INTERRELATIONSHIPS BETWEEN TEACHERS' CONTENT KNOWLEDGE OF RATIONAL NUMBER, THEIR INSTRUCTIONAL PRACTICE, AND STUDENTS' EMERGENT CONCEPTUAL KNOWLEDGE OF RATIONAL NUMBER

DISSERTATION

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Previous studies using quantitative methods have attempted to correlate teachers’ content knowledge with students’ acquisition of subject matter knowledge. Lack of highly significant correlations between teachers’ content knowledge and students’ achievement seems to indicate that increasing the level of teachers’ content knowledge has limited influence on students’ emergent subject matter knowledge. More recent studies using qualitative methods have shown teachers’ content knowledge influences teachers’ instructional practice. However, few studies using qualitative methods have examined the interrelationships between teachers’ content knowledge and students’ emergent subject matter knowledge.

This study was developed from a theory on the interrelationships between teachers’ content knowledge of rational numbers and students’ emergent conceptual knowledge of rational numbers. Two teachers were chosen from among four candidates based on the differences in their written responses on a test of rational number (fraction) knowledge and their location in the same school system. The case studies of these two teachers were generated from data collected through:

- observations and videotapes of each classroom as the teachers conducted their unit on rational numbers,
- interviews with the teachers and selected students from their classes,
• teachers’ and students’ responses on a test of rational number knowledge.

The cases were compared and contrasted to illuminate and illustrate the theoretical model of interrelationships and intervening contributions and limitations of interrelationships between teachers’ content knowledge of rational numbers and students’ emergent conceptual knowledge of rational numbers. The cases confirm the theoretical model that interrelationships between teachers’ content knowledge and students’ emergent conceptual knowledge are weak. The cases confirm the interrelationships suggested by the model between teachers’ content knowledge and instructional practice. Teachers’ content knowledge contributes to, but is not equivalent to, their pedagogical content knowledge. Teachers’ pedagogical content knowledge mediates/filters the impact of teachers’ content knowledge on their design of the instructional environment. As the model would predict, the varieties of students’ emergent conceptual knowledge across and within the instructional environments do not replicate the teachers’ content knowledge. The model suggests that students’ prior knowledge, predispositions, and/or experiences could mediate/filter the contributions of instructional practice to students’ emergent conceptual knowledge.
Dedicated to my mother, Irene Pace Millsaps, Ed.D.
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CHAPTER 1

INTRODUCTION

Since before Sputnik and the New Math there has been an interest in improving the outcomes of K-12 mathematics education. The last two decades of the 20th century saw the National Council of Teachers of Mathematics (NCTM) attempt to tackle the problem of improving K-12 mathematics education through the reform mechanism of the 1989 and 2000 Standards documents. These documents were intended to promote reform through an examination of and recommendations for the nature and scope of the K-12 mathematics curriculum. NCTM also published the Professional Standards for Teaching Mathematics (1991) that examined the important contributions of teachers to the implementation of reform in mathematics education. In their elaboration of the “Teaching Principle,” the authors of the Principles and Standards for School Mathematics (NCTM, 2000) observed that “to be effective, teachers must know and understand the mathematics that they are teaching and be able to draw on that knowledge flexibly in their teaching tasks” (p. 17). In the Professional Standards for Teaching Mathematics (NCTM, 1991), Standard 2 for the Professional Development of Teachers of Mathematics is “Knowing Mathematics and School Mathematics.” The authors explained, “Teachers’ comfort with, and confidence in, their own knowledge of mathematics affects both what they teach and how they teach it” (p. 132). They defined teachers’ mathematical knowledge as including
“understanding specific concepts and procedures as well as the process of doing mathematics” (p. 132). Thus, each document posited that the subject matter knowledge of teachers is an important contributor to effective mathematics teaching.

Statement of Problem

If the goal of effective mathematics teaching is student learning and if teachers’ subject matter knowledge contributes to effective mathematics teaching, are there patterns of interrelationship between teachers’ mathematical knowledge and student learning? We conjecture that the mechanism by which teachers' understanding of mathematics impacts students’ learning is the teachers’ instructional practice. As Fennema and Franke (1992) assert, there is a need to examine more closely the impact on students taught by knowledgeable teachers in the context of rich classroom environments. Our intuition tells us that teachers' knowledge of mathematics must certainly affect their students' learning of mathematics. The study described herein examines in depth two teachers' knowledge of the domain of rational number, their rational number instruction, and their students' learning of rational number concepts through observations of teachers’ and students’ interactions in teacher-designed instructional units on rational numbers and teachers’ and students’ responses to rational number test problems and interview tasks. Cases, developed from the descriptions of teachers’ knowledge, instruction, and their students’ construction of rational number knowledge, are examined for common patterns.
Focus Questions

Focus questions for the study include:

- Does teachers’ conceptual knowledge of a specific topic influence student learning?
- What patterns of interrelationship may be observed between a teacher's knowledge of mathematics as mediated through his/her instructional practice and her/his students’ emergent knowledge of mathematics?

Variables and Factors

The two central variables of the study that are examined for patterns are teachers' personal knowledge of rational numbers and students' emerging knowledge of rational numbers. Other factors that are examined for patterns that are related to the central variables include the nature of rational number knowledge and its construction by children, teachers’ knowledge for instruction including pedagogical content knowledge and beliefs, and classroom culture. Each of these variables is addressed in more depth in Chapter 2: Literature Review.

Rationale Justifying the Study

At the culmination of the New Math era, Begle (1979) examined studies of the effects of teacher attributes from the early 20th century up to 1979. Among those attributes was teachers' knowledge of mathematics. Nearly all of the studies investigated the mathematical knowledge of either preservice or inservice elementary school teachers. He noted that as of 1979 seventeen studies sought to describe how teachers' knowledge of mathematics contributed to teachers' effectiveness. From his study of the empirical
literature he concluded, "Once a teacher reaches a certain level of understanding of the subject matter, further understanding contributes nothing to student achievement" (p. 51). He corroborated his conclusion with the results of his own 1972 study of the relationship between ninth grade algebra teachers' knowledge of algebra and the relative achievement of their students. Five years later, Eisenberg (1977) replicated Begle's (1972) study and reported the same results. Both Begle's and Eisenberg's studies correlated the results of teachers' algebra knowledge as measured by a multiple choice test with a measure of teachers' effectiveness. Although Begle (1972) found a statistically significant but small correlation between teacher knowledge of algebra and students' growth in conceptual knowledge of the subject over the year, Eisenberg (1977) in his replication of Begle's study did not report a significant correlation for these two variables. Instead he found that the number of college mathematics courses—not grade point average in those courses—significantly correlated with the teachers' scores on the algebra test.

Ball, Lubienski, and Mewborn (2000) observed that in Begle’s (1979) analysis of the relationship between the number of courses teachers had taken past calculus and student performance, “He found that the extent of teachers’ mathematics course taking produced positive main effects on students’ achievement in only 10% of the cases and perhaps more jolting, negative main effects in 8%” (Ball et al., p. 442). He found that the relationship between a teacher having a major or minor in mathematics and their students’ performance “yielded positive main effects in 9% of the cases and negative main effects in 4%” (Ball et al., p. 442). Thus, the collegiate mathematics preparation of
teachers appeared to have little impact on their students’ performance and appeared as likely to be detrimental as beneficial.

Monk’s (1994) analysis of data from the Longitudinal Study of American Youth (LSAY) corroborated Begle’s work.

Teacher preparation was measured by a survey administered to teachers in which they reported the number of mathematics and science classes they took as undergraduates and in graduate programs. … In his [Monk’s] analysis, he found that five courses in mathematics, independent of the specific content covered, was the threshold beyond which few effects accrue (Ball et al., 2000, p. 442).

Monk also found that “whether a teacher had majored in mathematics had no effect on student performance. Whether a teacher had completed course work at the graduate level had either no effect or a negative effect on student performance” (Ball et al., 2000, p. 443).

In seeming contrast, the National Commission on Teaching and America’s Future (Darling-Hammond, 1996), citing various studies “showing that teacher knowledge makes a significant contribution to student achievement,” argued that “differences in teacher qualifications accounted for more than 90% of the variation in student achievement in reading and mathematics” (Armour-Thomas, Clay, Domanico, Bruno, & Allen, as cited in Darling-Hammond, 1996, p. 8). In another contrasting study, Ferguson (as cited in Ball et al, 2000)

found that when he controlled for family and community background influences, teachers’ scores on the state certification exam (Texas Examination of Current Administrators and Teacher [TECAT]) accounted for about 20-25% of the variation in students’ average scores on the state competency examination in mathematics and reading. Moreover, between third and seventh grade, teachers’ scores also predicted changes in students’ average scores over time. (p. 443)
The TECAT was a measure of basic skills and would not have measured the breadth or depth of teachers’ mathematical content knowledge. Neither Armour-Thomas et al. or Ferguson studies disputed the prior conclusion that teachers’ advanced study of mathematics at the collegiate level had limited influence on their students’ achievement.

All of the studies mentioned thus far have in common the assumption that the impact of teachers' subject matter knowledge on students' knowledge construction is a single measurable construct. Each study attempted to measure teacher knowledge by defining it as a number derived via a test, a grade point average, or a quantity of courses taken, and used similar methods to define student achievement. With such results we might be led to conclude with Begle (1979) that

> the effects of a teacher's subject matter knowledge . . . seem to be far less powerful than most of us had realized . . . Our attempts to improve mathematics education would not profit from further studies of teachers. (pp. 54-55)

Researchers who were dissatisfied with the previously described correlational studies suspected that “teachers’ knowledge of mathematics content mattered in ways that were masked by counting numbers of courses” (Ball et al., 2000, p. 443). For this and other reasons, subsequent studies examined teachers’ knowledge in different ways. For example, Ball et al. (2000) examined a collection of descriptive studies of teachers’ knowledge that looked at “specific mathematical topics rather than global conceptions of mathematical knowledge” (p. 443-444). Fennema and Franke (1992) cited descriptive studies that examined teaching:

> What teachers do in classrooms has been studied as a mediator between teachers' content knowledge and their students' learning. These studies have been, for the most part, conducted within the interpretive tradition, and have concentrated on providing rich descriptions of teachers in action in their own classrooms. (p. 149)
From such qualitative studies, inferences were drawn about the relationship between teachers' knowledge of subject matter and different aspects of the classroom by looking at case studies of particular teachers. In some studies, expert teachers were described or compared with less expert teachers, or a teacher was compared with him- or herself in domains where he or she had more or less knowledge.

To look at the interrelationships between teachers’ content knowledge and their practice, Fennema and Franke (1992) examined the case of a teacher who had been working intensively with the Cognitively Guided Instruction (CGI) group and was identified as an expert teacher. The study contrasted her practice when teaching addition and subtraction, for which she had rich pedagogical and content knowledge, with her practice when teaching rational numbers, for which she had comparatively less content knowledge. They found that her practice was substantially different with respect to the two topic areas, being less student-centered in the second case. In another study of the impact of teachers’ content knowledge on their instruction, Katzman (1996) found that during their lessons, teachers’ misconceptions and incomplete content understandings were apparent. Teachers with higher levels of knowledge and conceptual understanding of fraction content came closest to teaching conceptually, while the lower knowledge and procedurally-oriented teachers tended to teach procedurally. (abstract)

Ball et al. (2000) summarized research of prospective and practicing teachers’ knowledge of specific mathematical topics. Studies of preservice elementary and secondary teachers and inservice elementary teachers’ knowledge of whole number multiplication and place value (Ma, 1999) showed that teachers were typically successful in implementing the multiplication algorithm, but not in explaining the conceptual base for the procedure. Secondary teachers, who were mathematics majors, were no more able
to explain the algorithm than their elementary counterparts, though they were more successful in their application of it (Ball, 1990a). In the case of division, Ball et al. (2000) concluded “teachers use a predominately partitive (sharing) conception of division … Because those teachers did not tend to use the quotitive (measurement) interpretation of division, they often were not able to reason through problems involving division by zero, division of fractions and decimals, or dividing a smaller number by a larger number” (p. 446). A quarter of the teachers had difficulty computing with rational numbers in the Post, Harel, Behr, and Lesh (1991) study of middle school teachers’ rational number knowledge. Interviews of many who were able to compute correctly indicated that they had little facility in explaining their computations conceptually.

Leinhardt and Smith’s (1985) study found that teachers who thought that multiplication always resulted in a product larger than either factor failed to recognize the use of the multiplicative identity in forming equivalent fractions. The teachers in the study thought that the fraction resulting from the equivalent fraction algorithm was larger than the original fraction and similarly that the fraction resulting from simplification was smaller than the original. Studies of teachers’ Van Hiele levels (Mayberry, 1983; Swafford, Jones, & Thornton, 1997) report that most preservice elementary teachers surveyed functioned at the recognition or analysis levels and that most practicing middle school teachers surveyed functioned at or below informal deduction. Studies that examine teachers’ understanding of the relationship between area and perimeter (Ball & Wilson, 1990; Heaton, 1992; Ma, 1999) revealed that teachers believe that there is a direct relationship between the area and perimeter—as one increases the other does also and
vice versa. Finally, studies of teachers’ knowledge of proof show that “teachers are prone to accept inductive evidence, such as a series of empirical examples or a pattern, as sufficient to establish the validity of a claim (Ball & Wilson, 1990; Ma, 1999; Martin & Harel, 1989; Simon & Blume, 1996)” (Ball et al., 2000, p. 447).

In summary, research of U.S. teachers’ knowledge of mathematics seems to indicate that high level mathematics study (beyond five collegiate courses in mathematics) has little positive effect on student achievement. Notwithstanding, there does appear to be a positive correlation between teachers having received preparation to teach mathematics and student achievement. Studies that focus on teachers’ knowledge and instructional practice show teachers’ mathematical knowledge and beliefs about learning mathematics influence their instructional practice (Ball, 1991; Thompson, 1984). Finally, research of teachers’ knowledge of specific instructional topics indicates that U.S. teachers’ mathematical knowledge tends to be procedural. U.S. prospective and practicing teachers are often unable to give adequate conceptual accounts for the mathematical processes that they are or will be teaching. Case studies that focus on providing in-depth descriptions of teachers’ knowledge, knowledge for instruction, and their students’ construction of knowledge of content may illuminate patterns that could suggest areas of further investigation and potential relationships between teachers’ knowledge of content and students’ construction of content knowledge within the context of instruction.
CHAPTER 2

LITERATURE REVIEW

Because the intended goal of the study is to identify patterns that could suggest possible influence or interrelationships of teachers’ content knowledge and students’ learning or developing content knowledge, it is problematic to begin with a theoretical framework. Nevertheless, it is necessary to provide the background theory and research that informed the design of the study and through which the data were inevitably interpreted. In particular, it is necessary to discuss the theoretical underpinnings of the factors that were identified as possible significant contributors to the relationship between the two main variables. The factors that are examined as indicators of and contributors to the interrelationships between the dependent variable (students’ developing content knowledge) and the independent variable (teachers’ content knowledge) include the nature of rational number knowledge and its construction by children, teachers’ knowledge for instruction including pedagogical content knowledge and beliefs, and classroom culture.
Rational Number Knowledge: Internal Knowledge Structures and Knowledge Construction

To look at patterns within/between knowledge of rational numbers of both teachers and students requires a general theory of knowledge and knowledge acquisition and specific theories of knowledge and knowledge acquisition with respect to the content being learned.

A General Theory of Knowledge Structures

Several learning theories of knowledge and knowledge acquisition provide the foundation for the study. One contributing theoretical perspective is that knowledge is comprised of internal, structured representations as described by Hiebert and Carpenter (1992) from their examination of work in the field of cognitive science. Hiebert and Carpenter assert that we can assume that some relationship exists between external and internal representations and that these internal representations can be related or connected to one another in useful ways. Hiebert and Carpenter suggest,

the form of an external representation (physical materials, pictures, symbols, etc.) with which a student interacts makes a difference in the way the student represents the quantity or relationship internally. Conversely, the way in which a student deals with or generates an external representation reveals something of how the student has represented that information internally. (p. 66)

Thus, the external representations that teachers and students use in communicating mathematical concepts can contribute to the construction of and potentially reveal their internal representations, i.e., their understanding of a mathematical topic.

Another aspect of the theory of internal representations is that internal representations can be connected (Hiebert & Carpenter, 1992). Hiebert and Carpenter
propose that when relationships between internal representations of ideas are constructed, they produce networks of knowledge. Whether the networks of knowledge are described as vertical hierarchies (trees) or webs, a mathematical idea can be understood as a part of an internal network—more specifically, its representation is part of a network of representations. The number and the strength of the connections determine the degree of understanding. A mathematical idea is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. The goal of instructional can be defined as helping students build a coherent mental network in which all pieces are joined to others with multiple links. Constructing relationships within a representational form often increases the cohesion and structure of the network. Thus, the theory of connections underlies the assumption that one can compare teachers’ and students’ understanding of a topic as more and less sophisticated.

Another learning theory from cognitive science, schema theory, also contributes to the study (Davis, 1984; Riley, Greeno, & Heller, 1983; Schank & Abelson, 1977). Generally, schemas are relatively stable internal networks that are constructed at a relatively high level of abstraction or generality. Schemas serve as templates that are used to interpret specific events. That is, they are abstract representations to which specific situations are connected as special cases. Kieren’s (1988) portrayal of rational number knowledge is a special case that specifically contributes to the current study. Kieren describes "an image of ideal personal rational number knowledge" (see Figure 2.1) as a model of how experts might structure their knowledge of rational numbers. At the lowest level, closest to interactions with the external environment, are mental representations of
empirical data. These local representations are connected to more general principles or constructs that, in turn, are connected to higher-level constructs. Kieren suggests intuitive ideas of partitioning, of making quantities equivalent, and of forming units (that can be subdivided) as constructs at a medium level of generality, with a more formal notion of multiplicative structures as a highly general construct.

An alternative view of children’s knowledge construction to the networks of representations described by Hiebert and Carpenter (1992) that also contributes to the study is Piagetian scheme theory. Steffe (2001) describes von Glasersfeld’s reformulation of Piaget’s concept of scheme as consisting of three parts. The first part is an “experiential” or “activating” situation as perceived by the child. The second part is a particular activity that the child associates with the activating situation. The third part is the result or product of the child’s activity in response to the activating situation. Wadsworth (1996) describes two types of knowledge construction within Piagetian learning theory: assimilation and accommodation. Assimilation occurs when a learner perceives a new “experiential” or “activating” situation and uses an existing scheme to generate the product of that scheme. Accommodation occurs when a learner creates a new scheme or modifies an existing scheme because the “experiential” or “activating” situation could not be assimilated into existing schema. In assimilation, the category of perceptions that activate a scheme is modified/enlarged. In accommodation, new schemes and thus new products are created. There are similarities between Hiebert and Carpenter’s (1992) framework and Piagetian schema theory. Both propose that there are internal knowledge structures. Although Piagetian schemes do not use the language of
networks, nevertheless they can be seen as networks. For example, the product of one scheme can be the activating situation of another scheme. An activity that operates on the “activating” situation in one scheme may be used in another scheme with a different category of “activating” situation and produce a different type of product. Categories of “activating” situations may share specific perceptions.

Whether using the theory of internal representations or Piagetian scheme theory, it can be said that “students construct their own internal knowledge structures. A crucial aspect of students' constructive process is their inventiveness (Piaget, 1973; Resnick, 1980; Wittrock, 1974)” (Hiebert & Carpenter, 1992, p. 74). Inside or outside of school, whether students invent their own strategies or attempt to mimic those they observe others use to solve a variety of problems, they are applying their own internal knowledge structures in the process. Students' internal knowledge structures do not always reflect productive mathematics. If students are working with written symbols unconnected to richer networks of knowledge, their internal knowledge structures may produce flawed algorithms (Hiebert & Wearne, 1985). However, if students' internal knowledge structures reflect well-connected networks of viable mathematical processes and concepts, the resulting mathematics can be productive (Hiebert & Carpenter, 1992).

In summary, for the study described herein, constructivist theories of learning are used to identify patterns in teachers’ knowledge and students’ knowledge and knowledge acquisition. Constructivist theories posit that the internal knowledge structures of each individual differ and therefore it is anticipated that teachers’ and students’ internal knowledge structures will differ. Since each individual constructs knowledge based on
modifications of previous internal knowledge structures, it is anticipated that individual students may interpret the stimuli present in the instructional situation differently from each other and from those anticipated by their teachers. Therefore, observations of differences in researcher-constructed descriptions of teachers’ knowledge and students’ knowledge and knowledge acquisition would be attributed to differences in teachers’ and students’ construction of internal knowledge structures.

*Perspectives on Rational Number Knowledge*

Although the learning theory through which teachers’ and students’ knowledge and knowledge acquisition will be described has been discussed in the preceding section, an adequate description of teachers’ and students’ knowledge and knowledge acquisition requires an examination of the content that is being taught/learned. Again there are two perspectives for understanding the content (i.e., rational number knowledge): (a) an analysis of the concepts and procedures and their relationships that constitute the topic of rational numbers (in other words, content or knowledge of rational numbers) and (b) a study of children’s construction of rational numbers (acquisition of rational number knowledge). The former examines rational number knowledge in terms of a framework for an idealized personal knowledge of the content. It is idealized because the framework attempts to represent the content in all of its interconnected concepts and procedures from the perspective of a potentially completed product. The former (an examination of the content or knowledge of rational numbers) is more closely aligned to Hiebert and Carpenter’s (1992) framework for internal knowledge structures. The latter (the study of children’s construction of rational numbers) examines rational number knowledge as the
construction of children from their prior constructions of number. Rather than focusing on the content being constructed, the focus is on children making sense of (assimilating or accommodating) new experiences that relate to the content, thus building up their own sense of rational number from their prior number experiences. The latter (the study of children’s construction of rational numbers) makes use of Piagetian scheme development.

Both perspectives contribute to illuminating interrelationships between the teachers’ content knowledge and students’ learning (developing) content knowledge. The section “Structure of the Rational Number Domain” and its subsection (“The Ideal Network of Personal Rational Number Knowledge”) provide a theoretical foundation for examining rational number knowledge construction from the perspective of the content. The section “Children’s Construction of Rational Numbers” and its subsections (“Analysis of Schemes Contributing to the Construction of Rational Numbers,” “Integration of Numerical and Global Quantitative Schemas,” and “A Comparison of Unit Fraction Scheme Construction in Two Instructional Contexts”) provide a theoretical foundation for examining rational number knowledge from the perspective of children’s knowledge construction.

Structure of the Rational Number Domain

Rational numbers are mathematically defined as elements of an infinite quotient field consisting of infinite equivalence classes with rational numbers being defined as the elements of the equivalence classes. However, this definition is the culmination of centuries of work by mathematicians and is a highly abstracted point of view of the domain. It is the construction of a culture. As Freudenthal (1983) says, "our mathematical
concepts, structures, ideas have been invented as tools to organize the phenomena of the physical, social and mental world" (p. ix). What is needed by educators is an understanding of the rational number domain that shows how knowledge of rational number may be constructed by novices—a model that can inform the teacher how she/he can guide the learner to "step into the learning process of mankind" (Freudenthal, p. ix).

In their review of the work done in identifying the various elements of rational number understanding, Behr, Harel, Post and Lesh (1993) note that Kieren (1976) was the first to introduce the idea that rational number is a composite of several different constructs. Kieren suggests "to understand the ideas of rational numbers, one must have adequate experience with their many interpretations" (p. 102). Kieren's initial list of rational number constructs included the following: fractions; decimal fractions; equivalence classes of fractions; \( \frac{p}{q}, p, q \) integers, \( q \neq 0 \) (ratio numbers); multiplicative operators; and elements of an infinite ordered quotient field. Other analyses of rational numbers of several researchers since Kieren's initial one (Behr, Lesh, Post, & Silver, 1983; Ohlsson, 1988) are summarized by Behr et al. (1993) as distinguishing among the following concepts: "fractions as a part-whole relationship, rational numbers as the result of the division of two numbers, as a ratio, as an operator, and as a probability" (p. 997). A more recent analysis by Kieren (1988) presents the rational number concept as comprising four subconstructs: measure, quotient, ratio number, multiplicative operator.

Each of these lists of rational number subconstructs share many of the same characteristics. Although Ohlsson (1988) criticizes them for not being exhaustive and including things they should not, nevertheless they are informative in what they do note
and how they interconnect. They show that there is considerable agreement on what subconstructs may underlie the rational number construct. Behr et al. (1992) conclude that five subconstructs of rational number consistently appear in the literature: part-whole, quotient, ratio number, operator, and measure. These subconstructs "have stood the test of time and suffice to clarify the meaning of rational number" (p. 298).

The Ideal Network of Personal Rational Number Knowledge

Kieren (1988) has developed a theoretical model of mathematical knowledge building for rational numbers using the subconstructs that he and others have described as being fundamental to understanding rational number. Figure 2.1 (Kieren, 1993) illustrates how rational number knowledge can be connected and built up. The model is comprised of six levels. The first level, comprised of partitioning, equivalencing, and unit forming, is based upon constructs that are very local and close to the fact level. The second level is comprised of the four constructs of rational number: measure, quotient, ratio, and operator. These are four of the five subconstructs listed by Behr et al. (1992) as being foundational to understanding the rational number domain. The part-whole sub-construct has been subsumed within the first level and ratio constructs. Knowledge of the scalar and functional relationships of number comprise the third level upon which the fourth level of the more formal constructs of fraction and rational number equivalence depend. The fifth, and next-to-last level, synthesizes the lower levels to produce the general construct of the multiplicative conceptual field. The topmost level is the construction of the rational numbers as elements of an infinite quotient field.
Analysis of the Subconstructs of Rational Number

Level Zero: Initial Constructs. Kieren (1988) describes the first level of constructs as being very connected with the fact plane. These unnamed constructs represent local bits of situation-specific knowledge. They are by nature isolated from each other. Higher-level constructs are connected to the fact plane or referents through these constructs and are constructed by building on the invariants recognized in lower level constructs. In his discussion of the constructs from which the construct fraction is derived, Freudenthal (1983) makes many references to the referent, or represented world. How these constructs interact with the next level will be discussed in the following sections.
Level One: Partitioning, Quantitative Equivalence, and Unit Forming. Kieren (1988) described the three constructs of partitioning, quantitative equivalence and unit forming as being the first with which young learners are able to "solve" certain fractional relation problems. He conceives them as representable by an additive fractional language and as forming the basis for unit fraction knowledge. Behr, Lesh, Post, and Silver (1983) also refer to these three concepts as the basic thinking tools for understanding rational number. They are closely linked, as Carpenter, Fennema and Romberg (1993) note—“it is units of same kind that are partitioned” (p. 4) into equivalent quantities.

Partitioning. Streefland (1991), in developing his course in teaching about fractions, used two realistic models on which to build students' constructions of fraction—fair sharing and seating arrangements. Through his use of these models, he developed a course to guide students' connecting of their already developing schema about situations in which informal fraction knowledge is used with more sophisticated constructions of rational number. Key to this course is allowing students to use and explain how they partition in order to arrive at solutions to problems and connection of these partition schemes to symbol systems that the students develop, as illustrated in Figure 2.2 for the problem “divide three pizzas among four children.”
1. Partition each pizza one by one:

![Pizza diagrams](image)

Everyone gets 1/4 (pizza) + 1/4 + 1/4, that is 3 X 1/4, or 3/4.

2. Partition the first two pizzas, and then:

![Pizza diagrams](image)

Everyone gets 1/2 (pizza), and later 1/4 more, that is 1/2 + 1/4, or 3/4.

3. Partition all three pizzas at once:

![Pizza diagrams](image)

Two children get 1 – 1/4 (pizza) and two children get 1/2 + 1/4, or 3/4.

Figure 2.2. Children’s partitioning schemes (Streefland, 1993, p. 291).

Quantitative Equivalences. As asserted previously, it is necessary that the construct of quantitative equivalence be integrated with the partitive construct if learners are to construct the concept of fraction. Quantitative equivalence is based on the foundation of the construct of counting. This is a powerful construct that learners bring with them from their constructions of whole number and of its operations. However, with
the counting construct come other natural number constructs that learners may inappropriately apply to their rational number reasoning. Streefland (1991, 1993) terms these inappropriate applications of whole number constructs to rational number reasoning as N-distracters. He observes that N-distracters are most powerful when learners have not yet formed an adequate understanding of fractions. "Operating on a symbolic level with numerators and/or denominators without considering their conceptual relationship indicates that a student's concept of fractions is inadequately anchored" (1993, p. 300).

Resistance to N-distracters grows as learners find arguments to refute these incorrect operations. These arguments arise from connections that are made between the developing constructs and the represented world from which they are constructed. Informal strategies employed by children and adults "make heavy use of the situation or context with its concrete and visual supports, rather than depending on symbolic manipulations" (Hiebert & Behr, 1988, p. 9). At the beginning of a study by Mack (1993), sixth grade students, who initially could solve problems successfully in a real-life setting, consistently explained their solutions using informal reasoning. These same students, on the other hand, when given problems represented symbolically that were similar to the real-life problems, were unable to solve the problems. Their explanations were faulty and were given in terms of formal symbols and algorithmic procedures. Using an instructional method that emphasized symbolic representations to problems presented in the context of familiar situations that draw on informal knowledge, Mack's (1993) students began to relate fraction symbols and procedures to their informal knowledge. Using their informal knowledge they were able to correct misconceptions in
their symbol arguments. Researchers in other content domains (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Cobb, Yackel, & Wood, 1988; Lampert, 1986) have also found results that show matching of symbolic representations and familiar contexts characterizes the way learners build on their informal knowledge.

The equivalence construct is based on other proto-quantitative constructs than counting. Freudenthal's (1983) fraction as “fracturer” construct relies on equivalencies other than quantitative. He states that the experiential aspects of fractions are based on comparing quantities by sight, feel, folding, or weighing (measuring activity). According to Freudenthal (1983), equality of parts can be judged by sight or by feel or by more sophisticated means. Equivalence defined as congruence also plays a part in the informal construction of equivalence. Again, these are constructs that are closely tied to the learner's experience with the represented world.

Unit Forming. The construct of unit forming, like partitioning and equivalence, underlies informal reasoning about fractions and is fundamental to developing the more abstract concepts of rational number at the next level. Mack (1993) discusses the development of the unit construct. She observes that initially learners construct units as the single entities invoked in addition and subtraction of natural numbers. The next level of sophistication in the concept of unit arises when learners develop the construct of multiplication and division of natural numbers. Informal understandings of these operations rely on constructing composite units of natural numbers that are iterated to form the product or quotient (Steffe, 1988). There are changes in the unitizing construct that Mack (1993) notes as being necessary to the rational number construct. The first is

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the application of the concept of unit to continuous as well as discrete quantities. This requires the application of partitioning and equivalence as congruence. At this level of the construct, the symbol representation \( a/b, b \neq 0 \), is a number comprised of the unit \( 1/b \) counted \( a \) times. The next change in the unit concept relates to the quotient, ratio, and operator constructs of rational number knowledge. These require conceiving the comparison of units as being a unit. In other words, the numerator represents one unit and the denominator another unit, and their comparison is the unit that is constructed. This final unit construction corresponds with the intensive measure that Schwartz (1988) describes and the multiple proportion that Vergnaud (1983) describes.

The Rational Number Project in an unpublished study (Behr et al., 1992) designed a study to examine the flexibility of the unit concept across continuous and discrete contexts and complexity of numerical relationships. It focused on the knowledge structures that children use “to solve problems of the form “\( x \) is \( a/b \) of \( y \) where \( x \) and \( a/b \) are given and \( y \) is the unit whole; problems of the form \( y = c/b \) with \( c \) greater than \( b \)” (p. 306). Fifth-graders were given visual tasks in which, given a fractional part, they were asked to produce the unit whole, some part of the unit whole, or some multiple of the unit whole. The tasks varied in whether the fractional part was composed of continuous or discrete quantity types. Behr et al. (p. 306) categorized children’s responses into five solution styles:

1. “Unit fraction decomposition and composition”—children partitioned the given fractional part into unit fractions of the form \( 1/m \) (i.e., \( 3/8 \) decomposes
into three $\frac{1}{8}$-units) and reconstructed the unit whole by iterating the found unit fraction.

2. “Unit part decomposition and composition”—children partitioned the given fractional part into unit parts of the form $1$ of $m$ parts, written $1/m$ (i.e., $3/8$ decomposes into three $\left[1\text{ of }8\text{ unit parts}\right]$ parts) and reconstructed the unit whole by iterating the found unit part.

3. “No unit part of fraction or unit fraction decomposition”—children were unaware that the given fractional part was decomposable into either unit fractions or unit parts.

4. Children redefined the given fractional part as the whole.

5. Children iterated the fractional part as though it were the unit part.

Behr et al. observed that the solution styles 1 and 2 led to correct results and solution styles 3 to 5 did not. They note that reconstructing a unit whole given a fractional part is more difficult than the common task of finding the fractional part given the whole.

Students in their data set had less difficulty when the unit whole was composed of a discrete quantity type. Since this set of tasks reveals whether children can construct the meaning of $a/b$, $b$ not 0, as the unit $1/b$ counted $a$ times, it lies at the heart of unit forming and can indicate whether it is reasonable to expect children to begin constructing the next level of rational number constructs.

To summarize, the three constructs of partitioning, quantitative equivalence, and unit forming are close to the referent level of the rational number construct. They are
implicated in informal reasoning about rational number situations and link the referent level, children’s initial experiences, to higher-level rational number constructs.

*Level Two: Measure, Quotient, Ratio, Operator.* The constructs of measure, quotient, ratio, and operator are the middle level constructs of the rational number domain. In discussing rational number understanding, they are the most frequently mentioned of all the constructs in Kieren's (1988) model. They are the first level of formal rational number reasoning.

*Measure.* Kieren (1980) defines the measure construct as being the process by which a number is assigned to a length or region—planar or spatial. The assignment is made through the act of "covering" the length or region by an arbitrary unit. The measure construct requires an ability to conceive of a unit, to partition, and to recognize quantitative equivalence. The measure construct provides a natural setting for the development of the addition construct of rational numbers.

Freudenthal (1983) describes the construct of magnitude with reference to fraction that parallels Kieren's measure construct. Freudenthal sees the referent of this construct as being the process of distribution, but distribution of an arbitrarily divisible object. Requirements for the construction of magnitude include equivalence classes of units so that arbitrary partitionings can be made. Another requirement is a way of correlating units such that an addition can be defined. A third requirement is that there be an unrestricted availability of objects to represent the unit of the magnitude that will allow addition to be unrestricted. A final requirement is a way to arbitrarily partition the object into partial objects so that natural number division is defined.
Quotient. Kieren (1980) described the quotient construct as allowing for the quantification of the result of dividing a quantity into a given number of parts. Although learners may be able to partition a whole into four parts and recognize the grouping of three of these as representing the quantity 3/4, it is a different matter for them to recognize that three wholes partitioned into four parts would yield the same result if they are asked to divide three by four or fairly portion three items for four people.

The quotient construct as described by Behr, Harel, Post, and Lesh (1992) is composed of two possible subconstructs—partitive division and quotative division. The partitive division construct arises in one of two ways, both of which arise from the partitioning subconstruct. Parts #1 and #2 in the diagram from Streefland (1993) in Figure 2.2 demonstrates these two partitions. The quotative division construct is divided into four subconstructs based on two interpretations each of the numerator and denominator. In each case, either the numerator or the denominator is assigned to be a single composite unit and then is partitioned according to the other in order to derive the result. An example of a quotative division is #3 from Streefland's diagram in Figure 2.2. Behr et al. derived these constructs from operations with manipulative materials.

Ratio. Kieren's (1980) definition of the ratio is the comparison of a designated quantity of one set or measure space with that of another. This construct subsumes the part-whole construct whereby a whole is partitioned into n equal amounts and m of these are designated, thus yielding the ratio of parts to the whole as being m/n.

Operator. For Kieren (1980), the operator construct is a mechanism by which a mapping is constructed from one set or measure space onto another. For example 3/4
maps a set of 12 objects onto a set of 9 objects. Behr, Harel, Post, and Lesh (1992, 1993) describe the operator construct as being derived from three subconstructs or interpretations. These are designated: (a) the duplicator and partition reducer, (b) the stretcher and shrinker, and (c) the multiplier and divisor. They describe the duplicator and partition reducer as an exchange function. For example, the fraction $2/5$ operates on the operand and exchanges every set of 5 units with a set of 2 units. The stretcher and shrinker interpretation is the same as that described by Kieren. For the fraction $m/n$, it is a mapping of $x(m$-units) onto $y(n$-units). The description of the multiplier and divisor interpretation is as rich as the terms multiplier and divisor might imply. Behr et al. define multiplier as repeat adder, times as many factors, first factor in a cross-product and other ways. The divisor definition includes the partitive and quotative constructs as well as other ways to construct divisor. The numerator, as would be expected, is assigned the role of multiplier and the denominator as divisor.

**Level Three: The Function and Scalar Constructs.** The level three constructs in Kieren’s (1988, 1993) model contribute to the development of proportional reasoning, the multiplicative conceptual field (Vergnaud, 1983, 1988, 1994), in the process of developing rational numbers as a quotient field. Vergnaud (1988) identified the multiplicative conceptual field as “all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide” (p. 141). Figure 2.3 shows the four schemas that Vergnaud (1994, p. 50) uses to classify simple (direct) proportion problems. Schema 1 is the schema for multiplication—find $x$, given $a$ and $b$. 

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Figure 2.3. Schemas for simple (direct) proportion problems.

Schemas 2 and 3 represent partitive and quotitive division, respectively (find \(f(1)\) or \(x\), given \(a\) and \(b\)). Thus, 1-3 are schemas of the part-whole type of problem. Schema 4 represents “rule of three” problems—find \(x\), given \(a\), \(b\), and \(c\). Two solution processes that Vergnaud (1983, 1988) identified from children’s reasoning processes for solving each of the four simple proportion schemas are the two level-three constructs that Kieren (1988) identified—scalar and function.

*The Scalar Construct.* The scalar construct is the simplest construct to arise in children’s solution processes for simple (direct) proportion problems. According to Vergnaud (1988), it is an application of the isomorphic property of the general linear function, \(f(nx) = nf(x)\). Hence, \(n\) is the factor relating the numerator and denominator in each measure space of the proportion. It \((n)\) is a scalar that has no magnitude or dimension. Problems that use the scalar construct are those in which the scalars are interpreted as part-whole fractions (< 1) that relate two quantities of the same nature or from the same measure space and those in which the scalars are interpreted as part-part ratios (< 1 or > 1) that relate quantities that are from the same measure space but are not
included within each other (Vergnaud, 1988). Important to this construct is the fact that
the numerator and denominator of the fraction are the same type of unit, which is why the
scalar has no unit (dimension) as it cancels out. The scalar construct is related to the
operator, quotient, and ratio constructs described in level two of the model.

The Function Construct. The function construct is a more complex construct that
is applicable to simple (direct) proportion problems, but which children rarely apply to
problem situations unless they have abstracted proportionality. Kieren (1993) indicates
that the function construct is related to all four level two constructs in his model
(measure, quotient, ratio, operator). Vergnaud (1983) expresses the function construct as
\( f(x) = mx \). Hence, \( m \) is the factor (function operator) relating the numerator in the first
measure space (M1) with the numerator in the second measure space (M2) and similarly
for the denominators. Problems that use the function construct are those in which the
factors (\( m \)) are interpreted as rates (< 1 or > 1) that relate two quantities of different
natures or measure spaces (Vergnaud, 1988). Because the related quantities are from
different measure spaces, function operators (\( m \)) are quotients of dimensions. Vergnaud
(1983) hypothesizes that the composite dimensionality of function operations creates
conceptual difficulties for children. However, he observes that the function construct is
the natural situation in which the concept of an infinite class of order pairs (equivalence
class) arises.

Level Four: Formal Equivalence and Multiplicative Structure of Rational
Numbers. This level represents the synthesis of the more formal constructions of fraction
and rational number equivalence from the function and scalar constructs of the previous
level (Behr et al., 1992). Vergnaud (1983) theorizes that the concept of rational numbers (the set of equivalence classes of ordered pairs of whole numbers) is the synthesis of the measure (magnitude, from level two), scalar, and function interpretations of fractions or ratios. As can be seen from the discussion of level three constructs, the formal equivalence construct arises from the abstraction of the function construct. According to Kieren’s model (1993), the multiplicative structure of rational numbers is a synthesis of both of the level three constructs (scalar and function).

**Level Five: The Additive and Multiplicative Groups.** At this level, the subconstructs are the formal constructs of addition/subtraction and multiplication/division of rational number. Understanding at this level should be accompanied by the ability to create and relate symbolic rational number problems with one another and with possible referents.

**Summary**

Kieren's model (1993) of personal rational number knowledge is thorough and useful for examining the concepts that are necessary to a construction of the rational number concept. It provides a framework for describing the knowledge within an intended curriculum, available through instruction, and for comparing and contrasting teachers’ knowledge of content. However, it may not be sufficient for describing children’s construction of rational number concepts. The following section, “Children’s Construction of Rational Numbers,” suggests possible patterns of construction and the possibility that children’s construction of rational number concepts may be influenced by instruction.
Children's Construction of Rational Numbers

In contrast to examining knowledge construction from the perspective of the constructs and their relationships that comprise an idealized knowledge of rational numbers as presented in preceding section, this section examines knowledge construction from the perspective of children’s construction of knowledge, a Piagetian developmental model. Steffe (2001) and Moss and Case (1999) each propose theories that view the development of rational number knowledge as a series of assimilations and accommodations of children’s schemas for the natural numbers. Steffe (2001) theorizes that children’s fractional schemes emerge as accommodations to their numerical counting schemes. Moss and Case (1999) theorize that children’s construction of rational numbers parallels their construction of whole numbers in that it is the result of their coordination of numerical and global quantitative schemas. A comparison of Steffe’s (2001) and Moss and Case’s (1999) research suggests how varying instructional contexts may contribute to differences in children's knowledge construction.

Analysis of Schemes Contributing to the Construction of Rational Numbers

Steffe (2001) proposes that children’s fractional schemes can emerge as accommodations in their numerical counting schemes. He terms his proposal “the reorganization hypothesis because when a new scheme is established by using another scheme in a novel way, the new scheme can be regarded as a reorganization of the prior scheme” (p. 267). Steffe describes numerical schemes as consisting of three parts. The first part is an “experiential” or “activating” situation as perceived by the child. The second part is a particular activity that the child associates with the activating situation.
The third part is the result or product of the child’s activity in response to the activating situation. Steffe reiterates that the scheme is a construction of the child rather than the observer. He also clarifies that although “the first part of the scheme is established by assimilation,” it is not “a structure [that] somewhere ‘exists’ in the mind in its totality and as an object” (p. 268). Instead, he theorizes “operations used in past activity are used to assemble a recognition template that is used in creating an experiential situation that may have been experienced before” (p. 268). Steffe’s hypothesis arose from his analysis of observations of a teaching experiment on rational number development in which three children were instructed within a mathematical microworld (TIMA—Tools for Interactive Mathematical Activity) (Olive, 1999).

**Reorganizations of the Explicitly Nested Number Sequence Scheme: Connected Number Sequences**

From his observations of students, Steffe (2001) proposed three parallel schemes (the equi-partitioning scheme, the equi-segmenting scheme, and the simultaneous partitioning scheme) that reorganized the explicitly nested number sequence scheme to produce the connected number sequence. For each of the three schemes, the activation template of the simultaneous partitioning is the explicitly nested number sequence and the result is a connected number. Steffe highlights three operations as key to the reorganization of the explicitly nested number scheme: (1) the child’s ability to form an image of a particular number such as twenty-six that is both the iteration of twenty-six singleton units and a composite unit of twenty-six items; (2) the child’s ability to disembend an image of the continuation of a number such as seventeen in twenty-six and...
to join “the items of the image into a composite unit structure that contains it” (p. 269); (3) the *recursive operation* of taking the number sequence as an object of the counting operation. The first two operations are the operations of the tacitly nested number sequence scheme, and become the activation template for the explicitly nested number sequence that differs from the prior scheme by its ability to take the number sequence itself as an object for operating on. The result of the explicitly nested number scheme is a composite unit of units of discrete quantity. The three schemes (equi-partitioning, equi-segmenting, and simultaneous partitioning) that can produce a connected number sequence reorganize the explicitly nested number sequence scheme for application to a continuous quantity.

As just stated, the equi-partitioning, equi-segmenting, and simultaneous partitioning schemes reorganize the explicitly nested number sequence scheme for application to a continuous quantity. Counting schemes arise in the environment of discrete or countable elements. The explicitly nested number sequence scheme operates on abstract unit images and thus no longer depends on the presence of perceptual objects for counting. Nevertheless, the abstract unit items that are counted and form a composite unit of units by the explicitly nested number sequence scheme still have the quality of being discrete items if not perceptual items. Thus, the application of the explicitly nested number sequence scheme to a continuous quantity requires its reorganization. The activation template of the equi-segmenting and simultaneous partitioning schemes was the explicitly nested number sequence or the image of composite units applied to a continuous quantity, but the result was what Steffe (2001) terms a “connected number.”
A connected number differs from a composite unit of units, the result of the explicitly nested number sequence, in being a continuous quantity rather than a composite discrete quantity. Where the three schemes reorganize the explicitly nested number scheme to apply to a continuous quantity is in the operations or activity that produces the new connected number sequence. Steffe identifies the operation or activity of the equipartitioning scheme as the composite act of partitioning “a continuous unit into any-size part and iterating any of the parts to reconstitute the whole” (p. 272).

Children’s Construction of Unit Fractional Schemes

Steffe’s (2001) research indicates that development of the connected number sequence through the reorganization of the explicitly nested number scheme is an important step toward the development of rational numbers. Steffe explored several schemes for their possible contributions to the development of rational numbers. Among them were the often posited multiplicative schemes. Because children’s development of multiplication and division concepts is often suggested as a precursor to their development of rational number understanding, Steffe examined the possibility and concluded that the multiplication schemes often described as necessary are not directly involved in the construction of schemes related to rational numbers.

Partitive Unit Fraction Scheme: An Initial Fractional Scheme. In contrast to the multiplicative schemes, the initial unit fraction scheme that children often construct is the partitive unit fraction scheme (Steffe, 2001). The result of the partitive unit fraction scheme is an iterable fraction unit, which later in this study is termed a singleton discrete unit to differentiate it from a unit part or unit fraction. The “purpose of the scheme was to
divide the connected number, one, into so many equal parts, take one out of those parts, and establish a one-to-many relation between the part and the partitioned whole” (p.292). Essentially, it is the scheme for the often-identified initial part-to-whole relationship.

Steffe differentiates the partitive unit fraction scheme from the initial equi-partitioning scheme which it appears to use as a subschema by the “explicit numerical one-to-many comparison (one-to-ten [or one ‘out of’ ten]) and the explicit use of fractional language ‘one-tenth’ to refer to that relation” (p. 292). It can be considered a generalizing assimilation of children’s explicitly nested number sequence. It should not be construed that the partitive unit fraction scheme constitutes the recognition of a multiplicative relationship between the unit part and the connected number that is the whole. Students who have constructed the partitive unit fraction scheme may recognize that 1/5 is one of five parts of a whole and conversely that 5/5 makes the whole, but are unable to create a length five times which will be a given whole. The construction of the multiplicative relationship between a unit part and the whole thus appears to be a component of the construction of the concept of unit fraction, but not sufficient for it. Also, students who have constructed the partitive unit fraction scheme may be able to recognize that twice 4/5 is 8/5, but not that 8/5 is one whole and 3/5.

*Necessary Errors*. As a result of students’ construction of the partitive unit fraction scheme, attempts to engender equi-partitioning for connected numbers and other unit fraction producing schemes lead to what Steffe (2001) terms necessary errors. Steffe argues that the children are aware of the multiplicative relationship between the composite units that were being iterated to form the whole length and the number of
iterations required to form the whole. They know that the appropriate fraction language used the number of iterations to define the part size. Their construction is a modification of the partitive unit fraction scheme that utilized the explicit nested number sequence in its operation to produce the requested fraction language. Necessary errors do not occur as a result of misinterpretations of a situation; rather, they occur as a result of children’s operating schemes. They are an indication of the knowledge that children have thus far constructed.

Equi-Partitioning Scheme for Connected Numbers and the Fractional Connected Number Sequence (Unit Parts). Steffe’s (2001) conception of the equi-partitioning scheme for connected numbers corresponds to Confrey’s (1994) definition of a split of a split. The activating template would be a connected number, one or greater than one, that is to be partitioned. The activity would be splitting to form a partition of the connected number. The product would be the fractional connected number sequence. Steffe describes how a child might apply this scheme to a connected number such as five. The child could distribute a six-partition of the connected number across each of the five elements, observing that each part is 1/30 of the original connected number. The child might select 1/30 from each of the five partitioned parts and establish that the selection produced 5/30 of the original connected number. Steffe hypothesizes that “partitioning the elements of a connected number into composite units of equal numerosity is necessary in splitting a split” (p. 290).

Steffe (2001) hypothesizes that the scheme that produces the fractional connected number sequence is more than a generalizing assimilation of the explicitly nested number
sequence scheme. For example, there is a discontinuity between the initial equi-
partitioning scheme and the initial splitting operation that Steffe defined. The operations
of partitioning and iteration occur in both, but they are performed sequentially in the
initial equi-partitioning scheme and simultaneously in the initial splitting operation.
Steffe believes that the composition of partitioning and iterating is produced by an
interiorization of the two operations. Rather than reorganization, there is a metamorphic
accommodation of the two schemes being interiorized, producing vertical learning. Steffe
equates metamorphic accommodation with the strong form of Piaget’s reflective
abstraction. The strong form of reflective abstraction involves the interdependent but
distinct processes of “projection onto a higher plane of what is taken from the lower
level, hence a ‘reflecting,’ and of ‘reflection’ as a reorganization on the new plane”
(Steffe, p. 27). Although Steffe does not explicitly state it, the equi-partitioning scheme
for connected numbers is apparently a metamorphic accommodation that replaces the
sequential partitioning and iterating activity of the initial equi-partitioning scheme with
the initial splitting scheme.

Steffe (2001) defines the fractional connected number sequence as

a number sequence that can be used to imagine partitioning a connected number,
one, into any specific number of equal parts and, further, to imagine using any
one of these unit fractional parts to produce a particular connected number of
either more or fewer parts than the number of parts of the partition. (p. 303)

When the child can iterate the unit fractional part indefinitely, Steffe considers the child
to have constructed the fractional connected number sequence. Because the splitting
operation is the activity of the equi-partitioning scheme for connected numbers, the
fractional connected number sequence would show evidence of it. Thus, a child told to
partition an unmarked length into six parts should image the parts, know that the six parts form a whole or that the whole is six times longer than any of its part, and that any part is 1/6 of the whole. In other words, the child should recognize a multiplicative relationship between the whole and its parts. Another indication that the equi-partitioning scheme for connected numbers had been constructed would be the child’s production of improper fractions.

*Equi-Partitioning Scheme for Composite Units and the Fractional Composite Number Sequence.* Steffe (2001) defines the equi-partitioning scheme for composite units as having the activation template of a composite of composite units; that is, a sequence of composite units that form a composite unit. The child could disembed one of the composite units and iterate it the necessary number of times to form the composite of composite units. For example 24 is composed of eight 3-units. The product of the scheme is a sequence of iterable composite units. Steffe posits that the equi-partitioning scheme may require a splitting operation for composite numbers parallel to the splitting operation for connected numbers to produce the composite unit fraction sequence as a product. In other words, they may need to “establish a multiplicative relation between a composite unit and one of its hypothetical parts” (Steffe, p. 296).

Though Steffe (2001) does not discuss the fractional composite number sequence, it is the logical product of the equi-partitioning scheme for composite units. It is necessary for a meaningful interpretation of equivalent fractions. A child that had constructed the fractional composite number sequence would recognize that 2/12 is 1/6 of the composite unit because six 2/12-units comprise the composite unit, 12/12. The child
would also recognize that 4/12 and 2/6 are the same because each is twice the length of 2/12 of the composite unit.

**Distinct Learning Levels.** The third-grade students in the teaching experiment are considered to have constructed the number sequence schemes and initial multiplication schemes prior to the experiment. The fourth grade students evidenced construction of the partitive fractional sequence scheme, but Steffe (2001) considered all of the schemes, including the equi-partitioning scheme, plus the fractional number sequences that follow to be above their level of learning.

Steffe (2001) observes that there is a distinction between partitive fractions (fraction language applied to singleton discrete units), parts of fractional wholes (unit parts) and fractional numbers (unit fractions). The construction of fractional numbers indicates the realization of a multiplicative relationship between the unit fraction and the connected number or composite number that is the whole. Steffe theorizes that the construction of the multiplicative relationship requires constructing the splitting operation for connected numbers. Attempts to engender the construction of fractional numbers via students’ multiplication schemes for composite numbers and their equi-partitioning scheme for connected number promoted, or at least elicited, the students’ partitive unit fraction scheme (fraction language applied to singleton discrete units) rather than the fractional number sequence (unit fractions).

Steffe (2001) notes that the differentiation he observed has been reported in other studies. For example, Nik Pa observed from interviews of nine 10- and 11-year-olds that they were unable to identify 1/5 of 10 items because the children interpreted 1/5 as one of
five items. Hunting reported the case of Alan, which he considered a representative case of 9-year-olds, who thought that “sixths” meant piles of six. In each case the data corroborate that the children used the partitive unit fraction scheme. Washburne reported that it wasn’t until the “mental age” of 11 years 7 months that three of four children scored at the 80% level on a test assessing their understanding of “grouping” fractions (composite unit parts of composite discrete wholes). Steffe looks to Lamon’s study of 9-year-olds fair-sharing solution strategies as an indicator that the process of constructing fractional numbers requires the development of the equi-partitioning scheme. Lamon reports that roughly half of the 123 children in the study displayed incomplete, incomprehensible, or invalid strategies for sharing four pizzas among three children or four oatmeal cookies among six children. Steffe observes that Lamon’s description of successful students’ solution processes match his expectation of how children would construct the equi-partitioning scheme for connected numbers. These studies suggest that the construction of fractional numbers is not easily engendered and that the appearance of the partitive unit fraction scheme is a normal aspect of children’s development of meaning for fraction symbols. They also suggest the order of scheme construction indicated in the analysis is appropriate.

Integration of Numerical and Global Quantitative Schemas

The second theory of children’s development of rational number concepts that arises from Piagetian developmental theory is that of Moss and Case (1999). They and other researchers (Case & Okamoto, 1996; Resnick & Singer, 1993) propose that children’s development of rational number understanding parallels their development of
whole number understanding. Specifically, children initially develop numerical schemas and global quantitative schemas separately. As they move to higher level thought processes, children coordinate the two sets of schemas to allow them to understand the structure of the simplest numbers in the field and the notation that is used for representing them. The coordination of numerical and global schemas produces a core of understanding on which children build their understanding of more complex numbers and notational forms leading ultimately to understanding the structure of the field.

For rational numbers the coordinated schemas are a numerical structure for “splitting” or “doubling” (Case & Okamoto, 1996; Confrey, 1994) and a global structure for proportional evaluation (Resnick & Singer, 1993). Moss and Case (1999) observe that these two schemas are present by the time children are 9- to 10- years old. The initial coordination produces an understanding of relative proportion and simple fractions such as 1/2 and 1/4. As in the case of whole numbers, these initial understandings become the foundation for building understanding of more complex rational numbers and proportions and for rational number relationships and operations. Their summary of research on children’s rational number understanding seems to indicate that a general understanding of the rational number field often does not occur until or after high school.

The theory proposed by Moss and Case (1999) provided the foundation for the development of a rational number curriculum. They believed that “one of the most important roles that instruction can play is to refine and extend the naturally occurring process whereby new schemas are first constructed out of old ones, then gradually differentiated and integrated” (p. 125). Believing that the order in which rational number
forms (fraction, decimal, or percent) are learned is arbitrary, their curriculum began with percents. They used a length model, a beaker of water, as the primary visual representation because it allowed for the natural association of “global proportional terms” such as full, nearly full, half full, almost empty, and empty.

In their initial activities, “children were asked to assign a numerical value from 1 to 100 to water levels in various beakers in order to estimate their ‘fullness’” (Moss & Case, 1999, p. 126). Children were encouraged to illustrate the problems that they solved with narrow vertical rectangles shaded from the bottom by a portion corresponding to the percentage. This encouraged the coordination of children’s global proportion schema with their already well-developed concept of the number sequence 1 to 100. The lessons were designed to promote children’s development of percent concept schemas.

Subsequent lessons emphasized the strategies of “numerical halving, (100, 50, 25, etc.), which corresponds to visual-motor splits, and composition (e.g., 100 = 75 + 25), which corresponds to visual-motor addition of the results” (Moss & Case, 1999, p. 126). Children’s natural tendency was to use halving in their reasoning processes. Thus, their strategies often produced portions such as 50%, 25%, or 12.5%. Thus, they determined the location of 75% on their illustrations by recognizing the relationship 75 = 25 + 50. Such naturally occurring strategies were encouraged. Children were able to use these same strategies to solve numerical problems such as what “amount of liquid would be required to fill a 900-ml bottle 75% full” (p. 131). Initial numerical problems took advantage of children’s natural halving strategies, but subsequent problems such as finding 10% of an amount pushed children to extend their proportional reasoning
processes. Some children used their halving strategies to find 12.5% and then estimated 10%. Some children took advantage of their previously constructed schemas for money, 10% having a similar relationship to 100% or one whole as 10¢ has to 100¢ or a dollar. Gradually, 10% began to be defined as “divide by 10” and became an alternative to the halving strategy on some types of problems.

Children were introduced to two-digit decimals when it was clear that they had an intuitive understanding of the processes for computing percentage values based on the visual-motor operations of splits and composition (Moss & Case, 1999). The meaning for the decimal notation was developed in a measurement context, “explaining that a two-place decimal number indicates the percentage of the way between two adjacent whole number distances that an intermediate point lies (e.g., 5.25 is a distance that is 25% of the way between 5 and 6)” (p. 126). The researchers introduced the decimals using the visual representation of a large number line on the floor with each pair of consecutive numbers exactly a meter apart. The children were asked to walk to a position some percentage between one number and the next adjacent number. They were told that the total distance they walked “could be represented with a two-place decimal number in which the whole number represented the number of meters walked and the decimal number represented the percentage of the distance to the next meter mark” (pp. 131). For example, “when you pass the 2-m mark and walk 75% of the way to the 3-m mark, the point you reach can be written as 2.75m” (p. 131). The context allowed the immediate application of the children’s knowledge of percentage that they had developed in the beaker context. Building on this introduction, children were engaged in games and tasks that further
developed their understanding of decimal numbers and notation. Activities with a digital stopwatch that displayed seconds and centi-seconds allowed children to experience and compare time intervals represented by decimals such as 0.09 and 0.15. Twenty 10-cm number lines calibrated in tenths and hundredths constituted the route of a board game designed to encourage children to create their own decimal representations of lengths. Drawing a card with a two-digit number and a second card with either a “+” or “-,” they could place a “0” and the decimal to produce a longer or shorter length. As the researchers expanded the tasks from two-decimal to multi-place decimal problem situations, the children spontaneously invented a “double decimal notation” (p. 126). For example, 6.25.25 is a point 25% of the distance from 6.25 to 6.26.

The curriculum project culminated with exercises in which fractions, decimals and percents were used interchangeably (Moss & Case, 1999). Fraction notation was used throughout as an alternative form of representation for quantities understood via the percent and decimal contexts. Children already knew that a half, the first split, and 50% were equivalent forms for representing the same quantity and most also recognized that 25% was the same as a quarter, the split of a split. They were introduced to the fact that 1/8 was another representation for 12.5% or 12 1/2%. One of the last lessons was devoted to fraction notation. In subsequent lessons, students were engaged in solving and posing problems involving mixed expressions. Examples include: “true or false: 0.375 is equal to 3/8;” “stop the watch as close to the sum of (1/2 + 3/4) as possible, and then figure out the decimal value for how close you are;” and inventing mixed addition problems for others to solve such as “1/4 + 75% + 0.375 + 1/16” (p. 133).
A Comparison of Unit Fraction Scheme Construction in Two Instructional Contexts

Moss and Case’s (1999) integration of numerical and global quantitative schemas seems to provide a mechanism by which children’s development of Steffe’s (2001) equi-partitioning scheme for connected number could be engendered. Moss and Case’s curriculum sequence takes advantage of children’s existing connected number sequence from 1 to 100, their existing equi-portioning scheme, and provides situations that encourage the coordination of these two schemes with their existing splitting scheme of visual-motor halving. Since the students described in each study were fourth graders, it would seem reasonable that there might exist commonalities between the children’s rational number schemes. However, the different curricular experiences of students in Steffe’s teaching experiment and students in Moss and Case’s curriculum project could have produced a different sequence of schemes in their respective development of rational number knowledge. Although both curricula used length models, the curricula are significantly different. Steffe and his colleague used common fractions to develop rational number concepts and did not address either percent or decimal representations. In comparison, Moss and Case used percents and decimals to develop rational number concepts and encouraged fraction notation as an alternative representation.

The first question is whether the students in Moss and Case’s (1999) study ever developed an analog scheme to that of the partitive unit fraction scheme described by Steffe (2001). To review, the purpose of the partitive unit fraction scheme is to partition a connected number, one or the whole, into any given number of equal parts, take any one
of those parts and establish a “one-to-many” relation between the selected part and the whole. Moss and Case’s data indicate that children were able to partition a whole into parts such as halves, fourths, tenths, and thirds, suggesting a schema similar in purpose to that of the partitive unit fraction scheme. Moss and Case’s data, however, did not indicate the production of “necessary errors” such as Steffe observed when children used the partitive unit fraction scheme. It is quite possible that Moss and Case did not present their students with situations that would elicit the “necessary errors” observed by Steffe and his colleague. The errors arose in situations that allowed children to conflate the unit part size with the number of unit parts of a composite whole such as “Five is what fraction of 15?” Moss and Case’s students’ success on problems with misleading visual features from the assessment as compared to the control group suggests that they may not have been as likely to produce “necessary errors” and, hence, not using an analog of partitive unit fraction scheme in their production of fraction notation. One pair of problems asked students to provide a rational number representation of a portion of a whole: (a) “Here is Mary on her way to school. What fraction of the distance has Mary traveled from her home to school?” (b) “What percentage of the distance has she traveled?” (p. 138) The accompanying illustration showed a route divided into eight small units with Mary’s distance traveled indicated as five of the units. The percentage of the experimental group (n=16) that performed successfully on the two problems was 93% and 82% as compared to 31% and 38% of the control group (n=13), respectively. Most children who answered incorrectly gave the fraction as 1/5. However, the whole was not a composite unit as were the wholes that Steffe and his colleague observed to have elicited the production of
necessary errors and thus the errors produced by children in Moss and Case’s study do not provide a strong argument for the students’ development of a different unit fraction scheme from that used by the fourth graders in Steffe’s study. A problem that did require that students work with composite unit parts of a composite whole was, “Find 3/4 of a pizza (predivided into 8ths)” (p. 138). The entire experimental group succeeded with this problem as compared to 54% of the control group. Interviews reveal that some experimental group students did iterate three unit parts, two sections or 2 eighths. Others composed half of the whole and another fourth, recognizing that a fourth was two sections. The results of the two sets of questions suggest that indeed the experimental group in Moss and Case’s study were operating with a fraction schema that utilized splitting as its operation much as Steffe predicted should be the case with the equi-partitioning of a connected number scheme. Nevertheless, I would argue that the scheme was yet to be generalized to any possible partition of a composite whole. There was little evidence that children could move much beyond splits based on half and tenth relationships.

Summary

In summary, teachers’ and students’ knowledge will be examined from a constructivist perspective. The descriptions will integrate two perspectives for examining knowledge of rational numbers, the content (knowledge of rational numbers) and its construction (acquisition of knowledge of rational numbers). The discussion of Kieren’s ideal network of personal rational number knowledge provides a potential framework for examining and analyzing the content available for learning within each classroom. In
particular, it contributes to the design of instruments for gathering data about teachers’ and students’ knowledge of rational number. Steffe’s identification of children’s rational number schemes provides a framework for interpreting children’s attempts to construct rational number knowledge with the prior knowledge that they have already constructed such as the web of schemas that comprise children’s emerging whole number knowledge and the available experiences within the classroom. The comparison of Steffe’s study and Moss’ and Case’s study suggests potential patterns of difference in children’s construction of rational number knowledge could be attributable to the organization of concepts and activities within instruction. Together, these theories provide a context for interpreting the observations of teachers’ and students’ taken-as-shared knowledge of rational numbers both within the classroom and within the larger context of the mathematics education community.

Whereas, the preceding provides a foundation for describing teachers’ mathematical knowledge and students’ emergent mathematical knowledge—the primary variables of the study, the following discussion illuminates teachers’ pedagogical content knowledge. An understanding of the contribution of teachers’ pedagogical content knowledge to instruction is necessary for an adequate description of the context in which teachers’ mathematical knowledge and students’ construction of mathematical knowledge potentially interact and for the illumination patterns that could suggest areas of further investigation and potential relationships between teachers’ knowledge of content and students’ construction of content knowledge.
Interrelationships Between Teacher Knowledge and Instructional Practice

The literature on teachers' knowledge for instruction reveals a range of perspectives. For the purposes of this study, I will focus on the part of the literature that examines relationships between teachers’ content knowledge and instruction. The work of Shulman (1986, 1987), Peterson (1988), and Thompson (1992) provide insight and structure to the examination of the relationships between teachers’ content knowledge and instruction.

Shulman (1986) and Peterson (1988) both consider teachers' knowledge of content to be an important factor impacting teachers' instructional practice, but their definitions include more than just personal knowledge of a particular topic of study. Shulman (1986) describes three categories of content knowledge that would impact instruction: (a) subject matter (substantive or content—ways the concepts and principles of the discipline are organized; syntactic—ways validity is established); (b) pedagogical content knowledge (the most useful representations—analogy, illustration, example, explanation, demonstration—for making a topic comprehensible to others); and (c) curricular knowledge (programs and materials and their appropriateness for different pedagogical situations). Peterson (1988) proposes that teachers' knowledge comes in two forms:

- *cognitional knowledge for classroom learning*: That is, they have a knowledge of the mental processes or facilities by which learners acquire knowledge through classroom teaching.
• **metacognitive knowledge**: That is, teachers have self-awareness and an ability to reflect upon the cognitional knowledge that he or she has, both general and content specific.

While the two descriptions differ greatly, each moves beyond teachers’ personal experience of the subject matter to the engagement of subject matter with the act of teaching.

Although Shulman’s (1986) first category of content knowledge is related to a teachers’ personal hierarchical organization of a topic, he includes in his definition two other categories of content knowledge that are outside the context of a teacher’s personal knowledge of a topic of study. The categories of useful representations and curricular knowledge indicate a shift in focus from learner of a topic to teacher of a topic. It identifies the necessity of the teacher to move beyond a personal viewpoint on the topic of study to a viewpoint that values representations and curricula for their power to communicate the nature of the topic to and with others; in particular, novices that do not share the teacher’s expertise or sophistication in/with the topic. The attributes that Shulman (1986) lists as arising from content do not communicate the decision-making and organizational character of pedagogical content knowledge. Shulman’s (1987) later discussion of pedagogical knowledge recognizes that pedagogical content knowledge is the “blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented and adapted to the diverse interests and abilities of learners and presented for instruction” (p. 8).
Peterson’s (1988) two categories complement rather than overlap Shulman’s (1986) definitions of teachers’ content knowledge. She focuses on the teachers’ knowledge of how students or novices come to understand a topic of study. Although Shulman’s definition is larger than merely a teacher’s personal content knowledge, it still is centered on the subject matter. Shulman’s shift is within the teachers’ viewpoint on the subject matter from practitioner within a cognitive field (mathematician) to a teacher of a cognitive field (mathematics educator). Peterson’s categories signify a change in the teacher’s viewpoint from their own ways of organizing knowledge of the cognitive field to an understanding of how novices can and do acquire the knowledge of the cognitive field. Therefore, for this study personal content knowledge is defined to include Shulman’s substantive and syntactic knowledge and Peterson’s metacognitive knowledge. Similarly, a teacher’s pedagogical content knowledge is defined as being the knowledge of the content a teacher has constructed for subject matter. These are discussed in more detail in the following two sections.

Ball (1990b) makes the case that a discussion of how teachers’ content knowledge impacts their instruction requires the inclusion of teachers’ beliefs about the content and teaching the content. Her examination of the practice of three elementary school teachers contrasts the effect of teachers' beliefs about mathematics as well as their knowledge of the subject matter on their instructional choices. Her descriptions also reflect teachers’ beliefs about how children learn. Her observations of the classrooms of the teachers demonstrate that the teachers' instructional choices create differing classroom cultures. Thus, teachers’ beliefs about subject matter should be included in a discussion of
teachers’ knowledge and its relationship to instruction. The following three sections take
a closer look at the relationships between teachers’ content knowledge, pedagogical
content knowledge, and beliefs about content and teaching content, respectively.

Teacherr’s Personal Content Knowledge and Instructional Practice

As previously defined, teachers’ personal content knowledge includes three
aspects of their personal construction of subject matter knowledge: content knowledge,
syntactic knowledge, and metacognitional knowledge. Teachers’ content knowledge is
comprised of their personal organization of the concepts and principles as constructed via
their own experiences as a student of mathematics. One way to understand how teachers’
substantive knowledge may be related to their instructional practice is to look again at
how cognitive science has constructed the mental organization of knowledge as described
in the section on rational number construction. Cognitive researchers explore how
representations are developed, related to one another, and retrieved. One particular
knowledge structure that has been studied is the hierarchical organization. A different
description of this knowledge structure from that given in the preceding section is that
given appropriate experience, individuals are able to "chunk" information and organize it
in a hierarchical fashion (where specific knowledge is embedded within more generally
applicable knowledge). The model of a "chunk" allows one to envision that some
particularly complex bit of knowledge can be "chunked" in such a way that it can be
retrieved and used as one piece. This allows efficient access to information and increased
memory capacity as a "chunk" requires less space in working memory compared to many
disparate pieces of information. Yet any information that is embedded within the "chunk"
can be retrieved when needed since the "chunking" process maintains the connections (Fennema & Franke, 1992).

Studies of experts and novices as they solve problems have been one approach to understanding hierarchical knowledge structures. Researchers have found that experts efficiently search problem space, possess meta-statements to aid in decision-making and organize their knowledge based on properties rather than on individual instances (Chi, Feltovich, & Glaser, 1981; Chi, Glaser, & Rees, 1982; Voss, Greene, Post, & Penner, 1983). "For experts, knowledge is organized in specific ways: connections exist between ideas, the relationship between ideas can be specified, the links can differ among ideas, and the manner in which the knowledge is organized is relevant to understanding and application" (Fennema & Franke, 1992, p. 152).

When teacher knowledge of content has been defined in a way that is congruent with the nature of the content and/or when a conceptual organization of knowledge has been considered, a positive relationship has been found between content knowledge of teachers and their instruction (Even, 1993; Hashweh, 1986; Leinhardt & Smith, 1985). Studies of teachers teaching familiar and unfamiliar topics have shown that teachers’ content knowledge affected classroom discourse (Carlsen 1990, 1993; Lehrer & Franke, 1992). When teachers taught unfamiliar topics, they tended to dominate the discourse, talking more frequently, longer, and asking more low-level cognitive questions. Students were less likely to be the originators of questions or discussion topics. Teachers’ content knowledge also affects their assessment of students’ solution strategies (Van Dooren, Onghena, & Verschaffel, 2002). Finally, teachers’ schemas for a topic are not the same
(Hashweh, 1986; Lyons & Turner, 1993). Even expert teachers’ construction of a topic varies and those variations influence teachers’ organizations of the concepts in their lesson planning. It is reasonable to conclude that teachers are not monolithic in their schemas for any topic that they teach. Teachers’ organization of their knowledge of a topic is individual and reflects the connections that teachers have constructed between that topic and other parts of their knowledge base.

Thus, the previously cited research seems to indicate teachers’ content knowledge influences the organization of their content knowledge for teaching (Hashweh, 1986; Lyons & Turner, 1993), the content level and ownership of classroom discourse (Carlsen 1990, 1993; Lehrer & Franke, 1992), and teachers’ assessment of student thinking (Van Dooren, Onghena, & Verschaffel, 2002). Based on the studies discussed in the section on knowledge of rational numbers, teachers having a construct of rational number that approximates all levels of Kieren's model would be well prepared to teach students about rational numbers. However, their understanding of rational numbers may not need to be a rigorous axiomatic understanding of the concepts that define rational number, but consist of an understanding of the interrelationships between the constructs. Aspects of levels one through five of Kieren's model of personal rational number knowledge may represent the crucial knowledge for elementary teachers. With a richly interconnected understanding of the five levels, they could be more flexible in their teaching—able to take advantage of "teachable moments." They could utilize more varied instructional strategies that would provide their students with richer experiences of the subject matter. Being secure in their own knowledge of the subject matter they could be free to value
student viewpoints and trust in the meaningfulness of student constructions—able to
differentiate between fragile constructions and those that are different but highly viable.

However, there are other ways to have constructed rational number
knowledge—by memorization of rules and procedures. There is reason to believe that
many teachers may not have rich constructions of rational number knowledge. Post,
Cramer, Behr, Lesh, & Harel (1993) report in their findings from a study involving 221
intermediate-level teachers in the Rational Number Project (RNP) that 30% do not
understand a significant portion of the mathematics that they are teaching. Over 25%
scored below 50% on a 58-item instrument designed to assess multiplication, division,
part-whole, ratio, decimal, and proportionality with another 20% to 30% scoring between
50% and 70%. However, even expert teachers with rich connected knowledge of rational
numbers may differ in their rational number schemas. Since teachers’ personal schemas
of a topic may influence their organization of the topic for instruction, it is reasonable to
suggest that students’ experience of rational numbers is different in every classroom.

The second aspect of teachers’ personal content knowledge is teachers’ syntactic
knowledge—teachers’ construction of how the validity of concepts and principles is
established within mathematics. In other words, teachers’ syntactic knowledge is
synonymous with their understanding, recognition, and application of the principles of
mathematical reasoning and argumentation—it is “doing” mathematics. There is a dearth
of research examining teachers’ syntactic knowledge of mathematics and its relationship
to instruction as such. However, evidence for teachers’ syntactic knowledge of
mathematics can be found by looking at studies that describe teachers’ explanations for
concepts and/or procedures, teachers’ knowledge of problem solving, teachers’ implementations of classroom discourse and alternative symbolic systems, and teachers’ knowledge of proof and argumentation.

Studies that examine teachers’ explanations for concepts and/or procedures reveal a lack of mathematical or conceptual argumentation. Ma (1999) found that elementary teachers were not able to explain the reasoning behind the procedures for multiplying multi-digit numbers. Their explanations relied on non-mathematical descriptions of the procedure. Simon (1993) found that preservice elementary teachers in his study were unable to explain the reasoning for why the long division algorithm resulted in a correct solution. Their explanations focused on computational aspects of the procedure such as checking if the remainder is smaller than the divisor. Such results suggest that the preservice teachers constructed mathematical validity via external sources or careful execution of essentially meaningless procedures.

Schoenfeld (1992) identifies problem solving as “the work of mathematicians” (p. 339). He eschews the traditional sense of problem solving as a “task to be done” and embraces the sense of problem solving as “working problems of the ‘perplexing’ kind” (p. 338). To illustrate the type of instruction that engenders the kind of mathematical reasoning involved in problem solving, he offers the practices of Lampert and Fawcett. What is noticeable in each case is the teachers recognized that students’ must engage in mathematical argumentation to validate or refute conjectures proposed by their peers.

Lampert (1990) focused on fostering classroom discourse that would promote students’ engagement in mathematical argumentation. Cobb, Yackel and Wood (1995) in
describing their research on the development of constructivist teaching practices observed that the classroom teacher “initiated the development of classroom social norms” (p. 22). The teacher explicitly addressed the social norms for small group activity as including:

- persisting to solve personally challenging problems, explaining personal solutions to the partner, listening to and trying to make sense of the partner’s explanation, attempting to achieve consensus about an answer, and, ideally a solution process in situations in which a conflict between interpretations or solutions has become apparent. Social norms for whole class discussions included: explaining and justifying solutions, trying to make sense of explanations given by others, indicating agreement and disagreement, and questioning alternatives in situations in which a conflict between interpretations or solutions has become apparent. (p. 22)

The teacher did not write the norms as rules to be followed, but took advantage of opportunities to observe students’ instantiations or transgressions of the norms. Due to the special nature of the situation, it is difficult to determine whether to attribute the social norms to the teachers’ syntactic knowledge of mathematics or to her pedagogical knowledge of appropriate classroom discourse. Nevertheless, the social norms provided a context for the children to construct an appropriate syntactic knowledge of mathematics.

McClain and Whitenack (1995) describe the importance of notation as a part of mathematical discourse and argumentation. In a first grade classroom, a teacher introduced an informal notation to describe student’s use of partitioning of numbers in their solutions of arithmetic problems. For example, children often solve $7 + 8 = ?$ by partitioning the 8 into its components 7 and 1 in order to create a double 7 for which they know the sum. Then they add the one to the sum to produce their final answer. The children of their own initiative appropriated the teacher’s notation to explain their reasoning to their peers. The teacher herself recognized the importance of mathematical notation in the process of communicating mathematical reasoning and making arguments.
for the validity of conclusions. Hence, mathematical notation was an important aspect of her syntactical knowledge of mathematics.

Studies of teachers’ knowledge of proof have shown that teachers are prone to accept incomplete inductive evidence such as a series of empirical examples or a pattern as being sufficient for establishing validity (Ball & Wilson, 1990; Ma, 1999; Martin & Harel, 1989; Simon & Blume, 1996). Ma (1999) compared U. S. and Chinese teachers’ responses to the claim that if the perimeter of a rectangle is increased, its area must also increase. The claim was framed as the conjecture of a student and accompanied by a diagram of two rectangles for which the claim was true. The responses indicate a range of teacher syntactic knowledge. Some teachers accepted the claim as true on the basis of the examples provided. Some teachers required a textbook to verify the claim. Many U.S. teachers would have accepted the claim if given more examples. Some of the U.S. teachers and most of the Chinese teachers provided a counterexample to refute the claim. Further evidence of the nature of U. S. teachers’ syntactic knowledge of mathematics is their lack of motivation to seek alternative or more efficient and elegant solutions to problems. By contrast, Ma reports that Chinese teachers often provided multiple solutions without prompting, and were motivated to find efficient and elegant solutions.

The third aspect of teachers’ personal content knowledge is metacognitive knowledge. Metacognitive knowledge is teachers’ awareness of their own thinking processes whether mathematical or in general and their ability to use that awareness to facilitate their own learning and/or production of mathematics. Research on teacher content knowledge does not explicitly address teachers’ metacognitive knowledge of
mathematics. However, Thompson and Thompson (1994, 1996) provide an opportunity to contrast two teachers’ metacognitional knowledge of mathematics in the context of teaching one topic to the same student. The two teachers were using a curriculum that they had developed to teach children proportional reasoning. One teacher, Bill, began to have difficulty in his tutoring sessions with the student, Ann. Bill used “arithmetic representationally—he could read a situation into arithmetic expressions and he used arithmetic expressions to represent his understanding of situations” (Thompson & Thompson, 1994, p. 301). Bill could access his thinking about a problem situation and use it in his instruction. However, when his personal conceptualization was a mismatch for the student’s conceptualization, he was unaware of the mismatch and was unable to adjust his instruction to more effectively engage the student’s thinking processes. In contrast, the other teacher, Pat, had access to multiple representation systems for proportional problems, including an imagistic representational system. He could recognize that the student was applying a measurement scheme, an imagistic representation, to understand the problem situation. He was able to monitor his own thinking process as he made connections with the student to lead her to a more viable construction of the situation. He was also able to build models of both Bill’s and the student’s conceptualizations of the problem situation in order to develop a theory of how and why instruction “broke down.” Therefore, the nature of teachers’ metacognitional knowledge of mathematics can determine whether it will help or hinder their instructional practice.
In summary, teachers’ personal content knowledge includes not only their knowledge of concepts and procedures of mathematics, but how they construct “doing” mathematics, and their ability to monitor their thinking while doing mathematics. Although it represents their personal mathematics, it is an important influence on their teaching practice and, hence, has a dialectic relationship with their pedagogical content knowledge. The following section examines teachers’ pedagogical content knowledge and its relationship to their instructional practice.

*Teachers’ Pedagogical Content Knowledge and Instructional Practice*

An integration of Shulman’s (1987) and Peterson’s (1988) perspectives on teachers’ pedagogical content knowledge yields two aspects: content knowledge in teaching and cognitional knowledge for classroom learning. The first aspect of teachers’ pedagogical content knowledge (Shulman, 1986, 1987) blends content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners for the development of instruction. It takes as source material any representations that are used in teaching a topic to others such as analogies, illustrations, examples, explanations, demonstrations, and knowledge of instructional programs and materials and their appropriateness for different pedagogical situations. The second aspect of teachers’ pedagogical content knowledge, Peterson’s (1988) cognitional knowledge for classroom learning, includes teachers’ knowledge of how children learn mathematical topics and how teaching can impact that learning. Teachers’ pedagogical content knowledge is founded on teachers’ own experiences as students of mathematics. However, hopefully, it is expanded through
teacher preparation programs, preservice and inservice, through teachers’ practical experiences, and through teachers’ interactions with their peers.

Ball et al. (2000) offer Lampert’s research on teaching as indicative of knowledge of mathematics in teaching. It “reveals vividly the mathematical reasoning involved in choosing and using particular representations, in managing complex classroom discussion, and in designing a problem or in figuring out how to formulate a good question” (p. 450). Studies (Sowder, Phillipp, Armstrong, & Schappelle, 1998; Summers, Kruger, & Mant, 1998; Swafford, Jones, Thornton, Stump, & Miller, 1999) that describe professional development programs that engender enhanced teacher performance in teaching content as well as changes to teachers’ content knowledge highlight the need for teachers to develop the mathematical reasoning processes that can allow them to access meaningful representations and organize them in a way that promotes meaningful mathematical construction throughout the process of teaching. Summers et al. (1998) describe a professional development program for inservice elementary science teachers. Their observations of teacher participants revealed evidence that teachers used the knowledge and specific activities, representations, illustrations, and analogies that they learned in the program in their teaching. In particular, one teacher was able to adapt an analogy that was introduced in the professional development program to structure her teaching in a meaningful way, allowing her to address her children’s misconceptions and provide them with representations that could help them overcome the misconceptions as she had learned from the program. Other kinds of knowledge important for teaching identified by the research include: appropriate terminology and language, example and
counterexample, ways to simplify complex ideas appropriately that do not lead to
misconceptions, and technical knowledge that applies to activities that require
instrumentation (calculators, measuring instruments)—i.e. why an experiment does not
always reflect the concept.

Chinese teachers’ “knowledge packets” (Ma, 1999) provide another example of
the mathematical reasoning processes that can allow teachers to access meaningful
representations and organize them in a way that promotes meaningful mathematical
construction throughout the process of teaching. The knowledge packets organize
mathematical ideas in pedagogically viable ways. Concepts and procedures are
interwoven to provide meaningful engagement with both. Connections between topics
and multiple perspectives are emphasized. As more experienced teachers interact with
less experienced teachers they share the many representations and problems that help
instantiate the knowledge in the packets and help their younger peers to explore the
differing efficiencies of various models for instruction.

The studies describing classroom discourse (Cobb, Yackel, & Wood, 1995;
McClain & Whitenack, 1995) discussed in the section on teachers’ personal content
knowledge reveal a necessary blending of teachers’ syntactic knowledge of mathematics
and their knowledge of classroom situations. Studies of expert and novice teachers
(Borko et al., 1992; Livingston & Borko, 1989; Mapolelo, 1999) highlight the need for
teachers to develop the mathematical reasoning processes that can help them in designing
a problem or in figuring out how to formulate a good question. While some novice
teachers were able to write plans to communicate mathematics concepts and procedures
clearly and completely, they were not able to retrieve appropriate examples, analogies, and problems as needed to deal with students’ impromptu questions compared to more expert teachers. Even when novices responded with examples or problems to impromptu situations, they failed to make connections to related concepts within the lesson or other parts of the curriculum. Implicit in the discussion of content knowledge in teaching is teachers’ cognitional knowledge for classroom learning, or their knowledge of how children learn mathematical topics and how teaching can impact that learning. Finally, the work of the Cognitively Guided Instruction program (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996), Thompson & Thompson (1994, 1996) described in the section on metacognition, and Cobb, Yackel and Wood (1992) demonstrate the importance accounting for students’ thinking processes in organizing meaningful instruction.

In summary, teachers’ pedagogical content knowledge is a profound reorganization of teachers’ mathematical knowledge that must account for the interconnections between the varying aspects of mathematics (concepts, procedures, validation, etc.), the nature of the mathematics learner, and the nature of the mathematics classroom. It encompasses knowledge of the content, knowledge of learning and learners’ developmental processes, and knowledge of classroom social interaction. The following section examines the relationships between teachers’ beliefs about mathematics and mathematics teacher and instruction and in so doing reveals the connections between teachers’ beliefs, their content knowledge, and their pedagogical content knowledge.
Teachers’ Beliefs About Content and Teaching Content and Instructional Practice

The third section examines research on teachers’ beliefs for instruction. Teachers’ beliefs for instruction include teachers’ beliefs about the subject matter and about teaching the subject matter. Teachers’ beliefs about mathematics and mathematics teaching are each personal theories that are constructed from and influence the categories of teachers’ personal content knowledge and teachers’ pedagogical content knowledge. Thompson (1992) provides an overview of the research on teachers’ beliefs. She notes, “studies of mathematics teachers beliefs and conceptions have focused on beliefs about mathematics, beliefs about teaching and learning mathematics, or both” (p. 131). Thompson discusses the different categories of teachers’ beliefs about mathematics, about teaching and learning mathematics and how these are related to teachers’ instructional practice.

Thompson (1992) observes a number of viewpoints on beliefs about mathematics, some arising from mathematics and some from psychology. Two schemes for categorizing beliefs about mathematics arising from the discipline of mathematics are those of Ernest (1989) and Lerman (cited in Thompson, 1992). Ernest suggests that there are three categories of conceptions of mathematics based on philosophy of mathematics and empirical studies of teaching: the problem solving view, the Platonist view, and the instrumentalist view. Lerman describes two categories, absolutist and fallibilist, that correspond “to two competing schools of thought in the philosophy of mathematics: Euclidean and Quasi-empirical” (Thompson, 1992, p. 132). From their descriptions, Lerman’s absolutist category corresponds to Ernest’s Platonist category and Lerman’s
fallibilist category corresponds to Ernest’s problem solving category. The problem solving/fallibilist view is that mathematics is a human creation. It is an endless cycle of pattern generation, conjecture, proof and refutation. This perspective of mathematics emphasizes inquiry, open-endedness, and mathematics’ inherent uncertainty.

Mathematics is a structure being continuously renovated. Anyone can participate. The Platonist/absolutist view is that mathematics is a priori. It is a “static, but unified body of knowledge, a crystalline realm interconnecting structures and truths, bound together by filaments of logic and meaning” (as cited in Thompson, 1992, p. 132). This perspective of mathematics emphasizes abstraction, certainty, and the competing interests of discovery and authority. Conjecture, proof, and refutation reveal what already exists. Mathematics connections to the real world are of a platonic nature. Mathematics is the ideal waiting to be revealed. Only the gifted have the insight to see its form. Everyone else needs help. The difference between Lerman’s and Ernest’s categorizations is Ernest’s inclusion of the instrumentalist view. The instrumentalist view is that mathematics is a collection of facts and procedures. It emphasizes utility. Facts, skills, and procedures developed by expert craftsmen are learned, practiced, and used in specific applications. Mathematics is the quantitative toolbox. The tools are available to those willing to develop the skills necessary to use them.

Studies by Thompson and Lerman validate the usefulness of these categories for describing teachers’ beliefs (Thompson, 1992). In Thompson’s study of junior high teachers’ beliefs about mathematics she observed that each of the three categories (problem-solving, Platonist, and instrumentalist) could consistently describe the
expressed views and demonstrated instructional practices of teachers that she observed. In his study, Lerman gave pre-service secondary teachers an instrument designed to assess beliefs on a continuum from absolutist to fallibilist. Two teachers scored at the absolutist extreme and two at the fallibilist extreme. The four teachers at the extremes were asked to respond to a videotape of a mathematics lesson to assess the consistency of their views. The two absolutist teachers criticized the teacher in the video for not being directive enough and the two fallibilist teachers criticized the teacher in the video for being too directive.

Thompson (1992) describes two categorizations of beliefs about mathematics that came from psychology—a categorization developed by Copes based on the work of Perry and a categorization developed by Skemp. Copes viewed his categories as replicating the development of mathematics through history. He theorized that three categories could describe beliefs about mathematics—absolutism, multiplism, and relativism. Absolutism (not to be confused with Ernest’s absolutist category) prevailed from the Egyptians to the mid-nineteenth century and takes mathematics to be “a collection of facts whose truth is verifiable in the physical world” (Thompson, 1992, p. 133). Multiplism arose in response to the development of the non-Euclidean geometries and is characterized by the acceptance of the existence of multiple systems of mathematics that might contradict each other and are not verifiable by the physical world. Relativism was the response to the failure to prove the logical consistency and completeness of mathematics and the acceptance that there were equally valid contradictory systems of mathematics. “Dynamism is characterized by a commitment to a particular system or approach within
the context of relativism” (Thompson, 1992, p. 133). Copes suggested how his categorization of beliefs about mathematics could apply to mathematics teaching. For example, a teaching style that emphasized “transmission of mathematics facts, right versus wrong answers and procedures, and single solution approaches to solutions of problems” could represent absolutism (as cited in Thompson, 1992, p. 133). However, it is difficult to see how the other two categories, multiplism and relativism, could be differentiated. Stonewater and Oprea did a study of teachers’ beliefs about the nature of truth and the role of authority in mathematics and found that the teachers’ beliefs systems appeared to correlate with Perry’s scheme of intellectual and ethical development (cited in Thompson, 1992). This lends some credence to the usefulness of Copes’ categories to describing teachers’ beliefs about mathematics. From Thompson’s description of Copes’ absolutism category, an argument could be made that it has features that resonate with both the absolutist/Platonist and the instrumentalist categories previously described. The multiplism, relativism, and dynamism categories all appear to be degrees within the fallibilist/problem solving category.

The other categorization of beliefs about mathematics that Thompson (1992) describes from psychology is Skemp’s (1978) instrumental and relational understanding. Instrumental understanding is characterized by a collection of context-specific, disconnected, rigid schemas consisting of fixed step-by-step procedures. Each schema is constructed to accomplish a particular mathematical task. The schemas are applied when the conditions associated with their construction are present. Relational understanding is characterized by interconnected schemas that allow for the development of multiple plans
An overview of the various frameworks for categorizing beliefs about mathematics suggests that they could be reduced to three categories: absolutist/Platonist, fallibilist/problem solving, and instrumentalist. These are the categories that have demonstrated usefulness in describing teachers’ beliefs about mathematics in the literature. It also should be apparent that beliefs about mathematics describe what is the nature of mathematical knowledge or what is to be learned and how mathematical knowledge is acquired and perhaps even who is empowered to find the knowledge and who is empowered to use that knowledge. This seems to parallel Thompson’s (1992) description of Perry’s emphasis on belief systems as describing the nature of truth and the role of authority. It also suggests that beliefs about mathematics characterize a proto-theory of learning.

Thompson (1992) describes teachers’ beliefs about mathematics teaching as being composed of: “what a teacher considers desirable goals of the mathematics program, his or her role in teaching, the students’ role, appropriate classroom activities, desirable instructional approaches and emphases, legitimate mathematical procedures, and acceptable outcomes of instruction” (p. 135). She also notes that teachers’ beliefs about
mathematics teaching have relationships with teachers’ beliefs about mathematics since beliefs about mathematics speak to the nature of what is learned and how one comes to learn it—in other words, a teachers’ tacit learning theory. Teachers do not create explicit, well-defined learning theories. Rather, as Clark describes, “teachers implicit theories tend to be eclectic aggregations of cause-effect propositions from many sources, rules of thumb, generalizations drawn from personal experience, beliefs, values, biases, and prejudices” (as quoted in Thompson, 1992, p. 135). Despite this, Thompson suggests Kuhs’ and Ball’s four views of mathematics teaching as a useful framework for categorizing teachers’ beliefs about mathematics teaching: learner focused, content focused with emphasis on conceptual understanding, content focused with emphasis on performance, and classroom focused.

The learner focused belief system emphasizes students’ active involvement in exploring and formalizing mathematics—constructing a personal mathematical knowledge. Consequently, it has connections with constructivist learning theory and a fallibilist/problem solving view of mathematics. The content focused with emphasis on conceptual understanding belief system emphasizes students’ “understanding of the logical relations among various mathematical ideas and the concepts and logic underlying mathematical procedures” (Thompson, 1992, p. 136). Consequently, it has connections with Brownell’s (1935) meaning theory of instruction and an absolutist/Platonist view of mathematics. The content focused with emphasis on performance belief system emphasizes students’ performance and mastery of mathematical rules and procedures. Therefore, it reminiscent of what Brownell (1935) termed “drill theory” and an
instrumentalist view of mathematics. The *classroom focused* belief system emphasizes the provision of effective classroom environments. In its purest form this perspective is not content specific. It arises from the process-product models for effective instruction. Good, Grouws, and Ebmeier (1983) provide an example of this model applied to the mathematics classroom. Good, Grouws, and Ebmeier’s content-specific example more closely paralleled the content focused with emphasis on conceptual understanding belief system. Thus, the content focused with emphasis on conceptual understanding belief system subsumes the classroom focused belief system.

Thompson (1992) observed that the research on the relationships between teachers’ stated beliefs about mathematics or about mathematics teaching and their instructional practice are inconsistent. Some studies report high consistency (Grant, 1984; McGalliard, 1983; Shirk, 1973; Thompson, 1984). Others report inconsistency (Cooney, 1985; Kesler, 1985, McGalliard, 1983). Thompson notes that investigating teachers’ beliefs is problematic. Teachers’ are not always cognizant of their own beliefs—identifying one’s beliefs is a reflective activity. Belief systems have social and political costs and therefore to present themselves well to the questioner teachers may profess beliefs that they have not actualized in their instruction. Particularly in the current climate, teachers’ may hold beliefs that they do not yet have the skills or knowledge to implement or profess politically correct beliefs that are not correspondent to those that they have actually constructed through experience and practice (Fernandez, 1997; Shaw, 1989). One way to address the inconsistency is to not rely solely on teachers’ self reports of beliefs but to examine their beliefs in practice. However, Cobb, Wood and Yackel
(1990) question the causal relationship posited between beliefs and practice and offer a different perspective: “In our view, arguments about the direction of the assumed causality miss the point; the very nature of the relationship needs to be reconceptualized. Our current work with teachers is based on the alternative assumption that beliefs and practice are dialectically related” (p. 145).

In summary, teachers’ beliefs about mathematics and about mathematics teaching can be organized into a framework of corresponding beliefs (see Figure 2.4). Teachers’ may self-report one belief system and enact another in practice for many reasons.

<table>
<thead>
<tr>
<th>Beliefs about mathematics</th>
<th>Beliefs about mathematics teaching</th>
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<tbody>
<tr>
<td>Fallibilist/Problem Solving view</td>
<td>Learner-focused belief system</td>
</tr>
<tr>
<td>Absolutist/Platonist view</td>
<td>Content-focused with emphasis on conceptual understanding belief system</td>
</tr>
<tr>
<td>Instrumentalist view</td>
<td>Content-focused with emphasis on performance belief system</td>
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</tbody>
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Figure 2.4. Framework of teachers’ beliefs about mathematics and mathematics teaching.

**Summary**

In summary, teachers’ knowledge for teaching mathematics (or rational numbers) is a tripartite unit. It is composed of teachers’ personal knowledge of mathematics, teachers’ pedagogical mathematical content knowledge, and teachers’ beliefs about mathematics and mathematics teaching. Attempts to change any one part will not necessarily result in overall improvement of the whole (Heaton, 2000; Wilcox, Lanier, Schram, & Lappan, 1992). In particular, it appears that addressing teachers’ beliefs is an important part of changing other aspects of their knowledge for teaching mathematics.
An examination of teachers’ personal and pedagogical knowledge of mathematics is likely to also reveal their beliefs about mathematics and mathematics teaching. Since teachers’ beliefs are so tightly bound with teachers’ personal and pedagogical knowledge for mathematics, teachers’ beliefs about mathematics are included in the category of teachers’ personal knowledge of mathematics and teachers’ beliefs about teaching mathematics in the category of teachers’ pedagogical mathematical content knowledge.

It is not enough to have theories of the knowledge construction for rational numbers and theories of teachers’ knowledge to examine interrelationships between teachers’ knowledge of rational numbers and students’ emerging knowledge of rational number. It is also necessary to have a theory of where teachers’ knowledge and children’s knowledge interact, the classroom. The following section examines the nexus of instructional practice and the classroom, classroom culture development.

Interrelationships Between Instructional Practice and Classroom Culture

Work by Lave (1993), Bauersfeld (1992, 1995), and others shed light on how instructional practices may contribute to how students construct their understanding of a topic. Of note in their arguments is the situatedness of cognition. Hiebert and Carpenter (1992) reflect that all knowledge is situated and is partly the result of the activity, context, and culture in which it is developed (Brown, Collins, & Duguid, 1989). Not only is knowledge a partial result of these three interrelated components, but the component also remains a referent by which knowledge is retrieved, interpreted, and used.
Bauersfeld (1995) states, "Students arrive at what they know about mathematics mainly through participating in the social practice of the classroom rather than discovering external structures existing independent of the students" (p. 151). He argues that mathematical knowledge is more correctly recognized as a social accomplishment. It arises and has meaning in the context of a particular classroom environment. The observed similarities between different classrooms in different schools and communities and countries are coincidences of shared symbolism, but the meaning given to the symbols are those developed by the specific group using them. Thus, the individual conducts her/his mathematical activities according to her/his interpretation of mathematics within the classroom context. Together, teacher and students create their mathematics via their development of norms and regularities of speech and activity.

Doyle (1983) posits that the teacher plays a significant part in the development of the classroom culture. He argues that since students are, for expedience sake, grouped and assigned to teachers for specific periods of time, classrooms will always exist with their accompanying social processes necessitating some type of management on the part of the teacher.

Because classrooms are groups, teachers are faced with the task of organizing students into working groups and maintaining this organization across changing conditions for several months .... The immediate task of teaching in classrooms is that of gaining and maintaining the cooperation of students in activities that fill the available time. (p. 179)

The expectation then is that the teacher has primary responsibility for developing those mechanisms by which the group of students becomes a class culture. A teacher's success at achieving cooperation is a matter of the teacher's rapport with students (Kulik & McKeachie, 1975), the activities being used (Kounin & Gump, 1974), the students being
taught (Metz, 1978), the task students are required to accomplish (Morine-Dershimer, 1982; Redfield & Roenker, 1981), and the teachers skill at managing the activities as they are being carried out (Emmer, Evertson, & Anderson, 1980; Evertson & Emmer, 1982).

Cobb and Yackel (1995) have been exploring "ways to account for students' mathematical development as it occurs in the social context of the classroom." (p. 4). They critique accounts in which teachers are said to establish or specify the social norms for students.

To be sure, the teacher is necessarily an institutionalized authority in the classroom (Bishop, 1985). However, in our view, the most the teacher can do is express that authority in action by initiating and guiding the renegotiation process. The students also have to play their part in contributing to the establishment of social norms. (Cobb & Yackel, 1995, p. 7)

Thus, the interrelationships of teachers' constructions of mathematical knowledge and their students' construction of knowledge may be seen as being more complex. As a part of the research process Cobb and Yackel have developed an interpretive framework (see Figure 2.5) that they propose as being useful for examining not only students' mathematical development in the social context, but also for analyses of teachers' socially-situated activity. The framework illustrates the complementarity of the social perspective and psychological perspectives. Together these two theoretical perspectives "encompass the actively cognizant student, the local social situation of development and the established mathematical practices of the wider community" (Cobb, 1995, p. 381).
The construct of classroom social norms from the framework refers to the participation structure of the classroom. Classroom social norms are not the construction of any one individual but are a joint social construction. Cobb and Yackel (1995) contend that as teacher and students contribute to the establishment of the classroom social norms, they reorganize their individual beliefs about their own role, others' roles, and the general nature of mathematical activity. Thus, the students’ and teachers’ beliefs are the psychological correlates of the classroom social norms. Neither beliefs nor social norms are considered to have primacy over one another, but are reflexively related to each other in that neither exists without the other. Social norms evolve as students and teacher
reorganize their beliefs and, conversely, the reorganization of their beliefs is enabled and constrained by the evolving social norms.

Cobb and Yackel (1995) define sociomathematical norms to include what counts as a different mathematical solution, a sophisticated solution, an efficient mathematical solution, and an acceptable mathematical solution as negotiated by teacher and students. As students develop intellectual autonomy in the classroom, they do so only to the extent to which they have developed personal ways of judging when it is appropriate to make a mathematical contribution to the class and what constitutes an appropriate mathematical contribution. Students construct specific mathematical beliefs and values that enable them to act as increasingly autonomous members of the classroom mathematical community. These beliefs are the psychological correlates of the sociomathematical norms and as in the case of classroom social norms these are in a reflexive relation—one does not exist without the other.

Classroom mathematical practices constitute those practices which students participate in while doing mathematical activities. One example described by Cobb and Yackel (1995) from their research is that of second grade students' solution methods involving counting by ones that were established at the beginning of the school year. During the school year, some students developed solutions that involved the conceptual creation of units of ten and one, but they were obliged to explain and justify their interpretations of number words and numerals. Later in the school year, the classroom community took the solutions based on these interpretations as self-evident. Interpreting number words and numerals in this way had become an established mathematical practice
that no longer needed justification. From the students' viewpoint, numbers were simply composed of tens and ones—it is a mathematical truth.

Cobb and Yackel (1995) contend that detailed analyses of the development of classroom practices over a period of time, such as the one described, document instructional sequences as they are realized in interactions in the classroom. Such analyses are of theoretical significance as they reveal information about mathematical learning as it occurs in the social context of the classroom. Classroom mathematical practices constitute the immediate and local situations of students' development. Analysis of these practices can document the evolving social situations in which students participate and learn. The psychological correlates of these classroom mathematical practices are individual students' mathematical conceptions and activities. Again, as in the case of classroom social norms and sociomathematical norms, classroom mathematical practices are in a reflexive relationship with students' mathematical conceptions and activities. As students reorganize their individual mathematical activities, they actively contribute to the evolution of classroom mathematical practices and, conversely, students' individual reorganizations of their mathematical activities are enabled and constrained by their participation in the classroom mathematical practices. Psychological analyses of individual students' mathematical interpretations, as they participate in the same classroom mathematical activities, reveal qualitative differences. Social analyses of classroom mathematical practices conducted from the interactionist perspective reveal the joint establishment of meanings by teacher and students as they coordinate their individual activities. Taken together these analyses reveal the subtle
interplay between the development of individual understanding and the evolution of shared meaning within the local social world of the classroom.

The role this framework will play in the current study is to bring into focus those negotiations between students and teachers by which they come to define rational numbers and construct the practices for operating with and on rational numbers. A fundamental aspect of this study is the examination of how meanings and activities with rational numbers are constructed in classrooms which may have widely varying cultures and in which teachers have varying concepts of rational numbers as compared to the concept of rational number found in the larger mathematical culture represented by mathematicians and other "experts" on mathematics. An in-depth look at these negotiations of norms and practices may reveal the links between teachers' conceptual knowledge of rational number and their students' developing concepts (learning) of rational number.

Theoretical Framework

In summary, the preceding sections of the literature review, "Rational Number Knowledge: Internal Knowledge Structures and Knowledge Construction," "Interrelationships Between Teacher Knowledge and Instructional Practice," and "Interrelationships Between Instructional Practice and Classroom Culture" provide the basis for a theory of interrelationship between teachers’ knowledge of content and students’ emergent content knowledge. The diagram in Figure 2.6 illustrates this theory.

In the center of the diagram is the instructional environment. The instructional environment here connotes the social space where teaching and learning occurs. As noted
in the diagram, the instructional environment is a negotiated space in which “taken-as-shared” meanings occur. Both teacher and students are involved in making sense of the communications of their co-respondents within the instructional environment. Arrows on either side of the Instructional Environment box indicate the direction of influence that flows to and from the Instructional Environment. On the teacher side of communication teacher’s instructional decisions are shaped by his or her pedagogical content knowledge

Figure 2.6. Theory of interrelationship between teachers’ knowledge of content and students’ emergent content knowledge.

is a box labeled the teacher’s pedagogical content knowledge and decisions. The teachers’ instructional decisions are shaped by his or her pedagogical content knowledge and include the instructional themes that are developed by the teacher in the instructional environment. The teacher’s communication in the instructional environment is structured by his or her pedagogical content knowledge. Thus, the arrow pointing from the Teacher’s Pedagogical Content Knowledge box to the Instructional Environment box
indicates the shaping of the teacher’s communication within the instructional environment by his/her pedagogical content knowledge. On the student side of the instructional environment box is a box labeled students’ prior knowledge/predispositions/experiences. Students’ communications within the instructional space are shaped by their prior knowledge, predispositions, and internalized instructional experiences; hence, the arrow pointing from the Students’ Prior Knowledge box to the Instructional Environment. Thus, arrows pointing to the Instructional Environment box indicate the influences of teacher’s pedagogical content knowledge/decisions and students’ prior knowledge/predispositions/experiences on the instructional environment.

Similarly, arrows pointing away from the Instructional Environment box indicate the influence of the instructional environment on the communication of both students’ and teacher. On the teacher side of communication, the teacher is creating models of the students’ understanding of the content influenced by their observations of students’ behaviors and communications within the instructional environment; hence, the arrow from the Instructional Environment to the Teacher’s Pedagogical Content Knowledge/Decisions box. On the student side of communication, the student is making models of meaning based on their observations of teacher’s behaviors and communications including instructional themes. Both teacher and students’ test their models for viability in the instructional environment. Viable models are retained and may become part of their respective knowledge bases. Non-viable models are revised and tested again in the instructional environment.
Both the Teacher Pedagogical Content Knowledge and the Students’ Prior Knowledge/Predispositions/Experiences boxes indicate that each function as a filter/mediator of the influences of the communication occurring in the instructional environment and the respective content knowledge. In particular, the influence of a teacher’s personal content knowledge is filtered or mediated by his/her pedagogical content knowledge. Similarly, the influence of the instructional environment on a students’ emergent knowledge of content is filtered or mediated by their prior knowledge/predispositions/experiences. The use of the terms filter and mediator are used to indicate the limiting and focusing aspects of both the teacher’s pedagogical content knowledge and students’ prior knowledge/predispositions/experiences.

Another significant aspect of the diagram is that the Teacher’s Content Knowledge and the Outside Curricular Decisions are not included in the same box as the Teacher’s Pedagogical Content Knowledge. This is to indicate that each influence pedagogical content knowledge/decisions but are not synonymous with it. Likewise Students’ Emergent Content Knowledge is separated from Students’ Prior Knowledge/Predispositions to indicate that they are not synonymous. In the latter case, Students’ Emergent Content Knowledge will eventually move from the development phase to the knowledge base, i.e., prior knowledge. Prior to that, it may influence prior knowledge. However, there exists lag time between the development of content knowledge and its full appropriation into the knowledge base. Finally, a Teacher’s Pedagogical Content Knowledge can influence the Teacher’s Personal Content Knowledge but is not synonymous with it.
The theory has several interesting features. For example, it suggests an explanation for why the improvement of teachers’ personal content knowledge may be limited in its influence on students’ emergent content knowledge. There are two filtering intermediaries (i.e., Teachers’ Pedagogical Content Knowledge/Decisions and Students’ Prior Knowledge/Predispositions/Experiences) between any potential influences of changes in teachers’ personal content knowledge and changes in students’ emergent content knowledge. However, the theory still suggests that a teacher’s personal content knowledge plays an important role in the development of students’ content knowledge via its influence on a teacher’s pedagogical content knowledge.

The theory suggests that teachers’ pedagogical content knowledge is an important factor in influencing development of student knowledge. The theory itself does not suggest what the nature of that pedagogical content knowledge should be, but the case studies provide illustrations of teachers’ pedagogical content knowledge such as general knowledge of how children learn as well as knowledge of how children learn particular content. The theory allows that a change in either personal content knowledge or pedagogical content knowledge does not necessitate a commensurate change in the other. The linkage between the two is influential rather than direct. In addition, the theory allows for the change in teacher’s pedagogical content knowledge via the influence of instructional experience. It also suggests that the influence of the instructional environment on a teacher’s content knowledge is mediated by their pedagogical content knowledge.
The theory recognizes the equally, if not more, important influence of students’ prior knowledge on development of new knowledge. It suggests that students may attempt to make sense of the actions/communication of the teacher in an instructional environment based on their prior knowledge/predispositions/experiences. Usually, students will develop models that are viable within the instructional environment. If a student does not have an adequate knowledge base to make viable models of a teacher’s actions/communications, then they will make non-viable models. As students make sense of the instructional environment they may make non-viable models of emergent content knowledge based on teachers’ unintentional patterns of actions/communications that are insignificant or immaterial to the content being taught. If a child is predisposed to certain types of models of content, they will choose actions/communications in the instructional environment that best match their predispositions and put less emphasis on actions/communications that do not match their predispositions. Thus, the theory suggests that correspondence of communication may occur between teachers and students having similar predispositions and dissonance of communication between teachers and students having dissimilar predispositions.

In defining the instructional environment as a negotiated meaning-making space, the theory recognizes that teaching/learning is not an act of transmitting knowledge from expert to novice. Rather, it is a negotiation between goal-oriented actors who are endeavoring to create models of meaning based on the actions/communications of their co-respondents. Each, teacher and students, endeavor to make models that are viable within the limited context of the particular instructional environment that they cohabit.
However, the needs of the larger content-based community require that the models of both teacher and students be viable beyond the walls of the classroom. The theory recognizes that the larger learning community plays a limited role via the input of certain curricular expectations/decisions. However, the primary actors in the classroom are the teacher and the students.

The sections of the literature review, “Rational Number Knowledge: Internal Knowledge Structures and Knowledge Construction,” “Interrelationships Between Teacher Knowledge and Instructional Practice,” and “Interrelationships Between Instructional Practice and Classroom Culture” provided the framework for writing the cases of the study. Each addresses one of three overarching constructs considered to be important for illuminating patterns of potential relationships between teachers’ content knowledge and students’ emergent content knowledge: teachers’ knowledge of rational numbers and their students’ emergent rational number knowledge, teachers’ knowledge of rational numbers for instruction, and instructional environment. The theory of relationships between teachers’ content knowledge and students’ emergent content knowledge provides the lens through which the researcher has viewed the field and has collected and analyzed the data that have formed the cases. The overarching constructs and the theory of relationships between teachers’ content knowledge and students’ emergent content knowledge are evident in the methodology, the cases, and the concluding discussion that follows.
CHAPTER 3

METHODOLOGY

To reiterate, the study described herein is intended to illuminate patterns that suggest potential relationships between teachers' knowledge of the domain of rational number, their students' emergent knowledge of rational number concepts, teachers’ knowledge for rational number instruction, and the instructional environment. It is based on observations of teachers’ and students’ interactions during teacher-designed instructional units on rational numbers and teachers’ and students’ responses to rational number test problems and interview tasks. The methodology to accomplish the study is outlined below. The methodology includes: (a) the initial framework for organizing the study, (b) an examination of the researcher who has conducted the study, (c) the sample and site selection for the cases, (d) the organization of the data collection, and (e) the organization and trustworthiness of the data analysis.

Initial Framework

The research strategy chosen to accomplish the study was an instrumental case study (Stake, 1994). Cases were constructed to examine the interrelationships between teacher knowledge and students’ emergent knowledge of rational number. A maximum variation sampling of two elementary school teachers in one grade level and in one
school system who were utilizing a curriculum that purports to be in line with the NCTM Standards (1989, 2000) and who represent each of the following types (Patton, 1990) was used:

- One teacher having knowledge of the rational number domain that is conceptual and connected and whose belief system for teaching is learner-focused.
- One teacher having compartmentalized, primarily procedural knowledge of the rational number domain, and whose belief system for teaching is content-focused with emphasis on performance.

Each teacher category contrasts types of teacher knowledge of mathematics and types of teaching style that arise from the theories proposed in the section of “Interrelationships Between Teacher Knowledge and Instructional Practice” in Chapter 2.

The patterns of relationship between each teachers' conceptual understanding of rational number and the changes which occur in their students' conceptual understanding of rational number were observed and analyzed. Differences between teachers' instructional styles and the differences in the classroom environments or cultures were examined for their interrelationships with teachers' conceptual knowledge of rational numbers and with students' development of their own conceptual knowledge of rational numbers. From these case studies, a model was proposed to describe the interrelationships between teachers' knowledge of rational numbers and students' emergent knowledge of rational number. The following sections describe the research design in more detail.
The Researcher

Because the researcher is responsible for the decisions made in the design, data collection, and analysis of the study, it is important to understand the qualifications, experiences, and perspectives of the researcher (Patton, 1990). The researcher of the study is a student finishing a Ph.D. program in mathematics education at a Midwestern university in an urban area. She earned a Master’s degree in mathematics for secondary teachers at the same university. A Southern university, also in an urban area, awarded her undergraduate degree. However, her undergraduate major was English with a minor in mathematics and she was certified to teach secondary English and mathematics. Her first teaching experience, immediately following her graduation from college, involved teaching English and subjects other than mathematics to fourth graders in a rural Appalachian community close to her home. The next year she began teaching secondary English and mathematics at her alma mater, a rural Appalachian secondary school. Two years later, she moved from her home in the South to Texas and began her career as a mathematics teacher in a large secondary school in a small Southwestern border city. After five years teaching mathematics to bilingual Hispanic students, she moved to the Midwest to pursue her Masters’ degree in mathematics. There she spent another six years teaching mathematics and computer science in a secondary school in a small college town outside a major urban area. In each venue, she usually taught lower level courses: pre-algebra, algebra 1, and geometry.

The researcher has always been motivated to understand mathematics conceptually. Her personal mathematics experience led her to believe that students who
understood mathematics conceptually were more successful in pursuing further mathematics studies than those that focused on purely procedural understanding. Her peers in mathematics at the elementary and secondary levels had always relied on her to help them understand the mathematics that they were studying. Those that she worked closely with during her secondary education also became secondary mathematics teachers.

The researcher has had an interest in understanding mathematics learning from an early age since her mother had been a secondary mathematics teacher. Her mother had participated as a “New Math” instructor for the researcher’s teachers from elementary school. As a result of this knowledge, the researcher became very aware of her teachers’ mathematics teaching practices. The researcher had always been cognizant of her mathematics-learning processes from her first school experiences. She was able to recall how she learned her addition facts such as “8 + 5 is 13 because 5 is 2 + 3 and 8 + 2 + 3 is 13.” When her mother pursued a doctorate in mathematics education while the researcher was in her middle and secondary grades, the researcher read and edited her mother’s dissertation. This further whetted her interest in mathematics education. Her deep interest in understanding mathematics teaching/learning motivated her to continue her mathematics studies, certification, and career in mathematics education.

The researcher had considerable experience teaching mathematics to secondary students transitioning from arithmetic to algebra. Because of her personal mathematics experiences, she had been a teacher who was concerned that her secondary students would be able to understand the mathematics that they were learning conceptually. She
was never satisfied by a student’s request to “just show me how to do it.” She became aware of her own tendency to focus on procedure in some contexts and worked on her teaching to explore and reveal the meanings behind procedures to her students. Her discontent with her own practice motivated her to participate in professional development programs and further degrees in mathematics education. The researcher saw herself as the bridge between her students and the larger community of mathematics practitioners—the mathematics learning community.

In an exercise prior to beginning her dissertation, the researcher voiced her perspectives related to the study herein.

I am interested in studying teachers as the focal point of learning. I believe that teachers are the curriculum.

I believe that learning is a social act, a negotiation between teachers and students—teachers are the guides into a social group, the knowing group, those who share knowledge. I resonate with symbolic interactionism as a way of looking at meaning making, which is the essence of the teaching/learning act. Learning is enculturation.

I believe that mathematics is a social act. Mathematical acts are acts of meaning making.

I want to look at how knowledge is communicated from one generation to the next. I want to construct a model of what teachers are about—the act of teaching/learning as the creation/transfer of knowledge. How what the larger community might define as a body of knowledge about one aspect such as rational numbers is adopted/recreated/reinvented by novices. This requires knowing how the teacher is the embodiment/representative of the larger community, a lens through which students perceive the nature of the larger community.

I am concerned about the image of replacing the teacher with a computer program, a textbook, etc.—perceiving the teacher as an unnecessary middle person who should be replaced with some more “pure” form of knowledge transfer mechanism.
I am concerned about teachers’ lack of adequate knowledge as measured by others and felt by students.

I want to do this to develop a more humane/genuine/legitimate engagement of teachers into the culture of teaching/learning/doing mathematics.

I suspect that many teachers have not been incorporated legitimately/genuinely/humanely into the mathematics community, rather they have been marginalized and thus see themselves as other and lack affinity for mathematics and will not be as successful in their attempts to enculturate others.

These perspectives influenced the researcher’s decision-making processes in the sampling, data collection, and data analysis described in the following sections. They indicate potential limitations to the study in that the researcher may not be as aware of the curriculum and the pedagogical knowledge available as a researcher who has had personal experience teaching middle grades mathematics.

Site and Sample Selection

Because of the possible interference of differing scope and sequence, curricula and available curriculum materials between school systems, one of the criteria for choosing the teachers for the main part of the study was that they would be from the same school district. The sampling procedure used in the study was a mixed purposeful sampling (Patton, 1990). Many districts were contacted and three gave permission to study in their schools. The procedure that the researcher anticipated using to locate cases was the snowball sampling method. Peers provided names of fifth grade teachers who might be interested in the study, fifth grade teachers who had been contacted named other fifth grade teachers, and district personnel identified candidate fifth grade teachers. From the teachers contacted, those who were willing participated in the study. Thus, the sample was a convenience sample of five teachers, two from an urban school district and three
from a suburban school district. From the initial sample, two cases of maximum variation from the same school district were chosen to represent the extremes of the continuum of teacher knowledge and practice described in the “Introduction to Methodology.” Specifically, from among the five teachers who took the teachers’ test (Rational Number Test, Millsaps Version T, 12/18/96, see Appendix A), two teachers were chosen who were the most different in their responses and who would be teaching in the same district. The requirement that the teachers be from the same school district was made to manage the possible interference of differing scope and sequence, curricula and available curriculum materials between school systems.

Data Collection

Data were collected to describe teachers’ and students’ emergent conceptual knowledge of rational numbers, teachers’ knowledge for instruction including pedagogical content knowledge and beliefs related to rational numbers, and classroom culture. The following describe the methods employed to collect data for each.

**Describing Conceptual Knowledge of Rational Numbers**

To assure the validity of the case descriptions of student and teacher conceptual knowledge of rational number, data were triangulated (Stake, 1994). Three methods of data collection were used: rational number tests, standardized open-ended task interviews, and artifact collection. Each of the methods is described. Although not a method used to collect data on participants’ emergent conceptual knowledge of rational numbers, the observational data collected during the teachers’ rational number
instructional unit provided additional confirmation or disconfirmation of patterns constructed from primary data collection sources.

*Rational Number Tests*

The primary method for collecting data on conceptual knowledge of rational number was the development of rational number tests. Three factors constrained the development of the instruments for describing teachers' and students' rational number knowledge. First, describing an individual's conceptual knowledge of any mathematics topic using a complex definition that adequately addresses the nature of mathematics is not easy, and most instruments used have not been based on such definitions. A procedure was needed that could describe the interrelations of ideas, the applications of ideas, and the processes of mathematics (Webb, 1992). Second, since the study proposed to find interrelationships between teachers' and students' knowledge, it was necessary to assure that the instruments were in fact describing the same things. To that end, an attempt was made to use the same instrument for teachers and students. However, the third factor, time, made using the same instrument problematic. Teachers did not have the time to allow students to take a long test in class. The teachers’ test was too long for the students and the pretest student version was also too long. Therefore, the instrument was modified by reduction over the course of the research, and analysis of data was restricted to the data available on items parallel to those in the last version.

Because of anticipated time constraints and the availability of preexisting rational number instruments, the researcher decided to build a test from items developed by other researchers. Four instruments were examined for possible items to test rational number
knowledge: a written test developed by the Rational Number Project (Behr, Cramer, Post, & Lesh, 1996), Rational Number Thinking Test (Kieren, 1990), Grades 4 and 6 Pre- and Posttests, (Owens & Menon, 1991), and Test of Fraction Understanding (Niemi, 1994). The items from each test were compared and analyzed to form categories of items. The constructs of rational number knowledge up to level three from Kieren’s (1993) ideal network of personal rational number knowledge (see Figure 2.1: partitioning, equivalence, unit forming, measure, quotient, ratio, and operator) provided the main source of category types and names. Other category names for content topics (concepts/constructs) were developed as needed. The final list of category names included: measure, quotient (partitive), quotient (quotitive), operator (mapping), ratio, part/whole, computation (addition/subtraction), ordering/comparing, equivalence, concept of unit, and estimation. A second set of categories described the types of representational systems that were presented in each item such as visual, symbolic, and contextual. Three types of visual representations were identified: continuous area, continuous length, and discrete. Test items and categories were presented to experts to verify the validity of the constructs. Differences in responses were discussed until consensus was reached. The initial test, Rational Number Test, Millsaps Version T, 12/18/96 (see Appendix A), had 40 items. Since tests, especially multiple choice tests or short answer tests, have the shortcoming that they do not truly reveal what the participant is thinking as she/he is answering the questions, each item of the test asked for explanations for solutions provided. The teachers’ version was designed to capture teachers’ rich knowledge of a construct by asking for multiple solution paths to the same
result. It also was intended to capture some of the pedagogical knowledge by asking if the teacher thought the item would be appropriate for his or her students.

Since the teacher version of the test was deemed too long to give to students, a second version of the test was developed, Rational Number Test, Millsaps Version S, 3/4/97. It was composed of 25 of the original test problems. The items were chosen to represent the rational number content topics (concepts/constructs) and representations described previously and with consideration given to the teachers’ responses about appropriateness of test problems on their version of the test. The requests for multiple solution methods and appropriateness of item for instruction were not appropriate for the student test and were removed. However, since an important function of the test was to describe students’ emergent conceptual knowledge, it was necessary to provide children with a motive for revealing how they had formulated their solutions. Students could generate the same answer for very different reasons, some of which might not be at all related to the topic of the question or a conceptual understanding of the topic. For each item, students were asked to explain why they thought that their answer was correct. Rational Number Test, Millsaps Version S, 3/4/97 was a pretest given to students at the beginning of their teachers’ fraction instructional unit.

Although the test length had been reduced from the original 40 items for the student version, the teachers still thought the test was too long. The researcher modified the plan by suggesting that the teachers give the test in a manner that would be comfortable to them. Teachers chose to give the test as a take-home assignment and chose some of the items on the test for their students to do. As a result, there was not
parity between the two classes. Therefore, it was not useable as a pretest. Nevertheless, the initial student test did provide a tool for selecting items for the posttest and for suggesting relationships between students’ prior and post unit knowledge of rational numbers that confirmed and disconfirmed patterns of interrelationships between students’ emergent knowledge and teachers’ knowledge of rational numbers that other data may have suggested.

Because the teachers deemed the 25-item test too long, a third version of the test was developed, Rational Number Test, Millsaps Version S, 5/27/97 (Appendix B). The teachers administered the third version to their students as a posttest. Fourteen items were chosen from the Rational Number Test, Millsaps Version S, 3/4/97 for the third version. The items were chosen based on students’ responses to the pretest items, observations of the teachers’ fraction instructional units, and the provision that sufficient information would be available to describe students’ and teachers’ rational number knowledge. Figure 3.1 describes the rationale for the inclusion of each of the 14 test problems.
<table>
<thead>
<tr>
<th>Item No.</th>
<th>Test Problem</th>
<th>Rationale for Inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (1)</td>
<td>□</td>
<td>Given a non-standard continuous area model of the unit whole and fractional amount, can students represent the fractional amount with an appropriate fraction symbol? Elicits children's development of level one constructs (Behr et al, 1992; Kieren, 1988; Steffe, 2001): partitioning, quantitative equivalence, and unit forming.</td>
</tr>
<tr>
<td></td>
<td><img src="image" alt="" /> is one unit, what fraction <img src="image" alt="" /> is ______? Explain why you think your answer is correct.</td>
<td></td>
</tr>
<tr>
<td>2 (2)</td>
<td><img src="image" alt="" /> is two thirds of some length. Draw the whole length below and explain why it is the whole.</td>
<td>Given a continuous linear model of a fractional amount and an associated fraction symbol, can students represent the unit whole in terms of the given model. Elicits children's development of level one constructs: partitioning, quantitative equivalence, and unit forming.</td>
</tr>
<tr>
<td>3 (3)</td>
<td>Draw in the box to the right a set which has 2/3 as many circles as the set of circles in the box on the left. <img src="image" alt="" /> Explain why you think your answer is correct.</td>
<td>Given a discrete model of a whole and the fraction symbol of a fractional amount, represent the fractional amount in terms of the given model. Elicits children's development of level one constructs (partitioning, quantitative equivalence, and unit forming) and level two constructs (quotient and operator).</td>
</tr>
<tr>
<td>4 (6v)</td>
<td>Write a fraction to show what part is shaded. Explain why you think your answer is correct.</td>
<td>Given a circular continuous area model of a unit whole with a partition of the whole not directly related to the shaded fractional amount, represent the shaded fractional amount with a fraction symbol. Elicits children's development of level one constructs: partitioning, quantitative equivalence, and unit forming.</td>
</tr>
</tbody>
</table>

Note. Item numbers in parentheses are from corresponding items in Rational Number Test, Millsaps Version T, 12/18/96.

Figure 3.1. Items from Rational Number Test, Millsaps Version S, 5/27/97.
Figure 3.1 continued

<table>
<thead>
<tr>
<th>Item No.</th>
<th>Test Problem</th>
<th>Rationale for Inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 (7)</td>
<td>For each picture below, write a fraction to show what part is shaded. Choose one picture and explain why you think your answer is correct.</td>
<td>Given a rectangular continuous area model of a unit whole with a partition of the whole not directly related to the shaded fractional amount, represent the shaded fractional amount with a fraction symbol. Elicits children’s development of level one constructs: partitioning, quantitative equivalence, and unit forming.</td>
</tr>
<tr>
<td>6 (13)</td>
<td>Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.</td>
<td>Elicits children’s ability to compare extensive quantities. Elicits children’s development of level two constructs (quotient, ratio), level three constructs (scalar and function), and the level four construct of formal equivalence. Elicits children’s understanding of rational numbers in context.</td>
</tr>
<tr>
<td>7 (17, 18)</td>
<td>Three people are going to share these two pizzas equally. Color in one person’s part. Write a fraction that shows how much one person gets ____. Explain why you think your answer is correct.</td>
<td>Elicits children’s ability to find and represent fair shares and represent them with an appropriate fraction symbol. Elicits children’s development of level two constructs: measurement, quotient. Elicits children’s understanding of rational numbers in context.</td>
</tr>
</tbody>
</table>

Continued
<table>
<thead>
<tr>
<th>Item No.</th>
<th>Test Problem</th>
<th>Rationale for Inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 (19)</td>
<td><strong>Six people are going to share these five chocolate bars equally. Color in one person’s part.</strong>&lt;br&gt;<img src="image_url" alt="Image of chocolate bars" /></td>
<td>Elicits children’s ability to find and represent fair shares and represent them with an appropriate fraction symbol. Elicits children’s development of level two constructs: measurement, quotient. Elicits children’s understanding of rational numbers in context.</td>
</tr>
<tr>
<td>9 (22)</td>
<td>Each dark colored glass is chocolate syrup and each white colored glass is milk. Circle the mixture that has a stronger chocolate flavor: the mixture made using the glasses pictured in set A or the mixture made using the glasses pictured in set B. Circle both if the chocolate flavor is the same. Explain why you think your answer is correct.&lt;br&gt;<img src="image_url" alt="Image of glasses" /></td>
<td>Elicits children’s ability to compare extensive quantities. Elicits children’s development of level two constructs (quotient, ratio), level three constructs (scalar and function), and the level four construct of formal equivalence. Elicits children’s understanding of rational numbers in context.</td>
</tr>
</tbody>
</table>
| 10 (29) | For each row of fractions below, which fraction is the greatest and which fraction is the least? Explain why you think your answer is correct for two of the rows.<br>\[
\begin{array}{ccc}
\frac{1}{8} & \frac{1}{7} & \frac{1}{6} \\
\end{array}
\]
\[
\begin{array}{ccc}
\frac{6}{7} & \frac{8}{9} & \frac{7}{8} \\
\end{array}
\]
\[
\begin{array}{ccc}
\frac{3}{7} & \frac{4}{9} & \frac{4}{5} \\
\end{array}
\]| Elicits children’s construction of order of rational numbers. Elicit children’s development of the level two construct of measure as applied to rational numbers and the level four construct of formal equivalence. |
Figure 3.1 continued

<table>
<thead>
<tr>
<th>Item No.</th>
<th>Test Problem</th>
<th>Rationale for Inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>11 (31)</td>
<td>Write one fraction that is the same as each fraction below. Explain why you think your answer is correct for one of the columns. For example, $\frac{1}{2} = \frac{2}{4}$ $\frac{3}{9} = \frac{1}{7} = \frac{8}{12} = \text{____}$</td>
<td>Elicits children’s development of the level four construct of formal equivalence.</td>
</tr>
<tr>
<td>12</td>
<td>Name a fraction that is somewhere between the two given fractions. Explain why you think your answer is correct for one of the rows. $\frac{1}{3} = \text{<strong><strong>} \frac{3}{4}$ $\frac{3}{5} = \text{</strong></strong>} \frac{4}{5}$ $0 = \text{____} \frac{1}{7}$</td>
<td>Elicits children’s construction of order of rational numbers. Elicit children’s development of the level two construct of measure as applied to rational numbers and the level four construct of formal equivalence.</td>
</tr>
<tr>
<td>13 (35)</td>
<td>Liana ate $\frac{2}{3}$ of a small pizza. The next day she ate $\frac{1}{6}$ of a small pizza. How much of a pizza did she eat altogether? Explain why you think your answer is correct.</td>
<td>Elicits children’s understanding of computation with rational numbers. Elicit children’s development of the level two construct of measure, the level four construct of formal equivalence, and the level five construct of additive structures. Elicits children’s understanding of rational numbers in context.</td>
</tr>
<tr>
<td>14 (36)</td>
<td>Ann and Josie receive the same allowance. Josie spent $\frac{3}{5}$ of hers on CDs. Ann spent $\frac{1}{10}$ of her allowance on repairing her bicycle. Josie spent how much more of her allowance than Ann? Explain why you think your answer is correct.</td>
<td>Elicits children’s understanding of computation with rational numbers. Elicit children’s development of the level two construct of measure, the level four construct of formal equivalence, and the level five construct of additive structures. Elicits children’s understanding of rational numbers in context.</td>
</tr>
</tbody>
</table>
Interviews on Rational Number Knowledge

Standardized open-ended individual interviews (see Appendix C) were designed to gather additional data about teachers’ and students’ emergent knowledge of rational numbers (Patton, 1990). The use of prompts that mirrored Rational Number Test, Millsaps Version S, 5/27/97 (see Appendix B) items allowed for a check of the descriptions of teachers’ and students’ knowledge suggested by their test responses. Only the symbolic comparison items (i.e., test problems 10-12) were not replicated in the interview protocol. Because of time limitations, ten tasks were used for the interview protocol. The following table, Figure 3.2, describes the rationale for the inclusion of each interview task.

In addition to providing contexts for identifying knowledge of the identified rational number concepts, the interview protocols provided contexts for accessing different types of performance related to understanding of the concepts (Bisanz & LeFevre, 1992). As Bisanz and LeFevre note, there are several different contexts for understanding. There is (a) the type of understanding that is demonstrated when performing a task, (b) the type of understanding that is demonstrated when an individual can justify the procedures used in performing a task, and (c) the type of understanding that is demonstrated when an individual judges the validity of using a particular procedure on a given task. Each of these three contexts reveals a different aspect of the meaning of “to understand.” Though they may not be discrete, success in one context does not imply success in the other two. A fourth dimension of understanding is demonstrated: (d) when an individual recognizes how and when to modify or to use a
procedure on problems of varying similarity. The terms that Bisanz and LeFevre apply to these four contexts are respectively: (a) application of procedures, (b) justification of procedures, (c) evaluation of procedures and (d) generality. These four contexts for understanding constitute a profile of understanding. Videotaping, audiotaping, and the taking of field notes to record observations of what the students and teachers were doing as they participated in the interview task allowed the collection of multidimensional data.

<table>
<thead>
<tr>
<th>Item No.</th>
<th>Interview Task</th>
<th>Rationale for Inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Provide a set of fraction circles.</td>
<td>Given a circular continuous area model of a unit whole with multiple potential partitions of the whole, represent the fractional amount indicated by a given fraction symbol with the given model. Elicits children's development of level one constructs: partitioning, quantitative equivalence, and unit forming.</td>
</tr>
<tr>
<td></td>
<td>(a) Use the fraction circles to show the fraction 3/8.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(b) How do you know this is 3/8? Please draw a picture on this sheet to show what you did.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(c) If you didn't have these pieces [if they used 1/8 pieces in the previous problem, remove all but one 1/8 piece], could you use other fraction circle pieces to show 3/8? [If yes] How? [If they do] Explain how your two ways of showing 3/8 are alike and different.</td>
<td>Given a circular continuous area model of a unit whole with limited partitions of the whole, represent the fractional amount indicated by a given fraction symbol with the given model. Elicits children's development of level one constructs (partitioning, quantitative equivalence and unit forming), a level two construct (measure), the level four construct equivalence, and the level five construct of additive structure.</td>
</tr>
<tr>
<td>2</td>
<td>Display 16 tiles without counting or telling the child how many there are.</td>
<td>Given a rectangular discrete model of a unit whole, represent the fractional amount indicated by a given fraction symbol with the given model. Elicits children's development of level one constructs (partitioning, quantitative equivalence, and unit forming).</td>
</tr>
<tr>
<td></td>
<td>(a) Say: You can arrange the tiles any way you want to show me the fraction 3/8.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(b) Explain what you were thinking in order to solve this problem. Please draw a picture on this sheet to show what you did.</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3.2. Items from Formal Interview Guide.
Figure 3.2 continued

<table>
<thead>
<tr>
<th>Item No.</th>
<th>Interview Task</th>
<th>Rationale for Inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(b) Show me (\frac{3}{8}) using a different number of tiles. How are the two ways of using tiles alike?</td>
<td>Given a rectangular discrete model of a unit whole, represent the fractional amount indicated by a given fraction symbol with the given model. Elicits children's development of level two constructs (ratio and operator).</td>
</tr>
<tr>
<td></td>
<td>(d) How is this way of showing (\frac{3}{8}) like using the fraction circle pieces? How are they different?</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Display a 4 x 4 geoboard without counting or telling the child how many grids there are.</td>
<td>Given a rectangular area model of a unit whole, represent the fractional amount indicated by a given fraction symbol with the given model. Elicits children's development of level one constructs: partitioning, quantitative equivalence, and unit forming.</td>
</tr>
<tr>
<td></td>
<td>(a) If we count this area on the geoboard [bounded by a rubber band around the outside pegs] as one whole, show the fraction (\frac{3}{8}) using this rubber band.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(b) Explain what you were thinking in order to solve this problem.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(c) Show me (\frac{3}{8}) using a different arrangement of the rubber band on the geoboard. How are the two ways of showing (\frac{3}{8}) on the geoboard alike? different?</td>
<td>Given a rectangular area model of a unit whole, represent the fractional amount indicated by a given fraction symbol with the given model. Elicits children's development of level one constructs (partitioning, quantitative equivalence, and unit forming) and level two constructs (measure and operator).</td>
</tr>
<tr>
<td></td>
<td>(d) How is this way of showing (\frac{3}{8}) like using the fraction circle pieces or the tiles? How are they different?</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Three people are going to share these two pizzas equally. Color in one person's part. Explain what you were thinking in order to solve this problem.</td>
<td>Elicit children’s ability to find and represent fair shares and represent them with an appropriate fraction symbol. Elicits children’s development of level two constructs: measurement and quotient. Elicits children's understanding of rational numbers in context.</td>
</tr>
</tbody>
</table>

What fraction of a pizza is one person’s part? Explain how you know you are correct.
<table>
<thead>
<tr>
<th>Item No.</th>
<th>Interview Task</th>
<th>Rationale for Inclusion</th>
</tr>
</thead>
</table>
| 5       | Show and read the statement:  
Tim used 6 cubes to build 1/5 of a blue tower. How many cubes long will the whole tower be when it’s finished?  
(a) Provide unifix cubes and ask student to show you how to use these cubes to solve the problem. Ask students to talk aloud as they solve the problem.  
(b) [If correct repeat changing the data: 8 cubes; 2/3] | Given a discrete linear model of a unit fractional amount and an associated fraction symbol, represent the unit whole with the given model. Elicits children’s development of level one constructs: partitioning, quantitative equivalence, and unit forming. |
| 5       | [If correct repeat changing the data: 8 cubes; 2/3] | Given a discrete linear model of a non-unit fractional amount and an associated fraction symbol, represent the unit whole with the given model. Elicits children’s development of level one constructs (partitioning, quantitative equivalence, and unit forming) and level two constructs (measure and operator). |
| 6       | Show the 1/8 fraction piece.  
(a) Say: This is 1/6 of my unit. With your fraction circles show me the unit. Talk aloud as you solve the problem explaining each step. (Record answer.) | Given a non-standard continuous area model of a unit fractional amount and an associated fraction symbol, represent the unit whole with the given model. Elicits children’s development of level one constructs: partitioning, quantitative equivalence, and unit forming. |
| 6       | [If correct change data: 1/4 piece is 2/3; find the unit.] | Given a non-standard continuous area model of a non-unit fractional amount and an associated fraction symbol, represent the unit whole with the given model. Elicits children’s development of level one constructs (partitioning, quantitative equivalence, and unit forming) and level two constructs (measure and operator). |

Continued
<table>
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<tr>
<th>Item No.</th>
<th>Interview Task</th>
<th>Rationale for Inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.</td>
<td>Elicits children’s ability to compare extensive quantities. Elicits children’s development of level two constructs (quotient and ratio), level three constructs (scalar and function), and the level four construct of formal equivalence. Elicits children’s understanding of rational numbers in context.</td>
</tr>
<tr>
<td>8</td>
<td>Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.</td>
<td>Elicits children’s ability to compare extensive quantities. Elicits children’s development of level two constructs (quotient and ratio), level three constructs (scalar and function), and the level four construct of formal equivalence. Elicits children’s understanding of rational numbers in context.</td>
</tr>
<tr>
<td>9</td>
<td>Show and read this story to the student: Sally ate 2/3 of a pizza for dinner. The next morning she ate another 1/6 of a pizza. (a) Say: Without working out the exact answer, give me an estimate of How much pizza she ate altogether that is reasonable. (If needed, provide clues: Is the answer &gt; 1/2 or &lt; 1/2? Is the answer &gt; 1 or &lt; 1?) (b) Say: Tell me what you were thinking to reach this estimate. (c) Say: Using fraction circles, act out how you would find the exact answer. Talk aloud as you solve the problem. (d) If the student was successful, ask the student to record each step with the fraction circles with symbols.</td>
<td>Elicits children’s understanding of computation with rational numbers. Elicits children’s development of the level two construct of measure, the level four construct of formal equivalence, and the level five construct of additive structures. Elicits children’s understanding of rational numbers in context.</td>
</tr>
</tbody>
</table>
Student Interviews on Rational Number Knowledge

The interviews were conducted with selected students. Teachers were asked to select students for the interviews based on teachers' ranking of all the students in the class according to the teachers' perceptions of the students’ achievement during the rational number instructional unit—high, middle, low. Based on the observations of classroom interaction, the researcher also made suggestions for interviewees. Seven students were selected in one class and eight in the other, at least two for each achievement category per class. Each student was interviewed individually in a separate room.
Artifacts of Rational Number Knowledge

For the third aspect of triangulation of data to describe students’ emerging content knowledge, artifacts of student work were gathered at the end of the rational numbers unit. One teacher provided students’ worksheets and the other provided students’ portfolios of their work in class.

Describing Teachers’ Pedagogical Content Knowledge and Classroom Culture for Instruction

Observations of the teachers’ rational number instructional unit were used to collect data for describing teachers’ pedagogical content knowledge and for describing the classroom culture or instructional experiences of the teachers and the students. Standardized open-ended interviews were used to collect data for describing teachers’ pedagogical content knowledge and for describing teachers’ beliefs. Artifacts or the collection of students’ work provided a third source of data for describing teachers’ pedagogical content knowledge and the classroom culture or instructional experiences of the teachers and the students. The following two sections explain in more detail the observation and interview methods for collecting data to describe teachers’ pedagogical content knowledge and beliefs and the classroom culture or instructional experiences of the teachers and the students.

Observation of Teachers’ Rational Number Instructional Unit

To collect observational data for describing teachers’ pedagogical content knowledge and the classroom culture or instructional experiences of the teachers and the students, the classroom sessions of the selected teachers’ unit on rational numbers were
videotaped and audiotaped over the period of time that it took for them to complete their unit. The researcher also took field notes of significant happenings and other information during or immediately following the observations. The particular focus of the observation was the mathematics class time. Videotaping in each class was done at least twice a week when feasible. Teachers’ wore individual microphones to collect data on their teacher-student interactions. Another microphone in the classroom provided information about student-student interactions as well as additional instances of teacher-student interaction. The data from audiotaping were collected on a single tape recorder. Audio from the camera was separate and captured audio interaction that was within the range of the camera microphone. Videotaping focused primarily on teacher-student interactions, although other class interactions were also captured. The camera was set up in the middle of the rear of the classroom to capture as much of classroom interaction as possible. Videotaping and audiotaping captured evidence of classroom social norms, socio-mathematical norms, and classroom mathematics practices as well as evidence of the participants' (teachers' and students') taken-as-shared conceptual knowledge of rational number. Additional data for describing teachers’ pedagogical content knowledge were provided by the interview.

*Interviews About Teachers’ Pedagogical Content Knowledge and Rational Number Instructional Unit*

Although the protocol for the teacher began with the same task questions that were asked of the students (see Appendix C), additional questions were added to elicit their pedagogical content knowledge related to the concepts embedded in the tasks (see
Appendix C). In addition to describing their personal solution strategies for doing the task, teachers were asked to discuss their expectations for how their students would have done the task, how they would have taught their students the concepts embedded in the task, and whether they had done an activity during their instructional unit that was similar to the task. They were also asked if they were aware of whether their students had engaged in similar types of activities to those in each task during prior grades. Additional questions to illuminate their pedagogical content knowledge included opened-ended questions relative to: (a) what they thought was important to remember about their students and class as data were analyzed, (b) whether they covered what they had intended to cover in their unit, (c) what curriculum resources they used in their instruction and its relationship to the district’s mathematics curriculum, and (d) what they considered most important for their students to have learned in preparation for sixth grade. Following the interview tasks, teachers were asked to share their beliefs about teaching mathematics and to provide information about their personal mathematical histories. Throughout the observations, teachers also took part in informal conversations related to instruction, pedagogical beliefs, and assessments of students’ emergent knowledge. These conversations were noted and taken as part of the data for describing their pedagogical content knowledge for teaching rational numbers. The following section provides a chronological ordering of the data collection.

Data Collection Timeline

When the teachers chosen for the study began their rational number unit, the instruction of the rational number unit was observed two days a week using video, audio,
and field notes. The teachers’ rational number instructional units were approximately nine to ten weeks in length and filled most of the second semester of the school year. At the end of the rational number instructional units, the teachers gave their students the Rational Number Test, Millsaps Version S, 5/27/97 (Appendix B) as a posttest. Interviews with students were conducted following the rational number instructional units during the final weeks of the school year. Two to three interviews were conducted back-to-back on a single day as time allowed. Finally, teachers were interviewed after classes for the school year had ended.

Data Analysis

Multiple methods of analysis were employed to construct understandings of two main categories of the study: teachers’ and students’ emergent conceptual knowledge of rational number and the instructional experience of meaning making and/or the classroom culture. The following presents an approximate timeline for the analysis of data and the construction of the cases and cross-case analyses.

Constructing the Cases of Teachers’ Conceptual Knowledge of Rational Numbers

Items from the posttest and interviews themselves were categorized for the types of rational number concepts that they appeared to elicit from teachers and students. Descriptions of teachers’ test and interview responses were written. Then the responses were coded for concepts that were evident within the strategies that the teachers used to answer and/or explain their reasoning for each item. Of interest were concepts that were or were not anticipated by the construction of the rational number tests. Also of interest were concepts that were similar or different between the two teachers. After coding
teachers’ data, initial cases of teacher knowledge were written that described their thinking on the items of the teachers’ test and interview tasks that corresponded to the items found on the students’ posttest and interview tasks. Cases were eventually written that described teachers’ construction of rational numbers, teachers’ construction of equivalence of rational numbers, and teachers’ construction of operations on rational numbers. Examination of initial cases written to describe teachers’ test and interview responses were also coded to construct a description of teachers’ syntactic knowledge for rational number. Although teachers’ cases of personal rational number knowledge were completed prior to the completion of the cases describing students’ emergent rational number there was overlap in the analyses that contributed to each.

Constructing the Cases of Students’ Emergent Conceptual Knowledge of Rational Numbers

Although some analysis of student data began with the student pretest, the main analysis was begun soon after completion of the data collection. Student posttests were analyzed quantitatively as well as qualitatively. First, items of the posttests (Rational Number Test, Millsaps Version S, 5/27/97, Appendix B) were scored using a five-point rubric that took into consideration the answer and the explanation:

4: Items that had a correct answer and an appropriate strategy for answering the item.

3: Items that indicated an appropriate strategy for answering the problem and a minor error in finding the answer.
2: Items that indicated a somewhat appropriate strategy but had some evidence of misconception.

1: Items with only a correct answer.

0: Items for which students had neither a correct answer nor indicated an appropriate strategy for answering the item.

Qualitative analyses of students’ tests were based on students’ explanations for their answers and were begun shortly after scoring the students’ tests. Responses for each test and interview item were examined and coded across students within a class and responses of students were examined across items to identify patterns of emergent conceptual knowledge of rational numbers with respect to each item. Profiles of rational number understanding for individual students and for subgroups of students based on their rankings were generated and tested against observations from class and other data sets, such as interviews. As with teachers, cases of students’ emergent content knowledge of rational numbers were written that described their knowledge in terms of their emergent construction of rational number knowledge, their emergent construction of rational number equivalence, and their emergent construction of rational number operations.

Analysis of Instruction

Initial analysis of instruction began with watching and taking notes on each teacher’s instruction soon after initial analysis of teacher and student test and interview data. However, because of the necessity to watch and transcribe videotapes and audiotapes, the main analysis of instruction occurred well into the analysis process. The
interactions of teachers and students were coded to develop profiles of teacher instructional themes and students’ experience of instruction. Notes taken as videotapes were watched provided evidence for teachers’ instructional organization. Transcriptions of interactions of each class session were coded for types of interaction and, when applicable, types of rational number concepts/constructs/ schema that were “taken-as-shared.” These analyses give evidence of differing classroom cultures and the interrelationships between teachers’ personal content knowledge and their instructional choices.

Case Development

After the major analyses of teacher, student, and instructional data were completed, summary cases for each teacher were written. The cases described each of the following categories for each teacher: (a) their content knowledge of rational numbers and their syntactic knowledge of mathematics, (b) their pedagogical knowledge for rational numbers in terms of their instructional practice, (c) patterns of interrelationship between teachers’ content and syntactic knowledge and their instructional practice, (d) their students’ emergent knowledge of rational numbers and experience of the instructional environment, (e) interrelationships between students’ emergent knowledge of rational numbers and experience of the instructional environment.

Cross Analyses of the Teachers’ Cases

The cases of the two teachers were compared and contrasted within and among each of the five categories described in the preceding section to identify potential interrelationships between teachers' knowledge of rational number, their pedagogical
content knowledge of rational number, their contributions to the classroom culture, and their students’ emergent knowledge of rational number. As during all preceding analyses, original data were revisited to verify the patterns of interrelationship identified. Three cross-case analyses were developed that parallel the three segments of the theory described in the “Theoretical Framework” in Chapter 2. The five categories from the original cases were reorganized into the three categories of the cross-case analyses. The cross-case analysis of teachers’ personal rational number knowledge is described in Chapter 4. The cross-case analysis of the instructional environment and teachers’ enacted pedagogical rational number knowledge and beliefs about teaching rational numbers is described in Chapter 5. The cross-case analysis of students’ emergent rational number knowledge is described in Chapter 6.

**Trustworthiness**

The method used in this study to establish trustworthiness was the triangulation of data. Written assessments, videotapes, audiotapes, interviews, and other artifacts such as teacher produced tests and assignments provided ways to amplify, clarify and verify results that were found during analysis. Throughout and at each level of the analysis, observations, codes, patterns, and other objects of analysis were subject to verification and questioning by alternate data sets such as the comparison of the collection of artifacts and the classroom observations, the comparison of test explanations and interview explanations, the comparison explanations on related test problems, and the comparison of classroom observations and interview responses. Limitations are noted throughout the study, as they are relevant.
Ethical Consideration

The researcher attempted to abide by the ethical codes adopted by the Council of American Anthropological Association (Glesne & Peshkin, 1992). The researcher recognized her responsibility to do no harm to those participating in her research project. In line with this principle, she tried to communicate the purposes of the study and to provide the participants' anonymity as much as is feasible. In the use of videotaped data, she only shared that data with her advisor who was the principal investigator, directly involved in the study, and known to the participants and the person who helped to transcribe the tapes. The data that she gathered and analyzed was only used for the purposes of research and not for any personal gain. She endeavored to find ways that she could give fair return to those in the study for their participation based on who elects to participate and on what would be of value to them. She gauged how her study could affect the general population being studied and tried to do the study with sensitivity to its possible repercussions. Finally, she listened to the advice of wise others such as my committee and fellow doctoral students who have done research and are experienced in locating possible ethical pitfalls and how to avoid them.

Summary

In summary, the study is a multi-case study of two teachers who differ in their personal rational number knowledge and their beliefs about teaching rational numbers. The data collected include test and interviews with teachers and students about their emergent rational number knowledge and video and audio recordings of the teachers and students’ shared rational number instructional unit. Cases for each teacher and their
students were written from the data and then compared for relationships between teachers’ personal rational number knowledge and their students’ emergent rational number knowledge in the following cross-case analyses. The following three chapters report, respectively, the cross-case analyses of: (a) the teachers’ personal rational number knowledge; (b) the teachers’ enacted pedagogical rational number knowledge, beliefs about teaching rational numbers, and the instructional environment; and (c) their students’ emergent rational number knowledge.
CHAPTER 4

TEACHERS’ PERSONAL CONTENT KNOWLEDGE CROSS-CASE ANALYSIS

Two cases were written to describe each teacher’s rational number knowledge, their pedagogical content knowledge and beliefs as reflected in their teaching of rational numbers, the instructional environment and experiences related to rational numbers, and their students’ emergent rational number knowledge. The cases were structured based on categories of analysis that arose from the interplay between the theories outlined in Chapter 2: Literature Review and the data collected in the field. The theoretical design as illustrated in Figure 2.6, proposes that teachers and students are actors that meet in the instructional environment to negotiate meanings. These meaning are influenced by teachers’ knowledge and contribute to children’s emergent knowledge. Each case describes the data collected in terms of the teachers and in terms of their students. Teacher descriptions include their personal rational number knowledge and their rational number instruction. Student descriptions include their emergent rational number knowledge and their experience of rational number instruction.

Teachers’ Personal Mathematics Backgrounds

Mr. Kent and Mrs. Baker differed in their high school and college mathematics experiences. Mr. Kent observed, “I didn’t hit a wall in math until I took calculus when I
was in high school and then all of a sudden I was floored. That’s the only class I had to
work in.” Mrs. Baker related, “When I got to college, I discovered that I really like
calculus and I was good at it.” On her parents’ advice, Mrs. Baker earned a business
degree in college, taking at least two years of calculus and some statistics in the process.
Because Mr. Kent began college as a speech/communication major, he “had placed out of
all the math requirements for that particular field of study.” Later, he was required to take
“Math 150,” a college algebra class, as pre-requisite to mathematics for elementary
school teachers.

Both Mrs. Baker and Mr. Kent described their K-12 mathematical experiences as
successful. Mrs. Baker confessed, “I was never that great of a student,” acknowledging,
“I wasn’t your all A kid. I had that potential to get all A’s, but I just didn’t want to work
that hard. And if I had a B, I got a B, it wasn’t a big deal.” However, she recognized that,
“Math came early to me right away. I can remember in second grade learning parts of a
ruler and teaching it to kids.” She contrasted her own easy-going attitude with her
father’s “perfectionism” in an early mathematics memory of her father “making me
rewrite long multiplication problems at home. There were like 20 of them, three times
over, because they were sloppy.” But she also reported, “I always did well in math, all
through high school.” She took the college preparatory sequence, “geometry, Algebra I,
Algebra II, and pre-calculus.” Mr. Kent related, “when I was in school math, everything
just—it wasn’t that it was boring—but I just knew exactly how to do it.”

Mrs. Baker and Mr. Kent were required to take the mathematics-for-elementary-
school-teachers course offered by their college. Mrs. Baker reported not enjoying the
course, because “I didn’t agree with it. It was based on technology. It was using
calculators in the classroom. I was really bored.” Studying the syllabus the first day, Mr.
Kent determined that the course was a repeat of material with which he was already very
familiar. He received permission from the instructor to forgo class attendance, but to
participate in all aspects of assessment including a project. Mr. Kent reported,
“Conceptually, I was pretty solid unless I planned on teaching anything higher than
sophomore level high school math.”

Teachers’ Content Knowledge as Reflected on the Rational Number Test and Interview

The descriptions of teachers’ content (personal rational number) knowledge were
developed from their responses on the rational number tests and interviews. The
constructs and concepts that contributed to the design of the test and interview guide
formed the foundation for analyzing teachers’ rational number knowledge. Through the
process of analysis of the data, categories of rational number knowledge were revised and
integrated. Contributing to the revisions were the analysis of teachers’ and students’
answers and explanations across and within the tests and interviews and the analysis of
concepts evident in each instructional environment. The categories of rational number
knowledge that structured the teachers’ cases were: the part-whole construct and related
constructs, equivalence constructs, and addition or subtraction of fractions.

In each teacher’s case, the category of the part-whole construct and its related
concepts included the following schemata: composition/decomposition of the whole
from/into unit parts/fractions, composition of a non-unit fraction given the whole, and
composition of the whole given a non-unit fractional amount. These schemata were based on processes observed in explanations for answers to the first three problems in the Rational Number Test, Millsaps Version S, 5/27/97 and the first three tasks in the Rational Number Formal Interview, which in turn were based on similar items and responses from an unpublished study by Rational Number Project (Behr et al., 1992) described on p. 25. The concepts related to the part-whole schemata were “the whole” and “the unit fraction.” As per the Rational Number Project study, the concepts/constructs were observed across different types of quantities: continuous area (circle, square); continuous length (long rectangle); and discrete (unifix cubes, color tiles, set of small circles). One additional related concept (the complement of a fraction) is only described for the case of Mrs. Baker, the only one who emphasized it.

Except for the equivalence construct, its subschemata (equivalence models and equivalence algorithms), and its related concept (equivalence classes), the category of other rational number constructs and concepts differed in the two cases. For Mrs. Baker, the equivalence subschemata included a proportion schema. Mr. Kent’s case included other rational number constructs and concepts such as schemata related to geometry and/or measurement, the quotient construct, and the operator construct. Mr. Kent’s case described the concept of the fraction addition; whereas, Mrs. Baker’s case describes algorithm of fraction addition. The differences between the content knowledge categories of the two cases arose because of the differences observed between the two teachers’ personal rational number knowledge enacted in their instruction.
Because Mr. Kent and Mrs. Baker were adults, the discussion of their personal rational number knowledge does not include the rational number schemes described by Steffe (2001). It would be expected that each would have developed the equi-partitioning scheme for composite units and the fractional composite number sequence or equivalent or more sophisticated schemes for rational numbers.

For consistency, the categories of the cross-case analysis of the teachers’ cases adhere to the constructs/concepts that informed the test and interview guide design. The first two sections (“Area Models Versus Discrete Models of Fractional Amounts: Composition/Decomposition of Fractional Amounts From/Into Unit Fractions/Parts” and “Concept of Whole: Integration of Composition and Decomposition of a Whole From/Into Unit Parts/Fractions”) describe aspects of teachers’ rational number knowledge related to the part-whole concept of fractions arising from the sections of each teacher’s case describing teachers’ decomposition/composition of wholes and fractional amounts. The section “Concept of Fraction: Language and Definition” compares the teachers’ rational number knowledge based on cross-case analysis of the sections from the teachers’ cases about the concepts of wholes and fractions and other schemata such as measurement, operator, and quotient as well as from the sections on fraction and whole composition/decomposition. The section “Finding and Comparing Portions and Comparing Mixture Strengths: Reasoning with Fractional Amounts, Rates, and/or Ratios” re-examines the teachers’ responses to those test problems and interview tasks that involve rates and ratios such as fair sharing and strength of mixture problems. The section “Equivalence, Order, and Addition of Fractions: Models and Algorithms” re-
examines the teachers’ test and interview responses that involve equivalence, ordering, and addition of fractions in terms their use of models or algorithms to solve the problems.

*Area Models Versus Discrete Models of Fractional Amounts:*

*Composition/Decomposition of Fractional Amounts From/Into Unit Fractions/Parts*

Both Mr. Kent and Mrs. Baker were able to decompose or compose discrete and continuous wholes into/from unit parts. This included the ability to operate with composite discrete unit parts. Likewise, both were able to compose fractional amounts from discrete and continuous quantity unit parts, including composite discrete unit parts as well as singleton unit parts. Finally, each was able to compose a fractional amount given the whole or compose the whole given a fractional amount. Yet, there were differences in their processes and facility in accomplishing these tasks.

Mr. Kent and Mrs. Baker differed in their use of singleton versus composite units whether the units were discrete or continuous. Mr. Kent was more likely than Mrs. Baker to use composite continuous or area units in his explanations. Similarly, Mrs. Baker was more likely than Mr. Kent to use singleton or composite discrete units in her explanations. This difference was noticeable in their solution methods for interview tasks and test problems described in Figure 4.1. The error that Mrs. Baker initially made in answering interview Task 5b suggests a relationship between her use of composite discrete units and her algorithmic understanding of fractional amounts. However, her solution of test Problem 3 reinforces that her understanding of fractional amounts was related to a composite discrete model of fractional quantities.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 2</td>
<td></td>
<td>is three fourths of some length. Draw the whole length below and explain why it is the whole.</td>
<td>Mr. Kent partitioned the given length into three equal parts (singleton continuous units), labeling each as fourths. In his alternate solution, he did not decompose the fractional amount.</td>
</tr>
<tr>
<td>Problem 3</td>
<td></td>
<td>Draw in the box to the right a set which has 3/4 as many circles as the set of circles in the box on the left.</td>
<td>Mr. Kent multiplied the size of the composite discrete unit part by three to find the size of the fractional amount. In his alternate solution, he redrew the circles in an array and partitioned, labeling the appropriate number of sections to indicate the fractional amount.</td>
</tr>
<tr>
<td>Task 1c</td>
<td></td>
<td>Having already represented 3/8 once with a set of fraction circles, and with all but two 1/8 pieces removed, show 3/8. Explain how the two ways of showing 3/8 are alike and different.</td>
<td>He immediately chose a 1/4-circle piece to represent 2/8 in forming his representation of 3/8.</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 4.1. Items that feature teachers’ understanding of composition/decomposition of fractional amounts from/into unit fractions/parts.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task</td>
<td>2a</td>
<td>Using 16 square color tiles without counting or telling how many there are, arrange the tiles to show me the fraction 3/8. Explain the thinking used to solve this problem. Draw a picture to show what you did.</td>
<td>Mr. Kent represented 3/8 with three of eight contiguous tiles (singleton discrete or singleton continuous units) in a row of eight. He designated each tile as an eighth. Mrs. Baker represented 3/8 with three of eight non-contiguous color tiles (singleton discrete units).</td>
</tr>
<tr>
<td>Task</td>
<td>2c</td>
<td>Show 3/8 using a different number of tiles. How are the two ways of using tiles alike? How is this way of showing 3/8 like using the fraction circle pieces? How are they different?</td>
<td>Mr. Kent designated three pairs of tiles from a rectangular 2x8 array as an eighth. Although his representations were composite discrete units of the whole, their contiguous formation suggested a continuous quantity or composite continuous units. Mrs. Baker designated three columns of eight columns of 16 non-contiguous tiles to represent 3/8. In her observation, “you can build it,” she indicated that the size of the composite unit or column was arbitrary. The units were composite discrete units.</td>
</tr>
<tr>
<td>Task</td>
<td>3a</td>
<td>Using a 4 x 4 geoboard show the fraction 3/8 using rubber bands. Explain the thinking used to solve the problem.</td>
<td>Mr. Kent represented 3/8 with <em>three composite continuous units</em>. He envisioned each composite unit as the area of two squares. His first representation was a 3x2 rectangle for which he noted that each pair of squares represented an eighth. Mrs. Baker represented 3/8 with three of eight squares of the geoboard. Although the model was a continuous area, her units appeared to be <em>singleton discrete</em> from the way she formed and counted them.</td>
</tr>
</tbody>
</table>
Concept of Whole: Integration of Composition and Decomposition of a Whole From/Into Unit Parts/Fractions

Mr. Kent and Mrs. Baker differed in their facility in composing wholes. This was evident in their comfort with non-standard representations of a whole as illustrated in Figure 4.2 below. Note that there are overlaps in Figures 4.1 and 4.2 since the problems would require concepts of fractional amounts or unit fraction/part decomposition as well as composition of the whole. Mrs. Baker’s solution processes in interview Tasks 6a below and 1c (see Figure 4.1) suggest a preference for standard representations of a whole. By comparison, Mr. Kent’s responses to interview Tasks 6a and 6b and test
Problem 1 indicate his comfort with non-standard representations of a whole and his belief in the importance of reasoning with models other than the general standard forms of circle, square, or rectangle for developing children’s understanding rational numbers.

The difference in Mr. Kent’s and Mrs. Baker’s flexibility in representing wholes may be attributed to the difference in their fluency with composition and decomposition of whole and fractional amounts from/into unit parts/fractions illustrated by Figures 4.2 and 4.3. Mr. Kent had no difficulty with any of the composition interview tasks or test problems whether the unit part was singleton, composite, continuous, or discrete. Many times he identified units as unit fractions as well as unit parts. Although Mrs. Baker composed and decomposed wholes from/into unit parts, she did not demonstrate the same flexibility that Mr. Kent showed. Her initial misinterpretation of the problem on interview Tasks 5b, her comparative slowness to recognize that 1/4-circle piece could represent 2/8, and her preference for interpretations of models as singleton unit parts or composite discrete unit parts versus composite continuous unit parts suggests a predisposition to algorithmic interpretations of fractions versus model-based interpretations. The only case in which she composed a whole using a composite continuous quantity was in response to interview Task 6b. She identified fractional units almost exclusively as unit parts (see Figure 4.2, Problem 1) rather than unit fractions.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td></td>
<td>If is one unit, what fraction is ? Explain why you think your answer is correct.</td>
<td>Mr. Kent described the process of composing a partial circular whole through rotating the unit part. In his alternate solution, he described measuring the arc of the whole and the arc of the part and comparing the two.</td>
</tr>
<tr>
<td>Task 6a</td>
<td></td>
<td>Form a whole given that a 1/8-circle piece represents a sixth of the whole. Explain how you know the answer is correct.</td>
<td>Mr. Kent immediately formed the whole using six 1/8-circle pieces (singleton continuous unit parts) without comment.</td>
</tr>
<tr>
<td>Task 6b</td>
<td></td>
<td>Form a whole given that a 1/4-circle piece represents 2/3 of the whole. Explain how you know the answer is correct.</td>
<td>Mr. Kent observed that students “get so ingrained into thinking that the whole has to be a circle or a square.” He added a 1/8 piece to the 1/-4 piece to form the whole.</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 4.2. Items that feature teachers’ flexibility with non-standard representations of unit wholes.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 2</td>
<td>2</td>
<td>is three fourths of some length. Draw the whole length below and explain why it is the whole.</td>
<td>Mr. Kent appended one singleton continuous unit part onto a partitioned copy of the given fractional amount, labeling each as fourths. In his alternate solution, he copied the given length four times and partitioned it into three parts, each of which was a unit whole, labeling each whole.</td>
</tr>
<tr>
<td>Problem 3</td>
<td>3</td>
<td>Draw in the box to the right a set which has 3/4 as many circles as the set of circles in the box on the left. Explain why you think your answer is correct.</td>
<td>Mr. Kent counted the total number of circles and divided by four to find the size of the unit part. In his alternate solution, he redrew the circles in an array and partitioned the array by halving into four equal-size sections.</td>
</tr>
<tr>
<td>Problem 6v</td>
<td>6v</td>
<td>Write a fraction to show what part is shaded. Explain why you think your answer is correct.</td>
<td>Mr. Kent decomposed the circle as he “drew lines in the problem so the circle would be divided into 1/8s.”</td>
</tr>
<tr>
<td>Problem 7</td>
<td>7</td>
<td>For each picture below, write a fraction to show what part is shaded. Choose one picture and explain why you think your answer is correct.</td>
<td>Mr. Kent “visually determined their values by breaking down the entire square into equal shapes were the same size as the shaded area for each.”</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 4.3. Items that feature teachers’ understanding of composition/decomposition of unit wholes from/into unit fractions/parts.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task</td>
<td>2a</td>
<td>Using 16 square color tiles without counting or telling how many there are, arrange the tiles to show me the fraction 3/8. Explain the thinking used to solve this problem. Draw a picture to show what you did.</td>
<td>Mr. Kent’s representations of the whole on this task were contiguous starting with a long row of eight (interpretable as singleton continuous area or discrete units). He designated each tile as an eighth.</td>
</tr>
<tr>
<td>Task</td>
<td>2c</td>
<td>Show 3/8 using a different number of tiles. How are the two ways of using tiles alike? How is this way of showing 3/8 like using the fraction circle pieces? How are they different?</td>
<td>Mr. Kent doubled the length of eight tiles from Tasks 2a and then reformed it into a rectangular 2x8 array. He designated two tiles as an eighth — interpretable as either composite discrete units or composite continuous units.</td>
</tr>
<tr>
<td>Task</td>
<td>5a</td>
<td>Show a whole tower if 6 unifix cubes is 1/5 of a tower. Explain why you think your answer is correct.</td>
<td>Mr. Kent recognized the whole was composed of five unifix cube sticks (composite discrete units) of length six cubes.</td>
</tr>
<tr>
<td>Task</td>
<td>5b</td>
<td>Show a whole tower if 8 unifix cubes is 2/3 of a tower. Explain why you think your answer is correct.</td>
<td>Mrs. Baker recognized the whole was composed of five sets (composite discrete unit parts) of six cubes.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Mrs. Baker immediately responded “16/24.” She corrected her answer to 12 cubes when she modeled the problem with the unifix cubes using the composite discrete unit of four.</td>
</tr>
</tbody>
</table>
Mr. Kent and Mrs. Baker differed in their language describing fractional amounts and thus in the nature of their concepts of fractions. For example, Mr. Kent’s explanations of unit parts/fractions use geometric or measurement terminology that Mrs. Baker’s do not. For example, Mr. Kent’s explanation from test Problem 1 contains terminology that elicits transformational geometry such as “sliding” or “rotating.” It also has terminology that describes a measuring process using “perimeter” and “arc” to describe a method for determining a fractional amount of a given whole. His explanations or discussions of test Problem 7 and interview Tasks 3 refer to “area” and those in test Problem 2 refer to “length.”

Mr. Kent’s explanations also contain references indicating quotitive and operator constructs in defining rational number that Mrs. Baker’s did not. In several of his explanations on the test Mr. Kent used or implied division to form fractions. One example was his alternative explanation of test Problem 1, “measure the perimeter of the arc on the fraction piece and compare it to the measurement of the arc on the whole unit.” His construction of the length of the whole to solve test Problem 2 by duplicating the 3/4 strips four times and partitioning it into three equal lengths is an example of using the duplicator/division schema of the operator construct.

The basis of Mrs. Baker’s concept of fraction was the unit part and the ratio relationship between the number of unit parts in a designated fractional amount and in the whole. In addition to the concepts of fractions as operators or quotients or measurements, Mr. Kent’s concept of fraction included fractional amounts as composites of unit
fractions. Their discussions of interview Tasks 2c illustrate the unit part versus unit fraction difference. Mrs. Baker described “eight columns and you could pull out three” to designate the fractional amount. In discussing his solution to the interview Tasks 2c, Mr. Kent noted that “two tiles” formed “1/8.” Mr. Kent’s discussion of interview Task 5b, “Four is 1/3 and eight is 2/3, then 12 would be the whole,” further illustrates his integration of unit fraction and unit part concepts of fraction. Mrs. Baker’s observation, “If eight is 2/3, so it’s 12 total,” for interview Task 5b after forming two groups of four cubes is inconclusive evidence for or against a unit part concept of fraction as she was told that “eight cubes is 2/3.” There are several instances of Mrs. Baker counting unit parts in her explanations of problems and tasks, but no examples comparable to that of Mr. Kent’s counting of unit fractions.

Related to the unit part versus unit fraction difference in their explanations of problems or discussions of tasks was the difference in their argumentation and mathematical language or illustrations. Notable in Mrs. Baker’s case was the dearth of explanation for her test problems. By contrast, Mr. Kent’s explanations were detailed. For example, for his initial solution of test Problem 2, Mr. Kent carefully partitioned a copy of the given 3/4 length into three equal parts and labeled each “1/4.” He explained in detail, “I divided the 3/4-length into 3 equal parts and then added 1/4 length to the 3/4 length to create the whole. By dividing the 3/4 into three parts I knew what the 1/4 length would be and 4/4 = 1 whole.” Mrs. Baker similarly partitioned a copy of the given length and appended a fourth portion to form the whole. However, she bracketed and labeled the copy of the original “3” and the final partition “1.” She wrote in explanation, “3 + 1 = 4/4
= 1 whole” obviously meaning that $3/4 + 1/4 = 4/4 = 1$ whole. In addition to the
difference in the detail of their explanations for test Problem 2, it is noteworthy that Mr.
Kent’s explanation contains a more accurate symbolic representation of his reasoning
process than that of Mrs. Baker. This example also reconfirms the aforementioned
difference in their use of unit part versus unit fraction terminology.

Mrs. Baker’s explanations exhibited the use of the complements of fractional
amounts to determine either a fractional amount given the whole or the whole given a
fractional amount more often than Mr. Kent’s explanations did. In addition to appending
the unit part to the fractional amount on test Problem 2, she removed the unit part to find
the fractional amount on test Problem 3. Mr. Kent by comparison iterated the unit part to
find the fractional amount on test Problem 3.

Finding and Comparing Portions and Comparing Mixture Strengths: Reasoning With
Fractional Amounts, Rates, and/or Ratios

Whereas the preceding three sections provide a deep analysis of the teachers’
rational number knowledge from the perspective of the part-whole construct and related
concepts, the following three subsections examine their rational number knowledge from
the perspective of ratios and rates. Although mathematically the part-whole construct can
be seen as a special case of the ratio construct, developmentally the two constructs may
arise in parallel and integrate in the development of rational numbers. As discussed in
Chapter 2, ratio problems involve values from two different measure spaces that are
composed to form a third measure space such as “pizza per person” or portion. In part-
whole problems the two values (numerator/denominator) come from the same measure
space such as “3/4 of a line.”

Finding Portions

Profound differences in the thinking processes of Mrs. Baker and Mr. Kent are
found in the test problems and interview tasks that required finding or comparing
portions as illustrated in Figure 4.4. Mr. Kent used the same initial and alternative
solution methods across the fair sharing problems. His interview explanation revealed
that he equated the processes of shading one piece from each pizza and shading one
person’s portion on a single pizza. He identified the unit fraction value of the unit parts of
each pizza and counted the number of unit parts that would be given to one person to
form the fraction assigned to one person’s portion. Although Mrs. Baker used the same
partitioning method as Mr. Kent on the interview task, she used a composite partitioning
method on the test problems—first halving, assigning halves to each person, and then
partitioning any remaining half by the number of people and assigning one portion of the
remaining half to each person. Her assignment of the fractional value to one person’s
portion was inconsistent across the problems as illustrated in Figure 4.4. Oddly enough,
Mrs. Baker’s partitioning method and solution for test Problem 19 was the same as Mr.
Kent’s initial method on all of the fair sharing problems. Instead of using the total
number of pieces in all of the bars, she identified the fraction assigned to one person’s
portion as the total number of pieces that one person receives to the number of pieces in
one bar.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>17</td>
<td>Four people are going to share these two pizzas equally. Color in one person’s part.</td>
<td>Mr. Kent first partitioned each pizza into the same number of equal pieces as persons. He shaded one person’s portion on one of the two pizzas, but described giving each person one of the equal parts of each pizza. He identified the numerator as the number of pieces and the denominator as the size of the piece. His alternative method was to divide the number of pizzas by the number of people sharing the pizzas to find a ordinary fraction or decimal amount that represented one person’s share of pizza.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Mrs. Baker began by halving each of the pizzas. She assigned each person a half. If there were a half leftover, she divided it by the number of people and assigned each person one portion of the half. She identified the numerator as one and the denominator as the number of pieces across all pizzas.</td>
</tr>
<tr>
<td>Problem</td>
<td>18</td>
<td>Three people are going to share these two pizzas equally. Color in one person’s part.</td>
<td>Mr. Kent first partitioned each pizza into the same number of equal pieces as persons. He shaded one person’s portion on one of the two pizzas, but described giving each person one of the equal parts of each pizza. He identified the numerator as the number of pieces and the denominator as the size of the piece. He claimed that he would use the same alternate method as he had used for Problem 17.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Mrs. Baker began by halving each of the pizzas. She assigned each person a half. If there were a half leftover, she divided it by the number of people and assigned each person one portion of the half. The fraction that she identified as representing one person’s portion was 1 and 1/3—the number of halves assigned to each person.</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Continued

Figure 4.4. Items that feature teachers’ understanding of fractions formed by fair sharing.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 19</td>
<td>Six people are going to share these five chocolate bars equally. Color in one person’s part.</td>
<td>Mr. Kent partitioned each bar into six equal size pieces and shaded five parts of one candy bar for one person’s share. He claimed that he would use the same alternate method as he had used for Problem 17.</td>
<td>Mrs. Baker partitioned each bar into six equal size pieces and shaded five parts of one candy bar for one person’s share. Her alternative method was to divide the total number of candy bar pieces, 30, by the number of people, 6, to determine the number of candy bar pieces received by one person, 5. This is the only fair sharing problem for which she identified the same fraction as Mr. Kent for one person’s share.</td>
</tr>
<tr>
<td>Task 4</td>
<td>Three people are going to share these two pizza equally. Color in one person’s part.</td>
<td>Mr. Kent first partitioned each pizza into the same number of equal pieces as persons. He shaded one person’s portion on one of the two pizzas, but described giving each person one of the equal parts of each pizza.</td>
<td>Mrs. Baker partitioned each pizza by the number of people. She assigned a fraction for the amount of one person’s portion based on the ratio of the number of pieces in one person’s share to the total number of pieces in all of the pizzas.</td>
</tr>
</tbody>
</table>
Comparing Portions

The differences between Mr. Kent’s and Mrs. Baker’s reasoning process for comparing portions are illustrated in Figure 4.5 below. Mr. Kent’s reasoning was similar to that he used for fair sharing problems. In each case, he compared share sizes and Mrs. Baker compared ratios.

<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Kent</th>
<th>Baker</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>13</td>
<td>For the following situation decide whether the people in group A or the people in group B get more pizza. Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate. Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.</td>
<td>He determined a person’s share for each group by dividing the number of pizzas by the number of people in the group and compared the decimal amounts. His alternative solution method is the same as that described in Task 7.</td>
<td>Mrs. Baker wrote fractions based on the ratio of the number of pizzas to the number of people sharing the pizzas and compared the fractions. Her method for comparing the two fractions was not stated. She could have used her backward Z equivalence algorithm (a cross multiplication algorithm for finding equivalent fractions) for 8/10 = 4/5 or recognized the common factor between the numerators and denominators.</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Continued

Figure 4.5. Items that feature teachers’ understanding of comparing shared amounts.
Task 7
Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.

He partitioned each pizza by the number of people in the group and assigned each person one slice from each pizza. He wrote the fractional amount based on the unit fraction size of a slice. He then compared the two ordinary fractions by finding equivalents with common denominators.

Mrs. Baker wrote fractions based on the ratio of the number of pizzas to the number of people sharing the pizzas and compared the fractions. She immediately recognized that the two fractions were equal, but did not state why.

Task 8
Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.

He used the same strategy as that described in Task 7.

Mrs. Baker wrote fractions based on the ratio of the number of pizzas to the number of people sharing the pizzas and compared the fractions. She did not describe her method for comparing the two fractions. She could have simply known that 1/3 is greater than 1/5.

Comparing Strengths

Both Mrs. Baker and Mr. Kent used ratios to compare mixture strengths initially, but differed in their alternate solutions as illustrated in Figure 4.6.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
</table>
| Problem   | 22       | For the following diagram each dark colored glass represents chocolate syrup and each white colored glass represents milk. Circle the mixture that will have a stronger chocolate flavor: the mixture made using the glasses pictured in set A or the mixture made using the glasses pictured in set B.  
![Diagram](image)

Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate. Provide an alternative solution method for solving this problem which is also valid and explain your reasoning. | Kent: He found the ratios of amount of chocolate to amount of milk for each combination—"1 chocolate syrup to 2 milks for A and 2 chocolate syrups to 4 milks for B. 1:2 = 2:4." In his alternative solution, he divided "the number of glasses of chocolate syrup by the number of glasses of milk" to find the fraction and/or decimal rates as a comparison algorithm. Using this procedure, he found the common fraction 1/2 and/or decimal 0.5 for each mixture and thus determined that the rates were the same.  
Baker: She wrote the ratio of the number of chocolate glasses to the number of milk glasses for each combination in fraction form and recognized that they were equivalent fractions. Her alternative solution was to use shaded fraction strips to illustrate that the two fractions 1/3 and 2/6 are equivalent. Below the label “equal fractions,” each of two fraction strips drawn was partitioned by solid lines into 3 equal parts. Each third on the second strip was additionally partitioned into two equal parts by dotted lines to create sixths. Then the first third of each was shaded. |

Note. Tasks are interview items; problems are test items.

Figure 4.6. Items that feature teachers’ understanding of comparing strengths of mixtures.
Summary: Reasoning With Fractional Amounts, Rates, and/or Ratios

As a rate is the portion size for fair sharing problems, Mr. Kent was consistent in his reasoning across all problems in this category. In each case, he was concerned with finding the portion size or rate whether in decimal or ordinary fraction form. Each of his methods, division of the number of shared items by the number of sharers or the partitioning of each shared item by the number of sharers accomplished the same purpose. He used formal ratio notation only when finding the intensive quantity of mixture strength. In all other cases, the fractions or decimals that he wrote represented the amount of the item, a pizza or chocolate syrup, that was one share for a person or a glass of milk.

Mrs. Baker was notably inconsistent across the “finding portions” category of problems and tasks. Her schema was to make an equal partition of the items, distribute the resulting parts equally among the participants, and partition any remainders, if necessary. She formed the fraction value of a portion based on the ratio of the designated number of parts given to one participant and the total number of parts. The described process reflects her unit part schema of fraction previously described. However, she was not consistent in her identification of the total number of parts, nor was her schema viable in identifying an appropriate fraction representation of a portion. One could conclude that in the context of finding fair shares or one person’s portion she sometimes failed to identify the whole appropriately and thus could not construct an appropriate fraction that described one person’s fair share.
In the case of comparing portions or strengths, Mrs. Baker’s reasoning was more consistent and viable. Her schema was to form a ratio for the number of shared items and the number of sharers. She could then compare the resulting fractions using facts or processes that she understood well. Thus, she could predict that a group received more or less the same amount as the comparison group and that a mixture was stronger or the same as a comparison mixture. There was no evidence that she recognized that the resulting fraction in each case was the rate or the portion size for one participant.

_Equivalence, Order, and Addition of Fractions: Models and Algorithms_

The constructs of equivalence, order, and addition (or subtraction) of fractions are a level of understanding beyond the development of the initial concept of unit fraction or rate. They require development of the rational number knowledge described in the preceding sections. Each can be represented through models or through algorithms. The following sections describe the teachers’ rational number knowledge with respect to their use of models and algorithms in their solving of test problems and interview tasks involving each: equivalent fractions, ordering fractions, and adding or subtracting fractions.

_Models for Fraction Equivalence, Fraction Order, and Fraction Addition/Subtraction_

Figure 4.7 illustrates the ways in which Mr. Kent and Mrs. Baker indicated that they recognized equivalent models of fractions or models of equivalent fractions. The most powerful model of equivalence used by Mrs. Baker was the proportion schema that she employed in her solution of interview Task 2c.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>6v</td>
<td>Write a fraction to show what part is shaded. Explain why you think your answer is correct.</td>
<td>Mr. Kent used his recognition of a fourth circle as 2/8-circle and a half circle as 4/8-circle to justify his estimation of the shaded portion as “3/8” in his explanation. “By comparing the size of the area shaded to other figures’ shaded area, I determined it was between 2/8 and 4/8.”</td>
</tr>
<tr>
<td>Problem</td>
<td>7i</td>
<td>For each picture below, write a fraction to show what part is shaded. Choose one picture and explain why you think your answer is correct.</td>
<td>Mr. Kent “visually determined their values” by mentally moving two of the shaded eighths to form a shaded fourth of the square.</td>
</tr>
<tr>
<td>Task</td>
<td>1c</td>
<td>Having already represented 3/8 once with a set of fraction circles, and with all but two 1/8 pieces removed, show 3/8. Explain how the two ways of showing 3/8 are alike and different.</td>
<td>Mr. Kent immediately chose a 1/4-circle piece to represent 2/8 in forming his representation of 3/8.</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Continued

Figure 4.7. Items that feature teachers’ use of models to find equivalent fractions.
Figure 4.7 continued

<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task</td>
<td>2c</td>
<td>Show 3/8 using a different number of tiles. How are the two ways of using tiles alike? How is this way of showing 3/8 like using the fraction circle pieces? How are they different?</td>
<td>Mr. Kent doubled the length of eight tiles from Task 2a and then reformed it into a rectangular 2x8 array. He designated two tiles as an eighth—interpretable as either composite discrete units or composite continuous units. Mrs. Baker added a second row of eight non-contiguous tiles to the row from Task 2a. She observed, “Even though there is sixteen total here, there’s really eight,” indicating eight columns of two. In her observation, “you can build it,” she indicated that the size of the composite discrete units or columns was arbitrary.</td>
</tr>
<tr>
<td>Task</td>
<td>6b</td>
<td>Form a whole given that a 1/4-circle piece represents 2/3 of the whole. Explain how you know the answer is correct.</td>
<td>Mr. Kent recognized that if 1/4-circle piece represented 2/3 then 2/8 circle-pieces also represented 2/3, and hence, 1/8 represented 1/3. Mrs. Baker recognized that if 1/4-circle piece represented 2/3 then 2/8 circle-pieces also represented 2/3, and hence, 1/8 represented 1/3.</td>
</tr>
</tbody>
</table>

Figure 4.8 illustrates the differences in reasoning that Mr. Kent and Mrs. Baker used in ordering lists of fractions. Mr. Kent used processes related to internalized models of the fractions, but Mrs. Baker did not appear to use internalized models of the fractions from responses to the test problems and interview tasks in Figure 4.8.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>29i</td>
<td>For each row of fractions, determine which fraction is the greatest and which fraction is the least. Explain the reasons for your choices in the space provided. Provide an alternative solution method and explain your reasoning.</td>
<td>Kent ordered the list by ordering the numerators since the denominators were alike.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{2}{7}$ $\frac{5}{7}$ $\frac{4}{7}$</td>
<td>Mr. Kent ordered the list by the inverse relationship between the denominator size and the size of the fractional amount.</td>
</tr>
<tr>
<td>Problem</td>
<td>29ii</td>
<td>Greatest? Least?</td>
<td>Mr. Kent used the closeness to the whole related to the unit fraction size to order the fractions.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{1}{8}$ $\frac{1}{7}$ $\frac{1}{6}$</td>
<td>Kent used the strategies described in (ii) and (iii) to order them.</td>
</tr>
<tr>
<td>Problem</td>
<td>29iii</td>
<td>Greatest? Least?</td>
<td>Mr. Kent ordered them by whether they were greater, less or equal to a half.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{6}{7}$ $\frac{8}{9}$ $\frac{7}{8}$</td>
<td>Kent and Baker used the strategies described in (ii) and (iii) to order them.</td>
</tr>
<tr>
<td>Problem</td>
<td>29iv</td>
<td>Greatest? Least?</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{3}{7}$ $\frac{4}{9}$ $\frac{4}{5}$</td>
<td></td>
</tr>
<tr>
<td>Problem</td>
<td>29v</td>
<td>Greatest? Least?</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{4}{7}$ $\frac{3}{8}$ $\frac{1}{2}$</td>
<td></td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 4.8. Items that feature teachers’ use of models to order fractions.
Both Mr. Kent and Mrs. Baker used models in at least one of their solution strategies for adding and subtracting fractional amounts. This is illustrated by their responses to the test problems and interview tasks described in Figure 4.9.

<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 35</td>
<td>Liana ate 3/8 of a small pizza. The next day she ate 1/4 of a small pizza. How much of a pizza did she eat altogether? Describe how you found your solution and use a diagram if appropriate. Describe an alternative method for solving the problem.</td>
<td>He partitioned a circle by the common denominator, shaded and labeled the parts to illustrate the amounts of pizza eaten at different times described. His answer was the total fractional amount of the circle shaded given in terms of the common denominator.</td>
<td>In her initial solution, she drew two separate circles with a “+” between them to represent the two fractional amounts. She added dotted partition lines to one circle to show the common partition. She explained that she “added together shaded pieces” to find “total pieces” and gave an appropriate solution in terms of the common denominator.</td>
</tr>
</tbody>
</table>

| Problem 36 | Ann and Josie receive the same allowance. Josie spent 4/9 of hers on CDs. Ann spent 1/3 of her allowance on repairing her bicycle. Josie spent how much more of her allowance than Ann? Describe how you found your solution and use a diagram if appropriate. Describe an alternative method for solving the problem. | Mr. Kent drew two parallel fraction strips to compare the differing amounts of allowance described in the problem. He partitioned each strip by the common denominator, then shaded and carefully labeled the different fractional amounts and common denominator equivalents of allowance on each. His answer was the difference in terms of the common denominator. | Mrs. Baker drew two parallel fraction strips to compare the differing amounts of allowance described in the problem. She shaded the appropriate amounts on each strip, but did not show the common denominator partition. She drew a dotted line indicating where the larger amount of allowance would fall on the strip illustrating the smaller amount of allowance and asserted the difference. |

Note. Tasks are interview items; problems are test items.

Continued

Figure 4.9. Items that feature teachers’ use of models to add/subtract fractions.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task</td>
<td>9</td>
<td>Sally ate 2/3 of a pizza for dinner. The next morning she ate another 1/6 of a pizza. How much pizza did she eat altogether? What you were thinking to reach this answer?</td>
<td>Mr. Kent did not use models to solve addition problems in the interview. (See Figure 4.12) Mrs. Baker partitioned the circle that represented the pizza by the smaller denominator and shaded two parts. Then she used dotted lines to illustrate the common denominator partition and shaded the remaining part. Her answer was the total fractional amount of the circle shaded given in terms of the common denominator.</td>
</tr>
<tr>
<td>Task</td>
<td>10</td>
<td>Josie and Al went skateboarding down their block together. Al skated 6/10 of the way down the block before he stopped. Josie skated 4/5 of the way down the block before she stopped. How much further down the block did Josie skate than Al? What you were thinking to reach this answer?</td>
<td>Mr. Kent did not use models to solve subtraction problems in the interview. (See Figure 4.12) Mrs. Baker drew two parallel strips and partitioned the strip with larger number of parts first beginning at one end. As a result, the parts were not equal size. Although she did not partition the second strip by the same amount, she appeared to use the partition of the first strip as a guide to partition the second. However, the last partition line on the second strip was significantly displaced from its counterpart on the first strip. Nevertheless, after shading the two strips, second first, she was able to determine the appropriate difference.</td>
</tr>
</tbody>
</table>
Algorithms for Finding Equivalent Fractions, Ordering Fractions, and Adding/Subtracting Fractions

The different algorithmic processes that Mr. Kent and Mrs. Baker used to find equivalent fractions are illustrated in Figure 4.10. Of note are the unique processes that each used—Mrs. Baker’s “backward Z”, a mnemonic for finding equivalent fractions using cross multiplication, and Mr. Kent’s “principle of the rate series.” Each used a doubling schema for finding equivalent fractions that Mr. Kent described as having the effect of “multiplying by one,” but neither actually showed the concept of “multiplication by a fraction equivalent to one.”

<table>
<thead>
<tr>
<th>Item type</th>
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<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>3</td>
<td>Draw in the box to the right a set which has 3/4 as many circles as the set of circles in the box on the left. Explain why you think your answer is correct.</td>
<td>Mr. Kent did not interpret the problem as finding an equivalent fraction. However, his process of dividing the number of circles by four and multiplying the result by three is a process by which an equivalent fraction can be found. Mrs. Baker’s alternate solution was to use her “backward Z” algorithm to find an equivalent fraction to 3/4 with a denominator the size of the whole. Over the proportion 3/4 = __/8 she drew a “backward Z” connecting 4 to 8 to 3 to the missing amount. Below the proportion she wrote, “cross multiply.”</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 4.10. Items that feature teachers’ use of algorithms to find equivalent fractions.
Mr. Kent and Mrs. Baker both used algorithms as one of their solution methods for ordering fractions as illustrated by Figure 4.11.
### Item type | Item No. | Item description | Solution descriptions
--- | --- | --- | ---
**Problem** | 29 i | For each row of fractions, determine which fraction is the greatest and which fraction is the least. Explain the reasons for your choices in the space provided. Provide an alternative solution method and explain your reasoning. | Kent: As his alternate solution method for ordering the fraction lists, Mr. Kent found the decimal equivalents of each fraction, which he called quotients by division. (Original solution: see Figure 4.8)
Baker: Although she ordered the first list by ordering the numerators since the denominators were alike, the only method that she described for the other lists was the “backward Z” schema described in Figure 4.10, Problem 3.

|   |   | i) | 2
|   |   |     | 7
|   |   |     | 5
|   |   |     | 4
|   |   | ii) | 1
|   |   |     | 8
|   |   |     | 1
|   |   |     | 6
|   |   | iii) | 6
|   |   |     | 7
|   |   |     | 8
|   |   |     | 7
|   |   | iv) | 3
|   |   |     | 7
|   |   |     | 4
|   |   |     | 4
|   |   |     | 5
|   |   | v) | 4
|   |   |     | 7
|   |   |     | 3
|   |   |     | 8
|   |   |     | 2

**Note.** Tasks are interview items; problems are test items.

Figure 4.11. Items that feature teachers’ use of algorithmic methods to order fractions.

Figure 4.12 illustrates the algorithmic methods that Mr. Kent and Mrs. Baker used to add or subtract fractions. Mr. Kent indicated a preference for algorithmic methods for adding and subtracting fractions with unlike denominators for all tasks and problems. Mrs. Baker voiced a preference for visual reasoning methods.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 35</td>
<td>Lianna ate (\frac{3}{8}) of a small pizza. The next day she ate (\frac{1}{4}) of a small pizza. How much of a pizza did she eat altogether? Describe how you found your solution and use a diagram if appropriate. Describe an alternative method for solving the problem.</td>
<td>He found equivalent fractions with a common denominator and added the equivalents fractions.</td>
<td>Mrs. Baker found equivalent fractions with a common denominator and then added as her alternate solution method.</td>
</tr>
<tr>
<td>Problem 36</td>
<td>Ann and Josie receive the same allowance. Josie spent (\frac{4}{9}) of hers on CDs. Ann spent (\frac{1}{3}) of her allowance on repairing her bicycle. Josie spent how much more of her allowance than Ann? Describe how you found your solution and use a diagram if appropriate. Describe an alternative method for solving the problem.</td>
<td>Mr. Kent found equivalent fractions with a common denominator and subtracted the equivalent fractions.</td>
<td>Mrs. Baker found equivalent fractions with a common denominator using the “Backward Z” algorithm and then subtracted as her initial solution method.</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 4.12. Items that feature teachers’ use of algorithmic methods to add/subtract fractions.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item description</th>
<th>Solution descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task</td>
<td>9</td>
<td>Sally ate 2/3 of a pizza for dinner. The next morning she ate another 1/6 of a pizza. How much pizza did she eat altogether? What you were thinking to reach this answer?</td>
<td>Mr. Kent explained that considerable instructional time was spent adding fractions with the same denominator before adding fractions with unlike denominators was introduced. They “did a lot of problems to build the idea that they needed to find a common denominator where it might be a half plus fourths, thirds plus sixths, or stuff like that.” The effect was that he was very familiar with the equivalents for the fractions in each task and could readily restate the problems in terms of common denominators.</td>
</tr>
<tr>
<td>Task</td>
<td>10</td>
<td>Josie and Al went skateboarding down their block together. Al skated 6/10 of the way down the block before he stopped. Josie skated 4/5 of the way down the block before she stopped. How much further down the block did Josie skate than Al? What you were thinking to reach this answer?</td>
<td>See Task 9.</td>
</tr>
</tbody>
</table>
**Equivalence Classes**

Mrs. Baker’s proportion schema from interview Task 2c (see Figure 4.7) and Mr. Kent’s “rate series principle” on interview Task 5a (see Figure 4.10) were the equivalence processes that gave some indication of development of the concept of equivalence classes. Each implies an infinite series or class of equivalent fractions. Although a linkage between the two processes could be made, neither Mrs. Baker nor Mr. Kent described such a linkage. Thus, the two schemata represent their different understanding of equivalence classes.

**Summary: Equivalence, Order, and Addition of Fractions**

With respect to models, Mr. Kent used models to reason about fractions in each of the contexts (equivalence, order, and operations on fractions). Mrs. Baker used models when thinking about equivalence and in operating on fractions, but did not indicate reasoning with models in ordering fractions. Both used algorithms in all of the situations. Mr. Kent used algorithms that are commonly used in mathematics representations. In contrast, Mrs. Baker used representations of algorithms that were personal to her.

**Summary: Teachers’ Rational Number Knowledge**

In summary, the highest level of mathematics course did not predict the teachers’ performance on the Rational Number Test, Millsaps T, 12/18/96 (see Appendix A) or the interview (see Appendix C). Although Mrs. Baker reported the highest level of formal mathematics instruction (calculus) and both teachers self-reported success in their K-12 mathematics experience, Mr. Kent’s performance on the Rational Number Test appeared
more mathematically sophisticated. His explanations had connections to more constructs (geometric, measurement, operator), were more detailed, and were consistently mathematically appropriate. As described in the preceding sections, Mr. Kent and Mrs. Baker were different in the rational number knowledge exhibited on the test and in the interview. Chapter 5 examines how these differences were manifested in their rational number instructional unit.
CHAPTER 5

THE INSTRUCTIONAL ENVIRONMENT CROSS-CASE ANALYSIS: TEACHERS’ PEDAGOGICAL CONTENT KNOWLEDGE AND BELIEFS IN ACTION

In this chapter, the cross-case analysis of the two teachers’ cases focuses on their rational number instruction. The categories described in each teacher’s case were the organization and management for instruction and the instructional themes that comprised the teacher’s arrangement of content for instruction. The following cross-case analysis adds descriptions of their pedagogical background (experience and instruction in mathematics pedagogy). From evidence of teachers’ pedagogical content knowledge and beliefs embedded in the discussions of classroom culture and arrangement of content for instruction, the cross-case analysis compares their beliefs about mathematics and mathematics teaching. The last two sections of the cross-case analysis compares and describes their organization and management for fraction instruction (classroom culture) and their arrangement of content for instruction with attention to possible influences from their personal content knowledge.
Teachers’ Pedagogical Background

Data that contributed to the descriptions of teachers’ pedagogical background (experience and instruction in mathematics pedagogy) included the teachers’ interviews, and impromptu conversations that occurred during the taping of instruction.

Mathematics Pedagogy in College

Neither Mrs. Baker nor Mr. Kent began their college careers with an intention to teach. Mrs. Baker realized that she wanted to be a teacher after a year of work and returned to school for her elementary certification. Mr. Kent changed his major to elementary education during his college experience. In their background the only pedagogical course that focused on teaching mathematics was the required mathematics-for-elementary-school-teachers course offered by their college. As reported in the section on their mathematical background, neither was happy with the course. In addition to not agreeing with the course’s technology focus, Mrs. Baker reported, “I didn’t feel like there was any instruction in how to teach math.” Mr. Kent reported that the math-for-elementary-school-teachers course “was only about 30% teaching methods.” He complemented his professor, but bemoaned the fact that “the first seven weeks of class was making sure that we knew the math.” He complained that, “what I needed was a more methods class.” Mrs. Baker reported that her certification coursework did not include a course on teaching mathematics at the elementary level. She opined, “I wish there was more telling how you can fit small groups and how you can work one-on-one, how you can organize” teaching mathematics to elementary students.
Mrs. Baker and Mr. Kent differed in the length of their classroom experience. Mrs. Baker was more experienced, having taught seven years. Mr. Kent was relatively new, being in his second year of teaching. However, Mr. Kent reported some subsequent in-service work in teaching mathematics to elementary school children, which Mrs. Baker did not. He had taken *Math Their Way of Thinking*, an in-service course offered by his school district. However, he still felt that, “If I’m going to become a better math teacher I’m going to need more methods area stuff.” He observed, “I need to know why when someone doesn’t understand something. I need to have a better understanding of why they don’t understand it.” Mr. Kent contrasted his struggle with teaching mathematics with his confidence in teaching language arts. He confided, “I feel very confident that my kids are leaving with above average reading and writing skills from their grade level.” He felt that his success with teaching language arts was “because I struggled with that stuff when I was younger and so I know right where they are.” Later, he added,

I think that I have enough background and taken enough coursework and done enough things with language arts education that the kids that are doing well I can challenge them to go further. But the kids that aren’t doing well, I can still help.

With respect to math, he observed,

when a student is struggling with a (math) concept that is something that I think is really easy, it’s hard for me to at this point in my teaching career to understand where they are coming from. Especially things that have a definite pattern or a definite algorithm type set-up, like the multiplication, how you set that problem up. If they can't figure out how—that that's just how to do it—how to set it up all the time. I struggle mentally with how to get them to do it the same way over and over again.
His only recourse was to tell students, “You just got to do it,” but he bemoaned, “I don’t like being like that and so that’s why it’s a struggle.” He concluded, “It’s almost like the kids that are doing well I can challenge them to go further, but the kids that aren’t doing well, I struggle with how to get them to where they need to be.” In summary, the teachers’ preparation for certification was similar, but their subsequent pedagogical development differed in terms of the length of their teaching experience and their reported professional development participation.

Beliefs About Mathematics Instructional Goals and Teacher Roles

Data that contributed to the descriptions of teachers’ beliefs about mathematics instructional goals and teacher roles included their interviews, impromptu conversations, and observations of instruction. Mr. Kent and Mrs. Baker differed in their beliefs about mathematics instructional goals and teachers’ role in instruction.

Mathematics Instructional Goals

Mr. Kent observed that, as he understood it, “the major objective” for fractions in the course of study or curriculum for the school district “is just understanding what a fraction is and then to be able to apply that into addition and subtraction problems.” He “spent a great deal of time on the unit” since he wanted “to give them the opportunity to take the time that they need(ed) to.” His personal goals for his students were to be able to add and subtract fractions, to order fractions, and to recognize that fractions and decimals “are very similar, it’s just a different way of writing a particular number.” He emphasized “how to think about a problem and do the problems” instead of “a lot of drill and practice.” In describing her mathematics curricular objectives with respect to fractions,
Mrs. Baker reported that she wanted students to understand “the idea of what a fraction is and then how to apply it.” For example, “if you were eating 1/6 of a pizza, what does that look like?” On the other hand, she observed that “adding and subtracting were a major part of it.”

One focus of Mrs. Baker’s mathematics instruction was to provide students with practice with mathematical procedures. To this end, “every morning we’d do a multiplication, a division, adding or subtracting with decimals, and then fractions after we learned it.” She would “put problems on the board and they would have to do that as soon as they came in.” She added, “Their memories were incredible. They were so good at it.” One of Mrs. Baker’s students, who had been assigned to the resource class for mathematics, wanted to participate in this morning ritual and her peers “were teaching her.” The student “would get it. Then sometimes she was so proud of herself that she would raise her hand to” answer one of the problems. Despite their apparent success in the classroom, Mrs. Baker’s students were “weakest” in their performance on the mathematical procedures portion of the standardized test. Mr. Kent attributed the class’s performance on standardized testing to the emphasis on understanding over procedural skill, “They did well for their ability level in problem solving, but they didn’t do as well as they should have on procedures.” He anticipated that he would be adjusting his future curriculum to promote more procedural competence although “not throwing everything out that I’ve done for the last couple of years.” Rather he wished to automate their performance in producing an answer for “a simple problem that they understand exactly
“what to do.” For example, “If I said, ‘what’s four plus one?’ they would know ‘five’ immediately.”

Mr. Kent rarely used the textbook except for “review homework.” By comparison, Mrs. Baker used the textbook as a daily instructional source except for the first few lessons. Mr. Kent reported that the school district “heavily encourages us to use the ‘Seeing Fractions’ unit,” because “the unit does such a better job of helping the kids to understand what a fraction is.” Mrs. Baker did not seem to be aware of the “Seeing Fractions” supplemental unit despite being in the same district. Mr. Kent noted that in the textbook, “there’s four pages about understanding what a fraction is, then you’re leaping into other things about them.” Mr. Kent added, “I like being able to extend some of the stuff from the Seeing Fractions with different curriculum texts.” For example, “I’ve used some stuff out of the Math Their Way of Thinking book to kind of support the ‘Seeing Fractions’ because they do some stuff with unifix cubes and things like that.” His rationale was that

the more things we use the less likely there’s going to be a student that doesn’t grasp it. Like if they’re not understanding it on the geoboard then maybe they’ll get it with the unifix cubes; or if they are not getting it with the unifix cubes, maybe the geoboard is the way for them to go.

Teacher Roles

Mr. Kent described himself as “a facilitator.” He opined, “I don’t want to be someone that just stands there and tells them what to do. I want to help them get to the point where they can learn on their own.” He described two significant “ideas” that he wanted to communicate to his students that he felt would benefit them in the long term, “challenging themselves” in and “taking responsibility” for their learning. He mused, “I
look back at teachers that I had and the ones I learned the most from were the ones that I had to do the most work because I wasn’t being spoon-fed stuff.” He believed that if his students would “make the discovery themselves, or if it clicks in their minds instead of just telling them why it works then they’ll keep it for a much longer period of time.”

In comparison, Mrs. Baker confided,

I’ve always excelled in math my whole life. From elementary through college it came easy. I think that way. It used to be my favorite subject to teach. But, I’ve learned as I’ve been teaching other things that I love, and now it’s more social studies. That is my true passion right now.

Describing her social studies instruction, she observed that “I don’t use a textbook very much. It’s projects and going to the library and getting books and finding out things and sharing it and ‘how should I present this, maybe a book or maybe a mobile.’” She described integrating her social studies instruction with language arts. She also voiced her intention in the future to integrate other subject areas into her social studies units including mathematics. In describing her instructional methods for her least favorite subject, science, Mrs. Baker reported that “I really believe in inquiry and the whole self-exploration and learning things that you are interested in.” Therefore, despite her internal dialogue, “I hate this. I don’t want to teach this. How can I pull this one off?” she led the students in exploratory activities, such as, “Today we need to make a helicopter; let’s see if we can figure this out.” She organized her students into groups and encouraged them to develop their own ideas as they worked with the materials. She observed that her students “are coming to so many conclusions on their own that a lot of times I don’t have to stand up and teach it.” Thus, her beliefs and methodology for teaching social studies and science were reminiscent of how Mr. Kent described his beliefs and methodology for
mathematics instruction. Although she had created some non-text based lessons, Mrs. Baker relied on the textbook and worksheets for her mathematics lessons.

Organization and Management of Fraction Instruction

Mr. Kent and Mrs. Baker’s beliefs about teaching mathematics were evident in their organization and management of fraction instruction. They differed in their teaching styles, use of instructional resources, grouping of students for instruction, delivery of content, assessment practices and use of homework, and behavior management techniques.

Teaching Styles and Use of Instructional Resources

Mrs. Baker conducted her mathematics class in the same manner every meeting that was observed. She began with a presentation of the material followed by a student activity or assignment. She used a small chalkboard at the front of the room for her presentations with the exception of the geoboard lesson for which she used the overhead projector. She would draw diagrams and record important terms on the chalkboard as well as record the steps of the exercises that she demonstrated during her presentations. Mr. Kent’s lessons followed a similar pattern throughout, but were not exactly the same every day. Lessons could extend over several days, were composed of student activities in which they “explored” the concept that they were to be learning and were punctuated by student sharing times and teacher demonstrations. The overhead projector was the focal point of whole-class demonstrations and student sharing. The chalkboard was rarely used. He self-reported that his style of instruction was for students to “spend one or two
days with a concept and then they spend three or fours days on their own working through it.”

With the exception of her introductory lesson to fractions, Mrs. Baker’s lesson presentations often demonstrated the type of activity that the students would be doing following the lesson. Before the students’ textbook assignment, she sometimes demonstrated how to do each type of textbook problem in her presentation. Mr. Kent’s demonstrations began with student input and concluded with examples that students could use as guides to their own explorations. For example, during student sharing times with geoboard activities, Mr. Kent invited individual students to come forward and present their designs (decompositions of the geoboard whole into representations of a given fraction based on area) for their classmates’ approval. He asked students to explain why their designs represented the intended fractions. Student sharing times fell at significant junctures in an activity, such as after some students had completed their geoboard designs but before they had begun coloring. Mrs. Baker also invited students to participate in the lesson. However, students were not sharing their own work. They were responding to, participating in, or imitating the teacher’s demonstrations. Sometimes Mrs. Baker would ask an individual and sometimes she would select from volunteers. If an individual declined the invitation, she accepted their decision without argument or cajoling. When she was asking a student to explain an answer, she sometimes feigned ignorance in order to encourage the student.
**Grouping of Students for Instruction**

Mr. Kent’s class was organized into informal groups. Single student tables were organized to form tables of four during the time that students were working with the geoboards and unifix cubes. Similarly, in Mrs. Baker’s class four to five students sat at each of five tables. In her class, one or two students sometimes sat at an individual desk or on a beanbag chair that was available. Student groups in either class were not constant over time although it appeared that some students worked together more often than not.

In Mr. Kent’s class, students were encouraged to help one another and share their designs with one another. One student reported that she had based one of her sixteenth designs (decomposition of the geoboard whole into 16 sixteenths) on another’s work. Mr. Kent himself suggested that a student’s design presented to the whole class might be legitimately recorded by any student who did not already have a similar design in their fractions notebook. Although students worked together daily in either groups or pairs at the prompting of Mr. Kent, group work with role assignment was not apparent. Mr. Kent reported in the interview that “we spent a lot of time at the beginning of the year with the rules of that setup, that if your partner is not getting it, you can give them hints but you can’t do the work.” His rationale for using partners and groups was

if we try to do it all individually, not being able to work and to discuss things with each other, that they wouldn’t—they would give up on it. If they had to wait for me to come around every time they had a question, they wouldn’t do anything at all.

Although Mrs. Baker’s students often worked together during class activities, it was not prompted by Mrs. Baker. Sometimes she actually discouraged students from sharing their solutions. On one rare occasion, she designed an activity that required group
participation. She asked students to take turns in participating in the activity, but did not assign formal roles to each member of each group. She told the class the tasks of the activity and allowed each group to assign the roles. However, she did make an initial assignment of roles to one group in which a student was confused about what he should be doing during the assignment.

**Delivery of Content**

Mrs. Baker’s instruction occurred primarily through her presentations. During the presentations, she would ask short one-word-answer questions. When a student contributed the answer to a question, she would sometimes ask the class to verify the answer either by following her in a verification process or by their own thinking. However, she was more often the judge of whether an answer was correct or not. If a student made an error in answering a question, she usually allowed them to correct the error. However, Mrs. Baker sometimes would call on another student if the first did not give the anticipated response. Other times, if the question had been about a process, she would demonstrate the answer to the question herself. If a student made a correct answer, she would often repeat it and sometimes she would praise the student.

By contrast, most of Mr. Kent’s instruction occurred during his individual and group interactions with students. For example, he introduced and reinforced the idea of composing area units from half area units in interactions with individual students as described in the section on “Finding Areas of Regions Composed of Discrete and Partial Area Units.” He similarly introduced a method for finding the areas of irregular shapes with individuals, but he later took the opportunity presented by a student sharing a
triangular design to demonstrate the process to the whole class. Mr. Kent sometimes
simplified an idea or concept for students. An important concept of the geoboard unit was
the realization that equal area measure was a universal attribute of unit fractions of a
continuous quantity whole and congruence of the parts was not. The geoboard activities
were meant to reveal and reinforce the given realization. To help students to create
decompositions of the geoboard which used multiple non-congruent equal area regions,
Mr. Kent suggested that students use one representation of a unit fraction on half of the
geoboard and another on the other half.

In her individual instruction of students during the activity or assignment, Mrs.
Baker often recorded the work herself on the student’s paper using the student’s pencil.
Often she modeled the process that they were to use with a problem with which they were
having difficulty, but on one or two rare occasions, she modeled a problem that a student
already had answered. As she worked through a problem with various students, she
would streamline the demonstration, asking fewer questions and telling each student
more of the process.

Assessment Practices and Use of Homework

Mr. Kent integrated homework with classwork. Homework was often either the
completion of some subtask of the overall activity or a set-up task for the following day’s
major activity. For example, the homework on the first recorded day of class was to fill
the page of half designs that they had begun in class. On the fourth recorded day of class,
the students were to have begun the night before making designs that decomposed the
geoboard model into mixed unit fraction segments, an assignment that they completed
and used that day to explore addition and equivalence of various fractions. Mrs. Baker intended that students complete activities or assignments following a lesson by the end of class time, but some students would need to complete them later in the day or as homework. The activities or assignments did not extend more than one day.

Mr. Kent monitored students’ understanding and progress during the student activity period. He asked individual students to explain to him their thinking verbally as they engaged in creating their designs. He also encouraged students to monitor their own understanding at the same time. He encouraged them to recognize and avoid designs that were similar and to attempt challenging designs. Important to the process of self and teacher assessment was the process of recording the results of their explorations. In their initial work either with geoboard or unifix-cube models, students were encouraged to verify their designs using the model before recording them on their papers. Mr. Kent encouraged peer assessment as well in his habitual request that students’ peers verify shared work.

Mr. Kent used labeling, coloring, and explaining of students’ recorded geoboard explorations to help him assess students’ understanding fractional amounts of the whole. He used these different processes as subtasks in the activities and sometimes he acted as gatekeeper between sub-tasks. For example, he sometimes required students to summon him to verify their designs after they had been labeled, but before they were allowed to begin the coloring process. Mr. Kent also encouraged students to verify their own designs before coloring by counting the number of area units within each region to assure that it truly represented the fraction indicated by the label. In the case of the unifix cube
activity, Mr. Kent was able to use students’ recorded results both on their papers and on the chalkboard to assess their understanding of modeling fractions with discrete quantities and of modeling equivalent fractions.

Mr. Kent used students’ written explanations to help him as a final activity assessment to determine whether students adequately understood the idea of area measure as a method for determining a fractional amount of a continuous quantity whole. Students were encouraged to include at least one challenging design among those that they explained. In his assessment of written explanations, he promoted students’ use of precise language that highlighted the criteria determining the unit fractions illustrated by each design.

Although grading of assignments was observed only once in Mrs. Baker’s class, it appeared to be the common method of assessment. In that instance, Mrs. Baker had students mark their own textbook assignment as she read the answers from the textbook answer key. Reliance on the textbook appeared to interfere with her recognition of a correct answer for one item. In addition, examples of student work revealed grading by the teacher. Thus, self-assessment did not appear to be a common practice in her classroom. Nevertheless, Mrs. Baker demonstrated sensitivity to her students’ needs. She circulated during students’ work period to answer questions and observe their progress on the assignment. When a student introduced cross-multiplication ahead of her scheduled time for it, she introduced the algorithm to the class. At another time, she discontinued a personal method of recording the equivalence factor, because a student had misinterpreted it. She often praised students’ contributions and even their corrections.
She would praise a student for a correctly done exercise before working on an incorrectly done exercise. When she had asked the class if anyone was having difficulty with a new and unfamiliar problem, she chastised those who bragged about its “easiness,” reminding them that not everyone shared their experience of it. As a sidelight, Mrs. Baker had a habit of anthropomorphizing numbers, calling them “him” which may or may not be related to her desire to be sensitive to her students’ feelings.

Management of Behavior

Because Mr. Kent’s mathematics instruction was activity intensive, he was careful to organize students’ behavior and to communicate behavioral expectations prior to beginning the activities. In the case of the geoboard lessons, he focused his students on the appropriate use of the geoboard. He identified the rubber bands used with geoboards as “geobands,” emphasizing that they are “tools” and not toys. He gave his students two safety rules for working with geoboards. First, he introduced students to a method for avoiding the rubber bands flying out of control. Second, he warned that any student seen “intentionally shooting” a “geoband” would lose their privilege of working with the geoboard. In the case of the unifix cube activity, he had students work in pairs and he gave each pair a post-it note by which to indicate the number between 2 and 24 with which they were working.

Mr. Kent promoted students’ appropriate behavior using several methods. His constant monitoring and circulation among students promoted on-task behavior during activities. He praised students for their creativity in their work and answered their questions to help them continue their work as well as assessed their work. He also praised
individual students who were quick to act appropriately when he called the class to attention. He warned students of inappropriate behavior and when they failed to curb inappropriate behavior, his accountability procedure was to have students move their magnet from one side of a discipline board to another. Students’ whose magnets were moved would lose privileges such as free time at lunch.

Mrs. Baker used “stars” and candy rewards to motivate students to stay on task. During the student work portion of a lesson (group or individual), she would circulate and observe class behavior as well as answer questions. As the class began working on an assignment she might say, “This group can have a star for participation,” or “Chris’ table gets a star and an M&M for having their books out. Every person has them out to the right page.” Rewards also came at the end of an assignment, “Everybody can have an M&M on their way to lunch—you worked so hard. You can have a star also for working quietly.”

Teachers’ Fraction Instructional Unit and Its Relationship to Their Personal Content Knowledge

The category of organization of content for instruction was comprised of the instructional themes that had been identified from the observations of each teacher’s instructional unit for rational numbers. Each instructional theme was discussed in terms of its connections to the rational number constructs and concepts used to describe teachers’ personal rational number knowledge and students’ emergent rational number knowledge. As would be anticipated the instructional themes that were identified for each teacher differed significantly. The following sections of the cross-case analysis compare
and contrast the teachers’ instruction with respect to the major concepts/constructs that were addressed in their instruction: models of fractional amounts, fraction definitions, models of wholes, models of equivalent fractions, finding equivalent fractions algorithmically, finding equal portions (fair sharing), ordering fractions, and adding/subtracting fractions.

Models of Fractional Amounts

As is apparent from the discussion of their pedagogical beliefs and organization methods, Mr. Kent and Mrs. Baker differed in their instruction of fractions. The content of their fraction units differed as well despite the fact that they taught in the same school district. Both began their units with instructional activities intended to develop their students’ understanding of fraction concepts. Both teachers’ first lessons were designed to elicit students’ prior fraction knowledge. Mrs. Baker’s first fraction lesson began with the class reflecting on their personal definitions and experiences of fractions. Her students described pizza slices, pies, and parts of books. The teacher and students used pizza and pie slices to illustrate and discuss fractions as parts of wholes. Mr. Kent’s students began their first fraction lesson halving their geoboards using partitioning lines. A half as one of two congruent partitions of a whole was likewise a familiar concept to them.

Fraction Definitions and Models of Wholes

In these first lessons, each teacher also introduced their respective students to the teachers’ respective preferred fraction definitions and models of a whole. Mrs. Baker introduced her students to her definition of a fraction as the ratio of a designated number
of unit parts to the total unit parts present. The students’ first activity was to “build a fraction” using unifix cubes, discrete quantity model of a whole. She instructed her students to form unifix sticks from two colors and then to write on their papers the fraction formed. Beside each fraction on their paper, they were instructed to write “a word fraction.” The “word fractions” were composed of the color name of the cubes that they had counted as the fractional amount of the whole above the fraction line and the word “total” below it.

After their initial activity of forming a half on the geoboard using the concept of congruent partitioning, Mr. Kent introduced his students to the area model of a half on the geoboard. Thus, he used the geoboard as a continuous quantity model of a whole. With this model, Mr. Kent’s students were encouraged to define fractional amounts as composite continuous quantities. In back to back activities over the next two weeks, students decomposed their geoboards into multiple representations of halves, fourths, eighths, and sixteenths by defining each as a specific number of area units—eight, four, two, and one geoboard square(s), respectively. Their representations of each unit fraction were not limited to formations of discrete squares. Rather Mr. Kent’s students were encouraged to form any representation that had the requisite area including triangles and irregular polygonal figures. When the 4x6 grid was introduced as the whole to model thirds through twenty-fourths, students were asked to predict the areas of unit parts representing newly introduced unit fractions and to describe unit fractions in terms of their areas.
Throughout the lessons, Mr. Kent and his students discussed fraction representations using geometric as well as area measurement terminology. Congruence was an important concept that was reinforced in students’ explanations as well as being used in teacher-student discussions of fraction representations. Decomposing and composing wholes into/from unit fractions was daily fare. Mr. Kent required that his students record their geoboard compositions/decompositions on dot grids and carefully label each unit part with the appropriate unit fraction much as he had done in his answer to test Problem 2 (see Figure 4.1).

Mr. Kent consciously introduced other models of wholes in subsequent lessons to encourage his students to have flexible models of wholes and unit fractional amounts. When he introduced the 4x6 grid, he wanted his students to recognize that the areas of the representations of the unit fractions half, fourth, and eighth on this new model would be different from those on the geoboard. To reinforce this concept, he had his students decompose other irregular grid shapes into unit fractions. Mr. Kent’s students also modeled fractional amounts with unifix cube sticks, although they did not use the bicolor model that Mrs. Baker’s students had. Different lengths of sticks represented different sizes of wholes.

In the second observed lesson, Mrs. Baker used the geoboard as well to model fractional amounts. However, instead of treating the geoboard as the whole she introduced her students to the multiple polygonal shapes that can be modeled on the geoboard. Each square, triangle, rectangle, or other shape was interpreted as a potential whole. However, she did not introduce representing fractional amounts as composite
areas as Mr. Kent had. She instead asked her students, “How many equal parts are there?” in the given shape. Thus, she emphasized her definition of a whole as a composition of unit parts. Her students had little difficulty decomposing large squares or rectangles on the geoboard into unit squares. However, they were confused when asked to partition a large right isosceles triangle into equal parts. She used the situation to encourage students to recognize that unit triangles could be the unit part. She also introduced the idea of composing a unit square from two unit triangles. Thus, although she used a composite unit, perhaps even a composite continuous quantity unit, it was not an area unit. The emphasis was on the composition forming recognizably congruent figures (unit parts).

Although Mrs. Baker’s second lesson encouraged students to realize a whole can be represented in multiple ways, there was not the sense that she used or encouraged non-standard forms of a whole as Mr. Kent attempted.

Mrs. Baker reinforced her “parts to total” definition in this second lesson as well. Shading one or more “parts” of a shape on the geoboard, she asked students to name the fraction illustrated by counting the number of parts shaded and the total number of parts in the shape. She reinforced her definition as she recorded the fraction for all to see; e.g., saying “one out of four” as she wrote “1/4” on the overhead. As in the first lesson, students engaged in a subsequent activity of “making fractions” “out of shapes” using a geoboard. Thus, she encouraged her students to “make shapes” that could be decomposed/composed into/from unit parts. As was her pattern on the test, she relied on the concept of unit part rather than unit fraction in describing a fractional amount of a whole.
Modeling Equivalent Fractions

Mrs. Baker introduced modeling equivalent fractions in the second recorded lesson. She invited a student to decompose a 2x3 rectangle into equal parts on the overhead grid. The student formed the easily recognizable six unit squares. Shading two of the six squares, she asked her students to identify the fraction illustrated. They responded “two out of six” and “2/6.” She then suggested, “What if I said, ‘there’s another fraction there?’” She appropriately rejected a student’s offering of the ratio of shaded to unshaded parts, but she also rejected another student’s offering of 4/12. She sought the response “1/3.” Assuming the student could model how the two shaded sixths also formed a third of the rectangle she invited him forward. However, he had “reduced” and did not know what she wanted. She herself demonstrated a repartitioning of the rectangle into three equal parts that included the two shaded sixths as one of the three parts. She encouraged those students that understood the process to form models of equivalent fractions in their group activity with the geoboards. The following week she devoted a lesson to the repartitioning process of modeling equivalent fractions. Her students practiced the repartitioning process for modeling equivalent fractions with an activity taken from the text and on subsequent worksheets. The process that she demonstrated in class parallels that she used in modeling addition and subtracting of fractions with unlike denominators on the test and in the interview.

Mr. Kent waited until the second week of his unit to introduce modeling equivalent fractions. He included modeling fraction sums in the activity. Prior to the lesson, students had practiced decomposing the geoboard into multiple representations of
each of the common unit fractions: 1/2, 1/4, 1/8, and 1/16. Students began the activity by decomposing the geoboard into various combinations of these same unit fractions and recording their designs on grid paper. Mr. Kent then had students share designs that proved statements such as “1/4 + 1/4 = 1/2” or “1/4 = 2/8” or completed statements such as “1/4 + 1/4 + 1/4 = ?” or “1/2 = ?/16.” In a subsequent lesson with the 4x6 grid model, he asked his students, “What is the relationship between a sixth and a third? How many sixths does it take to make a third using one of those designs up there?” One student responded that a sixth was “half a third” and demonstrated for his peers that two representations of a sixth made a representation of third. The process that Mr. Kent encouraged his students to use parallels the one he used with the geoboard on interview Task 3 (see Figure 4.1).

Mr. Kent first introduced the fraction rate chain in this lesson. He asked his students to find designs that demonstrated that “1/2 = 2/4 = 4/8 = 8/16” and then asked his students to predict the extension of the proportion to thirty-seconds and sixty-fourths. This is reminiscent of his use of the “rate chain principle” in his solutions of test Problem 31 and interview Task 5a (see Figure 4.1). Mrs. Baker did not introduce extended proportions until after she had introduced her students to an algorithmic process of forming equivalent fractions. She did so then because it was included in the textbook exercise.

Mr. Kent also asked his students to model multiple equivalent fractions using the unifix cube sticks. Pairs of students shared the task of finding all unit fractions that sticks of lengths 3 to 23 could be decomposed/composed into/from. In the process, they were
required to find all equivalents of each unit fraction in terms of the other possible unit fractions of that length stick. When students recorded the equivalents for each unit fraction, the model that he suggested that they use was again an extended proportion or “rate chain.” In the subsequent group discussion, Mr. Kent emphasized all the possible sizes of wholes that could represent a particular unit fraction. This is reminiscent of his approach to finding the alternative representation for 3/8 in interview Task 2 (see Figure 4.3) by forming a larger whole that could be subdivided into eighths.

There is evidence from her discussion of interview Task 2 that Mrs. Baker may have used unifix cubes in an activity to demonstrate equivalent fractions as well. However, the lesson had not been videotaped. Similarities between her solution to interview Task 2c and those of two of her students support that they had had a common experience. The common reasoning process across the three solutions was to divide a whole into a specified number of groups of equal size and then designate a fractional amount of the whole. It was then observed that an equal number of cubes could be added to each of the groups without changing the designated fractional amount of the whole. The size of the whole itself and the size of each group were unimportant to defining the fractional amount.

A third model that Mr. Kent used to develop his students’ understanding of equivalent fractions was fraction strips. Students were given equal size lengths assigned the value of one whole. They first used the strips to measure various objects in the room. For objects having lengths that could not be measured exactly by whole strips, Mr. Kent asked his students to assign the fractional amount of a whole that they thought would best
approximate the actual length. After his students reported and discussed the lengths of the objects that they had measured, Mr. Kent had each student create “fraction rulers” with the strips. Students partitioned each into a different unit fractions—halves, thirds, fourths, etc. Mr. Kent emphasized that each partition line should be labeled with the fractional amount of the whole that it represented. For example, on the “fourths” ruler, the first mark would be labeled “1/4,” the second “2/4,” and the third “3/4.” During a subsequent lesson, Mr. Kent had the students put the strips in order from halves to ninths and identify equivalent fractions. This activity corresponds to Mr. Kent’s association of fraction and measurement concepts evidenced in his test answers as well as his knowledge of specific equivalence relationships. A comparable activity was not observed in Mrs. Baker’s class.

Finding Equivalent Fractions Algorithmically

As described in the preceding section, both Mr. Kent and Mrs. Baker introduced the fraction proportion chain in their development of the concept of equivalent fractions. For Mr. Kent writing the extended proportion “1/2 = 2/4 = 4/8 = 8/16” was initially a way to record symbolically what students were modeling. He next asked his students to look for a pattern in the extended proportion that would help them to predict other fractions that would follow in the extended proportion. He encouraged students’ observations of doubling. He emphasized doubling as multiplication by 2, but did not substantially discourage a definition of doubling as repeated addition. Thus, he introduced the algorithmic method of finding equivalent fractions of multiplying by the same amount via the extended proportion. Both the extended proportion and multiplication of the numerator and denominator by the same amount were evident from his test explanations.
The only other algorithmic method for finding equivalent fractions that Mr. Kent used in class was “reducing” by dividing the numerator and denominator by the same amount. He introduced “reducing” after developing the concept of equivalent fractions through discrete and area models and through observing the patterns of extended proportions.

Mrs. Baker began algorithmic processes for finding equivalent fractions comparatively early. During a lesson on modeling equivalent fractions, one of her students recalled for the class their exposure to the cross multiplication method of finding equivalent fractions. Mrs. Baker suggested that the class use the cross multiplication method as a “checking process” for their models of equivalent fractions. In the next observed lesson, she directly instructed students to find equivalent fractions by “multiplying the numerator and denominator by the same number.” Students were first given proportions and asked questions such as “how did they get from 2 to 6 and how did they get from 5 to 15?” Then they were asked to verify the truth of a proportion by observing whether it obeyed the rule. Finally, they were asked to find equivalent fractions by “choosing a number” and “using it on both” the numerator and denominator. Her instruction thus reflected the same reasoning process that she described for finding equivalents on her test.

Mrs. Baker’s use of the extended proportion in her lessons on equivalence appeared to be an artifact of the textbook that she used. She encouraged students to use patterns that they observed in order to complete the extended proportion in addition to emphasizing the algorithmic process that she had described for finding equivalents. She treated the numerator and denominator in extended proportions as two separate
sequences. She encouraged individual students to use skip counting to produce numerators and denominators of subsequent fractions in extended proportions. Therefore, it is not surprising that she did not use extended proportions in any of her arguments for finding equivalent fractions on her test as Mr. Kent had on his test.

Mrs. Baker introduced her unique method for completing a proportion, the “backward Z,” the third week of her unit. A whole lesson was devoted to the technique. Her next observed lesson was to introduce “reducing” via the greatest common factor method. By comparison, when Mr. Kent emphasized algorithmic thinking in his lesson on “reducing,” he merely encouraged students to divide both numerator and denominator by any common factor and then to repeat the process until there was not a common factor.

Finding Equal Portions or Fair Sharing

Mrs. Baker only discussed fair sharing or finding equal portions during her initial discussion of fractions in her first lesson. She and her students only discussed fair shares in terms of one whole. Mr. Kent, by comparison, had activities that relied on finding equal portions of multiple wholes shared by a number of people that would require fractional shares as a part of his development of his students understanding of fractional amounts. The sharing activities occurred during the fifth week of instruction, following his development of fraction concepts with the geoboard/dot grids and the unifix cubes, but before using fraction strips to measure objects in the room. The observed fair sharing activities included whole class example problems and student-developed problems. For example, Mr. Kent proposed that three people are sharing seven cookies, “so block off 7
One of his students demonstrated a solution on the overhead, explaining, “I have 2 1/3 because 7 divided by 3 is.” The student used letters to indicate that each person received two whole cookies and a third of another on the overhead. Thus, Mr. Kent’s instruction paralleled the process that he himself used in answering test Problems 17, 18 and 19 (see Figure 4.4).

**Ordering Fractions**

In her first lesson, Mrs. Baker used circle diagrams to review the inverse relationship between unit fraction size and the number of parts of the whole. This was a concept that she and Mr. Kent had demonstrated in their ordering of the first list of fractions on test Problem 29 (see Figures 4.8 and 4.11). In the fourth week, Mrs. Baker developed the concept of comparing fractional amounts. She began by comparing sizes of known fractional amounts such as “1/2” and “2/3” and reviewing the relational symbols, < and >. Then she instructed her students to compare fractions by comparing numerators once equivalent fractions with common denominators had been found. Of course, the method for finding equivalent fractions having a common denominator was her preferred method, “the backward Z.” This was the predominant method that she used to order the fraction lists on test Problem 29 (see Figures 4.8 and 4.11).

Mr. Kent compared fractional amounts during the fifth week of instruction. He used “candy company problems” to introduce the concept of comparing by asking the question “who is selling candy for less?” He encouraged his students to use extended proportions to solve the problems. Students were expected to use the extended proportion or “rate chain” for each fraction and compare the two equivalents that had a common
denominator. This process appears not to reflect either of the methods that he employed on the test to order the fraction lists. His preferred choice was to find decimal equivalents for each of the fractions. Finding decimal equivalents however was not part of the curriculum for the fifth grade. His other method was eclectic and dependent on the fractions being ordered.

Adding and Subtracting Fractions

Before introducing addition and subtraction of fractions, Mrs. Baker made a point of reviewing all the algorithms that she had taught for comparing and “reducing” fractions. Her first lesson on adding and subtracting fractions focused exclusively on adding or subtracting fractions with like denominators. This was in the eighth week of her fractions unit. She taught a procedure for adding or subtracting. She encouraged her students to write all addition and subtraction problems in vertical form and then to follow the procedure. The steps of the procedure were: (1) add or subtract the numerators, (2) write the answer for the numerator, (3) write the denominator, and then (4) “reduce” the answer. The following week she introduced adding or subtracting fractions with unlike denominators. She reviewed the procedure for adding/subtracting fractions with like denominators before demonstrating adding/subtracting fractions with unlike denominators. In demonstrating the latter, she emphasized the requirement that two fractions have the same denominator before they can be added or subtracted. She reinforced the process that she had developed for adding/subtracting fraction and inserted the finding equivalent fractions with common denominators as the second step before adding the numerators. Of course, the demonstrated process for finding equivalents was
her “backward Z” method. In her initial example, she suggested that students always check to see if the larger of the two denominators could be divided by the smaller in order to predict the common denominator. Oddly enough, the modeling process that Mrs. Baker personally used to solve addition and subtraction problems did not appear at all in her instruction. However, she did use the common denominator method that she had taught her students as one of her two solution methods on test Problems 35 and 36 (see Figures 4.12).

As described in an earlier section, Mr. Kent introduced the concept of adding fractions at the same time he introduced the concept of equivalent fractions, the second week of his fraction unit. To introduce the concept of addition of fractions he had his students “prove” addition statements via the area model, geoboard or dot grid. Most of the statements involved sums of repetitions of the same fraction, but at least one was the sum of two fractions with unlike denominators. A subsequent review activity had students use the dot grid to complete addition statements. Mr. Kent’s introduction of fraction subtraction was not videotaped. However, a later review lesson suggested that the subtraction of fractions had been introduced using students’ understanding that to add two fractions requires a common denominator. Students were expected to determine the equivalents of the fractions that they were asked to add. Then they were to restate the subtraction problem with equivalents that had common denominators. The method for finding fractions with a common denominator did not appear to be proscribed. However, students did appear to know that multiplying the numerator and denominator by the same amount produced equivalents. Mr. Kent’s instruction corresponds to the understanding of
addition and subtraction of fractions that he demonstrated on test Problems 35 and 36 and interview Tasks 9 and 10 (see Figure 4.12). However, his test and interview reasoning revealed a personal preference for an algorithmic approach to solutions of addition or subtraction problems since it was his first choice on the test and his only choice in the interview (see Figures 4.9 and 4.12).

Summary

As stated at the beginning of Chapter 5, Mr. Kent and Mrs. Baker differed in their beliefs about mathematics and mathematics teaching. Mr. Kent espoused beliefs that correspond to the description of the learner-focused teacher described in Chapter 2. His instructional organization and management appeared to be consistent with his beliefs. Mrs. Baker’s beliefs were not as clearly articulated, but her instructional organization and management indicated correspondence to the description of the content-focused teacher who emphasizes performance.

The fraction unit that Mr. Kent and Mrs. Baker each developed and executed reflected not only their beliefs about teaching mathematics and their role as teacher, but it also reflected their particular content knowledge. Their particular preferences for definitions of fractions and of wholes demonstrated in their responses on the test and in the interview are evident in their instruction. There are exceptions. Mrs. Baker introduced the topic of extended proportions, although she did not personally apply it in her own reasoning. It appeared to be an artifact of the textbook rather than a choice based on her personal knowledge of fractions. Another exception was her choice not to have her
students model addition and subtraction of fractions although it did appear to be part of her personal knowledge.

Mr. Kent’s instructional preferences differed from his personal knowledge preferences in two instances. In one case, that of ordering fractions by finding their decimal equivalents, the scope and sequence probably dictated his choice not to introduce decimal equivalents. His use of extended proportions, instead of conceptual models and benchmark numbers to order fractions, may have been an artifact of the curriculum unit that he used. In the other case, that of introducing addition and subtraction of fractions through area models instead of an emphasis on an algorithmic process was doubtless because of his espoused beliefs about how students best learn mathematical content. He developed the algorithmic process using the models to connect his students’ understanding of the nature of fractional amounts with the nature of their sums and differences.
CHAPTER 6

STUDENTS’ EMERGENT CONTENT KNOWLEDGE CROSS-CASE ANALYSES

Students’ test and interview responses provided the data for examining students’ emergent rational number knowledge across the two teachers’ cases. The focus for the analysis in the two teachers’ cases was the students’ test and interview explanations. The cross-case analysis of students’ emergent rational number knowledge as described in the teachers’ cases and the subsequent analysis of students’ test and interview explanations from each class are discussed.

Students’ Test and Interview Explanations Analysis

Nineteen students from Mrs. Baker’s class and seventeen from Mr. Kent’s class took the fraction posttest. However, Mrs. Baker had warned that her students were not average. She attracted students that had difficulties in other classes. Therefore, she had a disproportionate share of students that did not perform well in mathematics. By comparison, students appeared to be assigned randomly to Mr. Kent’s class and therefore, average in their overall mathematics performance. In an attempt to account for the class composition differences, I obtained the overall mathematics percentile scores for individual students and the mean score for each class from a standardized test administered in the district’s fifth grade. The students had taken the test toward the
beginning of their respective fraction units. The average standardized test mathematics score of the two classes corroborated Mrs. Baker’s observation.

In analyzing students’ explanations/responses to the posttest and interview items, two categories of contributors to the students’ conceptual development of fraction knowledge were evident at the time of the test or interview: instructional contributions and prior knowledge contributions. These were examined separately in the cross-case analysis. The explanations of all the students from both classes (Baker, n=11; Kent, n=17) who took the Rational Number Test, Millsaps S, 5/17/97 were examined in light of their instructional experiences. The explanations from both the test and interviews of all interviewed students were examined in light of their prior knowledge.

Students’ prior knowledge is defined as their quartile rank as well as constructions that students have applied to their solution explanations that have no apparent parallel in instruction or that were evident at the time of instruction. The quartile ranking was based on the students’ scores on a national standardized test administered by the district for measuring student achievement at the fifth grade level. Students whose standardized test percentile rank was above 75 were categorized as Quartile 1 (hereafter Upper Quartile) and students whose standardized test percentile rank was between 25 and 75 were categorized as Quartiles 2 and 3 (hereafter Middle Quartiles). Students whose percentile score was below 25 were not considered in the explanation analysis because all such students were in Mrs. Baker’s class and hence there could be no group with which to compare them. Students for whom no standardized test score could be found (Baker, n=2;
Kent n=3) were placed in quartile categories (Baker, two in middle quartiles; Kent three in upper quartile) based on the teachers’ assessment of the students’ performance.

*Interactions Between Students’ Experience of Instruction and Students’ Development of Fraction Concepts*

The descriptions of students’ instructional experiences and emergent knowledge of rational number from each teacher’s case provide the foundation for the cross-case analysis. The cases described students’ instructional experiences and emergent knowledge related to: (a) the part-whole construct and related concepts (whole and fraction), (b) equivalence (models, algorithms, and classes), (c) related constructs (geometry, measurement, quotient, operator), and (d) operations (addition or subtraction).

The part-whole construct, related concepts (whole and fraction), and related constructs (geometry, measurement, quotient, operator) were recategorized in the cross-case analysis as: (a) the relationship between the denominator of a fraction and the whole (number of unit parts in a whole and unit part size), (b) the relationship between a fractional amount and the unit part of a whole, and (c) flexibility in working with wholes and unit parts of wholes. Equivalence (models, algorithms, and classes) became in the cross-case analysis the concepts and processes of finding equivalent fractions.

Operations (addition and subtraction) became the concepts and processes of adding or subtracting fractions and the application of equivalent fraction concepts and processes. From additional test and interview data, cross-case categories were developed for the concepts and processes of fair sharing and of ordering fractions.
Although the interaction between students’ instructional experiences and their concept development seemed less powerful overall, those that exist suggest routes by which teachers’ personal fraction constructs could influence their students’ fraction constructs. Some concepts or processes related to fractions were introduced or reinforced in both classes. These concepts and processes were addressed through similar and differing instructional experiences. Some of the concepts or processes received varying degrees of emphasis in each classroom. Some concepts and processes related to fractions instantiated on the test were not addressed in either teacher’s instruction.

The Relationship Between the Denominator of a Fraction and the Whole: Number of Unit Parts in a Whole and Unit Part Size

In both classes, the concept that a denominator represented the number of parts by which a “whole” could be partitioned or could be comprised was reinforced as well as the concept that each part should be equal in size. However, the ways in which this concept and its related constructs and/or processes were evident in instruction differed. As described in Chapter 5, students in Mrs. Baker’s class initially “built” fractions from singleton units to form wholes using unifix cubes. Hence, their construct for the denominator of the “fraction” would have been an assimilation or accommodation of the number of cubes forming a stick or “total.” In contrast, Mr. Kent’s students were soon introduced to partitioning the 4x4 geoboard into non-congruent halves, fourths, eighths, and sixteenths having the same area as measured by the 16 unit squares of the geoboard. Their construct for the denominator of a unit fraction would have assimilated or accommodated the number of equal area parts into which the geoboard was partitioned.
As described in Chapter 5, students in both classes were introduced to alternative models that provided additional opportunities to connect the denominator in a fraction symbol with the number of unit parts in the represented whole. Mrs. Baker’s lesson on partitioning various polygonal regions on the geoboard or on graph paper into congruent parts exposed students to a variety of pre-partitioned figures for which they were to write a symbol for the illustrated fractional amount. All the examples that students encountered in Mrs. Baker’s class used congruent unit parts and her students heard repeatedly that the parts of a whole should be “equal size,” reinforcing the concept of unit parts as congruent figures. In addition, she emphasized complements of fractional amounts. In contrast, Mr. Kent introduced other standard and non-standard wholes based on dot grid forms and unifix cubes such as in the activity in which students represented various sizes of wholes with unifix cube sticks and explored all the possible fractions that could be represented with each whole. Thus, his students had the opportunity to construct unit parts as composite units of discrete units and to observe multiplicative relationships between the number of discrete units in a unit part, the number of unit parts in a whole, and the number of discrete units in a whole. The explanations and solution strategies from the test and interviews of students from both classes revealed viable and non-viable applications of the recognition that the denominator of the fraction represented the number of unit parts in a whole and the development of viable and non-viable constructs of the nature of unit part of a whole.

For problems and tasks that required finding the whole given a fractional amount (see Figure 6.1), students from each class applied two viable strategies for forming the
whole from identified unit parts: (a) iterate the unit amount or (b) add the complement amount based on the denominator of the fraction represented. An examination of the percents of students using given strategies to model a whole given a fractional amount (see Figure 6.2) show that Mr. Kent’s students were more likely to iterate a unit amount to describe the whole than to add the complement fraction amount to form a whole. In contrast, Mrs. Baker’s students were more likely to use a complement strategy as an iteration strategy to describe the whole. Thus, it could be posited that Mrs. Baker’s instructional emphasis on complementary relationships was connected to her students’ comparatively greater use of that strategy for forming wholes from unit parts.

<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>2</td>
<td>is two thirds of some length. Draw the whole length below and explain why it is the whole.</td>
</tr>
</tbody>
</table>
| Task      | 5a       | Show and read the statement:  
Tim used 6 cubes to build 1/5 of a blue tower. How many cubes long will the whole tower be when it’s finished?  
(a) Provide unifix cubes and ask student to show you how to use these cubes to solve the problem. Ask students to talk aloud as they solve the problem. |
| Task      | 5b       | (b) (If correct repeat changing the data: 8 cubes; 2/3) |
| Task      | 6a       | Show the 1/8 fraction piece.  
(a) Say: This is 1/6 of my unit. With your fraction circles show me the unit. Talk aloud as you solve the problem explaining each step. (Record answer.) |
| Task      | 6b       | (b) (If correct change data: 1/4 piece is 2/3; find the unit.) |

Note. Tasks are interview items; problems are test items.

Figure 6.1. Items that require finding the whole given a fractional amount.
<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n=10; 5)</td>
<td>Middle (n=6; 3)</td>
</tr>
<tr>
<td></td>
<td>Upper (n=3; 2)</td>
<td>Middle (n=7; 4)</td>
</tr>
<tr>
<td>Iteration</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 2</td>
<td>40% (20%)</td>
<td>50%.</td>
</tr>
<tr>
<td>Task 5a</td>
<td>80%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 5b</td>
<td>40%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 6a</td>
<td>80%</td>
<td>33 1/3%</td>
</tr>
<tr>
<td>Task 6b</td>
<td>20%*</td>
<td>0%</td>
</tr>
<tr>
<td>Complement</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 2</td>
<td>30%</td>
<td>16 2/3%</td>
</tr>
<tr>
<td>Task 5a</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 5b</td>
<td>20%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 6a</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 6b</td>
<td>60%*</td>
<td>0%</td>
</tr>
<tr>
<td>Visual approximation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 2</td>
<td>20%</td>
<td>16 2/3%</td>
</tr>
<tr>
<td>Task 5a</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 5b</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 6a</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 6b</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Rate series (unsuccessful application)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 2</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 5a</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 5b</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 6a</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 6b</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Other or no strategy</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 2</td>
<td>10%</td>
<td>16 2/3%</td>
</tr>
<tr>
<td>Task 5a</td>
<td>0%</td>
<td>(67 2/3%)</td>
</tr>
<tr>
<td>Task 5b</td>
<td>(20%)</td>
<td>(100%)</td>
</tr>
<tr>
<td>Task 6a</td>
<td>(20%)</td>
<td>(67 2/3%)</td>
</tr>
<tr>
<td>Task 6b</td>
<td>(40%)</td>
<td>(100%)</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. n=test; interview. Inappropriate symbolism or result in parenthesis.

*Overlap.

Figure 6.2. Percents of students from each quartile group in each class that use a given strategy for representing the unit whole on test Problem 2 and interview Tasks 5 and 6.
Viable iteration strategies described in each class included the use of multiplication to describe the iteration. When Mrs. Baker’s students did describe iteration in their explanations, it was in the context of a discrete quantity model and described multiplying a composite discrete unit part to find the size of the whole. Their explanations did not use multiplication to describe iteration on other items. By contrast, only one of Mr. Kent’s students used multiplication in the context of composite discrete units. However, he and another student did use multiplication to find the whole in the case of test Problem 2. Two students used multiplication to find the number of unit parts in the whole on test Problem 5 (see Figure 6.3). Thus, Mr. Kent’s students’ application of multiplication to the formation or description of a whole appeared more diverse. It could be posited that Mr. Kent’s students’ extensive experience with decomposing/composing wholes into composite unit parts without an explicit emphasis on multiplication throughout may have allowed students to develop more varied viable associations between unit parts and wholes. In the case of interview Tasks 5a and 5b, Mr. Kent’s students’ explanations included repeated addition and complement relationships.

Two types of problems and tasks required finding a particular fractional amount given the whole: (a) problems for which students were expected to determine the appropriate symbol to describe the indicated fractional amount (see Figure 6.3) and (b) problems for which students were expected to determine an appropriate representation of a fractional amount given its symbol and a representational medium (see Figure 6.6). In viable strategies for determining either the symbol or the representations students should recognize that the denominator of the fraction would enumerate the number of unit parts
contained in the whole. Thus, viable strategies for both problem types required determining an appropriate number of unit parts in the whole.

<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>1</td>
<td><img src="image1" alt="Image" /> If is one unit, what fraction is ? Explain why you think your answer is correct.</td>
</tr>
<tr>
<td>Problem 4</td>
<td>4</td>
<td>Write a fraction to show what part is shaded. Explain why you think your answer is correct.</td>
</tr>
<tr>
<td>Problem 5</td>
<td>5</td>
<td>For each picture below, write a fraction to show what part is shaded. Choose one picture and explain why you think your answer is correct.</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 6.3. Items that require finding the appropriate symbol to represent a given fractional amount of a whole.

Two viable strategies for determining the number of unit parts in the whole appeared in the students’ explanations of test Problem 1 (see Figure 6.3): (a) tiling the whole with the given fractional amount and (b) partitioning the whole into parts approximating the size of the given fractional amount. Although students from both classes used each of these processes in their explanations, Mr. Kent’s students were more likely to describe tiling and Mrs. Baker’s students were more likely to describe
partitioning (see Figure 6.4). Thus, it can be conjectured that Mr. Kent’s students’
practice covering a geoboard with different representations of various unit fractions may
have engendered a tiling process in his students’ thinking. By contrast, Mrs. Baker’s
emphasis on “cutting,” “breaking,” and “dividing” of wholes into unit parts in her
instruction encouraged a partitioning process.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tiling</td>
<td>Upper (n=10)</td>
<td>Middle (n=7)</td>
</tr>
<tr>
<td>success</td>
<td>80%</td>
<td>14%</td>
</tr>
<tr>
<td>not</td>
<td>0%</td>
<td>28%</td>
</tr>
<tr>
<td>Partitioning</td>
<td>success</td>
<td>Upper (n=3)</td>
</tr>
<tr>
<td>not</td>
<td>0%</td>
<td>10%</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>28%</td>
</tr>
<tr>
<td>Complement</td>
<td>success</td>
<td>Upper (n=3)</td>
</tr>
<tr>
<td>not</td>
<td>0%</td>
<td>10%</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Other</td>
<td>success</td>
<td>Upper (n=3)</td>
</tr>
<tr>
<td>not</td>
<td>0%</td>
<td>14%</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>14%</td>
</tr>
</tbody>
</table>

Figure 6.4. Percents of students from each quartile group in each class that use a given
strategy for interpreting the fractional amount with success (interpret fractional amount as
1/3 of whole) on test Problem 1.

For test Problems 4 and 5 (see Figure 6.3), students’ answers and explanations
indicated two viable strategies for describing the fractional amount: (a) assign a complex
fraction symbol based on the given partition of fourths or (b) repartition the whole based
on the shaded amount and assign an ordinary fraction symbol based on the repartition. On
test Problem 4, more successful students repartitioned to find an ordinary fraction symbol
for representing the shaded amount (see Figure 6.5). Similarly, roughly the same
percentage of successful students from each class used a complex symbol or fourth
partition to represent the shaded amount. On test Problem 5, all successful students, the
majority of the upper and middle quartiles of each class, repartitioned to match the shaded amount whether eighths or sixteenths (see Figure 6.5). Thus, the viable strategies used by the successful students did not appear to discriminate between the two classes on these two items.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n=10)</td>
<td>Middle (n=7)</td>
</tr>
<tr>
<td>Repartition to match shaded amount</td>
<td>30%</td>
<td>14%</td>
</tr>
<tr>
<td></td>
<td>80%</td>
<td>57%</td>
</tr>
<tr>
<td>Addition of fractions</td>
<td>10%</td>
<td>14%</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Complex fraction; given partition</td>
<td>20% (10%)</td>
<td>14% (14%)</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Combined fractions; given partition</td>
<td>20% (10%)</td>
<td>14% (14%)</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Mixed</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Other</td>
<td>0%</td>
<td>(14%)</td>
</tr>
<tr>
<td></td>
<td>(20%)</td>
<td>(14%)</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. Inappropriate symbolism or result in parenthesis.

Figure 6.5. Percents of students from each quartile group in each class that use a given strategy to interpret the fractional amount on test Problems 4 and 5.

However, the classes differed in the sophistication of non-viable answers. Many of Mr. Kent’s students that did not provide an appropriate symbol for the fractional amount on test Problem 4 at least appeared to describe the fractional amount in terms of fourths of the whole (see Figure 6.5). Mrs. Baker’s students who did not provide an appropriate symbol for the fractional amount gave answers that ignored the unit part. Their answers were based on a ratio of shaded number of parts to total number of parts, ignoring the difference in sizes. One middle quartiles student in Mrs. Baker’s class who used unit parts wrote the ratio of shaded to unshaded parts. The difference between the
explanations and answers of less successful students from Mr. Kent’s and Mrs. Baker’s classes may be related to their different instructional experiences. Mr. Kent’s students’ practice of using partial units in forming unit fraction amounts on the geoboard may have prepared them to interpret the given fractional amount as a collection of whole and partial unit parts. Mrs. Baker’s emphasis on counting the number of parts in the fractional amount and the number of parts in the whole did not provide an equal emphasis on the unit size requirement. Thus, her students who did not find a method for partitioning the whole equally were left to count the parts that they could see. Thus, the “identifying the size of the unit part” construct developed in Mr. Kent’s instruction appeared more viable in producing a meaningful understanding of the shaded portion of the whole than the “counting of parts to form a ratio” construct developed in Mrs. Baker’s instruction.

For test Problem 3 (see Figure 6.6), viable answers required that the whole be partitioned into appropriate composite discrete unit parts based on the denominator of the fraction. Two strategies for finding the size of the unit part that were apparent from students’ answers and explanations were: (a) division of the number of discrete units in the whole by the denominator of the fraction or (b) partitioning of the discrete units into equal groups based on the denominator. In the case of the second strategy, some students appeared to use the denominator as the unit size (see Figure 6.7). More students in Mr. Kent’s class from the middle quartiles group were successful in finding the unit part than middle quartiles students in Mrs. Baker’s class. This may be attributable to more practice with composite unit parts in his classroom. His students worked with composite units when decomposing geoboards, dot grids, and unifix cubes into unit parts.
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item</th>
</tr>
</thead>
</table>
| Problem   | 3       | Draw in the box to the right a set which has $\frac{2}{3}$ as many circles as the set of circles in the box on the left.  
|           |         | ![Diagram](image.png)  
|           |         | Explain why you think your answer is correct.                      |
| Task      | 1a      | Provide a set of fraction circles.  
|           |         | (a) Use the fraction circles to show the fraction $\frac{3}{8}$.  
|           |         | (b) How do you know this is $\frac{3}{8}$?  
|           |         | Please draw a picture on this sheet to show what you did.           |
| Task      | 2a      | Display 16 tiles without counting or telling the child how many there are.  
|           |         | (a) Say: You can arrange the tiles any way you want to show me the fraction $\frac{3}{8}$.  
|           |         | (b) Explain what you were thinking in order to solve this problem.  
|           |         | Please draw a picture on this sheet to show what you did.           |
| Task      | 2c      | (c) Show me $\frac{3}{8}$ using a different number of tiles. How are the two ways of using tiles alike?  
|           |         | (d) How is this way of showing $\frac{3}{8}$ like using the fraction circle pieces?  
|           |         | How are they different?                                           |
| Task      | 3a      | Display a 4 x 4 geoboard without counting or telling the child how many grids there are.  
|           |         | (a) If we count this area on the geoboard (bounded a rubber band around the outside pegs) as one whole, show the fraction $\frac{3}{8}$ using this rubber band.  
|           |         | (b) Explain what you were thinking in order to solve this problem.  |
| Task      | 3c      | (c) Show me $\frac{3}{8}$ using a different arrangement of the rubber band on the geoboard. How are the two ways of showing $\frac{3}{8}$ on the geoboard alike? different?  
|           |         | (d) How is this way of showing $\frac{3}{8}$ like using the fraction circle pieces or the tiles? How are they different?  |

Note. Tasks are interview items; problems are test items.

Figure 6.6. Items that require determining an appropriate representation of a fractional amount given its symbol and a representational medium.
<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent's students</th>
<th>Percent of Baker's students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n=10)</td>
<td>Middle (n=8)</td>
</tr>
<tr>
<td>Division of number of items in whole by denominator</td>
<td>20%</td>
<td>25%</td>
</tr>
<tr>
<td>Partition of set into number of groups by denominator</td>
<td>10%</td>
<td>25%</td>
</tr>
<tr>
<td>Use denominator as unit part size</td>
<td>(20%)</td>
<td>(25%)*</td>
</tr>
<tr>
<td>Know fraction relationship</td>
<td>12.5%</td>
<td>33 1/3%</td>
</tr>
<tr>
<td>Other or unknown</td>
<td>40% (10%)</td>
<td>(12.5%)</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. Inappropriate symbolism or result in parenthesis.

*Overlap.

Figure 6.7. Percents of students from each quartile group in each class that use a given strategy for representing a fraction with a model on test Problem 3.

Choice of solution strategies for determining the unit part did not appear to be related to the class that they attended or to their quartile group. Mr. Kent was not prescriptive in how his students’ identified unit parts as long as the method was viable. Thus, the processes and strategies for partitioning that his students employed were varied.

Although students were asked to represent the fraction 3/8 in two ways in each interview Task 1, 2 and 3 (see Figure 6.6), the different concrete media elicited different solution strategies. For interview Task 1, the initial viable strategy for representing 3/8 for both classes was to identify the appropriate size unit part by counting the number of parts of each size part and checking for a match with the denominator. Differences between students’ explanations and solution strategies did not appear to be related to instructional experience. Almost all from each quartile group from each class identified the appropriate unit part based on the match between the number of parts and the
denominator of the fraction (see Figure 6.8). Although two students used the less viable strategy of contrasting colors rather than unit part size to represent the fractional amount, a strategy reminiscent of Mrs. Baker’s first lesson on fractions using unifix cube sticks of two different colors to represent a fractional amount, one of the two was from Mr. Kent’s class in which there was no instructional analog that would have reinforced such a strategy (see Figure 6.7). Hence associations between students’ responses to interview Task 1a and their instructional experiences are difficult to identify.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n=5)</td>
<td>Middle (n=3)</td>
</tr>
<tr>
<td>Identify appropriate unit part of circle</td>
<td>80%</td>
<td>66 2/3%</td>
</tr>
<tr>
<td>Define parts by color contrast</td>
<td>0%</td>
<td>(33 1/3%)</td>
</tr>
<tr>
<td>Visual approximation</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Other</td>
<td>(20%)</td>
<td>0%</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. Inappropriate symbolism or result in parenthesis.

Figure 6.8. Percents of students from each quartile group in each class that use a given strategy to represent a fraction with a circle model on interview Task 1a.

For interview Task 2a (see Figure 6.6), the interviewees employed two viable strategies for representing 3/8 with the color tiles: (a) identify a single tile as the unit part and form the whole by counting out eight of them to match the denominator of the fraction or (b) form eight equal-size groups of tiles, composite discrete unit parts, to represent the whole. The first strategy was the most common strategy used by students from each class and quartile group. Most of these students resorted to a common factor equivalence algorithm to form an alternative representation with the tiles for interview
Task 2c (see Figure 6.9). One interviewee from each quartile group of Mrs. Baker’s class employed the second strategy. The two interviewees from Mrs. Baker’s class that used the second strategy simply increased the unit part size to form an alternative representation (see Figure 6.9). One upper quartile student from Mr. Kent’s class also increased the unit part size to form an alternative representation. In her interview, Mrs. Baker thought the unusual occurrence of students from two different quartile groups using exactly the same strategy might be related to a specific instructional experience with similar media. That one student from Mr. Kent’s class also produced composite discrete unit parts as an alternative representation can be attributed to his classroom experience of composite unit parts on the 4x4 geoboard and then the 4x6 dot grid in which 1/8 was represented by 2 squares and then 3 squares respectively.

In a significant contrast to the general success of interviewees from both classes in representing 3/8 on interview Task 2, two of Mr. Kent’s students, one each from the upper quartile and middle quartiles group, were not successful (see Figure 6.9). Each indicated that she could not represent the fractional amount unless the value of a color tile was identified. Their inability to do so can be traced to their experience in class. Students spent much time identifying various representations of the unit parts of the geoboard for the fractions 1/2, 1/4, 1/8, 1/16. Each unit part representation was based on knowing how many unit squares of the geoboard were required to form that fraction. Thus, 1/2 was 8 squares, 1/4 was 4 squares, 1/8 was 2 squares, and 1/16 was one square. Any configuration that had the appropriate area as measured by the unit squares could represent the indicated fraction. When other representations of a whole were explored
using various dot grids, students were encouraged to find the value of a unit fraction with that whole. Given a situation in which they had to form a whole with discrete units and no authoritative direction as to the whole or unit part, they were not sure that their choices would be deemed correct. Therefore, in response to their instructional experience, each may have developed the non-viable construct that the value of a unit part was pre-determined by an authority.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n=5)</td>
<td>Middle (n=3)</td>
</tr>
<tr>
<td>Singleton discrete unit part</td>
<td>Task 2a</td>
<td>60%</td>
</tr>
<tr>
<td></td>
<td>Task 2c</td>
<td>20%</td>
</tr>
<tr>
<td>Composite discrete unit part</td>
<td>Task 2a</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>Task 2c</td>
<td>20%</td>
</tr>
<tr>
<td>Common factor algorithm</td>
<td>Task 2c</td>
<td>40%</td>
</tr>
<tr>
<td>Other</td>
<td>Task 2a</td>
<td>(40%)</td>
</tr>
<tr>
<td></td>
<td>Task 2c</td>
<td>(20%)</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. Inappropriate symbolism or result in parenthesis.

Figure 6.9. Percents of students from each quartile group in each class that use a given strategy for representing a fraction with a model on interview Task 2.

Students employed several viable solution strategies for interview Task 3 (see Figure 6.6). The majority of Mr. Kent’s students employed one of two viable strategies in their representations of 3/8 on the geoboard: (a) showed two different representations of 3/8 based on the area model that each 1/8 was worth two unit squares or (b) used half the geoboard as the whole initially with each unit square being assigned as 1/8 of the whole and provided an alternative representation based on an equivalence algorithm. Most of
the upper quartile students emulated their classroom experience and used the first strategy (see Figure 6.10). Two upper quartile students and one middle quartiles student used the second strategy. One of the two upper quartile students that had used the second strategy provided a third representation using the first strategy with the geoboard as whole and two squares as 1/8.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n=5)</td>
<td>Middle (n=3)</td>
</tr>
<tr>
<td>Area model</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 3a</td>
<td>80%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 3c</td>
<td>80%</td>
<td>0%</td>
</tr>
<tr>
<td>Singleton congruent unit part</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(area)</td>
<td>Task 3a</td>
<td>20%</td>
</tr>
<tr>
<td></td>
<td>Task 3c</td>
<td>0%</td>
</tr>
<tr>
<td>Singleton unit part (discrete)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 3a</td>
<td>0%</td>
<td>33 1/3%</td>
</tr>
<tr>
<td>Task 3c</td>
<td>0%</td>
<td>(33 1/3%)</td>
</tr>
<tr>
<td>Common factor</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Algorithm</td>
<td>Task 3a</td>
<td>40%</td>
</tr>
<tr>
<td></td>
<td>Task 3c</td>
<td></td>
</tr>
<tr>
<td>Visual representation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 3a</td>
<td>0%</td>
<td>(33 1/3%)</td>
</tr>
<tr>
<td>Task 3c</td>
<td>0%</td>
<td></td>
</tr>
<tr>
<td>Other</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 3a</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 3c</td>
<td>0%</td>
<td>(33 1/3%)</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. Inappropriate symbolism or result in parenthesis.

Figure 6.10. Percents of students from each quartile group in each class that use a given strategy for representing a fraction with a model on interview Task 3.

Three interviewees from Mrs. Baker’s class (two upper quartile students and one middle quartiles student) used a viable strategy based on partitioning the geoboard as the whole into congruent unit parts (see Figure 6.10). One upper quartile student found two representations based on partitioning the geoboard into two different sets of eight congruent shapes. The other upper quartile student partitioned the geoboard into one set of eight congruent shapes and then based his second representation on an equivalence.
algorithm. The middle quartiles interviewee did not use the geoboard as the whole. She chose a shape and formed the whole from eight of them on the geoboard. Her alternative representation simply used eight of a second shape. The contrast between the strategies based on congruent unit parts that Mrs. Baker’s three interviewees used versus the strategies based on equal area unit parts that some of Mr. Kent’s students used might be related to their class experience. Mr. Kent’s students had ample practice using area measure representations of unit fractions with a geoboard as the whole. Mrs. Baker’s students had been limited in their classroom experience of partitioning continuous quantity wholes to the use of congruent unit parts.

Only middle quartiles students from each class used less viable or non-viable strategies on interview Task 3 (see Figure 6.6), which may be attributable to their instructional experiences. Most of Mrs. Baker’s middle quartiles students and one of Mr. Kent’s middle quartiles students used the less viable strategy of making eight pegs of the geoboard the unit parts of the whole (see Figure 6.10). A non-viable strategy used by one of Mr. Kent’s middle quartiles students was to create a three-part figure in the shape of a wedge on the geoboard to represent 3/8, a. The geoboard was used as a model much less in Mrs. Baker’s class than in Mr. Kent’s. Her students’ focus on the discrete elements of the geoboard might be related to the paucity of their instructional experience with this medium for representing fractional amounts as compared to Mr. Kent’s students and the focus in Mrs. Baker’s instruction on counting discrete elements to form fraction symbols.
The Relationship Between a Fractional Amount and the Unit Part of a Whole

The same instructional activities that were used to develop the relationship of the denominator to the number of unit parts in a whole were those that exposed students to constructs relating a fractional amount and unit parts of a whole. Hence, one would expect students’ processes and strategies for representing or identifying fractional amounts would differ in manners similar to those described in the preceding section. Mrs. Baker’s instruction was explicit in identifying that the numerator of a fraction symbol enumerated the number of unit parts in a given fractional amount of a whole. In the first activity, “building fractions” with unifix cubes, students counted the number of cubes of one color to identify the numerator of the fraction that they had “built.” They were encouraged to recognize that each color section of the unifix stick represented a fraction of the whole stick. In activities using congruent partitions of polygonal figures, students were encouraged to count the number of “shaded” or non-shaded parts to identify the numerator of a fraction represented by the respective shaded or unshaded portion of the whole. The emphasis of each activity was to write the appropriate fraction symbol by counting the number of unit parts indicated by the shading, color or other separating device and writing that number in the numerator location of the fraction symbol. Similarly, the total number of unit parts in the whole was to be counted and written in the denominator position of the fraction symbol.

Mr. Kent by contrast did not develop the concept of a fractional amount with such an explicit focus on the fraction symbol as Mrs. Baker used. Rather, the instruction in his class focused on developing the concept of unit part as unit fraction of a whole. He
focused on developing multiple representations of several unit fractions of a single whole, initially the geoboard and later various dot-grid shapes, unifix sticks, and paper strips. Fractional amounts that were comprised of multiple unit parts were almost an aside in his instruction. It was implicit in his instruction that 3/4 was represented by appending or “summing” three 1/4 units of the whole.

Students in both classes demonstrated both viable and non-viable interpretations of the numerator of a fraction and of the relationship between a fractional amount and the whole in response to their instructional experiences. In problems in which they were given the whole and expected to represent an indicated fractional amount (see Figure 6.6), students in both classes generally easily formed a representation of the fractional amount as long as they had identified an appropriate unit part of the whole. In the case of test Problem 3, neither the viable or non-viable strategies students employed to form the fractional amount lent themselves to distinguishing between their different instructional experiences in the two classes. For interview Tasks 1a (see Figure 6.7) and 2a (see Figure 6.8) those employing viable strategies for determining the unit part indicated the fractional amount by pulling away or otherwise separating three from a collection of eight unit parts. For interview Tasks 1c and 2c (see Figure 6.8), those employing viable strategies for determining unit parts of the whole either employed the same strategy as they had used for 1a or 2a or a strategy based on an equivalence algorithm. Any difference between students’ responses from the two classes based on instructional experiences was described in the discussion of determining a unit part given the unit whole.
However, in the case of interview Task 3 (see Figure 6.6), there were some distinguishing characteristics in the responses of most of Mr. Kent’s student interviewees related to their experience representing 1/8 on the geoboard in his classroom. Unique to Mr. Kent’s class and clearly related to his student’s instructional experience were representations based on six area units (or three when the whole was half the geoboard) in which partial unit squares were used (see Figure 6.8). These were non-standard configurations. Mr. Kent’s students did not partition the entire whole in order to represent the fractional amount. They only outlined the appropriate amount of the whole, knowing that if it contained the appropriate area then it could represent the fractional amount. This contrasts with the three student interviewees from Mrs. Baker’s class who had to totally partition the whole in order to represent 3/8 by identifying three of the eight congruent partitions.

Test Problem 4 (see Figure 6.3) represented a unique situation in that students were given a multi-unit fractional amount and had to determine the appropriate fraction symbol to represent it. Although the whole was partitioned, the partition was less than the shaded amount. Three viable strategies arose in the two classes in an attempt to describe the shaded amount: (a) name a numerator based on a found partition of the whole that coincided with the shaded amount, (b) create a complex numerator based on the given partition of the whole, or (c) describe the parts that comprised the shaded amount in terms of related unit fractions. Most, from either class, that gave an appropriate symbol for the fraction amount used the first strategy—repartitioning the whole and counting the number of unit parts in the shaded amount to form the fraction (see Figure 6.5). The
percentages of students from either quartile group that did so were not significantly different between the classes, although Mr. Kent’s middle quartiles students were nominally less likely to do so than those from Mrs. Baker’s class. Roughly half as many from each quartile group of each class used the second strategy correctly—representing the fractional amount with a complex symbol using the given partition. The difference between the two classes lay with those that used the third strategy—describing the fractional amount using a combination of unit fractions. With one exception, the remaining students from either quartile group of Mr. Kent’s class did this. In contrast, those students from Mrs. Baker’s class that did not use the first two strategies, both upper and middle quartiles, used non-viable strategies based on counting non-unit parts. Some of these non-successful students from Mrs. Baker’s class counted the number of shaded parts to create a numerator and the total number of parts to create the denominator of their inaccurate fraction symbol. One could surmise that given their experience with multi-unit fraction representations, Mr. Kent’s students were encouraged to be flexible in their identification of fractional amounts. In contrast, Mrs. Baker’s students’ experience of defining fractions as ratios of numbers of shaded parts to total parts encouraged some of them to identify fractions through counting parts rather than identifying unit amounts.

For problems in which students were given a multi-unit fractional amount and expected to form the whole (see Figure 6.1), students had to decompose the fractional amount into its unit part constituents in order to find the appropriate size unit part from which to compose the whole. They had to recognize that the numerator indicated the number of unit parts that would be contained in the given fractional amount. In the case
of interview Task 5b (see Figure 6.1), 2/3 of an unknown whole was represented by eight
unifix cubes. Students from either class that were successful in solving the problem used
two viable strategies for finding the unit part size. They either “broke in half” or “cut” the
fractional amount to find the unit part (see Figure 6.11) or they divided the fractional
amount or measure by the numerator. Some of Mr. Kent’s students may have related
these problems to their experience partitioning unifix cube wholes. In class, they “broke”
the various unifix sticks that represented different wholes into various unit fraction
representations. The linkages between Mrs. Baker’s students’ instructional experiences
and their interpretations of composite discrete unit parts are not explicit from the
available data. There is evidence from Mrs. Baker’s interview and two of her students’
answers to interview Task 2 that her students had an experience with wholes composed of
composite discrete unit parts. However, it is not clear whether Mrs. Baker explicitly
discussed how whole number operations informed the relationships between the unit
parts and the whole or if students’ may have made such connections themselves.

In the case of test Problem 2 (see Figure 6.1), the fractional amount of the whole
was represented by a length. All of the upper quartile students from Mrs. Baker’s class
and most from Mr. Kent’s class partitioned or described partitioning the fractional
amount into the number of unit parts indicated by the numerator (see Figure 6.11). The
successful middle quartiles students from each class did likewise. Three strategies for
partitioning were evident in upper and middle quartiles students’ explanations: (a)
drawing a line to form two equal-size parts by inspection, (b) describing such a partition,
or (c) describing the use of an external measurement unit such as inches or centimeters to
measure the fractional amount and then to divide the measurement by two to find the length of the unit part. The third strategy was used exclusively in Mr. Kent’s class.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Divide fractional amount or measure by the numerator</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem 2</td>
<td>20% 20%</td>
<td>0% 0%</td>
</tr>
<tr>
<td>Task 5b</td>
<td>0% 0%</td>
<td>0% 0%</td>
</tr>
<tr>
<td>Task 6b</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rate table</td>
<td>(20%) (33 1/3%)</td>
<td>0% 0%</td>
</tr>
<tr>
<td>Partition by the numerator (“break,” “draw lines”)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem 2</td>
<td>50% 50%</td>
<td>100% 14% (28%)</td>
</tr>
<tr>
<td>Task 5b</td>
<td>50% 50% (16 2/3%)</td>
<td>100% 14% (28%)</td>
</tr>
<tr>
<td>Task 6b</td>
<td>20% 0%</td>
<td>50% 25%</td>
</tr>
<tr>
<td>Recognize equivalent model (tile)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 6b</td>
<td>40% 0%</td>
<td>0% 0%</td>
</tr>
<tr>
<td>Other or no strategy</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem 2</td>
<td>(30%) (33 1/3%)</td>
<td>0% (57%)</td>
</tr>
<tr>
<td>Task 5b</td>
<td>(20%) (66 2/3%)</td>
<td>(25%)</td>
</tr>
<tr>
<td>Task 6b</td>
<td>(40%)* (66 2/3%)* (33 2/3%)</td>
<td>(50%)* (75%)*</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. n=test; interview. Inappropriate symbolism or result in parenthesis.

*Students who did not do Task 6b because they were unsuccessful on Task 6a.

Figure 6.11. Percents of students from each quartile group in each class that use a given strategy for determining a unit part on test Problem 2 and interview Tasks 5 and 6.

In their explanations, the determined unit part was described in four ways: (a) as a unit fraction (1/3), (b) as half of the fractional amount, (c) as “parts” or “pieces,” or (d) as a measurement unit (inch) or “length.” Only upper quartile students used the unit fraction to describe the unit part. Only Mr. Kent’s students used “length” or a measurement unit to describe the unit part. Mr. Kent’s students’ references to measurement and use of measurement processes to find the unit part can be attributed to instructional activities that connected ruler use and fraction strips. In contrast, Mrs. Baker’s instruction did not
include the use of measurement in the development of fraction concepts. Thus, it is not surprising that the use of measurement processes or terminology were not connected with fraction processes in Mrs. Baker’s class.

In the case of interview Task 6b (see Figure 6.1), interviewees were given a fourth circle unit as a representation of 2/3 of an unknown whole. Of those students who had the opportunity to do Task 6b and were successful, all but one upper quartile interviewee from either class and one middle quartiles student from Mrs. Baker’s class determined the unit part (see Figure 6.11). Apparent from their explanations, they employed two slightly different strategies to determine the unit part: (a) halve the given fractional amount based on the numerator’s being two, or (b) observe that two of an available unit part would fit on the given fractional amount. Mrs. Baker’s students only used the first strategy (see Figure 6.11). Mr. Kent’s successful interviewees employed both strategies. The contrast of these two strategies is reminiscent of the contrast between the instructional experiences with models of fractional amounts from the two classes. Mr. Kent’s students experience with forming multiple partitions of the geoboard or dot grid using combinations of various representations of the same unit part encouraged dual processes of partition and tiling with respect to unit part decomposition of a whole. In comparison, Mrs. Baker’s instruction only reinforced a partitioning process of decomposing a whole into unit parts.

*Flexibility in Working With Wholes and Unit Parts of Wholes*

In both classes, the students were exposed to various models of a whole. In Mr. Kent’s class, students used various models of a whole based on dot grids including the geoboard, a continuous quantity model. Mr. Kent in particular had students work with
decomposing odd-shaped dot grids into unit parts to encourage their flexibility in recognizing and working with different models of wholes. He also had them work with fraction strips, another continuous quantity model. Unifix cube sticks were a discrete quantity model that he used in his instruction. Mrs. Baker also used various models of wholes. Mrs. Baker used continuous quantity models through the geoboard in addition to the discrete quantity model of unifix cube sticks with which she introduced fractions. She had students form various polygonal shapes as wholes on the geoboard and on grid paper. Her students also had opportunities to decompose unusual shaped wholes on a worksheet.

On the test and in the interview, students were expected to understand and decompose standard wholes such as squares, circles, and collections of discrete units, and non-standard wholes such as parts of circles or indeterminate collections of discrete units. Although in absolute numbers, more of Mr. Kent’s students were successful in decomposing wholes whether standard or non-standard, in terms of percentages of each quartile group the difference was not significant. The only observable difference between the two classes in terms of their facility for working with types of wholes was in the case of discrete quantity wholes. More of Mrs. Baker’s interviewees, including middle quartiles, were able to meaningfully represent or identify fractional amounts when working with the discrete quantity wholes on interview Tasks 2 (see Figure 6.6) and 5 (see Figure 6.1). However, the performance of her students on test Problem 3 (see Figure 6.6) did not reinforce the appearance of a greater facility with discrete quantity wholes. It is difficult to identify how their success with composite discrete quantities was connected with their instruction as previously described. A hint might be taken from type of model
created on interview Task 2 by an upper and a middle quartiles interviewee. Although not recorded on the videotapes of instruction, there appears to have been an instructional occasion in which Mrs. Baker demonstrated representing fractional amounts with composite discrete quantities in a manner similar to that demonstrated by her two students on interview Task 2. This is a procedure that she also replicated in her solution of interview Task 2c during her interview.

*The Concepts and Processes of Fair Sharing*

In Mr. Kent’s class, students practiced solving fair sharing problems with multiple wholes. Mr. Kent encouraged his students to partition each whole by the number of people sharing the items. He noted that he was flexible in his acceptance of how his students represented one portion. He observed that his students usually represented one portion by shading one part per whole. Less often he observed his students shading all of one portion on one whole if possible. In Mrs. Baker’s class, the fair sharing concept was only reinforced for the partitioning of a single whole. She reviewed the process of fair sharing in her initial lecture in response to her students’ personal definitions of fractions. Her goal in the series of fair sharing situations that she described was to develop students’ understanding of the relationship between part size and the number of partitions of the whole. She appeared to take for granted that her students understood fair sharing processes.

Two types of problems from the test and interview encouraged the employment of fair sharing processes and concepts: (a) fair sharing problems in which the student was expected to find one person’s share if a number of people were to share a number of
items equally, pizza or candy (see Figure 6.12), and (b) comparison problems in which the student was expected to determine which of two groups of people sharing one or more pizzas received the larger amount (see Figure 6.13). Some students used fair sharing processes in their attempts to solve both types of problems. Those that used fair sharing processes for the second set of problems (see Figure 6.13) and tasks, used processes similar to those that were evident in the first set of problems (see Figure 6.12).

<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>7</td>
<td>Three people are going to share these two pizza equally. Color in one person’s part.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>![Pizza Diagram]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Write a fraction that shows how much one person gets. Describe how you solved the problem and explain your reasoning.</td>
</tr>
<tr>
<td>Problem</td>
<td>8</td>
<td>Six people are going to share these five chocolate bars equally. Color in one person’s part.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>![Chocolate Bars Diagram]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Write a fraction that shows how much one person gets. Describe how you solved the problem and explain your reasoning.</td>
</tr>
<tr>
<td>Task</td>
<td>4</td>
<td>Three people are going to share these two pizza equally. Color in one person’s part.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>![Pizza Diagram]</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Explain what you were thinking in order to solve this problem. What fraction of a pizza is one person’s part? Explain how you know you are correct.</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 6.12. Items that pose fair sharing problems.
There are three significant types of processes evident in students’ descriptions of solutions of all fair sharing problems. Two over-lapping types of processes are decisions about how to partition the whole(s) and how to indicate one person’s share. The third type of process is the identification of the whole or the determination of the denominator of the fraction. The three types of processes contribute to students’ determination of the fraction associated with one person’s share in the indicated situation.

<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>6</td>
<td>Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.</td>
</tr>
<tr>
<td>Task</td>
<td>7</td>
<td>Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.</td>
</tr>
<tr>
<td>Task</td>
<td>8</td>
<td>Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 6.13. Items that require the comparison of portions to determine which is larger.

Students from each class described four significant types of partitioning: (a) partitioning each shared item by the number of people sharing, (b) partitioning just one item by the number of people sharing, (c) partitioning each item by a multiple of the
number of people sharing the items, and/or (d) using a combination of partitions with ordinary fractions such as 1/2, 1/3, and 1/4, hereafter termed a combined partitions process. Most from each class described the first type of partitioning. Some from Mr. Kent’s class, upper and middle quartiles, described the second type (see Figure 6.14) on test Problems 7 and 8 (see Figure 6.12). Some from each class and quartile group described the third type of partitioning. The fourth type of partition process occurred primarily on test Problem 8 in solutions of Mr. Kent’s upper and middle quartiles students. Only one of Mrs. Baker’s middle quartiles students used a combined partitions process in his solutions of test Problems 7 and 8. However, only Mr. Kent’s

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent's students</th>
<th>Percent of Baker's students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n=10; 5)</td>
<td>Middle (n=7; 3)</td>
</tr>
<tr>
<td>Partition each item by number sharing</td>
<td>Problem 7</td>
<td>Problem 8</td>
</tr>
<tr>
<td></td>
<td>Problem 8</td>
<td>Task 4</td>
</tr>
<tr>
<td>Partition one item by number sharing</td>
<td>Problem 7</td>
<td>Problem 8</td>
</tr>
<tr>
<td></td>
<td>Problem 8</td>
<td>Task 4</td>
</tr>
<tr>
<td>Partition each item by multiple of number sharing</td>
<td>Problem 7</td>
<td>Problem 8</td>
</tr>
<tr>
<td>Combined partitions</td>
<td>Problem 7</td>
<td>Problem 8</td>
</tr>
<tr>
<td>Other or no strategy</td>
<td>Problem 7</td>
<td>Problem 8</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. n=test; interview. Inappropriate symbolism or result in parenthesis.

Figure 6.14. Percents of students from each quartile group in each class that use a given strategy for partitioning items for fair sharing Problems 7 and 8 interview Task 4.
students successfully applied the combined partitions process to determine the fractional amount of one person’s share. Over half of Mrs. Baker’s middle quartiles students, one of upper quartile student from each class, and two of Mr. Kent’s middle quartiles groups used inconsistent partitioning strategies on test Problem 8 that did not fall into either category.

Students who used one of the first three types of partitioning, used one of two processes for indicating one person’s share: (a) shading one piece of each shared item, or (b) shading all of the pieces of a portion on one of the shared items. Most students from each quartile group of Mr. Kent’s class shaded one piece of each shared item (see Figure 6.15). Mrs. Baker’s students from each quartile group were less consistent in their choice

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n=10; 5)</td>
<td>Upper (n=3; 2)</td>
</tr>
<tr>
<td>Shade one piece on each item</td>
<td>Problem 7 Problem 8</td>
<td>Problem 7 Problem 8</td>
</tr>
<tr>
<td></td>
<td>(50% (20%))</td>
<td>(43% (43%))</td>
</tr>
<tr>
<td></td>
<td>40% (10%)</td>
<td>66 2/3% (33 1/3%)</td>
</tr>
<tr>
<td></td>
<td>60% (20%)</td>
<td>(33 1/3%)</td>
</tr>
<tr>
<td></td>
<td>43%</td>
<td>50%* (50%)*</td>
</tr>
<tr>
<td>Shade all pieces on one item</td>
<td>Problem 7 Problem 8</td>
<td>Problem 7 Problem 8</td>
</tr>
<tr>
<td></td>
<td>(10% (20%))</td>
<td>(43%)</td>
</tr>
<tr>
<td></td>
<td>10% (10%)</td>
<td>(33 1/3%)</td>
</tr>
<tr>
<td></td>
<td>20%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>66 2/3% (33 1/3%)</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>(14%)</td>
<td>(14%)</td>
</tr>
<tr>
<td></td>
<td>14% (43%)</td>
<td>25%</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>(14%)</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>(50%)</td>
</tr>
<tr>
<td>Combined partition</td>
<td>Problem 7 Problem 8</td>
<td>Problem 7 Problem 8</td>
</tr>
<tr>
<td></td>
<td>Task 4</td>
<td>Task 4</td>
</tr>
<tr>
<td></td>
<td>(0% (20%))</td>
<td>(0%)</td>
</tr>
<tr>
<td></td>
<td>20% (10%)</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>14% (14%)</td>
<td>(14%)</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Other or no strategy</td>
<td>Problem 7 Problem 8</td>
<td>Problem 7 Problem 8</td>
</tr>
<tr>
<td></td>
<td>Task 4</td>
<td>Task 4</td>
</tr>
<tr>
<td></td>
<td>(0% (28%))</td>
<td>(0%)</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>(33 1/3%)</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>14% (14%)</td>
<td>(43%)</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. n=test; interview. Inappropriate symbolism or result in parenthesis.

*Overlap.

Figure 6.15. Percents of students from each quartile group in each class that use a given strategy for representing a share for fair sharing Problems 7 and 8 interview Task 4.
of strategy that they used for representing a share. Students who used the fourth type of partitioning, the combined partitions process, encountered difficulty in describing the shaded amount—similar to the difficulty that some of Mr. Kent’s students had encountered on test Problem 4. Students who did not identify a suitable ordinary fraction to represent one person’s portion could at least give a reasonable description of the portion in terms of the fractions that they determined. They indicated one person’s share by shading appropriate amounts of each type of piece. For example, a student who first partitioned into thirds and then sixths shaded 2/3 and 1/6 on test Problem 8.

Students of each class primarily used two processes for indicating the whole (determining the denominator of the fraction describing one person’s portion): (a) counting the number of parts in one item or by naming fractional amounts in relationship to one item being shared, or (b) counting all parts in the system of items being shared. Most of Mr. Kent’s students used the first process to determine the denominator, indicating that they had identified the whole as one of the shared items. Most of Mrs. Baker’s students used the second process to determine the denominator, indicating that, whether intentional or not, they had treated the system of items as a whole (see Figure 6.16). One of Mrs. Baker’s upper quartile students gave two answers to interview Task 4 (see Figure 6.12)—one for each process for indicating the whole.

All of the answers and explanations produced by Mr. Kent’s upper quartile students and all but one of those produced by Mrs. Baker’s upper quartile students on fair sharing problems, while not always correct, were produced using the aforementioned types of processes. Mr. Kent’s students were overwhelmingly successful in providing
viable answers as compared to Mrs. Baker’s students (see Figure 6.17). Mr. Kent’s students who succeeded showed a preference for the procedure: (1) identify one item as the whole, (2) partition each item by the number of people sharing, and (3) shade one part of each item to represent one person’s share. Exactly one of Mrs. Baker’s upper quartile

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n=10; 5)</td>
<td>Middle (n=7; 3)</td>
</tr>
<tr>
<td>Counting unit parts of one item</td>
<td>Problem 7</td>
<td>60% (40%)</td>
</tr>
<tr>
<td></td>
<td>Problem 8</td>
<td>70%</td>
</tr>
<tr>
<td></td>
<td>Task 4</td>
<td>60% (40%)</td>
</tr>
<tr>
<td>Counting unit parts of all items</td>
<td>Problem 7</td>
<td>0% (20%)</td>
</tr>
<tr>
<td></td>
<td>Problem 8</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>Task 4</td>
<td>0%</td>
</tr>
<tr>
<td>Combined partition</td>
<td>Problem 7</td>
<td>0% (10%)</td>
</tr>
<tr>
<td></td>
<td>Problem 8</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>Task 4</td>
<td>0%</td>
</tr>
<tr>
<td>Other or no strategy</td>
<td>Problem 7</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>Problem 8</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>Task 4</td>
<td>0%</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. n=test; interview. Inappropriate symbolism or result in parenthesis.
*Overlap.

Figure 6.16. Percents of students from each quartile group in each class that use a given strategy for determining the denominator of a share for fair sharing Problems 7 and 8 interview Task 4.

students used this procedure in his answer to interview Task 4 (see Figure 6.12).

However, he was unsure of his answer and hedged by providing an alternative answer based on identifying the whole as all of the items shared. Mr. Kent’s upper and middle quartiles students were also successful by partitioning only one item and making conjectures about the rest based on the one partition—a variation on the favored procedure. A corruption of the favored successful procedure led at least two of Mr.
Kent’s students to misidentify one person’s share as only one part of one item rather than a combination of all the parts across the shared items.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n=10; 5)</td>
<td>Middle (n=7; 3)</td>
</tr>
<tr>
<td>1) one item is unit whole; 2) partition by number of people; 3) shade one piece of each item</td>
<td>Problem 7 Problem 8 Task 4</td>
<td>50% (10%) 40% 60% (20%)</td>
</tr>
<tr>
<td>1) one item is unit whole; 2) partition by number of people; 3) shade share on one item</td>
<td>Problem 7 Problem 8 Task 4</td>
<td>10% 10% 20%</td>
</tr>
<tr>
<td>1) all items are unit whole; 2) partition by number of people; 3) shade one piece of each item</td>
<td>Problem 7 Problem 8 Task 4</td>
<td>(10%) (10%) 0%</td>
</tr>
<tr>
<td>1) all items are unit whole; 2) partition by number of people; 3) shade share on one item</td>
<td>Problem 7 Problem 8 Task 4</td>
<td>(10%) (10%) 0%</td>
</tr>
<tr>
<td>Combined partition</td>
<td>Problem 7 Problem 8 Task 4</td>
<td>0% 20% (10%) 0%</td>
</tr>
<tr>
<td>Other or no strategy</td>
<td>Problem 7 Problem 8 Task 4</td>
<td>(10%) 0% 0%</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. n=test; interview. Inappropriate symbolism or result in parenthesis.

*Overlap.

Figure 6.17. Percents of students from each quartile group in each class that use a given procedure for finding a fair share on Problems 7 and 8 interview Task 4.
All of the viable answers produced by Mrs. Baker’s students with the one exception were based on the procedure (see Figure 6.17): (1) partition by the number of people sharing, and (2) shade one person’s portion on one of the items. One each of Mr. Kent’s upper and middle quartiles students used the same combination successfully. They were also successful on test Problem 8 in using combined partitions although one student had to write the answer as the sum of two different ordinary fractions. By comparison, over half of the answers and explanations produced by Mrs. Baker’s middle quartiles students on fair sharing problems (see Figures 6.12, 6.17) did not fit into the categories previously described. While some of the same strategies were used, they were combined with other random strategies to produce answers that had little or no meaning. A fifth of the answers and explanations given by Mr. Kent’s middle quartiles students could be categorized similarly. The combinations of strategies that produced meaningless answers and explanations will hereafter be termed “other strategies.”

The distinguishing factor for the production of a viable answer for fair sharing problems was that the whole was identified as one of the items being shared rather than as the whole system of items being shared. The majority of answers and explanations given by Mrs. Baker’s upper and middle quartiles students that did not fall in the “other strategies” category were based on identifying the whole as all of the items being shared—the denominator of the representing fraction was found by counting all of the “parts” of all items shared (see Figure 6.17).

In examining the foregoing discussion of students’ processes for solving fair sharing processes, two interactions from Mr. Kent’s instruction can be identified. The
first interaction can be identified by the combination of strategies on which the preponderance of viable answers provided by Mr. Kent’s students was based. In their instructional activity on fair sharing, his students practiced appropriately identifying the whole and partitioning each by the number of people sharing. Although he did not require that students shade one person’s portion of each whole, he noted that it was his students’ preferred method of identifying one person’s share. A few non-viable applications of this combination of strategies can also be attributed to their instruction, albeit indicating a lack of understanding of the meaning of the denominator. The second interaction can be identified by his students’ successful use of combined partitioning. The second interaction is connected to his students’ experience with decomposing wholes into combinations of representations of ordinary fractions. Even his students’ use of combined partitions that produced inaccurate fractions were more meaningful than those produced by Mrs. Baker’s students’ attempts to use combined partitioning.

Other than the attribution of Mrs. Baker’s students’ notable lack of success with fair sharing problems to their relative lack of experience in class with such problems, one other instructional interaction can be identified. Mrs. Baker’s students’ predisposition to count all the parts in all of the items to identify the denominator of the representing fraction may be attributable to the “parts-to-total” definition of a fraction that pervaded Mrs. Baker’s instruction. Even Mrs. Baker made this error in her interview and on her test.
The Concepts and Processes of Finding Equivalent Fractions

In both classes, students were exposed to strategies for finding equivalent fractions. Mrs. Baker first introduced equivalent fractions using polygonal models. She demonstrated that a given partition of the whole could be modified to show an equivalent fraction. Her students’ predisposition in class was to halve given partitions to produce equivalent models. She also demonstrated combining parts to produce a new partition of the whole. Mrs. Baker soon introduced her students to a personal algorithmic model for producing equivalent fractions that she termed “backward Z.” Essentially, the method was to divide the new denominator by the given denominator and multiply the given numerator by the result to find the new numerator. Her “backward Z” method was reinforced when she introduced addition and subtraction of fractions not having a common denominator. Another method of finding equivalent fractions that she demonstrated was to multiply the numerator and denominator by the same factor. She had the habit of writing the common factor above the equal sign between the two equivalent fractions.

Mr. Kent introduced equivalent fractions during his students work with geoboard models of fractions. After his students had developed many different representations of the ordinary fractions 1/2, 1/4, 1/8, 1/16, he had them decompose the geoboard into combinations of representations of different fractions in this sequence. Using these various models of the whole he asked the students to choose an example that would “prove” that certain fractions were equivalent. He also used the same activity to introduce addition of fractions with unlike denominators. Specifically, they were asked to find the
solution to the sentence $\frac{1}{4} + \frac{1}{8} = ?$ In an activity with unifix cubes his students would expand their repertoire of models of equivalent fractions to include equivalents of thirds, fourths, fifths, sixths, and sevenths. They created a whole class table on which they showed for different lengths of unifix sticks from 3 to 23 what unit fractions could be modeled with a particular size whole and what their equivalents would be using the same whole. Mr. Kent’s intent was that his students would discover factor relationships that would facilitate the introduction of the concept of common denominators. Subsequent instruction encouraged students to recognize chains of equivalent fractions. Mr. Kent encouraged his students to build chains of equivalent fractions during the geoboard and unifix cube activities. Then he encouraged them to look for algorithmic patterns between the fractions in the equivalence chains. His goal was that they would recognize that a common factor was being used to produce the next fraction in the chain.

Several groups of test problems and interview tasks provided students with opportunities to demonstrate their knowledge of equivalent fractions. One group of items elicited equivalent models of fractions or models of fractions interview tasks (see Figure 6.18). For interview Task 1c, students that gave viable answers used two different strategies: (a) use the quarter piece to represent $\frac{2}{8}$; (b) an algorithmic argument of doubling the numerator and denominator to form an equivalent fraction. Two upper quartile students from Mr. Kent’s class and one upper and one middle quartiles student from Mrs. Baker’s class used the first strategy (see Figure 6.19). However, there appeared to be a difference between the respective solution descriptions of students in the two classes. Mr. Kent’s students emphasized the equivalence of $\frac{1}{4}$ and $\frac{2}{8}$ in their
reasoning process. Mrs. Baker’s students appeared to emphasize the repartitioning of the quarter into 2 eighths. An upper quartile student from Mr. Kent’s class provided the second strategy. Not having the appropriate pieces to show the equivalent model/fraction he drew a circle partitioned into sixteen parts and shaded six of them. One of Mrs. Baker’s upper quartile interviewees may have done likewise if she had been encouraged, having complained that an alternative representation was not possible since no pieces that represented half of an eighth were available.

<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task</td>
<td>1c</td>
<td>Having already represented 3/8 once with a set of fraction circles, and with all but two 1/8 pieces removed, show 3/8. Explain how the two ways of showing 3/8 are alike and different.</td>
</tr>
<tr>
<td>Task</td>
<td>2c</td>
<td>Show 3/8 using a different number of color tiles. How are the two ways of using tiles alike? How is this way of showing 3/8 like using the fraction circle pieces? How are they different?</td>
</tr>
<tr>
<td>Task</td>
<td>3c</td>
<td>Show 3/8 using a different arrangement of the rubber bands on the geoboard. How are the two ways of showing 3/8 on the geoboard alike? different? How is this way of showing 3/8 like using the fraction circle pieces or the tiles? How are they different?</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items

Figure 6.18. Items that ask for alternative representations of the fraction 3/8.

Other students that had successfully represented 3/8 with circle pieces (interview Task 1a) gave non-viable answers—one upper and two middle quartiles students from Mr. Kent’s class and two middle quartiles students from Mrs. Baker’s class. A middle quartiles interviewee from Mr. Kent’s class and two from Mrs. Baker’s class did not offer any solution—apparently not understanding equivalence. However, two of Mr. Kent’s
students identified 3/8 as equivalent to 1/3 apparently based on a perceived similarity in the shape of their models—one from each quartile group.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n= 5)</td>
<td>Middle (n= 3)</td>
</tr>
<tr>
<td>Half quarter is an eighth</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>1/4=2/8</td>
<td>40%</td>
<td>0%</td>
</tr>
<tr>
<td>Equivalence algorithm (3/8=6/16)</td>
<td>20%</td>
<td>0%</td>
</tr>
<tr>
<td>Visual representation (3/8 is “1/3”)</td>
<td>(20%)</td>
<td>(33 1/3%)</td>
</tr>
<tr>
<td>Not possible without three eighth</td>
<td>0%</td>
<td>(33 1/3)</td>
</tr>
<tr>
<td>pieces</td>
<td>(20%)</td>
<td>(33 2/3%)</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. n=test; interview. Inappropriate symbolism or result in parenthesis.

*Overlap.

Figure 6.19. Percents of students from each quartile group in each class that determine a given alternative representation 3/8 with a circle model on interview Task 1c.

In the case of interview Tasks 2c and 3c (see Figure 6.18), two strategies for producing alternative representations of 3/8 were apparent: (a) change the nature of the unit part described in the section on relationships between the unit part and the whole; (b) create a representation of an equivalent fraction that had been determined algorithmically.

The upper quartile student from Mr. Kent’s class that had used the second strategy for interview Task 1c (equivalent fraction) used it on interview Tasks 2c and 3c as well. One other upper quartile student from Mr. Kent’s class also used this strategy for both interview Tasks 2c and 3c. It is notable that no other student from Mr. Kent’s class used this strategy on interview Task 2c, whereas, the majority of Mrs. Baker’s middle quartiles
students did (see Figure 6.20). Changing the unit part size, an algorithmic strategy described in the section on relationships between the unit part and whole, allowed all of Mrs. Baker’s upper and middle quartiles students to succeed in providing alternative representations for 3/8 on interview Task 2c. In contrast, none of Mr. Kent’s middle quartiles students were able to provide an alternative representation for 3/8 on interview Task 2c. In contrast, only one upper quartile student from Mrs. Baker’s class thought to use the algorithmic strategy for interview Task 3c (see Figures 6.18). Most of Mrs. Baker’s middle quartiles students were not able to provide any solution for interview Task 3c (see Figure 6.20). Oddly enough, neither could Mr. Kent’s middle quartiles students. All of Mr. Kent’s upper quartile students solved interview Task 3c, but most based it on changing the shape of the region that encompassed 3/8 of the geoboard.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n= 5)</td>
<td>Middle (n= 3)</td>
</tr>
<tr>
<td>Change nature of unit part</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 2c</td>
<td>20%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 3c</td>
<td>80%*</td>
<td>0%</td>
</tr>
<tr>
<td>Represent equivalent fraction</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 2c</td>
<td>40%</td>
<td>0%</td>
</tr>
<tr>
<td>Task 3c</td>
<td>40%*</td>
<td>0%</td>
</tr>
<tr>
<td>Not possible another way</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 2c</td>
<td>(20%)</td>
<td>(33 1/3)</td>
</tr>
<tr>
<td>Task 3c</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Other or no strategy</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 2c</td>
<td>(20%)</td>
<td>(66 2/3%)</td>
</tr>
<tr>
<td>Task 3c</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. n=test; interview. Inappropriate symbolism or result in parenthesis.

*Overlap.

Figure 6.20. Percents of students from each quartile group in each class that use a given strategy for creating alternative representations of 3/8 on interview Tasks 2c and 3c.
It appears that Mr. Kent’s students’ use of equivalence to provide alternative representations of a fractional amount was limited to a few upper quartile students that had a predisposition to that particular strategy. Nevertheless, there was an instructional antecedent for Mr. Kent’s students’ use of an algorithmic strategy in their study of patterns in fraction equivalence chains. His students’ emphasis on the equivalence of 1/4 and 2/8 on interview Task 1c, likewise, can be attributed to their repeated exposure to such representations and the emphasis made on their equivalence in class. By this time, the relationship between 1/4 and 2/8 and other common equivalent fraction pairings were internalized.

Mrs. Baker’s students’ success on interview Task 2c (see Figures 6.18, 6.20) using an algorithmic strategy suggests a strong link to class experience. It is striking that it is only apparent when her students use discrete quantity models. She may have modeled equivalent fractions with discrete units. Perhaps such a demonstration is the source of the alternative representation strategy that two of her students use on interview Task 2c in which a tile is simply added to each stack. This cannot be verified since it occurred during a session that was not videotaped. It is clear from the video recordings that her students had much more experience with algorithmic processes for finding equivalents than they had in representing equivalents with models. If her students had simply been referencing their experience with algorithmic interpretations of fraction equivalence, it is surprising that they did not use the same algorithmic thinking on interview Task 3c as on interview Task 2c. One upper quartile student from Mrs. Baker’s class had apparently internalized the equivalence algorithm as two of Mr. Kent’s upper
quartile students apparently had since he used it to find alternative representations on both interview Tasks 2c and 3c (see Figure 6.18). The contrast between Mrs. Baker’s students’ performance on interview Task 2c and on interview Task 3c further suggests that a particular experience in Mrs. Baker’s class encouraged her students to tie a discrete model representation to their emergent algorithmic sense of fraction equivalence. With respect to continuous quantity models, the emphasis on repartitioning that two students described in their solutions of interview Task 1c (see Figure 6.18) do suggest a connection with the lesson in which she introduced equivalent fractions via re-partitioning strategies on continuous quantity models of a whole.

Only test Problem 11 (see Figure 6.21) on the test focused exclusively on finding equivalent fractions. Students from both classes used algorithmic processes exclusively as would be anticipated. Although there was little significant difference in the percentages of students from each quartile group of each class that were successful in giving equivalents for each fraction, there were differences in their explanation patterns.

<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>11</td>
<td>Write one fraction that is the same as each fraction below. Explain why you think your answer is correct for one of the rows. For example,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>i) $\frac{3}{9} = _____$ ii) $\frac{1}{7} = ____$ iii) $\frac{8}{12} = ____$</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 6.21. Items that ask for equivalent fractions.
Two major strategies for identifying equivalents can be identified: doubling and simplifying. Those that used doubling described three methods of doubling: repeated addition, multiplying by common factor, and counting. Most of Mr. Kent’s students that described their process, both upper and middle quartiles, described doubling the numerator and denominator by repeated addition (see Figure 6.22). A couple of his upper quartile students described a common factor approach. Only Mr. Kent’s students used a combination of simplifying and doubling. Exactly one attempted to use only simplifying which was not a viable strategy for finding an equivalent on one of the three items. More of Mr. Kent’s students described how they found their solutions as compared to Mrs. Baker’s students. There were no patterns among the explanations of Mrs. Baker’s students that described doubling to find equivalent fractions. One upper quartile student

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent's students Upper (n= 10)</th>
<th>Percent of Kent's students Middle (n= 6)</th>
<th>Percent of Baker's students Upper (n= 3)</th>
<th>Percent of Baker's students Middle (n= 6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double by repeated addition</td>
<td>40%*</td>
<td>50%</td>
<td>0%</td>
<td>16 2/3%</td>
</tr>
<tr>
<td>Double by multiplication</td>
<td>20%</td>
<td>0%</td>
<td>33 1/3%</td>
<td>0%</td>
</tr>
<tr>
<td>Double by counting</td>
<td>10% (10%)</td>
<td>0%</td>
<td>0%</td>
<td>(16 2/3%)</td>
</tr>
<tr>
<td>Double—unknown method</td>
<td>20%*</td>
<td>33 1/3%*</td>
<td>33 1/3%</td>
<td>33 1/3% (16 2/3)</td>
</tr>
<tr>
<td>simplify</td>
<td>0%</td>
<td>0%</td>
<td>(33 1/3%)</td>
<td>0%</td>
</tr>
<tr>
<td>Combination—simplifying and doubling</td>
<td>20%*</td>
<td>16 2/3%* (16 2/3%)*</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Other or no strategy</td>
<td>(10%)</td>
<td>(16 2/3%)</td>
<td>0%</td>
<td>(16 2/3%)</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. n=test; interview. Inappropriate symbolism or result in parenthesis.

*Overlap.

Figure 6.22. Percents of students from each quartile group in each class that use a given strategy for equivalent fractions on test Problem 11.
used a common factor. One middle quartiles student used repeated addition. One middle quartiles student described counting. An approximately equal percentage of her students from each quartile group described doubling without further description. One student from each quartile group of Mrs. Baker’s class described only simplifying and thus were unable to find an equivalent for at least one item.

Mr. Kent’s students’ preference for doubling by repeated addition of the numerator and the denominator to find equivalent fractions can be attributed to their pattern identification in class. In class they demonstrated a similar predisposition to identify patterns based on addition facts rather than on multiplication facts. Their flexibility in using strategies can be attributed to his encouragement of flexible strategy use in his class as compared to Mrs. Baker’s students’ compartmentalized experience of strategies. In one instance, Mrs. Baker discouraged a student from using an alternative strategy for finding an equivalent during a lesson because it did not fit her instructional goal.

*The Concepts and Processes of Adding or Subtracting Fractions and the Application of Equivalent Fraction Processes and Concepts*

The addition or subtraction of fractions with unlike denominators was an opportunity for students to apply their knowledge of equivalent fractions. After her initial introduction of adding and subtracting fractions with like denominators, Mrs. Baker introduced using the operations on fractions with unlike denominators. A notable feature of Mrs. Baker’s demonstration of addition or subtraction of fractions with unlike denominators was her use of the backward Z mnemonic for finding equivalent fractions
with a common denominator. Whether adding or subtracting, Mrs. Baker emphasized that students should operate on the numerators and should keep the denominator the same. Subsequently, she introduced adding and subtracting mixed numbers. Her final lessons on fractions demonstrated the process for subtracting mixed numbers in which the fractional part of the larger mixed number is smaller than the fractional part of the smaller mixed number. Specifically, she showed her students how to borrow one from the whole number in the larger mixed number and transform it into a fraction that can be added to the fraction portion of the larger mixed number to increase its size and to make it possible to subtract the fractional part of the smaller mixed number. Mrs. Baker emphasized as part of the process for adding and subtracting simplifying the fraction or mixed number to “lowest” terms.

Mr. Kent introduced addition of like and unlike denominator fractions in a lesson that used models of ordinary fractions on the geoboard. Students were not very successful with their first encounter with adding fractions with unlike denominators: 1/4 + 1/8. Although Mr. Kent hinted to his students that they should decompose a model of 1/4 into smaller fraction parts with which they were familiar, no student realized what the appropriate fractions should be. He had to guide them to decompose the 1/4 into eighths. However, over time his students had many experiences finding equivalents of familiar fractions. They were able to use their familiarity with equivalents of ordinary fractions when he began to focus on teaching the addition and subtraction of fractions with unlike denominators and mixed numbers. In the case of adding and subtracting mixed numbers,
he introduced converting each mixed number into an improper fraction before finding their common denominator equivalents.

For the test and interview items that elicited either addition (see Figure 6.24) or subtraction (see Figure 6.23) of fractions, it was necessary to find a common denominator and equivalent fraction to solve the problem. For the two subtraction situations, it was possible to simplify the resulting answer. For interview Task 10 (see Figure 6.23), it was possible to simplify one of the given fractions before solving the problem. Upper quartile students from both classes were successful at the same rate on each pair of problems, addition and subtraction. Mrs. Baker’s upper quartile students scored perfectly on the two subtraction problems. Mr. Kent’s upper quartile students would have except that one individual described how to compare the fractions but did not find the difference on test Problem 14.

<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>14</td>
<td>Ann and Josie receive the same allowance. Josie spent 3/5 of hers on CDs. Ann spent 1/10 of her allowance on repairing her bicycle. Josie spent how much more of her allowance than Ann? Explain why you think your answer is correct.</td>
</tr>
<tr>
<td>Task</td>
<td>10</td>
<td>Josie and Al went skateboarding down their block together. Al skated 6/10 of the way down the block before he stopped. Josie skated 4/5 of the way down the block before she stopped. How much further down the block did Josie skate than Al? What you were thinking to reach this answer?</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 6.23. Items that describe situations requiring subtraction of fractions.
Upper quartile students were not as successful with addition (see Figures 6.24, 6.25). Although both groups of upper quartile interviewees scored perfectly on interview Task 9, some from each group did not accurately respond to test Problem 13. The nature of their errors differed individually. Mrs. Baker’s unsuccessful student used a diagram of each fraction with the common sixth partition, but stated an inaccurate answer. One of Mr. Kent’s students found the common denominator but subtracted. Another drew a diagram of each fraction but did not use a common partition and hence could not determine the answer. Another described using a common denominator but did not adjust the numerator to find an equivalent fraction before adding.

<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>13</td>
<td>Liana ate 2/3 of a small pizza. The next day she ate 1/6 of a small pizza. How much of a pizza did she eat altogether? Explain why you think your answer is correct.</td>
</tr>
<tr>
<td>Task</td>
<td>9</td>
<td>Sally ate 2/3 of a pizza for dinner. The next morning she ate another 1/6 of a pizza. How much pizza did she eat altogether? What you were thinking to reach this answer?</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 6.24. Items that describe situations requiring addition of fractions.

Middle quartiles students from each class were less successful in finding solutions to the addition and subtraction problems. As in the case of their upper quartile peers, they were more successful with the subtraction problems than with the addition problems. With the exception of the interview tasks, Mr. Kent’s students success rate with addition and subtraction problems was higher than Mrs. Baker’s students.
Successful students in each quartile group of the two classes used finding an equivalent fraction as their primary means for solving the addition and subtraction problems (see Figure 6.25). However, there were strategy differences that differentiated

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent's students</th>
<th>Percent of Baker's students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n= 10; 5)</td>
<td>Upper (n= 3; 2)</td>
</tr>
<tr>
<td>Equivalent fractions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem 13 Task 9</td>
<td>70% (10%)</td>
<td>33 1/3%</td>
</tr>
<tr>
<td>Problem 14 Task 10</td>
<td>0%</td>
<td>50%</td>
</tr>
<tr>
<td></td>
<td>100%</td>
<td>16 2/3%</td>
</tr>
<tr>
<td></td>
<td>0% (33 1/3%)</td>
<td>(25%)</td>
</tr>
<tr>
<td>Equivalent fractions—“Backward Z”</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem 13 Task 9</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Problem 14 Task 10</td>
<td>0%</td>
<td>50%</td>
</tr>
<tr>
<td></td>
<td>0% (33 1/3%)</td>
<td>16 2/3%</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>(25%)</td>
</tr>
<tr>
<td>Diagram</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem 13 Task 9</td>
<td>(10%)</td>
<td>(33 1/3%)</td>
</tr>
<tr>
<td>Problem 14 Task 10</td>
<td>0%</td>
<td>(50%)</td>
</tr>
<tr>
<td></td>
<td>0% (33 1/3%)</td>
<td>(25%)</td>
</tr>
<tr>
<td>Common denominator error</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem 13 Task 9</td>
<td>(10%)</td>
<td>0%</td>
</tr>
<tr>
<td>Problem 14 Task 10</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>0% (33 1/3%)</td>
<td>0%</td>
</tr>
<tr>
<td>Other or no answer</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem 13 Task 9</td>
<td>0%</td>
<td>33 1/3%</td>
</tr>
<tr>
<td>Problem 14 Task 10</td>
<td>0%</td>
<td>(16 2/3%)</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>(25%)</td>
</tr>
<tr>
<td></td>
<td>0%</td>
<td>(50%)</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items. n=test; interview. Inappropriate symbolism or result in parenthesis.

*Overlap.

Figure 6.25. Percents of students from each quartile group in each class that use a given strategy for adding (test Problem 13 and interview Task 9) or subtracting (test Problem 14 and interview Task 10).

Several instances of unusual notations that Mrs. Baker had used were found among Mrs. Baker’s students’ explanations. This included but was not limited to the use of the backward Z mnemonic. Students also used a notation for simplifying fractions that was
identical to notation that Mrs. Baker had used in class. This is not to say that Mr. Kent’s students were not also mimicking his notational style. The preferences for writing number sentences horizontally or vertically are indicative of students parroting their teachers’ notational styles. However, one middle quartiles students’ focus on notation over substance from Mrs. Baker’s class may have contributed to her lack of success. Although she used the backward Z mnemonic to find equivalents and added and subtracted appropriately, she did not provide correct answers to at least three of the problems because she insisted on interpreting the problems as mixed number problems. She found some number to interpret as a whole and added or subtracted accordingly.

between the two classes. Mr. Kent’s students were more likely to write their operation sentences horizontally. Mrs. Baker’s were more likely to write them vertically. Mr. Kent’s students were more likely to state that they were finding the “common denominator.” Mrs. Baker’s students were more likely to describe their process for finding equivalent fractions whether by “doubling,” “multiplying,” or using the backward Z mnemonic. Mr. Kent’s students appeared to just “know” the required equivalent for each fraction. More of Mrs. Baker’s students simplified their answers than Mr. Kent’s. Most of those who drew diagrams in order to solve the problems were from Mrs. Baker’s middle quartiles group. Those who drew diagrams as part of their solution process whether from Mrs. Baker’s class or from Mr. Kent’s class were notably unsuccessful. Those from either class who drew diagrams and were successful appeared to have drawn the diagrams after they had determined the solution via another method. Successful diagram drawers were primarily from Mr. Kent’s upper quartile group.
Content rather than notational differences between the two classes can be also be traced but are less apparent. The inability of students from either class to appropriately use diagrams in finding solutions can be attributed to instructional emphases in each class. The focus of the balance of their lessons on adding and subtracting fractions was algorithmic. Mr. Kent’s students knew how to find the appropriate answers using the algorithms and then could retrace how that might look on a model. Mrs. Baker’s middle quartiles students’ attempt to use a model was likely due to the unfamiliarity of the problems and the encouragement to do so in the directions. However, they were not able to interpret their models. Mr. Kent’s students had more experience with situational problems and thus were not distracted. They were able to interpret the problem and use the appropriate algorithm to solve it. Then they drew their models. Mr. Kent’s students lack of explanation of how they found their equivalent fractions relates to the amount of practice and the lack of emphasis on a particular process for finding equivalent fractions in addition and subtraction problems. By contrast, Mrs. Baker’s instruction was about developing appropriate processes for adding and subtracting fractions and thus her students were more likely to show the particular processes that they used.

*The Concepts and Processes of Ordering Fractions*

In neither class were students exposed to a significant degree to the ordering of unequal fractions. However, in both classes, teachers reinforced the concept that the denominator size was inverse the size of the unit part of a whole. Mrs. Baker did so explicitly in her initial lesson on fractions using the pizza or pie fair sharing model. Mr. Kent did so implicitly through having his students model progressively smaller fractions.
on the geoboard and the dot grid. Soon his students were able to predict that if a new
denominator were twice the size of a given denominator then the representation of the
new fraction would be half the size of the given fraction.

On test Problem 10 (see Figure 6.26), the responses from students of both classes
indicated that they tend to use the same strategy in ordering lists of fractions, no matter
the nature of the list. Strategies that produced a correct ordering for the first list did not
necessarily produce a correct ordering for subsequent lists. Strategies that were used
included: (a) recognizing the unit part size of the fraction; (b) ordering by the reverse of
the denominator size; (c) finding equivalent fractions.

<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>10</td>
<td>For each row of fractions below, which fraction is the greatest and which fraction is the least? Explain why you think your answer is correct for two of the rows.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>i) $\frac{1}{8}$, $\frac{1}{7}$, $\frac{1}{6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Greatest? ____  Least? ____</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ii) $\frac{6}{7}$, $\frac{8}{9}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Greatest? ____  Least? ____</td>
</tr>
<tr>
<td></td>
<td></td>
<td>iii) $\frac{3}{7}$, $\frac{4}{9}$, $\frac{4}{5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Greatest? ____  Least? ____</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 6.26. Item that requires students to order lists of fractions.
Recognizing the unit part size of the fraction was the most successful strategy. A higher percentage of Mr. Kent’s upper quartile students applied the strategy than did upper quartile students from Mrs. Baker’s class (see Figure 6.27). Roughly the same percentage of middle quartile students from each class applied the strategy. However, 30% of Mr. Kent’s upper quartile students did not apply the unit part size strategy appropriately to the subsequent lists (see Figure 6.26). They simply ordered by part size instead of recognizing that the part size would predict distance from the whole. They ignored the contribution that numerators make to the size of the fraction. In contrast, all middle quartile students from both classes that used the unit part size strategy applied it appropriately to the other two lists. Nevertheless, few students that were successful in ordering the first two lists were equally successful in ordering the last list. They could predict the largest fraction based on unit part size and distance from the whole, but they could not order the two fractions close to the half. Two students from Mrs. Baker’s class

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognizing the unit part size of the fraction</td>
<td>70% 43%</td>
<td>33 1/3% 43%</td>
</tr>
<tr>
<td>Ordering by the reverse of the denominator size</td>
<td>20% 14%</td>
<td>66 2/3% 0%</td>
</tr>
<tr>
<td>Finding equivalent fractions</td>
<td>10% 0%</td>
<td>0% 0%</td>
</tr>
</tbody>
</table>

Figure 6.27. Percentages of students from each class in each quartile group that use a given strategy.
and one from Mr. Kent’s did provide the correct ordering, but how they accomplished the ordering is not clear. The unit part size strategy was not viable for ordering the two fractions close to a half.

With one exception upper quartile students from either class that did not use the unit part size strategy used the less sophisticated strategy of ordering by the reverse of the denominator size. This strategy was also used by one of Mr. Kent’s middle quartiles student. Other middle quartiles students from Mr. Kent’s class used a combination strategy of ordering by the size of the numerator and reverse the size of the denominator. In point of fact the number of middle quartiles students from Mr. Kent’s class that used a unit part strategy to order the fractions was the same as the number that ordered based on reversing the order of the denominators. In contrast, the number of middle quartiles students from Mrs. Baker’s class that ordered based on the size of the denominator was the same as the number of students that ordered based on unit part size. Only one of Mr. Kent’s students ordered by the size the denominator—a middle quartiles student.

Exactly one upper quartile student who was from Mr. Kent’s class attempted to apply equivalent fraction concepts to ordering each list. She was successful on the first list. However, she did not succeed on the other two lists because she did not find a common denominator for all three fractions. She attempted to order the fractions two at a time and made some technical errors. One middle quartiles student from Mr. Kent’s class created an unusual strategy for ordering based on counting how many unit parts for each fraction in a list were left in the whole. She successfully ordered the first two lists and partially ordered the last list appropriately.
The test problems that elicited students’ understanding of betweenness (see Figure 6.28) did not elicit the same strategy throughout. In the case of the first pair of fractions, the success of upper quartile students from the two classes was similar, but slightly more of Mrs. Baker’s middle quartiles students were successful than Mr. Kent’s middle quartiles students (see Figure 6.29). Students’ viable strategies for the first pair fall into two categories (see Figure 6.30): (a) recognizing an ordinary fraction between the two given fractions; (b) finding equivalent fractions. There was some overlap between students’ use of the two strategies. Two of Mr. Kent’s upper quartile students and one of Mrs. Baker’s middle quartiles students found equivalents of the given fractions to solve the problem. One upper quartile and a middle quartiles student from each class identified a ordinary fraction that was known to be greater than the smaller given fraction and verified through an equivalent with a common denominator that it was less than the larger given fraction. One other upper quartile student from Mr. Kent’s class identified a fraction known to be between the two given fractions and verified that it was between the two given fractions by finding its equivalent with a common denominator that could accommodate all three fractions. One middle quartiles student from Mr. Kent’s class found equivalent fractions for the two given fractions but gave the difference between the fractions as his solution. He used the same strategy for the second pair of fractions as well.
Item type | Item No. | Item  
--- | --- | ---  
Problem 12 | 12 | Name a fraction that is somewhere between the two given fractions. Explain why you think your answer is correct for one of the rows.  
   i) \( \frac{1}{3} \) | \( \frac{3}{4} \)  
   ii) \( \frac{3}{5} \) | \( \frac{4}{5} \)  
   iii) 0 | \( \frac{1}{7} \)  

Note. Tasks are interview items; problems are test items.

Figure 6.28. Items that require students to name a fraction between two given fractions.

| Problem 12 subitem | Percent of Kent’s students | Percent of Baker’s students |
| --- | --- | --- | --- |
| | Upper (n=10) | Middle (n=6) | Upper (n=3) | Middle (n=6) |
| i) \( \frac{1}{3} \) | 70\% | 50\%. | 66 2/3\% | 83 1/3\%. |
| ii) \( \frac{3}{5} \) | 20\% | 14\% | 66 2/3\% | 0\% |
| iii) 0 | 10\% | 0\% | 0\% | 0\% |

Figure 6.29. Percentages of students from each class in each quartile group that successfully named a fraction between the given pair.
Figure 6.30. Percentages of students from each class in each quartile group that successfully named a fraction between the given pair on test Problem 12i.

<table>
<thead>
<tr>
<th>Strategies used for Problem 12 subitem</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n=10)</td>
<td>Upper (n=3)</td>
</tr>
<tr>
<td></td>
<td>Middle (n=6)</td>
<td>Middle (n=6)</td>
</tr>
<tr>
<td>Recognized an ordinary fraction between the two</td>
<td>40%</td>
<td>66 2/3%</td>
</tr>
<tr>
<td>Finding equivalent fractions</td>
<td>40%</td>
<td>0%</td>
</tr>
<tr>
<td>Adding the numerators and denominators of the boundary fractions</td>
<td>10%</td>
<td>0%</td>
</tr>
<tr>
<td>Add 1/2 to the lower boundary numerator</td>
<td>0%</td>
<td>16 2/3%</td>
</tr>
<tr>
<td>Non-viable strategies</td>
<td>20%</td>
<td>33 1/3%</td>
</tr>
</tbody>
</table>

For the second pair of fractions most successful students adjusted the numerator of the smaller fraction by a half to achieve a fraction between the two given fractions (see Figure 6.31). This strategy was used by the same percentage of upper quartile students in each class, but only by middle quartiles students from Mrs. Baker’s class. One upper quartile student from each class used finding equivalents via a common factor to generate their correct solution. One middle quartiles student from Mrs. Baker’s class attempted to use the same strategy but made a technical error. No middle quartiles student from Mr. Kent’s class found a fraction between the second pair.
### Strategies used for Problem 12 subitem

<table>
<thead>
<tr>
<th>ii) (\frac{3}{5} \quad _____ \quad \frac{4}{5})</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper (n=10)</td>
<td>Upper (n=3)</td>
</tr>
<tr>
<td>adjusted the numerator of the smaller fraction by a half to achieve a fraction between the two given fractions</td>
<td>30%</td>
<td>33 1/3%</td>
</tr>
<tr>
<td>Finding equivalent fractions</td>
<td>10%</td>
<td>33 1/3%</td>
</tr>
<tr>
<td>Adding the numerators and denominators of the boundary fractions</td>
<td>10%</td>
<td>16 2/3%</td>
</tr>
<tr>
<td>Non-viable strategies</td>
<td>50%</td>
<td>83 1/3%</td>
</tr>
</tbody>
</table>

Figure 6.31. Percentages of students from each class in each quartile group that successfully named a fraction between the given pair.

For the pair, zero and a unit fraction (see Figure 6.32), only one student from Mrs. Baker’s class gave a viable answer—a middle quartiles student. One middle quartiles student and roughly a third of the upper quartiles students from Mr. Kent’s class also gave a viable answer. In each case, the student increased the denominator of the given unit fraction. Half or more of each quartile group from Mr. Kent’s class actually decreased the denominator of the unit fraction and thus made an inappropriate answer. Roughly a third of each quartile group from Mrs. Baker’s class also decreased the denominator as their solution.
<table>
<thead>
<tr>
<th>Strategies used for Problem 12 subitem</th>
<th>Percent of Kent’s students</th>
<th>Percent of Baker’s students</th>
</tr>
</thead>
<tbody>
<tr>
<td>iii) 0 _____ 1/7</td>
<td>Upper (n=10)</td>
<td>Upper (n=3)</td>
</tr>
<tr>
<td></td>
<td>Middle (n=6)</td>
<td>Middle (n=6)</td>
</tr>
<tr>
<td>increased the denominator of the given unit fraction</td>
<td>30%</td>
<td>0%</td>
</tr>
<tr>
<td></td>
<td>16 2/3%</td>
<td>16 2/3%</td>
</tr>
<tr>
<td>Non-viable strategies</td>
<td>70%</td>
<td>100%</td>
</tr>
<tr>
<td></td>
<td>83 1/3%</td>
<td>83 1/3%</td>
</tr>
</tbody>
</table>

Table 6.32. Percentages of students from each class in each quartile group that successfully named a fraction between the given pair.

Because neither Mr. Kent nor Mrs. Baker spent time in their instruction on ordering fractions or finding fractions between a pair of given fractions, it is not surprising that there are relatively few characteristics of students solutions that differentiate between the two classes. One difference is that Mr. Kent’s students that did not use the unit part strategy for ordering at least ordered by reversing the denominator. One might posit that in their instructional experiences even those students that had not developed an appropriate unit part concept had at least reinforced or developed a recognition that the size of a denominator was inverse the size of a unit fraction. Their difficulty with multi-unit fractions may be attributed to the relative dearth of practice representing multi-unit fractions on the geoboard or dot-grid as compared to the extensive time they spent representing unit fractions. A related difference is that although almost no student was able to name a fraction between zero and a unit fraction, those that could were from Mr. Kent’s class and used the unit part size strategy. Another difference that is apparent is the ability of Mrs. Baker’s students to name a fraction between two consecutive fractions by appending 1/2 to the numerator of the smaller fraction and the inability of Mr. Kent’s students to do so. This occurrence is counterintuitive since Mr.
Kent’s students had been exposed to partial units and named similar complex fractions in their answers to other test problems. Although this occurrence does differentiate the two classes, a link to Mrs. Baker’s instruction is not apparent.

For the comparison problems from the test and interview (see Figure 6.33), students exhibited two strategies: (a) find a fraction, ratio, or rate to represent each amount described; (b) compare the amounts based on a non-fraction strategy. Those that found fractions, ratios, or rates to represent the fair shares compared them using known fraction relationships or equivalence relationships. Strategies available to those that used equivalence relationships included finding common denominators, recognizing a doubling relationship between the fractions, simplifying one or both fractions, or finding the rate. Those that did not find a fraction chose to compare either the leftovers from giving each “person” a portion of the “pizza” or the total amounts. A few chose the non-viable strategy of comparing unit part sizes.

Because comparison problems of the type indicated in Figure 6.33 were not available in students’ instructional experiences, there were few identifiable linkages between their solution methods and their instructional experiences. There were few identifiable patterns overall in the strategy combinations that students chose. The exceptions were some problems appeared to elicit certain types of strategies across all groups and that there were identifiable differences between quartile groups success in choosing viable strategies. Those patterns that appear to connect with instructional experiences include the slightly higher incidence of using fair sharing reasoning among Mr. Kent’s students. With respect to comparing, Mr. Kent’s students had a higher
<table>
<thead>
<tr>
<th>Item type</th>
<th>Item No.</th>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem</td>
<td>6</td>
<td>Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.</td>
</tr>
<tr>
<td>Problem</td>
<td>9</td>
<td>Each dark colored glass is chocolate syrup and each white colored glass is milk. Circle the mixture that has a stronger chocolate flavor: the mixture made using the glasses pictured in set A or the mixture made using the glasses pictured in set B. Circle both if the chocolate flavor is the same. Explain why you think your answer is correct.</td>
</tr>
<tr>
<td>Task</td>
<td>7</td>
<td>Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.</td>
</tr>
<tr>
<td>Task</td>
<td>8</td>
<td>Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.</td>
</tr>
</tbody>
</table>

Note. Tasks are interview items; problems are test items.

Figure 6.33. Items that ask students to compare apportioned amounts or mixture strengths.
incidence of using known facts. If finding equivalents, Mr. Kent’s students had a higher incidence of identifying the common denominator or simplifying. In contrast, Mrs. Baker’s students were more likely to recognize fractions that had a doubling relationship.

**Summary: Instruction and Concept Development: The Influence of Instructional Themes**

An examination of the interactions between instruction and students’ concept development as revealed through students’ answers and explanations on the posttest and interview reveal several instructional themes that appeared to influence the concepts, strategies, and processes that students’ employed to solve the items. Some instructional themes were unique to individual classrooms and others were common. The instructional themes had explicit and implicit influences on students’ concepts, strategies, and processes employed on the test and in the interview. Individual students’ interpretations of their instructional experiences within each theme led to both viable and non-viable concepts, strategies, and processes. Some instructional themes were more likely to produce non-viable concepts, strategies, or processes and other instructional themes were more likely to produce viable ones. The instructional themes that were unique to each classroom and the differences in interpretations of common instructional themes available in each classroom bore the influence of each teacher’s personal fraction knowledge.

A common instructional theme did not necessarily indicate the same experience in each classroom. For example, both teachers addressed equivalent fractions as one of the themes in their classroom, but their manner of addressing the theme was different. Mr. Kent introduced equivalence through unit fraction decompositions of the geoboard whole. Students were expected to combine representations of a unit fraction to represent
another unit fraction such as two \( \frac{1}{8} \) is the same as \( \frac{1}{4} \)—e. g., \( \frac{2}{8} = \frac{1}{4} \). Mrs. Baker introduced equivalent fractions via repartitioning. For example, a figure partitioned into sixths could also be partitioned into thirds. Students were expected to recognize that the third-partition contained two \( \frac{1}{6} \)-partitions—e. g., \( \frac{2}{6} = \frac{1}{3} \). Mr. Kent continued equivalent fraction instruction by introducing a proportion chain of known equivalents and asking students to examine the chain for a pattern. Mrs. Baker continued her equivalent fraction instruction by introducing the concept of an equivalence factor. Later, she introduced the backward Z mnemonic device in conjunction with finding equivalent fractions for addition or subtraction. She also required all answers to be in simplest form. Mr. Kent encouraged his students to find “common denominators” when finding equivalent fractions for addition or subtraction. His students were expected to use the processes for finding equivalents that they had developed earlier through pattern recognition and knowledge of common equivalents to supply the equivalent fractions. He did not require the use of an equivalence factor or similar process for finding the equivalent fractions with a common denominator.

The differences in instruction of equivalent fractions were evident in students’ concepts, processes, and strategies employed on the test and interview. By encouraging his students to find patterns of equivalence rather than explicitly instructing them in a particular procedure for finding equivalents, Mr. Kent’s instruction had the implicit influence of reinforcing his students’ flexibility in their strategies and processes for finding equivalent fractions. The pattern relationship that his students observed was repeated addition of numerators and denominators, a subset of doubling. Hence, this was
the process that they most used to find equivalent fractions. Strong evidence for his students’ flexibility was provided by a few who were able to apply “doubling” and “reducing” appropriately to individual items in test Problem 11 (see Figure 6.21). His instructional methods also had the implicit influence of encouraging his students simply to know certain equivalents. An implicit influence of Mr. Kent’s instruction on using equivalent fractions when adding or subtracting was that his students were aware and noted that they were finding “common denominators.” His students were also less likely to describe or show a particular process for finding equivalents when adding or subtracting as compared to their peers in Mrs. Baker’s class.

Mrs. Baker’s instruction had both explicit and implicit influences on her students’ concepts, processes and strategies related to equivalent fractions. One implicit influence was that her students exhibited a reliance on particular strategies in finding equivalent fractions that replicated their classroom experiences. They doubled or they simplified to find an equivalent fraction, but they did not use a combination of the processes on test Problem 11 (see Figure 6.22). Their processes or strategies appeared compartmentalized as compared to Mr. Kent’s students’ flexible strategy use. An explicit influence of Mrs. Baker’s instruction was her students’ predisposition to simplify fraction answers to addition or subtraction problems. They were also more likely than Mr. Kent’s students to show a method for finding the equivalent fractions when adding or subtracting. Although the explicit and implicit influences of her instruction produced viable process and strategies, in at least one instance, a non-viable strategy of one student could be traced to explicit instruction. This student misapplied the process that Mrs. Baker had
demonstrated for the addition or subtraction of mixed numbers to the addition and subtraction of simple fractions.

The addition or subtraction of fractions was another common instructional theme as indicated in the preceding paragraphs. Again Mr. Kent and Mrs. Baker differed in their instruction. Since for both teachers finding equivalent fractions with common denominators was the most important component of instruction on addition or subtraction of fractions, the differences in each teacher’s instruction described in the preceding paragraph also applied. However, additional differences arose due to the situational nature of the addition and subtraction problems that students were presented on the test and the interview. Mrs. Baker did not as a rule provide contexts for the problems that her students were expected to solve. In contrast, Mr. Kent’s students were expected to solve problems set in contexts during some of their instruction. Mr. Kent’s students also had more practice making diagrams that represented fractional amounts than did Mrs. Baker’s students. Thus, the comparative success of Mr. Kent’s students with the addition and subtraction problems on the test and in the interview may have arisen from the implicit influence of his having employed situational problems and diagram drawing in his instruction versus Mrs. Baker’s lack of doing so. Several of Mrs. Baker’s students attempted to use diagrams to interpret the addition problem on the test and were confused by the diagrams. In contrast Mr. Kent’s students solved the problems and then drew a diagram if at all.

The partitioning of a discrete whole into composite unit parts was another instructional theme common to both classes. Two different instructional experiences in
Mr. Kent’s class explicitly and implicitly influenced students’ development of concepts, processes, or strategies related to this instructional theme. The area model developed on the geoboard in which each unit part was composed of area units implicitly influenced students’ ability to partition a whole into composite discrete units. The partition of discrete unit wholes represented by unifix cube sticks into composite discrete unit parts explicitly influenced students’ ability to partition discrete wholes into composite units. In contrast Mrs. Baker’s students experiences with discrete wholes began with “building” fractions from unifix cubes. The whole was the completed stick of singleton discrete units. Although it was not recorded, Mrs. Baker demonstrated in her interview with color tiles a whole representing a fraction composed of composite unit parts that she recalled having demonstrated in class with a different but comparable medium. Her students’ experience of composite discrete units of a whole was an explicit influence on their ability to compose a whole from composite discrete units on interview Task 2. Beside the aforementioned explicit influence on a couple of Mrs. Baker’s students’ process for representing a fractional amount with composite discrete units, a more pervasive difference of influence of the instruction from each class can be found in Mr. Kent’s students’ predisposition to partition a discrete whole by “breaking” as opposed to Mrs. Baker’s students predisposition to partition one by “dividing” the number of discrete units of the whole to determine the unit part size.

Notational conventions of each teacher, though a minor consideration, are a final instructional theme with explicit influence on students’ processes. For example, students replicated their respective teacher’s notational style with respect to how addition and
subtraction was represented. Mr. Kent’s students primarily wrote horizontal sentences and Mrs. Baker’s students wrote vertical sentences. Even more notable was the replication by some of Mrs. Baker’s students of her simplification notation and her backward Z notation.

Three instructional themes were primarily developed in Mrs. Baker’s class although Mr. Kent’s instruction also touched on them in some respects. For example, the instructional theme of congruent unit part partitioning of a region was a major theme in Mrs. Baker’s classroom. Mr. Kent also began his instruction of fractions with congruent unit part partitioning of a region, but he quickly moved to unit part partitioning using area units. Mrs. Baker exclusively used congruent unit part partitioning with region models of a whole as compared to Mr. Kent’s students’ use of both congruent and area models. The explicit influence of Mrs. Baker’s exclusive use of congruent part partitioning of region models of a whole was that her students’ only viable strategy for partitioning region models of wholes on the test and interview was congruent part partitioning.

However, only Mrs. Baker’s upper quartile students were able to partition the geoboard to form unit parts. Some of her middle quartiles students partitioned region models such as the circle of test Problem 4 (see Figure 6.3) and interview Task 1 (see Figure 6.6) and square of test Problem 5 (see Figure 6.3), but were not able to appropriately partition the geoboard on interview Task 3 (see Figure 6.6) although they did have one experience modeling fractions and wholes on the geoboard. Instead they used the pegs of the geoboard as singleton unit parts. The contrast in her middle quartiles students’ ability to solve the former as compared to the latter can be attributed in part to
the presence of partition examples in each of the former items. For interview Task 3, they had to both define a whole and define a unit part. They chose the pegs as recognizable singleton units as opposed to the square regions. Although Mr. Kent’s middle quartiles students were not very successful with interview Task 3, both interviewees used region models and one was semi-successful. She used the squares of the geoboard as singleton units. Mrs. Baker’s middle quartiles students’ choice of the pegs of the geoboard to represent singleton units can be attributed to their lack of experience with the square as an area unit as compared to Mr. Kent’s students’ extensive experience. Their lack of experience with the area model interacted with a preference for singleton unit part concept of fraction models. In other words, the implicit influence of Mrs. Baker’s instruction was a combination of reinforcement of singleton or discrete fraction models in her class and the comparative lack of development of region or area models of fractions.

The parts-to-total definition of a fraction was the primary instructional theme from Mrs. Baker’s classroom. Although not a major instructional theme in Mr. Kent’s class, a parts-to-total definition of fraction could have been reinforced by some of Mr. Kent’s instruction. When students were asked to represent fractions such as 3/4 on their geoboards they could interpret the fraction as the ratio of three unit parts out of four total unit parts on the geoboard. In addition, when students were asked to decompose a geoboard into multiple representations of a particular unit fraction, one method of checking their success was to count the total number of regions they had formed. Yet much less emphasis was placed on multi-unit fractions in Mr. Kent’s classroom, and hence, there was less reinforcement for the parts-to-total definition of fraction.
In comparison, Mrs. Baker’s instruction on modeling fractions primarily emphasized the parts-to-total definition of fraction and multi-unit fractions. She began by having her students “build” fractions using two-color unifix cube sticks and continued with region models of fractional amounts on the geoboard and other media. Over and over she reinforced counting the number of selected unit parts and the “total” number of unit parts on the model. The explicit influence of the parts-to-total instructional theme was identifiable in concepts, strategies, or processes used by her students on the test and in the interview. The influence encouraged some non-viable strategies and processes as well as quasi-viable strategies and processes for identifying fractional amounts. One quasi-viable strategy was the identification of a fractional amount by producing the ratio of parts shaded or otherwise selected to the total number of parts in the whole. As long as the parts were units of the whole, the strategy produced an appropriate fraction identifier. However, some students had not developed the restriction of their parts definition to unit parts and sometimes would use non-unit parts to produce their fraction symbol. The parts-to-total instructional theme also had a quasi-viable influence on students’ relationship of the whole to the fraction symbol. As long as a problem involved the partition of a single whole for identification of a fractional amount, students were able to write an appropriate fraction symbol for the amount by recognizing that the total number of unit parts in the whole would be the denominator of that symbol. However, in the case of the fair sharing problems and tasks in which multiple wholes were being shared between a given number of people, students’ strategy or process for identifying the fractional amount of one portion based on the parts-to-total definition was non-viable.
Students appeared to identify the total as all parts across the number of wholes shared. Thus, they had difficulty relating the “total” aspect of the definition to the number of unit parts in a single “whole.”

Mrs. Baker’s emphasis on complementary fraction relationships was a minor instructional theme in her class. However, it did seem to have an explicit influence in that many of her students used complementary fraction relationships in their reasoning on the test and in the interview.

Instructional themes that were unique to Mr. Kent’s classroom include area of a unit part, unit fraction representation, solving situational problems, and fraction strips. The primary instructional themes were unit part area and unit fraction representation. These were the themes that he developed through the decomposition of the geoboard and 4x6 dot grid as a whole into multiple representations of given unit fractions beginning with halves and ending with twenty-fourths. Both contributed explicitly to Mr. Kent’s students’ development of the unit part area or size concept and implicitly to their flexibility in using strategies or processes to solve problems. The unit part area instructional theme contributed implicitly to students’ development of the composite discrete unit part concept. The unit fraction representation instructional theme contributed explicitly and implicitly to students’ development of the tiling concept of unit parts with respect to the whole.

The influence of the unit part area and unit fraction representation instructional themes usually produced viable concepts and strategies or processes. However, there were some instances of non- or quasi-viable strategies developed by particular students.
For example, one influence of instruction related to unit part area on two upper quartile students’ development of the unit part area concept was the quasi-viable strategy of requiring an outside authority to provide the representation of a whole or of a unit fraction or other fractional amount before they could represent a fractional amount with a given medium. An influence of the unit fraction representation theme and the lack of equal emphasis in instruction on multi-unit fraction representation was that some students sometimes had difficulty producing appropriate fraction symbols for multi-unit fractional amounts. The quasi-viable strategy that they used instead was to describe the fractional amount with a combination of unit fractions. As a result they were better able to semi-appropriately describe multi-unit fractional amounts than their peers in Mrs. Baker’s class who also had difficulty repartitioning a whole to determine the appropriate description of a designated fractional amount. An explicit influence of the unit fraction representation instructional theme was that Mr. Kent’s students were better able to create diagrams of fractional amounts than their peers from Mrs. Baker’s class. Their strategy for drawing diagrams was quasi-viable in that they could draw the diagram of a known fractional amount but were not able to use diagrams as reasoning tools to solve problems. Finally, the unit fraction representation instructional theme may have had an implicit influence in Mr. Kent’s students’ development or reinforcement of the size concept of unit fractions as reverse the size of the denominator.

Fair sharing is another major instructional theme exclusive to Mr. Kent’s class. It had an explicit influence in Mr. Kent’s development of the concept of fair sharing and viable strategies and processes for finding one person’s portion. In addition, it had an
explicit influence in Mr. Kent’s students’ ability to recognize that the denominator of the fraction symbol representing one person’s share of multiple wholes was the number of unit parts in one whole as compared to Mrs. Baker’s students’ failure to do so.

Solving situational problems and creating fraction strips are two other instructional themes that were exclusive to Mr. Kent’s class. Solving situational problems had an implicit influence in Mr. Kent’s students’ comfort with solving the addition and subtraction problems from the test and interview since each was set in a context. Creating fraction strips explicitly influenced Mr. Kent’s students’ use of measurement concepts to solve test Problem 2.

In summary, the common instructional themes represented curricular considerations. The common themes (partitioning a discrete quantity whole into composite unit parts, equivalent fractions, addition and subtraction of fractions) are those that are suggested by outside sources such as the district’s scope and sequence. However, each common theme was addressed in the classroom in ways that suggest the influence of each teacher’s personal/pedagogical fraction knowledge. Unique instructional themes (notational conventions, congruent unit part partitioning, equal area unit part partitioning, parts-to-total definition of fraction, complementary fraction relationships, unit fraction representation, solving situational problems, fraction strips) appeared to arise either from a teacher’s personal fraction knowledge or from supplementary curriculum materials chosen by the teacher. Explicit and implicit influences of instruction on students’ strategies/processes/concepts primarily occurred in relationship to teachers’ instructional styles—directive vs. facilitative, respectively. However, in particular instances, implicit
influences suggest interactions between instructional themes and students’ predispositions. Most instructional themes were associated with students’ development/use of viable processes/strategies/concepts. However, some instructional themes such as algorithmic equivalence and parts-to-total definition of fraction could be linked to non-viable processes/strategies/concepts. Whether or not instructional themes were linked to viable, quasi-viable, or non-viable strategies/processes/concepts could be dependent on particular students’ interpretations of the instructional themes, i.e., the particular students’ predispositions.

Summary: Students’ Emergent Content Knowledge

The test scores analysis suggests that students’ prior knowledge is a strong influence on their emergent content knowledge. Test item analysis and analysis of teachers’ instructional units suggest that students’ opportunity to learn (exposure to a concept such as fair sharing as a significant instructional item) is another strong influence on students’ emergent content knowledge. Finally, analysis of students’ explanations on test problems and interview tasks and their instructional experiences suggest that teachers’ instructional themes can have explicit and implicit influence on students’ emergent content knowledge, but that those influences may be mediated by the students’ prior knowledge and predispositions.
In this chapter, the cases are discussed in light of the theory. From the discussion, an interpretation of the theory is generated. Based on the discussion and interpretations of the theory, implications are identified for K-12 and teacher education. Finally, suggestions for further research that build on the cases and theory are made.

**Discussion**

Case studies examine the particular and as such do not generalize to a population. However, they can illustrate and examine theory. Figure 7.1 (same as Figure 2.6) illustrates the theory that informed the case studies. It provides a visual representation of a theory of influence/interrelationship between teachers’ content knowledge and students’ emergent content knowledge. The case studies of the two teachers provide “real” representations of the theory. The preceding cross-case analyses of the case studies of the two classrooms provide a basis for confirming and/or questioning the theory through identifying commonalities and contrasts between the cases. Beginning with an examination of the teachers’ content knowledge of rational numbers, the cases interrogate the teacher aspect of the theory through examining the interrelationships between teachers’ content knowledge, other factors, and their instruction. Similarly, they
interrogate the student aspect of the theory through examining the interrelationships between students’ emergent knowledge and their instructional experiences.

Figure 7.1. Theory of interrelationship between teachers’ knowledge of content and students’ emergent content knowledge.

**Teachers’ Personal Content Knowledge**

As discussed in “Chapter 2: The Literature Review,” there are three aspects of teachers’ personal content knowledge that contribute to their rational number instruction: content, syntactic, and metacognitive knowledge related to rational numbers.

**Teachers’ Content Knowledge of Rational Numbers**

With respect to their content knowledge of rational numbers interpreted through Kieren’s (1988, 1993) model of personal rational number knowledge and the knowledge structures described by The Rational Number Project (Behr et al., 1992), the cases revealed that Mr. Kent and Mrs. Baker differed at every level. In terms of rational
number knowledge at *level one* (partitioning, quantitative equivalence, and unit forming), Mrs. Baker and Mr. Kent differed in their comfort in using different types of quantities (continuous or discrete) in representing fractional amounts and in interpreting fractional amounts of non-standard representations of a whole. Mr. Kent was more likely to use composite continuous (area) units in his explanations and was more fluent in decomposing/composing representations of fractional amounts and wholes into unit parts/fractions. He quickly and accurately interpreted fractional amounts situations in which there was a non-standard representation of a whole. Mrs. Baker was more likely to use singleton or composite discrete units in her explanations. She was hesitant or made initial errors when confronted with situations involving non-standard representations of a whole. In describing the relationship between wholes and fractional amounts, Mr. Kent often expressed the whole as an iterative composition of the unit fraction and Mrs. Baker sometimes expressed the whole as the sum of complementary fractional amounts.

In terms of rational number knowledge at *level two* (measure, quotient, operator, ratio), Mrs. Baker and Mr. Kent differed in the evidence of constructs other than part-whole in their interpretations of fractions. Mr. Kent’s discussions of fractions included geometric and measurement terms and used operator and quotitive processes. Mrs. Baker’s discussion of fractions focused on part-whole and ratio relationships.

In terms of rational number knowledge at *level three* (scalar and function constructs), Mr. Kent can be said to have the function construct available based on his alternative responses to the regular fraction problems in which he found the quotient representation of a fraction. Since Mrs. Baker gave few or no alternative responses to the
regular fraction problems, it cannot be said with any certainty whether or not she had the
function construct as a part of her rational number reasoning. Because they imply
proportional relationships, fair sharing and mixture strength problems also reveal
differences in Mr. Kent’s and Mrs. Baker’s rational number reasoning related to level
three constructs (Kieren, 1980; Streefland, 1993; Vergnaud, 1983, 1988). Mr. Kent
consistently solved such problems by finding unit rates in decimal or ordinary fraction
form—quotient or function constructs. Mrs. Baker was inconsistent in her interpretation
of the whole in fair sharing situations leading to inconsistency in her interpretation of one
person’s share across such problems. However, in comparisons of shares or mixture
strengths, she was successful through interpreting each situation in terms of ratios of
different measures that she could compare using proportional reasoning (scalar
construct).

In terms of rational number knowledge at *level four* (formal equivalence and
multiplicative structures) and *level five* (additive and multiplicative groups), only the
teachers’ applications of algorithmic equivalence (formal equivalence) and addition or
subtraction of fractions (additive group) were observed, respectively. Mr. Kent used
models and algorithms in solving problems of equivalence, ordering, or
addition/subtraction of fractions, suggesting that he had available several aspects of the
constructs for formal equivalence and additive groups. Mrs. Baker used models and
algorithms in solving all but ordering of fractions. She appeared to use proportional
reasoning exclusively in ordering fractions. Her reasoning suggests that she had available
some aspects of the constructs for formal equivalence and additive groups. Mr. Kent used
common representations of mathematical algorithms and Mrs. Baker used a personal algorithmic representation (see the “Backward Z,” Figure 4.10) in solving equivalence, ordering, and addition/subtraction problems. The test that Mr. Kent and Mrs. Baker took was not adequate to explore the sophistication of their levels three to five constructs of Kieren’s (1988, 1993) model of personal rational number knowledge. To say that they had available constructs at these levels is not to say that their constructs had the same sophistication as the constructs of a practicing mathematician at those levels.

*Teachers’ Syntactic Knowledge Related to Rational Numbers*

With respect to syntactic knowledge related to rational numbers interpreted as the knowledge of how the validity of concepts and principles is established within a content area (Shulman, 1986), Mrs. Baker and Mr. Kent also differed. In particular, the differences in their syntactic knowledge were evident in the symbol systems and visual representations they used and in their written arguments for why a solution was appropriate. Mrs. Baker had very little explanation in her responses to the problems on the test and in some cases her written explanations contained mathematical inaccuracies such as “3 + 1 = 4/4 = 1 whole.” Her explanations contained personal symbol systems such as the backward Z mnemonic. Her visual illustrations were not always precise. Mr. Kent provided detailed explanations in which he was careful to express ideas with mathematical precision, using common mathematical language and symbol systems. His explanations were sometimes illustrated with carefully drawn and labeled visuals.
Teachers’ Metacognitional Knowledge Related to Rational Numbers

Although the study did not explicitly collect and analyze data on teachers’ metacognition, defined as their self-awareness of and ability to reflect on their reasoning processes (Peterson, 1988), Mr. Kent’s and Mrs. Baker’s responses to the test and interview items suggest differences. Mr. Kent’s articulate expression of his mathematical thinking suggests that he had spent time thinking about and verbalizing his solution processes. In contrast, Mrs. Baker’s lack of detailed explanation suggests at the least that she had not practiced expressing her personal mathematical thinking verbally. In their interviews each spoke to their awareness of aspects of their mathematical thinking. Mr. Kent observed that when he was more successful with a topic, particularly procedural or automated knowledge of topics, it was more difficult for him to access concepts that might underlie the topic. Mrs. Baker observed that she often used visualization in her personal fraction reasoning.

Interrelationships Between Teachers’ Personal Content Knowledge and Instruction

Given two teachers who differed in their content, syntactic, and metacognitive knowledge of rational numbers (Peterson, 1988; Shulman, 1986), the cases reveal interrelationships between teachers’ personal content knowledge of rational numbers, other possible factors, and their instruction of rational numbers. The interrelationships appear in the organization of content knowledge for instruction (Fennema & Franke, 1992), the nature of classroom discourse (Carlsen, 1990, 1993; Lehrer & Franke, 1992), and teachers’ assessment of students’ thinking (Thompson & Thompson, 1994, 1996; Van Dooren, Onghena, & Verschaffel, 2002).
The cases suggest interrelationships between teachers’ content knowledge and their organization of content knowledge for instruction. For example, regarding different types of quantities (continuous or discrete) in representing (modeling) and interpreting (defining) fractional amounts, Mr. Kent’s instructional activities and his test responses focused on composite continuous (area) quantities. To develop the relationship between wholes and fractional amounts, the activities that Mr. Kent’s students engaged in often involved decomposing a whole into a composition (iteration) of the unit fraction, emphasizing the number of unit fractions that compose the whole. This paralleled Mr. Kent’s use of iteration to describe composition of a whole from the unit fraction on the test. Mr. Kent’s discussions during instruction included geometric and measurement terms as had his discussions of fractions in his test and interview responses.

Mrs. Baker’s discussion of fractions in class and in her test and interview responses focused on part-whole and ratio relationships. She used complementary fractions both in her test solutions and in her classroom discussions, whereas Mr. Kent did not in either situation. Mrs. Baker’s personal mnemonic device for finding equivalent fractions in solving addition/subtraction problems was evident both in her test responses and in her classroom presentations. Her use of repartitioning to find equivalent fractional amounts in using models to add and subtract in the interview was evident in her instructional activity for modeling equivalent fractions.
The Nature of the Teachers’ Classroom Discourse

The cases suggest interrelationships between the nature of classroom mathematical discourse and teachers’ syntactic knowledge of mathematics. Mr. Kent, whose knowledge of rational numbers appeared to be “more explicit, better connected, and more integrated” (Brophy, 1991, p. 352) encouraged more open discourse between students and between students and himself as they worked on rational number activities. He encouraged his students to share their solutions with one another and to evaluate each other’s work, offering corrections and suggestions for alternative viewpoints (Cobb, Yackel, & Wood, 1995; Lampert, 1990). He also required his students to explain their reasoning both in writing and when sharing their creations with the class. Mrs. Baker, in contrast, tended “to depend on the text for content, de-emphasize interactive discourse in favor of seatwork assignments, and in general, portray the subject as a collection of static” procedural knowledge (Brophy, 1991, p. 352). Mrs. Baker did not require written explanations for solutions during observed lessons and did not foster students’ sharing their thinking except during the lesson presentations.

Interrelationships between Mr. Kent’s and Mrs. Baker’s use of notation and their syntactic knowledge of mathematics (McClain & Whitenack, 1995) were also evident in their instruction. Mr. Kent was careful that his students used standard mathematical notation. He required that his students carefully label and record their visuals of wholes decomposed into unit fractions with the standard fraction symbols. In contrast, Mrs. Baker used personally meaningful notation in her instruction such as the Backward Z and
placed the common factor that related equivalent fractions over the equal sign and encouraged students to use similar notation if they wished.

**Teachers’ Assessment of Students’ Thinking**

The cases suggest interrelationships between teachers’ assessment of student thinking (or rating students’ solution processes) and understanding (constructing an internal model of) students’ reasoning processes, and teachers’ content (Van Dooren, Onghena, & Verschaffel, 2002), syntactic, and metacognitive knowledge of rational numbers (Thompson & Thompson, 1994,1996). Observations of the types of evaluation that teachers used in the classroom suggests the nature of their evaluation of students. Mrs. Baker’s emphasis on use of the text answer key or other predetermined lists of expected answers to evaluate student understanding suggests that she would rate students according to the number of accurate answers that they could provide over the spectrum of activities or formal assessments that she gave them. Mr. Kent’s emphasis on explanations as an evaluation tool suggests that he rated students’ understanding according to the appropriateness of their reasoning to the solution of the task and their use of mathematically appropriate language.

In the case of teachers’ models of student thinking, there was evidence that Mrs. Baker used her own reasoning processes to anticipate students’ thinking about a mathematical situation. An example of this was the situation in which she assumed that a student who observed that 1/3 was an equivalent fraction symbol for a model that was partitioned and shaded to show 2/6 had combined every two unit parts to form a partition of the whole by three as she would have done. Thus, Mrs. Baker’s personal reasoning
(metacognition) appeared to influence her model of a particular students’ thinking. Mr. Kent voiced in his interview that he used own personal reasoning (metacognition) as a model for anticipating student reasoning. He also realized the problems that this engendered. He noticed that if he had particular success with a mathematical topic as a student, he did not have an appropriate model for the thinking of students who were having difficulty. Nevertheless, as he became aware of students’ errors in thinking, he consciously designed instruction to address those errors. He also used teaching techniques that allowed him to access children’s thinking processes such as requiring students to make explanations for their reasoning. Thus, the teachers’ metacognitional knowledge of mathematics did inform their models of students’ reasoning. However, as evident in Mr. Kent’s practice, pedagogical content knowledge could modify their models.

Other Factors and Instruction

The cases also suggest interrelationships between Mrs. Baker’s and Mr. Kent’s instructional units and other factors such as the district’s curriculum guide, curriculum resources, and student placement. Since the two teachers taught in the same district, their instructional units show evidence of addressing many of the same topics (models of fractions, fraction definitions, models of wholes, models of equivalent fractions, finding equivalent fractions algorithmically, ordering fractions, and adding/subtracting fractions). They addressed topics in a similar order, moving from models of fractions and connecting the definition of fraction with fraction symbols, to equivalent fractions, to addition and subtraction of fractions. Curriculum resources and student placement were
factors that played a differential role and were arguably within the respective teachers’ control. Mr. Kent used other curriculum materials than the text because he chose to do so. Mrs. Baker had more low-performing students in her classroom than the norm, because she chose to take them. Additional factors that interrelated with Mr. Kent and Mrs. Baker’s rational number instruction were: (a) their teaching experience, (b) their exposure to mathematics pedagogical instruction, and (c) their beliefs about mathematics and mathematics teaching.

Teachers’ Experience and Pedagogical Instruction

The cases suggest interrelationships between Mr. Kent’s and Mrs. Baker’s teaching experience and pedagogical instruction and their pedagogical content knowledge for rational numbers (Sowder et al., 1998; Summers et al., 1998; Swafford et al., 1999). Although Mrs. Baker had been teaching longer than Mr. Kent, 7 and 2 years, respectively, Mr. Kent reported the influence of additional mathematics methods instruction through a professional development course encouraged by his district, Math Their Way of Thinking. Mr. Kent appeared to have more knowledge of activities for developing conceptual knowledge of rational numbers, attributable to his participation in a professional development program. In contrast, Mrs. Baker appeared to use personally created and text-based lessons and focused more on developing procedural knowledge for working with rational numbers. Because Mrs. Baker did not report any other pedagogical training for teaching mathematics than her one course in college—a course that she did not feel prepared her to teach mathematics—there is little reason to believe that she experienced any other opportunities for improving her pedagogical knowledge for
teaching rational numbers than her personal teaching experience, possible discussions with peers, and exposure to curriculum materials (Saxe, Gearhart, & Nasir, 2001).

Teachers’ Beliefs About Teaching Content

The cases suggest interrelationships between teachers’ beliefs, defined as including “what a teacher considers desirable goals of the mathematics program, his or her role in teaching, the students’ role, appropriate classroom activities, desirable instructional approaches and emphases, legitimate mathematical procedures, and acceptable outcomes of instruction” (Thompson, 1992, p.135) and Mr. Kent’s and Mrs. Baker’s instruction. They voiced similar instructional goals for students’ understanding fractions and applying that understanding to addition and subtraction of fractions. Both were concerned that their students’ procedural skill performance on the standardized test was not as strong as they thought would be appropriate. However, they were very different in their mathematics instructional practices.

The classroom social norms (teachers’ beliefs about their and their students’ role in instruction, appropriate classroom activities, and approaches to instruction), and sociomathematical norms (teachers’ and students’ negotiations of what counts as different, sophisticated, efficient, and acceptable mathematical solutions) as revealed through the participation structure of each classroom described in the cases are also revelatory of interrelationships between Mr. Kent and Mrs. Baker’s emergent beliefs about teaching mathematics (Cobb & Yackel, 1995; Lampert, 1990). In Mr. Kent’s class, students were organized into small groups for activities. Students might work individually or in pairs, but they were encouraged to share ideas with one another. Mr.
Kent would walk around the classroom to observe students’ work and ask them questions to explore their thinking. He often introduced new ideas to small groups of students as the students appeared ready to move to a more difficult concept before he explored the new ideas with the whole class. Students were encouraged to generate their own ideas and concepts that were shared with the whole class. Activities often lasted more than one day. Culmination of an activity would be a class sharing time when individual students would come forward and explain their work to the whole class using the overhead projector.

Although Mrs. Baker’s students were seated at tables that could be identified as groups, a lesson usually began with a teacher-guided demonstration. Students were invited to participate through answering various procedural or conceptual questions at appropriate places in instruction. When a student introduced an idea during class discussion, she would sometimes incorporate it into the lesson. Sometimes students worked on a hands-on activity as a group, but more often they were expected to work individually on a textbook assignment or handout after the lesson had been given. They were sometimes discouraged from sharing their ideas or understanding with one another. Mrs. Baker would walk around the class and help individual students who were struggling. She would often work through the problem with the student and explain to them the procedure that they could use to complete the exercise.

How solutions were evaluated in each classroom reveals differences in the sociomathematical norms engendered by each teacher. In Mr. Kent’s class, students were encouraged to monitor their own understanding. Although he required students to verify their work with him before moving to the next subtask, he would ask them to explain to
him their reasoning and ask them questions to guide them to recognize errors and success. He also encouraged peer assessment, habitually requesting that peers verify the shared work. Students discussed with him and each other what constituted similar or challenging solutions and together established a consensus. In Mrs. Bakers’ class, she or the textbook answer key established what constituted an appropriate solution.

Summary: Interrelationships Between Teachers’ Personal Knowledge, Other Factors, and Instruction

In summary, teachers’ personal content knowledge and their instructional practices appeared to be interrelated. Patterns of interrelationship were found between: (a) the teachers’ content knowledge and their organization of instruction, (b) the sophistication of teachers’ content knowledge and the types of discourse present in their classroom, (c) the syntactic knowledge that appeared in their personal mathematical knowledge and the syntactic knowledge that they promoted in their classrooms, and (d) their metacognition of rational number concepts and their models of students’ thinking about rational number concepts. However, there were indications of patterns of interrelationship between other factors, their personal content knowledge of rational numbers, and their instruction. Notably, in the case of Mr. Kent, additional pedagogical instruction for teaching rational numbers may have contributed to his personal content knowledge, his beliefs about mathematics and mathematics instruction, as well as his knowledge for instruction. In addition, his pedagogical content knowledge in relation to emergent students’ syntactic knowledge may have provided additional access to understanding students’ thinking about rational number concepts. It is not possible to say
the existence, nature or direction of the influence of the pedagogical instruction, but the differences between his and Mrs. Baker’s instruction are those that would be anticipated given the instruction that he received.

Interrelationships Between Students’ Emergent Content Knowledge and Instruction

The cases suggest interrelationships between students’ experience of classroom instruction and their emergent content knowledge. Kieren’s (1988, 1993) model of personal rational number knowledge, the knowledge structures described by The Rational Number Project (Behr et al., 1992), and the rational number schemes described by Steffe (2001) contributed to illuminating the interrelationships.

Level One: Partitioning, Quantitative Equivalence, and Unit Forming Constructs

With respect to constructs and schema at level one (Kieren, 1988, 1993), all three constructs (partitioning, quantitative equivalence, and unit forming) contribute to and develop through one overarching instructional theme for both classes: the composition/decomposition of wholes (discrete and continuous quantity) from/into unit parts (discrete and composite) and the assigning of a fraction symbol to a quantity represented or the representation of a quantity given the fraction symbol. Partitioning corresponds to the composition/decomposition of wholes (discrete or continuous quantity) from/into unit parts. Quantitative equivalence contributes to and arises through the formation and recognition of unit parts (singleton discrete or composite) as having the attribute of being the same “size” as each other whether size is based on the congruence of parts, on the measure of the parts with an area unit, or some other perceptual process (Freudenthal, 1983; Mack, 1993). Unit forming contributes to and arises through the formation and
recognition of unit parts (singleton discrete or composite) as countable quantities (Mack, 1993; Steffe, 1988). In other words, unit forming contributes to and arises from the development of understanding the relationship between the fraction symbol assigned to a fractional quantity and the unit part(s) that comprise the fractional quantity. That is, \(\frac{a}{b}\), \(b \neq 0\), is a number comprised of the unit \(\frac{1}{b}\) counted \(a\) times.

Within the overarching instructional theme, both classes experienced composition/decomposition of discrete quantity wholes from/into composite unit parts. However, they had differing experiences with respect to composition/decomposition of continuous quantity wholes from/into unit parts. Instruction in Mrs. Baker’s class emphasized decomposition of continuous quantity wholes (regions) into congruent unit parts. In contrast, instruction in Mr. Kent’s class emphasized composition/decomposition of continuous quantity wholes (geoboards and various dot grid regions) from/into composite unit parts composed of area units. Only in Mr. Kent’s class did students compose/decompose continuous quantity wholes from/into composite unit parts that were lengths through creating and using fraction rulers.

The cases suggest interrelationships between students’ instructional experiences and their emergent knowledge of rational numbers at level one through three problem situations: (a) determine the appropriate numerical symbol to describe the representation of a fractional amount of a given whole, (b) determine the appropriate representation of a fractional amount of a whole given the numerical symbol, (c) find the whole given a representation of a fractional amount and its numerical symbol.
With respect to problem situations in which students were to determine the appropriate numerical symbol to describe the representation of a fractional amount of a given whole, interrelationships between instruction and students’ emergent knowledge were revealed in their partitioning strategies for producing unit parts (quantitative equivalence) and their conception of the relationship between a fraction symbol, unit parts of a unit whole, and a fractional amount (unit forming). In problem situations where no partition was provided, interrelationships were noted between instructional activities in Mr. Kent’s class that reinforced covering a whole with unit fraction representations and his students’ preference for describing a tiling process as compared to Mrs. Baker’s students’ preference for describing a splitting process in partitioning. In problem situations where partial partitions were available, interrelationships were not apparent between instruction and students’ choices to repartition the whole to produce unit parts commensurate with the shaded portion or to use the given partition to interpret the indicated fractional amount. However, interrelationships between instruction and unit forming were apparent. Of students who chose not to repartition the whole, but to use an indicated partial partition, some from both classes could form a complex fraction to represent the shaded fractional amount. Yet, only students from Mr. Kent’s class identified the fractional amount as composed of two unit fractions—an accurate if quasi-viable description. Students in Mrs. Baker’s class were more likely than those in Mr. Kent’s class to form a fraction symbol based on non-unit parts. This was taken as an implicit influence of the instruction in Mr. Kent’s class that encouraged multiple representations of fractional amounts.
With respect to situations in which students were to determine the appropriate numerical symbol to determine the appropriate representation of a fractional amount of a unit whole given the numerical symbol, interrelationships between students’ emergent knowledge and instruction was revealed in students’ composition/decomposition (partitioning) of a whole from/into unit parts (quantitative equivalence) and identification/composition of a set of unit parts to form a representation of the given fraction symbol (unit forming). In problem situations for which neither a whole nor a unit part or partition was provided, interrelationships were apparent between students’ instructional experiences and the types of unit parts chosen. Although students from both classes often chose a singleton discrete unit and iterated it by the denominator to form a whole, in the case of a discrete quantity whole some of Mrs. Baker’s students formed the fractional representations with composite discrete unit parts as they may have experienced in class. In the case of the continuous quantity whole (geoboard), Mr. Kent’s students formed the fractional representations with unit parts based on area units as they had in class.

Implicit interrelationships were apparent between students’ instructional experiences and their viable and non-viable strategies for partitioning/determining a unit part. In one problem situation, one of Mr. Kent’s students was able to form a composite unit part ostensibly because of his experience with composite unit parts formed from area units in class. However, two of Mr. Kent’s students were unable to form a unit part apparently because they were unfamiliar with the representational medium. They required that an authority identify a representation of either a whole or a unit part before a
solution could be obtainable. In a different problem situation, more of Mr. Kent’s than Mrs. Baker’s students were able to form a composite unit part partition. Mrs. Baker’s students either formed singleton discrete unit parts from the pegs of a geoboard or formed singleton congruent continuous unit parts ostensibly because they did not have the many experiences representing fractional amounts through composing/decomposing geoboards into composite continuous unit parts that Mr. Kent’s students had.

With respect to situations in which students were to find the whole given a representation of a fractional amount and its numerical symbol, interrelationships between students’ emergent knowledge and instruction were revealed in students’ conceptions of a relationship between the fraction symbol and the fractional amount (unit forming) such that they could compose/decompose (partitioning) a fractional amount from/into unit parts (quantitative equivalence) and iterate/compose the identified unit part(s) to form a set of unit parts that would represent the unit whole (partitioning). In problem situations in which students determined the unit whole given a fractional amount, interrelationships between instruction and students’ emergent knowledge of rational numbers were less apparent. One implicit interrelationship between instruction and students’ viable solutions was that Mrs. Baker’s students were more likely than Mr. Kent’s to append unit parts to a given fractional amount to form a representation of the whole. This is likely attributable to her emphasis on complementary fractions. Mr. Kent’s students were more likely to iterate the unit part to form a representation of the whole. This might have been an influence of Mr. Kent’s emphasis on checking a partition of the whole by counting the number of unit parts constructed.
Level Two: Measure, Quotient, Ratio, Operator Constructs

With respect to constructs and schema at level two (Kieren, 1988, 1993), the application of the constructs (measure, quotient, ratio, operator) within instructional themes differed between the classes. Level two constructs appear to be more situational and differentiated in their applications to rational number problems (Kieren, 1993). Although more than one construct may be applied in a given situation, they do not often appear together in an application. The constructs of quotient, ratio, and operator were more evident in the instructional themes of Mrs. Baker’s class. In her development of the common instructional theme of composition/decomposition of wholes (discrete and continuous quantity) from/into unit parts (discrete and composite) and the assigning of a fraction symbol to a quantity represented or the representation of a quantity given the fraction symbol, Mrs. Baker emphasized the symbolization of the “part-to-total” relationship of the unit parts in the fractional amount to the unit parts in the whole by the numerator and denominator relationship of the fraction symbol. The quotient, ratio, and operator constructs were present in the common instructional theme of unit part composition/decomposition of wholes in Mr. Kent’s class. However, the measure construct was also present through the use of area units in the formation of composite unit parts—a theme unique to his classroom. The measurement and other level two constructs in service of rational number knowledge development were also evident in Mr. Kent’s instructional themes of creating fraction strips and fair sharing. In Mrs. Baker’s class, the measurement construct was present in her instructional emphasis on the complement of a fractional amount.
The cases suggest interrelationships between students’ instructional experiences and their development of measurement constructs related to their emergent knowledge of rational numbers through problem situations that related to the composition/decomposition of wholes and/or fractional amounts and the ordering of fractions. Upper quartile students in both classes “covered” continuous quantity wholes or fractional amounts with equal size unit parts in finding fraction symbols, representations of fractional amounts, or representations of whole as in Kieren’s (1980) description of the measurement construct. Mrs. Baker’s upper quartile students used congruent unit parts exclusively, but were able to partition wholes with different shapes. Mr. Kent’s students, upper and middle quartiles, were able to represent fractional amounts with more flexible representations of unit parts that were equal area without relying on congruence. Such flexibility in the partitioning of a whole into unit parts exhibited by Mrs. Baker’s upper quartile students and Mr. Kent’s students suggests the arbitrariness of partitioning that Freudenthal (1983) attributes to the measure construct. The different types of partitioning exhibited by Mr. Kent’s and Mrs. Baker’s students suggest implicit influences of the different instructional themes with respect to partitioning in each classroom.

Students’ use of the complement to form a whole given a fractional amount and their use of unit part size to order fraction symbols in problem situations also suggest relationships between instruction and students’ development of the measurement construct. The complement-of-a-fractional-amount strategy is related to informal addition, which is described by both Kieren (1980) and Freudenthal as arising from the measurement construct. Mrs. Baker’s students were more likely than Mr. Kent’s students
to use the complement, suggesting an explicit influence of Mrs. Baker’s instruction theme of emphasizing complements of fractional amounts. However, informal addition may have played a role in many of Mr. Kent’s students recognizing and using the knowledge that 1/4 is the same as 2/8, a fact reinforced in his instruction on informal fraction addition—a topic that Mr. Kent addressed and that Mrs. Baker did not. The unit part size strategy for ordering fractions is related to measurement in that partitions by different natural numbers (denominators) produce unit parts of differing magnitude (Freudenthal, 1983). Mr. Kent’s students were more likely to employ the unit part size strategy in ordering fractions than Mrs. Baker’s students, suggesting that their added practice working with representations of fractions of differing magnitude had an implicit influence on their reasoning about fraction order.

The cases suggest interrelationships between students’ instructional experiences and their development of quotient constructs (partitive and quotitive) (Behr, Harel, Post, & Lesh, 1992; Kieren, 1980) related to their emergent knowledge of rational numbers through problem situations that describe fair sharing situations. The cross-case analysis shows that fair sharing was an instructional theme in Mr. Kent’s class and not in Mrs. Baker’s class. It also shows that Mr. Kent’s students were more successful in their solutions of the fair sharing problems than Mrs. Baker’s students. However, students in both classes primarily used partitive schemes of division in their attempts to solve the problems. Of the two models of partitive division (Streefland, 1993), students preferred to partition each whole by the number of people sharing and giving each sharer one piece of each whole. However, some students in Mrs. Baker’s class began with an arbitrary
partition, giving each one share until all have received one, then partition the remaining amount with a smaller partition and doling out each until all have received one of the second size, and continue this process until all is exhausted. Some of Mr. Kent’s students appeared to use quotitive partitioning (Streefland, 1993)—partition across all wholes so that each partition represents one person’s share. However, as described by Mr. Kent, some students whose solutions appeared to exemplify a quotitive scheme may have actually used partitive reasoning as in the first model and moved the pieces from each whole to one of the wholes.

The cases suggest interrelationships between students’ instructional experiences and their development of ratio constructs (Kieren, 1980) through problems situations in which two quantities, which may be in different measure spaces, are compared. Students’ ratio reasoning had viable and non-viable aspects. The part-whole construct as a special case of the ratio construct was particularly encouraged through Mrs. Baker’s “part-to-total” instructional theme and the models that she used to reinforce it. It may have contributed: (a) implicitly to her students’ use of a non-viable solution based on the ratio of circle pieces of one color to circle pieces of another rather than a unit part partition of the circle and (b) explicitly to her students’ viable solutions and greater success in modeling fractional amounts with discrete quantity models. Problem situations in which quantities were compared (fair sharing and mixture strength) also provided a venue for students’ application of ratio reasoning (number of items to number of sharers and amount of solvent to amount of solution, respectively), but no interrelationships between
students’ instructional experiences and their ratio reasoning were apparent since neither class provided instruction in comparing such quantities.

The cases suggest interrelationships between students’ instructional experiences and their development of operator constructs (duplicator/partition reducer, stretcher/shrinker, and multiplier/divisor) (Behr, Harel, Post, & Lesh, 1992, 1993; Kieren, 1980) through problem situations that involve the composition/decomposition of wholes and fractional amounts using discrete quantity representations. Students, who appeared to use an operator construct in their solutions, used types of the multiplier/divisor construct exclusively. Instruction in both classes emphasized the multiplicative relationships between unit parts, fractional amounts, and wholes in discrete quantity representations and little or no instructional activities in either class appeared to use the other operator constructs in defining fractions.

Levels Three and Four: Scalar and Function Constructs and Formal Equivalence

With respect to constructs and schema at levels three and four (Kieren, 1988, 1993), the level three constructs (scalar and function) and one level four construct (formal equivalence) likely developed through the common instructional theme of equivalent fractions in each class and in the unique instructional themes of part-to-total from Mrs. Baker’s class and of fair sharing in Mr. Kent’s class. Mrs. Baker’s instructional theme of part-to-total relationships reinforced scalar relationships between fractions. Fractions were conceived as part-wholes in the same measure space and isomorphic scalar operators related their numerators (parts) and denominators (wholes). Mr. Kent also used instructional activities that similarly encouraged students to recognize that equivalent
fractions had isomorphic scalar relationships between pairs of numerators and denominators. Although Mr. Kent’s fair sharing problems provided a context in which the function construct might arise, the problems did not necessitate it. Shares could be conceived as rates of number of items shared and number of sharers. However, rather than interpret a share as a quotient (amount per person), students could interpret a share simply as a quantity (amount of item). The development of formal equivalence in both classes was addressed through instructional activities that introduced extended proportions of equivalent fractions (e.g., \( \frac{a}{b} = \frac{2a}{2b} = \frac{3a}{3b} = \ldots = \frac{na}{nb} \)) and the common denominator algorithm. Both encouraged students to look for patterns of relationship between pairs of fractions in the extended proportion. The patterns that students recognized were based on scalar relationships. Students recognized the isomorphic relationships between numerators and denominators (the same operator related the two numerators and the two denominators). Each of the processes rests on scalar relationships. Hence, the development of the formal equivalence construct of level four was addressed via the scalar construct of level three.

The cases suggest interrelationships between students’ instructional experiences and their development of level three and four constructs through problem situations that had an underlying assumption of proportionality: equivalence, fair sharing, and comparison or ordering. Across all types of problems, students from both classes primarily used a scalar approach to interpret proportionality or equivalence, as it was common to instruction in both classes. Students’ approaches that used “doubling” were essentially isomorphisms of measure, where the same scalar operator (two) related the
numerators and denominators. Students’ approaches that used a potential infinite process for identifying equivalents of a fraction such as increasing the unit part size in discrete models suggest the development of the scalar construct rather than the function construct as a precursor to formal equivalence. Only a few students from both classes expressed solution strategies that used rates in the comparison and mixture situations, suggesting initial development of function relationships. Only a few students, slightly more students from Mrs. Baker’s class than from Mr. Kent’s class, used the common denominator method for solving ordering, betweenness, or equivalence problems although the common denominator method was a common feature of the instruction of formal equivalence in both classes. Ordering and betweenness problems per se were not a feature of either class.

*Level Five: Additive Group and Multiplicative Group*

With respect to constructs and schema at level five (Kieren, 1988, 1993), only constructs related to the additive group were addressed through the common instructional theme of addition and subtraction of fractions. In both classes instruction on addition and subtraction of fractions with unlike denominators utilized prior instruction for finding equivalent fractions (common denominator algorithm). Both teachers instructed students to find equivalent fractions with common denominators before adding or subtracting the given rational numbers. Only Mr. Kent introduced addition of fractions via representational models suggesting the connection of the measure construct (Kieren, 1993; Vergnaud, 1988).
The cases suggest interrelationships between students’ instructional experiences and their development of addition and subtraction of fractions. The emphasis on the common denominator algorithm for adding and subtracting fractions in both classes and the lack of significant difference in students’ success in solving addition and subtraction problems between the two classes suggests a relationship. Students from both classes who were successful used their knowledge of equivalent fractions to find fractions with common denominators for adding or subtracting as would be expected from the nature of instruction in both classes. Differences between students’ solution processes also suggest relationships between instruction and students’ developing knowledge of addition and subtraction of fractions. For example, some of Mr. Kent’s students could provide accurate visual representations although they relied on finding equivalent fractions with common denominators and some of Mrs. Baker’s attempted to solve the problems using visual representations without success. There were differences in the solutions of those who used equivalent fractions with common denominators, successful or not. Explanations from some students from Mr. Kent’s class described “finding common denominators” as compared to explanations from some students from Mrs. Baker’s class described “doubling,” “multiplying,” or using Mrs. Baker’s “backwards Z” mnemonic. The differences in instructional emphases in the two classes that are reflected in the differences in students’ solutions support the observation that the measure construct from level two was more available to Mr. Kent’s students for developing addition and subtraction of fractions and the scalar construct from level three was Mrs. Baker’s students’ principal resource for developing addition and subtraction of fractions.
Summary: Interrelationships Between Students’ Emergent Content Knowledge and Instruction

In summary, the cases suggest interrelationships between students’ instructional experiences and their developing knowledge of rational numbers at each level of the model of personal rational number knowledge (Kieren, 1988, 1993). Students’ reasoning processes appeared more different when their instructional experiences differed and appeared more similar when their instructional experiences were similar.

Interpretations

The discussion of the theory in light of the cases suggests several interesting interpretations of the theory. For example, it suggests that the improvement of teacher’s personal content knowledge could be limited in its influence on students’ emergent content knowledge by two filtering intermediaries—teachers’ pedagogical content knowledge and students’ prior knowledge/predispositions. Changes in teachers’ personal content knowledge would be mediated by their pedagogical content knowledge. The case studies confirm that teachers’ personal content knowledge does interact with their pedagogical content knowledge defined as their instruction practices, including teachers’ organization of instruction for learning a topic, promotion of classroom discourse about a topic, and informal assessment of student understanding of a topic. The cases also illustrate that the differences in teachers’ pedagogical content knowledge and/or their beliefs about mathematics and mathematics teaching can mediate between their personal content knowledge and their instruction. Other factors such as the curriculum guide are
also observed to shape instruction. Thus, all play a role in shaping the instructional environment in which students’ engage.

The theory allows that a change in either personal or pedagogical content knowledge would not necessitate a commensurate change in the other. The diagram in Figure 7.1 illustrates a linkage between the two that is influential rather than direct. However, the cases are limited in their ability to confirm or disconfirm this possibility. In addition, the theory allows for the change in teacher’s pedagogical content knowledge via the influence of instructional experience. The cases do speak to this possibility through Mr. Kent’s instructional practices that allowed him to capture models of students’ understanding as it was emerging. The theory also suggests that the influence of the instructional environment on a teacher’s content knowledge is mediated by their pedagogical content knowledge. The case of Mr. Kent could be cited to confirm this aspect of the theory, but it does not disconfirm the potential for other avenues of influence between teachers’ personal content knowledge and their participation in the instructional environment.

The theory suggests that the influence of teachers’ personal and pedagogical content knowledge shapes the instructional environment, thereby influencing the development of student knowledge. The cases appear to confirm that teachers’ personal and pedagogical content knowledge can influence students’ emergent knowledge through the opportunity to learn such as the employment of fair sharing activities in Mr. Kent’s class and lack thereof in Mrs. Baker’s class and through instructional emphases such as
the backwards Z mnemonic in Mrs. Baker’s class and the measurement concept in Mr. Kent’s class.

The theory recognizes the important influence of students’ prior knowledge/predispositions on development of new knowledge. The cases suggest that students attempt to make sense of the actions/communication of the teacher in an instructional environment based on their prior knowledge/predispositions/experiences. Usually, students will develop models that are viable within the instructional environment. If a student does not have an adequate knowledge base to make viable models of a teacher’s actions/ communications, then they will make non-viable models. As students make sense of the instructional environment they may make non-viable models of emergent content knowledge based on teachers’ unintentional patterns of actions/communications that are insignificant or immaterial to the content being taught.

In defining the instructional environment as a negotiated meaning-making space, the theory recognizes that teaching/learning is not an act of transmitting knowledge from expert to novice. Rather, it is a negotiation between goal-oriented actors who are endeavoring to create models of meaning based on the actions/communications of their co-respondents. Each teacher and student endeavors to make models that are viable within the limited context of the particular instructional environment that they cohabit. However, the needs of the larger content-based community require that the models of both teacher and students be viable beyond the walls of the classroom. The theory recognizes that the larger learning community plays a limited role via the input of certain curricular expectations/decisions. However, the primary actors in the classroom are the
teacher and the students. The experiences outlined in the cases appear to confirm this aspect of the theory.

Implications for the Classroom

The cases and theory described in the study have implications for both the K-12 classroom and for the education of teachers.

K-12 Classroom

The cases suggest some implications for teaching rational numbers. Students in the middle grades appear to construct the equi-partitioning scheme for connected numbers and the fractional connected number sequence (the first rational number scheme) differentially. There will be more students in a middle grades class who will be working with the partitive unit fraction scheme and the partitive unit fraction sequence that precedes the construction of rational numbers. Some students may not have developed even the partitive unit fraction scheme. Students who have yet to construct equi-partitioning scheme for connected numbers will likely produce necessary errors (Steffe, 2001) based on their partitive unit fraction scheme. Middle grades teachers should engage their students in a variety of activities at level one of Kieren’s (1988, 1993) model of personal rational number knowledge (partitioning, quantitative equivalence, and unit forming) to promote the construction of the equi-partitioning scheme for connected numbers. They should recognize that when students produce errors that it is indicative of students’ having not constructed the equi-partitioning scheme for connected numbers. In an environment of discussion and inquiry, students who have constructed the equi-partitioning scheme for connected numbers will engage with those
who have not and provide for disequilibrium, thus, promoting the construction of the new scheme. It is not suggested that all students will. Kamii (1992) observed that with respect to the construction of conservation, the social environment provided a motivation for students who had the capacity to construct conservation to do so, but it did not result in all students being able to do so.

The use of a variety of models (continuous, discrete, standard, and non-standard) is suggested by the cases, since reference to one model over time tends to produce non-viable constructions. Mr. Kent used the geoboard almost exclusively in his initial fraction lessons and inadvertently encouraged non-viable schemes for the definition of the fractions 1/2, 1/4, 1/8, etc. The use of a variety of situations and activities that can promote students’ construction of level two of Kieren’s (1988, 1993) model of personal rational number knowledge (measure, quotient, operator, ratio) is also suggested by the cases. The differences between the constructs evident in the explanations of the students in the two classes appeared to be related to the types of activities available in their classes and the level two constructs that those activities elicited.

The cases also suggest that the scalar construct, level three of Kieren’s (1988, 1993) model of personal rational number knowledge, provides a by-pass to constructing a scheme for producing equivalent representations of fractions before the construction of rational numbers. Students use this by-pass before they construct rational numbers to produce quasi-viable solutions to addition and subtraction problems. The by-pass may explain the limitations of students’ understanding of the process since they are not really adding or subtracting rational numbers. As indicated in Kieren’s model, understanding of
addition and subtraction requires linkage to the measurement concept of rational numbers—what Vergnaud (1988) describes as “rational number as quantity” versus “rational number as operator.” Viable construction of the measurement concept related to rational numbers probably requires the construction of the equi-partitioning scheme for connected numbers. Meaningfully teaching addition and subtraction of fractions to students who have not constructed the equi-partitioning scheme for connected numbers would seem to be limited to known quantities—available on the fact level but not generalizable to any number.

The cases and theory suggest some implications for teaching mathematics generally. The cases suggest that teachers choose activities and examples that cover the scope of concepts and models that comprise/represent a topic. Offering instructional variety insures opportunity-to-learn since students have different prior knowledge/predispositions that can promote or discourage particular constructs related to a topic. Instructional variety also addresses the concern that students’ constructions will be viable for the topic and that quasi- or non-viable constructions that are model- or activity-specific will be discouraged or avoided. The cases also suggest that teachers should be aware that some constructs provide alternative routes to understanding a topic that may be quasi-viable and thus interfere with students’ construction of viable schemes with respect to a topic. This is not to say that alternative constructions must never occur, but that they should be taken into account when assessing students’ understanding of a topic. Finally, using a variety of instructional and assessment tasks provides more accurate assessments of students’ understanding of a topic.
The theory and cases suggest implications for teacher education. The comparison of the two cases demonstrates that teachers’ personal content knowledge interacts with the instructional environment that teachers’ shape. Teachers’ personal knowledge of a topic potentially influences the organization of their instruction, the promotion of discourse and the development of students’ syntactic knowledge of mathematics, and teachers’ assessment of students’ emergent mathematical knowledge. The cases suggest that attention should be given to teachers’ content, syntactic, and metacognitive knowledge of the mathematics that they will teach. Each of these aspects of teachers’ personal content knowledge has the potential to influence some aspect of their instruction.

The cases suggest that teachers’ pedagogical content knowledge is an important contributor to teachers’ instruction. Mr. Kent’s additional pedagogical experiences provided material for the creation of meaningful instructional activities. However, his inexperience with how the activities would or should interact with students’ emergent knowledge may have contributed to an organization of instruction that promoted non- or quasi-viable constructs even as they contributed to more diverse and interconnected concepts of rational number. This suggests that pedagogical instruction should address how students’ learn particular topics and how instructional activities may be organized to facilitate students’ attention to key concepts and to reduce students’ quasi- or non-viable constructions. Such attention to teachers’ pedagogical content knowledge is suggested by Ma’s (1999) observation of the contribution of learning packets in Chinese teaching to
teachers’ profound understanding of fundamental mathematics (PUFM) and by Simon’s (2002) call for finding key development understandings in mathematics.

The cases suggest that teachers’ personal content knowledge, their pedagogical content knowledge, and their beliefs about mathematics and teaching mathematics interact in teachers’ construction of knowledge for instruction and their implementation of instruction. For this reason, pedagogical instruction must take into account these competing and interlocking aspects of teachers’ instructional personas. In summary, teachers’ personal content knowledge interrelated with their pedagogical content knowledge and their beliefs about mathematics and mathematics teaching shapes the instructional environment in which students engage. Students’ emergent knowledge arises through their negotiations with their teachers in the instructional environment. Students’ prior constructions and predispositions are the media through which students interpret the instructional environment and their teachers’ communications and expectations. Increasing the effectiveness of the instructional environment requires an investment in shaping teachers’ personal content knowledge, their pedagogical content knowledge, and their beliefs about mathematics and teaching mathematics.

Implications for Future Research

The cases and theory do not complete a program of research. They raise questions that require future research and the integration of current research. The questions include: questions specific to this dissertation; questions about the construction of rational number knowledge and the related pedagogical content knowledge with respect to rational numbers; questions about the interrelationships between teachers’ personal content
knowledge, teachers’ pedagogical content knowledge, and teachers’ beliefs about mathematics and mathematics teaching; and questions about what changes to teacher knowledge could have the greatest influence for positive changes in students’ emergent knowledge.

Questions specific to the dissertation include how well do the cases illustrate the interrelationships between teachers’ personal content knowledge, their pedagogical content knowledge, and their beliefs about mathematics and mathematics teaching and their students’ emergent knowledge? If other pairs of teachers were observed, how would they differ from the current pair and what would that say about the interrelationships observed herein? In the present study, the teacher who has more sophisticated content knowledge as measured by the test, however, has the least teaching experience. What might be observed if the two teachers who were observed were from other categories based on the types suggested by the discussion in Chapter 2? Particularly, what might be observed if both teachers were similar in their knowledge of the topic but differed in their pedagogic styles and beliefs about mathematics and mathematics teaching? What might be observed if both teachers were similar in their pedagogic styles and beliefs about mathematics and mathematics teaching but differed in their knowledge of the topic?

Questions about the construction of rational number knowledge include what are the relationships between the construction of the measure construct, addition and subtraction of rational numbers, the function construct, the quotient construct and the construction of equi-partitioning scheme for connected numbers? Does the alternative construction of addition and subtraction of fractions through the use of the scalar
construct create non-viable schemes that interfere with constructions of subsequentrational number constructs and the solving of rational number related problems? Do those
students who are predisposed to view fractions as the concatenation of whole number
operations (scalar or operator constructs) and use that to construct their schemes for
fraction addition and subtraction subsequently construct the equi-partitioning scheme for
connected numbers? If yes, are they able to integrate their equi-partitioning scheme for
connected numbers with their previously developed algorithmic process for addition and
subtraction of rational numbers?

Questions about pedagogical content knowledge related to rational numbers
include both new ideas and the integration of current and past research on rational
numbers. When should rational number instruction begin and what should it include?
What are optimum activities for promoting the construction of the equi-partitioning
scheme for connected numbers? Are there activities that are not currently emphasized in
the curricula that students should do to construct pre-cursor schemes to the rational
number schemes such as the splitting operation (Confrey, 1994; Kieren, 1994; Steffe,
2001)? What is the optimum organization of rational number instruction to encourage
students’ viable constructions rather than quasi- or non-viable ones?

Questions about the interrelationships between teachers’ personal content
knowledge, teachers’ pedagogical content knowledge, and teachers’ beliefs about
mathematics and mathematics teaching include whether the interrelationships are
directional? Can changes be effectively made in one of the three and how might that
affect the others? Must all be addressed simultaneously for effective change to be made in any one?

Finally, what changes to teacher knowledge have the greatest potential to influence positive changes in students’ emergent knowledge? Which is more influential in creating an effective learning environment: teachers’ personal content knowledge, teachers’ pedagogical content knowledge, or teachers’ beliefs about mathematics and mathematics teaching? Embedded in the previous question is what are the most effective learning environments? What is an effective learning environment? For this researcher, an effective learning environment must be defined as one in which the most students have the greatest opportunity to construct the most viable schemes/concepts for any topic that they are required to learn.


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APPENDIX A

RATIONAL NUMBER TEST, MILLSAPS VERSION T
Rational Number Test
Millsaps Version T
1/3/96

2.

a) __________ is three fourths of some length. Draw the whole length below and explain why it is the whole.

b) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________
Rational Number Test
Millsaps T
8/2/05

3. Draw in the box to the right a set which has 3/4 as many circles as the set of circles in the box on the left.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
</table>

a) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness: ________

Rational Number Test
Millsaps T
8/2/05

4. Which of the following figures has 1/4 shaded?

a) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness: ________
5. Circle all the pictures that show 4/5.

6. For each picture below, write a fraction to show what part is shaded.

i. 
ii. 
iii. 
iv. 
v. 

a) For each diagram above describe how you determined your answer and explain your reasoning. Use the diagram where appropriate.

b) Provide alternative solution methods for solving each which are also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness.
7. For each picture below, write a fraction to show what part is shaded.

i.  

ii.  

iii.  

a) For each diagram above describe how you determined your answer and explain your reasoning. Use the diagram where appropriate.

b) Provide alternative solution methods for solving each which are also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness.
9. Give two fraction names for the shaded amount.

a) Describe how you solved each problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving each problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its appropriateness. ________
11. Give two fraction names for the shaded amount.

a) Describe how you solved each problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving the problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1=inaappropriate, 5=appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________

12. Circle the fractions that show what part of each circle below is grey:

i. $\frac{1}{4}$  $\frac{3}{5}$  $\frac{9}{10}$

ii. $\frac{1}{6}$  $\frac{1}{2}$  $\frac{2}{5}$

a) Describe how you determined an answer for each diagram and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving these problems which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1=inaappropriate, 5=appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________
13. For the following situation decide whether the people in group A or the people in group B get more pizza.

A

B

a) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 = inappropriate, 5 = appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness.
15. For the following situation decide whether the people in group A or the people in group B get more pizza.

A

B

a) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness.
17. Four people are going to share these two pizzas equally. Color in one person’s part.

Write a fraction that shows how much one person gets ________.

a) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________
19. Six people are going to share these five chocolate bars equally. Color in one person’s part.

Write a fraction that shows how much one person gets ________.

a) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________

20. Three people are going to share these four-sectioned chocolate bars equally. Color in one person’s part.

Write a fraction that shows how much one person gets ________.

a) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________
21. The following is a packaging machine. The diagram shows that when 12 widgets are put into the machine 4 packages come out.

Here is more information about this machine:

<table>
<thead>
<tr>
<th>Widgets In</th>
<th>Packages Out</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>5</td>
</tr>
<tr>
<td>27</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>18</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>31</td>
</tr>
</tbody>
</table>

Using the information give a fill in the following blanks:

<table>
<thead>
<tr>
<th>Widgets In</th>
<th>Packages Out</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td></td>
</tr>
<tr>
<td>96</td>
<td></td>
</tr>
<tr>
<td>81</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td></td>
</tr>
</tbody>
</table>

a) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriate
23. For the following diagram each dark colored glass represents chocolate syrup and each white colored glass represents milk. Circle the mixture that will have a stronger chocolate flavor: the mixture made using the glasses pictured in set A or the mixture made using the glasses pictured in set B.

a) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriate.
25. For each of the following explain how you decided where to locate the arrow.

i. Mark with an arrow (✓) where $\frac{3}{2}$ goes on the number line below.

ii. Mark with an arrow (✓) where $\frac{1}{2}$ goes on the number line below.

iii. Mark with an arrow (✓) where $\frac{1}{2}$ goes on the number line below.

iv. Mark with an arrow (✓) where $\frac{2}{3}$ goes on the number line below.

a) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________
27. Mark an X on the number line where 1 2/3 should be.

\[ \begin{array}{cccc}
0 & 1 & 2 & 3 \\
\end{array} \]

a) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning. Use the number line provided to illustrate your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________

28. Mark the following line to show 2 3/4 and explain how you made your decision.

\[ \begin{array}{cccc}
0 & 1 & 2 & 3 \\
\end{array} \]

a) Describe how you solved the problem and explain your reasoning. Use the diagram if appropriate.

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning. Use the number line provided to illustrate your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________
a) For each row of fractions below, determine which fraction is the greatest and which fraction is the least. Explain the reasons for each of your choices in the space provided. Use a diagram if appropriate.

i) \( \frac{2}{7}, \frac{5}{7}, \frac{4}{7} \)  
Greatest? _____  
Least? _____

ii) \( \frac{1}{8}, \frac{1}{7}, \frac{1}{6} \)  
Greatest? _____  
Least? _____

iii) \( \frac{6}{7}, \frac{8}{9}, \frac{7}{8} \)  
Greatest? _____  
Least? _____

iv) \( \frac{3}{7}, \frac{4}{9}, \frac{4}{3} \)  
Greatest? _____  
Least? _____

v) \( \frac{4}{7}, \frac{3}{8}, \frac{1}{2} \)  
Greatest? _____  
Least? _____

b) Provide an alternative solution method for solving this problem which is also valid and explain your reasoning.

b) Provide other valid solution methods for solving any of these and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________
a) Write one fraction that is the same as each fraction below and explain how you arrived at your solution. For example, \( \frac{1}{2} = \frac{2}{4} \).

i) \( \frac{2}{6} = \)

ii) \( \frac{1}{5} = \)

iii) \( \frac{12}{16} = \)

iv) \( \frac{7}{6} = \)

b) Provide other valid solution methods for solving any of these and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________
For each of the following write a fraction that is somewhere between the two given numbers and explain how you found your solution. Use a diagram if appropriate.

i) \( \frac{1}{3} \) \quad \quad \frac{3}{4}

ii) \( \frac{3}{5} \) \quad \quad \frac{4}{5}

iii) \( 0 \) \quad \quad \frac{1}{7}

b) Provide other valid solution methods for solving any of these and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________
35. Liana ate 3/8 of a small pizza. The next day she ate 1/4 of a small pizza. How much of a pizza did she eat altogether?

Rational Number Test
Miltaps Version T
8/205

36. Ann and Josie receive the same allowance. Josie spent 4/9 of hers on CDs. Ann spent 1/3 of her allowance on repairing her bicycle. Josie spent how much more of her allowance than Ann?

Rational Number Test
Miltaps Version T
8/205

325
37. Bert's father cuts a cake into 8 pieces. He is going to take three fourths of the cake to the party. How many pieces of cake will he take with him? ________

a) Describe how you found your solution and use a diagram if appropriate.

b) Describe an alternative method for solving the problem.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriate use: ________
iii. \[ \frac{1}{2} - \frac{3}{8} \]

Method

vi. \[ \frac{2}{3} - \frac{1}{2} \]

Method

b) Provide other valid solution methods for solving any of these and explain your reasoning.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is noted as inappropriate, please list the reasons for its inappropriateness.
Rational Number Test
Millsaps Version T

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39.
a) Circle the best estimate for the answer to $12/13 + 7/8$. Explain your reasoning.

i) 1

ii) 2

iii) 1/2

iv) 19

v) 21

b) Describe an alternative method for solving the problem.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________

40.
a) For each of the following estimate the answer by recording in the box the whole number closest to the answer. Explain your reasoning.

i) $3/8 + 5/12$

ii) $8/9 - 7/8$

b) Describe an alternative method for solving these problems.

c) Rate this problem as to its appropriateness for use with your students on a scale of 1 to 5 (1 inappropriate, 5 appropriate). If the problem is rated as inappropriate, please list the reasons for its inappropriateness. ________
APPENDIX B

RATIONAL NUMBER TEST, MILLSAPS VERSION S
1. If \( \square \) is one unit, what fraction is \( \square \)? ______. Explain why you think your answer is correct.

2. \( \square \) is two thirds of some length. Draw the whole length below and explain why it is the whole.

3. Draw in the box to the right a set which has \( 2/3 \) as many circles as the set of circles in the box on the left. Explain why you think your answer is correct.

4. Write a fraction to show what part is shaded. Explain why you think your answer is correct.
5. For each picture below, write a fraction to show what part is shaded. Choose one picture and explain why you think your answer is correct.

i. ______________________

ii. ______________________

iii. ______________________

6. Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.

8. Six people are going to share these five chocolate bars equally. Color in one person's part.

Write a fraction that shows how much one person gets ________. Explain why you think your answer is correct.
9. Each dark colored glass is chocolate syrup and each white colored glass is milk. Circle the mixture that has a stronger chocolate flavor: the mixture made using the glasses pictured in set A or the mixture made using the glasses pictured in set B. Circle both if the chocolate flavor is the same. Explain why you think your answer is correct.

A

B

11. a) Write one fraction that is the same as each fraction below. Explain why you think your answer is correct for one of the rows. For example, \( \frac{1}{2} = \frac{2}{4} \)
   i) \( \frac{1}{9} = \)
   ii) \( \frac{1}{7} = \)
   iii) \( \frac{8}{12} = \)

12. Name a fraction that is somewhere between the two given fractions. Explain why you think your answer is correct for one of the rows.
   i) \( \frac{1}{3} \) (______) \( \frac{3}{4} \) (______)
   ii) \( \frac{3}{5} \) (______) \( \frac{4}{3} \) (______)
   iii) \( \frac{6}{7} \) (______) \( \frac{7}{9} \) (______)
   iv) \( \frac{3}{7} \) (______) \( \frac{4}{5} \) (______)
   v) \( \frac{5}{9} \) (______) \( \frac{7}{5} \) (______)
   vi) \( \frac{1}{7} \) (______) \( \frac{1}{4} \) (______)

10. For each row of fractions below, which fraction is the greatest and which fraction is the least? Explain why you think your answer is correct for two of the rows.
   i) \( \frac{1}{8} \) \( \frac{1}{7} \) \( \frac{1}{6} \) Greatest? ______ Least? ______
   ii) \( \frac{6}{7} \) \( \frac{8}{9} \) \( \frac{7}{8} \) Greatest? ______ Least? ______
   iii) \( \frac{3}{7} \) \( \frac{4}{9} \) \( \frac{4}{5} \) Greatest? ______ Least? ______
13. Liana ate $\frac{2}{3}$ of a small pizza. The next day she ate $\frac{1}{6}$ of a small pizza. How much of a pizza did she eat altogether? _____ Explain why you think your answer is correct.

14. Ann and Josie receive the same allowance. Josie spent $\frac{3}{5}$ of hers on CDs. Ann spent $\frac{1}{10}$ of her allowance on repairing her bicycle. Josie spent how much more of her allowance than Ann? Explain why you think your answer is correct.
APPENDIX C

FORMAL INTERVIEW GUIDE
Formal Interview Guide

I am going to ask you some questions about fractions. I am very interested in how you come up with the answers so it is important for you to tell me what you are thinking about. The interview will not be graded so you do not have to worry about wrong answers. Are you ready?

1. Provide a set of fraction circles.
   (A) Use the fraction circles to show the fraction 3/8.

2. Display 16 tiles without counting or telling the child how many there are.
   (A) Say: You can arrange the tiles any way you want to show me the fraction 3/8.

   (B) Explain what you were thinking in order to solve this problem

   (B) How do you know this is 3/8?

   Please draw a picture on this sheet to show what you did.

   (C) Show me 3/8 using a different number of tiles. How are the two ways of using tiles alike?

   (C) If you didn’t have these pieces [if they used 1/8 pieces in the previous problem], could you use other fraction circle pieces to show 3/8? [If yes]

   How?

   (D) How is this way of showing 3/8 like using the fraction circle pieces? How are they different?

   [If they do] Explain how your two ways of showing 3/8 are alike and different.
3. Display a $4 \times 4$ geoboard without counting or asking the child how many grids there are.

(A) If we count this area on the geoboard (bounced a rubber band around the outside pegs) as one whole, show the fraction $\frac{3}{8}$ using this rubber band.

(B) Explain what you were thinking in order to solve this problem.

What fraction of a pizza is one person's part? Explain how you knew you were correct.

4. Three people are going to share these two pizzas equally. Color in person's part.

Show and read the statement:

If the circle is complete, how many cubes must the whole be when it is finished?

5. (A) Provide notecards labeled with phrases and ask students to put them on the geoboard to solve the problem. Ask students to talk about how they solve the problem.

(B) If correct, repeat changing the data: 8 cubes, 2 cakes.
6. Show the $1/8$ fraction piece.

Say: This is $1/6$ of my unit. With your fraction circles show me the unit. Talk aloud as you solve the problem explaining each step. (Record answer.)

[If correct change data: $1/4$ piece is $2/3$; find the unit.]

7. Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.

8. Do the people in group A or the people in group B get more pizza or do both groups get the same amount? Explain why you think your answer is correct.

9. Show and read this story to the student:

Sally ate $2/3$ of a pizza for dinner. The next morning she ate another $1/6$ of a pizza.

(A) Say: Without working out the exact answer, give me an estimate of How much pizza she ate altogether that is reasonable. (If you needed provide clues: Is the answer $>1/2$ or $<1/2$? Is the answer $>1$ or $<1$?)

(B) Say: Tell me what you were thinking to reach this estimate.

(C) Say: Using fraction circles, act out how you would find the exact answer. Talk aloud as you solve the problem.

(D) If the student was successful, ask the student to record each step with the fraction circles with symbols.

10. Read this story to the student:

Josie and Ali went skateboarding down their block together. Ali skated $6/10$ of the way down the block before he stopped. Josie skated $4/5$ of the way down the block before she stopped.

(A) Say: Without working out the exact answer, give me an estimate of How much further down the block Josie skated than Ali that is reasonable. (If you needed provide clues: Is the answer $>1/2$ or $<1/2$? Is the answer $>1$ or $<1$?)

(B) Say: Tell me what you were thinking to reach this estimate.

(C) Say: Using fraction circles, act out how you would find the exact answer. Talk aloud as you solve the problem.

(D) If the student was successful, ask the student to record each step with the fraction circles with symbols.
11. Read this story to the student and present on card:

Jon and Lara each ordered a small pizza at Dominoes. Jon ate 5/8 of his pizza and Lara ate 2/6 of her pizza. Did they eat the same amount, or did one eat less?

Explain your reasoning. [ex: Do you imagine or picture something in your mind to help you tell which is less?]

12. Read this story to the student and present on card:

Mark and William both had bags of M&M peanut candies. The bags held the same number of candies. Mark ate 2/5 of his bag. William ate 3/4 of his bag. Did they eat the same amount or did one eat less?

Explain your reasoning. [ex: Do you imagine or picture something in your mind to help you tell which is less?]

13. Read this story to the student and present on card:

Mark and Jenny walk home from school. Mark walks 3/7 of a mile. Jenny walks 6/9 of a mile. Do they walk the same amount or does one walk less?

Explain your reasoning. Did you picture anything in your mind as you thought about these fractions?

14. Read this story to the student and present on card:

Mary and Jose both have some money to spend. Mary spends 1/4 of hers and Jose spends 1/4 of his. Is it possible that Mary and Jose spent the same amount of money?

Tell me what you are thinking.

[If the answer is yes] Is it possible that they spent different amounts of money?
TEACHER INTERVIEW GUIDE

Here are the tasks that I gave the students during the interviews and I would like to go through them with you to get your input about them.

• How would you do this task?
• How would you expect your students to do this task?
• How would you teach your students to do this?
• Have you done something similar to this task in class? Describe it.
• Do you know or are you aware of any opportunities that any of your students may have had to do a task like this in prior grades?

I would like to ask you some other more general questions if that is all right with you.

What are some things that I should remember about your class or particular students as I analyze the data?

• Did you cover what you wanted in your fraction unit?

What curriculum are you following? Do you have a copy I could use? Is there a district mathematics curriculum?

• What are the most important ideas about fractions you hope your students will be taking with them to sixth grade?