THE PROCESSES OF LEARNING IN A COMPUTER ALGEBRA SYSTEM (CAS) ENVIRONMENT FOR COLLEGE STUDENTS LEARNING CALCULUS

DISSERTATION

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ABSTRACT

This study is a qualitative case study focusing on the question "What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus?" The study is designed to research the impact on student learning of new and revolutionary software available for mathematics education.

Research in CAS has focused on outcome-based studies and more philosophical work on the curriculum and assessment implications of CAS. Missing are studies which look at what exactly happens when students try to learn mathematics in a CAS environment: What does CAS learning look like on a day-to-day basis? and What does CAS learning look like across the period of a ten-week course? The importance of investigating these processes lies in understanding the impact of CAS on learning so that an appropriate balance can be found in the integration of technology into teaching ensuring that the merits of traditional methods are not lost and that the merits and demerits of CAS use are fully understood.

The primary data for the study consists of audio and video of tape of a group of students in a college course learning calculus using CAS software. This data was supplemented by interviews with the students as well as analysis of the students' homework and tests. Analysis of the data via Rotman Model of
Mathematical Understanding provides a lens through which to view the place of technology in the CAS classroom and through which to view the journey of each student across the quarter. The Pirie-Kieren Model for the Growth of Mathematical Understanding is then mobilised as a lens through which to examine specific learning episodes as they occur in the classroom.

As well as providing examples of student learning, the broader conclusions of the study are (i) that the framing and introduction of technology at the beginning of an instruction period impacts crucially on student behaviour and use of technology throughout that period and (ii) that while students will naturally experiment in a CAS environment intervention is probably required for them to develop sophistication in their experimental behaviour and strategies.
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**FIELDS OF STUDY**

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CHAPTER 1

INTRODUCTION AND THEORETICAL FRAMEWORKS

Computer Algebra Systems (CAS) are software which provide advanced calculating technology which allows the user to perform algebraic manipulations such as factoring, solving equations, multiplying polynomials, taking derivatives, calculating limits etc. There are several computer-based CASs such as Maple, Mathematica, and DERIVE as well as calculator-based versions built into the TI-89, TI-92, and Voyage 200 calculators. Since its introduction in education empirical studies in the use of CAS have been primarily concerned with comparing summative achievement between control groups taught without CAS and an experimental group taught with CAS. An aspect under-researched is the effect of learning in a CAS environment on student’s conceptual development, as opposed to their outcome achievement. It is important to understand the nature of conceptual development in a CAS environment so that appropriate use and integration of the technology can be effected.

This study is designed to research the impact of CAS-enabled technology on how students approach the learning of calculus. The particular CAS is a
computer program called Mathematica and the students are college students taking their first college course in calculus. The study took place in a computer lab-based class at a large Midwestern university and focused on three students who worked together as a group for the duration of the class. The study will consist of a set of qualitative case studies focusing on the question: “What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus?” The processes of the students’ learning is analysed both from the point of view of the students as a group and from the point of view of the three students as individuals. This set of case studies aims to provide insight into how students work on a day-to-day basis in a technology-rich environment and to provide insight into the place of technology in a mathematics classroom.

Background to the Study

Calculating technology in mathematics has recently taken the fourth step in an evolution which started with four-function calculators. Following the four-function machines, computer-based plotters and scientific calculators developed in parallel and integrated in the form handheld graphing calculators. Now we have calculators and computers with CAS capability. The advent of CAS-enabled technology, which allows a great deal of the problems in a standard algebra or calculus text book to be solved at the push of a few buttons, is a quantum leap in technology and has generated a great deal of debate in the mathematics education community as to whether this is one of the most exciting or most frightening developments in the history of mathematics education (Kutzler, 2000;
Waits & Demana, 1999). For centuries now the emphasis in algebra and calculus curricula has been on the development of technical skills through practicing algorithms: the overarching aim has been to make students into efficient machines for adding algebraic fractions, factoring quadratic expressions, solving equations, differentiating by various rules, etc. Machines with CAS capability automatise the most common algorithms taught in school and early college mathematics, and have brought into focus fundamental questions about the purpose, curriculum content, and assessment of mathematics in schools and colleges. Is the content of mathematics courses outdated? Does the use of technology suggest different teaching approaches? What is the place of paper-and-pencil skills in a modern mathematics curriculum? etc.

Research on CAS has developed in two main strands. The first strand has concentrated on showing the effectiveness of technology in supporting the learning of specific topics (e.g. solving equations, differentiation, integration, application in optimisation, etc.) most of which are part of a traditional curriculum (Judson, 1990; Mayes, 1995; Palmiter, 1991; Runde, 1997). The second strand has examined technology-enhanced curriculum design suggesting new topics (cryptography, chaos theory, etc.), changes to the order in which topics are introduced and assessments which emphasise problem solving ability over technical skill in implementing algorithms (Brown, 2001; Drijvers, 1998; Heid, 1988; Heid 1997; Herget et al, 2000; Kokol-Voljc 1999a, 1999b; McCrae & Flynn, 2001; McCrae and Stacey, 2000).

The area least researched since the introduction of CAS is the precise nature of student learning in a CAS environment and the merits and demerits of
such an environment relative to more traditional environments. Most of the research cited above has emphasised comparisons on performance in summative assessment between a CAS group and a non-CAS control group. The aim of the current project is to examine the nature of student learning on a day-to-day basis during a college-level mathematics course in which CAS-enabled technology, Mathematica, is at the core of the course design. My approach uses a particular theoretical model as a means for understanding the place of technology in the mathematics classroom. With this understanding in place I then analyse particular learning episodes to capture students’ activities, approaches and conceptual development as they learn calculus. These episodes, while focusing on particular moments in the class provide exemplars of student behaviours and approaches in the CAS environment. Both the macro and micro analyses are intended to provide a step to filling the gap in the literature outlined above.

Specifically the overarching research question for the project is:
“What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus?”

Sub-questions addressed are:
• What strategies do students adopt in a CAS environment?
• How do students negotiate between different mathematical representations in a CAS environment?
• What are students’ perceptions of the role of technology in their learning?
• What is the students’ relationship to Mathematica?
• What is the role of experimentation in mathematical learning in a CAS environment?
How do students approach formalisation (in the sense explained in the Pirie-Kieren model below) of mathematics in a CAS environment?

What is the effect of learning in a CAS environment on students’ conceptions of mathematics, as a subject?

What is the relationship between the pedagogy in the classroom and the use of technology?

Theoretical Frameworks

The relationship between technology, mathematics, and mathematics education has a long history. For centuries, many mathematical techniques depended entirely on the carrying out of calculations now automated by various technologies with the consequence that technical competence in calculation, or procedure, has been seen as intrinsic to the development of mathematical knowledge. However, from the wide availability of the earliest four function calculators in school, a dichotomy has arisen between the tools considered proper for mathematicians to use in their work and the tools considered proper for students to use in their learning (Waits & Demana, 1999). Computers and calculators have been used in various ways over the years to facilitate learning, but the designers and developers of the most innovative technologies have sought to give learners access to mathematical objects (numbers, variables, lines, shapes, etc.) for the purposes of creating learning environments in which learners can operate mathematically. It can be argued that, technology in general, and CAS in particular, can allow students a different access to mathematics and mathematical ideas than is possible in other environments. Furthermore,
technology can provide them with immediate feedback on their operations with those objects. As will be shown below, technology also can be a conduit for the introduction of experimentation into the learning of school mathematics.

In the rest of this chapter two theoretical frameworks will be introduced. The Rotman Model of Mathematical Reasoning (1993, 1995) is presented as a macro-framework for understanding the place of technology in the learning of mathematics. This framework will be useful for addressing the question of the effect of technology on learning by positioning technology in the activity of mathematical reasoning. The framework will provide a lens for interpreting the separate relationships that students have with mathematics qua mathematics and the technology they are using to learn mathematics. The framework is also useful in addressing questions of students’ conceptions of mathematical objects and mathematics as a subject. The Pirie-Kieren Model of Mathematical Understanding will then be introduced as a micro-framework and as a lens through which to interpret and analyse specific learning episodes as they take place in the classroom. This frame for the analysis of learning episodes will help in providing existence proofs of certain processes of learning as they manifest themselves in a CAS environment, as well as answering questions about formalisation of mathematics in a CAS environment. The two frameworks together will provide a vehicle for understanding learners’ mathematical activity, mathematical reasoning and mathematical development in a CAS environment across a period of time.
The Rotman Model of Mathematical Reasoning

Drawing on the work of C. S. Pierce, Rotman’s (1993, 1995) semiotic reading of mathematics discourse constitutes mathematical reasoning as the unison of three agencies: the Subject, the Person, and the Agent (Fig. 1.1).

Figure 1.1: The Rotman Model of Mathematical Reasoning

Rotman’s (1993, 1995) theoretical model is concerned with ontological questions about mathematical objects and processes. Its principal value for my study is its tripartite nature with the student, the mathematics, and the technology at the vertices of the triangle. Nevertheless, it is worth looking in some detail at what Rotman has to say because of his understanding of agency in mathematics and where that agency resides. In his model the Subject is “the agency [which] reads/writes mathematical texts and has access to all and only those linguistic means allowed by the Code” (p.396), the Code being, essentially, mathematics as sanctioned by the mathematical community. The Person is the agency who works within the Metacode “the entire matrix of unrigorous mathematical
procedures” (p.396), such as stories, ideograms and pictures. Finally the Agent is the agency who “acts on mathematical objects in a purely formal way” (p.397), i.e. with a Virtual Code. Mathematical reasoning in this formulation is considered to arise from the interplay between a learner, mathematics as a subject, and a disembodied agent who carries out mathematical procedures accurately and without prejudice. For the purposes of this study the disembodied agent is the CAS. For Rotman, mathematical reasoning is “an irreducibly tripartite activity in which the Person … observes the Subject … imagining a proxy – the Agent … - of him/herself” (p.397), and is persuaded by the closeness of the Subject and the Agent of the validity of the mathematical activity. For this project the CAS will be interpreted as the Agent of mathematics with which students interact to gain access to calculus. This adaptation of the Rotman model is seen in Figure 1.2:

![Figure 1.2: The Adapted Rotman Model of Mathematical Reasoning](image-url)
Thus one avenue for the development of mathematical reasoning could be engaged in creating or facilitating intellectual space where the learner can experiment and play within the realm of the Metacode i.e. the “stories, motives, pictures, diagrams, and other so-called heuristics” (p.396), through which the learner gains access to the Code but with reference to the Virtual Code i.e. the acting on of mathematical objects in a formal way.

One of the goals of this project is to show that technology can furnish just such intellectual space in mathematics education. It is argued that CAS allows students to take experimental steps in working with mathematical objects. Which is to say that CAS provides a Virtual Code (Mathematica) allowing students to act experimentally and observe the consequences of those actions within the Virtual Code (Mathematica) to examine, or negotiate, a relationship with the Code (Calculus) itself. The following example from algebra illustrates the process: A student is attempting to solve the equation $3x - 5 = x + 9$ using a CAS-enabled machine. One approach to solving equations involves a student choosing an equivalence transformation (e.g. adding the same number to both sides) and then applying that transformation before choosing another equivalence transformation and so on until the equation is of the form $x = k$ (a number). In this example the student might choose the transformation “add 5 to both sides” and the CAS, given appropriate input then applies this transformation accurately (something students often have trouble doing) and the equation becomes $3x = x + 14$. The transformation “subtract x from both sides” further transforms the equation to $2x = 14$. Both of these transformations are deemed “good” in that they simplify the equation. A common error for students to make at this stage is,
in an attempt to “get rid of the 2,” to choose the transformation “subtract 2 from each side.” Using the CAS to apply this transformation results in the equation $2x - 2 = 12$. To use the terminology of Rotman’s model and the Adapted model, the Virtual Code (Mathematica), provided by the CAS, is at odds with students’ experience of the Code (Algebra) because the equation was not simplified and has not taken the expected form. The CAS has, of course, applied the transformation correctly by the rules of the algebra Code [one might use the term “microworld” (see below) here] and the student must continue to think about alternative transformations to achieve the desired result. It is argued that when used as in this example the CAS can allow students to experiment fruitfully with mathematical objects, without being hampered by a lack of technical skill, as they try to achieve usable results.

The Rotman model as described above provides a mechanism in this project for understanding the effects of learning in a CAS environment on college students learning of calculus by providing a way for understanding the role of technology in learning. As we will see in Chapter 4, the Rotman model will also be used as an interpretive frame for understanding the role of experimentation in learning mathematics with CAS and for understanding the relationship learners have with the technology.

**Microworlds and Experimentation**

The role of CAS outlined above can be put in a broader context of technologies designed to facilitate mathematical learning. The “microworlds” of Papert’s LOGO project (1980) are designed to allow learners to make connections
between their physical experience of the world and the creation of mathematical objects such as squares or circles with the rules of the microworld representing the Code in Rotman’s formulation. Dynamic geometry systems such as Cabri and Geometer’s Sketchpad extend the microworld of LOGO to the microworld of Euclidean Geometry and CAS provides, among others, arithmetic, algebra, and calculus microworlds.

Bourbakian structuralism translated into mathematics education as what Freudenthal called the “anti-didactical inversion” of starting with definitions and proceeding: to examples in uses which may be purely mathematical or real-world applications (Cobb, Yackel, & Wood, 1993). Drawing on the work of Buchberger, Kutzler (2000) describes the following learning cycle: “Applying known algorithms produces examples. From the examples we observe properties, which are expressed as a conjecture. Proving the conjecture yields a theorem, i.e. guaranteed knowledge. The theorem’s algorithmically usable knowledge is implemented in a new algorithm. Then the algorithm is applied to new data, yielding new examples, which lead to new observations” (p. 7). Kutzler goes on to argue that the Bourbakian codification of mathematics which took primacy in mathematics discourse at the beginning of the twentieth century and became a guide for mathematics teaching omitted the phase of experimentation, since it is not needed for the codification, and developed the system of: definition -> theorem -> proof -> corollary -> next definition -> next theorem -> … and reduced Kutzler’s three phases of experimentation, exactification, and application to the latter two phases. This phase of experimentation is crucial to how mathematicians work to extend mathematical
theories, but school mathematics has grown very distant from any real world activity, and in particular, that of mathematicians.

*The Pirie-Kieren Model of the Growth of Mathematical Understanding*

Drawing philosophically on the ontological assumptions of Von Glaserfeld, Maturana and Varela, and Tomm, the Pirie-Kieren model of mathematical understanding is an attempt, arising from the work of Pirie (1988), to characterise understanding as an holistic, dynamic, non-linear, and transcendent process. This characterisation is in contrast with views of understanding as a series of discrete states or knowledge categories.

There are three main features of the Pirie-Kieren Model for the Growth of Mathematical Understanding (1991), (see diagram below). One is the eight levels that learners move through (although not in a linear, hierarchical manner) in developing understanding. The second is the, so-called, “don’t need” boundaries signified by the bold rings in the model which characterise the fact that learners who have reached certain levels will not need to revert to lower levels in order to develop understanding. The third critical feature is “folding back” which occurs when learners return to inner levels of the model but do not operate at the inner levels in the same way as when they first encountered them, their thinking now being informed by the outer levels. Each of these features will be discussed in detail below with examples from arithmetic, algebra, and calculus.

The model provides a frame for analysing particular learning episodes in order to understand, in general, the unique effects of learning in a CAS environment on learning of calculus, and, in particular, for understanding the
process of learning in such an environment. The episodes are both particular examples of student processes but are each representative examples of such processes as observed across the time of the study.

Figure 1.3: The Pirie-Kieren Model of the Growth of Mathematical Understanding

The eight levels of the model for the theory are Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring,
and Inventising. Pirie and Kieren (1991) stress the transcendent nature of understanding in stating that “there is no reductionism – the new concept is not reducible to the old. There is transcendence with compatibility” (p.2). They further stress that the levels do not constitute understanding in and of themselves but are named parts of the phenomenon.

**Primitive Knowing**: this level involves physical actions, symbols, graphs etc. It does not necessarily involve low-level mathematics but rather represents the starting point for growth in understanding, the initial state. In terms of fractions such knowing might involve the ability to divide a rectangle into equal parts. In the case of limits in calculus it may involve the recognising that a graph tends towards a certain value reading the graph from left to right.

**Image Making**: this is the level at which the learner begins to make distinctions between the actions at the previous level and do, for example “sharing problems.” In the case of fractions this may involve sharing three pizzas between four people and saying that each person gets three quarters of a pizza. In the case of limits, it may involve observing that \(1/\text{n}\) tends to zero as \text{n} gets large.

**Image Having**: at this level the learner is released from any particular physical action. Learners at this stage have images as mental objects. In arithmetic, they may say that “fractions show the amounts that come from dividing up,” or in the case of limits say that limits represent the process of a function approaching a fixed value.
Property Noticing: at this level learners begin to notice distinctions between, commonalities within, and connections across certain images. For example, learners may notice that equivalent fractions can be generated by multiplying the numerator and denominator of a given fraction by the same number. In the case of limits, learners may notice that $n/(n+1)$ tends to the same limit as $1/(1+(1/n))$. At this level the experimentation discussed in the previous section can come into play as learners experiment with different cases to investigate various properties of the mathematical objects with which they are working.

Formalising: this is the level at which learners abstract common qualities of the noticed properties at the previous level. At this level learners are capable of making statements involving “for all fractions” since fractions are, at this level, sets of numbers rather than physical objects. In the domain of functions could, similarly, talk about the set of all functions having a certain property. Among of the affordances of CAS technology, understood as an algebra or calculus microworld, are mathematical objects that can be used for formalising such as equations, graphs, and algebraic expressions. Thus the metamorphic transformations necessary to move through the levels of the Pirie-Kieran model can be facilitated through interaction and exploration with mathematical objects as in the example of solving the equation $3x = x + 14$ discussed above.

Observing: this is the level at which learners organise their mental structures and are aware of being aware of these structures. At this level students may be able to
observe that no smallest fraction is possible or may be able to establish epsilon values for a limit.

Structuring: this is the level at which learners may become aware of axiomatic structures and may be ready to make rigorous arguments in support of claims. At this level calculating limits is a logical procedure following from axiomatic claims.

Inventising: this is the level at which students may move to the creation of new objects or structures by breaking away from given structures in appropriate dimensions. This is the level at which Hamilton invented the quaternions.

It is important to remember that outer levels grow recursively from inner levels and that each of the levels can be considered as embedded within a given level. For example, for a learner at the inventing level all previous understanding can be considered as a new primitive knowing.

The second feature of the Pirie-Kieren model to be considered are the “don’t need” boundaries. This term represents the notion that once a learner has an image of a mathematical concept they no longer need to engage in primitive doing or image making in order to work with the mathematical concept. For example a person at the structuring level for fractions does not need to cut up pieces of paper or draw pictures in order to do arithmetic of fractions. The role of CAS in taking over the burden of calculations allows “don’t need” boundaries to be operationalised without a complete technical ability in calculation having
been developed. CAS might thus be useful in accelerating progress towards formalisation.

The third important feature of the Pirie-Kieren model is the idea of “folding back” which is concerned with the process of learners moving back to inner levels from outer levels in order to resolve particular cognitive dissonances. This return to an inner level is not identical to the learner’s original action at this level since it is now informed by some work at the outer level and the particular goal of the problem to be solved at the outer level. CAS can be useful in instigating cognitive dissonances since the output from a Mathematica command may be at odds with students expectations and force them to return to inner levels with the outside constraint of the computer’s feedback. An example of this was seen in the discussion of the Rotman model when 2 was subtracted from each side of the equation $2x = 14$ with the unexpected result $2x - 2 = 12$.

The following example to illustrate the “folding back” process is due to Pirie & Kieren (1991). Consider the formal process of being asked to add $\frac{1}{2}$ and $\frac{2}{3}$. The rule “find a common denominator and cross multiply to find the numerators” presents an action but does not manifest understanding of the process involved. A learner may think about the objects called fractions, remember that they came from cutting things up and may form the images:
A learner may then use his/her understanding of equivalence to form the following image:

![Image of one-half and one-third](image1)

![Image of three-sixths and four-sixths](image2)

The learner may then count the pieces to get $\frac{7}{6}$ or 1 and $\frac{1}{6}$.

The learner in this process folded back from formalisation ($\frac{1}{2} + \frac{2}{3}$) to image having to make sense of the problem and then used equivalence of fractions at the property noticing level. Note that the second step here is informed by the overarching goal of adding the two fractions so the inner level understanding is reconstructed as a foundation for outer level understanding.
This process is illustrated on the model as follows:

![Pirie-Kieren map of the problem](image)

**Figure 1.6: Pirie-Kieren map of the problem**

The recursive, non-linear, dynamic nature of mathematical understanding is clear in this process. Pirie and Kieren posit “formalising” as the first formal level of understanding which embeds less formal understanding in the form of primitive knowing, image making, image having, and property noticing.
According to Pirie and Kieren, students, having established formalisations, are in a “position to reflect on them, observing theorems through creating and proving them, and then structuring them into a mathematical system” (Pirie & Kieren, 1994, p.39). As stated above, CAS technology can provide mathematical objects for use in formalising such as equations, graphs, and algebraic expressions. It is possible that an acceleration towards formalisation can be effected using these affordances.

The Pirie-Kieren model for mathematical understanding can thus be used to examine the unique effects of learning in a CAS environment on students’ learning of calculus. Specific learning episodes can be examined with particular attention paid to the role technology is playing the process of understanding and the effect of CAS providing mathematical objects with which students can work in developing mathematical understanding.

A Word on Methodology

Methodological considerations of the study will be considered in detail in Chapter Three. As will be explained in detail in that chapter an approach grounded in the traditions of qualitative research provides the best opportunity of describing, analysing, and understanding the specific impact of learning in a CAS environment on student conceptual development. Philosophically, a hermeneutic approach to interpretation, an approach which aims to go beyond binary logic of this and that, and which does not seek objective truth but, rather, viability and reasonableness of interpretation was employed. The traditional
qualitative methods of observation, interview, grounded survey, and document analysis were employed.

Conclusion

The goal of this project is to answer the question “What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus?” Two theoretical frames are offered through which these effects can be interpreted. The Rotman model of mathematical reasoning provides a macro-framework for understanding the role of CAS in the activity of mathematical reasoning and a tool for understanding the operationalising of experimentation in mathematical learning in which CAS can have an important role. The Pirie-Kieren model of mathematical understanding provides a micro-framework through which particular processes can be analysed.
CHAPTER 2

REVIEW OF LITERATURE

In the previous chapter I argued that there is a gap in the research on the use of Computer Algebra Systems (CAS) and identified a need to research in greater detail the process of learning in a CAS environment. Research on the process of learning will help answer the question “What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus?” In this chapter I will present some background in the literature on the place of technology in education and present a detailed review of the literature on CAS research with the aim of making the gap in the literature clear and highlighting the importance of answering my research question.

Technology

A number of years ago in a newsletter article I wrote about the use of TI-92s (a hand-held calculator with Computer Algebra System (CAS) capability) I stated:
Thirty years ago mathematicians used computers that could perform thousands of calculations faster and more accurately than any person ever will so that they were free to concentrate on large scale patterns leading to Chaos Theory. Three hundred years ago John Napier developed a system of logarithms so that he didn’t have to spend hours doing tedious calculations and had more time to concentrate on modelling astronomical movements. Three thousand years ago the Greeks engraved the results of important calculations on stone to avoid having to repeat them endlessly and so leave them time to contemplate important problems in their geometry. The future has always been here.

This paragraph is full of deliberately rhetorical flourishes but it does highlight a central issue about the use of technology in general namely the difference between what Moursund (2002) has called first-order use of technology (amplification) and second-order use. Moursund argues that, in the first instance, we use technology to do what we can do already, but to do these things faster and more accurately. Indeed, it is a truism in theorising about technology that the only conception we can make of new technologies in the first instance is in terms of their first order, amplifying effects. The classic example of this is the car which was first thought of as a “horseless carriage.” Therefore, the car was first conceived of as a technology which would move people from A to B just as a carriage does, only faster. The second-order effects of cars are now well-known to us: complete transformation of the very conception of the city, dismantling of public transport, rampant pollution, raise in the age for legal drinking, etc., but were certainly not part of the design and conception of cars.
The three examples in the long quotation above are examples of first order usage in mathematics, and very successful examples they are too. However, mathematics and mathematics education are not the same thing and while the first-order effects of introducing technology in either field are, more or less the same, the second-order effects can be quite dramatically different.

For the reasons outlined above the benefits to mathematicians of calculating, or perhaps algorithm implementing, machines are obvious. This is to say that if someone is extremely competent at calculating and implementing algorithms then having a machine to do the work is a clear benefit. It is not, on the other hand, at all obvious that using such machines is a good way to learn mathematics. Indeed, one might argue that mathematics educators have, historically, been forced to accept technological devices in their work because they exist in the non-school world, and it is certainly not the case, until very recently, that any technology has been developed at the behest of educators and specifically with pedagogy in mind.

Calculators in Education

As discussed in Chapter 1 calculating technology in mathematics has evolved from simple four-function calculators to calculators and computers with CAS capability. CAS-enabled technology is so powerful that its use is highly controversial in the mathematics curriculum (Kutzler, 2000; Waits & Demana, 1999). Like other technologies the first order of understanding CAS-enabled technology is in terms of amplification since virtually every algorithm learned in school and early college mathematics can now be automatised. In fact the
amplification here is so enormous that it has brought into focus fundamental questions about the purpose, curriculum content, and assessment of mathematics in schools and colleges.

A common model for problem-solving in mathematics is as follows:

![Figure 2.1: A model for problem-solving in mathematics](image)

Kutzler (2000) estimates that as much as 75% of time in school mathematics is spent on the calculating step above. However, this is precisely the step at which technology is most effective since calculators and computers can perform algorithms more quickly, more accurately, and more efficiently than any human being. Moreover, it is precisely the step that CAS can perform for almost all of school mathematics. So, if a machine can do most of school mathematics more accurately and more efficiently than our students, what exactly is mathematics curriculum trying to achieve?
A partial answer to the question above is illuminated by a different question asked by Dugdale et al. (1995) about algebra, but applicable in other areas of mathematics:

To what extent do our positive feelings toward “intuition-building” software arise because this genre of software expresses or enriches an algebra that we already know, but which students (and perhaps teachers) do not know? Can we turn the edifice of our understanding on its head, expecting students to build rich understandings of something that is understood by us via a set of cognitions that our students will never have?

(p. 347)

In other words, are the technical procedures and algorithms that form so much of the focus of school mathematics necessary conduits to a full and deep understanding of mathematics? Or, if the process of learning mathematics is substantially altered by that learning taking place in a CAS environment what will be the scale and parameters of the resulting change in students mathematical development?

Research on Computer Algebra Systems

Since the introduction of CAS, research on its impact on mathematics education has developed in two main strands. The first has concentrated on showing the effectiveness of technology in supporting the learning of specific topics (e.g. solving equations, differentiation, integration, application in optimisation etc.) most of which are part of a traditional curriculum (Judson, 1990; Mayes, 1995; Palmiter, 1991; Runde, 1997). The second strand examines
technology-enhanced curriculum design suggesting new topics (cryptography, chaos theory, etc.), changes to the order in which topics are introduced and assessments which emphasise problem solving ability over technical skill (Brown, 2001; Drijvers, 1998; Heid, 1988; Heid 1997; Herget, Heugl, Kutzler & Lehmann, 2000; Kokol-Voljc 1999a, 1999b; McCrae & Flynn, 2001; McCrae & Stacey, 2000). To phrase it somewhat simplistically:

- **Strand I**: Leave the curriculum largely as is but use CAS to deliver the curriculum more effectively
- **Strand II**: Now that we have CAS technology what should a curriculum that assumes its use look like and how do we assess students with access to CAS?

A missing strand here is research studies which address the particular nature of student learning in a CAS environment. The research cited above does not look closely at what happens on a day-to-day basis in a classroom where students are learning using CAS. The lack of such research makes it hard to answer, admittedly difficult, but important research questions such as What are the merits and demerits of learning with CAS compared to more traditional methods? Do students approach learning differently in a CAS environment? Do students develop a different concept of mathematics if they learn in a CAS environment? How do theories of learning become operationalised in a CAS environment?

**Strand I: Research Studies**

This research has sought to establish the case for the use of CAS as an effective tool in supporting the teaching and learning of mathematics within a
“traditional content” paradigm. In one sense the use of CAS in these studies is in Moursund’s first-order in that the new technology is used to achieve old goals. On the other hand as will be seen in the discussion below the introduction of the new technology has precipitated a shift in approach to traditional goals with an emphasis on understanding rather than technical competence.

Exemplars of the Strand I approach include Palmiter (1991), Runde (1997), Mayes (1995), and Judson (1990). Palmiter compared the performance of university students (n=81) in an engineering integral calculus course who volunteered to be taught calculus using a Computer Algebra System to the performance of students taught using paper-and-pencil computations. Students in the traditional course covered the material in the usual 10-week period, whereas the experimental group, using a computer algebra system called MAC's SYmbolic MAnipulator (MACSYMA), covered the material in five weeks. The researcher investigated whether there was a significant difference in the two approaches in (a) knowledge of calculus concepts, (b) knowledge of calculus procedures, and (c) grades in subsequent calculus courses. At the conclusion of the courses identical conceptual and computational examinations were administered to both classes. Students who were taught calculus using the CAS had higher scores on a test of conceptual knowledge of calculus than the students taught by traditional methods. Students in the CAS class also had higher scores on a calculus computational exam using the CAS than the students in the traditional class using paper and pencil. Students in the experimental group performed at least as well as the traditional group in subsequent calculus courses.
Runde’s (1997) study was conducted to investigate whether TI-92 (a hand-held calculator with CAS capability) based instruction could improve students’ ability to solve word problems. Two intact sections of basic college algebra at a community college were used to form an experimental group (n=16) and a control group (n=22). Both groups were taught by the author, with the control group taught via heuristic instruction only and the experimental group taught via heuristic instruction along with the use of a TI-92. Data were gathered in the form of a pre-test and post-test as parallel forms of the same test. No students were allowed to use TI-92s in the pre-test, whereas students in the experimental group had ongoing access to TI-92s throughout the course and were allowed to use them in the post-test. All students improved in their ability to solve word problems between the pre-test and the post-test, with students in the experimental group showing significant improvement over those in the control group.

A common conclusion of studies using technology is that, compared, to a traditional classroom, students do no worse in traditional tests but gain in other areas such as problem solving ability, and ability to connect representations. Mayes (1995) discusses one such research project in which the author compared a traditional lecture-based college algebra course to an experimental algebra course. The experimental course stressed active student involvement and the use of the computer as a tool to explore mathematics. The purpose of the study was to determine the effect of a technology-rich environment on the ability of college algebra students to solve problems using analytical and graphical methods. Three experimental groups and four control groups of college algebra students
(n=137) were randomly formed from intact classes with students in the experimental group given the option of transferring to a traditional group. Data were gathered via a pre-test, a post-test, and interviews on attitude. The groups were found to be statistically equivalent at the beginning of the study. Subjects in the experimental group scored significantly higher than the control group on a final measure of inductive reasoning, visualisation, and problem solving while maintaining an equivalent level of manipulation and computation skills. The attitude of the subjects in the experimental group towards the use of the computer in learning mathematics declined significantly primarily due to the extra time spent in computer labs necessitated by their course.

Similarly, Judson (1990) describes an experimental study conducted using a Computer Algebra System (CAS) in teaching first-semester calculus. The purpose of the study was to determine whether or not the instruction of mainstream calculus could be improved by using CAS. Two experimental sections (n=40) were taught a course in which they were encouraged to concentrate on the concepts and applications of calculus, relegating computational tasks to the computer. One control section (n=20) was taught in the traditional manner. Students in the experimental classes completed ten homework assignments above and beyond the common assignments for all groups. Data were gathered in the form of scores on a common final examination and interviews with students in the experimental group. The final test, consisted of questions on skill acquisition (40%), conceptual understanding (30%), and applications (30%). Analysis of data showed no significant differences in achievement between the experimental groups and the control group in any of
these areas. Based on the interviews the author found that there were fewer students in favour of using CAS than there were opposed. This was principally because of the extra time involved in taking the CAS version of the course. Students with positive responses claimed that they had a richer understanding of topics as a result of using CAS.

Moving towards a second-order use of technology, other studies have emphasised a different pedagogical approach to be used in combination with CAS enabled calculators. Keller & Russell (1997) report on a study designed to examine the ability of students to solve problems symbolically if instructed in a technology-rich environment with an emphasis on a sense-making approach towards mathematics. The authors posit that the use of technology is a key component of reform calculus efforts because (a) technology allows students to spend more time on meta-cognitive behaviours, (b) technology provides an intermediate language in which the students can work, and (c) technology creates cognitive objects which students can flexibly manipulate. The study took place over two semesters during the first of which detailed information was collected from 9 sections with sizes of approximately 30 students. Students in one of these sections used a prototype TI-92 unit (which incorporates a computer algebra system) while the other eight sections used parallel materials for the TI-85 (which does not). In the second semester, students in two sections used the TI-92s including some students who had used them in the first semester. A two-hour final exam was designed to involve questions perceived as base-line expectations for any standard calculus student and was divided into two parts, the first of which consisted of ten short-response questions and the second part
of which consisted of seven long-response questions. The authors analysed, in
detail, not just the scores of the control and experimental groups in the
departmental exam but also made an analysis of error to establish whether
student errors, where they occurred, were either minor, major or there was an
uninterpretable response. For the purposes of analysis the test items were
classified into these three types, the goal of which was to factor questions by the
extent to which the TI-92 could have been responsible for the solution of the
problem. Students who learned calculus using the TI-92 and used it in the exam
scored significantly higher than the students in the control group in all three
aspects in both semesters. An item-by-item analysis indicated that almost
without exception, the students using the TI-92 produced more correct solution
methods than students in the control group. The authors state that the superior
performance of students in the experimental group on straightforward
computational problems was to be expected but provide a number of
explanations for the fact that students in the experimental group fared
significantly better even on problems where technology played less of a role.
Among these explanations are the suggestions (a) that confidence in the
computational power of the machine empowered the students and (b) that
students in the experimental group were able to spend more time problem
solving and less time computing than control students. The authors suggest that,
if true, the latter reason is highly significant in terms of the potential benefits of a
technology rich teaching environment. They also concluded that there is
evidence that an accelerated development of symbol sense has been fostered in
the technology-rich environment coupled with the sense-making approach.
Other studies which have worked primarily in the realm of first-order use in contexts as various as junior high school algebra classes and college business calculus classes have come to similar conclusions about the use of CAS to support a traditional curriculum: CAS students tend to do no worse than traditional students in traditional tests; there are benefits to CAS students in terms of reasoning ability, ability to make connections between representations, and attitude towards mathematics; CAS classes tend to facilitate multiple learning styles (Brown, 1998; Beckmann, Browning, Hart, & Irwin in Dunham, 2000; Hillel, Lee, Laborde, & Linchevski, 1992; Hollar & Norwood, 1999; Porzio, 1995; Skarke & Koenig, 1998; Soto-Johnson, 1998; Vlachos & Kehagis 2000).

Strand II: Research on Curriculum

The second of the aforementioned strands examines the question of what a technology-enhanced curriculum should look like and how it might be assessed. Curriculum models and assessment of curricula are the structures that schools and examining boards create to educate students. The two are deeply connected and research efforts are beginning to focus on the development of curriculum models in which the use of CAS is an integral part, not just of the practice and assessment of the curriculum, but of the conception of the curriculum. This has some far-reaching implications for mathematics, not just in a pedagogical, but also in an epistemological sense. With the primacy of algorithms and technical competence challenged, there are significant questions to be asked and answered which go beyond what mathematics our students
should learn to more fundamental questions of what mathematics is and how it should be represented to learners.

Curriculum Issues

Research on curriculum in the age of CAS is in its infancy, but already some trends are emerging. A conclusion that, perhaps, precedes all discussion on the matter is that of Leinbach et al. (1997), to wit “the nature, value, and importance of the mathematics curriculum will not be changed by the introduction of a CAS.” Their intent here is to propose that the fundamental goals of educators in teaching mathematics are independent of the means employed to achieve those goals. The authors, therefore, seem to suggest that the nature of mathematical proof is impervious to the development of human discourse. This is a particular philosophical position which is certainly not universal but it does highlight the fact that the emergence of CAS technology has been the catalyst for the re-ignition of the perennial debate about what mathematics we want our students to learn. The debate is not just between those who favour the use of CAS in teaching and learning and those who oppose it, but also within the “CAS camp” between those who value the relationship of CAS to applied mathematics and those who are more interested in its relationship to pure mathematics. A clear example of the latter can be found in the papers of Dubinsky et al. (1994) and Burn (1996). Dubinsky and his colleagues reported on the use of a computer software programme in the teaching of group theory and prompted basic disagreement from Burn about the fundamental concepts of
group theory and whether logically prior notions are necessarily psychological prerequisites for learning the subject.

Burn’s (1996) analysis, distinguishing between mathematics qua mathematics and the learning of mathematics, is echoed in the work of researchers who have looked at the sequencing of certain topics in a CAS environment (Cabezas & Roanes-Lozano, 1998; Heid, 1988). Heid, in particular, has looked at a reformulation of a traditional algebra curriculum to make Function the central concept; this in contradistinction to a traditional approach which stresses symbolic manipulation and solving equations. The fact that these latter processes are automated by a CAS is central to the debate of what topics are relevant for a modern mathematics curriculum and how their teaching should be approached. Heid (1997) takes a strongly constructivist approach to the teaching and learning of mathematics emphasising sense-making at the introduction of every new concept and the facilitation of the construction of student knowledge.

Curriculum content

One approach to imagining a CAS curriculum is to conduct a deep and thorough sifting of topics in order to decide what really has to stay for conceptual reasons and what remains extant only because of tradition. A very controversial addition to this debate is the paper arising out of the discussion between four prominent educators about what exactly they consider to be the essential skills in arithmetic and algebra in the CAS age (Herget et al., 2000). They imagine an environment of a technology-free examination and classify
problems into essentially two types: those questions which a student would be expected to answer in such a technology-free examination, i.e. without any calculator or computer, and those questions which would not be asked in such an examination. (There is, in fact, a third type in the classification, namely, those questions about which the authors have doubts as to where to classify them.) The authors stress the point that those questions that they would not ask in a technology-free examination they find inappropriate for any examination, even one where powerful technology is allowed. As they say “we would not ask these questions in a technology-supported exam either, because these questions appear useless as such, their best use might be to test how well a student can operate a calculator” (p. 4). The paper is intentionally controversial, and as the authors themselves say, “we deliberately wanted to be provocative and shake the mainstay of traditional mathematics teaching” (p. 5).

Another attempt to define essential skills emanates from Sweden and examines the question from the point of view of asking what paper-and-pencil skills should students who will go on to study mathematics at university have (Björk & Brolin in Usiskin, 1999). This latter paper is interesting inasmuch as it focuses on what Björk and Brolin deem to be the prerequisite skills necessary for mathematical training, as opposed to all students. This contribution, then, takes the mathematicians point of view in preference to that of the educational researcher.

McCrae and Stacey (2000) report on a large research group in Australia that is conducting a study to “investigate the changes that regular access to CAS calculators may have on senior secondary mathematics subjects and to explore
the feasibility of offering new subjects that use CAS extensively." Students in three volunteer schools have been taking part in the study which will result in formal assessment, probably in a CAS-active environment only. Their initial results suggest that basic algebraic skills will remain necessary and that there is much to be gained from the use of CAS in terms of student facility with multiple representations (Stacey & Ball, 2001). In a case study from the same project, researchers found that students who normally had little success with algebra were able to attempt a particular problem under investigation (Garner & Ball, 2001). The members of the Computer Algebra Systems – Curriculum Assessment and Teaching Project (CAS-CAT) project do not appear, as yet, to be taking strong ideological positions on the nature of curriculum and are, for the most part, engaged in an existence proof for the workability of a CAS-rich curriculum culminating in a high-stakes examination.

As explicitly mentioned in the case of Heid (1988) above, most researchers who have taken an interest in CAS adopt a constructivist approach. This is, perhaps, a reflection of the fact that, since mathematics education has established itself as a distinct discipline, curriculum design and reform has been led by researchers in that discipline to a greater degree, and not to the same degree by mathematicians as had been the case. (A post-structuralist influence which might reflect, for example, on the nature of power structures in a technology-enriched class setting, while emerging as a significant one in mathematics education generally, is not discernible in the literature on CAS). It is no coincidence then that the predominant influence on curriculum ideas in the age of CAS is, implicitly, that of Poincaré (a scientist’s mathematician) rather than that of
Hilbert (a mathematician’s mathematician); the difference being the adoption of simple intuitions, as opposed to simple axioms, as the logical starting point for mathematics (Gray, 1998). This has serious implications for future generations of mathematicians who may have a very different epistemological notion of mathematics from currently practising mathematicians.

Student learning

The issue of epistemology is manifest in the way that researchers have started to focus on the ways in which student learning has changed in the age of CAS (Berry, 1999; Drijvers, 2000; Heugl, Barzel & Furukawa, 1997; Heugl, 2000; Noss, 1999). Berry (1999) and Noss (1999) take the long view, with the former considering the role of CAS in the apprenticeship of a mathematician, and the latter suggesting a re-evaluation of the meaning of abstraction. Noss’ approach touches on a very important dichotomy arising in views of mathematics in the CAS age. Noss problematises the notion of proof in mathematics reporting on mathematics students in England who see mathematics as a problem-solving enterprise and do not acknowledge the relevance of proof in helping them to solve problems. In these ways the issue of the effects of CAS on mathematics is a subplot in a greater debate about the nature of mathematics and the relationship of mathematics in an epistemological sense to mathematics as presented to students in schools. Kutzler (1999) envisages, in a CAS environment, the re-establishment of experimentation as a crucial part of mathematical learning in a post-Bourbaki paradigm. Kutzler argues that the codification of mathematics under the influence of Dieudonné and the Bourbaki school gradually influenced
teaching and learning to the point that the “definition-theorem-proof” model became the standard method of presenting mathematics to learners. The reintroduction of experimentation and the adoption of a Poincaré-like approach to mathematics is deemed to avoid what Freudenthal termed an anti-didactical inversion where applications of mathematics are presented after the introduction of formal mathematical structures (Cobb, Yackel, & Wood, 1993).

Drijvers (1997) suggests three domains in which further research on the role and effects of CAS are necessary, namely, foundations, cognition, and social/affective. He poses twelve specific questions which remain open and are suggested as important areas for further research. In the first domain lie questions about student prerequisites before using a CAS and teaching methodology in a CAS environment. The cognitive domain on understanding and student construction of knowledge presents questions about obstacles to learning specific to a CAS environment, the possibility of identifying student misconception, and the possibility of creating more effective pathways to higher-order skills. Finally, questions on student motivation and the changing role of the teacher belong to the social/affective domain.

Many subsequent papers, not least by Drijvers himself (2000), can be viewed as attempts to answer these questions. Heugl, Barzel, and Furukawa (1997) propose four new principles and methods for teaching and learning:

- the White Box/Black Box Principle
- the Black Box/White Box Principle
- the Window Shuttle Method
- the Module Method
The first of these principles and methods involves students developing a concept or algorithm with all calculations done by hand (the White Box phase) and later applying the algorithms in a problem-solving situation while using a CAS to perform calculations (the Black Box phase). In applying the Black Box/White Box principle CAS is used as an experimentation tool to allow students to discover concepts and algorithms (the Black Box phase) before proving conjectures made (the White Box phase). The Window Shuttle Method takes advantage of the fact that student learning is no longer restricted to progressing in a serial way. Multiple windows available in a CAS environment allow the possibility of working with algebraic and graphical representations simultaneously and observing, instantaneously, the effect of changes in one representation on the other. Finally, the Module Method explores the programming possibilities of CAS to involve students in creating programs in the White Box phase, or teachers in creating programs to be used by students in the Black Box phase.

Heugl (2000) has distilled seven algebraic competencies in the age of CAS starting from a definition, owing to Bruno Buchberger, of mathematics as “the technique, refined throughout the centuries, of problem solving by reasoning” (p. 29). Those competencies are

• finding terms or formulas
• recognising structures and recognising equivalence of terms
• testing
• calculating
• visualising
• working with modules
• using the chosen CAS

In a related effort to understand the relationship between CAS and student learning, Drijvers (1999) studied the impact of the integration of CAS in a mathematics program in the Netherlands. Arising out of the study he identifies five specific obstacles students encountered when using CAS as a tool for learning mathematics. They are:

• The difference between numerical and exact algebraic calculations
• Algebraic representations provided by a CAS that are different to what students expect and conceive as “simple”
• The limitations of a CAS and the consequent need to frame calculator use differently
• The matter of when and how computer algebra can be useful
• The insight into variables and parameters necessary to use a CAS

It seems likely that the issue of what students should know will be prominent engine of research as technology becomes more integrated into mathematics curricula in the 21st century. This knowledge/skill issue may be approached from the very broad perspective of Heugl (2000) and Berry (1999), from the more topic specific perspective of Herget et al. (2000) and Lagrange (1999), or with consideration of the specific difficulties arising from the teaching and learning of mathematics in a CAS environment (Drijvers, 2000, 2001).
Theories informing use

The most recent research on CAS is explicitly attempting to draw connections between particular theories of learning and the use of CAS in mathematics education. Heid (2002) exemplifies this approach in discussing how theories related to mathematical learning and teaching can inform the use of CAS. Heid posits that CAS impacts on school mathematics in three ways and asks “First, what is the impact of the CAS on the learning of school algebra, however defined. Second, what is the impact of the CAS in curricula designed to foster students’ movement from informal to formal strategies. Third, what is the impact of CAS on the level of mathematical object students study” (Heid, 2002, p. 7). In the first of these instances, school algebra, Heid argues that a constructivist taking a functions approach to algebra would have students exploring the effects of parameters on different families of functions whereas a social constructivist may have students working in groups with a physical model exploring motion and using CAS to get an exact symbolic rule to describe and understand the motion.

Heid’s discussion on the movement from formal to informal strategies focuses on the difference between what Doerr (1995) has called “expressive” and “exploratory” notation. The latter approach starts with students’ own notations and aims to move them towards “mature” mathematical notation. The latter approach, more suited to CAS use, starts with accepted notation which students must then adopt. This issue is complicated by the fact that some CAS software (e.g. Mathematica, DERIVE) has its own non-standard notation. However as Heid acknowledges, it remains an open question as to what effect CAS use will have
on students understandings and abilities to interpret and reason from formal mathematical notation.

Heid’s third question focuses on the effect of CAS on the old question of differences between conceptual and procedural knowledge. A typical use of CAS in mathematics is to allow concentration on interpretation and understanding while “outsourcing” computation to the machine. Artigue (2001), however, argues, that using technology does not eliminate technical work but rather requires a reorganisation of the relationship between technical and conceptual work. She identifies the work done by CAS as opening up a discursive space because of, for example, unexpected output from the machine or the necessity of discussion of differences between exact and approximate results.

This recent research is deliberately focussed on second-order effects on the use of CAS both in the sense of identifying how different pedagogical philosophies will effect decisions about how CAS can and should be used as well as examining unexpected consequences of the use of CAS on students sense of mathematical objects and their representation.

Conclusion

The first phase of use of Computer Algebra System technology has followed the common course of other technologies in being used, in the first instance, for the purposes of amplification. This first-order approach is exemplified by research which sought to examine the use of CAS to deliver a traditional curriculum. Second-order consideration of CAS has developed quickly with a great deal of scholarly work now focusing on what a twenty-first
century curriculum, informed in its design, by the affordances and obstacles of CAS technology, might look like and what assessment models are appropriate for such a curriculum. As the curriculum debate has developed various theories and models of learning are being examined for their relationship to the implementation of CAS in mathematics teaching and learning. The area least researched since the introduction of CAS remains the particular nature student cognitive activity and concept development in a CAS environment. Work such as that of Artigue (2001), Drijvers (2000), and Ruthven (2001) are steps in this direction and point the way for the next phase of research on CAS. The goal of my research project is to contribute to this new direction of CAS research by exploring the research question “What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus?” By examining in detail the actions and behaviours of a small group of students every day in a CAS-rich environment and by using two theoretical frameworks to analyse the meaning of those actions and behaviours, I aim to provide insight into how students work and learn using CAS and to provide a model for understanding the place of technology in the learning of mathematics.

My project has the potential to contribute to the gap outlined above in a number of ways because of the detailed descriptions of student behaviour and strategising in the CAS environment. Such detailed description will, for example, allow insight into how students operationalise the Black Box and White Box principles of Heugl, Barzel, and Furukawa (1997). This approach will also allow us to see whether the way in which students use the CAS does indeed bring them closer to the way mathematicians work and whether experimentation can
become a significant element in student work. An extensive and detailed picture of student work in the CAS environment will also allow glimpses to emerge of the second-order effects of the use of CAS going beyond mere amplification.
CHAPTER 3

METHODS AND METHODOLOGY

There is an infamous study in Computer Algebra Systems research which may well be an urban myth (I’ve never come across the paper) in which four different teachers use four different CASs to teach a course. At the end of the course the students who learned using Mathematica performed significantly better than the other classes in a summative assessment. In a follow-up study the same four teachers taught the same course to four new classes. On this occasion the students who learned using Maple did the best. However, the Maple students in the follow-up study had the same teacher as the Mathematica students in the first study. Naturally, this “story” is not to say that teaching with a CAS, or any technology for that matter, has no effect on teaching but one has to very careful in judging what is being attributed to the technology.

This chapter is divided into two sections the first of which consists of a detailed description of the setting for my research project. The second section outlines my philosophical disposition to research in the social sciences and then explores in some detail how this disposition plays into the specifics of my
research methodology for my project. The chapter will explain why my methodologies are the best way to address the question “What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus?”

Part I: The Research Setting

*Calculus & Mathematica*

In the large Midwestern university at which my study takes place, students taking their first calculus class have the option of attending a traditional lecture/recitation class or can choose to take a computer-based class where calculus is taught with the aid of a CAS called *Mathematica*. The course is called *Calculus & Mathematica* (C&M), lasts for a ten-week term, and uses materials developed at The Ohio State University and The University of Illinois (Urbana-Champaign) and now sold commercially as an introductory course in Calculus. The structure of a C&M course is such that it requires self-directed learning with students using a CD as a interactive textbook. While there is a lecturer and a teaching assistant in the laboratory, they are there to respond to student questions rather than to lecture. Students in the C&M classes work, mostly in groups of three, through one module a week of material. Each module consists of four sections: (i) Basics; (ii) Tutorial; (iii) Give it a Try; and (iv) Literacy. Suggested practice is that students work through the first two of these sections by themselves in order to learn and practice the basic concepts of the module. The “Give it a Try” section consists of more challenging questions which students work on in their groups with the aid of *Mathematica* and submit for
grading. Finally, the “Literacy” section consists of problems similar to the problems in “Give it a Try” but which the students work on without the aid of the CAS and submit for grading. The other forms of assessment are individual and consist of quizzes, examinations during the term, and a final examination, all of which are done by hand without use of the CAS.

Design of the Study

My research project is a case study of three students in a Calculus & Mathematica (C&M) class as individuals and as a group. I have chosen this number of students because the group size in the class I studied is three. Much of the work in the class is group work so it makes sense to study an entire group although the analysis considers the individuals within the group as well as the group as a whole. The data collection phase of the study took place in a ten-week term at the beginning of the academic year. The primary data is audio tape and video capture of computer screens of the group’s discussions and collaborations representing learning episodes in the C&M classroom. Analysis and interpretation of this data, through the Rotman (1993, 1995) and Pirie-Kieren (1990) frameworks, is the primary source of answers to questions about the processes of student learning in the CAS environment including the place of experimentation, and formalisation in mathematical reasoning. As stated in Chapter One the Rotman model will be a macro frame for understanding the place of technology in the students learning of calculus using CAS and the Pirie-Kieren model will provide a frame for a detailed analysis of learning episodes as recorded on audio and video.
This data was supplemented by interviews with the students and analysis of their “Give it a Try” and “Literacy” responses. The first two levels of analysis are looking closely at the students group work through particular episodes viewed with Pirie-Kieren micro-framework and looking at the students individual journeys during the ten-week term within the Rotman macro-framework. In an attempt to take an even broader view and to place the analysis of the three students in the context of the class as a whole, a grounded survey was administered to all students in the C&M class. Early analysis of some learning episodes and reflection on the initial and midterm interviews with the students provided the basis for the questions in the grounded survey. As discussed in more detail below, an approach grounded in the traditions of qualitative research provides the best opportunity of describing, analysing, and understanding the specific impact of learning in a CAS environment on student conceptual development. As discussed in Chapter 2, analyses focusing on measuring increase in student achievement have been done in similar contexts in the past (Judson, 1990; Mayes, 1995; Palmiter, 1991; Runde, 1997) but do not provide sufficient insight into students’ specific conceptual development over a period of time.
Significance

While there have been many benefits to the introduction of technology to classroom environments it has not always been the case that we fully understand the impact of technology on such environments in terms of student learning, pedagogical strategies, curriculum, and assessment. The use of CAS by teachers (as opposed to by research mathematicians) is still relatively rare but the growing body of research and the interest expressed by such organisations as the National Council of Teachers of Mathematics (Cuoco et al., 2003) suggests that its extended use is imminent. It is important, therefore, that there be a firm research base for the implementation of CAS-capable technology in schools which has addressed the conceptual development of students in this new environment as well as recognising the merits and demerits of learning in the environment. As was argued in Chapter Two the empirical research heretofore on CAS use has focused on the outcome achievement of students learning with CAS. The case study methodology of this study with an emphasis on analysis of learning episodes will allow a more detailed analysis of what happens in terms of conceptual development and group interaction with technology in a CAS environment than is possible through studying outcomes in tests and analyzing self-reporting by students of their experiences. Therefore, this project aims to contribute to filling a gap in the research base by providing existence proofs of student learning processes and behaviour patterns in a CAS-rich class and by demonstrating the usefulness of two theoretical frameworks in providing understanding of the place of technology in mathematics education.
Timeline of the Study

The data collection period of the study was mostly covered by the ten-week term of the class with some final interviews taking place in the month after the term. A full timeline of the study follows:

Mid to late September: identification of students willing to take part in the study.

Late September: initial interviews with the participants.
- The goal of these interviews was to develop rapport with the students and get some initial impressions of what they were expecting from the course, how they analyse their own learning style, their knowledge of and attitude to mathematics, and their experience with and attitude to technology.

Late September to early December: daily recording of students working in the C&M class.
- The goal of these recordings, which were supplemented by observations, was to provide me with a set of learning episode vignettes to be analysed within the Pirie-Kieren framework.

Early November: mid-term interviews with participants.
- The goal of these interviews was to follow up on questions at the initial interview about, for example, how the students’ expectations were playing out in the class, how their relationship to the technology was developing, and their reflections on the learning process they experienced in the class.

Early December: administration of grounded survey.
• The goal of this survey was to take issues about the class and learning with technology and see whether they are common experiences across the class in general.

Mid-December: Final interviews with participants.

• The goal of these interviews was to gain summative evidence of the students’ experiences learning in a CAS environment. Each individual’s three interviews, taken together, provide a context for understanding each students’ learning. These interviews complement the recordings of learning episodes.

Spring: Follow-up interviews with participants.

• The purpose of these interviews, which followed some initial analysis, was to continue my commitment to the participants while pursuing reflections and perspectives they had on learning in the CAS environment after some time had passed, including how well they felt prepared for subsequent classes.
Part II: Philosophy, methodology, and methods

“Ring the bells that still can ring.
Forget your perfect offering.
There is a crack in everything.
That’s how the light gets in.”
- Leonard Cohen

Perhaps the greatest challenge of doing non-positivist research in social sciences is not that there is so much to learn as that there is so much to unlearn. Positivist assumptions that are challenged and argued away in paradigm talk reemerge as positivist reflexes in the field. The researcher is faced with questions such as “How can I move the last paragraph of the interview to the beginning of the story and still be honest?” and “Part of my document analysis confirmed what I noticed in the observation, this means I must be converging on truth, right?” and “87% of respondents agreed with that statement in my grounded survey and they all meant the same thing, right?” Academic discourse is so deeply imbued with “methodolatry” (Janesick, 2000) that it is hard work to begin to question the ontological and epistemological baggage they carry. Any researcher in the field will constantly struggle with these issues and I see this, ultimately as struggle not to do research but to be a researcher.

In this study I used the standard methods of qualitative research. I did observations, I did interviews, I did a grounded survey, and I did some document analysis. For what else is there to do? But the key turn in methodology
in modern social sciences is to be concerned not with what a researcher does but rather to ask who is this researcher? What is his or her disposition to their research? What is his or her relationship to the topic of their research? What is his or her relationship to their participants? What is his or her relationship to the site of their research? What is his or her relationship to knowledge? To interpretation? To being a researcher?

**Validity**

The validity of positivism is located in method (Janesick, 2000; Scheurich, 1996). Ontologically a “capturable” real is posited, and epistemologically the “Scientific Method,” correctly applied, guarantees (offers) access to this real. Method is designed to remove researcher influence and correct practices facilitates convergence on truth. Method allows the researcher to seal the cracks in knowledge and make a perfected offering. There are many strands and hues of non-positivist research all of which reject the ontology of the “capturable,” quantifiable real (Yon, 2000; Erickson, 1986; Guba & Lincoln, 1989) all of which raise important questions about the place of method. Non-positivist research is, perhaps, about locating cracks and about opening them up, worrying them and letting light in rather than sealing the cracks. The goal is no longer facilitating convergence on truth but rather divergence to truths. This reorientation has deep implications for the relationship between epistemology and validity and perhaps co-implicates epistemology with ethics (Lather, 2001; Scheurich, 1997) or relocates validity to the realm of ethics (Smith & Deemer, 2000). As Werner Heisenberg said, “What we observe is not nature itself but nature exposed to our
method of questioning” (quoted in Davis, 1996, p.24). This study does not, therefore, claim that a true reality will be captured and displayed. The worth and acceptability of the study will arise from an attempt to provide an analysis and interpretation which is self-conscious about the researchers’ relationship to the participants, the technology, and the research situation, but which seeks to offer an analysis and interpretation which is viable, reasonable, and relevant both in terms of its reflection of the data and its contribution to our understanding of learning in a CAS environment. The analysis aims at viability and reasonableness inasmuch as, while it is my interpretation of the data, that interpretation is based solidly in the data and is supported in the (re)presentation of the data. Viability, reasonableness, and most importantly relevance are judged also by how the study and its interpretation are received. Those reading the study ranging from writing groups to committee members to audiences at presentations to reviewers for publications will judge as to whether the interpretation is assessed as reflecting the data and being a relevant contribution to the study of the use of technology in education.

Methodology

The historical primacy of method in the conduct and validation of academic research is in many ways responsible for the false dichotomy between what are seen as quantitative methods and qualitative methods. The cliché that quantitative methods tell a researcher what is going on and that qualitative methods tell the researcher why it is going on, not only falsely represents both types of research, but betrays ontological and epistemological assumptions about
method in and of itself. As Kvale (1996), Scheurich (1997), and Janesick (2000) have pointed out that while interviews in some literal sense may be qualitative in nature they most certainly do not necessarily guarantee a non-positivist approach to research. The rub here lies in the language of qualitative vs. quantitative wherein method may well be the key issue. It is my contention that the action is elsewhere: perhaps in a positivist vs. non-positivist dichotomy but certainly not so much in the doing as in the being. There is nothing inherently qualitative about any particular method and that it is what the researcher, as a researcher and as a person, brings to bear on the method that determines whether the approach is qualitative in nature. Therefore, the challenge that I face in using what are seen as traditionally qualitative methods (interviews, observations, document analyses, and grounded surveys) is to use them in a qualitative manner. This goes far beyond mere resistance to quantitative strategies of data analysis and requires active recognition and refusal of positivist assumptions. It requires a way of being.

Doing versus Being

Guba and Lincoln’s (1989) “classic” attempt to provide criteria for judging the quality of “fourth generation evaluation” provides part of a bridge on the way from positivist to non-positivist validity and, therefore, reflects part of the journey from doing to being. Their construction of a set of parallel criteria to traditional positivist criteria may have been grounded in the political necessity of competing with positivist validity criteria but it remains nevertheless steeped in the necessity of doing. What do you need to do to validate non-positivist
research? You need to do your research over a prolonged period of time, you need to do observations persistently, you need to do peer debriefs, you need to do negative case analysis, and you need to do member checks. Recognising that the parallel criteria are “primarily methodological criteria” (p. 245) Guba and Lincoln attempt to find criteria which arise from and honour a different set of ontological assumptions. Whether they are successful in doing so, and whether their use of language (for example, constructing their criteria almost exclusively around the word “authentic”) is always reflective of non-positivist assumptions, what Guba and Lincoln certainly achieve is a co-implication of the researcher, in relationship and prejudice, with the validity of research.

Towards Enactivism

There remains the danger of creeping realism manifested as “neo-realism” (Smith & Deemer, 2000) in the judgement of non-positivist research. Lather’s (2001) blurring of the lines between ethics and epistemology and the proposal of a shift from regulatory to constitutive practices of validity in the work of Lather (2001) and Scheurich (1997) call for interruption of assumptions and a constant interrogation of method as a implicit carrier of positivist assumptions. An ontological shift to an enactivist view (Varela, 1987; Davis, 1999) can provide a next step along the road from doing to being. Enactivist theory sees the world as performed not preformed. It is in being and relationship that the world is constructed and validity can only be constitutive as the world in constantly co-constructed and co-imagined in relationship. This worldview echoes the re-interpretation of evolution not as “natural selection,” which suggests
progression, convergence and idealism, to “natural drift” which suggests non-directed change, locality of solutions and meaning, and ongoing construction of reality. Garratt and Hodkinson (1998) argue against the possibility of there being criteria for validity criteria recognising the emergent nature of validity within any non-positivist research process and the assumptions about reality implicit, and explicit, in prescriptiveness.

An interpretive framework consistent with the enactivist view is that of hermeneutics. Hermeneutics has historically been associated with the interpretation of biblical texts and might be more broadly defined as the art of interpretation. Davis (1996) argues that hermeneutics involves a recognition that we are “always thrown into the middle of things; we cannot extricate ourselves to gaze upon the objects of our inquiry” (p.18) and seeks to move beyond the binary this-or-that logic that serves as the foundation for much of modern Western thought. In this sense, a hermeneutic approach is one which seeks to worry the cracks in view of what light they may let in. Hermeneutics should not, in its rejection of a capturable objective truth, be thought of as an idiosyncratic, subjective approach. Rather it is an approach which is concerned with viability, reasonableness, and relevance, in place of validity, reliability, and generalisability (Davis, 1996).

Who is this researcher?

At the beginning of Part II of this Chapter I listed a series of important questions facing a researcher operating in a non-postivist paradigm (who is this researcher? What is his or her disposition to their research? What is his or her
relationship to the topic of their research? What is his or her relationship to their
participants? What is his or her relationship to the site of their research? What is
his or her relationship to knowledge? To interpretation? To being a researcher?)
The previous sections are offered as an answer to the question of disposition to
research. The following sections of the Chapter will address the remaining
questions including an item-by-item analysis of my approach to the specific
methods I will employ in my study.

My mathematical autobiography in relation to technology

It was not until I was a teacher that I used calculating technology
significantly in mathematical activity. In school I fitted the mould of the
traditionally good student in that I was able to perform algorithms quickly an
efficiently with some sense of what I was doing to evaluate answers. Until the
end of middle school I rarely used calculators of any kind because I wasn’t
allowed to use them in exams and thought it be better for me to practice without
them. In university, very little of the mathematics we did had a computational
aspect, so while I became aware what Mathematica was, it wasn’t used in any
course I took.

My first teaching position was in a very traditional environment where a
strict curriculum was imposed and the use of calculators was discouraged. When
I moved to my second teaching position the environment was one much more
engaged in the use of technology in teaching. Students were expected,
particularly in the higher grades, to have graphing calculators and there was
both an interest in using calculators to teach and to develop curriculum materials
in which calculator use was integrated. During my time in this teaching position over five years various professional development activities took place which significantly increased the use of calculator technology (primarily graphing calculators) to the point that the ideology of the department was strongly in favour of incorporating technology into teaching at all levels. This practice extended to the use of CAS with the school buying 50 hand-held CAS enabled calculators (TI-92s) and two classes using the calculators on an experimental basis.

Much of my interest in using CAS-enabled technology was the novelty value of the machines but as I read literature associated with the use of calculators I came to believe that calculators, like manipulatives, can be effective vehicles for students to construct their own knowledge as they can experiment with mathematical objects and get immediate feedback on the relationship between their experimentation and the correct technical implementation of their experimentation in a mathematical world. My interest in technology was always in students getting closer to pure mathematics rather than applied mathematics. To some the benefit of having technology adopt the burden of symbolic manipulation and calculation is the access afforded to authentic, real-world problems, whereas I have always preferred the possibility that the structures of mathematics and their meanings are more visibly important since the difficulty of becoming mired in calculation has been scaffolded.

My initial approach to my research, early in my graduate career, was therefore, as something of an advocate of technology. (It is rare for opponents of technology to research its use in learning). This position has been tempered
somewhat as I am now still very interested in the effect of technology on learning but am more skeptical about how well it can be implemented in schools and whether, through both the expense of the hardware and the nature of the interfaces used, it is a source of inequity in education.

My relationship to the research setting

My initial intention was to sit with the three participants in my study taking notes as they do their work without formally intervening. As it turned out I did not sit with them for the first week or more because I wanted them to develop their group dynamic and their habits of work without my being there. It is, of course, naïve to imagine that students did not behave differently in their classroom knowing that I was recording what they are doing and taking notes. That said, my non-intervention is important since any intervention on my behalf would undoubtedly result in the imposition of a pedagogical approach possibly designed to provoke certain answers to my research questions. One of my fears in taking this approach is that the participants would see me as sitting in judgment on their work and be considerably inhibited in their working habits. However, it is my experience that people being recorded quickly get used to the presence of a microphone and it seemed that my daily and prolonged presence (over a ten-week term) resulted in the participants quickly becoming used both to being recorded and to my presence. During the midterm interviews I asked the students if they perceived any effect I was having on how they work and the students are reported that they found my presence a neutral factor. Adler and Adler (1994) argue that all researchers go through stages from covert to overt,
passive to active, outside to inside. A further challenge in my observations was that the students assumed that I had mathematical knowledge or knowledge of Mathematica and might be willing to help them. I was explicit in saying that I was not there as a resource for them in the class and that they will not see me as such. The fact that they worked in a group was an advantage to me in this regard and generally the participants became caught up in their interactions with one another rather than looking to me for intervention of some sort. Acceptance of these shifting roles in a research setting is, to a certain extent inevitable. The aim remains, nevertheless, not to “alter the flow of interaction unnaturally” (Adler & Adler, 1994). I did, however, find it very difficult on occasion not to be a teacher. There were many potentially extremely valuable “teaching moments” which occurred during the ten-week term and it took quite a combination of discipline and desire to adhere to my concept of the fidelity of my research not to intervene.

**Expectations and coding categories**

Two very specific theoretical frameworks were delineated in Chapter Two to answer the question “What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus?” Despite the specificity of these models it was to be expected that some themes of analysis would be emergent. The sub-questions presented in Chapter One served as initial coding categories and framed the initial analysis, in situ, throughout the data collection.

These categories were:

- Student strategising
• Negotiating between multiple representations
• Perceptions of technology
• Relationship to Mathematica
• Experimentation
• Conceptions of mathematics
• Formalisation
• Pedagogy and technology

Prior to data collection I wrote that “Evidence of experimentation may include the participants trying several parameters for solving a problem and noticing patterns in the CAS output which allows them to find an appropriate response. Evidence of the process of learning may include episodes similar to the example in Chapter One which show participants changing levels in the Pirie-Kieren model to achieve understanding of a concept. Evidence of formalisation may include participants speaking or writing in formal mathematical terms by trying to construct definitions or make statements such as “for all functions.” Evidence of perceptions of technology may include explicit comments about Mathematica or difficulties participants have with the technology, as opposed to the mathematics. Evidence of conceptions of mathematics may include statements participants make about usefulness or applicability of the mathematics they are learning or how they go about answering questions which require justification, i.e. what they consider as acceptable justification in mathematics. Evidence of the relationship between pedagogy and technology may arise from explicit statements from participants about how the class is run and how it compares to other pedagogies they have experienced.” Precisely how
this played out in the data collection and analysis can be seen in subsequent chapters.

*Methods employed*

In the following few pages I outline the methods employed for data collection in this study together with my reflection on the employment of these methods.

*Observations*

Observations formed the main data source for my analysis and interpretation. Specifically, the participants were recorded on audio and video (software on their machine capturing their on-screen work) for 48-minute classes five days a week for ten weeks (excluding two examination days). I considered the forty-five, or so, observations to be enough to reach data saturation in the sense of seeing patterns repeat themselves. Initial coding was done under the rubric outlined above. I recorded in my field notes if an instance in the class seemed to be a good example of any of these categories and, subsequently reviewed the video and audio tape for confirmation. The methodology outlined here provided a few examples of exemplary learning episodes. A second iteration occurred during the transcription phase. The final two iterations of episode choice were a combination of reading through the transcripts of the interviews with the students for further themes and then looking through the class transcripts to find examples of these themes and to establish that the themes appeared multiple times.
Interpreting my participants’ learning through such particular theoretical frameworks is something of a double-edged sword. On the one hand, the frameworks offer a focus within the multiplicity of observables in any situation, but on the other hand there will be an inevitable tendency for me to want to see any given episode as interpretable within the frameworks. However, a sense, or even conviction, that a particular theoretical framework is applicable and appropriate is not the same as an investment in that framework which is likely to override the interpretation that the data will bear.

A number of studies (Pirie & Kieren, Towers, ??) have used the Pirie-Kieren model although none have used it in the context of learning using technology, let alone a CAS. There is evidence that the model is an appropriate one for characterising mathematical learning and for analysing learning episodes by tracing a student’s, or group of students’, movement through the levels. It seems to me that the model will be particularly appropriate for the 151C class based, as it is, on experimentation by use of the CAS to explore parameters given problems and interpret the meaning of the CAS’s response.

Interviews

The participants in my study were a volunteer group. They chose to take the course for a variety of reasons but all had a somewhat positive disposition to technology. I interviewed the students at the beginning of the ten-week term to get a sense, biographically more than anything else who these people are. From a pilot study that I had done, as well as my knowledge of previous studies using C&M classes at this institution as their site of research, I had some
preconceptions as to the kind of students who take this course. The sum of this bias is that the students are not mathematics majors, are not, in traditional terms, among the stronger students among their peers (their peer group being students at the same stage of degree progress at the university), and they are not inclined to talk especially freely about their mathematical experiences. These preconceptions were somewhat misplaced and while the students I interviewed and studied were not mathematics majors (two were pre-engineering and one was a life sciences major) they would all have been regarded as strong students in their high schools and were quite able to reflect on their mathematical experiences.

Perhaps my strongest influence in how I like to conduct interviews is the work of Holloway and Jefferson (1997) in their interviews with inhabitants of a town in England. They express their dissatisfaction with a first round of interviews they conduct because they feel they aren't getting the stories of the people they are interviewing. An analysis of their interview transcripts allows them to see that the problem is in their interview questions which, rather than being designed to elicit stories and personal experiences from the interviewees, in fact invite the interviewees to co-theorise with the interviewers in an abstract sense. Not that this is without value as such but it is not what Holloway and Jefferson sought from their interviews and not what they value in their participants.

My motivation, certainly in the pre-term interviews was not in co-theorising with my participants but in hearing their stories and experiences, particularly with mathematics and their attempts to learn mathematics as well as
what they are looking forward from the class and college in general, since they are likely to be incoming “freshpersons.” The initial interviews were one of my first opportunities to develop rapport. Therefore, my interview approach was one where I would try to establish a dialogue with each participant rather than a structured, or even semi-structured, interview as a way both to establish rapport with my participants and as a way of eliciting a more story/experiences-based response (Scheurich, 1996; Rhodes, 19??). This approach is consistent with an enactivist approach whereby meaning emerges from the performance of world through, for example, dialogue.

Grounded Survey

The final contributing element is to the description of the context surrounding the group “case” study of my participants through a grounded survey. My experience in constructing surveys of any type is that it is a very seductive enterprise. Whenever I have constructed a survey I find myself drifting off in flights of fancy of what results I will find and how they will confirm my own ideas and investments. In an interview you can have similarly hoped for responses but you don’t have the control of language available in the more static format of a survey. As the writer of the survey the danger is that I have an interpretation of each node (Strongly disagree, disagree, agree, strongly agree) in mind for each question and will happily see that interpretation in the respondent’s choice of node. Had I been more experienced in survey design I would have constructed the survey more from the point of view of imagining what responses might occur from the survey takers and making sure that there
are enough, and appropriate nodes to reflect a full range of responses. For example, in research which is so exploratory in nature it was probably a mistake not to include either a “neither agree nor disagree” node or “no answer” node. My decision to force respondents to choose one way or the other probably cost me some data in terms of understanding where opinions have not solidified on certain issues.

The writing of a grounded survey helps both to interrupt and to reinforce this practice in interesting ways. It being the case that grounded survey questions come from the data rather than my initial dispositions and prejudices. Such surveys are posited as a way of guarding against questions that I have too much personal investment in by including questions which are relevant but only became apparent to me after spending some time with the data. Of course it will happen, as it has happened for me in previous grounded surveys, that some issues I care about did arise in the data and are easy to find because I am looking for them. The interpretive act of choosing grounded questions thus reintroduces my prejudices into the grounded survey process.

The grounded survey turned out to be useful in situating the three students in the study relative to the group as a whole. Many of the survey questions arose directly out of the observations and early analysis of the learning episodes data as well as from issues which arose in the interviews. The results show that the three students are not anomalous relative to the group but also show important differences from the entire class.
**Document Analysis**

There is a very large amount of documentation associated with the Calculus & *Mathematica* (CM) program of which the class I studied is a part. The documentation is on the program’s website and details the provenance of the program both in terms of the people who developed the program (and continue to work on it) and in terms of the philosophical underpinnings of the program in general and the use of *Mathematica* in teaching and learning in particular. The front page of the website associated with the course contains the following statement “"Curious? We are working to change the way math is taught. Come inside and discover a new way to think about and teach mathematics." [http://socrates.math.ohio-state.edu/](http://socrates.math.ohio-state.edu/). Further statements on the website explain something of this “new” approach which emphasises teaching for understanding, group work, collaboration, and visualization. Specific statements from the website are used in some of the analysis to situate the goals and materials of the course relative to processes and behaviours of the students.

**Conclusion**

The main source of data for the analysis in my project is the observations but the participant interviews, the grounded survey, and the document of the CM site provide background and context contributing to a “thick description” (Denzin, 19??; Geertz, 1983) of the participants in the study and the environment and context in which their learning took place. My interpretation of “thick description” is that analysis of learning episodes will neither be offered as free-floating, contextless vignettes, nor will they be interpreted, in the first instance,
without reference to the people involved and the knowledge of their background elicited from the interviews, and their “official” context of the C&M program.

Philosophically, I see my journey on the way to locating myself theoretically and methodologically and as a struggle to imagine practices that will embody a non-positivist worldview. To oversimplify, it not as important whether I do, for example, an interview as how I develop relationship through interview, how I (re)present that relationship in writing and how I honour the complication, complexity and natural drift of the world in my work. From oversimplification to brevity: how can I locate cracks and let more light in.
CHAPTER 4

ANALYSIS VIA THE ADAPTED ROTMAN MODEL OF MATHEMATICAL REASONING

In Chapter 2 there were two different theoretical frameworks proposed through which data would be analysed to answer the question: What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus? The Pirie-Kieren Model for the Growth of Mathematical Understanding (Pirie & Kieren, 1990; Pirie & Kieren, 1994) will be used in Chapter 5 as a micro-framework for analysing specific learning episodes and for establishing important processes of learning manifested by the students in their day-to-day activity in the CAS environment. In this Chapter, I will turn to the Adapted Rotman Model of Mathematical Reasoning (Rotman, 1993; Rotman 1995) as a macro-framework for understanding the place of technology in the learning environment and the interstices between the students, the mathematics, and the technology. The analysis of Chapter 5 will treat the students as a group whereas the analysis in this Chapter will look more closely at the individuality of the students.
As well as viewing the data through the lens of the Adapted Rotman Model, I will also report on a grounded survey administered to all twenty-four student in the Calculus & Mathematica class. This analysis will serve to situate the three students’ processes and behaviours in the context of the larger group. The analysis shows similarities and differences between the three students and the larger group. Finally, this Chapter includes sections on the implications for teaching and the implications for research arising out of the analysis of the data via the two theoretic al frameworks.

Understanding the place of technology

When I first taught with CAS it was in an Algebra I class where the students had hand-held devices (TI-92s) with inbuilt CAS capability. The beginning of one of the worksheets was as follows:

Type 3a + 2a. Press enter and record the result
Type 4b + 5b. Press enter and record the result
Type 3x + 2y. Press enter and record the result
What do you think will be the result of 3m + 2m? Record your guess and check.
What do you think will be the result of 3r + 2q? Record your guess and check.
Do you observe a pattern or rule? State the pattern or rule.

The students worked through this sheet with great success and their attempts to articulate the rule provided a good opportunity for discussion of definitions, vocabulary, and concision in mathematical descriptions.

What struck me as curious about all of this is when I thought to myself: would this be different if I, as the teacher, went to the blackboard and wrote “3a
+ 2a = 5a; 4b + 5b = 9b” etc. and asked the students to do the examples and tell me what the pattern/rule is. Having taken both approaches I can attest that each approach plays out very differently in a classroom. Students regard the mathematical authority of a calculator/computer differently to how they regard the authority of a teacher. There is a clear sense in which students felt they were discovering something when getting feedback from technology rather than responding to my probing. It was not clear to me why this was happening but it did beg many questions about how students relate to a calculator/computer (as opposed to a person) and, in particular, how students relate to the authority of a calculator/computer. The way that I found to characterize the interaction that was going on was as a “trialogue” between the students, the technology and the mathematics. It was this tripartite relationship which led me to the Rotman Model of Mathematical Reasoning.

Rotman revisited

The Rotman Model of Mathematical Reasoning was presented in Chapter One. The Model as I am using it provides a way of understanding the place of technology in the mathematical activity of the learner. The important part of the model is its triangular nature which affords separate agency to three entities: the learner, the technology, and the mathematics. What I mean by agency here is that none of these entities is neutral but that each has a relationship with the other two, effecting and being effected by the other two in a manner such that each contributes continually to the students mathematical reasoning. Each of these entities acts on the other two and thus has agency.
I will briefly discuss a traditional model and then discuss each “side” of the Adapted Rotman Model: Technology $\leftrightarrow$ Mathematics; Student $\leftrightarrow$ Technology; and Student $\leftrightarrow$ Mathematics

*A Traditional Model*

A traditional model relative to the Adapted Rotman Model for understanding the place of technology in the learning of mathematics locates technology in between the student and the mathematics where it acts purely as a mediating influence:

![Diagram of traditional model](image)

*Figure 4.2: A Traditional Model of Technology in Mathematics Education*
This is, of course, an important part of the place of technology in the learning process, indeed, unidirectionally, it forms two “sides” of the Adapted Rotman triangle. However, the Traditional Model is not sophisticated enough as it fails to take into account the relationship between the technology and the mathematics and fails to take into account the direct relationship students have to the mathematics. Furthermore, the Traditional Model, as it is usually used, is its unidirectionality. Classic examples of the model being used, at least implicitly, in this way can be seen in many text books commonly used in Middle and High Schools. An examination of successive revised editions of textbooks will reveal that at some point, let’s say between the 7th and 8th editions, technology appears. It appears in the form of certain portions of the text having a calculator icon beside them and students being given a “calculator alert” that technology will be used in this part. There will also be some extra questions in which students are encouraged to use their calculators.

What is remarkable about these texts is how little has actually changed with the introduction of the technology. The topics covered, the sequence of the topics, the achievement goals of the book, the philosophy/approach outlined, and the forms of assessment tend to be virtually identical. This being the case the Traditional Model outlined above is entirely appropriate since technology has, literally, been inserted between the student and the mathematical goals of the course. The visibility of the technology is such that a calculator icon has been strategically placed to tell the student to “see” the technology.
When I first conceptualized the “trialogic” relationship between the student, the technology, and the mathematics this side of the triologue seemed the least interesting. I described the relationship simply as “the machine has been programmed to perform accepted algorithms.” As I have studied the students’ work with Mathematica and examined other forms of CAS available it has become apparent that the relationship of the mathematics to the technology is rather more complicated.

This becomes apparent if one looks at the different inputs necessary to “ask” different CASs the same question, and one looks at the different outputs from different CAS systems in response to those questions. It becomes quickly apparent that, while the mathematics and algorithms might be agreed upon by the different programmers, the interfaces of the programs and the presentation of results are radically different.

Mathematica, a program originally written when memory was at more of a premium than it is nowadays, is a program which depends on the user learning a very particular syntax. This syntax is not reflective of mathematics as it is written in textbooks or mathematical journals and cleaves more closely to high level
programming languages. Another memory saving device of Mathematica’s is the rendering of graphs. Graphs are regularly displayed in a non-standard format (for example the axes do not intersect at (0, 0) in a way that is efficient but may not be easy for the user to interpret.

We will say many of the difficulties confronted by a learner with regard to syntax and graphs in Chapter 5.

DERIVE is a program which was designed, particular in later versions, which considerable attention paid to the fact that the user may be trying to learn mathematics using DERIVE. The program is essentially menu driven, looking more like graphing calculator interfaces and word processing programs than Mathematica. In addition, graphs in DERIVE always include the origin and error messages tend not be as intimidating as those in Mathematica.

As we shall see in subsequent sections the comfort students have with the interface of the technology with which they are working has considerable impact on their learning.
In a course such as Calculus & *Mathematica* the relationship of the students to technology and the impact of the technology on student work is crucial. The technology is not an add on to the course, or a possible feature of the course depending on the instructor. Rather the course is designed, constructed, and conducted with technology at its core: “The program is devoted to teaching calculus in a new and effective manner through the use of technology, the socratic teaching style, and a learning environment that emphasizes communication and student learning.” [http://socrates.math.ohio-state.edu/about/](http://socrates.math.ohio-state.edu/about/) The success or otherwise of the course is heavily dependent on the relationship that the students develop with the technology. CAS, in this context, can become the springboard from which students learning of calculus is infused with, and enhanced by, the affordances of technology, or can become an obstacle that students must overcome to learn calculus. I will talk in Chapter 6 about ways in which I feel this can be done effectively.
This side of the Adapted Rotman triangle bears examination in both directions and it bears examination individually in the cases of the three students participating in this study.

**Student → Technology**

The student relationship to technology and to mathematics has been analysed to a large extent in group form through the learning episodes in Chapter 5. However, each of the students brought separate mathematical histories and separate relationships to technology to the class. It is interesting to look at this history and to see how the dispositions to technology in mathematics that the students brought to the class. All three of the students in this study expressed at the beginning of the course an interest in and comfort with using technology both in their lives in general and in their learning of mathematics; but, the transition to the use of technology in Calculus & *Mathematica* proved to less than smooth. In the following few pages I will deal with each of the students separately in their relationship to technology.
Student A

Student A came to the class with a strong interest and background in technology. At the orientation students were informed of the different calculus sequences they could take “and when the advisor … mentioned 151C and said that it was taught using computers and my dad [a Microsoft Systems engineer] and I both looked at each other and said that’s the class I’m taking” (A, 1st interview, Sept 26th). A’s role in the group quickly became that of technology expert: “I learned to program and I learned it really quickly and so I’ve sort of been taking the lead on the programming aspect. On how do we program this function into the computer to get the results we want. Or here’s the program’s results but I know that’s wrong. What did we do wrong with the equation” (A, 2nd interview, Nov 6th). A had programmed his calculator when he was in high school to perform various mathematical functions such as solving quadratic equations. During the term he was likely to use his calculator, with which he felt very comfortable, but was also the student most interested in learning how to use Mathematica:

“I’m finding the program incredibly interesting. It’s very odd to try and work with some of the different programming styles. I believe Mr. Wolfram designed it himself, the software and he put in some very unusual things, like before closing an f/x statement you have to put an underscore after the x. Just little things like that I have finally figured out but I think the program overall is intriguing with as intricate and as powerful as it is. I’m surprised at how quick it works. I’m surprised at how much it can do, as quickly as it does” (A, 2nd interview, Nov 6th).
Student A was, far and away, the student who was happiest with the course and the only one of the three who chose to take the calculator version of the next course in the sequence “[in the computer based classes] you learn how to do [calculus] and the ideas behind it. And with computer itself it makes it a little bit easier to get the ideas instead of just how it works” (A, 3rd interview, Feb 25th). We see here, and again with students B & C, that there is an element of self-fulfilling prophesy at work here. A’s considerable interest in computers was fed by what the class had to offer and there is determination in his responses to interview questions to see things in a positive light relative to the class. The students all experienced the same difficulties from the trivial (computer crashes) to the more obstructive (frustration with Mathematica’s code and graph rendering) but A was far and away the most sanguine about such interruptions. A commitment to technology and a desire to learn the technology (perhaps more than a desire to learn the mathematics) drives this approach. In any case, A recognises technology as a tool which he can use to learn mathematics and focus on concepts to the exclusion of technical drudgery. A also has a high level of self-efficacy with computer technology which, perhaps, makes him more tolerant of its limitations.

Student B

At the beginning of the term B seemed to bring an attitude to technology, inherited from his classroom experience, that using computers and calculators was less valuable than doing mathematics by hand: “They always tell you to use your calculator then check it by hand but usually when I’ve done it, I’ve done it
and then I checked it with the calculator” (B, 1st interview, Sept 29th). B brought considerable self-efficacy in doing mathematics but exhibits some nervousness about technology: “I’m not worried about learning calculus. I’m just worried about learning the program that runs calculus” (B, 1st interview, Sept 29th). As the term developed B became more and more dissociated from the technology aspect of the course reporting that “[Mathematica is] just not my type of program” (B, 2nd interview, Nov 6th). In contrast to A, he makes a point of expressing frustration with the technical difficulties presented by the use of Mathematica “We get error messages all the time. It’s from dumb stuff like not capitalizing (inaudible). You can have an entire equation and if you don’t capitalize that “e” you don’t get the right answer” (B, 2nd interview, Nov 6th). This dissociation with the technology was reflected in the physical set-up as the students worked in class. C was always the one typing (he felt this was the best way for him to stay involved) with A in close attendance as the expert in Mathematica syntax. B sat more to one side and was often reading notes from a previous mathematics class or working with a calculator. This was not always the case but there was a certain distance he often held from the locus of mathematical activity in the group. B was not interested in taking the next computer-based course in the sequence and felt that he would be much better off in a more traditional environment “I’m actually almost sure I’m not taking [the next computer-based course]. I can’t afford for my grade to drop just so I can learn a computer-based program that really I’m not going to ever need again” (B, 2nd interview, Nov 6th). This statement was made with several weeks to go in the ten-week term and this attitude was reflected in B’s lack of involvement with the Mathematica aspects of
questions although he continued to contribute to the mathematical discussions. Making the transition from calculator-based technology to computer-based technology was difficult for all the students, especially with the lack of introduction to Mathematica. It is a transition that B never quite made “I wouldn’t use Mathematica any time if I had to choose between my calculator and Mathematica because I’m so comfortable with [my calculator] . . . I can do the problems faster on that than they can do on Mathematica” (B, 2nd interview, Nov 6th). At the end of the term it is fair to say that B would be the student least proficient in the use of Mathematica.

Student C

Some apprehension about the technology in the course was also exhibited by Student C at the beginning: “I guess my biggest concern in this class is the Mathematica program … the problem though was that I could sit there and see the problem and I was sort of like confused. Should I work it out and show that I know how to do it or should I just click here for the answer” (C, 1st interview, Sept 26th). We see here that C is exhibiting some of the same anxiety as B in that he is not sure about how he should be doing mathematics with the technology and how much should be what he sees as his own responsibility. C also entered the course with a somewhat ambivalent approach to the technology “I thought it would just be interesting - see how it goes and if I don’t like the computer thing I’ll switch over to a standard lecture and recitation. It's only ten weeks” (C, 1st interview, Sept 26th). He saw the value of the technology in certain instances “there are some parts where I think Mathematica is great making these points …
for example when you’re looking at the relationship between the derivative and the original function,” (C, 2\textsuperscript{nd} interview, Nov 10\textsuperscript{th}) but overall felt that it did not suit his learning style “I just think that some of that is stuff that needs to be taught like on a board and explained by people” (C, 2\textsuperscript{nd} interview, Nov 10\textsuperscript{th}). This statement reflects C’s attitude to what constitutes authentic mathematical activity which is further reflected in his statement that “I don’t think [Mathematica is helping me to learn]. I think it’s almost a short cut around doing work. We don’t want to—we don’t exactly know how to solve for this so we let Mathematica do it” (C, 2\textsuperscript{nd} interview, Nov 10\textsuperscript{th}). We will see more of this questioning of what it really means to be doing/learning mathematics in the Mathematics section.

We can see in each of these stories that the relationship each student developed with Mathematica had a great bearing on how they felt about their work in the course and their ability to learn mathematics. Student A embraced the technology and was not dissuaded by the challenges the technology presented. B and C, whose school experience with calculators made them more skeptical about technology to begin with, found it much harder to make the technological transition and ended up, especially in B’s case, trying to survive the course while minimising the influence of the technology. The students’ initial feelings about technology set the path they would follow in the course. This was certainly likely, and perhaps, inevitable since the transition from graphing calculator to CAS was never facilitated and the transcripts of the classes, particularly the first few class meetings, show that the students were never
“sold” the CAS in a way which might convince them of its power and its possibilities for helping them learn calculus.

Technology → Student

Calculus & Mathematica, as the students in this study experienced it, is a course which is dominated by computer-based technology. The students spent the entirety of their in class time sitting in front of a computer. The lack of direct instruction meant that the computer was situated as the primary locus of mathematical authority. Therefore, it is important to look at the impact of the technology on the students particularly in the sense of Mathematica acting as an authority.

The impact of Mathematica as authority can be seen in the fact that the vast majority of interventions by the teaching assistant with this group were technical issues related to the working of Mathematica rather than discussions of a mathematical concept. A typical exchange is as follows:

A: [to TA] Question. What am I doing wrong here? I’ve got ... I’m storing m, I’m storing x, I’m storing ... well I wasn’t sure to store y.
A: I’m trying to solve for ...
TA: Yeah, you can’t do that …
A: Is there a way I can?
C: Why is it?
TA: Now say f(270) is it? Instead of x put in
A: Yeah. Ok
TA: What point is it?
A: 1081 and then … so clear variables. Ok. That’s what we were looking for.
C: [to A] We got that. Look we got that there. [to TA] If we take those semi-colons away it shows up but why won’t it show up in this plot? Does it not store? It figures out m. Does it not store it for down here or?
TA: No because you haven’t defined f(x) anywhere. Yeah. You can’t do that. If you would just take away f(x) = it might work better. You have to store b too. So you’ll probably want to do it in a couple of different steps. You have to say b equals whatever, because it doesn’t set b equal to this b.

This is a purely technical discussion of how to get Mathematica to solve a problem that the students could have done fairly easily by hand. They end up getting frustrated with Mathematica and, while acknowledging the power of the machine, their comfort with the authority of the machine is compromised by their inability to get Mathematica to perform mathematical tasks that they can either do by hand (or, often, using their calculators).

Further evidence of the hold of technology on the students’ processes is the fact that transcripts of the students work in class show that on almost two thirds of the days in the term there was no intervention at all from either the TA or the Instructor.

What effect does this prominence of Mathematica have on the student behaviour and their processes of learning? One of the most immediate effects of the technology on the students is that they spend a great deal of their time doing
Mathematica rather than doing mathematics. We will see several incidents in the learning episodes of Chapter 5 (and there were many more besides) where the students spend inordinate amounts of time trying to reconcile the output of Mathematica with one of their expectations, their by-hand work, or the output of their calculators. There is a certain value in students having to think through the relationship between their expectations and the feedback provided by Mathematica. Indeed it can provide many important learning experiences. However, there is a point at which this can cease to be fruitful and simply be a source of frustration for the students and can lead to their developing an antagonistic attitude to the technology. This can result then in the students having a somewhat dysfunctional relationship with the technology in which they are unable to take full advantage of its affordances or feel comfortable exploring/experimenting with the technology. This is unfortunate as it seems to have been almost completely avoidable if some effort had been made to bridge the students’ old experience with calculator technology to their new experiences with the computer. The students were given virtually no training in using what is commonly acknowledged as a very difficult piece of software to use and were, it seems, expected to “pick it up” from modeling the Mathematica code used in the Basics and Tutorials. It is interesting that the students’ level of comfort and sense of power in their relationship to Mathematica lies in stark contrast to their level of comfort and sense of power in their relationship to their calculators. This means that the issue is not discomfort with technology but, rather, discomfort with the particular technology they were using in this class.
Calculators versus Computers

All three students had experience with graphing calculator technology in high school although those experiences varied. It is worth looking in depth at some of these experiences the students with calculators that the students brought to the Mathematica class.

A had used TI-83 calculators extensively in both middle and high school and started writing programs as early as the 8th grade. However, he was in 10th grade before calculators (anything from a TI83 to a TI86 but no calculator with CAS capability) were used extensively in his classes. “Basically the teachers would stand in the front and they had this device that they plugged their calculator into—that they could put it on an overhead and the students were able to see what the teacher was doing on their calculator and do the same on their own, and I had actually done the vast majority of what the teachers were showing and so if people were having trouble with their calculators I’d go and help them” (A, 1st interview, Sept 29th). Calculators were allowed in the assessment, and A’s teacher in his senior year was an advocate of calculator use “She really knew what she was doing on the calculators and actually she was the teacher that did more of having us put them away. She had a sort of a pedagogy to using the calculator” (A, 1st interview, Sept 29th). A lot of A’s use of calculators was to check work done by hand rather than generate work.

Calculators were not used much in B’s class until very late in high school. “The teacher [didn’t even] use one as a demonstration. We always drew our graphs” (B, 1st interview, Sept 29th). B had high self-efficacy for mathematics “Basic math I can do in my head. I’m really good with your standard
things—I’ve pretty much always been good at math” (B, 1st interview, Sept 29th).
There was not, even in the senior classes an atmosphere of using technology in class: “Definitely not. We were allowed to use calculators as long as we showed the work” (B, 1st interview, Sept 29th). B agreed with this approach in doing his own mathematics: “it’s a good way to check yourself but you still need to know how to do it to get there” (B, 1st interview, Sept 29th). However, through using calculators to aid his own learning, B came to the Mathematica class with good calculator skills: “Sometimes it’s easier once you get the answer though because then you can work backwards. So I’ve used that to my advantage a couple of times. Once I see what the answer is, I can figure out what step I’m missing or if I’m doing something backwards. So that’s why I’ve become pretty fluent with using my graphic calculator” (B, 1st interview, Sept 29th).

C’s experience was not dissimilar to B’s although calculators were used even less. In C’s junior and senior years there was not an atmosphere of using technology. The teacher was “a very traditional teacher with his textbook, and there is an atmosphere of we’re doing math here and we’re doing work and we’re going to learn this the way that this book has it” (C, 1st interview, Sept 26th). This suggests that C had the greatest adjustment to make to the goals of the course and the use of technology in the course. However, calculators were readily available in the class: “there was a calculator on the corner of everyone’s desk and being used constantly” (C, 1st interview, Sept 26th).

It is clear from these interview excerpts that the students have, at the very least, a decent level of familiarity with the capabilities of a standard graphing calculator and some experience of using them in their mathematical activity.
Therefore, the transition from calculator to computer was a crucial part of their learning process early in the ten-week term. This transition, as noted above, was not at all facilitated in the class. It became clear that it took the students a long time to understand that Mathematica was, at a basic level, simply a much more powerful version of the hand-held calculators that they had been accustomed to using in their mathematics for several years.

A different approach by the instructor to the nature of Mathematica and how the students understand its place and use might have made a great difference in the class. The instructor could have taken some time to make the bridge from calculators to computers by spending some time explicitly showing Mathematica’s capabilities. For example, some time could have been taken on the first or second day of the course to say “You use your calculators to multiply 523 by 611 and this is how you do that on Mathematica. You use your calculators to find the sine of 48 degrees. This is how Mathematica does that. You use your calculators to graph y = 4x + 2. Here’s the Mathematica version. And now here are some things that Mathematica can do that your calculator can’t.” This may have had the three-fold effect of helping students to understand that Mathematica is just like their calculators but has greater capability; would have been an early introduction with a chance for some discussion of the peculiarities of Mathematica’s syntax; and may have gotten the students somewhat excited about the power of the Mathematica (e.g. the ability to solve equations and simplify expressions).

Some exchanges in the early weeks of the term show that the students’ loyalty remained with the calculator. For example:
A: “Use the Mathematica interpolation instruction to come up with a function whose plot runs through all these points. Confirm with a plot.”

B: Copy. . . . Oh man.

A: I can do it on my calculator.

B: No I don’t think you’ll be able to get that line.

…

A: I can do it on my calculator.

B: No you can’t. You can’t plot all those points. Well you can. It would just take forever.

A: It won’t take that long but I’d rather not do it if we can do it on the computer.

…

C: OK. Come up with an equation for that plot. Can you give me the function?

B: All right.

A: I can do this on my calculator. It’s really easy but I don’t know how to do it in this program.

B: Then use your calculator and get an answer and we can figure it out.

This is a fairly typical exchange among the students and shows that the students are unsure of Mathematica's capabilities and are unsure of how to take advantage of those capabilities. When B says, “No I don’t think you’ll be able to get that line,” there is a sense of awareness of differences between the capabilities of the calculator and the computer which stymies the students. The students take a long time to answer the question they are working on in the exchange above as they negotiate between the technologies available.

Early in the term there is an episode in which the students are shown a capability of Mathematica that extends beyond what their calculators can do. The students are trying to find the intersection of two lines. This can be approximated on a graphing calculator by graphing the two lines and having the calculator
perform a numerical procedure to find the best approximation it can to the point of intersection. *Mathematica*, however, can find a solution analytically. The TA shows the students how to do this in the following exchange:

A: [asks TA] Is there a way to find out what the intercept of the two is?
C: He said there is.
A: I can do it on my calculator like I just did.
TA: I bet I can do it faster in *Mathematica*. [Types in the solve function]
A: Wow. That’s really easy. That’s the first thing that’s made total sense to me so far.
TA: You just have to remember to use the two equals signs when it’s an equation. That’s the only tricky part.

Two aspects of this exchange are noteworthy about the exchange. First, the students are genuinely impressed by the power of *Mathematica* to solve a linear equation analytically and efficiently. Unfortunately this willingness to be impressed was not taken advantage of either at the beginning or through the course. Secondly, we see the TA warn the students that there is a “tricky part.”

To solve the equation $3x + 2 = 5x – 1$ in *Mathematica*, the command is `Solve [3 x + 2 == 5 x – 1, x]`. Only in the world according to *Mathematica* are two equals signs involved in solving an equation. The other quirks of the code are that the `Solve` command must be written with an upper case `S` and that the students have to specify that they are solving for `x`. (Many educators would say that the latter quirk is actually a good thing since the students’ attention is drawn to the fact that solving an equation means finding a value of `x` which makes the equation true.)

We will see other incidences in the learning episodes of Chapter 5 where the dissonance between computer representations and calculator representations.
This was most clear in the case of graphs, which *Mathematica* presents with axes that do not generally intersect at (0, 0). There is a reason why *Mathematica* does this (a combination of aesthetics and what *Mathematica* is programmed to consider the “interesting” part of the graph), one might even accept it as a good one but rather than providing a learning opportunity for the students the effect is for them to lose faith in *Mathematica* relative to their calculators. The computer’s representation does not, by the students’ standards, make a lot of sense and they never develop a facility in manipulating the “window” through which they look at a graph to the level that they do with their graphing calculators.

As we shall see in the Student ↔ Mathematics section the dysfunctional relationship in fact goes further with the students often believing that *Mathematica* is doing the work not the students themselves.

A final word in this section should be said about *Mathematica* notebooks. To do work on a particular question the students work in what is called a *Mathematica* notebook. This means that the text of the question as well as some instructions and, possibly, some *Mathematica* code has been pre-programmed for the students to interface with. The materials, therefore, that the students form an interactive electronic text. The students have this text on a CD and also have a hard copy of all the text and pre-run code. It was interesting to observe how the students treated this text, particularly in relation to the possibilities for experimentation in *Mathematica*.

Despite the interactive nature of the textbook the students treated the space provided by the *Mathematica* notebooks as fixed, inasmuch as they were reluctant to try an approach, erase the attempt and try again, even to the point
they sometimes worked something out on paper before entering the relevant code into Mathematica. In this sense, the students often treated the notebooks as they would treat a textbook and were as reluctant to use the notebook as a sketchpad for working out possibilities as they would be reluctant to work out questions on the pages of a printed textbook.

*Student ↔ Mathematics*

\[\text{CALCULUS}\]
\[\text{STUDENT}\]
\[\text{MATHEMATICA}\]

The use of technology is a powerful, perhaps overpowering, influence in the Calculus & Mathematica course but the students come with, and develop in the course, a separate relationship with mathematics/calculus. Sometimes this relationship is in harmony with the use of technology and is sometimes more dissonant. As with the relationship to technology each of the students’ stories bears telling.

*Student A*

Student A had a very strong record in school mathematics. He got high school credit for algebra while still in the 8th grade and had completed all the
mathematics classes that his school had to offer before his senior year. “I wanted to take more math classes. I wanted to maybe go through it a little bit quicker. We had post secondary option but unfortunately they wouldn't work with me to set up a schedule so that I could actually get through all of the courses offered in high school and take further courses in college” (A, 1st interview, Sept 29th). At the beginning of the course he had a positive attitude to mathematics. He found himself bored in his early years of mathematics but once he encountered algebra he found mathematics much more interesting. “Once I got into algebra I started to see where some of it might be able to be used for something that actually happens, and I started to get into [mathematics]” (A, 1st interview, Sept 29th). A sees that creativity may be possible in mathematics but probably at a level beyond which he is operating: “I think that in order for calculus or any of those really to be created there had to be a level of that artistic genius, but now that it’s where it is, and until we get to a point where we’re able to create or define something like that, it’s almost just very mechanical” (A, 1st interview, Sept 29th).

A was very successful in the course. He got an A minus overall and felt that he was learning calculus not just on a mechanical level but on a conceptual level.

Student B

Student B took all the mathematics classes available at his High School. At the beginning of the term he had high self-efficacy in his mathematics: “Basic math I can do in my head. I'm really good with your standard things” (B, 1st interview, Sept 29th). Although he felt that his persistence could be better: “problems that - not that I can't figure out - it's just I give up too easily so I
usually wait for somebody to help me a bit” (B, 1st interview, Sept 29th). B had some exposure to calculus in school and part of his motivation for taking the computer-based course was to have “maybe a more in depth look at why different procedures are done the way they are. I know the rules but [I didn’t] get to sit and learn the basics on why the rule is the way it is” (B, 1st interview, Sept 29th). He felt that in school he learned the how but not the why: “They’ll show you the example—okay, this proves it but it doesn’t really explain it so possibly [with the computer-based approach] I can learn more of why it’s like that or maybe something that will help me remember it better” (B, 1st interview, Sept 29th). In fact he was worried not so much about learning calculus as about learning Mathematica. B’s attitude to mathematics was that work had to be done by hand to prove that you could really do it. Checking by calculator was fine but one still needs to know how to do the calculations. B’s attitude to creativity in mathematics is similar to A’s: “Well—I guess somebody would have to get creative to figure out all these rules … There’s always something, but I mean all the basics are pretty much founded” (B, 1st interview, Sept 29th). By the middle of the term B was starting to regret the decision to take Calculus & Mathematica “because [if I had done the regular calculus course] I’d definitely have a higher grade” (B, 2nd interview, Nov 6th). He was clearly conscious of his grade and felt was a little frustrated that the emphasis on the class was on interpretation and application rather than mechanical procedures: “In calculus there aren’t a whole lot of difficult word problems that they don’t give you all the numbers you need. I’m learning to apply the stuff better but at the expense of my grade. That’s something I don’t like” (B, 2nd interview, Nov 6th). That said, B had a somewhat
rigid notion of mathematics. For example, B was bothered by the fact that
*Mathematica* would give answers in unexpected forms which were equivalent to
the from B expected (e.g. giving an answer as \(-(x – 3x^2)\) rather than \(3x^2 – x\):
“it always comes out with the right answer obviously, but sometimes you’re just
not expecting it and you have to look it over before you realize: okay, they just
did this or they did it a different way” (B, 2\(^{nd}\) interview, Nov 6\(^{th}\)). In fact, it
became clear that B’s idea of authentic mathematical activity involved working
with numbers and expressions to the exclusion of interpretation and explanation.
“That was the whole thing back when we had geometry. When we had theorems
and postulates. I completely thought they were pointless. It’s like somebody
already said this so we just get to the point and use what we’re going to use it”
(B, 2\(^{nd}\) interview, Nov 6\(^{th}\)). This attitude was clear in the workings of the group. B
was involved in solving the problems but generally absented himself from the
articulation of explanations and interpretations: “If it's a small expression and
you have in a paragraph of words, it's just like this is too much English for me”
(B, 2\(^{nd}\) interview, Nov 6\(^{th}\)). B’s dissatisfaction with the Calculus & *Mathematica*
course and his performance in the course (he got a B in the end) was an
expression of this dissonance between what he values in his mathematics and
what the course assigned value to.

*Student C*

C’s secondary education was in a rural school district and, while he took
the classes available to him, his background was not as strong as A and B
entering the Calculus & *Mathematica* class. The teaching in his classes was quite
traditional, and he was looking forward to “more discussion about what to do between the students we would normally see in a more traditional classroom setting” (C, 1st interview, Sept 26th). At the beginning of the term C felt that his learning style was to “learn a lot through demonstration—the detailed demonstration of a problem being done” (C, 1st interview, Sept 26th). C adapted this style during the term to the dynamic of the group which was often A and B convincing C of a certain strategy, approach, or result. C’s other strategy for staying involved was that he was always the group member typing. At the beginning of the term, C had a very positive attitude to mathematics and problem solving: “I really enjoy math. I enjoy sitting, because when you sit there and you stare at a problem, you try it one way, it doesn’t work, and you try it again and it doesn’t work and you try it again the third time, and it comes out right. It’s a good feeling and I enjoy doing them” (C, 1st interview, Sept 26th). C expressed a concern that using Mathematica may change his attitude to mathematics since there would not be as much demonstration of steps and that he may stop enjoying mathematics. That this is how things turned out is a testament to the way the students used Mathematica and the materials. The Basics and Tutorials sections of the Calculus & Mathematica are designed to provide many of the kinds of demonstrations that would harmonise with C’s learning style. However, the working habits of the group, discussed in Episode II: Working habits of the next chapter, meant that C became disadvantaged in this regard. C got a B in the course and took the traditional version of the next course in the sequence the subsequent term.
The Adapted Rotman Model

We have now looked at all sides of the triangular model. The evidence of the observations together with the interviews is that technology must be understood as a separate agency in understanding the place of technology in learning mathematics. The choice of technology and software has an impact on the mathematics, and the students have distinct and separable relationships with mathematics and with Mathematica. There will be further discussion of these results in Chapter 6.

The Adapted Rotman Model was a mechanism for analysing the students as individuals in the research setting. By way of triangulation and to place the students in the context of the class as a whole a grounded survey was administered to the entire class in the last week of the ten-week term.
The survey

The survey was administered to the entire class of twenty-four students. The idea of the survey was to triangulate the ongoing observations and analysis of the students learning. The survey results would provide another layer of insight: the Pirie-Kieren Model serves as a micro-picture, the Adapted Rotman Model as a macro-picture and the survey serves as a super macro-picture. The survey was grounded in observations and analysis of the data ongoing through the ten-week term. Most of the survey questions reflected concerns or issues expressed by the three students in their first two interviews. For example, common concerns such as whether the students felt they were learning calculus and the amount of time spent using computers in the class appeared as questions in the survey. The survey was administered in the last week of the ten-week term.

The full survey with results follow. The four numbers with each question represent the number of students giving the relevant response to each question in the order Strongly Disagree (SD), Disagree (D), Agree (A), Strongly Agree (SA).
<table>
<thead>
<tr>
<th>Statement</th>
<th>SD</th>
<th>D</th>
<th>A</th>
<th>SA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Using <em>Mathematica</em> is a good way to learn Calculus</td>
<td>3</td>
<td>9</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>I feel that I am learning Calculus effectively in 151C</td>
<td>2</td>
<td>10</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>When I ask questions in class they are more about how <em>Mathematica</em> works than about Calculus</td>
<td>-</td>
<td>5</td>
<td>11</td>
<td>7</td>
</tr>
<tr>
<td>The biggest difference between 151C and other mathematics classes is the use of <em>Mathematica</em></td>
<td>2</td>
<td>7</td>
<td>12</td>
<td>3</td>
</tr>
<tr>
<td>The biggest difference between 151C and other mathematics classes is the groupwork</td>
<td>2</td>
<td>6</td>
<td>14</td>
<td>2</td>
</tr>
<tr>
<td>Using <em>Mathematica</em> to learn meant that I was able to concentrate on the meaning of Calculus</td>
<td>5</td>
<td>7</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>Learning using <em>Mathematica</em> meant that I didn’t spend enough time practicing algebraic skills (e.g. solving equations, differentiating functions etc.)</td>
<td>4</td>
<td>4</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>I feel I would be able to give a good answer to the question “What is Calculus?”</td>
<td>3</td>
<td>9</td>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>The lecturer in 151C should spend more time teaching the whole class together</td>
<td>2</td>
<td>7</td>
<td>11</td>
<td>4</td>
</tr>
<tr>
<td>We shouldn’t have to use <em>Mathematica</em> all the time during the class</td>
<td>3</td>
<td>3</td>
<td>15</td>
<td>3</td>
</tr>
<tr>
<td>Learning Calculus using <em>Mathematica</em> means we should be allowed to use <em>Mathematica</em> in quizzes and tests</td>
<td>2</td>
<td>13</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>Learning Calculus using <em>Mathematica</em> in 151C meant the lecturer was not very important</td>
<td>4</td>
<td>9</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>I am interested in doing 152C next quarter</td>
<td>10</td>
<td>7</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Having done 151C, I feel that I am well-prepared for a regular 152 lecture/recitation class</td>
<td>2</td>
<td>13</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>Using <em>Mathematica</em> allowed us to experiment with mathematics (e.g. we could try something get a response from <em>Mathematica</em> then change a parameter and try again)</td>
<td>-</td>
<td>5</td>
<td>14</td>
<td>5</td>
</tr>
<tr>
<td>I feel I would be able to give a good answer to the question “What does a gradient measure?”</td>
<td>6</td>
<td>12</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>A successful calculus student can differentiate the function $f(x) = x^3$ w.r.t $x$, without using <em>Mathematica</em></td>
<td>2</td>
<td>-</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>A successful calculus student can differentiate the function $f(x) = \sin x(x^2 + 2)^3$ w.r.t $x$, without using <em>Mathematica</em></td>
<td>2</td>
<td>-</td>
<td>15</td>
<td>7</td>
</tr>
<tr>
<td>Compared to other mathematics classes I talk more about mathematics in 151C</td>
<td>3</td>
<td>7</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>Compared to other mathematics classes I think more about mathematics in 151C</td>
<td>5</td>
<td>7</td>
<td>10</td>
<td>2</td>
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</tbody>
</table>

Table 4.1: Group survey results

101
Group results

In analysing the results of the survey I have aggregated the positive and negative responses by adding together the SA and A results and adding together the SD and D results. Of the twenty survey items 6 were divided 11 – 13, 12 –12 or 13 – 11 when aggregated. The overall interpretation of the survey seems to be that there is a symmetric distribution in attitudes on many aspects of the class. Those results that are not symmetric tend to view the class and the learning experience somewhat negatively.

There is an overall feeling evident in the survey that *Mathematica* is interfering with some traditional goals and values of mathematics education that the students hold. For example, the question “Learning using *Mathematica* meant that I didn’t spend enough time practicing algebraic skills (e.g. solving equations, differentiating functions etc.)” has a response of 16 versus 8 (A vs. D). Students clearly feel that technical competence is an important goal of mathematics education and that they have been short-changed in this regard. This attitude is further reflected in the fact that an overwhelming majority of the students (22 vs. 2) feel that a successful student of calculus is able to differentiate both basic and not-so-basic functions without the support of technology.

One of the goals of the course is to deemphasise calculations and emphasise meaning and interpretation in the learning of calculus: “Typical mathematics classes place emphasis on memorization and isolation. Students do not need to learn the material, they simply have to memorize how to perform specific symbolic manipulations in specific contexts. In C&M, we feel the focus should be on understanding concepts.” (http://socrates.math.ohio-
The students’ experience of the course suggests that this transition was not accomplished. This is clear in light of the results above about what students value in their mathematics education, and is quite dramatically brought out in the response to the question “I feel I would be able to give a good answer to the question ‘What does a gradient measure?’”: 16 versus 8 (D vs. A). The course also fails in another of its goals since students feel that they are being somewhat disadvantaged taking the course: Having done 151C, I feel that I am well-prepared for a regular 152 lecture/recitation class: 15 versus 9 (D vs. A).

It is clear from the survey results that a good deal concern with student learning involves *Mathematica* “getting in the way” and the logistics of the class. In relation to the visibility of the technology *Mathematica* performs badly: “When I ask questions in class they are more about how *Mathematica* works than about Calculus”: 18 versus 5 (A vs. D). Clearly students found it difficult to get past the idiosyncrasies of *Mathematica* to focus on the calculus. The complete de-emphasis on direct instruction was another area that students did not respond to well: “The lecturer in 151C should spend more time teaching the whole class together”: 15 versus 9 (A vs. D) and “We shouldn’t have to *Mathematica* all the time during class”: 18 versus 6 (A vs. D). The students are clearly looking for more direction and the frustrations of working with *Mathematica* are not helping.

The students final judgment on the class comes in the response to the question: “I am interested in doing 152C next quarter”: 17 versus 7 (D vs. A). This judgment is hardly surprising given the other results above. The overall reason for the dissatisfaction with the class is that the students found it difficult
to make the adjustment to learning mathematics in a new way. If the three students studied in detail are in any way typical of the general population of the class, then the students have spent their entire mathematics career being taught in a particular way and are asked to change the mindset and approach they have developed in a ten-week term. This transition is not helped by the fact that Mathematica is not an easy program to work with, and as we discussed in several places above, this difficulty was not made easier for the students. The difficulty of the transition was exacerbated by the way the class was run with the almost complete absence of direct instruction and the insistence on the students working at the computer at all times. The result was that the students held on to their traditional goals of success and achievement in a mathematics class and did not feel that learning with Mathematica aided them in those goals. Nor did they embrace new, alternative goals of the Calculus & Mathematica course.

It is worth looking in some detail as well at the questions on which the students in the survey were divided.

The question of what the students valued in the learning and what they felt they learned was addressed above. The results seem, at least at first blush, to be contradicted by the following two responses: “Using Mathematica to learn meant that I was able to concentrate on the meaning of calculus”: 12 versus 12 and “I feel I would be able to give a good answer to the question ‘What is calculus?’ “: 12 versus 12. My interpretation is that these responses are divided because the questions are quite abstract and general in nature. It is significant that when the students are asked the much more specific question: “I feel I would be able to give a good answer to the question ‘What does a gradient
measure?”, the response changes to 16 versus 8 (D vs. A). The response to “Using Mathematica is a good way to learn calculus”: 12 versus 12, seems also not to be in tune with other responses and may reflect an attitude among a few students that learning using Mathematica is in principle a good idea even if it does not necessarily reflect their experience in this class.

Finally, it is interesting to look at the following two responses: “Compared to other classes I talk more about mathematics in 151C”: 10 versus 14 (A vs. D), and “Compared to other classes I think more about mathematics in 151C”: 12 versus 12. The first response reflects the more collaborative environment of a Calculus & Mathematica class when compared to other mathematics classes, although it is a little surprising that the result is not even more pronounced. The latter response is another reflection of the overall failure of the course that seeks to involve students more in their learning and facilitate students in thinking about mathematics rather than just memorising algorithms.

*Individual results for the students A, B, and C in the study*

Looking at the individual results we see that on 14 of the 20 survey items B and C are on the same side of the SA and A versus D and SD dividing line. A is on his own on 8 of the 20 items.

The full results are shown in Table 4.2 below:
Using *Mathematica* is a good way to learn Calculus           SD   D   A   SA
I feel that I am learning Calculus effectively in 151C        -   B, C   A   -
When I ask questions in class they are more about how *Mathematica* works than about Calculus -   A   -   B, C
The biggest difference between 151C and other mathematics classes is the use of *Mathematica* -   -   A, B   C
The biggest difference between 151C and other mathematics classes is the groupwork -   C   A, B   -
Using *Mathematica* to learn meant that I was able to concentrate on the meaning of Calculus C   -   B   A
Learning using *Mathematica* meant that I didn’t spend enough time practicing algebraic skills (e.g. solving equations, differentiating functions etc.) A   -   B   C
I feel I would be able to give a good answer to the question “What is Calculus?” -   C   A, B   -
The lecturer in 151C should spend more time teaching the whole class together A   B, C   -   -
We shouldn’t have to use *Mathematica* all the time during the class -   -   A, C   B
Learning Calculus using *Mathematica* means we should be allowed to use *Mathematica* in quizzes and tests A   B, C   -   -
Learning Calculus using *Mathematica* in 151C meant the lecturer was not very important A   B, C   -   -
I am interested in doing 152C next quarter B, C   -   -   A
Having done 151C, I feel that I am well-prepared for a regular 152 lecture/recitation class -   B, C   A   -
Using *Mathematica* allowed us to experiment with mathematics (e.g. we could try something get a response from *Mathematica* then change a parameter and try again) -   C   B   A
I feel I would be able to give a good answer to the question “What does a gradient measure?” B, C   A   -   -
A successful calculus student can differentiate the function \( f(x) = x^3 \) w.r.t x, without using *Mathematica* C   -   -   A, B
A successful calculus student can differentiate the function \( f(x) = \sin x(x^2 + 2)^3 \) w.r.t x, without using *Mathematica* C   -   -   A, B
Compared to other mathematics classes I *talk* more about mathematics in 151C B   C   -   A
Compared to other mathematics classes I *think* more about mathematics in 151C B, C   -   A   -

<table>
<thead>
<tr>
<th>Table 4.2: Individuals survey results</th>
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The fact that A is on a different side of the Agree versus Disagree line so often is consistent with the analysis via the Adapted Rotman Model in this chapter which suggested that A had a different experience of the class than B and C. A’s responses to the survey question are almost uniformly positive and reflect what I perceive to be his determination that the course and learning with Mathematica were valuable. I am not suggesting that A is being disingenuous. He was one of a small number of students to continue in the computer-based sequence but he does put a consistently positive spin on his experiences in the class.

In B’s case only two of his responses could be seen as reflecting positively on the class and on Mathematica. His responses to the questions which I consider to be the most important indicators of response to learning with CAS are almost all negative:

Using Mathematica is a good way to learn Calculus
I feel that I am learning Calculus effectively in 151C
When I ask questions in class they are more about how Mathematica works than about Calculus
Using Mathematica to learn meant that I was able to concentrate on the meaning of Calculus
Learning using Mathematica meant that I didn’t spend enough time practicing algebraic skills (e.g. solving equations, differentiating functions etc.)
I feel I would be able to give a good answer to the question “What is Calculus?”
Compared to other mathematics classes I talk more about mathematics in 151C
Compared to other mathematics classes I think more about mathematics in 151C
On these major issues of the experience of the class and relationship to Mathematica, C’s responses were almost identical to B’s. The one question on which they differ is the question of whether the student feels that they would be able to give a good answer to the question “What is Calculus?” My feeling is that B’s self-efficacy in this case comes from the exposure he had to calculus in high school.

It is also interesting to look at the questions on which C is an outlier from the other two students. An examination of those results shows a strong typist/data entry effect. For example, in response to the question “The biggest difference between 151C and other mathematics classes is the groupwork” A and B agree but C disagrees. This may be in large part because he has not had the same experience of groupwork as the others because of his concentration on typing. A starker example is the responses to the question “Using Mathematica to learn meant that I was able to concentrate on the meaning of Calculus.” Again A and B agree (A strongly) and C disagrees strongly. Using Mathematica involved him concentrating on data entry not calculus. There are other examples of this typist/data entry operator effect on C’s experience of the class which shows poor management by the student’s of their own groupwork.

The examination of the three students responses to the survey shows when compared with that of the group overall shows that A’s positive experience was not anomalous and there were other students who had a positive experience in the class while learning with Mathematica. This comparison also shows that B and C’s experiences were also reflected in the group as a whole.
Overall the results of the survey show that many students had a successful experience with learning calculus using Mathematica. However, many of the students were alienated from their mathematics and did not make the transition to experiencing the affordances of Mathematica in supporting their learning and did not wish to continue learning using CAS and would feel more comfortable in a traditional environment.
This study is designed to address the question “What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus?” I argued in Chapter 2 that there is a gap in the literature on research in CAS between outcome-based studies and more philosophical work on the curriculum and assessment implications of CAS. Missing are studies which look at what exactly happens when students try to learn mathematics in a CAS environment: What does CAS learning look like on a day-to-day basis? and What does CAS learning look like across the period of a ten-week course?

In order to address these micro and macro questions I have mobilised two theoretical frameworks. In the previous chapter I showed that the Rotman Model of Mathematical Understanding (Rotman 1993; Rotman 1995) provides a lens through which to view the place of technology in the CAS classroom and through which to view the journey of each student across the ten-week term. In this chapter I will use the Pirie-Kieren Model for the Growth of Mathematical
Understanding (Pirie & Kieren, 1990; Pirie & Kieren 1994) as a lens through which to examine specific learning episodes as they occur in the classroom. The unit of analysis for the learning episodes is the group of three students as a whole whereas each student is a unit of analysis for the broader picture across the ten-week term.

Methodological considerations

This chapter consists of analysis (a close reading if you will) of a series of learning episodes. This begs the questions of how the episodes were chosen and what they are meant to illustrate.

The episodes were chosen through an iterative process. During my classroom observations I noted interactions which were deemed to be interesting under the broad rubric of looking at the sides of the Modified Rotman triangle: Student ↔ Technology, Technology ↔ Mathematics, Mathematics ↔ Student. Arising from these notes specific sub themes emerged as patterns across several episodes e.g. the problems posed by the syntax of Mathematica; how the students used the instant feedback possibilities of Mathematica; and issues raised by the output representations of Mathematica (e.g. graphs).

The second iteration in the choosing of episodes was during the transcription of the audio and videotapes of the classes. More episodes which were exemplars of the students processes throughout the term were noted and tallied to establish that they provided themes across many classes. Themes emerging from this iteration included the visibility of technology; and the working habits of the students in relation to the text.
The final two iterations of episode choice were a combination of reading through the transcripts of the interviews with the students for further themes and then looking through the class transcripts to find examples of these themes and to establish that the themes appeared multiple times. Finally, some themes emerged from the presentation of some early analysis to a seminar reading group. Themes emerging from these final iterations included issues related to the power of CAS; and the possible effect of using different types of CAS.

The Learning Episodes

The outcome of these various iterations is nine learning episodes, each chosen as an exemplar of student interactions, approaches, discussions, and strategising the students engaged in through the quarter, that is, to show the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus.

The episode titles together with a brief annotation are:

• Episode I: Visibility of technology
  How does Mathematica present an obstacle to student learning by making it hard to see the mathematics?

• Episode II: Working habits
  How do the students use the text and materials of the course?

• Episode III: The syntax of Mathematica
  What is the effect of the Mathematica’s unique syntax on the students’ work?

• Episode IV: Using CAS for what it is really good at
How well do students see the power that CAS affords them?

- **Episode V: Instant feedback**
  
  How do the students’ use the feedback provided by the interactive nature of CAS?

- **Episode VI: Too much power?**
  
  Does the power of *Mathematica* take over student strategies and divert them from their goals?

- **Episode VII: Multiple representations**
  
  How do students link the algebraic and graphical representations afforded by CAS?

- **Episode VIII: The Role of Metacognition**
  
  Does working in a CAS environment help to develop the students’ metacognitive skills?

- **Episode IX: A conjunction of episodes**
  
  What happens when a number of the issues from the other episodes happen together?

The format of each episode report is a paragraph introducing the theme of the episode followed by the episode itself which is a combination of screen captures from the students’ activity, dialogue from their conversations, and analysis of the students’ processes in the context of the theme. Each of the episodes has letters in the margin, for example $F_2$. These letters are a guide to the Pirie-Kieren map of the episode with the letters representing the level and the number representing the time that level has been visited. Therefore, $F_2$, in the margin means that at that stage of the episode the students are operating at the
Formalisation level for the second time in that episode. The students are referred to by their usual labels A, B, and C; I stands for Instructor and TA stands for Teaching Assistant.

*Episode I: Visibility of technology*

There is a strand of discourse on technology in mathematics education which revolves around the visibility of the technology. There are those who say that technology is at its most effective when it is invisible, which is to say that the technology works so well that it almost seems that it is not there: the students see only the mathematics and not the technology. Another way of positioning technology suggests that it should be highly visible, in that the students should see very clearly how the technology is able to support their mathematical activity and, therefore, their learning of mathematics. In any case, regardless of visible the technology is it is certainly present and certainly impacts, one way or another, on student learning. It is important that the locus and impact of technology be understood, hence the triangular model modified from Rotman is a better model for understanding the place of technology than the linear model outlined on page 74, which simply inserts technology in between the student and the mathematics.

The visibility argument does, however, hold some sway when, rather than being blind to the technology, the student is blinded by the technology. When the technology actually interferes with student learning then it becomes hard to support its use. We will see in this episode that the visibility of the technology in the learning environment that gives issue to that visibility is a crucial component of the process of student learning in a CAS environment.
Episode I, in which the students fail to make the leap from the calculator (which they love) to the computer (which they do not love).

This episode takes place on the second day of class. On the first day the lecturer read through the syllabus but did not, crucially as it turns out, introduce the students to Mathematica. The questions the students attempt are in the section “Give it a Try” and are close mirrors of the questions in the “Basics” and “Tutorials.” The students began as they went on by largely ignoring the “Basics” and “Tutorials” in their work. On occasion they had read them in advance but mostly they were referred to, if at all, only if the students were having difficulty with the question in front of them.

In this set of exercises the students explore linear models starting with:

A function \( f(x) \) starts at \( x = 0 \) with a value of 7.9 and goes up at a constant rate of 0.3 units on the y-axis for each unit on the x-axis. Give a formula for this function and plot it.

The students are easily able to see that the required function is \( f(x) = 0.3x + 7.9 \) and produce the following plot:
The students are happy with this and do not notice that y-axis is running from approximately 7.8 to 9.4 and assume that the intersection of the axes shown above is at (0, 0). This is not a problem at the moment but we will see in a subsequent problem that this quirk of how Mathematica renders graphs will become an obstacle to learning.

The next problem is:

A function \( f(x) \) starts at \( x = 1.1 \) with a value of 7.3 and goes up at a constant rate of -0.3 units on the y-axis for each unit on the x-axis. Give a formula for this function and plot it.

This prompts the following dialogue:

B: That’s not fair.
C: Why?
B: Because they don’t give you the y-intercept.
C: But it’s easy to figure out.

The students do some pencil-and-paper work to figure out that the function is \( f(x) = 7.63 - 0.3x \) by multiplying -0.3 by 1.1 and subtracting the result from 7.3. The trouble starts when they plot the function:
Once again the plot does not have (0, 0) as the intersection of the axes and so the x-axis shown is not the standard x-axis (which passes through (0, 0)) that the students are used to seeing in a graph. This causes a great deal of confusion because one of the students has graphed the line on a graphing calculator and the result looks different from the graph *Mathematica* has produced. This could be a learning opportunity but in fact it merely sidelines the students for more than ten minutes as they try to discover what they have “done wrong” in asking *Mathematica* to plot.

A: Something’s wrong with the graph. It shouldn’t look like that.
B: It is “Zoom Standard.”
A [to himself]: Zoom Standard. That’s a standard zoom. Now, look …
C: It’s because …
A: We’re seeing an x-intercept of something near 2.
A: Here’s our intercept. That’s what the intercept should be on that formula. That’s what it is there. [pointing to calculator]
C: For which formula? This is the 7.63.
A: Yeah.
B: Yeah. The slope’s wrong.
A: Something’s wrong with the computer set up.
C: 7.63 minus point 3 and x is 2.
A: We must have typed something wrong somewhere. Let’s see. That’s what our graph looks like. The same as yours. Our intercept should not be 2. It should be more like 25.4.

[silence]
A: The formula we put down is right because it gives the right answer with the calculators but we did something wrong with the plotting because we got an entirely wrong graph.
B: We didn’t do something wrong with the plotting. We did something wrong with the … Something’s wrong somewhere because we did the same formula in both places. I’m not going to say it’s necessarily the computer but definitely not the graph on my calculator.
A: No. My calculator is right. I can guarantee that. And, I even tried it this way.
C: No, no. You’re right. If you do it and you solve for y …
B: It’s not like 2.
A: We got the right answer on the calculator. We got the right answer on paper. But our graph looks wrong.
B: Let’s put in our graphs from calculator to …
T: That’s the right formula.
B: Yeah. It’s the wrong graph though.
C: No it’s not. Because this isn’t zero guys. This is 7. So …

There are a number of important points in this exchange. One key point is the difference between the students’ attitude to computers and calculators at this stage: the very beginning of the quarter. The students trust the calculator and are confident in their use of it. They are not confident about Mathematica and have not made the transition to understanding and feeling comfortable with the computer software as a calculating tool. They are, in a sense, doing their work on the calculator and then attempting to get the computer to conform to this. In the early part of the quarter we hear the students constantly refer to “programming” the computer. (Importantly they are talking about using the basic interface of the software). The interface that students have with a calculator, which, like the computer, has software running on a machine, is analogous to the interface they have with the calculator but they never do, and never would, refer to their use of the basic interface of the calculator as “programming.”

In fairness, the computer is not being helpful. It is presenting the students with an unorthodox view of a linear graph and not manifesting the x-axis in a way they have seen before (except in the previous problem). What Mathematica has shown them is correct but is not problematic until the second problem when
they are trying to reconcile the output with standard calculator output. This issue of the “false” graph arises on several more equations through the quarter and presents a noticeable hindrance to learning on each occasion. The students, in fact, never get used to it. The design of the computer software is very much less intuitive than the calculator software. Furthermore the design of the computer software is much less in tune with mathematical representations that the students are used to seeing both in textbooks and in classrooms. The transition to the computer software is, therefore, inherently problematic, but is exacerbated by the instructor’s omission in not even addressing, let alone helping, the transition from calculator (a domain of comfort and familiarity to the students) to the “hostile” domain of the computer. We shall in several subsequent episodes the hindrance this imposes on student learning.

In Pirie-Kieren terms we see the difficulty students have in the developing from Image Making to Image Having. The students are able to generate plots but have difficulty in correctly interpreting the image they have generated. Granted the computers representation is, at best, unhelpful the students are not good at interpreting graphical information in part because they have not developed good habits of expectation about what the graphical representation of algebraic information should be. The result is that instead of reflecting on multiple graphical representations of information the students waste ten minutes trying to see what they did wrong and all they learn at the end is that they don’t like how Mathematica presents graphs.
Also, the students have noticed the property of the link between the algebraic representation and the x-intercept of the graph but have not formalised this concept to the extent that it solves the problem for them.

Figure 5.1: The Pirie-Kieren Map of Episode I
Episode II: Working Habits

As mentioned before the materials with which the students were working are split into four sections: (i) Basics; (ii) Tutorials; (iii) Give it a Try; and (iv) Literacy. The first two of these, which the students have on CD and in hard copy (a print out of the CD) largely serve the function of a traditional textbook: the introduction of concepts and the illustration of those concepts through examples. The “text” is, however, interactive in that the students read examples, are supposed to think about what might happen when some Mathematica code is run, and then run the code. A number of issues arise here which will play out in various ways in the students work through the quarter: (i) the questions in the “Give it a Try” section (i.e. the questions that the students work on together) are often very similar, if not identical, to examples in the Tutorial section. However, the students rarely use the text as a resource in this way; (ii) an obvious advantage of using CAS is that you can get instant feedback on mathematical procedures you implement (drawing a graph of a function etc.). A possible disadvantage of being able to do this so quickly and easily is a lack of student reflection before they have the technology implement the procedure on what the outcome might look like.

We will see in this episode that the students often do not use resources well in both of the senses outlined above. This pattern persisted throughout the quarter.
Episode II, in which the students establish their working pattern and still love their calculators.

The students are given the question:

*Here are two points *(1, 3) and *(3, 2). Find a formula for the line function whose plot hits both points. Confirm your answer with a plot.*

The students are able to answer this question fairly easily but they do so outside the realm of Mathematica and, in fact, only run into trouble when they start to work with Mathematica. In the Tutorial section the students are introduced to a built-in function of Mathematica which will calculate a linear function given two points. There is no mention of this example or this function in the conversation the students have about their approach and solution. What they do use is a program that one of them wrote on his calculator to solve exactly such a problem which they have encountered in their high school mathematics.

B: Just do it on paper. \( y – y_1 \) etc.
A: This is a brand new calculator so I don’t have the program written in yet.
C: I wrote this program myself.
A: You sound like me. That’s what I’ve always done.
B: What?
A: Writing programs.
B: [seeing that C has a result] Plug in into the computer.
C: OK. The slope is negative point 5.
A: The \( y \)-intercept is …
C: The \( y \)-intercept is 3.5. The slope is negative point 5.
A: That looks like it should be right.

It is also worth noting in this example that the students are now sensitive to the window that Mathematica is offering and are somewhat flexible in using
the graphical representation to check if the points are on the line. This sensitivity comes and goes through the quarter. The students do not, however, use the techniques outlined in the equivalent problem of the Tutorial section to do an algebraic check of the points. We see here, therefore, that the students already lacking in a proper introduction to Mathematica are also not educating themselves by using the materials at hand well. This habit of work persists through the quarter as we shall see in subsequent episodes.

In Pirie-Kieren terms we see that students are able to stay on the formal level and solve the problem. They do so in a technology context but without using Mathematica.
Figure 5.2: The Pirie-Kieren Map of Episode II
**Episode III: The Syntax of Mathematica**

There are many programs which incorporate CAS: `Mathematica`, DERIVE, and Maple are just a few. Of these three `Mathematica` and Maple were designed as research tools for mathematicians and scientists. DERIVE, on the other hand, had a pedagogical perspective built into its design from an early stage. In other words `Mathematica` was not designed for use in learning mathematics but rather for use in doing mathematics and was designed to be used by expert mathematicians. Later versions of `Mathematica` have become more user friendly (error messages are not in bright red anymore) but it clearly remains designed to be used by someone who knows the mathematics. The use of `Mathematica` is highly syntax driven: the user not only has to ask `Mathematica` what s/he wants to do but has to ask with precisely the correct syntax (DERIVE, by way of contrast, is much more menu driven). Consequently, `Mathematica` is full of quirks, which once you get used to them shouldn’t interfere with your work. For example most functions in `Mathematica` have to be capitalised: if you want to graph a function you type “Plot” not “plot.” If you write the latter the following happens:

```
In[7]:= plot[2 x + 3, {x, -5, 5}]

Out[7]= plot[3 + 2 x, {x, -5, 5}]
```

`Mathematica` simply doesn’t plot the function. You don’t even get an error message saying why the plot was not displayed.

To define a function f(x) you have to write f[x_] = 2 + 3x but to graph the same function you write `Plot[f[x]]` without the underscore.
In all of mathematics $e$, referring to the base of the natural logarithm is written in the lower case except in Mathematica where the user has to write “E.”

There are countless incidences in the course of the quarter where the students execute a Mathematica function, get an error message, and realise that they need to change a letter to upper case. Sometimes the students find the syntax errors more difficult to fix. In the episode which follows we see the Mathematica syntax defeat the students.

*Episode III, in which the peculiar syntax of Mathematica defeats the students.*

The students start with the following problem:

*A certain airplane uses $k$ gallons of fuel for each combined take-off and landing. If this airplane uses 1081 gallons of fuel on a 270-mile trip and uses 1575 gallons on a 435-mile trip, then estimate the miles per gallon delivered by the airplane and estimate the value of $k$.***

After some discussion and some working on paper the students come up with the following:
This is actually a full solution, if interpreted correctly, with $k$ being $\frac{2999}{11} = 272.6$ and the miles per gallon being $\frac{494}{165} = 2.99$. However, the focus of the student’s discussion becomes why the graph does not appear as they expect even though they are not asked to provide a graph.

B: What happened to the graph?
A: I’m trying to solve for ...
TA: Yeah, you can’t do that ...
A: Is there a way I can?
TA: Yeah. Yeah. Get rid of the ...
C: Why is it?
TA: Now say $f(270)$ is it? Instead of $x$ put in ...
A: Yeah. Ok
B: Our line’s not on that graph that’s why we don’t see it.
The attempt at correction is to define the variables of the function to be plotted based on the successful calculations from the first attempt and to define a function f:

```math
\text{b} = \frac{2999}{11} \\
\text{m} = \frac{1575 - 1001}{135 - 270} \\
\{f[x] = m \times x + b, \{x, 0, 2000\}\}
```

This comes after a prolonged discussion including some time spent looking back at previous successful plot syntax to try and model. However, the students have gotten somewhat confused. They try to define a function f(x) but need to write f[x_] instead of f(x) and then give lower and upper bounds for a plot despite not having asked *Mathematica* to plot anything. Naturally this fails to provide a plot and results in an error message.

At this point the students give up prompted by B complaining that they don’t need to make a plot anyway.

B: Why are we plotting things? Because it just asks “estimate the miles per gallon” and “estimate the value of k.”
C: We don’t need to plot it?

...  
A: Our slope is our miles per gallon, k is the take off and landing.
C: He’s exactly right.
A: we have the slope, the miles per gallon and k.
B: So why are we graphing anything?
A: We don’t have to but it’s not that hard.
B: Yeah we just spent in excess of ten minutes when we could have been
done with this problem if we weren’t worried about how a graph looked.
A: We’re learning the program which is useful.

The students in fact spent almost half an hour on this problem, at least
half of which was spent discussing the graph. Despite what A says they do not
end up “learning the program” because they never satisfactorily address the
problem of how to draw the graph. What is frustrating the student learning here
is the peculiar, non-intuitive, syntax of Mathematica which represents
mathematical objects in a logical but non-standard way and interferes with the
student learning.

In many subsequent episodes we see the students spend inordinate
amounts of time trying to work through why Mathematica is not helping them
solve a problem and the root cause turns out to be that they are using “e” instead
of “E” as the base of the natural logarithm or they don’t clear a previously stored
value so Mathematica is using the wrong number in a calculation.

In this episode it is not clear that there has been any growth in
mathematical understanding so the Pirie-Kieren model does not apply.

Episode IV: Using CAS: for what it is good at

Advocates of CAS in education claim various advantages and possibilities
for the technology. One of the most cited possibilities for CAS is to take the
burden of calculations away from students and allow them to focus on the
concepts involved. For example, if a problem involves finding turning points of a
quite complicated function the students can focus on the concept that the
differential of the function can help in finding the turning points while getting
the CAS to do the differentiating and solving. The students then accept the
burden of interpreting the results the CAS produces safe in the knowledge that
the CAS has not made any errors in performing the differentiating and solving
algorithms.

As we have seen above, the students in this study had considerable
difficulty shifting from graphing calculator technology to CAS technology. Part
of this has to do with the attachment that they retained to their hand-held
calculators and part of the story is the students never really understanding what
the CAS can do for them. As a result of this the students spent time and effort
performing algebraic manipulations that the CAS is designed to do for them. We
see an example of this in the next episode.

Episode IV, in which the students don’t understand what the use of CAS is in the
first place.

In this set of exercises we see that the students have little appreciation of
the symbolic manipulation functionality of Mathematica and do not use it as a
“scrapbook” on which to explore mathematical ideas.

The students are working on the problem: Fill in the blank: If you go with a
line function $f(x) = ax + b$ and $h$ is any number then $f(x+h) – f(x) = \_\_\_\_$

The students have been prepped for this problem with a sample problem
with the following output:
This is followed by an example of how the function \( f(x) = 2x - 3 \) increases by 2 units for every increase of 1 unit in \( x \). Therefore, in answer to the question above the students should either know straight away from their knowledge of linear functions that the answer is “ah units” OR be able to use the template above to work out that the answer is “ah units.”

In fact, in what follows C immediately says that “ah units” is the answer but in his reasoning he works with the function \( f(x) = 2x + 3 \) and choosing numerical example which fits his, unexplained theory, that the answer is “ah units.” A takes a similarly numerical approach:

A: Yeah. \( ah \) is right. Well, I wanted to see it on paper?
B: What mathematics did you use to find that?
A: Here’s what I did. Exactly as they said, only I plugged in a number for \( h \). So I plugged in 4 for \( h \) the first time and I plugged \( x + 4 \) in for \( x \) which gave me \( 2x + 5 \). Then I subtracted the function \( x \) which was negative \( 2x + 3 \) and 8 came out.
B: Which is the same as 4 times 2.
A: And then I did. Yeah. The same as 4 times 2. And then I did the same thing with 5 just to make sure that it wasn’t a fluke.

Since the thinking that the students are employing here is numerical rather than algebraic it perhaps not surprising that they do not use Mathematica as an algebraic calculator. This episode occurring as early in the quarter as it does shows that the students still do not understand the algebraic capabilities of Mathematica (having not been introduced to these capabilities) and that they trust their own paper-and-pencil abilities above Mathematica. As the episode develops a conflict emerges as they reach the limit of their paper-and-pencil abilities but are unable to use Mathematica to help them.

In the next exercise the students work on a problem analogous to the one above except this time working with an exponential rather than a linear function. As above they work through a problem showing the nature of the growth which has the following output:

```
Exponential functions do not exhibit a steady growth rate.
Look at the exponential function f[x] = 10 e^{1.2x}.

In[59]:= Clear[f, x];
f[x_] = 10 e^{1.2 x}
Out[59]= 10 e^{1.2 x}

In[61]:= Factor[f[x + 1] - f[x]]
Out[61]= 2.3 10 e^{1.2 x}

The bigger x is, the faster f[x] grows.

In[65]:= Table[f[x + 1] - f[x], {x, 0, 5}]
Out[65]= {23.012, 77.0306, 255.751, 849.122, 2819.18, 9368.02}
That's big-time exponential growth.
Now look at this table:

In[66]:= Table[100 (f[x + 1]/f[x] - 1), {x, 0, 6}]
```
In this case the template the students are given for solving the problem is more important than in the previous problem since, arguably, it is not as easy to see the answer without some calculation. The students do opt for an algebraic approach but, without any discussion of how `Mathematica` might help them they start to work on paper.

A: What? Yeah. Does this want us to create our own function or does it want us to just use the letters?

B: That’s a good question. I think the letters. I think the letters would answer.

C: Yeah we have to use the letters.

A: Can I have a sheet of paper so that I can do this myself also? Thanks.

The students spend the next twenty minutes with paper and pencil attempting to simplify the expression $100 \left(\frac{f(x+1)}{f(x)} - 1\right)$. While they, finally, do this correctly the irony is that it is precisely their lack of skill and confidence in algebraic manipulation which causes the students trouble in this work and it is precisely the performing of this algebraic manipulation which is one of `Mathematica`'s greatest strengths.

In the first part of the problem the students work from the formal level down to the property noticing associated with the numerical patterns they generate and quickly make a successful return to a formal solution of the problem because of their ability to work formally with the numerical patterns. This is similar to what we see in a later episode when students can “make” an image but do not “have” the image. In the second problem they spend a long
time stuck on the level of Property Noticing because they do not have the same command of the properties associated with algebra, its rules and its patterns, as they do in the numerical realm. They do eventually return to the formal level but are unable to get Mathematica to help them in this task.

Figure 5.3: The Pirie-Kieren Map of Episode IV
Episode V: Instant feedback

Another claim of use and effectiveness of a CAS in learning is the possibility of instant feedback. Too often when they are learning students are working through problems and calculations without knowing whether what they are doing is correct or not. Of course the possibility for feedback has to be managed in a way which benefits the learner. If the student does not make a genuine attempt at a question before using the CAS to do some work then the opportunity for learning is lost.

We see in the following episode the students using the possibilities of CAS quite effectively to practice basic rules of differentiation.

Episode V, in which the students use the CAS quite effectively.

The writers of the materials for Calculus & Mathematica state at the beginning of this set of questions “Rapid hand calculation for its own sake is not the goal of this course, but if you expect to be successful in understanding the myriad of application of the derivative, you should be in a position to obtain these derivatives without much work or thought.” (Calculus & Mathematica, 19??) All fifteen questions in the set are straightforward applications of the power rule, the product rule, and the chain rule. The students develop an impressive discipline of discussing among themselves what the answer should be before asking Mathematica.

The first question is to differentiate $x^5 + 6$. (We see as well that there is some trouble with the Mathematica syntax).

C: [reads question]
B: 5 x to the fourth.

A: Uh-huh.

B and A: No. No.

B: No. The 6 is constant.

C: Right, right, right.

A: That’s fine. It can stay exactly as it is [pause] space plus space six. Hold up.

A: There it is [pause] that’s not it. Just delete the little x at the end then.

B: Oh. Capital D.

A: Yeah. That’s it.

B: And you need that x at the end also because it’s with respect to x.

A: There it is. 5 x to the fourth.

The next problem: 3x^5 – 4 x^2 again causes no problems and prompts the following comment:
B: 15 x to the fourth minus 8 x. Disagree? I think you can do them faster with your mouth than you can with your hand.

The fourth question allows the students to work with the product rule: e^(2x) * Sin (3x). We see in the following a useful exchange about commutativity as it relates to the product rule but more importantly we see the students working together and agreeing on an answer before checking on the CAS. It is also interesting to see the power of Mathematica in changing their linearly written answer into standard form.

B: Alright. I can’t do this one in my head. It’s going to be e to the 2x times 3 Cosine of 3x plus …
C: Wait. Wait. Wait. It’s 2x e …
B: What?
A: 2x e to, errr, 2 e to the 2x.
B: What are you doing?
A: This.
B: Yeah. What are you doing?
A: Derivative of the first … Is this not?
B: No this is a product rule. First times the derivative of the second
A: Yes
A and B have different mnemonics for remembering the product rule which causes some temporary confusion.

B: Plus second times the derivative of the first. e to the 2x …
A: Well, you can do it either way because it’s addition.
B: Yeah.
A: e to the power of 2x. You’re going to need that little power sign. Power 2x.
...
B: 3 Cosine 3x.
A: You’re going to have to have that in parentheses. 3x
B: plus
B: 2 e to the 2x.
…
A: See if we’re right. Oh.
They don’t get quite the feedback they expect but they have a strong
enough command of what they’re doing to adjust.

B: You better go up and readjust the one right above it.
A: Run it and see if it works. OK. Now run the actual one that they give
us. Just highlight and copy and paste the one that they give us. Capital E
remember … parentheses … space times space … Sin 3x … close bracket.
C: Yeah.
A: Well done. Yes beautiful. We’re on the right—Oooh. Wait a second.
We’ve got a little change there. Ah. I see. Put those in parentheses right
you understand the product rule?

What is important here is that the students work hard on an answer, have
genuinely thought about the question, but are able to get feedback instantly on
their work and correct a minor error. It is clear that this gives them good sense of
achievement and a good deal of motivation.

In the final question I will look at (the ninth in the set) we see the students
make a more fundamental error but the feedback they get from the computer
allows them to solve the problem correctly:

B: Is that the chain rule?
A: Yeah.
A: Because you’ve got g of x which is this whole function to the third
power and then you’ve f of x
…
B: 5 x
A: Close that. To the squared.
B: You forgot to put a 3 out front. Is that what you were thinking?
A: Yes. A 3 out front and then there’s more.
B: Yeah. Then you’ve got to take the derivative of what’s inside. So you’re going to have: negative 5 Cosine of 5 x. Negative 25 Cosine of 5x.

What has happened here is that the students have gotten confused between the power rule and the chain rule and applied them incorrectly. In a non-CAS environment they would probably move on to the next question at this point but the CAS allows them to find out instantly that there is a problem and they are able to use the CAS’ response to fix their solution.

B: What are you thinking it is?
A: I think that that’s doing the second derivative. I think that we just—what’s here outside it.
B: Why? See don’t you. See what the answer turns out to be. Run the deal, and run what we got and see.
[C runs them both]

A: Well, first of all I’m a little confused where that number comes from … B: 75? [laughs]
[silence as they look at the screen]
B: Well, I was somewhat right with the whole derivative of the Sin of 5x … So far what we have there is right up until the Sine of 5x.

[silence]
B: Yeah. We should just multiply those together and get negative 15. And then we take the derivative of …

A: I see what we did.
B: You just have to do a power rule. You don’t have to do any chain rule.
A: No. It is chain rule.
B: It’s a power rule.
In fact it is a combination of both. B and C again have some conflict because they are in between the Property Noticing and Formalising stages. They have enough command properties to be using the correct concepts but have not gained enough command to avoid conflict over what rule is being applied and how to correctly apply the rules.

A: You do the outside and then the inside. See we have—look at the answer negative 15 Cosine 5x squared. Sin of 5x. What you’ve got is—
B: Right.
A: So first we do 3 times the Cosine of 5 x squared.
B: That’s the power rule.
A: times 5 or times negative 5 Sin of 5x.


A: Hold up. Hold up. 3 Cosine 5 x. What?
C: … times … Sin of 5x. negative Sin of 5x.
A: Good.
C: Is that right?
A: Run it. There we go. Do you get it?
We see here that the students got there in the end and that the instant feedback possible in the CAS environment was instrumental in their being able to see and correct their mistake. The disciplined way in which the students used the CAS (generating their own answer before checking) allowed this to be an important learning experience for them.

As in the previous episode we see that the students are sometimes able to move quickly from the property Noticing to the Formalising level when they have reified the property to the extent that they can solve the problem without conflict. We also see examples where they have noticed properties but have insufficient command of the property to move directly back to the Formal level. The CAS helps them a great deal in negotiating between these levels since it implements the algorithms related to the properties without error and helps the students to see where they are making errors.
Figure 5.4: The Pirie-Kieren Map of Episode V
Episode VI: Too much power?

Much of the debate about technology in mathematics education, and CAS in particular, is that it gives students too much power, or rather, that it does too much for them. Part of the reason this has some truth, at least in relation, to the standard mathematics curriculum, is that too much of that curriculum is based on algorithms which are easily programmed by computers. An arguably far more important problem than technology doing too much for the students is that they do not develop sophisticated control over what it can do. There are good arguments to say that use of CAS promotes a positive shift in the mathematics curriculum away from mechanics, algorithms, and repetition towards a more meaning-based conception of mathematics. However, it is not enough to just use technology it must be used in a thoughtful and meaningful way, which is to say that it must help to generate mathematical activity not overwhelm that activity; and it should lead students to learn mathematics rather than doing the mathematics for them.

We see many instances in these learning episodes where a problem of the technology is not so much that it does too much for the students but rather that it does so much so quickly that the technology becomes the engine of strategies rather than the students’ own thinking. A particular example of this is the way the students have a tendency to generate graphs sometimes without any, let alone careful, consideration of what they are looking for and, therefore, what might make an appropriate window. There is no cost to the students in generating many graphs (it takes just a few seconds) and so they do so without much thought. (In the early days of computing the cost of generating a graph
might be literally $1.00 or might be several hours of waiting time.) We see in the next two episodes examples of where the power of the technology takes over and the students get distracted from their task because the momentum provided by the technology’s power seems to become the focus of activity.

*Episode VI, in which the students show the limitations of their knowledge of limits.*

In this set of exercises we see the students struggle with the meaning of limit. The exercises call on the students to state the limiting value of a function based on their knowledge of the relative dominance of polynomials, exponentials and logs. In most cases the students are asked to illustrate the result with a plot.

In the first problem the students decide after some discussion and with the intervention of a Teaching Assistant that the limiting value of $f(x) = \frac{2x^4 + 50 \log|x|}{x^4 + 3x^2 + 1}$ as $x$ tends to infinity is 2:

C: Well. My guess is 2. Just the number 2.  
B: That’s my guess actually.  
...  
C: Does the Log affect the limiting value of this function?  
I: It’s exponents, then powers, and then Logs.  
C: So it’s still going to be 2.  

As part of the discussion the students show a lack of perspective on how a visual approach may help them:

C: I don’t know what Log does to that but I’m going to guess 2.  
...  
C: Does the Log affect this? That’s really the question we have to determine.  
A: We could plot it.
B: No. We don’t need to plot it.
A: But we can.
B: There’s no reason to plot it.

This issue of the relationship between a plot and the limiting value arises in the next problem where the students are asked to find the limiting value of \( f(x) = \frac{x^{0.8} + 4 \log(x)}{3x^{0.8} + 2 \log(x)} \) as \( x \) tends to infinity. The students are happy that the polynomials dominate the logs and that the limiting value is \( \frac{1}{3} \). They move on to the next problem and then remember that they’re supposed to illustrate this limiting value with a plot. The students produce the following plot:

This plot does not, at all, illustrate that the limiting value of the function is \( \frac{1}{3} \). However, the only problem that occurs to the students is that the plot does not seem to be from –10 to 10 as they had coded. Their response to this is to change the boundaries of the plot to 0 and 5:

A: Drop it down to negative … drop it down to zero.
B: negative zero doesn’t work either.
A: Drop it to zero to 5 because that will be much more easy to look at.
B: There we go.
A: That’s prettier.

Therefore, they end with a plot which is less illustrative of the limiting value of the function, missing the connection between the algebraic and graphical representations.

This lack of sensitivity to the visual meaning of the limiting value is further illustrated in the next question which is to find the limiting value of \( f(x) = (45x^8 - 123 \cos[x] + 6x^6) / (e^{0.04x}) \) as \( x \) tends to infinity. Having correctly identified the limit as zero due to the dominance of the exponential function the students graph the function from \(-10\) to \(10\), boundaries which are unlikely to show them anything useful and, indeed, this is the case:

![Graph of f(x) from -10 to 10](image)

A: That’s not right.

B: The limit does not seem to be looking like zero because it’s not getting closer to …

A: It’s a parabola.

B: A parabola but it never touches.

C: It gets close to zero but never touches it?
This question poses a particular problem for the students since the dominance of the exponential factor is very slow to manifest itself so the interval –10 to 10 does not show them the long-term of the function unlike the first example in this episode. The students “got away” with graphing the interval –10 to 10 in the first example because of the relative speed with which the dominant term dominates. With no good idea of how they can illustrate the limit and thrown by the fact that this graph appears to show a minimum at zero the students then change the boundaries of the graph to –0.001 and 0.0001 to get the following graph:

B: Well then zoom in. Don’t have such a big interval. Make it from .1 to -.1. Like a graph this big we’re going to see it never touching zero.

B: What did you do?
A: I don’t think that’ll do. Let’s try –1 to 1.
C: Sorry guys. I tried.
B: Go for –1 to 1.
C: [changes the parameters]
B: That does not approach zero.
A: See. It’s a parabola.
B: But it’s not approaching zero.
...
A: It’s not approaching zero.
B: No that’s approaching –120.

At this point the instructor intervenes and says, with surprise, that 1, the upper boundary of their graph, is nowhere near infinity. This solves the problem and the students end up with the following graph:

The students are now able to marry their knowledge of the dominance of terms with a strategy for illustrating the limiting value. The students know, from the dominance, that the limiting value of \((\log x)/x^{0.03}\) is zero and this time
when their initial plot, with boundaries from 0 to 100 is not showing this, they are able to see how they can illustrate the result as they change their boundaries:

![Plot](image1.png)

![Plot](image2.png)

![Plot](image3.png)

We see in this episode that the students are able to combine analytic knowledge of functions with visual representations of functions to identify
limits. However, this only happens as a result of the intervention of the Teaching Assistant and, furthermore, the students never employ a numerical strategy to identify limits by substituting large values for \( x \) in the function and evaluating.

In Pirie-Kieren terms the students have trouble connecting the Formal level of knowing which terms dominate in an expression with the Image Making level. This dissonance is caused by their not reaching Image Having because they do not know which part of the plot they should be looking at. They are not really Formalising because they have not connected information about dominance of terms with numerical or graphical behaviour of those terms for large values of \( x \).
Figure 5.5: The Pirie-Kieren Map of Episode VI
Episode VII: Multiple representations

The importance of multiple representations has been long recognised in mathematics education not just to provide different modalities of instruction for different kinds of learners but as a way of presenting the richness and complexity of connections between algebraic, visual, and numerical representations of mathematical objects. CAS, which adds algebraic representations, is a breakthrough in technology which allows another dimension of representation which graphing calculators cannot provide.

There are many possibilities for using graphs to compare, for example, the behaviour of functions given in algebraic form or for the solutions of equations for the intersection of functions to be compared to behaviour before and after the intersection points to establish dominance or relative growth. We see in the following episode a situation which is ripe to be explored in various representations.

Episode VII, in which the students fail to use the multiple representation possibilities of the software to help them.

The students are given the following question:

Rank the following functions in order of dominance as x --> infinity

x^52, 0.0004e6(0.01x), (e^(0.02x))/x, xLog[x], 89x^2, Sqrt[x], 100 Log[x], 17x, 0.08x^3, 0.0000013e^2x, 100 x^0.004

The students make good progress moving the exponential terms which they recognise as most dominant to one side and ordering them and then ordering the powers of x including the observation that Sqrt[x] = x^(1/2). The
trouble begins when they have to consider the Log functions. The students are not sure which of $x\log[x]$ and $100 \log[x]$ dominates the other and they are not sure where they go in relation to the exponential and polynomial functions:

A: $100 \log(x)$. Where does that go in the rank? It’s either going to be before or after the $x \log(x)$.

In the Basics and Tutorials of this section and in previous problems the students have done, including some demonstration by the instructor, dominance had been addressed visually. For example, the students have seen graphs of $x^4$ and $e^x$ to see that $e^x$ eventually dominates. However, the students do not take a visual approach to the problem at any stage. In fact they don’t have a great deal of discussion at all and end up getting the wrong answer by considering the Log functions as having the least dominance without considering the relationship, for example, of $x\log[x]$ to $17x$.

There are numerical, visual, and algebraic possibilities open to the students in approaching this problem, but they do not take full, or in some cases any, advantage of the approaches afforded by CAS and solve the problem with a basic rubric of exponential beats polynomial which beats logarithmic. This rubric is flawed and they fail in the question taking either a multiple representation approach or treating any representation effectively. Unfortunately, this error was never followed up on by intervention from instructor or teaching assistant.

As in Episode III it is not clear that there has been any growth in mathematical understanding so the Pirie-Kieren model does not apply.
Episode VIII: The Role of Metacognition

One of the major goals of modern education is that students should learn how to learn; that is, students should develop meta-cognitive skills. An argument can be made for CAS that it can help in the development of meta-cognitive skills by allowing students to separate the mechanical/algorithmic parts of questions from the conceptual parts. Thus students can develop, through practice, a sense of how to take a broad approach to a question without getting bogged down in details.

However, the skill of metacognition is unlikely to develop without nurturing and we see in the following episode that the students have difficulty taking precisely the broad approach needed and have difficulty separating two different goals in the question. The episode below highlights again the absence of teacher intervention to take advantage of moments where students have engaged considerably with a question and might be considered ripe for a teaching moment.

Episode VIII, in which the students have difficulty separating two different goals in the question: the role of metacognition

In this set of questions, the students are asked to find good representative plots of various functions and say what is happening to the function as \( x \) gets very large.

The students are first prompted in the text on what makes a good representative plot and explicitly asked to explain why “if your plot of a function includes all points at which the derivative is zero … you can be sure that your
plot cannot miss any of the dips and crests in the graph of the function.” This is answered with the response in the text that “if you include all points that are zero [in derivative] you include all the maximum and minimums. Therefore you can not miss a crest or valley.”

The first function to be addressed is \( f(x) = e^{\frac{x}{-x/5}} \cdot (x + 3)^2 \). Despite having just written that solving for stationary points helps in getting a good representative plot, the students ignore this and plot the function from 0 to 100:

![Graph of f(x) = e^{\frac{x}{-x/5}} \cdot (x + 3)^2 from 0 to 100]

Again, without any consideration for stationary points the lower boundary for the plot is changed to -100 yielding:

![Graph of f(x) = e^{\frac{x}{-x/5}} \cdot (x + 3)^2 from -100 to 100]
This prompts the following dialogue:

[plot above appears]

A: Really? Any ideas?

C: [changes it to –25] It’s going to zero.

A: What if we plot the derivative of the f of x to where the maxs and mins are, then we can see what we need to include.

C: I just think that it’s going to zero; that’s—as x goes to infinity. As x goes to infinity the f of x goes to zero.

A: Sounds right. I do have an idea. If we were to solve for the f derivative of x when y equals zero then we would know what all of the maxs, what all of the zeros would be.

C: How does that help?

A: Because then we know if there are any more of those points that we need to include in our plot.

C: We want to include those right?

A: It wants a good representative plot so we want to include everything, every one of the dips and valleys and all of that.

This discussion shows the lack of metacognition in how the students approach their work. First of all A and C are discussing different aspects of the question “a good representative plot” versus “what happens as x tends to infinity.” Secondly, C, who typed “if you include all points that are zero you include all the maximum and minimums. Therefore you can not miss a crest or valley,” now asks how doing so will help.

Having found the x-values of the stationary point the students find a good representative plot and are satisfied that the function tends to zero as x tends to infinity although there is no discussion on this latter point either in terms of the visual representation of the function or the algebraic properties of the function.
For the next function the students now include solving for the stationary points of the function in their initial coding, which is to say they incorporate this strategy but not in such a way that it can influence the boundaries of their initial plot. Again this demonstrates a lack of metacognition as well as showing how easy it can be to employ strategies in a CAS environment without giving due consideration to the meaning of employing those strategies.

The students settle on the following as a good representative plot:

A: Ah, we need to say what it does as x approaches zero.
C: [types in the response]
A: No. It’s not approaching zero. I don’t think [pause] because you remember when it was 100 it definitely didn’t look like zero.
C: This zero point 2.
A: Well, maybe you’re right. It works for me.
C: [changes upper limit to 500000]
A: Yeah. It approaches zero.

What is interesting here is that the argument for \( f(x) \to 0 \) as \( x \to \infty \) is made by C purely visually. He plots the function with an upper boundary of 500000 and the visual evidence of the plot is considered, tacitly, by the entire group, to be convincing evidence that the function value tends to zero. There is no analytic consideration of the function and no reference to section 1.01 of the course where the students answered many questions about, for example, the relative growth rates and dominance of \( \log(x) \) and \( x \).

For the final function the students again involve solving for stationery points in their initial coding but not to influence their choice of boundaries for the plot until after the fact. On the metacognitive level, thinking carefully about everything the question is asking them, and whether multiple approaches might be necessary, the students again have difficulty separating the “good representative plot goal” from the “as \( x \) tends to infinity/global scale” goal. At one point in their work the students have the following plot:
But offer the following as their “good representative plot”:

In Pirie-Kieren terms, the students are stuck on the Image Making level and have difficulty ascending to image having in the sense that there is no evidence that they know what makes a good representative plot. They have achieved some Property Noticing in that they know to find the turning points but certainly have not reached a formal level because they (a) are unclear on why the turning points serve the problem, and (b) they make no formal argument about the limit of the function as x tends to plus or minus infinity.
Figure 5.6: The Pirie-Kieren Map of Episode VIII
Episode IX: A conjunction of episodes

The last episode in the set is a particularly rich example of the issues of the other episodes coming together. In this episode we see the technology getting in the way with Mathematica’s idiosyncratic graphs; we see power to generate many graphs quickly interfering with the learning process; and we see the students’ struggling with metacognitive issues related to focusing on what they are being asked to do in the question.

Episode IX, in which the students have good strategies but use them badly.

The students are posed the following question:

Find the highest point on the graph
\[ f(x) = e^{-x^2}(2 + \cos x + \frac{\sin x}{2}) \]

Is there a lowest point on the graph?

A: OK. Is there a lowest point on the graph?
C: I don’t know. Let’s plot it and find out.
C: Let’s solve for the maxs and mins here as well.

(silence)

A: Did you put the “Solve” function in wrong or is something that’s too complex for it?
The students initial approach to this problem is to look for a good picture to get an idea of whether they can “see” where a lowest or highest point would be. The suggestion that they use Mathematica to solve for maxs and mins (as they have done in previous problems) is offered as something of an afterthought and is rejected in favour of “Image Making” following the feedback from Mathematica that there is a problem with finding an algebraic solution.

The students try to make a better image by changing the lower and upper limits of the graph from –10 to 10 to –100 to 100.

This doesn’t seem to be helping in finding a maximum point but rather than making more images the students switch tactics to trying to get Mathematica to find a solution again. They try to get a numerical answer from the input

\[ \text{NSolve}[f'[x] == 0, x] \]

They give up on the solution approach at this point although they will return to a numerical approach at a later stage.
For the moment they return to a visual approach and change the limits of the graph to –1 to 1

It should be clear to the students now, at least what an approximate solution to the problem is since they now have an excellent representation of the function. However, and it’s not clear why they do this, they then get progressively worse images, with limits of –5 to 5; –25 to 25. Then C remembers the properties of the derivative function:

C: Try “f'(0)” and see if that works.

[Output: 1/2]
T: Go to the right and see if it’s still increasing.
C: f'(1). Wow.
T: Put a 1.

[Output –2.38879]
C: Yeah. So it’s decreasing at 1.
T: So it’s between 0 and 1.
C: Well, it’s between 1/2 and 1. You’re right 0 and 1. Closer (as A tries other values).
This exchange represents the mathematical breakthrough of the episode. C has noticed the property of the relationship of the value of the gradient function to the maximum value of the function. The noticing of this property is constrained by, or informed by, the formal problem of finding the maximum value of a function.

Having established earlier that Mathematica is unable to offer an exact solution to the equation $f'(x) = 0$. The students then proceed to find better and better numerical approximations of the answer. They spend quite a lot of time on this and don’t seem to have a good notion of how accurate an acceptable answer is. B speculates on whether the answer they are converging on is the square root of a “nice” decimal like 0.5 without offering any algebraic/analytic suggestion as to why this should be so.
The students begin their analysis on a formal level but find that Mathematica is unable to help them find an algebraic solution. They descend to the Image Making level in order to get a picture of what the solution might be. When this fails they return to the formal level of a numerical solution. When this in turn fails they return again to Image Making. The successful representation of the function inspires a move upwards helped by the Property Noticing by C of the values of the gradient function and culminates in a numerical solution correct...
We see in this episode that in Week 7 of the 10 week course the students are able to implement graphical, numerical, and algebraic solution strategies but that there is a lack of sophistication in how they use Mathematica. We see this in the creation of less successful graphical representations after they have made an excellent representation, as well in the lack of a good sense of the acceptable level of accuracy of answers and in the speculation on “coincidence” in the decimal representation they achieve.

Summary

We have seen in these nine episodes the most prominent processes and behaviours of the students learning while using CAS. In Episode I: “Visibility of technology” we saw that the technology can get in the way of student learning because of the way that it represents graphs and equations. In the related Episode III: “The syntax of Mathematica“ we saw that input required of the CAS can impact learning as well as the output. Episode II: “Working habits” examined how the way the students negotiated the fact that they were working in a group affected their learning. Examples of the affordances of Mathematica, and how well the students were able to take advantage of those affordances, were presented in Episode IV: “Using CAS for what it is really good at,” and Episode V: “Instant Feedback.” Examples of how powerful and flexible technology can be a hindrance to student learning were seen in Episode VI: “Too much power?” and Episode VII: “Multiple representations.” Finally, issues of
how well the students developed necessary learning skills and mathematical strategies in a CAS environment were explored in Episode VIII: “The role of metacognition” and Episode IX: “A conjunction of episodes.”

Some conclusions

The nine episodes analysed above are offered as existence proofs of students’ processes while working in a CAS environment. In looking at the episodes we can see something of their working habits, their understanding of what the technology has to offer them and their ability to take advantage of the affordances of the technology.

The value of the Pirie-Kieren Model of the Growth of Mathematical Understanding is as an analytic tool with which particular learning episodes can be examined. The Model is particularly interesting in pinpointing where students are on the cusp of understanding. In a number of the episodes we see where students are involved in Image Making but have not reached the stage of Image Having. This is an important distinction and the Model is useful in articulating what the difference between these levels looks like. Other episodes showed a similar difference between the levels of Property Noticing and Formalising. The episodes also show that occasionally students are able to solve problems while remaining on the Formal level but most commonly have to descend levels in order to be able to solve problems. Perhaps most importantly for the purposes of this study we see occasions on which Mathematica is instrumental in allowing students to move up through levels and other occasions where if they had a good grasp of what Mathematica can do for them they would be able to be even more
successful in solving problems. This is a key part of why the model is useful: the most interesting aspect of the model is to ask how do students move between the levels and occasions where the CAS is the engine of this movement are, therefore, particularly important.

The model also has limitations. While it provides a way of landmarking the stages students go through a learning episode it is not so strong in characterising how they move through those stages. We saw some instances in Episode V: “Instant Feedback” where the CAS can be the engine moving students from Property Noticing to Formalising. However, in most instances the Pirie-Kieren Model does not account that well for how students move from, for example, Image Making to Image Having so much as it describes what it means for students to be in each state.
The place of technology in mathematics education has been controversial as long as mathematics has been taught. Every technological innovation effects how mathematics is recorded, processed, thought of, and conceived but the gap between technology, as it is used by mathematicians and other professionals employing mathematics, and how technology is regarded in mathematics education has grown with each innovation. Mathematicians and other professionals have simply proceeded with using technology as best they can to enhance their work (admittedly not without some controversy about the nature of proof) whereas educators have struggled with profound questions of whether and how technology can enhance education and whether and which pedagogical approaches make this possible.

There is now a considerable research base indicating that the use of technology in mathematics education can have considerable beneficial effect but that those benefits can only be carefully gleaned if the use of the technology is
judiciously managed. Rather than obviate the need for teacher involvement and a pedagogical philosophy the use of technology enhances that need.

Computer Algebra System (CAS) technology is the logical next step following from all the technologies that came before it and is a boon to mathematicians and their work but has been a bane to many educators. This is the technology that, even for many educators who are advocates of technology, goes too far. This is unfortunate since CAS is an excellent catalyst to demand of mathematics educators to deeply consider the goals and purposes of a mathematics curriculum. Much of the standard mathematics curriculum in schools, particularly the High School, is based on algorithms and mechanical procedures which are easily performed by CAS-enabled technology. It is for this reason that CAS is seen as posing such a threat. This, allied to the fact that CAS-enabled machines are banned from use in most high-stakes tests in the United States means that CAS has not had a great deal of traction in schools.

The body of research that exists is, therefore, mostly at the college level but as the technology becomes more affordable and more accessible it is important that there be a research base in place so that the technology is used in a manner which benefits the learning process. Of the research that has taken place there is a strand which is oriented around outcome achievement: some students in a college learn using CAS and others learn using traditional methods. The outcomes of the students taking the same final exams are compared. Another strand is oriented around more philosophical questions about the make-up and ordering of topics in the curriculum.
There is a gap in this research base which relates to how students work in a CAS-rich environment. Rather than focusing on student outcomes there is a need for research which looks at how students relate to and work with technology on a day-to day basis. Thus, the primary question of this research project was “What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus?”

Sub-questions addressed were:

• What strategies do students adopt in a CAS environment?
• How do students negotiate between different mathematical representations in a CAS environment?
• What are students’ perceptions of the role of technology in their learning?
• What is the students’ relationship to Mathematica?
• What is the role of experimentation in mathematical learning in a CAS environment?
• How do students approach formalisation (in the sense explained in the Pirie-Kieren model) of mathematics in a CAS environment?
• What is the effect of learning in a CAS environment on students’ conceptions of mathematics, as a subject?
• What is the relationship between the pedagogy in the classroom and the use of technology?
Summary and Conclusions

In this chapter I will address some limitations of the study, make some general observations about the overarching question of the study, and then deal with each subquestion individually. This will be followed by some discussion of the implications for research, for curriculum and for teaching.

Limitations

It should be clear to the reader at this stage that this study makes no large claims to generalisation. Rather, the hope is that the study will provide a basis for further study, in different situations and contexts, of the use of CAS in mathematics education. Some of the specific limitations of the study are discussed below.

The first, and most obvious, limitation is that the study focuses on just three students. In a study of such an exploratory nature, it is justified to examine a small sample. However, a study restricted to three subjects runs the risk of finding anomalous results. Other factors which make the study very particular are: the type of software used, the role of the instructor in the course, the fact that the students worked in groups, and the way that the students worked in groups.

It is, I believe, unusual to have an environment in which the instructor is so absent in terms of direct instruction of mathematics or the use of the technology. A more active instructor might construct the CAS environment differently and, therefore, claims made in this study are limited by the particular way the instructor operated.
It is very difficult to know how the students would have worked in a different CAS environment, that is, one which employed a different software. Mathematica is very syntax driven, other CAS software is very menu-driven. Yet others are somewhere in between. There is certainly a limitation to this study in the fact the claims and conclusions may be particular to a Mathematica environment.

One aspect of the way that the students worked as a group is that the same student in the group was always the typist and this clearly coloured his experience in learning mathematics in a CAS environment. This provides a limitation in that claims made may be particular to the way group work was constructed in this particular CAS environment.

These limitations granted there is much evidence to support the claims and conclusions of this study as has been seen already and will be seen in the remainder of this chapter.

What are the processes of learning in a Computer Algebra System (CAS) environment for college students learning calculus?

In attempting to answer this question I mobilised two theoretical frameworks. The Adapted Rotman Model of Mathematical Reasoning was used to take a broad look at the place of technology in students’ learning and their relationship to mathematics across a period of time. I used the Pirie-Kieren Model of the Growth of Mathematical Understanding to take a narrow look at students’ processes and behaviours in particular instances in the students’ learning.
In Chapter 4 it was shown that the Adapted Rotman Model (Rotman 1993; Rotman, 1995) provides a framework which stresses the importance of technology as an active agent in the process of learning. The evidence of Chapter 4 was that the technology is not simply a blameless intermediary between the student and the mathematics. In fact, the relationship is much more complicated and the technology must be understood as having its own agency and own relationships with the mathematics and with the students. As we saw in Chapter 4 students develop a relationship with technology which is separate from their relationship with mathematics; they have to learn how to communicate with the technology and the technology often circumscribes their thinking on a mathematical problem. As we saw, in some cases this relationship serves to enhance student learning but in some cases the relationship inhibited learning. It is important that this separate relationship students have with technology is acknowledged so that research provides a more textured picture of learning with technology. It is also important that the relationship to the technology is nurtured, by teaching the technology directly, so that the technology deepens students’ relationship to mathematics rather than impeding this relationship.

Chapter 5 consisted of a series of learning episodes offered, in part, as existence proofs of student behaviours and processes in a CAS environment. We saw in that Chapter, for example, how students use materials in such an environment; how the technology proves too powerful for students on occasion; how the type of CAS used is important; and how the students are able to use the instant feedback possibilities of CAS effectively. These and other processes shown in the Chapter can be serve as a guide for approaching teaching with CAS
as they highlight behaviours and processes to look for in other students learning with CAS.

Chapter 5 employed the Pirie-Kieren Model for the Growth of Mathematical Understanding (Pirie & Kieren 1990; Pirie & Kieren, 1994) to give some insight into how students work in particular episodes. The evidence of Chapter 5 highlights some areas in which students are on the cusp of some greater understanding (e.g. moving from Image Making to Image Having) and how CAS can be effective in bringing the students to that higher level if used appropriately. The Model also highlights common processes employed by the students and strategies that they repeatedly fall back on circumscribed by the affordances of the technology such as generation of multiple graphs.

What strategies do students adopt in a CAS environment?

We have seen in a number of the episodes in Chapter 5 that the student strategising is highly driven by the CAS environment in which they found themselves. In Episode IX: “A conjunction of episodes” we saw the students use a number of strategies (graphical, numerical, analytical) to try and solve a problem. The students had quickly developed this multi-strategy approach on exposure to such strategies in the example problems and previous problems they had solved. However, the students did not develop very sophisticated use of the strategies and allowed the CAS to take over. What we see happening in this episode and in “Episode VI: Too much power?” is that the students are in an environment where everything is organised around the computer and the CAS and the CAS takes over. The CAS environment in which the students find
themselves is a computer lab. the computer is privileged as a powerful agent in this environment. Furthermore, the calculus as it is presented to the students is in text and exercises based solely on the computer. Often, then, the students’ strategies, certainly by the second half of the ten-week term are organised around what the computer can do for them.

On the more positive side, we can see in Episode V: “Instant feedback” that the students are able to develop strategies of using instant feedback. Their use of instant feedback to successfully learn the implementation of combinations of the product rule and power rule is evidence that there are situations in which the knowledge of the students together with the CAS is enough for them to achieve success without an instructor or Teaching Assistant (TA) intervention. This episode also serves as an example illustrating that when students are on the cusp of a new understanding, CAS can help them achieve the next level by providing them appropriate feedback in timely fashion for them to use the feedback. In Pirie-Kieren terms the students are on the Property Noticing level of the Pirie-Kieren scale with some of the rules of differentiation but are not quite at the Formalising level since they are still making some errors both in deciding which rules apply in which situations and in applying the rules. The chance to practice the rules in a situation where they are getting instant feedback as well as a situation where they are discussing the results with one another, articulating their reasoning enables them to tip over, to some degree, into the Formalising level with this piece of mathematics.
How do students negotiate between different mathematical representations in a CAS environment?

For many years mathematics educators have been arguing for the importance of multiple representations in teaching. And not just from a Gardner Multiple Intelligences perspective (Gardner, 1983) but also from the point of view of doing honour to the importance of the connections between different representations of mathematical objects and how each helps in understanding of concepts. Four-function calculators allowed users to automate a great number of numerical processes; graphing calculators extended these numerical possibilities and added visualisation; CAS-enabled takes the next step and allows the user to do algebra and calculus through technology.

The most significant representations in mathematics (numerical, graphical, algebraic) can now be instrumentalised with technology, and pedagogy aiming to use technology can take advantage of this instrumentalisation. We saw in Chapter 4 that the students in this study were familiar with the possibilities of graphing calculators although they tended to use them in a unimodal style. That is, they were comfortable with taking numerical or graphical advantage of their calculators but didn’t give the sense that they were sensitive to connections between those representations. For example, if they wanted to find a value at a point of a function they had graphed they didn’t use the “find value” functionality in the graphing arena, or use the table of values associated with the function but, rather, moved to the separate arena of numerical calculation and calculated the relevant value. This says something about their previous training in the use of technology and it is apparent in the
students’ histories that those connections have not been well developed. On the one hand then the foundations to build upon are far from complete in the case of each of these students but it is also apparent in many of the learning episodes of Chapter 5 that sophistication with such connections is not automatic and does not necessarily “just happen” in a technology-rich environment.

There are almost no instances in the learning episodes of Chapter 5 (there were almost none during the ten-week term) of the students changing modality without being explicitly asked to do so. In other words, the students don’t generate a graph or picture to get some insight into a problem unless the question asks them to do so. They are able to make connections between representations when it is a necessary part of the problem they have been asked to solve, but the students do not develop a sense of the connections as meta-cognitive problem solving strategy in and of itself.

We saw in Episode VII: “Multiple representations” where the students chose not to use graphical or numerical modalities to help them. However, there were instances in which the connection between multiple representations was required by the question. In such cases the students were able to learn something from these connections. Student C saw this in particular in regards to differentiation as he reports “there are some parts where I think Mathematica is great making these links, for example, when you’re looking at the relationship between the derivative and the original function” (C, 2nd Interview, November 10th).
What are students’ perceptions of the role of technology in their learning?

One of the reasons that constructivism in all its guises (Ernest, 1991) has had such traction in educational circles is its promotion of the idea of student ownership of their learning. Rather than content being simply delivered to the student constructivism embraces the notion that learning must be related to students individual conceptions and their personal schema of organisation of their own knowledge. Technology arrives into this situation in an interesting way. Does technology allow students to work with concepts and objects towards developing their knowledge? or Does technology become another deliverer of information? It is particularly interesting to look at this question of ownership in mathematics education in relation to how students regard the work that the computer does for them. In his final interview student C, when asked to describe what he remembered and felt about the class said, “I tell people when they ask what [the class] is, I said I learned how to make a program. I learned how to program a computer to do calculus and didn’t master it myself. I didn’t want to go out and be able to tell a computer how to do it. I wanted to be able to do it myself.” (C, 3rd Interview, February 20th).

The issue, of course, is what students consider to be the substance of their mathematics education. Students B and C in this study, as we saw repeatedly in Chapter 4, were uncomfortable that Mathematica was often doing for them what they considered to be the “real” mathematics. As has been noted before, the students took a long time to make the transition to feeling comfortable with using Mathematica in any sense, but further, they did not make the transition to the ideology of the Calculus & Mathematica course. As can be seen from the
survey results many of their classmates did not either. It is interesting to observe that for many students the reaction to CAS technology is very similar to that of teachers. Some are very positive about CAS but many students, including those who are comfortable with graphing calculator technology have this sense that CAS is finally technology that does too much mathematics to be appropriate for education. This feeling is exemplified in the quote of student C above and positions the students, with many teachers, as feeling that they are very much outside mathematics, that there are well-known ways to master mathematics, and, at a certain point, if some technology is doing the calculations (algebraic or numerical) then they are in some sense either cheating their way to mastery of mathematics or will not achieve said mastery because they are emasculated in their mathematics by technology.

*What is the students’ relationship to Mathematica?*

The students’ relationship to Mathematica in particular, as opposed to technology in general, is split between student A on the one hand and students B and C on the other. A was very much convinced, as we saw in his responses in Chapter 4, that Mathematica was taking the burden of calculation from him and allowing him to concentrate on the concepts of calculus. Students B and C, however, wanted the burden of calculation as this is where they saw the “true” mathematics as residing.

Challenging the students in this course with a new approach to their learning of mathematics is likely to cause some difficulty. If they then have to learn a piece of software that is new to them, and they are given no direct
training in using the software, it is not surprising that the students develop some resistance. In Episode III: “The syntax of Mathematica” we saw that Mathematica is a somewhat idiosyncratic program with a very particular syntax. Student A was intrigued by the challenge of learning Mathematica becoming the expert within the group in the writing of Mathematica code. The reaction of students B and C was to regard Mathematica as an obstacle. Their comments in Chapter 4 expressed frustration that they were having to learn Mathematica as well as learning calculus. It is impossible to know if they would have reacted differently to a menu driven CAS with an interface more like that which they are familiar with from using graphing calculators or if CAS was on a piece of hand-held technology, but it is certainly the case that they were not happy about the amount of separate work gaining mastery of Mathematica needed.

The lesson here is that the students’ relationship with Mathematica needs to be nurtured. Little or no effort was made to teach Mathematica itself. The assumption from the instructor and TA seemed to be that students pick up Mathematica by modeling problems in the Tutorial sections and in their group work on the Give it a Try sections. However, the students took a long time to gain any comfort with Mathematica and because of the way that Mathematica is embedded within the materials the students worked on as part of the Calculus & Mathematica it took the students some time to understand Mathematica as a calculating device which is a more powerful version of other calculating devices such as graphing calculators that they are used to. The students needed Mathematica to be related to their previous experience with technology in mathematics education and they needed to be shown how the power of
Mathematica can help them in their mathematics. This can be done without great difficulty at the beginning of the course by paying attention to the fact that the students are not necessarily advocates of technology and need to be helped with this new relationship. It was clear from the survey data that students felt that too much time was spent on Mathematica not mathematics and that too many of their questions were technical (about Mathematica) rather than conceptual (about mathematics).

What is the role of experimentation in mathematical learning in a CAS environment?

If mathematics education is to move away from a transition model of teaching and honour the mathematical activity of students there must be a range of ways for students to generate mathematics. Hence, the development of, for example, manipulatives which can allow students the possibility of working with mathematical objects. Exactly this possibility is the second great promise of calculating technology in mathematics education. As discussed above, calculating technology can be used to take the burden of calculation or representation (numerically, graphically, or algebraically), but technology can also be used for students to experiment with mathematical objects (numbers, graphs, or algebraic expressions), to investigate mathematical relationships, and to experiment with the effect of changing parameters in a problem. An example in the realm of graphing calculator technology is students graphing \( y = ax + b \) for various values of \( a \) and \( b \) to explore the effect of on the graph changing these parameters.
In practice what seems to happen is that if students are told to explore a certain situation, like the graphing of $y = ax + b$ above, they will do so but there was very little evidence that students developed this approach as a metacognitive skill. B felt that they had done this occasionally in some numerical exploration of problems whereas C reported that this approach was a technique that they had not used. Part of what was holding the students back from embracing this approach more was the rather prescriptive nature of the materials and the time pressures on the students. Despite the claim of some self-directed, more exploratory learning in materials (“In C&M classes, questions are answered with more questions, rather than just telling the student the answer. By doing this, the students are encouraged to discover concepts for themselves, rather than just accept what they are told” [http://socrates.math.ohio-state.edu/about/environment/teaching.php3](http://socrates.math.ohio-state.edu/about/environment/teaching.php3). The problems the students were asked to answer were written with specific outcomes in mind, and there was no longer-term project/portfolio type work that the students were asked to do. The questions were largely self-contained and intended to be answerable in a class setting in a few minutes. It is unlikely that students will develop the sort of metacognitive skills we might desire simply because they are possible with the technology. The time pressures of the course also did not help in giving the students some space not to feel under constant pressure to finish the particular section they were working on. This is, unfortunately, typical of college mathematics courses which are highly content oriented and much as Calculus & Mathematica might claim difference this content orientation remains in place. This has something to do with political pressures of Calculus & Mathematica to
conform with the larger program of undergraduate mathematics education at the institution but also has a lot to do with the final conception of mathematics that the authors have.

How do students approach formalisation (in the Pirie-Kieren sense) of mathematics in a CAS environment?

It has been mentioned a number of times already that the approach to mathematics taken in the Calculus & Mathematica course (problem-based, explanation oriented, etc.) is somewhat unusual. One characteristic that the course does share with many or most mathematics courses is that it is one of the goals of the course to have students engage with mathematics in a formal way, in the sense that Pirie and Kieren use the term. That is to say, learners are expected to abstract common qualities of noticed properties. The students in this study were certainly capable in many instances of solving problems on a formal level. They were capable of solving problems with drawing pictures or generating many examples from which to abstract properties. Several examples of the students solving problems in this way were presented in the learning episodes of Chapter 5.

Of more interest, and where the Pirie-Kieren Model becomes useful, is observing when students are on the Property Noticing level, whether they are able to get to the Formalising level, and whether CAS has a role to play. The evidence of the learning episodes in Chapter 5 is that there were occasions when the technological affordance of generating examples or making attempts at solutions while getting instant feedback, was a factor in moving students from
the Property Noticing to Formalising level. We saw in particular how this movement worked in Episode V: “Instant feedback” where the students were able to work through the errors of their attempts to differentiate functions and get a stronger sense of the formal rules of differentiation.

To gain further advantage of the instant feedback affordance of CAS-enabled technology it might be beneficial to design questions that involve students exploring properties and using the correct instant feedback to their explorations in order abstract those properties and think of the mathematical objects in a formal way.

What is the effect of learning in a CAS environment on students’ conceptions of mathematics, as a subject?

One of the challenges that CAS presents in mathematics education is that much of the mathematics curriculum in schools, particularly in secondary schools is based on technical algebraic skills. Success in standard courses such as Algebra I, Algebra II, PreCalculus, and Calculus depends on an ability to perform such algorithmic activities as solve equations, factor algebraic expressions and combine algebraic fractions. CAS-enabled technology automates all of these tasks and, therefore, one of the effects of CAS on mathematics education has been to pose some fundamental questions about the nature and goals of the mathematics curriculum.

The students in this study were all successful students of mathematics in their high schools. All three students took all the mathematics classes available to them in their respective schools and all three tested into the calculus level of the
university which is the highest level students can test into. It was clear that the students had learned the lessons of their school mathematics well. Not only were they very good at performing algebraic manipulations, they valued such activity as important in their sense of their own mathematical success. When asked at the beginning of the quarter how they learned mathematics the students’ answers frequently mentioned such issues as seeing all the steps of model problems and working through problems step-by-step.

The use of CAS, certainly as it used in the Calculus & Mathematica environment, interrupts this high school dynamic and goes some way to redefining mathematical success. In most of the problems the students are asked in Calculus & Mathematica, the emphasis is on explanation and interpretation rather than calculation. Therefore, success is supposed to be a function of an understanding of the mathematical processes involved rather than simply an ability to perform manipulations and computations correctly. Student A took to this adjustment very well and enjoyed what he saw as an emphasis on concepts over computation. His survey response and comments he made in his interviews make it clear that he felt that Mathematica’s taking care of the computations made him feel that he was able to concentrate on learning the substance of the mathematics.

The reaction of students B and C differed from that of student A. Having Mathematica do the computations had the effect of alienating them from the mathematics. Students B and C both reported that they felt Mathematica was doing too much for them. Because they were not writing out every step they were not happy that they were not, in some sense, really doing the problem.
Student B, in particular, expressed frustration on a number of occasions about the amount of writing and explanation involved in answering the questions. Rather than challenging his ideas about what is important mathematical activity, using CAS had the effect of making him dismiss much of the activity of the class. At one point in the class he said, “I don’t understand why we have to do this. This isn’t an English class.”

The evidence of this research is that students do not naturally make a transition to a new way of thinking about mathematics. Even if the change required is explicit in the materials, students still need guidance in the transition. There is an onus on the pedagogy to help students to see this different approach and to convince them that it is worthwhile rather than to assume the transition would occur naturally. In addition, the students were never given the opportunity to discuss this change in approach or reflect upon the change the course demanded of them.

What is the relationship between the pedagogy in the classroom and the use of technology?

A question that can easily be asked about both of the theoretical frameworks used in this study is “What is the role/place of the teacher?” In fact, the set up of the Calculus & Mathematica class in this study was such that the role of the instructor was very minimal. The materials are designed with the intention that students work through the “Basics” and “Tutorials” by themselves and work through the “Give It A Try” section as a group in class. The instructor almost never taught the class as a whole. His expectation was that his
interventions would be group-by-group based on questions that had arisen for the students in their work. The nature of these interventions was not entirely satisfactory. In the survey 18 students out of 23 agreed or strongly agreed with the statement “When I ask questions in class they are more about how Mathematica works than about Calculus.” This hearkens again to the issue of the students’ issues in making the transition from technology with which they are familiar (e.g. graphing calculators) to a new technology.

The pedagogical decision to make the work of the class self-directed may be supportable. However, it is subverted by the decision not to introduce Mathematica to the students in a way which both addresses technical problems of how to use the software but also gives students a sense of the power of the technology and how it can help them to do and learn mathematics. Therefore, the pedagogy in this class failed in the sense that many students were alienated from the technology they were asked to use in the class. Further, many students were alienated from their mathematics because of their lack of understanding and their lack of acceptance of the nature and goals of the class. Responsibility for this alienation on two fronts lies, at least in part, with the instructional model.

Overall Conclusions

This dissertation began with an acknowledgement of the growing importance of technology in mathematics education. The case was made that CAS, the latest step in technology, represents a new and different order of technological possibility. It was then argued in the literature review that there is a gap in the research between studies which focus on the outcomes of learning
with CAS and studies which speculate towards the future. These studies have not looked at the quotidian life of a class in which CAS is being used and do not provide insight into the processes of student learning in a CAS environment. This study was designed to acknowledge this gap and contribute some work towards filling the gap. This study contributes some insight into how two theoretical models can be used to understand the lessons of CAS technology’s use in the classroom. The Adapted Rotman Model of Mathematical Reasoning (Rotman, 1993; Rotman, 1995) is presented as a way of understanding the place of technology in the student learning and is important in making the claim that technology acts as a distinct agency in the learning process. The technology is shown to both act on and be acted on by the students in their learning. The technology also acts on the mathematics and is acted on by the mathematics.

The Pirie-Kieren Model of the Growth of Mathematical Understanding (1990, 1994) is a well-established model which is used in this study as a lens for analysing specific learning episodes. The idea of looking at those episodes presented in Chapter 5 in such depth is to argue them as exemplars of specific issues and processes which arise for students when learning mathematics in a CAS environment. Taken as a whole the nine episodes provide a dense picture of a CAS learning environment which will be of use to educators and researchers in the future.
Implications for Research

Research on CAS in mathematics education has matured considerably in the last few years but it remains somewhat in its nascent stages. What is the case with research on technology in general applies all the more to research in CAS. There is an absence of theoretical frameworks through which to interpret the place of technology (and CAS in particular) in the classroom. There is also an absence of empirical studies looking at how the use of CAS plays out in the classroom. This study contributes to both of these deficits by showing how the Rotman Model of Mathematical Reasoning (Rotman, 1993; Rotman 1995), and the Pirie-Kieren Model for the Growth of Mathematical Understanding (Pirie & Kieren, 1990; Pirie & Kieren, 1994) are useful frameworks for theorising technology in mathematics education. Furthermore, this study provides empirical evidence of CAS use in the classroom.

Early research studies in CAS have shown that students learning using CAS did, at least, no worse than traditional students. Furthermore, there were positive affective outcomes for students learning with CAS: they felt more ownership of their learning, felt they had a good understanding of the material etc. There has also been quite a lot of scholarly material taking a broader, more philosophical, approach to the question of CAS in education. Such writing focuses on questions of curriculum, assessment and pedagogical strategy. Would the introduction of CAS mean that certain topics are not necessary anymore? Would the order of topics change? Would the introduction of CAS invite particular pedagogical strategies? How would the introduction of CAS affect assessment, particularly in high stakes tests? Should technology be banned for
exams or parts of exams? These are very difficult questions to answer, the more so because of the lack of maturity of the research base in CAS.

The gap identified in Chapter 2 between the outcomes of students learning in a CAS environment and the more philosophical approach outlined above remains and needs to be filled. This study has attempted to contribute to filling this gap by asking specific questions, not about outcomes, and not about curricular implications but, rather, about the day-to-day processes of students engaged in learning in a CAS environment. This study makes an attempt to answer such questions as: Are there particular strategies and/or habits of work that arise in such an environment? What relationship do the students develop to the technology in such an environment? What relationship do the students develop to mathematics in such an environment? However, much more research needs to be done to provide answers to these questions in different settings and with different populations. Answering these questions in different settings holds the promise of the research community having a more informed position from which to approach pedagogical and curricular questions. A strong sense of student strategising and behaviour in a CAS environment may give direction to which teaching strategies may best take advantage both of the affordances of the technology and of student processes in the CAS environment.

In this study the instructor was virtually an absent element in the learning process. The responsibility for learning was handed over almost exclusively to the students in their groups. Direct instruction took place in the class almost exclusively in the returning of quizzes and exams when the instructor perceived common areas of difficulty for the students and addressed the concerns with the
group as a whole. Since the assessments were done without Mathematica these discussions did not include CAS. Therefore the teacher is not a part of either of the models used in this study for understanding CAS. Therefore, an area for further research is how to develop a pedagogy which helps students to overcome some of difficulties pointed out in the learning episodes of Chapter 5. We saw many instances in the episodes (and there were others besides) where a teacher intervention to take advantage of the learning opportunity that had arisen from the students’ work could have contributed enormously to the students’ development.

The models used in this study are useful for analysing student interactions and development of understanding on a micro level and for understanding the place of technology in a trialogue of student learning with technology. However, the models, particularly the Adapted Rotman model, need modification to incorporate an instructor/teacher dimension or new models would need to be developed which address the place of teaching in the process. In the case of the Pirie-Kieren model there may be a place for the teacher to be instrumental in moving students from one level of the model to the other.

The use of CAS in this course and the resistance of some of the students to the goals of the course has interesting implications students ideas of success in mathematics. The students in the Calculus & Mathematica course would not have placed into the course unless they were, by common standards of schools, successful at mathematics. Much of their sense of their own success was attached to an ability to perform easily automatisable mathematical procedures efficiently and accurately. That this ability is not especially valued in a CAS environment
was a challenge for many of the students. An interesting area of research would be to look into what students consider to be valuable mathematical activity and what they consider as success and then to investigate how learning in a CAS environment may affect these attitudes and perhaps change them. The attitude and activity of the Calculus & Mathematica course is so anomalous in the students’ mathematical biographies that it is, perhaps, easy for them to close off as unimportant. It remains important to investigate how CAS can, like graphing calculators, become a more “normal” part of students’ mathematical activity.

Implications for curriculum and teaching

As stated early calculators/computers with CAS capability represent the fourth generation of calculating technology in mathematics education. Four function calculators were supplanted by scientific calculators which were, in their turn, supplanted by graphing calculators. CAS capable technology is the latest step on the technological road but has had much less traction in education for two reasons, one philosophical and one practical. The philosophical reason is that there is a huge emphasis on technical manipulation in high school algebra, pre-calculus, and calculus. To impose CAS capable technology would necessitate a complete overhaul of the mathematics curriculum, something which is logistically difficult and politically almost impossible. The more practical reason is that CAS capable calculators are not, for the most part, allowed in high stakes exams in the United States and, in the instances where they are permitted (e.g. in AP Calculus) the questions are designed to be calculator neutral meaning that the questions are designed so that the capabilities of the calculator do not help in
answering the question. These are the main reasons why a CAS approach has not taken hold in the United States.

There are many instances of CAS capable technology becoming a much more integral part of mathematics education in other parts of the world. The Austrian government has funded a longitudinal project more than a decade old on the integration of DERIVE into mathematics teaching; the board Victorian Certificate of Education in Australia initiated a CAS version of their University Matriculation course in 2002; and the International Baccalaureate is planning to introduce a CAS element in the next revision of one of their mathematics syllabi. There are many other instances in countries such as Germany, France, and Denmark where CAS is being used in both secondary schools and universities in teaching and learning mathematics.

The use of CAS and the effect that it has on a traditional curriculum does beg the larger philosophical questions about curriculum and pedagogy. A research question, far beyond the scope of this study, but one of great importance as technology develops over time becoming smaller, more affordable and more accessible is that supposing all students 6th grade and above had a CAS enabled machine and was capable of using it at their level then what should the curriculum look like? This question is important because, suggested by the power of CAS technology it asks, fundamentally, what is a mathematics curriculum trying to achieve? What mathematics do we want students to know and in what way do we want them to know that mathematics?

In many ways graphing calculators have been relatively easy to integrate into mathematics education. The dimension of visualisation that they introduce
goes beyond making a mathematics classroom a more productive environment for students with different learning styles but allows multiple representations of mathematical objects and concepts much easier. Students learning the concept of slope and investigating the effect of the parameter $a$ on the graph of $y = a x$ would generate several graphs by hand and observe a pattern. Not only is this very labour intensive but many students will make errors in calculating points on the graphs and will, therefore, not observe the correct pattern because there data does not manifest the pattern. Of course students need to learn to draw graphs and establish a physical relationship to the graph as well as a conceptual relationship to the connection between the algebraic equation $y = a x$ and its graph. However, if the burden of the calculations and the rendering can be taken by the technology then students can focus on the effect of the parameter.

Much the same argument can be made for the use of CAS and yet, rather than enhancing the traditional curriculum as the use of graphing calculators outlined above does, the use of CAS calls that very curriculum into question. The emphasis in most mathematics curricula in high schools and universities does not emphasise mathematisation of problems to solved or the interpretation of solutions to those problems but rather emphasises the technical solutions of already mathematised problems. In other words the curricula emphasise the very part of the mathematics curriculum that can be programmed into a computer or calculator.

Therefore, the challenge facing a teacher looking to incorporate CAS technology into his or her teaching is (i) standard curricula as they currently exist do not invite the use of CAS technology, indeed using CAS in them does not, at
some level, make sense; and (ii) the students learning using CAS will have to make a considerable adjustment from the approach to mathematics learning that they have likely experienced all their lives. The Calculus & Mathematica class examined in this study and the group of students case studied provide much evidence of the nature of this challenge.

The materials used in Calculus & Mathematica were specifically designed to be used in a CAS context. There is a clear attempt to reimagine an approach to calculus which takes advantage of the possibilities for exploration and experimentation afforded by the technology. Allied to this is a clear attempt to deemphasise mechanical procedures and memorisation/algorithmic approaches. It was interesting to see in the study that this approach ran into the problem that for many students their notion of what mathematics is about and their self-efficacy in studying mathematics comes, if not from memorisation, then certainly from an ability to implement algorithms correctly and to perform mechanical operations efficiently and accurately. As we have seen students who have been successful at mathematics in school do not necessarily place a great deal of value on the meaning of their mathematical activity or on their ability to explain and interpret mathematical results. Rather, they value the ability to find correct answers to fixed problems. Finding good materials to use in teaching with CAS and convincing students of the value of the approach that must follow from appropriate use of CAS are major challenges for teachers seeking to implement the technology in their classroom.

The pedagogical challenge presented by the fact that students learning using CAS will have to make a considerable adjustment from the approach to
mathematics learning needs to be addressed in various ways. The most important aspect of this is that attention needs to be paid to bridging the students’ experience with technology in the past (graphing calculators, etc.) to this new technology. Showing students how to perform functions they are familiar with, such as plotting graphs and doing arithmetic, on the CAS before showing them some of what the CAS is capable of that is new is an important bridge to build in order for students to feel comfortable with the new technology. Teachers should also teach the technology directly rather than assume that students will discover various functionalities of the CAS.

Having become familiar with the affordances of CAS students must adjust to a different sense of what mathematics is in a CAS environment. Again, teachers should be careful not to assume that this will simply happen because of the environment. We saw in the experiences of students B and C that there is the possibility of resistance to this adjustment if the reasons behind the adjustment are not clear to the students. Teachers must continually strive, not just at the beginning of the term, to explain this adjustment to students and to convince them of the value of this new approach. Students should also be given the chance to reflect on this adjustment and work through frustrations they may be having with this adjustment. Journaling and class discussion are possible mechanisms for such reflection and adjustment.

There are many pedagogical challenges to using advanced technology but as we have seen in Chapters 4 and 5 many rewards are possible for students learning of mathematics: a mathematics that does not focus simply on algorithms but is more reflective of the work of professional mathematicians.
LIST OF REFERENCES


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