A NON-CONFORMAL DOMAIN DECOMPOSITION METHOD FOR SOLVING LARGE ELECTROMAGNETIC WAVE PROBLEMS

DISSERTATION

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ABSTRACT

It is well known that the scope and application of numerically-rigorous techniques for full-wave electromagnetic characterization is limited to problems of moderate electrical size and simplified complexity. These limitations stem from the vast computational resources required by numerical methods such as finite element method (FEM), boundary element method (BEM) or finite difference method (FDM). During the last decade a number of fast and memory efficient numerical algorithms such as Multigrid methods and the Fast Multipole Algorithm (FMA), have been proposed to further reduce storage and computational requirements of full-wave methods.

In this dissertation an alternative proposition will be presented, that is a fast and efficient Domain Decomposition (DD) methodology, appropriately tailored for the solution of time-harmonic Maxwell's equations. The DD method proposed here is a non-conforming one, namely it allows for different mesh on either side of domain interface. This not only relaxes and speeds up automatic mesh generation algorithms, but at the same time opens the road of efficient and robust adaptive field computations. The DD technique is based on a divide-and-conquer philosophy. Instead of tackling a large and complex problem directly (as a whole), it divides the computational domain into smaller, possibly repetitive, and easier to solve partitions called domains. Such domains can be solved with a variety of numerical methods, e.g. finite elements, boundary elements, etc.
The algorithm proceeds iteratively by appropriately communicating information across domains and ultimately reaching the solution for the original (whole) problem.

A detailed presentation of the proposed DD method for electromagnetic problems will be given, along with a novel methodology called "cement" finite elements, for the coupling of domains with non-matching meshes. In addition, a variant of the Finite Element Tearing and Interconnecting (FETI) sub-structuring algorithm will be introduced. Numerical results for a number of challenging real-life engineering electromagnetic applications, ranging from large antenna arrays to novel engineered materials, photonic crystals, and large object scattering, will be given. As a result of this research, complex problems with upwards of 50 to 900 million finite element unknowns have been solved on personal computers without the need of parallelization. Finally, further applications of the cement and DD methods on infinite periodic problems without periodic meshing and a very promising DD based FEM-BEM hybrid are also proposed and studied.
To my Father and Mother, to Golsa

and to all the teachers and students who passionately pursue scientific knowledge
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CHAPTER 1

INTRODUCTION

1.1 Problem Statement

The core of this research is to propose and investigate the use of domain decompositions (DD) for the solution of Maxwell’s equations in the time-harmonic regime. In particular, a class of domain decompositions will be proposed to efficiently analyze electrically large and geometrically complicated scattering, radiating and waveguiding structures. The vehicle that will be used to demonstrate the power of the proposed DD algorithms will be the Tangential Vector Finite Element Method (TVFEM), but the methodologies can be also extend to other Partial Differential Equation (PDE) or Integral Equation (IE) based numerical methods. All the domain decomposition methods and algorithms developed and studied in this research will share one very important common characteristic; they will be non-overlapping and non-conforming in nature. That is, the original computational domain is decomposed into a set of disjoint domains, whereas the mesh on either side of a domain interface can be completely different.

The first question that this thesis will try to address is why use domain decomposition methods to analyze electromagnetic problems? The answers to this question are four:
1. It is well known that, as the electrical size of the computational domain in a PDE or IE based method increases, the performance of iterative solution methods dramatically deteriorates. DD methods try to overcome this effect using a “divide and conquer” strategy. Namely, instead of solving the large and hard to solve problem in the entire computational domain, they divide it into smaller and easier to solve sub-problems. This strategy can be used to create effective preconditioners in a Krylov based solvers, or as a direct DD method in a stationary iteration scheme.

2. In the quest to solve larger electromagnetic (EM) problems using discretization based methods, the amount of memory needed to setup and solve a problem scales, in the best case, as O(N), where N is the number of unknowns in the problem. Naturally, this complexity poses an inherent limitation related to the size of memory utilized by a single computer. To overcome this problem, large scale parallel processing needs to be considered. Fortunately, DD methods are inherently parallelizable. Specifically, their parallel versions are easy to implement in numerical codes, and moreover both memory and computational time scale linearly with the number of processors.

3. Most of the large EM problems encountered possess certain repetitions, or symmetries of some geometrical features. For example large antenna arrays or metamaterial structures, exhibit many repetitions, whereas most of the scattering targets of interest do have at least one plane of symmetry. DD methods easily exploit these duplications and symmetries, which numerically translate into a tremendous reduction of computational resources.
4. Last but not least, solving large EM problems with FEM or even Boundary Element Methods (BEM), requires some sort of discretization of the geometry into a mesh. The generation of that volume or surface mesh, can be an extremely challenging task, especially when the size of the problem is large and at the same time exhibits fine geometrical features. Using the proposed non-conforming DD methods the meshing of such geometries tremendously relaxes since each domain is kept relatively small, and the conformity constraints across domains are absent. Moreover, adaptive mesh refinement strategies become more flexible, since they can be based on subdomains alone. In addition, the non-conformity of the mesh creates new potentials for fast optimization strategies.

Needless to say, the efficient and accurate simulation of large and geometrically complicated EM problems is of vital importance in many areas of electrical engineering. To this end, it suffices to mention the evaluation of Radar Cross Section (RCS) from airplanes, ships or other targets. Till today, the RCS computations of large structures has been dominated by fast BEM or Method of Moments (MoM) simulators, such as the Fast Multipole Method (FMM) [1], Multi-Level Fast Multipole Algorithm (MLFMA) [2], Adaptive Integral Method (AIM) [3], precorrected-FFT, (pFFT) [4] etc. Even though most of these methods are extremely fast when the scatterers are perfectly electric conducting (PEC), their efficiency and accuracy tends to deteriorate when dielectric or magnetic materials are included in the computational domain. Instead, the proposed FEM domain decomposition, can very efficiently and accurately analyze such scatterers. Furthermore, the nearly linear processor scalability of the method makes it, a valuable rival of MLFMA on very large scale RCS computations.
Another very important class of large and geometrically complicated EM
problems is antenna arrays. Arrays are used on every wireless communication base
station, RADAR systems, and nearly every satellite communication link. The proposed
method is tailored to take advantage the repetitions among array elements, leading to
extremely efficient EM analysis.

Due to that very fact of exploiting repetitions, the proposed method can be
adapted for the analysis of finite photonic crystals (PC), photonic band gap (PBG) or
electromagnetic band gap (EBG) structures, and any other type of finite in extent
metamaterials. Since the proposed DD is built on top of an FEM kernel, the incorporation
of lumped L and C elements used in many metamaterials is relatively easy. In addition,
complex materials such as ferrites, garnets, ferromagnetics, ferroelectrics, and other
exotic materials used to form metamaterials are easily incorporated into the proposed
method. In light of the abovementioned comments, novel microwave and antenna
technologies printed on metamaterial substrates can be efficiently simulated.

Having established the reasons and the significance of DD methods in solving EM
problems, it is time to address the specific DD methodologies that will be used to
accomplish that goal. The key ingredients of the proposed research can be grouped into
the following three categories:

1. *Non-Overlapping Schwarz Algorithm* is a domain decomposition algorithm,
   where each subdomain does not overlap with its neighboring ones, but only at an
   interface. The Schwarz Algorithm proceeds by solving each subdomain problem
   alternatively, and passing information across domain interfaces through an
   appropriate transmission condition (TC). As it will be emphasized later, the
communication of information through the TC’s is of crucial importance in obtaining a stable, optimal and convergent algorithm. The algorithm is iterative, namely starting from a false initial guess for each domain, the subdomain solutions and communication across subdomains proceeds until a certain equilibrium in the solution has been achieved. The non-overlapping Schwarz algorithm has been used directly as an iterative solver, or as a additive or multiplicative preconditioner on Krylov subspace methods. The non-overlapping variant of the Schwarz algorithm is more elegant and efficient than its overlapping counterpart, unfortunately it pose more mathematical challenges in achieving stability, optimality and convergence.

2. **Cement Finite Elements** is a methodology of solving finite element problems with non-conforming or non-matching triangulations across an interface. In a more general setting, cement elements can be used to interface between different types of elements, for example finite elements with spectral elements etc. Usually, the appropriate continuity conditions on the solution across the different triangulations are weakly enforced through the use of Lagrange multiplier set.

3. **Finite Element Tearing and Interconnecting (FETI) Algorithm** is a class of dual iterative substructuring algorithms based on Lagrange multipliers. Like all substructuring algorithms, FETI tries to establish an iterative process based on a reduced number of unknowns, usually the ones on the domain-interface skeleton. Many different variants of the algorithm can be found in the literature, e.g. 2-level-FETI, FETI-2LM, FETI-H, FETI-DP, etc dedicated to a specific application. The main idea among all FETI algorithms is to solve each subdomain problem
independently (tearing process), and then use a set of extra Lagrange multiplier unknowns to appropriately constrain the solutions across interfaces, in an iterative fashion (interconnecting process). In comparison to other substructuring algorithms, FETI exhibits superior parallelization scalability, and when used as a preconditioner, it can result in faster and scalable algorithms with respect to the number of domains and the size of the overall problem.

1.2 Literature Review

Domain decomposition methods have been among the favorite areas of research in the applied mathematics, computational mechanics and fluid dynamics communities, during the last ten years. Even though DD is a relative new field, a number of specialized scientific symposiums has been organized and dedicated to DD methods, among them the most prominent is the annual International Conference on Domain Decomposition Methods. In the general literature three books stand out, the Domain Decomposition, Parallel Multilevel Methods for Elliptic Partial Differential Equations by B. Smith et.al. [5], the Domain Decomposition Methods for Partial Differential Equations by Quarteroni and Valli [6], and the recent book by Toselli and Widlund entitled Domain Decomposition Methods–Algorithms and Theory [7]. Two more monographs of Springer-Verlag series on Lecture Notes in Computational Science and Engineering have been devoted in DD methods, namely the Discretization Methods and Iterative Solvers Based on Domain Decomposition by B. Wohlmuth [8], and the edited monograph of L. Pavarino and A. Toselli entitled Recent Developments in Domain Decomposition Methods [9]. In addition, a number of general DD tutorial articles have been published, among them we cite [10] and [11].
Unfortunately, the electrical engineering community, and especially the engineering EM community, has meticulously overlooked the possibility of utilizing DD methods in solving EM related problems. The only really DD method used to analyze two dimensional scalar and three dimensional vector wave problems was proposed by Stupfel in [12], [13], respectively. In both works an FEM non-overlapping onion like domain decomposition was used to analyze scattering problems. In Microwave community a number of DD methods have been used to analyze interconnects and Printed Circuit Boards (PCBs) in the quasi static regime using the FEM in [14],[15] and [16]. DD has been also used in the context of Finite Difference in Time Domain (FDTD) on both 2D and 3D scattering problems [17], [18]. At this point, it is important to address the contribution of Prof. Gedney’s group in [19] and [20]; In the former paper the FETI method was used to solve 3-D wave guiding problems in frequency domain in parallel machines. In the latter paper, the method was extended to 3-D Finite Element in Time Domain (FETD) to solve cavity problems. Apart from Stupfel’s and Gedney’s work all other works were based on overlapping domain decomposition. In EM literature there have been a number of other domain decomposition attempts, more precisely these are domain partitioning schemes that resorted direct matrix solvers [21-23]. Recently, Kindt and Volakis proposed a Finite Element-Boundary Integral (FEBI) based array decomposition methodology (ADM) to analyze finite antenna arrays, [24],[25]. Even though, this approach generally classifies into the classical domain decomposition approaches, it is more like a domain partitioning scheme that transfers information across domains globally, through the use of the appropriate Green’s function. Certain
translational symmetries (Toeplitz matrix structure) can be exploited through the use of FFT or even FMM.

1.2.1 Non-Overlapping Schwarz Domain Decomposition

The classical alternating Schwarz algorithm is over a century old. In 1896 Schwarz [26] developed the overlapping variant of the algorithm to prove uniqueness of the Laplace problem in non-separable domains. The first to recognize the numerical implications of the algorithms was Miller in 1965 [27], and later, Loins in [28] and [29]. The first non-overlapping Schwarz Algorithm was proposed by Loins in [30]. In these early studies of the Schwarz algorithms for elliptic (static) problems were considered. Apart from the non-overlapping variant of Loins, all the information communication was done through the use of a Dirichlet-to-Dirichlet or Neumann-to-Neumann or even Dirichlet-to-Neumann type interface transmission conditions. Alternatively, a Robin type TC had to be imposed to ensure convergence for the non-overlapping variant of [30]. For that reason the non-overlapping Schwarz algorithm with Robin TCs is often called Lions-Schwarz Algorithm. The first to extend and study the non-overlapping Schwarz Algorithm on Helmholtz (dynamic) problems was Despres in [31] and [32]. Till today, a large number of the literature has been devoted in analyzing and optimizing these first DD algorithms, either in the context of alternating Schwarz or as preconditioner accelerators in a Krylov method. To this, we refer to [33-38].

The first to tailor the non-overlapping Schwarz algorithm for Maxwell equations and time-harmonic EM problems was Despres, Collino and Robles in their early work [39]. The appropriate Robin TCs were extended to account for the vector nature of the problem. This modified Schwarz algorithm is often termed as Despres-Schwarz
Algorithm. Following the footsteps of Despres, Collino in [40] and Stupfel in [13] introduced new TCs that further improve the convergence and overall efficiency of the Despres-Schwarz algorithm.

1.2.2 Non-Conforming Finite Element Methods

As in the case of non-overlapping Schwarz methods, non-conforming/non-matching element techniques have their origin in the French school of applied mathematics. Almost a decade ago Bernardi, Maday and Patera in [41] and [42] proposed the Mortar elements to solve domain decomposition problems with non-matching interfaces. In these original works the authors used the constrained multiplier approach to enforce a weak continuity of the solution across the non-conforming interfaces. The method was applied to two dimensional elliptic problems that led to a symmetric positive definite variational statement. The extension of the method on three dimensional nodal finite elements for elliptic problems was done in [43] and [44]. Note that in these early works and all the following ones based on the constrained multiplier space, no extra unknowns were introduced, but the finite support of the basis functions on the non-mortar interfaces was lost. The unconstrained multiplier approach of realizing Mortar elements was proposed in 1999 by Ben Belgacem in [45]. The extensions of the constrained Mortar element method to low-frequency Maxwell’s problem, mainly eddy-current applications was done in [46], [47]. In these studies the optimality and error estimates of the Mortar method were assessed. The second-family of Nedelec’s elements [48] were used in this study. Hoppe in [49] studied the uniqueness and stability of the mortar edge elements based on the first-family of Nedelec’s elements [50] that are widely used in high-frequency FEM computations. So far, the only numerical demonstrations of the
Mortar element method in EM involves eddy-current problems of sliding meshes in electrical machines [51].

Even though mortar element methods have been very successful in treating non-conforming grids with Dirichlet-to-Neumann or Neumann-to-Neumann domain decomposition algorithms, they can not be easily used for DD algorithms based on Robin or other transmission conditions. For such transmission conditions, a new class of non-conforming methods has been proposed, and it is termed *cement element method*. The method was originally proposed by Arbogast and Yotov in [52] for elliptic and parabolic problems. The term “cement” is due to Achdou et.al. [53], that used the methods in the context of finite volume and later in the context of nodal finite elements in [54]. Independently, Lee, Vouvakis and Lee developed a variant the cement method for Maxwell’s equation in [55], [56] and [57]. The method is based on a mixed finite element formulation that does not require the mortar and non-mortar sides. The two sides of the nonconforming triangulations are symmetric and local adaptive mesh refinement techniques can be used in each side e.g. mortar side, without the need of modifying the non-mortar side’s mesh. Along with other advantages, Robin or other higher order transmission conditions can be enforced weakly through the use of a double valued Lagrange multiplier (on either side of the interface) representing the electric current.

1.2.3 *Finite Element Tearing and Interconnection (FETI) Algorithms*

The method was introduced by Farhat and Roux in [58] in the early nineties. FETI was originally used to solve elliptic problems arising in structural mechanics. Due to its superior scalability and parallel performance over other substructuring methods, FETI quickly gain popularity on other areas of computational science, such as elasticity [59],...
acoustic scattering [60], Stokes equations [61] and Maxwell’s equations [19], [20]. Along with the various applications, a number of different variants of the method started emerging. To that, we refer to the Two level FETI-H in [60, 62, 63] and the FETI-DP in [64] and [65]. In the former case “H” stands for Helmholtz, and it is an adaptation of the method to account for the wave nature of Helmholtz problems. The method is used in the context of a 2-level (one local one global) preconditioner to achieve scalability and superior convergence. Note that one and two (one on either side of the interface) Lagrange multiplier sets have been previously proposed in the framework of FETI-H. The mathematical study of the FETI preconditioner on 2-D Maxwell problems was done in [66], whereas [19], [20] numerically study the possibility of the FETI in 3-D wave problems. Lastly, the Mortar elements of FETI for elliptic was introduce by Stefanica in [67]. The marriage of the Mortar element and FETI method for 2-D Maxwell problems was done in [68].

1.3 Contributions and Summary

Before describing the proposed methodology, it would be beneficial to state the contributions and where this research stands in regards to the already published literature. The proposed DD is based on the original Despres-Schwarz algorithm [39], for time-harmonic vector wave propagation problems. However, herein an elegant and novel way of analyzing the Despres-Schwarz algorithms and the transmission conditions in general is proposed. The method is based on the Fourier analysis, and its inspiration was the work of Gander, Magoules and Nataf in [38]. Due to the difference in the nature of the problem Maxwell versus Helmholtz equations, the analysis significantly deviates for the one in [38]. The significance of the proposed analysis lays on the fact that it is the first step on
designing and optimizing transmission conditions that exhibit superior convergence properties.

The proposed cement element method is significantly different from all the mortar methods summarized in section 1.2.2. It is more in line with the cement methodology of [52-54], but with the significant difference it is appropriately modified to account for vector time-harmonic electromagnetic problems. Unlike the classical mortar method that enforces a Dirichlet-to-Dirichlet, Neumann-to-Neumann or Dirichlet-to-Neumann type continuity across non-matching grids, the proposed cement method allows for the more flexible, and efficient Robin-to-Robin map. In a way, the proposed method can be viewed as a generalization of the classical mortar method. From practical point of view, the proposed formulation requires simpler and less expensive iterative solvers. Moreover, it simplifies on local (subdomain) adaptive meshing strategies since there is no need for mortar and non-mortar sides.

Finally, the concept of FETI in this work is different from the versions proposed in the literature. Unlike most of the existing algorithms, FETI will be used exclusively as a substructuring algorithm and not as a preconditioner in a Krylov subspace method. The “tearing” process will be done beforehand in the pre-processing step, since only a small number of different domains will be considered. This will allow for efficient treatment of multiple excitations and optimization, since the time consuming pre-processing can be easily reused. In contrast to Farhat’s FETI, where the tearing process is incorporated in the outer-loop or “interconnection” process, as a subdomain based preconditioner. Herein, FETI uses a economical outer-loop stationary iteration method, because of the large size of the unknown vector. The key point of using FETI in this work is to reduce the volume
FEM unknowns to surface unknowns in the interfaces of the domains, thus save greatly in memory storage.

In CHAPTER 5 a new approach of analyzing infinite periodic structures using FEM is proposed. To the best of the author’s knowledge, up to now all FEM simulations on infinite periodic structures require a periodic mesh to impose the periodic boundary conditions. Namely, the mesh on opposite sides of the computational domain (Floquet’s cell) must be identical. In this work the cement element method is used to remedy this restriction. Naturally, the method can be easily incorporated into existing adaptive meshing strategies without the need of constraining the periodicity of the mesh.

Last but not least a new formalism of hybridizing or coupling finite element and boundary element methods (BEM) is proposed. The approach is inspired from the non-conforming domain decompositions described above but, instead of FEM domains alone FEM and BEM domains are coupled using the cement element technique. The unique features of the approach can be summarized in three points. The method is variational and symmetric, it is free of the notorious internal resonance problem [69], and it is non-conforming allowing different mesh or basis functions on FEM and BEM sides.
CHAPTER 2

NOTATIONS

Before start developing the theory and formulation it is essential to introduce some notations and definitions that will ease the reading of the next chapters. Throughout the manuscript, boldface capital letters will represent matrices and operators, except when explicitly stated otherwise; boldface lowercase letters represent column/row vectors and vector fields and over hat e.g., \( \hat{a} \) indicates unit vectors in \( \mathbb{R}^3 \). Fourier or spectral variables will be denoted with an overhead cup e.g. \( \hat{u} \). Position vectors \( \mathbf{r} \) and \( \mathbf{r}' \) refer to the observation and source, respectively. It should be noted that throughout the document \( j \) denotes the imaginary unit. The free space wave number will be denoted by \( k = \omega \sqrt{\mu_0 \varepsilon_0} \), where \( \omega = 2\pi f \) is the radial frequency of operation, and \( \varepsilon_0 \) and \( \mu_0 \) are the free space permittivity and permeability, respectively. Note that \( \varepsilon_r = \varepsilon / \varepsilon_0 \) and \( \mu_r = \mu / \mu_0 \) will represent the relative permittivity and permeability of dielectric and magnetic materials, respectively. The free space intrinsic impedance will be represented by \( \eta = \sqrt{\mu_0 / \varepsilon_0} \). The fundamental solution (Green’s function) for the scalar Helmholtz equation for free-space will be denoted by

\[
g(\mathbf{r}|\mathbf{r}') = \frac{e^{-j|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad \mathbf{r} \neq \mathbf{r}' \tag{2.1}
\]
Most of the definitions in this section will be loosely presented for the sake of brevity and clarity; for rigorous definitions the interested reader is referred to the PhD dissertation of Buffa [70]. First let’s define the surface integral of two complex-valued vector functions as

\[
\langle u, v \rangle_\Gamma = \int_\Gamma (u \cdot v) \, d\Gamma.
\]  

(2.2)

Similarly, the volume integral of two complex valued functions in a domain \( \Omega \) is denoted by

\[
\langle u, v \rangle_\Omega = \int_\Omega (u \cdot v) \, d\Omega.
\]  

(2.3)

Several spaces need to be defined beforehand. The following convention will be adopted: spaces of scalar valued functions will be denoted by \( H \), whereas for spaces of vector valued functions the boldface letter \( \mathbf{H} \) will be used. One of the most important spaces in electromagnetics is that of curl-conforming functions in a subdomain \( \Omega \)

\[
\mathbf{H} (\text{curl}, \Omega) = \left\{ u \left| \int_\Omega \left( |\nabla \times u|^2 + |u|^2 \right) \, d\Omega < \infty \right. \right\}.
\]  

(2.4)

This is the space where electric and magnetic field reside; the physical meaning of the space \( \mathbf{H} (\text{curl}, \Omega) \) is that in subdomain \( \Omega \), the electric and magnetic energies are finite.

Now it is time to introduce some trace operators and spaces of tangential vector fields on the surface \( \Gamma = \partial \Omega \). First let’s define the **tangential surface trace** \( \gamma^\prime \) acting on \( u \) as

\[
\gamma' u := \hat{n} \times (u \times \hat{n}),
\]  

(2.5)

where \( \hat{n} \) denotes the outward (with respect to \( \Omega \) ) pointing unit vector on \( \Gamma \). In other words, \( \gamma' u \) contains the tangential components of the vector field \( u \) on the surface \( \Gamma \). A second trace necessary is the **twisted tangential trace** \( \gamma^\times \) which is defined as
\[
\gamma^u := \hat{n} \times u,
\]  
which again implies that \( \gamma^u \), contains the tangential components of \( u \) on surface \( \Gamma \), the same as \( \gamma^u \), but twisted 90° around \( \hat{n} \).

Of particular interest in this work will be the spaces \( H^{1/2}_{\parallel} (\Gamma) \) and \( H^{1/2}_{\perp} (\Gamma) \) of surface vector functions on \( \Gamma \). Informally speaking, these spaces contain the tangential surface vector fields with “weak tangential continuity” and “weak normal continuity”, respectively, across the edges of a faceted surface \( \Gamma \). At this point it is important to give the meaning of the superscript number that appears in the definition of the Hilbert spaces. In simple terms, it represents the order regularity, namely the degree of continuity in derivatives or the order of singularity, such that the functions are integrable. Note that \( \frac{1}{2} \) is the order of integrable singularity on surfaces; that is why the superscript in the above tangential surfaces is \( \frac{1}{2} \). The exact definitions of these spaces can be found in [70]. It is important to emphasize the weak nature of the continuity, since the surface can be faceted; in that case, strong continuity along the edges and corners cannot be rigorously defined. The corresponding dual spaces of \( H^{1/2}_{\parallel} \) and \( H^{1/2}_{\perp} \) will be denoted by \( H^{-1/2}_{\parallel} \) and \( H^{-1/2}_{\perp} \), respectively; again for the definition of the dual space we suggest the readers to consult any functional analysis book such as [71]. Finally, and most important for this work, the following two spaces need to be defined

\[
H^{-1/2}_{\parallel} (div, \Gamma) = \left\{ u \in H^{-1/2}_{\parallel} (\Gamma), div u \in H^{-1/2} (\Gamma) \right\},
\]  
(2.7)

and

\[
H^{-1/2}_{\perp} (curl, \Gamma) = \left\{ u \in H^{-1/2}_{\perp} (\Gamma), curl u \in H^{-1/2} (\Gamma) \right\}.
\]  
(2.8)
where $\text{div}_{\Gamma}$ and $\text{curl}_{\Gamma}$ are the surface divergence and curl operators defined in [72]. From engineering point of view, it is sufficient to say $H_{1}^{-1/2}(\text{div}_{\Gamma}, \Gamma)$ is the space that the Whitney 2-forms belong to, and some of the most famous vector basis functions in EM community such as the RWG [73]; similarly, $H_{\perp}^{-1/2}(\text{curl}_{\Gamma}, \Gamma)$ contains the surface Whitney 1-form (or edge elements) [50]. The following theorem, taken directly from [70], helps to establish the trial and test function spaces for the symmetric coupling of the FEM and IE.

**Trace Theorem**

The trace mapping $\gamma' : H(curl, \Omega) \mapsto H_{\perp}^{-1/2}(\text{curl}_{\Gamma}, \Gamma)$ and $\gamma'' : H(curl, \Omega) \mapsto H_{1}^{-1/2}(\text{div}_{\Gamma}, \Gamma)$ are linear and continuous.

It is very important at this point to note that $H_{\perp}^{-1/2}(\text{curl}_{\Gamma}, \Gamma)$ and $H_{1}^{-1/2}(\text{div}_{\Gamma}, \Gamma)$ are dual to each other through a duality pairing defined in (2.2). That is to say

$$\left( H_{1}^{-1/2}(\text{div}_{\Gamma}, \Gamma) \right)' = H_{\perp}^{-1/2}(\text{curl}_{\Gamma}, \Gamma), \tag{2.9}$$

where prime indicates the dual space.
3.1 Boundary Value Problem Statement

The electromagnetic scattering or radiation in unbounded domains is governed by Maxwell equations for the electric and magnetic fields \((\mathbf{E}, \mathbf{H})\), subject to the Silver-Müller radiation condition \([72]\). The formal statement of such boundary value problem (BVP) can be found elsewhere \([74]\). Here only an approximate one will be considered for the sake of simplicity. Namely, the unbounded domain is replaced by a bounded one \(\Omega\), with first order absorbing boundary conditions (ABC) imposed on the finite truncation boundary \(\partial \Omega \equiv \Gamma_{\text{ext}}\). Under conditions of time-harmonic excitations, and in the absence of non-linear materials, the Maxwell’s system reduced into the following BVP statement:

\[
\text{Find the electric field } \hat{\mathbf{E}} \in \mathbf{H}(\text{curl};\Omega) \text{ such that }
\]
\[
\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\n\nThe excitation is assumed an impressed electric current \( \mathbf{J}^{\text{imp}} \) within the domain of interest. The Neumann and Dirichlet boundary conditions are imposed on perfect magnetic conductors (PMC) on surface \( \Gamma_{\text{PMC}} \), and electric conductors (PEC), respectively. The last boundary condition represents the first order ABC imposed on the finite truncation boundary \( \Gamma_{\text{ext}} \). It should be noted that the theory and formulation in this paper is not restricted by the use of the ABC, this was only done for brevity and simplicity reasons. The same formulation holds for PML, boundary integral or any other boundary truncation method.

### 3.2 Domain Decomposition for Maxwell Equations

In the case of the non-overlapping domain decomposition, the computational domain \( \Omega \) is decomposed into a number of disjoint sub-domains. For the sake of brevity and simplicity and without any loss of generality, only two sub-domains \( \Omega_1 \) and \( \Omega_2 \) will be considered. In each sub-domain, let us denote the restrictions of the unknown vector field \( \mathbf{E} \) as \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \) for sub-domain \( \Omega_1 \) and \( \Omega_2 \), respectively. With respect to Figure 3.1 the decomposed continuous BVP reads:

*Given an arbitrary initial value \( (\mathbf{E}_1^{(0)}, \mathbf{E}_2^{(0)}) \) find \( (\mathbf{E}_1^{(n)}, \mathbf{E}_2^{(n)}) \) through the iteration*
\begin{align*}
\nabla \times \frac{1}{\mu_r} \nabla \times \mathbf{E}_1^{(n)} - k^2 \varepsilon_r \mathbf{E}_1^{(n)} &= - j \omega \mu_0 \mathbf{J}_{imp}^{(n)} & \text{in } \Omega_1 \\
\gamma' \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}_1^{(n)} \right) &= 0 & \text{on PMC} \\
\gamma' \left( \mathbf{E}_1^{(n)} \right) &= 0 & \text{on PEC} \\
\gamma' \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}_1^{(n)} \right) - j k \gamma' \left( \mathbf{E}_1^{(n)} \right) &= 0 & \text{on } \partial \Omega_1 \cap \Gamma_{ex} \\
\gamma' \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}_1^{(n)} \right)_{\Gamma_{12}^{(1)}} - S_1 \left( \gamma' \left( \mathbf{E}_1^{(n)} \right) \right) &= - \gamma' \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}_2^{(n-1)} \right)_{\Gamma_{12}^{(2)}} - S_1 \left( \gamma' \left( \mathbf{E}_2^{(n-1)} \right) \right) & \text{on } \Gamma_{12} \\
\text{and} \\
\nabla \times \frac{1}{\mu_r} \nabla \times \mathbf{E}_2^{(n)} - k^2 \varepsilon_r \mathbf{E}_2^{(n)} &= - j \omega \mu_0 \mathbf{J}_{imp}^{(n)} & \text{in } \Omega_2 \\
\gamma' \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}_2^{(n)} \right) &= 0 & \text{on PMC} \\
\gamma' \left( \mathbf{E}_2^{(n)} \right) &= 0 & \text{on PEC} \\
\gamma' \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}_2^{(n)} \right) - j k \gamma' \left( \mathbf{E}_2^{(n)} \right) &= 0 & \text{on } \partial \Omega_2 \cap \Gamma_{ex} \\
\gamma' \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}_2^{(n)} \right)_{\Gamma_{12}^{(2)}} - S_2 \left( \gamma' \left( \mathbf{E}_2^{(n)} \right) \right) &= - \gamma' \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}_1^{(n-1)} \right)_{\Gamma_{12}^{(1)}} - S_2 \left( \gamma' \left( \mathbf{E}_1^{(n-1)} \right) \right) & \text{on } \Gamma_{12} \\
\text{We note that } \gamma' \left( \mathbf{u} \right)_{\Gamma_{12}^{(n)}} = \mathbf{n}_i \times \mathbf{u} ; \text{ where } \mathbf{n}_i \text{ is the unit normal on the interface boundary } \Gamma_{12} \text{ and pointing away from sub-domain } \Omega_i \text{ (refer to Figure 3.1). Moreover } \square^{(n)} \text{ denotes iteration count. Two important remarks should convey the decomposed continuous BVP of (3.2) and (3.3). It is apparent that the divergence free condition was dropped from both sub-domain problems BVPs. This was only done due to space-limitation reasons, at this moment the reader should assume that this condition is carried with the vector curl-curl equation until the definition of discrete problem statement. There, its enforcement will be done through a graph partitioning (tree-cotree) gauge [75, 76]. The second and most important remark refers to the “artificial” transmission condition used to communicate information across domains. As it was noted by many researchers in the past, to name a}
few, Després in [39] and Gander in [37], the choice of the transmission condition has tremendous impact on the success and speed of the Domain Decomposition algorithm. The important principles to “design” a transmission condition are two: (a) to ultimately enforce the tangential continuity of the electric and magnetic fields across the sub-domain interface; and (b) to provide a mean of accelerating the convergence for the DD iteration loop. Looking at the last equation of (3.2) and (3.3) both principles have been taken into consideration. The continuity of the tangential magnetic field is imposed by the first term in each side of the two equations, whereas the continuity of the electric field is imposed through the tangential operators $S_i(\bullet)$ where $i=1,2$. It is vital for the operator to be non-singular, and strictly positive-definite; otherwise the electric field continuity can not be imposed. In the next section it will be discussed how to choose the tangential operators $S_i(\bullet)$ such that, not only it leads to the correct physics of the problem, but also accelerates the convergence.
3.3 Convergence Analysis

In this section, a detailed analysis of the TCs used in the present DDM will be presented. As it was stated in the introduction, a judicious choice of the TCs “artificially” imposed across subdomain interfaces, not only ensures uniqueness, but also accelerates convergence on a DD algorithm. Even though in practical DDM implementations TCs can be imposed on completely arbitrarily shaped interfaces the general analysis of such situations can be very difficult or impossible. For that reason, we will restrict our analysis...
on planar interfaces only. Moreover, infinite lateral extent (half-space subdomains) will be assumed, in order to facilitate the mathematical apparatus provided by Fourier theory. At this point, it is important to stress that although Fourier analysis is restricted to rather idealistic geometries; it has been found that its convergence estimates predict very accurately quite complicated interface boundaries, at least for two dimensional scalar Helmholtz problems [37].

For reasons given in the previous paragraph, the simplified model problem of Figure 3.1 will be considered. The electromagnetic boundary value problem of interest consists of the vector **curl-curl** equation of in \( \Omega = \mathbb{R}^3 \), and the Silver-Müller radiation condition at infinity. Note, that no PEC, PMC or material in homogeneities are present in \( \Omega \). The decomposed problem, now is that of (3.2) and (3.3) on \( \Omega_1 = (-\infty, 0] \times \mathbb{R}^2 \) and \( \Omega_2 = [0, +\infty) \times \mathbb{R}^2 \), respectively. The planar interface \( \Gamma_{12} = \Gamma = \mathbb{R}^2 \) of the model problem is shown in Figure 3.2.

With respect to the Figure 3.2 let us first introduce the following 2-dimensional (partial) Fourier transform pair:

\[
\tilde{f}(k_x, k_y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) e^{i(k_x x + k_y y)} dx dy, \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k_x, k_y, z) e^{-i(k_x x + k_y y)} dk_x dk_y
\]

(3.4)

where \( k_x, k_y \) are the Fourier variables. In a straightforward way, the Fourier analysis of the BVP described in (3.2) and (3.3) will require the application of the Fourier transform on both vector **curl-curl** equation subject to the Silver-Müller radiation condition on each domain, and each of the TCs. Even though this is a simple concept, the underlying algebraic manipulations are very lengthy and require analytic evaluations of the
eigenvalues and eigenvectors of a 2×2 matrix, which leads to rather complicated final expressions for the convergence factors.

![Diagram of domain decomposition](image)

Figure 3.2: Idealized domain decomposition used in the Fourier analysis.

Alternatively a considerably easier approach will be formulated based on the concept of Transverse Electric (TE) or H and Transverse Magnetic (TM) or E modal decomposition of electromagnetic fields [77], [78]. The validity of the approach is based on the fact that any electromagnetic field produced by an arbitrary oriented electric or magnetic source in a “layered” environment, like the mode problem, can be written as a superposition of TE and TM modes with respect to the “layered” interface normal [78]. The term “layered” implies geometry with multiple planar parallel interfaces that extend laterally at infinity. It is apparent that the model problem of Figure 3.2 belongs to the abovementioned category. The second key ingredient of the following derivation is the use of a rotation transformation of the Fourier variables \( k_x, k_y \) to the “natural” spectral
coordinates \( u \) and \( v \), which significantly simplifies the expressions of the electromagnetic fields in the Fourier domain.

The analysis starts by expressing both electric and magnetic fields in terms of scalar “wave” functions, \( \psi \) for TE and \( \phi \) for TM. Specifically, for TE(H) modes [78]:

\[
E = -\hat{z} \times \nabla \psi = \hat{x} \frac{\partial \psi}{\partial y} - \hat{y} \frac{\partial \psi}{\partial y},
\]

\[
H = -\frac{\eta}{jk} (\nabla \times \nabla \times \hat{z} \psi) = -\frac{\eta}{jk} \left[ \hat{x} \frac{\partial^2 \psi}{\partial x \partial \zeta} + \hat{y} \frac{\partial^2 \psi}{\partial y \partial \zeta} - \hat{z} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \right].
\]

For TM(E) modes [78]:

\[
E = \nabla \times \nabla \times \hat{z} \phi = \left[ \hat{x} \frac{\partial^2 \phi}{\partial x \partial \zeta} + \hat{y} \frac{\partial^2 \phi}{\partial y \partial \zeta} - \hat{z} \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \right],
\]

\[
H = -\frac{jk}{\eta} (\hat{z} \times \nabla \phi) = -\frac{jk}{\eta} \left[ \hat{x} \frac{\partial \psi}{\partial y} - \hat{y} \frac{\partial \psi}{\partial x} \right].
\]

where \( \psi \) and \( \phi \) are scalar and satisfy Helmholtz equation:

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) f(x, y, z) = 0, \quad f = \psi, \phi
\]

The goal now is to express all fields in Fourier (spectral) domain. By taking the Fourier transform of (3.7) leads to

\[
\left[ \frac{d^2}{d\zeta^2} + \left( k^2 - \beta^2 \right) \right] \hat{f}(k_x, k_y, \zeta) = 0, \quad f = \psi, \phi
\]

where \( \beta^2 = k_x^2 + k_y^2 \) is the radial Fourier variable. It is convenient to further define the z-directed wave number as

\[
k_z^2 = \beta^2 - k^2 \Rightarrow k_z = \begin{cases} j\sqrt{k^2 - \beta^2} & k > |\beta|, \text{ propagating modes} \\ \sqrt{\beta^2 - k^2} & k < |\beta|, \text{ evanescent modes} \end{cases}
\]

Before start simplifying (3.5) and (3.6) by applying the Fourier transform, it would be useful to introduce the following transformation to the “natural” Fourier coordinate
system \((k_x,k_y) \rightarrow (k_u,k_v)\), since it will significantly simplify the field representations. The transformation is

\[
\begin{bmatrix}
  u \\
v
\end{bmatrix} =
\begin{bmatrix}
  \sin \alpha & -\cos \alpha \\
  \cos \alpha & \sin \alpha
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\] (3.10)

where the rotation angle is given by \(\cos \alpha = \frac{k_x}{\beta}\) or \(\sin \alpha = \frac{k_y}{\beta}\), which more importantly leads to

\[
\begin{align*}
k_u &= k_x \sin \alpha - k_y \cos \alpha = 0 \\
k_v &= k_x \cos \alpha + k_y \sin \alpha = \beta
\end{align*}
\] (3.11)

It should be noted here that the abovementioned transformation can be also viewed as a transformation to the polar Fourier domain with radial variable \(\beta\), and angular Fourier angle \(\alpha\). This is a fairly standard approach in dealing with the derivation of “layered” Green’s functions in electromagnetics [79], [80].

Having defined the auxiliary Fourier wave functions \(\tilde{\psi}\) and \(\tilde{\phi}\) and the rotation transformation, it is now time to apply the Fourier transform on the TE and TM modal fields of (3.5) and (3.6). In doing so, the following expressions are obtained for the TE modes:

\[
\begin{align*}
\tilde{E} &= -j \left[ \hat{x} k_x \hat{\psi} - \hat{y} k_y \hat{\psi} \right] \\
\tilde{H} &= \frac{\eta}{jk} \left[ \hat{x} j k_x \frac{d\hat{\psi}}{dz} + \hat{y} j k_y \frac{d\hat{\psi}}{dz} - \hat{z} \beta \hat{\psi} \right].
\end{align*}
\] (3.12)

Applying the rotation transformation of (3.10), the Fourier fields for the TE modes are further simplified to

\[
\begin{align*}
\tilde{E} &= -\hat{u} j \beta \hat{\psi} \\
\tilde{H} &= \frac{\eta}{jk} \left[ \hat{v} j \beta \frac{d\hat{\psi}}{dz} - \hat{z} \beta \hat{\psi} \right].
\end{align*}
\] (3.13)

In a similar fashion the TM Fourier fields in the rotated system becomes
\[
\mathbf{E} = -\hat{\psi} j \beta \frac{d\phi}{dz} + \hat{\beta} z \phi
\]
\[
\mathbf{H} = \mathbf{u} \frac{k}{\eta} \beta \phi
\] (3.14)

where both \( \psi \) and \( \phi \) satisfy the one dimensional homogeneous wave equation
\[
\frac{df}{dz} - k^2 z f = 0, \quad f = \phi, \psi.
\] (3.15)

The general solutions of the ordinary differential equation in (3.15) are of the form
\[
\hat{f} = A^+ e^{-k_z z} + A^- e^{k_z z}.
\] (3.16)

Referring to the interface problem of Figure 3.2 subject to the Silver-Müller radiation on each appropriate sides of \( \Omega_1 \) and \( \Omega_2 \), only outgoing waves ought to be considered thus the solutions in each subdomain become
\[
\hat{f}_1(k_x, k_y, z) = F^-_1(k_x, k_y, 0) e^{k_z z}, \quad \text{in } \Omega_1
\]
\[
\hat{f}_2(k_x, k_y, z) = F^+_2(k_x, k_y, 0) e^{-k_z z}, \quad \text{in } \Omega_2
\] (3.17)

where \( F^-_1 \) and \( F^+_2 \) are the Fourier modal excitation coefficients to be determined by the enforcement of the TC across interface \( \Gamma_{12} \). Note that the following part of the section, \( F^-_1 \) and \( F^+_2 \) will be replaced with \( A_1 \) and \( A_2 \), respectively, when referring to TM modes, and with \( B_1 \), \( B_2 \) when referring to TE modes.

Putting together (3.13), (3.14) and (3.17) the Fourier field representation of Table 3.1 are obtained.
Table 3.1: TE and TM Fourier field representations on the two sub-domains.

<table>
<thead>
<tr>
<th></th>
<th>$\Omega_1$</th>
<th>$\Omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE</td>
<td>$\mathbf{\hat{u}}\left(-B_1^{(n)} j \beta e^{z k z; z}\right)$</td>
<td>$\mathbf{\hat{u}}\left(-B_2^{(n)} j \beta e^{-z k z; z}\right)$</td>
</tr>
<tr>
<td>TM</td>
<td>$A_1^{(n)} (-\hat{\mathbf{k}} j z k + \hat{\mathbf{z}} \beta) e^{z k z; z}$</td>
<td>$A_2^{(n)} (-\hat{\mathbf{k}} j z k + \hat{\mathbf{z}} \beta) e^{-z k z; z}$</td>
</tr>
</tbody>
</table>

The objective of this section is to find the convergence rates $\rho^{TE}$ and $\rho^{TM}$ of the TE and TM modes respectively, as functions of $k_z$. To achieve that, it is necessary to find a relation between the excitation coefficients $A^{(n)}$ and $A^{(n-2)}$ for TM modes, along with $B^{(n)}$ and $B^{(n-2)}$ for the TE modes. This is done through the enforcement of the TC on both domains, namely through the last equations on (3.2) and (3.3).

\[
\begin{align*}
\mathbf{j}_1^{(n)} - \mathbf{S}_1^{(n)}(\mathbf{e}_1^{(n)}) &= -\mathbf{j}_2^{(n-1)} - \mathbf{S}_1^{(n-1)}(\mathbf{e}_2^{(n-1)}) & \text{at } z = 0^- \\
\mathbf{j}_2^{(n)} - \mathbf{S}_2^{(n)}(\mathbf{e}_2^{(n)}) &= -\mathbf{j}_1^{(n-1)} - \mathbf{S}_2^{(n-1)}(\mathbf{e}_1^{(n-1)}) & \text{at } z = 0^+.
\end{align*}
\tag{3.18}
\]

The analysis proceeds by finding $\mathbf{j}_i$ and $\mathbf{e}_i$, $i = 1, 2$ with the use of Table 3.1. After some very simple and short algebraic manipulations, the tangential Fourier fields of Table 3.2 are obtained. Note, that with the use of the rotation transformation in the Fourier domain, the tangential TE and TM fields are completely decoupled. This is something that is reflected on Table 3.2, since the TE representation involves only $\mathbf{u}$ coordinate whereas the TM representation involves $\mathbf{v}$ coordinate only.
<table>
<thead>
<tr>
<th></th>
<th>$\Omega_1$</th>
<th>$\Omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{e}^{(n)}(z=0^-)$</td>
<td>$j^{(n)}(z=0^-)$</td>
<td>$\mathbf{e}^{(n)}(z=0^-)$</td>
</tr>
<tr>
<td>TE</td>
<td>$\hat{u}(-B_1^{(n)}j\beta)$</td>
<td>$\hat{u}(B_1^{(n)}j\beta k_z)$</td>
</tr>
<tr>
<td>TM</td>
<td>$\hat{v}(-A_1^{(n)}j k_z \beta)$</td>
<td>$\hat{v}(-A_1^{(n)}j k^2 \beta)$</td>
</tr>
</tbody>
</table>

Table 3.2: TE and TM Fourier tangential field representations on the domain interface.

With this observation in mind it is now straightforward to substitute the values of Table 3.2 on (3.18) and obtain the following convergence factors

$$
\rho^{TE}(k_z) = \frac{|B_1^{(n)}|}{|B_1^{(n-2)}|} = \frac{|k_z - \tilde{S}_{2u}|}{|k_z + \tilde{S}_{1u}|},
$$

(3.19)

for TE modes and

$$
\rho^{TM}(k_z) = \frac{|A_1^{(n)}|}{|A_1^{(n-2)}|} = \frac{|k^2 - \tilde{S}_{2u}k_z|}{|k^2 + \tilde{S}_{1u}k_z|},
$$

(3.20)

for TM modes. Were $\tilde{S}_{iu}$ and $\tilde{S}_{iv}$ are the two orthogonal Fourier components of the operator $S$. From (3.19) and (3.20) it is apparent that in Fourier domain choosing

$$
\tilde{S}_{1u} = \tilde{S}_{2u} = k_z
$$

(3.21)

and

$$
\tilde{S}_{1v} = \tilde{S}_{2v} = -\frac{k^2}{k_z}
$$

(3.22)

will lead to $\rho^{TE}(k_z) = \rho^{TM}(k_z) = 0$; therefore a stationary iteration based DD algorithm will converge in two iterations. In practice, an algorithm with such a transmission condition would require the inverse Fourier transform of $k_z$ and $1/k_z$. This task requires
the inverse Fourier transform of the square root, which would lead to a non-local vector function in xyz domain. This is similar to the two dimensional Helmholtz equation, but in Maxwell case matters are more complicated due to (3.22).

Rather than trying to invert the square root and its reciprocal, several local (sparse) approximations of (3.21) and (3.22) can be found. In this paper only two will be presented.

### 3.3.1 First Order (Robin) Transmission Condition

This is the simplest approximation, easy to implement and theoretical analyze. The operator $S$ is chosen as

$$S = \gamma, \quad \gamma \in \mathbb{C}. \quad (3.23)$$

It is the goal of this section to find if choice of (3.23) can potentially lead to a convergent stationary iteration DD algorithm. It is easy to see that $\tilde{S}_{1u} = \tilde{S}_{1v} = \tilde{S}_{2u} = \tilde{S}_{2v} = \gamma$ thus, (3.19) and (3.20) simplify to

$$\rho^{TE}(k_z) = \frac{|B_{1}^{(n)}|}{|B_{1}^{(n-2)}|} = \frac{k_z - \gamma}{k_z + \gamma}, \quad (3.24)$$

and

$$\rho^{TM}(k_z) = \frac{|A_{1}^{(n)}|}{|A_{1}^{(n-2)}|} = \frac{k_z^2 + \gamma k_z}{k_z^2 - \gamma k_z}. \quad (3.25)$$

In a more convenient way, the convergence estimates of (3.24) and (3.25), can be written in terms of the transverse spectral variable $\beta$, with the aid of (3.9). Therefore, for each mode set, two subset of modes need to be defined, the propagating and the evanescent identified as
\[
\rho_{\text{TE}}(\beta) = \begin{cases} 
\frac{j\sqrt{k^2 - \beta^2 - \gamma}}{j\sqrt{k^2 - \beta^2 + \gamma}} & k > |\beta|, \text{ propagating modes} \\
\frac{\sqrt{\beta^2 - k^2 - \gamma}}{\sqrt{\beta^2 - k^2 + \gamma}} & k < |\beta|, \text{ evanescent modes} 
\end{cases} 
\] (3.26)

and

\[
\rho_{\text{TM}}(\beta) = \begin{cases} 
\frac{k^2 + j\gamma\sqrt{k^2 - \beta^2}}{k^2 + j\gamma\sqrt{k^2 - \beta^2}} & k > |\beta|, \text{ propagating modes} \\
\frac{k^2 + \gamma\sqrt{\beta^2 - k^2}}{k^2 + \gamma\sqrt{\beta^2 - k^2}} & k < |\beta|, \text{ evanescent modes.} 
\end{cases} 
\] (3.27)

In a convergent DD scheme that employs a stationary outer loop iteration, all four modes in (3.26) and (3.27) should have \( \rho < 1 \). This implies that our analysis and convergence rates are valid for both Jacobi and Gauss-Seidel iteration schemes, or in terms of the Schwarz theory, both additive and Multiplicative Schwarz. On the other hand, if each of the above schemes is used as a preconditioning accelerator on a Krylov type method, a spectral radius \( \rho > 1 \) does not necessarily imply that the method diverges.

In light of (3.26), (3.27) and the comments of the previous paragraph, a number of convergent regions can be identified for each set of modes, based on the choice of the Robin constant \( \gamma \). From (3.26), it is clear that choosing \( \text{Im}\{\gamma\} > 0 \) will lead to convergent TE propagation modes, whereas the \( \text{Re}\{\gamma\} > 0 \) would lead to convergent TE evanescent modes. These convergence regions for the TE modes are depicted at Figure 3.3(a) in the complex \( \gamma \)-plane. It is clear that the first quadrant will be the choice of preference since both evanescent and propagation modes converge. This is exactly the case described in
[38] for the two-dimensional Helmholtz equation. Furthermore, like the Helmholtz case, even if $\gamma$ is chosen on the appropriate quadrant, the cut-off mode $\beta = k$ (plane-wave incidence parallel to interface) will never converge, since that choice makes $\rho^{TE}(\beta = k) = 1$. On the other hand, the situation for the TM modes is quite different. From (3.27), the propagating TM modes will converge in the region where $\text{Im}\{\gamma\} > 0$, but the evanescent ones will converge on the region $\text{Re}\{\gamma\} < 0$, which is complementary to the evanescent TE mode case. Again the regions of convergence for the TM modes are plotted in the complex $\gamma$-plane in Figure 3.3(b). This is a situation unique to the Maxwell equations. Unfortunately, this complicates the convergence behavior of a stationary iteration DD. As it is apparent from Figure 3.3, there is no region where all four different modes are convergent. In first quadrant all TE modes together with the propagating TM modes are convergent, but the evanescent TM are divergent $\rho^{TM}_{\text{evan}} > 1$; this situation is depicted in Figure 3.4(a). Similarly, in second quadrant, all TM and propagating TE modes are convergent, but the evanescent TE diverge, $\rho^{TE}_{\text{evan}} > 1$ as shown in Figure 3.1(b). The situation becomes more severe in the other two quadrants where new two sets of modes do not converge at the same time. It is interesting to observe in Figure 3.4 the duality among TE and TM as we switch quadrants, this is true wherever the locations of $\gamma$ are symmetrically positioned with respect to the imaginary axis.

Fortunately, for the case of the present paper, our primary concern is radiating applications such as antenna arrays where the primary quantities of interest are quantities related to the propagating modes. In this case, the choice where all the propagating modes
converge and $\rho_{\text{evan}}^{TE} = \rho_{\text{evan}}^{TM} = 1$ will be sufficient. Such choice was provided by Després et.al. in the original paper [39, 81]. In that work $\gamma = jk$ was used; this choice is also depicted in Figure 3.5. In this paper the same $\gamma$ has been used.

![Regions of convergence of the propagation and evanescent modes](image)

Figure 3.3: Regions of convergence of the propagation and evanescent modes: (a) TE, and (b) TM modes, for the 1st order (Robin) TCs.

### 3.3.2 Second Order Transmission Condition

In this section the transmission condition proposed by Collino et.al. [40] is considered. According to their experimental data and their experience as Absorbing Boundary Condition on FEMs, there is strong evidence that their choice would successfully work for both TE and TM modes. The operator now is approximated by
\[ S = \gamma I(\bullet) + \delta \text{curl}_t \text{curl}_t(\bullet), \quad \gamma, \delta \in \mathbb{C}, \]  
\[ (3.28) \]

where \( I \) denotes identity operator and \( \text{curl}_t \) and \( \text{curl}_t \) denote the vector and scalar tangential rotational operators [72]. With this choice Table 3.3 can be updated as:

<table>
<thead>
<tr>
<th>( \Omega_1 )</th>
<th>( \Omega_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{e}^{(n)}(z = 0^-) )</td>
<td>( \hat{e}^{(n)}(z = 0^+) )</td>
</tr>
<tr>
<td>( \text{curl} \hat{e}^{(n)}(z = 0^-) )</td>
<td>( \text{curl} \hat{e}^{(n)}(z = 0^+) )</td>
</tr>
</tbody>
</table>

\begin{align*}
\text{TE} & & \hat{u}(\cdot) & & \hat{u}(\cdot) & & \hat{u}(\cdot) & & \hat{u}(\cdot) \\
& & \hat{u}(-\mathcal{B}^{(n)} j\beta) & & \hat{u}(\mathcal{B}^{(n)} j\beta k) & & \hat{u}(\mathcal{B}^{(n)} j\beta k) & & \hat{u}(\mathcal{B}^{(n)} j\beta k) \\
\text{TM} & & \hat{v}(\cdot) & & \hat{v}(\cdot) & & \hat{v}(\cdot) & & \hat{v}(\cdot) \\
& & \hat{v}(-\mathcal{A}^{(n)} jk\beta) & & \hat{v}(\cdot) & & \hat{v}(\cdot) & & \hat{v}(\cdot) \\
& & 0 & & 0 & & 0 & & 0
\end{align*}

Table 3.3: TE and TM Fourier tangential field representations on the domain interface.

With the use of (3.28), (3.18) becomes

\[ \hat{j}_e^{(n)} - \gamma \hat{e}_e^{(n)} - \delta \text{curl} \text{curl} (\hat{e}_e^{(n)}) = -\hat{j}_e^{(n-1)} - \gamma \hat{e}_e^{(n-1)} - \delta \text{curl} \text{curl} (\hat{e}_e^{(n-1)}) \quad \text{at} \quad z = 0^- \]

\[ \hat{j}_e^{(n)} - \gamma \hat{e}_e^{(n)} - \delta \text{curl} \text{curl} (\hat{e}_e^{(n)}) = -\hat{j}_e^{(n-1)} - \gamma \hat{e}_e^{(n-1)} - \delta \text{curl} \text{curl} (\hat{e}_e^{(n-1)}) \quad \text{at} \quad z = 0^+. \]  
\[ (3.29) \]

Again after simple algebraic manipulations that involve the substitution of 0 in each interface condition of (3.29) the following result is obtained

\[ \rho^{\text{TE}}(k_z) = \frac{k_z - \gamma + \delta \beta^2}{k_z + \gamma - \delta \beta^2} \]  
\[ (3.30) \]

for the TE modes, whereas the TM modes remain same as in the first order case, namely given by equations (3.25) or (3.27). The TE convergence rate can be further written as

\[ \rho^{\text{TE}}(k_z) = \frac{k_z + \delta k_z^2 - (\gamma - \delta k_z^2)}{k_z - \delta k_z^2 + (\gamma - \delta k_z^2)}, \]  
\[ (3.31) \]

or as a function of the transverse spectral variable as:
\[
\rho^{TE}(\beta) = \begin{cases} 
\frac{j\sqrt{k^2 - \beta^2 + \delta \beta^2 - \gamma}}{j\sqrt{k^2 - \beta^2 - \delta \beta^2 + \gamma}} & k > |\beta|, \text{ propagating modes} \\
\frac{\sqrt{\beta^2 - k^2 + \delta \beta^2 - \gamma}}{\sqrt{\beta^2 - k^2 - \delta \beta^2 + \gamma}} & k < |\beta|, \text{ evanescent modes.} 
\end{cases}
\tag{3.32}
\]

Even though it is not easily recognizable, equations (3.27) for TM modes and (3.32) for TE modes can lead to convergent stationary iteration algorithm, if \( k \neq \beta \). The main idea is to choose the Robin constant \( \gamma \) in such a way that TM modes converge for both propagating and evanescent modes. This was done in previous section by choosing \( \text{Im}\{\gamma\} > 0, \text{Re}\{\gamma\} < 0 \) (refer to Figure 3.4). In the next step the parameter \( \delta \) is chosen such that both ratios in (3.32) are less than one. At this point no optimization has been tried to the second order parameter, but as it is shown in Figure 3.5(b), both propagating and evanescent modes converge favorably. In contrast, the Robin transmission condition with Després’ choice shown in Figure 3.5(a), fails to converge the evanescent modes. Note that for these two non-optimized examples, the Robin transmission condition has superior convergence for propagating modes. As it was mentioned previously, in this paper we are primarily interesting in open radiating structures, therefore only the Robin condition with Després’ choice was used in the numerical results.
Figure 3.4: Convergence rates versus spectral variable (incident field angle) of TE and TM modes for the 1st order transmission condition. (a) when $\gamma$ is location in the first complex quadrant, (b) when $\gamma$ is located in the second complex quadrant.

Figure 3.5: Convergence rates versus spectral variable (incident field angle) of TE and TM modes for (a) when $\gamma=jk$ (Despres’ choice), (b) 2nd order transmission condition.
3.4 Non-Conforming (Cement) Finite Element Method

The DDM suggested by equations (3.2) and (3.3) requires a matching grid between sub-domains $\Omega_i$ and $\Omega_j$. Practically, this requirement can tax significantly on the numerical analyst when he/she tries to analyze large finite antenna arrays, or any other large and geometrically complicated structure. In this paper, we propose a novel approach, similar to the mortar techniques employed in the literature, for DDM with non-matching grids. With the proposed technique, each sub-domain can be meshed independently without consideration of conformity to adjacent sub-domains.

![Figure 3.6: Non-conforming triangulation on either side of an inter-domain interface.](image)

With the introduction of a Lagrange multiplier representing the dual (electric current) variable

---

37
\[ \mathbf{j}_i = \gamma_i \left( \frac{1}{\mu_i} \nabla \times \mathbf{E}_i \right) \in \mathbf{H}_0^{1/2}(\text{div}, \Gamma_i) \]  

(3.33)

at interfaces \( \Gamma_1 \) and \( \Gamma_2 \), the two sub-domain problem of Figure 3.6 decouples into two independent problems. Now the decomposed BVP is compactly written as

\[
\nabla \times \frac{1}{\mu_r} \nabla \times \mathbf{E}_1^{(n)} - k^2 \epsilon_r \mathbf{E}_1^{(n)} = -j \omega \mu_0 \mathbf{J}_{\text{imp}}^{(n)}, \quad \text{in } \Omega_1,
\]

\[
\gamma' \left( \mathbf{E}_1^{(n)} \right) = 0, \quad \text{on } \Gamma_{\text{PEC}},
\]

\[
\gamma' \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}_1^{(n)} \right) = 0, \quad \text{on } \Gamma_{\text{PMC}},
\]

\[
\gamma' \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}_1^{(n)} \right) - j k \gamma' \left( \mathbf{E}_1^{(n)} \right) = 0, \quad \text{on } \Gamma_{\text{ext}}
\]

\[
\mathbf{j}_1^{(n)} - \gamma \mathbf{e}_1^{(n)} = -\mathbf{j}_2^{(n-1)} - \gamma \mathbf{e}_2^{(n-1)}, \quad \text{on } \Gamma_1,
\]

and,

\[
\nabla \times \frac{1}{\mu_r} \nabla \times \mathbf{E}_2^{(n)} - k^2 \epsilon_r \mathbf{E}_2^{(n)} = -j \omega \mu_0 \mathbf{J}_{\text{imp}}^{(n)}, \quad \text{in } \Omega_2,
\]

\[
\gamma' \left( \mathbf{E}_2^{(n)} \right) = 0, \quad \text{on } \Gamma_{\text{PEC}},
\]

\[
\gamma' \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}_2^{(n)} \right) = 0, \quad \text{on } \Gamma_{\text{PMC}},
\]

\[
\gamma' \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}_2^{(n)} \right) - j k \gamma' \left( \mathbf{E}_2^{(n)} \right) = 0, \quad \text{on } \Gamma_{\text{ext}}
\]

\[
\mathbf{j}_2^{(n)} - \gamma \mathbf{e}_2^{(n)} = -\mathbf{j}_1^{(n-1)} - \gamma \mathbf{e}_1^{(n-1)}, \quad \text{on } \Gamma_2,
\]

where

\[
\mathbf{e}_i = \gamma' \left( \mathbf{E}_i \right) \in \mathbf{H}_0^{1/2}(\text{curl}, \Gamma_i)
\]

(3.36)

is the tangential component of the electric field on interface \( \Gamma_i \). Note that the right-hand side of the transmission conditions in both (3.34) and (3.35) is a known quantity, since the \((n-1)\) fields are readily available. These known right-hand sides are defined by
$$g_i^{(n-1)} = j^{(n-1)}_{\text{neig}(i)} - \gamma e^{(n-1)}_{\text{neig}(i)},$$  

(3.37)

where $\text{neig}(i)$ indicates the neighboring domain of domain $i$. Although in principle, the triangulations in $\Gamma_i$ and $\Gamma_j$ can be drastically different, the rule of thumb is that the spatial resolutions $h_i$ (the characteristic element size in triangulation $T_{h_i}$) and $h_j$ (the characteristic element size in triangulation $T_{h_j}$) should be of the same order to assure accuracy. We cite the work of Chen et al. [82] for the rigorous analysis of such situations, as well as the discontinuous coefficient case. The present way treating non-conforming meshes differ from other approaches in two important points: (1) The Lagrange multiplier unknowns carry significant physical meanings; namely, they are proportional to the electric current density on the surfaces; and, (2) The proposed cement technique will not result in any zero diagonal blocks in the final matrix equations, thus regularizing the resulting mixed problem.

### 3.5 Galerkin Statement

To implement the domain decomposition iteration described by Eqs. (3.34) and (3.35), we need employ finite dimensional descriptions for $\Omega_i$ and $\Omega_2$. In the current implementation, we have chosen to discretize $\Omega_i$ and $\Omega_2$ into unions of tetrahedral. The corresponding spaces for the fields and the mortar variables are

$$\begin{align*}
E_1 &\in S_1^h \subset H_0^1(\text{curl}; \Omega_i) & E_2 &\in S_2^h \subset H_0^1(\text{curl}; \Omega_2) \\
J_1 &\in A_1^h \subset H^{-1/2}_1(\text{div}; \Gamma_1) & J_2 &\in A_2^h \subset H^{-1/2}_1(\text{div}; \Gamma_2)
\end{align*}$$

(3.38)

Specifically, the basis functions for $E_1, E_2$ within each tetrahedron are the $p = 2, 1^{st}$ kind Nedelec $H(\text{curl})$ vector elements. Moreover, for the mortar variables, $J_1$ and $J_2$, 39
the basis functions are the $p = 2$, 1st kind Nedelec $H(\text{div})$ vector elements on triangles.

Consequently, the corresponding Galerkin weak statements for Eqs. (3.34) and (3.35) are:

Seek \( (\mathbf{E}_1^{(n)}, \mathbf{E}_2^{(n)}) \in S_1^h \times S_2^h \) and \( (\mathbf{J}_1^{(n)}, \mathbf{J}_2^{(n)}) \in \Lambda_1^h \times \Lambda_2^h \) such that

\[
\begin{align*}
    a_1 \left( \mathbf{v}_1^h, \mathbf{E}_1^{(n)} \right) + \gamma k \left( \mathbf{v}_1^h, \mathbf{J}_1^{(n)} \right)_{\partial \Omega} + \left( \gamma k \mathbf{v}_1^h, \mathbf{J}_1^{(n)} \right)_{\Gamma_1} &= -\frac{\gamma k}{\eta} \left( \mathbf{v}_1^h, \mathbf{J}_1^{mp} \right)_{\Omega_1}, \\
    a_2 \left( \mathbf{v}_2^h, \mathbf{E}_2^{(n)} \right) + \gamma k \left( \mathbf{v}_2^h, \mathbf{J}_2^{(n)} \right)_{\partial \Omega} + \left( \gamma k \mathbf{v}_2^h, \mathbf{J}_2^{(n)} \right)_{\Gamma_2} &= -\frac{\gamma k}{\eta} \left( \mathbf{v}_2^h, \mathbf{J}_2^{mp} \right)_{\Omega_2},
\end{align*}
\]

\begin{equation}
\begin{align*}
    \left\langle \lambda_1^h, \mathbf{e}_1^{(n)} \right\rangle_{\Gamma_1} + \frac{1}{\gamma} \left\langle \lambda_1^h, \mathbf{j}_1^{(n)} \right\rangle_{\Gamma_1} &= \left\langle \lambda_2^h, \mathbf{e}_2^{(n-1)} \right\rangle_{\Gamma_1} - \frac{1}{\gamma} \left\langle \lambda_2^h, \mathbf{j}_2^{(n-1)} \right\rangle_{\Gamma_1}, \\
    \left\langle \lambda_2^h, \mathbf{e}_2^{(n)} \right\rangle_{\Gamma_2} + \frac{1}{\gamma} \left\langle \lambda_2^h, \mathbf{j}_2^{(n)} \right\rangle_{\Gamma_2} &= \left\langle \lambda_1^h, \mathbf{e}_1^{(n-1)} \right\rangle_{\Gamma_2} - \frac{1}{\gamma} \left\langle \lambda_1^h, \mathbf{j}_1^{(n-1)} \right\rangle_{\Gamma_2},
\end{align*}
\end{equation}

\[
\forall (v_1^h, v_2^h) \in S_1^h \times S_2^h \text{ and } (\lambda_1^h, \lambda_2^h) \in \Lambda_1^h \times \Lambda_2^h
\]

where the bilinear form $a_i(\mathbf{u}, \mathbf{v})$ is defined by

\[
a_i(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \left[ (\nabla \times \mathbf{u}) \cdot \frac{1}{\mu_r} (\nabla \times \mathbf{v}) - k^2 \mathbf{u} \cdot \mathbf{e} \right] \, dx.
\]

Notice that the communication or coupling between domains comes through the interfaces and more precisely through the right hand side term $g_{\text{neig}(i)}^{(n-1)} = \mathbf{g}_{\text{neig}(i)}^{(n-1)} + \gamma \mathbf{e}_{\text{neigh}(i)}^{(n-1)}$, that relates the solution of the neighboring sub domains at the previous DD iteration to the $i$th domain in the present DD iteration. The statement in (3.39) was obtained through a standard Galerkin treatment together with the scaling of the transmission equations in (3.34) and (3.35) by $-1/\gamma$. Notice that with these modifications the symmetry each sub domain problem is retained.

As a result of using a Robin type transmission condition, the resulting mixed formulation of (3.39) is a generalized one that avoids of the zero diagonal block on the dual unknowns. For the finite element discretization of the mixed problem (3.39), $p = 2$.
tetrahedral elements of the 1st Nédélec family [50] are used. After finite dimensional
discretization, the final linear system of equations is of the form

\[ \mathbf{K}_i \mathbf{u}_i^{(n)} = y_i + \mathbf{g}_i^{(n-1)} \quad \forall i = 1, \ldots, I \]  \hspace{1cm} (3.41)

where \( \mathbf{K}_i, \mathbf{u}_i, y_i \) and \( \mathbf{g}_i \) are of the form

\[
\mathbf{K}_i = \begin{pmatrix}
\mathbf{A}^{II}_i & \mathbf{A}^{IR}_i & 0 \\
\mathbf{A}^{RI}_i & \mathbf{A}^{RT}_i & \mathbf{D}_i \\
0 & \mathbf{D}_i^T & \mathbf{T}_i
\end{pmatrix},
\]

\[
y_i = \begin{pmatrix}
\mathbf{b}_i \\
0 \\
0
\end{pmatrix},
\quad \mathbf{u}_i^{(n)} = \begin{pmatrix}
\mathbf{E}_i^{(n)} \\
\mathbf{e}_i^{(n)} \\
\mathbf{j}_i^{(n)}
\end{pmatrix}
\quad \text{and} \quad \mathbf{v}_i^{(n)} = \begin{pmatrix}
\mathbf{e}_i^{(n)} \\
\mathbf{j}_i^{(n)}
\end{pmatrix},
\]

\[
\mathbf{g}_i^{(n)} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & \mathbf{D}_i^T & \mathbf{T}_i
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
\mathbf{j}_i^{(n)}
\end{pmatrix},
\]

note that all matrices involved in (3.42)-(3.44) are sparse. Note that \( \mathbf{A} \) matrices are finite
elements matrices with first order ABC on the exterior surfaces, and can be found in
standard FEM books for EM [83, 84]. Moreover \( \mathbf{D} \) and \( \mathbf{T} \) are given by

\[
\begin{pmatrix}
\mathbf{T}_{ij}
\end{pmatrix}_{mn} = \int_{\Gamma_i} \left( \mathbf{\hat{n}}_i \times \mathbf{w}_m \cdot \mathbf{\hat{n}}_j \times \mathbf{w}_n \right) dx^2,
\]

\[
\begin{pmatrix}
\mathbf{T}_i
\end{pmatrix}_{mn} = \int_{\Gamma_i} \left( \mathbf{\hat{n}}_i \times \mathbf{w}_m \cdot \mathbf{\hat{n}}_j \times \mathbf{w}_n \right) dx^2,
\]

and

\[
\begin{pmatrix}
\mathbf{D}_{ij}
\end{pmatrix}_{mn} = \int_{\Gamma_i} \left( \mathbf{w}_m \cdot \mathbf{\hat{n}}_j \times \mathbf{w}_n \right) dx^2,
\]

\[
\begin{pmatrix}
\mathbf{D}_i
\end{pmatrix}_{mn} = \int_{\Gamma_i} \left( \mathbf{w}_m \cdot \mathbf{\hat{n}}_j \times \mathbf{w}_n \right) dx^2.
\]
3.6 FETI Iterative Substructuring Algorithm

As it was pointed in the introduction, the classes of problems we consider have a large number of repeated domains. In a straightforward manner, the solution of (3.41) would require the solution of every sub-domain problem on each iteration step. An elegant way to overcome this is to rewrite (3.41) as

\[ u^{(n)}_i = K_i^{-1} y_i + K_i^{-1} g_i^{(n)} \quad \forall i = 1, \cdots I \]  

(3.49)

Before proceeding with the rest of the derivation, let’s first define two useful restriction operators: one that restricts the solution vector on inter-domain interface unknowns

\[ v = R_{DP} u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \tilde{E}_i^{(n)} \\ \tilde{c}_i^{(n)} \\ \tilde{g}_i^{(n)} \end{pmatrix}, \]  

(3.50)

and another that restricts into just the dual part of the solution,

\[ \tilde{j}_i^{(n)} = R_D u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \tilde{E}_i^{(n)} \\ \tilde{c}_i^{(n)} \\ \tilde{g}_i^{(n)} \end{pmatrix}. \]  

(3.51)

Applying the surface restriction on both sides of (3.49) results on

\[ R_{DP} u_i^{(n)} = R_{DP} K_i^{-1} y_i + R_{DP} K_i^{-1} g_i^{(n-1)}. \]  

(3.52)

Utilizing the well known property of a restriction matrix,

\[ R_D^T R_D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}, \]  

(3.53)

the following equation is obtained

\[ R_{DP} u_i^{(n)} = R_{DP} K_i^{-1} y_i + R_{DP} K_i^{-1} R_D^T R_D g_i^{(n-1)}, \]  

(3.54)
this can be compactly written as
\[ v_i^{(n)} = x_i^0 + Z_i g_i^{(n-1)} \quad \forall i = 1, \cdots, I, \]

where
\[ x_i^0 = R_{DP} K_i^{-1} y, \]
\[ Z_i = R_{DP} K_i^{-1} R_D, \]
\[ g_i = R_{DP} \tilde{g}_i. \tag{3.56} \]

The iteration scheme of (3.56) can be solved with a Jacobi or Gauss-Seidel stationary iteration method. Note that the iteration matrix is a dense one, and for its construction requires the solution of the sub-domain as many times as the dual unknown’s space kernel. Note that the construction of \( Z_i \) is independent of the iteration, thus it can be done once in the preprocessing step. For general domain decomposition this may not be an efficient way to proceed, but in the case of repeating sub-domains the computational cost can be greatly decrease.

The following algorithm summarizes the most important steps of the process.

\begin{algorithm}
Let:
I = number of domains.
K (\(<<I\)) = number of different building blocks.
M_k = number of dual unknowns of block \( k \)
TOL = a prescribed tolerance.

Let denote the row partition of matrices
\[ Z_i = \begin{bmatrix} z_1 \mid z_2 \mid \cdots \mid z_{M_i} \end{bmatrix} \quad \text{and} \quad R_D = \begin{bmatrix} r_1 \mid r_2 \mid \cdots \mid r_{M_f} \end{bmatrix} \]

1. Pre-Processing (Elimination of internal primal unknowns, generation of iteration matrix)
   for \( k = 1, K \)
   assemble \( K_k, y_k, G_{kj}, l = \text{neigh}(k) \)
   solve \( K_k \tilde{x}_k^0 = y_k \), restrict \( x_k^0 = R_{DP} \tilde{x}_k^0 \)
   for \( m = 1, M_k \)
   solve \( K_k \tilde{z}_m = r_m \), restrict \( z_m = R_{DP} \tilde{z}_m \)
\end{algorithm}
2. Solution (Gauss-Seidel iteration on interface unknowns $v$)

while error $\geq$ TOL

for $i=1,I$

initialize $v_i^{(n)} \leftarrow x_i^0$, update $g_i^{(n-1)} \leftarrow G_{ij}^{(n-1)}, l = \text{neigh}(i)$

update $v_i^{(n)} \leftarrow Z_{k(i)}^l g_i^{(n-1)}$

$$e_i^{(n)} = \frac{\|v_i^{(n)} - v_i^{(n-1)}\|}{\|v_i^{(n)}\|_\infty}$$

end

error $\leftarrow \max(e_i)$

$n++$

end

3. Post-Processing (Recovery of internal primal unknowns)

for $i=1,I$

update $g_i \leftarrow G_{ij}^{(n-1)}, l = \text{neigh}(i)$

solve $K_i \hat{x}_i = y_i + g_i$

end

### 3.6.1.1 Discussion and complexity of FETI algorithm

For the proposed algorithm the complexity of the memory, assuming $I \gg K$ is

$$\text{memory} \propto \sum_{k=1}^{K} N_k (M_k + 1) + 2M_k I \propto M_k I,$$

where $I$, $K$ and $M_k$ were defined in the above-mentioned algorithm, $N_k$ is the total number of FEM unknowns on building block $k$. Note that for large number of subdomains the memory scales proportional to the number of surface unknowns. On the other hand the computational time is

$$\text{time} \propto \sum_{k=1}^{K} (f_1 M_k + 2f_2 M_k^2 I),$$

(3.58)
where the factors $f_1$ and $f_2$ depend on the number of iterations of the inner PCG loop and outer Gauss-Seidel loop, respectively.

There is a number of attractive features inherent with the proposed methodology. The method is particularly efficient for optimizations. Namely, if one domain needs to be optimized, only that domain should be modified and independently re-meshed. In the cases of adaptive-mesh refined FEMs, the adaptive strategy can be greatly improved, since it is now based on a number of domains rather than the whole computational space. In the case of electromagnetic scattering, or array radiation, the DP-FETI algorithm greatly improves the computation, since the different incident or scanning angles do not affect the computationally intensive pre-processing step. Finally, the proposed method is extremely suitable for parallel computations.
CHAPTER 4

FEM DOMAIN DECOMPOSITION RESULTS AND NUMERICAL STUDIES

4.1 Introduction

In this chapter the numerical results for the FEM based domain decomposition described in CHAPTER 3 will be presented. Primarily, radiation and scattering problems from large arrays will be considered. On the latter part of the chapter waveguiding and cavity resonating phenomena in photonic crystals will be also studied. Most of the examples examined are real life industrial problems that involve large electrical size as well as geometric and material complexity. The full wave analysis of some problems considered here is for the first time attempted on a personal computer without parallelization.

The purpose of this chapter is to demonstrate the validity, accuracy, performance and versatility of the method. For that reason comparisons with measurements, and other already validated methods and codes will be given. For example in section 4.1.1.1, comparisons with measurements and full FEM results will be given for radiating problem. Both near and far-field quantities are considered. The accuracy and performance of the proposed methods will be compared against the multilevel fast multipole algorithm (MLFMA) for scattering problems by finite frequency selective surface and arrays. A
large part of the chapter will be devoted to the analysis of conformal array and printed antennas in both radiation and scattering mode. Even though such finite antenna arrays have been previously analyzed with spectral or spatial domain Method of Moments e.g. for printed dipoles on dielectric coated cylinders refer to [85] or for planar patches in [86], in both case an infinite dielectric substrate was considered. Here the real-life problem of an finite array on a finite-size mounting platform will be considered. To the best of our knowledge the only published work on finite printed array with finite dielectric substrates or superstrates is that of Jandhyala in [87]. The latter part of the chapter will be devoted into photonic crystal and electromagnetic band gap (EBG) structures in a cavity or wave guiding mode. Of particular interest will be the radiation of a monopole antenna over a high impedance EBG.

All computations were performed in a PC with single 2.4GHz Intel® Xeon™ processor with 512KB L2 cache and 2GB RAM. Double precision complex arithmetic is assumed. The computational codes were implemented in object-oriented C++ using the GNU g++ compiler with optimization level -O9.

4.1.1 Radiation

4.1.1.1 Radar Arrays

The first antenna radiation problem is shown in Figure 4.1(a). It consists of an array of 9×12 monopole radiators excited by a coaxial feeder at their base. A detail of each monopole and its coax-like feed is shown in Figure 4.1(b). The array resides on a finite perfect electric conductor (PEC) plate connected to 4 wedges on each side. The array is excited with constant amplitude and linear progressive phase along the x
direction. This excitation ideally results on a radiation peak at the elevation angle $\theta = 90^\circ, \phi = 90^\circ$, where $(\theta, \phi)$ are the polar angles with respect to $\hat{z}$ and $\hat{x}$ axes. The frequency of operation was assumed $f = 1.3$ GHz.

The configuration is simulated with both proposed method and full FEM. In both cases the boundaries of the computational domain are placed $\lambda/2$ away from the structure, where $\lambda = c/f$ is the wavelength in free space. The two main parameters of interest are the shape of the radiation pattern and the directivity of the antenna, which is the ratio of the peak over the total radiated power of the antenna. The x-y cut of the radiation pattern is shown in Figure 4.1(c). In the same plot, the solid red line represent the full FEM results while the circle/solid blue line represent the proposed method’s results. Much of the dominant component field agrees favorably with the full FEM. At this point it should be noted that the reference full FEM result was obtained using an h-adaptive mesh refinement process. As can be seen in Figure 4.1(c), only minor discrepancies are present at low intensity field levels. It should be noted that both full FEM and the DD method sub-domain solve accuracies were set to $10^{-3}$, whereas the outer loop for the DD method was set to $10^{-2}$. For that residual the DD method converged in 32 outer loop iterations. In Figure 4.2 the active reflection coefficient of each array element are considered. It is well-known that the computation of near-field quantities such as input impedance, and active reflection coefficient are more sensitive to errors in a simulation. As it is shown in Figure 4.2 both magnitude and phase agree favorably with full FEM. Finally, a summary of the computational statistic and other important simulation merits for both DD with FETI and full FEM are presented in Table 4.1. Here it should be noted (as it will become evident from later examples) that, for small number of
repetitions like the present array problem, DD without FETI would outperform the FETI in terms of memory savings. Despite that fact, the FETI algorithm was chosen because larger arrays of the same geometry were of interest.

<table>
<thead>
<tr>
<th></th>
<th>DD with FETI</th>
<th>Full FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Meshing Time</td>
<td>32 s</td>
<td>38 m 12 s</td>
</tr>
<tr>
<td>Total Solution Time</td>
<td>5 h 20 s</td>
<td>19 h 32 s</td>
</tr>
<tr>
<td>Memory (GB)</td>
<td>1.5</td>
<td>1.8</td>
</tr>
<tr>
<td>Directivity (dBi)</td>
<td>22.01</td>
<td>22.02</td>
</tr>
</tbody>
</table>

Table 4.1: Computational statistic of DD with FETI versus the full FEM for the 9×12 monopole array. Computations performed on a double precision arithmetic on an Intel® Xeon™ 2.4GHz processor with 512KB L2 cache and 2GB RAM.

The second array considered, consist of a 24×12 monopole array on a flat ground plane. For this experiment the proposed DD method is compared with measured data. The measured configuration is shown in Figure 4.3(a), whereas the monopole and coax feeder are shown in Figure 4.3(b). The challenge in this simulation is to accurately represent the thin flat ground plane. Even though the ground plane is thin enough (≈λ/100), it is not modeled as infinitely thin. The partitioning scheme of the domain decomposition is chosen to be slice-wise, similar to that of the Figure 4.1. As it can be shown in Figure 4.3(c) and (d), both elevation and azimuth far-field patterns agree very well with measurements. Note that the gain and directivity were predicted within 0.05dB of accuracy. The phase and magnitude of the active reflection coefficient at the coax...
ports of each array element are plotted in Figure 4.4. Before closing this section, it is interesting to stress that the full FEM model was impossible to fit in a 2GB RAM PC, whereas the DD method used only 550MB. The DD method did not utilize the FETI substructuring algorithm and it converged to a residual of $10^{-2}$ within 26 outer loop iterations. Here and in all computations the residual will be defined as

$$
\mathbf{r} = \frac{\| \mathbf{x}^{(n)} - \mathbf{x}^{(n-1)} \|_2}{\| \mathbf{x}^{(n)} \|_2}
$$

(4.1)

where $\mathbf{x}$ is the solution vector, in case of DD it involves the entire primal (volume) unknowns and the dual (interface surface) unknowns of the problem. Note that in case of the FETI the solution vector refers to the dual unknowns only. The total solution time for this problem was less than 1 h 42 m, and the computations involved 4 million total unknowns.
Figure 4.1: A 9×12 monopole antenna array mounted on a wedge-tail ground plane; (a) geometry and domain partitioning, (b) radiating monopole detail, (c) azimuth far-field radiation pattern, comparison with full FEM.
Figure 4.2: Active reflection coefficient, comparisons with full FEM. Magnitude across center elements in the (a) scanning direction (b) non-scanning direction. Phase across center elements in (c) scanning direction and (d) non-scanning direction.
Figure 4.3: DD vs. measurements for a 24×12 monopole array on a flat finite ground plane. (a) Photo of the measured configurations. (b) A detail of the monopole radiator. (c) Elevation far-field pattern. (d) Azimuth far-field pattern.
4.1.1.2 Finite Conformal Arrays

Conformal antenna arrays are of vital importance for both military and commercial applications since the large majority of the arrays mounted on aircrafts, spacecrafts, ships or missiles have to conform on the body of the vehicle for aerodynamic purposes [88-91]. Despite the common use of conformal antennas only small amount of research has been done for the development of analysis and design tools for such applications. Majority of the conformal array designs are based on planar approximations, experimental or empirical intuition and a handful of analysis methods based on rigorous Green’s function approaches that use the layered cylindrical Green’s function [92]. Among other it is worth mentioning the work of Da Silva et.al. [93], and the more resent and rigorous work of Erturk et.al. in [85]. In [85] rigorously analyzed finite arrays of axially oriented printed diploes on large cylinder using a MoM approach with hybrid
spatial/spectral domain Green’s function. The analysis is based on a computationally intensive element-by-element approach that rigorously incorporates the mutual coupling effects.

Unlike most of the existing approaches the proposed DD method is based on the FEM approach thus it is able to analyze complex antenna elements with complex feeding and excitations. More importantly because the method is not based on the knowledge of the Green’s function and its appropriate representation it works equally well for large and small cylindrical curvatures. More importantly, effects of a finite/truncated cylindrical mounting platforms and/or substrates and can be accounted for.

The conformal patch arrays considered here will be coax-fed rectangular patch arrays. The geometry of the elements can be found in the insert of Figure 4.18, whereas the elements are arranged in a rectangular periodic lattice with $0.5\lambda_0 \times 0.5\lambda_0$ period. The radius of the cylindrical sector where the array is printed on is chosen $8\lambda_0$ where $\lambda_0$ is the free space wavelength. In this study a series of arrays will be analyzed in order to further understand the effects of curvature. Namely four arrays with $11\times11$, $21\times21$, $31\times31$, $41\times41$ elements are considered by keeping the same cylinder radius and increasing the cylinder coverage by adding elements in both axial and circumferential directions. The four cylindrical array arrangements are shown in Figure 4.5 on a perspective and side view. Each array and its planar counterpart are analyzed in the radiating mode under various scan angle excitations. The near fields at a surface $1.5\lambda_0$ around the antenna are shown in Figure 4.6 for the $21\times21$ patch array under $0^\circ$, $30^\circ$ and $60^\circ$ scan in the circumferential H-plane. Under the same scan directions the near fields of the $31\times31$ and $41\times41$ arrays are shown in Figure 4.7 and Figure 4.8, respectively.
Figure 4.5: Various patch array arrangements printed on a finite cylindrical grounded dielectric slab. (a) 11×11. (b) 21×21. (c) 31×31. (d) 41×41.
Figure 4.6: Near fields for a $21 \times 21$ patch array on finite cylindrical grounded dielectric. (a) $0^\circ$ scan. (b) $30^\circ$ scan. (c) $60^\circ$ scan.

Figure 4.7: Near fields for a $31 \times 31$ patch array on finite cylindrical grounded dielectric. (a) $0^\circ$ scan. (b) $30^\circ$ scan. (c) $60^\circ$ scan.
It is more insightful to make a comparison between the present cylindrical patch arrays with their planar counterparts. In Figure 4.9 the H-plane far field patterns of an 11×11 array are considered. Both planar (blue line) and cylindrical (red line) are plotted together for 0°, 30°, 60° and 90° H-plane span. In both planar and cylindrical arrays a simple uniform amplitude and appropriate phasing was employed. Under all four excitations the effect of the curvature is apparent predominately in the side lobes especially around grazing angles. For broadside excitation the main beam region is mainly unaffected even though a close look will reveal a 0.16 dB peak directivity reduction due to the curvature beam spreading. This effect will become apparent for larger arrays. Notice that there is a slight shift in the location of the side lobe peaks and nulls between the planar and cylindrical array. These effects are expected since each patch element pattern in the cylindrical array has different peak radiation direction, namely at an angle normal to the cylinder tangent plane That leads to spreading of the
overall far-field pattern that translates to spreading of the main beam and shift of the sidelobes. Notice that in order to compensate this spreading effect elaborate pattern synthesis algorithms have to be developed that incorporate both mutual coupling and element orientation. The broadside and H-plane scanned far-field patterns of the 21×21 array configurations are shown in Figure 4.10, while the 31×31 and 41×41 are plotted in Figure 4.11 and Figure 4.12, respectively. It is observed that as the size of the array increases the peak directivity of the cylindrical array increases much less compared to the increase in the peak directivity of the planar counterpart. The effect is more dominate for close to broadside scan angles. In all four figure clusters the effects of the edge diffractions from the finite platform are visible close to grazing angles in the form of highly oscillating ripple. Another interesting observation concerns the directivity front-to-back ratio at broadside excitation. In every arrangement the front-to-back ratio of the planar configuration was in the order of 32 dB and always less that the equivalent of the cylindrical one. Specifically the ratio for the cylindrical arrays was 32.3, 40.5, 37.2 and 36.8.

In order to further quantify and understand the trends, differences and similarities between planar and cylindrical arrays, a table with the peak directivities and angle of maximums is given in Table 4.2. The first and most prominant observation concerns the difference between the peak directivities of planar and cylindrical arrays as the size increases. The planar array almost always have higher directivity (except at 90° for 31×31 and 41×41 arrays), and more importantly as the size increase the difference becomes larger. This behavior can be explained using the array effective aperture concept. It is known that the directivity is directly proportional to the projected electrical area of
the antenna from the angle of interest (scan angle). In the planar arrangement the
effective aperture quadruples every time the array size in each direction doubles. This
effect is observed if one compares the broadside directivity difference between 21×21
and 11×11 from Table 4.2. In theory it should be four times larges, thus 6 dB higher
whereas in practice is 5.11 dB. In case of 21×21 and 41×41 the computed number is 5.29
dB. Both numbers are close to the ideal 6dB, but slightly smaller since mutual coupling
effects and edge diffractions from the array and dielectric edges have not been taken into
account in the simple projected effective aperture approximations. Using the same
effective aperture principle, it can be seen from Figure 4.5 that the projected area of the
cyllindrical sector will not quadruple every time the elements are double in each direction
due to the curvature. From Figure 4.5 and some elementary calculations involving the
ration of the projected areas of the 11×11 and 21×21 arrays it results that the directivity
of the 21×21 should be 32.9 dB, whereas the calculations predict 31.47 dB. The
difference can be attributed to diffraction and mutual coupling. If the same calculation is
done for the 41×41, the approximation will lead to 36.1 dB which is almost 6 dB more
that the full wave computations. To explain these discrepancies somebody has to
compensate the effective aperture theory for the misalignment of the element radiation
pattern, that lead to the extra beam spreading factor as it was discussed above. On the
other hand, for grazing angles the cylindrical arrays should radiate better since the
effective area is larger than that of the planar ones. The effect is clearly observed in the
last row of Table 3.1.
Figure 4.9: Far-field patterns for 11×11 patch arrays printed on a finite planar and cylindrical grounded dielectric. (a) 0° scan. (b) 30° scan. (c) 60° scan. (d) 90° scan.
Figure 4.10: Far-field patterns for 21×21 patch arrays printed on a finite planar and cylindrical grounded dielectric. (a) 0° scan. (b) 30° scan. (c) 60° scan. (d) 90° scan.
Figure 4.11: Far-field patterns for 31×31 patch arrays printed on a finite planar and cylindrical grounded dielectric. (a) 0° scan. (b) 30° scan. (c) 60° scan. (d) 90° scan.
Figure 4.12: Far-field patterns for 41×41 patch arrays printed on a finite planar and cylindrical grounded dielectric. (a) 0° scan. (b) 30° scan. (c) 60° scan. (d) 90° scan.
<table>
<thead>
<tr>
<th></th>
<th>0° scan</th>
<th>30° scan</th>
<th>60° scan</th>
<th>90° scan</th>
</tr>
</thead>
<tbody>
<tr>
<td>11×11</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Planar</td>
<td>27.2 (0°)</td>
<td>21.78 (30°)</td>
<td>16.72 (57°)</td>
<td>10.89 (71°)</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>27.13 (0°)</td>
<td>21.67 (29°)</td>
<td>16.36 (57°)</td>
<td>10.77 (71°)</td>
</tr>
<tr>
<td>Difference</td>
<td><strong>0.16</strong></td>
<td><strong>0.11</strong></td>
<td><strong>0.36</strong></td>
<td><strong>0.12</strong></td>
</tr>
<tr>
<td>21×21</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Planar</td>
<td>32.34 (0°)</td>
<td>24.60 (30°)</td>
<td>18.76 (59°)</td>
<td>11.10 (75°)</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>31.47 (0°)</td>
<td>24.00 (29°)</td>
<td>18.27 (58.5°)</td>
<td>10.91 (82°)</td>
</tr>
<tr>
<td>Difference</td>
<td><strong>0.87</strong></td>
<td><strong>0.6</strong></td>
<td><strong>0.49</strong></td>
<td><strong>0.19</strong></td>
</tr>
<tr>
<td>31×31</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Planar</td>
<td>35.42 (0°)</td>
<td>26.31 (30°)</td>
<td>20.11 (595°)</td>
<td>10.35 (78°)</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>32.70 (0°)</td>
<td>24.86 (30°)</td>
<td>19.76 (59°)</td>
<td>12.39 (87.5°)</td>
</tr>
<tr>
<td>Difference</td>
<td><strong>2.72</strong></td>
<td><strong>1.45</strong></td>
<td><strong>0.35</strong></td>
<td><strong>-2.04</strong></td>
</tr>
<tr>
<td>41×41</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Planar</td>
<td>37.63 (0°)</td>
<td>27.51 (30°)</td>
<td>21.13 (59.7°)</td>
<td>8.08 (76.3°)</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>30.97 (0°)</td>
<td>25.33 (29.7°)</td>
<td>20.56 (59.7°)</td>
<td>13.89 (87.7°)</td>
</tr>
<tr>
<td>Difference</td>
<td><strong>6.66</strong></td>
<td><strong>2.18</strong></td>
<td><strong>0.57</strong></td>
<td><strong>-5.81</strong></td>
</tr>
</tbody>
</table>

Table 4.2: Peak directivity in dB versus scan angle for both planar and cylindrical arrays under consideration. The number inside the parenthesis indicates the angle where the peak directivity occurs.

It is now time to study the input impedance characteristics of the two array configurations. Before start describing the results it is important to stress that the antenna is fed with a 50 ohm coaxial cable in all cases, no broadside matching was attempted. The active reflection coefficient magnitude at the center rows and column elements of 11×11, 21×21, 31×31, 41×41, and infinite planar arrays are plotted in Figure 4.13-Figure 4.16 for
$0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$, respectively. It should be noted that the infinite array results were computed using the theory described in the next chapter. As it will become clear there, the computational mode of the element in both infinite array simulations and DD with FETI were identical in terms of geometry, excitation, and meshing. In the left-hand-side (subfigures (a) and (c)) of each figure cluster the cylindrical array active reflection coefficient magnitude is plotted. Next to those, in the right-hand side, the equivalent plots for the planar array are given. Moreover, the top subfigures ((a) and (b)) refer to the center elements along the E-plane (non-scanning/cylinder axis plane), whereas the bottom subfigures refer to the center elements parallel to the H-plane or scanning plane. Referring to the planar array results, it is apparent that in all cases but the $90^\circ$ scan the as the array size increases the reflection coefficient for the majority of the interior elements approach the infinite array limit. Depending on the excitation the edge element effect can affect one row of elements in case of $0^\circ$ scan, or more in case of scanning more than $60^\circ$. The discrepancies between the infinite array predictions and the large finite ones at $90^\circ$ scan can be explained from the strong surface waves exited by the finite dielectric slab truncation edges. The active reflection coefficients of the cylindrical arrays resemble little the planar ones. The observation becomes more prominent for the elements along the cylinder circumference. Apart from the broadside case, the active reflection coefficient for all cylindrical arrays tend to look similar to the planar one, along the axial elements (because they are planar), but the values tend to be smaller. A possible explanation can be the reduced coupling effects per element in the circumferential direction due to the curvature. Along the circumferential elements the reflection coefficient tends to vary over a wide range of values, this makes the matching of cylindrical arrays difficult since every
element row needs to be matched differently, increasing the overall array cost and feeding-network size. The possible explanation of the large differences among circumferential elements can be the different environment (with respect to the excitation) that each element experiences. For example in case of 60° scan the elements directed towards that scanning angle experience a much different electromagnetic environment (fields) than the elements on the back side of the 60°.

Before closing this section it is useful to demonstrate the efficiency and versatility of the DD method with FETI. In these runs only one element block and one air block were discretized independently for each array configuration, namely planar and cylindrical. The meshing and creation of the FETI “transfer function” matrix $Z$ was created only once in the preprocessing and it took total of approximately 2.6 hours to assemble for each configuration. The mesh of each antenna element was discretized with an initial mesh of $h=\lambda_0/4$ and 9 $h$-adaptive mesh refinements with 0.038 error based on the indicators and adaptive process described in [94]. The total number of unknowns, memory and domain decomposition outer loop iterations are documented in Table 4.6 for the various configurations and array sizes. Among all the different excitations the maximum solution time was 2.9h for the 11×11, 8.1h for the 21×21, 36.4h for the 31×31 and 33.6h for the 41×41 cylindrical configurations.
Figure 4.13: Magnitude of the active reflection coefficient under $0^\circ$ H-scanning for different size cylindrical and planar arrays. (a) Along the center elements parallel to the E-plane (axially) of cylindrical arrays. (b) Along the center elements parallel to the E-plane of planar arrays. (c) Along the center elements parallel to the H-plane (circumference) of cylindrical arrays. (d) Along the center elements parallel to the H-plane of planar arrays.
Figure 4.14: Magnitude of active reflection coefficient under 30° H-scanning for different size cylindrical and planar arrays. (a) Along the center elements parallel to the E-plane (axially) of cylindrical arrays. (b) Along the center elements parallel to the E-plane of planar arrays. (c) Along the center elements parallel to the H-plane (circumference) of cylindrical arrays. (d) Along the center elements parallel to the H-plane of planar arrays.
Figure 4.15: Magnitude of active reflection coefficient under 60° H-scanning for different size cylindrical and planar arrays. (a) Along the center elements parallel to the E-plane (axially) of cylindrical arrays. (b) Along the center elements parallel to the E-plane of planar arrays. (c) Along the center elements parallel to the H-plane (circumference) of cylindrical arrays. (d) Along the center elements parallel to the H-plane of planar arrays.
Figure 4.16: Magnitude of active reflection coefficient under 90° H-scanning for different size cylindrical and planar arrays. (a) Along the center elements parallel to the E-plane (axially) of cylindrical arrays. (b) Along the center elements parallel to the E-plane of planar arrays. (c) Along the center elements parallel to the H-plane (circumference) of cylindrical arrays. (d) Along the center elements parallel to the H-plane of planar arrays.
<table>
<thead>
<tr>
<th>Array Size</th>
<th># Unknowns Cylindrical</th>
<th># Unknowns Planar</th>
<th>Memory (MB)</th>
<th>#Iteration (tol=10^{-1})</th>
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<td>6.3 million</td>
<td>6.9 million</td>
<td>680</td>
<td>56</td>
</tr>
<tr>
<td>21×21</td>
<td>23.9 million</td>
<td>25.3 million</td>
<td>731</td>
<td>62</td>
</tr>
<tr>
<td>31×31</td>
<td>50.2 million</td>
<td>55.0 million</td>
<td>794</td>
<td>65</td>
</tr>
<tr>
<td>41×41</td>
<td>87.8 million</td>
<td>96.2 million</td>
<td>853</td>
<td>68</td>
</tr>
</tbody>
</table>

Table 4.3: Summary of the computational requirements of the conformal array radiation simulations using the DD with FETI. All computations were performed in double precision arithmetic.

### 4.1.2 Scattering

#### 4.1.2.1 Finite Frequency Selective Surfaces

The scattering from a finite frequency selective surface on a finite PEC backed dielectric is considered next. The geometry and dimensions of the scatterer are show in Figure 4.17(a). The bistatic RCS comparisons of the proposed DD with the PMCHWT method of moments formulation utilizing MLFMA acceleration of Jandhyala et.al reported in [87] are shown in Figure 4.17(b). The structure is under a normal plane wave incidence, with respect to the plane of the FSS. Good agreement is obtained in almost the whole angle spectrum. It is believed that some of the discrepancies in angles around 60° are due to a small difference in the size of the ground plane between FETI and MLFMM simulations. Finally, the near-field distribution in the surface 1\(\lambda\) away from the FSS is plotted in Figure 4.17(c). The corner end edge diffraction effects of both FSS and finite dielectric truncations are prominent. The summary of the computational statistics of this example are provided in Table 4.4. It should be noted that for the present example the MLFMA solution required 1.2 GB of memory which is comparable to that of the
proposed method. It should be noted that the MLFMA method used only 120 thousand first order surface unknowns whereas the proposed method topped the 10 million second order volume FEM unknowns.

Figure 4.17: A finite frequency selective surface with 11×11 elements, on finite dielectric grounded slab. (a) FSS arrangement and detail of the element. (b) Bistatic RCS, comparisons with MLFMA (PMCHWT). (c) Near-fields.

<table>
<thead>
<tr>
<th># Domains</th>
<th># Ununknowns</th>
<th># Dual Unknowns</th>
<th>Memory (MB)</th>
<th>Z assemble Time (hh:mm:ss)</th>
<th>Total Time (hh:mm:ss)</th>
<th>#Iterations</th>
</tr>
</thead>
<tbody>
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<td>440</td>
<td>11.3 million</td>
<td>3.2 million</td>
<td>1,239</td>
<td>03:22:03</td>
<td>02:59:41</td>
<td>28</td>
</tr>
</tbody>
</table>

Table 4.4: Summary of the computational requirements of the finite FSS scattering simulations using the DD with FETI. All computations were performed in double precision arithmetic and an Intel® Xeon™ 2.4GHz processor with 512KB L2 cache and 2GB RAM.
4.1.2.2 Finite Planar Patch Arrays

In this section the scattering behavior of a planar finite square patch array on a
finite grounded dielectric substrate is studied. The same geometry, but with infinite in
extent substrate, has been analyzed before by Pozar in [86]. The array and its geometric,
material and excitation details is shown if Figure 4.18. In the same figure a detail of the
coax-fed patch is also shown. It is important to stress the very detailed geometric
modeling of the finite thickness probe and coax since effects of the loading on the RCS
will be presented. For the same reason each element is meshed with an initial mesh of
$h=\lambda_0/4$ and 8 $h$-adaptive mesh refinement (AMR) steps. Majority of the extra $h$-AMR
unknowns are added at the regions around the radiating slots of the patch and the
sensitive region of the coax. Three different array sizes will be considered. The
configurations along with the domain decomposition scheme are shown in Figure 4.19.
The bistatic scattering of the 11×11, 21×21 and 31×31 arrangements under E-polarized
incident plane wave in the H-plane ($\phi=90^\circ$) are shown in Figure 4.19 (a), (b) and (c) for
$0^\circ$, $30^\circ$ and $60^\circ$ incident elevation angles, respectively. Before proceeding into the
interpretation of these results it is necessary to highlight that each array coax is
terminated into a 50 ohm load. With that in mind, the results of Figure 4.19 can be new
interpreted. As the size of the array increases, especially for incident angles close to
broadside, the backscattering decreases. For the moment this may not sound correct, but
since each array element is matched into a load and because a ground place is present,
these results make sense. For a very thorough and deep understanding of such phenomena
the interested reader is referred to the recent book by Munk on finite antenna arrays [95].
A very simplified explanation comes from the fact that: the array is not completely
mismatched, thus a portion of the incident energy will be scattered by the structure, yet another part will be absorbed by the antenna elements on the load. The net result will be reduced backscattering levels at the secular angles. As it is expected the minimum backscattering should occur for the broadside incident since the matching of the array elements is better. On the other hand, as the incident wave impinges from grazing the array size does not affect any more since the elements are almost completely mismatched, thus the structural scattering of the configuration dominates. Note that the structural scattering is small because the projected optical area for these angles of incidence is small.

In the same line of thought, a second series of experiments is performed. This time the array size is kept fixed to $21 \times 21$ elements, but the loading and feeding structure are changing. First the patch configuration is considered without any probe and coaxial feed, and then the load is considered $0, \infty, 50 \, \text{ohm}$ and conjugate matched. The meaning of conjugate matched should be understood as conjugate with respect to the matching of an infinite array of the same elements under broadside excitation. Such conjugate load condition is found by simulating the antenna element structure using the method proposed in CHAPTER 5. The resulted impedance is found $Z_{c.m.} = 41.2143 + j32.2626 \, \Omega$. To be precise and to achieve better RCS reduction, the full finite array problem should be first solved by exciting all elements with $0^\circ$ scan and then find the appropriate conjugate matching for each element. Since this approach is time consuming and in practice expensive this is avoided. Instead the infinite array approximation for the conjugate matching condition is used. As it was shown in section 4.1.1.2, for array larger than $11 \times 11$ elements and angles close to broadside the finite array active reflection coefficient matches very well the infinite one. Here it should be emphasized that for the above
statement to be true the array periodicity should be kept larger than 0.3 λ such that array
guided surface waves are not excited [95-98]. Only few rows of edge elements deviate
from the infinite array result. The effect of the loading is shown in Figure 4.21 for 0°, 30°
and 90° incident angles. As it was expected the backscattering is significantly reduced in
the broadside case by 17 dB, when the array is conjugate matched. In theory if the array
was infinite or the edge effects where negligible or appropriately treated, the
backscattering can be reduced to almost zero. The cases of short and open loadings are
almost similar to the no probe results. The only minor differences occur at the grazing
angles. As the scan angle increases, as shown in (b) and (c), the loading effects become
less prominent. Finally at 90° incidence the structure without probe results into lower
backscattering RCS. This is understandable since for this angle and wave polarization the
thick probes are visible to the incident wave.

Since the antenna is single polarized, it was found interesting to study the effects
of the incident wave polarization. Till this point an E-polarized wave in the φ=90° plane
was considered. Since the antenna in linearly polarized it is natural that only the co(E)-
polarization will be absorbed under appropriate matching conditions. Throughout this
study the conjugate matching condition described in the previous paragraph is used. On
the other hand in a cross (H)-polarized wave will be completely reflected since the
antenna is incapable of receiving such polarization. That very behavior is observed in the
results of Figure 4.22 for the φ=0° observation plane in the left and the φ=90° observation
plane in the right. Among others, one of the very interesting phenomena can be observed
in Figure 4.22(f), where the incident angle is at grazing. The E_θ incident polarization is
completely reflected back while the E_φ-polarization is not. This is because the E_φ-
polarization interacts strongly with the vertical polarized probes in the structure; whereas the other polarization is normal to the probes thus it interacts weakly. Finally the computational statistics for the present set of simulations with the DD method with FETI are presented in Table 4.5.

Before closing this section the DD method with FETI acceleration will be used to study the effect of the array shape when mounted on the same grounded dielectric slab. The two array configurations are shown in Figure 4.23. On the left side a square array with lateral size of 31 elements is plotted whereas on the right a 31 diameter circular shape array of the same patches is considered. The bistatic scattering patterns under broadside incident angle and H-pol wave of both configurations is depicted in Figure 4.24. Both arrays are conjugate matched. Since the number of elements of the circular array is less the energy absorbed at the loads will be less, thus the backscattering will be higher. The difference in the broadside direction is approximately 10dB. Moreover the overall shape of the bistatic pattern is considerably different.
Figure 4.18: A coax-fed finite patch array on finite grounded dielectric slab. The array is exposed to an incident plane wave in the H-plane. (a) Perspective view and dimensions. (b) Side view, incident wave angle/polarization and loading.
Figure 4.19: Finite patch arrays and domain partitioning for the scattering experiments. (a) 11×11. (b) 21×21. (c) 31×31.
Figure 4.20: Effect of the array size on the bistatic radar cross section under various incident angles. (a) 0° incident angle. (b) 30° incident angle (c) 90° incident angle.
Figure 4.21: Effect of array loading on the bistatic radar cross section under various incident angles. (a) 0° incident angle. (b) 30° incident angle (c) 90° incident angle.
Figure 4.22: Effect of the incident field polarization on a 21×21 patch array. (a) Incident angle 0°, observation at E-plane. (b) Incident angle 0°, observation at H-plane. (c) Incident angle 30° on H-plane, observation at E-plane. (d) Incident angle 30° on H-plane, observation at H-plane. (e) Incident angle 90° on H-plane, observation at E-plane. (f) Incident angle 90° on H-plane, observation at H-plane.
<table>
<thead>
<tr>
<th>Array Size</th>
<th># Unknowns</th>
<th># Dual Unknowns</th>
<th>Memory (MB)</th>
<th>#Iteration (tol=5×10^{-2})</th>
</tr>
</thead>
<tbody>
<tr>
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<td>9.4 million</td>
<td>2.2 million</td>
<td>981</td>
<td>44</td>
</tr>
<tr>
<td>21×21</td>
<td>25.6 million</td>
<td>4.9 million</td>
<td>1072</td>
<td>43</td>
</tr>
<tr>
<td>31×31</td>
<td>50.2 million</td>
<td>9.1 million</td>
<td>1202</td>
<td>43</td>
</tr>
</tbody>
</table>

Table 4.5: Summary of the computational requirements of the patch array scattering simulations using the DD with FETI. All computations were performed in double precision arithmetic.
Figure 4.23: Finite patch arrays with 31 elements in the maximum lateral direction, mounted on the same grounded dielectric platform. (a) Rectangular arrangement. (c) Circular arrangement.

Figure 4.24: Effect of the patch array shape on the bistatic RCS.
4.2 Photonic and Electromagnetic Band Gap Structures

In this section we present some real life problems solved with the proposed algorithm. Three different applications were chosen, the light confinement inside photonic crystal nanocavities, the wave guiding in sharp bend PBG channel waveguide, and the monopole radiation in the present of an EBG based Artificial Magnetic Conductor (AMC). Through these examples, the convergence, accuracy and computational properties of the proposed method are illustrated.

All examples involve the solution of 3D unbounded Maxwell’s problems using the first order vector Absorbing Boundary Condition (ABC) were used to approximate the radiation condition. Always the truncation boundary was placed at least $2\lambda_0$ way from any structure.

4.2.1 PBG nanocavity

The example used in this section has been previously analyzed using the FDTD method in [99]. A PBG nanocavity is formed in a finite thickness 2-D photonic crystal through a defect or impurity. As it is shown in Figure 4.25, a GaAs substrate is drilled with air holes on a triangular stencil. These holes at certain frequencies act like barriers, so the light can not propagate through the structure. In the center of the substrate, a single defect is created by leaving the substrate intact. This forms a nanocavity that traps the light inside. This particular arrangement is widely used in laser technology. In this example it will be shown that the trapping of the light gets more and more effective as the number of air hole layers around the nanocavity increases. Unfortunately, this increases the computational domain size, thus the memory and time requirements. The present
method is ideal for analyzing this type of problems, since the preprocessing cost is paid only once, regardless of the number of layers around the nanocavity. This structure was simulated using only two building blocks, as shown in Figure 4.25(a). These are the defect block that contains only the GaAs substrate, the other building block is the air-cylinders.

Figure 4.25: A 3D-laser nanocavity surrounded by a photonic crystal. (a) Geometry, domain partitioning, material and excitation configuration. (b) Magnitude of the electric field at the mid-plane, white, yellow, red, blue and green indicate the field strength in deciding order.

The magnitude of the electric field distribution on the transverse mid-plane of the substrate is plotted in Figure 4.25(b). The lowest cavity mode is excited with a y-directed
infinitesimal dipole embedded at the center of the GaAs substrate. The field plot resembles the one plotted in [99], only a limited field concentration is achieved because the number of photonic crystal layers is small. To further confine the near-fields in the defect region, 7 and 14 layer structures are considered. The mid-plane section of each structure is shown in Figure 4.26(a) and (c), together with the domain partition. Same frequency and excitation are assumed. The magnitude of the electric field is shown in Figure 4.26(b) and (d) for the 7 and 14 layer nanocavity, respectively. It is clear that the field is trapped tighter around the defect as the number of air-cylinder layers increases.
The computational statistics of the DD method are tabulated in Table 4.6 for the three simulations. In the first column of the table the first number represent the number of PBG layers, whereas the second is the number of buffer substrate layers before the

Figure 4.26: Top view and field distributions on 3D laser nanocavities. (a) A 7 layer PBG nanocavity. (b) Magnitude to the electric field on the transverse mid-plane of the 7 layer nanocavity. (c) A 14 layer PBG nanocavity. (d) Magnitude of the electric field at the transverse mid-plane of the 14 layer nanocavity.
truncation boundary. Among others, it is interesting to observe the memory consumption; in all cases the memory was virtually unaffected, since it scales linearly with the number of dual unknowns. On the contrary, the solution time dramatically increases as the problem size increases.

<table>
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<th># Layers</th>
<th># domains</th>
<th># dual unknowns</th>
<th># total unknowns</th>
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<th>Z assemble (hh:mm:ss)</th>
<th>Solve (hh:mm:ss)</th>
</tr>
</thead>
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<td>29,900</td>
<td>510,000</td>
<td>131</td>
<td>00:31:25</td>
<td>00:01:07</td>
</tr>
<tr>
<td>7+3</td>
<td>331</td>
<td>78,100</td>
<td>2,252,100</td>
<td>142</td>
<td>00:31:25</td>
<td>00:06:22</td>
</tr>
<tr>
<td>14+3</td>
<td>919</td>
<td>216,800</td>
<td>8,365,700</td>
<td>156</td>
<td>00:31:25</td>
<td>03:12:41</td>
</tr>
</tbody>
</table>

Table 4.6: Computational statistics of the proposed method for the PBG nanocavity examples. The computations were preformed on a single 2.4GHz Intel® Xeon™ processor with 512KB L2 cache and 2GB RAM.

The convergence of the Gauss-Seidel iteration is plotted in Figure 4.27 for the three nanocavity examples. The three examples 3, 7 and 14 layer nanocavities are plotted with blue dashed-dotted, green solid, and red dashed lines, respectively.
The second example simulated was a PBF waveguide bend. PBGs are known to guide the light if a defect channel is formed in the crystal [100, 101] and [102]. In this example a 60º bend is studied, the geometry same as the one used in [99]. A cross-section of the geometry is shown in Figure 4.28(a), where the white circles represent the air holes drilled in the GaAs substrate (red). The radius of each cylindrical hole is \( r = 0.44a \) whereas the height of the substrate is \( h = 0.44a \); note that \( a \) is the
characteristic length of the triangular grid and is equal to \( a = 0.42\lambda_0 \). The free space wavelength \( \lambda_0 \) is computed at frequency \( f_0 = 140 \) THz. In this frequency both TE and TM surface waves are in the bandgap region. The field is excited with a \( y \)-directed infinitesimal dipole, as shown in Figure 4.28(a). The magnitude of the electric field at the mid-plane of the GaAs substrate is plotted in linear scale in Figure 4.28(b). It is apparent that much of the energy has propagated at the output in the top right side. At this point it should be pointed out that almost the same near-field pattern have been found in [99] using a full FDTD simulation. To gain more insight, the near fields in the air region 1.5\( \lambda_0 \) above the PBG structure, are plotted in Figure 4.29. It is apparent that the fields are strong at the region right above the abrupt bend discontinuity. This is expected since discontinuities are known to give rise to higher-order (below waveguide cut-off) modes that translate to radiation effects. Notice that in this example the radiation leakage shown in Figure 4.29 is undesirable, because it acts like a loss mechanism, thus reduces the transmission of the bend. Note that such effects cannot be accounted for with the fast and widely used 2D simulations.
Figure 4.28: A 60° PBG waveguide bend. (a) Cross-section of the geometry, and excitation. (b) Magnitude of the electric field in the mid-plane. White, yellow, red, blue and green are the color map color in descending order.
Figure 4.29: Near-field distribution on the surrounding air region of the PBG. In this figure a top view of the magnitude of the electric field is plotted in the air region $1.5\lambda_0$ above the PBG mid-plane.
4.2.3 Monopole above artificial magnetic conductor (AMC) ground plane

In this section the radiation of a monopole in the presence of an electromagnetic band gap surface will be considered. The experimental setting has been inspired from the high-impedance surfaces described in the work of Sievenpiper in [103, 104]. As meticulously described in [104], metalo-dielectric electromagnetic bandgap structures operating at the bandgap region, act as high-impedance surfaces, in some cases and narrow frequency bands they approximate artificial magnetic conductors (AMC).

The arrangement, dimensions and materials of the monopole and AMC surface are shown in Figure 4.30. For the sake of simplicity only a small portion of the AMC ground plane is shown in Figure 4.30(a), from the top view, and a cross section, on the bottom for the figure. In the same figure the domain partitioning scheme used in the DD simulations is also depicted. The monopole is fed by a 50 Ω coaxial cable at the back of the ground plane. The operating frequency is set at $f=35$ GHz. A close-up top view of the tri-legged mushroom-like metallic protrusions is depicted in the right side of Figure 4.30. Note that the black circle in the center indicates the location of a thick shorting cylindrical pin. The cell and dimensions have been chosen such that the band gap is around the operation frequency. The actual lateral size of the ground plane is 50mm×50mm as showing in Figure 4.31. The surface contains 1455 tri-legged protrusions arranged in a hexagonal lattice.

For the domain decomposition computation the entire geometry is enclosed in a rectangular box at least 1.5 $\lambda$ away from any geometry feature, where $\lambda$ is the free space wavelength. The entire computational domain is decomposed into three building blocks, the monopole block, the AMC cell block, and air block surrounding the ground plane.
laterally. Each domain in the geometry is an instance of these building blocks. Each building block is meshed independently with an initial mesh of $h=\lambda/10$, second order tangential vector finite elements on tetrahedrons. Moreover, the final mesh for the monopole and AMC cells were obtained through 6 steps of h-version adaptive mesh refinements [94].

Figure 4.30: Geometry, domain partitioning and material parameters of a monopole mounted on an high impedance surface. On the left a detail of the metallic protrusions are also shown.
Having performed the meshing of each individual block, the domain decomposition results are obtained through the FETI algorithm. Figure 4.32 shows the magnitude of the electric fields on the surface of the AMC ground plane in the left, and in

Figure 4.31: Top view of the actual artificial magnetic conductor surface. The monopole is mounted at the center.
surrounding air region in the right. It is apparent that the field in the AMC ground plane is confined around the monopole, and only small values have reached the edges of the ground plane. As it is advocated in [103], the AMC surface does suppress the diffractions from the edges of the ground plane, because the surface waves excited in the AMC are in the band gap region. Before establishing a more concrete view by plotting the far-field pattern, it would be of value to compare the above result with a PEC ground plane of the same size. The near fields at the same locations as in the AMC example are plotted in Figure 4.33. It is apparent in Figure 4.33(a) that relatively large values of the field reach the edges of the PEC ground plane and strongly diffract. The same effect is visible in the plot of Figure 4.33(b).

To assess the accuracy of the proposed method, the computational domain of the PEC ground plane is analyzed with a brute-force FEM with second order tetrahedral elements. Unfortunately, the AMC ground plane problem is impossible to be solved in a 2 GB memory computer with full FEM, even for an initial mesh of \( h \approx \lambda/2 \). For that reason only the PEC ground plane will be considered in the comparisons. The initial FEM mesh is discretized with \( h \approx \lambda/6 \). The far-field comparisons of the proposed DD with FETI and the full FEM are plotted in the elevation pattern. The two patterns agree favorably for the entire angle spectrum. As for the near field quantities, the proposed method predicted reflection coefficient of

\[
S_{11} = 0.74 \angle 8.8^\circ, \tag{4.2}
\]

whereas the full FEM predicted

\[
S_{11} = 0.73 \angle 2.1^\circ. \tag{4.3}
\]
It is apparent that both methods agree very well, and it is believed that the proposed DDM may lead into more accurate results due to the better discretization.

Figure 4.32: Currents and near fields on AMC ground plane under monopole excitation. (a) Magnitude of the electric current on AMC. (b) Magnitude of the electric field 1.5\(\lambda\) above the AMC surface.
Figure 4.33: Surface current and near-fields on the top of a monopole mounted on a PEC ground plane. (a) Electric current magnitude on the PEC plate. (b) Electric field magnitude $1.5\lambda$ above the PEC plate.
Before closing this section it is very interesting to present some of the computational statistics of the simulations performed. A summary of the memory and computational time are given in Table 4.7 Note that for such a complicated geometry and fine geometrical features, the number of unknowns almost topped 40 million. It is also important to point that only 1 GB of memory was utilized. Unfortunately, due to the dense matrix-vector multiplications on the solver part, the solution time is slightly more that 24 hours. In the second and third rows of the table, the PEC ground plane example is described. It is interesting to point out that even though the number of unknowns for the DDM and full FEM are dramatically different; the amount of memory is almost the same. As for the memory and assemble time of the $Z$ matrix for this example only one building
block had to be re-constructed since the other two were readily available from the AMC example calculations. This very point is one of the most beautiful characteristics of the DD with FETI method, because it pre-solves for the EM transfer function matrix $Z$, a library of reusable blocks can be created and saved into disk, therefore saving large amount of time. Moreover, this approach is inherently suitable for geometry and material optimization, since only small domains need to be re redesigned and re-meshed.

<table>
<thead>
<tr>
<th>Example/Method</th>
<th>Domain number</th>
<th>Total unknowns</th>
<th>Dual unknowns</th>
<th>Memory (MB)</th>
<th>Z assemble (hh:mm:ss)</th>
<th>Solve (hh:mm:ss)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMC/DD</td>
<td>2083</td>
<td>38,978,438</td>
<td>5,915,946</td>
<td>1,054</td>
<td>02:02:56</td>
<td>24:01:21</td>
</tr>
<tr>
<td>PEC/DD</td>
<td>2083</td>
<td>30,740,228</td>
<td>5,598,756</td>
<td>994</td>
<td>00:33:30</td>
<td>17:42:36</td>
</tr>
<tr>
<td>PEC/FEM</td>
<td>1</td>
<td>1,114,794</td>
<td>-</td>
<td>929.4</td>
<td>-</td>
<td>00:18:43</td>
</tr>
</tbody>
</table>

Table 4.7: Computational statistics of the proposed method for the monopole radiation on the top of an AMC and PEC ground plane examples. The computations were preformed on a single 2.4GHz Intel® Xeon™ processor with 512KB L2 cache and 2GB RAM in double precision arithmetic.

Finally, the far-field pattern comparison of the two ground plane arrangements is attempted. The elevation plane of the far-field is plotted in a polar plot in Figure 4.35. The blue solid line represents the AMC ground plane, whereas the dashed red line corresponds to the PEC ground plane. The AMC ground plane design not only removes the ripple in both forward and backward directions but at the same time improves the directivity by almost 2 dB. At 90° (grazing angles) the AMC ground far-field has a 10 dB lower value than the PEC. This can be attributed to the surface wave suppression by
the AMC. To further elaborate the back-radiation reduction and diffraction effects, the near-field at the back of the ground plane are plotted in Figure 4.36. In the left (Figure 4.36(a)), the AMC fields are plotted, and found to be almost negligible levels, on the contrary the fields due to the PEC ground plane are quite strong as shown in Figure 4.36(b).
Figure 4.35: Polar plot of the elevation ($\phi=0^\circ$) far-field patterns for the monopole radiation mounted on AMC and PEC ground planes.
Figure 4.36: Reduction of the edge diffractions using AMC ground plane. (a) Near-fields 1.0λ away from the bottom of AMC ground plane. (b) Near-fields 1.0 λ away from the bottom of PEC ground plane.
CHAPTER 5

A NON-CONFORMING VARIATIONAL FORMULATION FOR INFINITE PERIODIC STRUCTURES

Periodic structures have a prominent role in electromagnetic and radio frequency (RF) regimes. They possess unique properties such as backward waves, broadband passband or stopband characteristics as well as resonance and absorption phenomena etc. Many of these unique features are utilized in practical applications such as phased arrays, photonic crystal, metamaterials such as Negative refractive index materials (NIM), filters, electromagnetic band gap (EBG) structures, frequency selective surfaces (FSS), hybrid radomes, absorbers etc. Although all practical applications of periodic strictures are finite in size, they require large number of periodic cells to function properly. Without any loss of generality, large and finite periodic structures can be very accurately approximated by infinite ones. In fact, most of the unique features of finite periodic structures are inherited from their infinite counterparts.

Apart from the practical engineering and design point of view, infinite periodic structures offer a very attractive feature to the computational scientist. The computational effort required for the analysis of infinite periodic structures, in comparison to large finite ones, is orders of magnitudes smaller. This characteristic is consequence of the Floquet’s theorem which is applicable to infinite periodic structures only. In a very simplistic
explanation, Floquet’s theorem [105] allows the reduction of an infinite periodic domain to a finite one (Floquet cell) subject to appropriate boundary conditions. In practice solving an infinite antenna array problem requires approximately the same memory as solving one element of the array alone. This very feature makes infinite periodic simulators very attractive and efficient design tools.

In the past decades, large amount of research effort has been placed in the development of both differential and integral equation (IE) based methods for infinite periodic structures. When Floquet’s theorem is used in conjunction with IE based methods such as Method of Moments (MoM) only the kernel of the IE (Green’s function) need to be appropriately modified. Namely the Green’s function is written as an infinite series in terms of Floquet harmonics [106]. On the other hand PDE based methods such as FEM, incorporate the periodicity by enforcing suitable periodic boundary conditions (PBCs). Such boundary conditions relate the electromagnetic quantities on two opposite boundaries of Floquet’s cell through an appropriate phase shift indicated by the periodicity and the excitation [107], [108]. Very often the two approaches are combined together in hybrid formulations such as Finite Element Boundary Integral (FEBI). The first FEBI approach that fully incorporated periodic boundary conditions in two dimensional problems was proposed by Gedney, Lee and Mittra in [109]. Later, three dimensional vector finite element extensions of the periodic FEBI were proposed in [110], [111]. Similar approaches have been proposed in [112] and [113] for FSS and photonic band gap (PBG) structures.

Till today, all finite element based approaches face a fundamental restriction. The finite element mesh on opposite sides of Floquet’s cell has to be exactly the same; that is
termed a *periodic mesh*. Consequently, for single periodicity two of the opposite
boundary triangulations need to be identical in order to enforce the PBCs. On the other
hand, for double and triple periodic structures the situation becomes even more restrictive
since four and six surface triangulations need to be constrained in pairs. That may sound
like a minor CAD modeling detail, but in practice is an implementation nightmare. The
constrained meshing problem becomes further more difficult if unstructured tetrahedron
meshes are involved. Moreover, such constrained meshes on complicated geometries tend
to result in poor quality factors. This can be translated into highly distorted and
anisotropic elements that result into poorly conditioned FEM matrices. Furthermore,
FEM solution becomes unreliable since approximation errors usually increase, leading to
inaccurate results. Till today, only a handful of commercial mesh generators are able to
tackle the problem of periodic mesh. It is worth mentioning that in [113] a clever yet
restrictive approach was used to by-ass the periodic meshing problem. The analysis was
based on prismatic vector finite elements that lead to analytic rectangular meshes on the
PBC interfaces. The approach is not general since only layered type geometries are
tractable.

The present chapter proposes a finite element methodology for solving infinite
periodic electromagnetic problems without the need of periodic mesh. To the best of our
knowledge, this is the first time that an infinite periodic problem is solved without
periodic meshing. The idea behind the present development is to use a mixed-hybrid
variational formulation to weakly enforce the periodic boundary conditions on the
Floquet cell. Note that the mixed-hybrid terminology used here should not be confused
with the engineering community’s use of the term hybrid. The mixed-hybrid term is more
in line with the mathematics community use of the term and implies a dual/primal variational problem [114]. The primal unknown will be considered the electric field within the entire Floquet cell, whereas the dual unknown will represent the electric current on the PBC interfaces. The enforcement of the periodic boundary conditions will be done through a linear combination of the electric field and electric current on opposite side interfaces. A “straight-forward” enforcement of the PBC will readily lead to a non-Hermitian system of equations even in the lossless case. Following the essence in the symmetric FEBI coupling of [115], [116] and [74], a similar coupling is derived for the infinite array mixed-hybrid FEM problem. The resulting system of equations leads to a Hermitian matrix for the PBC interactions in case of loss-less materials. Since this is not a symmetric system of equations iterative solution techniques other than Conjugate Gradient (CG) or Conjugate Residual Methods (CGR) need to be resorted. An efficient method for solving the matrix equation is proposed. The solution process follows an inner/outer loop iteration scheme. The outer loop is based on a memory inexpensive stationary Richardson iteration [117], with a CG inner solution loop. The method can be viewed as matrix splitting preconditioner, where the original Hermitian matrix is split into a symmetric and a Hermitian part.

5.1 Boundary Value Problem Statement

Before start describing the formal statement of the problem, it should be noted that throughout the chapter the term periodic will imply infinitely periodic (periodicity in all three principal directions) or quasi-periodic structures (periodicity on one or two directions). Referring to Figure 5.1 let $\Omega$ be a regular unbounded domain in $\mathbb{R}^3$ that
contains the scatterer or/and antenna of interest. The geometry is unbounded in the
direction of the infinitely periodicity. Moreover, as it is shown in the same figure, both
geometry and internal excitation are considered infinitely periodic with respect to a
certain direction or directions, thus

\[ \Omega = \bigcup_{m=-\infty}^{\infty} \Omega_m, \quad \Omega_j = \Omega_i \forall i, j. \]  

(5.1)

For simplicity, the present analysis will be based on a singly infinite periodic geometry as
shown in Figure 5.1. The methodology can be straightforwardly extended to double or
triple periodic structures.

Figure 5.1: A generic singly infinite periodic structure under near or far field excitation.

In the case of radiation the sources of electromagnetic fields are \( J_m^{imp} \) and \( M_m^{imp} \)
residing inside each \( \Omega_m \). On the other hand, in the scattering case the sources are assumed
to exist somewhere in \( \Omega_e = \mathbb{R}^3 \setminus \Omega \) and produce locally on \( \partial \Omega \) a plane wave of the form
\( \mathbf{E}^{\text{inc}}(\mathbf{r}) = \mathbf{E}_0 e^{-jk^{\text{inc}} \cdot \mathbf{r}}. \) \( \mathbf{E}_0 \) and \( k^{\text{inc}} \) are the incident electric field vector and vector wave number, respectively. The material surrounding the scatterer is assumed to be air, with wave number \( k_0 = \omega \sqrt{\mu_0 \varepsilon_0}. \)

The Boundary Value Problem (BVP) statement associated to the configuration of Figure 5.1 reads as:

Find an electric field \( \mathbf{E} \in \mathbf{H}(\text{curl}; \Omega) \) that satisfies the system

\[
\begin{align*}
\nabla \times \frac{1}{\mu_r} \nabla \times \mathbf{E} - k_0^2 \varepsilon_r \mathbf{E} &= -j\omega \mu_0 \mathbf{J}^{\text{imp}}, & \text{in } \Omega, \\
\n\nabla \times (\varepsilon_r \mathbf{E}) &= 0, & \text{in } \Omega, \\
\n\hat{n} \times \mathbf{E} &= 0, & \text{on } \Gamma_{\text{PEC}}, \\
\n\hat{n} \times \nabla \times (\mathbf{E} - \mathbf{E}^{\text{inc}}) - jk \hat{n} \times (\mathbf{E} - \mathbf{E}^{\text{inc}}) \times \hat{n} &= 0, & \text{on } \partial \Omega,
\end{align*}
\]

(5.2)

where the radiation excitation \( \mathbf{J}^{\text{imp}} \) is assumed

\[
\mathbf{J}^{\text{imp}} = \sum_{m=-\infty}^{\infty} \mathbf{J}^{\text{imp}}_m
\]

(5.3)

and similarly for \( \mathbf{M}^{\text{imp}}. \) In the last equation of (5.2) the Silver-Müller radiation condition at infinity has been explicitly approximated by the first order Absorbing Boundary Condition (ABC). This is a simplification that serves the compactness of the derivations. It is noted that the proposed method can be straightforwardly extended to other higher order ABC, PML absorbers or even “exact” integral equation based truncations. Before proceed into further details of the formulation, it is very important to further comment on the last equation of (5.2). In an infinite periodic structure, the Silver-Müller radiation condition needs to be appropriately modified to account for the physics of the infinite problem. Namely, in an infinite periodic structure, the outgoing waves are not spherical waves, as implied by Silver-Müller condition. In contrast, propagating outgoing plane
waves and/or evanescent decaying waves need to be considered. In this study, the first order ABC is used to absorb only the dominant outgoing propagating waves (Floquet’s mode) and will not be appropriate for evanescent waves. Therefore, for good accuracy, the 1\textsuperscript{st} order ABC needs to be placed sufficiently far away from the infinite structure. Moreover, the impedance for the 1\textsuperscript{st} order ABC should be properly chosen, namely\
\[ k_x = \sqrt{k_0^2 - (k_z^2 + k_y^2)} , \] where \( k_z \) and \( k_y \) are determined by the excitation.

From Figure 5.1, periodicity implies that a certain part of the geometry, in this case \( \Omega_0 \), repeats itself. Therefore, the coefficients (material parameters) of the curl-curl, and divergence equation in (5.2) are of the form\
\[ \varepsilon_r (\mathbf{r} + m \mathbf{D}) = \varepsilon_r (\mathbf{r}), m = -\infty, \cdots, -1, 1, \cdots, \infty, \]
\[ \mu_r (\mathbf{r} + m \mathbf{D}) = \mu_r (\mathbf{r}), m = -\infty, \cdots, -1, 1, \cdots, \infty, \]
\[ (5.4) \]
where \( \mathbf{D} = \text{displacement} (\Omega_m, \Omega_{m+1}) \).

Instead of directly applying the results of Floquet’s theorem to reduce the infinite periodic problem into a finite one, let’s continue work with the infinite domain, but now decomposed into an infinite number of Floquet’s cells. The situation is depicted in Figure 5.2, where a two dimensional cross section of the geometry of Figure 5.1 is considered. In the same figure the mesh of the structure is also plotted, to indicate the non-matching triangulations on each interface of two neighboring Floquet’s cells. The key idea is to treat this problem, at least conceptually, in a similar manner assuming the structure finite periodic problem, thus use domain decomposition and cement element methods described in CHAPTER 3. With this in mind, let’s introduce a set of electric field currents on each interface \( \Gamma \), namely
\[ j = \hat{n} \times \frac{1}{\mu_r} \nabla \times E = \gamma \left( \frac{1}{\mu_r} \nabla \times E \right) \in H^1_{1/2}(\text{div}; \Gamma), \quad (5.5) \]

notice that the quantity \( j \) belong to the div-conforming tangential trace space. To reach this conclusion one has to observe that

\[ E \in H(\text{curl}; \Omega) \xrightarrow{\nabla \times} \nabla \times E = B \in H(\text{div}; \Omega) \xrightarrow{\frac{1}{\mu_r}} \frac{1}{\mu_r} \nabla \times E = H \in H(\text{curl}; \Omega), \quad (5.6) \]

where \( B \) is the magnetic flux density vector and \( H \) is the magnetic field vector. Thus, from the trance theorem of CHAPTER 2, the result of (5.6) is obtained.

Referring to the decomposition of Figure 5.2 the original BVP can be re-stated as:

For each \( \Omega_m, m = -\infty, \ldots, -1, 0, 1, \ldots, \infty \) seek

\[ (E_m \cdot j_{l,m} \cdot j_{r,m}) \in H(\text{curl}; \Omega_m) \times H^{1/2}_1(\text{div}; \Gamma_{L,m}) \times H^{1/2}_1(\text{div}; \Gamma_{R,m}) \text{ s.t.} \]
\[ \nabla \times \left( \frac{1}{\mu_{r,m}} \nabla \times E_m - k_0^2 \epsilon_{r,m} E_m \right) = -j \omega \mu_j \mathbf{j}_{\text{imp}}, \quad \text{in } \Omega_m, \]

\[ \nabla \cdot \left( \epsilon_{r,m} E_m \right) = 0, \quad \text{in } \Omega_m, \]

\[ \hat{n} \times E_m = 0, \quad \text{on } \Gamma_{\text{PEC,m}}, \]

\[ \hat{n} \times \nabla \left( E_m - E_{m}^{\text{inc}} \right) + j k z \hat{n} \times (E_m - E_{m}^{\text{inc}}) \times \hat{n} = 0, \quad \text{on } \partial \Omega_m \setminus \left( \Gamma_{L,m} \cap \Gamma_{R,m} \right), \]

\[ j_{L,m} - \gamma_L e_{L,m} = -j_{R,m-1} - \gamma_L e_{R,m-1}, \quad \text{on } \Gamma_{L,m}, \]

\[ j_{R,m} - \gamma_R e_{R,m} = -j_{L,m+1} - \gamma_R e_{L,m+1}, \quad \text{on } \Gamma_{R,m}, \]

(5.7)

where

\[ e_{i,m} = \hat{n}_{i,m} \times E_m \times \hat{n}_{i,m}, \]

\[ j_{i,m} = \frac{1}{\mu_{r,i,m}} \hat{n}_{i,m} \times \nabla \times E_m, \quad i = L \text{ or } R \]  

(5.8)

are the tangential electric field and electric current on each PBC interface. Note that in (5.7), the required continuity of tangential electric and magnetic fields across domains is enforced through in the last two equations. These equations can be recognized as Robin transmission condition, with \( \gamma_L \) and \( \gamma_R \) been a complex numbers.

The boundary value problem of (5.7) is now in the right form to apply Floquet’s theorem. Floquet’s Theorem [105] states that the solutions of an ordinary differential equation with periodic coefficients are also periodic functions with the same period. Extensions of the same theorem to infinitely periodic boundary value problems (Bloch’s theorem) state that, given a periodic excitation, all observable quantities will have the same periodicity as the structure and will also have a cell-to-cell phase shift equivalent to that that of periodic structure [109]. The phase shift has the form

\[ \mathbf{u}_m = \mathbf{u}_0 e^{i m \phi}, \quad m = -\infty, \cdots, -1, 0, 1, \cdots \infty, \]

(5.9)

where \( \mathbf{u} \) is any observable quantity such us the electric field \( \mathbf{E} \), electric current \( \mathbf{j} \), etc. The phase term \( \phi \) is determined by:
\[ \phi = \begin{cases} \mathbf{k}^{inc} \cdot \mathbf{D}, & \text{scattering}, \\ \mathbf{\beta} \cdot \mathbf{D}, & \text{radiation}, \end{cases} \quad (5.10) \]

where and \( \mathbf{\beta} \) is the scanning direction of the infinite phased array.

Utilizing Floquet’s Theorem the BVP on a single cell can be obtained. For the shake of simplicity lets assume that the Floquet cell is the \( m=0 \) domain in Figure 5.2. Then the TC equations of (5.7) become

\[
\begin{align*}
\mathbf{j}_{L,0} - \gamma_L \mathbf{e}_{L,0} &= -\mathbf{j}_{R,-1} - \gamma_R \mathbf{e}_{R,-1}, & & \text{on } \Gamma_{L,0}, \\
\mathbf{j}_{R,0} - \gamma_R \mathbf{e}_{R,0} &= -\mathbf{j}_{L,1} - \gamma_L \mathbf{e}_{L,1}, & & \text{on } \Gamma_{L,0}.
\end{align*}
\quad (5.11)
\]

At this point the TCs of (5.11) imply that the tangential electric and magnetic fields on two opposite sides of a Floquet’s cell should be continuous. This is no different from any other EM problem. To incorporate the periodicity, a phase shift indicated by Floquet’s Theorem should be also enforced. This is done by the incorporation of (5.9) into the TCs of (5.11), leading to

\[
\begin{align*}
\mathbf{j}_{L,0} - \gamma_L \mathbf{e}_{L,0} &= e^{-j\phi} \left(-\mathbf{j}_{R,0} - \gamma_R \mathbf{e}_{R,0}\right), & & \text{on } \Gamma_{L,0}, \\
\mathbf{j}_{R,0} - \gamma_R \mathbf{e}_{R,0} &= e^{j\phi} \left(-\mathbf{j}_{L,0} - \gamma_L \mathbf{e}_{L,0}\right), & & \text{on } \Gamma_{L,0}.
\end{align*}
\quad (5.12)
\]

From this point on, only the Floquet’s cell problem is consider, thus all subdomain indexes \( m \) will be dropped for convenience. Thus, the Floquet’s cell BVP becomes:

\[
\text{Seek } (\mathbf{E}, \mathbf{j}_L, \mathbf{j}_R) \in \mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}^{-1/2}_1(\text{div}_1; \Gamma_L) \times \mathbf{H}^{1/2}_1(\text{div}_1; \Gamma_R) \text{ s.t.}
\]
\[ \nabla \times \frac{1}{\mu_r} \nabla \times \mathbf{E} - k_0^2 \varepsilon_r \mathbf{E} = -j \omega \mu J^{imp}, \quad \text{in } \Omega, \]
\[ \nabla \cdot (\varepsilon_r \mathbf{E}) = 0, \quad \text{in } \Omega, \]
\[ \hat{n} \times \mathbf{E} = 0, \quad \text{on } \Gamma_{PEC}, \]
\[ \hat{n} \times \nabla \times \left( \mathbf{E} - \mathbf{E}^{inc} \right) + jk \hat{n} \times \left( \mathbf{E} - \mathbf{E}^{inc} \right) \times \hat{n} = 0, \quad \text{on } \partial \Omega \setminus (\Gamma_L \cap \Gamma_R), \]
\[ \mathbf{j}_L - \gamma_{L} \mathbf{e}_L = e^{-i\phi} \left( -\mathbf{j}_R - \gamma_{L} \mathbf{e}_R \right), \quad \text{on } \Gamma_L, \]
\[ \mathbf{j}_R - \gamma_{R} \mathbf{e}_R = e^{i\phi} \left( -\mathbf{j}_L - \gamma_{R} \mathbf{e}_L \right), \quad \text{on } \Gamma_R, \]

where all the quantities are defined on a Floquet’s cell, as shown in Figure 5.3.

Figure 5.3: Floquet’s cell along with the associated variable definitions.
5.2 Galerkin Statement

Having defined the Floquet’s cell continuous boundary value problem, it is necessary to cast its variational form (Galerkin statement):

\[
\begin{align*}
\text{Seek } (E, j_L, j_R) & \in \mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}^{1/2}_i(\text{div}; \Gamma_L) \times \mathbf{H}^{1/2}_i(\text{div}; \Gamma_R) \text{ s.t.} \\
& a(u, E) + \langle u, j_L \rangle_{\Gamma_L} + \langle u, j_R \rangle_{\Gamma_R} = -j\omega \mu \langle u, J^{\text{imp}} \rangle_{\Omega}, \quad \forall u \in \mathbf{H}(\text{curl}; \Omega), \\
& \langle \lambda, e_L \rangle_{\Gamma_L} - \frac{1}{\gamma_L} \langle \lambda, j_L \rangle_{\Gamma_L} - e^{-j\phi} \langle \lambda, e_R \rangle_{\Gamma_L} - \frac{e^{-j\phi}}{\gamma_L} \langle \lambda, j_R \rangle_{\Gamma_L} = 0, \quad \forall \lambda \in \mathbf{H}^{1/2}_i(\text{div}; \Gamma_L), \\
& \langle \lambda, e_R \rangle_{\Gamma_R} - \frac{1}{\gamma_R} \langle \lambda, j_R \rangle_{\Gamma_R} - e^{+j\phi} \langle \lambda, e_L \rangle_{\Gamma_R} - \frac{e^{+j\phi}}{\gamma_R} \langle \lambda, j_L \rangle_{\Gamma_R} = 0, \quad \forall \lambda \in \mathbf{H}^{1/2}_i(\text{div}; \Gamma_R), 
\end{align*}
\]

with bilinear form \( a(\bullet, \bullet) : \mathbf{H}(\text{curl}; \Omega) \times \mathbf{H}(\text{curl}; \Omega) \rightarrow \mathbb{C} \) is given by

\[
a(u, E) = \int_{\Omega} \left\{ \nabla \times u \cdot \frac{1}{\mu_r} \nabla \times E - k_0^2 \varepsilon_r u \cdot E \right\} d\chi. \quad (5.15)
\]

It should be pointed out that in (5.14) the factors \( \gamma_L \) and \( \gamma_R \) were divided in both sides of the second and third equations, respectively. As it will become clear when the solution process is described, this modification naturally leads to an elegant Hermitian preconditioner for the problem.

5.3 Galerkin Statement for Hermitian System of Equations

As it is seen from the statement of (5.14), the formulation will lead to a non-symmetric system of equations. In this section appropriate modifications will be presented in order to cast the variational problem of (5.14) in a Hermitian manner. Such a development not only is more elegant and reflects the physics of the problem, but improves the efficiency of linear system solvers.
The majority of this section follows the techniques used in [115], [116] and [74] to symmetrize FEBI problems. The main idea in restoring the symmetry is to split the surface integral terms \( \langle u, j_L \rangle_{\Gamma_L} \) and \( \langle u, j_R \rangle_{\Gamma_R} \) in first equation of (5.14) into two halves, and substitute the electric current by the TCs of (5.12). This will lead to

\[
\langle u, j_L \rangle_L = \frac{1}{2} \langle u, j_L \rangle_L + \frac{1}{2} \left( u \left( \gamma_L e_L - \frac{e^{-j\phi}}{\mu_{r,L}} j_L - \gamma_L e^{j\phi} e_L \right) \right)_{L} \\
= \frac{1}{2} \langle u, j_L \rangle_L + \frac{\gamma_L}{2} \langle u, e_L \rangle_L - \frac{e^{-j\phi}}{2} \langle u, j_L \rangle_L - \frac{\gamma_L e^{j\phi}}{2} \langle u, e_L \rangle_L ,
\]

(5.16)

and

\[
\langle u, j_R \rangle_R = \frac{1}{2} \langle u, j_R \rangle_R + \frac{1}{2} \left( u \left( \gamma_R e_R - \frac{e^{j\phi}}{\mu_{r,R}} j_R - \gamma_R e^{j\phi} e_R \right) \right)_{R} \\
= \frac{1}{2} \langle u, j_R \rangle_R + \frac{\gamma_R}{2} \langle u, e_R \rangle_R - \frac{e^{j\phi}}{2} \langle u, j_R \rangle_R - \frac{\gamma_R e^{j\phi}}{2} \langle u, e_R \rangle_R .
\]

(5.17)

Finally, the Hermitian structure of the problem is recovered by substituting (5.16) and (5.17) into the first equation of (5.14), and scaling by half the last two equations in (5.14).

The final Galerkin statement reads as:

Seek \( (E, j_L, j_R) \in H(\text{curl}; \Omega) \times H_{-1/2}^{1/2}(\text{div}; \Gamma_L) \times H_{-1/2}^{1/2}(\text{div}; \Gamma_R) \) s.t.

\[
a(u, E) + \frac{1}{2} \langle u, j_L \rangle_L + \frac{\gamma_L}{2} \langle u, e_L \rangle_L - \frac{e^{-j\phi}}{2} \langle u, j_L \rangle_L - \frac{\gamma_L e^{j\phi}}{2} \langle u, e_L \rangle_L + \forall u \in H(\text{curl}; \Omega),
\]

\[
\frac{1}{2} \langle u, j_R \rangle_R + \frac{\gamma_R}{2} \langle u, e_R \rangle_R - \frac{e^{j\phi}}{2} \langle u, j_R \rangle_R - \frac{\gamma_R e^{j\phi}}{2} \langle u, e_R \rangle_R = -j\omega \mu \langle u, J^{\text{imp}} \rangle_{\Omega},
\]

\[
- \frac{1}{2\gamma_L} \langle \lambda, j_L \rangle_L + \frac{1}{2\gamma_L} \langle \lambda, e_L \rangle_L - \frac{e^{-j\phi}}{2\gamma_L} \langle \lambda, j_L \rangle_L - \frac{e^{-j\phi}}{2\gamma_L} \langle \lambda, e_L \rangle_L = 0, \quad \forall \lambda \in H_{-1/2}^{-1/2}(\text{div}; \Gamma_L) , (5.18)
\]

\[
- \frac{1}{2\gamma_R} \langle \lambda, j_R \rangle_R + \frac{1}{2\gamma_R} \langle \lambda, e_R \rangle_R - \frac{e^{j\phi}}{2\gamma_R} \langle \lambda, j_R \rangle_R - \frac{e^{j\phi}}{2\gamma_R} \langle \lambda, e_R \rangle_R = 0, \quad \forall \lambda \in H_{-1/2}^{-1/2}(\text{div}; \Gamma_R).
\]
5.4 Discrete Galerkin Statement

For the description of the discrete problem statement the Floquet’s cell $\Omega$ is discretized in a conforming but unstructured tetrahedron mesh $\mathcal{T}$ such that

$$\Omega = \bigcup_i K_i, \quad K_i \cap K_j = 0, \text{ if } i \neq j,$$  \hspace{1cm} (5.19)

where $K_i \in \mathcal{T}$ is a tetrahedron. The tetrahedral elements in the volume of the Floquet’s cell, induce simplecial triangulations $\Delta_L$ and $\Delta_R$ on the left and right PBC surfaces of the domain, such that $\Delta_L \neq \Delta_R$. For the construction of the discrete counterpart of $\mathbf{V}$ the $p=2$ first Nedelec family hierarchical finite elements [74] on tetrahedron mesh will be used and will be denoted by $\mathcal{ND}_2^1(\mathcal{T})$. In the abovementioned notation, the superscript describes the family of the Nedelec elements, whereas the subscript 2 denotes the order. On the other hand, the discrete subspace of $\mathcal{M}$ will be constructed as a span of $p=2$ hierarchical Raviart-Tomas finite elements of surface triangulations, and will be denoted by $\mathcal{RT}_2(\Delta)$. Based on these definitions, the discrete spaces

\[ \mathcal{V}_h = \{ \mathbf{u}_h \in H(curl; \Omega), \mathbf{u}_h|_K \in \mathcal{ND}_2^1(K), \forall K \in \mathcal{T} \}, \]  \hspace{1cm} (5.20)

and

\[ \mathcal{M}_{h,i} = \{ \mathbf{u}_h \in H^{-1/2}_i(div; \Gamma_i), \mathbf{u}_h|_K \in \mathcal{RT}_2(K), \forall K \in \Delta_i \}, \]  \hspace{1cm} (5.21)

are constructed, where the subscript $i$. denotes L or R, for left or right PBC surface, respectively. The discrete Galerkin statement is now stated as:

\[ \text{Seek} \left( \mathbf{E}_h, \mathbf{j}_{h,L}, \mathbf{j}_{h,R} \right) \in \mathcal{V}_h \times \mathcal{M}_{h,L} \times \mathcal{M}_{h,R} \text{ s.t.} \]
where \( \mathbf{u}_h \) and \( \lambda_j \) are the testing function for the primal and dual unknowns, respectively.

Expanding each discrete space into a set of bases functions such as

\[
\mathbf{E}_h = \sum_{i=1}^{I} \tilde{E}_i \mathbf{w}_i, \quad \mathbf{w}_i \in \mathcal{N}\mathcal{D}_2^1(K), K \in T, \\
\mathbf{j}_{L,h} = \sum_{i=1}^{I} \tilde{j}_{i,L} \mathbf{a}_i, \quad \mathbf{a}_i \in \mathcal{R}\mathcal{T}_2(K), K \in \Delta, \\
\mathbf{j}_{R,h} = \sum_{i=1}^{I} \tilde{j}_{i,R} \mathbf{a}_i, \quad \mathbf{a}_i \in \mathcal{R}\mathcal{T}_2(K), K \in \Delta,
\]

where \( \tilde{E}, \tilde{j}_L \) and \( \tilde{j}_R \) are the unknown coefficients of the electric field and electric current on the PBC interfaces. Similarly expanding the testing spaces in \( \mathbf{u}_h = \text{span}\{\mathbf{w}_i\} \) and \( \lambda_j = \text{span}\{\mathbf{a}_i\} \), results into the following system of equations

\[
\begin{pmatrix}
\mathbf{A}_{ii} & \mathbf{A}_{IL} & \mathbf{0} & \mathbf{A}_{IR} & \mathbf{0} \\
\mathbf{A}^T_{IL} & \mathbf{A}_{LL} & \frac{1}{2} \mathbf{T}_{LL} & \frac{1}{2} \mathbf{D}_{LL} & \frac{1}{2} \mathbf{T}_{LR} & \frac{1}{2} \mathbf{D}_{LR} \\\n\mathbf{0} & \frac{1}{2} \mathbf{D}_{LL} & -\frac{1}{2} \mathbf{T}_{LL} & -\frac{1}{2} e^{i\theta} \mathbf{D}_{LR} & -\frac{1}{2} e^{i\theta} \mathbf{T}_{LR} & \frac{1}{2} e^{i\theta} \mathbf{D}_{RR} \\\n\mathbf{0} & -\frac{1}{2} e^{i\theta} \mathbf{D}_{LL} & \frac{1}{2} e^{i\theta} \mathbf{T}_{LR} & \frac{1}{2} \mathbf{D}_{LL} & \frac{1}{2} \mathbf{T}_{RR} & \mathbf{0} \\\n\end{pmatrix}
\begin{pmatrix}
\tilde{\mathbf{E}} \\
\tilde{\mathbf{e}}_L \\
\tilde{\mathbf{j}}_L \\
\tilde{\mathbf{e}}_R \\
\tilde{\mathbf{j}}_R \\
\end{pmatrix}
= \begin{pmatrix}
g \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

\[
(\mathbf{A}_{ii})_{mn} = \int_{\Omega} \left( \nabla \times \mathbf{w}_m \cdot \frac{1}{\mu_r} \nabla \times \mathbf{w}_n - k^2 \varepsilon_r \mathbf{w}_m \mathbf{w}_n \right) dx^3, \quad (5.25)
\]
\begin{align}
(A_{ii})_{mn} &= \int_{\Omega} \left( \nabla \times \mathbf{w}_m \cdot \frac{1}{\mu_r} \nabla \times \overline{\mathbf{w}}_n - k^2 \varepsilon_r \mathbf{w}_m \overline{\mathbf{w}}_n \right) d^3 x, \\
(A_{ii})_{mn} &= \int_{\Gamma_i} \left( \nabla \times \overline{\mathbf{w}}_m \cdot \frac{1}{\mu_r} \nabla \times \overline{\mathbf{w}}_n - k^2 \varepsilon_r \mathbf{w}_m \overline{\mathbf{w}}_n \right) d^2 x, \\
(T_{ij})_{mn} &= \int_{\Gamma_i} \left( \frac{1}{\gamma_i} a_{i,m} \cdot a_{j,n} \right) d^2 x, \\
(D_{ij})_{mn} &= \int_{\Gamma_i} (\overline{\mathbf{w}}_{i,m} \cdot a_{j,n}) d^2 x,
\end{align}

the subscripts $i$ and $j$ indicate left and right side, and $m$ and $n$ indicate the matrix element locations. In (5.26)-(5.29) the notation

$$\overline{\mathbf{w}} = \gamma \left( \mathbf{w} \right),$$

was adopted to indicate the surface (triangle) trace of the volume (tetrahedron) basis functions. Before proceeding into the iterative solution of the final system of equations, it is important to note if the Robin constants $\gamma_L$ and $\gamma_R$ are chosen such

$$\gamma_L = (\gamma_R)^*,$$

where star denotes complex conjugate, the system of equations is block-Hermitian. Moreover, based on the discussions in section 3.3, if the system is solved with an iterative solver, the convergence can be granted only with a purely imaginary value on the Robin constants. In that case though, the Hermitianity of the system is broken because the $T$ blocks that are symmetric, and multiplied by pure complex Robin constants. In this work the Robin constants will be chosen $\gamma_R = jk = (\gamma_L)^*$, which leads to a block-Hermitian system, but not Hermitian for the reasons described above.
5.5 Iterative Solution Method

In this section an efficient and easy to implement iterative solution of (5.24) is proposed. A straight-forward solution via Krylov subspace methods would require non-symmetric matrix solvers such as BiCG, BiCGStab, QMR or GMRES. In generally these solvers are relatively inefficient in storage and number of matrix vector multiplies per iteration. To avoid such solvers, the present method is tailored such that only minor modifications need to be done from a “conventional” symmetric FEM solver based on CG. The main idea is to split the matrix of (5.24) into two parts, one symmetric and one Hermitian. If the sparse matrix of (5.24) is denoted by \( A \), the following splitting is introduced

\[
Ax \equiv (K + C)x = y.
\]  

(5.32)

where \( x \) and \( y \) are the solution vector and right-hand side, respectively. Moreover, matrices

\[
K = \begin{pmatrix}
A_{il} & A_{il} & 0 & A_{ir} & 0 \\
A_{il}^T & A_{ll} + \frac{\gamma_l}{2}T_{ll} & \frac{1}{2}D_{ll} & 0 & 0 \\
0 & \frac{1}{2}D_{ll} & -\frac{1}{2\gamma_l}T_{ll} & 0 & 0 \\
A_{ir}^T & 0 & 0 & A_{rr} + \frac{\gamma_r}{2}T_{rr} & \frac{1}{2}D_{rr} \\
0 & 0 & 0 & 0 & \frac{1}{2}D_{rr} - \frac{1}{2\gamma_r}T_{rr}
\end{pmatrix}
\]  

(5.33)

and
\[
\mathbf{C} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\gamma_L e^{-i\phi}}{2} \mathbf{T}_{LR} - \frac{e^{-i\phi}}{2} \mathbf{D}_{LR} \\
0 & 0 & 0 & -\frac{\gamma_R e^{i\phi}}{2} \mathbf{T}_{RL} - \frac{e^{i\phi}}{2} \mathbf{D}_{RL} \\
0 & -\frac{\gamma_R e^{i\phi}}{2} \mathbf{T}_{RL} - \frac{e^{i\phi}}{2} \mathbf{D}_{RL} & 0 & 0 \\
\end{pmatrix},
\]

(5.34)

are the symmetric and Hermitian part of the splitting, respectively. It is interesting to observe that \( \mathbf{K} \) in nothing but the FEM matrix of the Floquet’s cell with first order ABC truncations in place of the PBCs; thus obtaining a matrix vector product with \( \mathbf{K}^{-1} \) is nothing but the solution of the Floquet’s cell problem with ABC as truncation. The Hermitian part, \( \mathbf{C} \), of the splitting represent the coupling between two opposite side interfaces due to the periodicity of the problem. In light of the above comments, the splitting of (5.32) can be written as

\[
\mathbf{K}^{-1} (\mathbf{K} + \mathbf{C}) \mathbf{x} = \mathbf{K}^{-1} \mathbf{y} \Rightarrow \\
(\mathbf{I} + \mathbf{K}^{-1} \mathbf{C}) \mathbf{x} = \mathbf{K}^{-1} \mathbf{y} \Rightarrow \\
\mathbf{x} + \mathbf{K}^{-1} \mathbf{C} \mathbf{x} = \mathbf{K}^{-1} \mathbf{y}.
\]

(5.35)

Isolating \( \mathbf{x} \) on the left-hand side, and adding and subtracting \( \mathbf{K} \mathbf{x} \) on the right-hand side, leads to

\[
\mathbf{x} = -\mathbf{K}^{-1} \left[ (\mathbf{K} + \mathbf{C}) \mathbf{x} - \mathbf{K} \mathbf{x} - \mathbf{y} \right] \\
= \mathbf{x} - \mathbf{K}^{-1} \left[ (\mathbf{K} + \mathbf{C}) \mathbf{x} - \mathbf{y} \right].
\]

(5.36)

The last equation in (5.36) suggests the following relaxed stationary (fixed point) iteration scheme

\[
\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} - \alpha \mathbf{K}^{-1} \left( \mathbf{A} \mathbf{x}^{(n)} - \mathbf{y} \right), \quad n \geq 0, \alpha \in \mathbb{R},
\]

(5.37)
where \( \alpha \) is the relaxation factor. The form in (5.37) is usually called second normal form and converges only if

\[
\rho \left( I - K^{-1}A \right) < 1.
\]  

(5.38)

Two important remarks are in place of the solution suggested in (5.37). First, it involves only one solution of a symmetric FEM-ABC system and on matrix vector multiplication with a Hermitian matrix per iteration. On the other hand, the iteration of (5.37) readily suggest a preconditioning scheme of a non-symmetric Krylov solver.

### 5.6 Numerical Results

All computations were performed in a PC with single 2.4GHz Intel® Xeon™ processor with 512KB L2 cache and 2GB RAM. Double precision complex arithmetic is assumed. The computational codes were implemented in object-oriented C++ using the GNU g++ compiler with optimization level -O9.

To demonstrate the validity of the proposed method, an infinite array of rectangular patches printed on a thick GaAs substrate is considered. The patches are fed on the radiating slot by a coax probe, as shown in Figure 5.4. All geometry dimensions and excitation parameters can be found in [110]. The present array has been studied initially by Pozar and Schaubert with the spectral domain Method of Moment (MoM) [118]. The same antenna was analyzed in [110] with a hybrid FEM and BI method. In both studies, the parameter of interest was the normalized active reflection coefficient, as function of the scanning angle. The E-Plane active reflection coefficient comparisons of the proposed mixed-hybrid FEM with both MoM and FEBI methods are plotted in Figure 5.5. In almost the entire angle spectrum, the proposed method predicts the reflection
coefficient quite accurately. It is believed that the minor discrepancies may be attributed to the thickness of the probe or the distance of truncation boundary. It is important to stress that the exact thickness and location of the probe and coax was not known. For the present problem, the total unknown number was 98,472, among those 93,532 where primal unknowns. Finally it should be remarked that the mesh of the proposed method was obtained through a 10-step h-adaptive mesh refinement (h-AMR) [94]. From Figure 5.4(b), it is visible the effect of the adaptive mesh refinement in the regions around the patch and the coax feeder.

![Figure 5.4](image)

**Figure 5.4**: Geometry and mesh of the infinite array of rectangular patches. (a) Geometry and dimensions. (b) H-adaptive mesh refined mesh on the substrate plane.
The double infinite periodic array of X-band waveguides on an infinite PEC ground plane is considered next. The geometry and dimensions of the structures are shown in Figure 5.6. The return loss for this array under H-plane scan is plotted in Figure 5.7. The proposed method agrees very well with the results obtained by ANSOFT’s HFSS infinite array simulator.

Figure 5.5: Comparisons of the normalized active reflection coefficient versus scan angle.
Figure 5.6: A doubly periodic infinite array of X-band waveguides.
Figure 5.7: Return loss as functions of the H-plane scan angle, comparisons with HFSS.
CHAPTER 6

A DOMAIN DECOMPOSITION BASED FINITE ELEMENT BOUNDARY ELEMENT COUPLING

The hybrid Finite Element Method-Boundary Element Method (FEM-BEM) is one of the most appealing approaches for the analysis of unbounded electromagnetic radiation and scattering from heterogeneous structures. The method not only combines the FEM’s versatility to model geometrically complex structures and materials, but at the same time remedies spurious reflections from the truncation boundary by enforcing the “exact” boundary condition though the boundary integral coupling.

A direct, and widely used, hybridization of FEM and BEM [83, 84, 119, 120] is based on a non-variational setting that leads to a non-symmetric complex system of equations, even when the actual physical problem involves only reciprocal materials. These non-symmetric matrices do not reflect the reciprocal nature of actual physical problem and at the same time are considerably more difficult and computationally expensive to solve with iterative or even direct solvers. Apart from the $O(N_s^2)$ computational complexity, where $N_s$ is the number of unknowns on the BEM boundary, the infamous internal or interior resonance or forbidden frequency problem [69] is probably the most serious drawback of FEM-BEM couplings. In this dissertation, the reduction of the $O(N_s^2)$ computational complexity will not be addressed, since it can be
considered as a strictly BEM issue and its remedies can be found elsewhere [121]. It suffices to say that the present numerical implementations have incorporated the ACA [122] and IE-FFT [123, 124] compression/acceleration schemes for the BEM domain. Last but not least, the majority of FEM-BEM couplings, except the works in [125, 126], lack modularity namely, FEM and BEM have to be consistent with each other in term of mesh, basis functions and solver. Finally, till today no FEM-BEM formulation has been able to guarantee the convergence of iterative solvers for FEM-BEM systems.

The proposed method is a continuation of [74]. Unlike [74], the present method manages to decouple FEM and BEM as two separate computational domains that can be modularly treated in terms of meshing, basis functions, matrix assembling and solution process. The overall coupling and matrix solution is then sought via inner-outer loop iteration scheme between FEM and BEM domains. Similar to [74], this approach is variational and it retains the much needed symmetry of the final matrix system. Unlike the work in [74], the present method is free of internal resonance problems since the BEM part, employs a Direct Boundary Integral Equation (BIE) formulation [127], and a Robin-to-Robin map instead of the Dirichlet-to-Neumann map. In addition, the method is extremely modular, namely the FEM and BEM mesh and approximation spaces can be completely different. This not only allows different order or type of basis functions on the FEM and BEM part allowing the easy integration of existing FEM and BEM or fast BEM implementations, but at the same time it paves the road for versatile and efficient adaptive mesh refinement techniques. In summary, the proposed method offers five distinct features: (a) symmetry, (b) internal resonance free, (c) modularity, (d) non-
conformity between FEM and BEM domains, and (e) natural and very effective 
preconditioning schemes that guarantee spectral radius less or equal to one.

The plan of this chapter will be organized as follows. In section 6.1 the general 
statement of an unbounded EM radiation and/or scattering problem will be given. The 
transmission problem and its Robin-to-Robin modified form, that is the basis of the 
proposed domain decomposition FEM and BEM coupling will be presented. The 
treatment of the exterior unbounded problem in integral and variational form will be 
discussed in section 6.3. Since it has been described in earlier chapter the interior FEM 
based problem will be just summarized in section 6.4. The coupling between FEM and 
BEM will be given in section 6.5 together with is fully discrete and ready-to-implement 
final matrix form. A natural preconditioning scheme based on the FEM and BEM domain 
decomposition in conjunction with Krylov space iterative methods will be outlined in 
section 6.6. Finally the chapter will conclude in section 6.7 with a number of numerical 
results and statistics that demonstrate the accuracy, versatility, robustness and efficiency 
of the present method.

6.1 Boundary Value Problem Statement

The reduced form an unbounded EM radiation or scattering problem is described 
in (6.1), where the time-harmonic form of Maxwell’s equations has been reduced to a 
curl-curl equation subject to the divergence free condition.
\[ \nabla \times \frac{1}{\mu} \nabla \times \mathbf{E} - k^2 \varepsilon \mathbf{E} = -j \omega \mu_0 \mathbf{J}, \quad \text{in } \mathbb{R}^3. \]

\[ \nabla \cdot (\varepsilon \mathbf{E}) = 0, \quad \text{in } \mathbb{R}^3. \]

\[ \gamma^\prime (\mathbf{E}) = 0, \quad \text{on } \Gamma_{PEC}, \]

\[ \gamma^\prime \left( \frac{1}{\mu} \nabla \times \mathbf{E} \right) = 0, \quad \text{on } \Gamma_{PMC}, \]

\[ \lim_{|\mathbf{r}| \to \infty} \left( \nabla \times (\mathbf{E} - \mathbf{E}^\text{inc}) \times \mathbf{r} - j k |\mathbf{r}| (\mathbf{E} - \mathbf{E}^\text{inc}) \right) = 0 \]

where \( \mathbb{R}^3 \) denotes the unbounded domain and \( \Gamma_{PEC} \) and \( \Gamma_{PMC} \) are the surface of perfect electric and magnetic conductors respectively. The near-field excitation is accounted through an impressed electric current \( \mathbf{J} \), whereas the incident plane wave is accounted by an incident electric field \( \mathbf{E}^\text{inc} = \mathbf{E}_0 e^{-j k_{inc} \cdot \mathbf{r}} \) where \( \mathbf{E}_0 \) and \( k_{inc} \) denote the incident electric field vector and wave vector, respectively. It should be emphasized here that \( \mathbf{E} = \mathbf{E}^\text{inc} + \mathbf{E}^{sw} \) is the total electric field.

Due to the difficulties arising from the discretization of an unbounded problem like the one suggested in (6.1), it is beneficial to split the unbounded domain into two. One simply connected and bounded domain \( \Omega \subset \mathbb{R}^3 \) that contains the localized heterogeneous scatterer and/or antenna, and its complement \( \Omega' = \mathbb{R}^3 \setminus \Omega \). Note that \( \Omega' \) is unbounded homogeneous free space region, thus suitable boundary element or integral equation approaches. Based on this domain decomposition, the above BVP can be written as the following transmission problem.
\[ \nabla \times \frac{1}{\mu_r} \nabla \times \mathbf{E} - k^2 \varepsilon_r \mathbf{E} = -j \omega \mu_0 \mathbf{J} , \quad \text{in } \Omega , \]

\[ \nabla \times (\varepsilon_r \mathbf{E}) = 0 , \quad \text{in } \Omega , \]

\[ \gamma^+ (\mathbf{E}) = 0 , \quad \text{on } \Gamma_{PEC} , \]

\[ \gamma^+ \left( \frac{1}{\mu_r} \nabla \times \mathbf{E} \right) = 0 , \quad \text{on } \Gamma_{PMC} \]

\[ \nabla \times \nabla \times \mathbf{E} - k^2 \mathbf{E} = 0 , \quad \text{in } \Omega' \]

\[ \lim_{|\mathbf{r}| \to \infty} \left( \nabla \times (\mathbf{E} - \mathbf{E}^{\text{inc}}) \times \mathbf{r} - j k |\mathbf{r}| (\mathbf{E} - \mathbf{E}^{\text{inc}}) \right) = 0 \]

\[ \gamma^+ (\mathbf{E})_{\Gamma^+} = \gamma^+ (\mathbf{E})_{\Gamma^-} , \quad \text{on } \Gamma \]

\[ \gamma^+ \left( \frac{1}{\mu_r} \nabla \times \mathbf{E} \right)_{\Gamma^+} = \gamma^+ \left( \frac{1}{\mu_r} \nabla \times (\mathbf{E}) \right)_{\Gamma^-} , \quad \text{on } \Gamma \] (6.4)

where \( \Gamma = \partial \Omega \) is the closed surface containing the scatterer, and \( \Gamma^+ \) and \( \Gamma^- \) are the exterior and interior sides of the bounding surface \( \Gamma \), as shown in Figure 6.1.

**6.2 Transmission Problem**

The key feature of the proposed FEM-BEM coupling is the Robin-to-Robin transmission problem, and the “cement” finite element coupling of non-conforming grids proposed in CHAPTER 3.
With the picture of Figure 6.1 in mind, and following the spirit described in previous chapters, the transmission problem of (6.2)-(6.4) can be modified as:

$$\nabla \times \frac{1}{\mu_r} \nabla \times \mathbf{E} - k^2 \varepsilon_r \mathbf{E} = -j \omega \mu_0 \mathbf{J}, \quad \text{in } \Omega,$$

$$\nabla \cdot (\varepsilon_r \mathbf{E}) = 0, \quad \text{in } \Omega,$$

$$\gamma^{\times} \left( \mathbf{E} \right) = 0, \quad \text{on } \Gamma_{PEC},$$

$$\gamma^{\times} \left( \frac{1}{\mu_r} \nabla \times \mathbf{E} \right) = 0, \quad \text{on } \Gamma_{PMC}.$$
\[ \nabla \times \nabla \times E - k^2 E = 0, \quad \text{in } \Omega, \]
\[ \lim_{|r| \to \infty} \left( \nabla \times (E - E^{\text{inc}}) \right) \times r - jk \frac{E - E^{\text{inc}}}{|r|} = 0, \quad (6.6) \]

\[
\gamma' \left( \frac{1}{\mu_e} \nabla \times E \right)_{\Gamma^{-}} - \alpha \gamma' (E)_{\Gamma^{-}} = -\gamma' \left( \frac{1}{\mu_e} \nabla \times E \right)_{\Gamma^{+}} - \alpha \gamma' (E)_{\Gamma^{+}}, \quad \text{on } \Gamma^{-},
\]
\[
\gamma' \left( \frac{1}{\mu_e} \nabla \times E \right)_{\Gamma^{+}} - \alpha \gamma' (E)_{\Gamma^{+}} = -\gamma' \left( \frac{1}{\mu_e} \nabla \times E \right)_{\Gamma^{-}} - \alpha \gamma' (E)_{\Gamma^{-}}, \quad \text{on } \Gamma^{+}, \quad (6.7)
\]

The continuity of the Dirichlet and Neumann traces of the electric field in (6.4) have been replaced by Robin-to-Robin map. In this work the complex Robin constant \( \alpha \) is chosen \( \alpha = jk \), in light of the convergence analysis of section 3.3. Here it should be emphasized that Hoppe et.al. in [126] has proposed a symmetric and yet modular coupling which is based on the pure Dirichlet or Neumann coupling. The present approach significantly differs for that of [126] since it is based on the Robin-to-Robin coupling, thus it alleviates any internal resonance problems and offers better convergence properties. Before proceeding further into the derivation let introduce the following electric and magnetic currents for the interior side of \( \Gamma \)

\[
j^- (r') = \gamma' \left( \frac{1}{\mu_e} \nabla \times E (r') \right)_{\Gamma^{-}}, \quad m^- (r') = \gamma' (E (r'))_{\Gamma^{-}}, \quad (6.8)
\]

and for the exterior side of \( \Gamma \)

\[
j^+ (r') = \gamma' \left( \frac{1}{\mu_e} \nabla \times E (r') \right)_{\Gamma^{+}}, \quad m^+ (r') = \gamma' (E (r'))_{\Gamma^{+}}, \quad (6.9)
\]

moreover, the interior and exterior electric filed tangential traces will be denoted by \( e^- = \gamma' (E)_{\Gamma^{-}} \) and \( e^+ = \gamma' (E)_{\Gamma^{+}} \), respectively. With these notation in place the transmission conditions in (6.7) simplify to

\[
j^- - \alpha e^- = -j^+ - \alpha e^+, \quad \text{on } \Gamma^{-},
\]
\[
j^+ - \alpha e^+ = -j^- - \alpha e^-, \quad \text{on } \Gamma^{+}. \quad (6.10)
\]
The important and distinct characteristic of the transmission problem of (6.5)-(6.10) is the Robin-type transmission condition of equation (6.10). Notice this modification ensures continuity of both tangential electric and magnetic fields and at the same time guarantees well-posed of both interior and exterior problems at all frequencies.

6.3 Exterior Unbounded Region Problem

In this section the treating of the exterior/unbounded EM radiation and/or scattering problem will be given along with its variational form.

6.3.1 Representation Formulae

Having establish the appropriate transmission problem based on the Robin-to-Robin map it is now time to proceed with the formulation of the exterior region. Instead of dealing with the differential equation form of (6.6), it is better suited to converted it into an integral form. Starting from (6.6) and applied the Green’s theorem on $\Omega'$ the following representation formulae for the electric and magnetic field are obtained [72, 128]

$$\mathbf{E}(r) = \mathbf{E}^{inc}(r) + \mathbf{A}\left(j^+\right) + \frac{1}{k^2} \nabla \Phi\left(\text{div}_j^+ \mathbf{j}^+\right) + \nabla \times \mathbf{A}\left(m^+\right), \quad \text{in } \Omega \setminus \Gamma$$

$$j k \eta \mathbf{H}(r) = j k \eta \mathbf{H}^{inc}(r) - k^2 \mathbf{A}\left(\text{div}_j^+ \mathbf{m}^+\right) - \nabla \Phi\left(m^+\right) + \nabla \times \mathbf{A}\left(j^+\right), \quad \text{in } \Omega \setminus \Gamma$$

(6.11)

where $\mathbf{A} : H^{1/2}_\Gamma (\Gamma) \mapsto H^1_\text{loc} (\mathbb{R}^3)$ denotes the vector single layer potential, $\Phi : H^{-1/2}(\Gamma) \mapsto H^1_\text{loc} (\mathbb{R}^3)$ is the scalar single layer potential given by

$$\mathbf{A} (v) = \int_{\Gamma} v(r') g(r | r') \, dx^2, \quad r \notin \Gamma,$$

$$\Phi(u) = \int_{\Gamma} u(r') g(r | r') \, dx^2, \quad r \notin \Gamma,$$

(6.12)
6.3.2 Integral Equations

A set to integral equations for the exterior BEM domain can be written by taking
the $\gamma'$ trace of the first and $\gamma'$ of the second equations in (6.11) and letting the
observation points to approach $r \to \Gamma'$. This leads to:

$$
\begin{align*}
\frac{1}{2} e^e (r) &= e^{inc} (r) + \gamma' \left( R(m^+) \right) + \frac{1}{k^2} \text{div} \Phi \left( \text{div} j^e \right) + \gamma' \left( A(j^e) \right) \\
\frac{1}{2} j^e (r) &= j^{inc} (r) + k^2 \gamma' \left( A(m^+) \right) + \gamma' \left( \nabla \Phi \left( \text{div} m^+ \right) \right) + \gamma' \left( R(j^e) \right)
\end{align*}
$$

(6.13)

where the integral operator $R : H^{1/2}_i (\Gamma) \to H^{1/2}_m (\mathbb{R}^3)$ was introduced, and it is given by

$$
R (v) = \text{pv} \int \nabla' g (r | r') \, dv
$$

(6.14)

here pv indicates integration in principal value sense. The formulas in (6.13) are the
expressions corresponding to the exterior Calderon projector [72].

Naturally, from the trace theorem of CHAPTER 2 we conclude
that $e^e \in H^{1/2}_- \left( curl, \Gamma^+ \right)$, and $j^e \in H^{1/2}_i \left( \text{div}, \Gamma^+ \right)$. Equations (6.13) are the well known
Electric Field Integral Equations (EFIE) and Magnetic Field Integral Equations (MFIE),
respectively.

6.3.3 Variational From

The weak statement of the above integral equations is obtained by testing each
equation with the appropriate set of testing functions. Following the spirit of duality
pairing the appropriate space to test $e^e \in H^{1/2}_- \left( curl, \Gamma^+ \right)$ is its dual space
$H^{-1/2}_i \left( \text{div}, \Gamma^+ \right)$; whereas the appropriate testing space for $j^e \in H^{1/2}_i \left( \text{div}, \Gamma^+ \right)$ will be its
dual space $H^{1/2}_- \left( curl, \Gamma^+ \right)$. For that reason, the EFIE form will be tested by functions
that belong to the div-conforming trace space, which is the space of the well-known RWG, or Raviart-Thomas space. On the other hand MFIE form is tested by surface curl-conforming basis functions or Whitney 1-forms, on $\Gamma$. Following that spirit the subsequent variational form is obtained

$$\frac{1}{2} \langle \lambda^+, e^+ \rangle_{\Gamma^+} = \langle \lambda^+, e^{inc} \rangle_{\Gamma^+} + \langle \lambda^+, R(m^+) \rangle_{\Gamma^+} - \frac{1}{k^2} \langle div, \lambda^+, \Phi(div, j^+) \rangle_{\Gamma^+} + \langle \lambda^+, A(j^+) \rangle_{\Gamma^+},$$

$$\frac{1}{2} \langle v^+, j^+ \rangle_{\Gamma^+} = \langle v^+, j^{inc} \rangle_{\Gamma^+} - \langle \hat{n} \times v^+, R(j^+) \rangle_{\Gamma^+} + \langle div, \hat{n} \times v^+, \Phi(div, m^+) \rangle_{\Gamma^+} - k^2 \langle \hat{n} \times v^+, A(m^+) \rangle_{\Gamma^+},$$

(6.15)

$$\forall \lambda^+ \in H^{-1/2}(div, \Gamma^+) \text{ and } v^+ \in H^{-1/2}_l(curl, \Gamma^+).$$

### 6.4 Interior Problem

In the interior domain, the standard variational form of (6.5) reads as:

$$\text{Seek } E \in H(curl; \Omega) \text{ such that}$$

$$a(v, E) + j \omega \mu_0 \langle v, J \rangle_{\Omega} = -j \omega \mu_0 \langle v, J \rangle_{\Omega} \quad \forall v \in H(curl; \Omega)$$

(6.16)

where the dentitions of bilinear form $a(v, E)$, $\langle \cdot, \cdot \rangle_{\Gamma}$ and $\langle \cdot, \cdot \rangle_{\Omega}$ can be found in section 5.2.

Notice that the divergence-free condition in (6.5) has not been explicitly discretized in terms of a mixed problem because it will be incorporated into the finite element spaces through the inexact Helmholtz decomposition of the discrete spaces in conjunction with the appropriate scaling. From the implementation point of view the above gauge will be incorporated for the edge elements through a tree-cotree mesh partitioning scheme [76, 129, 130].
6.5 Coupled Problem

The coupling of the interior and exterior problems will be accomplished through the variational form of the Robin-to-Robin map described in (6.10). The variational statement of the transmission problem in (6.10) now reads as:

Seek \( (e^-, j^-) \in H_\text{curl}_r; \Gamma \times H^{1/2}(\text{div}_r; \Gamma) \) and \( (e^+, j^+) \in H_\text{curl}_r; \Gamma \times H^{1/2}(\text{div}_r; \Gamma) \) such that

\[
\begin{align*}
\langle \lambda^-, e^- \rangle_{\Gamma^-} &- \frac{1}{\alpha} \langle \lambda^-, j^- \rangle_{\Gamma^-} = \frac{1}{\alpha} \langle \lambda^-, j^+ \rangle_{\Gamma^-} + \langle \lambda^-, e^+ \rangle_{\Gamma^-}, \text{ on } \Gamma^- , \\
\langle v^-, j^- \rangle_{\Gamma^-} & = \alpha \langle v^-, e^- \rangle_{\Gamma^-} - \langle v^-, j^+ \rangle_{\Gamma^-} - \alpha \langle v^-, e^+ \rangle_{\Gamma^-}, \text{ on } \Gamma^- , \\
\langle \lambda^+, e^+ \rangle_{\Gamma^+} & = \frac{1}{\alpha} \langle \lambda^+, j^+ \rangle_{\Gamma^+} + \frac{1}{\alpha} \langle \lambda^+, j^- \rangle_{\Gamma^+} + \langle \lambda^+, e^- \rangle_{\Gamma^+}, \text{ on } \Gamma^+ , \\
\langle v^+, j^+ \rangle_{\Gamma^+} & = \alpha \langle v^+, e^+ \rangle_{\Gamma^+} - \langle v^+, j^- \rangle_{\Gamma^+} - \alpha \langle v^+, e^- \rangle_{\Gamma^+}, \text{ on } \Gamma^+ ,
\end{align*}
\]

\( \forall (v^-, \lambda^-) \in H_\text{curl}_r; \Gamma \times H^{1/2}(\text{div}_r; \Gamma) \) and \( (v^+, \lambda^+) \in H_\text{curl}_r; \Gamma \times H^{1/2}(\text{div}_r; \Gamma) \)

Using the fact that \( \gamma'(w_{\text{res}})_{\Gamma^-} = v^- \) the above variational statement is combined with the interior problem variational statement. In detail, the surface integral term in (6.16) is divided into two halves, and one half is substituted from the second equation of (6.17). That leads to

\[
\begin{align*}
a(v, E) + \frac{1}{2} \langle v^-, j^- \rangle_{\Gamma^-} + \frac{\alpha}{2} \langle v^-, e^- \rangle_{\Gamma^-} - \frac{1}{2} \langle v^-, j^+ \rangle_{\Gamma^-} - \frac{\alpha}{2} \langle v^-, e^+ \rangle_{\Gamma^-} = -j \omega \mu_0 \langle v, J \rangle_{\Omega},
\end{align*}
\]

(6.18)

Similarly the exterior variational problem and transmission equations are combined by substituting the right-hand side terms in the last two equations of (6.17) by the equations in (6.15). That gives
\[
\frac{1}{2\alpha}\langle \lambda^+, j^+ \rangle_{\Gamma^-} + \frac{1}{2\alpha}\langle \lambda^+, j^- \rangle_{\Gamma^-} + \frac{1}{2}\langle \lambda^+, e^- \rangle_{\Gamma^-} = \langle \lambda^+, e^{inc} \rangle_{\Gamma^-} + \\
\langle \lambda^+, R(m^+) \rangle_{\Gamma^-} - \frac{1}{k^2}\langle \text{div}_t \lambda^+, \Phi(\text{div}_t j^+) \rangle_{\Gamma^-} + \langle \lambda^+, A(j^+) \rangle_{\Gamma^-}, \\
(6.19)
\]

Equations (6.18) and (6.19) together form final variational form of the coupled problem becomes:

Seek \( E \in H(\text{curl}; \Omega), \ j^* \in H^{1/2}_\perp(\text{curl}_r; \Gamma^-), \ e^- \in H^{1/2}_\perp(\text{div}_r; \Gamma^-), \ j^* \in H^{1/2}_\perp(\text{curl}_r; \Gamma^+) \),

\[
e^+ \in H^{1/2}_\perp(\text{div}_r; \Gamma^+) \text{ such that }
\]

\[
a(v, E) + \frac{1}{2}\langle v^-, j^- \rangle_{\Gamma^-} + \frac{\alpha}{2}\langle v^-, e^- \rangle_{\Gamma^-} - \frac{1}{2}\langle v^-, j^+ \rangle_{\Gamma^-} - \frac{\alpha}{2}\langle v^-, e^+ \rangle_{\Gamma^-} = -j_0\mu_0 \langle v, J \rangle_{\Omega} \\
\frac{1}{2\alpha}\langle \lambda^+, j^- \rangle_{\Gamma^-} + \frac{1}{2}\langle \lambda^+, e^- \rangle_{\Gamma^-} + \frac{1}{2\alpha}\langle \lambda^+, j^+ \rangle_{\Gamma^-} + \\
\frac{1}{k^2}\langle \text{div}_t \lambda^+, \Phi(\text{div}_t j^+) \rangle_{\Gamma^-} - \langle \lambda^+, A(j^+) \rangle_{\Gamma^-} - \langle \lambda^+, R(m^+) \rangle_{\Gamma^-} = \langle \lambda^+, e^{inc} \rangle_{\Gamma^-}, \\
(6.20)
\]

\[
\forall v \in H(\text{curl}; \Omega), \lambda^- \in H^{1/2}_\perp(\text{div}_r, \Gamma^-), \ v^- \in H^{1/2}_\perp(\text{curl}_r, \Gamma^-) \lambda^+ \in H^{1/2}_\perp(\text{div}_r, \Gamma^+) \text{ and } v^+ \in H^{1/2}_\perp(\text{curl}_r, \Gamma^+) .
\]

The discretization process proceeds by approximating both \( \Omega \approx \Omega_h = \bigcup_{m=0}^{M_\Omega} T_{m,m} \) and \( \Gamma^+ = \Gamma_h^+ = \bigcup_{m=0}^{M_\Gamma} \Delta^+_h \) by a set of conformal \( (T_{h,i} \cap T_{h,j} = 0, i \neq j) \) and the intersection of \( T_{h,j} \cap T_{h,j} \) is a vertex, edge of face of both \( T_{h,i} \) and \( T_{h,j} \), similarly for \( \Delta^+_h \) tetrahedrons set \( T_h \) and triangles \( \Delta^+_h \) of average edge length \( h \). Notice that the discretization of \( \Omega \) will induce a
surface tessellation $\mathcal{D}_h$ on $\Gamma$ that will be denoted by a set of $\Delta_h$ triangles. As it was
tioned in the introduction the two triangulations on the interior $\mathcal{D}_h$ and exterior $\mathcal{D}_h$
do not have to be conforming. The finite dimensional weak formulation of the couple
FEM-BEM problems can be stated as:

$$\text{Seek } E_h \in \mathcal{V}_h \subset \mathbf{H} (\text{curl}, \Omega), \ j_h \in \mathbf{A}_h^- \subset \mathbf{H}_i^{1/2} \left(\text{div}, \Gamma^- \right), \ j^* \in \mathbf{A}_h^* \subset \mathbf{H}_i^{1/2} \left(\text{div}, \Gamma^+ \right),$$

$$\mathbf{e}_h^+ \in \mathcal{V}_h^+ \subset \mathcal{H}_-^{1/2} \left(\text{curl}, \Gamma^+ \right)$$

such that (6.20) is satisfied for all $v_h \in \mathcal{V}_h \subset \mathbf{H} (\text{curl}, \Omega)$,

$$\lambda_h \in \mathbf{A}_h^- \subset \mathbf{H}_i^{1/2} \left(\text{div}, \Gamma^- \right), \ \lambda^* \in \mathbf{A}_h^* \subset \mathbf{H}_i^{1/2} \left(\text{div}, \Gamma^+ \right), \ v_h^* \in \mathcal{V}_h^+ \subset \mathcal{H}_-^{1/2} \left(\text{curl}, \Gamma^+ \right).$$

Where the finite dimensional spaces are given by

$$\mathcal{V}_h = \left\{ v \in \mathbf{H} (\text{curl}, \Omega) : v|_T \in \mathcal{N}\mathcal{D}_1^1(T) \forall T \in \mathcal{T}_h \right\},$$

(6.21)

$$\mathbf{A}_h^- = \left\{ \lambda \in \mathbf{H}_i^{1/2} \left(\text{div}, \Gamma^- \right) : \lambda|_T \in \mathcal{R}\mathcal{T}_2 (\Delta) \forall \Delta \in \mathcal{D}_h \right\},$$

(6.22)

$$\mathbf{A}_h^* = \left\{ \lambda^* \in \mathbf{H}_i^{1/2} \left(\text{div}, \Gamma^+ \right) : \lambda^*|_T \in \mathcal{R}\mathcal{T}_1 (\Delta) \forall \Delta \in \mathcal{D}_h \right\},$$

(6.23)

$$\mathcal{V}_h^+ = \left\{ v^* \in \mathcal{H}_-^{1/2} \left(\text{curl}, \Gamma^+ \right) : v^*|_T \in \mathcal{R}\mathcal{T}_1 (\Delta) \forall \Delta \in \mathcal{D}_h \right\},$$

(6.24)

where $\mathcal{N}\mathcal{D}_1^1(T)$ denotes the $p=2$ first Nedelec family hierarchical finite element [74] on
tetrahedrons. In the abovementioned notation, the superscript describes the first family of
the Nedelec elements, whereas the subscript 2 denotes the order. On the other hand,
$\mathcal{R}\mathcal{T}_i (\Delta)$ denotes $p=i^{th}$ order hierarchical Raviart-Tomas finite elements of surface
triangulations. Each space is spanned by the appropriate set of basis functions $w$, $\hat{n} \times w^-$, $\hat{n}^+ \times w^+$ and $w^+$ respectively. That is
where $\vec{e}_n$ refers to the electric field expansion coefficients inside $\Omega$, whereas $\vec{e}_n^-$, $\vec{j}_n^-$ and $\vec{e}_n^+$, $\vec{j}_n^+$ are the electric field and current expansion coefficients on boundaries $\Gamma^-$ and $\Gamma^+$, respectively. Substituting the above expansion for both trial and testing spaces the following final system of equations is obtained

$$
E_h = \sum_{n=0}^{N_{j}} w_n(r) \vec{e}_n^+ + \sum_{n=N_{j}}^{N_{j}+N_{-}} w_n(r) \vec{e}_n^-, \quad w_n \in \mathcal{V}_n,
$$

$$
\vec{j}_n^+ = \sum_{n=0}^{N_{j}} \hat{n} \times w_n^+(r) \vec{j}_n^+ = \hat{n} \times w_n^+ \in \Lambda_n^+,
$$

$$
\vec{j}_n^- = \sum_{n=0}^{N_{-}} \hat{n} \times w_n^-(r) \vec{j}_n^- = \hat{n} \times w_n^- \in \Lambda_n^-,
$$

$$
\vec{e}_n^+ = \sum_{n=0}^{N_{j}} w_n^+(r) \vec{e}_n^+,
$$

where the matrix sub-blocks are given by
\[
(P_E)_{ij} = (P_M)_{ij},
\]
\[
(T_{r-r})_{ij} = \int_{\Gamma'_{is_j}^i} \hat{n} \times w_i(r) \cdot \hat{n} \times w_j(r) \; dx^2,
\]
\[
(T_{r-r})_{ij} = \int_{\Gamma'_{is_j}^i} \hat{n} \times w_i(r) \cdot \hat{n} \times w_j^+(r) \; dx^2.
\]
\[
(D_{r-r})_{ij} = \int_{\Gamma'_{is_j}^i} w_i^+(r) \cdot \hat{n} \times w_j^+(r) \; dx^2,
\]
\[
(D_{r-r})_{ij} = \int_{\Gamma'_{is_j}^i} w_i^-(r) \cdot \hat{n} \times w_j^+(r) \; dx^2.
\]

The excitation vector blocks are given by
\[
(y_i) = \int_{\Omega} w_i(r) \cdot J \; dx^3,
\]
\[
(y_M) = -\int_{S_i^r} w_i^+(r) \cdot j^{inc}(r) \; dx^2,
\]
\[
(y_E) = \int_{S_i^r} \hat{n} \times w_i^+(r) \cdot E^{inc}(r) \; dx^2.
\]

where \(S_i\) is the finite support associated with the \(i^{th}\) basis function. Note that from all four dense sub matrices \(Q_M, Q_E, P_M,\) and \(P_E,\) and only one from each pair needs to be computed and stored in memory. Moreover, each sub matrix is symmetric; therefore an additional factor of two can be saved.

### 6.6 Iterative solution

In this section an efficient preconditioning for iterative solution for the FEM-BEM system will be proposed. The concept is very similar to that used in section 5.5, but unlike section 5.5, the method will be based on a Krylov solver rather than a stationary
iteration. In other words a block diagonal preconditioner will be employed where the two
block sub matrices represent the FEM-ABC problem and the BEM problem, respectively.

6.7 Numerical Results

For all the examples in this section, the FEM unknown space is spanned by the
$p=2$ 1st kind Nedelec tetrahedral elements [130], while BEM utilizes $p=1$ RWG (Rao-
Wilton-Glisson) basis functions [73] over a triangular support. Double precision
arithmetic was used. The codes were designed in a completely modular object oriented
fashion, and the C++ compiler used was the GNU g++ compiler with –O9 optimization
level. All the computations, except for the cases explicitly stated otherwise, were
performed on a Pentium IV 1.8 GHz with 2GB RAM and 256KB L2 cache.

6.7.1 Internal Resonance and Numerical Stability Study

Before start describing some real-life examples it is important to first verify some
of the theoretical clams stated in the previous sections. First the internal resonance issue
will be studied. For the study, a square box air computational domain with one meter side
is considered. The air box is descretized with approximately $h=\lambda_0/4$ tetrahedral elements.
For the present geometry the internal resonance (which will be both TE and TM due to
the degeneracy) should occur around 212 MHz. To identify the presence or absence of
internal resonances the spectral condition number of the proposed DD FEM-BEM matrix
is considered in the neighborhood of the resonance frequency. The condition number
was estimated using the open source SPARSE software. The results of the study are
shown in Figure 6.2(b) and (c) with solid blue line. As it is apparent neither the diagonal
scaled nor the non-diagonal scaled systems show signs of condition number increase
around the resonance. To further support the theory that the direct boundary integral formulation utilized here does not suffer forbidden frequencies the BEM sub matrix is plotted in black line. Notice that when the sparse matrix $T$ is added into the BEM sub matrix, the condition number improves by almost one order of magnitude. In this case $T$ acts as a regularizer. On the other hand the symmetric formulation proposed in [74, 115, 116] does suffer internal resonances as it is shown in Figure 6.2(a).

The same one meter air box is now considered but the discretization size is increased by keeping the frequency constant and equal to 300 MHz. The spectral radius distribution of the preconditioned system is plotted in Figure 6.3. Each subfigure represents the spectral radius for double the number of unknowns each time. As it is observed all the eigenvalues are within the unity circle, which implies that even a stationary iteration method would lead to convergent and stable solution process. It should be noted that the similar spectral radius plots are observed for all the DD based methods described in this dissertation, namely the FEM domain decomposition of CHAPTER 3 and the infinite periodic solver of CHAPTER 5. It is apparent that the spectrum has three accumulation centers one for low frequencies at 1 and one for high frequencies (evanescent modes) around -1 and one for the at zero. Notice that the frequency here needs to be interpretive as the rate of oscillation of field on the plan of the interface since the eigenvalues represent the $M^{-1}A$ which is

$$I - M^{-1}A = \begin{pmatrix} 0 & A_1^{-1}C \\ A_2^{-1}C & 0 \end{pmatrix}$$

(6.38)

where $A_1$ and $A_2$ are the sub domain matrices, in this case the FEM-ABC and Direct Boundary Integral BEM matrix. Therefore high oscillating modes on the interface
(transverse direction) are evanescent in the normal direction since \( k_t = \sqrt{k_0^2 - k_i^2} \) is purely imaginary if \( k_0^2 > k_i^2 \). It is apparent that as the mesh size decreases, the accumulation points become more clustered and at the same time the clusters approach closer to 1 and -1. The clustering of the eigenvalues is a desirable behavior, but the tendency towards -1 increases the condition number since \( \rho(I - M^{-1}A) = 1 + \kappa(M^{-1}A) \).
Figure 6.2: Condition number of the system of equation in the neighborhood of the “internal” resonance. (a) Symmetric FEBI formulation. (b) Proposed DD FEM-BEM approach without diagonal scaling. (c) Proposed DD FEM-BEM approach with diagonal scaling.
6.7.2 Dielectric Sphere Scattering

The first example consists of a dielectric sphere of $\varepsilon_r=2$ and $\mu_r=1$, with 1m radius under a monochromatic plane wave incidence. The computational domain is that of the sphere with the truncation boundary paced right at the dielectric-to-air interface, the discretization size is kept at $h=\lambda_0/5$, where $\lambda_0$ is the free space wavelength at each frequency. The bi-static RCS in the plane of incident for the frequencies of 300 MHz and 600 MHz is shown in Figure 6.4 (a) and (b), respectively. The FEM-BEM results are in very close agreement to the exact Mie series solution. More specifically, at 300 MHz leads to RMS error in the RCS of approximately 1.75% while for 600 MHz the error is 3.25%. For these examples, 155,720 and 363,846 FEM unknowns were used, for each frequency. The BEM unknowns used where 5,280 and 24,576, respectively. The BEM matrix is assembled with ACA acceleration [122, 131, 132] requires 174s and 1,577s for

Figure 6.3: Eigenvalue distribution of the preconditioned system ($\mathbf{I} - \mathbf{M}^{-1} \mathbf{A}$) for: (a) $N = 1076$ unknown problem. (b) $N = 2708$ unknown problem. (c) $N = 4824$ unknown problem. The frequency is kept constant $f=300\text{MHz}$.
each frequency. The memory storage of the BEM portion is 32MB and 263MB. The convergence history of the DD FEM-BEM is reported in Figure 6.5. It should be stressed that a plain Conjugate Gradient solver has been used, where the residual error refers to the infinite norm of the relative residual

\[
\mathbf{r}^{(n)} = \frac{\| \mathbf{Ax}^{(n)} - \mathbf{b} \|}{\| \mathbf{b} \|} \quad (6.39)
\]

where \( \mathbf{A} \), \( \mathbf{b} \) are the final matrix and right-hand side of (6.26) and \( \mathbf{x}^{(n)} \) is the \( n \)th approximate of the solution vector. As it is expected the convergence deteriorates as the frequency increases. This should not be alarming due to the wave nature of the problem and the lack of a global preconditioner for the interface problem.

![Figure 6.4: Bistatic scattering from dielectric spheres, comparisons with Mie series solution. (a) 1\( \lambda_0 \) diameter sphere. (b) 2\( \lambda_0 \) diameter sphere.](image)

Figure 6.4: Bistatic scattering from dielectric spheres, comparisons with Mie series solution. (a) 1\( \lambda_0 \) diameter sphere. (b) 2\( \lambda_0 \) diameter sphere.
Figure 6.5: Convergence history of the DD FEM-BEM based on a CG solver with block diagonal preconditioning, for different frequencies.
6.7.3 Coated PEC Sphere Scattering

To compare the performance of the present approach with other existing symmetric FEBI methods a dielectric coated PEC sphere is considered. The example has been borrowed from [74], where the symmetric FEBI is used. The sphere consists of a 0.3423\(\lambda_0\) inner PEC shell radius coated by a dielectric of \(\varepsilon_r=4.0\) and \(\mu_r=1.0\) of 0.444 radius. The dimensions are chosen such that the internal resonance occurs at the frequency of operation. The bistatic scattering pattern obtained by the proposed DD FEM-BEM is overlaid with the analytical solution and the symmetric FEBI of [74] in Figure 6.6(a). It is clear that the DD FEM-BEM is more accurate, and at the same time it converges 10 times faster than the symmetric FEBI of [74], as indicated by Figure 6.6(b). In this example only 93MB of memory were used and the solution time was 51 seconds, the assemble time of both FEM and BEM was 381 seconds. The problem was discretized with 60 thousand FEM unknowns and 8 thousand BEM unknowns.
6.7.4 Generic Battleship Scattering

To demonstrate the versatility of the method a more complicated example is considered; that is the scattering by a generic battleship. The geometry, dimensions, excitation and FEM-BEM computational domain are shown in Figure 6.7. Notice that the BEM surface is quite complicated, non-convex, and the ship is in the free-space. The bistatic scattering patterns for 30, 60 and 120 MHz are plotted in Figure 6.8. In each figure the solid line represents the DD FEM-BEM solution and red dashed line the method of moments solution with IE-FFT acceleration [123]. Notice that since the battleship is perfect electric conducting an efficient PEC based EFIE MoM solution can be computed. For all frequencies the comparison between DD FEM-BEM and MoM is
very good. The minor discrepancies can be due to the different mesh densities or dispersion error due to the FEM part in the hybrid approach. The DD FEM-BEM mesh was obtained from an initial discretization of \( h = \frac{\lambda_c}{6} \) with 3 h-adaptive mesh refinements. The computational statistics of the DD FEM-BEM simulations are reported in Table 6.1 for a 64bit dual AMD® Opteron™ 246 with 1024KB L2 cache and 16GB RAM. Moreover, it should be stressed that the BEM computations of the proposed DD FEM-BEM where accelerated by the IE-FFT algorithm of [124]. In the last row of Table 6.1 the number of iterations reported refers to \( 10^2 \).

Figure 6.7: A generic battleship. (a) Geometry and dimensions. (b) Computational domain.
Table 6.1: Computational statistics of the generic battleship simulations using the DD FEM-BEM. Simulations performed on a 64bit dual AMD® Opteron™ 246 with 1024KB L2 cache and 16GB RAM.
CHAPTER 7

EPILOGUE

7.1 Summary

The rationale of employing non-conforming domain decomposing methods for time-harmonic electromagnetic problems and the importance of the problem was given in CHAPTER 1. A detailed literature review for the three main ingredients of this work was presented. Namely, the existing work on domain decompositions for Maxwell’s equations was given along with non-conforming mesh methodologies and FETI substructuring algorithms. Finally a clear statement of the contributions of this dissertation closed the chapter.

The second chapter is an introduction to notations and conventions used throughout the document. Some important and widely used function spaces, trances and definitions, and theorems were also given.

CHAPTER 3 is the core of this dissertation. The basic ideas of the proposed non-conforming domain decomposition concepts are described. The boundary value statement of the continuous problem is stated and the importance of the transmission condition and the appropriate functions spaces are emphasized. The convergence properties of the iterative non-overlapping domain decomposition (Schwarz) algorithm are analyzed using the Fourier domain and the TE/TM decomposition. The main idea on the newly
developed cement element method for coupling problems with non-matching grids across interfaces in given. Finally the chapter ends with a variant of the FETI substructuring algorithm that allows fast computations of periodic structures with repetitions.

The results of the FEM based domain decomposition are described in CHAPTER 4. Radiation scattering and waveguiding problems are analyzed. Starting with radiation, the method is first compared with other well established methods and against measured data for large finite arrays on finite mounting platforms. The efficiency and memory saving of the method are also documented. The radiation from cylindrically conformal finite patch arrays on finite cylinders is, for the first time in the numerical literature, presented. Various parametric studies and radiation behaviors are documented and compared with the planar counterparts. Next, performance of the proposed method in the analysis of scattering problems is studied. The method is validated against method of moments published data for finite frequency selective surfaces and passive arrays. Having established the validity and accuracy, a number of case studies are considered for the scattering by finite antenna arrays with loading. The reminding of the chapter is devoted to artificial electromagnetic materials and photonic crystal computations. Various photonic crystal nanocavities and waveguide bends are simulated including the full 3D radiation loss effects. Then the effect of artificial high-impedance surfaces on the radiation of simple antennas is studied.

The extension of the cement method, to incorporating periodic boundary condition in FEM computations of infinite periodic structures with non-periodic mesh is presented in CHAPTER 5. The Floquet’s theorem is combined with the cement element method to impose periodic boundary conditions on the opposite sides of a general
unstructured grid. The variational problem is formulated similarly to that in CHAPTER 3 but leads to a matrix with non Hermitian structure. The problem is appropriately reformulated in order to be Hermitian and retain the lossless nature of PBCs. An elegant and efficient iterative solution process is proposed based on the Richardson iteration and a symmetric/Hermitian matrix splitting. Numerical results and comparisons with existing methods demonstrate the accuracy of the method.

Finally this dissertation proposes a new approach for solving unbounded EM problems using the coupled FEM and BEM. The method is based on the domain decomposition and its cornerstone ideas are the theories developed in CHAPTER 3. The method leads to symmetric, internal resonance free matrices that allow for non-conforming triangulations and approximations in the truncation boundary. Moreover, proposed iterative solution guarantees spectral radius less or equal to one. The idea is to treat the FEM and BEM problems as a domain decomposition domain. Unlike CHAPTER 3 the problem is formulation in a symmetric manner and it is solved using an efficient preconditioned conjugate gradient method. The numerical results show a very promising performance in terms of accuracy, versatility and convergence properties.

7.2 Conclusions

This dissertation described a comprehensive domain decomposition-based methodology and its successful application to time-harmonic electromagnetic problems. Overall, very encouraging results, numerical behaviors and computational statistics were produced making this domain decomposition method a valuable alternative for future state-of-the-art computational engines. Several applications of interest to industry were presented.
Among others, this first attempt to use the proposed non-conforming non-overlapping domain decomposition method for electromagnetic computations revealed a number of issue and challenges. It was concluded (see CHAPTER 3) that the present Robin transmission condition works well for radiating problems and problems with rich preparative error-mode spectrum, but the convergence tends to saturate for evanescent modes. With the aid of the TE/TM decomposition and Fourier analysis proposed herein, higher order transmission conditions need to be developed for better convergence. Alternatively Krylov solvers such as CG or GMRES need to be resorted, with a domain decomposition based preconditioner.

The cement element method is a promising mixed variational method for imposing various constraints like non-conforming meshes. It works surprisingly well even for complicated meshes, as long as the mesh ratios on either side of the interface were of compatible order. Unlike the well established mortar method, one of the biggest advantages of the cement method is its symmetric treatment of both sides of the interface. Moreover, since no constraints are enforced on the Lagrange multiplier sets, the support of the basis functions remains local and subdomain adaptive meshing strategies are completely localized to that domain alone. A complication in the implementation of the cement, as well as any other non-conforming method, arises from the integration across non-conforming mesh interactions. Here, it should be emphasized that only a careful integration on the union mesh of each non-conforming interface would lead to efficient and accurate integration strategies.

The proposed FETI algorithm is similar to the two-Lagrange multiplier FETI used in mechanical engineering. The method was conceived and understood as a systematic
method of obtaining “numerical” Green’s or EM transfer functions of domains. Even though a direct factorization could achieve that goal, in this work an iterative process, more in line with the aforementioned insight, was utilized. In particular, for problems with large number of identical subdomains such as antenna arrays and metamaterials it was found that method tends to be faster, but potentially more memory hungry than a direct application of the domain decomposition. In addition, the FETI algorithm it was found very efficient for modular optimization. Among other, a contribution of this dissertation is the combination of the cement method with the FETI substructuring algorithm.

The mixed formulation for infinite periodic structures without periodic mesh is another example of applying the cement element method. The method was found versatile, efficient and accurate. Among others, the great prospect of this approach is on adaptive meshing and error conodeled FEM computations for infinite structures. It should be stressed that the method is compatible with FEM-BEM or FEI computations of infinite structures.

The coupling of FEM and BEM using the domain decomposition and cement techniques is also a promising approach. The method was found extremely efficient, stable and versatile. Even though the convergence of the method was not optimal in terms of frequency scaling (increasing the frequency results into larger number of iterations), it performed better than other existing formulations. It should be however explicitly stated that although the method employed more unknowns that is counterparts it tends to be more efficient when iterative solvers are used. This is because it converges faster and the preconditioning strategies are more efficient and effective. It should also be noted that
because a Krylov solver was used, the method tends to be less susceptible to the convergence issues reported for in CHAPTER 3. Similar to the FEM domain decomposition, it is believed that better transmission conditions will directly improve the convergence of DD FEM-BEM.


