1, more numerical solutions should be expected, especially on the hardening portions of the primary and subharmonic resonance peaks. A main difficulty is that the numerical integration method is inefficient in terms of computation time particularly for lightly damped systems. In addition, in the regions where multiple stable solutions coexist, one needs to search for proper initial conditions by trial-and-error to obtain the desired steady state solutions. As a result, the numerical solutions presented in Figure 3.1 are incomplete, while HBM solutions are not subject to such difficulties.

In Figure 3.2, the harmonic components for period-$\eta$ motions from HBM and numerical integration are shown. These figures also illustrate that period-2 and period-3 subharmonic motions are dominated by $A_{\eta/2}$ and $A_{\eta/3}$ components, while the primary resonance of period-1 motion is governed by $A_1$, and superharmonic resonances are controlled by $A_j, j=2, 3\ldots$ respectively. In Figures 3.2 and 3.3, a very good agreement is observed between these two methods with respect to the harmonic content in frequency domain. Moreover, the dominant Fourier components at orders 1/2 and 1/3 in Figures
Figure 3.2. Harmonic contents of the data shown in Figure 3.1; (a) $A_1$, (b) $A_2$, (c) $A_3$, (d) $A_4/2$, (e) $A_3/2$, (f) $A_5/2$, (g) $A_1/3$, (h) $A_2/3$, (i) $A_4/3$, (j) $A_5/3$, (k) $A_7/3$, and (l) $A_8/3$; (—) stable HBM solutions, (—) unstable HBM solutions, and (○) numerical integration solutions. (Continues)
Figure 3.2. Continues.

(c)

(d)
Figure 3.2. Continues.

(e)

(f)

(Continued)
Figure 3.2. Continues.

(g) $A_{1/3}$

(h) $A_{2/3}$

(Continued)
Figure 3.2. Continues.

(i) $A_{4/3}$

(j) $A_{5/3}$
Figure 3.2. Continues.

(k)

(l)
Figure 3.3. Frequency spectrum of a (a) period-1 (b) period-2 and (c) period-3 motion at $\Lambda = 0.85, 1.75$ and $2.9$, respectively. (a1, b1, c1) time history, (a2, b2, c2) HBM solutions, and (a3, b3, c3) numerical integration results.
3.3(b) and (c) confirm again that the corresponding motions are indeed period-2 and period-3 type motions, respectively.

3.4 Parametric Studies

A parameter set of \([\alpha_1, \alpha_2, \alpha_3, f_i, f_i, w_i, \zeta, \Lambda]\) is considered to investigate the subharmonic response of the systems governed by equation (2.1). In chapter 2, the influence of this set on period-1 motions was studied in detail for \(\Lambda \in (0, 1.5]\). In this section, period-\(\eta\) motions to a wider range of \(\Lambda \in (0, 6]\) are investigated. In order to limit the parametric study within a reasonable size, only period-2 and period-3 motions are considered. In equation (3.2), the maximum harmonic index of Fourier series is chosen as \(R = 6\), which results in sufficiently accurate period-1 solutions as shown in Figure 3.2. Therefore, the number of harmonic terms for period-2 and period-3 motions are defined as \(\eta R = 12\) and 18 terms, respectively. The RMS and mean values of the response \(u_{rms}\) and \(u_1\) are used to represent \(u(\tau)\). In the figures provided in following sections, solid and dashed lines represent stable and unstable motions, separately.

3.4.1 Influence of \(\alpha_2\) and \(\alpha_3\)

In Figures 3.4 and 3.5, the effect of cubic nonlinear term \(\alpha_3\) on period-2 and period-3 motions represented by \(u_1\) and \(u_{rms}\) values is shown. In this case, quadratic nonlinearity is not included, \(\alpha_2 = 0\). One softening condition with \(\alpha_3 = -0.1\), and two hardening conditions with \(\alpha_3 = 0.2\) and 0.5 are considered in addition to the PL case of
Figure 3.4. Influence of $\alpha_3$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$ and $\alpha_2 = 0$, given $\zeta = 0.01$, $f_1 = 0.5$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$; period-2 motions. (—) Stable and (−−) unstable HBM solutions.
Figure 3.5. Influence of $\alpha_3$ on (a) $u_{rms}$ and (b) $u_i$ of an oscillator with $\alpha_1 = 1$ and $\alpha_2 = 0$, given $\zeta = 0.01$, $f_i = 0.5$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$; period-3 motions. (—) Stable and (—) unstable HBM solutions.
\( \alpha_3 = 0 \). Period-2 and period-3 motions of the PL system agree well with the results of Kahraman and Blankenship [107]. In Figure 3.4, \( u_{rms} \) of resonance peak is reduced with an increment of \( \alpha_3 \) for the softening case, and the resonance frequency range is shifted to the left slightly. The SSI branch bends to the left further, and there are DSI motions for \( \alpha_3 = -0.1 \). However, for the hardening cases, the peak of period-2 resonance moves to the right on the frequency axis with the increment of \( \alpha_3 \). A pair of softening type SSI response curves (one stable and one unstable) is followed by a pair of hardening type DSI curves bending to the right. The stable SSI motions bifurcate into unstable DSI motions at certain transition frequencies while unstable SSI motions turn into stable DSI motions. The frequency of this transition increases with the value of \( \alpha_3 \) as shown in Figure 3.4. In summary, the influence of \( \alpha_3 \) on period-2 response is quite similar to that on period-1 motions near primary resonance.

As in the case of PL systems [107], the \( u_1 \) and \( u_{rms} \) values of period-3 response shown in Figure 3.5 is mostly an isolated closed curve, and another stable branch of period-1 motions coexists at lower amplitude that is not shown here. This period-3 resonance is associated with the second parametric resonance peak of the corresponding linear TV system near \( \Lambda = 3 \). Similar to Figure 3.4, the curves of period-3 motions also move to the left when \( \alpha_3 < 0 \), and to the right when \( \alpha_3 > 0 \). For softening cases, DSI motions do not exist while a wide range of DSI motions exists for the hardening cases. As the upper DSI branch is completely stable for \( \alpha_3 = 0 \), a portion of it becomes unstable for \( \alpha_3 > 0 \). Numerical integration solutions indicate that higher order
subharmonic motions such as period-6 motions exist at these frequencies of unstable DSI motions. The frequency range of this unstable upper branch DSI motions is expanded by increasing $\alpha_3$. As shown in Figure 3.5, DSI motions are unstable when $\Lambda \in [2.08, 2.31]$ for $\alpha_3 = 0.2$, and $\Lambda \in [2.19, 2.92]$ for $\alpha_3 = 0.5$. Moreover, unlike the period-2 motions of Figure 3.4, the value of $\alpha_3$ affects how low $u_{rms}$ of period-3 motions can reach. For $\alpha_3 < 0$, increasing the magnitude of $\alpha_3$ causes the response curve of period-3 motions shrink, and eventually disappear. For $\alpha_3 > 0$, a larger $\alpha_3$ makes the SSI portion of the closed curve approach the stable branch of period-1 motions underneath.

In Figures 3.6 and 3.7, the effect of $\alpha_2$ on $u_1$ and $u_{rms}$ values of period-2 and period-3 motions is shown for the same system as in Figures 3.4 and 3.5. The comparison of these two figures illustrates that the impact of a hardening $\alpha_2$ is quite similar to that of a hardening $\alpha_3$, except that the resonance peaks for $\alpha_3 > 0$ bend more severely than those for $\alpha_2 > 0$. For the softening case of $\alpha_2 = -0.1$, both period-2 and period-3 motions exhibit stable DSI behavior, which could not be observed in Figures 3.4 and 3.5 for $\alpha_3 = -0.1$. With such similarities in mind, only $\alpha_3$ is considered for the hardening PN cases in the following sections, while both of $\alpha_2$ and $\alpha_3$ are considered for the softening cases.

### 3.4.2 Influence of damping ratio $\zeta$

The effect of damping ratio $\zeta$ on the $u_1$ and $u_{rms}$ values of period-2 and period-3 motions is illustrated in Figures 3.8 to 3.13 for PN systems with $f_1 = 0.5$, $f_i = 0$ ($i \geq 2$),
Figure 3.6. Influence of $\alpha_2$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$ and $\alpha_3 = 0$, given $\zeta = 0.01$, $f_1 = 0.5$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$; period-2 motions. (—) Stable and (−−) unstable HBM solutions.
Figure 3.7. Influence of $a_2$ on (a) $u_{\text{rms}}$ and (b) $u_1$ of an oscillator with $a_1 = 1$ and $a_3 = 0$, given $\zeta = 0.01$, $f_1 = 0.5$, $f_i = 0 \ (i \geq 2)$, $w_3 = 0.3$; period-3 motions. (--) Stable and (--) unstable HBM solutions.
$w_3 = 0.3, \alpha_1 = 1$, and different nonlinearities. In Figures 3.8 and 3.9, a softening case of cubic nonlinearity is considered: $\alpha_3 = -0.1$ and $\alpha_2 = 0$. For period-2 motions shown in Figure 3.8, the amplitude of resonance peaks is reduced slightly with the increasing of $\zeta$ as expected. However, the frequency range of stable SSI motions is affected significantly by changing $\zeta$. When $\zeta = 0.01$, the stable SSI motions exist with $\Lambda \in [1.27, 2.01]$, and the range increases to $\Lambda \in [1.12, 1.97]$ for $\zeta = 0.05$, which is the maximum one in the figure. Beyond that, the response curve starts to diminish, and reduces to an isolated island as shown in Figure 3.8 for $\zeta = 0.07$. Increasing $\zeta$ further causes the period-2 motions to disappear. In Figure 3.9, the influence of $\zeta$ on period-3 motions is shown. The damping ratio $\zeta$ has the same qualitative effect on stable period-3 motions, which do not to exist for $\zeta > 0.02$ in this case.

In Figures 3.10 and 3.11, a hardening PN system with $\alpha_3 = 0.2$ and $\alpha_2 = 0$ is examined. Unlike the softening cases, both SSI and DSI type period-2 motions coexist here as illustrated in Figure 3.10. When $\zeta$ is as low as 0.01, these two types of motions are connected at saddle-node bifurcation points. Increasing $\zeta$ cannot eliminate the DSI motions, but separate them from the SSI motions. In Figure 3.10, the SSI type period-2 motions disappear for $\zeta > 0.075$, and the amplitudes of the DSI motions increase significantly. Similarly, in Figure 3.11 for period-3 motions, both of SSI and DSI motions are present for the same system when $\zeta \leq 0.05$. Within a boundary defined by the response curve for $\zeta = 0.005$, increasing $\zeta$ makes the response curve to shrink towards the middle as in the softening case of Figures 3.8 and 3.9. Further increasing $\zeta$,
Figure 3.8. Influence of $\zeta$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -0.1$, given $f_1 = 0.5$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$; period-2 motions. (—) Stable and (−−) unstable HBM solutions.
Figure 3.9. Influence of $\zeta$ on (a) $u_{rms}$ and (b) $u_i$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -0.1$, given $f_1 = 0.5$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$; period-3 motions. (---) Stable and (---) unstable HBM solutions.
Figure 3.10. Influence of $\zeta$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = 0.2$, given $f_1 = 0.5$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$; period-2 motions. (—) Stable and (−−) unstable HBM solutions.
Figure 3.11. Influence of $\zeta$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = 0.2$, given $f_1 = 0.5$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$; period-3 motions. (—) Stable and (––) unstable HBM solutions.
first the DSI motions disappear, followed by the SSI motions. Like the hardening cases in Figure 3.5, there is an unstable portion at the beginning of upper DSI branch. Increasing $\zeta$ reduces the frequency range of this unstable DSI section.

Finally, a softening PN system with quadratic nonlinearity is considered in Figures 3.12 and 3.13 for $\alpha_2 = -0.1$ and $\alpha_3 = 0$. The comparison between Figures 3.8 and 3.12, and 3.9 and 3.13 shows that these two types of softening PN systems are influenced in a similar way by $\zeta$. One difference is that DSI motions exist in Figures 3.12 and 3.13 for both period-2 and period-3 motions provided $\zeta$ is not too large. In addition, Hopf bifurcation points appear on the DSI branches of period-2 motions as shown in Figure 3.12. Two types of stable DSI motions are exhibited: one is a continuation of the SSI motions while the other bifurcates from the unstable DSI motions. In short, increasing the value of $\zeta$ diminishes subharmonic motions due to its well-known effect on parametric instabilities. In addition, it can be stated that period-3 motions appear to be more sensitive to $\zeta$ than period-2 motions.

3.4.3 *Influence of stiffness $w(\tau)$*

Figures 3.14 to 3.17 show the effect of $w(\tau)$ on period-2 and period-3 motions of the same PN systems for $\alpha_3 = -0.1$ and 0.2. A harmonic form of $w(\tau)$ is considered in this section. Therefore, only the value of $w_3$ is varied, and other Fourier components $w_i$ of $w(\tau)$ are assumed zero. Decreasing $w_3$ has the similar effect as increasing $\zeta$, both of which tend to reduce the amplitude of parametric resonance. The system response $u_{rms}$
Figure 3.12. Influence of $\zeta$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = -0.1$ and $\alpha_3 = 0$, given $f_1 = 0.5$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$; period-2 motions. (---) Stable and (--) unstable HBM solutions.
Figure 3.13. Influence of $\zeta$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = -0.1$ and $\alpha_3 = 0$, given $f_1 = 0.5$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$; period-3 motions. (-----) Stable and (---) unstable HBM solutions.
changes gradually with $w_3$ in contrast to that of $\zeta$, as shown in Figures 3.8 to 3.13. Small changes in $w_3$ do not result in drastic variations in period-2 and period-3 motions. For period-2 motions, the isolated islands of SSI motions shown in Figures 3.8 and 3.12 for large $\zeta$ are not evident in Figure 3.14. As the value of $w_3$ is reduced, stable and unstable SSI branches come closer, and period-2 motions disappear altogether with further reductions in $w_3$. Finally, a portion of the SSI branch of period-3 motions becomes unstable in Figure 3.17 when $w_3 = 0.4$. This suggests that even though a larger $w_3$ increases the amplitudes of subharmonic response, stable period-η motions may yield to higher order subharmonic motions through period-doubling bifurcations [15].

3.4.4 Influence of mean load $f_1$

The effect of mean load $f_1$ on period-2 and period-3 motions of PN systems with cubic nonlinearity is illustrated here. In Figures 3.18 and 3.19, the results of a softening PN case with $\alpha_3 = -0.1$ and $\alpha_2 = 0$ are shown for different values of $f_1$. Similar to period-1 solutions, the amplitudes of period-2 motions near the parametric resonance are influenced significantly by $f_1$ as shown in Figure 3.18. The amplitudes of the stable SSI motions increase first when $f_1$ is changed from 0.25 to 0.5, and then decrease drastically until disappear for $f_1 > 1$. DSI motions may exist for a lightly loaded oscillator, say $f_1 = 0.25$ in Figure 3.18. For period-3 motions shown in Figure 3.19 for the same system, a similar qualitative effect of $f_1$ is observed. Again, period-3 motions vanish for a heavily loaded system, say $f_1 > 1$. 

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Figure 3.14. Influence of $w_3$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -0.1$, given $\zeta = 0.01$, $f_i = 0.5$, $f_i = 0$ ($i \geq 2$); period-2 motions. (—) Stable and (−−) unstable HBM solutions.
Figure 3.15. Influence of $w_3$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = -0.1$, given $\zeta = 0.01$, $f_i = 0.5$, $f_i = 0$ ($i \geq 2$); period-3 motions. (—) Stable and (—) unstable HBM solutions.
Figure 3.16. Influence of $w_3$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 0.2$, given $\zeta = 0.01$, $f_i = 0.5$, $f_i = 0$ ($i \geq 2$), $w_i = 0$ ($i \geq 2$ and $i \neq 3$); period-2 motions. (—) Stable and (−−) unstable HBM solutions.
Figure 3.17. Influence of \( w_3 \) on (a) \( u_{rms} \) and (b) \( u_1 \) of an oscillator with \( \alpha_1 = 1 \), \( \alpha_2 = 0 \), and \( \alpha_3 = 0.2 \), given \( \zeta = 0.01 \), \( f_i = 0.5 \), \( f_i = 0 \) \((i \geq 2)\), \( w_i = 0 \) \((i \geq 2 \text{ and } i \neq 3)\); period-3 motions. (—) Stable and (––) unstable HBM solutions.
Figure 3.18. Influence of $f_1$ on (a) $u_{rms}$ and (b) $u_i$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = -0.1$, given $\zeta = 0.01$, $f_i = 0 \ (i \geq 2)$, $w_3 = 0.3$; period-2 motions. (---) Stable and (−−) unstable HBM solutions.
Figure 3.19. Influence of $f_1$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = -0.1$, given $\zeta = 0.01$, $f_i = 0 \ (i \geq 2)$, $w_3 = 0.3$; period-3 motions. (—) Stable and (−−) unstable HBM solutions.
In Figures 3.20 and 3.21, the subharmonic response of a hardening type PN oscillator with $\alpha_3 = 0.2$ and $\alpha_2 = 0$ is shown for different $f_1$ values. In this case, the influence of $f_1$ is gradual. For both period-2 and period-3 motions, the parametric resonance peaks and the saddle-node bifurcation points connecting SSI and DSI motions move to the right on the scale of $\Lambda$ by increasing $f_1$. At the same time, the amplitudes of stable SSI motions are increased significantly, while the amplitudes of DSI motions are decreased slightly. Therefore, the influence of $f_1$ on large amplitude DSI motions is quite insignificant. In addition, unstable motions exist in the portion of the stable DSI branch as of period-3 motions when $f_1$ is relatively low, as in the case for $f_1=0.25$ and 0.5 in Figure 3.21. In summary, $f_1$ has also a significant influence on both period-2 and period-3 motions.

In Figures 3.22 and 3.23, a softening case for $\alpha_2 = -0.1$ is shown. For period-2 motions, the plots in Figure 3.22 are as complicated as those in Figure 3.18. For period-3 motions in Figure 3.23, increasing $f_1$ moves up the stable section of SSI motions for each case before period-3 motions vanish when $f_1 > 1$.

3.4.5 Influence of external excitation $f_3$

The effect of external excitation $f_3$ on period-2 and period-3 motions of a PN system with $\alpha_2 = 0$ and $\alpha_3 = 0.2$ is illustrated in Figures 24 and 25, respectively. A harmonic external force defined by $f_3$ is considered in addition to a parametric excitation of $w_3 = 0.2$. As demonstrated in both figures, an increase of $f_3$ tends to diminish
Figure 3.20. Influence of $f_1$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 0.2$, given $\zeta = 0.01$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$; period-2 motions. (—) Stable and (−−) unstable HBM solutions.
Figure 3.21. Influence of $f_1$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 0.2$, given $\zeta = 0.01$, $f_i = 0 \ (i \geq 2)$, $w_3 = 0.3$; period-3 motions. (—) Stable and (−−) unstable HBM solutions.
Figure 3.22. Influence of $f_1$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = -0.1$, and $\alpha_3 = 0$, given $\zeta = 0.01$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$; period-2 motions. (—) Stable and (––) unstable HBM solutions.
Figure 3.23. Influence of $f_1$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = -0.1$, and $\alpha_3 = 0$, given $\zeta = 0.01$, $f_i = 0$ ($i \geq 2$), $w_3 = 0.3$; period-3 motions. (−−) Stable and (−−−) unstable HBM solutions.
Figure 3.24. Influence of $f_3$ on (a) $u_{rms}$ and (b) $u_{i}$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 0.2$, given $\zeta = 0.01$, $f_1 = 0.5$, $f_i = 0$ ($i = 2,4,5...$), $\omega_3 = 0.2$; period-2 motions. (---) Stable and (− −) unstable HBM solutions.
Figure 3.25 Influence of $f_3$ on (a) $u_{rms}$ and (b) $u_1$ of an oscillator with $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 0.2$, given $\zeta = 0.01$, $f_1 = 0.5$, $f_i = 0$ ($i = 2, 4, 5, \ldots$), $w_3 = 0.2$; period-3 motions. (—) Stable and (--) unstable HBM solutions.
subharmonic resonances that are excited primarily by \(w_3\). This is because \(w_3\) and \(f_3\) are defined as positive numbers so that \(w(\tau)\) and \(f(\tau)\) are in phase. The same type of cancellation of in-phase internal and external excitations was repeated for the period-1 motions as well. When internal and external excitations are out of phase, subharmonic resonances are increased by external excitation amplitude \(f_3\).

### 3.5. Conclusions

In this chapter, the dynamic response of a piecewise-nonlinear oscillator is investigated near the parametric resonance frequencies. This oscillator is subjected to a mean load and combined parametric and external excitations, as well as a restoring function \(g[u(\tau)]\) formed by clearance and continuous nonlinearities. Multi-term HBM is used in conjunction with the Newton-Raphson method and DFT to predict period-\(\eta\) (\(\eta = 2, 3\)) subharmonic motions. The stability analysis of steady state response is performed by applying Floquet theory. The HBM predictions of period-1, period-2 and period-3 motions are shown to agree well with the direct numerical integration solutions.

A parametric study on the influence of \(\alpha_2, \alpha_3, \zeta, w_i, f_1\) and \(f_i\) on period-2 and period-3 subharmonic motions is also included. The following conclusions can be made from these parametric studies:

- Both quadratic and cubic nonlinearities defined by \(\alpha_2\) and \(\alpha_3\) influence the system behavior near parametric resonance frequencies significantly. For hardening cases, \(\alpha_2\) and \(\alpha_3\) have a similar effect qualitatively. For softening cases, however, major
differences are noted in terms of the impact of $\alpha_2$ and $\alpha_3$ on the subharmonic response.

- The unstable DSI period-3 motions diminish, as $\zeta$ or $f_1$ increases, or $w(\tau)$ decreases.

- The cubic nonlinearity $\alpha_3$ has a more significant effect on the system response compared to quadratic nonlinearity $\alpha_2$, especially for hardening type systems.

- The subharmonic motions are very sensitive to the value of three key parameters: damping ratio $\zeta$, TV stiffness $w(\tau)$ and mean load $f_1$.

- Parametric resonance peaks become extremely significant for very lightly damped systems excited heavily by $w(\tau)$.

- The mean load $f_1$ also has a substantial effect on subharmonic motions. Parametric resonances are eliminated for large $f_1$ when the nonlinearity is of the softening type, while the amplitudes of SSI motions are increased for hardening type PN systems.

- Finally, an external excitation $f(\tau)$ that is in phase with $w(\tau)$ tends to reduce the subharmonic resonance of the steady state response.
4.1. Introduction

The vibratory motions at contact interfaces of elastic solids have been of interest to several investigators. Dynamic motions at such interfaces influence fatigue and wear performance of contact surfaces. Some examples of such contact interfaces include gear mesh, rolling element bearings, and railroad wheel-rail contacts. Hertzian theory has been employed extensively to model the flexibility of the contact interfaces. One major theoretical study on contact dynamics is by Nayak [6] who proposed a SDOF dynamic model, and obtained analytical solutions by using a single-term HBM. As the theoretical predictions matched experimental results qualitatively, a heuristic approach was employed to describe the dynamic behavior further. Hess and Soom [7] studied the same problem by using the method of multiple scales, and quantified the amount of friction reduction due to contact vibrations. These earlier studies assumed that the surfaces are maintained in contact all the time. In two recent studies, Perret-Liaudet [8, 9] investigated a sphere-plane contact problem that allows the separation of contacting
surfaces (contact loss). The existence of subharmonic and superharmonic responses was demonstrated by using numerical methods. In a later work, Sabot, et al. [10] reported the results of an extensive experimental study on the same sphere-plane contact problem. They demonstrated that the steady state response predicted by a dynamic model based on the Hertzian contact formulation agrees well with experimental data for both free and harmonically excited vibrations. Focusing on vibro-impacts, Perret-Liaudet and Sabot [11] investigated the same model of Hertzian contact by numerical methods, and confirmed the significance of clearance at the contact on super- and subharmonic resonances. In two recent studies, Rigaud and Perret-Liaudet [12], and Perret-Liaudet and Rigaud [13] solved an impacting Hertzian contact problem by using the shooting method in conjunction with parametric continuation technique.

These previous studies used either numerical or experimental methods to investigate contact vibration problems, and very little analytical treatment of this problem is available, especially when the contact loss is included. Accordingly, this chapter focuses on the dynamic analysis of a flat surface in contact with a sphere by using multi-term HBM. The HBM solutions are compared with corresponding numerical solutions from the shooting method. Results of an experimental study are also presented to validate both the dynamic model and the HBM predictions. The condition for contact loss to occur is obtained by using a single-term HBM approximation. At the end, the influence of external force $f(\tau)$ and damping ratio $\zeta$ on steady-state response is quantified.
4.2. Multi-term HBM formulations

4.2.1 Equation of Motion

A double sphere-plane Hertzian contact can be modeled as a SDOF system as shown in Figure 1.4 [12], which is governed by equation (1.1). In this equation, \( g[u(\tau)] \) is defined by equation (1.2), which represents a Hertzian contact, allowing contact losses. \( d[u(\tau)] \) in equation (1.1) is a damping function subject to a power index \( m \), which is defined by equation (1.3).

4.2.2 Multi-term HBM Solution

In previous studies, the method of multiple scales was used extensively to analyze contact vibration problems with continuous nonlinearity only [8-9]. When contact loss is considered, such kind of perturbation methods is no longer applicable. Single-term HBM was utilized by Nayak [6], and some obvious discrepancies were observed between HBM results and experiments. Multi-term HBM was used in previous studies to investigate similar discontinuous problems successfully [105, 106, 136]. The same method combined with DFT is employed here to obtain the steady-state solutions of equation (1.1). First, let \( \theta = \frac{\Lambda \tau}{\eta} = ? \tau \) where \( \Lambda \) is the fundamental excitation frequency, and \( \eta \) is a subharmonic index, and write equation (1.1) as

\[
?^2 \frac{d^2 u}{d \theta^2} + 2\zeta \omega \left\{ d[u(\theta)] \right\}^m \frac{du}{d \theta} + g[u(\theta)] = 1 + f(\theta). \tag{4.1}
\]
The periodic forcing function is given in the form of a Fourier series as

\[
    f(\theta) = \sum_{r=1}^{R} [f_{2r} \cos(r\eta \theta) + f_{2r+1} \sin(r\eta \theta)],
\]

(4.2a)

and the periodic steady state response is assumed as

\[
    u(\theta) = u_1 + \sum_{r=1}^{R} [u_{2r} \cos(r\theta) + u_{2r+1} \sin(r\theta)].
\]

(4.2b)

In order to maintain a harmonic balance, the nonlinear functions \(d[u(\theta)]\) and \(g[u(\theta)]\) must be periodic as well, i.e.

\[
    d[u(\theta)] = d_1 + \sum_{r=1}^{R} [d_{2r} \cos(r\theta) + d_{2r+1} \sin(r\theta)],
\]

(4.2c)

\[
    g[u(\theta)] = g_1 + \sum_{r=1}^{R} [g_{2r} \cos(r\theta) + g_{2r+1} \sin(r\theta)].
\]

(4.2d)

By substituting equations (4.2) into equation (4.1) and enforcing a harmonic balance with \(m = 1\), a vector equation \(S = 0\) is obtained where \(S = [S_1 \ S_2 \ S_3 \ ... \ S_{2\eta R} \ S_{2\eta R+1}]^T\), and
\[ S_1 = g_1 - 1 + \zeta \eta \sum_{r=1}^{R} \left( d_{2r} u_{2(\eta+r)} - d_{2r+1} u_{2\eta} \right), \]  \hspace{1cm} (4.3a)

\[ S_{2i} = -2i^2 u_{2i} + 2\zeta^2 d_{1i} u_{2i+1} + g_{2i} - f_{2i} \eta \]

\[ -\zeta \sum_{r=1}^{R} d_{2r+1} \left[ (r\eta - i)u_{2(r\eta-i)} - (i - r\eta)u_{2(i-r\eta)} + (r\eta + i)u_{2(\eta+i)} \right] \]  \hspace{1cm} (4.3b)

\[ +\zeta \sum_{r=1}^{R} d_{2r} \left[ (r\eta - i)u_{2(\eta-i)+1} + (i - r\eta)u_{2(i-r\eta)+1} + (r\eta + i)u_{2(\eta+i)+1} \right] \]

\[ S_{2i+1} = -2i^2 u_{2i+1} - 2\zeta^2 d_{1i} u_{2i} + g_{2i+1} - f_{2i+1} \eta \]

\[ +\zeta \sum_{r=1}^{R} d_{2r+1} \left[ (r\eta - i)u_{2(\eta-i)+1} + (i - r\eta)u_{2(i-r\eta)+1} - (r\eta + i)u_{2(\eta+i)+1} \right] \]  \hspace{1cm} (4.3c)

\[ +\zeta \sum_{r=1}^{R} d_{2r} \left[ (r\eta - i)u_{2(\eta-i)} - (i - r\eta)u_{2(\eta+i)} \right] \]

where \( i \in [1, R] \) in equations (4.3b) and (4.3c). These algebraic equations can be reduced to the case of constant viscous damping \((m = 0)\) by letting \( d_1 = 1 \), and \( d_{2r} = d_{2r+1} = 0 \) for \( r \in [1, R] \).

Meanwhile, coefficients \( g_i \) and \( d_i \) in equations (4.2c) and (4.2d) can be expressed in terms of unknown Fourier coefficients of the response \( \mathbf{u} = [u_1 \ u_2 \ u_3 \ ... \ u_{2R} \ u_{2R+1}]^T \) by utilizing DFT. The values of \( \mathbf{u}(\theta) \) at discrete values of \( \theta_n = nh \) are

\[ \mathbf{u}^{(n)} = u_1 + \sum_{r=1}^{R} \left[ u_{2r} \cos \left( \frac{2\pi r n}{N} \right) + u_{2r+1} \sin \left( \frac{2\pi r n}{N} \right) \right], \quad n \in [0, N - 1] \]  \hspace{1cm} (4.4)
where \( h = 2\pi \eta/(NA) \) and \( N \geq 2R \). Using equation (1.2b), \( n \)-th discrete value of \( g[u(\theta)] \) is given as

\[
g^{(n)} = \begin{cases} (1 + \rho u_n)^{\frac{1}{p}}, & u_n > -\frac{1}{\rho}, \\ 0, & u_n \leq -\frac{1}{\rho}. \end{cases}
\] (4.5)

The coefficients \( g_i \) are calculated by taking the inverse DFT of equation (4.5) as

\[
g_1 = \frac{1}{N} \sum_{n=0}^{N-1} g^{(n)}, \quad \text{(4.6a)}
\]

\[
g_{2r} = \frac{2}{N} \sum_{n=0}^{N-1} g^{(n)} \cos \frac{2\pi rn}{N}, \quad \text{(4.6b)}
\]

\[
g_{2r+1} = \frac{2}{N} \sum_{n=0}^{N-1} g^{(n)} \sin \frac{2\pi rn}{N}. \quad \text{(4.6c)}
\]

For the case of a constant viscous damping model with \( m = 0 \) in equation (1.1), \( d_n = 1 \).

For a nonlinear damping model \( (m = 1) \), \( n \)-th discrete value of \( d[u(\theta)] \) is given as

\[
d^{(n)} = \begin{cases} \left[1 + \rho u^{(n)}\right]^p, & u_n > -\frac{1}{\rho}, \\ 0, & u_n \leq -\frac{1}{\rho}. \end{cases}
\] (4.7)
and \( d_i \) are defined as

\[
d_1 = \frac{1}{N} \sum_{n=0}^{N-1} d^{(n)},
\]

\[ (4.8a) \]

\[
d_{2r} = \frac{2}{N} \sum_{n=0}^{N-1} d^{(n)} \cos \frac{2\pi r n}{N},
\]

\[ (4.8b) \]

\[
d_{2r+1} = \frac{2}{N} \sum_{n=0}^{N-1} d^{(n)} \sin \frac{2\pi r n}{N}.
\]

\[ (4.8c) \]

Having \( g_i \) and \( d_i \) defined, the vector equation \( S = 0 \) can be solved for \( u \) by using Newton-Raphson method as defined in equation (2.10).

### 4.3. Experimental Setup

In order to check the validity of the equation of motion as well as the solution method, a set of experimental data obtained from Ecole Centrale de Lyon in France through collaboration is used here. The experimental setup consists of a double sphere-plane contact as shown in Figure 4.1. A spherical ball is preloaded between two horizontal plane surfaces. One plane is attached to a heavy rigid frame while the other plane is on a vertically moving cylinder. These two contacts can be approximated by Hertzian theory when the surface roughness of contacting surfaces is not excessive (In this case, for the plane \( Ra \leq 0.4 \mu m \), and for the ball surface \( Ra \leq 0.03 \mu m \)). In addition, other critical assumptions for Hertzian theory are met including negligible friction effects.
Figure 4.1. The test rig: (1) vibration exciter, (2) force transducer, (3) moving cylinder, (4) accelerometer, (5) ball, (6) tri-axial force transducer, and (7) rigid frame.
and elastic deformations. The elastic bodies in contact are made of SAE 52100 steel. The ball has a diameter of 25 mm and a mass of 0.064 kg, which is significantly lower than the weight of the moving cylinder (6.48 kg). The mass of the moving cylinder corresponds to an applied static load equal to 63.6 N. Since the ball mass is negligible in comparison with that of the moving cylinder, the experimental system acts as a SDOF oscillator. Thus, assuming identical mechanical properties for the ball and the planes, the restoring elastic force expression can be deduced from the double sphere-plane Hertzian contact as \( F = C z^2 \), where \( C = E \sqrt{R}/[3\sqrt{2}(1-\nu^2)] \). Given the modulus of elasticity \( E = 205 \text{ GPa} \), Poisson ratio \( \nu = 0.29 \) and the radius of the ball \( R = 12.5 \text{ mm} \), one finds \( C = 5.9(10)^9 \text{ Nm}^{-1/2} \), \( z_s = 4.9 \mu m \), and \( f_0 = 276 \text{ Hz} \). Here, \( z_s \) corresponds to the static displacement of the moving cylinder due to its weight, and \( f_0 \) is the calculated linearized natural frequency.

The contact surface is excited in normal direction by a suspended vibration shaker connected to a signal generator and a power amplifier. Therefore, a harmonic normal force is applied to the moving cylinder in addition to the static load. The excitation force, the acceleration of the moving cylinder, and the normal force transmitted to the rigid frame through the contact are measured by piezoelectric transducers. Conventional charge amplifiers are used for all responses. Each harmonic component of the signal is analyzed using a lock-in amplifier. This one is based on a phase sensitive detection in order to single out the components of the signal (frequency, amplitude and phase).

The linear experimental contact natural frequency of 269 Hz and equivalent viscous damping ratio of nearly 0.7 percent are measured from the resonant peaks under very
small external input amplitudes. This measured natural frequency is within 3 percent of the calculated one. Moreover, the measured damping ratio is also in good agreement with previous studies [10].

4.4. Results and Discussion

4.4.1 Comparison of HBM with a Numerical Solution Method and Experiments

The predictions of multi-term HBM are compared with those of a numerical solution method to determine the accuracy of HBM solutions. The shooting method in conjunction with a continuation procedure is used as the numerical method here. Details in regards to the application of this method to contact problems can be found in [140]. This method employs a simple pseudo arc length continuation scheme to predict periodic solutions. The case of linear damping \((m=0)\) with \(\zeta = 0.008\) is considered for this comparison. The amplitudes of the first two harmonics of \(\ddot{u}(\tau)\) are defined as

\[
H_1 = \Lambda^2 [u_2^2 + u_3^2]^{1/2} \quad \text{and} \quad H_2 = 4\Lambda^2 [u_4^2 + u_5^2]^{1/2},
\]

which are used as the parameters for the comparison. In Figure 4.2, the predictions of \(H_1\) and \(H_2\) from HBM and the shooting method are compared for a harmonic external excitation of \(f_0 = 0.04\) (other \(f_i = 0\)) within the frequency range of \(\Lambda \in [0.5, 1.5]\). First twelve harmonic terms are included in equation (4.2b) that corresponds to \(R = 6\). It is evident that two methods match each other well for both \(H_1\) and \(H_2\). The combined softening effect of continuous contact nonlinearity and contact losses results in a significant bend of the primary resonant peak to the left, forming a region of double stable motions within \(\Lambda \in [0.75, 0.96]\). The same
Figure 4.2. Comparison of the acceleration response predicted by HBM and the shooting method for $\zeta=0.008$, $f_3=0.04$. (a) $H_1$, and (b) $H_2$; (--) stable HBM, (---) unstable HBM, and (○) shooting method. (Continued)
Figure 4.2. Continues.
level of agreement is obtained for other values of $f_3$ and $\zeta$ as well, which suggests that multi-term HBM is capable to solve equation (1.1) accurately.

Similarly, comparisons between the predictions of HBM and measured response are provided in Figures 4.3 and 4.4 for two different excitation amplitudes in order to demonstrate the validity of equation (1.1) in representing an actual sphere-plane contact. A linear damping model with $\zeta = 0.007$ is used for these comparisons. In Figure 4.3, the amplitudes of predicted response $H_1$ and $H_2$ are compared with measured ones for $f_3 = 0.007$, which is not large enough to cause any contact loss happen. Therefore, only a very slight softening type resonance peak is obtained, and the predictions of $H_1$ and $H_2$ by HBM match the measured values quite well for this continuously nonlinear case.

In Figure 4.4, another case is shown to compare the theoretical predictions and the measured data from the same experimental set up for $f_3 = 0.04$. In this case, contact loss is observed, which results in a more significant softening behavior than the previous case in Figure 4.3. The values of predicted and measured $H_1$ and $H_2$ are again in good agreement, which suggests that not only HBM is suitable to investigate the problem of sphere-plane contact, but also equation (1.1) is accurate enough to model the actual system in hand.
Figure 4.3. Comparison of measured acceleration response to HBM prediction for $\zeta = 0.007$, $f_3 = 0.018$.
(a) $H_1$, and (b) $H_2$; (---) stable HBM, (---) unstable HBM, and (□) measurements. (Continued)
Figure 4.3. Continues.
Figure 4.4. Comparison of measured acceleration response to HBM prediction for $\zeta = 0.008$, $f_3 = 0.04$. (a) $H_1$, and (b) $H_2$; (---) stable HBM, (---) unstable HBM, and (□) measurements. (Continued)
Figure 4.4. Continues.
4.4.2 Criterion for Contact Loss to Occur

When the amplitude of the external harmonic excitation \( f_3 = f_k \) exceeds a threshold value, and the fundamental frequency of external excitation is near the primary resonant frequency \( \Lambda = 1 \), the contacting surfaces of the sphere and plane might be separated during the certain portion of a vibration period [12]. The condition for contact loss to occur can be predicted approximately by using a single-term HBM formulation. Using a linear damping model (\( m = 0 \)) and expanding \( g[u(\tau)] \) into a truncated Taylor series [10], one can write

\[
g[u(\tau)] = \begin{cases} 
1 + \frac{3}{2} \rho u(\tau) + \frac{3}{8} [\rho u(\tau)]^2 - \frac{1}{16} [\rho u(\tau)]^3, & u(\tau) > -\frac{1}{\rho}, \\
0, & u(\tau) \leq -\frac{1}{\rho}.
\end{cases} \tag{4.9}
\]

This approximation of \( g[u(\tau)] \) does not satisfy the boundary condition \( g[-\frac{1}{\rho}] = 0 \) for \( u(\tau) = -\frac{1}{\rho} \). In order to remedy this problem, the Taylor series approximation is modified slightly as

\[
g[u(\tau)] = \begin{cases} 
1 + \frac{3}{2} \rho u(\tau) + \frac{3}{8} [\rho u(\tau)]^2 - \frac{1}{14} [\rho u(\tau)]^3, & u(\tau) > -\frac{1}{\rho}, \\
0, & u(\tau) \leq -\frac{1}{\rho}.
\end{cases} \tag{4.10}
\]
In Figure 4.5, the comparison of different definitions of \( g[u(\tau)] \) is illustrated, which shows that equation (4.10) is quite accurate to replace the original one when \( u(\tau) < 2 \).

Assume harmonic forms of \( u(\tau) \) and \( g[u(\tau)] \) as

\[
u_1 = 1 + \frac{3}{2} \rho u_1 + \frac{3}{7} \rho^2 u_1^2 + \frac{3}{14} \rho^2 u_k^2 - \frac{1}{144} \rho^3 u_1^3,
\]

(4.13a)

\[
v_k = \frac{3}{2} \rho u_k + \frac{6}{7} \rho^2 u_1 u_k - \frac{3}{14} \rho^3 u_1^2 u_k - \frac{9}{56} \rho^3 u_k^3.
\]

(4.13b)

Substituting equations (4.11) and (4.12) into equation (1.1) with \( m = 0 \), one obtains

\[
v_1 - 1 = 0,
\]

and

\[
\left( v_k - \kappa^2 \Lambda^2 u_k \right) \cos \alpha_k - 2 \zeta \kappa \Lambda u_k \sin \alpha_k - f_k \cos \Gamma_k = 0,
\]

(4.14a)

\[
\left( v_k - \kappa^2 \Lambda^2 u_k \right) \sin \alpha_k + 2 \zeta \kappa \Lambda u_k \cos \alpha_k - f_k \sin \Gamma_k = 0,
\]

(4.14b)

where \( \Gamma_k \) is the phase angle between external excitation and the system response. By imposing the condition at the point of contact loss.
Figure 4.5. Comparison of power series approximations to exact $G[u(\tau)]$, (a) equation 91b), (b) equation (11a), and (c) equation (11b).
\[ u_1 - u_\kappa = -\frac{1}{\rho}, \quad (4.15) \]

to equations (4.13) and (4.14), a cubic polynomial in \( u_1 \) is obtained as

\[ \rho^3 u_1^3 - 9 \rho^2 u_1^2 - 27 \rho u_1 - 3 = 0. \quad (4.16) \]

This equation has three real roots that can be solved numerically, and only one of them is suitable by considering the physical system in hand. With the value of \( u_1 \) is known at the separation point, the magnitudes of \( u_\kappa \) and \( v_\kappa \) can be calculated from equations (4.15) and (4.13b). Eliminating \( \Gamma_\kappa \) and \( \alpha_\kappa \) from equations (4.14a, b) yields

\[ u_\kappa^2 \kappa^4 \Lambda^4 - 2u_\kappa (v_\kappa - 2 \xi^2 u_\kappa) \kappa^2 \Lambda^2 + v_\kappa^2 - f_\kappa^2 = 0. \quad (4.17) \]

The transition frequencies at which the contact loss is initiated are given as

\[ \Lambda_{1,2}^2 = \frac{1}{\kappa^2} \left[ \frac{v_\kappa}{u_\kappa} - 2 \xi^2 + \frac{1}{u_\kappa} \sqrt{4 \xi^2 u_\kappa (\xi^2 u_\kappa - v_\kappa) + f_\kappa^2} \right]. \quad (4.18) \]

For the contact loss to happen, the discriminant of equation (4.18) should be positive. Hence, a separation criterion is obtained as

\[ f_\kappa > 2 \xi \sqrt{u_\kappa (v_\kappa - \xi^2 u_\kappa)} . \quad (4.19a) \]
For a lightly damping system, $\zeta << 1$, equation (4.19a) is reduced as

$$f_k > 2\zeta \sqrt{u_k v_k}. \quad (4.19b)$$

For the experimental setup considered in this study, the values of $u_k$ and $v_k$ at the separation point are obtained as $u_k = 1.3266$, and $v_k = 1.1252$, such that the condition for the contact loss to happen for this system is $f_k > 2.44\zeta$. Nayak [6] used a set of heuristic arguments to arrive at a similar threshold value. He stated that the average dissipated power may be equal to the average input power at the downward jump frequency, and that the forced response is the same as the undamped free response at the same frequency. By considering the undamped free response possessing an amplitude grazing contact loss, one obtains a contact loss condition of $f_k > 2.66\zeta$, which is in reasonably good agreement with the condition obtained by the single-term HBM.

In Figure 4.6, the effect of harmonic excitation $f(\tau)$ on $H_1$ is shown for a linearly damped case of $\zeta = 0.008$. Here, three response curves for the different values of $f_k$ are illustrated. The curve for $f_k = (2.44)(0.008) = 0.0195$ represents the threshold value. When $f_k < f_t$, no contact loss is observed. When $f_k > f_t$, such as $f_k = 0.03$, the contact loss takes place. Similarly, in Figure 4.7, damping ratio $\zeta$ is varies for constant $f_3 = 0.03$. The threshold value of the damping ratio is $\zeta = 0.03/2.44 = 0.0123$. When
Figure 4.6. The influence of $f_k$ on $H_1$ for $\zeta = 0.008$ and $f_3 = 0.014, 0.0195, \text{ and } 0.03$. 

- $f_3 = 0.03$, contact loss
- $f_3 = 0.01952$
- $f_3 = 0.014$, no contact loss
Figure 4.7. The influence of $\zeta$ on $H_1$ for $f_3 = 0.03$ and $\zeta = 0.008, 0.0123, \text{ and 0.02}$. 
\( \zeta \) is as low as 0.008, contact loss is predicted. For \( \zeta > 0.0123 \), say \( \zeta = 0.02 \), the softening behavior is reduced greatly since there is no contact loss.

4.5. Summary

In this chapter, a sphere-plane contact problem with contact loss is investigated theoretically and experimentally. The physical system is modeled by a SDOF oscillator with piecewise nonlinear damping and stiffness. Multi-term HBM is used to obtain the steady state solution of the governing equation. The HBM solutions are confirmed by comparison to those of the shooting method. An experimental setup is devoted to measure the steady state response of the physical system excited harmonically. A very good agreement between theoretical prediction and measurements are demonstrated. Moreover, a separation criterion, which is defined by \( f(\tau) \) and \( \zeta \), is derived by using a one-term HBM formulation, and examined for different values of \( f(\tau) \) and \( \zeta \).

For the contact problem in hand, a viscously linear damping model yields results that are in good agreement with the experiments. Appyling current work on other contact problems might require a nonlinear damping model. The HBM methodology can also expanded to study MDOF contact problems. Such cases exist for the experimental setup considered here when the mass of the ball is not negligible compared to the mass of the preload cylinder.
5.1. Introduction

In this chapter, the same solution method used in previous chapters to study SDOF systems is applied to a nonlinear MDOF system with multiple clearances and PLTV stiffness. The \( N \)-DOF system is subjected to viscous damping and PN restoring functions \( g^{(i)}(t, u) \) \((i \in [1, N])\) that are formed by a combination of clearances and continuous nonlinearities. The system is excited by a set of periodically TV stiffnesses \( w^{(i)}(\tau) \) and external forces \( f^{(i)}(\tau) \) \((i \in [1, N])\). The equations of motion of this system are given in dimensionless matrix form as

\[
M\ddot{u}(\tau) + C\dot{u}(\tau) + K\dot{u}(\tau) = f(\tau).
\]

(5.1)

Here, overdot denotes derivative with respect to dimensionless time \( \tau \), and \( M \), \( C \) and \( K \) are constant coefficient matrices defined as
\[ M = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1N} \\ \vdots & \ddots & \vdots \\ \gamma_{N1} & \cdots & \gamma_{NN} \end{bmatrix}, \quad C = \begin{bmatrix} \zeta_{11} & \cdots & \zeta_{1N} \\ \vdots & \ddots & \vdots \\ \zeta_{N1} & \cdots & \zeta_{NN} \end{bmatrix}, \quad K = \begin{bmatrix} \omega_{11} & \cdots & \omega_{1N} \\ \vdots & \ddots & \vdots \\ \omega_{N1} & \cdots & \omega_{NN} \end{bmatrix}. \quad (5.2a-c) \]

In equation (5.1), \( \mathbf{u}(\tau) \) is displacement vector, \( \mathbf{f}(\tau) \) is the external excitation vector, and \( \mathbf{g}(\tau) \) is the vector of PN restoring functions with TV stiffness, which are defined as

\[
\mathbf{u}(\tau) = \begin{bmatrix} u^{(1)}(\tau) \\ \vdots \\ u^{(N)}(\tau) \end{bmatrix}, \quad \mathbf{f}(\tau) = \begin{bmatrix} f^{(1)}(\tau) \\ \vdots \\ f^{(N)}(\tau) \end{bmatrix}, \quad (5.2d,e)
\]

\[
\mathbf{g}(\tau) = \begin{bmatrix} w^{(1)}(\tau)g^{(1)}[u^{(1)}(\tau)] \\ \vdots \\ w^{(N)}(\tau)g^{(N)}[u^{(N)}(\tau)] \end{bmatrix}. \quad (5.2f)
\]

In this study, \( w^{(i)}(\tau) \) and \( f^{(i)}(\tau) \) (\( i \in [1, N] \)) are both assumed to be periodic functions corresponding to parametric and external excitations of the system, respectively. PN restoring functions \( g^{(i)}[u^{(i)}(\tau)] \) (\( i \in [1, N] \)) are defined as

\[
g^{(i)}[u^{(i)}(\tau)] = \begin{cases} \sum_{k=1}^{3} \alpha^{(i)}_k [u^{(i)}(\tau) - 1]^k, & u^{(i)}(\tau) > b^{(i)} \\ 0, & |u^{(i)}(\tau)| \leq b^{(i)} \\ \sum_{k=1}^{3} (-1)^{i-1} \alpha^{(i)}_k [u^{(i)}(\tau) + 1]^k, & u^{(i)}(\tau) < -b^{(i)} \end{cases}, \quad (5.3)
\]
where \( b^{(i)} \) is half clearance magnitude. As illustrated in Figure 1.2(a), \( g^{(i)}[u^{(i)}(\tau)] \) consists of three segments: a clearance (dead-zone) segment for \( |u^{(i)}(\tau)| \leq b^{(i)} \), and two continuously nonlinear segments for \( u^{(i)}(\tau) > b^{(i)} \) and \( u^{(i)}(\tau) < -b^{(i)} \). These two nonlinear segments are defined by a linear stiffness component of slope \( \alpha_1^{(i)} \), a quadratic nonlinearity term with coefficient \( \alpha_2^{(i)} \), and a cubic nonlinearity term with coefficient \( \alpha_3^{(i)} \).

Previous studies considered PL version of this MDOF system with [108] or without [14, 112, 113] TV stiffness. Numerical integration methods were generally used to solve the motion equations [14, 112, 113]. As mentioned in chapter one, analytical treatment on PN systems were limited to SDOF, TV models [18, 121, 128-130, 135, 136]. Therefore, in this chapter, the same method used earlier for SDOF PN systems is generalized to obtain the steady-state response of forced MDOF systems with multiple clearances and periodically PNTV stiffness. First, the general HBM formulation to obtain the steady-state period- \( \eta \) (\( \eta \): positive integer) solutions of equation (5.1) is presented. Then, the application of the solution method is demonstrated by considering a 3-DOF system. Instead of providing a complete understanding of the nonlinear dynamics of MDOF systems with PNTV stiffness, the solution method is proved to be suitable for such kind of systems.
5.2. Multi-term HBM Response to Periodic Excitations

Before implementing the multi-term HBM to solve equation (5.1), the excitations \( w^{(j)}(\tau) \) and \( f^{(i)}(\tau) \) \((i, j \in [1, N])\) should be expressed in the form of truncated Fourier series as

\[
\begin{align*}
  w^{(j)}(\tau) &= 1 + \sum_{\kappa=1}^{K} \left[ w_{2\kappa}^{(j)} \cos(\kappa \Lambda \tau) + w_{2\kappa+1}^{(j)} \sin(\kappa \Lambda \tau) \right], \\
  f^{(i)}(\tau) &= f_1^{(i)} + \sum_{\mu=1}^{M} \left[ f_{2\mu}^{(i)} \cos(\mu \Lambda \tau) + f_{2\mu+1}^{(i)} \sin(\mu \Lambda \tau) \right],
\end{align*}
\]

(5.4a)

(5.4b)

where \( \Lambda \) is the dimensionless fundamental excitation frequency. \( w_1^{(j)} = 1 \) is the mean component of the stiffness function, and \( w_{2\kappa}^{(j)} \) and \( w_{2\kappa+1}^{(j)} \) are the amplitudes of \( \kappa \)-th harmonic of \( w^{(j)}(\tau) \). Similarly, \( f_1^{(i)} \) is the mean load (preload) applied to the unit mass, and \( f_{2\mu}^{(i)} \) and \( f_{2\mu+1}^{(i)} \) are the amplitudes of \( \mu \)-th harmonic of \( f^{(i)}(\tau) \). Indices \( \kappa \) and \( \mu \) are integer-valued to ensure that \( w^{(j)}(\tau) \) and \( f^{(i)}(\tau) \) are commensurate to each other.

By defining \( \theta = \frac{\Lambda \tau}{\eta} = H \tau \) for \( H = \frac{\Lambda}{\eta} \), equation (5.1) becomes

\[
H^2 \mathbf{M} \ddot{\mathbf{u}}(\theta) + \mathbf{H} \mathbf{C} \dot{\mathbf{u}}(\theta) + \mathbf{K} \mathbf{g}(\theta) = \mathbf{f}(\theta).
\]

(5.5)
where \(^{(\prime)}\) denotes differentiation with respect to \(\theta\). The vector of unknown steady-state, period-\(\eta\) displacements \(\mathbf{u}(\theta)\) becomes

\[
\mathbf{u}(\theta) = \begin{bmatrix}
\mathbf{u}^{(1)}(\theta) \\
\vdots \\
\mathbf{u}^{(N)}(\theta)
\end{bmatrix}
\]  

(5.6a)

where

\[
u^{(i)}(\theta) = u_1^{(i)} + \sum_{r=1}^{\eta R} \left[u_{2r}^{(i)} \cos(r\theta) + u_{2r+1}^{(i)} \sin(r\theta) \right], \quad i \in [1,N].
\]  

(5.6b)

In equation (5.6b), \(R\) is the number of harmonics that would be sufficient to describe the steady-state period-1 response. For subharmonic motions, \(\eta R\) harmonic terms are considered. The vector of nonlinear restoring functions \(\mathbf{g}(\theta)\) is given as

\[
\mathbf{g}(\theta) = \begin{bmatrix}
w^{(1)}(\theta)g^{(1)}[\mathbf{u}^{(1)}(\theta)] \\
\vdots \\
w^{(N)}(\theta)g^{(N)}[\mathbf{u}^{(N)}(\theta)]
\end{bmatrix},
\]  

(5.6c)

where
\[ w^{(j)}(\theta) = 1 + \sum_{k=1}^{K} \left[ w_{2k}^{(j)} \cos(\kappa \eta \theta) + w_{2k+1}^{(j)} \sin(\kappa \eta \theta) \right], \quad (5.6d) \]

\[ g^{(j)}[u(\theta)] = v_1^{(j)} + \sum_{r=1}^{R} \left[ v_{2r}^{(j)} \cos(r \theta) + v_{2r+1}^{(j)} \sin(r \theta) \right], \quad j \in [1,N]. \quad (5.6e) \]

Moreover, the vector of external forces is expressed as

\[
\mathbf{f}(\theta) = \begin{cases} 
  f^{(1)}(\theta) \\
  : \\
  f^{(N)}(\theta)
\end{cases}, \quad (5.6f)
\]

where

\[
f^{(i)}(\theta) = f_1^{(i)} + \sum_{\mu=1}^{M} \left[ f_{2\mu}^{(i)} \cos(\mu \eta \theta) + f_{2\mu+1}^{(i)} \sin(\mu \eta \theta) \right], \quad i \in [1,N]. \quad (5.6g)
\]

By substituting equations (5.6) into equation (5.5) and enforcing harmonic balance, a vector equation \( \mathbf{S} = \mathbf{0} \) is obtained where vector \( \mathbf{S} \) has the following elements (\( i \in [1,N] \), \( r \in [1,R] \))

\[
S_1^{(i)} = \sum_{j=1}^{N} \omega_j v_1^{(j)} - f_1^{(i)} + \frac{1}{2} \sum_{K=1}^{K} \sum_{j=1}^{N} \omega_j \left[ w_{2k}^{(j)} v_1^{(j)} + w_{2k+1}^{(j)} v_{2k+1}^{(j)} \right], \quad (5.7a)
\]
\[
S_{2r}^{(i)} = \sum_{j=1}^{N} \left[ -H_{ij}^2 r^2 u_{2r}^{(j)} + H_{ij} r u_{2r+1}^{(j)} + \omega_{ij} v_{2r}^{(j)} \right] + \sum_{j=1}^{N} \left[ \omega_{ij} v_{1}^{(j)} w_{2r+1}^{(j)} \right] - f_{2r}^{(i)} + \frac{1}{2} \sum_{k=1}^{K} \sum_{j=1}^{N} \left\{ \omega_{kj} w_{2k}^{(j)} \left[ v_{2(k\eta-r)}^{(j)} + v_{2(k\eta+r)}^{(j)} + v_{2(r-\eta)}^{(j)} \right] \right\} \]
\]
\[
+ \frac{1}{2} \sum_{k=1}^{K} \sum_{j=1}^{N} \left\{ \omega_{kj} w_{2k+1}^{(j)} \left[ v_{2(k\eta-r)+1}^{(j)} + v_{2(k\eta+r)+1}^{(j)} - v_{2(r-\eta)+1}^{(j)} \right] \right\}. \tag{5.7b}
\]
\[
S_{2r+1}^{(i)} = \sum_{j=1}^{N} \left[ -H_{ij}^2 r^2 u_{2r+1}^{(j)} - H_{ij} r u_{2r}^{(j)} + \omega_{ij} v_{2r+1}^{(j)} \right] + \sum_{j=1}^{N} \left[ \omega_{ij} v_{1}^{(j)} w_{2r+1}^{(j)} \right] - f_{2r+1}^{(i)} + \frac{1}{2} \sum_{k=1}^{K} \sum_{j=1}^{N} \left\{ \omega_{kj} w_{2k}^{(j)} \left[ v_{2(k\eta-r)+1}^{(j)} + v_{2(k\eta+r)+1}^{(j)} + v_{2(r-\eta)+1}^{(j)} \right] \right\} \]
\[
+ \frac{1}{2} \sum_{k=1}^{K} \sum_{j=1}^{N} \left\{ \omega_{kj} w_{2k+1}^{(j)} \left[ v_{2(k\eta-r)}^{(j)} - v_{2(k\eta+r)}^{(j)} + v_{2(r-\eta)}^{(j)} \right] \right\}. \tag{5.7c}
\]

The coefficients of \( g^{(j)}[u^{(j)}(\theta)] \) can be expressed in terms of unknown Fourier coefficients of the response by utilizing discrete Fourier transforms. The values of \( u^{(j)}(\theta) \) at discrete values of \( \theta_{(m)} = mh \) are
\[
u_{(m)}^{(j)} = u_{1}^{(j)} + \sum_{r=1}^{R} \left[ u_{2r}^{(j)} \cos \left( \frac{2\pi rm}{M} \right) + u_{2r+1}^{(j)} \sin \left( \frac{2\pi rm}{M} \right) \right], \tag{5.8a}
\]
where \( h = 2\pi/(MA) \), \( M \geq 2R \), and \( m \in \left[0, M-1\right] \). Likewise, the discrete value of \( g^{(j)}[u^{(j)}(\theta)] \) at \( \theta_{(m)} \) is obtained from equation (5.3) as,
\begin{equation}
\mathcal{g}_{(m)}^{(j)} = \begin{cases} 
\sum_{k=1}^{3} \alpha_k^{(j)}(u_{(m)}^{(j)} - b^{(j)})^k, & u_{(m)}^{(j)} > b^{(j)}, \\
0, & |u_{(m)}^{(j)}| \leq b^{(j)}, \\
\sum_{k=1}^{3} (-1)^{k-1} \alpha_k^{(j)}(u_{(m)}^{(j)} + b^{(j)})^k, & u_{(m)}^{(j)} < -b^{(j)}. 
\end{cases} \tag{5.8b}
\end{equation}

Fourier coefficients of \( g^{(j)}[u^{(j)}(\theta)] \) in equation (5.6) are calculated by taking inverse discrete Fourier transform [135] of equation (5.8) as

\begin{equation}
v_1^{(j)} = \frac{1}{M} \sum_{n=0}^{M-1} \mathcal{g}_{(m)}^{(j)}, \tag{5.9a}
\end{equation}

\begin{equation}
v_{2r}^{(j)} = \frac{2}{M} \sum_{n=0}^{M-1} \mathcal{g}_{(m)}^{(j)} \cos \frac{2\pi nm}{M}, \tag{5.9b}
\end{equation}

\begin{equation}
v_{2r+1}^{(j)} = \frac{2}{M} \sum_{n=0}^{M-1} \mathcal{g}_{(m)}^{(j)} \sin \frac{2\pi nm}{M}. \tag{5.9c}
\end{equation}

Having these coefficients, the vector equation \( \mathbf{S} = \mathbf{0} \) can be solved for unknown displacement amplitudes \( u_i^{(j)} \ (i \in [1, 2r+1], \ j \in [1, 3]) \) by employing the Newton-Raphson method as

\begin{equation}
\mathbf{U}^{(k)} = \mathbf{U}^{(k-1)} - \left[ \Pi^{-1} \right]^{(k-1)} \mathbf{S}^{(k-1)}. \tag{5.10}
\end{equation}

In equation (5.10), vectors \( \mathbf{U} \), \( \mathbf{S} \), and matrix \( \Pi \) are defined as following
\[
U = \begin{pmatrix}
\mathbf{u}^{(1)} \\
\vdots \\
\mathbf{u}^{(j)} \\
\vdots \\
\mathbf{u}^{(N)}
\end{pmatrix}, \quad S = \begin{pmatrix}
\mathbf{s}^{(1)} \\
\vdots \\
\mathbf{s}^{(j)} \\
\vdots \\
\mathbf{s}^{(N)}
\end{pmatrix}, \quad (5.11a, b)
\]

\[
\mathbf{II} = \begin{pmatrix}
\mathbf{J}^{(11)} & \cdots & \mathbf{J}^{(1j)} & \cdots & \mathbf{J}^{(1N)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\mathbf{J}^{(i1)} & \cdots & \mathbf{J}^{(ij)} & \cdots & \mathbf{J}^{(iN)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\mathbf{J}^{(N1)} & \cdots & \mathbf{J}^{(Nj)} & \cdots & \mathbf{J}^{(NN)}
\end{pmatrix}, \quad (5.11c)
\]

where

\[
\mathbf{u}^{(j)} = \begin{pmatrix}
\mathbf{u}_1^{(j)} \\
\vdots \\
\mathbf{u}_q^{(j)} \\
\vdots \\
\mathbf{u}_Q^{(j)}
\end{pmatrix}, \quad S^{(i)} = \begin{pmatrix}
S_1^{(i)} \\
\vdots \\
S_q^{(i)} \\
\vdots \\
S_Q^{(i)}
\end{pmatrix}, \quad (5.12a, b)
\]

\[
\mathbf{J}^{(ij)} = \begin{pmatrix}
J_{11}^{(ij)} & \cdots & J_{1q}^{(ij)} & \cdots & J_{1Q}^{(ij)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
J_{q1}^{(ij)} & \cdots & J_{qq}^{(ij)} & \cdots & J_{qQ}^{(ij)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
J_{Q1}^{(ij)} & \cdots & J_{Qq}^{(ij)} & \cdots & J_{QQ}^{(ij)}
\end{pmatrix}, \quad Q = 2\eta R + 1. \quad (5.12c)
\]

The elements of the Jacobian matrix \( J_{rq}^{(ij)} = \frac{\partial S_r^{(i)}}{\partial u_q^{(j)}} \) are defined as
\[
\frac{\partial S^{(i)}}{\partial u_q^{(j)}} = \omega_{ij} \frac{\partial v_1^{(j)}}{\partial u_q^{(j)}} + \omega_{ij} \sum_{k=1}^{K} \left[ w_{2k}^{(j)} \frac{\partial v_{2k}^{(j)}}{\partial u_q^{(j)}} + w_{2k+1}^{(j)} \frac{\partial v_{2k+1}^{(j)}}{\partial u_q^{(j)}} \right],
\]

(5.13a)

\[
\frac{\partial S_{r}^{(j)}}{\partial u_q^{(j)}} = -H^2 \gamma_{ij} r^2 \delta_{2r,q} + H \zeta_{ij} r \delta_{2r+1,q} + \omega_{ij} \frac{\partial v_{2r}^{(j)}}{\partial u_q^{(j)}} + \omega_{ij} w_{2r+1}^{(j)} \frac{\partial v_{1}^{(j)}}{\partial u_q^{(j)}}
\]

+ \sum_{k=1}^{K} \frac{\omega_{ij} w_{2k}^{(j)}}{2} \left[ \frac{\partial v_{2k}^{(j)}}{\partial u_q^{(j)}} + \frac{\partial v_{2k+1}^{(j)}}{\partial u_q^{(j)}} \right]

(5.13b)

\[
\frac{\partial S_{2r+1}^{(j)}}{\partial u_q^{(j)}} = -H^2 \gamma_{ij} r^2 \delta_{2r+1,q} - H \zeta_{ij} r \delta_{2r,q} + \omega_{ij} \frac{\partial v_{2r+1}^{(j)}}{\partial u_q^{(j)}} + \omega_{ij} w_{2r+1}^{(j)} \frac{\partial v_{1}^{(j)}}{\partial u_q^{(j)}}
\]

+ \sum_{k=1}^{K} \frac{\omega_{ij} w_{2k+1}^{(j)}}{2} \left[ -\frac{v_{2(k\eta-r)+1}^{(j)}}{\partial u_q^{(j)}} + \frac{v_{2(k\eta+r)+1}^{(j)}}{\partial u_q^{(j)}} \right]

(5.13c)

where \( r \in [1, R] \), \( q \in [1, Q] \), \( i, j \in [1, N] \) and \( \delta_{i,j} \) is the Kronecker delta, which is defined as \( \delta_{i,j} = 1 \) for \( i = j \) and \( \delta_{i,j} = 0 \) otherwise. The partial derivatives in equations (5.13) are given as

\[
\frac{\partial v_1^{(j)}}{\partial u_1^{(j)}} = \frac{1}{2} \sum_{m=0}^{M-1} E_{(m)}^{(j)},
\]

(5.14a)

\[
\frac{\partial v_1^{(j)}}{\partial u_2^{(j)}} = \frac{1}{2} \sum_{m=0}^{M-1} E_{(m)}^{(j)} \cos \frac{2\pi q m}{M},
\]

(5.14b)
In these equations

\[ E_{(m)}^{(j)} = \frac{2}{N} \Phi_{(m)}^{(j)} \left\{ C_1^{(j)} - \Phi_{(m)}^{(j)} C_2^{(j)} u_{(m)}^{(j)} + C_3^{(j)} [u_{(m)}^{(j)}]^2 \right\}, \]  

(5.15a)

\[ C_1^{(j)} = \alpha_1^{(j)} - 2b_2^{(j)} \alpha_2^{(j)} + 3[b_2^{(j)}]^2 \alpha_3^{(j)}, \]  

(5.15b)

\[ C_2^{(j)} = 2\left[ \alpha_2^{(j)} - 3b_2^{(j)} \alpha_3^{(j)} \right], \]  

(5.15c)

\[ C_3^{(j)} = 3\alpha_3^{(j)} , \]  

(5.15d)
for \( j \in [1, N] \) and \( r \in [1, R] \). The value of \( U^{(k)} \) at the \( k \)-th iteration is calculated from the values of \( S^{(k-1)} \) and \( U^{(k-1)} \), and \( J \) is the Jacobian matrix. The stability of the steady-state response is determined by utilizing Floquet theory [19, 135].

### 5.3. An Application Example

#### 5.3.1 The Physical System and Dynamical Model

In order to illustrate the application of the multi-term HBM procedure discussed in previous section, a dynamic model of a spur gear pair mounted on compliant bearings is considered. In Figure 5.1, a 4-DOF semi-definite model of this system is illustrated.

Kahraman and Singh [14] studied the system by using a 3-DOF definite model with PLTI stiffness and a numerical integration method focusing primarily on the effect of clearances. However, both gear mesh and bearing support have PNTV characteristics. In reference [14], it was also shown that the actual bearing restoring function is piecewise nonlinear, and can be described by dimensionless equation (1.7).

In this section, PN restoring functions are used to represent both bearings and gears. The gear mesh stiffness is considered to be TV while the bearing stiffness is assumed to be TI. The governing equations of this 3-DOF definite system is given by

\[
\Phi_{(m)}^{(j)} = \begin{cases} 
-1, & u_{(m)}^{(j)} > b^{(j)}, \\
0, & -b^{(j)} \leq u_{(m)}^{(j)} \leq b^{(j)}, \\
1, & u_{(m)}^{(j)} < -b^{(j)},
\end{cases} \tag{5.15e}
\]
Figure 5.1. A torsional dynamic model of a geared rotor-bearing system.
Here, \( u^{(1)}(\tau) \) and \( u^{(2)}(\tau) \) represent the transverse displacements of gears 1 and 2. The third variable \( u^{(3)}(\tau) \) is defined as

\[
u^{(3)}(\tau)=\frac{1}{b_c} \left[ r^{(1)} \theta^{(1)} + r^{(2)} \theta^{(2)} \right] + u^{(1)}(\tau) - u^{(2)}(\tau) - e(\tau),
\]

where \( r^{(1)} \) and \( r^{(2)} \) are base circle radii of gears 1 and 2, \( \theta^{(1)} \) and \( \theta^{(2)} \) are the torsional displacements of two gears, \( b_c \) is the characteristic length, and \( e(\tau) \) is the static transmission error of the gear mesh. By defining variable \( u^{(3)}(\tau) \), the original 4-DOF semi-definite model is reduced to the 3-DOF definite model defined in equation (5.16a) effectively from eliminating the rigid body mode.

In equation (5.16a), \( \gamma_{ij} \) are unit mass elements, which are defined as \( \gamma_{11} = \gamma_{22} = \gamma_{32} = \gamma_{33} = 1 \), and \( \gamma_{31} = -1 \). \( \zeta_{ij} \) are dimensionless viscous damping ratios,
defined as $\zeta_{ii} = c_i / m_i \omega_n \ (i=1, 3)$, $\zeta_{13} = c_3 / m_1 \omega_n$, and $\zeta_{23} = -c_3 / m_2 \omega_n$. Here, $c_i \ (i=1, 3)$ are viscous damping coefficients of two bearings and gear mesh, respectively. $m_1$ and $m_2$ are gear masses, and $m_3$ is equivalent gear pair mass, and $\omega_n = \sqrt{k_3 / m_3}$ that $k_3$ is the average gear mesh stiffness [14]. $\omega_j$ are defined as $\omega_{ii} = m_j c_i / m_j k_j \ (i=1, 3)$, $\omega_{13} = m_3 / m_1$, $\omega_{23} = -m_3 / m_2$, where $k_1$ and $k_2$ are the stiffness of bearings.

In order to analyze the stability of the solutions from equation (5.16), small variation variables $\Delta u^{(i)}(\tau)$ are introduced, and variational equations are obtained as

$$
\begin{bmatrix}
\gamma_{11} & 0 & 0 \\
0 & \gamma_{22} & 0 \\
\gamma_{31} & \gamma_{32} & \gamma_{33}
\end{bmatrix}
\begin{bmatrix}
\Delta u^{(1)}(\tau) \\
\Delta u^{(2)}(\tau) \\
\Delta u^{(3)}(\tau)
\end{bmatrix} +
\begin{bmatrix}
\zeta_{11} & 0 & \zeta_{13} \\
0 & \zeta_{22} & \zeta_{23} \\
0 & 0 & \zeta_{33}
\end{bmatrix}
\begin{bmatrix}
\Delta u^{(1)}(\tau) \\
\Delta u^{(2)}(\tau) \\
\Delta u^{(3)}(\tau)
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
$$

(5.17a)

where $\bar{g}^{(j)}[u^{(j)}(\tau)] \ (j \in [1,3])$ are defined as

$$
\bar{g}^{(j)}[u^{(j)}(\tau)] =
\begin{cases}
\sum_{i=1}^{3} i \alpha_i [u^{(j)}(\tau) - 1]^{i-1}, & u^{(j)}(\tau) > b^{(j)}, \\
0, & |u^{(j)}(\tau)| \leq b^{(j)}, \\
\sum_{i=1}^{3} [-1]^{i-1} i \alpha_i [u^{(j)}(\tau)] + 1]^{i-1}, & u^{(j)}(\tau) < -b^{(j)}.
\end{cases}
$$

(5.17b)
Defining a state vector

\[
\mathbf{y} = [\Delta u^{(1)}(\tau) \Delta u^{(2)}(\tau) \Delta u^{(3)}(\tau) \Delta \hat{u}^{(1)}(\tau) \Delta \hat{u}^{(2)}(\tau) \Delta \hat{u}^{(3)}(\tau)]^T, \tag{5.18}
\]

equation (5.17a) can be transformed into matrix form similar to equation (2.15c). Here, periodic state matrix \( \mathbf{H}(\tau) = \mathbf{H}(\tau + T) \) becomes

\[
\mathbf{H}(\tau) = \begin{bmatrix}
\mathbf{H}_{11} & \mathbf{H}_{12} \\
\mathbf{H}_{21}(\tau) & \mathbf{H}_{22}
\end{bmatrix}, \tag{5.19a}
\]

where

\[
\mathbf{H}_{11} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \mathbf{H}_{12} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \tag{5.19b, c}
\]

\[
\mathbf{H}_{21}(\tau) = \begin{bmatrix}
-\frac{\omega_{11} w_1(\tau) \tilde{g}_1(\tau)}{\gamma_{11}} & 0 & -\frac{\omega_{13} w_3(\tau) \tilde{g}_3(\tau)}{\gamma_{11}} \\
0 & -\frac{\omega_{22} w_2(\tau) \tilde{g}_2(\tau)}{\gamma_{22}} & -\frac{\omega_{23} w_3(\tau) \tilde{g}_3(\tau)}{\gamma_{22}} \\
0 & 0 & -\frac{\omega_{33} w_3(\tau) \tilde{g}_3(\tau)}{\gamma_{33}}
\end{bmatrix}, \tag{5.19d}
\]
\[
H_{22} = \begin{bmatrix}
-\zeta_{11} & 0 & -\zeta_{13} \\
\gamma_{11} & -\zeta_{22} & \gamma_{11} \\
0 & -\frac{\zeta_{23}}{\gamma_{22}} & -\zeta_{33} \\
0 & 0 & -\frac{\zeta_{33}}{\gamma_{33}}
\end{bmatrix}.
\] (5.19e)

In equations (5.19d) and (5.19e), \(\hat{\phi}_{33}\) and \(\hat{\zeta}_{33}\) are defined as

\[
\hat{\phi}_{33} = \frac{\gamma_{22}\gamma_{31}\hat{\phi}_{13} + \gamma_{11}\gamma_{31}\hat{\phi}_{23} - \gamma_{11}\gamma_{22}\hat{\phi}_{33}}{\gamma_{11}\gamma_{22}} w^{(3)}(\tau) \tilde{g}^{(3)}[u^{(3)}(\tau)],
\] (5.19f)

\[
\hat{\zeta}_{33} = \frac{\gamma_{22}\gamma_{31}\hat{\zeta}_{13} + \gamma_{11}\gamma_{31}\hat{\zeta}_{23} - \gamma_{11}\gamma_{22}\hat{\zeta}_{33}}{\gamma_{11}\gamma_{22}}.
\] (5.19g)

Therefore, applying the same procedure as the one described in section 2.2.2 with the new state vector of equation (5.18) and the new periodic state matrix of equation (5.19a), the stability of the solution is determined by the Floquet multipliers obtained from the monodromy matrix \(F\) that can be estimated by equation (2.17). Here, all related matrices have a dimension of 6. As a result, there are six eigenvalues \(\lambda_i\) for \(i \in [1, 6]\). If all \(\lambda_i\) are less than unity, the solution is stable. Otherwise, the solution is unstable.

5.3.2 *Comparison of HBM with Direct Numerical Integration Results*

In order to evaluate the accuracy of solutions of multi-term HBM, equation (5.1) is solved by a numerical integration method. The results from these two methods are
compared as shown in Figure 5.2. A special case of gear pair is considered here, which has very stiff bearings. For this 3-DOF model, elements of damping and stiffness matrices in equation (5.16) are $\zeta_{11} = \zeta_{22} = 100$, $\zeta_{13} = \zeta_{23} = 0.0125$, $\zeta_{33} = 0.05$, $\omega_{11} = \omega_{22} = 125$, $\omega_{13} = \omega_{23} = 0.25$, and $\omega_{33} = 1.0$. The coefficients of periodical TV stiffness for each DOF is chosen as $w^{(1)}_3 = w^{(2)}_3 = w^{(3)}_3 = 0.3$, $w^{(1)}_5 = w^{(2)}_5 = w^{(3)}_5 = 0.15$, $w^{(1)}_7 = w^{(2)}_7 = w^{(3)}_7 = 0.1$, and all other $w^{(i)}_{ji} = 0$. The quadratic and cubic nonlinear terms in equation (5.3) are chosen as $\alpha^{(1)}_2 = \alpha^{(2)}_2 = \alpha^{(3)}_2 = 0.1$, $\alpha^{(1)}_3 = \alpha^{(2)}_3 = \alpha^{(3)}_3 = 0.2$. The coefficients of external excitation are $f^{(1)}_1 = f^{(2)}_1 = f^{(3)}_1 = 0.5$, and $f^{(i)}_j = 0$ for $i \in [1,3]$, $j > 1$. To predict period-1 motions of this model by using multi-term HBM, six harmonic terms ($R = 6$) in equation (5.6b) are assumed.

In Figure 5.2, only the predicted $u^{(3)}_{rms}$ and $u^{(3)}_1$ values of gear mesh are illustrated since $u^{(1)} \approx 0$ and $u^{(2)} \approx 0$ due to the large values used for the stiffness of two bearings. The two solutions methods, multi-term HBM and the numerical integration method, agree very well with each other, suggesting that the multi-term HBM solutions of the 3-DOF system are indeed accurate. Comparisons with more compliant bearings are also performed, and very good agreement was observed between numerical and multi-term HBM solutions for these cases as well.
Figure 5.2(a) $u_{rns}^{(3)}$ and (b) $u_1$ values of forced response ($R = 6$) as a function of dimensionless frequency $\Lambda$. (—) stable HBM solutions, ( — ) unstable HBM solutions, and (○) numerical integration solutions.
5.3.3 *Period-one Motions of a Two Degree-of-freedom System*

In this section, another set of parameters for the 3-DOF system shown in Figure 5.1 is considered, which has one rigid and one compliant bearing. The nonzero elements of damping and stiffness matrices are $\zeta_{11}=500$, $\zeta_{22}=0.01$, $\zeta_{33}=0.05$, $\zeta_{13}=\zeta_{23}=0.0125$, $\omega_{11}=500$, $\omega_{33}=1.0$, $\omega_{13}=\omega_{23}=0.25$, and $\omega_{22}=0.25$ or 1.25. The quadratic and cubic nonlinear terms in equation (5.3) are chosen as $\alpha_2^{(1)} = \alpha_2^{(2)} = \alpha_2^{(3)} = \alpha_3^{(1)} = 0$, and $\alpha_3^{(2)} = \alpha_3^{(3)} = 0.1$. The external forces are assumed to be constant with $f_1^{(1)} = f_1^{(2)} = f_1^{(3)} = 0.5$ and all other $f_j^{(i)} = 0$. The stiffness of bearings is assumed to be constant as well, i.e. $w_j^{(1)} = w_j^{(2)} = 0$ ($j > 1$). The parametric excitation at gear mesh, which is assumed to be harmonic with $w_3^{(3)} = 0.2$ and $w_j^{(3)} = 0$ ($j \neq 1$ and $j \neq 3$), is the only excitation for this case. Given these values, the 3-DOF system should act as a 2-DOF dynamical system approximately with $u^{(1)} \approx 0$.

In Figures 5.3 and 5.4, the dynamic response of a system with $\omega_{22}=0.25$, is illustrated, which includes two primary resonant peaks for both $u^{(2)}$ and $u^{(3)}$. As mentioned in reference [14], the natural frequencies of the corresponding linear system for the two motions considered are $\omega_n^{(2)}=0.44$ and $\omega_n^{(3)}=1.14$. However, both resonant peaks of nonlinear response are offset to the right relative to $\omega_n^{(2)}$ and $\omega_n^{(3)}$, especially the resonance near $\omega_n^{(2)}$. The reason of this shift is mainly due to the cubic nonlinear terms $\alpha_3^{(2)}$ and $\alpha_3^{(3)}$. Shapes associated with these two natural modes exhibit coupled
Figure 5.3. (a) $u_{\text{rms}}^{(2)}$ and (b) $u_1^{(2)}$ values of forced response ($R = 12$) as a function of dimensionless frequency $\Lambda$ for $\omega_{22} = 0.25$ and $\nu_3^{(3)} = 0.2$. 

(-----) stable HBM solutions, (---) unstable HBM solutions.
Figure 5.4. (a) $u_{\text{rms}}^{(3)}$ and (b) $u_1^{(3)}$ values of forced response ($R = 12$) as a function of dimensionless frequency $\Lambda$ for $\omega_{22} = 0.25$ and $w_3^{(3)} = 0.2$. (---) stable HBM solutions, (---) unstable HBM solutions.
transverse-torsional motions. The mode near $\Lambda = 0.44$ has more translational component than the other one near $\Lambda = 1.14$. $u^{(2)}$ exhibits a very large primary resonance near $\Lambda = 0.44$ with a very slight softening behavior, while $u^{(3)}$ is dominated by the softening type primary resonance near $\Lambda = 1.14$.

In Figures 5.5 and 5.6, the results for $\omega_{22} = 1.25$ are presented. Here, two natural frequencies of the corresponding linear system are closer than previous case, and the coupling between two modes becomes more of an issue. The effect of $w_3^{(3)}$ on the first primary resonant peak near $\Lambda = 0.78$ is more obvious than those in Figures 5.3 and 5.4. As a result, $u^{(2)}$ is not dominated by the first primary resonant peak that becomes the largest resonance of the response of $u^{(3)}$. Two super-harmonic resonant peaks appear near $\Lambda = 0.4$ and 0.7 in Figures 5.6 and 5.7, which are not observed in previous case. For both cases, there are frequency ranges in which period-1 motions become unstable, such as the one for $\Lambda \in [1.16, 1.23]$ in Figures 5.3 and 5.4. This suggests that subharmonic motions should dominate these ranges of frequency.
Figure 5.5. (a) $u_1^{(2)}$ and (b) $u_{rms}^{(2)}$ values of forced response ($R = 12$) as a function of dimensionless frequency $\Lambda$ for $\omega_{22} = 1.25$ and $\omega_3^{(3)} = 0.2$. ($\text{---}$) stable HBM solutions, (−−) unstable HBM solutions.
Figure 5.6. (a) $u_{1}^{(3)}$ and (b) $u_{rms}^{(3)}$ values of forced response ($R = 12$) as a function of dimensionless frequency $\Lambda$ for $\omega_{22} = 1.25$ and $\omega_{3}^{(3)} = 0.2$. 

(---) stable HBM solutions, (---) unstable HBM solutions.
5.4. Concluding Remarks

In this chapter, a generalized solution method for MDOF dynamical system having PNTV stiffness is proposed. This method is based on multi-term HBM in conjunction with DFT, and is capable of predicting harmonic, superharmonic and subharmonic motions of systems with clearance and continuous type of nonlinearities.

An application of this solution method is demonstrated through a 3-DOF gear-bearing system. The stability of HBM solutions is estimated by Floquet theory, and the accuracy of these solutions is confirmed through comparison to numerical integration results. A limited number of cases are analyzed to demonstrate the capability of this method. Further investigation and detailed parametric studies of this class of MDOF dynamical systems can be performed using the method proposed. Such studies are beyond the scope of this research activity.
6.1. Summary

In this study, the dynamic behavior of a class of nonlinear oscillators with piecewise nonlinear element and periodically time-varying parameters was investigated. First, a general SDOF piecewise nonlinear mechanical oscillator with parametric and external excitations was considered. A multi-term harmonic balance formulation was used in conjunction with DFT and a parametric continuation scheme to determine steady-state period-1 and period-$\eta$ ($\eta > 1$) response of the system. The accuracy of the multi-term HBM was demonstrated by comparing the solutions to those from direct numerical integration. Floquet theory was applied to determine the stability of the steady-state harmonic balance solutions. Detailed parametric studies were presented to quantify the combined influence of clearance, quadratic and cubic nonlinearities within typical ranges of all other system parameters. A comparison between time-invariant and time-varying systems was also provided to demonstrate the influence of the parametric and external excitations on the response of a piecewise nonlinear system.
As an application of SDOF piecewise nonlinear systems, a generic elastic sphere-plane contact was considered next. The SDOF dynamical model of this system includes both a continuous nonlinearity associated with the Hertzian contact and a clearance-type nonlinearity due to contact loss. The same multi-term harmonic balance solution method was applied to find the steady state motions. The accuracy of the dynamic model and solution methods was demonstrated through comparisons with experimental data as well as numerical solutions. A single-term harmonic balance approximation was used to derive a criterion for contact loss to occur. The influence of harmonic external excitation $f(t)$ and damping ratio $\zeta$ on the steady state response was also demonstrated.

Finally, a general MDOF system with multiple clearances and PNTV components was considered. A generalized form of solution method was provided for analyzing this kind of MDOF system. A 3-DOF gear-bearing system was considered as an example application of this formulation. The accuracy of the multi-term HBM solution was confirmed by comparison to numerical integration results. A number of limiting cases were analyzed to demonstrate the applicability of this solution method.
6.2. Contributions

This study contributes to the literature on nonlinear dynamical systems with time-varying components and piecewise nonlinear characteristics in a number ways:

• The solution method proposed in this research is comprehensive, reasonably accurate and computationally efficient. Therefore, this method is available for future studies on systems having different forms of piecewise nonlinear characteristics and excitation schemes.

• This study describes the combined influence of continuous and clearance types of nonlinearities on a parametrically excited system, including the similarities and differences between piecewise linear and piecewise nonlinear systems. Therefore, the fidelity of previous studies that employed piecewise linear or continuous nonlinear approximations can be assessed.

• The behavior near the parametric instability regions of corresponding linear time-varying systems is of special interest since piecewise nonlinear systems do not experience such instabilities while the frequency ranges are occupied by large amplitude stable subharmonic resonance peaks. This study describes such phenomenon of a SDOF piecewise nonlinear system in detail allowing potential
implementation of piecewise nonlinear stiffness models to prevent such instabilities.

- The application of proposed analysis methodology to Hertzian contact problems brings much needed analytical understanding to such systems with contact loss.

- The analytical treatment of a generalized MDOF piecewise nonlinear system provides a generalized method beyond the numerical methods used in previous studies.
6.3. Recommendations for Future Work

Based on this study, following issues can be identified for future work:

- Investigation of SDOF piecewise nonlinear systems by numerical methods, focusing on the ranges within which quasi-period and chaotic motions might exist.

- Examination of multiple sphere-plane contact interfaces within a MDOF model, and investigation of the existence of parametric resonances in contact problems.

- Detailed parametric study of the MDOF system with PNTV components to quantify combined influences of continuous nonlinearities, parametric and external excitations, and key system parameters, such as damping and stiffness values.

- Application of this solution methodology to various gear-bearing systems including other nonlinear effects such as friction.

- Experimental validation of the solution method using specific rotating machine components with piecewise nonlinear characteristic.
BIBLIOGRAPHY


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