BOUNDARY BEHAVIOR OF THE BERGMAN KERNEL FUNCTION ON STRONGLY PSEUDOCONVEX DOMAINS WITH RESPECT TO WEIGHTED LEBESGUE MEASURE

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By
Lauren R. Kennell, A.B., M.A.

* * * * *

The Ohio State University
2005

Dissertation Committee:
Dr. Jeffery D. McNeal, Advisor
Dr. Gerald A. Edgar
Dr. Yuan Lou

Approved by

Advisor
Graduate Program in Mathematics
ABSTRACT

The author proves pointwise estimates for the weighted Bergman kernel and its derivatives near the boundary of a smoothly bounded, strongly pseudoconvex domain. The estimate is obtained by relating the Bergman kernel to the Neumann operator, and estimating the Neumann operator using certain biholomorphic coordinate changes chosen to take advantage of the boundary geometry. The result obtained says, essentially, that a weight function which is smooth up to the boundary of the domain neither improves nor worsens the singularity of the kernel near the boundary diagonal.
Do you not know? Have you not heard?

Has it not been declared to you from the beginning?

Have you not understood from the foundations of the earth?

Lift up your eyes on high

And see who has created these stars,

The one who leads forth their host by number,

He calls them all by name;

Because of the greatness of His might and the strength of His power,

Not one of them is missing.

Isaiah 40:21,26. (NAS)
ACKNOWLEDGMENTS

To my advisor, Jeffery McNeal, how do I thank you as you deserve? Your love of mathematics and for people is an inspiration. Everything I know about complex analysis comes from you, and you shaped the way I think about mathematics, mentoring, and many other things. In addition to being a terrific advisor (twice!), you have been a loyal friend throughout my adult life. I cannot repay what you have given to me, but I will do my best to imitate your dedication to learning, and to be as generous to others as you have been to me.

To my mother and father, I am so thankful for your continual love. You have always supported me unconditionally and self-sacrificially. This degree is as much yours as it is mine.

To Bob, Eileen, Sam, Sam-Squared, and all my friends from Mars Hill and InterVarsity Christian Fellowship, thank you for supporting my graduate school journey so faithfully with our chats, and your prayers, phone calls, e-mails, advice, meals, and movie nights. You are my Ohio family.

To Ron Walker, thank you for support and advice, and for your help with LaTex.

God bless you all. You have my love and gratitude forever.
VITA

1997-Present  . . . . . . . . . . . . . . . . . . . . . . . . . . Graduate Teaching Associate,
The Ohio State University

1997  . . . . . . . . . . . . . . . . . . . . . . . . . . M.A. in Mathematics,
The University of California, Los Angeles

1995  . . . . . . . . . . . . . . . . . . . . . . . . . . A.B. in Mathematics,
Princeton University

FIELDS OF STUDY

Major Field: Mathematics

Specialization: Several Complex Variables
# TABLE OF CONTENTS

Abstract ................................................................. ii
Dedication ................................................................. iii
Acknowledgments ......................................................... iv
Vita .......................................................................... v

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2 Preliminaries</td>
<td>5</td>
</tr>
<tr>
<td>2.1 Differential Operators and Forms</td>
<td>5</td>
</tr>
<tr>
<td>2.2 The Fourier Transform</td>
<td>10</td>
</tr>
<tr>
<td>2.3 The Operator $A^m$</td>
<td>11</td>
</tr>
<tr>
<td>2.4 Sobolev Spaces</td>
<td>12</td>
</tr>
<tr>
<td>2.5 Operator Orders and Commutators</td>
<td>12</td>
</tr>
<tr>
<td>2.6 Tangential Norms</td>
<td>27</td>
</tr>
<tr>
<td>2.7 Weighted Spaces</td>
<td>28</td>
</tr>
<tr>
<td>2.8 The Operators $A$ and $A^*$</td>
<td>31</td>
</tr>
<tr>
<td>3 Scaling</td>
<td>33</td>
</tr>
<tr>
<td>3.1 Definitions of $\Phi_{z'}$ and $S_\delta$</td>
<td>33</td>
</tr>
<tr>
<td>3.2 Bounds on the Weight Function</td>
<td>37</td>
</tr>
<tr>
<td>4 The Basic Estimate</td>
<td>39</td>
</tr>
<tr>
<td>5 The Subelliptic Estimate</td>
<td>45</td>
</tr>
<tr>
<td>6 A Bound on the Neumann Operator</td>
<td>50</td>
</tr>
<tr>
<td>6.1 Two Lemmas for $Q_{z',\delta}$</td>
<td>50</td>
</tr>
</tbody>
</table>

vi
CHAPTER 1
INTRODUCTION

Let Ω be a domain (an open connected set) in \( \mathbb{C}^n \), and let \( A^2(\Omega) \) be the space of holomorphic \( L^2 \) functions on \( \Omega \). The Bergman kernel function \( K_\Omega(z^1, z^2) \) is the integral kernel for the Bergman projection operator \( P \), which projects \( L^2(\Omega) \) orthogonally onto \( A^2(\Omega) \):

\[
P f(z^1) = \int_\Omega K_\Omega(z^1, z^2) f(z^2) \, dV(z^2), \quad f \in L^2(\Omega)
\]

where \( z^j = (z^j_1, z^j_2, \ldots, z^j_n) \).

An overview of the properties of the kernel can be found in [Kr] and [CS]. One important property is that the kernel usually blows up when \( z^1 \) and \( z^2 \) approach a common boundary point. In [McN], Jeffery McNeal constructed a scaling map to prove a pointwise estimate for the Bergman kernel and its derivatives on smooth bounded pseudoconvex domains in \( \mathbb{C}^2 \) near boundary points of finite type. The goal of this thesis is to use scaling to prove estimates for the weighted Bergman kernel on locally strongly pseudoconvex domains in \( \mathbb{C}^n \). The weighted projection is defined for a real-valued function \( \gamma \) by

\[
P_\gamma f(z^1) = \int_\Omega K_{\Omega, \gamma}(z^1, z^2)f(z^2) e^{-\gamma(z^2)} \, dV(z^2),
\]

which projects the weighted space \( L^2(\Omega, e^{-\gamma}) \) onto \( A^2(\Omega, e^{-\gamma}) \).
In a weighted or unweighted situation, the idea is to map $\Omega$ onto a family of domains on which we know something about the kernel, then pull the information back through the map, and obtain an estimate for the kernel on the original domain. This strategy works because the Bergman kernel transforms nicely under biholomorphic maps: if $F : \Omega_1 \to \Omega_2$ is biholomorphic and $\det J_C(\cdot)$ denotes the determinant of the complex Jacobian matrix, then

\[ (*) \quad K_{\Omega_1}(z^1, z^2) = \det J_C F(z^1) \cdot K_{\Omega_2}(F(z^1), F(z^2)) \cdot \det J_C F(z^2). \]

This equation also holds for weighted kernels.

The result we obtain says, essentially, that a weight function $\gamma$ which is smooth up to the boundary of $\Omega$ neither improves nor worsens the singularity of the kernel near the boundary diagonal. The theorem is the following:

**Theorem 1.** Let $\Omega$ be a smooth bounded domain which is strongly pseudoconvex near a point $p \in \partial \Omega$, and let $\gamma$ be a smooth, real-valued function on $\overline{\Omega}$. Let $U$ be a small neighborhood of $p$, $z^1, z^2 \in U \cap \Omega$, and $z'$ be the boundary point closest to $z^1$. Then we can construct a biholomorphic map $\Phi_{z'} : \zeta \to z$, depending on $z'$ but such that the Jacobian of $\Phi_{z'}$ is uniformly nonsingular on $U$, and show

\[ |D^{\alpha}_{\zeta} D^{\beta}_{\zeta^2} K_{\Omega_{z'}, \gamma_{z'}}(\zeta^1, \zeta^2)| \leq c(\gamma) \cdot \delta^{-(n+1)} \delta^{-\alpha_n - \beta_n - \frac{1}{2}} \sum_{j=1}^{n-1} (\alpha_j + \beta_j). \]

Here

\[ \delta = |r(\Phi_{z'}(\zeta^1))| + |r(\Phi_{z'}(\zeta^2))| + |\zeta^1_n - \zeta^2_n| + \sum_{j=1}^{n-1} |\zeta^1_j - \zeta^2_j|^2, \]

and $\Phi_{z'}(\Omega_{z'}) = \Omega$, $\gamma_{z'} = \gamma \circ \Phi_z$, and $r$ is a strictly plurisubharmonic defining function for $\Omega$. 2
Without a weight function, the estimate contains the same power of $\delta$. The introduction of the weight $\gamma$ produces a great many constants involving $\gamma$ and its derivatives, but they appear as terms (not factors) within $c(\gamma)$.

**Outline of the Proof**

For a point $z' \in b\Omega$, we construct the local coordinate map $\Phi_{z'}$ so that $z'$ gets mapped to the origin, and the complex normal to the boundary at the origin becomes the $n$th coordinate direction. The coordinate change depends on $z'$, so we get a new family of domains $\Omega_{z'}$. The second coordinate change is the scaling map: we scale the coordinates by a factor of $\sqrt{\delta}$ in the tangential directions, and by a factor of $\delta$ in the normal direction. This is the scaling in [McN], but simplified for strongly pseudoconvex domains.

The choice of the powers on $\delta$ and the boundedness of $\gamma$ allow us to prove a uniform subelliptic estimate (“uniform” meaning the constants are independent of $\delta$), and from there to get a bound on the weighted Neumann operator (also independent of $\delta$). The connection to the Bergman projection comes from a weighted version of Kohn’s Formula:

$$P_{\gamma_{z'},\delta} = I - \overline{\partial}_{\gamma_{z'},\delta} N_{\gamma_{z'},\delta} \overline{\partial}$$

where $\gamma_{z',\delta}$ is the weight function after the two coordinate changes, and $N_{\gamma_{z'},\delta}$ is the weighted Neumann operator.

A theorem of Kerzman says that, under certain conditions, the Bergman kernel is bounded by a constant if the two variables are restricted to disjoint compacts (see
In chapter 7 we use the bound on $N_{\gamma z',\delta}$ to prove a version of Kerzman’s theorem for the weighted kernel on the scaled domains $\Omega_{z',\delta}$. To make use of it, we choose $\delta$ to be the special “distance” defined in Theorem 1, which measures how close two points $\zeta^1$ and $\zeta^2$ are to each other and to the boundary. With this choice of $\delta$, the scaled images $w^1$ and $w^2$ of $\zeta^1$ and $\zeta^2$ are pulled apart, so that Kerzman’s theorem applies on $\Omega_{z',\delta}$. The estimate on the kernels for $\Omega_{z'}$ follows from the transformation property ($\ast$).

In chapter 2 we set up the notation for the various operators and function spaces. We also prove some facts about pseudodifferential operators in unweighted and weighted $L^2$ spaces. In chapter 3 we write down the coordinate maps, and in subsequent chapters use them to bound the Neumann operator and estimate the Bergman kernel.
CHAPTER 2
PRELIMINARIES

The first section in this chapter is devoted to setting up notation and basic facts about differential forms and the Dolbeault complex, which are required for a subelliptic estimate (in chapter 5) and eventually the bound on the Neumann operator (in chapters 6-7). Subsequent sections 2.2-2.8 are devoted to pseudodifferential operators in weighted and unweighted spaces. These sections make for a rather long (and tedious!) chapter, so the reader wishing a faster read is invited to scan the results and hop over the proofs.

2.1 Differential Operators and Forms

The inner product of two functions $f$ and $g$ is defined

$$(f, g)_\gamma = \int fg \ dV,$$

and the $L^2$ functions are those with finite $L^2$ norm,

$$L^2(\mathbb{R}^N) = \{ f : ||f||^2 = (f, f) < \infty \}.$$

Similarly, the weighted norm $|| \cdot ||_\gamma$ and space $L^2(\mathbb{R}^N, e^{-\gamma})$ correspond to the inner product

$$(f, g) = \int f\bar{g}e^{-\gamma} \ dV.$$
Next, for \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n, z_k = x_k + iy_k \), we define the differential operators
\[
\frac{\partial}{\partial z_k} = \frac{1}{2} \left( \frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_k} = \frac{1}{2} \left( \frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k} \right)
\]
and dual forms
\[
dz_k = dx_k + idy_k \quad \text{and} \quad d\overline{z}_k = dx_k - idy_k.
\]
For a \( C^1 \) function \( u \) we also define
\[
du = \sum_{k=1}^{n} \frac{\partial u}{\partial z_k} dz_k + \sum_{k=1}^{n} \frac{\partial u}{\partial \overline{z}_k} d\overline{z}_k
\]
\[
def = \frac{\partial u + \overline{\partial} u}{2}.
\]
Note that a function \( f \) is holomorphic if \( \overline{\partial} f \equiv 0 \).

**The Dolbeault Complex**

The absolute value of an \( n \)-index \( I \) is defined by \( |I| = i_1 + \cdots + i_n \). The space \( \Lambda^{0,0}(\Omega) \) of smooth \((0,0)\) forms is the space of smooth functions on \( \Omega \). The space of general \((p,q)\) forms is defined
\[
\Lambda^{p,q}(\Omega) = \left\{ u = \sum_{|I|=p, |J|=q} u_{IJ} \ dz^I \wedge d\overline{z}^J, \quad u_{IJ} \in C^\infty(\Omega) \right\}
\]
where \( dz^I = dz_1^{i_1} \wedge \cdots \wedge dz_n^{i_n} \) and similar for \( d\overline{z}^J \). At each form level the operator \( \overline{\partial} \) maps \( \Lambda^{p,q}(\Omega) \) into \( \Lambda^{p,q+1}(\Omega) \), which produces the Dolbeault complex, shown here for \( p = 0 \):
\[
\Lambda^{0,0}(\Omega) \xrightarrow{\overline{\partial}} \Lambda^{0,1}(\Omega) \xrightarrow{\overline{\partial}} \Lambda^{0,2}(\Omega) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Lambda^{0,n}(\Omega).
\]
For instance, on a \((0, 1)\) form \(u = \sum_{j=1}^{n} u_j \, d\bar{z}_j\),

\[
\bar{\partial} u = \sum_{j,k=1}^{n} \frac{\partial u_j}{\partial \overline{z}_k} \, d\overline{z}_k \wedge d\overline{z}_j = \sum_{k<j} \left( \frac{\partial u_j}{\partial \overline{z}_k} - \frac{\partial u_k}{\partial \overline{z}_j} \right) \, d\overline{z}_k \wedge d\overline{z}_j.
\]

By equality of mixed partials, the operator \(\overline{\partial}^2 = 0\) at each form level.

To discuss \(\overline{\partial}\) as an operator on a Hilbert space, the \(L^2(\gamma)\) inner product on forms is defined componentwise:

\[
(u, v)_\gamma = \int \langle u, v \rangle e^{-\gamma} \, dV.
\]

Then define the domain of \(\overline{\partial}\) on \((0,0)\) forms as

\[
\text{Dom } \overline{\partial} = \{ u \in L^2(\Omega, e^{-\gamma}) : \overline{\partial} u \in L^2_{0,1}(\Omega, e^{-\gamma}) \}
\]

where \(\overline{\partial} u\) is in the sense of distributions, and \(L^2_{0,1}\) is the space of \((0,1)\) forms with finite \(L^2\) norm. The extension of \(\overline{\partial}\) to \((0,q)\) forms is defined similarly.

We also need adjoints of a \((0,1)\) form. First, \(v \in \text{Dom } \overline{\partial}^*_\gamma\) means that \(\overline{\partial}^*_\gamma v\) satisfies the equation

\[
(u, \overline{\partial}^*_\gamma v)_\gamma = (\overline{\partial} u, v)_\gamma \quad \forall u \in \text{Dom } \overline{\partial},
\]

which is equivalent to the condition

\[
\text{Dom } \overline{\partial}^*_\gamma = \{ v \in L^2_{0,1}(\Omega) : |(\overline{\partial} u, v)_\gamma| \leq c ||u||_\gamma, \forall u \in \text{Dom } \overline{\partial} \}
\]

because of Riesz Representation.
The formal adjoint $\overline{\partial}_1'$ of a $(0, 1)$ form should satisfy the inner product equation for $u, v \in \Lambda_{0}^{0,1}(\Omega)$, that is, for smooth $(0, 1)$ forms with compact support. The formal adjoint can be computed by integration by parts, which in complex notation has form

$$
\left( \frac{\partial a}{\partial z_j}, b \right) = - \left( a, \frac{\partial b}{\partial z_j} \right) + \int_{\partial \Omega} a \overline{b} \frac{\partial r}{\partial z_j} d\sigma
$$

where $a$ and $b$ are smooth functions and $r$ is a defining function for $\Omega \subset \subset \mathbb{C}^n$,

$$
\Omega = \{ z : r(z) < 0 \}
$$

which is also normalized,

$$
|\nabla r| = 1 \quad \text{on} \quad b\Omega = \{ z : r(z) = 0 \}.
$$

Applying integration by parts to $\overline{\partial} u$ and $v$, the boundary integral vanishes, but we have to incorporate the weight function:

$$
(\overline{\partial} u, v)_{\gamma} = \int_{\Omega} \left( \sum_{j=1}^{n} \frac{\partial u}{\partial z_j} \cdot \overline{v_j} \right) e^{-\gamma} dV
$$

$$
= - \int_{\Omega} u \cdot \sum_{j=1}^{n} \frac{\partial}{\partial z_j} (\overline{v_j} e^{-\gamma}) dV
$$

and since $\gamma$ is real valued,

$$
= - \int_{\Omega} u \cdot e^{\gamma} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} (v_j e^{-\gamma}) \cdot e^{-\gamma} dV,
$$
so we obtain the formal adjoint

$$
\overline{\partial}_\gamma v = -e^\gamma \sum_{j=1}^n \frac{\partial}{\partial z_j} (v_j e^{-\gamma}) \\
= \sum_{j=1}^n \left( \frac{\partial \gamma}{\partial z_j} v_j - \frac{\partial v_j}{\partial z_j} \right).
$$

Using integration by parts we can also establish a condition for smooth forms to be in $\text{Dom } \overline{\partial}_\gamma$.

**Lemma 2.1.1.** Let $r(z)$ define $\Omega$ and let

$$
\mathcal{D}^{0,1}(\Omega) = \Lambda^{0,1}(\Omega) \cap \text{Dom } \overline{\partial}_\gamma.
$$

Then $v \in \mathcal{D}^{0,1}(\Omega)$ if and only if $\sum_{j=1}^n \frac{\partial r}{\partial z_j} v_j = 0$ on $b\Omega$.

**Pf.** For $u, v \in \Lambda^{0,1}(\Omega)$, we have

$$
(\overline{\partial} u, v)_{\gamma} = (u, \overline{\partial}_\gamma v)_{\gamma} + \int_{b\Omega} \sum_{j=1}^n u \overline{\bar{v}}_j \frac{\partial r}{\partial z_j} e^{-\gamma} \, d\sigma.
$$

The only way to ensure $|(\overline{\partial} u, v)_{\gamma}| \leq c ||u||_{\gamma}$ is if the boundary integral vanishes, which means $\sum_{j=1}^n (\partial r/\partial z_j) \bar{v}_j = 0$ on $b\Omega$. ■

For further discussion of the material in this section, we recommend [CS] and [FK].

We devote the next several sections to the Fourier transform, Sobolev spaces, and pseudodifferential operators. In sections 2.2-2.6 we state (and sometimes prove) results for unweighted $L^2$ spaces, most of which can be found in [F] and/or [FK]. Then we discuss how these results are adjusted for weighted spaces.
2.2 The Fourier Transform

The Schwarz functions are the smooth functions which decrease rapidly at infinity:

\[ S(\mathbb{R}^N) = \left\{ f \in C^\infty(\mathbb{R}^N) : \forall \alpha, \beta, \sup_{x \in \mathbb{R}^N} |x^\alpha D^\beta f(x)| < \infty \right\} \]

where \( \alpha \) and \( \beta \) are \( N \)-indeces and

\[ D^\beta = \left( \frac{\partial}{\partial x} \right)^\beta = \left( \frac{\partial}{\partial x_1} \right)^{\beta_1} \cdots \left( \frac{\partial}{\partial x_N} \right)^{\beta_N}. \]

The Fourier transform \( \mathcal{F} : S \to S \) is defined by

\[ (\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-2\pi i \langle x, \xi \rangle} \, dx \]

and its inverse \( \mathcal{F}^{-1} : S \to S \) is

\[ (\mathcal{F}^{-1}f)(x) = \int_{\mathbb{R}^N} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} \, d\xi. \]

The Fourier transform enjoys the following five properties.

**Lemma 2.2.1 (Fourier Transform Properties).** For \( f \in S \) we have

1. **Fourier Inversion Formula:** \( \mathcal{F}^{-1}(\hat{f}) = f \)
2. **Plancherel’s Identity:** \( \|f\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2 \)
3. **Parseval’s Identity:** \( (f, g) = (\hat{f}, \hat{g}) \)
4. **Transform of a Derivative:** \( \mathcal{F}(D^\beta f)(\xi) = (2\pi i)^{\beta} \xi^\beta \hat{f}(\xi) \)
5. **Transform of a Product:** \( \mathcal{F}(f \cdot g)(\xi) = \hat{f} \ast \hat{g}(\xi) \)
2.3 The Operator $\Lambda^m$

Define the pseudodifferential operator $\Lambda^m : S \to S$ by
\[
\Lambda^m f(x) = \mathcal{F}^{-1} \left( (1 + |\xi|^2)^{\frac{m}{2}} \hat{f}(\xi) \right).
\]
The operator $\Lambda^m$ has the following two properties, similar to derivatives of order $m$.

**Lemma 2.3.1 (Self-Adjointness of $\Lambda^m$).**
\[
(\Lambda^m f, g) = (f, \Lambda^m g).
\]

**Pf.** From Parseval’s Identity,
\[
(\Lambda^m f, g) = ((1 + |\xi|^2)^{\frac{m}{2}} \hat{f}, \hat{g})
= (\hat{f}, (1 + |\xi|^2)^{\frac{m}{2}} \hat{g})
= (f, \Lambda^m g).
\]

**Lemma 2.3.2.** The orders can be added:
\[
\Lambda^{m_1} \Lambda^{m_2} = \Lambda^{m_1+m_2}.
\]

**Pf.**
\[
\Lambda^{m_1} \Lambda^{m_2} f(x) = \mathcal{F}^{-1} \left( (1 + |\xi|^2)^{\frac{m_1}{2}} \mathcal{F} (\Lambda^{m_2} f)(\xi) \right)
= \mathcal{F}^{-1} \left( (1 + |\xi|^2)^{\frac{m_1+m_2}{2}} \hat{f}(\xi) \right)
= \Lambda^{m_1+m_2} f(x).
\]
2.4 Sobolev Spaces

For $m \in \mathbb{R}$, the space $H_m(\mathbb{R}^N)$ is the completion of $\mathcal{S}$ under the $m$-norm given by

$$||f||_m^2 = ||\Lambda^m f||_{L^2}^2.$$ 

In particular, $H_0(\mathbb{R}^N)$ is $L^2(\mathbb{R}^N)$. Also, if $m$ is a positive integer, the $m$-norm of $f$ is equivalent to the sum of the $L^2$ norms of the derivatives of $f$ up to order $m$:

$$||f||_m^2 \sim \sum_{|\alpha| \leq m} ||D^\alpha f||_{L^2}^2.$$ 

We also need the following two lemmas (see [FK]).

**Lemma 2.4.1 (Generalized Schwarz Inequality).** For $f, g \in H_0$,

$$(f, g) \leq ||f||_s ||g||_{-s}.$$ 

**Lemma 2.4.2 (Sobolev Lemma).** If $m > k + \frac{1}{2}N$, then $H_m(\mathbb{R}^N) \subset C^k(\mathbb{R}^N)$ and

$$\sup_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^N} |D^\alpha f(x)| \leq c_{m,k} ||f||_m.$$ 

2.5 Operator Orders and Commutators

An operator $T$ is of order $m$ if

$$||Tf||_s \leq c_s ||f||_{s+m} \quad \forall s \in \mathbb{R}, f \in \mathcal{S}.$$ 

For instance, derivatives of order $m$ and the operator $\Lambda^m$ are both order $m$ operators.

To establish some other orders we need three inequalities.
Lemma 2.5.1 (Young’s Inequality). Suppose $F(x, y)$ is bounded and measurable on $\mathbb{R}^N \times \mathbb{R}^N$, and

$$\int |F(x, y)| \, dx \leq c \quad \text{for all } y$$

and

$$\int |F(x, y)| \, dy \leq c \quad \text{for all } x.$$

Then the operator $T$ defined by

$$Tg(x) = \int F(x, y)g(y) \, dy$$

is bounded on $L^2(\mathbb{R}^N)$.

Pf. We have

$$||Tg||^2_{L^2} = \int |Tg(x)|^2 \, dx$$

and by the Schwarz Inequality,

$$|Tg(x)|^2 \leq \left( \int |F(x, y)||g(y)| \, dy \right)^2 \leq \int |F(x, y)| \, dy \int |F(x, y)||g(y)|^2 \, dy,$$

and so

$$||Tg||^2_{L^2} \leq c \int \int |F(x, y)||g(y)|^2 \, dy \, dx \leq c^2 \int |g(y)|^2 \, dy \leq c^2 ||g||^2_{L^2}.$$
Lemma 2.5.2 (The "lc/sc" Inequality). For any positive numbers $A$, $B$, and $\epsilon$,

$$AB \leq \epsilon A^2 + \frac{1}{4\epsilon}B^2,$$

or,

$$AB \lesssim (\text{large constant}) \cdot A^2 + (\text{small constant}) \cdot B^2.$$

\textbf{Pf.} Expand the expression $\left(\sqrt{\epsilon}A - \frac{1}{2\sqrt{\epsilon}}B\right)^2 \geq 0$. \hfill $\blacksquare$

Lemma 2.5.3. For all $x, y \in \mathbb{R}^N$ and $s \in \mathbb{R}$,

$$\left(\frac{1 + |x|^2}{1 + |y|^2}\right)^s \leq 2^{|s|}(1 + |x - y|^2)^{|s|}.$$

\textbf{Pf.} Start with the triangle inequality,

$$|x| \leq |x - y| + |y|$$

and square both sides and use the $lc/sc$ Inequality,

$$|x|^2 \leq 2(|x - y|^2 + |y|^2).$$

Adding one to the left side and more than two to the right side,

$$1 + |x|^2 \leq 2(|x - y|^2 + |y|^2 + |x - y|^2|y|^2 + 1)$$

$$= 2(1 + |x - y|^2)(1 + |y|^2).$$

If $s \geq 0$, we are finished by raising both sides to the $s$ power. If $s < 0$, use the same argument, but switch $x$ and $y$ and replace $s$ with $-s$. \hfill $\blacksquare$
The first lemma we prove with the above inequalities shows that multiplication by a
Schwarz function is an order zero operator.

**Lemma 2.5.4.** For \( a \in S \),

\[
||af||_s \lesssim ||f||_s
\]

uniformly for \( f \in S \).

**Pf.**

\[
\mathcal{F}(\Lambda^s(af))(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}(af)(\xi)
\]

\[
= \int (1 + |\xi|^2)^{\frac{s}{2}} \hat{a}(\xi - \tau) \hat{f}(\tau) \, d\tau
\]

\[
= \int \frac{(1 + |\xi|^2)^{\frac{s}{2}}}{(1 + |\tau|^2)^{\frac{s}{2}}} \hat{a}(\xi - \tau) \hat{f}(\tau)(1 + |\tau|^2)^{\frac{s}{2}} \, d\tau.
\]

Let

\[
F(\xi, \tau) = \left( \frac{1 + |\xi|^2}{1 + |\tau|^2} \right)^{\frac{s}{2}} \hat{a}(\xi - \tau)
\]

and

\[
g(\tau) = (1 + |\tau|^2)^{\frac{s}{2}} \hat{f}(\tau),
\]

so

\[
||af||^2_s = \int \left| \int F(\xi, \tau) g(\tau) \, d\tau \right|^2 \, d\xi.
\]

But by Lemma 2.5.3,

\[
|F(\xi, \tau)| \lesssim (1 + |\xi - \tau|^2)^{\frac{|s|}{2}} |\hat{a}(\xi - \tau)|,
\]
and $a$ is a Schwarz function, so apply Young’s Inequality to obtain

$$\|af\|_{s}^{2} \lesssim \|g\|_{L^{2}}^{2} = \|f\|_{s}^{2}.$$ 


### Operators with Commutators

The commutator $[A, B]$ of operators $A$ and $B$ is defined as $AB - BA$. The next several lemmas are about the orders of various operators involving commutators. The proofs are from [FK].

**Lemma 2.5.5.** For $a \in S$, the operator $[\Lambda^{m}, a]$ is of order $m - 1$, that is

$$\|[\Lambda^{m}, a]u\|_{s}^{2} \lesssim \|u\|_{s+m-1}^{2}.$$ 

**Pf.** This is equivalent to

$$\|[\Lambda^{m}, a]\Lambda^{-s-m+1}\Lambda^{s+m-1}u\|_{L^{2}}^{2} \lesssim \|\Lambda^{s+m-1}u\|_{L^{2}}^{2}$$

or

$$\|[\Lambda^{s}[\Lambda^{m}, a]\Lambda^{-s-m+1}f\|_{L^{2}}^{2} \lesssim \|f\|_{L^{2}}^{2}$$

where $f = \Lambda^{s+m-1}u$.

In order to use Plancherel’s identity (and eventually Young’s Inequality) on the left side, we need the Fourier transform of $\Lambda^{s}[\Lambda^{m}, a]\Lambda^{-s-m+1}f$. But first, since we will use it here and later on, we compute the transform of the expression $[\Lambda^{m}, a]g$. We
use coordinates $x = (x_1, \ldots, x_N)$, and two transform variables, $\xi = (\xi_1, \ldots, \xi_N)$ and $\tau = (\tau_1, \ldots, \tau_N)$. We have

$$F([\Lambda^m, a]g)(\xi) = F(\Lambda^m ag - a\Lambda^m g)(\xi)$$

$$= (1 + |\xi|^2)^{\frac{m}{2}} (\hat{a} \ast \hat{g})(\xi) - (\hat{a} \ast F(\Lambda^m g))(\xi)$$

$$= (1 + |\xi|^2)^{\frac{m}{2}} \int \hat{a}(\xi - \tau)\hat{g}(\tau) \, d\tau - \int \hat{a}(\xi - \tau)(1 + |\tau|^2)^{\frac{m}{2}}\hat{g}(\tau) \, d\tau$$

$$= \int \left( (1 + |\xi|^2)^{\frac{m}{2}} - (1 + |\tau|^2)^{\frac{m}{2}} \right) \hat{a}(\xi - \tau)\hat{g}(\tau) \, d\tau.$$

Then

$$F(\Lambda^s[\Lambda^m, a]\Lambda^{-s-m+1} f)(\xi)$$

$$= (1 + |\xi|^2)^{\frac{s}{2}} F([\Lambda^m, a]\Lambda^{-s-m+1} f)(\xi)$$

$$= (1 + |\xi|^2)^{\frac{s}{2}} \int \left( (1 + |\xi|^2)^{\frac{m}{2}} - (1 + |\tau|^2)^{\frac{m}{2}} \right) \hat{a}(\xi - \tau)(\Lambda^{-s-m+1} f)(\tau) \, d\tau$$

$$= \int (1 + |\xi|^2)^{\frac{s}{2}} \left( (1 + |\xi|^2)^{\frac{m}{2}} - (1 + |\tau|^2)^{\frac{m}{2}} \right) \left( (1 + |\tau|^2)^{-\frac{s-m+1}{2}} \hat{a}(\xi - \tau) \hat{f}(\tau) \right) \, d\tau.$$

Now define

$$F_1(\xi, \tau) = (1 + |\xi|^2)^{\frac{s}{2}} \left( (1 + |\xi|^2)^{\frac{m}{2}} - (1 + |\tau|^2)^{\frac{m}{2}} \right) \left( (1 + |\tau|^2)^{-\frac{s-m+1}{2}} \hat{a}(\xi - \tau) \right).$$

To use Young’s Inequality we have to bound $|F_1(\xi, \tau)|$, so we prove the following inequality.

**Lemma 2.5.6.** For $\xi, \tau \in \mathbb{R}^N$ and $m \in \mathbb{R}$,

$$|(1 + |\xi|^2)^{\frac{m}{2}} - (1 + |\tau|^2)^{\frac{m}{2}}| \lesssim |\xi - \tau| \left( (1 + |\xi|^2)^{\frac{m-1}{2}} + (1 + |\tau|^2)^{\frac{m-1}{2}} \right).$$
Pf. By Taylor’s Theorem on the function $h(ξ) = (1 + |ξ|^2)^{\frac{m}{2}}$,

$$(1 + |ξ|^2)^{\frac{m}{2}} - (1 + |τ|^2)^{\frac{m}{2}} \lesssim |ξ - τ| \sup_j \left| \frac{∂h}{∂ξ_j} \right| \text{ on } B(τ, |ξ|)$$

where $B(τ, |ξ|)$ is the ball in $\mathbb{R}^N$ with center $τ$ and radius $|ξ|$. Note that

$$\frac{∂h}{∂ξ_j} = m(1 + |ξ|^2)^{\frac{m-2}{2}} ξ_j,$$

and also $|ξ| \leq \sqrt{1 + |ξ|^2}$ for all $ξ$. This means

$$\left| \frac{∂h}{∂ξ_j} \right| \leq (1 + |ξ|^2)^{\frac{m-1}{2}}$$

and therefore

$$\sup_j \left| \frac{∂h}{∂ξ_j} \right| \text{ on } B(τ, |ξ|) \lesssim (1 + |ξ|^2)^{\frac{m-1}{2}} + (1 + |τ|^2)^{\frac{m-1}{2}}$$

which proves Lemma 2.5.6.

Returning to the proof of Lemma 2.5.5,

$$|F_1| \lesssim \left( (1 + |ξ|^2)^{\frac{s+m-1}{2}} (1 + |τ|^2)^{\frac{s-m+1}{2}} + (1 + |ξ|^2)^{\frac{s}{2}} (1 + |τ|^2)^{-\frac{s}{2}} \right) |ξ - τ| |⟨a(ξ - τ)|$$

and by Lemma 2.5.6 this is

$$\lesssim \left( (1 + |ξ - τ|^2)^{\frac{s+m-1}{2}} + (1 + |ξ - τ|^2)^{\frac{s}{2}} \right) |ξ - τ| |⟨a(ξ - τ)|.$$

Since $a ∈ S$, and the rest of the expression is a polynomial in $|ξ - τ|$, the operator

$$T : \hat{f}(ξ) \rightarrow \int F_1(ξ, τ) \hat{f}(τ) \ dτ$$

18
is bounded on $L^2$, that is

$$\left\| \int \int F_1(\xi, \tau) \hat{f}(\tau) \, d\tau \right\|^2 \, d\xi \lesssim \int |\hat{f}(\xi)|^2 \, d\xi.$$ 

Returning to (†) and using Plancherel’s identity twice, we have

$$\left\| \Lambda^s [\Lambda^m, [\Lambda^m, a]] \Lambda^{-s-m+1} f \right\|_{L^2}^2 = \int \left| \int F_1(\xi, \tau) \hat{f}(\tau) \, d\tau \right|^2 \, d\xi$$

$$\lesssim \int |\hat{f}(\xi)|^2 \, d\xi$$

$$= \int |f(x)|^2 \, dx$$

$$= \int |\Lambda^{s+m-1} u(x)|^2 \, dx$$

$$= \left\| u \right\|^2_{s+m-1}$$

and the proof is finished.

\[\square\]

**Lemma 2.5.7.** The operator $[\Lambda^m, [\Lambda^m, a]]$ is of order $2m - 2$.

**Pf.** We need to show

$$\left\| \Lambda^s [\Lambda^m, [\Lambda^m, a]] \Lambda^{-s-2m+2} f \right\|_{L^2}^2 \lesssim \left\| f \right\|^2_{L^2}.$$ 

Let $g = \Lambda^{-s-2m+2} f$ and expand the commutator,

$$[\Lambda^m, [\Lambda^m, a]] g = \Lambda^{2m} a g - 2 \Lambda^m a \Lambda^m g + a \Lambda^{2m} g,$$
and compute the Fourier transform of $\Lambda^*[\Lambda^m, [\Lambda^m, a]]g$,

$$\mathcal{F}[\Lambda^m, [\Lambda^m, a]]g(\xi)$$

$$= (1 + |\xi|^2)^{\frac{1}{2}} \int \left( (1 + |\xi|^2)^m - 2(1 + |\xi|^2)^{\frac{m}{2}}(1 + |\tau|^2)^{\frac{m}{2}} + (1 + |\tau|^2)^m \right) \hat{a}(\xi - \tau) \hat{g}(\tau) d\tau$$

$$= (1 + |\xi|^2)^{\frac{1}{2}} \int \left( (1 + |\xi|^2)^{\frac{m}{2}} - (1 + |\tau|^2)^{\frac{m}{2}} \right)^2 \hat{a}(\xi - \tau) \hat{g}(\tau) d\tau$$

$$= (1 + |\xi|^2)^{\frac{1}{2}} \int \left( (1 + |\xi|^2)^{\frac{m}{2}} - (1 + |\tau|^2)^{\frac{m}{2}} \right)^2 \hat{a}(\xi - \tau)(1 + |\tau|^2)^{\frac{s-2m+2}{2}} \hat{f}(\tau) d\tau$$

$$= \int F_2(\xi, \tau) \hat{a}(\xi - \tau) \hat{f}(\tau) d\tau,$$

where

$$|F_2(\xi, \tau)| \lesssim \frac{(1 + |\xi|^2)^{\frac{1}{2}}}{(1 + |\tau|^2)^{\frac{s+2m-2}{4}}} \left( |\xi - \tau| \left[ (1 + |\xi|^2)^{\frac{m+1}{2}} + (1 + |\tau|^2)^{\frac{m+1}{2}} \right] \right)^2$$

$$= |\xi - \tau|^2 \left( \frac{(1 + |\xi|^2)^{\frac{s+2m-2}{4}}}{(1 + |\tau|^2)^{\frac{s+2m-2}{4}}} + \frac{(1 + |\xi|^2)^{\frac{1}{4}}}{(1 + |\tau|^2)^{\frac{1}{4}}} \right)^2$$

$$\lesssim |\xi - \tau|^2 \left( (1 + |\xi - \tau|^2)^{\frac{s+2m-2}{4}} + (1 + |\xi - \tau|^2)^{\frac{1}{4}} \right)^2$$

and we are done as before. ■

Next we establish the order of the operator $[[\Lambda^m, a], a]$, for which we need another inequality.

**Lemma 2.5.8.** Let

$$R(\xi, \eta) = (1 + |\xi|^2)^{\frac{1}{2}} - (1 + |\eta|^2)^{\frac{1}{2}} - \sum_{j=1}^{N} (\xi_j - \eta_j) \frac{\partial}{\partial \xi_j} (1 + |\xi|^2)^{\frac{1}{2}}.$$
Then

\[ |R(\xi, \eta)| \lesssim |\xi - \eta|^2 \left( (1 + |\xi|^2)^{\frac{m-2}{2}} + (1 + |\eta|^2)^{\frac{m-2}{2}} \right). \]

\textbf{Pf.}  As before let

\[ h(\eta) = (1 + |\eta|^2)^\frac{m}{2} \]

and use Taylor’s Theorem:

\[ (1 + |\eta|^2)^\frac{m}{2} = (1 + |\xi|^2)^\frac{m}{2} + \sum_{j=1}^{N} \frac{\partial}{\partial \xi_j} (1 + |\xi|^2)^\frac{m}{2} \cdot (\eta_j - \xi_j) + R_{1,\xi}(\eta) \]

where

\[ |R_{1,\xi}(\eta)| \lesssim \left\{ \sup_{j,k} \left| \frac{\partial^2 h}{\partial \eta_j \partial \eta_k} \right| \text{ on } B(\xi, |\eta|) \right\} \cdot |\eta - \xi|^2. \]

To bound the second partial derivatives notice that

\[ \frac{\partial^2 h}{\partial \eta_j \partial \eta_k} = m\delta_{jk}(1 + |\eta|^2)^{\frac{m-4}{2}} + m(m-2)\eta_j \eta_k (1 + |\eta|^2)^{\frac{m-4}{2}} \]

and since \(|\eta_j \eta_k| \leq |\eta|^2 \leq 1 + |\eta|^2|\), we have

\[ \left| \frac{\partial^2 h}{\partial \eta_j \partial \eta_k} \right| \lesssim (1 + |\eta|^2)^{\frac{m-2}{2}}, \]

and so

\[ \sup_{j,k} \left| \frac{\partial^2 h}{\partial \eta_j \partial \eta_k} \right| \lesssim (1 + |\xi|^2)^{\frac{m-2}{2}} + (1 + |\eta|^2)^{\frac{m-2}{2}} \]

on \( B(\xi, |\eta|) \). Thus

\[ \left| (1 + |\xi|^2)^\frac{m}{2} - (1 + |\eta|^2)^\frac{m}{2} - \sum_{j=1}^{N} \frac{\partial}{\partial \xi_j} (1 + |\xi|^2)^\frac{m}{2} \cdot (\xi_j - \eta_j) \right| \]

\[ \lesssim |\xi - \eta|^2 \left( (1 + |\xi|^2)^{\frac{m-2}{2}} + (1 + |\eta|^2)^{\frac{m-2}{2}} \right) \]

as desired.  \[ \blacksquare \]
Lemma 2.5.9. The operator $[[\Lambda^m, a], a]$ is of order $m - 2$.

Pf. Rewrite $[[\Lambda^m, a], a]$ as

$$[[\Lambda^m, a], a] = [[\Lambda^m, a] + T, a] - [T, a]$$

where $T$ is defined by

$$\mathcal{F}(Tu)(\xi) = - \int F_3(\xi, \tau) \hat{f}(\tau) \, d\tau$$

and

$$F_3(\xi, \tau) = \sum_{j=1}^{N} \frac{\partial}{\partial \xi_j} (1 + |\xi|^2)^{\frac{m}{2}} (\xi_j - \tau_j) \hat{a}(\xi - \tau)$$

$$= m \sum_{j=1}^{N} (1 + |\xi|^2)^{\frac{m-2}{2}} \xi_j (\xi_j - \tau_j) \hat{a}(\xi - \tau).$$

First we show that $-T$ is the principal part of $[\Lambda^m, a]$, so that $[\Lambda^m, a] + T$ is of order $m - 2$.

$$\mathcal{F}(\Lambda^s([\Lambda^m, a] + T)\Lambda^{2-m-s} f)(\xi)$$

$$= (1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}(([\Lambda^m, a] + T)g)(\xi).$$
where \( g = \Lambda^{2-m-s}f \). In the next line we borrow the previous result for the Fourier transform of \([\Lambda^m, a]g\), and obtain the above

\[
= (1 + |\xi|^2)^{\frac{1}{2}} \left[ \int ((1 + |\xi|^2)^{\frac{m}{2}} - (1 + |\eta|^2)^{\frac{m}{2}}) \hat{a}(\xi - \eta) \hat{g}(\eta) \, d\eta \\
- \int F_3(\xi, \eta) \hat{g}(\eta) \, d\eta \right] \\
= (1 + |\xi|^2)^{\frac{1}{2}} \int R(\xi, \eta) \hat{a}(\xi - \eta)(1 + |\eta|^2)^{\frac{2-m-s}{2}} \hat{f}(\eta) \, d\eta.
\]

Using the bound on \( R(\xi, \eta) \) from Lemma 2.5.8, we are done by Young’s inequality.

Now we have to show that \([T, a]\) is order \( m - 2\):

\[
\mathcal{F}(\Lambda^s[T, a]\Lambda^{2-m-s}f)(\xi) \\
= (1 + |\xi|^2)^{\frac{1}{2}} \mathcal{F}[T, a]\Lambda^{2-m-s}f)(\xi) \\
= (1 + |\xi|^2)^{\frac{1}{2}} \mathcal{F}(T(a\tilde{g}) - aT\tilde{g})(\xi)
\]

where again \( g = \Lambda^{2-m-s}f \). So we need to compute the transform of \( T(a\tilde{g}) \) and \( aT\tilde{g} \).

Let

\[
\ell(\alpha, \beta) = m(\alpha, \beta)(1 + |\beta|^2)^{\frac{m-2}{2}} \hat{a}(\alpha)
\]

and notice

\[
F_3(\xi, \tau) = \ell(\xi - \tau, \xi).
\]
Then we have

\[
\mathcal{F}(aTg)(\xi) = (\hat{a} \ast \mathcal{F}(Tg))(\xi)
\]

\[
= \int \hat{a}(\xi - \tau) \mathcal{F}(Tg)(\tau) \, d\tau
\]

\[
= -\int \hat{a}(\xi - \tau) \int F_3(\tau, \eta) \hat{g}(\eta) \, d\eta \, d\tau
\]

\[
= -\int \int \hat{a}(\xi - \tau) \ell(\tau - \eta, \tau) \hat{g}(\eta) \, d\eta \, d\tau,
\]

and

\[
\mathcal{F}(T(a)g)(\xi) = -\int F_3(\xi, \tau) \mathcal{F}(ag)(\tau) \, d\tau
\]

\[
= -\int \ell(\xi - \tau, \xi) \int \hat{a}(\tau - \eta) \hat{g}(\eta) \, d\eta \, d\tau
\]

and replacing \( \tau \) by \( \xi - \eta - \tau \) we obtain

\[
= -\int \int \ell(\tau - \eta, \xi) \hat{a}(\xi - \tau) \hat{g}(\eta) \, d\eta \, d\tau.
\]

Therefore

\[
\mathcal{F}(\Lambda^s[T,a]\Lambda^{2-m-s}f)(\xi)
\]

\[
= -\frac{(1 + |\xi|^2)^{\frac{s}{2}}}{(1 + |\eta|^2)^{\frac{4s + m - 2}{2}}} \int \int \hat{a}(\xi - \tau) (\ell(\tau - \eta, \xi) - \ell(\tau - \eta, \tau)) \hat{f}(\eta) \, d\tau \, d\eta
\]

Let

\[
F_4(\xi, \eta) = \frac{(1 + |\xi|^2)^{\frac{s}{2}}}{(1 + |\eta|^2)^{\frac{4s + m - 2}{2}}} \int \hat{a}(\xi - \tau) (\ell(\tau - \eta, \xi) - \ell(\tau - \eta, \tau)) \, d\tau,
\]

24
and work on bounding the $\ell$ expression. From the definition of $\ell$ we have

$$|\ell(\tau - \eta, \xi) - \ell(\tau - \eta, \tau)|$$

$$\sim \left| \langle \tau - \eta, \xi \rangle (1 + |\xi|^2)^{\frac{m-2}{2}} \hat{a}(\tau - \eta) - \langle \tau - \eta, \tau \rangle (1 + |\tau|^2)^{\frac{m-2}{2}} \hat{a}(\tau - \eta) \right|$$

$$\lesssim |\tau - \eta| |\xi - \tau| \cdot |(1 + |\xi|^2)^{\frac{m-2}{2}} + (1 + |\tau|^2)^{\frac{m-2}{2}}| \cdot |\hat{a}(\tau - \eta)|.$$ 

Now use the triangle inequality to split the above expression, and therefore also $|F_4(\xi, \eta)|$. Then the first piece is

$$\lesssim \frac{(1 + |\xi|^2)^{\frac{s}{2}}}{(1 + |\eta|^2)^{\frac{s+2}{2}}} \int |\hat{a}(\xi - \tau)||\tau - \eta||\xi - \tau|(1 + |\xi|^2)^{\frac{m-2}{2}} |\hat{a}(\tau - \eta)| d\tau$$

$$= \left( \frac{1 + |\xi|^2}{1 + |\eta|^2} \right)^{\frac{s+m-2}{2}} \int |\hat{a}(\xi - \tau)||\tau - \eta||\xi - \tau||\hat{a}(\tau - \eta)| d\tau$$

$$\lesssim (1 + |\xi - \eta|^2)^{\frac{s+m-2}{2}} \int |\hat{a}(\xi - \tau)||\tau - \eta||\xi - \tau||\hat{a}(\tau - \eta)| d\tau$$

and replacing $\tau$ by $\tau + \eta$ this becomes

$$\lesssim (1 + |\xi - \eta|^2)^{\frac{s+m-2}{2}} \int |\hat{a}(\xi - (\tau + \eta))||\hat{a}(\tau)||\xi - \tau - \eta||\tau| d\tau.$$ 

Since $\hat{a}$ is a Schwartz function, the above can be bounded as

$$\lesssim (1 + |\xi - \eta|^2)^{\frac{s+m-2}{2}} \int (1 + |\xi - \eta|^2)^{-p} (1 + |\tau|^2)^{-p} d\tau$$

where $p$ is any positive integer. By homogeneity we then bound this as

$$\lesssim (1 + |\xi - \eta|^2)^{\frac{s+m-2}{2}} \cdot (1 + |\xi - \eta|^2)^{-p+\frac{N}{2}}.$$ 

25
Taking $p$ large enough and applying Young’s inequality, we’re done with this piece. The second piece is similar:

\[
\frac{(1 + |\xi|^2)\frac{1}{2}}{(1 + |\eta|^2)^{1 + m - 2}} \int |\tau - \eta||\xi - \tau||\hat{a}(\tau - \eta)||\hat{a}(\xi - \tau)|(1 + |\tau|^2)^{\frac{m - 2}{2}} \, d\tau \\
\lesssim (1 + |\xi - \tau|^2)^{\frac{1 + m - 2}{2}} \int |\tau - \eta||\xi - \tau||\hat{a}(\tau - \eta)||\hat{a}(\xi - \tau)|(1 + |\xi - \tau|^2)^{\frac{m - 2}{2}} \, d\tau \\
= (1 + |\xi - \tau|^2)^{\frac{1 + m - 2}{2}} \int |\tau||\xi - \eta - \tau||\hat{a}(\tau)||\hat{a}(\xi - \eta - \tau)|(1 + |\xi - \eta - \tau|^2)^{\frac{m - 2}{2}} \, d\tau
\]

and we use Young’s inequality the same way as before, which finishes the proof of the lemma.

**Commutators and Derivatives**

The hard commutator lemmas are done, but we need a lemma about commuting differential operators with $\Lambda^m$.

**Lemma 2.5.10.** A derivative $D^\beta$ commutes with $\Lambda^m$, and if $a_\beta$ are Schwarz functions and $L = \sum_{|\beta| \leq k} a_\beta D^\beta$ is a differential operator of order $k$, then $[\Lambda^m, L]$ is of order $m + k - 1$, $[\Lambda^m, [\Lambda^m, L]]$ is of order $2m + k - 2$, and $[[\Lambda^m, L], L]$ is of order $m + 2k - 2$.

**Pf.** That $D^\beta$ commutes with $\Lambda^m$ follows directly from the definitions. We will prove the first of the commutator orders for the general term $a_\beta D^\beta$ of $L$.

\[
[a_\beta D^\beta, \Lambda^m] = a_\beta D^\beta \Lambda^m f - \Lambda^m a_\beta D^\beta f \\
= a_\beta D^\beta \Lambda^m f - a_\beta \Lambda^m D^\beta f - [\Lambda^m, a] D^\beta f.
\]

The first two terms cancel, and the remaining term is of order $m + |\beta| + 1$. ■
2.6 Tangential Norms

Most of the functions we will be using to prove Theorem 1 will be in a class denoted \( C_0^\infty(U \cap \overline{\Omega}) \). These are the functions which have compact support in a neighborhood \( U \) around a boundary point of \( \Omega \), and are smooth on \( U \cap \overline{\Omega} \). (So, their support can intersect \( b\Omega \).)

If \( r \) is a defining function for \( \Omega \), then \( \nabla r \neq 0 \) on \( b\Omega \), so we can locally flatten \( b\Omega \) to identify \( U \cap \Omega \) with \( U' \cap \mathbb{R}^N \), using coordinates \((t_1, \ldots, t_{N-1}, r)\). The tangential Fourier transform \( \mathcal{F}^t \) is then defined by

\[
\mathcal{F}^t f(\tau, r) = \tilde{f}(\tau, r) = \int_{\mathbb{R}^{N-1}} f(t, r) e^{-2\pi i \langle t, \tau \rangle} dt.
\]

The operator \( \Lambda^m_t \) is given by

\[
\mathcal{F}^t (\Lambda^m_t f)(\tau, r) = (1 + |\tau|^2)^m \tilde{f}(\tau, r),
\]

and the tangential Sobolev norm ||| \cdot |||_m is

\[
|||f|||_m^2 = |||\Lambda^m_t f|||^2_{L^2} = \int_{\mathbb{R}^{N-1}} (1 + |\tau|^2)^m |\tilde{f}(\tau, r)|^2 d\tau dr.
\]

An operator \( T \) is of tangential order \( m \) if

\[
|||Tf|||_s \lesssim |||f|||_{m+s} \quad \forall s \in \mathbb{R}.
\]

The operators \( \Lambda^m_t \) is of tangential order \( m \), and so is any \( D^\beta \), when \(|\beta| \leq m \) and the derivatives are in the first \( N - 1 \) directions only.
The same commutator orders from section 2.5 apply to tangential orders of commutators with $\Lambda^m_t$. This is because we can work through the same proofs with the functions $|\tilde{a}|$ replaced by $|\tilde{a}(\tau, r)|$, and prove the result for each fixed $r$. The second integral which arises with respect to $r$ doesn’t change anything since $\tilde{a}$ is a Schwarz function and decreases rapidly as $r \to -\infty$.

2.7 Weighted Spaces

The weighted $m$ norms are defined

$$||f||_{m, \gamma} = ||\Lambda^m f||_{\gamma}$$

$$|||f|||_{m, \gamma} = ||\Lambda^m_t f||_{\gamma}.$$  

An operator $T$ is of order $m$ relative to $\gamma$ if

$$||Tf||_{s, \gamma} \leq c_{s, \gamma} ||f||_{s+m, \gamma}$$

or tangential order $m$ if

$$|||Tf|||_{s, \gamma} \leq c_{s, \gamma} |||f|||_{s+m, \gamma}, \quad \forall s \in \mathbb{R}.$$  

The weighted norm $|||\cdot|||_{m, \gamma}^2$ is still equivalent to the sum of the norms of the derivatives up to order $m$:

$$||f||_{m, \gamma}^2 \sim \sum_{|\alpha| \leq m} ||D^\alpha f||_{L^2(\gamma)}^2.$$  

This follows from using a commutator to interchange the order of the weight function with either $D^\alpha$ or $\Lambda^m$, using the unweighted norm equivalence, and commuting the weight function back to the other side of the operator.
Next, the orders of operators involving commutators are the same as in the unweighted case. Provided that $\gamma$ is bounded, we use the unweighted result and show that inserting a weight produces only an extra lower order term. Since the idea is always the same, we will prove only one of the lemmas.

**Lemma 2.7.1.** If $a \in \mathcal{S}(\mathbb{R}^N)$, the operator $[\Lambda^m, a]$ is of order $m - 1$ with respect to weighted norms:

$$||[\Lambda^m, a]u||_{s,\gamma}^2 \lesssim ||u||_{s+m-1,\gamma}^2,$$

for any $s \in \mathbb{R}$.

**Pf.**

$$||[\Lambda^m, a]u||_{s,\gamma}^2 = ||e^{-\frac{\gamma}{2}}\Lambda^s[\Lambda^m, a]u||_{L^2}^2$$

$$= ||e^{-\frac{\gamma}{2}}\Lambda^s[\Lambda^m, a]\Lambda^{-s-m+1}\Lambda^{s+m-1}u||_{L^2}^2$$

and commute $e^{-\frac{\gamma}{2}}$ with $\Lambda^s[\Lambda^m, a]\Lambda^{-s-m+1}$,

$$\lesssim ||\Lambda^s[\Lambda^m, a]\Lambda^{-s-m+1}e^{-\frac{\gamma}{2}}\Lambda^{s+m-1}u||_{L^2}^2$$

$$+ ||[e^{-\frac{\gamma}{2}}, \Lambda^s[\Lambda^m, a]\Lambda^{-s-m+1}]\Lambda^{s+m-1}u||_{L^2}^2.$$
For the first term, the norm is unweighted so we know that $\Lambda^s[\Lambda^m, a] \Lambda^{-s-m+1}$ is of order 0. Therefore the first term is

$$\lesssim \| e^{-\gamma \frac{s}{2}} \Lambda^{s+m-1} u \|^2_{L^2}$$

$$= \| \Lambda^{s+m-1} u \|^2_{L^2, \gamma}$$

$$= \| u \|^2_{s+m-1, \gamma}.$$ 

For the second term, we insert a clever choice of 1 to put a weight function in the right place, and the second term

$$= \| [e^{-\gamma \frac{s}{2}}, \Lambda^s[\Lambda^m, a] \Lambda^{-s-m+1}] e^{\gamma \frac{s}{2}} \Lambda^{s+m-1} u \|^2_{L^2}.$$ 

Since $e^{\gamma \frac{s}{2}}$ is bounded, $[e^{-\gamma \frac{s}{2}}, \Lambda^s[\Lambda^m, a] \Lambda^{-s-m+1}] e^{\gamma \frac{s}{2}}$ is of order $-1$, so the above

$$\lesssim \| e^{-\gamma \frac{s}{2}} \Lambda^{s+m-1} u \|^2_{-1}$$

$$\leq \| u \|^2_{s+m-1, \gamma}$$

which finishes the proof. 

The last thing to notice about the weighted spaces is that $\Lambda^m$ is not self-adjoint with respect to the weighted inner product.

**Lemma 2.7.2.** Moving $\Lambda^m$ to the other side of the weighted inner product produces a commutator:

$$\langle \Lambda^m u, v \rangle_\gamma = \langle u, \Lambda^m v \rangle_\gamma + \langle u, e^{\gamma \Lambda^m} v \rangle_\gamma.$$ 

30
Pf. Bring $e^{-\gamma}$ into the inner product, and commute it with $\Lambda^m$:

$$(\Lambda^m u, v)_\gamma = (\Lambda^m u, e^{-\gamma}v)$$

$$= (u, e^{-\gamma} \Lambda^m v) + (u, [\Lambda^m, e^{-\gamma}]v)$$

$$= (u, \Lambda^m v)_\gamma + (u, e^\gamma[\Lambda^m, e^{-\gamma}]v)_\gamma.$$

\[ \blacksquare \]

### 2.8 The Operators $A$ and $A^*$

For Schwarz functions $\zeta$ and $\zeta_1$, let $A = \zeta \Lambda_t^m \zeta_1$. Since we use this operator in later sections, we establish a few of its properties here.

First, the formal adjoint of $A$ with respect to the unweighted inner product is

$$A^* = \zeta_1 \Lambda_t^m \zeta.$$

This follows from the self-adjointness of $\Lambda^m$. However, moving $A$ to the other side of the weighted inner product produces a commutator term:

$$(Au, v)_\gamma = (u, A^* v)_\gamma + (u, e^\gamma[A^*, e^{-\gamma}]v)_\gamma$$

Next, in both weighted and unweighted norms, the operators involving commutators which were discussed above have the same orders when we replace $\Lambda_t^m$ with $A$ or $A^*$.

For instance, the operator $[A, a]$ is of tangential order $m - 1$:

$$[A, a]u = \zeta \Lambda_t^m \zeta_1 au - a \zeta \Lambda_t^m \zeta_1 u$$

$$= \zeta \Lambda_t^m a \zeta_1 u - a \zeta \Lambda_t^m \zeta_1 u$$

31
and commute $\Lambda_t^m$ with $a$:

$$a = \zeta a \Lambda_t^m \zeta_1 u + \zeta [\Lambda_t^m, a] \zeta_1 u - a \Lambda_t^m \zeta_1 u$$

$$= \zeta [\Lambda_t^m, a] \zeta_1 u$$

and the commutator $[\Lambda_t^m, a]$ is of tangential order $m - 1$.

Lastly, we point out that the operator $A - A^*$ is of order $m - 1$, not $m$:

$$A - A^* = \zeta \Lambda_t^m \zeta_1 - \zeta_1 \Lambda_t^m \zeta$$

$$= \zeta [\Lambda_t^m, \zeta_1] - \zeta_1 [\Lambda_t^m, \zeta]$$

and the commutators $[\Lambda_t^m, \zeta]$ and $[\Lambda_t^m, \zeta_1]$ have order $m - 1$. 
CHAPTER 3
SCALING

Now we start the proof of Theorem 1.

3.1 Definitions of $\Phi_{z'}$ and $S_{\delta}$

Let $\Omega \subset \subset \mathbb{C}^n$ be smoothly bounded domain, and let $\gamma$ is smooth on $\overline{\Omega}$. Assume $\Omega$ strongly pseudoconvex near a point $p \in b\Omega$. This means there exists a neighborhood $U$ of $p$ and a strictly plurisubharmonic defining function $r(z)$ on $U$, meaning the Levi form of $r$ satisfies

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(z) t_j \overline{t}_k \geq c|t|^2, \quad \forall z \in U, t \in \mathbb{C}^n.$$  

We can also assume

$$(\dagger\dagger) \quad \begin{cases} 
\frac{\partial r}{\partial z_1}(p) = 0 \\
\vdots \\
\frac{\partial r}{\partial z_{n-1}}(p) = 0 \\
\frac{\partial r}{\partial z_n}(p) = 1
\end{cases} \quad \text{and } |\frac{\partial r}{\partial z_n}| >> 0 \text{ on } U.$$
The Taylor expansion of \( r \) around \( z' \in U \cap b\Omega \) is

\[
r(z) = r(z') + 2\text{Re}\left\{ \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(z_j - z'_j) + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(z_j - z'_j)(z_k - z'_k) \right\}
\]

\[
+ \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(z_j - z'_j)(\bar{z}_k - \bar{z}'_k) + O(|z - z'|^3).
\]

Now we construct the two coordinate changes advertised in chapter 1. For each \( z' \), define the biholomorphic map \( \Phi_{z'} : \zeta \rightarrow z \) by translating the first \( n - 1 \) coordinates, and letting \( \zeta_n \) be the quantity above in braces \( \{\cdots\} \):

\[
\begin{align*}
\zeta_1 &= z_1 - z'_1 \\
\vdots \\
\zeta_{n-1} &= z_{n-1} - z'_{n-1} \\
\zeta_n &= \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(z_j - z'_j) + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial z_k}(z_j - z'_j)(z_k - z'_k).
\end{align*}
\]

Then we get a family of domains \( \Omega_{z'} \) defined by

\[
R_{z'}(\zeta) = r(\Phi_{z'}^{-1}(\zeta)) = 2\text{Re} \zeta_n + \sum_{j,k=1}^{n-1} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(z'_j)\zeta_j \bar{\zeta}_k
\]

\[
+ 2\text{Re} \sum_{j=1}^{n-1} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_n}(z'_j)\zeta_j \bar{\zeta}_n \cdot \left( \frac{\partial r}{\partial \bar{z}_n}(z'_n) \right)^{-1}
\]

\[
+ \frac{\partial^2 r}{\partial z_n \partial \bar{z}_n}(z'_n)\zeta_n \bar{\zeta}_n \cdot \left( \frac{\partial r}{\partial \bar{z}_n}(z'_n) \right)^{-2}
\]

\[
+ O(|\zeta_n||\zeta| + |\zeta|^2 + |\zeta|^3).
\]
Under this map the point $z'$ is moved to 0, and the normal at 0 is $\nabla_C R_{z'}(0) = (0, \ldots, 0, 1)$. The Jacobian of the map is uniformly nonsingular if $z'$ is restricted to a small neighborhood of $p$. (By the transformation formula $(\ast)$ in chapter 1, this means the singularity of the kernel on domains $\Omega_{z'}$ will be the same as that for $\Omega$. We will prove the transformation formula in chapter 7 when we compute the singularity of the kernel through the scaling map below.)

Also, notice that the new defining functions $R_{z'}$ are still plurisubharmonic in a small neighborhood of 0,

$$\sum_{j,k=1}^{n} \frac{\partial^2 R_{z'}}{\partial \zeta_j \partial \zeta_k}(\zeta) t_j t_k \geq \tilde{c}|t|^2, \quad \forall \zeta \text{ near } 0 \in \Omega_{z'}, \ t \in \mathbb{C}^n,$$

where the neighborhood of 0 and $\tilde{c}$ can be chosen independently of $z'$. This is because the coefficients on the $|\zeta||\zeta_n|$ and $|\zeta|^2$ error terms contain factors of $|\partial r / \partial z_j(z')|$, $j < n$, which by $(\dagger\dagger)$ can be taken sufficiently small (by a small $U$).

The second coordinate change is the scaling map. For small $\delta > 0$ we define the map $S_\delta : \zeta \to w$ by

$$w_1 = \frac{1}{\sqrt{\delta}} \zeta_1$$
$$\vdots$$
$$w_{n-1} = \frac{1}{\sqrt{\delta}} \zeta_{n-1}$$
$$w_n = \frac{\zeta_n}{\delta}$$

35
and the local defining functions for the domains $\Omega_{z',\delta}$ are

$$R_{z',\delta}(w) = \frac{1}{\delta}R_{z'}(S_{\delta}^{-1}(w))$$

$$= 2\text{Re} \ w_n + \sum_{j,k=1}^{n-1} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(z') w_j \bar{w}_k$$

$$+ \delta^{\frac{1}{2}} \cdot 2\text{Re} \ \sum_{j=1}^{n-1} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_n}(z') w_j \bar{w}_n \cdot \left( \frac{\partial r}{\partial \bar{z}_n}(z') \right)^{-1}$$

$$+ \delta \cdot \frac{\partial^2 r}{\partial \bar{z}_n \partial z_n}(z') w_n \bar{w}_n \cdot \left( \frac{\partial r}{\partial \bar{z}_n}(z') \right)^{-2}$$

$$+ O(\delta^{\frac{3}{2}}(|w_n||w| + |w|^2 + |w|^3)).$$

The complex normal at 0 is still $(0, \ldots, 0, 1)$. The defining functions $R_{z',\delta}$ are still strictly plurisubharmonic, but the eigenvalues depend on $\delta$. To fix that, replace $R_{z',\delta}$ with a different defining function

$$\rho_{z',\delta}(w) = e^{R_{z',\delta}(w)} - 1.$$ 

The functions $\rho_{z',\delta}$ define the same domains $\Omega_{z',\delta}$, but the Levi forms have an extra positive term:

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho_{z',\delta}}{\partial w_j \partial \bar{w}_k}(w)t_j \bar{t}_k = e^{R_{z',\delta}(w)} \left[ \sum_{j,k=1}^{n} \frac{\partial^2 R_{z',\delta}}{\partial w_j \partial \bar{w}_k}(w)t_j \bar{t}_k + \sum_{j=1}^{n} \left| \frac{\partial R_{z',\delta}}{\partial w_j}(w) t_j \right|^2 \right].$$

Writing $t' = (t_1, t_2, \ldots, t_{n-1})$ and $t = (t', t_n)$, the Levi form at $w = 0$ can be bounded as

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho_{z',\delta}}{\partial w_j \partial \bar{w}_k}(0)t_j \bar{t}_k \gtrsim c|t'|^2 - 2\delta^{\frac{3}{2}} \sum_{j=1}^{n-1} |r_{z_j \bar{z}_n}(z')| |t|^2 + (1 + \delta r_{z_n \bar{z}_n}(z')) |t_n|^2.$$
which is $\gtrsim |t|^2$ for small $\delta$. By continuity the estimate also holds for points $w$ in a small neighborhood $V$ of 0, so

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho_{z',\delta}}{\partial w_j \partial w_k}(w) t_j t_k \gtrsim C|t|^2, \quad \forall w \in V \cap \Omega_{z',\delta}, \ t \in \mathbb{C}^n.$$ 

### 3.2 Bounds on the Weight Function

The weight functions on $\Omega_{z'}$ and $\Omega_{z',\delta}$ are obtained by pulling $\zeta$ or $w$ back to the $z$-coordinates:

$$\gamma_{z'}(\zeta) = \gamma(\Phi_{z'}(\zeta))$$

$$\gamma_{z',\delta}(w) = \gamma(\Phi_{z'}(S_{-1}^{-1}(w))).$$

The minima and maxima of $e^{-\gamma_{z'}}$ and $e^{-\gamma_{z',\delta}}$ are the same as those for $e^{-\gamma}$. For the derivatives, for $\zeta$ in a small neighborhood of 0, the chain rule shows

$$\left| \frac{\partial \gamma_{z'}}{\partial \zeta_j} \right| \gtrsim \left| \frac{\partial \gamma}{\partial z_j} \right|, \quad j = 1, \ldots, n;$$

$$\left| \frac{\partial^2 \gamma_{z'}}{\partial \zeta_j \partial \zeta_k} \right| \gtrsim \left| \frac{\partial^2 \gamma}{\partial z_j \partial z_k} \right|, \quad j, k = 1, \ldots, n.$$

The constants swept under the rug in “$\gtrsim$” arise because of the relatively complicated definition for $\zeta_n$.

After scaling, various positive powers of $\delta$ land on the first derivatives,

$$\left| \frac{\partial \gamma_{z',\delta}}{\partial w_j} \right| \lesssim \sqrt{\delta} \left| \frac{\partial \gamma}{\partial z_j} \right|, \quad j = 1, \ldots, n - 1;$$

$$\left| \frac{\partial \gamma_{z',\delta}}{\partial w_n} \right| \lesssim \delta \left| \frac{\partial \gamma}{\partial z_n} \right|.$$
and on the second derivatives,

\[
\left| \frac{\partial^2 \gamma}{\partial w_j \partial w_k} \right| \leq \delta \left| \frac{\partial^2 \gamma}{\partial z_j \partial z_k} \right|, \quad j, k = 1, \ldots, n - 1;
\]

\[
\left| \frac{\partial^2 \gamma}{\partial w_j \partial w_n} \right| \leq \delta^{\frac{3}{2}} \left| \frac{\partial^2 \gamma}{\partial z_j \partial z_n} \right|, \quad j = 1, \ldots, n - 1;
\]

\[
\left| \frac{\partial^2 \gamma}{\partial w_n \partial w_n} \right| \leq \delta^2 \left| \frac{\partial^2 \gamma}{\partial z_n \partial z_n} \right|.
\]
CHAPTER 4

THE BASIC ESTIMATE

We begin by defining the operator $\square_{\gamma}$ and the associated Dirichlet form $Q_{\gamma}$.

**Definition 4.1.** For a smooth function $\gamma$, the “box” operator $\square_{\gamma}$ on $(0, 1)$ forms is

$$\square_{\gamma} = \bar{\partial}_{\gamma} \bar{\partial} + \partial \bar{\partial}_{\gamma}^*.$$  

A $(0, 1)$ form $u$ is in $\text{Dom} \, \square_{\gamma}$ if $u \in \text{Dom} \, \bar{\partial}_{\gamma} \cap \text{Dom} \, \bar{\partial}$, $\bar{\partial} u \in \text{Dom} \, \bar{\partial}_{\gamma}^*$, and $\bar{\partial}_{\gamma}^* u \in \text{Dom} \, \bar{\partial}$ at the $(0, 1)$, $(0, 2)$, and $(0, 0)$ form levels, respectively.

When $\gamma \equiv 0$, the operator $\square$ is just a multiple of the Laplacian acting componentwise on the form. With a weight function there are some extra lower order terms, and we will exploit the relationship between $\square_{\gamma}$ and $\Delta$ in chapter 6.

**Definition 4.2.** For two $(0, 1)$ forms $u$ and $v$, the Dirichlet form $Q_{\gamma}(u, v)$ is defined

$$Q_{\gamma}(u, v) = (\bar{\partial} u, \bar{\partial} v)_{\gamma} + (\bar{\partial}_{\gamma}^* u, \bar{\partial}_{\gamma}^* v)_{\gamma}.$$  

The goal of this chapter is to prove the basic estimate in Lemma 4.4, relating $Q_{\gamma}(u, u)$ to the barred derivatives of $u$ and some other controlled terms. In the next lemma and beyond, we say $u \in C_0^\infty(V \cap \Omega_{\gamma}^*, \delta)$ if $u$ is smooth on $V \cap \Omega_{\gamma}^*, \delta$ and has compact
support in $V$. In other words, the support of $u$ can intersect $b\Omega_{z',\delta}$. Similarly, $u \in \Lambda^0_0(V \cap \overline{\Omega}_{z',\delta})$ if its components are in $C^\infty_0(V \cap \overline{\Omega}_{z',\delta})$.

**Lemma 4.3 (The Basic Identity).** Let $u \in \Lambda^0_0(V \cap \overline{\Omega}_{z',\delta}) \cap \text{Dom } \overline{\partial}^*_{\gamma_{z',\delta}}$. Then

$$Q_{\gamma_{z',\delta}}(u, u) = \sum_{j,k=1}^n \left| \frac{\partial u_j}{\partial w_k} \right|^2_{L^2(V \cap \Omega_{z',\delta}, \gamma_{z',\delta})} + \int_{\Omega_{z',\delta} \cap V} \sum_{j,k=1}^n \frac{\partial^2 \rho_{z',\delta}}{\partial w_j \partial w_k} u_j \overline{u}_k e^{-\gamma_{z',\delta}} d\sigma$$

$$+ \int_{V \cap \Omega_{z',\delta}} \sum_{j,k=1}^n \frac{\partial^2 \gamma_{z',\delta}}{\partial w_j \partial w_k} u_j \overline{u}_k e^{-\gamma_{z',\delta}} dV.$$

**Pf.** Since $u$ has compact support in $V$, we can prove the identity on $\Omega_{z',\delta}$ and restrict to $V \cap \Omega_{z',\delta}$ at the end. To simplify notation, we drop the subscripts on $\gamma_{z',\delta}$, $\Omega_{z',\delta}$, and $\rho_{z',\delta}$. Let

$$u = \sum_{j=1}^n u_j d\overline{w}_j$$

which means

$$\overline{\partial} u = \sum_{j,k=1}^n \frac{\partial u_j}{\partial w_k} d\overline{w}_k \wedge d\overline{w}_j$$

and

$$\overline{\partial}_{\gamma} u = -e^{\gamma} \sum_{j=1}^n \frac{\partial}{\partial w_j} (e^{-\gamma} u_j)$$

$$= \sum_{j=1}^n \left( \frac{\partial \gamma}{\partial w_j} u_j - \frac{\partial u_j}{\partial w_j} \right)$$

and let

$$(\overline{\partial}_{\gamma} u)_j \overset{def}{=} \frac{\partial \gamma}{\partial w_j} u_j - \frac{\partial u_j}{\partial w_j}.$$
Start with the $\bar{\partial}$ term:

$$||\bar{\partial}u||_\gamma^2 = (\bar{\partial}u, \bar{\partial}u)_\gamma = \sum_{j,k=1}^n \left| \frac{\partial u}{\partial w_k} \right|_\gamma^2 - \sum_{j,k=1}^n \left( \frac{\partial u_j}{\partial w_k}, \frac{\partial u_k}{\partial w_j} \right)_\gamma.$$

The first term is what we want, so work with the second by integration by parts:

$$- \sum_{j,k=1}^n \int_{V \cap \Omega} \frac{\partial u_j}{\partial w_k} \frac{\partial u_k}{\partial w_j} e^{-\gamma} dV$$

$$= \sum_{j,k=1}^n \int_{\Omega} \frac{\partial}{\partial w_j} \left( \frac{\partial u_j}{\partial w_k} e^{-\gamma} \right) \frac{u_k}{e^\gamma} dV - \sum_{j,k=1}^n \int_{\partial \Omega} \frac{\partial u_j}{\partial w_k} \frac{\partial}{\partial w_j} \frac{u_k}{e^\gamma} d\sigma$$

$$= \sum_{j,k=1}^n \int_{\Omega} \frac{\partial^2 u_j}{\partial w_j \partial w_k} \frac{u_k}{e^\gamma} dV - \sum_{j,k=1}^n \int_{\partial \Omega} \frac{\partial u_j}{\partial w_k} \frac{\partial u_k}{\partial w_j} e^{-\gamma} d\sigma$$

$$- \sum_{j,k=1}^n \int_{\partial \Omega} \frac{\partial u_j}{\partial w_j} \frac{\partial}{\partial w_j} \frac{u_k}{e^\gamma} d\sigma.$$

To change this last integral, we use the fact that $u \in \text{Dom } \bar{\partial}^*_{\gamma}$. We have

$$\sum_{j=1}^n \frac{\partial \rho}{\partial w_j} u_j = 0 \quad \text{on } b\Omega,$$

which means

$$\sum_{j=1}^n u_k \frac{\partial}{\partial w_k}$$

is a tangential derivative. Applying a tangential derivative to a function which vanishes on the boundary gives us zero, so

$$\sum_{k=1}^n u_k \frac{\partial}{\partial w_k} \left( \sum_{j=1}^n \frac{\partial u_j}{\partial w_j} \right) = 0 \quad \text{on } b\Omega,$$

41
which means
\[ \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial w_j \partial w_k} u_j u_k = - \sum_{j,k=1}^{n} \frac{\partial \rho}{\partial w_j} \frac{\partial u_j}{\partial w_k} \bar{u}_k \quad \text{on } b\Omega. \]

Substituting into the integral and collecting what we have so far,
\[
||\bar{\partial}u||^2_\gamma = \sum_{j,k=1}^{n} \left| \frac{\partial u_j}{\partial w_k} \right|^2_\gamma + \sum_{j,k=1}^{n} \int_{\Omega} \frac{\partial^2 u_j}{\partial w_j \partial w_k} \bar{u}_k e^{-\gamma} dV
- \sum_{j,k=1}^{n} \int_{\Omega} \frac{\partial u_j}{\partial w_k} \frac{\partial \gamma}{\partial w_j} \bar{u}_k e^{-\gamma} dV + \sum_{j,k=1}^{n} \int_{b\Omega} \frac{\partial^2 \rho}{\partial w_j \partial w_k} u_j \bar{u}_k e^{-\gamma} d\sigma.
\]

Now looking at the \( \bar{\partial}_\gamma^* \) piece,
\[
||\bar{\partial}_\gamma^* u||^2_\gamma = \sum_{j,k=1}^{n} ((\bar{\partial}_\gamma^* u)_j, (\bar{\partial}_\gamma^* u)_k)_\gamma
- \sum_{j,k=1}^{n} \int_{\Omega} (\bar{\partial}_\gamma^* u)_j e^\gamma \frac{\partial}{\partial w_k} (e^{-\gamma} \bar{u}_k) e^{-\gamma} dV
= \sum_{j,k=1}^{n} \int_{\Omega} \frac{\partial}{\partial w_k} ((\bar{\partial}_\gamma^* u)_j) \bar{u}_k e^\gamma dV - \sum_{j,k=1}^{n} \int_{b\Omega} (\bar{\partial}_\gamma^* u)_j \bar{u}_k \frac{\partial \rho}{\partial w_k} e^{-\gamma} d\sigma.
\]
The boundary integral vanishes since \( \sum u_k(\partial / \partial w_k) \) is tangential. Continuing, we have the above

\[
= - \sum_{j,k=1}^{n} \int_{\Omega} \frac{\partial}{\partial w_k} \left( e^\gamma \frac{\partial}{\partial w_j} (e^{-\gamma} u_j) \right) \overline{u}_k e^{-\gamma} dV \\
= - \sum_{j,k=1}^{n} \int_{\Omega} \frac{\partial}{\partial w_k} \left( \frac{\partial u_j}{\partial w_j} - \frac{\partial \gamma}{\partial w_j} u_j \right) \overline{u}_k e^{-\gamma} dV \\
= - \sum_{j,k=1}^{n} \int_{\Omega} \frac{\partial^2 u_j}{\partial w_j \partial w_k} \overline{u}_k e^{-\gamma} dV + \sum_{j,k=1}^{n} \int_{\Omega} \frac{\partial^2 \gamma}{\partial w_j \partial w_k} u_j \overline{u}_k e^{-\gamma} dV \\
+ \sum_{j,k=1}^{n} \int_{\Omega} \frac{\partial u_j}{\partial w_j} \frac{\partial \gamma}{\partial w_k} \overline{u}_k e^{-\gamma} dV.
\]

Combining everything and restricting again to \( V \cap \Omega_{z',\delta} \), we get

\[
Q_{\gamma_z',\delta}(u, u) = \sum_{j,k=1}^{n} \left\| \frac{\partial u_j}{\partial w_k} \right\|_{L^2(\gamma_z',\delta)}^2 + \int_{\Omega_{z',\delta} \cap V} \sum_{j,k=1}^{n} \frac{\partial^2 \rho_{z',\delta}}{\partial w_j \partial w_k} u_j \overline{u}_k e^{-\gamma_z',\delta} d\sigma \\
+ \int_{\Omega_{z',\delta} \cap V} \sum_{j,k=1}^{n} \frac{\partial^2 \gamma_{z',\delta}}{\partial w_j \partial w_k} u_j \overline{u}_k e^{-\gamma_z',\delta} dV.
\]

From the bound on the Levi form of \( \rho_{z',\delta} \) in chapter 3, and the bounds on the derivatives of \( \gamma_{z',\delta} \), we obtain an inequality for \( Q_{\gamma_z',\delta}(u, u) \):

\[
Q_{\gamma_z',\delta}(u, u) \geq \sum_{j,k=1}^{n} \left\| \frac{\partial u_j}{\partial w_k} \right\|_{L^2(\gamma_z',\delta)}^2 + C \int_{\Omega_{z',\delta} \cap V} |u|^2 e^{-\gamma_z',\delta} d\sigma \\
- \delta \left\{ \max_{j,k} \left| \frac{\partial^2 \gamma}{\partial z_j \partial z_k} \right| \text{ on } U \cap \Omega \right\} \cdot \int_{V \cap \Omega_{z',\delta}} |u|^2 e^{-\gamma_z',\delta} dV.
\]

\[43\]
Abbreviating the $\gamma$ derivatives with

$$C_1 = \max_{j,k} \left| \frac{\partial^2 \gamma}{\partial z_j \partial z_k} \right| \text{ on } U \cap \Omega$$

we obtain the basic estimate for small $U$ and small $\delta$:

**Lemma 4.4 (The Basic Estimate).** For $u \in \Lambda_0^{0,1}(V \cap \Omega_{z',\delta}) \cap \text{Dom } \bar{\partial}^{z'}_{\gamma',\delta'}$,

$$Q_{\gamma',\delta'}(u, u) + \delta C_1 \|u\|_{L^2(V \cap \Omega_{z',\delta} \cap \gamma_{z',\delta})}^2 \gtrsim \left\{ \min e^{-\gamma} \text{ on } U \cap \Omega \right\} \cdot \left( \sum_{j,k=1}^n \left| \frac{\partial u_k}{\partial w_j} \right|_{L^2}^2 + C \int_{\partial \Omega_{z',\delta} \cap V} |u|^2 \, d\sigma \right).$$
A subelliptic estimate of order $\epsilon > 0$ holds on $V \cap \Omega$ if

$$||u||^2 \lesssim ||\bar{\partial} u||^2 + ||\bar{\partial}' u||^2, \quad \forall u \in \mathcal{D}^{0,1}(V \cap \Omega).$$

We want to get a weighted subelliptic estimate of order $1/2$ on $V \cap \Omega_{z',\delta}$, and such that the constant is independent of $\delta$. From the basic estimate, the right hand side (plus a harmless term $\delta||u||^2_{L^2}$) dominates $\sum_{j,k} ||\partial u_k/\partial \overline{w}_j||^2_{L^2}$. But the barred derivatives are only half of all possible first derivatives, so they do not dominate $||u||^{1/2}_{1/2}$.

To get the rest of the $1/2$-norm, notice that

$$0 << |\nabla_C \rho_{z',\delta}(w)| = \left| \sum_{j=1}^n \frac{\partial \rho_{z',\delta}}{\partial \overline{w}_j}(w) \right| \quad \text{on } V \cap b\Omega_{z',\delta},$$

where the inequality comes from the choice of powers in the scaling map, and the equality is true since $\rho_{z',\delta}$ is real-valued. This means the barred derivatives $\partial/\partial \overline{w}_j$ contain a derivative in the normal direction to the boundary. Therefore, to get control of the $1/2$-norm, we should focus our attention on the tangential norm $|||u|||^{2}_{1/2}$.

We will use the notation “$D$” to mean

$$|||Du|||^{2}_{s} \overset{def}{=} \sum_{k=1}^{2n} |||D_k u|||^{2}_{s} + |||u|||^{2}_{s}$$

$$\sim |||u|||^{2}_{s+1} + |||D_\rho u|||^{2}_{s}$$

45
where $D_\rho$ is the derivative is in the normal direction, and $D_k = D_{t_k}$ when $k = 1, \ldots, 2n - 1$ or $D_\rho$ when $k = 2n$.

To get the subelliptic estimate we relate the right side of the basic estimate to $|||Du|||^2_{-\frac{1}{2}}$.

**Theorem 5.1.** For $u \in \Lambda^{0,1}_0(V \cap \Omega_{\tau', \delta})$,

$$|||Du|||^2_{-\frac{1}{2}} \lesssim \sum_{j=1}^{n} \left( \left| \frac{\partial u}{\partial w_j} \right| \right|_{-\frac{1}{2}}^2 + \int_{b\Omega_{\tau', \delta} \cap V} |u|^2 d\sigma + |||u|||^2_{-\frac{1}{2}}.$$

**Pf.** It is enough to prove the theorem on functions, so take $u \in C^\infty_0(V \cap \Omega_{\tau', \delta})$. Also, extend $u$ to be identically zero on $\Omega_{z', \delta}^c$, so that $u$ has compact support on $\mathbb{R}^{2n}$.

Now write

$$u = v + (u - v)$$

where

$$\mathcal{F}_t v(\tau', \rho) = e^{\rho(1+|\tau'|^2)^{\frac{1}{2}}} \mathcal{F}_t u(\tau', 0),$$

so that $u = v$ on $b\Omega_{z', \delta} \cap V$. Then we have

$$|||Du|||^2_{-\frac{1}{2}} = \sum_{k=1}^{2n} |||D_k u|||^2_{-\frac{1}{2}} + |||u|||^2_{-\frac{1}{2}}$$

$$(\star) \quad \lesssim \sum_{k=1}^{2n} |||D_k(u - v)|||^2_{-\frac{1}{2}} + \sum_{k=1}^{2n} |||D_k v|||^2_{-\frac{1}{2}} + |||u|||^2_{-\frac{1}{2}}.$$

To deal with the first term, we use $\rho_{z', \delta}$ as one of the coordinates, and let $\tau = (\tau', \tilde{\rho}) =$
\((\tau'_1, \ldots, \tau'_{2n-1}, \tilde{\rho})\) where \(\tilde{\rho}\) is the Fourier transform variable for \(\rho_{\tau', \delta}\). In the proof we drop the subscripts on \(\rho_{\tau', \delta}\) for convenience. From Plancherel’s identity we have

\[
\sum_{k=1}^{2n} |||D_k(u - v)|||_{L^2}^2 = \sum_{k=1}^{2n} |||\Lambda_{\tau'}^{-\frac{1}{2}} D_k(u - v)|||_{L^2}^2
\]

\[
\sim \int (1 + |\tau'|^2)^{-\frac{1}{2}} \sum_{k=1}^{2n} |\mathcal{F}(D_k(u - v))(\tau)|^2 d\tau
\]

\[
\sim \int (1 + |\tau'|^2)^{-\frac{1}{2}} |\tau|^2 |(\hat{u} - \hat{v})(\tau)|^2 d\tau
\]

\[
= \sum_{j=1}^{n} \int (1 + |\tau'|^2)^{-\frac{1}{2}} |\mathcal{F}(L_j(u - v))(\tau', \tilde{\rho})|^2 d\tau
\]

where \(L_j = \partial/\partial w_j\),

\[
= \sum_{j=1}^{n} |||\Lambda_{\tau'}^{-\frac{1}{2}} L_j(u - v)|||_{L^2}^2
\]

\[
= \sum_{j=1}^{n} |||L_j(u - v)|||_{L^2}^2
\]

\[
\lesssim \sum_{j=1}^{n} |||L_j u|||_{L^2}^2 + \sum_{j=1}^{n} |||L_j v|||_{L^2}^2
\]

The first sum is what we wanted, and the second is \(\lesssim \sum_{k=1}^{2n} |||D_k v|||_{L^2}^2\), a duplicate of the second term in \((\star)\). To estimate it, address the tangential and normal derivatives separately.
For \( k = 1, \ldots, 2n - 1 \) we have
\[
|||D_k v|||^2_{L^2} \sim \int_{\mathbb{R}^{2n-1}} \left( \int_{-\infty}^{0} (1 + |\tau'|^2)^{-\frac{1}{2}} |\tau'_k \mathcal{F}_t v(\tau', \rho)|^2 \, dr \, d\tau' \right)
= \int_{\mathbb{R}^{2n-1}} \left( \int_{-\infty}^{0} (1 + |\tau'|^2)^{\frac{1}{2}} e^{2\rho(1 + |\tau'|^2)^{\frac{1}{2}}} \, d\rho \right) (1 + |\tau'|^2)^{-\frac{1}{2}} |\tau'_k|^2 |\mathcal{F}_t u(\tau', 0)|^2 \, d\tau'.
\]

Computing the inside integral, let
\[
s(\rho) = (1 + |\tau'|^2)^{\frac{1}{2}} \rho
\]
and we obtain
\[
\int_{-\infty}^{0} (1 + |\tau'|^2)^{\frac{1}{2}} e^{2\rho(1 + |\tau'|^2)^{\frac{1}{2}}} \, d\rho = \int_{-\infty}^{0} e^{2s} \, ds = \frac{1}{2}.
\]
Then
\[
|||D_k v|||^2_{L^2} \sim \frac{1}{2} \int_{\mathbb{R}^{2n-1}} (1 + |\tau'|^2)^{-\frac{1}{2}} |\tau'_k|^2 |\mathcal{F}_t u(\tau', 0)|^2 \, d\tau'
\leq \frac{1}{2} \int_{\mathbb{R}^{2n-1}} |\mathcal{F}_t u(\tau', 0)|^2 \, d\tau'
\sim \frac{1}{2} \int_{b\Omega_{\epsilon', \delta} \cap V} |u|^2 \, d\sigma
\]
as desired. Finally, for the case when \( k = 2n \) we have
\[
|||D_\rho v|||^2_{L^2} \sim \int_{\mathbb{R}^{2n-1}} \left( \int_{-\infty}^{0} (1 + |\tau'|^2)^{-\frac{1}{2}} |\mathcal{F}_t (D_\rho v)(\tau', \rho)|^2 \, dr \, d\tau' \right)
= \int_{\mathbb{R}^{2n-1}} \left( \int_{-\infty}^{0} (1 + |\tau'|^2)^{-\frac{1}{2}} \left(1 + |\tau'|^2\right)^{\frac{1}{2}} e^{\rho(1 + |\tau'|^2)^{\frac{1}{2}}} |\mathcal{F}_t u(\tau', 0)|^2 \right) \, dr \, d\tau'
\]
and using the same \( s \) as before,
\[
= \int_{\mathbb{R}^{2n-1}} \left( \int_{-\infty}^{0} e^{2s} \, ds \right) |\mathcal{F}_t u(\tau', 0)|^2 \, d\tau'
\sim \int_{b\Omega_{\epsilon', \delta} \cap V} |u|^2 \, d\sigma
\]
48
and the proof is finished.

Since \( \|\|u\|\|^2 \lesssim (\text{small const.}) \cdot \|\|u\|\|^2 \) on small neighborhoods of 0, we can combine Theorem 5.1 with the basic estimate to obtain

**Theorem 5.2 (The Subelliptic Estimate).** For \( u \in \Lambda^{0,1}_0 (V \cap \Omega_{z',\delta}) \cap \text{Dom} \, \overline{\mathcal{S}}_{z',\delta} \),

\[
\|\|u\|\|^2 \lesssim Q_{z',\delta} (u, u)
\]
CHAPTER 6
A BOUND ON THE NEUMANN OPERATOR

The subelliptic estimate is enough to guarantee the existence of a self-adjoint operator $N_{\gamma_{z',\delta}}$ such that the range of $N_{\gamma_{z',\delta}}$ is contained in Dom $\Box_{\gamma_{z',\delta}}$ and

$$\Box_{\gamma_{z',\delta}} N_{\gamma_{z',\delta}} = N_{\gamma_{z',\delta}} \Box_{\gamma_{z',\delta}} = I$$
on Dom $\Box_{\gamma_{z',\delta}},$

and

$$||N_{\gamma_{z',\delta}} f||_{L^2(V \cap \Omega_{z',\delta})} \lesssim ||f||_{L^2(V \cap \Omega_{z',\delta})},$$

where supp $f \subset V$ and the constant is independent of $\delta$ (see [CS]).

The goal of this chapter is to obtain a bound on the $s + 1$ norm of $N_{\gamma_{z',\delta}} f$, which is Corollary 6.2.4. We start with a couple lemmas about moving functions and the operator $A$ (defined in section 2.8) through $Q_{\gamma_{z',\delta}}$. These help us prove the inequality in Lemma 6.2.2 involving $\Box_{\gamma_{z',\delta}} u$, which we apply to $u = N_{\gamma_{z',\delta}} f$ to obtain Corollary 6.2.4.
6.1 Two Lemmas for $Q_{\gamma_{z',\delta}}$

**Lemma 6.1.1.** For $u \in \mathcal{D}^{0,1}(V \cap \Omega_{z',\delta})$ and real-valued functions $\zeta, \zeta_1 \in C_0^\infty(V \cap \bar{\Omega}_{z',\delta})$ with $\zeta_1 = 1$ on $\text{supp} \ \zeta$, we have

$$Q_{\gamma_{z',\delta}}(\zeta u, \zeta u) = \text{Re} \ Q_{\gamma_{z',\delta}}(u, \zeta^2 u) + \mathcal{O}(\|\zeta_1 u\|_{\gamma_{z',\delta}}^2).$$

**Pf.** As usual, we will drop the subscripts on $\gamma_{z',\delta}$. From the product rule on the $\bar{\partial}$ piece,

$$\langle \bar{\partial} \zeta u, \bar{\partial} \zeta u \rangle_\gamma = \langle \bar{\partial} \zeta \wedge u, \bar{\partial} \zeta \wedge u \rangle_\gamma + \langle \bar{\partial} \zeta \bar{\partial} u, \zeta \bar{\partial} u \rangle_\gamma + 2\text{Re} \ \langle \zeta \bar{\partial} u, \bar{\partial} \zeta \wedge u \rangle_\gamma.$$

In the first term we can change each $u$ to $\zeta_1 u$, and so the first term is $\mathcal{O}(\|\zeta_1 u\|_{\gamma}^2)$. Also, the second term is real-valued, so the above

$$= 2\text{Re} \ \langle \zeta \bar{\partial} u, \bar{\partial} \zeta \wedge u \rangle_\gamma + \text{Re} \ \langle \zeta \bar{\partial} u, \zeta \bar{\partial} u \rangle_\gamma + \mathcal{O}(\|\zeta_1 u\|_{\gamma}^2)$$

$$= \text{Re} \ \langle \bar{\partial} u, 2\zeta \bar{\partial} \zeta \wedge u \rangle_\gamma + \text{Re} \ \langle \bar{\partial} u, \zeta^2 \bar{\partial} u \rangle_\gamma + \mathcal{O}(\|\zeta_1 u\|_{\gamma}^2)$$

and use the Chain Rule,

$$= \text{Re} \ \langle \bar{\partial} u, \bar{\partial}(\zeta^2 u) \rangle_\gamma + \mathcal{O}(\|\zeta_1 u\|_{\gamma}^2)$$

$$= \text{Re} \ \langle \bar{\partial} u, \bar{\partial}(\zeta^2 u) \rangle_\gamma + \mathcal{O}(\|\zeta_1 u\|_{\gamma}^2).$$
For the $\partial^*_\gamma$ piece notice that

$$\partial^*_\gamma(\zeta u) = \sum_{j=1}^{n} \left( \frac{\partial \gamma}{\partial w_j} \zeta u_j - \frac{\partial}{\partial w_j}(\zeta u_j) \right)$$

$$= \zeta \sum_{j=1}^{n} \left( \frac{\partial \gamma}{\partial w_j} u_j - \frac{\partial u_j}{\partial w_j} \right) - \sum_{j=1}^{n} u_j \frac{\partial \zeta}{\partial w_j}$$

$$= \zeta \partial^*_\gamma u - \sum_{j=1}^{n} u_j \frac{\partial \zeta}{\partial w_j},$$

and so

$$(\partial^*_\gamma \zeta u, \partial^*_\gamma \zeta u)_\gamma = (\zeta \partial^*_\gamma u, \zeta \partial^*_\gamma u)_\gamma - 2\text{Re} \left( \zeta \partial^*_\gamma u, \sum_{j=1}^{n} u_j \frac{\partial \zeta}{\partial w_j} \right)_\gamma$$

$$+ \left( \sum_{j=1}^{n} u_j \frac{\partial \zeta}{\partial w_j}, \sum_{k=1}^{n} u_k \frac{\partial \zeta}{\partial w_k} \right)_\gamma.$$
Lemma 6.1.2. For $u \in \mathcal{D}^{0,1}(V \cap \Omega_{z',\delta})$,

$$Q_{\gamma_{z',\delta}}(Au, Au) \lesssim |Q_{\gamma_{z',\delta}}(u, A^*Au)| + \mathcal{O}(|D u|_{m-1, \gamma_{z',\delta}}^2)$$

where $\zeta, \zeta_1 \in C_0^\infty(V \cap \Omega_{z',\delta})$ and $A = \zeta_1 \Lambda_t^m \zeta$.

Pf. The idea of the proof is to interchange as many operators and functions as we need to get the terms we want, and throw all the resulting commutator terms into the $\mathcal{O}(\cdot)$ term. Dropping subscripts on $\gamma_{z',\delta}$, we start with the $\bar{\partial}$ piece,

$$(\bar{\partial} Au, \bar{\partial} Au)_\gamma = (A \bar{\partial} u, \bar{\partial}Au)_\gamma + ([\bar{\partial}, A]u, \bar{\partial}Au)_\gamma$$

and move $A$ in the first inner product at the expense of an extra term:

$$= (\bar{\partial} u, A^* \bar{\partial} Au)_\gamma + (\bar{\partial} u, [A^*, e^{-\gamma}] \bar{\partial} Au) + ([\bar{\partial}, A]u, \bar{\partial}Au)_\gamma$$

and commute $\bar{\partial}$ and $A^*$ in the first term:

$$= (\bar{\partial} u, A^* \bar{\partial} Au)_\gamma + (\bar{\partial} u, [A^*, \bar{\partial}]Au)_\gamma$$

$$+ (\bar{\partial} u, [A^*, e^{-\gamma}] \bar{\partial} Au) + ([\bar{\partial}, A]u, \bar{\partial}Au)_\gamma$$

and commute $[A^*, \bar{\partial}]$ and $A$ in the second term:

$$= (\bar{\partial} u, A^* \bar{\partial} Au)_\gamma + (\bar{\partial} u, [A^*, \bar{\partial}]Au)_\gamma + (\bar{\partial} u, A[A^*, \bar{\partial}]u)_\gamma$$

$$+ (\bar{\partial} u, [A^*, e^{-\gamma}] \bar{\partial} Au) + ([\bar{\partial}, A]u, \bar{\partial}Au)_\gamma$$

$$= (i) + (ii) + (iii) + (iv) + (v).$$
We want to keep \((i)\) and fold everything else into \(O(||Du||^2_{m-1,\gamma})\) or back into \(||\overline{\partial} u||_{\gamma}^2\). To dispose of \((ii)\), the operators \(\overline{\partial}\) and \([[A^*, \overline{\partial}], A]]\) have tangential order 1 and \(2m - 1\) respectively. Change them to operators of order \(m\) by inserting \(\Lambda_t^{m-1}\) and \(\Lambda_t^{-(m-1)}\),

\[
(\overline{\partial} u, [[A^*, \overline{\partial}], A] u)_{\gamma} = (\overline{\partial} u, \Lambda_t^{m-1} \Lambda_t^{-(m-1)} [[A^*, \overline{\partial}], A] u)_{\gamma}
\]

and moving \(\Lambda_t^{m-1}\) to the other side:

\[
= (\Lambda_t^{m-1} \overline{\partial} u, \Lambda_t^{-(m-1)} [[A^*, \overline{\partial}], A] u)_{\gamma}
+ (e^\gamma \Lambda_t^{m-1}, e^{-\gamma} \overline{\partial} u, \Lambda_t^{-(m-1)} [[A^*, \overline{\partial}], A] u)_{\gamma}.
\]

In absolute value these are

\[
\leq ||\Lambda_t^{m-1} \overline{\partial} u||_\gamma \cdot ||\Lambda_t^{-(m-1)} [[A^*, \overline{\partial}], A] u||_\gamma
+ ||e^\gamma \Lambda_t^{m-1}, e^{-\gamma} \overline{\partial} u||_\gamma \cdot ||\Lambda_t^{-(m-1)} [[A^*, \overline{\partial}], A] u||_\gamma.
\]

All the operators are of tangential order \(m\) except \(e^\gamma \Lambda_t^{m-1}, e^{-\gamma} \overline{\partial}\), which is order \(m - 1\). So the above \(\lesssim ||Du||^2_{m-1,\gamma}\) as desired.

For \((iii)\), the operator \(A[[A^*, \overline{\partial}]\) has order \(2m\), so move \(A\) to the other side,

\[
(\overline{\partial} u, A[[A^*, \overline{\partial}] u)_{\gamma} = (A^* \overline{\partial} u, [A^*, \overline{\partial}] u)_{\gamma} + (e^\gamma [A^*, e^{-\gamma}] \overline{\partial} u, [A^*, \overline{\partial}] u)_{\gamma}
\]

and since \(A^* \overline{\partial}\) is order \(m + 1\) we rewrite it as \([[A, \overline{\partial}] + \overline{\partial} A + (A^* - A) \overline{\partial}\):

\[
= ([A, \overline{\partial}] u, [A^*, \overline{\partial}] u)_{\gamma} + (\overline{\partial} Au, [A^*, \overline{\partial}] u)_{\gamma}
+ ([A^* - A] \overline{\partial} u, [A^*, \overline{\partial}] u)_{\gamma} + (e^\gamma [A^*, e^{-\gamma}] \overline{\partial} u, [A^*, \overline{\partial}] u)_{\gamma}.
\]

54
Now all the operators are of tangential order $m$ except $\partial Au$. Taking absolute values and using a $lc/sc$ on the second term we get the above

\[ \lesssim |||Du|||_{m-1,\gamma}^2 + (lc)|||Du|||_{m-1,\gamma}^2 + (sc)|||\partial Au|||_\gamma^2 \]

as desired.

For (iv), just move the commutator to the left and reweight the norm:

\[
(\partial u, [A^\ast, e^{-\gamma}]\partial Au) = ([e^{-\gamma}, A^\ast]\partial u, \partial Au)
\]

\[ = (e^{\gamma}[e^{-\gamma}, A^\ast]\partial u, \partial Au)_\gamma. \]

With a cutoff on $e^{-\gamma}$, the commutator has tangential order $m-1$ so in absolute value the above

\[ \lesssim (lc)|||Du|||_{m-1,\gamma}^2 + (sc)|||\partial Au|||_\gamma^2, \]

Finally, (v) is the easiest to deal with:

\[ |([\bar{\partial}, A]u, \bar{\partial}Au)_\gamma| \lesssim (lc)|||Du|||_{m-1,\gamma}^2 + (sc)|||\partial Au|||_\gamma^2 \]

as desired, and we’re finished with the $\bar{\partial}$ piece.

For the $\partial^s_\gamma$ piece, insert commutators to get five terms as before:

\[
(\partial^s_\gamma A u, \partial^s_\gamma A u)_\gamma = (\partial^s_\gamma u, \partial^s_\gamma A^\ast A u)_\gamma + (\partial^s_\gamma u, [[A^\ast, \partial^s_\gamma], A]u)_\gamma
\]

\[ + (\partial^s_\gamma u, A[A^\ast, \partial^s_\gamma]u)_\gamma + (\partial^s_\gamma u, [A^\ast, e^{-\gamma}]\partial^s_\gamma A u)
\]

\[ + ([\partial^s_\gamma, A]u, \partial^s_\gamma Au)_\gamma
\]

\[ = (i) + (ii) + (iii) + (iv) + (v), \]

55
and again we want to fold everything except (i) into $O(|||Du|||_{m-1,\gamma}^2)$ or $|||D^\gamma Au|||_{\gamma}^2$.

Recall that

$$D\gamma f = \sum_{j=1}^n \left( -\frac{\partial f_j}{\partial w_j} + f_j \frac{\partial \gamma}{\partial w_j} \right)$$

is defined as $\bar{\partial} f + \sum_{j=1}^n \delta_j f$.

This time we start with the easiest term and write (v) in absolute value as

$$||(D\gamma^*, A)u, D\gamma^* Au|||_{\gamma} \lesssim (lc)|||D\gamma, A|||_{\gamma}^2 + (sc)|||D\gamma^* Au|||_{\gamma}^2$$

as before. And

$$|||D\gamma^*, A|||_{\gamma}^2 \lesssim |||D\gamma, A|||_{\gamma}^2 + |||\sum_{j=1}^n [\delta_j, A]u|||_{\gamma}^2$$

$$= |||D\gamma, A|||_{\gamma}^2 + |||\sum_{j=1}^n \left( \frac{\partial \gamma}{\partial w_j} (Au)_j - A \frac{\partial \gamma}{\partial w_j} u_j \right) |||_{\gamma}^2$$

$$\lesssim |||D\gamma, A|||_{\gamma}^2 + \sum_{j=1}^n \left( |||\frac{\partial \gamma}{\partial w_j} (Au)_j |||_{\gamma}^2 + \sum_{j=1}^n \left( |||A \frac{\partial \gamma}{\partial w_j} u_j |||_{\gamma}^2 \right) \right)$$

$$\lesssim |||Du|||_{m-1,\gamma}^2 + \sum_{j=1}^n \max \left| \frac{\partial \gamma}{\partial w_j} \right| |||(Au)_j |||_{\gamma}^2 + \sum_{j=1}^n \left| \frac{\partial \gamma}{\partial w_j} u_j \right| |||_{m,\gamma}^2$$

$$\lesssim |||Du|||_{m-1,\gamma}^2 + \sum_{j=1}^n \max \left| \frac{\partial \gamma}{\partial w_j} \right| |||Du|||_{m-1,\gamma}^2 + |||\gamma|||_{m+1,\gamma}^2 \cdot |||Du|||_{m-1,\gamma}^2.$$
Handling \((iv)\) as before we have
\[
(\bar{\partial}_\gamma^* u, [A^*, e^{-\gamma}]\bar{\partial}_\gamma^* u, \bar{\partial}_\gamma^* Au) = ([e^{-\gamma}, A^*]\bar{\partial}_\gamma^* u, \bar{\partial}_\gamma^* Au)
\]
\[
\leq (e^\gamma [e^{-\gamma}, A^*] \bar{\partial}_\gamma^* u, [A^*, e^{-\gamma}]\bar{\partial}_\gamma^* Au) = (e^\gamma [e^{-\gamma}, A^*] \bar{\partial}_\gamma^* u, [A^*, e^{-\gamma}]\bar{\partial}_\gamma^* Au)
\]
so we only have to track down the order of (and constants from) \(e^\gamma [e^{-\gamma}, A^*] \bar{\partial}_\gamma^* u\):
\[
||e^\gamma [e^{-\gamma}, A^*] \bar{\partial}_\gamma^* u||^2 \gamma \lesssim ||e^\gamma [A^*, e^{-\gamma}] \partial^\gamma u||^2 \gamma + \left|\left|\sum_{j=1}^n \frac{\partial \gamma}{\partial w_j} u_j\right|\right|^2 \gamma
\]
\[
\lesssim ||Du||^2_{m-1,\gamma} + ||\gamma||^2_{m,\gamma}||Du||^2_{m-2,\gamma}.
\]

We also take care of \((iii)\) as before, with a little more work:
\[
(\bar{\partial}_\gamma^* u, A[A^*, \bar{\partial}_\gamma^*]u) = (A^* \bar{\partial}_\gamma^* u, [A^*, \bar{\partial}_\gamma^*]u) + (e^\gamma [A^*, e^{-\gamma}] \bar{\partial}_\gamma^* u, [A^*, \bar{\partial}_\gamma^*]u)
\]
\[
= ([A, \bar{\partial}_\gamma^*] u, [A^*, \bar{\partial}_\gamma^*] u) + (\bar{\partial}_\gamma^* Au, [A^*, \bar{\partial}_\gamma^*] u)
\]
\[
+ ([A^* - A] \bar{\partial}_\gamma^* u, [A^*, \bar{\partial}_\gamma^*] u) + (e^\gamma [A^*, e^{-\gamma}] \bar{\partial}_\gamma^* u, [A^*, \bar{\partial}_\gamma^*] u).
\]

The first, second, and fourth terms are fine because of \(lc/sc\) and the arguments from \((iv)\) and \((v)\). As for the third term, \(A^* - A\) is order \(m - 1\) so
\[
||(A^* - A) \bar{\partial}_\gamma^* u||^2 \gamma \lesssim ||\bar{\partial}_\gamma^* u||^2_{m-1,\gamma}
\]
\[
\lesssim ||\bar{\partial}_\gamma^* u||^2_{m-1,\gamma} + \left|\left|\sum_{j=1}^n \frac{\partial \gamma}{\partial w_j} u_j\right|\right|^2_{m-1,\gamma}
\]
\[
\lesssim ||Du||^2_{m-1,\gamma} + ||\gamma||^2_{m,\gamma}||Du||_{m-2,\gamma}.
\]

57
As for \((ii)\), insert \(\Lambda_t^{m-1}\) and \(\Lambda_t^{-(m-1)}\) as in the \(\overline{D}\) piece:

\[
(\overline{\partial}_\gamma u, [[A^*, \overline{\partial}^*_\gamma], A]u)_\gamma = (\Lambda_t^{m-1}\overline{\partial}^*_\gamma u, \Lambda_t^{-(m-1)}[[A^*, \overline{\partial}^*_\gamma], A]u)_\gamma
\]

\[
+ (e^\gamma [\Lambda_t^{m-1}, e^{-\gamma}]\overline{\partial}^*_\gamma u, \Lambda_t^{-(m-1)}[[A^*, \overline{\partial}^*_\gamma], A]u)_\gamma.
\]

The three operators that appear here can be bounded appropriately:

\[
||\Lambda_t^{m-1}\overline{\partial}^*_\gamma u||^2_\gamma \lesssim ||Du||^2_{m-1,\gamma} + ||\gamma||^2_{m,\gamma}||Du||^2_{m-1,\gamma};
\]

\[
||e^\gamma [\Lambda_t^{m-1}, e^{-\gamma}]\overline{\partial}^*_\gamma u||^2_\gamma \lesssim ||Du||^2_{m-2,\gamma} + ||\gamma||^2_{m-1,\gamma}||Du||^2_{m-2,\gamma};
\]

\[
||\Lambda_t^{-(m-1)}[[A^*, \overline{\partial}^*_\gamma], A]u||^2_\gamma \lesssim ||\gamma||^2_{m,\gamma}||Du||^2_{m-1,\gamma};
\]

so

\[
|(\overline{\partial}_\gamma u, [[A^*, \overline{\partial}^*_\gamma], A]u)_\gamma| \lesssim ||Du||^2_{m-1,\gamma} + ||\gamma||^2_{m,\gamma}||Du||^2_{m-1,\gamma}
\]

which finishes the proof.

6.2 Inequalities for \(\Box_{\gamma,z'},\delta\) and \(N_{\gamma,z',\delta}\)

Lemma 6.2.1. Define a family of cutoff functions \(\{\zeta_k\}, k \geq 0\), so that

\[
\begin{cases}
\zeta_k \in C_0^\infty(V) \\
0 \leq \zeta_k \leq 1 \\
\zeta_k = 1 \text{ on a small neighborhood of } 0 \\
\zeta_k = 1 \text{ on } \text{supp } \zeta_{k+1}.
\end{cases}
\]
Then for \( u \in \mathcal{D}^{0,1}(V \cap \Omega_{z', \delta}) \) we have

\[
|||D\zeta_k u|||_{2^{\frac{2}{3}k-1, \gamma, \delta}}^2 \lesssim |||\zeta_1 \square_{\gamma, \delta} u|||_{L^2(\gamma, \delta)}^2 + |||\zeta_1 u|||_{L^2(\gamma, \delta)}^2.
\]

**Pf.** We prove this with the cutoff \( \zeta_0 \) in the last term. At the end we can re-index, letting the “new” \( \zeta_1 \) be the “old” \( \zeta_0 \). As usual, we drop the subscripts on \( \gamma, z, \delta \).

Starting with \( k = 1 \), since \( \zeta_1 u \) is also in \( \mathcal{D}^{0,1} \), apply the subelliptic estimate:

\[
|||D\zeta_1 u|||_{2^{\frac{2}{3}, \gamma}}^2 \leq \max e^{-\gamma} |||D\zeta_1 u|||_{2^{\frac{2}{3}, \gamma}}^2 \lesssim Q_\gamma(\zeta_1 u, \zeta_1 u)
\]

and apply Lemma 6.1.1,

\[
\lesssim |Q_\gamma(u, \zeta_1^2 u)| + |||\zeta_0 u|||_{L^2(\gamma)}^2
= |(\square u, \zeta_1^2 u)| + |||\zeta_0 u|||_{L^2(\gamma)}^2
= |(\zeta_1 \square u, \zeta_1 u)| + |||\zeta_0 u|||_{L^2(\gamma)}^2
\lesssim |||\zeta_1 \square u|||_{L^2(\gamma)}^2 + |||\zeta_0 u|||_{L^2(\gamma)}^2
\]

which finishes the case when \( k = 1 \).

For higher \( k \), let \( \Lambda = \Lambda^{\frac{1}{2}(k-1)} \) and proceed by induction:

\[
|||D\zeta_k u|||_{2^{\frac{2}{3}k-1, \gamma}}^2 = |||\Lambda D\zeta_k u|||_{2^{\frac{2}{3}, \gamma}}^2 = |||\Lambda D\zeta_1 \zeta_k u|||_{2^{\frac{2}{3}, \gamma}}^2
\]

using the facts that \( \zeta_1 \zeta_k = \zeta_k \) and \( \Lambda \) commutes with \( D \). To manipulate \( \Lambda D\zeta_1 \zeta_k u \), first expand \( \Lambda \zeta_1 \) with a commutator,

\[
\Lambda D\zeta_1 \zeta_k u = D\zeta_1 \Lambda \zeta_k u + D[\Lambda, \zeta_1] \zeta_k u.
\]
Since $\zeta_k \zeta_{k-1} = \zeta_k$, we can insert $\zeta_{k-1}$ into the second term, then expand it with more commutators, and get

$$D \Lambda \zeta_k u = D \zeta_1 \Lambda \zeta_k u + D[\Lambda, \zeta_1] \zeta_k \zeta_{k-1} u$$

$$= D \zeta_1 \Lambda \zeta_k u + [\Lambda, [D, \zeta_1]] \zeta_k \zeta_{k-1} + [\Lambda, \zeta_1][D, \zeta_k] \zeta_{k-1} + [\Lambda, \zeta_1] \zeta_k D \zeta_{k-1}.$$ 

So in norms we now have

$$|||D \zeta_k u|||_{2^{k-1}, \gamma} \lesssim |||D \zeta_1 \Lambda \zeta_k u|||_{2^{-\frac{1}{2}, \gamma}} + |||[\Lambda, [D, \zeta_1]] \zeta_k \zeta_{k-1} u|||_{2^{-\frac{1}{2}, \gamma}} + |||[\Lambda, \zeta_1][D, \zeta_k] \zeta_{k-1} u|||_{2^{-\frac{1}{2}, \gamma}} + |||[\Lambda, \zeta_1] \zeta_k D \zeta_{k-1} u|||_{2^{-\frac{1}{2}, \gamma}}.$$ 

Now since $[\Lambda, [D, \zeta_1]]$ and $[\Lambda, \zeta_1]$ both have tangential order $\frac{1}{2}(k-3)$ and $[D, \zeta_k]$ is just multiplication by $D \zeta_k$, we have the above

$$\lesssim |||D \zeta_1 \Lambda \zeta_k u|||_{2^{-\frac{1}{2}, \gamma}}^2 + |||D \zeta_{k-1} u|||_{2^{-\frac{1}{2}, \gamma}}^2 + |||D \zeta_{k-1} u|||_{2^{k-2}, \gamma}^2.$$ 

So, consolidating the last two terms and taking a higher Sobolev norm, we have

$$|||D \zeta_k u|||_{2^{k-1}, \gamma}^2 \lesssim |||D \zeta_1 \Lambda \zeta_k u|||_{2^{-\frac{1}{2}, \gamma}}^2 + |||D \zeta_{k-1} u|||_{2^{k-2}, \gamma}^2$$

$$\lesssim |||D \zeta_1 \Lambda \zeta_k u|||_{2^{-\frac{1}{2}, \gamma}}^2 + |||D \zeta_{k-1} u|||_{2^{(k-1)-1}, \gamma}^2.$$ 

The second term is okay by the induction hypothesis, so concentrate on the first term. Let

$$A = \zeta_1 \Lambda \zeta_k = \zeta_1 \Lambda_1^{\frac{1}{2}(k-1)} \zeta_k$$
and use the subelliptic estimate,
\[
|||D\zeta \Lambda u|||_{\frac{1}{2}, \gamma}^2 = |||DAu|||_{\frac{1}{2}, \gamma}^2 \\
\leq \max e^{-\gamma} |||DAu|||_{\frac{1}{2}, \gamma}^2 \\
\lesssim Q_\gamma(Au, Au).
\]

We will fold the right hand side into terms $|||\zeta \Box u|||_{\frac{3}{2}, \gamma}^2$ and $|||D\zeta_{k-1} u|||_{\frac{3}{2}, (k-1), \gamma}^2$. First we apply Lemma 6.1.2 to get the above
\[
\lesssim |Q_\gamma(u, A^* Au)| + |||D\zeta_{k-1} u|||_{\frac{3}{2}, (k-1), \gamma}^2 \\
= |(\Box u, A^* Au)_\gamma| + |||D\zeta_{k-1} u|||_{\frac{3}{2}, (k-1), \gamma}^2 \\
= |(A\Box u, Au)_\gamma + ([A, e^{-\gamma}]\Box u, Au)| + |||D\zeta_{k-1} u|||_{\frac{3}{2}, (k-1), \gamma}^2 \\
= |(\Lambda_{A, \frac{1}{2}} A\Box u, A^* Au)_\gamma + (\Lambda_{A, \frac{1}{2}} A\Box u, e^\gamma [\Lambda_{A, \frac{1}{2}}, e^{-\gamma}] Au)_\gamma \\
+ (\Lambda_{A, \frac{1}{2}} [A, e^{-\gamma}]\Box u, A^* Au)| + |||D\zeta_{k-1} u|||_{\frac{3}{2}, (k-1), \gamma}^2.
\]

We can insert $\zeta_1$ in front of $\Box u$ in the third inner product and get the above
\[
\lesssim |||A\Box u|||_{\frac{3}{2}, \gamma}^2 + |||Au|||_{\frac{3}{2}, \gamma}^2 + |||\zeta_1 \Box u|||_{\frac{3}{2}, k-2, \gamma}^2 + |||D\zeta_{k-1} u|||_{\frac{3}{2}, (k-1), \gamma}^2.
\]

We can also insert $\zeta_1$ and $\zeta_k$ for free by the definition of $A$:
\[
= |||A\zeta_1 \Box u|||_{\frac{3}{2}, \gamma}^2 + |||A\zeta_k u|||_{\frac{3}{2}, \gamma}^2 + |||\zeta_1 \Box u|||_{\frac{3}{2}, k-2, \gamma}^2 + |||D\zeta_{k-1} u|||_{\frac{3}{2}, (k-1), \gamma}^2 \\
\lesssim |||\zeta_1 \Box u|||_{\frac{3}{2}, k-1, \gamma}^2 + |||D\zeta_k u|||_{\frac{3}{2}, k-1, \gamma}^2 + |||\zeta_1 \Box u|||_{\frac{3}{2}, k-2, \gamma}^2 + |||D\zeta_{k-1} u|||_{\frac{3}{2}, (k-1), \gamma}^2.
\]
So we have

$$|||D\zeta_k u|||^2_{2^{k-1},\gamma} \lesssim |||\zeta_1 \Box_{\gamma} u|||^2_{2^{k-1},\gamma} + |||D\zeta_{k-1} u|||^2_{2^{(k-1)-1},\gamma}$$

which finishes the proof. ■

**Lemma 6.2.2.** Let $\lambda, \tilde{\lambda} \in C_0^\infty(V \cap \Omega_{z', \delta})$, $\tilde{\lambda} = 1$ on supp $\lambda$, and let $u \in \text{Dom} \Box_{\gamma_{z', \delta}} \cap \Lambda^{0,1}(V \cap \Omega_{\gamma_{z', \delta}})$. Then for nonnegative integers $s$, we have

$$||\lambda u||^2_{s+1,\gamma_{z', \delta}} \lesssim ||\tilde{\lambda} \Box_{\gamma_{z', \delta}} u||^2_{s,\gamma_{z', \delta}} + ||\tilde{\lambda} u||^2_{L^2(\gamma_{z', \delta})}.$$

**Pf.** As usual, we drop the subscripts on $\gamma_{z', \delta}$. Starting with $s = 0$, set $\lambda = \zeta_2$ and $\tilde{\lambda} = \zeta_1$, and use the previous lemma with $k = 2$:

$$||\lambda u||^2_{1,\gamma} \sim ||D\lambda u||^2_{0,\gamma} \lesssim ||\tilde{\lambda} \Box_{\gamma} u||^2_{0,\gamma} + ||\tilde{\lambda} u||^2_{L^2(\gamma)} + ||\tilde{\lambda} u||^2_{L^2(\gamma)} \lesssim ||\tilde{\lambda} \Box_{\gamma} u||^2_{L^2(\gamma)} + ||\tilde{\lambda} u||^2_{L^2(\gamma)}.$$

For higher $s$ we use induction. First we separate the order $s + 1$ derivatives,

$$||\lambda u||^2_{s+1,\gamma} \sim ||\lambda u||^2_{s,\gamma} + \sum_{|\alpha|=s+1} ||D^\alpha \lambda u||^2_{L^2(\gamma)}$$

and use the induction hypothesis,

$$\lesssim ||\tilde{\lambda} \Box_{\gamma} u||^2_{s-1,\gamma} + ||\tilde{\lambda} u||^2_{L^2(\gamma)} + \sum_{|\alpha|=s+1} ||D^\alpha \lambda u||^2_{L^2(\gamma)}.$$

62
The first two terms are fine, so concentrate on the sum term. If $D^\alpha$ are all tangential derivatives, then

$$||D_\nu^\alpha \lambda u||_{L^2(\gamma)}^2 \lesssim ||D\lambda u||_{s,\gamma}^2, \quad |\alpha| = s + 1.$$  

Letting $k = 2s + 2$, $\lambda = \zeta_k$, and $\tilde{\lambda} = \zeta_1$, we obtain

$$= ||||D\lambda u|||_{\frac{1}{2} k-1, \gamma},$$

$$\lesssim ||||\tilde{\lambda} \Box_\gamma u|||_{s, \gamma}^2 + ||||\tilde{\lambda} \Box_\gamma u|||_{L^2(\gamma)}^2 + ||\tilde{\lambda} u||_{L^2(\gamma)}^2$$

$$\lesssim ||||\tilde{\lambda} \Box_\gamma u|||_{s, \gamma}^2 + ||\tilde{\lambda} u||_{L^2(\gamma)}^2$$

which finishes this case.

When $D^\alpha$ contains one derivative $D_\rho$ in the normal direction, we can use $k = 2s + 2$ as above,

$$||D_\nu^\alpha D_\rho \lambda u||_{L^2(\gamma)}^2 \lesssim ||D_\rho \lambda u||_{s,\gamma}^2, \quad |\alpha| = s$$

$$\lesssim ||D\lambda u||_{\frac{1}{2} k-1, \gamma}$$

and we’re done as before.

But when $D^\alpha$ contains two or more normal derivatives we use a relationship between the Laplacian and $\Box_\gamma$.

**Lemma 6.2.3.** The operator $\Box_\gamma$ on a $(0,1)$ form is the Laplacian plus lower order derivatives:

$$\Box_\gamma f = \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \left( -\frac{\partial^2 f_k}{\partial w_j \partial \overline{w}_j} + \frac{\partial^2 \gamma}{\partial w_j \partial \overline{w}_k} f_j + \frac{\partial f_k}{\partial \overline{w}_j} \frac{\partial \gamma}{\partial w_j} \right) \right) d\overline{w}_k.$$  

63
Pf. Let $f = \sum_{j=1}^{n} f_j \, d\bar{w}_j$. Then

$$\partial^*_{\gamma} f = \sum_{j=1}^{n} \left( -\frac{\partial f_j}{\partial w_j} + \frac{\partial \gamma}{\partial w_j} f_j \right)$$

and

$$\bar{\partial} \partial^*_{\gamma} f = \sum_{k=1}^{n} \frac{\partial}{\partial \bar{w}_k} \left( \sum_{j=1}^{n} \left( -\frac{\partial f_j}{\partial w_j} + \frac{\partial \gamma}{\partial w_j} f_j \right) \right) d\bar{w}_k$$

$$= \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \left( -\frac{\partial^2 f_j}{\partial w_j \partial \bar{w}_k} + \frac{\partial^2 \gamma}{\partial w_j \partial \bar{w}_k} f_j + \frac{\partial \gamma}{\partial w_j} \frac{\partial f_j}{\partial \bar{w}_k} \right) \right) d\bar{w}_k.$$ 

To find $\partial^*_{\gamma} \bar{\partial} f$, we need $\partial^*_{\gamma}$ on a $(0, 2)$ form. Let

$$g = \sum_{j,k=1}^{n} g_{jk} d\bar{w}_j \wedge d\bar{w}_k$$

$$= \sum_{j<k} (g_{jk} - g_{kj}) d\bar{w}_j \wedge d\bar{w}_k.$$ 

and let

$$\alpha = \sum_{k=1}^{n} \alpha_k d\bar{w}_k$$

be a smooth $(0, 1)$ form with compact support, with

$$\bar{\partial} \alpha = \sum_{j,k=1}^{n} \frac{\partial \alpha_k}{\partial \bar{w}_j} d\bar{w}_j \wedge d\bar{w}_k$$

$$= \sum_{j<k} \left( \frac{\partial \alpha_k}{\partial \bar{w}_j} - \frac{\partial \alpha_j}{\partial \bar{w}_k} \right) d\bar{w}_j \wedge d\bar{w}_k.$$
The adjoint satisfies

\[(\bar{\partial} \alpha, g)_\gamma = (\alpha, \bar{\partial}^* g)_\gamma,\]

and computing the left side,

\[(\bar{\partial} \alpha, g)_\gamma = \sum_{j<k} \int \alpha_k \left( \left( -\frac{\partial \bar{g}_{jk}}{\partial \bar{w}_j} + \frac{\partial \bar{g}_{kj}}{\partial \bar{w}_j} \right) + (\bar{g}_{jk} - \bar{g}_{kj}) \frac{\partial \gamma}{\partial \bar{w}_j} \right) e^{-\gamma} \, dV\]

\[= \sum_{j<k} \int \alpha_k \left( \left( \frac{\partial \bar{g}_{jk}}{\partial \bar{w}_j} - \frac{\partial \bar{g}_{kj}}{\partial \bar{w}_j} \right) - (\bar{g}_{jk} - \bar{g}_{kj}) \frac{\partial \gamma}{\partial \bar{w}_j} \right) e^{-\gamma} \, dV\]

\[= \sum_{j,k} \int \alpha_k \left( \left( \frac{\partial \bar{g}_{jk}}{\partial \bar{w}_j} + \frac{\partial \bar{g}_{kj}}{\partial \bar{w}_j} \right) + (\bar{g}_{jk} - \bar{g}_{kj}) \frac{\partial \gamma}{\partial \bar{w}_j} \right) e^{-\gamma} \, dV\]

\[= \sum_{k=1}^n \left( \alpha_k \sum_{j=1}^n \left( -\frac{\partial \bar{g}_{jk}}{\partial \bar{w}_j} + \frac{\partial \bar{g}_{kj}}{\partial \bar{w}_j} + (\bar{g}_{jk} - \bar{g}_{kj}) \frac{\partial \gamma}{\partial \bar{w}_j} \right) \right) \gamma\]

So

\[\bar{\partial}^* g = \sum_{k=1}^n \left( \sum_{j=1}^n \left( -\frac{\partial \bar{g}_{jk}}{\partial \bar{w}_j} + \frac{\partial \bar{g}_{kj}}{\partial \bar{w}_j} + (\bar{g}_{jk} - \bar{g}_{kj}) \frac{\partial \gamma}{\partial \bar{w}_j} \right) \right) d\bar{w}_k,\]

and applying this to \( g = \bar{\partial} f \),

\[\bar{\partial}^* \bar{\partial} f = \sum_{k=1}^n \left( \sum_{j=1}^n \left( -\frac{\partial^2 f_k}{\partial \bar{w}_j \partial \bar{w}_j} + \frac{\partial^2 f_j}{\partial \bar{w}_j \partial \bar{w}_k} + \left( \frac{\partial f_k}{\partial \bar{w}_j} - \frac{\partial f_j}{\partial \bar{w}_k} \right) \frac{\partial \gamma}{\partial \bar{w}_j} \right) \right) d\bar{w}_k.\]

Thus

\[\Box f = \sum_{k=1}^n \left( \sum_{j=1}^n \left( -\frac{\partial^2 f_k}{\partial \bar{w}_j \partial \bar{w}_j} + \frac{\partial^2 \gamma}{\partial \bar{w}_j \partial \bar{w}_k} f_j + \frac{\partial f_k}{\partial \bar{w}_j} \frac{\partial \gamma}{\partial \bar{w}_j} \right) \right) d\bar{w}_k\]

as desired.
Returning to the proof of Lemma 6.2.2, we can use the Laplacian to express $D^2_\rho(\lambda u)$ locally as

\[(\dagger) \quad D^2_\rho(\lambda u) = a \left( \Box_{\gamma}(\lambda u) + \sum_{j,k=1}^n \left( \frac{\partial^2 \gamma}{\partial w_j \partial \overline{w}_k} \lambda u_j + \frac{\partial (\lambda u)_k}{\partial \overline{w}_j} \frac{\partial \gamma}{\partial w_j} \right) \right) \]

\[+ \sum_{j=1}^{2n-1} p_j D_\rho D_{t_j}(\lambda u) + \sum_{i,j=1}^{2n-1} q_{ij} D_{t_i} D_{t_j}(\lambda u) \]

where $a(t, r), p_j(t, r)$, and $q_{ij}(t, r)$ are $C^\infty$ functions and $a \neq 0$. Now suppose $D^\alpha$ contains two normal derivatives. Then

\[\|D_t^\alpha D^2_\rho \lambda u\|_{L^2(\gamma)}^2 \lesssim \|D^2_\rho \lambda u\|_{s-1, \gamma}^2, \quad |\alpha| = s - 1 \]

\[\lesssim \left\| a \left( \Box_{\gamma}(\lambda u) + \sum_{j,k=1}^n \left( \frac{\partial^2 \gamma}{\partial w_j \partial \overline{w}_k} \lambda u_j + \frac{\partial (\lambda u)_k}{\partial \overline{w}_j} \frac{\partial \gamma}{\partial w_j} \right) \right) \right\|_{s-1, \gamma}^2 \]

\[+ \sum_{j=1}^{2n-1} \left\| p_j D_\rho D_{t_j}(\lambda u) \right\|_{s-1, \gamma}^2 + \sum_{i,j=1}^{2n-1} \left\| q_{ij} D_{t_i} D_{t_j}(\lambda u) \right\|_{s-1, \gamma}^2. \]

Within the first term, the $a \Box_{\gamma}(\lambda u)$ is what we want (just insert $\tilde{\lambda}$ in front), and the sum terms are handled by previous cases. Likewise the $q_{ij}$ terms contain only tangential derivatives and are fine. For the $p_j$ terms, use the product rule to move $D_{t_j}$ to the front:

\[\sum_{j=1}^{2n-1} \left\| p_j D_\rho D_{t_j}(\lambda u) \right\|_{s-1, \gamma}^2 \lesssim \left\| (D_{t_j} p_j) D_\rho(\lambda u) \right\|_{s-1, \gamma}^2 + \left\| D_{t_j} (p_j D_\rho(\lambda u)) \right\|_{s-1, \gamma}^2 \]

\[\lesssim \left\| D_\rho(\lambda u) \right\|_{s-1, \gamma}^2 + \left\| p_j D_\rho(\lambda u) \right\|_{s, \gamma}^2 \]

\[\lesssim \left\| D_\rho(\lambda u) \right\|_{s-1, \gamma}^2 + \left\| D_\rho(\lambda u) \right\|_{s, \gamma}^2 \]
and we're back to the case of one normal derivative.

Finally, when $D^\alpha$ contains three or more normal derivatives, take $D_\rho$ of equation (\ref{eq:induction}) and use induction, and the lemma is finished.

\[ \]

**Corollary 6.2.4.** Letting $u = N_{\gamma_{\alpha}, \delta} f$ in Lemma 6.2.2, we obtain

\[
||\lambda N_{\gamma_{\alpha}, \delta} f||_{s+1, \gamma_{\alpha}, \delta}^2 \lesssim ||\tilde{\lambda} f||_{s, \gamma_{\alpha}, \delta}^2 + ||\tilde{\lambda} f||_{L^2(\gamma_{\alpha}, \delta)}^2.
\]
CHAPTER 7

ESTIMATE ON THE KERNEL FUNCTION

The relationship between the Neumann operator $N$ and the Bergman kernel is given by Kohn’s formula ([Ko]). The next theorem is a weighted version of Kohn’s formula, but the proof is the same as in the unweighted case.

**Theorem 7.1 (Weighted Kohn’s Formula).** For a smooth function $\gamma$ and $f \in L^2(\gamma)$ we have

$$P_\gamma f = (I - \overline{\partial}_\gamma^* N_\gamma \overline{\partial}) f.$$

**Pf.** We prove the formula for smooth functions, and the result for $L^2$ functions follows by taking a limit.

Let $R(f) = (I - \overline{\partial}_\gamma^* N_\gamma \overline{\partial}) f$. First we show that $R(f)$ is holomorphic by showing that it is annihilated by $\overline{\partial}$:

$$\overline{\partial}(I - \overline{\partial}_\gamma^* N_\gamma \overline{\partial}) f = \overline{\partial} f - \overline{\partial} \overline{\partial}_\gamma^* N_\gamma \overline{\partial} f,$$
and use the fact that $\Box_\gamma N_\gamma = I$, so that $\overline{\partial} \partial^*_\gamma N_\gamma = I - \overline{\partial}^*_\gamma \partial N_\gamma$, and substitute into the above,

$$= \overline{\partial} f - (I - \overline{\partial}^*_\gamma \partial N_\gamma) \overline{\partial} f$$

$$= \overline{\partial}^*_\gamma \partial N_\gamma \overline{\partial} f$$

$$= \overline{\partial}^*_\gamma N_\gamma \overline{\partial}^2 f$$

$$= 0.$$

In the next to last line we used the fact that $\overline{\partial}$ commutes with $N_\gamma$, which is true because

$$N_\gamma \overline{\partial} = N_\gamma \overline{\partial} \Box_\gamma N_\gamma$$

$$= N_\gamma \overline{\partial}(\partial \partial^*_\gamma + \overline{\partial}^*_\gamma \partial) N_\gamma$$

$$= N_\gamma \overline{\partial} \partial^*_\gamma \partial N_\gamma,$$

and replacing $N_\gamma \overline{\partial} \partial^*_\gamma$ by $I - N_\gamma \overline{\partial}^*_\gamma \overline{\partial}$ we obtain

$$= (I - N_\gamma \overline{\partial}^*_\gamma \overline{\partial}) \overline{\partial} N_\gamma$$

$$= \overline{\partial} N_\gamma.$$

We also have to show orthogonality, that $R(f) - f$ is orthogonal to the space of holomorphic functions. Since $R(f) - f$ simplifies to $-\overline{\partial}^*_\gamma N_\gamma \overline{\partial} f$, we have for any holomorphic function $h$,

$$(\overline{\partial}^*_\gamma N_\gamma \overline{\partial} f, h)_\gamma = (N_\gamma \overline{\partial}, \overline{\partial} h)_\gamma$$

$$= 0.$$
which finishes the proof of Kohn’s formula.

A Weighted Version of Kerzman’s Theorem

Armed with Corollary 6.2.4, we can show that the Bergman kernels for the domains \( \Omega_{z',\delta} \) are bounded by a constant if the two variables are restricted to disjoint compacts. The strategy is due to Kerzman ([Ke]): we express the kernel as the projection of a polyradial function and use Kohn’s formula to express the projection in terms of the Neumann operator. Then we use Corollary 6.2.4 to get the bound. We prove the theorem on the scaled domains \( \Omega_{z',\delta} \) with weights \( e^{-\gamma_{z',\delta}} \), but the constant we obtain is independent of \( \delta \).

**Theorem 7.2.** Let \( K_1 \) and \( K_2 \) be disjoint compacts and \( U_1 = K_1 \cap \Omega_{z',\delta} \) and \( U_2 = K_2 \cap \Omega_{z',\delta} \). If \( \alpha \) and \( \beta \) are \( n \)-indices and \( w^1 \in U_1 \) and \( w^2 \in U_2 \), then for small \( \delta \) we have

\[
|D_{w^1}^\alpha D_{w^2}^\beta K_{\Omega_{z',\delta}; \gamma_{z',\delta}}(w^1, w^2)| \lesssim c(\alpha, \beta, \gamma).
\]

**Pf.** Choose a cutoff function \( \phi \) such that

\[
\begin{align*}
\phi &\in C_0^\infty(P(0,1)), \text{ where } P(0,1) \text{ is the unit polydisc in } \mathbb{C}^n; \\
\phi &\geq 0; \\
\int \phi = 1; \\
\phi \text{ is polyradial: } \phi(w_1, \ldots, w_n) = \phi(|w_1|, \ldots, |w_n|).
\end{align*}
\]

Then for small \( \epsilon > 0 \), \( w^2 \in U_2 \), and any \( w^1 \in \Omega_{z',\delta} \), let

\[
\phi_{w^2,\epsilon}(w^1) = \frac{1}{\epsilon^{2n}}\phi \left( \frac{w^1 - w^2}{\epsilon} \right)
\]
which is supported in a polydisc $P(w^2, \epsilon)$. We can take $\epsilon$ small enough so that $P(w^2, \epsilon) \subset U_2$.

Since $\phi_{w^2, \epsilon}$ is polyradial and the Bergman kernel is harmonic (in the second variable), use the Mean Value Property to write

$$K_{\gamma'_{\delta}, \delta}(w^1, w^2) = \int_{P(w^2, \epsilon)} K_{\gamma'_{\delta}, \delta}(w^1, t) \phi_{w^2, \epsilon}(t) \, dV(t).$$

Taking derivatives in $\bar{w}^2$ we have

$$D_{\bar{w}^2}^\beta K_{\gamma'_{\delta}, \delta}(w^1, w^2) = \int_{P(w^2, \epsilon)} K_{\gamma'_{\delta}, \delta}(w^1, t) D_{\bar{w}^2}^\beta \phi_{w^2, \epsilon}(t) \, dV(t)$$

$$= (-1)^{|\beta|} \int_{P(w^2, \epsilon)} K_{\gamma'_{\delta}, \delta}(w^1, t) D_{t}^\beta \phi_{w^2, \epsilon}(t) \, dV(t)$$

$$= (-1)^{|\beta|} \int_{P(w^2, \epsilon)} K_{\gamma'_{\delta}, \delta}(w^1, t) D_{t}^\beta \phi_{w^2, \epsilon}(t) e^{\gamma_{\delta}} \cdot e^{-\gamma_{\delta}} \, dV(t),$$

and by the support of $\phi_{w^2, \epsilon}$ we can expand the domain of integration,

$$= (-1)^{|\beta|} \int_{\Omega_{\gamma'_{\delta}}} K_{\gamma'_{\delta}, \delta}(w^1, t) D_{t}^\beta \phi_{w^2, \epsilon}(t) e^{\gamma_{\delta}} \cdot e^{-\gamma_{\delta}} \, dV(t)$$

$$= (-1)^{|\beta|} P_{\gamma'_{\delta}, \delta}(D_{t}^\beta \phi_{w^2, \epsilon} \cdot e^{\gamma_{\delta}})(w^1).$$

Now apply Kohn’s formula,

$$= (-1)^{|\beta|} (I - \overline{\partial}_{\gamma'_{\delta}, \delta} N_{\gamma'_{\delta}, \delta} \overline{\partial})(D_{t}^\beta \phi_{w^2, \epsilon} \cdot e^{\gamma_{\delta}})(w^1).$$

When $w^1 \in U_1$, $\phi_{w^2, \epsilon} \equiv 0$, so the above

$$= (-1)^{|\beta|+1} \overline{\partial}_{\gamma'_{\delta}, \delta} N_{\gamma'_{\delta}, \delta} \overline{\partial}(D_{t}^\beta \phi_{w^2, \epsilon} \cdot e^{\gamma_{\delta}})(w^1).$$
Below we work on bounding the $s$ norm, in the variable $w^1$, of this expression. From now on we omit the subscripts on $\gamma_{\zeta',\delta}$, and also define the $(0,1)$ form $g$ by

$$g(w^1) = \partial(\bar{D}^\beta \phi_{w^2,\epsilon} \cdot e^{\gamma_{\zeta',\delta}})(w^1).$$

Then

$$||D_{\bar{w}^2}^2 K_{\gamma}(w^1, w^2)||^2_{H_s(U_1, e^{-\gamma})} = ||\bar{D}^\gamma N_\gamma g(w^1)||^2_{H_s(U_1, e^{-\gamma})} \quad \text{(norms in } w^1)$$

$$= \left|\left| \sum_{j=1}^{n} \left( -\frac{\partial}{\partial w_j^1}(N_\gamma g)_j + \frac{\partial \gamma}{\partial w_j^1} \cdot (N_\gamma g)_j \right) \right|_{H_s(U_1, e^{-\gamma})} \right|^2$$

$$\lesssim \sum_{j=1}^{n} \left|\left| \frac{\partial}{\partial w_j^1}(N_\gamma g)_j \right|_{H_s(U_1, e^{-\gamma})} \right|^2 + \sum_{j=1}^{n} \left|\left| \frac{\partial \gamma}{\partial w_j^1} \cdot (N_\gamma g)_j \right|_{H_s(U_1, e^{-\gamma})} \right|^2$$

From here we would like to combine the two terms into an $s + 1$ norm of $N_\gamma g$. Remembering that $\gamma$ is really $\gamma_{\zeta',\delta}$, the second term can be bounded above by a constant times

$$||\gamma_{\zeta',\delta}||^2_{H_{s+1}(U_1, e^{-\gamma_{\zeta',\delta}})} \cdot ||N_{\gamma_{\zeta',\delta}} g||^2_{H_s(U_1, e^{-\gamma_{\zeta',\delta}})}.$$  

The only case we will need is when $s$ is a positive integer. From section 3.2, taking a derivative of $\gamma_{\zeta',\delta}$ produces a positive power of $\delta$ times a corresponding derivative of $\gamma$. The $s + 1$ norm also includes a term with no derivatives, so $||\gamma_{\zeta',\delta}||_{s+1} \sim ||\gamma||_{s+1}$ with no factors of $\delta$. Combining the two terms (and dropping subscripts again), we have

$$||D_{\bar{w}^2}^2 K_{\gamma}(w^1, w^2)||^2_{H_s(U_1, e^{-\gamma})} \lesssim (1 + ||\gamma||^2_{H_{s+1}(U_1, e^{-\gamma})}) \cdot ||N_\gamma g||^2_{H_{s+1}(U_1, e^{-\gamma})}.$$
To bound $\|N_{\gamma}g\|_{H^{s+1}(U_1, e^{-\gamma})}^2$, define four cutoff functions:

$$\begin{cases}
\lambda_1, \lambda_2 \in C^\infty_0(U_1), & \lambda_2 = 1 \text{ on supp } \lambda_1; \\
\xi_1, \xi_2 \in C^\infty_0(U_2), & \xi_2 = 1 \text{ on supp } \xi_2, \text{ and } \xi_1 = 1 \text{ on supp } g.
\end{cases}$$

Then applying Corollary 6.2.4,

$$\|\lambda_1 N_{\gamma}g\|_{H^{s+1}(U_1, e^{-\gamma})}^2 \lesssim \|\lambda_2 N_{\gamma}g\|_{L^2(U_1, e^{-\gamma})}^2 + \|\lambda_2 g\|_{H^s(U_1, e^{-\gamma})}^2.$$  

Since $\lambda_2 = 0$ near the support of $g$, the second term vanishes. To bound the first term, let $\psi \in A^{0,1}_0(\Omega_{z', \delta})$ with $\|\psi\|_{L^2(\Omega_{z', \delta}, e^{-\gamma z', \delta})} = 1$. Then

$$\left(\lambda_2 N_{\gamma}g, \psi\right)_\gamma = (N_{\gamma}g, \lambda_2 \psi)_\gamma$$

$$= (g, N_{\gamma} \lambda_2 \psi)_\gamma$$

$$= (g, \xi_1 N_{\gamma} \lambda_2 \psi)_\gamma$$

$$= (g, e^{-\gamma} \xi_1 N_{\gamma} \lambda_2 \psi),$$

so in absolute value we have

$$\left|\left(\lambda_2 N_{\gamma}g, \psi\right)_\gamma\right| \lesssim \|g\|_{H^{-s}(U_2)} \cdot \|e^{-\gamma} \xi_1 N_{\gamma} \lambda_2 \psi\|_{H^s(U_2)}.$$

To bound $\|g\|_{H^{-s}(U_2)}$,

$$\|g\|_{H^{-s}(U_2)} = \|\bar{\mathcal{D}}^\beta (\bar{\mathcal{D}}^\beta \phi_w^2 \cdot e^\gamma)\|_{H^{-s}(U_2)}$$

$$\lesssim \|\bar{\mathcal{D}}^\beta \phi_w^2 \cdot e^\gamma\|_{H^{-s+1}(U_2)}$$

$$\lesssim \|\Lambda^{-s+1}(\bar{\mathcal{D}}^\beta \phi_w^2 \cdot e^\gamma)\|_{L^2(U_2)}.$$

73
and pull the exponential to the front,

\[ \lesssim ||e^\gamma \Lambda^{-s^+1} D^\beta \phi_{w^2,\epsilon}||_{L^2(U_2)} + ||[\Lambda^{-s^+1}, e^\gamma] D^\beta \phi_{w^2,\epsilon}||_{L^2(U_2)} \]

\[ \lesssim \{ \max e^\gamma \text{ on supp } \phi_{w^2,\epsilon} \} \cdot ||\Lambda^{-s^+1} D^\beta \phi_{w^2,\epsilon}||_{L^2(U_2)} + ||D^\beta \phi_{w^2,\epsilon}||_{H^{-s}(U_2)} \]

\[ \lesssim \{ 1 + \max e^\gamma \text{ on supp } \phi_{w^2,\epsilon} \} \cdot ||\phi_{w^2,\epsilon}||_{H^{-s+|\beta|+1}(U_2)}. \]

Now use duality:

\[ ||\phi_{w^2,\epsilon}||_{H^{-s+|\beta|+1}(U_2)} = \sup \left\{ \left| \int \phi_{w^2,\epsilon} \cdot h \right| : h \in C_0^\infty, ||h||_{H^{-|\beta|-1}} \leq 1 \right\} \]

\[ \leq \sup \left\{ \sup |h| \cdot \left| \int \phi_{w^2,\epsilon} \right| : h \in C_0^\infty, ||h||_{H^{-|\beta|-1}} \leq 1 \right\}. \]

By the Sobolev Lemma, if \( s - |\beta| - 1 > n \) we have

\[ \sup |h| \leq c_s ||h||_{H^{-|\beta|-1}}, \]

so we get

\[ ||g||_{H^{-s}(U_2)} \lesssim 1 + \max e^\gamma \text{ on supp } \phi_{w^2,\epsilon}. \]

For \( ||e^{-\gamma} \xi_1 N_\gamma \lambda_2 \psi||_{H_s(U_2)} \), we need to re-weight the norm with a commutator:

\[ ||e^{-\gamma} \xi_1 N_\gamma \lambda_2 \psi||_{H_s(U_2)} \leq ||e^{-\gamma} \Lambda^s \xi_1 N_\gamma \lambda_2 \psi||_{L^2(U_2)} + ||[\Lambda^s, e^{-\gamma}] \xi_1 N_\gamma \lambda_2 \psi||_{L^2(U_2)} \]

\[ = ||\Lambda^s \xi_1 N_\gamma \lambda_2 \psi||_{L^2(U_2, e^{-\gamma})} + ||e^{-\gamma} [\Lambda^s, e^{-\gamma}] \xi_1 N_\gamma \lambda_2 \psi||_{L^2(U_2, e^{-\gamma})} \]

\[ \lesssim ||\xi_1 N_\gamma \lambda_2 \psi||_{H_{s}(U_2, e^{-\gamma})} \]

and apply Corollary 6.2.4 again,

\[ \lesssim ||\xi_2 \lambda_2 \psi||_{H_{s-1}(U_2, e^{-\gamma})} + ||\xi_2 N_\gamma \lambda_2 \psi||_{L^2(U_2, e^{-\gamma})}. \]
The first term vanishes because of the disjoint supports, and since $N_\gamma$ is a bounded operator, we get the above
\[
\lesssim ||\lambda_2\psi||_{L^2(U_2,e^{-\gamma})} \\
\leq 1.
\]

Putting it all together and remembering that $\gamma$ is really $\gamma_{z^\prime,\delta}$ we get
\[
||D_\overline{w}^\beta K_{\gamma_{z^\prime,\delta}}(w^1, w^2)||^2_{H_s(U_1,e^{-\gamma_{z^\prime,\delta}})} \lesssim M
\]
where $s - |\beta| - 1 > n$ and
\[
M = \left(1 + ||\gamma_{z^\prime,\delta}||^2_{H_{s+1}(U_1,e^{-\gamma_{z^\prime,\delta}})}\right) \cdot \{1 + \max e^{\gamma_{z^\prime,\delta}} \text{ on supp } \phi_{w^2,\epsilon}\}.
\]
Finally, by the Sobolev Lemma,
\[
|D_{w^1}^\alpha D_{\overline{w}^2}^\beta K_{\gamma_{z^\prime,\delta}}(w^1, w^2)| \lesssim M_{\alpha,\beta,\gamma}
\]
for $s > n + |\beta| + |\alpha| + 1$. ■

To obtain the estimate on the kernel for $\Omega_{z^\prime}$, we need the transformation formula relating kernels through coordinate changes.

**Lemma 7.3.** Let $F : \Omega_1 \rightarrow \Omega_2$ be biholomorphic, and $w^j = F(\zeta^j)$, $j = 1, 2$. Let $\gamma$ be smooth on $\overline{\Omega}_1$, $\tilde{\gamma} = \gamma \circ F$, and let $K_{1,\gamma}$ and $K_{2,\tilde{\gamma}}$ denote the weighted kernels on the two domains. Then
\[
K_{1,\gamma}(\zeta^1, \zeta^2) = \det J_\mathbb{C} F(\zeta^1) K_{2,\tilde{\gamma}}(w^1, w^2) \det J_\mathbb{C} \overline{F}(\zeta^2),
\]

75
where the complex Jacobian is defined

\[ J_C F(z) = \left( \frac{\partial F_k}{\partial z_l}(z) \right)_{k,l=1}^n. \]

Pf. Let \( g \in A^2(\Omega_1, e^{-\gamma}) \). We have to show that the right side of the above equation reproduces \( g \):

\[
\int_{\Omega_1} \det J_C F(\zeta^1) K_{2,\tilde{\gamma}}(w^1, w^2) \det J_C F(\zeta^2) g(\zeta^2) e^{-\gamma(\zeta^2)} dV(\zeta^2)
\]

and change coordinates to integrate in \( w^2 \),

\[
= \int_{\Omega_2} \det J_C F(\zeta^1) K_{2,\tilde{\gamma}}(w^1, w^2) \det J_C F(\zeta^2) g(\zeta^2) e^{-\gamma(\zeta^2)} dV(\zeta^2) \cdot e^{-\gamma(F^{-1}(w^2))} \det J_F F^{-1}(w^2) dV(w^2).
\]

To combine two of the determinants, notice that

\[
\det J_C F(F^{-1}(w^2)) = \left| \frac{\det J_C F(F^{-1}(w^2))}{\det J_C F(F^{-1}(w^2))} \right|^2
\]

and in general, \( |\det J_C(\cdot)|^2 = \det J_F(\cdot) \), so we can rewrite this as

\[
= \frac{\det J_F F^{-1}(w^2)}{\det J_C F(F^{-1}(w^2))} \det J_F F^{-1}(w^2)
= \frac{1}{\det J_C F(F^{-1}(w^2))}.
\]
Then the integral
\[
= \det J_{C} F(\zeta^{1}) \int_{\Omega_{2}} K_{\zeta}(F(\zeta^{1}), w^{2}) \frac{1}{\det J_{C} F(F^{-1}(w^{2}))} g(F^{-1}(w^{2})) e^{-\gamma(w^{2})} dV(w^{2}).
\]

Since
\[
\frac{1}{\det J_{C} F(F^{-1}(w^{2}))} g(F^{-1}(w^{2}))
\]
is holomorphic on $\Omega_{2}$, the above expression
\[
= \det J_{C} F(\zeta^{1}) \frac{1}{\det J_{C} F(F^{-1}(F(\zeta^{1})))} g(F^{-1}(F(\zeta^{1})))
\]
\[
= g(\zeta^{1})
\]
as desired. \[\square\]

Finally, we estimate the kernel for the unscaled domains $\Omega_{z'}$. Let $\zeta^{1}$ be a point in $V \cap \Omega_{z'}$ whose closest boundary point is 0, and let $\zeta^{2}$ also be in $V \cap \Omega_{z'}$. Let
\[
\delta = |R_{z'}(\zeta^{1})| + |R_{z'}(\zeta^{2})| + \sum_{j=1}^{n-1} |\zeta_{j}^{1} - \zeta_{j}^{2}|^{2} + |\zeta_{n}^{1} - \zeta_{n}^{2}|
\]
which makes
\[
1 = |\rho_{z',\delta}(w^{1})| + |\rho_{z',\delta}(w^{2})| + \sum_{j=1}^{n-1} |w_{j}^{1} - w_{j}^{2}|^{2} + |w_{n}^{1} - w_{n}^{2}|.
\]
As $\zeta^{1}$ and $\zeta^{2}$ approach $b\Omega_{z',\delta}$, both $\rho_{z',\delta}(w^{1})$ and $\rho_{z',\delta}(w^{2})$ approach zero, so that eventually
\[
\sum_{j=1}^{n-1} |w_{j}^{1} - w_{j}^{2}|^{2} + |w_{n}^{1} - w_{n}^{2}| \geq \frac{1}{2}.
\]
So, \( w^1 \) and \( w^2 \) are apart from each other, and from Theorem 7.2 we have a bound on 
\[ D_{w^1}^\alpha D_{w^2}^\beta K_{\gamma',\delta}(w^1, w^2). \]
To pull back to the unscaled domains, first apply Lemma 7.3,
\[
K_{\Omega_{\gamma',\gamma}}(\zeta^1, \zeta^2) = \det J_{\Omega_{\gamma',\gamma}}(\zeta^1) \cdot K_{\Omega_{\gamma',\gamma}}(w^1, w^2) \cdot \det J_{\Omega_{\gamma',\gamma}}(\zeta^2)
\]
\[
= \left( \delta^{-1 - \frac{1}{2}(n-1)} \right)^2 \cdot K_{\Omega_{\gamma',\gamma}}(w^1, w^2)
\]
\[
= \delta^{-n-1} \cdot K_{\Omega_{\gamma',\gamma}}(w^1, w^2).
\]

By the chain rule, taking derivatives on both sides increases the singularity,
\[
D_{\zeta^1}^\alpha D_{\zeta^2}^\beta K_{\Omega_{\gamma',\gamma}}(\zeta^1, \zeta^2) = \delta^{-n-1} \cdot \delta^{-\frac{1}{2}(|\alpha'|+|\beta'|)-|\alpha_n|-|\beta_n|} D_{w^1}^\alpha D_{w^2}^\beta K_{\Omega_{\gamma',\gamma}}(w^1, w^2),
\]
where \( \alpha' = \alpha_1 + \cdots + \alpha_{n-1} \) and similarly for \( \beta' \). Thus we obtain
\[
|D_{\zeta^1}^\alpha D_{\zeta^2}^\beta K_{\Omega_{\gamma',\gamma}}(\zeta^1, \zeta^2)| \lesssim \delta^{-(n+1)} \cdot \delta^{-\frac{1}{2}(|\alpha'|+|\beta'|)-|\alpha_n|-|\beta_n|} \cdot M_{\alpha,\beta,\gamma}
\]
and the proof of Theorem 1 is finished. \( \blacksquare \)
BIBLIOGRAPHY


