COLLEGE STUDENTS' INTUITIVE UNDERSTANDING OF THE CONCEPT OF LIMIT AND THEIR LEVEL OF REVERSE THINKING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in the Graduate School of The Ohio State University

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This thesis addresses the relationship of students’ intuitive understanding of the limit of a sequence to their reversibility, which is an ability to reverse the order between $\varepsilon$ and $N$ as required in the rigorous definition of limit. The subjects of this research were students who had not had any experience with rigorous proofs using the $\varepsilon-N$ definition of a limit. Eleven students completed a series of 1-hour semi-structured, task-based interviews once a week for 5 weeks. Monotone bounded, unbounded, constant, oscillating convergent, or oscillating divergent sequences were tested. Students represented each sequence numerically as well as graphically in determining convergence of the sequence. Students also used tools, called $\varepsilon-\text{strips}$, specially developed for this study to explore the $\varepsilon-N$ relation in defining limits through hands-on activities with the $\varepsilon-\text{strips}$. Finally, students were presented the following $\varepsilon-\text{strip}$ definitions, and were asked to evaluate the propriety of the definitions as statements of the limit of a sequence.

$\varepsilon-\text{Strip Definition A:}$ A certain value $L$ is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any epsilon strip as long as the epsilon strip covers $L$.

$\varepsilon-\text{Strip Definition B:}$ A certain value $L$ is a limit of a sequence when only finitely many points on the graph of the sequence are NOT covered by any epsilon strip as long as the epsilon strip covers $L$. 
It was found that students’ understanding of the definition of the limit of a sequence was associated with not only their conception of limit but also their level of reversibility. In addition, there was improvement in students’ reversibility and/or their conception of limit through the \( \varepsilon - \) strip activity, even though there was no procedure for indicating students’ errors, correcting students’ misconceptions about limit, or confirming the propriety of the \( \varepsilon - \) strip definitions to students during the interviews. This study implies that the \( \varepsilon - \) strip activity, combined with the various types of sequences used in this teaching experiment, is one instructional method for helping students develop and accommodate their conception of the limit of a sequence.
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CHAPTER 1

STATEMENT OF THE PROBLEM

This research examined how college students understand the concept of limits and develop their conception of limits in the case of sequences. The subjects of this research were students who had no experience with any mathematically rigorous processes using the \( \varepsilon - N \) definition or proofs related to limits. The study focused on how they applied their conception of limits to determine the convergence of a sequence and its limit. In particular, the students were asked to participate in activities using some tools for measuring how many terms of a sequence are or are not close to the limit within given differences. This research investigated how students accommodated and developed their conception of limits after participating in the activities.

The Problem

Importance of Limits

Teaching and learning the concept of limit has long been a very important and interesting research subject to mathematics educators. First, the concept of limit plays an important role in mathematics. From the time that Isaac Newton and Gottfried W. Leibniz formulated calculus in the 17th century, the concept of limit has been regarded as one of the foundational concepts in mathematics. In fact, it is necessarily used in defining and
deploying various other concepts and theories in mathematics. Of course, the concept of limit is important in itself, as in problems of determining the convergence of a sequence and calculating the value of the limit. However, the main importance of the concept of limit is that many other mathematical concepts depend upon it, and it plays a crucial role in applying various mathematical theories.

For example, the sum of an infinite series, continuity of a function, the derivative of a function, and the integral of a function all involve the concept of limit in defining them. In defining an infinite series \( \sum_{n=1}^{\infty} a_n \), we consider the sum \( \sum_{k=1}^{n} a_k \) of the first finite number of values of \( a_n \), called the partial sum, and determine whether the sequence of the partial sums has a limit or not. As another example, when we say a function \( f(x) \) is continuous at a point \( x_0 \), we mean that the limit of the function \( f(x) \) exists as \( x \) approaches \( x_0 \), and coincides with the value \( f(x_0) \), i.e., \( f(x_0) = \lim_{x \to x_0} f(x) \). Therefore, finding the \( \lim_{x \to x_0} f(x) \) is essential in determining continuity of the function \( f(x) \).

Besides the cases mentioned above, many other concepts in advanced mathematics are based on the concept of limit. Ferrini-Mundy and Lauten (1993) regard the concept of limit as one of the most fundamental in calculus. Tall (1992) points out that the concept of limit is one of the foundational ideas in understanding advanced mathematics beyond calculus.

Not only is the concept of limits dealt with in advanced mathematics, it is a phenomenon easy to find in the world around us. For instance, if two mirrors are placed opposite each other, uncountably many mirror images are formed by the reflection of
mirrors, as we see in many barbershops and hair salons. By observing them in detail, one can easily see that such mirror images have a limit point. Moreover, one can find the concept of limit in K-12 mathematics, even though it may not be definite mathematically. For example, infinite decimal representations of real numbers and the process of deriving the formula for the area of a circle are typical cases using the idea of limits. Thus, the limit is not only a tool used to develop theories in advanced mathematics, but also a concept to analyze problems encountered in daily life.

Examples like those above show that the concept of limit is experienced and used earlier than one might expect. NCTM (1989) points out in *Curriculum and Evaluation Standards for School Mathematics* that the concept of limit is one of the most important topics studied in high school. Such a viewpoint is in line with those of mathematics educators who insist on the necessity of teaching and learning the concept of limit (Ferrini-Mundy & Lauten, 1993; Tall, 1992).

**Misconceptions About Limits**

In general, students who start to study limits are led to conceptualize the idea of a limit in the same order as they read the symbol of the limit. To be precise, students are taught that the limit of a sequence $a_n$ is a certain value which, as the index $n$ goes to infinity, the sequence is “approaching” or “close to.” That thinking is identical to the order of reading $\lim_{n \to \infty} a_n$ as “the limit of the sequence $a_n$ as $n$ goes to infinity.” However, thinking of limits in this order may interfere with understanding mathematically the concept of limit. According to the formal definition of a limit, we start with the limit and then see if all terms in the sequence after some $n$ are close to the limit. This requires a type of reverse thinking to be discussed in more detail below. No
wonder that Ferrini-Mundy and Graham (1991) point out that even students who answered limit problems correctly often encountered discordance between the answer perceived intuitively and that obtained through algebraic manipulations.

One of the common misconceptions about limits is that students believe that the limit is the endless dynamic motion itself. Students possessing such a misconception tend to disregard the value of the limit as the result of the dynamic motion, and in fact, such students often conclude that a specific value could not be the limit of the sequence (Szydlik, 2000; Williams, 1991).

Another misconception of limits is related to the limit of constant sequences or constant functions. According to the pilot studies preceding the present research, some students responded that the limit of a constant sequence did not exist on the ground that they could not find any dynamic motion in its value.

Another misconception related to limits is to assume the dynamic motion of a sequence is restricted when the sequence is convergent. For instance, if a value is regarded as the limit of a given sequence, some students believe no term of the sequence should exceed the value on one side, as if the limit were an upper bound or a lower bound of the sequence. For instance, some students in the pilot studies meant by the word *approaching* that the terms of the sequence were getting close to the limit either from above or else from below. These students believed the alternating sequence \( a_n = (-1)^n / n \) did not converge to zero because even-numbered terms of the sequence are getting close to zero from above whereas odd-numbered terms are approaching from below. Due to this misconception, they categorized alternating sequences as divergent sequences.
Finally, it should be pointed out that some students in the pilot studies did not accurately distinguish *getting close to* from *clustering to*. This caused the misconception that a sequence may have several limit values and any cluster point can be a limit point. Actually, some students in the pilot studies regarded each of the cluster points, 1 and −1, in the case of the alternating sequence \( a_n = (-1)^n (1 + 1/n) \) as a limit point and stated as their the reason that infinitely many terms of the given sequence are densely close to each of the values, 1 and -1.

As mentioned above, most misconceptions about limits are presumed to be caused by the meaning of the words used in expressing the limit symbol \( \lim_{n \to \infty} a_n \). In particular, such expressions as *approaching* or *getting close to* used in reading the limit symbol do not precisely convey the mathematical meaning of the concept of limit. Moreover, they also convey an everyday meaning slightly different from the mathematical one, which may affect students’ conceptualization of limits. The usual meaning may interfere with students’ understanding of the concept of limit by leaving afterimages different from the mathematical meaning (Tall & Vinner, 1981).

Misconceptions about limits mentioned above may reflect on studying other related topics. Consequently, they land students in difficulties in learning subsequent mathematics (Benzuidnhout, 2001; Cornu, 1991; Merenluoto & Lehtinen, 2000; Sierpińska, 1987; Tall & Schwarzenberger, 1978). Such difficulties are continued in negative experiences in solving relevant mathematics problems. Repetition of such negative experiences will eventually deprive students of their desire to solve mathematical problems and will lead them not to trust their own mathematical thinking.
Precise Understanding of the Concept of Limit

It is instructive to review the historical development of the concept of limit. In fact, mathematicians used limits in somewhat obscure ways until the rigorous definition of the limit was suggested by Karl Weierstrass (1815–1897). It is important to note that the definition of a limit prior to that time implied the possibility of errors similar to the students' misconceptions mentioned above.

For example, in the 17th century when Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) formulated calculus, a limit was described as a quantity which a variable approaches but never exceeds. In particular, a limit was regarded as a value that certain quantities can approach “nearer than by any given difference, but never go beyond, nor in effect attain to, until the quantities are diminished in infinitum” (Grabiner, 1981, p. 82).

This viewpoint of the concept of limit still appeared in the 18th century when algebraic techniques for calculating limit values had been well developed. A mathematician Jean Le Rond d'Alembert (1717–1783) described the concept of limit as follows:

One magnitude is said to be the limit of another magnitude, when the second can approach nearer to the first than a given magnitude, as small as that given magnitude may be supposed; nevertheless, without the magnitude which is approaching ever being able to surpass the magnitude which it approaches [italics added], so that the difference between a quantity and the limit never coincide, or never become equal, to the quantity of which it is a limit, but the latter can always approach closer and closer, and can differ from it by as little as desired. (Grabiner, 1981, p. 8)

In this definition, the limit was a certain value that quantities approach closer and closer to but never coincide with or exceed. However, the concept of limits in those days was
defined somewhat indefinitely, hence such definitions might make the concept of limit rather to be uncertain. For example, the expression "approaching" implied "not to be equal" and "not to go beyond." On the other hand, the restriction that each term of a convergent sequence should not be equal to the limit value causes an error that a constant sequence never converges. Moreover, according to such a definition, no alternating sequence can be convergent. It is remarkable that the definitions of limit used in the 17th and 18th centuries caused the same problems that students had in the pilot studies.

The present research investigated the precise understanding of the concept of the limit of a sequence as described by the current, rigorous \( \varepsilon - N \) definition as follows:

We call a sequence \( a_n \) to be convergent to a real number \( A \) if for any positive number \( \varepsilon \), there is a natural number \( N \) such that \( |a_n - A| < \varepsilon \) for all \( n \geq N \).

\[(Apostol, 1974, p. 70)\]

This definition of the limit first suggested by Weierstrass resolves the ambiguity of definitions used in earlier eras, and is the commonly accepted definition of the limit nowadays. It should be noted that the index \( N \) in the above rigorous definition is properly chosen dependent on the given error \( \varepsilon \) measuring the difference from the limit value. This implies that the index \( N \) varies as the positive number \( \varepsilon \) moves around. In particular, the smaller \( \varepsilon \) is chosen, the larger \( N \) becomes. Hence the above rigorous definition reflects the dynamic motion implied in "as \( n \) goes to infinity, the sequence approaches its limit."

As noted above, an index number is properly chosen later according to the error determined as the difference between each term of the sequence and the limit. However, in contrast, students typically first choose an index number and next determine how close the term corresponding to the index is to a certain value, just as they read the limit
symbol. Comparing the process of students' thinking (for $N$, see what the error $\varepsilon$ is) with that of $\varepsilon-N$ definition (given $\varepsilon$, find $N$), one sees that the order of finding $\varepsilon$ and $N$ is reversed. If we accept that the order of intuitive thinking is the same as in reading the limit symbol, then the order of students' intuitive thinking seems natural, whereas the order required in the $\varepsilon-N$ definition can be regarded as a reverse process.

In line with this viewpoint, throughout the present research, we call the process of thinking implied in the $\varepsilon-N$ definition reverse thinking in the context of the limit of a sequence. Throughout this dissertation, the ability to reverse one’s thinking, or reversibility, means the ability to use such a reverse process by reconstructing the problem and finding the solution by thinking backwards.

Reverse thinking plays a crucial role in the rigorous definition of the limit. To be precise, a positive number $\varepsilon$ is first chosen as a bound dominating the absolute value of the difference between terms of a given sequence and the limit, and afterwards the index number $N$ is determined depending on $\varepsilon$. Such a process requires that as $\varepsilon$ varies, $N$ can be always determined such that for each term corresponding to indices after $N$, $\varepsilon$ chosen in advance can play a role as a bound of the difference between the sequence and its limit value. Therefore, the rigorous definition of the limit naturally represents the dynamic motion implicit in the concept of limit.

At the same time, the rigorous definition of the limit overcomes various unnecessary restrictions that appeared in definitions used during the 17th and 18th centuries. For example, in the case of constant sequences, the difference between each term and the constant itself is always zero; hence it is less than any chosen positive number $\varepsilon$. Consequently, every constant sequence is convergent and the constant itself
is the limit of the constant sequence. Also, by comparing the absolute value of the difference with an error $\varepsilon$, we can deal with convergence even in the case of sequences oscillating around their limits, in particular, alternating sequences.

The Role of Meaningful Intuitive Cognition in Understanding the Concept of Limit

Mathematically accurate intuition plays an important role in conceptualization of mathematical knowledge as well as in rigorous proofs. Usually, intuitive cognition is regarded as a certain immediate cognition directly accepted prior to any kind of rigorous justification, or without any feeling of necessity of rigorous proof (Fischbein, 1994). In line with this viewpoint, throughout the present research the term intuitive understanding means immediately cognizing some concepts without any rigorous proof.

Intuitive cognition proves its worth in directly and precisely gripping the essence of a given problem. Actually, mathematicians often attempt to recognize the essence of a given problem via their intuitive understanding before a rigorous proof. Also, as we know, an intuitive idea often opens a new area in mathematics and provides a clue to the development of a mathematical topic. One such example is the following concept of limit intuitively defined by Augustin-Louis Cauchy (1789–1857), one of the most important mathematicians contributing to the development of rigorous calculus:

When the successively attributed values of one variable approach indefinitely a fixed value, finishing by differing from that fixed value by as little as desired, that fixed value is called the limit of all the others. (Grabiner, 1981, p. 7)

In the above, Cauchy also used the term approach in representing the dynamic motion in the limit process as did mathematicians in the 17th and 18th centuries and as do students nowadays. Nevertheless, Cauchy's definition is distinguished from previous ones in the sense that the term approach was used in describing the relation between $\varepsilon$ and $N$ in
real problems related to the limit (Grabiner, 1981). Such an $\varepsilon - N$ method intuitively described by Cauchy would provide the principal idea behind the definition to be rigorously established by Weierstrass later on.

One of the merits of intuitive understanding is that it gives conviction and certainty toward the derived result from a given problem. Pointing this out, Fischbein (1987) also insists that when a student has a problem in accepting a certain concept intuitively, he or she cannot be sure of his or her own reasoning, hence feels difficulties in understanding problems related to the concept. For example, to possess intuitive understanding is very important in accurately comprehending the rigorous definition of the limit. As reported by Cottrill et al. (1996), only a few students understood the rigorous definition of limits, and even these students had difficulties in applying the rigorous definition to solve problems related to the limit. On the other hand, Szydlik (2000) found that many students had difficulties in understanding the rigorous definition of limits, whereas students who understood intuitively the rigorous definition tended to trust their decision in solving related problems. As mentioned above, the intuitive understanding of limits is regarded as a crucial factor in understanding the rigorous definition of the limit and solving related problems.

It should be noted that mathematical experiences seem to be very important in developing intuition that is accurate and compatible with objective mathematical concepts. In general, mathematicians seem to possess mathematically well-developed intuition. Fischbein (1987) considers accurate intuition in a professional area, such as mathematics, as a factor distinguishing experts from novices. Also, he asserts that a learner can develop his or her intuition, depending on what kind of experiences he or she
has. Such viewpoints mentioned above suggest that intuition compatible with the rigorous definition of limit can be developed through adequate experiences.

**Research Questions**

The present research investigated college students’ intuitive understanding in determining convergence and the limit of a sequence. Also, it studied a number of students to investigate how they developed and accommodated their intuitive cognition of the limit of a sequence. Students were asked to participate in activities in which they determined the convergence and the limit values of typical sequences. For each task activity, the students were asked to represent a sequence numerically as well as graphically. Some tools, called $\varepsilon$–strips, were specifically developed for the research to measure how close the terms of the sequence were to the limit. By using the $\varepsilon$–strips, students were asked to determine a proper index number $N$ corresponding to each given $\varepsilon$–strip. Finally two arguments, called $\varepsilon$–strip definitions, interpreting the reverse relation between $\varepsilon$ and $N$ in the context of sequences were shown to students, and they were asked to compare their own conception of the limit of a sequence with the $\varepsilon$–strip definitions.

The research addressed the following specific question: “Does an activity with the graphical illustration of a sequence, along with statements describing the reverse relationship between $\varepsilon$ and $N$, influence development and accommodation of students’ intuitive understanding of the concept of limit?” In order to investigate this research question, the following sub-questions were addressed:
1. How do students explain their understanding of convergence and the limit of a sequence?

2. How do students explain the $\varepsilon - N$ relationship in the context of the limit of a sequence?

3. How are the levels of the development of students’ reversibility, that is, an ability to understand the $\varepsilon - N$ relationship, associated with students’ intuitive understanding of the limit of a sequence?

4. How different is students’ intuitive understanding of the limit of a sequence after the teaching experiment? How different is students’ reversibility after experiencing the $\varepsilon –$ strip activity?

**Theoretical Framework**

Many philosophers and psychologists in education consider intuition as a mental strategy or a method that enables humans to ascertain the essence of phenomena (Bergson, 1944; Descartes, 1967; Spinoza, 1967). Poincaré (1920/1956) even argued that no genuine creative activity is possible in science and mathematics without intuition. On the other hand, there is also a negative perspective toward using intuition in conceptualization. For the philosophers and psychologists opposed to using intuition, intuition is regarded as an elementary and primitive form of knowledge, as opposed to an advanced and scientific conception and interpretation. For Kant (1911/1980), intuition refers to the ability through which objects are directly grasped and is distinguished from the ability of understanding through which we achieve conceptual knowledge. Hahn (1956a, 1956b) even treated intuition as the source of misconceptions and asserted that intuition should be eliminated from a serious scientific study.
This section proposes that intuitive cognition is important in the acquisition of the concept of limit. It is likely that students utilize their intuition first, and then justify their intuitive ideas via the definition of limit (Szydlik, 2000; Tall & Vinner, 1981). Even though the rigorous definition of the limit provides mathematically valid ideas about limits, it is not easy to resolve discrepancies between most students' own intuitive ideas about limits and those of mathematicians. Also, mathematicians have some well developed ways to refine their intuitive idea, but most students still need to have more training in these methods. Consequently, most students’ intuitive idea may remain vague, and students have difficulty constructing proper images of limits. The ways that mathematicians have been able to refine their intuitive ideas about limits into an efficient and accurate concept may not parallel students’ understanding of limits. Consequently, students’ intuition may remain vague, self-contradictory, and too diffuse to constitute any reliable image of limits (Williams, 1991).

This section also focuses on reversibility in understanding mathematics. First, important aspects of reversibility asserted by Inhelder and Piaget (1958) and independently by Krutetskii (1969) will be dealt with in this section. Discussion of reversibility as related to the definition of limit and its association with intuitive cognition of limits is then provided.

**Formal Cognition and Intuitive Cognition**

Once mathematics is assumed to be a formal and rigorous body of knowledge as conveyed in mathematics textbooks, mathematical knowledge consists of axioms, definitions, and theorems, with a logical structure (Ervynck, 1991). Logical and analytic thinking processes then play crucial roles in representing the logically structured body of
mathematical knowledge. These sorts of mathematical thinking are produced through conscious mental processes, which are essential in mathematics. However, pursuing precision and utilizing formalism tend to be applied only to the final products of mathematical activity. Rather, the process of formulating the mathematical knowledge includes imagery of mathematical structures that unconsciously produces recognition of certainty or uncertainty, verification or refutation without proofs, as well as interaction among them (Fischbein, 1994; Kossak, 1996). Therefore, in order to articulate the two different types of mental activities involved in mathematical concept formation, we assume that human mental activities consist of formal cognition and intuitive cognition of mathematical knowledge.

Formal cognition refers to cognition controlled by mathematical logic and proofs via mathematical induction or deduction (Fischbein, 1994). Utilizing the definition of the limit and the theorems related to limits corresponds to formal cognition. Formal cognition provides a rigorous way to understand mathematical knowledge. It allows us to reflect on our mental activities and to convince ourselves of the credibility of mathematical knowledge obtained from logic or from already established mathematical knowledge. Formal cognition is also necessary for mathematicians to communicate with each other in mathematical society. Similarly the perception of formal knowledge is absolutely necessary for students to move toward a higher level of mathematical knowledge.

However, formal cognition does not explain every step of the thinking involved in mathematical activities. Development of the ability to understand and to use formal knowledge is unlikely to guarantee mathematical creativity, which is important in
“doing” mathematics, such as making conjectures or new knowledge claims. Furthermore, it is not clear whether we can develop mathematical creativity through developing only formal cognition. Students might become very logical and quite highly confident with their reasoning abilities in proving mathematical statements. However, only a few students who have been well trained in actively using their formal knowledge are likely to become creative in mathematical thinking.

We therefore presume that there is a different kind of mental activity from the formal cognition operating in mathematical activities. We call it *intuitive cognition*, or *intuition*. Fischbein defines intuition as the “immediate cognition that is accepted directly without the feeling that any kind of justification is required” (Fischbein, 1994, p. 232). In a similar way, Beth and Piaget (1966) regarded intuition as a cognition which is directly grasped without, or prior to, any need for explicit justification or interpretation.

The present study, in relation to the conception of limits, regards *intuitive cognition* or *intuition* as cognition directly accepted without or prior to any rigorous justification or interpretation. The term *intuitive understanding* in this study simply means immediately cognizing some concept without any kind of rigorous process.

![Figure 1.1 Formal cognition versus intuitive cognition](image-url)
Role of mathematical intuition. First of all, intuition is closely related to creativity or inventiveness in mathematical science (Courant & Robbins, 1963; Erwynck, 1991; Fischbein, 1994). It is likely that many talented mathematicians and scientists, such as Henri Poincaré or Albert Einstein, possessed mathematical or scientific sense about the essence of an idea before they used formal knowledge. Indeed, this mathematical sense seems to be related to the intuitive cognition that produces creative ideas (Poincaré, 1913). Their intuition did not lead them to false results but revealed the essence of mathematical knowledge (Fischbein, 1994). For people like Poincaré and Einstein, the function of formal cognition is likely to verify and formulate the idea more precisely as the very last step of the creative process. Albert Einstein, the originator of the theory of relativity, wrote a letter to Jacques Hadamard, a distinguished mathematician, stating that formal knowledge seemed not to play a role in his thinking mechanism, even though he used mathematically formal theories of Riemannian geometry to justify his relativity theory (Einstein, 1955; Hadamard, 1945). Thus, these creative ideas seem to lie, not in a conscious mental process that uses formal knowledge, but in unconscious ideas germinating over a long period of time, and eventually merging intuitive ideas into consciousness.

It should be noted that mathematically talented people, such as Henri Poincaré, are convinced that their intuitive ideas lead them to mathematical truth. In fact, intuition of two-dimensional or three-dimensional spaces often helps mathematicians construct abstract mathematical structures, such as higher dimensional vector spaces, non-Euclidean geometries, and topological spaces of various kinds (Kossak, 1996). Even though those structures may not have concrete interpretations in our real world, the very
simple intuitive images enable mathematicians to develop such abstract concepts to a remarkable degree.

An intuitively clear solution in mathematical problem solving leads to a more direct and much deeper involvement of the individual than an analytic solution without any intuitive basis. Suppes (1966) emphasized the importance of developing intuitions for formulating and presenting mathematical proofs. If a student’s intuitive representation is mathematically adequate, the student may fruitfully build further conceptual structures. On the other hand, if the intuitive basis is not adequate, conflict may be generated. In order to build a formal, non-intuitive and sometimes counter-intuitive proof, what are first needed are adequate and efficient intuitions.

Classification of intuition via its origin. Our perspective toward mathematical intuition is that it is not only a kind of gifted capacity but also a kind of cognitive capacity obtained by experiencing proper examples and lasting practice based on mathematical activities. It is likely that intuition has been accomplished through an individual’s accumulated experience in society (Bruner, 1960; Fischbein, 1987). For instance, the novice in solving plane geometry problems draws supplementary lines cautiously recalling individual rules, whereas the experienced mathematician often absorbs a complicated situation at a glance without feeling the necessity of supplementary lines. In like manner, this research assumes that mathematical intuition tends to grow with experience. In particular, intuitive cognition in this theoretical framework is classified via its origin in accordance with the approach of Fischbein (1987) into primary intuition and secondary intuition. This classification depends on whether the
experience affecting the formulation of intuition is related to the general human experience, the particular instruction, or the practice of individuals.

*Primary intuitive cognition* refers to intuitive cognition that “develops in individuals independently of any systematic instruction as an effect of their personal experience” (Fischbein, 1987, p.64). All representations and interpretations naturally developed during his or her childhood and generally shared by all the members in his or her community belong to primitive intuitive cognition. All common and basic conceptions, such as the three-dimensional representation of space and the idea that every event must have a cause, are examples of primary intuitive cognition (Fischbein, 1975). Individuals may personally acquire primary intuitive cognition through their natural and normal life experiences.

*Secondary intuitive cognition,* on the other hand, implies that new intuitions may be developed through educational circumstances. Such intuitions are not produced by the natural and usual experience of an individual. For instance, the statement “the sum of the angles of a triangle is 180 degrees” may not be self-evident. For some students, a proof is required to accept the fact that, no matter what shape the triangle has, the sum of its angles remains constant. Once students are able to see directly that the sum must necessarily remain constant, and the statement becomes a belief and a self-evident conception so that the student directly conceives it without needing further justification, we then consider that a new, secondary intuition has been acquired by the student.¹

¹ Fischbein (1987) remarked that secondary intuition might not be clearly distinguished from primary intuition: It may be rather considered a continuum ranging from very elementary and naturally acquired intuitive cognitions to very complex and genuinely counter-intuitive notions, or, primary intuition for one individual may actually be secondary intuition for others.
In this classification of intuition, the point to be aware of is that intuition, in particular, secondary intuition, can be obtained and developed by appropriate training. Bruner (1960) pointed out that intuition can be developed and has been developed by repeating similar experiences using a specific idea or procedure. Keisler (1976) argued that intuition of the concept of limit should help students as they repeatedly experience this development. Even Hahn (1956a, 1956b), who criticized the use of intuition in mathematics, admitted that higher-order intuitions could be formed by adequate instruction. Similar experiences, either consciously or unconsciously, seem to foster a student’s confidence in using the idea or procedure. Eventually, the student will be immediately reminded of a specific idea, and feel confident using the idea. Therefore, the intuition, whether it is primary or secondary, should be considered as an essential component in teaching and learning mathematical concepts.

Figure 1.2 Formal cognition versus Two kinds of intuitions: Primary intuition and secondary intuition
The Role of Reverse Thinking

In general, reverse thinking changes the direction of the process of thinking in a problem from the desired conclusion toward the given assumptions of the problem (Driscoll & Moyer, 2001). As an information processing model of mathematical abilities, Krutetskii (1969) described reversibility as follows: “By reversible (two-way) associations (and series of associations) we mean those associations in which the thought or realization of the second element (or of the last element) evoke the thought or realization of the first element” (Krutetskii, 1969, p. 51). In a more general sense, Krutetskii (1969) also described reversibility as “an ability to restructure the direction of a mental process from a direct to a reverse train of thought” (p. 143).

Concerning reversible thinking processes, Piaget (1967) asserted that children in the concrete operational stage begin to understand reversibility, which indicates the capacity to understand the relationship between things, in other words, the recognition that one thing can turn into another and back again. According to Inhelder and Piaget (1958), once children are in the concrete operational stage, which starts around seven years of age, they begin to understand that every logical and mathematical procedure can be offset by a reverse procedure.

Reverse thinking in the context of limits. In this dissertation, reverse thinking means to change the direction of the process of thinking in a limit problem from the error bounds in values of the sequence to the index defining the values of the sequence. Thus, reversibility in the context of the limit of a sequence, in this dissertation, means the ability to think of the infinite process in defining the limit in terms of the index and
simultaneously to reverse the process by finding an appropriate index in terms of an arbitrarily chosen error bound.

When reading the symbol \( \lim_{n \to \infty} a_n \), it is natural to first consider the index \( n \) to check the statement “\( n \) goes to infinity,” and then the corresponding term \( a_n \) to see if \( a_n \) approaches a certain number, say \( L \). By understanding limits in this order, students might think that they should first choose \( N \), and then take a sufficiently small \( \varepsilon \), depending on the chosen \( N \): That is,

\[
\text{(1) \ For any positive integer } N, \text{ there exists } \varepsilon > 0 \text{ such that if } n > N, \text{ then } |a_n - L| < \varepsilon.
\]

However, in the rigorous definition of limit, the order of steps is reversed. The index \( N \) is dependent on \( \varepsilon \):

\[
\text{(2) \ For any } \varepsilon > 0, \text{ there exists a positive integer } N \text{ such that if } n > N, \text{ then } |a_n - L| < \varepsilon.
\]

Furthermore, in order to utilize the \( \varepsilon - N \) definition in solving limit problems or other calculus problems involving limits, students must first choose an arbitrarily small margin \( \varepsilon \) around \( L \) and then determine whether the sequence satisfies the condition by taking a sufficiently large index \( N \). For instance, the sequence \( a_n \) defined by

\[
a_n = \frac{1 + (-1)^n}{2}
\]

\[
= \begin{cases} 
0 & \text{if } n \text{ is odd,} \\
1 & \text{if } n \text{ is even,}
\end{cases}
\]

is not a convergent sequence. However, according to statement (1), which is incorrect, this sequence seems to be convergent. We can even determine the limit of this sequence
as either .5, 1, or 0. This is an example indicating how important it is to apply correctly the reverse thinking process in utilizing the $\varepsilon - N$ definition in limit problems.

Courant and Robbins (1963) point out the counter-intuitive nature of the reverse thinking process as one factor in students’ difficulties in conceptualizing limits:

There is a definite psychological difficulty in grasping this precise definition of limit. Our intuition suggests a “dynamic” idea of a limit as the result of a process of “motion”: we move on through the row of integers 1, 2, 3, …, $n, \ldots$ and then observe the behavior of the sequence $a_n$. We feel that the approach $a_n \rightarrow a$ should be observable. But this “natural” attitude is not capable of clear mathematical formulation. To arrive at a precise definition we must reverse the order of steps; instead of first looking at the independent variable $n$ and then at the dependent variable $a_n$, we must base our definition on what we have to do if we wish actually to check the statement $a_n \rightarrow a$. (p. 292)

Concerning the rigorous definition of the limit of a function, Kidron and Zehavi’s (2002) study indicates that some students who failed to express the rigorous definition of limit tend to have difficulty in utilizing the reverse thinking process. While students who understood the rigorous definition of the limit of a function translated it as “to every $\varepsilon_n$, there is $\delta_n$” (Kidron & Zehavi, 2002, p. 223), students who failed to express the rigorous definition tended to remember the order in which they worked with reading the limit symbol. These students often explained “$\delta$ is not dependent on $\varepsilon$; $\varepsilon$ is dependent on $\delta$,” or “$\delta$ is not dependent on the error, since $\delta$ fixes the error” (Kidron & Zehavi, 2002, p. 223). These excerpts demonstrate that students have difficulty in reversing the order between $\varepsilon$ and $\delta$ in the rigorous definition of limits, since in their intuitive

\footnote{2 For instance, $\lim_{n \to \infty} a_n = 1/2$ since no matter how large an index $N$ is given, we can choose any positive number $\varepsilon$ which is greater than 1/2 so that if $n > N$, then $|a_n - \lim_{n \to \infty} a_n| = |a_n - 1/2| = 1/2 < \varepsilon$.}
cognition they began with the domain $x$ and the corresponding $\delta$ and then found the error $\varepsilon$.

The reverse thinking process appears necessary for a precise understanding of limits. Tall and Schwarzenberger (1978) point out the lack of precision as a cause of students’ misconceptions about limits. In general, the instructional approach to the concept of limit at the secondary school level uses a definition of limit designed to appeal to students’ intuitive cognition. In reading the limit symbol, there is no specification of how close $a_n$ is to the limit, nor of the relationship between the index number $n$ and the corresponding term $a_n$. On the other hand, the limit in the $\varepsilon - N$ definition appears to be manageable and the closeness is specified in terms of the algebra of inequalities. By properly choosing $N$ depending on any given $\varepsilon$, the difference between the terms of a given sequence and the limit must be smaller than the allowed error $\varepsilon$. Consequently, the reverse thinking process in describing the quantitative but statically represented relationship between $\varepsilon$ and $N$ plays a crucial role in representing the concept of closeness inherent in the qualitative but dynamic description of the limit of a sequence.
CHAPTER 2

LITERATURE REVIEW

This chapter reviews literature related to teaching and learning the concept of limit. Studies of students’ readiness for instruction, of misconceptions, and of instructional approaches to the concept of limit will be discussed. Much of the work on limits in mathematics education research has been focused in four major areas. One body of work makes use of Piaget’s theory of cognitive development and/or focuses on students’ readiness for instruction related to limits. A second body of work focuses on students’ misconceptions in understanding limits and investigates how their understanding is different from the mathematical idea embedded in the $\varepsilon – \delta$ definition of limit. A third area investigates cognitive obstacles in learning the concept of limit, and a fourth area proposes instructional strategies to help students overcome the cognitive obstacles.

Readiness for Instruction on Limits

Relatively early studies of students’ understanding of the concept of limit explored the role of maturation in students’ thinking. In particular, these studies investigated the age at which children can perceive the notion of infinite processes as one of the important concepts involved in limits that cannot be directly observed by them.
The contributions of this type of study include designing curricular sequences and deciding specific grade levels to teach the concept of limit.

It seems desirable to teach about limits when children reach the formal operational stage of Piaget’s cognitive development theory. According to Piaget’s *Psychology of Intelligence* (1967), cognitive development and mental processes leading to the capacity for adult thought consist of four major stages: the sensorimotor stage, the stage of preoperational thought, the stage of concrete operations, and the stage of formal operations. Studies related to limits focus on children in the formal operational stage, which is characterized by the ability to think abstractly, to reason deductively, and to define concepts. One of the main attributes of the formal operational stage appears to be that of “hypothetico-deductive” thinking, which is the highest organization of cognition, and enables people to make a hypothesis or proposition and test it against reality (Piaget, 1967). Students possess the ability to work with hypothetical situations when they reach the formal operational stage (Sinclair, 1971). Empirical studies indicate that middle school students in the formal operational stage possess the ability to work with the hypothetical situation of infinite processes and the limit as the ultimate result of the processes (Piaget & Inhelder, 1967; Taback, 1975).

Piaget and Inhelder (1967) examined 11-12 year old children’s understanding of the concept of limit. In particular, they investigated whether or not 11-12 year old children understood the notion of convergence in infinite subdivisions. The children participating in interviews were asked to insert as many points as possible between two given points or to subdivide a geometric figure. In this manner, students’ ability to
conceptualize the infinite subdivision of line segments, squares, circles, and triangles was
explored as well as the shape of the end product of the subdivision. The former shed light
on students’ ability to understand the infinite process, while the latter revealed students’
ability to understand the ultimate result of the process, which is the limit. Piaget and
Inhelder reported that children could understand subdivision as an infinite process by the
formal operational stage of cognitive development, which begins around 11-12 years of
age.

Similar to Piaget and Inhelder’s study, Taback (1975) explored the conception of
limit at three age levels, namely 8-, 10-, and 12-year-olds. Whereas Piaget & Inhelder
relied only on the infinite process of subdivision to obtain their results, Taback provided
students with various infinite processes, along with monotone increasing, monotone
decreasing, and alternating infinite sequences. Students were asked to explain the
convergence and/or divergence and the limit in non-mathematical terms and contexts.
Students’ responses were evaluated in terms of a predetermined rating scheme with three
categories: clear understanding, uncertain understanding, or no understanding. The
responses were compared to others in terms of their age levels. The results demonstrated
that the 12-year-olds performed slightly, but consistently, better than the 10-year-olds. On
the other hand, there were substantial differences in the performance of the 8-year-olds
compared to the 10-year-olds and 12-year-olds. Furthermore, only students in the 12-
year-old group answered questions on convergence, most of which demanded the ability
to operate in terms of hypothetical situations.
Like Piaget and Inhelder, Taback concluded that children in the formal operational stage are free from a dependence on physically presented real objects, and have the ability to perform with clear understanding of the concept of limit at the abstract level. These two studies imply that children’s ability to understand the concept of limit corresponds to their cognitive development, and that instruction on limits should be given to children during the formal operational stage.

On the other hand, Brackett (1991) pointed out that there are various contexts for infinite processes leading to convergence and divergence, and that even young children’s abilities depend on the context in which the infinite process is given. Whereas the previous studies of Piaget and Inhelder and Taback relied on mainly geometric objects and situations, Brackett examined students’ intuitive knowledge of infinite processes in tasks that reflected various combinations of three paired contexts of infinite processes:

3 Numerical/geometric, 4 convergent/divergent, and 5 aggregate/serial. Thirty-one 6th-grade students volunteered for two 20-minute semi-structured interviews. Students’ responses to tasks in the $2 \times 2 \times 2$ different contexts of infinite processes were rated in terms of their intuitive knowledge of infinite processes. The findings showed that 6th-graders possessed the concept of infinite processes before it was taught in school.

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3 The *numerical* context is illustrated by the context of numbers, where numbers are considered objects in their own right, and not as descriptions of other mathematical objects. On the other hand, a line, circle, triangles, and so on illustrate the *geometrical* context, which are considered as a purely geometric construction, without reference to any numerical measurement.

4 The *convergent* context and *divergent* context are illustrated by the behaviors of numbers or geometric constructions in their infinite extent – converging or diverging, respectively.

5 The *serial* context describes any set that is conceptualized as “on and on” whereas the *aggregate* context is illustrated by the density of numbers in any interval on the real number. Aggregate infinity describes infinite sets conceptualized as “bounded.”
Since 6th-graders are often at the formal operational stage, Brackett’s results appear consistent with previous studies indicating that children in the formal operational stage have the ability to learn infinite processes. However, a different aspect pointed out by Brackett was that the students’ perception of infinite processes varied according to the context of the process. The students’ intuitive approach to infinite processes was more likely to be useful in a task situation involving the numerical, serial, and divergent contexts of infinite processes than in a situation involving the geometric, aggregate, and convergent contexts of infinite processes.

Once students reach the formal operational stage, their experiences and instruction seem to play a more important role than chronological age in understanding the concept of limit. Brackett’s study notes the significance of instruction in 6th-grade students’ understanding of limits. Smith (1959) also argues for the importance of instruction in acquiring the concept of limits. In investigating the role of maturation in acquisition by secondary school students of the concept of limit, Smith measured each student’s maturity in terms of his/her chronological age, mental age, and grade point average. The relative importance of maturity was then compared to experience in acquiring the concept of limit. The correlation between chronological age and limit test performance scores was very low. On the other hand, when three hours of instruction and experience with problems involving limits were provided, significant gains were made by students in the extent to which they conceptualized the limit at each of the grade levels, 7, 9, and 11.
**Summary.** It is likely that students possess the aptitude to learn the concept of limit beginning at the formal operational stage, which starts at approximately 11-12 years of age. The research findings indicate that each student in the formal operational stage had the ability to understand the concept of limit. However, instruction was an important factor in determining the extent to which students actually conceptualized the limit. Limitations of representational systems may contribute to students’ difficulties in understanding limits in the geometric context. By providing a variety of experiences and representations, teachers can empower students to better understand infinite processes and limits.

**Misconceptions About Limits**

The second body of studies focuses on students’ misconceptions about limits. Studies report various types of students’ misconceptions as follows: self-assumption of monotonicity of sequences or functions; self-assumption of the last term of sequences; confusion of the limit with a bound of a sequence or a function; and confusion of the limit with the value of the function.

As a preface to students’ typical misconceptions about limits, research has investigated the dynamic motion inherent to the concept of limit (Cornu, 1991; Davis & Vinner, 1986; Szydlik, 2000; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Williams, 1991, 2001). The dynamic motion refers to the infinite process involved in the concept of limit, which induces a feeling of motion. For instance, when reading the limit symbol \( \lim_{x \to x_0} f(x) = b \), the dynamic motion is described by such expressions as “close to,” “tends to,” or “approaches” (Tall & Vinner, 1981). That is, we say “\( b \) is the limit of a
function $f$ at a point $x_0$ if $f(x)$ approaches $b$ as $x$ approaches $x_0$” (Marsden & Hoffman, 2000, p. 178); or, if we can make $f(x)$ as close to $b$ as we please by making $x$ sufficiently close to $x_0$. On the other hand, in the rigorous definition of limit, the dynamic motion is described by quantification such as choosing a positive number $\delta$ corresponding to every positive number $\epsilon$ (Cottrill et al., 1996). Mathematicians accept the $\epsilon$-$\delta$ definition of limit, which translates the mathematically ambiguous word approach into mathematical rigor in the following manner:

[We say] $b$ is the limit of a function $f$ at a point $x_0$, denoted as $\lim_{x \to x_0} f(x) = b$, if given any $\epsilon > 0$ there exists $\delta > 0$ [italics added], which possibly depends on $f$, $x_0$, and $\epsilon$, such that for all $x$ satisfying $x \neq x_0$ and $|x - x_0| < \delta$, we have $|f(x) - b| < \epsilon$. (Marsden & Hoffman, 2000, p. 177)

In this section, we discuss two misconceptions that students often have regarding the dynamic motion of limits. The first misconception is that students do not understand the infinite process inherent in a limit. The second misconception is that students only conceptualize the dynamic motion of limits without conceptualizing the limit as the ultimate result of the dynamic process.

**Misconception 1: No Conception of Infinite Processes**

One part of the misconceptions related to the dynamic motion of limits occurs when students do not perceive the infinite processes but just apply the finite notion of processes to solve problems of limits. Those students who do not understand infinite processes appear to confuse the limit of a function at a point with the value of the function at the point or an approximation of the limit (Davis & Vinner, 1986). This
phenomenon was described in detail by Dubinsky (2000) and his colleagues (Cottrill et al., 1996). Dubinsky and his colleagues designed instruction based on their analysis of students’ understanding of limits. Twenty college students participated in interviews after going through this instruction. Results showed that students tended to regard the limit of a function at a point as the same as the value of the function at the point, that is, \( \lim_{x \to a} f(x) = f(a) \), or the value of the function at a point very close to the given point. The following excerpt from Cottrill et al.’s (1996) study indicates a student’s misconception of the equality of \( f(a) \) and the limit at \( a \):

Charles: Yeah, after you plug in a to the function [italics added], you come out with what \( L \) equals.
Interviewer: OK, if it’s…will that always happen, or are there cases where…
Charles: Uh, well, you can plug in and get no limit for infinity.
Interviewer: Well, which ok, um, if \( f(a) \), does \( f(a) \) have to equal the limit?
Charles: No, Well, \( f \) …Yeah, \( f(a) \) , yeah it does.
Interviewer: It does?
Charles: Yeah, it would have to equal the limit. (p. 178)

Such students do not understand the limiting process, which is the ongoing process or motion without end, but they tend to use finite computation by plugging a point, or at most a finite number of points, into the given function in deciding the value of the limit (Cottrill et al., 1996).

**Misconception 2: Limit as the Infinite Process Itself**

Another misconception about limits is to regard a limit as the infinite process itself. When students’ understanding of limits reflects only the infinite process implied by the limit, they tend not to accept the idea of the limit as the result of motion (Kidron,
For instance, a typical example used in teaching the concept of limit is $\lim_{n \to \infty} \frac{1}{n}$, and students learn the limit of this sequence is 0 because 0 is the value that this sequence approaches but does not reach. It should be noted that students often use the word *reach* along with words like *approach* or *close to* in order to describe limits as endless processes. According to pilot studies preceding the present research, some students used the word *reach* in determining whether the sequence $a_n = \frac{1}{n}$ has a limit. In particular, some students used simultaneously two criteria “close” and “reachable.” When students use the expressions *approach*, *close to*, or *reach* to describe limits, they do not capture the result of the process, as the completion of the process, because literally speaking, there is no such thing as the result of an endless process. Therefore, it might be reasonable that students regard “A sequence does not reach a certain value” as a criterion for convergence. However, students who use such a criterion for convergence may restrict the class of convergent sequences as shown in the following examples.

**Example 1: A value $L$ is not the limit of a sequence $a_n$ if $a_N = L$ for some $N$.** Some students may conclude that a specific value could not be the limit of a sequence if the sequence actually reaches the value in the sense that $a_N = L$ for some index number $N$. For instance, some students in the pilot studies responded that the alternating sequence $a_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{n} & \text{if } n \text{ is odd,} \end{cases}$ does not have a limit:
Paige: Zeros are already attained here, and a limit you can’t actually reach that value. It’s just getting closer and closer and closer to it, so it couldn’t be there. But this one, it wouldn’t, it wouldn’t have a limit.

**Example 2: A value** \( L \) **is not the limit of a sequence** \( a_n \) **unless** \( a_N = L \) **for a large** \( N \). Some students who regard a limit as an infinite process itself tend to conclude that a specific value could not be the limit of a sequence unless the sequence reaches the value in the sense that \( a_N = L \) for a large index \( N \). For instance, some students consider \( 0.999\cdots \) as an approximation of the natural number 1 (Tall & Schwarzenberger, 1978). In considering the notation “…” as denoting only the indefinite and ongoing process, students often regard \( 0.999\cdots \) as not being 1 but less than 1 (Sierpińska, 1987; Tall & Schwarzenberger, 1978). However, in mathematics, the notation “…” is taken as representing a fixed, indefinitely long sequence of particular digits, a case of the actual result of the infinite process. The infinite decimal expression \( 0.999\cdots \) is thus equal to the number 1 by a standard computational procedure:

\[
\begin{align*}
x &= 0.999\cdots, \\
10x &= 9.999\cdots, \\
10x - x &= 9.999\cdots - 0.999\cdots, \\
9x &= 9.000\cdots, \text{ and hence} \\
x &= 1.000\cdots.
\end{align*}
\]

Here, the substitution of \( 0.999\cdots \) as \( x \) means that \( 0.999\cdots \) is conceptualized as a number so that multiplication, subtraction, and division can be performed on the number \( 0.999\cdots \). This conceptualization of \( 0.999\cdots \) involves the infinite process as well as the actual result of the process, which is 1.
In line with the above misconceptions about limits, research shows that students who have such misconceptions about limits describe limits as “approaching but not reaching” (Davis & Vinner, 1986; Szydlik, 2000; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Williams, 1991). Tall and Vinner (1981) found that the most common misconception among students regarding limits was that they tended to perceive a limit as a dynamic process but as unreachable. Williams (2001) also asserted the concept of limit as “approaching but not reaching” was a coherent image that students had, and described the motion metaphor as having a powerful influence on students’ conceptions of limit.

The misconception of limit as “approaching but not reaching” results from students’ misconception of the dynamic motion described by the word “approaching.” When students do not possess the concept of limit as the final result of the limiting process, they appear to acquire the concept of the infinite processes but not to conceptualize the actual limit as the result of limiting processes. On the other hand, in mathematics, the intuitive idea of “approaching” the limit is characterized as the metaphorical final state of the process (Lakoff & Núñez, 2000). Mathematically, the limiting process, which proceeds indefinitely, is defined as a process having an end and an ultimate result. For instance, the irrational number \( \pi \) is not approaching a number but is a precise number characterized by an infinitely long string of particular digits to the right of the decimal point. This is not just a sequence that gets longer and longer but an infinitely long fixed sequence. As one of the expressions of the irrational number whose product, \( 2\pi \), denotes the circumference of a unit circle, \( \pi \) is a particular fixed number represented by an infinitely long fixed sequence. Since the infinite process occurs,
students seem to think that the limit should be in motion and different from the value of any term of the sequence. This misconception is thus related to the typical perception that a function is not equal to the limit. In contrast, when mathematicians describe the limit, they do not care whether the function \( f \) ever equals the limit or not. Their only concern is with numbers close to, or approaching, the limit (Szydlik, 2000).

**Cognitive Obstacles Influencing Misconceptions About Limits**

Cognitive obstacles in learning the concept of limit have also been explored. In particular, daily experiences and language related the concept of limit (Cornu, 1991; Lakoff & Núñez, 2000; Kossak, 1996; Oehrtman, 2002; Tall & Vinner, 1981; Williams, 1991) and students’ beliefs about mathematical knowledge (Sierpińska, 1987; Szydlik, 2000) have been investigated to explain the discrepancy between students’ concept images and the formal definition.

**Daily Experience and Language**

Studies reveal that daily experiences lead to conflicting images in the mind, and that the images often show a tendency which does not disappear even in a mathematics lesson. Students’ daily experiences are likely to be combined with newly acquired mathematical knowledge and to constitute a part of the student’s personal conceptions (Tall & Vinner, 1981). The model of limits induced from daily experiences tends to affect mathematical understanding of the concept of limit by remaining as a certain type of image (Fischbein, 1987). In order to articulate the difference between formal mathematical knowledge and the image remaining in the student’s mind, Tall and Vinner suggested the terms *concept image* and *concept definition*. The term *concept image* refers
to the total cognitive structure associated with the concept, whereas the *concept definition* is a formulation used to specify that concept. According to Tall and Vinner, the visual representations, the mental pictures, the impressions, and the experiences associated with the name of the concept can be a part of memories evoked in a given context, and later they can be translated into verbal forms as the formal mathematical definition.

Based on the theoretical framework of concept images and concept definitions, Tall and Vinner investigated cognitive conflicts by identifying students’ intuitive conceptualizations of limit. The results demonstrated that many students had great difficulty in manipulating the definition of limit even though they seemed to understand the statements of theorems on limits. This implies that when a student was given the concept definition of limit, the student formed some intuitive ideas in his/her cognitive structure which were not consistent with the concept definition, and this intuitive idea caused the student cognitive conflict. Tall and Vinner’s study is valuable for revealing that knowing a concept definition does not appear to guarantee understanding of the concept.

Cornu (1991) explored students’ use of the expression *tends to* in their daily experiences as well as in interpreting limiting processes in mathematical contexts. The results showed that students equated *tends to* with *approaching but not reachable* in understanding the concept of limit in their daily experiences. This result indicates that the ordinary use of the expression *tends to* may cause the typical misconception of limit as *approaching but not reachable*. Cornu concluded that students had a certain number of ideas, intuitions, images, and knowledge of the concept of limit which came from their
daily experience. Students’ daily experience of using the same terminology appeared to interfere with their understanding of the mathematical meaning of various terms.

Students with their own coherent concept image often have more difficulty understanding the central ideas of the concept of limit. Williams (1991) showed that it is hard to change students’ perception of limits once it is established based on their experience. Williams used the following six statement to describe a limit: (1) dynamic-theoretical, that is, a limit describes how a function moves as \( x \) moves toward a certain point; (2) acting as a boundary, that is, a limit is a number or a point past which a function cannot go; (3) formal, that is, a limit is a number that the \( y \)-values of a function can be made arbitrarily close to by restricting \( x \)-values; (4) unreachable, that is, a limit is a number or point the function gets close to but never reaches; (5) acting as an approximation, that is, a limit is an approximation that can be made as accurate as you wish; and (6) dynamic-practical, that is, a limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

These six statements describing limits were given to 341 students of two second-semester calculus classes. Among the students participating in the first part of Williams’ survey, 10 students who had clearly presented mathematically imperfect viewpoints of limits were selected for the purpose of remedying their understanding. Four of the 10 students had a dynamic view of limits, four viewed limits as unreachable, one viewed the limit as a boundary, and one viewed limits as approximations. Over a period of seven weeks, these 10 students met with an investigator for five sessions, which were designed to help students understand their own models of limit as well as other models. In order
for students to change their incomplete viewpoints of limit to the formal conception, the
students compared various models of limit and discussed the problem of limits in terms
of incomplete viewpoints. At the end of most sessions, theses students were asked if they
wanted to alter their original definition of limit to any other definitions. The results
indicated that most of the students did not adopt a more formal view of limits.

In particular, when dealing with the limit of a function, \( \lim_{x \to a} f(x) \), it is hard to
develop a mental image faithful to the dynamic motion of an infinite process as a
continuous process that has no ends and no results (Courant & Robbins, 1963; Kossak,
1996; Lakoff & Núñez, 2000). Courant and Robbins (1963) describe this obstacle as
resulting from the counter-intuitive nature of infinite processes on continuous variables.
In dealing with a continuous variable \( x \) that ranges over a whole interval of the number
axis, there is no next point after a given point has been reached. Therefore, there must
remain a discrepancy between the intuitive idea and the mathematical language designed
to describe the scientifically relevant features of our intuition of infinite processes in
precise logical terms.

Lakoff and Núñez (2000) suggest the “basic metaphor of infinity” as a mental
image of infinite processes. The basic metaphor of infinity focuses on the fact that
visualization of a sequential infinite process is relatively less difficult to understand than
a continuous infinite process. In proceeding step by step through a discrete sequence of
values \( a_1, a_2, a_3, \ldots \), or using alternative expressions such as “going on and on,” it is
possible to conceptualize the sequential infinite process. The infinitely iterating step-by-
step process is again applicable for finding the limit of a function \( f(x) \) at a point \( a \) (Lakoff
and Núñez, 2000). Infinite processes that include all possible sequences approaching the point \( a \) is the mathematicians’ very intuitive thinking of continuous infinite processes represented in the theorem, “sequential convergence as convergence,” from introductory calculus:

\[
\lim_{{n \to \infty}} f(x) = q \text{ if and only if } \lim_{{n \to \infty}} f(p_n) = q \text{ for every sequence } \{ p_n \} \text{ such that } p_n \neq x_0, \quad \lim_{{n \to \infty}} p_n = x_0 \quad (\text{Rudin, 1976, p. 84}).
\]

This theorem is mathematically true for the real numbers. Even though it is impossible for students to say how \( x \) shall approach the fixed value \( a \) in such a way as to assume consecutiveness and by order of magnitude for all the values, students’ intuitive thinking of the limit of a sequence is capable of generalizing to the continuous case. In this manner, students may intuitively conceptualize infinite continuous processes as a number of iterative processes. Furthermore, having emphasized the sequentially convergent aspect of limits, which is more familiar than continuous contexts, students might be more comfortable accepting mathematically formal ideas of limit.

On the other hand, Oehrtman (2002) points out that dynamic motion language might not be sufficient to properly develop students’ conception of limit. Through analysis of students’ written and verbal responses to limit problems, Oehrtman found that use of motion language did not help students. Instead, it was found that relevant metaphors for limit fell into the following five major categories: (1) a collapse in dimension, (2) approximation, (3) closeness in a spatial domain, (4) infinity as a number, and (5) physical limitation. Students used these metaphors productively in determining
the limits of given problems properly. These metaphors also appeared to play a role as a relevant conceptual tool in developing a dynamic motion image of the concept of limit.

Beliefs About Mathematical Knowledge

Students’ obstacles in learning the concept of limit also appear to be related to their beliefs and perspectives on mathematical knowledge. Studies indicate that students who conceive of mathematical truth in the same manner that mathematicians do tend to accept the idea of limit more readily.

Szydlik (2000) investigated the relationship between students’ beliefs about mathematics and their conceptual understanding of the limit of a function. Participants in this study were taught via a traditional calculus course using the rigorous $\varepsilon - \delta$ definition of limit and the limit processes in functions and sequences. The results indicated that students whose content beliefs about limits were similar to those of mathematicians also conceived of limits as mathematicians rigorously define limits. For these students, their convictions were derived from internal sources such as students’ intuition or logical thinking based on their experience. On the other hand, most students participating in the study conceived of limits only as dynamic motion, and their conviction appeared to result from external sources such as authority of mathematical truth supplied by textbooks or mathematics teachers. Szydlik’s study indicates as far as the phenomenon that students who understand the rigorous definition of limits are convinced from their experiences is concerned, it appears that those students had appropriate experience to link their experience with the abstract mathematical definition. In contrast, those whose conception of limit was different from mathematicians’ seemed to show discrepancies between the
intuition obtained from their experiences and the knowledge gained from their mathematics teachers or textbooks. Repetition of contradictory experiences appears to cause students to distrust their own intuitive thinking in mathematical tasks.

Sierpińska (1987) also pointed out the relationship between students’ attitudes toward mathematical knowledge and their conception of limit. A group of 17-year-old humanities students participated in four 45-minute instructional sessions dealing with exercises related to limits. The main purpose of the study was to analyze changes in students' attitudes toward geometric infinite series through instruction. The analysis of students’ responses through these sessions indicated that the important factor that determines students’ attitudes toward the result of an infinite series was their attitudes towards mathematical knowledge and infinity. For instance, students who accepted calmly that $1-1+1-\cdots=\frac{1}{2}$ regarded mathematics as being completely abstract and far from reality, and with mathematical techniques, they believed one could prove either $1-1+1-\cdots=\frac{1}{2}$, $1-1+1-\cdots=0$, or $1-1+1-\cdots=1$.

Sierpińska’s study demonstrates the possibility of students’ conceptual changes through instruction designed to illustrate situations that conflict with their misconceptions. Mathematics educators even argue that it is necessary to present situations in which students are made aware of the difficulties in learning limits and to reflect on their own ideas and epistemological obstacles (Brousseau, 1997; Cornu, 1991; Fischbein, 1987). Through experiences of constructing their own mental images of limits, encountering obstacles in understanding the rigorous definition of limit via mental images, and
reflecting on their thinking, students are expected to not only change their conception of limits but also to overcome obstacles in understanding the concept of limit.

**Instructional Approaches**

As seen from the previous section, students have difficulty in advancing from finite to infinite processes as a way of thinking. The difficulty that students encounter in understanding infinite processes further hinders their ability to learn the limit of a function at a point, \( \lim_{x \to a} f(x) \), and infinite decimals such as \( 0.9 = 0.999\ldots \), each of which requires students’ acquisition and application of thinking skills including infinite processes (Brackett, 1991). Mathematics educators point out that the difficulty in understanding infinite processes results in part from the problems in learning unfamiliar topics and the negative influence of students’ daily experience in which intuitive understanding conflicts with mathematical meaning. In this section, we review literature related to the corresponding instructional implications.

To think in terms of infinite processes may be unfamiliar to many high school students as well as college students. The *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) introduces the infinite process inherent in the concept of limit as a different mode of mathematical thinking that high school students are not expected to have conceptualized before. Standard 13, Grades 9-12, makes the following case:

Most of the mathematics described in the other 9-12 standards involves finite processes… In contrast, the concept of a limit and its connection with the other mathematical topics in this standard is based on infinite processes [italics added]. Thus, explorations of the topics proposed here not only extend students’ knowledge of function characteristics but also introduce them to another mode of mathematical thinking (NCTM, 1989, p. 180).
Although the *Standards* describes the case of high school students, it seems plausible that college students may have the same difficulty in conceptualizing infinite processes (Cottrill et al., 1996; Davis & Vinner, 1986; Tall & Schwarzenberger, 1978). It is likely that students feel unfamiliar with the infinite motion of the limiting process since they have no real-world experience with infinite processes.

How to provide students with proper and sufficient experience with sequentially infinite processes is therefore crucial for familiarizing them with the dynamic motion of limits. As an efficient means of teaching the dynamic motion of limits, computer technology has been recommended in mathematics education communities. For recent decades, computers and graphing calculators have become an educational tool to complement traditional mathematics teaching techniques (Anderson et al., 1999; Porzio, 1999; Weigand & Weller, 2001).

Dubinsky (2000) and his colleagues (Cottrill et al., 1996) suggest an instructional treatment structured around the Activities-Class-Exercises (ACE) teaching cycle. The instructional focus of the ACE teaching cycle is to have students make specific mental constructions in learning the dynamic process by using a computer program. Although cooperative learning and alternatives to lecturing are also part of the suggested pedagogical strategies, the most crucial part of the ACE cycle is to use computers in teaching the dynamic processes inherent in the concept of limit. Using computer software is suggested in order to help students create mental constructions in their mind corresponding to the constructions on the computer. The researchers had students write computer code to investigate what happened to the value of a function as the numerical
difference between two values of $x$ became increasingly small. Through these computer-aided activities, students had numerous chances to experience the dynamic processes as well as to mentally construct the dynamic process. Dubinsky and his colleagues’ studies are valuable for suggesting various instructional strategies using computer technology.

Friedrich Froebel (Banchoff, 1990) emphasizes the importance of visual images in students’ learning, and states that if students could be stimulated to observe objects from the primary stages of their education, ideas from these observations would return to them again and again during the course of their schooling, deepening with each new level of sophistication. Pinto and Tall (2002) showed that college students’ visual images played a crucial role in learning calculus and real analysis. Alcock and Simpson (2004) pointed out that students should internalize links between visualization and symbolic representations of sequences in order for students to efficiently use their visual images in proving convergence of infinite sequences or infinite series.

Among various computer technologies, computer graphics or animation techniques are regarded as efficient for helping students visualize the image of motion in the limiting process. By using graphing technology in computers, students are expected to view visual images and receive continuous support on the visual aspects of the dynamic motion of the limiting process (Arcavi, 2003; de Guzmán, 2002). Tall (1992) recommends using graph magnification technology rather than directly teaching the rigorous definition of limits as well as for teaching any related topics such as derivatives and integrals. Teo (2002) proposes a graphical method for composing functions to provide students with geometric insight into the theorem on limits of a composite
function. The graphical approach appears to improve students’ recall of limit theorems, thus allowing students to visually perceive why the continuous condition for the function \( f \) should be assumed in the limit theorem of the composition function \( f \circ g \) but continuity of \( g \) is not necessarily required to satisfy the theorem.

Kidron & Zehavi’s (2002) study demonstrates how animation generated by Computer Algebra System (CAS) software improves students’ understanding of the concept of limit. Eleventh grade high school students (N=78) explored the approximation and interpolation of functions using Taylor polynomials in connection with the concept of limit. The Mathematica CAS was used to generate dynamic graphs to enable visualization of the process of convergence as well as give meaning to the definition of limit. In particular, an animation technique provided by Mathematica was used to illustrate the dynamic process of convergence. The students’ achievements in learning the concept of limit indicated that they visually confirmed the results obtained by computation of the approximating Taylor polynomials and were able to view the dynamic process of convergence as a general property. The students used computation and algorithms to clarify the visual illusion of completing the ongoing process of convergence and to build animations illustrating converging processes. With the aid of computer animation, most of the students (81%) were able to visualize the process described by the rigorous definition of the limit and were aware of infinite processes as well.

---

6 The theorem on the limit of a composite function is as follows: If \( f \) is continuous at \( b \) and \( \lim_{x \to a} g(x) = b \), then \( \lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(b) \).
As seen in this section, the use of graphing technology is regarded as an effective tool to visualize the dynamic motion of infinite processes. However, the effects of using computers on teaching the dynamic process of the limit seem to rely on teachers’ attitudes toward technology as well as the way they use the technology. Kendal and Stacey (2001) investigated how different teaching perspectives about technology were reflected in what their students learned. Two teachers of 11th-grade students were compared in terms of their teaching style and their perspectives on technology use in mathematics classes. The 20 units of calculus lessons relied on CAS facilities of Texas-Instruments’ graphing calculators (TI-92), which were observed and audiotaped. Teachers were interviewed individually before and after the program. The 33 female 11th-grade students learned how to use the TI-92. At the end of the program, 15 students among them were interviewed using task-based questionnaires. Students’ competence in calculus was assessed in terms of numerical, graphical, and symbolic representation. The results showed that each teacher had very different notions of mathematical knowledge as well as very different attitudes toward technology use. One teacher, who focused on the conceptual understanding of mathematical ideas and students’ construction of meaning, often used graphical representations to interpret concepts in a visual way. The other teacher, who emphasized performance, employed a lecture style of teaching focusing on teaching routine procedures and rules. Consequently, the latter teacher had a strong personal preference for symbolic and numerical representation and therefore adopted the CAS more frequently. His students made better use of CAS for solving routine problems than for understanding of meaning.
**Conclusion**

We have reviewed studies investigating students’ understanding of the concept of limit and have explored its association with intuitive cognition. The reviewed studies focus on cognitive development and/or students’ readiness for instruction, students’ misconceptions, and their cognitive obstacles with regard to the concept of limit. Based on this discussion, we now summarize what the research seems to show (or not show) about students’ conceptualizations of limit, and we suggest some implications for further research directions in investigating students’ understanding of the concept of limit.

First, studies exploring the role of maturation of students’ abilities in understanding the concept of limit have contributed to curricular sequence design and grade level specifications for teaching about limits. These studies imply that curriculum developers and mathematics teachers should remember that instruction on limits should be given to children when they have the ability to understand hypothetical situations such as infinite processes. Furthermore, the experience given in instruction is an important factor in determining students’ actual conceptualization of limits. Therefore, providing experiences with limits in various contexts is likely important to empower students to conceptualize limits.

For students whose understanding is confined to finite processes, infinite processes will appear unfamiliar. The use of graphing technology is regarded as an effective tool to provide such students with experience of infinite processes. By capturing the imagination of students with the use of graphing technology, direct experiences of
infinite processes can not only improve students’ visual abilities but also enhance their mathematical skills.

In particular, studies suggest that the most typical misconception is that of perceiving limits as “approaching but not reaching” which indicates that students conceptualize the dynamic motion but do not conceptualize the limit as the final result of the dynamic process. As one possible solution for this misconception, studies suggest extracting intuitive ideas from the rigorous definition of limits and designing instruction in which students face situations that conflict with their misconception.

The present research focuses on students’ understanding of the concept of limits and its association with the reverse thinking process mentioned in Chapter 1. This direction may provide an explanation of why students encounter difficulty in understanding mathematical ideas embedded in the formal $\varepsilon-N$ definition of limits. Furthermore, research on the role of reverse thinking in the context of limits is necessary to account for why students who have no problem understanding the concept of limit in an informal way, who have no common misconceptions about limits, still have difficulty making the transition to the advanced mathematical thinking processes involved in the $\varepsilon-N$ definition of limit.
CHAPTER 3

RESEARCH METHODOLOGY

The research design of the study is classified as a collective case study, based on a Soviet-style teaching experiment (Krutetskii, 1969, 1976), in which the investigator engages students in instructional activities that also serve as tasks to gauge their conceptual understanding. As descriptive research, this study focused on analysis of student conceptualizations to make inferences for the development of instructional activities. The teaching experiment focused on moving students’ intuitive conception of limit more toward the formal definition.

Preparation of the Research

The researcher worked with a number of college calculus students, each of whom was examined to provide insight and better understanding of their concept of limit. The subjects of this research were students who were acquainted with and had used the limit symbol, but had not had any experience with rigorous proofs using the $\varepsilon - N$ definition of limit. A sample of students was selected to investigate their intuitive grasp of convergence and the limit of a sequence. The researcher served as an interviewer as well as an observer so that the interactions between each individual student and the researcher could be investigated.
Field Site

This research was conducted in a Midwestern university with a fairly diverse department of mathematics. Instructors in the mathematics department who were going to teach calculus courses in autumn 2004 were contacted in spring 2004 and were asked to cooperate in allowing interviews with some of their students. Four instructors volunteered to participate in this research. The courses they taught were first quarter calculus for biology majors, third quarter calculus for engineering majors, and fourth quarter calculus for engineering majors. Pre-calculus and algebra are pre-requisites for these courses. Therefore, the researcher expected to find students who had already encountered limits through an intuitive way of reading the limit symbol but not by rigorous definition.

Selection of Participants

A survey was used as a source of data for purposive sampling. Based on a certain set of criteria, a subset of students participating in the survey was selected as participants in the teaching experiment. The criteria for the selection of participants included experience in working with limits. In particular, the participants were undergraduate students who had already been taught the concept of limit as well as topics in algebra and precalculus. A total of 21 students in the calculus courses for engineering majors and 12 students in the calculus course for biology majors voluntarily took the survey. Of them, 12 students were selected for a series of 1-hour semi-structured, task-based interviews once per week for 5 weeks; and 11 of these students completed the whole series of interviews.
Time Schedule

The time frame for the data collection extended from September to November 2004. The data collection and preliminary analysis of data fed off each other and were closely intertwined. Table 3.1 delineates the phases and methods of data collection used in the study. A detailed description of each of the activities and the methods used to collect data is provided later in this chapter.

Table 3.1 Time schedule of the study

<table>
<thead>
<tr>
<th>Period</th>
<th>Research Activity</th>
<th>Methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winter Quarter 2004</td>
<td>Human Subject Exemption</td>
<td>Submission of the Research Proposal</td>
</tr>
<tr>
<td></td>
<td>1st Pilot Study - Interviews</td>
<td>Voluntary Sampling</td>
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<tr>
<td></td>
<td></td>
<td>Structured, Task-Based</td>
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<tr>
<td></td>
<td></td>
<td>Audio-and Video Taping</td>
</tr>
<tr>
<td>Spring Quarter 2004</td>
<td>2nd Pilot Study – Survey and</td>
<td>Open-ended Questionnaires</td>
</tr>
<tr>
<td></td>
<td>Interviews</td>
<td>Purposive Sampling</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Structured, Task-Based</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Audio- and Video Taping</td>
</tr>
<tr>
<td></td>
<td>Initial Contact with an Instructor</td>
<td>Voluntary and Request Cooperation</td>
</tr>
<tr>
<td>1st week of Autumn Quarter 2004</td>
<td>Initial Contact with Students</td>
<td>Voluntary Involvement with Consent</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Request Cooperation</td>
</tr>
<tr>
<td>2nd week of quarter</td>
<td>Survey</td>
<td>Open-ended</td>
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<tr>
<td></td>
<td>Analysis of Survey</td>
<td>Survey Rubric</td>
</tr>
<tr>
<td></td>
<td>Selection of Cases</td>
<td>Purposive Sampling</td>
</tr>
<tr>
<td>3rd week of quarter</td>
<td>Pretest with Individual Students</td>
<td>Structured, Task-Based</td>
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<tr>
<td></td>
<td></td>
<td>Audio- and Video Taping</td>
</tr>
<tr>
<td>4th week of quarter – 6th week of quarter</td>
<td>1st Interview ~ 6th Interview</td>
<td>Structured, Task-Based</td>
</tr>
<tr>
<td></td>
<td>with Individual Students</td>
<td>Audio- and Video Taping</td>
</tr>
<tr>
<td>7th week of quarter</td>
<td>Posttest with Individual Students</td>
<td>Structured, Task-Based</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Audio- and Video Taping</td>
</tr>
<tr>
<td>8th week of quarter</td>
<td>Document Analysis</td>
<td>Homework, Quizzes, and Midterm &amp; Final</td>
</tr>
<tr>
<td></td>
<td>Conduct Preliminary Data Coding</td>
<td>Use Field Notes, Interview Transcripts and Reflection, &amp; Document Analysis</td>
</tr>
<tr>
<td>10th week of quarter</td>
<td>Interview with an Instructor</td>
<td>Semi-Structured</td>
</tr>
<tr>
<td></td>
<td>Construct a Model</td>
<td>Audio Taping and Field Note</td>
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<tr>
<td></td>
<td></td>
<td>Triangulation from collected data</td>
</tr>
</tbody>
</table>
Ethical Issues

Each student was provided with the topic of this research, orally as well as in written form (see Appendix A). Letting participants know the research topic is often helpful in qualitative research because participants can furnish valid information to the researchers (Crane & Angrosino, 1992). Students were also told that the study would not influence their grades. Instead, for students who participated in the interviews, the researcher provided free tutoring. The design of the study did not require that participants be deceived. In order to protect the field site and the participants from the risk of identification in public, information obtained from the pretest, interviews, posttest, and document analysis was recorded and interpreted in such a manner that subjects were not identified directly. Pseudonyms were used in interview transcripts and field notes.

Outline of the Teaching Experiment

A Soviet-style teaching experiment (Krutetskii, 1969, 1976) was used to foster a conceptually rich environment to optimize the chances that relevant development would occur in students’ understanding of the concept of limit in forms that the researcher could observe. The teaching experiment aspect of this research refers to the method of organizing and presenting data through the series of tasks for each case. The primary purpose for using teaching experiment methodology in this research was for the researcher to experience and describe each individual student’s intuitive understanding in determining convergence as well as the limit of a sequence. Without the experiences afforded by teaching, there would be no basis for coming to understand the mathematical concepts students construct or even for suspecting that these concepts may be distinctly
different from those of the researcher (Steffe, Thompson, & von Glasersfeld, 2000). The teaching experiment is primarily an exploratory tool not only to explore students’ current response to given tasks but also to examine the ways that the tasks influence students’ understanding (Lesh & Kelly, 2000).

Figure 3.1 summarizes the whole procedure of the teaching experiment for this research. To design the teaching experiment, the researcher conducted pilot studies in
Winter 2004 and Spring 2004. The teaching experiment was conducted from the third week of autumn quarter through the 7th week of autumn quarter 2004 (see Table 3.1).

**Data Collection Methods**

The data collection procedure for the study consisted of four phases as follows: (1) the survey with 33 students; (2) task-based interviews with 12 students; (3) document analysis; and (4) semi-structured interviews with 4 instructors. During the first phase of the study, 21 students in calculus courses for engineering majors and 12 students in a calculus course for biology majors volunteered (see Appendix B) to take a survey (see Appendix C). After analyzing the survey results, 12 students were selected as interviewees and asked to participate in special sessions of the teaching experiment. During the second phase of data collection, semi-structured task-based interviews, including a pretest, special sessions with limit tasks, and a posttest, were conducted over five weeks. Among the 12 interviewees, 11 of them completed the whole set of interviews. Students’ paperwork, such as quizzes, homework, and exams were analyzed, and students’ answers to coursework were also compared to their responses to the tasks in the interviews. Once the preliminary data coding was complete, semi-structured interviews with the interviewees’ instructors were conducted.

**Survey**

The survey was designed to identify characteristics of the target students as well as select interviewees for the teaching experiment (see Appendix C). All students who were willing to participate in the teaching experiment took this survey. The survey questions were modified from those used in the pilot study conducted during spring 2004.
According to the pilot study, every student participating in the survey responded to all questions they could answer within one hour. Therefore, one hour was expected as the reasonable time for most students to complete the survey. However, there was no time limit if a student requested more time for the survey.

The survey consisted of structured questions in which all respondents received the same set of questions in the same order as follows: (1) students’ background; (2) basic knowledge related to sequences; (3) basic knowledge related to graphs; and (4) basic knowledge of the limit of a sequence (see Appendix C). The following checklist provided a guideline for selecting participants for the teaching experiment:

- Is the student older than 18 years old?
- Has the student worked with limits before?
- Does the student have a basic knowledge of sequences? For instance, can the student determine the value of a specific term of a sequence? Can the student determine the number of a term in the sequence when the value of the term is given?
- Is the student familiar with reading graphs and plotting graphs?
- Does the student express his/her thinking clearly and informatively so that the researcher can obtain rich information about the student’s ideas and can easily transcribe it?

**Semi-Structured Task-Based Interviews with Students**

The semi-structured task-based interviews involved an individual student as a respondent and the interviewer as the clinician, interacting in relation to several limit tasks (see Appendices D, E, and F). The interviews were structured in the sense that each task was introduced to the student by the researcher in a preplanned way providing a structured mathematical environment that could be controlled to some extent. The
interviews were also categorized as being task-based because students’ interactions were not merely with the interviewer but with the task environments.

Goldin (2000) points out interventions as a part of the task environment are a general feature of semi-structured task-based interviews. For contingencies that occurred as the interview proceeded, explicit provision was made for branching sequences of interview questions or interventions by the researcher. Through the semi-structured task-based interviews, the interviewer observed students’ responses as well as aspects of their intuitive cognitions and affect in the presence of the interventions. Through the interventions, the task-based interviews made it possible to focus attention more directly on students’ responses to determining convergence as well as the limit of a sequence, rather than just the pattern of correct and incorrect answers for the limit.

Even though important structures were imposed in the choice of the tasks, the interview contingencies were unstructured. For instance, after answering the question “Does this sequence have a limit?” the student was asked, “How can you tell this?” In particular, this research used the questioning technique of providing heuristic suggestions that might guide the students to overcome obstacles in understanding the concept of limit. Therefore, the main question of each task proceeded to further exploration using the questioning techniques suggested by Goldin (2000):

- Posing the question with sufficient time for the child to respond and nondirective follow-up questions, such as, “Can you tell me more why you think this way?”
- The guided use of heuristic suggestions when the requested description or anticipated behavior does not occur spontaneously, such as, “Do you see any pattern in the distribution of points inside the strip and outside the strip?”
• Exploratory, metacognitive questions, such as “Can you explain how you thought about the problem?” (p. 523)

Each interview was conducted for an hour each week. According to the pilot study in Spring 2004, students generally completed each task in 20 to 30 minutes. Therefore it was regarded as reasonable that each student met weekly with the interviewer to complete two tasks, so the one-hour, semi-structured, task-based interviews were conducted for five successive weeks. The following describes the aims and the procedures of the pretest, the set of six tasks, and the posttest.

**Pretest.** The aim of the pretest (see Appendix D) was to probe students’ initial understanding of the concept of limit before participating in the teaching interview sessions. The subjects of this pretest were students who had taken the survey and been selected as participants for the interview sessions. The pretest collected background information on students’ conceptions or misconceptions about the limit of a sequence. Thus the researcher could predict contingent situations which might occur during the interview sessions and prepare appropriate follow-up questions.

The pretest consisted of questions asking students to determine convergence and the limit of five different types of sequences that were going to be dealt with in the subsequent interview sessions (see Appendix D). (1) Items 1 and 5 involved constant sequences; (2) items 2 and 6 involved monotone increasing or monotone decreasing, convergent sequences; (3) items 3 and 8 involved monotone increasing or monotone decreasing, divergent sequences; (4) item 9 involved an oscillating convergent sequence; and (5) items 4 and 7 involved oscillating divergent sequences. In the pretest, each sequence was given as a formula in terms of the index number \( n \). Through the pretest
results, the interviewer observed how students determined convergence and the limit of each sequence. Moreover, when observing each student’s responses, the interviewer paid attention to which representation the student first set up to interpret the sequence—the numerical values of the sequence or the graph of the sequence. These preferences were compared with their readiness in using graphs or other visual representations in determining convergence and the limit of a sequence in the following interview sessions.

Finally, the researcher compared the criteria that the students used in determining limits. For this analysis, the researcher showed the following two informal statements describing sufficient and necessary conditions for having a limit:

A: If a sequence is approaching a certain value \( L \), then \( L \) is a limit of the sequence.
B: If a sequence is approaching but does not reach a certain value \( L \), then \( L \) is a limit of the sequence.

Then the researcher asked what the students thought about these statements as statements to describe a limit. If the student responded that neither of these statements properly describes a limit, the researcher asked the student to suggest the most appropriate description of a limit. Once the student chose one statement to describe a limit, the researcher asked the student to re-interpret convergence and the limit of each sequence, items 1-10 in the pretest. From the student’s responses, the researcher investigated if there was any difference between the student’s initial conception of the limit of a specific sequence (that is, the student’s primary intuitive cognition) and how the student later described the limit.

**Task Activities.** The teaching experiment was designed to investigate students’ responses to a sequence of task activities (see Appendix E). The task activities were
developed to help students intuitively but accurately cognize the rigorous definition of the limit of a sequence. Six tasks were administered, two per interview, over a period of three weeks. In this section, what steps constituted each task activity and what types of sequences were dealt with in the activities are described.

The first task began with the mirror activity, which was designed to provide a physical example of infinity and a convergent sequence in a concrete way. For the mirror activity, the researcher prepared two mirrors, A and B, one of which had a small spot scratched out in the back side to make the spot transparent. Then the researcher checked to see if the student could see the other mirror through the transparent spot. In the mirror activity, students have a chance to observe a sequence of mirror images created by reflections of the pair of mirrors placed in parallel. The mirror activity shows that an infinite number of images do not have to take up a great deal of space. Such an activity is designed to promote a positive environment for students to engage in the teaching experiment with objects of interest.

![Figure 3.2 Mirror activity](image-url)
The remaining five tasks were designed to investigate how students determined convergence and the limit of a sequence for various cases. In this teaching experiment, sequences in five categories were used as follows: (1) monotone bounded; (2) unbounded; (3) constant; (4) oscillating convergent; and (5) oscillating divergent. These categories of sequences were identified in terms of students’ misconceptions about convergence and the limit of a sequence, as reported from previous research as well as from the pilot study preceding this research (Roh, 2004). For instance, a constant sequence seems simple to conceptualize because the limit is the value of the sequence itself. However, according to the pilot studies, some students responded that there is no limit for this type of sequence, indicating that constant sequences might be an interesting example to investigate to see how students cognize the limit of this type of sequence.

Table 3.2 Task activities

<table>
<thead>
<tr>
<th>Types of Sequences Used in Each Task</th>
<th>Limit Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Infinity</td>
<td>Infinite Mirror Reflections</td>
</tr>
<tr>
<td>2 Monotone Bounded</td>
<td>$\lim_{n \to \infty} \frac{1}{n} = 0$</td>
</tr>
<tr>
<td>3 Unbounded</td>
<td>$\lim_{n \to \infty} \sqrt{n} = \infty$</td>
</tr>
<tr>
<td>4 Constant</td>
<td>$\lim_{n \to \infty} \frac{n}{n} = 1$</td>
</tr>
<tr>
<td>5 Oscillating Convergent</td>
<td>$\lim_{n \to \infty} \frac{(-1)^n}{n} = 0$</td>
</tr>
<tr>
<td>6 Oscillating Divergent</td>
<td>$\lim_{n \to \infty} \frac{(-1)^n}{1 + 1/n}$</td>
</tr>
</tbody>
</table>
Table 3.2 lists the categories of sequences, the order of the tasks used in the teaching experiment, and specific examples of sequences that the students worked with in the task-based interviews. To investigate any differences in students’ intuitive understanding of the limit for each sequence in Table 3.2, the following representations of sequences were identified by the interviewer to use in determining convergence/divergence in the task-based interviews: (1) a symbolic representation; (2) a numerical representation; (3) a graphical representation; and (4) the graphical representation of the sequence with $\varepsilon$ – strips.

$$a_n = \frac{1}{2n+5}$$

Figure 3.3 An example of a symbolic representation of a sequence: EMMA’s case at pretest

A symbolic representation means determination of the limit of a sequence directly from the algebraic formula, containing the index number $n$, without referring to any numerical values of terms or any graphical representation. For instance, ELENA concluded that the limit of the sequence $a_n = 1/(2n+5)$ is 0 because

$$\frac{1}{2n+5} = \frac{1/n}{2n/n+5/n} = \frac{1/n}{2+5/n} \quad \text{and} \quad 1/n \to 0, \quad \text{therefore,} \quad \frac{1}{2n+5} \to 0 = 0.$$  

Similarly,
EMMA determined the limit of this sequence by substituting infinity for $n$ as seen in Figure 3.2. For EMMA, the limit of the sequence $a_n = \frac{1}{2n+5}$ was 0 because the index $n$ goes towards infinity, therefore, $\frac{1}{2n+5} \rightarrow \frac{1}{2 \cdot \infty + 5} = \frac{1}{\infty} = 0$.

The numerical representation of a sequence meant displaying a sequence as its numerical values. After observing a student’s first representation of a sequence in each task-based interview, the interviewer asked the student to describe numerical values of some terms of the sequence and, based on those numerical values of the sequence, to determine convergence or divergence and the limit of the sequence as follows:

Q: What is the value of the first term? What is the value of the second term? What is the value of third term? …What is the 100th term? Why do you expect this result?

Q: Does this sequence have a limit? How can you tell?

For instance, EVE determined the limit of the sequence $a_n = \frac{1}{n}$ after listing numerical values of some terms of the sequence (see Figure 3.4)

\[
a_n = \frac{1}{n}, \quad n: \text{positive integers}
\]

\[
\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{10}, \frac{1}{100}
\]

Figure 3.4 An example of a numerical representation of a sequence: EVE’s case at Task 2
A graphical representation of a sequence meant displaying a sequence as its graph. Students who responded by graphical representation determined the limit of a sequence mainly by plotting some points of the graph of the sequence. The researcher asked students to sketch the graph of the sequence by plotting some points to see if they comprehended the sequence and its graphical representation.

Q: Plot the first term of the sequence on this plane. Plot the second term of the sequence on this plane…. Plot the 10th term of the sequence on this plane.

Figure 3.5 shows an example of a student’s sketch of the graph of the sequence $a_n = \frac{1}{n}$.

![Graph of sequence $a_n = \frac{1}{n}$](image)

Figure 3.5 An example of a graphical representation of a sequence: ELISA's case at Task 2
Then, the researcher showed a graph of the sequence drawn by the computer software, *Mathematica*, and asked the student to determine convergence and the limit of the sequence, based on the graph of the sequence, as shown in Figure 3.6:

(After showing the graph of the sequence)

Q: Where would the 70th point on the graph fall? Where would the 100th point on the graph fall?

Q: Does this sequence have a limit? How can you tell?

![Figure 3.6 An example of the graph of a sequence by software at Task 2](image)

As the last step of each interview, the graphical representation of a sequence with \(\varepsilon\)–strips meant displaying \(\varepsilon\)–strips as well as the graph of the sequence. The strips, named \(\varepsilon\)–strips, were designed for this teaching experiment and were strips of constant width 2\(\varepsilon\), made of translucent paper so that the student could observe the graph of the sequence through the \(\varepsilon\)–strip. In the middle of each \(\varepsilon\)–strip, a red line was drawn so as to mark a possible limit point. These strips were devised to represent pictorially the \(\varepsilon\)–\(N\) relation in the rigorous definition of limit. The researcher asked the student to...
put an $\epsilon$–strip on the graph of the sequence to cover a proposed limit value as well as some points on the graph. By determining the number of points inside and outside the given $\epsilon$–strip, the student had the opportunities to intuitively understand the $\epsilon-N$ relation in the rigorous definition of limits and to practice reverse thinking in the context of the limit of a sequence. Figure 3.7 illustrates the graph of the sequence $a_n = \frac{1}{n}$ along with an $\epsilon$–strip:

![Graph of sequence with $\epsilon$-strip](image)

**Figure 3.7 An example of the graphical representation of a sequence along with an $\epsilon$–strip**

While engaging in this step, the student might observe that the strip covered all but a finite number of points on the graph if the sequence was a convergent one. Similarly, in case of the constant sequence given in Task 4 or the alternating but convergent sequence given in Task 5, students might observe that the strip covered all but a finite
number of points on the graph. Figure 3.8 illustrates the graphs of sequences (a) $a_n = 1$

and (b) $a_n = (-1)^n \cdot \frac{1}{n}$ along with an $\varepsilon$-strip:

![Graphical representation of sequences along with $\varepsilon$-strips: Convergent cases](image)

Figure 3.8 Graphical representation of sequences along with $\varepsilon$-strips: Convergent cases

On the other hand, students might observe that in divergent cases such as the sequence in Task 3, which diverges to infinity, and the sequence in Task 5, which oscillates between 1 and $-1$, the strip covers infinitely many points on the graph, but also fails to cover infinitely many points on the graph. Figure 3.9 illustrates the graph of the divergent sequence $a_n = (-1)^n \cdot \left(1 + \frac{1}{n}\right)$ along with an $\varepsilon$-strip:

![Graphical representation of the divergent sequence](image)
Figure 3.9 Graphical representation of sequences along with $\varepsilon$—strips: A divergent case

In order to investigate students’ intuitive understanding of the $\varepsilon-N$ relation in the rigorous definition of limits, the interviewer asked the student to describe the distribution of points on the graph of the given sequence in terms of whether they lay outside or inside the strip. The interviewer also asked the student to count the number of points covered by strips and the number of points not covered by strips. (Any strip covering the limit point should NOT cover only a finite number of points in the sequence no matter how small the width of this strip is.) The following questions show the procedures involved in probing the visual context of reverse thinking:

Q: Assume that the $-\varepsilon$—strip as well as the red line in the $-\varepsilon$—strip is extended horizontally but remaining at a constant width. Place the red line of the $-\varepsilon$—strip on the x-axis and cover as many points on the graph of the sequence as you can. Can you describe the distribution of points inside the strip and outside the strip?

Q: How many points on the graph are outside the strip?

Q: How many points on the graph are inside the strip?

Moreover, by repeating these questions with different widths of $\varepsilon$—strips, the student could observe a certain pattern in the index number $N$ dynamically chosen.
corresponding to the values of $\varepsilon$. The interviewer observed if the student intuitively used reverse thinking in determining the index $N$ depending on the value of $\varepsilon$ through the following questions:

Q: If the width of strips is getting smaller and smaller, how many points on the graph of this sequence will the strips cover? How can you tell this?

Q: If the width of strips is getting smaller and smaller, how many points on the graph of this sequence will the strips NOT cover? How can you tell this?

Q: Does this sequence have a limit? How can you tell this? Can you explain why the sequence has or does not have a limit by using $\varepsilon$–strips?

In particular, the interviewer examined whether the student accurately conceptualized the limit of a sequence and whether s/he understood the reverse relation between $\varepsilon$ and $N$ as in the rigorous definition. For this analysis, the interviewer showed the following two different statements, called $\varepsilon$–strip definitions in this research, which describe sufficient and necessary conditions for having a limit according to the reverse thinking process:

$\varepsilon$–strip definition A: A sequence has a limit when infinitely many points on the graph of the sequence are covered by any $\varepsilon$–strip as long as the $\varepsilon$–strip covers a certain number. Then the certain number is called a limit of the sequence.

$\varepsilon$–strip definition B: A sequence has a limit when only finitely many points on the graph of the sequence are not covered by any $\varepsilon$–strip as long as the $\varepsilon$–strip covers a certain number. Then the certain number is called a limit of the sequence.

Students who accurately conceptualized the limit of a sequence were expected to recognize $\varepsilon$–strip definition B as the correct interpretation of the concept of limit. On
the other hand, those who did not accurately conceptualize limits might not recognize the difference between $\varepsilon$–strip definition A and $\varepsilon$–strip definition B. Or they might even choose $\varepsilon$–strip definition A as the correct interpretation of the limit. According to the pilot study, students who regarded $\varepsilon$–strip definition A as the more accurate interpretation of the limit revealed a misconception that led them to identify an oscillating divergent sequence as being convergent with two limits.

From the second to the sixth tasks, the students determined convergence under the numerical and the graphical representation of sequences with or without $\varepsilon$–strips, respectively. By analyzing procedures that students used to come to their conclusion in each context, students’ intuitive understanding of the concept of limit in the visual context of reverse thinking was compared with that in the numerical and the graphical contexts, respectively. Through such comparisons of students’ intuitive understanding in three different contexts, the teaching experiment explored the roles of reverse thinking in understanding the limit of a sequence intuitively as well as formally.

**Posttest.** The aim of the posttest was to investigate any influence of the visual context of reverse thinking on students’ intuitive understanding of the concept of limit. In the posttest, the researcher explored any changes in the students’ concept of limit since the students had had experience with intuitively understanding convergence and the limit of a sequence using graphical representations with $\varepsilon$–strips. The interviewer showed the following two $\varepsilon$–strip definitions of the limit of a sequence, and then asked the student to choose one of them as being more accurate for the concept of limit and more understandable:
$\varepsilon$–strip definition A: A sequence has a limit when infinitely many points on the graph of the sequence are covered by any $\varepsilon$–strip as long as the $\varepsilon$–strip covers a certain number. Then the certain number is called a limit of the sequence.

$\varepsilon$–strip definition B: A sequence has a limit when only finitely many points on the graph of the sequence are not covered by any $\varepsilon$–strip as long as the $\varepsilon$–strip covers a certain number. Then the certain number is called a limit of the sequence.

These two statements had already been repeatedly shown to the student during Task 2 to Task 6. Once the student selected one of the two $\varepsilon$–strip definitions as an appropriate and understandable one for the concept of limit, the interviewer asked the student to explain why she/he chose the statement.

The rest of the posttest consisted of limit problems to determine convergence and the limit of sequences. The interviewer showed a general expression for each sequence and asked the student to sketch the overall shape of the graph of the sequence. Then the student was asked to determine convergence and the limit of the sequence by using $\varepsilon$–strip definition A or B chosen by the student. The researcher might guide the student to plot several terms of the sequence and then show the student the graph of the sequence so she/he could continue to perform the rest of the posttest. The results of the posttest were used in comparing the students’ conceptions of limit and their stage of development of reversibility before and after participating in the task-based interviews.

**Document Analysis**

Document analysis involved analysis of the students’ answer sheets from their regular coursework, such as quizzes, midterm, final exam, and homework. Since the
students were working with limits in their regular coursework, the researcher wanted to distinguish the influence of coursework from that of the teaching experiment on the students’ conceptual changes. First, the researcher compared problems themselves in the students’ course materials with those in the teaching experiment. Such comparisons helped the researcher recognize through what examples the students had learned the concept of limit, whether from the course or from the teaching experiment. Second, the researcher compared students’ answers in the coursework to those in the teaching experiment. If students’ responses to limit tasks during the teaching experiment were not clear, the researcher used data from their answers in the coursework to clarify students’ conceptions. Finally, the researcher looked for changes in students’ approaches to solving limit problems after they participated in the teaching experiment. In particular, since the teaching experiment guided students to think of the concept of limit using the graphical representation of a sequence with $\varepsilon$ – strips, the researcher checked to see if any student began using such a visual context in determining convergence and the limit of a sequence. In this way, data from the document analysis played a supplementary role in establishing the development of students’ intuitive understanding of the concept of limit.

A Semi-Structured Interview With the Instructor

After completing the preliminary coding of the data from the teaching experiment and document analysis, a semi-structured interview was conducted with each instructor who was teaching interviewees in class during the time of the teaching experiment. These interviews explored any effects from the class on the way students intuitively understood
the concept of limit. The following questions were the main focus during the interviews with the instructors:

- How does the instructor describe the concept of limit? How much does the instructor expect students to understand about limits?
- How does the instructor help students understand better?
- Is there any special emphasis in teaching students about limits?
- What is the instructor’s perspective on the role of visualization in teaching and learning the concept of limit? Has the instructor ever used graphs of a sequence in determining convergence and the limit of the sequence?

**Data Analysis**

This section is a detailed account of how each stage of the data analysis and the numerous factors considered helped in the evolution of the theory about students’ intuitive understanding of the concept of limit and their development of reversibility in the context of sequences. The overall theoretical framework for the data analysis was based on a way of building grounded theory from a collective case study, as suggested by Lincoln and Guba (1985) and Stake (2000). In order to derive a grounded theory explaining students’ intuitive understanding of limit and its association with the development of reversibility, the researcher treated each interviewee as a case, and analyzed the data in the following four steps:

1. Find main themes emerging from the preliminary coding procedure.
2. Triangulate the collected data in an effort to build grounded theory around the main themes.
3. Construct a model of the case studies with major attention to the necessity of specifying all the theoretical elements and their connections with each other.
4. Afterwards, build in illustrative data, selected according to the required validity, reliability, and ethical issues.

The following subsections explain each step of the data analysis in detail.
Preliminary Data Coding and Emerging Themes

As the initial step of data analysis, the written responses to the survey problems and the full text of all interview transcripts were coded and analyzed by Microsoft Excel using “Sort” and “Filter.” Brief descriptions of students’ behavior from video tapes and their written responses to the interview questions were also entered into Excel with indices for reference to the actual documents through the coding process.

For the initial data coding, the researcher read interviewees’ responses several times, and made notes of any noticeable diagrams, problem-solving strategies, language expressions, and logical structures used by the interviewees. Any misconceptions that hindered students in figuring out the given limit problems were also marked, even if those misconceptions were not directly related to the concept of limit. Based on the researcher’s notes, students’ responses were coded for the emerging categories. Such a preliminary data coding consisted of two phases: within-case data coding and cross-case data coding. For the within-case analysis, the researcher focused on all data about one student, from his/her survey response to the posttest response, and coded any themes from the student each time. In order to answer the research questions in Chapter 1, the within-case data analysis focused on each student’s responses to the researcher’s questions as shown in Table 3.3.

The whole set of data, from the survey to the posttest, for each individual student was coded before moving to the next student. Passages where coding decisions were made:

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7The written responses were students’ response made during the interviews on paper. Students made a table to look at any pattern of the numerical value of or the graph of a given sequence. Also, the interviewer sometimes asked each student to state his/her own conception of limit, or to graph a given sequence, and so on.
ambiguous were marked and reviewed in the following cross-data analysis procedure.

Second, in the cross-data coding step, themes that emerged from each individual student were evaluated across all interviewees to see if the theme applied to other cases.

Table 3.3 Interview questions focused on preliminary data coding

| Q1  | Does this sequence have a limit? How can you tell this? |
| Q2  | Can you explain the limit of the given sequence by using the numerical values of the sequence? |
| Q3  | Can you explain the limit of the given sequence by using the graph of the sequence? |
| Q4  | Can you explain the limit of the given sequence by using the graph of the sequence along with \( \varepsilon \)–strips? |
| Q5  | What does it mean for a sequence to have a limit in general? |
| Q6  | What do you think about \( \varepsilon \)–strip definition A and B as descriptions for the limit of a sequence? |
| Q7  | Can you explain what it means for the definition of the limit of sequence you have learned in class? |
| Q8  | Which \( \varepsilon \)–strip definition, among A and B, is better to describe the limit of a sequence? How can you tell? |
| Q9  | Which epsilon strip definition, among A and B, do you prefer to use in determining the limit of a sequence? How can you tell this? |
| Q10 | How can you explain that this value is (or is not) a limit of a sequence by using the \( \varepsilon \)–strip definition that you chose as correct? |
| Q11 | Can you compare your thought of the meaning of the limit of a sequence to that of the definition you have been taught in class? |
| Q12 | Can you compare your thought of the meaning of the limit of a sequence to that of the \( \varepsilon \)–strip definition that you chose as being correct? |
| Q13 | Can you compare \( \varepsilon \)–strip definitions and the definition of the limit of a sequence you have been taught in class? |

Triangulation of Collected Data Around the Main Themes

Triangulation of data in this research started with the aim of properly answering the research questions in Chapter 1. Through the preliminary data coding, students’ responses were categorized around several main themes. Such preliminarily coded data were triangulated according to the 4 major topics in Table 3.4:
Table 3.4 Focused items in triangulation of data

<table>
<thead>
<tr>
<th>Focus Item I:</th>
<th>The representation of the sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Focus Item II:</td>
<td>The interviewee’s conception of the convergence of the sequence</td>
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<tr>
<td>Focus Item III:</td>
<td>The interviewee’s image of the limit of the sequence</td>
</tr>
<tr>
<td>Focus Item IV:</td>
<td>The interviewee’s conception of the $\varepsilon - N$ relationship implied by the $\varepsilon - \delta$ definitions</td>
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</tbody>
</table>

Table 3.5 shows the overall triangulation procedure, focusing on which of the main themes discovered from preliminary data coding were grouped in terms of the focused items.

Table 3.5 Interview questions used in triangulation

<table>
<thead>
<tr>
<th>Focus Item I</th>
<th>Focus Item II</th>
<th>Focus Item III</th>
<th>Focus Item IV</th>
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</thead>
<tbody>
<tr>
<td>Interview Q1</td>
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<td>Interview Q2</td>
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<td>Interview Q6</td>
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<tr>
<td>Interview Q7</td>
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<tr>
<td>Interview Q8</td>
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<tr>
<td>Interview Q9</td>
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<tr>
<td>Interview Q10</td>
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<tr>
<td>Interview Q11</td>
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<tr>
<td>Interview Q12</td>
<td></td>
<td></td>
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<tr>
<td>Interview Q13</td>
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</tbody>
</table>

Note: See Table 3.3 for Interview Q1~Q13

**Focus Item I: Representation of the sequence.** For each given sequence, students’ responses to interview questions (Q1) ~ (Q4) in Table 3.5 were used in deriving a theory of students’ preferred representation and the accuracy of their answers. The
The researcher categorized representation types that the student used in responding to the above interview questions. In addition, the researcher also categorized any representation types suggested by the interviewer in asking the interview questions. For instance, during Task 2, the interviewer repeatedly asked the students how to explain the limit of a sequence: At first, the interviewer did not suggest using any particular type of representation of the sequence. Later on, the interviewer suggested using numerical or graphical representation (with or without $\varepsilon$ – strips) in explaining why a particular value was (or was not) a limit of the sequence. The researcher coded the representation suggested by the interviewer and the representation actually used by the student.

Table 3.6 illustrates an example of triangulation of data for two cases: When representation of a sequence was suggested by the interviewer and when representation of a sequence was not suggested by the interviewer. Comparing a student’s answers when a particular representation was suggested by the interviewer and when it was not, the researcher analyzed which representation seemed preferred by each student in obtaining the answer to limit problems.

**Focus Item II: The interviewee’s conception of the convergence of a sequence.** Students’ responses to interview questionnaires (Q1) ~ (Q3), (Q5), (Q11), and (Q12) were used in deriving a theory of students’ conceptions of the convergence of a sequence. By triangulating students’ answer to these questions, the researcher examined the reliability of identified categories in data coding.
Table 3.6 An example of triangulation: Representation used

<table>
<thead>
<tr>
<th>Transcripts</th>
<th>Written responses</th>
<th>Limit</th>
<th>Suggested Representation</th>
<th>Used Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>I: Does this have a limit? ELENA: Umm, Yeah. [pause] It’s [pause] zero. I: Umm, how can you tell this? ELENA: Okay. I divided everything by n which is the highest power of n. Then the limit [pause] as n goes to infinity anything over n is zero so it is ended up zero which is [pause] zero.</td>
<td>$\frac{1}{2n+5}$ $\frac{1}{n}$ $\frac{1}{n} = \frac{2n/n + 5/n}{2 + 5/n}$ $\frac{2 + 0}{0} = 0$</td>
<td>$\lim_{n \to \infty} \frac{1}{2n+5}$ $\frac{1}{2n+5}$ [= 0 ]</td>
<td>None</td>
<td>Symbolic</td>
</tr>
<tr>
<td>I: What would the tenth term be? What would the twentieth term be? What would the fiftieth term be? ELENA: [writing] I: Does this sequence have a limit? ELENA: Umm [pause], this is, it’s gonna get very very close to zero. It just keep going, infinitely close to zero. Umm because as the numbers get smaller and smaller and smaller, but that won’t be zero.</td>
<td>$1/10$, $1/20$, $1/50$</td>
<td>$\lim_{n \to \infty} \frac{1}{n}$ $\frac{1}{n}$ [= 0 ]</td>
<td>Numerical</td>
<td>Numerical</td>
</tr>
</tbody>
</table>

**Focus Item III: The interviewee’s image of the limit of a sequence.** During the preliminary data coding, it was found that each student tended to have his/her own image of the limit of a sequence no matter what expression of limit they used – either their own conception or the rigorous definition of limit or the $\varepsilon – \delta$ strip definition of limit. Due to such an emerged theme, students’ images of the limit of a sequence were categorized from their responses to all of interview questions listed in Table 3.5. By triangulating students’ answers to these questions, the researcher examined the reliability of identified categories of images of the limit of a sequence.
Focus Item IV: The interviewee’s conception of the $\varepsilon - N$ relation implied in the $\varepsilon - \text{strip definitions}$. During the preliminary data coding, it was also found that each student cognized the meaning of the $\varepsilon - \text{strip definition}$ in various ways. The main difference in their understanding was in their conception of the reverse relation between $\varepsilon$ and $N$. Moreover, there were several stages in the understanding of the $\varepsilon - N$ relation implied by the $\varepsilon - \text{strip definitions}$. Interview questions (Q6) ~ (Q10) and (Q12) in Table 3.5 were used in examining the reliability of identified categories in the stages of understanding the $\varepsilon - N$ relation.

Construct a Model

Through the preliminary data coding and triangulation of the coded data, this research derived a model to interpret the relation of students’ intuitive understanding of the limit of a sequence to their stages of reversibility in the context of the limit of a sequence. Chapter 4 introduces this model constructed using the categories of conceptions of convergence, images of the limit value(s) of convergent sequences, and stages of reversibility in the context of the limit of a sequence. These categories were discovered in particular from the focus items I, II, III, and IV. Then Chapter 5 shows how this model supports students’ development of reversibility while participating in the teaching experiment.
CHAPTER 4

RESULTS

This chapter analyzes students’ understanding of the limit of a sequence, students’ reversibility in the context of limits, and the relation between students’ understanding and their reversibility in the context of limits. First, an overview of students’ intuitive understanding of limit is given in terms of tasks, from the pretest through six interview tasks to the posttest, and any notable answers are described to show students’ misconceptions. Second, how students understand convergence of sequences is analyzed and their conception of the limit value of a sequence is classified into three major categories: asymptotes, cluster points, and limit points. Finally, five levels of reversibility in the context of the limit of a sequence are proposed in this chapter.

Summary of Results by Interview Task

Through the detailed research on task-based interviews, this study identified categories of students’ understanding of limits of five types of sequences: Monotone bounded, unbounded, constant, oscillating convergent, and oscillating divergent sequences. In the next section, students’ main responses to the convergence and the limit of a given sequence are overviewed by tasks. Because students were asked to evaluate $\varepsilon-\delta$ strip definitions as a definition of the limit of a sequence, and to determine the limit
of a sequence using $\varepsilon$– strips, we follow that discussion with a report on students’ main responses to the $\varepsilon$– strip definitions as ways of describing the limit of a sequence.

**Students Responses to the Limit of a Given Sequence**

Table 4.1 shows the types of sequences dealt with in each task-based interview from the pretest to the posttest. The columns show the types of sequences used in this study, and the rows represent the specific interview in order from the pretest to the posttest. Diagonally shaded cells show which type of sequence was dealt with in each task.

<table>
<thead>
<tr>
<th></th>
<th>Monotone bounded sequence</th>
<th>Unbounded sequence</th>
<th>Constant sequence</th>
<th>Oscillating divergent sequence</th>
<th>Oscillating convergent sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pretest</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Task 1</strong></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td><strong>Task 2</strong></td>
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</tr>
<tr>
<td><strong>Task 3</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td><strong>Task 4</strong></td>
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<tr>
<td><strong>Task 5</strong></td>
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<tr>
<td><strong>Task 6</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Posttest</strong></td>
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</tr>
</tbody>
</table>

When a given sequence was not familiar to the students, they explained the convergence and limit of the sequence based on their own conception of limit. In that case, some of them applied the definition of limit that they remembered, regardless of the correctness of the remembered definition. Some other students determined convergence
of the sequence by images or properties of typical types of convergent sequences, without recalling the definition of limit. Such students often called the sequence divergent if the sequence was not the same as any existing images or did not have the properties of typical convergent sequences.

**Students’ responses to given monotone bounded sequences.** Monotone bounded sequences were dealt in the pretest, Task 1, Task 2, and the posttest. Table 4.2 shows monotone bounded sequences used in the task-based interviews by task. Seven monotone bounded sequences were used altogether, three in both the pretest and the posttest and one in both Task 1 and Task 2.

<table>
<thead>
<tr>
<th>Sequence ID</th>
<th>General form of a monotone bounded sequence</th>
<th>Task dealing in the sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mono Bdd-1</td>
<td>( a_n = \frac{1}{(2n + 5)} )</td>
<td>Pretest</td>
</tr>
<tr>
<td>Mono Bdd-2</td>
<td>( a_n = \frac{1}{(1-n)} )</td>
<td>Pretest</td>
</tr>
<tr>
<td>Mono Bdd-3</td>
<td>( a_n = \frac{1}{n^2} )</td>
<td>Pretest</td>
</tr>
<tr>
<td>Mono Bdd-4</td>
<td>Sequence of mirror reflections</td>
<td>Task 1</td>
</tr>
<tr>
<td>Mono Bdd-5</td>
<td>( a_n = \frac{1}{n} )</td>
<td>Task 2 &amp; Posttest</td>
</tr>
<tr>
<td>Mono Bdd-6</td>
<td>( a_n = \frac{n}{n+1} )</td>
<td>Posttest</td>
</tr>
</tbody>
</table>
| Mono Bdd-7  | \( a_n = \begin{cases} 
1/n & \text{if } n \leq 10, \\
1/10 & \text{if } n > 10. 
\end{cases} \) | Posttest                     |

Students determined convergence of monotone bounded sequences correctly by and large. However, several students seemed to have difficulty determining and
explaining the convergence of this type of sequence. One of the main reasons was that such students got confused between infinite sequences versus a limit of infinity. For instance, students were unacquainted with a sequence [Mono Bdd-7] which, for them, looks decreasing bounded, but actually is neither decreasing/increasing bounded nor constant. In particular, students showed misconceptions about the limit of such sequences. In this case, students who cognized the limit of a sequence as a value that the sequence is continually approaching but never reaches pointed out that the sequence [Mono Bdd-7] was continually approaching 1/10 and eventually became and reached 1/10. Such reasoning led these students to the conclusion that the sequence [Mb-7] was not convergent.

**Students’ responses to given unbounded sequences.** Unbounded sequences were dealt with in the pretest, Task 3, and the posttest. Four unbounded sequences were used in total; two in the pretest and one in both Task 3 and the posttest (see Table 4.3).

<table>
<thead>
<tr>
<th>Sequence ID</th>
<th>General form of an unbounded sequence</th>
<th>Task dealing in the sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unbdd-1</td>
<td>( a_n = n )</td>
<td>Pretest</td>
</tr>
<tr>
<td>Unbdd-2</td>
<td>( a_n = 10^n )</td>
<td>Pretest</td>
</tr>
<tr>
<td>Unbdd-3</td>
<td>( a_n = \sqrt{n} )</td>
<td>Task 3</td>
</tr>
<tr>
<td>Unbdd-4</td>
<td>( a_n = \frac{n^2}{n+1} )</td>
<td>Posttest</td>
</tr>
</tbody>
</table>
Students correctly determined divergence of unbounded sequences by and large. They identified such sequences as divergent because they could not find any real number that the sequence could continually approach. However, several students were not sure whether the limit of unbounded sequence was infinity or no limit. Indeed, they did not recognize that approaching a real number is different from approaching infinity in the number system. Hence, such students expressed confusion in saying “the sequence approaches infinity” but has no limit.

On the other hand, several students did not recognize that diverging to infinity is not equivalent to saying that a finite real number is not a limit. Such students could explain why infinity was the limit of an unbounded sequence or why an unbounded sequence was divergent to infinity, but they could not well explain why a finite real number was not the limit. Instead, they seemed to be satisfied saying that an unbounded sequence is divergent to infinity when asked, for instance, “Why is 4 not a limit of the sequence $a_n = \sqrt{n}$?”

Table 4. 4 Constant (Const.) sequence used in tasks

<table>
<thead>
<tr>
<th>Sequence ID</th>
<th>General form of a constant sequence</th>
<th>Task dealing in the sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Const.-1</td>
<td>$a_n = 0$</td>
<td>Pretest</td>
</tr>
<tr>
<td>Const.-2</td>
<td>$a_n = n/n$</td>
<td>Pretest &amp; Task 4</td>
</tr>
<tr>
<td>Const.-3</td>
<td>$a_n = 1$</td>
<td>Task 4</td>
</tr>
<tr>
<td>Const.-4</td>
<td>$a_n = 1/5$</td>
<td>Posttest</td>
</tr>
<tr>
<td>Const.-5</td>
<td>$a_n = n/(5n)$</td>
<td>Posttest</td>
</tr>
</tbody>
</table>
**Students’ responses to given constant sequences.** Constant sequences were dealt with in the pretest, Task 4, and the posttest. Five constant sequences were used in total; two in the pretest, one in Task 4, and one in the posttest (see Table 4.4). One sequence $a_n = n/n$ was used twice in both the pretest and Task 4.

Several students correctly determined convergence of constant sequences. However, a large number of interviewees improperly responded to these problems. First of all, these students did not recognize sequences like [Const.-1], [Const.-3], or [Const.-4] as sequences. For them, sequences required an algebraic expression including the index $n$ that described a relationship between $n$ and $a_n$. From this point of view, [Const.-1], [Const.-3] and [Const.-4] were displayed without having the index variable $n$ in them. In particular, there were several students who regarded one point as the trajectory of [Const.-1], which convinced them that such a trajectory did not represent the graph of a sequence. Consequently, they asked if such an algebraic expression has a limit because it was not even a sequence.

Although students did not treat $a_n = 0$ as a sequence, they regarded [Const.-2] and [Const.-5] as sequences and determined the convergence. In those cases, students’ responses were varied. Several students determined that these sequences converge to the values that defined the sequences. Some other students referred to the graph of such constant sequences, which was a straight line going towards infinity, and therefore answered infinity as the limit of this type of sequence.

**Students’ responses to given oscillating convergent sequences.** Oscillating convergent sequences were dealt with in the pretest, Task 5, and the posttest. Four
oscillating convergent sequences were used in total, one sequence in the pretest and in Task 4 and two sequences in the posttest (see Table 4.5). A sequence \( a_n = 1 + (-1)^n \cdot 1/n \) was used in the posttest only for some students who completed all other tasks in the posttest due to their speed in figuring out the problems.

Several students had an incorrect understanding that no term of a sequence should be equal to the limit value of the sequence. Such students pointed out that a large portion of \([\text{Osc. Conv.-1}]\) and \([\text{Osc. Conv.-3}]\) were equal to the predicted limit value of the sequence 0 and 1, respectively; therefore, neither of the values, 0 or 1, was a limit of the sequence \([\text{Osc. Conv.-1}]\) and \([\text{Osc. Conv.-2}]\), respectively. As stated above, some of those who said constant sequences have no limit also said that the limit of \([\text{Osc. Conv.-2}]\) was 0 because it approached but did not equal 0.

<table>
<thead>
<tr>
<th>Sequence ID</th>
<th>General form of an oscillating convergent sequence</th>
<th>Task dealing in the sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Osc. Conv.-1</td>
<td>( a_n = \begin{cases} 0 &amp; \text{if } n \text{ is odd,} \ 1/n &amp; \text{if } n \text{ is even.} \end{cases} )</td>
<td>Pretest</td>
</tr>
<tr>
<td>Osc. Conv.-2</td>
<td>( a_n = (-1)^n \cdot 1/n )</td>
<td>Task 5</td>
</tr>
<tr>
<td>Osc. Conv.-3</td>
<td>( a_n = \begin{cases} 1 &amp; \text{if } n \text{ is odd,} \ 1-1/n &amp; \text{if } n \text{ is even.} \end{cases} )</td>
<td>Posttest</td>
</tr>
<tr>
<td>Osc. Conv.-4</td>
<td>( a_n = 1 + (-1)^n \cdot 1/n )</td>
<td>Posttest</td>
</tr>
</tbody>
</table>

On the other hand, those who incorrectly believed that oscillating behavior always leads to divergence identified all sequences in Table 4.5 as divergent sequences.
Among these students were several who considered the graph of a sequence, not as a set of discrete points, but a continuous curve, and such a misconception about the graph of a sequence had an inappropriate influence on their decision about the convergence of oscillating sequences. In particular, the graph of an oscillating sequence [Osc. Conv.-2: \( a_n = (-1)^n \cdot 1/n \) ] was considered continuous even if the interviewer indicated the sequence was defined on positive integers, which implied to them that 0 was actually a value of the sequence. Consequently, the combination of both misconceptions that (1) the graph of a sequence is continuous and (2) no term of a sequence should equal the limit, led these students to conclude that 0 was not a limit of the sequence [Osc. Conv.-2].

**Students’ responses to given oscillating divergent sequences.** Oscillating divergent sequences were dealt with in the pretest, Task 6, and the posttest. Eight oscillating divergent sequences were used altogether, two sequences in the pretest, one sequence in Task 6, and six sequences in the posttest (see Table 4.6). The sequence [Osc. Div.-2] was used in both the pretest and the posttest, and the sequence [Osc. Div.-3] was used in both Task 6 and the posttest. The sequences [Osc. Div.-5] ~ [Osc. Div.-8] were used only for some students who completed all other tasks in the posttest due to their speed in figuring out the problems.

Students who perceived the meaning of convergence as having a unique limit value identified the oscillating sequences in Table 4.6 as divergent. However, several students considered the possibility of multiple limits for a sequence. Among such students, those who regarded convergence as “approaching” determined that the sequence [Osc. Div.-3] had two limit values because they could find two subsequences converging
to 0 and 1, respectively. By the same reasoning, they determined that 1 and -1 were limits of the sequence [Osc. Div.-4]. Moreover, several students who considered constant sequences as convergent responded that the sequence [Osc. Div.-1] had limit values of 0 and 1, and the sequence [Osc. Div.-4] had limit values of 1 and -1. Finally, several students among those who classified both constant sequences and monotone bounded sequences as convergent determined 1 and 0 as limit values of the sequence [Osc. Div.-2].

Table 4.6 Oscillating divergent (Osc. Div.) sequences used in tasks

<table>
<thead>
<tr>
<th>Sequence ID</th>
<th>General form of an oscillating divergent sequence</th>
<th>Task dealing in the sequence</th>
</tr>
</thead>
</table>
| Osc. Div.-1 | \( a_n = \begin{cases} 
0 & \text{if } n \text{ is odd}, \\
1 & \text{if } n \text{ is even.} 
\end{cases} \) | Pretest                     |
| Osc. Div.-2 | \( a_n = \begin{cases} 
1 & \text{if } n \text{ is odd}, \\
1/n & \text{if } n \text{ is even.} 
\end{cases} \) | Pretest & Posttest             |
| Osc. Div.-3 | \( a_n = (-1)^n \cdot (1 + 1/n) \) | Task 6 & Posttest           |
| Osc. Div.-4 | \( a_n = \begin{cases} 
-1 & \text{if } n \text{ is odd}, \\
1 & \text{if } n \text{ is even.} 
\end{cases} \) | Posttest                    |
| Osc. Div.-5 | \( a_n = n + (-1)^n \) | Posttest                    |
| Osc. Div.-6 | \( a_n = (-1)^n \cdot n \) | Posttest                    |
| Osc. Div.-7 | \( a_n = (-1)^n + 1/n \) | Posttest                    |
| Osc. Div.-8 | \( a_n = (-1)^n \left(1 - 1/n\right) \) | Posttest                    |

Students who believed that no term of a sequence should equal its limit value pointed out that the sequence [Osc. Div.-1] already had 0 and 1 as term values, therefore, neither 0 nor 1 should be a limit of the sequence, regardless of their understanding of the
uniqueness of the limit value of a convergent sequence. Similarly, they determined that -1 and 1 were not limit values of the sequence [Osc. Div.-4]. Furthermore, among those who misconceived the possibility of multiple values for the limit, several students who also believed that no term of a sequence should equal its limit value pointed out that there were already terms having a value of 1 in the sequence [Osc. Div.-2], but the odd terms of the sequence were approaching 0 but never equal to 0. Consequently, they determined that 1 should not be a limit but 0 should be a limit of the sequence [Osc. Div.-2].

Another notable answer among students’ responses was observed when students perceived no anticipated limit values due to the unfamiliarity of a given sequence. For instance, several students responded that they could not suggest any value as a possible limit of some sequences because they did not remember such sequences being taught in class. In the absence of a value to compare with the sequence to see if the sequence was approaching it, they tended to compare two consecutive terms of the oscillating sequence, $a_n$ and $a_{n+1}$. Once they observed that the difference between two consecutive terms was getting smaller and seemed to be approaching 0, they determined the sequence was convergent. In particular, several students could not decide what value to start with to check convergence of an oscillating sequence. A detailed discussion related to this finding is given in the next section.

**Students’ Responses to the $\varepsilon$-strip Definitions**

The following $\varepsilon$-strip definitions, A and B, were proposed to investigate students’ ability to intuitively reverse the order of the quantifier $\varepsilon$ and the index $N$ in the context of the limit of a sequence, and to apply the reverse process to limit problems.
\(\varepsilon\) – strip definition A: A certain value \(L\) is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any \(\varepsilon\) – strip as long as the \(\varepsilon\) – strip covers \(L\).

\(\varepsilon\) – strip definition B: A certain value \(L\) is a limit of a sequence when only finitely many points on the graph of the sequence are not covered by any \(\varepsilon\) – strip as long as the \(\varepsilon\) – strip covers \(L\).

These \(\varepsilon\) – strip definitions were introduced to students during Task 2, and all through the tasks from Task 2 on, students evaluated the \(\varepsilon\) – strip definitions A and B to properly describe the limit of a sequence. Also, they explained which definition was better and chose a preferred definition to apply to sequences. In case neither of the \(\varepsilon\) – strip definitions was regarded as a good description of the limit of a sequence, students modified the \(\varepsilon\) – strip definitions to make a better statement. Finally, students compared their own conception of the limit of a sequence, the chosen \(\varepsilon\) – strip definition, and the definition of the limit of a sequence taught in class. Through these activities, students evaluated the propriety of using the \(\varepsilon\) – strip definition in determining the limit of a sequence.

Most interviewees answered that \(\varepsilon\) – strip definition A was easier to understand than \(\varepsilon\) – strip definition B regardless of the appropriateness of these definitions as descriptions of the limit of a sequence. In particular, students who considered \(\varepsilon\) – strip definition A as easier to understand, as well as to properly describe the limit, responded that definition A was equivalent to their own conception of limit and to what they had been taught in class. They thought definition B was difficult to understand, and they also doubted its propriety for describing the limit of a sequence.
One of the noticeable things here is that students differed in their extent of recognizing the propriety of the $\varepsilon$–strip definitions. The difference in their recognition was related to the extent to which students understood the dynamic nature of the reversal in the $\varepsilon$–N relation in describing the limit of a sequence. To be precise, students who saw the dynamic feature in the $\varepsilon$–N method understood the $\varepsilon$–strip definitions to a certain extent, whereas those who saw the $\varepsilon$–N relation only as a static feature experienced difficulty in understanding the $\varepsilon$–strip definitions.

Furthermore, students’ conceptions as to which $\varepsilon$–strip definition was proper to describe the limit of a sequence were associated with their conceptions of limit. Students who seemed to properly understand the concept of limit responded that $\varepsilon$–strip definition B described the limit of a sequence. Nonetheless, most students who preferred definition B as a correct statement did not distinguish its different meaning from definition A. Rather than recognizing the impropriety of definition A as a description of the limit, these students misconceived that definition A was appropriate and implied definition B; therefore, these students thought definition B was also appropriate to describe the limit.

These students continued to show much the same response until they actually applied definition A and definition B to oscillating divergent sequences in Task 6. During Task 6 some students realized that definition A did not imply definition B. Students who perceived the uniqueness of the limit value responded that definition B was correct but definition A was incorrect. On the other hand, those who considered the possibility of
multiple values for the limit considered definition A correct and definition B incorrect, or neither of them correct.

Students’ overall understanding of the $\varepsilon$–$N$ relationship in the $\varepsilon$–strip definitions tended to evolve throughout the interviews. Such a result shows the possibility of developing an effective instructional approach from the instruments used in this teaching experiment.

**Students’ Conception of the Convergence of a Sequence**

This section considers criteria that students used in determining the convergence of sequences represented symbolically, numerically, or graphically (without $\varepsilon$–strip). According to this analysis, the following criteria used by students played an important role in determining the convergence of a sequence as shown in Table 4. 7: (1) Continuing endlessly, hence no limit; (2) completing the index process; (3) getting close to, but not equal to; (4) getting close to or equal to (possibly multiple values for limit); (5) getting close to or equal to (a unique value for the limit); (6) one of the criteria (1) ~ (3) along with some exceptions.

These criteria for convergence used by students were not necessarily hierarchically structured, and students’ distribution across categories was not mutually exclusive either. In other words, students could use several criteria in determining the convergence of sequence, and their choice of a criterion could change according to the type of sequence (monotone bounded, unbounded, constant, oscillating convergent, or oscillating divergent).
Table 4.7 Students’ criteria used in testing convergence

Criteria used in testing convergence of a sequence

- Continuing endlessly, hence no limit
  Considering only the index process and not the values of the terms in the sequence, hence regarding infinite sequences as divergent.
- Completing the index process
  Checking if \( a_n \) is a real number
- Getting close to, but not equal to
  Checking if \( |a_n - L| = \varepsilon_n \) is not equal to 0 but decreasing to 0 for a value \( L \)
- Getting close to or equal to (possibly multiple values for limit)
  Checking if \( |a_n - L| = \varepsilon_n \) is equal to 0 or decreasing to 0 for one or more values of \( L \)
- Getting close to or equal to (a unique value for the limit)
  Checking if \( |a_n - L| = \varepsilon_n \) is equal to 0 or decreasing to 0 for a unique value \( L \)
- Difference between consecutive terms is getting smaller
  Checking if \( |a_n - a_{n+1}| = \varepsilon_n \) is decreasing to 0

The following subsections show the characteristics of each criterion used by interviewees in determining the convergence of a sequence and describe related misconceptions.

Conception of Convergence I: Continuing Endlessly, Hence No Limit

The category “continuing endlessly, hence no limit” refers to a conception that considers only the index \( n \) process but not the values of the \( a_n \)’s, hence regards infinite sequences as divergent. Students in this category, “continuing endlessly, hence no limit” understand that for any positive integer \( n \), no matter how big the value of \( n \) is, there should always be a next term \( a_{n+1} \) after \( a_n \). Although students in this category perceive only the fact that the index \( n \) will keep going without bound, they do not perceive that they need to consider the limit of a sequence as the result of such an infinite process.
Table 4.8 shows students who determined the convergence of a given sequence using the criterion “continuing endlessly, hence no limit.” In Table 4.8, the first column lists interview tasks—i.e., the pretest, Task 1–Task 6, and the posttest—and the following columns list students’ common answers, students’ pseudonym, and the number of students who determined convergence using the criterion, “continuing endlessly, hence no limit.” (The tables in the following subsections are constructed in the same format as Table 4.8.)

Table 4.8 Conception of convergence: Continuing endlessly, hence no limit

<table>
<thead>
<tr>
<th>Task</th>
<th>Types of Sequences</th>
<th>Students’ Responses</th>
<th>Students’ pseudonym</th>
<th># of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest</td>
<td>All types</td>
<td>Diverge</td>
<td>BRIAN, BRIGID, EMILY, EMMA, ERICA, EVE</td>
<td>6</td>
</tr>
<tr>
<td>Task 1</td>
<td>Monotone bounded</td>
<td>Diverge</td>
<td>BEN, BECKY, BRIAN, BETH, BRIGID ELISA, ELENA, EMILY, ERICA, EVE</td>
<td>10</td>
</tr>
<tr>
<td>Task 2</td>
<td>Monotone bounded</td>
<td>Diverge</td>
<td>BRIGID, ELISA, EVE</td>
<td>3</td>
</tr>
<tr>
<td>Task 3</td>
<td>Unbounded</td>
<td></td>
<td>ELISA</td>
<td>1</td>
</tr>
<tr>
<td>Task 4</td>
<td>Constant</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Task 5</td>
<td>Oscillating convergent</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Task 6</td>
<td>Oscillating divergent</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Posttest</td>
<td>All types</td>
<td></td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

When imagining sequences, students in this category conjured up visions of an infinite process of the index going on forever through the x-axis. Based on this imagery,
they determined that an infinite sequence was divergent to infinity as shown in the following excerpt from an interview with BRIAN:

**BRIAN on Pretest:** \( a_n = 1/(2n + 5) \)

**I:** Does this sequence have no limit? Is it what you are saying?

**BRIAN:** [pause] I think [pause] hmm [pause] yeah, there will be no limit, right?

**I:** There will be no limit? How can you tell this?

**BRIAN:** The greater \( n \) gets [pause] would be [pause] would just make the number smaller and smaller, but it will never [pause]. It will continue that way. The denominator will just get larger and larger.

**I:** So if a sequence has infinitely many fractions, then the sequence does not have a limit?

**BRIAN:** I guess so, yeah.

Students in this category, “continuing endlessly, hence no limit”, regarded convergent sequences as ones having a last term in the sense that the sequence could not continue to the next term because it was undefined. For instance, ELISA regarded an infinite sequence \( a_n = \sqrt{n} \) (for any positive integer \( n \)) as divergent but a finite sequence \( a_n = \sqrt{n} \) (for any positive integer \( 1 \leq n \leq 25 \)) as having a limit.

**ELISA on Task 5:** \( a_n = \sqrt{n} \)

**I:** Can we say a limit of this sequence is five?

**ELISA:** [pause] if there are or cut off points?

**I:** With this sequence, square root of \( n \)?

**ELISA:** Uh, if \( n \) couldn’t be greater than 25, then we’ll say that is the limit of this sequence.

**I:** I see. If we consider all positive integers?

**ELISA:** No!

**I:** Oh, we cannot?

**ELISA:** Yeah.

Such a tendency was shown in the survey which was used to select interviewees from the volunteers for this study. For instance, to one of the survey questions, “an example of a sequence having a limit”, BRIAN and EMMA answered “1/5, 1/4, 1/3, 1/2, 1/1” is an
example of a sequence having a limit because it should end at 1/1 but not continue to the
next term 1/0 which is undefined (see Figure 4.1).

![Figure 4.1 BRIAN's response from the survey: Continuing endlessly, hence no limit.](image)

In this sense, they determined that infinite sequences, which continue to the next
term $a_{n+1}$ after any term $a_n$ no matter how large $n$ is, should always diverge to infinity.

In particular, during Task 1 in which a sequence produced by an infinite number
of mirror reflections was considered, students determined the sequence of mirror images
as divergent. Such a trend was observed from almost all interviewees (10 out of 11
interviewees). These students who called the infinite sequence of mirror images a
divergent sequence conceived that, once two mirrors were in parallel and facing each
other, infinitely many mirror images would be generated due to the endless reflection of
mirrors. They even recognized that the size of the mirror image would keep getting
smaller and smaller. However, they determined the sequence of mirror images generated
by the above situation should be divergent to infinity. For example, ELENA understood that when two mirrors were facing each other in parallel, there would be an infinite number of mirror images, and also recognized the size of the mirror image would keep decreasing.

ELENA on Task 1: The sequence of mirror images

ELENA: [looking at a mirror through a hole on another mirror] Oh, I can see that. It’s like [drawing the whole mirrors, see Figure 4.7] you will keep seeing them [laugh]
I: Okay. What do you think about the end of this procedure?
ELENA: What do I think about the end?
I: Yes.
ELENA: I don’t think there is an end.
I: There isn’t end? How can you tell this?
ELENA: Umm. [pause] Because it just kind of gets smaller and smaller that you can’t see with. There is no way to count how many there are. It is just gonna keep bouncing, you know? [pause] So, so I know, there are infinitely many mirrors [laugh].

It also appeared that ELENA understood the infinite sequence of mirror images from the diagram which ELENA drew during Task 1.

![Figure 4.2 ELENA's response from Task 1: Infinite mirror images generated by mirror reflections](image)
In order to investigate which criterion ELENA used in determining the convergence of this sequence, the interviewer suggested generating a sequence of mirror images as follows:

ELENA on Task 1: The sequence of mirror images (continued from the above excerpt)
I: Umm. Okay. Great! Let’s say the first mirror image as a sub one, the second mirror image a sub two, the third mirror image a sub three. We can just count, okay? In our math class, we just used numbers to make a sequence. But this case, instead of numbers, we are going to just assign each mirror image to each term of a sequence. So the first term of the sequence is the first mirror image, and the second term of the sequence is the second mirror image. Okay?
ELENA: Okay.
I: Okay. So we can produce a sequence by using the mirror images. Does it make sense?
ELENA: Right. That makes sense.

When the sequence of mirror images was suggested in the above way, ELENA only considered the infinite process of generating the mirror images and determined this sequence to be divergent to infinity.

ELENA on Task 1: The sequence of mirror images (continued from the above excerpt)
I: Okay. Does this sequence have a limit?
ELENA: Umm, no. Umm, because [pause] like [pause]. There is always another mirror. There is no stop because there is no end.
I: All right. So if a sequence in general does not stop, then the sequence does not have a limit?
ELENA: Right.
I: Okay. Then in which case does a sequence have a limit?
ELENA: Umm, when it stops like [pause] at a number.

As shown in Table 4.8, in Task 1, 10 interviewees out of 11 students responded that the sequence of mirror images was divergent to infinity even though they recognized the size of the mirror images was getting smaller. But it should be noted that even
students who fairly well understood convergence of sequences determined the sequence
generated by mirror reflections as divergent to infinity. Furthermore, this tendency to call
convergent sequences divergent seemed to occur when students were confronted with a
sequence in which they could not anticipate any candidate for a limit value of the
sequence. For instance, ELENA did not usually determine the convergence of a sequence
via “continuing endlessly, hence no limit” once a sequence was represented in algebraic
form. ELENA’s conception of the limit of a sequence was pretty accurate as well.
ELENA only resorted to the criterion, “continuing endlessly, hence no limit” when
working with the sequence of mirror images in which she could not predict any value for
its limit. The following dialogue between ELENA and the interviewer showed that
ELENA applied the criterion “continuing endlessly, hence no limit” not for a sequence of
numbers but just for a sequence of mirror images.

**ELENA on Task 1: The sequence of mirror images (continued from the above excerpt)**

I: Can you give an example of a sequence having a limit?
ELENA: Okay. Umm a sequence [1, 1/2, 1/4, …] has a limit, Umm. then it
would be umm [pause] it will like one half, one fourth, and so on,
umm that has a limit of zero.
I: Umm, how can you tell?
ELENA: Because it’s gonna slowly [pause] umm. It’s not gonna stop, but it’s
going to reach a point or won’t get any lower [pause] than zero, like it
will never reach zero, [pause] and so that’s a limit.
I: Then what about this [mirrors] case?
ELENA: Since, umm [pause], there is no way where, umm, it, umm, appears
still like slow down or get close. You know just get real growth. It
keeps going at a static pace.
I: How can you tell this sequence [1/n] has a limit?
ELENA: Because this one umm [pause] the numbers [in the sequence 1, 1/2,
1/4, …] won’t reach zero, and it won’t reach zero or reach negative
one. Because that [the sequence of mirror images] doesn’t have a
number to reach in the sequence, it will keep going there will be the
tenth mirror, the hundredth mirror, and the thousandth mirror, and so on and so forth.

I: I see. Then what about the shape of the mirrors? Or the size of the mirrors?

ELENA: The size of the mirrors? Well, they are gonna be very small at the end. But there is still gonna be another mirror. It will still be [pause] very, very small.

It is likely that students who mainly use the criterion “continuing endlessly, hence no limit” in determining the convergence of a sequence experienced cognitive dissonance while learning the concept of limit in regular mathematics classes. For instance, EVE perceived that the infinite sequence \( a_n = \frac{1}{n} \) defined for any positive integer \( n \) would consist of an infinite number of terms, and would be getting close to 0. Nonetheless, when determining the convergence of this sequence, EVE clung to the fact that the index \( n \) increased to infinity and, therefore, the limit of the sequence should be infinity.

**EVE on Task 2:** \( a_n = \frac{1}{n} \)

I: All right. Does this sequence have a limit when you see the graph?

EVE: Umm [pause]. [laugh] Umm. [pause] Honestly, I guess I am [pause] I would be just confused to ask to what is getting [pause], like what would be zero and what would be infinity. Because, here, like [pause], the graph itself is like [pause], it is getting closer and closer to zero, like the x axis, but it is also like going [pause] towards infinity. I see infinity as umm [pause] like the graph keeps going straight out. So I am confused. Like [pause]. That confuses me. I guess. Because how would I know whether or not it is getting close to zero [pause] or getting close to infinity. I guess they will be the same thing to me.

I: Because \( n \) is getting larger and larger, so actually this sequence is going toward infinity? Is that what you mean?

EVE: That’s what I see because infinity is out there. Do you know what I mean? But I know, I know from class that we learned one over \( n \) is, the limit is zero.

I: So you learned it in class as zero? But somehow you think about infinity also?

EVE: I did ask him [the instructor] about this one because I was confused. But I don’t remember what he said.
Students’ tendency to use this criterion “continuing endlessly, hence no limit” was also reported from other studies (Szydlik, 2000; Williams, 1991), which means that such a tendency occurs not only in this circumstance fostered by the present researcher but also in general circumstances in regular mathematics classes. In addition, the results of the present research suggest that students used the criterion “continuing endlessly, hence no limit” when it was not easy for them to see any anticipated limit values. It even happened to students who properly determined convergence of a familiar sequence when they could not predict the limit value of an unfamiliar sequence, like the infinite sequence of mirror images.

Conception of Convergence II: Completing the Index Process

The category, “completing the index process” refers to the case when students determined the convergence of a sequence by terminating the infinite process of \( n \) at infinity and then plugged in infinity for \( n \). For instance, EMMA considered the limit of a sequence as the last term of the sequence and evaluated the value of \( a_n \) by plugging infinity in for \( n \).

**EMMA-Pretest:** \( a_n = 1/(2n + 5) \)

**I:** Does this sequence have a limit?

**EMMA:** Umm. [pause] so zero? [pause] Or infinity? You should have [pause]. Yeah, zero because you end up one over infinity.

**I:** Umm. What do you mean one over infinity?

**EMMA:** Like [pause] Umm [pause] Do you have a certain one over two, basically what times infinity [writing \( \frac{1}{2 \cdot \infty + 5} \); See Figure 4.3] then?

**I:** Keeps going. [Writing \( =\frac{1}{\infty} = 0 \); See Figure 4.3] So basically one over infinity, eventually, because it keeps going. So I mean basically one over infinity and that is zero.
In particular, students in this category seemed to determine the convergence of a sequence without considering the intermediate dynamic changes in values of the sequence as the index increases. Instead, they focused on the static image of a sequence at infinity and just plugged in infinity for \( n \) so as to check whether the corresponding term value of \( a_n \) was a real number or not. For example, EMMA checked if the value of \( a_\infty \) at infinity was a real number or infinity, and said the sequence converged if it was a real number.

**EMMA on Pretest:**

EMMA: [pause] I will say that this \([ a_n = n \] doesn’t have a limit and this \([ a_n = 1/(2n+5) ] \) would have a limit.

I: Umm. Because?

EMMA: Umm, because [pause] that the others \([ a_n = 1/(2n+5) ] \ and \( a_n = 1/(1-n) \) will just [pause] stops at zero basically. I mean when we talk about infinity something like ten [confused] [pause] can you give me more time? [pause]

I: Take your time. That’s fine.

EMMA: [pause] All right, because the last term of the sequence \([ a_n = 1/(2n+5) ] \) is going to be a number, it’s going to be zero. That is [pause] a last term. And here \([ a_n = n \], there is no last term [as a real number].
Indeed, it was likely that students who used this criterion “completing the index process” in determining the convergence of a sequence applied it to limit problems only procedurally, in the sense that they did not understand the convergence in a conceptually proper way. In particular, EMMA could determine the limit value of the following sequence, but did not know what it meant.

**EMMA on Pretest:** (continued from the above dialogue)

I: Okay. How can you say one over infinity is zero?
EMMA: Uh [pause] I don’t know. I can’t think of it. I remembered [laugh].
Somebody said it before, okay? [laugh] I mean I guess.

This excerpt from the dialogue between the interviewer and EMMA shows that EMMA seemed to feel difficulty in conceptually understanding the convergence of a sequence even if she successfully got the answer for the limit problem. Rather, when a student did not conceptually understand the criterion “completing the index process” but only procedurally applied it to a sequence in determining its convergence, she or he often came to an incorrect conclusion along with mistakes in their calculation. For instance, EMMA started to evaluate the sequence by plugging in infinity for $n$, and determined the limit of a sequence without noticing mistakes in calculation.

**EMMA on Pretest:** $a_n = n/(1-n)$

I: $a_n$ equals one over one minus $n$ for any positive integer $n$.
EMMA: Okay. [writing $1/(1-\infty) = 1/0 = \infty$] one over one minus hundred, but it will get infinite.

Table 4.9 shows the students who determined the convergence of a given sequence using the criterion “completing the index process.” As seen in Table 4.9, the number of students who applied the criterion “completing the index process” to determine convergence of a sequence seemed to decline as the teaching experiment.
continued. Such a reduction suggests that students began to internalize other ways to determine convergence of a sequence while participating in this teaching experiment.

Table 4.9 Conception of convergence: Completing the index process

<table>
<thead>
<tr>
<th>Task</th>
<th>Types of Sequences</th>
<th>Students’ Responses</th>
<th>Students’ Pseudonym</th>
<th># of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest</td>
<td>All types</td>
<td>Various answer</td>
<td>BRIAN, ELENA, EMMA, EVE</td>
<td>4</td>
</tr>
<tr>
<td>Task 1</td>
<td>Monotone bounded</td>
<td></td>
<td></td>
<td>0</td>
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<tr>
<td>Task 2</td>
<td>Monotone bounded</td>
<td>Diverge</td>
<td>EMMA</td>
<td>1</td>
</tr>
<tr>
<td>Task 3</td>
<td>Unbounded</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Task 4</td>
<td>Constant</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Task 5</td>
<td>Oscillating convergent</td>
<td>Converge</td>
<td>ELISA, EMILY</td>
<td>2</td>
</tr>
<tr>
<td>Task 6</td>
<td>Oscillating divergent</td>
<td>Diverge</td>
<td>ELENA</td>
<td>1</td>
</tr>
<tr>
<td>Posttest</td>
<td>All types</td>
<td></td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

Conception of Convergence III: Getting Close to, but Not Equal to

Students in this category, “getting close to, but not equal to” are those who determined convergence by seeing if the difference between the term value of a sequence $a_n$ and a value $L$, $|a_n - L|$, decreased to, but did not equal 0 as $n$ increased to infinity. When students applied this criterion to a sequence, they properly determined its convergence. However, they identified a sequence as divergent in the case that the difference between the value of the terms and the predicted limit value $L$ was neither getting close to 0 nor was equal to 0. Table 4.10 shows the students who determined the convergence of a given sequence using the criterion “getting close to, but not equal to.”
Students in this category considered that in order for a sequence \( a_n \) to be convergent to a value \( L \), \( |a_n - L| \) should decrease to, but not equal 0. ELISA, one of those who mainly used this criterion, believed she could give an accurate account for the dynamic motion of convergent sequences and could determine convergence as well.

**ELISA on Posttest**

ELISA: I just always thought that a sequence like went out to a certain value, like it kept going out but never actually reached it. And so if that’s the case, then this sequence wouldn’t have a limit because it actually reaches it.

Indeed, as shown in Table 4.10, students extensively applied this criterion, “getting close to, but not equal to” to monotone bounded sequences in describing the pattern of the sequence as well as determining its convergence. For instance, students
could properly determine the convergence of the sequence \( a_n = 1/n \) during Task 2. However, this criterion cannot be applied to every sequence to determine convergence. In particular, it cannot be applied to sequences that become constant. By inappropriately applying this criterion to a constant sequence, BRIAN came to the incorrect conclusion that a constant sequence did not converge to any value because the sequence was not getting close to any real number.

**BRIAN on Task 4:** \( a_n = 1 \)

I: Does this sequence have a limit?

BRIAN: [pause] I want to say no, just because it would be a straight line, always the values are gonna be one.

I: How can you tell this?

BRIAN: Because [pause] it is not approaching any number.

I: What do you mean by approaching?

BRIAN: The value is just always 1, so there is [pause] Guess my definition of limit is approaching a number but not reaching it.

I: Okay.

BRIAN: This \([a_n = 1]\) is not approaching a number, but it is already there. So, and then it is never changing the value from the value of 1.

Moreover, students in this category considered the condition of “not equal to” as indispensable for convergence. For instance, ELISA insisted on the divergence of the following sequence due to the fact that a fairly large portion of the sequence was already equal to 0 which was the only possible value for its limit.

**ELISA on Pretest:** \( a_n = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ 1/n & \text{if } n \text{ is even}. \end{cases} \)

I: Does this sequence have a limit?

ELISA: I don’t think it does. Because it approaches zero [pause] for the even numbers, and like the even numbers will never actually equal zero. But the odd numbers will make it, so that it equals like, it equals zero. So it doesn’t ever [pause], it doesn’t ever approach something, and not equal it. Does that make sense?

I: Can you say one more time?
ELISA: Umm, like this \(1/n\) part approaches zero but that it, it will never equal it. But these \(0\) part do equal zero, so umm I would say that if we want odd part, that would equal zero because it never approaches it, or it approaches but never reaches it. So, [pause] this umm so it doesn’t have a limit because it approaches something, but then it equals it. It reaches what it was approaching. So I don’t think it has a limit.

Some students in this category determined convergence of a sequence by finding any subsequence that satisfied the condition “getting close to, but not equal to.” For instance, ERICA called an oscillating divergent sequence convergent due to the fact that a subsequence consisting of only even terms of the original sequence was getting close to but not equal to a value of 0:

\[
ERICA \text{ on Pretest: } a_n = \begin{cases} 
1 & \text{if } n \text{ is odd,} \\ 
\frac{1}{n} & \text{if } n \text{ is even.}
\end{cases}
\]

I: Does this sequence have a limit?
ERICA: [pause] It has a limit [pause] at zero.
I: Limit at zero?
ERICA: At zero because these points, the even points [pause] have a limit at zero because they are [pause] never gonna reach zero [pause], but they are gonna come to closer to zero. As n gets larger, it’s gonna get closer and closer to zero.

Such an image of convergence attached to this criterion of “getting close to but not equal to” caused cognitive conflict in students who had not internalized but remembered the convergence of constant sequences. For instance, ERICA identified a constant sequence \(a_n = n/n\) as convergent in the pretest. Nonetheless, she seemed confused because the sequence appeared to not get close to 1, but was equal to 1.

\[
ERICA \text{ on Pretest: } a_n = n/n
\]

ERICA: Hmm [pause]. This one is just weird because you will always get the same answer. So no matter what you [pause] no matter what number you plug in, you will always get 1. [pause] So [pause] I don’t know [pause]. I don’t know if that means there isn’t a limit or there is a limit of 1.
I: Can you say again?
ERICA: Well [pause], I think that means the limit is 1. But [pause] it also confuses me because [pause] like [pause] no matter what number you plug in, you are always gonna get 1. But [pause] because it is never gonna be some other point over here [points not on the y-axis] something like that. But again [pause] it doesn’t mean that it isn’t a limit because it is always 1. You know, it is not getting close to one, you know, like that [see the following Figure 4.4 suggested by ERICA].

I: Umm. The reason you are confused is that this sequence is not getting closer to a certain number 1?

ERICA: Right, it’s [pause] like [pause] not because it’s not like [pause]. There is no clear like [pause] whether. Like if there is a line, then you can say, oh, the limit is 1. You know. But it’s always on this line. It’s hard for me to say whether that means that is a limit, or [pause] there isn’t a limit.

![Diagram from ERICA's sketch during pretest](image)

Figure 4.4 Diagram from ERICA's sketch during pretest

It was found that the largest portion of students conceptualized convergence as “getting close to, but not equal to,” suggesting that this misconception about convergence of a sequence may be the most common one. A more serious problem in teaching and learning the concept of limit is that several students were convinced that they had been taught this criterion in school to determine convergence of a sequence. The grounds for
such a conviction would be that examples of sequences used in mathematics classes for the concept of limit, such as $a_n = 1/n$, were sequences that satisfied such a criterion. Using a wider variety of examples should be considered in developing curricula and instruction for the concept of limit. Considerations for teaching and learning the limit of a sequence are dealt with in Chapter 5 in relation to the results of this research.

**Conception of Convergence IV: Getting Close to or Equal to (Possibly Multiple Values for the Limit)**

This category, “getting close to or equal to” refers to the case when students determine that the difference between the term values of the sequence $a_n$ and the value $L$ decreases to or equals 0 as $n$ increases to infinity. Students in this category called sequences convergent not only when $|a_n - L|$ was getting close to, but did not equal 0, but also when $|a_n - L|$ equaled 0 for infinitely many terms of $a_n$. The students in this category used "getting close to" or "staying at", for monotone bounded sequences and constant sequences, respectively. For instance, BEN determined properly the convergence of a constant sequence $a_n = n/n$ by applying this criterion as follows:

**BEN on Pretest:** $a_n = n/n$

I: Okay. Think about this sequence: $A_{sub n}$ equals $n$ over $n$.

BEN: [pause] And I say as $n$ [pause] gets larger, umm both denominator and numerator will increase by the same amount. And it will be 1 because anything divided by the one so limit will be 1.

I: Okay. So in your words?

BEN: [pause] Umm [pause] I guess the value that it reaches is the limit. And because it is not changing, then it is going to be a limit.

Table 4.11 shows the students who determined convergence of a given sequence using the criterion “getting close to or equal to.”
Table 4. 11 Conception of convergence: Getting close to or equal to (Possibly multiple values for limits)

<table>
<thead>
<tr>
<th>Task</th>
<th>Types of Sequences</th>
<th>Students’ Responses</th>
<th>Students’ Pseudonym</th>
<th># of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest</td>
<td>All types</td>
<td>Converge</td>
<td>BEN, BRIAN, BRIGID, EMILY, ERICA</td>
<td>5</td>
</tr>
<tr>
<td>Task 1</td>
<td>Monotone bounded</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Task 2</td>
<td>Monotone bounded</td>
<td>Converge</td>
<td>ELENA, EMILY</td>
<td>2</td>
</tr>
<tr>
<td>Task 3</td>
<td>Unbounded</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Task 4</td>
<td>Constant</td>
<td>Converge</td>
<td>BECKY, BRIGID, ELISA, EMILY</td>
<td>4</td>
</tr>
<tr>
<td>Task 5</td>
<td>Oscillating convergent</td>
<td>Converge</td>
<td>BEN, BECKY, ELENA, EMILY, EMMA</td>
<td>5</td>
</tr>
<tr>
<td>Task 6</td>
<td>Oscillating divergent</td>
<td>Converge</td>
<td>EMILY, EMMA, ERICA</td>
<td>3</td>
</tr>
<tr>
<td>Posttest</td>
<td>All types</td>
<td></td>
<td>BEN, BRIAN, BETH, BRIGID, ELISA, EMMA, ERICA</td>
<td>7</td>
</tr>
</tbody>
</table>

Instead of considering the whole sequence in determining its convergence, students in this category tended to look for a subsequence which was getting close to or equaling a real number. In addition, if they could find two subsequences, each of which was getting close to a different value, they regarded the original sequence as convergent with two limit values. For instance, EMMA determined inappropriately the convergence of a sequence after finding two subsequences that converged to -1 and 1, respectively.

**EMMA on Task 6:** \( a_n = (-1)^n \cdot (1+1/n) \)

I: All right. Does this sequence have a limit?
EMMA: [pause] Umm, it has [pause] umm [pause] two limits.
I: Two limits?
EMMA: The all even terms of the sequence has a limit of 1, and the all odd terms of the sequence has a limit of -1.
I: Can you explain one more time why 1 and -1 are limits of the sequence?
EMMA: Because the odd terms are getting closer and closer to -1; the even terms are getting closer and closer to 1.
I: How can you tell this?
EMMA: Umm, because the, this part, the upper part of the graph will [pause], it is getting closer and closer to 1 which is, you know [pause]. And these will never, these points [odd terms] will never [pause] umm exceed -1, so you know that it will never touch the strip.

In this way, students who used the criterion, “getting close to or staying at”, tended to improperly consider the possibility of multiple values for the limit of a convergent sequence.

Consequently, students in this category were a little better at determining the convergence of a sequence than those who applied the criterion “getting close to but not equal to” in the sense that they could determine convergence of constant sequences. However, they still did not conceptualize the uniqueness of the limit value for a convergent sequence. The following subsection focuses on students who recognized the uniqueness of the limit of a convergent sequence and describes their responses in determining the convergence of a sequence.

**Conception of Convergence V: Getting Close to or Equal to a Unique Value**

Students in this category, “getting close to or equal to a unique value,” are those who determined convergence by seeing if the difference between the term values of a sequence $a_n$ and a value $L$, $|a_n - L|$, was equal to or decreased to 0. In line with this perspective, ELENA explained the meaning of convergence of a sequence as follows:

**ELENA on Pretest**

I: All right. The next question I prepared is about your understanding “what does that mean for the sequence has a limit”.

ELENA: Umm, my understanding of that would be umm. I use the term converges which umm you say that the limit of a sequence, or a function, umm, converges means that has a real number limit. And I would say that converging is also like that practical sense, closer and closer to, you know, and like a physical sense. Umm. So, I would say
that a limit, if a sequence has a limit, then [pause] that sequence umm is converging upon a, a number, a real number. Umm.

I: So converging means?
ELENA: Getting close to OR being that number.

I: Umm. Or being that number? Okay. Can you state what you were saying?
ELENA: (Writing "In order for a sequence to have a limit, the terms of that sequence must either be converging or have converged on a real number." [see Figure 4.5])

I: What do you mean ‘converging’ here?
ELENA: Umm, [pause] converging that will mean approaching.

I: Okay. Then having converged means then?
ELENA: Well, I would say that’s already there.

What does it mean for a sequence to have a limit?

In order for a sequence to have a limit, the terms of that sequence must either be converging or have converged on a real number.

Figure 4.5 ELENA’s criterion used in determination of the convergence of a sequence: Getting close to or equal to a unique value

Table 4.12 shows the students who determined convergence of a given sequence using the criterion “getting close to or equal to a unique value.”
Table 4.12 Conception of convergence: Getting close to or equal to a unique value

<table>
<thead>
<tr>
<th>Task</th>
<th>Types of Sequences</th>
<th>Students’ Responses</th>
<th>Students’ Pseudonym</th>
<th># of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest</td>
<td>All types</td>
<td>BECKY, (ELENA)</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Task 1</td>
<td>Monotone bounded</td>
<td>Converge</td>
<td>BECKY, ELENA</td>
<td>2</td>
</tr>
<tr>
<td>Task 2</td>
<td>Monotone bounded</td>
<td>Converge</td>
<td>BECKY, ELENA</td>
<td>2</td>
</tr>
<tr>
<td>Task 3</td>
<td>Unbounded</td>
<td>Diverge</td>
<td>ELENA</td>
<td>1</td>
</tr>
<tr>
<td>Task 4</td>
<td>Constant</td>
<td>Converge</td>
<td>ELENA</td>
<td>1</td>
</tr>
<tr>
<td>Task 5</td>
<td>Oscillating convergent</td>
<td>Converge</td>
<td>BECKY, ELENA</td>
<td>2</td>
</tr>
<tr>
<td>Task 6</td>
<td>Oscillating divergent</td>
<td>Diverge</td>
<td>BEN, BECKY, ELENA</td>
<td>3</td>
</tr>
<tr>
<td>Posttest</td>
<td>All types</td>
<td>BEN, BECKY, BRIGID, ELENA</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

When students applied this criterion to a sequence, they properly determined its convergence, even for constant sequences. For example, BECKY described convergence of a constant sequence like this:

BECKY on Task 6: \( a_n = 1 \)

I: What about this sequence? If the graph looks like all the same as 1? (Graphing \( a_n =1 \); [see Figure 4.7])

BECKY: Oh, yeah, they can, the points can, they can reach it also, or they can [pause] like only be that point and would still be the limit. So they are either approaching it, or they can approach and reach it, or they can be the point that is the limit.
Figure 4.6 BECKY’s conception of convergence of sequences

ELENA, who was one of the students in this category from the pretest to the posttest, pointed out that constant sequences are convergent even if they are not approaching any value in the sense of getting close to but not equal to; in that sense, the expression “approaching” is not appropriate for describing convergence of a sequence.

ELENA on Pretest: \( a_n = 0 \)

I: Does this sequence have a limit?
ELENA: Yes, it has a limit. That will be 0 umm.
I: Umm, how can you tell this?
ELENA: Uh, that’s because every, every, every a sub n is gonna equal 0. So n [pause] I think you can still have the limit of a constant equal to constant.
I: Does this sequence approach 0?
ELENA: Umm, [pause] I don’t I don’t. I think approaching may be the wrong term for this. Umm. Because there is that number, you know? Approaching indicates that [pause], approaching indicates that umm the [pause] I don’t wanna call function, umm but the set, the umm, [pause]
the sequence, [laugh] the sequence, umm, gets umm infinitely close to the number, but never quite hits it. You know what I am saying? Umm, you have a couple of examples of that. But umm, this \([a_n = 0]\), on the other hand, this is a completely different story because it is that number for whole time. Umm. This one \([a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1/n & \text{if } n \text{ is even.} \end{cases}]\) half the time that is that number. Umm.

Students in this category also understood the uniqueness of the limit value of a convergent sequence. When there were several subsequences of a sequence, each of which was getting close to or equal to a value, students in this category called the sequence divergent if there was no unique value to which the whole sequence was getting close to or equal to.

\[
\text{BECKY on Pretest: } a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1/n & \text{if } n \text{ is even.} \end{cases}
\]

I: Does this sequence have a limit?
BECKY: Umm [pause] I don’t think so because it is just like this one [pointing out the sequence \(a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}\)]

I: Can you explain more?
BECKY: Umm [pause] like every other number like for the odd number it is gonna be 1, and then for the even number, it is gonna be a fraction, and there will be 1 again, and fraction again [pause]. And it will just keep going [pause] between the two.

I: All right.

BECKY: And for this one [pointing out the sequence \(a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1/n & \text{if } n \text{ is even.} \end{cases}\], like if you don’t separate this one, \([1/n]\) will approach 0. It is getting smaller and smaller. But since these two limits are not equal [pause], there isn't one overall limit.

As shown below, it was clear that BECKY bore in mind the image of both oscillating convergence sequences and oscillating divergent sequences and checked the uniqueness of the limit value in determining convergence.
BECKY on Task 6

I: What is your criterion in deciding if it is limit or not?
BECKY: Umm [pause], there has to be like, as x approaches infinity, all the umm values of the sequence have to be approaching [pause] that number, and [pause] like all the points. They can't be like some of them, half approaching like 1 and half of them approaching -1. They all have to be approaching the same number. It does not matter if there were positive or negative as long as it is approaching that number.

It was found that only 2 students (BECKY, ELENA) out of 11 interviewees used this criterion, “getting close to or equal to a unique value” when they thought forward, so to speak, to see what happened to $|a_n - L|$ as $n$ increased.

**Conception of Convergence VI: Measuring the Difference Between Consecutive Terms**

There were only a few students who used differences between consecutive terms in determining convergence of a sequence. Students in this category called a sequence convergent if the difference between consecutive terms was getting smaller and ultimately becoming 0. Table 4.13 shows the students who determined convergence using the difference between consecutive terms.

<table>
<thead>
<tr>
<th>Task</th>
<th>Types of Sequences</th>
<th>Students’ Responses</th>
<th>Students’ Pseudonym</th>
<th># of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest</td>
<td>All types</td>
<td>ELENA</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Task 1</td>
<td>Monotone bounded</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Task 2</td>
<td>Monotone bounded</td>
<td>BETH</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Task 3</td>
<td>Unbounded</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Task 4</td>
<td>Constant</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Task 5</td>
<td>Oscillating convergent</td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Task 6</td>
<td>Oscillating divergent</td>
<td>ELISA</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Posttest</td>
<td>All types</td>
<td>EVE</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
It was noticed that students looked at the differences between consecutive terms mainly when they could not predict any value for a limit, which often occurred with oscillating sequences in which no pattern was apparent. For instance, ELENA applied this criterion to an oscillating sequence as an alternative tool for determining convergence/divergence.

**ELENA on Pretest**

ELENA: If you talk about the distance between the points, and umm, the, the umm, the actual, [pause] the actual terms of the sequence if we talk about the distance between two terms.
I: Two terms? You mean?
ELENA: Two terms. Right! Two consecutive terms!
I: Oh, I see.
ELENA: Umm. So if we talk about the distance between this term \([a_i]\) and this term \([a_j]\), and this term \([a_j]\) and this term \([a_k]\), and so on and so forth.
I: Okay, two consecutive terms?
ELENA: Exactly umm.

For some students, consecutive terms did not mean \(a_n\) and \(a_{n+1}\) but meant the nearest points among points plotted on the graph of the sequence. For instance, when the interviewer asked EVE to plot points on the graph of the sequence \(a_n = n + (-1)^n\), she plotted points only for \(n = 5, 10, 15, 20, 25, 30\) (see Figure 4. 8), and by checking differences in the term values between \(a_{5n}\) and \(a_{5(n+1)}\) plotted on the graph, determined the convergence of the sequence.

**EVE on Posttest:** \(a_n = n + (-1)^n\)

EVE: So one [making a table for \(n = 5, 10, 15, 20, 25, 30\) and then plotting points (see Figure 4.8)] so it is always gonna be one [pause] okay [working] so it is going from less than [working] I would say that it does not have a limit because it is not approaching it constantly.
I: What do you mean by approaching constantly?
EVE: Like it is going the difference between this and this are not equal, so seven, and fourteen minus [pause], wait, is three, [writing "constantly" and calculating $a_5-a_{10}$, $a_{10}-a_{15}$, and $a_{15}-a_{20}$] (see Figure 4.8)

![Figure 4.7 EVE's response: Measuring differences between consecutive terms](image)

The point here is that students seemed to regard checking the difference between consecutive terms as an alternative but effective method in determining the convergence of a sequence, in particular when they could not predict and choose a value for testing the convergence of a sequence. For instance, ELENA usually checked differences between the term values of a sequence $a_n$ and an anticipated value $L$ for the limit, $|a_n-L|$, to see if the sequence was convergent. But when dealing with oscillating sequences, ELENA found that the difference between consecutive terms of the following divergent sequence was increasing, and therefore, determined it as divergent.
ELENA on Pretest:  \( a_n = \begin{cases} 
1 & \text{if } n \text{ is odd}, \\
1/n & \text{if } n \text{ is even}. 
\end{cases} \)

ELENA: If you do it for this, this question 8, the distances [between two consecutive terms] keep getting larger and larger. Okay? Umm [pause], and so [pause] there is no way this is approaching this, there is no way this [pause] is approaching any kind of real line here because the distance between themselves keep getting farther and farther.

In addition, ELENA also examined an oscillating sequence to show how to determine divergence when the differences between consecutive terms were always the same:

ELENA on Pretest:  \( a_n = \begin{cases} 
1 & \text{if } n \text{ is odd}, \\
0 & \text{if } n \text{ is even}. 
\end{cases} \)

ELENA: Now this one, the distance between each other is at static 1 so they are [pause], you know, not converging anything, either. Okay? You know? As far as you are concerning converging.

On the other hand, ELENA examined another oscillating convergent sequence, saw that the difference between consecutive terms decreased to 0, and identified it as convergent:

ELENA on Pretest:  \( a_n = \begin{cases} 
0 & \text{if } n \text{ is odd}, \\
1/n & \text{if } n \text{ is even}. 
\end{cases} \)

ELENA: But this one, the distance between these terms, two consecutive terms you can see, you know, one half, if you keep getting absolute values, one fourth, one sixth, and so on and so forth. Umm. So it has to approach [pause] some number. Okay?

In this way, students often applied this criterion, “difference between consecutive terms is getting smaller,” in determining convergence when it was not easy for them to predict any value for the limit.

Judging from ELENA, it is likely that students considered this criterion an acceptable method for determining convergence of sequences. Nonetheless, such a method can cause quite serious misconceptions about limits. First of all, these students might fail to note that there are divergent sequences whose differences between
consecutive terms decrease to 0. Although questions related to this issue were not investigated in this study, in the pilot study, students inappropriately responded that a sequence of partial sums \( \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} \) was convergent by checking the differences between consecutive terms. One of the students who participated in the pilot study, PAIGE, first applied this criterion to a sequence \( a_n = \sqrt{n} \) to see if the differences between consecutive terms were getting smaller:

**PAIGE in the pilot study:** \( a_n = \sqrt{n} \)

PAIGE: As you continue to go from one step to the next step, the difference between [pause] the member of the sequence, like the difference between these two [the first and the second terms] is greater than the difference between the second and the third. So the difference between, umm, like the square root of 100 and the square root of 101 is gonna be much smaller. But even though you were getting, [pause] even though the difference in between the values is becoming smaller in between the successive set, it [the sequence itself] is still gonna get greater and greater and greater. The entire graph, like, it’s gonna continue to increase in values. It’s just gonna be a bit smaller in between each term. But it won’t approach any value.

As shown above, PAIGE first examined the differences between consecutive terms to see if they were getting smaller, though she did not conclude convergence of this sequence \( a_n = \sqrt{n} \) based on the fact that the differences were getting smaller. However, this was probably because she already perceived the divergent character of the sequence \( a_n = \sqrt{n} \). Consequently, PAIGE did not explain why \( a_n = \sqrt{n} \) was not approaching a limit even though the difference between consecutive terms was getting
smaller. Instead, she applied this criterion to other sequences such as \( \lim_{n \to \infty} \sum_{k=1}^{n} 1/k \) and improperly inferred convergence.

**Students’ Conception of the Limit Value of a Sequence**

In this section, we analyze how students who participated in this research understood the limit value of a sequence. Through the qualitative data analysis mentioned in Chapter 3, these students’ conceptions of the limit value of a sequence were classified into three main categories: (1) asymptotes; (2) cluster points; and (3) limit points. This classification was extracted from the data through analysis of students’ responses to questions asking them to determine the limit values of given sequences. Table 4.14 shows students’ responses by interview task, grouped according to their conception of the limit as an asymptote, cluster point, or limit point, respectively. Incorrect answers are shaded to distinguish them from correct answers.

In analyzing students’ responses to the limit problems, it was noted that students seemed to possess different images of the limit of a sequence. Several students determined the limit through an image of asymptotes of a function or cluster points of a sequence rather than the limit of a sequence. Table 4.15 shows students’ conception of the limit of a sequence by task when mainly the forward thinking process was used in determining the limit. The categories describe each student’s main conception of the limit value of the given sequence.
Table 4. 14 Students’ responses to limit problems by their conceptions of the limit value

<table>
<thead>
<tr>
<th>Sequences</th>
<th>Types of Sequences</th>
<th>Asymptote</th>
<th>Cluster Point</th>
<th>Limit Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pretest</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_n = n/n$</td>
<td>Constant</td>
<td>no limit</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$a_n = \begin{cases} 1 &amp; \text{if } n \text{ is odd,} \ 0 &amp; \text{if } n \text{ is even.} \end{cases}$</td>
<td>Oscillating Divergence</td>
<td>no limit</td>
<td>1 and 0</td>
<td>no limit</td>
</tr>
<tr>
<td>$a_n = \begin{cases} 1 &amp; \text{if } n \text{ is odd,} \ 1/n &amp; \text{if } n \text{ is even.} \end{cases}$</td>
<td>Oscillating Divergence</td>
<td>0</td>
<td>1 and 0</td>
<td>no limit</td>
</tr>
<tr>
<td>$a_n = \begin{cases} 0 &amp; \text{if } n \text{ is odd,} \ 1/n &amp; \text{if } n \text{ is even.} \end{cases}$</td>
<td>Oscillating Convergence</td>
<td>0 or no limit</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

| Task 4 | $a_n = 1$ | Constant | no limit | 1 | 1 |

| Task 5 | $a_n = (-1)^n \cdot 1/n$ | Oscillating Convergence | 0 or no limit | 0 | 0 |

| Task 6 | $a_n = (-1)^n \cdot (1+1/n)$ | Oscillating Divergence | 1 and -1 | 1 and -1 | no limit |

| Posttest | $a_n = \begin{cases} 1/n & \text{if } n \leq 10, \\ 1/10 & \text{if } n > 10. \end{cases}$ | Bounded Convergence | no limit | 1/10 | 1/10 |
| $a_n = 1/5$ | Constant | no limit | 1/5 | 1/5 |
| $a_n = \begin{cases} -1 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$ | Oscillating Divergence | no limit | 1 and -1 | no limit |
| $a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1-1/n & \text{if } n \text{ is even.} \end{cases}$ | Oscillating Convergence | 1 or no limit | 1 | 1 |

Note: Shaded cells indicate incorrect answers for limit problems; unshaded cells indicate correct answers.
Table 4.15  

Students' conceptions of the limit value of a sequence by the forward thinking process

<table>
<thead>
<tr>
<th></th>
<th>Pretest</th>
<th>Task 2 (a_n = 1/n)</th>
<th>Task 3 (a_n = \sqrt{n})</th>
<th>Task 4 (a_n = n/n)</th>
<th>Task 5 (a_n = (-1)^n \frac{1}{n})</th>
<th>Task 6 (a_n = (-1)^n \left(1 + \frac{1}{n}\right))</th>
<th>Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td>BEN</td>
<td>Cluster points</td>
<td>Cluster points</td>
<td>NA</td>
<td>Cluster points</td>
<td>Cluster points</td>
<td>Limit points</td>
<td>Limit points</td>
</tr>
<tr>
<td>BECKY</td>
<td>Limit points</td>
<td>Limit points</td>
<td>NA</td>
<td>Limit points</td>
<td>Limit points</td>
<td>Limit points</td>
<td>Limit points</td>
</tr>
<tr>
<td>BRIAN</td>
<td>Cluster points</td>
<td>Asymptotes</td>
<td>NA</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
</tr>
<tr>
<td>BETH</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
<td>NA</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
</tr>
<tr>
<td>BRIGID</td>
<td>Cluster points</td>
<td>Asymptotes</td>
<td>NA</td>
<td>Cluster points</td>
<td>Asymptotes</td>
<td>Limit points</td>
<td>Limit points</td>
</tr>
<tr>
<td>ELISA</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
<td>NA</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
</tr>
<tr>
<td>ELENA</td>
<td>Limit points</td>
<td>Limit points</td>
<td>NA</td>
<td>Limit points</td>
<td>Limit points</td>
<td>Limit points</td>
<td>Limit points</td>
</tr>
<tr>
<td>EMILY</td>
<td>Cluster points</td>
<td>Asymptotes</td>
<td>NA</td>
<td>Cluster points</td>
<td>Cluster points</td>
<td>Cluster points</td>
<td>Cluster points</td>
</tr>
<tr>
<td>ELENA</td>
<td>Asymptotes</td>
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<td>Cluster points</td>
<td>Cluster points</td>
</tr>
<tr>
<td>ERICA</td>
<td>Cluster points</td>
<td>Asymptotes</td>
<td>NA</td>
<td>Cluster points</td>
<td>Cluster points</td>
<td>Asymptotes</td>
<td>Cluster points</td>
</tr>
<tr>
<td>EVE</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
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<td>Asymptotes</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
<td>Asymptotes</td>
</tr>
</tbody>
</table>

As shown in Table 4.15, most students did not change their conception through tasks while understanding limits as they read the limit symbol, that is, their forward thinking process. On the other hand, it should be noted that students better understood limits of sequences through the \(\epsilon - \)strip activity than through reading the limit symbol. See Table 5.2 and compare it with Table 4.15.

BETH considered a unique asymptote as the limit of a sequence. Even though BETH perceived the uniqueness of the limit of a sequence, this student was classified as one of those who regarded asymptotes as limit because of misconception about limit due to this conception. Similarly, it was shown ELISA at the posttest when ELISA used representations other than graphical one with epsilon strips for oscillating divergent sequences.
Regarding Asymptotes as Limits

Students in this category regarded the limit of a sequence as a straight line (asymptote) to which the graph of the sequence was getting arbitrarily close but not surpassing nor intersecting. Students who considered asymptotes as limits could explain properly the limit value of a monotone bounded sequence like \( a_n = 1/n \) given in Task 2. However, they did not properly determine or had difficulty determining the limits of other types of sequences.

First, students who were looking for an asymptotic line to determine the limit of a sequence responded that a constant sequence had no limit. For instance, EVE explained that the constant sequence \( a_n = n/n \) did not have a limit because the graph intersected the straight line \( y = 1 \) which was the only possible limit value, and therefore, there was no limit value for this sequence.

EVE on Task 4: \( a_n = n/n \)

I: Does this sequence have a limit?
EVE: No.
I: How can you tell this?
EVE: Because it hits 1.

Second, students who regarded asymptotes as limits did not properly determine the limit value if there was a constant subsequence of the given sequence regardless of the type of the originally given sequence. For instance, BETH was not confident to say 0 was a limit of the following sequence.

BETH on Pretest: \( a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1/n & \text{if } n \text{ is even.} \end{cases} \)

I: Does this sequence have a limit?
BETH: [pause] [laugh] [pause] I don’t think that it does. Because. Although these are moving towards 0, [pause] these [odd terms] are already set at 0. So [pause] these [odd terms] aren’t changing at all. Even though [pause] even though these [even terms] are moving towards 0, so the limit for like these three numbers will be 0, since these are set 0, and they will not be moving. Then there is no limit.

Also, BRIAN seemed to experience cognitive conflict from the fact that a straight line \( y = 1 \) could be an asymptote of a subsequence generated by only even terms but not be considered an asymptote of the sequence as the whole. Consequently, BRIAN responded that 1 was not the limit of the following sequence whose odd terms generated a constant subsequence of the same value as what the even terms of the sequence were approaching.

**BRIAN on Posttest:**

\[
a_n = \begin{cases} 
1 & \text{if } n \text { is odd}, \\
1 - 1/n & \text{if } n \text { is even}. 
\end{cases}
\]

I: A sub \( n \) equals 1 if \( n \) is odd number; a sub \( n \) equals 1-1/\( n \) if \( n \) is even integer.

BRIAN: [pause] Umm, even [pause] values of \( n \) will approach [pause] 1.

I: How can you tell this?

BRIAN: As \( n \) gets, [pause] as the value of \( n \) becomes a larger and larger even number, the fraction [1/\( n \)] here will be smaller and smaller, and 1 minus a small fraction will be close to 1. [pause] So this [even terms] will approach 1, and this one will remain at 1. So if you were allowing a straight line to be a limit, the limit will be 1. If not, then I guess you would say no limit because 1....

I: What do you think?

BRIAN: I think that I don’t understand this problem. I would say, [pause] I don’t mean I don’t understand the problem. I don’t understand [pause] whether a straight line is a limit or not. I guess I would say no limit just because all the values are gonna be on a straight line at 1.

Third, students in this category improperly believed the meaning of “\( n \to \infty \)” in the limit symbol \( \lim_{n \to \infty} \) as continuous change of the value of \( n \) in the set of all real numbers, not just the positive integers. While graphing the sequence \( a_n = (-1)^n \cdot 1/n \), EVE plotted
points on the graph for some first positive integers \( n \), then connected them to draw a continuous piecewise linear graph.

\[
EVE \text{ on Task 5: } a_n = (-1)^n \cdot 1/n
\]

I: Does this sequence have a limit?
EVE: [pause] Umm. No.
I: How can you tell this?
EVE: Because [pause] umm [pause] let’s see [pause]. Because it is going to cross [pause] whatever the limit might be, if I said that limit was 0, then it would be wrong because it crosses 0. The graph crosses 0. So like here. Like even if like if I said that as \( n \) approaches infinity and that is included in this part so that will be wrong because it crosses it.
I: Can you explain one more time by using the values?
EVE: We can tell that this sequence does not have a limit because it goes from -1 to 1/2 which is going from negative to positive, and in order to get from negative to positive [pause] on a graph, it should cross the \( x \)-axis.
I: Can you say one more time by using the graph?
EVE: If you look at the graph, you can see that [pause] umm it crosses the \( x \)-axis.

For EVE, oscillating sequences were always divergent sequences because there was no asymptote such that the graph generated by connecting consecutive terms of the sequence was arbitrarily close to but never surpassed or intersected it.

\[
EVE \text{ on Posttest: } a_n = 1+(-1)^n \cdot 1/n
\]

I: So you have a criterion that if the sequence is approaching \( L \)?
EVE: Constantly.
I: What do you mean constantly?
EVE: It does not go back and forth.
I: Oh, so constantly means without back and forth?
EVE: Right.
I: If the graph approaches \( L \) without back and forth, then \( L \) is the limit of the sequence?
EVE: Yes.
I: What about this sequence?
EVE: [making a table for \( n = 5, 10, 15 \); see Figure 4.9] It is not constant.
I: This is not constantly approaching?
EVE: Right.
I: So this sequence does not have a limit?
EVE: No!
Several students graphed subsequences first, and then looked for asymptotes of each subsequence. For instance, BRIAN considered the sequence $a_n = (-1)^n \cdot (1 + 1/n)$ as having two values for the limit because of two subsequences approaching two different asymptotes.

**BRIAN on Task 6:** $a_n = (-1)^n \cdot (1 + 1/n)$

I: Does this sequence have a limit?

BRIAN: Umm [pause] Okay, this will go, this one will be positive and negative 1. This is gonna be one [pause] point a certain number, that will be smaller and smaller, so it is gonna be [writing +1, -1, then drawing a continuous graph; see Figure 4.9] so I am not quite [pause]

I: Is this the graph of the sequence?

BRIAN: I think so.

I: Can you explain why the graph of the sequence will look this way?

BRIAN: [pause] positive 1 times 1 plus a fraction will be closer to the y axis fraction is gonna be [pause] so and it will never be above 2. [pause] So.

I: What about this part [graph under the x-axis]?

BRIAN: Okay. That will be for odd numbers. [pause] And any odd number will be negative 1 times 1 plus a fraction. [pause] So the fraction will be [pause] closer, will be closer to negative 1.

I: Does this sequence have a limit?

BRIAN: It is approaching both 1 and -1, and will never reach either of them.
I: What do you mean by approaching but never reaching?
BRIAN: Even numbers are approaching [pause] +1, but they will never reach 1.

As shown above, such a misconception about graphing a sequence caused students to consider a sequence as being defined on all real numbers rather than examining changes in the term values of $a_n$, letting the index $n$ increase by 1. Furthermore, one of the problems in understanding limits as asymptotes is likely that these students do not take the meaning of $n \to \infty$ into account when determining the limit of a sequence. For instance, ERICA pointed out that the function $y = \tan x$ has two asymptotes, both $x = -\pi / 2$ and $x = -\pi / 2$, and based on images of the graph of this function, sequences could similarly have multiple values for the limit.

ERICA on Task 4
ERICA: I think, like umm [pause] like, other, like, any kind of sequence, like [pause] I think, well, [pause]. As long as your sequence was going
horizontally, [pause] if you have a horizontal limit, and then you can use, you can take this one this way [graphing \( y = \tan x \) and drawing its asymptotes; See Figure 4.10]. If your limit is vertical. [pause] You know. If you were some [pause] Umm. Oh, just today in class he [the instructor] just showed us the graph of tangent or something like that, and that has its limit \( \pi/2 \) on each side.

I: Does this sequence have a limit?
ERICA: Yeah, \( \pi \) over two.

I: \( \pi \) over two is a limit?
ERICA: Yes. And then negative \( \pi \) over two.

I: Oh. Does this sequence have two limits then?
ERICA: Yes.

As seen in the case of ERICA, the graph of a function and its asymptotes, such as a function \( y = \tan x \), seems to remain a sharp image of a mathematical object having multiple asymptotes. Regarding multiple asymptotes as limits is likely to be derived from the image of a function with multiple vertical asymptotes.

**Regarding Cluster Points as Limits**

The notion of a cluster point is defined as follows in the mathematical context:
A point \( x \) is called a \textit{cluster point} of the sequence \( x_n \) if for every \( \varepsilon > 0 \) there are infinitely many value of \( n \) with \( |x_n - x| < \varepsilon \) (Marsden & Hoffman, 1993, p.53).

For instance, both 1 and -1 are cluster points of the sequence \( a_n = (-1)^n \) even though neither of these values is a limit of the sequence. Students who regarded cluster points as limits tended to determine the limit of a sequence by looking at which values the terms of the sequence were crowded around. Students in this category properly determined the limit of a monotone bounded sequence as well as a sequence that was approaching and eventually equaling a value since infinitely many terms of such a sequence were close to a certain value.

\textbf{EMMA on Posttest:} \[
a_n = \begin{cases} 
1/n & \text{if } n \leq 10, \\
1/10 & \text{if } n > 10.
\end{cases}
\]

I: Okay. What about this sequence? \( A \) sub \( n \) equals \( 1/n \) if \( n \) is less than or equal to 10; \( A \) sub \( n \) equals \( 1/10 \) if \( n \) is greater than 10.

EMMA: The limit would be umm \( 1/10 \).

I: The limit would be \( 1/10 \)? How can you tell this?

EMMA: Umm, because to be umm this first part, \( 1/n \), is approaching and then reaches \( 1/10 \). And then for to be the second condition greater than 10, \( A \) sub \( n \) equals \( 1/10 \). All the terms will be \( 1/10 \), so the sequence is getting closer to \( 1/10 \) until it reaches it, and it equals \( 1/10 \) thereafter.

In this way, students who regarded limits as cluster points conceived the limit of a sequence as a value for which there were an infinite number of terms close to it, while students who regarded limits as asymptotes focused on the image of a sequence along with an asymptotic line that the sequence never intersected.
Students in this category properly determined limits of oscillating convergent sequences. For instance, the sequence \( a_n = (-1)^n \cdot \frac{1}{n} \) was determined as having a limit of 0 because infinitely many terms of the sequence were close to the value of 0.

![Diagram of oscillating sequence](image)

**Figure 4.11 ERICA’s response: Regarding cluster points as limit**

However, students who regarded cluster points as limits improperly responded that a sequence could have multiple values for the limit, as did students who regarded asymptotes as limits. They considered all values for the limit as long as a significant portion of the sequence was crowded around these values. For instance, ERICA identified both 1 and \(-1\) as limits of the following oscillating divergent sequence due to the fact that terms of the sequence clustered around these values.

\[
a_n = (-1)^n \left( \frac{1}{n} \right), \text{ for any positive integer } n
\]

ERICA on Pretest: \( a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \)

ERICA: [pause] And I kind of think like [pause] maybe 1 and 0 are [pause] are limits.

I: Umm
ERICA: This is 0 here, and this is 1 here.
I: Okay.
ERICA: They kind of are all the limits because no matter what \( n \) you plug in, you would have some huge around here. The points always are gonna be on these two lines here. They are never gonna be out there.

**Regarding Limit Points as Limits**

The notion of a limit point is defined as follows in the mathematical context:

We call a sequence \( a_n \) to be convergent to a real number \( L \) if for any positive number \( \varepsilon \), there is a natural number \( N \) such that \( |a_n - L| < \varepsilon \) for all \( n \geq N \).

(Apostol, 1974, p.70)

Students in this category were those who conceptualized the limit of a sequence properly so that they could determine the limit of any given sequence. Sometimes they described the limit of a sequence as an asymptote or cluster point. However, they recognized that a convergent sequence may have a limit value to which infinitely many terms of the sequence are equal, and they also perceived the uniqueness of the limit value. Therefore, students in this category properly determined the limit of every sequence among monotone bounded, unbounded, constant, oscillating divergent, and oscillating convergent sequences. Furthermore, these students could determine properly the limit of a sequence even in the more complicated cases. For instance, BECKY figured out some extra limit problems during the posttest after completing the first part of the tasks that were commonly given to other students. BECKY said that such problems were not familiar enough.

**BECKY on Posttest:** \( a_n = 1 + (-1)^n \cdot 1/n \)

I: \( A \) sub \( n \) equals 1 plus negative 1 to the \( n \)th power times \( n \) for any positive integer \( n \).
BECKY: Oh, my!
I: You will like this.

Nonetheless, BECKY properly determined the limit of the sequence based on her conception of limit points.

BECKY on Posttest: \( a_n = 1 + (-1)^n \cdot \frac{1}{n} \) (continued from the above dialogue.)

BECKY: [graphing this sequence; see Figure 4.12]
I: Can you explain why the graph of the sequence should look this way?
BECKY: When \( n \) is odd, it is gonna be 1 minus [pause] 1 over \( n \), so the numbers are gonna be smaller than 1, but they will be approaching 1 because 1 over \( n \) approaches 0. When \( n \) is even, it would be 1 plus 1 over \( n \), so they will be, they will start out larger than 1, and approach 1 because 1 over \( n \) approaches 0.
I: Great. Does this sequence have a limit?
BECKY: Umm [pause] 1.
I: Can you explain why 1 is the limit of the sequence?
BECKY: Umm [pause] because umm all the values of the sequence are approaching 1 even though like part of them are approaching from umm greater than 1 and part of them are from less than 1. But they are still approaching the same number which is 1. So that is the limit.

\[
a_n = 1 + (-1)^n \cdot \frac{1}{n} \quad \text{for any positive integer} \ n
\]

Figure 4.12 A graph of \( a_n = 1 + (-1)^n \cdot \frac{1}{n} \) drawn by BECKY at posttest
In contrast, EVE, whose conception of the limit of a sequence was an asymptote, determined the sequence was not convergent due to its oscillating pattern.

**EVE on Posttest:** \[ a_n = 1 + (-1)^n \cdot \frac{1}{n} \]

I: What about this sequence?
EVE: [making a table for \( n=5, 10, 15 \); see Figure 4.13] Hmm [working] Nope.
I: Can you tell me why this sequence does not have a limit?
EVE: Because it goes back and forth.
I: So this sequence does not approach constantly?
EVE: Right.

![Figure 4.13 A table of numerical values of a sequence \( a_n = 1 + (-1)^n \cdot \frac{1}{n} \) drawn by EVE at posttest](image)
Among the 11 interviewees, there were only 2 students (ELENA and BECKY) who conceptualized limits correctly at the pretest. The other students initially conceptualized limits as either asymptotes or cluster points. On the other hand, it is noteworthy that at least one student (BEN), who at first was confused between the notion of cluster point and the limit of a sequence, later recognized the difference between them and even changed his conception of the limit from cluster point to limit point after working with $\varepsilon$–strip definitions during the interview tasks.

**Relationship Between Concept of Limit and Definition of Convergence**

In this section, students’ ability to perceive the relationship between $\varepsilon$ and $N$ described in $\varepsilon$–strip definitions and their ability to apply these definitions to sequences to determine convergence are analyzed. Students first experienced the following $\varepsilon$–strip definitions in Task 2, and thereafter, until the posttest, they used them to determine convergence and limit of a sequence.

$\varepsilon$–strip definition A: A certain value $L$ is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any epsilon strip as long as the epsilon strip covers $L$.

$\varepsilon$–strip definition B: A certain value $L$ is a limit of a sequence when only finitely many points on the graph of the sequence are not covered by any epsilon strip as long as the epsilon strip covers $L$.

According to the results of this study, there was a relationship between how students evaluated the correctness of the $\varepsilon$– strip definitions and how they conceptualized the limit of a sequence. For instance, even though students understood the reversal between $\varepsilon$ and $N$ that was represented in the the $\varepsilon$–strip definitions A and B, their response to the question asking about the correctness of the $\varepsilon$–strip definitions...
varied according to their conception of the limit of a sequence as asymptote, cluster point, or limit point. For instance, students’ whose conception of the limit of a sequence was as an asymptote disagreed with both definitions A and B; students who understood cluster points as limit regarded definition A but not definition B as a proper description for a limit of a sequence. On the other hand, among students who properly understood the limit of a sequence as a limit point, several students did not recognize the difference between cluster points and limit points, and they considered both $\varepsilon$–strip definitions A and B correct to describe the limit of a sequence. Finally, students who understood the limit of a sequence as a limit point and recognized the difference between cluster points and limit points responded that definition B but not definition A was correct to describe the limit of a sequence. Table 4.16 shows the relationship between students’ conceptions of the limit of a sequence and their responses to the question, “Which $\varepsilon$–strip definition between A and B properly describes the limit of a sequence?”

<table>
<thead>
<tr>
<th>Conception of Limit</th>
<th>Asymptotes</th>
<th>Cluster points</th>
<th>No distinction between Cluster points &amp; Limit points</th>
<th>Limit points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conception of Convergence</td>
<td>Getting close to but not equal to</td>
<td>Getting close to or being equal to</td>
<td>Getting close to or being equal to a (unique) value</td>
<td>Getting close to or being equal to a unique value</td>
</tr>
<tr>
<td>A is Correct</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>B is Correct</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Such a result was suggested to some extent when this study was designed and developed through the pilot studies. For example, when students regarded asymptotes as
limits of a sequence, it was expected that they would determine neither definition A nor B as proper to describe the notion of asymptotes. The main reason was that neither definition describes the most important criterion for asymptotes, which was that a convergent sequence should never equal its limit. On the other hand, when students regarded cluster points as limits of a sequence, it was expected that they would say definition B was not correct because there were examples of sequences, such as \( a_n = (-1)^n \cdot (1 + 1/n) \), having multiple cluster points which were limits for these students; they would have had to call such a sequence divergent if they applied definition B.

**Students’ Reversibility in the Context of the Limit of a Sequence**

There were several levels in students’ understanding of the reversal between \( \varepsilon \) and \( N \) in the \( \varepsilon - \) strip definitions. That is, there was not a dichotomy between no-reversibility and full-reversibility in the context of the limit of a sequence, but rather, a continuum of levels of reversibility. Table 4.17 shows five levels of reversibility that were identified as crucial in the continuum of development of reversibility in the context of limit. These levels of reversibility emerged from analyzing students’ conception of the \( \varepsilon - \) strip definitions and their ways of applying these definitions to determining limits of sequences. The following subsections discuss the main characteristics of each of the five levels of reversibility, report how students at each level evaluated \( \varepsilon - \) strip definitions, and describe possible reasons for such evaluations in terms of students’ conceptions of the limit of a sequence.
It should be noted that when neither of the $\varepsilon$–strip definitions was evaluated as correct by students, the interviewer suggested modifying at least one of the definitions for a better description of the limit. In the data analysis, each student’s level of reversibility was coded by the response to the initial task rather than the modification of these $\varepsilon$–strip definitions to minimize any possible influence by the interviewer when suggesting modifications.

Table 4. 17 Levels of reversibility in the context of the limit of a sequence

<table>
<thead>
<tr>
<th>Level of Reversibility</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reversibility Level 0</td>
<td>No reversibility</td>
</tr>
<tr>
<td>Reversibility Level 1</td>
<td>Completing the $\varepsilon$ process</td>
</tr>
<tr>
<td>Reversibility Level 2</td>
<td>For some fixed $\varepsilon &gt; 0$, finding $N_\varepsilon$</td>
</tr>
<tr>
<td>Reversibility Level 3</td>
<td>For any fixed $\varepsilon &gt; 0$, finding $N_\varepsilon$ (Static)</td>
</tr>
<tr>
<td>Reversibility Level 4</td>
<td>For any fixed $\varepsilon &gt; 0$, hereafter as $\varepsilon \to 0$, find $N_\varepsilon$ (Dynamic)</td>
</tr>
</tbody>
</table>

It was found that students’ levels of reversibility were related to their conception of the limit of a sequence. Further analysis of this relationship is considered in Chapter 5 to see if students developed in their reversibility by participating in this teaching experiment.

**Reversibility Level 0: No Reversibility**

Students at this level do not understand the reversal process in describing the concept of limit. That is, they always think forward, beginning with the terms of the sequence ($n$) and see how close ($\varepsilon$) they get to the limit. They may find limits of some sequences by applying their own conception of limit, whether or not it is complete. They
may (or may not) recognize that their own conception implies one of the $\varepsilon$–strip definitions. However, they do not believe that either of the $\varepsilon$–strip definitions is sufficient to describe their own conception of limit, and therefore have difficulty explaining their conception of limit in terms of the $\varepsilon$–strip definitions (see Figure 4.14).

![Figure 4.14](image)

**Figure 4. 14 Reversibility level 0: Regarding own conception as implying the $\varepsilon$–strip definitions, but not vice versa.**

Consequently, students at level 0 consider the $\varepsilon$–strip definitions as irrelevant for describing the limit of a sequence. Rather, they add their own conception to the $\varepsilon$–strip definitions to modify them for a better description of the limit (see Figure 4.15).

![Figure 4.15](image)

**Figure 4. 15 Modification of $\varepsilon$–strip definitions by students at reversibility level 0**
Table 4.18 shows students who were at reversibility level 0 listed according to their concept of limit. It can be seen that all students at reversibility level 0 regarded limits as asymptotes. It can be seen that no one among those who regarded cluster points or limit points as limits was at reversibility level 0 in this research. Hence, the following discussion focuses on students at reversibility level 0 among those who regarded asymptotes as limit.

Table 4.18 Students' distribution: Reversibility level 0

<table>
<thead>
<tr>
<th>Reversibility 0</th>
<th>Asymptotes</th>
<th>Cluster points</th>
<th>No distinction: Cluster points &amp; Limit</th>
<th>Limit points</th>
<th># (students)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task 2</td>
<td>EMILY, ERICA, EVE</td>
<td></td>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>Task 3</td>
<td>BETH, EVE</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Task 4</td>
<td>BETH, EVE</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Task 5</td>
<td>BETH, EVE</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Task 6</td>
<td>BETH, EVE</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Posttest</td>
<td>BETH, EVE</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>#(students)</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Students at reversibility level 0 and regarding asymptotes as limit. Students who regarded asymptotes as limits of a sequence and who were at reversibility level 0 said that the $\varepsilon$–strip definitions did not make sense to them. The main reason was that for them the $\varepsilon$–strip definitions did not describe divergence of oscillating sequences or constant sequences, as found in this dialogue with EVE:
EVE on Posttest

I: Student A said a certain value L is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any epsilon strips as long as the epsilon strip covers L.

EVE: But they didn’t include the fact when it crosses [pause] or when the graph goes from negative to positive.

I: That means student A is not correct?

EVE: Right.

I: Student B said a certain value L is a limit of a sequence when only finitely many points on the graph of the sequence are not covered by any epsilon strip as long as the epsilon strip covers L.

EVE: So, there wasn’t [pause]. Neither of them said anything about [pause] according to this graph the fact that it crosses [pause] L. The graph goes from [pause] like negative to positive.

Similarly, ERICA said that definition B was not relevant as a statement describing the limit of a sequence because looking outside the $\varepsilon$ – strips was not related to finding any line that a sequence was getting close to.

ERICA Task 2: $a_n = 1/n$

ERICA: I think student B is just kind of complicated because [pause] you are talking about points that are not covered, and it is just like what about all the points covered by the strip. I think student B is talking about the part that doesn’t even matter.

I: Oh, doesn’t even matter?

ERICA: Right, because the points not covered the strip [in B] isn’t really the part of that the sequence is getting close to the limit.

Another student EMILY, who determined the convergence of a sequence by the criterion, “getting close to, but not equal” was also at reversibility level 0. This student recognized that if a sequence was getting close to but not equal to a certain value, then there would be infinitely many points inside a (or any) strip as long as the red line of the $\varepsilon$ – strip were on the certain value. However, EMILY was not sure whether the converse (definition A) was also true, that is, if infinitely many points on the graph of a sequence
were covered by any $\varepsilon$–strip as long as the $\varepsilon$–strip covered $L$, then the sequence would be getting close to but not equal to $L$.

**EMILY on Task 2**: $a_n = 1/n$

I: Okay. Then can you explain why student A is correct?

EMILY: [pause] Umm, because [pause] all the strips cover [pause] umm 0, and [pause] you, we see that there is an infinite amount of umm points that approach 0, they are again covered by strips, but these points never touch 0, I mean, closer and closer and closer and approach 0, but never reach 0. So the limit will be 0.

I: So you mean when you use, the limit of a sequence is 0, that means, the sequence is getting closer and closer and closer to 0, but never reaches 0? Am I correct?

EMILY: Yeah.

I: That is described by this sentence [student A]? The same meaning?

EMILY: [reading A] Umm, I don’t know, that [A]’s not very [pause]. Maybe if I, I mean, I guess, like I see where the strips [pause] like, how they work on paper, but I guess like the whole concept of epsilon, maybe if I better understand that, I guess, I could say that this is correct but umm [pause]. Because they didn’t say about the points approaching 0, so.

In the same line of reasoning, EMILY responded that definition B was not a proper description of the limit of a sequence because it did not interpret the convergent sequence as having an asymptotic line.

**EMILY on Task 2**: $a_n = 1/n$

I: Okay. What do you think about student B?

EMILY: Umm [reading] Umm, no, I don’t, I don’t think this is very [pause] accurate, just because [pause] umm, I don’t think this is accurate just because it doesn’t say anything about these, you know, infinite amount of points approaches 0 which [pause]. I mean [pause] it doesn’t say anything about that.

Indeed, the students at reversibility level 0 who regarded asymptotes as limits often expressed how hard it was to understand the $\varepsilon$–strip definitions.

---

10EMILY was on reversibility level 3 at the end of Task 2. However, EMILY is classified into reversibility level 0 at Task 2 because that was the initial understanding EMILY brought to the task.
EVE on Posttest: \( a_n = \frac{n}{n+1} \)

I: Can you tell me why 1 is the limit of the sequence by using \( \varepsilon \)-strips?
EVE: [pause] [laugh] I don’t know how. I really don’t know how. I don’t know why.
I: So you mean infinitely many points covered by any \( \varepsilon \)-strip is not related to the limit of the sequence?
EVE: No, I never learn that way so I can’t apply that. Like [pause] I don’t know exactly how to use that definition for a limit.
I: You mean this definition or the definition you learned in class?
EVE: I don’t know how to use \( \varepsilon \)-strip to describe the definition of the limit. Now, when we are dealing with definition, do we have to use \( \varepsilon \)-strip definition? Do we have to use \( \varepsilon \)-strip in the definition?

As seen above, EVE had responded that the \( \varepsilon \)-strip definitions were not clear to her beginning with Task 2 through the posttest. This student did not want to apply these definitions to sequences to determine convergence and the limit value of a sequence even at the posttest. On the other hand, students who responded that they understood the \( \varepsilon \)-strip activity understood the \( \varepsilon \)-strips could mark asymptotic lines by the red line in the middle of the strips.

BETH on Task 2: \( a_n = \frac{1}{n} \)
I: Can you explain the limit of the sequence by using the strip activity? Is there any idea?
BETH: Umm [pause], I think you can tell that it is going towards 0 with the strip activity because [pause] the points are getting like closer and closer to the red line when you place it [the red line on the \( \varepsilon \)-strip] down on the x-axis. And you can tell that [pause] the points are never gonna go back above the previous number, so you can tell that they are all going towards 0, based on where there are the relation to the red line.

As shown above, students who regarded asymptotes as limits of sequences and who were at reversibility level 0 maintained it was not proper to describe their own conception of limit by using the \( \varepsilon \)-strips. Instead, these students tended to determine the limit of a sequence using their own conception of limit.
EVE on Task 2: \( a_n = 1/n \)

I: Do you think after because the part of the sentence explains why the limit of the sequence is 0?

EVE: I would use the equation \[ a_n = 1/n \] as part of my reasoning.

I: What do you mean?

EVE: The, a sub n equals 1 over n [pause] I would use this equation in my reasoning. I would say that limit of a sub n equals 1 over n for n a positive integer is 0 because as n [pause] as n is [pause] approaching infinity, 1 over n is getting closer to 0.

Or, like BETH, they tended to consider modifications of \( \varepsilon \) – strip definition A by adding their own conception of the limit as an asymptote (see Figure 4.16).

BETH on Posttest

I: Is there any way you can modify so that we can apply to this sequence?

BETH: Umm, you would be able to modify student A [pause] by saying umm [pause] something about [pause] like the points need to be moving towards the red line, but not covered by the red line.

I: Can you modify it?

BETH: Yes. [adding "and the points are moving towards the red line and are not covered by the red line" to A; see Figure 4.16]

---

**Student A**

A certain value \( L \) is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any epsilon strip as long as the epsilon strip covers \( L \). and the points are moving towards the red line and are not covered by the red line.

---

Figure 4. 16 BETH’s response: Reversibility level 0 and regarding asymptotes as limits
Reversibility Level 1: Completing the $\varepsilon$-process

Students in this level of reversibility regarded “any epsilon strip” in the $\varepsilon$–strip definitions as including a strip of width 0 or infinity. In other words, these students completed the process of increasing or decreasing the value of $\varepsilon$, and then examined the relationship between $\varepsilon$ and $N$ in the $\varepsilon$–strip definition when the value of $\varepsilon$ becomes ultimately infinity or 0. Table 4.19 shows students’ understanding of the $\varepsilon$–strip definitions A and B at reversibility level 1.

Hence, students in this category may point out that applying definition A, one could conclude that every sequence should be convergent because the value of $\varepsilon$ can get larger and larger, and ultimately become infinite, then all terms of the infinite sequence would be inside the strip and no terms of the sequence would be outside the strip; or, one could conclude that no sequence should be convergent because the value of $\varepsilon$ can get smaller and smaller, and ultimately become 0, then no terms of the infinite sequence would be inside the strip and infinitely many terms of the sequence would be outside the strip.

Table 4.19 Students’ conception of $\varepsilon$–strip definitions at reversibility level 1

<table>
<thead>
<tr>
<th>Students’ conception of $\varepsilon$–Strip Definition A:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A certain value $L$ is a limit of a sequence when infinitely many points are covered by any epsilon strip, therefore, by an epsilon strip of width 0 (or infinity), as long as the epsilon strip covers $L$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Students’ conception of $\varepsilon$–Strip Definition B:</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A certain value $L$ is a limit of a sequence when only finitely many points are not covered by any epsilon strip, therefore, by an epsilon strip of width 0 (or infinity), as long as the epsilon strip covers $L$.</td>
<td></td>
</tr>
</tbody>
</table>
Due to the consideration of the width of 0 or width of infinity, students in this category considered $\varepsilon$–strip definitions as inappropriate for describing the limit of a sequence. In line with this reasoning, students in this category pointed out that none of the $\varepsilon$–strip definition would be implied by their own conception of the limit of a sequence (see Figure 4.17).

![Figure 4.17 Reversibility level 1: Own conception does not imply any one of $\varepsilon$–strip definitions.](image)

Table 4.20 shows students’ distribution at reversibility level 1 by their own concept of limit of a sequence. It can be seen that no one who regarded limit points as limits was at reversibility level 1 in this study. Hence, the following discussion considers students at reversibility level 1 who regarded asymptotes or cluster points as limits.
Table 4. 20 Students' distribution: Reversibility level 1

<table>
<thead>
<tr>
<th>Reversibility 1</th>
<th>Asymptotes</th>
<th>Cluster points</th>
<th>No distinction: Cluster points &amp; Limit points</th>
<th>Limit points</th>
<th>#(students)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task 2</td>
<td>BRIGID</td>
<td>BRIAN</td>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>Task 3</td>
<td>ERICA</td>
<td>BEN</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Task 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Task 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Task 6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Posttest</td>
<td>(ERICA)</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>#(students)</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Students at reversibility level 1 who regarded asymptotes as limits. Students who regarded an asymptote as a limit of a sequence considered neither of the $\varepsilon$–strip definitions as a proper description of a limit. They may have considered the $\varepsilon$–strip definitions as almost correct except for the fact that they misconceived the width of the strip as possibly 0 or infinity. For instance, when applying definition A to show divergence of an unbounded sequence, ERICA imagined using an $\varepsilon$–strip whose width was infinity and then found that the unbounded sequence was convergent according to definition A:

ERICA on Task 3: $a_n = \sqrt{n}$

I: Can you explain it by using student A’s idea?

ERICA: Okay. Umm. Since there isn’t an infinite number of points [pause] umm [pause] covered by the epsilon strip, then the certain four is not the limit. [pause] I think it would be, if it says [pause] any epsilon strip. If someone can imagine epsilon strip that was [pause] infinitely greater than for each side, then you could say oh, it would cover anything. But not case then [pause] you could imagine epsilon strip [pause] however

---

11 ERICA was at reversibility level 4 for almost all sequences except the unbounded sequence during the posttest.
big you would want it, like any sequence have in it anything. You don’t have it. Everything because we can have any limit.

I: So we have to assume that the width of the strip is not infinite?

ERICA: Right

Consequently, ERICA decided the $\varepsilon$–strip definition was inappropriate as a definition of limit. Indeed, ERICA developed reversibility up to level 4 for sequences other than unbounded sequences. However, when dealing with an unbounded sequence in Task 3 or the posttest, ERICA remained at reversibility level 1.

**ERICA on Posttest:** $a_n = \frac{n^2}{n+1}$

I: Okay. If somebody, like me, says 20 is a limit of this sequence, can you explain why 20 is not a limit of this sequence by using student A’s idea?

ERICA: Okay. Umm [pause] I can say that if I want to choose any of these $\varepsilon$–strips, I will choose this one [small strip]. [I: Okay] I am gonna say that put that on 20 [pause] and [pause] I will say, look, there aren’t infinitely many points inside this strip because [pause] there aren’t infinitely many points this side and this side, these points will continue up and up and up and up there [pause] and not inside the strip down here. [I: I see.] So that is why I say. [pause] And [pause] your counter argument could be [pause] look, choose a different strip. If I chose an even larger $\varepsilon$–strip [pause], then I have got more points inside of it, like I can move it over here, then I have got more and more points inside the strip. And [pause] you will say well, what if I have an $\varepsilon$–strip that was higher, you know, then I will say well, [laugh] you wouldn’t be wrong [laugh] but.

I: What do you mean?

ERICA: I mean like, if you are, I mean, like [pause] if you said that you want to pick any $\varepsilon$–strip, and your $\varepsilon$–strip is, and you want to have an imaginary $\varepsilon$–strip that has a mile on each side. And then [pause]. Therefore you have got infinitely many points inside your strip because your $\varepsilon$–strip goes up and up and up to right and right and right. And you have got a huge imaginary $\varepsilon$–strip [pause], and then [pause] I would say [smile] it is kind of ridiculous. But, I mean [pause].

I: Then 20 is a limit of the sequence?

ERICA: Then 20 could be but [pause] I don’t [pause] I mean according to this, 20 would be that [pause]. But I don’t [pause] 20 could be a limit but [pause].
Generally, regardless of the level of reversibility, students who considered asymptotes as limits modified definition A. In particular, ERICA changed the part “any epsilon strip” to “an epsilon strip” in order to exclude an $\varepsilon$–strip whose width is infinite (see Figure 4.18).

ERICA on Task 3: $a_n = \sqrt{n}$ (continued from the above dialogue)

I: In that case, student A’s idea is correct?
ERICA: Yes. Where else the width of the strip is infinite, then student A is, I think.
I: Do you want to modify it? Or do you think student A is okay?
ERICA: Umm. I think it is okay as long as you, just don't say any [cross out “y” of the word “any” in definition A; see Figure 4.17]. I think you say an [emphasizing] $\varepsilon$–strip, covered by an $\varepsilon$–strip, and that case is most of the cases, oh, an $\varepsilon$–strip when we have here. You know, not, you know, any $\varepsilon$–strip they can imagine.
I: Okay. So if I change it like “a certain number L is a limit of a sequence when infinitely many points on the graph of the sequence are covered by an epsilon strip as long as the strip covers L”?
ERICA: Right.

As indicated above, students in reversibility level 1 thought the $\varepsilon$–strip definitions might not be proper to describe the concept of limit due to their allowance of
$\varepsilon$– strips of width infinity or 0. Hence, they tended to modify the $\varepsilon$– strip definitions to describe the concept of limit better. As seen in ERICA’s argument, students at reversibility level 1 tended to modify the part “any epsilon strip” into “one epsilon strip” so they could exclude the extreme cases of $\varepsilon$– strips. There was another group of students who understood “any epsilon strip” as “one (or at most some) epsilon strips” in the $\varepsilon$– strip definition. These students were classified as being at reversibility level 2 and are discussed in the next subsection.

**Students at reversibility level 1 who regarded cluster points as limits.** Students who regarded a cluster point as a limit thought $\varepsilon$– strip definition A was partially correct but definition B was not correct at all. They perceived definition A as correct except for the fact that they misconceived the width of the $\varepsilon$– strip as possibly 0 or infinity. For instance, EMMA, who regarded cluster points as limits, considered the case of $\varepsilon$ eventually becoming 0 and concluded that definition A did not properly describe the limit of a sequence.

**EMMA on Task2:** $a_n = 1/n$

EMMA: Well [pause], I think saying that ‘the strips are covering 0’ is kind of [pause] misleading because, I mean, you can say that your strips [pause] get so thin that it’s just like a line and then does not include anything other then 0.

I: So that is the reason to say that as long as the strip covers 0?

EMMA: See, like, but I am saying like yeah that’s not good enough because if it’s only covering 0, then it’s not including the planes.

I: Umm.

EMMA: I mean, you were assuming that, like, well, it depends on how you consider this strip, you know what I mean?
Reversibility Level 2: For Some Fixed $\varepsilon > 0$, Find the Corresponding $N_\varepsilon$

Students in reversibility level 2 understood the phrase “any epsilon strip” in the $\varepsilon$–strip definition as meaning “some fixed epsilon strips.” While students at reversibility level 1 examined $\varepsilon$–strip definitions with a strip of width 0 or infinity, those at reversibility level 2 examined them with some fixed $\varepsilon$–strips of real-number widths greater than 0 but less than infinity. However, they perceived the $\varepsilon$–strip definitions as meaning it is enough to check only one or at most some $\varepsilon$–strips, rather than all $\varepsilon$–strips, to see if the sequence is convergent. Table 4.21 shows how students in this category inappropriately interpreted the $\varepsilon$–strip definitions.

Table 4. 21 Students' conception of $\varepsilon$–strip definitions at reversibility level 2

<table>
<thead>
<tr>
<th>Students’ conception of $\varepsilon$–Strip Definition A:</th>
</tr>
</thead>
<tbody>
<tr>
<td>A certain value $L$ is a limit of a sequence when infinitely many points are covered by some epsilon strip as long as the epsilon strip covers $L$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Students’ conception of $\varepsilon$–Strip Definition B:</th>
</tr>
</thead>
<tbody>
<tr>
<td>A certain value $L$ is a limit of a sequence when only finitely many points are not covered by some epsilon strip as long as the epsilon strip covers $L$.</td>
</tr>
</tbody>
</table>

Table 4.22 shows students’ distribution at reversibility level 2 in terms of their own conception of the limit of a sequence. It should be noted that students at this level seem to have a static image of the limit of a sequence with an $\varepsilon$–strip. They do not perceive the notion of the infinite process of $\varepsilon$–values to be examined in determining
the limit of a sequence. Consequently, once there is at least one $\varepsilon$–strip satisfying the $\varepsilon$–strip definitions, they regard that sequence as convergent (see Figure 4.19).

Table 4.22 Students’ distribution: Reversibility level 2

<table>
<thead>
<tr>
<th>Reversibility 2</th>
<th>Asymptotes</th>
<th>Cluster points</th>
<th>No distinction: Cluster points &amp; Limit</th>
<th>Limit points</th>
<th>#(students)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task 2</td>
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<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Task 3</td>
<td>ELISA</td>
<td>BRIGID</td>
<td>ELISA</td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Task 4</td>
<td>ELISA</td>
<td></td>
<td>ELISA</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Task 5</td>
<td>ELISA</td>
<td>EMILY</td>
<td>BEN</td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>Task 6</td>
<td>ELISA</td>
<td>EMILY</td>
<td>BEN</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>Posttest</td>
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<tr>
<td>#(students)</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.19 Reversibility level 2: None of $\varepsilon$–strip definitions implies own conception.

The following subsections describe how students at reversibility level 2 understand and apply the $\varepsilon$–strip definition to see if a certain value is (or not) a limit of a sequence.
Students at reversibility level 2 who regarded asymptotes as limits. Students who regarded asymptotes as limits believed that constant sequences were not convergent because the sequence actually reached a value which was the only possible limit value.

ELISA on Task 4: \( a_n = 1 \)

I: Okay. Does this sequence have a limit?
ELISA: [pause] No, I still think it doesn’t.
I: Okay. Can you explain why the sequence does not have a limit?
ELISA: Because it never, it, it’s always 1 so it never reaches. Umm.
I: What do you mean by reaching?
ELISA: It \( [a_n = 1] \) never, umm gets close to. Look, it never gets close to something that doesn’t reach. Like one when we looked at last week \( [a_n = 1/n] \), it would get close to 0 but never actually reach 0, so therefore 0 was the limit. But in this one \( [a_n = 1] \), it is always 1. The answer is always 1.

Among those who regarded asymptotes as limits, students at reversibility level 2 believed they could determine convergence of a sequence using \( \varepsilon - \) strip definition A along with some finite number of \( \varepsilon - \) strips. However, this reasoning implied that constant sequences were convergent, which was not true according to their conception of convergence. Therefore, they did not agree with either definition A or B.

For instance, ELISA showed how to determine convergence of a constant sequence using both definitions A and B and some \( \varepsilon - \) strips.

ELISA on Task 4: \( a_n = 1 \) (continued from the above dialogue)

I: What do you think?
ELISA: Umm [pause], I guess I have to change my answer about the limit by using these [definition A & B]. The limit would have to be 1 in this case \( [a_n = 1] \).
I: Can you explain why you have to change your answer if you use these ideas?
ELISA: Umm. Because [pause] umm, if you place the epsilon strip on, umm [pause] on \( L \), in this case it is 1, then there are infinitely many points in
the \( \varepsilon \)–strip, so therefore they [definition A] are saying that 1 would be
the limit of the sequence.

ELISA: And this one [definition B], they are saying um there are finitely many
points outside the \( \varepsilon \)–strip, and in this case \( [a_n = 1] \) is 0 and that’s a
finite number so umm 1 would be the limit.

I: What do you think student A and student B?

ELISA: [pause] Yes, If they are both right, then in this sequence \( [a_n = 1] \), the
limit will be 1.

I: Do not just assume that they are correct.

ELISA: Correct? Okay.

I: They might be wrong. They just explained and I just picked up some of
the ideas. So tell me what you think.

ELISA: Okay. Well, it doesn’t, since it doesn’t ever actually like kind of ever try
to reach something. It is always 1. Then I will tell that they don’t have a
limit.

As seen in the above dialogue, ELISA did not accept the \( \varepsilon \)–strip definitions because
they gave a result contradictory to her own conception of limit, which was an asymptote.

On the other hand, students at reversibility level 2 had some cognitive obstacles in
understanding \( \varepsilon \)–strip definition B. Rather than checking for finitely many points
outside some \( \varepsilon \)–strips, ELISA preferred seeing infinitely many points inside some
\( \varepsilon \)–strips. However, this was not because ELISA perceived some convergent sequence
whose limit could be equal to some terms of the sequence as did those who regarded
cluster points as limits. Instead, ELISA imagined that infinitely many points inside the
strips represented a sequence getting close to but not being equal to its asymptote.

**ELISA on Task 5:** \( a_n = (-1)^n 1/n \)

I: Umm [pause]. Okay. Can you tell me why you think student A is better?

ELISA: Umm [pause], I don’t like these [definition B] like ‘not’. It is still
confusing.

I: I see.

ELISA: Umm, but infinitely many inside like it that’s definitely, if there is
infinitely many points inside the strip that means, it’s not converge
[meant not reach] to a certain value. It is gonna keep going forever.
I: Umm, can you say one more time? Does this sequence not converge?
ELISA: If it like, [pause] Like when they say infinitely many points that means that, it’s not like reach a certain value, or stay in there.
I: Umm,
ELISA: It’s gonna keep going on.
I: I see.
ELISA: Yeah, that implies that it never reaches a certain value. Well this one [definition B] doesn’t say anything about that.
I: Can you state your standard when you determine a limit of a sequence?
ELISA: Well, what do you mean?
I: What do you mean for a sequence to have a limit?
ELISA: Uh, Umm. [pause]
I: What do you usually use?
ELISA: That it goes towards a point but never actually reach a certain point.

As seen above, ELISA preferred to use $\varepsilon$–strip definition A due to the fact that looking at infinitely many points inside an $\varepsilon$–strip implied to her that the sequence kept going but never reached its asymptotic line, whereas looking outside an $\varepsilon$–strip did not guarantee that aspect.

**Students at reversibility level 2 who regarded cluster points as limits.** Among students who regarded cluster points as limits, those at reversibility level 2 understood $\varepsilon$–strip definitions A and B as examining some, but not all, $\varepsilon$–strips. In line with this reasoning, they determined a sequence convergent when infinitely many points on the graph of the sequence would be covered by some $\varepsilon$–strips.

**EMILY on Task 6:** $a_n = (-1)^n(1 + 1/n)$

I: If we use student A, 1 is the limit of the sequence, or not?
EMILY: Yeah we could say that. If we were using student A, we could say that 1 is. Because umm [pause] when infinitely many points on the graph of the sequence are covered by an $\varepsilon$–strip as long as the $\varepsilon$–strip covers $L$. So if we were covering $L$ which is 1 and there is an infinite amount of points inside the $\varepsilon$–strip, that is true. So we can say that [pause] 1 is the [pause] limit by using this definition [definition A].
On the other hand, these students tended to reject $\varepsilon$–strip definition B in determining the limit of a sequence. The main reason was that they accepted multiple values for limits, regarding cluster points as limits. However, when applying $\varepsilon$–strip definition B, they could not conclude multiple values for the limit, say, of an oscillating divergent sequence. For instance, EMILY examined an oscillating divergent sequence to see if $\varepsilon$–strip definition B was good enough to determine convergence of the sequence.

**EMILY on Task 6:** $a_n = (-1)^n(1+1/n)$

I: What about student B? If we use student B’s definition, 1 is a limit of the sequence, or not?

EMILY: [pause] Umm. No, we can’t use this [B] because [pause] there isn’t a finite amount of points outside.

I: What do you mean that we cannot use student B?

EMILY: Umm, because it says [pause] a certain value $L$ is a limit of a sequence [pause] when only finitely many points on the graph of the sequence [pause] are not covered by $\varepsilon$–strip [pause] as long as the strip covers $L$. So we were covering $L$ which [pause] I am saying, is 1 [laugh], I don’t know. Umm [pause] that says [pause] it’s, $L$ is the limit when only finitely many points of the graph of the sequence are not covered by $\varepsilon$. Well [pause], in this case $L$ would be 1 and 1 wouldn’t be the limit because there is an infinite amount of points outside the strip. So we can’t use it but.

I: What do you think? Which one is correct?

EMILY: Student A.

I: Student A is correct?

EMILY: Yes.

As seen in the above dialogue with EMILY, students in this category found that definition B contradicted their own conception of limit which was as a cluster point. Consequently, they did not agree with definition B as a proper definition of limit of a sequence.

On the other hand, students in this category saw a contradiction between their own conception and the result implied from $\varepsilon$–strip definition A. It was not clear to them that a convergent sequence was getting close to or equal to a value according to
definition A. Such a contradiction made students not accept definition A as describing
t heir conception of limit. For instance, EMILY responded that definition A did not imply
that convergent sequences were getting close to a value.

**EMILY on Task 5:** \(a_n = (-1)^n 1/n\)

I: I see. Does this sequence have a limit?
EMILY: Zero.
I: Zero is the limit? Okay.
EMILY: Right? [laugh]
I: All right. I am going to show you other students’ ideas, how they were thinking. Let’s just think about whether they are correct or not.
EMILY: Umm.
I: Student A said a certain value \(L\) is a limit of the sequence when infinitely many points on the graph of the sequence are covered by any \(\varepsilon\) – strip as long as the \(\varepsilon\) – strip covers \(L\).
EMILY: [pause] I don’t know, it doesn’t seem very [pause] really. I don’t know maybe you should say something about like [pause], how the points are approaching [pause] the limit. And [pause] I don’t know [pause], when they talk about the \(\varepsilon\) – strip, it just like the points are covered. [pause] Like all my thinking was like, oh, it [\(\varepsilon\) – strip] was up here, you know, limits still covered [pause], umm [pause] and some of the points are also covered, but the line is other up here. So I don’t know.

The main reason that students in this category did not recognize that definition A described cluster points was that they considered it with only some \(\varepsilon\) – strips but not all \(\varepsilon\) – strips. However, without seeing the difference between using any \(\varepsilon\) – strip and using some \(\varepsilon\) – strips, they even pointed out that the definition A might mistakenly identify 0.1 as a limit of the sequence \(a_n = (-1)^n 1/n\) even though the sequence was not clustered around 0.1.

**EMILY on Task 5:** \(a_n = (-1)^n 1/n\)

EMILY: Like, that \([A]\) should how sounds like [pause], when infinitely many points on the graph of the sequence are covered by any \(\varepsilon\) – strip as long as the \(\varepsilon\) – strip covers \(L\). So, yeah, if we have it on [pause] 0.1, if it [an \(\varepsilon\) – strip] extends all the way to right, we will still have infinitely many,
Students at reversibility level 2 who do not distinguish cluster points from limit points. Students in this category understood the reverse relation between $\varepsilon$ and the index $N$ described in both $\varepsilon$–strip definitions A and B, and they could apply them to determine the limit of a sequence. However, they did not recognize the difference between the definitions A and B. Indeed, students in this category thought that definition A not only describes properly the concept of limit but also implies definition B; therefore definition B must also be correct. That is, they misconceived that having infinitely many points inside a strip would imply having only finitely many points outside the strip. Following this line of argument, these students were classified as those who did not recognize the difference between cluster points and limit points.

As other students at reversibility level 2 did, students in this category also examined the $\varepsilon$–strip definitions with only some $\varepsilon$–strips but not all. Based on examination with some $\varepsilon$–strips, BEN in this category responded that definitions A and B both described the concept of limit to some extent. However, this did not mean that he regarded either $\varepsilon$–strip definition as a proper description of the limit. Because of using some $\varepsilon$–strips and not all, students could find problems with both definitions for determining the limit of a sequence. For instance, BEN disagreed with definition A because there were some sequences which were not convergent but infinitely many points would be inside some $\varepsilon$–strips. In other words, BEN pointed out that there was a value for the sequence $a_n = (-1)^n \cdot 1/n$ that was not a cluster point but which satisfied the requirement that: infinitely many points were covered, whether or not it was the limit.
BEN on Task 5: $a_n = (-1)^n \cdot 1/n$

BEN: Because again, our anticipated limit is covering this, -.2, and there are infinite amount of points inside here. So according to this [A], -.2 is the limit.
I: How can we say that?
BEN: Because, umm, the strip is covering the value $L$, and an infinite amount of points are inside there.

As shown in the above, BEN conceived the meaning of the $\varepsilon - \text{strip}$ definition as “if there were infinitely many points inside an $\varepsilon - \text{strip}$, then the sequence would be convergent.” However, BEN did not recognize that it should be examined not for some fixed $\varepsilon - \text{strips}$ but for all $\varepsilon - \text{strips}$, and by doing this, it would not be true that infinitely many points would be covered by every $\varepsilon - \text{strip}$. Consequently, BEN responded that definition A meant that $-.2$ would be a limit of the sequence $a_n = (-1)^n \cdot 1/n$ because of finding an $\varepsilon - \text{strip}$ that covered infinitely many points on the graph of the sequence when the red line in the middle of the $\varepsilon - \text{strip}$ was on $y = -.2$ as a limit value.

It was noticed that students at reversibility level 2 were not able to use the $\varepsilon - \text{strip}$ definitions properly due to lack of logic related to the word any in the statements. In order for students to properly understand the role of $\varepsilon - \text{strips}$ in the definitions, it is necessary to imagine infinitely many graphical representations of the sequence, one for each fixed $\varepsilon - \text{strip}$. Since it was hard to imagine an infinite set of $\varepsilon - \text{values}$, students like BEN seemed to examine only some $\varepsilon - \text{strips}$ but not all, so they often identified limits that were not actually limits of a sequence. Consequently, students in this category seemed to disagree with both definitions A and B.
Students at reversibility level 2 who conceptualized limit properly. By Task 6, BEN understood the concept of limit properly and recognized that $\varepsilon$–strip definition A did not represent the limit of a sequence properly. In particular, BEN pointed out that the definition did not take into account the case of divergent sequences with multiple cluster points.

BEN on Task 6: $a_n = (-1)^n (1 + 1/n)$

I: Can you explain why student A does not work?
BEN: [pause] I don’t know not be a limit, but umm according to them I place it on which we think of limit, there is an infinite amount of points in this strip which there are, then $L$ would be the limit. But [laugh] [pause].

I: You do not think 1 is the limit?
BEN: No.
I: Why not?
BEN: I don’t think it has a limit because I don’t think there is a limit.

On the other hand, BEN seemed to agree with the $\varepsilon$–strip definition when putting the red line of an $\varepsilon$–strip on $y = 1$ which was small enough to show that infinitely many points appeared outside the strip. Then BEN pointed out that infinitely many terms of the sequence around $y = -1$ were not covered by the $\varepsilon$–strip, hence, according to student B’s definition, 1 should not be a limit of a sequence. In this way, BEN perceived what student B meant by examining definition B with some $\varepsilon$–strips.

BEN on Task 6: $a_n = (-1)^n (1 + 1/n)$

I: Can you explain it [$\varepsilon$–strip definition B] one more time?
BEN: Okay. This one [definition B] say if there is a finite amount of points outside the strip, which there aren’t. [pause]. Like it says that if there are, [pause] Okay. It says if there are, wait [laugh] [pause] Okay [pause]. Is it saying that when a finite amount of points aren’t covered by the strip, then it is the limit? [pause] Well, in this case there are an infinite amount of points outside the strip [pause] which means that is not the limit which mean what I am saying.

I: Student B is correct?
BEN: [pause] Yes. [laugh]

However, once examining \( \varepsilon - \) strip definition B using an \( \varepsilon - \) strip large enough to cover all but finitely many points while the red line on the strip was on \( y = -1 \), BEN pointed out that 1 would also be a limit of the sequence when definition B was applied to the sequence.

**BEN on Task 6:** \( a_n = (-1)^n(1+1/n) \) (continued from the above dialogue)

I: If we use this strip, [putting an \( \varepsilon - \) strip on 1 covering all points but finitely many]

BEN: [laugh]

I: Then there will be one point outside the strip. Or, if you use a little bit smaller strip, then finite number of points outside the strip?

BEN: Then I guess it [definition B] does not work. [laugh]

As seen above, BEN made a determination of the limit of a sequence based on observation of the graph of the sequence with some, but not all, \( \varepsilon - \) strips. Therefore, proper determination of the divergence of the sequence \( a_n = (-1)^n(1+1/n) \) depended on the width of the \( \varepsilon - \) strip used.

It is noted that students at reversibility level 2 supposed the given \( \varepsilon - \) strip definitions to represent a static image of the graph of a sequence along with an \( \varepsilon - \) strip. As the interviews went on, these students who were in level 2 often tried to modify the \( \varepsilon - \) strip definitions to better describe the concept of limit. In the case of BEN, it was modified to specify the range of widths to be used in determining divergence of the sequence \( a_n = (-1)^n(1+1/n) \). In other words, since BEN misconceived that it should be said that 1 was a limit of this sequence when using an \( \varepsilon - \) strip whose width was large
enough to cover all but finitely many points, BEN suggested modifying the definition to specify the possible widths of the $\varepsilon$–strips as small enough.

**BEN on Task 6:** $a_n = (-1)^n(1+1/n)$  (continued from the above dialogue)

BEN: The only question is depending on only as long as the strip umm [pause] as long as there is one point outside the strip, like umm as long as the strip [pause] does not cover all points. I am trying to limit the [pause] maximum that does not cover every single point.

I: Is there any way that we can modify student B?

BEN: I think so. [Writing modification; See Figure below]

I: Can you read it?

BEN: A certain value $L$ is a limit of a sequence when only finitely many [pause] points on the graph of the sequence are not covered by epsi-[pause], by any $\varepsilon$–strip as long as the $\varepsilon$–strip covers $L$ [pause] and as long as the $\varepsilon$–strip is small enough that at least one point lies outside the strip.

I: Umm. At least one point outside the strip?

BEN: You can’t take it bigger than any value that the sequence can take on. That is what I wanted to say.

I: Because finitely many should be outside?

BEN: Yeah, so if you have to make it smaller but at least one point outside, and then [pause] as long as at least one point outside, then you can apply it [definition B] to say whether there is a limit or not.

In this way, BEN modified definition B to determine the divergence of the sequence.

Nonetheless, he did not recognize that the definition assumed all $\varepsilon$–strips.

**Reversibility Level 3: For Any Fixed $\varepsilon > 0$, Find $N_\varepsilon$ (Static)**

Students at reversibility level 3 understand that the phrase “any $\varepsilon$–strip” in the $\varepsilon$–strip definitions means “any fixed $\varepsilon$–strip.” While students at reversibility level 2 examined $\varepsilon$–strip definitions with only some $\varepsilon$–strips, students at level 3 understood they should consider all $\varepsilon$–strips in determining the limit of a sequence. That is, for any fixed $\varepsilon$–strip, they should be able to find infinitely many points inside the strip or only finitely many points outside the strip when using definition A or B, respectively.
Table 4.23 shows how students at level 3 inappropriately understand the $\varepsilon$–strip definitions.

**Table 4.23 Students' conception of $\varepsilon$–strip definitions at reversibility level 3**

<table>
<thead>
<tr>
<th>Students’ conception of $\varepsilon$–strip definition A:</th>
<th>Students’ conception of $\varepsilon$–strip definition B:</th>
</tr>
</thead>
<tbody>
<tr>
<td>A certain value $L$ is a limit of a sequence when infinitely many points are covered by any fixed epsilon strip as long as the epsilon strip covers $L$.</td>
<td>A certain value $L$ is a limit of a sequence when only finitely many points are not covered by any fixed epsilon strip as long as the epsilon strip covers $L$.</td>
</tr>
</tbody>
</table>

It should be noted that students in reversibility level 3 perceive just a collection of static images of the graphical representation of a sequence for any fixed value of $\varepsilon$, but they do not construct a continuum of these images. Even though they imagine infinitely many static images of the graphical representation of a sequence, along with all $\varepsilon$–strips, such a collection of static images still makes it difficult for students to grasp the role of the $\varepsilon$–strip. In addition, they do not understand that the $\varepsilon$–strip definition can be used to reduce the error bound as much as they want. As a result, students at reversibility level 3 do not believe the $\varepsilon$–strip definitions properly describe the concept of limit (see Figure 4.20), and often modify the definitions to reflect reversibility level 2, or reversibility level 4 which is dealt with in the next subsection. Thus, we regard students’ conceptions at this level as different from the proper conception of the limit of a sequence, which should be compatible with the rigorous definition of limit.
Figure 4.20 Reversibility level 3: None of $\varepsilon$–strip definitions implies own conception.

Table 4.24 shows students’ distribution at reversibility level 3 in terms of their own conceptions. The following subsections describe how students at reversibility level 3 responded differently in terms of their conception of limits.

Table 4.24 Students’ distribution: Reversibility level 3

<table>
<thead>
<tr>
<th>Reversibility 3</th>
<th>Asymptotes</th>
<th>Cluster points</th>
<th>No distinction: Cluster points &amp; Limit</th>
<th>Limit points</th>
<th>#(students)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task 2</td>
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<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>Task 3</td>
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<td></td>
</tr>
<tr>
<td>Task 6</td>
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<td>EMILY</td>
<td></td>
<td>BRIGID</td>
<td>2</td>
</tr>
<tr>
<td>Posttest</td>
<td></td>
<td></td>
<td></td>
<td>BEN BECKY BRIGID</td>
<td>5</td>
</tr>
<tr>
<td>#(students)</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

**Students at reversibility level 3 who regarded asymptotes as limits.** Students who regarded asymptotes as limit and who were at reversibility level 3 considered if they
could determine that a convergent sequence was getting close to but not equal to its limit when applying the \( \varepsilon \)–strip definitions for all \( \varepsilon \)–strips. They tended to think definition A would better describe the limit of a sequence because they could see the pattern of a convergent sequence in that it was getting close to but not equal to its asymptotes.

**ELISA on Task 6:** \( a_n = (-1)^n(1+1/n) \)

I: Can you explain why you think student A would be better?

ELISA: Umm, because umm in both of these examples \([ a_n = (-1)^n1/n \text{ & } a_n = (-1)^n(1+1/n) ]\), umm when you place these \( \varepsilon \)–strips, there is infinitely many points inside the strip making that converge to that \( \varepsilon \)–strip but not actually ever reach what is covering.

Furthermore, they also examined the limit values of a sequence using not some \( \varepsilon \)–strips but all \( \varepsilon \)–strips. For instance, ELISA responded that definition A with all \( \varepsilon \)–strips would be better to describe limits of oscillating sequences each of which was getting close to two asymptotic lines.

**ELISA on Task 6:** \( a_n = (-1)^n(1+1/n) \)

I: Umm. Okay. So student A would be better? I can use it, right?

ELISA: Yeah, *if I were you, I wouldn’t be able to use just this [one larger strip] in this case to explain this. I think you work through different all the strip.*

Although ELISA recognized the necessity of examining \( \varepsilon \)–strip definition A with all \( \varepsilon \)–strips, ELISA did not perceive that the assumption of all \( \varepsilon \)–strips would imply dynamic changes in the value of \( \varepsilon \) so as to move its value towards 0 when determining the limit of a sequence. Since she did not recognize that the \( \varepsilon \)–strip definitions already implied that the width of the \( \varepsilon \)–strips could be made to decrease to 0, ELISA tried to modify definition A to better describe this phenomenon.
ELISA on Task 6: \( a_n = (-1)^n (1 + 1/n) \)

I: Okay. Great! Suddenly I thought if I place it [the largest \( \varepsilon \) – strip] here like that on 0 [putting red line on the x-axis], infinitely many points are inside the strip, aren’t they?

ELISA: Yes.

I: Then 0 is a limit of this sequence if we use student A? What do you think?

ELISA: [pause] Well, \( \varepsilon \) – strip covers both 1 and -1, umm [pause]. So that like [pause], I don’t think 0 is a limit to that function.

I: Umm, but if you use student A, placing the red line on the x-axis, which is \( y = 0 \), then infinitely many points are inside the strip, aren't they?

ELISA: I guess if you are using this big one, then like .5 could be one [limit], and -.5 could be one [limit], umm, because it is too broad of like a strip. If they [\( \varepsilon \) – strips] get smaller, then you will see that like [pause] umm 1 and -1. I guess that means that if that strip doesn’t work, then the definition [A] isn’t [pause] gonna be good enough, but I don’t know how to change it [A].

Similarly, BETH also responded that definition A would be almost correct except for the fact that it should include some description of the \( \varepsilon \) – strips to show that the sequence got close to its asymptotic line. From this point of view, BETH actually suggested modifying the definition by including a description of how the \( \varepsilon \) – strips got smaller and smaller.

BETH on Task 2: \( a_n = 1/n \)

I: Oh, you do not like student A?

BETH: I, I mean [pause] I agree that the limit is 0 [pause]. But I think that you need to say something about [pause] that they can actually like see the points going closer and closer to 0, not necessarily just because the points inside the strip that the limit is 0.

I: So you mean that even using smaller width of the strip does not work to say that the limit of the sequence is 0? So actually student A is not correct?

BETH: [pause] I think that they are kind of correct. Like I think that [pause] it needs to be explained more like about strips, and that they are using strips that are getting smaller and smaller.
As seen in the above, students in this category tried to modify definition A to say that the value of $\varepsilon$ can dynamically change and decrease to 0. Figure 4.21 shows BETH’s modification of definition A to describe the dynamic change in the value of $\varepsilon$.

As seen in the above excerpts, students in this category regarded definition A as a better description of the limit of a sequence than definition B when the given sequence appeared to have its asymptotic line. Nonetheless, they tended to disagree with both definition A and B if the given sequence was constant so that there was no asymptotic line for the sequence to get close to but not equal to.

![Figure 4.21 BETH’s response to the limit of a sequence](image)

**Figure 4.21 BETH’s response to the limit of a sequence $a_n = 1/n$: Reversibility level 3 and regarding asymptotes as limits**

**Students at reversibility level 3 who regarded cluster points as limits.** EMILY considered $\varepsilon$–strip definition A as better to describe the concept of limit than definition B because A could describe a sequence in which infinitely many terms of the sequence were getting close to or equal to its cluster points. Moreover, EMILY recognized that all $\varepsilon$–strips should be examined when using the $\varepsilon$–strip definition to determine the limit of
a sequence. Nonetheless, EMILY did not perceive why it would be better to use smaller widths of $\varepsilon$-strips if a bigger one worked.

EMILY on Posttest: $a_n = \begin{cases} 1/n & \text{if } n \leq 10, \\ 1/10 & \text{if } n > 10. \end{cases}$

I: Great!
EMILY: And I, I, after going back and forth, I want to say the limit is one tenth. [laugh]
I: Umm,
EMILY: So using student A’s definition, I can line up the epsilon strip on the limit which is one fifth, and we can see there is an infinite amount of points inside the epsilon strip so I am gonna say one fifth.
I: One fifth?
EMILY: I mean, one tenth. I am sorry.
I: Is there any reason that you use this strip not other strips?
EMILY: Umm, I like using these smaller ones just because
I: So you mean any epsilon strip means this one strip?
EMILY: No, you can use any, you can use this one, too because infinite amount of points inside that epsilon strip. I don’t know why I use this smaller strip.

As seen above, EMILY did not grasp the dynamic notion in using all $\varepsilon$-strips, therefore, could not explain how definition A described cluster points appropriately.

On the other hand, students who regarded cluster points as limits also responded that definition B was not a correct description of limits. The main reason was that they had to classify sequences having multiple cluster points as divergent according to definition B. For instance, EMMA pointed out that infinitely many points on the graph of the sequence $a_n = (-1)^n(1 + 1/n)$ were not within some thin $\varepsilon$-strips when the red line in the middle of the the $\varepsilon$-strips was on $y = 1$, and this would mean that 1 was not a limit of this sequence. However, EMMA regarded 1 as one of limit values of the
sequence because infinitely many points on the graph of the sequence were getting close
to 1. Hence EMMA regarded definition B as improper for describing limits.

**EMMA on Task6:** $a_n = (-1)^n(1 + 1/n)$

EMMA: The student [B] is saying that for a value to be a limit of a sequence
finitely many, only finitely many points [pause] should not be covered
by any $\varepsilon$–strip. And [pause] here, by this argument, 1 would not be a
limit of the sequence.

I: Oh, can you explain what you mean by any $\varepsilon$–strip?

EMMA: An $\varepsilon$–strip with any width which includes the $\varepsilon$–strips that have a
smaller width than this one [wide one], and so while [pause] this
definition works for wide one but doesn’t work for smaller ones. So you,
you can’t say that that 0 is the limit of the sequence because it doesn’t
work for all $\varepsilon$–strips.

**Students at reversibility level 3 who regarded limit points as the limit of a sequence.** Students in this category understood definition B and could apply it to
determine the limit of a sequence. They may even have understood that definition B
should be examined for all $\varepsilon$–strips. However, they only understood the meaning of
“for any $\varepsilon$” as “for any fixed $\varepsilon$” and did not recognize that “any $\varepsilon$” implies the
dynamic motion of the infinite process of reducing the size of $\varepsilon$ to determine the limit
of a sequence. Instead, as seen in the following excerpt, BRIGID suggested modifying
definition B to describe the role of the size of $\varepsilon$–strips in determining the limit of a
sequence.

**BRIGID on Task 6:** $a_n = (-1)^n(1 + 1/n)$

I: You said here finitely many points should be outside the strip to get a
limit right?

BRIGID: Yep.

I: If I used this strip then, and placed the red line on the x axis, then there
are only finitely many points outside the strip, right?

BRIGID: Yeah.

I: So can we say 0 is the limit of the sequence? If Student B is correct?
BRIGID: Umm [pause] no [pause]
I: How can you tell that?
BRIGID: [pause] I guess you can’t really tell from that explanation, because 0 is not the limit. Umm [pause] yeah I don’t, I mean it works for some cases, but not when the $\varepsilon$ – strip is that big. Maybe if it said something about no matter how small the strip gets, there are still going to be a finite amount of points.

**Reversibility Level 4: For any $\varepsilon > 0$, Hereafter, as $\varepsilon \to 0$, $N_\varepsilon$ Can Be Found**

(Dynamic)

Students at reversibility level 4 understand that the phrase “any $\varepsilon$ – strip” in the $\varepsilon$ – strip definitions means “any fixed $\varepsilon$ – strips, that is, as $\varepsilon \to 0$”. While students at reversibility level 3 perceive just a collection of static images of the graphical representation of a sequence for any fixed value of $\varepsilon$, students at reversibility level 4 can imagine a continuum of these images. Such a dynamic image of the graph of a sequence with all $\varepsilon$ – strips can help students to conceptualize that the $\varepsilon$ – strip definition could be used to reduce the error bound as much as they want. As a result, students at reversibility level 4 can consider the $\varepsilon$ – strip definitions as properly describing their own conception of the limit of a sequence (see Figure 4.22).

![Diagram](image)

**Figure 4.22 Reversibility level 4: None of $\varepsilon$ – strip definitions implies own conception.**

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Table 4.25 shows students’ distribution at reversibility level 4 in terms of their own conception of limit. As far as the level of reversibility goes, students in this category properly understood the role of $\varepsilon$–strips of any size. However, their evaluation of the $\varepsilon$–strip definitions varied according to their conception of limit. The following subsections describe how students at reversibility level 4 responded differently in terms of their conception of limit.

<table>
<thead>
<tr>
<th>Reversibility 4</th>
<th>Asymptotes</th>
<th>Cluster points</th>
<th>No distinction: Cluster points &amp; Limit</th>
<th>Limit points</th>
<th>#(students)</th>
</tr>
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<tr>
<td>Task 2</td>
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<td>BECKY</td>
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<td>ELISA</td>
<td>BECKY</td>
<td>ELENA</td>
<td>4</td>
</tr>
</tbody>
</table>

**Students at reversibility level 4 who regarded asymptotes as limits.** Students in this category looked to see if they could determine a sequence as getting close to but not
equal to a certain value when applying the $\varepsilon$–strip definitions for all $\varepsilon$–strips. They also pointed out that testing the $\varepsilon$–strip definition with any fixed $\varepsilon$–strip implied the dynamic movement of the value of $\varepsilon$ towards 0. They seemed to understand the reverse relationship between the value of $\varepsilon$ and the index $n$ in the descriptions of the $\varepsilon$–strip definitions A and B.

ELISA on Posttest: $a_n = 1/n$

I: Can you explain why 0 is the limit of this sequence by using student A’s idea?
ELISA: Umm, yeah, uh, as long as you are covering 0 [covering the x-axis by a strip], there will be infinitely many points inside the strip no matter the width of the strip. [testing with another strip] There is always gonna be an infinitely many amount if they cover 0. And they cover 0.

In particular, BRIAN pointed out that “any $\varepsilon$–strip” in definition A meant all sized $\varepsilon$–strips in the continuum from infinitely large to infinitely small ones.

BRIAN on Posttest
I: So you mean this case any $\varepsilon$–strip, and that case?
BRIAN: Any $\varepsilon$–strip you choose should have infinitely many points inside of it.
I: Can you explain any?
BRIAN: Okay. You can use any size of the $\varepsilon$–strip, and an infinitely large $\varepsilon$–strip [pause] or an infinitely small $\varepsilon$–strip, and there will still be [pause] infinitely many points [pause] covered by that strip.

Nonetheless, among these students at reversibility level 4, those who regarded asymptotes as limits responded that neither of the $\varepsilon$–strip definition was proper in describing the limit of a sequence. The main reason was that the definitions were not compatible with their own conception of limit. That is, neither of the $\varepsilon$–strip definitions could properly interpret asymptotic lines as values to which the given sequence would be getting close but not equal. For instance, ELISA, who was at reversibility level 4 through all the tasks from Task 3 on, determined that the oscillating divergent sequence
\( a_n = (-1)^n (1 + 1/n) \) had two limits, 1 and -1, due to the fact that this sequence has two asymptotic lines. However, ELISA had to call this sequence divergent according to definition B because there was an \( \varepsilon - \)strip where there were infinitely many points on the graph of the sequence outside the \( \varepsilon - \)strip. Such a contradiction between this student’s own conception of limit and the result from definition B made it hard for ELISA to regard definition B as a proper description of the limit.

**ELISA on Posttest**

I: I see. I see. Why do you not like student B?

ELISA: Umm. Because of the last week’s \( a_n = (-1)^n (1 + 1/n) \), we did it and we found out that there were, umm [pause] there could be infinitely many points outside umm of the \( \varepsilon - \)strip, and there were a limit. Or the sequence still had a limit.

ELISA did not agree with definition A as a proper description for a limit of a sequence, either. She pointed out that according to definition A, even a sequence that was getting close to and eventually equaling a certain value should be called convergent. It caused her cognitive dissonance when determining the limit of a sequence as follows.

**ELISA on Posttest:**

\[
\begin{align*}
a_n = \begin{cases} 
1/n & \text{if } 1 \leq n \leq 10 \\
1/10 & \text{if } n > 10
\end{cases}
\end{align*}
\]

I: Great. Does this sequence have a limit?

ELISA: [pause] Umm, no.

I: How can you tell this?

ELISA: Because umm [pause] hmm [pause] I guess by that definition [A], this would be, the limit would be 1/10.

I: Can you explain if you follow student A’s definition, why 1/10 should be the limit of the sequence?

ELISA: Umm, because the, when you place the \( \varepsilon - \)strips [placing an \( \varepsilon - \)strip] over 0.1, umm there is infinitely many points [pause] umm going out, because the rest of points are all gonna equal 1/10. So I guess the limit to this would be 1/10.

I: What do you think?
ELISA: I just always thought that a sequence like went out to a certain value, like it kept going out but never actually reached it. And so if that’s the case, then this sequence wouldn’t have a limit. Because it actually reaches it.

I: Okay. What about student A?

ELISA: Then that [A] would not be [pause] right, the right definition. But I don’t know if I am right, so.

In order to find a better description of the limit of a sequence than the given ε–strip definitions, students in this category modified the ε–strip definition by adding some phrase to describe a sequence which did not reach its asymptotic line (see Figure 4.23).

**ELISA on Posttest**

I: Is there any way you can modify student A or student B?

ELISA: [pause] umm maybe if you said a certain value L is a limit of a sequence when infinitely many points on the graph of the sequence are covered by but do not reach.

I: Umm, do not reach?

ELISA: Umm. Do not reach L. I think it wouldn’t make sense if you put it right there. But It [sequence]’s covered, like the strip covered L but they [points on the graph] don’t actually, they don’t actually equal L.

---

**Student A**

A certain value L is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any epsilon strip as long as the epsilon strip covers L.

Epsilon strip covers L and there are infinitely many points covered by the epsilon strip, but do not actually equal L.

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Figure 4. 23 ELISA’s response: Reversibility level 4 and regarding asymptotes as limits
Once modifying definition A in this way, students in this category could apply such a modification to sequences to determine asymptotic lines of a sequence in terms of $\varepsilon$.

\[
BRIAN \text{ on Posttest } a_n = \begin{cases} 
1/n & \text{if } 1 \leq n \leq 10 \\
1/10 & \text{if } n > 10
\end{cases}
\]

I: Can you explain one more time what that [BRIAN’s modification of A] means?

BRIAN: Umm. [pause] A certain value L is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any [emphasizing] $\varepsilon$−strip as long as the $\varepsilon$−strip covers L and no points [pause] umm on the graph equal value L.

I: So on the red line, there should be no points on the graph?

BRIAN: Right.

I: Can you explain why this sequence does not have a limit by using your last version?

BRIAN: Because [pause] we were approaching one tenth here at one tenth here. So the only way that there can be infinitely many points inside any epsilon strip is if this strip is on one tenth here.

I: You mean the red line of the strip?

BRIAN: Right! Because if you move the red line of the strip down, just an [pause] infinitely small fraction [pause] to make no points equal to the value L, there is gonna be a point where the strip gets smaller than that small fraction you made it down, and there will no longer be [pause] infinitely many points inside of the strip.

Students at reversibility level 4 who regarded cluster points as limits. When applying the $\varepsilon$−strip definitions for all $\varepsilon$−strips, students in this category looked to see if they could identify subsequences each of which was close to one of the cluster points. They also pointed out that testing the $\varepsilon$−strip definition with any fixed $\varepsilon$−strip implied the dynamic movement of the value of $\varepsilon$ towards 0. They seemed to understand the reverse relationship between the value of $\varepsilon$ and the index $N$ in the descriptions of $\varepsilon$−strip definitions A and B. However, among students at reversibility level 4, those who regarded cluster points as limits of a sequence tended to respond definition A but not definition B was a proper description of the limit of a sequence.
For instance, EMMA said that whenever the width of the \( \varepsilon \)–strip was fixed, no matter how small the width of the strip was, infinitely many points on the graph of the sequence were getting close to the value of 1 or -1. EMMA could recognize that even if \( \varepsilon \) were getting smaller and close to 0, infinitely many terms would be getting close to a certain number \( L = 1 \), which was a limit of this sequence. In this manner, EMMA conceived that definition A properly described how the sequence was getting close to its cluster points 1 and -1.

\[
\text{EMMA on Task 6: } a_n = (-1)^n \left(1 + \frac{1}{n}\right)
\]

I: I like your idea. By the way, 0 is a limit of the sequence or not?
EMMA: No.
I: No? Why is 0 not a limit of the sequence? Can you explain why 0 is not a limit of the sequence by using student A’s idea?
EMMA: Umm, because student A says that \( \varepsilon \)–strip, the value \( L \) is a limit of a sequence when infinitely many points on the graph are covered by any \( \varepsilon \)–strip. And that’s not true. When you put the \( \varepsilon \)–strip on 0, it does not cover infinitely many points.
I: But if I place this one [large strip], infinitely many points are inside the strip, aren’t there?
EMMA: Yes. But their definition [A] says umm any \( \varepsilon \)–strip. So it has to include smaller ones also.
I: Oh, can you explain what you mean any \( \varepsilon \)–strip?
EMMA: An \( \varepsilon \)–strip with any width which includes the \( \varepsilon \)–strips that have a smaller width than this one, and so while this definition works for wide one but doesn’t work for smaller ones. So you, you can’t say that that 0 is the limit of the sequence because it doesn’t work for all \( \varepsilon \)–strips.

However, students in this category did not regard definition B as a proper description of the limit of a sequence, because they could determine neither 1 nor -1 as a limit of the sequence \( a_n = (-1)^n(1+1/n) \) even though both 1 and -1 should be limits according to their own conception.
EMMA on Task 6: \( a_n = (-1)^n \left( 1 + \frac{1}{n} \right) \)

I: Umm. Can you explain it [\( \varepsilon - \) strip definition B] one more time?
EMMA: Umm [pause]. A certain value \( L \) is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any \( \varepsilon - \) strip as long as it covers \( L \). And that’s true there are infinitely many points and this is on the sequence. And here [pause] this [student B] is not true umm there are infinitely many points outside the \( \varepsilon - \) strip [pause] while the \( \varepsilon - \) strip is in fact umm a limit of the sequence. So this isn’t true here.
I: Because?
EMMA: Because the student [B]’s saying that for a value to be a limit of a sequence finitely many, only finitely many points [pause] should not be covered by any \( \varepsilon - \) strip. And [pause] here, by this argument, 1 would not be a limit of the sequence. So, well, student B is wrong.
I: But 1 is a limit of the sequence?
EMMA: Right.

**Students at reversibility level 4 who were confused between cluster point and limit.** Students in this category properly understood the reverse relationship between \( \varepsilon \) and the index \( N \) described in both \( \varepsilon - \) strip definitions A and B, and they could apply these definitions to determine the limit of a sequence. They also regarded both \( \varepsilon - \) strip definitions as compatible with their own conception of the limit of a sequence. For instance, BRIGID could explain why 1 was the limit of a constant sequence \( a_n = n/n \) by applying definition A.

BRIGID on Task 4: \( a_n = n/n \)

I: Okay, does this sequence have a limit?
BRIGID: Um, yes, one
I: How can you tell this? Can you explain the limit of the sequence using the strips?
BRIGID: It’s one because if you place the red line on one, um, no matter how small the strips get all the points are still going to be included. I mean you could just have the red line, or just barely have the \( \varepsilon - \) strip and it would still be included.
ERICA in this category also properly applied definition B in determining the convergence of an oscillating sequence.

**ERICA on Task 5:** \( a_n = (-1)^n \frac{1}{n} \)

I: Can you explain why student B is correct?
ERICA: Umm, because if you have [pause] like this graph here. Umm [pause]. If we put our \( \varepsilon \) – strip on [pause] on our limit, then we have got [pause] a finite, we can count the number of points that is the sort of idea. We have got finite points [pause], a finite number of points outside the strip. Even if the width of the strip is smaller and smaller, we can still know how many points are out here. Like, here, I have got thirty points. Thirty points. Umm. [pause]. I think, and then like, if I put the [pause], if I messed up and then say, oh, well, my limit is that [pause] .2 [putting the strip on .2]. [pause] But that is obviously not right because [pause] there is an infinite amount of points that are not inside the strip. [pause] So I think student B has it. [pause] I think it is just easier to see that [pause] since there are [pause] an infinite amount of points inside the strip [pause], then it is a limit than say that [pause] because finite number of points outside the strip. [I: I see.] Umm. Because I see there is the [pause] the points [pause] are inside the strip because they are approaching or coming closer and closer to the limit. And so they are always gonna be inside the strip. And I think that is, I think that better [pause] shows the idea that they are approaching the limit, and they are getting closer and closer, but they are never gonna farther away than inside the strip. Like, B, I think it [pause] you know, there is only finitely many points outside the strip, but that doesn’t say that, you know, there isn’t a point here.

It was noticeable that students at this level used any \( \varepsilon \) – strip and could imagine the value of \( \varepsilon \) moving towards 0 as they tried to describe the limit value more accurately.

**ELENA on Task 4:** \( a_n = \frac{n}{n} \)

I: Okay. Can you try student B?
ELENA: Student B? Okay. Umm. The \( \varepsilon \) – strip and says that only finitely many points on the graph are not covered. But there are no points on the graph that will not be covered. So that’s definitely, that 0 is a finite number.

I: Because 0 is a finite number?
ELENA: Zero is a finite number [laugh].

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I: Makes sense. So, you just used the smallest width of the strip. Why you do not try other strips?
ELENA: Umm, because [pause] it’s more accurate to use like you will get closer. You, you will be more like to be on the limit if you use a smaller strip. And so there is less room for [pause] error [laugh] than if I use a big one.
I: Umm, I see. But is there any possibility if you use a smaller width than?
ELENA: Well, there is a possibility that you could be wrong. I guess you should start with a big one.
I: Big one?
ELENA: And then working a way down and then, this one \[ a_n = n/n \], it was kind of obvious [laugh] so I just started with a little one. But if one that is little obvious, then I would start with a big one.

Although students in this category recognized the dynamic nature of the reverse thinking process reflected in the \( \varepsilon \)-strip definitions, they did not see the difference between definition A and definition B. Rather, they regarded both of these definitions as describing the limit of a sequence properly in different ways. In this research these students were classified as those who did not recognize the difference between cluster points and limit points.

BECKY on Task 5: \( a_n = (-1)^{n+1} / n \)

BECKY: [pause] They both explain it umm [pause] they both explain it like what we were saying because when umm [pause] you have a strip covering the limit which is 0. There is [pause] like they are both true. There is an infinite amount of umm points in it because they are getting closer to it. [pause] So, and if it extends out, all of them are gonna be [pause] like covered by it. And then also for B, there is only a finite number outside because like there are only gonna be outside but [pause] up until the point where it starts getting like small strip. And all would be [pause] inside the strip, so.

Indeed, students in this category thought that definition A not only described properly the concept of limit but also implied definition B, therefore definition B was also correct. For instance, ELENA misconceived that having infinitely many points inside a strip would imply having only finitely many points outside a strip.
**ELENA on Task 5**

I: I see. Do you mean student A and student B both are correct?
ELENA: Yes.
I: But for me those descriptions look different.
ELENA: There is difference. But they kind of imply each other.
I: You mean student A implies student B, and student B implies student A?
ELENA: Yes [pause], because what you are saying there is infinitely many points on the [pause] inside the graph, then it kind of says that has to be finitely many points outside, you know.

It appeared that BECKY, who was in this category since Task 3, could understand what definition B meant by first understanding definition A and then concluding B with the misleading perception that subtraction of an infinite set from an infinite set should be finite. Since BECKY could understand definition B from definition A, definition A was regarded as easier to understand and therefore preferred in determining the limit of a sequence.

**BECKY on Task 3**

BECKY: They are kind of the same thing, but this one [A] is easier to [pause] the words are easier [laugh].
I: If you were a teacher, which one would you think better for your students to understand?
BECKY: A. I think it [A] is easier.
I: Okay. Then what about student B?
BECKY: Umm [pause] I would only use B if you are gonna use with student A. Don't only use B because it might be harder. Like after you hear A, and then you hear B, then it is easier to understand. But if you only hear B, it might be kind of hard.

**Students at reversibility level 4 who regarded limit points as limits.** Students in this category perceived the limit value of a convergent sequence as a unique value to which the sequence is getting close to or equal to. They also perceived the necessity of using any $\varepsilon$-strip in determining the limit of a sequence via the reverse relation between the value of $\varepsilon$ and the index $N$. Furthermore, they even understood that
\( \varepsilon \)-strip definition B meant that by reducing the value of \( \varepsilon \) towards 0, they could explain their conception of limit, i.e., the limit of a convergent sequence is the unique value that the sequence is getting close to or equal to. The significant difference between students in this category and other students at reversibility level 4 was their conception of definition A. Whereas students who regarded asymptotes or cluster points as limits preferred to use definition A in determining the limit, students who regarded limit points as limits believed that definition A was not appropriate as a definition of the limit of a sequence.

**ELENA on Task 6:** \( a_n = (-1)^n (1 + 1/n) \)

I: What do you think?
ELENA: Well, now this one [A] keeps, keeps presenting wrong. Because umm [pause] because \( L \), this is covering our \( L \), our supposed \( L \), our hypothetical \( L \), and there still, there is an infinite amount of points inside the strip. But there is also an infinite amount of points outside the strip as well. And so even though it [A] says that it’s not taking into these points.

On the other hand, students in this category regarded definition B as properly describing the limit of a sequence. ELENA in this category even recognized that definition B should imply definition A, and therefore, it was unnecessary to describe what happened to points covered by any \( \varepsilon \)–strip.

**ELENA on Posttest**

I: What do you think?
ELENA: Umm, I think student B is right, umm, completely right. Umm.
I: In what sense? Can you explain why student B is completely right?
ELENA: Umm [pause], because [pause] yeah, for any, umm, for a sequence has a limit umm, that is true that only few points would be outside the \( \varepsilon \)–strip, and umm [pause] all the rest of the point which would be infinitely many points would be inside the strip and umm, and it also says any \( \varepsilon \)–strip so it doesn’t matter how small the \( \varepsilon \)–strip is, umm there is only finite amount outside.
I: Okay. Can you tell me why student A you did not choose?
ELENA: Umm. Because of the umm [pause], the, the sequence where it was 1 for odd, and I think -1 for even, there were infinitely many points inside the strip and there were also infinite points outside the strip so that number is not the limit, so that number was not the limit of the sequence.

Moreover, ELENA also pointed out the significant role of using any $\varepsilon$–strip to describe the uniqueness of the limit value.

**ELENA on Task 6:** $a_n = (-1)^n(1 + 1/n)$

I: Okay. What about student B?
ELENA: Okay. [pause] I have to say that one [B] is [pause] correct.
I: Oh, student B is correct?
ELENA: Well, you see that it says any $\varepsilon$–strip, so yeah.
I: Can you explain one more time with this sequence why student B is correct?
ELENA: Because umm, [pause]. Umm, because even though, [pause] even though like if you put a big strip on 0, and it covers, and there is only finite, finitely many points outside the strip, this [B] also says that any $\varepsilon$–strip, so I should be able to take this $\varepsilon$–strip [small $\varepsilon$–strip] and put it on there [the x-axis], and still cover, well still cover infinitely many points but there is no way.

That is, ELENA found that there were several $\varepsilon$–strips in which only finitely many points on the graph of a sequence $a_n = (-1)^n(1 + 1/n)$ would not be covered when an $\varepsilon$–strip were on the value of 0. However, ELENA recognized that this did not mean that 0 was a limit of the sequence since infinitely many points would remain uncovered by at least a small $\varepsilon$–strip. Existence of such a small $\varepsilon$–strip, where finitely many points were not covered in it, played a role as a counterexample in arguing against 0 as a limit when applying $\varepsilon$–strip definition B. Since 0 was not a limit of the sequence according to ELENA’s own conception of limit, she concluded $\varepsilon$–strip definition B described the limit of the sequence properly.
This chapter discussed the main results of this research in terms of students’ conception of convergence and the limit of a sequence, students’ reversibility levels, and the association between students’ conception of limit and their reversibility levels. Based on such results, the next chapter deals with a possible effect of this research on development and accommodation of students’ intuitive understanding of the limit of a sequence, implications of this research for teaching and learning of limits, and future extensions of this research.
CHAPTER 5

CONCLUDING REMARKS AND DISCUSSION

This research explored students’ conception of limit through the criteria used in determining the convergence and limit values of sequences represented symbolically, numerically, and graphically. In addition, an activity of measuring with $\varepsilon$–strips how many terms of a sequence were (or were not) close to a certain value was looked at using the graph of a sequence. Through this $\varepsilon$–strip activity, this research examined how students conceptualized the reverse relation between $\varepsilon$ and $N$ in the definition of limit and how students’ reversibility was related to their conception of the limit of a sequence. Finally, the kinds of changes that occurred in students’ conception of the limit of a sequence while participating in the $\varepsilon$–strip activity were explored.

It was found that various students regarded asymptotes, cluster points, or limit points as limits of sequences. In addition, students’ ways of relating the value of $\varepsilon$ to the corresponding index $N$ fell into five major levels in terms of their understanding of the role of $\varepsilon$ as an arbitrarily chosen value and their recognition of the dependent role of $N$. Furthermore, the results of this research show that students’ understanding of the rigorous definition of the limit of a sequence is associated not only with their conception of limit but also with their reversibility. The better they recognized the dynamic feature of
the intuitively reverse relation between $\varepsilon$ and $N$ in the $\varepsilon-N$ strip definitions, the better they understood the limit of a sequence, and vice versa. In addition, it should be noted that there was improvement in students’ reversibility through the $\varepsilon-N$ strip activity even though there was no procedure in the interviews for indicating students’ errors, correcting misconceptions about limits, or confirming the propriety of the $\varepsilon-N$ strip definitions.

Results of this research were obtained through investigating the following research questions:

5. How do students explain their understanding of convergence and the limit of a sequence?
6. How do students explain the $\varepsilon-N$ relationship in the context of the limit of a sequence?
7. How are the levels of the development of students’ reversibility, that is, an ability to understand the $\varepsilon-N$ relationship, associated with students’ intuitive understanding of the limit of a sequence?
8. How different is students’ intuitive understanding of the limit of a sequence after the teaching experiment? How different is students’ reversibility after experiencing the $\varepsilon-N$ strip activity?

Based on discussion of students’ conceptions of the limit of a sequence and development in their level of reversibility, the answer to the following overall research question is suggested in this chapter:

Does an activity with the graphical illustration of a sequence, along with statements describing the reverse relationship between $\varepsilon$ and $N$, influence the development and accommodation of students’ intuitive understanding of the concept of limit?

In addition, the role of the $\varepsilon-N$ strip activity in teaching and learning the limit of a sequence is discussed in terms of Bruner’s three modes of representation. Finally, further discussion suggests directions for future research related to students’ concept of limit and their level of reversibility.
Results in Terms of Research Questions

Question 1: How do students explain their understanding of convergence and the limit of a sequence?

In order to examine students’ conception of convergence and limit of a sequence, students were asked to explain their own criteria for determining the limit of a sequence and to apply their criteria to given sequences. Throughout this investigation, students’ conceptions of convergent sequences were classified as follows: (1) Continuing endlessly, hence no limit; (2) Completing the index process; (3) Getting close to, but not equal to; (4) Getting close to or equal to (Possibly multiple values for limit); (5) Getting close to or equal to a unique value; and (6) Difference between consecutive terms is getting smaller. Among such criteria, the fifth one, “Getting close to or equal to a unique value” appeared to lead students to properly determine the convergence and limit of a sequence. Students who applied other criteria to sequences showed misconceptions in determining limits, particularly when the given sequences were constant or oscillating.

“Continuing endlessly, hence no limit” describes the case in which students recognize that the process of defining a sequence continues endlessly, so they believe the sequence does not have a limit value. This misconception has been pointed out in other literature as one of the serious misconceptions about limits (Szydlik, 2000; Williams, 1991). Students who determined convergence by applying this criterion obtained results that conflicted with what they had learned in class. Due to such conflicts, students often got confused psychologically and distrusted results they learned in class (Szydlik, 2000) even when they properly responded to limit problems. Students who determined properly
the convergence of a sequence given as an algebraic expression often improperly applied
the criterion “continuing endlessly, hence no limit” to sequences not given as an algebraic
expression or for which it was hard to predict the limit value.

Students who used the criterion “completing the index process” determined the
limit value of a sequence by increasing the index \( n \) to infinity and plugging infinity for \( n \)
into the algebraic expression for the sequence. Such students might figure out limit
problems procedurally correctly. Nevertheless, they often did not relate the definition of
limit or any theorems about limits to explain why such a value should be the limit of a
given sequence. What is worse, they did not recognize their answers for limit problems
were incorrect even when they made errors in symbolic manipulation while applying the
criterion “completing the index process” procedurally.

The number of students who used mainly the above two criteria for the limit of a
sequence, (1) continuing endlessly, hence no limit, and (2) completing the index process,
decreased as the interviews proceeded (see Table 4.8 and Table 4.9). It should be noted
that students began to account for the convergence of any given sequence in more
concrete ways as they repeatedly represented sequences numerically as well as
graphically. For instance, BRIGID mainly used the criterion “continuing endlessly, hence
no limit” in determining the convergence of a sequence on the pretest. However, she used
different criteria while working with \( \varepsilon - \) strips throughout the following interviews.
Consequently, students tried to understand and explain the limit of a sequence
conceptually as the interviews went on. Such a result suggests that activities used in the
task-based interviews throughout this research helped students internalize and justify their conception of limit beyond getting the right answers.

On the other hand, most students who made errors in determining convergence of sequences mainly used the criteria (3) getting close to but not equal to or (4) getting close to or equal to. First, students could properly determine convergence of monotone bounded sequences and divergence of unbounded sequences by applying the criterion “getting close to but not equal to.” However, these students called constant sequences and oscillating convergent sequences divergent. They also said oscillating divergent sequences had multiple values for limits. Second, students who applied the criterion “getting close to or equal to” could properly determine convergence of monotone bounded, constant, or oscillating convergent sequences and divergence of unbounded sequences. However, these students also incorrectly responded that oscillating divergent sequences had multiple values for limits.

In view of the results obtained from analysis of students’ criteria used in determining convergence of sequences throughout the pretest, Task 1 to Task 6, and the posttest, students’ conceptions of the limit of a sequence were classified into the following three main categories: (1) regarding asymptotes as limits; (2) regarding cluster points as limits; (3) regarding limit points as limits. Students who regarded asymptotes as limits tended to apply “getting close to but not equal to” in determining convergence of sequences. Students who regarded cluster points as limits tended to apply “getting close to or equal to” in determining convergence of sequences. Only students who properly
conceptualized limits used the criterion “getting close to or equal to a unique value” in determining convergence of a sequence.

**Question 2:** How do students explain the $\varepsilon - N$ relationship in the context of the limit of a sequence?

This research also investigated students’ ability to understand the relationship between $\varepsilon$ and $N$ that appears in the $\varepsilon - N$ definition of limit and to apply this reverse relationship in determining convergence and limits. This research did not investigate students’ formal cognition in which they used the $\varepsilon - N$ definition of limits in rigorous proof. Instead, it explored how students intuitively understood the reverse relationship between $\varepsilon$ and $N$ in the $\varepsilon - N$ definition of limits.

It should be noted that the $\varepsilon - N$ definition of limit was neither given nor taught to students during the interviews. Instead, students were asked to use graphs of sequences and $\varepsilon - \text{strips}$ in determining and explaining limit values of sequences. The $\varepsilon - \text{strips}$ were specially developed for this teaching experiment so students could describe the $\varepsilon - N$ relationship in the rigorous definition of limit. Each $\varepsilon - \text{strip}$ was made of translucent paper so that students could observe the graph of the sequence through the $\varepsilon - \text{strip}$; furthermore, each $\varepsilon - \text{strip}$ had constant width, and in the middle of the strip, a red line was drawn so as to mark a possible limit point. During the teaching experiment, students were also asked to determine the propriety of $\varepsilon - \text{strip}$ definitions A and B as definitions of the limit of a sequence.

- **$\varepsilon - \text{strip definition A:}$** A certain value $L$ is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any epsilon strip as long as the epsilon strip covers $L$. 
\(\varepsilon\)–strip definition B: A certain value \(L\) is a limit of a sequence when only finitely many points on the graph of the sequence are not covered by any epsilon strip as long as the epsilon strip covers \(L\).

The \(\varepsilon\)–strip activities played an important role in investigating students’ intuitive understanding of the reverse relationship between \(\varepsilon\) and \(N\) through their determination of the propriety of the \(\varepsilon\)–strip definitions. Based on students’ responses to the \(\varepsilon\)–strip definitions, the levels of understanding the reverse relationship between \(\varepsilon\) and \(N\) in the context of limits were classified as follows.

Students at reversibility level 0 did not recognize the reverse relationship between \(\varepsilon\) and \(N\) described in the \(\varepsilon\)–strip definitions. That is, these students were not aware of the reversal of the order of determining \(\varepsilon\) and \(N\) in \(\varepsilon\)–strip definitions from the way of reading the limit symbol \(\lim_{n \to \infty} a_n\). What is worse, they conceived the relationship between \(\varepsilon\) and \(N\) in the \(\varepsilon\)–strip definitions as meaning that the value of \(\varepsilon\) was determined after selecting a value of \(N\). Such a misconception about the direction between \(\varepsilon\) and \(N\) in the context of limits has been shown in other research as well (Kidron & Zehavi, 2002; Pinto & Tall, 2000).

Students at reversibility level 1 started to recognize the order of determining \(\varepsilon\) and \(N\) in the \(\varepsilon\)–strip definitions, which is reversed from the order of reading the limit symbol \(\lim_{n \to \infty} a_n\). However, they regarded consideration of “any \(\varepsilon\)” as meaning either that the value of \(\varepsilon\) decreases to 0 and hence ultimately becomes 0, or that the value of \(\varepsilon\) increases to infinity and hence ultimately becomes infinite. That is, as the completion of
an infinite process, students at reversibility level 1 substituted 0 or infinity for the value of $\varepsilon$ used in the $\varepsilon$–strip definitions.

Students at reversibility level 2 understood that the use of “any $\varepsilon$” in the $\varepsilon$–strip definitions did not mean to complete the infinite process of decreasing or increasing the value of $\varepsilon$. Indeed, they perceived that the value of $\varepsilon$ used in determining convergence of limits should not be either 0 or infinity. Rather, they conceptualized the meaning of “any $\varepsilon$” as “some values of $\varepsilon$.” That is, students at reversibility level 2 understood $\varepsilon$–strip definition A as “$L$ is a limit of a sequence if there is some $\varepsilon$–strip such that there are infinitely many points on the graph of the sequence inside the strip,” and $\varepsilon$–strip definitions B as “$L$ is a limit of a sequence if there is some $\varepsilon$–strip such that there are finitely many points on the graph of the sequence outside the strip.” Due to such a conception of the reverse relationship between $\varepsilon$ and $N$, students at reversibility level 2 pointed out that even for a value which is not a limit of a sequence, there could be some $\varepsilon$–strips covering infinitely many points or uncovering only finitely many points on the graph of a sequence. Consequently, they responded that the $\varepsilon$–strip definitions were not proper statements in describing the limit of a sequence. Using only “some values” of $\varepsilon$ in the rigorous $\varepsilon$–$N$ definition of limits is a misconception also found in other literature (Pinto & Tall, 2000).

Students at reversibility level 3 recognized that the $\varepsilon$–strip definitions did not mean to test with just some $\varepsilon$–strips but with all $\varepsilon$–strips. To be precise, students at reversibility level 3 understood $\varepsilon$–strip definition A as “$L$ is a limit of a sequence if infinitely many points on the graph of a sequence are inside any fixed $\varepsilon$–strip,” and
\( \varepsilon \)-strip definition B as “\( L \) is a limit of a sequence if there are only finitely many points on the graph of the sequence outside any fixed \( \varepsilon \)-strip.” Moreover, they understood that a certain value cannot be the limit of a sequence if there is at least one \( \varepsilon \)-strip which does not satisfy the conditions for the \( \varepsilon \)-strip definitions no matter how many \( \varepsilon \)-strips do satisfy the conditions. Nonetheless, students at reversibility level 3 regarded neither of the \( \varepsilon \)-strip definitions as equivalent to their own conception of the limit, mainly because they did not consider the definitions as describing the dynamic motion of a convergent sequence whose terms are getting close to its limit value. This was because students at reversibility level 3 did not recognize that the value of \( \varepsilon \) can be decreasing to 0. Some students even wanted to add a condition to the definitions specifying that the value of \( \varepsilon \) should move towards 0.

Finally, students at reversibility level 4 not only conceptualized the meaning of “any \( \varepsilon \)” in the \( \varepsilon \)-strip definitions as “any fixed \( \varepsilon \)” but also understood that \( \varepsilon \)-strip definitions imply that, no matter how small the value of \( \varepsilon \) is, there should be a corresponding value for \( N \) in the reverse relationship between \( \varepsilon \) and \( N \). From such understanding, they regarded their own conception of the limit of a sequence as equivalent to determining convergence using the reverse relationship between \( \varepsilon \) and \( N \).

It would be presumptuous to consider the above five levels of reversibility as the only levels in students’ conception of the \( \varepsilon - N \) definition of limit. Instead, these five levels of reversibility should be regarded as a classification of students’ intuitive cognition of the relationship between \( \varepsilon \) and \( N \) as well as the meaning of “any positive value of \( \varepsilon \)” in the \( \varepsilon - N \) definition of limit. As seen above, there were differences of
degree in intuitively understanding the relationship between $\varepsilon$ and $N$ in the context of limits. Actually, such differences in students’ understanding were found throughout a series of processes of evolving and developing intuitive cognition which should be compatible with the rigorous $\varepsilon-N$ definition of limits.

Furthermore, these reversibility levels are not only categories but also seem to be hierarchically structured conceptually. That is, in order to develop cognition compatible with the formal definition of limit, students need to conceptualize the following two notions: (1) the expression “any $\varepsilon > 0$” means “any positive value of $\varepsilon$ to be chosen arbitrarily”; and (2) such chosen values can be rearranged to decrease to 0. These two notions are implicit in the definition of limit. In particular, reversibility level 4 implies both of these ideas, so students at this level are regarded as intuitively understanding the reverse relationship between $\varepsilon$ and $N$ in the formal definition of limit. Other levels of reversibility either fail to include both of the above notions or include different notions.

For instance, reversibility level 3 includes the case where students conceptualize the first notion “any positive value of $\varepsilon$ to be chosen arbitrarily”, but not the second notion “such chosen values can be rearranged to decrease to 0.” In this sense, it is valid to regard reversibility level 4 as higher than reversibility level 3 in interpreting the concept of limit.

Similarly, reversibility level 3 is higher than reversibility level 2. Level 2 describes the case where students perceive the first notion “any positive value of $\varepsilon$ to be chosen arbitrarily” as “some positive value of $\varepsilon$ to be chosen.” However, since students in this level of reversibility regard some positive value as any positive value,
they are not able to conceptualize that terms of a sequence should be within any small error bound of the limit if the sequence is to be convergent. In addition, due to misconstruing “any positive value” as “some positive value,” conceptualization of the second notion “such chosen values can be rearranged to decrease to 0” becomes impossible for students at this level of reversibility. Therefore, it is reasonable to regard reversibility level 3 as higher than reversibility level 2 in interpreting the concept of limit.

Continuing, at reversibility level 1, it is assumed that any positive value of $\varepsilon$ can be ultimately substituted to 0 or infinity, and as a consequence, limit values of a sequence are found by replacing 0 or infinity for $\varepsilon$. Speaking differently, students at level 1 complete the $\varepsilon$–process preferentially so as to fix the value of $\varepsilon$ at 0 or infinity, and then explore the reverse relation between $\varepsilon$ and $N$. Considering the fact that, from reversibility level 2 to 4, the reverse relation between $\varepsilon$ and $N$ is explored before completing the $\varepsilon$–process, reversibility level 1 can be regarded as lower than reversibility levels 2-4.

Finally, reversibility level 0 describes the case of no recognition of the reverse relationship between $\varepsilon$ and $N$, whereby a proper index $N$ corresponds to a given value of $\varepsilon$. Conversely, students at level 0 of reversibility select a value of $N$ first and then determine the value of $\varepsilon$. Therefore, reversibility level 0 should be regarded as the lowest level in conceptualizing the reverse relationship between $\varepsilon$ and $N$ in the formal definition of limit. As seen above, the reversibility levels in the context of limits implicitly reflect a conceptually hierarchical structure.
Question 3: How are the levels of the development of students’ reversibility, that is, an ability to understand the $\varepsilon-N$ relationship, associated with students’ intuitive understanding of the limit of a sequence?

It appeared that students’ own conception of limit played a crucial role in determining their sense of the propriety of $\varepsilon-\text{strip}$ definitions A and B. For instance, students at reversibility level 4 were those who understood the reverse relationship between $\varepsilon$ and $N$ in the rigorous $\varepsilon-N$ definition of limit. These students’ selection of one of the $\varepsilon-\text{strip}$ definitions A and B differed according to their conception of limits as asymptotes, cluster points, or limit points.

First, students who conceptualized limits as asymptotes rejected both $\varepsilon-\text{strip}$ definitions A and B, because they determined convergence using the criterion “a convergent sequence is getting close to but not equal to its limit value.” Students in this category thought the number of points outside $\varepsilon-\text{strips}$ described in $\varepsilon-\text{strip}$ definition B was irrelevant to the criterion for convergence of sequences, “getting close to but not equal to.” In addition, these students thought that knowing infinitely many points were inside $\varepsilon-\text{strips}$ did not guarantee that none of the terms of the sequence was equal to the limit.

Second, students who regarded cluster points as limits of sequences did not accept $\varepsilon-\text{strip}$ definition B but considered $\varepsilon-\text{strip}$ definition A as a proper description for limits. These students used the criterion, a convergent sequence is “getting close to or is equal to its limit values.” In this case, they considered knowing that finitely many points on the graph of a given sequence were outside the $\varepsilon-\text{strips}$ was irrelevant for
determining whether the sequence was getting close to or equal to its limit value. Furthermore, if they were to accept \( \varepsilon \)–strip definition B as proper, they would have to call sequences having multiple cluster points, which they believed were all limits, as divergent sequences. Hence, these students thought \( \varepsilon \)–strip definition B was not proper as a description for the limit of a sequence.

Third, students in the category of “no distinction between cluster points and limit points” showed the same understanding as those in the category of regarding cluster points as limits of sequence in the sense that they believed that a sequence should get close to or be equal to its limit. However, students in this category had an additional criterion in determining limits of sequences: The limit value of a sequence should be unique. Consequently, students in this category could exclude sequences having more than two cluster points from a set of convergent sequences. Nevertheless, these students thought not only \( \varepsilon \)–strip definition A but also \( \varepsilon \)–strip definition B was a proper description for the limit of a sequence. Actually, they considered the existence of a finite number of points outside an \( \varepsilon \)–strip as equivalent to the existence of an infinite number of points inside the \( \varepsilon \)–strip. Because of such a misunderstanding about subtraction of an infinite number from infinity, they regarded each of \( \varepsilon \)–strip definitions as implying the other. Some students in this category later recognized only \( \varepsilon \)–strip definition B as a proper description for limits by correcting their misconception as tasks went on and they worked on Task 6 with the sequence \( a_n = (-1)^n \cdot (1 + 1/n) \). That is, a conceptual change from regarding cluster points as limits to regarding limit points as limits occurred through working with various types of sequences.
Finally, students who regarded limit points as limits used the criterion “a convergent sequence is getting close to or equal to a unique value” in determining convergence of sequences. These students perceived $\varepsilon$–strip definition B as proper to describe the limit of a sequence. Further, they pointed out that if they applied $\varepsilon$–strip definition A to sequences, they would have to call a sequence that had subsequences getting close to or equal to two different values as convergent. Hence, students in this category considered $\varepsilon$–strip definition A as an improper description for the limit of a sequence. As seen above, there was a close association between students’ conception of limit and their recognition of the propriety of $\varepsilon$–strip definitions A and B. This association was very strong among students at reversibility level 4. Those who did not achieve reversibility level 4 showed a somewhat weaker association between their conception of limit and their perception of the propriety of $\varepsilon$–strip definitions A and B (see Table 5.1).

As seen in Table 5.1, students who regarded asymptotes as limits ranged across reversibility levels 0 to 4 regardless of the type of sequence. Students whose conception of limits was as cluster points or limit points had reversibility levels closer to level 4. For instance, there were no students at reversibility level 0 among those who regarded cluster points or limit points as limits, whereas there were always at least two students at reversibility level 0 among those who regarded asymptotes as limits. In addition, all students who regarded limit points as limits had a reversibility level higher than or equal to 2. Such results imply that the more properly students conceptualize the limit of a sequence, the better they understand the reverse relationship between $\varepsilon$ and $N$ in the
Table 5.1 Association of students' conception with reversibility in the context of limit: Under the graphical representation with $\varepsilon - \delta$ strips

<table>
<thead>
<tr>
<th>Task 2: $a_n = 1/n$</th>
<th>Asymptote Cluster point Cluster point &amp; Limit point Limit point</th>
<th>Task 5: $a_n = (-1)^n 1/n$</th>
<th>Asymptote Cluster point Cluster point &amp; Limit point Limit point</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reversibility 0</td>
<td>EMILY ERICA EVE</td>
<td>Reversibility 0</td>
<td>BETH EVE</td>
</tr>
<tr>
<td>Reversibility 1</td>
<td>BRIGID BRIAN EMMA</td>
<td>Reversibility 1</td>
<td></td>
</tr>
<tr>
<td>Reversibility 2</td>
<td>ELISA</td>
<td>Reversibility 2</td>
<td>ELISA EMILY BEN</td>
</tr>
<tr>
<td>Reversibility 3</td>
<td>BETH</td>
<td>Reversibility 3</td>
<td></td>
</tr>
<tr>
<td>Reversibility 4</td>
<td>BECKY ELEN A</td>
<td>Reversibility 4</td>
<td>BRIAN ERICA BECKY BRIGID ELEN A</td>
</tr>
<tr>
<td>Task 3: $a_n = \sqrt{n}$</td>
<td>Asymptote Cluster point Cluster point &amp; Limit point Limit point</td>
<td>Task 6: $a_n = (-1)^n (1+1/n)$</td>
<td>Asymptote Cluster point Cluster point &amp; Limit point Limit point</td>
</tr>
<tr>
<td>Reversibility 0</td>
<td>BETH EVE</td>
<td>Reversibility 0</td>
<td>BETH EVE</td>
</tr>
<tr>
<td>Reversibility 1</td>
<td>ERICA BETH ELEN A</td>
<td>Reversibility 1</td>
<td></td>
</tr>
<tr>
<td>Reversibility 2</td>
<td>BRIGID ELISA BRIAN</td>
<td>Reversibility 2</td>
<td>EMILY BEN</td>
</tr>
<tr>
<td>Reversibility 3</td>
<td>EMMA BRIAN BRIGID</td>
<td>Reversibility 3</td>
<td>ELISA ERICA BRIGID</td>
</tr>
<tr>
<td>Reversibility 4</td>
<td>BRIAN EMILY ELEN A</td>
<td>Reversibility 4</td>
<td>BRIAN ELEN BECKY</td>
</tr>
<tr>
<td>Task 4: $a_n = n/n$</td>
<td>Asymptote Cluster point Cluster point &amp; Limit point Limit point</td>
<td>Posttest</td>
<td>Asymptote Cluster point Cluster point &amp; Limit point Limit point</td>
</tr>
<tr>
<td>Reversibility 0</td>
<td>BETH EVE</td>
<td>Reversibility 0</td>
<td>BETH EVE</td>
</tr>
<tr>
<td>Reversibility 1</td>
<td></td>
<td>Reversibility 1</td>
<td></td>
</tr>
<tr>
<td>Reversibility 2</td>
<td>ELISA BRIAN BRIGID</td>
<td>Reversibility 2</td>
<td>ELISA BEN</td>
</tr>
<tr>
<td>Reversibility 3</td>
<td></td>
<td>Reversibility 3</td>
<td>EMILY BECKY BRIGID</td>
</tr>
<tr>
<td>Reversibility 4</td>
<td>BRIAN EMILY ERICA BRIAN ELEN A</td>
<td>Reversibility 4</td>
<td>BRIAN ELEN BECKY</td>
</tr>
</tbody>
</table>
definition of limit.

**Question 4:** How different is students’ understanding of the limit of a sequence after the teaching experiment? How different is students’ reversibility after experiencing the $\varepsilon$–strip activity?

One of the most important results of this research is that students’ reversibility was developing toward level 4 as the interviews progressed from Task 2 to Task 6. This result was found through analysis of students’ responses to devising better descriptions for limits than were given in the $\varepsilon$–strip definitions. That is, students at reversibility level $n$ ($n = 0, 1, 2, 3$) modified at least one of the $\varepsilon$–strip definitions A or B to what students at reversibility level $n + 1$ understood. Moreover, they conceived such a modification as proper in describing the limit of a sequence using the relationship between $\varepsilon$ and $N$.

As shown in Table 5.2, there were six students at reversibility level 0 or 1 on Task 2, in which the $\varepsilon$–strip was first introduced, and only four students at these levels during Task 3. By Task 4, all but two students understood the reverse relation between $\varepsilon$ and $N$ at reversibility level 2 or higher. Furthermore, some students, such as EMMA, ERICA, BRIAN, and BRIGID, continued developing their reversibility level throughout the tasks. It should be noted that this research did not intend or implement any sort of instruction for developing students’ reversibility. There was no procedure for indicating students’ errors, correcting students’ misconceptions about limit, or confirming the propriety of the $\varepsilon$–strip definitions to students during any of the interviews. Nevertheless, there was progress in students’ understanding of the reverse relation between $\varepsilon$ and $N$, which suggests that $\varepsilon$–strip activity plays a positive role in
<table>
<thead>
<tr>
<th></th>
<th>Pretest</th>
<th>Task 2: $a_n = 1/n$</th>
<th>Task 3: $a_n = \sqrt{n}$</th>
<th>Task 4: $a_n = n/n$</th>
<th>Task 5: $a_n = (-1)^n \frac{1}{n}$</th>
<th>Task 6: $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$</th>
<th>Posttest</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BEN</strong></td>
<td>Cluster pts</td>
<td>Reversibility 4, Cluster points &amp; Limit points</td>
<td>Reversibility 1, Cluster points</td>
<td>Reversibility 4, Cluster points</td>
<td>Reversibility 2, Cluster points &amp; Limit points</td>
<td>Reversibility 2, Cluster points &amp; Limit points</td>
<td>Limit points</td>
</tr>
<tr>
<td><strong>BECKY</strong></td>
<td>Limit points</td>
<td>Reversibility 4, Cluster points</td>
<td>Reversibility 4, Cluster points &amp; Limit points</td>
<td>Reversibility 4, Cluster points &amp; Limit points</td>
<td>Reversibility 4, Cluster points &amp; Limit points</td>
<td>Limit points</td>
<td>Limit points</td>
</tr>
<tr>
<td><strong>BRIAN</strong></td>
<td>Cluster points</td>
<td>Reversibility 1, Cluster points</td>
<td>Reversibility 4, Asymptotes</td>
<td>Reversibility 4, Asymptotes</td>
<td>Reversibility 4, Asymptotes</td>
<td>Reversibility 4, Asymptotes</td>
<td>Limit points</td>
</tr>
<tr>
<td><strong>BETH</strong></td>
<td>Asymptotes</td>
<td>Reversibility 3, Asymptotes</td>
<td>Reversibility 0, Asymptotes</td>
<td>Reversibility 0, Asymptotes</td>
<td>Reversibility 0, Asymptotes</td>
<td>Reversibility 0, Asymptotes</td>
<td>Limit points</td>
</tr>
<tr>
<td><strong>BRIGID</strong></td>
<td>Cluster points</td>
<td>Reversibility 1, Asymptotes</td>
<td>Reversibility 2, Cluster points &amp; Limit points</td>
<td>Reversibility 4, Asymptotes</td>
<td>Reversibility 4, Asymptotes</td>
<td>Reversibility 3, Asymptotes</td>
<td>Limit points</td>
</tr>
<tr>
<td><strong>ELISA</strong></td>
<td>Asymptotes</td>
<td>Reversibility 2, Cluster points</td>
<td>Reversibility 2, Cluster points &amp; Limit points</td>
<td>Reversibility 2, Asymptotes</td>
<td>Reversibility 2, Asymptotes</td>
<td>Reversibility 3, Asymptotes</td>
<td>Limit points</td>
</tr>
<tr>
<td><strong>ELENA</strong></td>
<td>Limit points</td>
<td>Reversibility 4, Cluster points &amp; Limit points</td>
<td>Reversibility 4, Cluster points &amp; Limit points</td>
<td>Reversibility 4, Asymptotes</td>
<td>Reversibility 4, Asymptotes</td>
<td>Reversibility 4, Asymptotes</td>
<td>Limit points</td>
</tr>
<tr>
<td><strong>EMILY</strong></td>
<td>Cluster points</td>
<td>Reversibility 0, Asymptotes</td>
<td>Reversibility 4, Cluster points</td>
<td>Reversibility 4, Cluster points</td>
<td>Reversibility 2, Cluster points</td>
<td>Reversibility 2, Cluster points</td>
<td>Limit points</td>
</tr>
<tr>
<td><strong>EMMA</strong></td>
<td>Asymptotes</td>
<td>Reversibility 1, Cluster points</td>
<td>Reversibility 2, Cluster points &amp; Limit points</td>
<td>Reversibility 4, Cluster points</td>
<td>Reversibility 3, Cluster points</td>
<td>Reversibility 4, Cluster points</td>
<td>Limit points</td>
</tr>
<tr>
<td><strong>ERICA</strong></td>
<td>Cluster points</td>
<td>Reversibility 0, Asymptotes</td>
<td>Reversibility 4, Cluster points</td>
<td>Reversibility 4, Cluster points</td>
<td>Reversibility 4, Cluster points</td>
<td>Reversibility 4, Cluster points</td>
<td>Limit points</td>
</tr>
<tr>
<td><strong>EVE</strong></td>
<td>Asymptotes</td>
<td>Reversibility 0, Asymptotes</td>
<td>Reversibility 0, Asymptotes</td>
<td>Reversibility 0, Asymptotes</td>
<td>Reversibility 0, Asymptotes</td>
<td>Reversibility 0, Asymptotes</td>
<td>Limit points</td>
</tr>
</tbody>
</table>
developing cognition of the reverse relationship between $\varepsilon$ and $N$ in the formal definition of limit.

Another important result of this research is that students’ conceptions of limit improved constantly as interviews proceeded. While there were nine students who regarded asymptotes or cluster points as limits during Task 2, the number of such students was reduced to six at Task 6. Conversely, there were no students who regarded limit points as limits during Task 2, but four students who regarded limit points as limits at Task 6. Furthermore, a large number of students who did not distinguish between cluster points and limit points grew to understand their difference and eventually regarded limit points as limits. These results suggest that the $\varepsilon$–strip activity used in this research and the sequences, which were carefully selected for the interviews, played a positive role in accommodating and developing students’ conceptions of limit.

**Main Question:** Does an activity with the graphical illustration of a sequence, along with statements describing the reverse relationship between $\varepsilon$ and $N$ influence development and accommodation of students’ intuitive understanding of the concept of limit?

Most students who participated in this research developed their reversibility, which is necessary to construct meaningful intuitive understanding of the limit of a sequence. Furthermore, it was shown that various types of sequence, representations of sequences, and $\varepsilon$–strip activity played crucial roles in helping students develop their intuitive understanding of limits to be compatible with the rigorous $\varepsilon–N$ definition of
limit. Consequently, the following conclusions about students’ intuitive cognition of the limit of a sequence and the effect of this teaching experiment can be drawn:

1. Reversibility in the context of limits plays a crucial role in achievement of intuitive cognition which is compatible with the rigorous definition of limit.
2. As students’ conception of limit develops from regarding asymptotes as limits, to cluster points as limits, to limit points to limits, students’ reversibility level increases.
3. The $\varepsilon$–strip activity, combined with the various types of sequences used in this teaching experiment, is one instructional method for helping students develop and accommodate their conception of the limit of a sequence.

**Implications of the Research for Teaching and Learning**

This section discusses the implications of the research for teaching and learning the concept of limit. It is focused on discussion of the role of various types of sequences, the role of representations of sequences, and the $\varepsilon$–strip activity, each of which was used in this research for conceptualizing the limit of a sequence. Finally, the importance of relational reversibility in the context of limits is contrasted with the operational reversibility used in arithmetic and algebra.

**The Role of Various Types of Sequences**

While the teaching experiment was proceeding, students were asked to determine and explain convergence of various types of sequences: monotone bounded sequences, unbounded sequences, constant sequences, oscillating convergent sequences, and
oscillating divergent sequences. Due to their unfamiliarity with these kinds of sequences, students revealed uncertainty in their conceptions of limit.

For instance, among interviewees, there were several students who imagined only monotone bounded sequences as convergent sequences. Based on this image of convergent sequence, they considered convergent sequences to be sequences getting close to but not equal to their limit values. While exploring limits of constant sequences, these students experienced cognitive obstacles in the contradiction between their own criterion for convergence and the fact that constant sequences are convergent.

Another example of students’ cognitive obstacles appeared when students dealt with oscillating divergent sequences. In particular, for students who did conceptualize the uniqueness of the limit values of sequences, oscillating divergent sequences caused students’ recognition of their uncertainty about the uniqueness of the limit of a sequence. Consequently, throughout the process of determining convergence and limit values of various types of sequences, students realized which part of the concept of limit they did not understand. Furthermore, once students found the solution to their misconception about limits through the interviews, they could develop their intuitive understanding of limit toward properly conceptualizing the limit of a sequence.

The Role of Representations of Sequences and the \( \epsilon - \text{strip} \) Activity

In this research, students determined convergence and limit values of given sequences under symbolic, numerical, and graphical representations. What is important to note is that it was possible to develop students’ intuitive cognition compatible with formal cognition using these various modes of representation. This result shows that, as
Fischbein (1987) and Bruner (1960) have asserted, mathematical intuition can be developed throughout a well designed curriculum. Furthermore, the series of representations of each sequence used in the teaching experiment implies a model of instruction to teach the $\varepsilon - N$ definition of limit.

In order for students to construct new ideas or concepts based on their current knowledge, Bruner (1960) emphasizes the process of learning and asserts that a curriculum should reflect not only the nature of knowledge itself but also the nature of the process of acquiring the knowledge. Bruner describes three modes of representing knowledge, which emerge in a developmental sequence, namely, the “enactive,” “iconic,” and “symbolic” modes of representation. First, the “enactive” mode of representation is the earliest stage, in which children represent objects in terms of their immediate sensation of them, as a way to manipulate the environment. Secondly, the “iconic” mode of representation involves the use of pictorial images that stand for certain objects or events. Finally, the “symbolic” mode of representation is the stage at which students use a symbolic system such as mathematical notation or verbalization of the meaning of mathematical notation to encode knowledge. Bruner points out that curricula should provide learning environments in which students can evolve through the above three modes of representation.

In this research, task-based interviews with students seemed to provide such a learning environment as follows: First, while determining the limit of a sequence in numerical form, students recognized and got familiar with limits of sequences as mathematical objects in the enactive mode of representation. Secondly, graphing a given
sequence to determine its limit value was the iconic mode of representation in this research. The $\varepsilon$–strip activity allowed students to begin reversing their thinking while still in the iconic mode, thus bridging the gap between the “forward” direction of their intuitive thinking (first $N$, then $\varepsilon$) and the reverse thinking (first $\varepsilon$, then $N$) in the formal definition. Throughout the $\varepsilon$–strip activity, students verbalized their understanding of how and why the reverse relation between $\varepsilon$ and $N$ should be used to describe the concept of limit.

The $\varepsilon$–strip activity used in this research played an important role in helping students transfer from one mode of representation to the other. In particular, it assisted students to transfer from the iconic mode to the symbolic mode which can be summarized as follows: First, the $\varepsilon$–strip activity along with graphs of sequences provided a tool for students to remember appropriate pictorial images in determining limits of sequences. Through such an activity, students could internalize the reverse relation between $\varepsilon$ and $N$ in intuitive cognition. The $\varepsilon$–strip activity plays the role of a tool in imagining the convergent situations pictorially as well as in verbally explaining the limit values obtained from the pictorial images. Actually, before carrying out the $\varepsilon$–strip activity, students who participated in this teaching experiment had experienced difficulty explaining why and how a given sequence was convergent. In view of this difficulty, the $\varepsilon$–strip activity seemed effective in transferring from the iconic mode to the symbolic mode of representation, the last step in Bruner’s three modes of representation suggested for internalization of mathematical concepts (Bruner, 1960). By internalizing the reverse relationship between $\varepsilon$ and $N$, in the iconic mode of the $\varepsilon$–strip activity, students
could more easily grasp the reverse relationship between $\varepsilon$ and $N$ in the symbolic mode of the $\varepsilon-N$ definition.

In the meantime, as mentioned above, types of sequences that were unfamiliar to students in conceptualizing their convergence caused cognitive dissonance, especially when students found that the result from their primary intuition was different from that of the $\varepsilon$–strip activity. Students had to decide which of the results was correct. While students proceeded with the $\varepsilon$–strip activity, they could clearly recognize circumstance of such cognitive dissonance. In addition, throughout the $\varepsilon$–strip activity, they could not only verbalize this confusion but also gradually modify their conception of limits of sequences.

Furthermore, according to the results of this research, students’ ability to properly understand the reverse relation between $\varepsilon$ and $N$ played an important role in building up their conception of limit. In this point of view, the graphical mode of representation and the $\varepsilon$–strip activity played an important role of an effective learning environment in building up and accommodating meaningful intuitive cognition compatible with the rigorous $\varepsilon-N$ definition of limits of sequences.

**Relational Reversibility in the Context of Limits**

Reverse thinking plays an important role in finding connections between different algebraic operations such as addition and subtraction, multiplication and division, or multiplication of polynomials and factorization (Krutetskii, 1969, 1976). Such a viewpoint is also expressed by Inhelder and Piaget (1958). In fact, Inhelder and Piaget (1958) classified reverse thinking processes in terms of inversion (or negation), and
compensation (or reciprocity). *Inversion* means reversing an operation by undoing it. For instance, given a balance at equilibrium, when an object is placed on one of its sides, disequilibrium occurs. In order to restore equilibrium, one may remove the object, that is, “undo” the placing of the object on the balance. This is reversing an operation by negation. On the other hand, *compensation* refers to reversing an operation by canceling its effects (Inhelder & Piaget, 1958). Considering the balance example described above, placing an equivalent weight on the other side of the balance to regain equilibrium is regarded as compensation. Reversibility of operations has long been regarded as an important problem-solving technique in arithmetic (Adi 1978; Bryant et al., 1999; Ferrandez-Reinisch, 1985; Inhelder & Piaget, 1958) and an efficient strategy in algebra (Krutetskii, 1969, 1976; Rachlin, 1981; Wagner et al., 1984).

One important result of the present research is that reversibility in the context of limits is a relational reversibility, rather than an operational one. That is, this reversibility reverses the order relationship between two variables $\varepsilon$ and $N$, rather than the inversion or compensation of a single operation. For most students, their primary intuition of limits leads them to think first in terms of the index $N$ and then to see how large the error $\varepsilon$ is, the same order reflected in reading the limit symbol. As illustrated in this research, reversibility of this dual relationship is the only way to correctly understand the rigorous definition of limit. In fact, if a student is not able to achieve this relational reversibility, the student has difficulty determining limits of many types of sequences.

**Future Research**

This research used a teaching experiment to investigate how students develop their intuitive understanding of the reverse relation between $\varepsilon$ and $N$ in the context of
limits of sequences and how such an intuitive understanding is associated with students’ conception of limits. It was found that students could develop their understanding of the reverse relation between $\varepsilon$ and $N$ through the teaching experiment, regardless of whether or not students had learned the rigorous $\varepsilon-N$ definition of limits. However, in order to use the results of this research in developing proper instructional methods for limits of sequences, research on the following problems should be conducted.

**Generalization of Findings**

As mentioned above, there was close association between students’ conceptions of limit and development of their reversibility in the context of limits. However, since this research investigated such an association based on qualitative research methodology with a small group of students, it was not designed to generalize the grounded theory. The whole population of the teaching experiment consisted of only 3 classes of 40 students, and rather than random sampling, voluntary sampling was used in selecting interviewees. Secondly, there were only 11 students who participated in the task-based interviews. Thirdly, students to be selected for task-based interviews were above average students whose final grades were A or B in calculus course. Furthermore, they were all successful in previous mathematics course so that they could draw and read graphs of basic functions. Such selective participants preclude meaningful generalization. Therefore, the results of this research might change if it were applied to a wider range of students.

There were several reasons that this research could not use methodology to support generalization. One of the most crucial reasons was that there was no prior research interpreting students’ reversibility in the context of limits. Thus, there was no
theoretical background to classify levels of reversibility in the context of limits. Furthermore, there was relatively little research to show categories of students’ conceptions of limits as asymptotes, cluster points, and limit points. Therefore, the present research was, of necessity, theory-building rather than theory-testing.

Future research should deal with generalizing the results of this research, such as the association of students’ conceptions of limit with development of their reversibility levels, as well as the instructional effect of $\varepsilon$–strip activity in improving students’ understanding of the $\varepsilon$–$N$ definition of limit.

**Secondary Mathematics Teachers’ Conception of the Limit of a Sequence**

Another idea to be investigated is research on mathematics teachers’ conceptions of limit and their practice in class. Activities used in teaching and learning play important roles in properly understanding the concept of limit. It appeared that students who improperly remembered limits as asymptotes or cluster points had considerable difficulty in accommodating their concept images into proper images for the concept of limit.

There were possibilities that students acquired incorrect concepts in learning limits of sequences. To be more precise, students might have been taught either implicitly or explicitly to determine convergence of a sequence by applying the criterion of “getting close to but not equal to.” Although previous instructors might not mean to explain the concept of limit in such a way, students may perceive it that way, regardless of the instructor’s intention. Indeed, some instructors mentioned that they tried to guide their students to understand limits in terms of the notion of asymptotes. These instructors would expect that students could better understand limits by remembering the image of
asymptotes. However, such a misconception about limits—regarding asymptotes as limits—causes problems with sequences that are not bounded, monotonic sequences.

As seen above, in order to understand how students’ misconceptions about limits are shaped and to find methods of resolving them, it is necessary to investigate terminology and expressions that teachers use in teaching the limit of a sequence. Furthermore, research on mathematics teachers’ conceptions of limit and their practice in class is essential in developing proper instructional methods. Therefore, investigating how teachers conceive the $\varepsilon$–strip activity and use it in class would be meaningful research.

**Effect of the $\varepsilon$–strip Activity in Development of Students’ Formal Cognition**

Another direction for further research is investigation of the effect of the $\varepsilon$–strip activity in developing students’ formal cognition. According to the results of the present research, there was improvement in students’ reversibility through the $\varepsilon$–strip activity. It was also found that such reversibility played an important role in understanding $\varepsilon$–strip definitions. However, this research did not investigate how students used the rigorous definition of limits in the proof of problems involving the concept of limit. For instance, while participating in this research, students were not asked to justify whether or not monotone bounded sequences were convergent by using the formal definition of limit. Instead, this research explored how students intuitively understood the reverse relation between $\varepsilon$ and $N$ in the rigorous definition of limit. On the other hand, this research did not report whether the $\varepsilon$–strip activity was effective in obtaining formal cognition, in particular, as related to the proof of theorems and properties of limits.
Therefore, the effect of this teaching experiment on developing students’ formal cognition should be examined as well.

**Conclusion**

This research focused on students’ understanding of the concept of limit and its association with reversibility in the context of limits. This research suggests an explanation of why students encounter difficulty in understanding mathematical ideas embedded in the formal $\varepsilon - N$ definition of limit. Furthermore, this research helps explain why students who have no problem understanding the concept of limit in an informal way may still have difficulty making the transition to the advanced mathematical thinking processes involved in the $\varepsilon - N$ definition of limit.
REFERENCES


Merenluoto, K., & Lehtinen, E. (2000, April 24-28). *Do theories of conceptual change explain the difficulties of enlarging the number concept in mathematics learning?* Poster session presented at the AREA annual meeting, New Orleans, LA.


APPENDICES
Appendix A. Description of Research Project

Dr. Sigrid Wagner and Dr. Kyeong Hah Roh are interested in students’ understanding of the concept of limits. We invite you to participate in a research study that will attempt to evaluate the educational impact of an intuitive approach to students’ understanding of the concept of limits. This study will allow us to identify necessary changes to develop more effective curricula and instruction for future students.

In order to conduct the study, we would like each of you to complete a survey. As a part of this research, some of your class work and exams may be copied and used to analyze your understanding. Some portions of the documents may be presented at professional meetings but all identification will be removed. In addition we will be conducting a 50-minute interview once a week for five weeks with a subset of the class. These interviews will be videotaped and/or audiotaped and analyzed. Some portions of videotaped written work may be presented at professional meetings but no pictures of you will be made public without your consent. The data collected is not part of your regular course work, and will not be made available to your instructor, thus poses no risks or discomfort.

We would very much appreciate your agreement to participate in this study. Our analysis will provide insight into students’ thinking in relation to the concept of limits, and thus suggest ways to enhance the effectiveness of instruction in college calculus courses. Other instructors will be able to use the results to enhance their own instruction in similar calculus courses.

Your participation in this study is completely voluntary. Your class activities and interview will be used only for this study. The responses to your class work and exams will be kept strictly confidential. The papers, videotapes and audiotapes will be kept secure in one of the principal investigator's offices. Pseudonyms will be used in all professional presentations and written papers related to this research. Your real identity will not be revealed in any description or publication of this research, unless you wish it. Participation or non-participation in the research in no way influences the grade in your homework, quizzes, midterms, and final. You may stop your participation in the study at any time by telling Dr. Kyeong Hah Roh. If you agree to be in the study, please sign your name on the attached consent form.
The consent form should be returned to Dr. Kyeong Hah Roh. If you have any questions about this study, feel free to contact either Dr. Kyeong Hah Roh or Dr. Sigrid Wagner. The contact information is provided below.

Respectfully,

Sigrid Wagner Ph.D.  Kyeong Hah Roh Ph. D

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Appendix B. Student Consent Form

I consent to participating in research project on students’ understanding of calculus. Dr. Sigrid Wagner, Principal Investigator, or her authorized representative Dr. Kyeong Hah Roh has explained the purpose of the study, the procedures to be followed, and the expected duration of my participation. Possible benefits of the study have been described, as have alternative procedures, if such procedures are applicable and available.

I acknowledge that I have had the opportunity to obtain additional information regarding the study and that any questions I have raised have been answered to my full satisfaction. Furthermore, I understand that I am free to withdraw consent at any time and to discontinue participation in the study without prejudice to me.

Finally, I acknowledge that I have read and fully understand the consent form. I sign I freely and voluntarily. A copy has been given to me.

Date:_________________________________ Signed:___________________________

(Principal Investigator or her authorized representative)

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Appendix C. Survey

Directions:

As you may already know, I would like to conduct a research project on students’ understanding of calculus. My hope is that this study will provide insight into students’ thinking in calculus, and allow me to identify necessary changes to develop more effective curriculum and instruction for future students.

The information obtained from this survey will be used to select interviewees. After analyzing the survey results, some of you will be selected and be asked to participate in interview sessions. I appreciate your agreement to participate in this study. Thank you!

Dr. KyeongHah Roh
Ohio University

Name ________________________________

E-mail ________________________________

1. Gender ( ) Female ( ) Male
2. Age
3. Major
4. I took at least one Calculus Courses from high school.

( ) Yes ( ) No

List all mathematics courses you took in your high school.

5. I took at least one calculus course in a previous quarter since entering Ohio University.

( ) Yes ( ) No

If Yes, list the course(s) along with instructor’s name and the year you took them.
The following values are the first four terms of an arithmetic sequence:
27, 31, 35, 39, …
Write the next two terms and explain how you found them.

Find the 25th term and explain how you found it.

What is the number (index) of the term that has value 683? How can you tell you?
The following values are the first six terms of a sequence

\[4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\]

Write the next two terms and explain how you found them.

Find the 25th term and explain how you found it.

What is the number (index) of the term that has value \(\frac{1}{1024}\)? How can you tell?

Draw the graph of a function \(f(x) = 3x - 2\). Explain why it looks this way.
9. Draw the graph of a function \( f(x) = \begin{cases} 
    x - 1 & (x > 1) \\
    0 & (x = 1) \\
    1 - x & (x < 1) 
\end{cases} \). Explain why it looks this way.

10. Draw the graph of a function \( f(x) = \sqrt{x} \), \( x \geq 0 \). Explain it looks this way.

11. Draw the graph of a function \( f(x) = \frac{1}{x} \), \( x \neq 0 \). Explain why it looks this way.
12 Find a function that would be pictured by the following graph. Explain how you found it.

13 Find a function that would be pictured by the following graph. Explain how you found it.
14 Show an example of a sequence that has a limit. Explain how you can tell the sequence has a limit.

15 Show an example of a sequence that does not have a limit. Explain how you can tell the sequence has no limit.
Appendix D. Pretest

ID:

Date:

Write the first four terms of each of the following sequences. Does each of the following sequence have a limit? Explain how you can tell.

1. \( a_n = \frac{1}{2n+5} \) for any positive integer \( n \).
2. \( a_n = \frac{1}{1-n} \) for any integer \( n \) greater than or equal to 2.
3. \( a_n = n \) for any positive integer \( n \).
4. \( a_n = 10^n \) for any positive integer \( n \).
5. \( a_n = 0 \) for any positive integer \( n \).
6. \( a_n = \frac{n}{n} \) for any positive integer \( n \).
7. \( a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \) for any positive integer \( n \).
8. \( a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even.} \end{cases} \) for any positive integer \( n \).
9. \( a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even.} \end{cases} \) for any positive integer \( n \).

10. What does it mean for a sequence to have a limit?

11. Let’s think about this sequence \( a_n = \frac{4}{n^2} \). Does this sequence have a limit? If so, how far out would you have to go to get within 1/100 of the limit? How far out would you have to go to get within 1/1000 of the limit?
12. There are two students who answered question 10 this way:

   Student A: “If a sequence is approaching a certain value $L$, then $L$ is a limit of the sequence.”
   Student B: “If the sequence is approaching but does not reach a certain value $L$, then $L$ is a limit of the sequence.”

Which student between student A and student B describes a limit most correctly for all types of sequences? How can you tell this?

Can you explain sequences having a limit among the above examples of sequences given in #1 ~ #9 by using the student’s answer you choose as correct?

13. What does it mean for a sequence to have no limit?

14. There are two students who answered question 13 this way:

   Student C: If a sequence is not approaching a certain value $L$, then $L$ is not a limit of the sequence.
   Student D: “If a sequence is approaching but does not reach a certain value $L$, then $L$ is not a limit of the sequence.”

Which student between student C and student D describes the limit most correctly for limits of all types of sequences? How can you tell this?

Can you explain sequences having no limit among the above examples of sequences given in #1 ~ #9 by using the student’s answer you choose as correct?
Appendix E. Task Activities & Interview Protocols

Task 1: Infinitely Many Mirror Reflections

**ID #**

**Date**

Q: Have you ever heard about infinity?
If Yes,
Q: Could you explain what does infinity mean?
Q: Have you ever experienced some physical examples of infinity?
  If Yes, Q: Describe one of them and explain how it is related to infinity.
  If No, Q: If you have not, nonetheless, what is the reason infinity is necessary in math?

The following mirror activity will show an infinite amount does not have to take up a great deal of space. The interviewer will prepare two rectangular papers of the same dimensions in one of which a rectangle is drawn on as follows:

The interviewer will introduce those papers to a student as mirrors, and will ask to locate them in parallel on a flat table. Then the interviewer will ask the following questions:

Q1: What would you expect to see on the mirror B? Draw what you expect to see on the mirror B. Can you explain why you can draw it this way?
Q2: What would you expect to see on the mirror A? Draw what you expect to see on the mirror A. Can you explain why you can draw it this way?
Now, the interviewer will show the student two actual mirrors, of one of which a small spot in the backside would be scratched out a small spot and the part is made to be transparent as the following diagram.

![Diagram of mirrors with a scratch](image)

The interviewer will ask the student to place these two mirrors in parallel on a flat table, and to look at a mirror through the scratched part of the other mirror. In order to make sure that the student look at the whole reflected mirror images, the interviewer will ask the student if the student could see the whole boundary of mirror as an image of the other mirror.

![Diagram of mirrors placed in parallel](image)

Q: What do you actually see on a mirror through the other mirror?  
Q: How many mirror images would be there? Explain.

Then the interviewer will ask to slightly incline the mirror in the opposite side and watch the change of images in the mirror.  
Q: What do you actually see on a mirror through the other mirror?  
Q: How many mirror images would be there? Explain.

Now the interviewer will ask the student what to expect to happen at the end.  
Q: What would we have left at the end?  
Q? Is there a limit of mirror images? How can you tell this?
Task 2: \( \lim_{n \to \infty} \frac{1}{n} = 0 \)

ID #
Date

Step 1: The Numerical Context
The interviewer will show a sequence \( a_n = \frac{1}{n} \), for \( n \in \mathbb{N} \), and will ask the value of several term numbers of the sequence.

Q: There is a sequence \( a_n = \frac{1}{n} \), for \( n \in \mathbb{N} \). What is the value of the first term? What is the value of the second term? What is the value of third term? What is the value of fourth term? What is the 10\(^{th}\) term? What is the 100\(^{th}\) term? Why do you expect this result?

Q: Does this sequence have a limit? How can you tell?

Step 2: The Graphical Context
The interviewer will give a coordinate plane and will ask to plot \( a_1, a_2, a_3 \), and \( a_{10} \).

Q: Plot the first term of the sequence on this plane.
Q: Plot the second term of the sequence on this plane.
Q: Plot the third term of the sequence on this plane.

Then the interviewer will show the following graph of the sequence \( a_n = \frac{1}{n} \).

Q: Plot the 10\(^{th}\) term of the sequence on this plane.
Q: Plot the 50\(^{th}\) term of the sequence on this plane.
Now, the interviewer will show another graph of the sequence in which the first 50 terms are plotted and the range of x-axis is about [0, 100].
Q: Where would the 70th point on the graph fall?
Q: Where would the 100th point on the graph fall?
Q: Does this sequence have a limit? How do you tell this?

If students’ answer is different from the answer in Step 1,

Q: You said this sequence had (did not have) a limit when you see the list of numbers, but after seeing the graph of this sequence you say that this sequence does not (has) a limit. Why do you think you have a different conclusion from your first conclusion?

**Step 3: Forward Thinking vs Reverse Thinking**

The interviewer will give the student a rectangular paper, called the N-plane which is made by translucent patty paper so that the student could look the graph through it. The interviewer will ask to put the N-plane on the graph of a sequence

\[ a_n = \frac{1}{n} \]

to cover all points on the graph in which term numbers are greater than N.

Q: Let’s assume that the N-plane is extended to all directions except the left-hand side, that the x-axis is extended, and the graph of the sequence is drawn more and more as the x-axis is extended. Place the N-plane on the graph of the sequence and cover all points where the term numbers are greater than 10. Can you determine the maximum of the differences between the value of 0 and each value of the sequence covered by the N-plane?

Q: Can you describe the distribution of points inside the plane and outside the plane?
Q: How many points on the graph are outside the N-plane?
Q: How many points on the graph are inside the N-plane?

Q: Place the N-plane on the graph of the sequence and cover all points where the term numbers are greater than 50. Can you determine the maximum of the differences between the value of 0 and each value of the sequence covered by the N-plane?
Q: Can you describe the distribution of points inside the plane and outside the plane?
Q: How many points on the graph are outside the N-plane?
Q: How many points on the graph are inside the N-plane?

Q: Place the N-plane on the graph of the sequence and cover all points where the term numbers are greater than 100. Can you determine the maximum of the differences between the value of 0 and each value of the sequence covered by the N-plane?
Q: Can you describe the distribution of points inside the plane and outside the plane?
Q: How many points on the graph are outside the N-plane?
Q: How many points on the graph are inside the N-plane?

The above set of questions will establish the pattern and show what “greater” means in the following question:

Q: If the value N, which is the term number of the point that the N-plane starts to cover, is greater and greater, how many points on the graph of this sequence will N-planes cover? How can you tell this?
Q: If the value N, which is the term number of the point that the N-plane starts to cover, is greater and greater, how many points on the graph of this sequence will N-planes NOT cover? How can you tell this?
Q: Does this sequence have a limit? How can you tell this?

Now, the interviewer will give the student an epsilon strip which is made with patty paper and is of a constant width. A red line is drawn in the middle of the strip so as to mark a possible limit value. The interviewer will ask to put an epsilon strip on the graph of a sequence \( a_n = \frac{1}{n} \) to cover \( y = 0 \) as well as some points on the graph.
Q: Assume that the epsilon strip as well as the red line in the epsilon strip is extended horizontally but remaining the constant width. Place the red line of the epsilon strip on the x-axis and cover points on the graph of the sequence as many as you can. Can you describe the distribution of points inside the strip and outside the strip? 
Q: How many points on the graph are outside the strip? 
Q: How many points on the graph are inside the strip? 

Now, the interviewer will give the student another epsilon strip of a smaller constant width than the previous one. 
Q: Place the red line of the epsilon strip on the x-axis and cover points on the graph of the sequence as many as you can. Can you describe the distribution of points inside the strip and outside the strip? 
Q: How many points on the graph are outside the strip? 
Q: How many points on the graph are inside the strip? 

Again, the interviewer will give the student another epsilon strip of a smaller constant width than the previous one. 
Q: Place the red line of the epsilon strip on the x-axis and cover points on the graph of the sequence as many as you can. Can you describe the distribution of points inside the strip and outside the strip? 
Q: How many points on the graph are outside the strip? 
Q: How many points on the graph are inside the strip? 

The above set of questions will establish the pattern and show what “smaller” means in the following question:
Q: If the width of this strip is getting smaller and smaller, how many points on the graph of this sequence will strips cover? How do you tell this? 
Q: If the width of this strip is getting smaller and smaller, how many points on the graph of this sequence will NOT be covered by strips? How do you tell this? 
Q: Does this sequence have a limit? How can you tell this? 

Q: Student E says, “The limit of this sequence is 0 because infinitely many points are covered by strips as long as the strips are covering 0.” On the other hand, Student F says, “A sequence has the limit if only finitely many points are NOT covered by strips as long as the strips are covering 0.” What do you think? 
   If the student answers that they are both correct.
Q: Can you explain why they are both correct? 
Q: Which student’s description is better for you to understand? How can you tell this?
Task 3: $\lim_{n \to \infty} \sqrt{n} = \infty$

ID #
Date

Step 1: Numerical Context
The interviewer will show a student a sequence $a_n = \sqrt{n}$ for $n \in \mathbb{N}$, and will ask the value of several term numbers of the sequence.

Q: There is a sequence: $a_n = \sqrt{n}$ for $n \in \mathbb{N}$. What is the value of the first term? What is the value of the second term? What is the value of the third term? What is the value of the fourth term? How can you tell this? What is the $10^{th}$ term? What is the $100^{th}$ term? How can you tell this?

Q: Does this sequence have a limit? How can you tell?

Step 2: Graphical Context
Now, the interviewer will give the student a coordinate plane, and will ask to plot $a_1$, $a_2$, $a_3$, and $a_{10}$.

Q: Plot the first term of the sequence on this plane.
Q: Plot the second term of the sequence on this plane.
Q: Plot the third term of the sequence on this plane.
Q: Plot the 10th term of the sequence on this plane.

Now, the graph of a sequence $a_n = \sqrt{n}$ will be visually shown.

Q: Where does the $10^{th}$ point on the graph fall?
Q: Where would the $50^{th}$ point on the graph fall?
Q: Where would the $100^{th}$ point on the graph fall?
Q: Does this sequence have a limit? How can you tell this?

If the student’s response is different from the response in Step 1,
Q: You said this sequence had (did not have) a limit when you see the list of numbers, but after seeing the graph of this sequence you say that this sequence does not (does) have a limit. Why do you make different conclusion from your previous conclusion?

Step 3: Forward Thinking vs Reverse Thinking
The interviewer will give a rectangular paper, called the N-plane, which is made by translucent patty paper so that students look the graph through it. The N-plane will be assume to be extended to cover right half coordinate plane. The interviewer will then ask to put the N-plane on the graph of the sequence...
\( a_n = \sqrt{n} \). The interviewer will ask the student to put the N-plane on the graph of a sequence \( a_n = \sqrt{n} \) to cover all points on the graph in which term numbers are greater than N.

Q: Let’s assume that the N-plane is extended to all direction except the left-hand side, that the x-axis is extended, and the graph of the sequence is drawn more and more as the x-axis is extended. Place the N-plane on the graph of the sequence and cover all points where the term numbers are greater than 20. Can you describe distribution of points inside the plane and the outside the plane?
Q: How many points on the graph are outside the N-plane?
Q: How many points on the graph are inside the N-plane?

Q: Place the N-plane on the graph of the sequence and cover all points where the term numbers are greater than 50. Can you describe distribution of points inside the plane and the outside the plane?
Q: How many points on the graph are outside the N-plane?
Q: How many points on the graph are inside the N-plane?

Q: Place the N-plane on the graph of the sequence and cover all points where the term numbers are greater than 100. Can you describe distribution of points inside the plane and the outside the plane?
Q: How many points on the graph are outside the N-plane?
Q: How many points on the graph are inside the N-plane?

The above set of questions will establish the pattern and show what “greater” means in the following questions:
Q: If the value N, which is the term number that each N-plane starts at to cover, is greater and greater, how many points on the graph of this sequence will N-planes cover? How can you tell this?
Q: If the value N, which is the term number of the point that each N-plane starts at to cover, is greater and greater, how many points on the graph of this sequence will N-planes NOT cover?
Q: Does this sequence have a limit? How can you tell this?

Now, the interviewer will give the student an upper half plane made by patty paper, so called M-plane, and will ask to put the upper half plane on the graph to cover points in which values of the terms are greater than M.

Q: Assume that the M-plane is extended all direction but not extended to below. Place the M-plane to cover all points of the sequence whose values are greater than 4. Can you describe distribution of points inside and outside of the M-plane?
Q: How many points on the graph are outside the upper half plane?
Q: How many points on the graph are inside the upper half plane?

Again, the interviewer asks to put the upper half plane on the graph points in which values of terms are greater than 6.
Q: Place the M-plane to cover all points of the sequence greater than 6. Can you describe distribution of points inside and outside of the M-plane?
Q: How many points on the graph are outside the upper half plane?
Q: How many points on the graph are inside the upper half plane?

Again, the interviewer asks to put the upper half plane on the graph points in which values of terms are greater than 10.
Q: Place the M-plane to cover all points of the sequence greater than 6. Can you describe distribution of points inside and outside of the M-plane?
Q: How many points on the graph are outside the upper half plane?
Q: How many points on the graph are inside the upper half plane?

The above of questions will establish the pattern and show what “getting larger” means in the following question:
Q: If M is getting larger and larger, how many points on the graph of this sequence will the upper half planes cover? How do you tell this?
Q: If M is getting larger and larger, how many points on the graph will the upper half
planes NOT cover? How do you tell this?

Q: There are two students. Student A says, “This sequence is divergent to infinity because infinitely many points on the graph are covered by any M-plane, \( y \geq M \), no matter how \( M \) is large.” Student B says, “This sequence is divergent to infinity if only finite number of points are NOT by any M-plane, \( y \geq M \), no matter how \( M \) is large.” What do you think?

If the student responds that both are correct,
   Q: Can you explain why they are both correct?
   Q: Which students’ description is better for you to understand? How can you tell this?
Task 4: \( \lim_{n \to \infty} \frac{n}{n} = 1 \)

**ID #**

**Date**

**Step 1: Numerical Context**

The interviewer will show a sequence \( a_n = \frac{n}{n} \) for \( n \in \mathbb{N} \), and will ask the value of several term numbers of the sequence.

Q: There is a sequence \( a_n = \frac{n}{n} \) for \( n \in \mathbb{N} \). What is the value of the first term? What is the value of the second term? What is the value of the third term? What is the value of the fourth term? What is the 10\(^{th}\) term? What is the 100\(^{th}\) term? How can you tell this?

Q: Does this sequence have a limit? How can you tell this?

**Step 2: Graphical Context**

The interviewer will give a coordinate plane and will ask to plot \( a_1, a_2, a_3, \) and \( a_{10} \).

Q: Plot the first term of the sequence on this plane.

Q: Plot the second term of the sequence on this plane.

Q: Plot the third term of the sequence on this plane.

Q: Plot the 10\(^{th}\) term of the sequence on this plane.

Then the interviewer will show a graph of the sequence, in which the first 20 terms are plotted and the range of the x-axis is about [0, 40].

Q: Where would the 40\(^{th}\) point on the graph fall?

Q: Where would the 50\(^{th}\) point on the graph fall?

Q: Where would the 100\(^{th}\) point on the graph fall?

Q: Does this sequence have a limit? How do you tell this?

If the student’s response is different from the response in Step 1,

Q: You said this sequence had (did not have) a limit when you saw the list of values, but after seeing the graph of this sequence you say that this sequence does not have (has) a limit. Why do you think you have a different conclusion this time from your previous conclusion?

**Step 3: Forward Thinking vs Reverse Thinking**

The interviewer will give the student a rectangular paper, called the N-plane which is made by translucent patty paper so that students look the graph through
it. The interviewer will ask the student to put the N-plane on the graph of the sequence \( a_n = \frac{n}{n} \) to cover all points on the graph in which their indices are greater than N.

\[ Q: \text{Let’s assume that the N-plane is extended to all direction except the left-hand side, that the x-axis is extended, and the graph of the sequence is drawn more and more as the x-axis is extended. Place the N-plane on the graph of the sequence and cover all points where the term numbers are greater than 10. Can you determine the maximum of the differences between the values of 1 and each value of the sequence covered by the N-plane?} \\
Q: \text{How many points on the graph are outside the N-plane?} \\
Q: \text{How many points on the graph are inside the N-plane?} \\
Q: \text{Place the N-plane on the graph of the sequence and cover all points where the term numbers are greater than 30. Can you determine the maximum of the differences between the values of 1 and each value of the sequence covered by the N-plane?} \\
Q: \text{How many points on the graph are outside the N-plane?} \\
Q: \text{How many points on the graph are inside the N-plane?} \\

The above set of questions will establish the pattern and show what “greater” means in the following questions:

Q: If the value N, which is the term number of the point that the N-plane starts to cover, is greater and greater, how many points on the graph of this sequence will N-planes cover? How can you tell this?
Q: If the value N, which is the term number of the point that the N-plane starts to cover, is greater and greater, how many points on the graph of this sequence will N-planes NOT cover? How can you tell this?
Q: Does this sequence have a limit? How can you tell this?

Now the interviewer will give the student an epsilon strip which is made with patty paper and is of constant width. A red line is drawn in the middle of the strip so as to mark a possible limit point. The interviewer will ask to put an epsilon strip on
the graph of the sequence \( a_n = \frac{n}{n} \) to cover \( y=1 \) as well as some points on the graph.

Q: Assume that the epsilon strip is extended horizontally but remaining the width, and the red line on the epsilon strip is extended as well as the graph of the sequence. Place the red line of the epsilon strip on the horizontal line \( y=1 \) and cover points on the graph of the sequence as many as you can. Can you describe distribution of points inside and outside the strip?
Q: How many points on the graph are outside the strip?
Q: How many points on the graph are inside the strip?

Now, the interviewer will give the student another epsilon strip of a smaller constant width than the previous one.
Q: This time we also assume that the epsilon strip is extended horizontally but remaining the width, and the red line on the epsilon strip is extended as well as the graph of the sequence. Place the red line of the epsilon strip on the horizontal line \( y=1 \) and cover points on the graph of the sequence as many as you can. Can you describe distribution of points inside and outside the strip?
Q: How many points on the graph are outside the strip?
Q: How many points on the graph are inside the strip?

The above set of questions will establish the pattern and will show what “smaller” means in the following question:

Q: If the width of strips is getting smaller and smaller, how many points on the graph of this sequence will strips cover? How can you tell this?
Q: If the width of strips is getting smaller and smaller, how many points on the graph of this sequence will strips NOT cover? How can you tell this?
Q: Does this sequence have a limit? How can you tell this?

Q: Student A says “The limit of this sequence is 1 because infinitely many points on the graph of this sequence are covered by epsilon strips as long as the strips are covering 1.” On the other hand, Student B says “The limit of this sequence is 1 because only finitely many points on the graph of this sequence are NOT covered by epsilon strips as long as the strips are covering 1.” What do you think?
If the student responds that both are correct,
Q: Can you explain why they are both correct?
Q: Which student’s description is better for you to understanding? How can you tell this?
Task 5: \( \lim_{n \to \infty} \frac{1}{n} = 0 \)

ID #
Date

Step 1: Numerical Context
The interviewer will show a sequence \( a_n = (-1)^n \frac{1}{n} \) for \( n \in \mathbb{N} \), and will ask a student the value of several term numbers of the sequence.

Q: There is a sequence \( a_n = (-1)^n \frac{1}{n} \) for \( n \in \mathbb{N} \). What is the value of the first term?

What is the value of the second term? What is the value of the third term? What is the value of the fourth term? What is the value of the 10th term? What is the value of the 11th term? What is the value of the 100th term? What is the value of the 101st term? How can you tell this?

Q: Does this sequence have a limit? How can you tell this?

Step 2: Graphical Context
Now, the interviewer will give the student a coordinate plane and will ask to plot several first terms of the sequence.

Q: Plot the first term of the sequence on this plane.
Q: Plot the second term of the sequence on this plane.
Q: Plot the third term of the sequence on this plane.
Q: Plot the fourth term of the sequence on this plane.

Then the interviewer will show the graph of a sequence \( a_n = (-1)^n \frac{1}{n} \) is visually shown.

Q: Where does the 10th point on the graph fall?
Q: Where does the 11th point on the graph fall?
Q: Where would the 25th point on the graph fall?
Q: Where would the 26th point on the graph?
Q: Does this sequence have a limit? How can you tell this?

If the student’s response were different from the response in Step 1,
Q: You said this sequence had (did not have) a limit when you see the list of values of the sequence, but after seeing the graph of this sequence you say that
this sequence does not have (has) a limit. Why do you think you have a different conclusion from your first conclusion?

**Step 3: Forward Thinking vs Reverse Thinking**
The interviewer will give the student a rectangular paper, called the N-plane which is made by translucent patty paper so that the student can look the graph through it. The interviewer will ask to put the N-plane on the graph of a sequence
\[ a_n = (-1)^n \frac{1}{n} \]
to cover all points on the graph in which term numbers are greater than N.

Q: Let’s assume that the N-plane is extended all direction except the left-hand side, that the x-axis is extended, and the graph of the sequence is drawn more and more as the x-axis is extended. Place the N-plane on the graph of the sequence and cover all points where the term numbers are greater than 10. Can you determine the maximum of the difference between the value of 0 and each value of the sequence covered by the N-plane?

Q: Can you describe distribution of points inside the plane and outside the plane?

Q: How many points on the graph are outside the N-plane?

Q: How many points on the graph are inside the N-plane?

Q: Let’s assume that the N-plane is extended all direction except the left-hand side. Place the N-plane on the graph of the sequence and cover all points where the term numbers are greater than 25. Can you determine the maximum of the difference between the value of 0 and each value of the sequence covered by the N-plane?

Q: Can you describe distribution of points inside the plane and outside the plane?

Q: How many points on the graph are outside the N-plane?

Q: How many points on the graph are inside the N-plane?

The above set of questions will establish the pattern and show what “greater” means in the following question:
Q: If the value $N$, which is the term number of the points that the $N$-plane starts to cover, is greater and greater, how many points on the graph of this sequence will $N$-planes cover? How can you tell this?

Q: If the value $N$, which is the term number of the points that the $N$-plane starts to cover, is greater and greater, how many points on the graph of this sequence will $N$-planes NOT cover? How can you tell this?

Q: Does this sequence have a limit? How can you tell this?

Now, the interviewer will give the student an epsilon strip, which is made with patty paper and is of a constant width. A red line is drawn in the middle of the strip so as to mark a possible limit value. The interviewer will ask to put the strip on the graph of a sequence $a_n = (-1)^n \frac{1}{n}$ to cover the x-axis as well as some points on the graph.

Q: Assume that the epsilon strip is extended horizontally but remaining the constant width, the red line on the strip is extended as well as the x-axis, and more points are plotted as the x-axis is extended. Place the red line of the epsilon strip on the x-axis and cover points on the graph of this sequence as many as you can. Can you describe the distribution of points inside and outside the strip?

Q: How many points on the graph are outside the strip?

Q: How many points on the graph are inside the strip?

Now, the interviewer will give students another epsilon strip of a smaller width than the previous one.

Q: Assume that as similar to the previous epsilon strip, this epsilon strip is extended horizontally but remaining the constant width, the red line on the strip is extended as well as the x-axis, and more points are plotted as the x-axis is extended. Place the red line of the epsilon strip on the x-axis and cover points on the graph of this sequence as many as you can. Can you describe the distribution of points inside and outside the strip?

Q: How many points on the graph are outside the strip?

Q: How many points on the graph are inside the strip?

The above set of questions will establish the pattern and show what “smaller” means in the following question:

Q: If the width of this strip is getting smaller and smaller, how many points on the graph would be covered by this strip? How do you tell this?
Q: If the width of this strip is getting smaller and smaller, how many points on the graph would the strip NOT cover? How do you tell this?
Q: Does this sequence have a limit? How can you tell this?

Q: There are two students. Student A says, “The limit of this sequence is 0 because infinitely many points around 0 are covered by strips no matter how small width strips have.” On the other hand, Student B says, “The limit of this sequence is 0 because only finite number of points are NOT covered by strips covering 0 no matter how small width strips have.” What do you think?

If the student responds that they are both correct,
Q: Can you explain why they are both correct?
Q: Which student’s description of the limit is better for you to understand?
Step 1: The Numerical Context
The interviewer shows a sequence \( a_n = (-1)^n \left( 1 + \frac{1}{n} \right) \) for \( n \in \mathbb{N} \) to the student, and will ask the value of several term numbers of this sequence.

Q: There is a sequence \( a_n = (-1)^n \left( 1 + \frac{1}{n} \right) \) for \( n \in \mathbb{N} \). What is the first term of this sequence? What is the second term? What is the third term? What is the fourth term? What is the 10th term? What is the eleventh term? How do you tell this?

Q: Does this sequence have a limit? How can you tell this?

Step 2: The Graphical Context
Now, the interviewer will give the student a coordinate plane and will ask to plot several first terms of the sequence.

Q: Plot the first term of the sequence on this plane.
Q: Plot the second term of the sequence on this plane.
Q: Plot the third term of the sequence on this plane.
Q: Plot the fourth term of the sequence on this plane.

Then the interviewer will show the graph of a sequence \( a_n = (-1)^n \left( 1 + \frac{1}{n} \right) \) is visually shown.

Q: Where does the 20th point on the graph fall?
Q: Where does the 21st point on the graph fall?
Q: Where would the 50th point on the graph fall?
Q: Where would the 51st point on the graph fall?
Q: Does this sequence have a limit? How can you tell this?

If the student's response were different from the response in Step 1,

Q: You said this sequence had (did not have) a limit when you see the list of values of the sequence, but after seeing the graph of this sequence you say that
this sequence does not have (has) a limit. Why do you think you have a different conclusion from your first conclusion?

**Step 3: Forward Thinking vs Reverse Thinking**

The interviewer will give the student a rectangular paper, called the N-plane which is made by translucent patty paper so that the student can look the graph through it. The interviewer will ask to put the N-plane on the graph of a sequence

\[ a_n = (-1)^n \left( 1 + \frac{1}{n} \right) \]

to cover all points on the graph in which term numbers are greater than N.

Q: Let’s assume that the N-plane is extended all direction except the left-hand side, that the x-axis is extended, and the graph of the sequence is drawn more and more as the x-axis is extended. Place the N-plane on the graph of the sequence and cover all points where the term numbers are greater than 10. Can you determine the maximum of the difference between the value of 0 and each value of the sequence covered by the N-plane?

Q: Can you describe distribution of points inside the plane and outside the plane?

Q: How many points on the graph are outside the N-plane?

Q: How many points on the graph are inside the N-plane?

Q: Let’s assume that the N-plane is extended all direction except the left-hand side. Place the N-plane on the graph of the sequence and cover all points where the term numbers are greater than 25. Can you determine the maximum of the difference between the value of 0 and each value of the sequence covered by the N-plane?

Q: Can you describe distribution of points inside the plane and outside the plane?

Q: How many points on the graph are outside the N-plane?

Q: How many points on the graph are inside the N-plane?

The above set of questions will establish the pattern and show what “greater” means in the following question:
Q: If the value $N$, which is the term number of the points that the $N$-plane starts to cover, is greater and greater, how many points on the graph of this sequence will $N$-planes cover? How can you tell this?

Q: If the value $N$, which is the term number of the points that the $N$-plane starts to cover, is greater and greater, how many points on the graph of this sequence will $N$-planes NOT cover? How can you tell this?

Q: Does this sequence have a limit? How can you tell this?

Now, the interviewer will give the student an epsilon strip, which is made with patty paper and is of a constant width. A red line is drawn in the middle of the strip so as to mark a possible limit value. The interviewer will ask to put the strip on the graph of a sequence $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$ to cover the $y = 1$ as well as some points on the graph.

Q: Assume that the epsilon strip is extended horizontally but remaining the constant width, the red line on the strip is extended as well as the $x$-axis, and more points are plotted as the $x$-axis is extended. Place the red line of the epsilon strip on $y = 1$ and cover points on the graph of this sequence as many as you can. Can you describe the distribution of points inside and outside the strip?

Q: How many points on the graph are outside the strip?

Q: How many points on the graph are inside the strip?

Now, the interviewer will give the student another epsilon strip of a smaller width than the previous one.

Q: Assume that as similar to the previous epsilon strip, this epsilon strip is extended horizontally but remaining the constant width, the red line on the strip is extended as well as the $x$-axis, and more points are plotted as the $x$-axis is extended. Place the red line of the epsilon strip on $y = 1$ and cover points on the graph of this sequence as many as you can. Can you describe the distribution of points inside and outside the strip?

Q: How many points on the graph are outside the strip?

Q: How many points on the graph are inside the strip?

The above set of questions will establish the pattern and show what “smaller” means in the following question:

Q: If the width of this strip is getting smaller and smaller, how many points on the graph
would be covered by this strip? How do you tell this?
Q: If the width of this strip is getting smaller and smaller, how many points on the graph would the strip NOT cover? How do you tell this?
Q: Is the value one a limit of this sequence? How can you tell this?

Now the interviewer will ask to put the strip on the graph of a sequence
\[ a_n = (-1)^n \left(1 + \frac{1}{n}\right) \]
to cover \( y = -1 \) as well as some points on the graph.

Q: Assume that the epsilon strip is extended horizontally but remaining the constant width, the red line on the strip is extended as well as the x-axis, and more points are plotted as the x-axis is extended. Place the red line of the epsilon strip on \( y = -1 \) and cover points on the graph of this sequence as many as you can. Can you describe the distribution of points inside and outside the strip?
Q: How many points on the graph are outside the strip?
Q: How many points on the graph are inside the strip?

Now, the interviewer will give the student another epsilon strip of a smaller width than the previous one.
Q: Assume that as similar to the previous epsilon strip, this epsilon strip is extended horizontally but remaining the constant width, the red line on the strip is extended as well as the x-axis, and more points are plotted as the x-axis is extended. Place the red line of the epsilon strip on \( y = -1 \) and cover points on the graph of this sequence as many as you can. Can you describe the distribution of points inside and outside the strip?
Q: How many points on the graph are outside the strip?
Q: How many points on the graph are inside the strip?

The above set of questions will establish the pattern and show what “smaller” means in the following question:
Q: If the width of this strip is getting smaller and smaller, how many points on the graph would be covered by this strip? How do you tell this?
Q: If the width of this strip is getting smaller and smaller, how many points on the graph would the strip NOT cover? How do you tell this?
Q: Is the value negative one a limit of this sequence? How can you tell this?
Once again, the interviewer will ask to put the strip on the graph of a sequence 
\[ a_n = (-1)^n \left(1 + \frac{1}{n}\right) \] to cover the x-axis as well as some points on the graph.

Q: Assume that the epsilon strip is extended horizontally but remaining the constant width, the red line on the strip is extended as well as the x-axis, and more points are plotted as the x-axis is extended. Place the red line of the epsilon strip on the x-axis and cover points on the graph of this sequence as many as you can. Can you describe the distribution of points inside and outside the strip?
Q: How many points on the graph are outside the strip?
Q: How many points on the graph are inside the strip?

Now, the interviewer will give the student another epsilon strip of a smaller width than the previous one.
Q: Assume that as similar to the previous epsilon strip, this epsilon strip is extended horizontally but remaining the constant width, the red line on the strip is extended as well as the x-axis, and more points are plotted as the x-axis is extended. Place the red line of the epsilon strip on the x-axis and cover points on the graph of this sequence as many as you can. Can you describe the distribution of points inside and outside the strip?
Q: How many points on the graph are outside the strip?
Q: How many points on the graph are inside the strip?

The above set of questions will establish the pattern and show what “smaller” means in the following question:
Q: If the width of this strip is getting smaller and smaller, how many points on the graph would be covered by this strip? How do you tell this?
Q: If the width of this strip is getting smaller and smaller, how many points on the graph would the strip NOT cover? How do you tell this?
Q: Is the value zero a limit of this sequence? How can you tell this?

Q: There are two students. Student A says, “The limit of sequence is 1 because infinitely many points around the limit are covered by strips no matter how small width strips have.” Another Student B says “1 is not a limit of this sequence. If 1 is the limit of this sequence, any epsilon strip covering 1 cover all points on the graph but not only finitely many points as long as the epsilon strip covers 1. However, my epsilon strip does not
cover infinitely many points. What do you think? How can you tell this?

If the student responds that they are both correct,

Q: Can you explain why they are both correct?

Q: Which student’s description of the limit is better for you to understand?
Appendix F. Posttest

ID:
Date:

Does each of the following sequence have a limit? Explain how you can tell this.

1. \( a_n = \frac{1}{n} \) for any positive integer \( n \).

2. \( a_n = \begin{cases} 
\frac{1}{n} & \text{if } n \leq 10 \\
\frac{1}{10} & \text{if } n > 10
\end{cases} \) for any positive integer \( n \).
3. \( a_n = \frac{n}{n+1} \) for any positive integer \( n \).

4. \( a_n = \frac{n^2}{n+1} \) for any positive integer \( n \).

5. \( a_n = \frac{n}{5n} \) for any positive integer \( n \).
6. \( a_n = \begin{cases} -1 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases} \) for any positive integer \( n \).

7. \( a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even} \end{cases} \) for any positive integer \( n \).

8. \( a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1 - \frac{1}{n} & \text{if } n \text{ is even} \end{cases} \) for any positive integer \( n \).
9. There are two students who described limit this way:

Student A: A certain value L is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any epsilon strip as long as the epsilon strip covers L.

Student B: A certain value L is a limit of a sequence when only finitely many points on the graph of the sequence are NOT covered by any epsilon strip as long as the epsilon strip covers L.

Which student most accurately describe limit? How can you tell?

10. Can you explain the above examples of sequences given in #1 ~ # 15 by using the student you choose as accurate?

11. [For Math 263 C students] The following statement is a definition of limit of a sequence in your textbook:

For any positive value \( \varepsilon \), there exists a positive integer \( K \) such that whenever \( k \geq K \), \( |a_k - L| < \varepsilon \).

Compare this definition to Student A, to Student B, and to your statement. Does each of the following sequence have a limit? Explain how you can tell this.
12. $a_n = n + (-1)^n$ for any positive integer $n$.

13. $a_n = (-1)^n \cdot n$ for any positive integer $n$.

14. $a_n = 1 + (-1)^n \cdot \frac{1}{n}$ for any positive integer $n$. 

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15. $a_n = (-1)^n + \frac{1}{n}$ for any positive integer $n$.

16. $a_n = (-1)^n \left( 1 + \frac{1}{n} \right)$ for any positive integer $n$.

17. $a_n = (-1)^n \left( 1 - \frac{1}{n} \right)$ for any positive integer $n$. 


18. $a_n = \frac{n^3}{2^n}$ for any positive integer $n$. 