ON LOW LYING ZEROS OF AUTOMORPHIC L-FUNCTIONS

DISSERTATION

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By

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* * * *

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ABSTRACT

In this thesis we study assuming the generalized Riemann Hypothesis the distribution of low-lying zeros, zeros at or near the central point \( s = \frac{1}{2} \), for the family of symmetric power \( L \)-functions associated with cusp forms on \( \text{GL}(2) \) and for those of Hecke \( L \)-functions associated with cubic characters.

By studying the one-level scaling density statistics for these families of \( L \)-functions we provide further evidence to the truth of the density conjecture of Katz and Sarnak which asserts that the statistics of low-lying zeros of a family of \( L \)-functions should coincide with the corresponding statistics of the eigenvalues of matrices from a suitable family of classical compact groups determined by the symmetry type of the family.

Following the approach of Iwaniec, Luo and Sarnak we determine the corresponding symmetry types for the family of symmetric \( r \)th power \( L \)-functions where we use the recent works of Kim and Shahidi for \( r = 3 \) and \( r = 4 \) while we assume the truth of Langlands Functoriality Conjecture for \( r > 4 \).

As for the family of Hecke \( L \)-functions we identify the symmetry type as unitary and our computations depend on an analog of Polya-Vinogradov inequality due to Heath-Brown and Patterson and the generalized Riemann Hypothesis as well as Patterson’s work on the distribution of cubic Gauss sums at prime arguments.
Dedicated to my parents
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CHAPTER 1
INTRODUCTION

1.1 Distribution of Zeros and Random Matrix Theory

In 1973 Montgomery [26] proved assuming the Riemann Hypothesis that for any Schwartz function \( \phi \) whose Fourier transform has compact support in \((-1, 1)\),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq j \neq k \leq N} \phi(\gamma_j - \gamma_k) = \int_{-\infty}^{\infty} \phi(x) r_2(x) \, dx \tag{1.1}
\]

where \( \gamma_j > 0 \) is the ordinate of the \( j \)th scaled non-trivial zero of the zeta function and

\[
r_2(x) = 1 - \left( \frac{\sin \pi x}{\pi x} \right)^2 \tag{1.2}
\]

is the pair correlation function that he found. Montgomery then conjectured that (1.1) holds without any restriction on the support of \( \hat{\phi} \).

This result made a major contribution to the study of the distribution of zeros of the zeta function and its generalizations. What makes it so significant is the link it establishes between the distribution of zeros and Random Matrix Theory due to Dyson’s crucial observation that the pair correlation distribution \( r_2(x) \) is the same as the pair correlation \( r_2(\text{GUE})(x) \) for the eigenvalues of a large random matrix from the Gaussian Unitary Ensemble, a certain probability distribution on the linear space of \( N \times N \) Hermitian matrices which are unitary invariant. It’s worth noting that Montgomery’s result is thus the first to provide significant insight to the hypothesis 1.
of Hilbert and Pólya which suggests that the non-trivial zeros of the Riemann zeta function might be the eigenvalues of a Hermitian operator.

The ensembles GUE as well as GOE (Gaussian Orthogonal Ensemble) and various others were introduced in the 1950’s by Wigner [37] to model complex energy levels of heavy nuclei in quantum mechanics. Later in 1962, Dyson [7] introduced his three related circular ensembles COE, CUE, CSE as well as the associated Gaussian Symplectic Ensemble, GSE. He showed that the local spacing distributions of the eigenvalues of matrices in these ensembles agree as \( N \to \infty \) with those from the corresponding ensembles GOE, GUE and GSE. Below in Table 1.1 are the ensembles CUE, which is the first ensemble \( U(N) \), COE, CSE and some others. Katz and Sarnak called the first four ensembles Type II symmetric spaces. They show in [21] that all these spaces have the same limiting measure as \( N \to \infty \), namely the GUE measure. This result together with what Dyson found about the agreement of GUE and CUE on the local spacing distributions show that Type II local spacings are all GUE.

<table>
<thead>
<tr>
<th>Symmetry Type G</th>
<th>Realization Of G(N) As Matrices</th>
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<tbody>
<tr>
<td>U</td>
<td>( U(N), \ N \times N ) unitary matrices</td>
</tr>
<tr>
<td>SO(even)</td>
<td>( SO(2N), \ 2N \times 2N ) unitary ( A, A^t = I )</td>
</tr>
<tr>
<td>SO(odd)</td>
<td>( SO(2N+1) ) see above</td>
</tr>
<tr>
<td>Sp</td>
<td>( USp(2N), \ 2N \times 2N ) unitary ( A, \quad A^t J A = J, J = \begin{bmatrix} 0 &amp; I_N \ -I_N &amp; 0 \end{bmatrix} )</td>
</tr>
<tr>
<td>COE</td>
<td>( U(N)/O(N), ) symmetric unitary ( N \times N ) matrices</td>
</tr>
</tbody>
</table>

identified with the above cosets via \( B \to B^t B \)

| CSE             | \( U(2N)/USp(2N), \ 2N \times 2N \) unitary matrices               |

satisfying \( J H^t J = H \) identified by \( B \to B J B^t J^t \).

Table 1.1: Type II Symmetric Spaces
Inspired by Montgomery’s conjecture, Odlyzko [27] has made extensive numerical computations since 1987 finding a favorable agreement of the local spacing distributions of zeros of the zeta function with the GUE measure. In 1994 Hejhal [11] showed that the triple correlation of the zeros is the GUE triple correlation as determined by Dyson. Rudnick and Sarnak [32] established in 1996 that all the \( n \geq 2 \) correlations not only for the zeros of the Riemann zeta function but also of quite general automorphic \( L \)-functions over \( \mathbb{Q} \) are all given by the GUE measure. Unfortunately, all the results above are restricted to test functions whose Fourier transforms have compact support.

Katz and Sarnak call the phenomenon that the high zeros of any fixed \( L \)-function associated to a cusp form on \( GL_m/\mathbb{Q} \) obey GUE spacing laws the Montgomery-Odlyzko Law. In 1998 they established this law for wide classes of zeta and \( L \)-functions associated with curves over finite fields. They showed that most curves of sufficiently large genus over a sufficiently large finite field satisfy the Montgomery-Odlyzko Law to an arbitrary degree of precision. In the function field setting, the Riemann Hypothesis was proved by Weil [36] and one has a spectral interpretation for the zeros of the zeta functions in terms of the reciprocals of the eigenvalues of Frobenius acting on the first cohomology group of the curve. Moreover, their study reveals the source of the GUE phenomenon as the monodromy (or the symmetry group) of the family and its scaling limits combined with the universality of the spacing statistics for type II symmetric spaces.

### 1.2 One Level Density and Density Conjecture

Although the statistics above satisfy for most general \( L \)-functions the Montgomery-Odlyzko Law, at least for a restricted class of test functions, these statistics are unfortunately not sensitive enough to distinguish the symmetry relations that are
peculiar to a certain ensemble. For instance, the functional equations for the characteristic polynomials of unitary matrices mimic those of the usual $L$-functions whereas orthogonal and symplectic matrices show a resemblance to the self-contragredient $L$-functions. Hence Katz and Sarnak introduced another statistic called the n-level scaling density. They showed that for any of the type II spaces $G(N)$, the limiting measure as $N \to \infty$ depends on the particular symmetry group $G$.

In this thesis we shall only consider one-level scaling density statistics which we describe below. Let $A$ be in one of the type II spaces $G(N)$ and $[a, b] \subset \mathbb{R}$. Define

$$\Delta(A)[a, b] = \{ \theta(A) \mid e^{i\theta(A)} \text{ eigenvalue of } A, \frac{\theta(A)N}{2\pi} \in [a, b]\}$$

and denote by $W$ the average of $\Delta(A)$

$$W(G(N)) = \int_{G(N)} \Delta(A) \, dA.$$  \hspace{1cm} (1.4)

In [21] Katz and Sarnak show that

$$\lim_{N \to \infty} W(G(N))[a, b] = \int_a^b W(G(x)) \, dx$$ \hspace{1cm} (1.5)

where

$$W(G)(x) = \begin{cases} 
1 & \text{if } G = U \text{ or } SU \\
1 - \frac{\sin 2\pi x}{2\pi x} & \text{if } G = Sp \\
1 + \frac{\sin 2\pi x}{2\pi x} & \text{if } G = SO(\text{even}) \\
\delta_0 + 1 - \frac{\sin 2\pi x}{2\pi x} & \text{if } G = SO(\text{odd}) \\
1 + \frac{1}{2}\delta_0(x) & \text{if } G = O.
\end{cases}$$ \hspace{1cm} (1.6)

Here $\delta_0(x)$ is Dirac’s delta function.

From the number theoretical aspect, the one-level scaling density is concerned with the distribution of the low-lying zeros, zeros at or near the central point $s = \frac{1}{2}$, for a certain family of $L$-functions. In other words, Katz and Sarnak suggested that the statistics of the low-lying zeros for a certain family of $L$-functions should coincide with
the corresponding statistics of eigenvalues for a suitable family of classical compact
groups. This idea has been carried out and tested both analytically and numerically
for various families of $L$-functions, examples of which can be found in [21, 22].

To describe this in more detail, we start with a family $\mathcal{F}$ of $L$-functions. We
assume the generalized Riemann Hypothesis for all $L$-functions to appear. For a
given $L$-function in $\mathcal{F}$, we order its non-trivial zeros by their ordinates
\begin{equation}
\ldots \leq \gamma_{-1} \leq 0 \leq \gamma_1 \leq \gamma_2 \leq \ldots
\end{equation}
and scale them using the so-called analytic conductor $c_L$ so that the zeros have unit
mean spacing. For any even Schwartz function $\phi(x)$ whose Fourier transform has
compact support, we shall consider the sum
\begin{equation}
\sum_{\gamma_L} \phi \left( \frac{\gamma_L \log R}{2\pi} \right).
\end{equation}
where $R$ is to be chosen later. This sum captures essentially a bounded number
of zeros within a distance $O(1/\log R)$ of the central point $s = \frac{1}{2}$. Hence to get an
asymptotical result we have to average over the $L$-functions in $\mathcal{F}$ ordered by their
conductors. In other words, we shall consider
\begin{equation}
\frac{1}{|\{L : c_L \leq X\}|} \sum_{c_L \leq X} \sum_{\gamma_L} \phi \left( \frac{\gamma_L \log R}{2\pi} \right).
\end{equation}
The density conjecture of Katz and Sarnak states that (1.9) converges as $X \to \infty$
without any restriction on the test function to
\begin{equation}
\int_{-\infty}^{\infty} \phi(x) W(G)(x) \, dx
\end{equation}
where $W(G)(x)$ is a distribution which depends on the family $\mathcal{F}$ through a symmetry
group $G$.

Recently Iwaniec, Luo and Sarnak [16], Conrey [2], Royer [31], Rubinstein [33],
Miller [25], Fouvry and Iwaniec [8], Hughes and Rudnick [15], Young [38], Hughes
and Miller [13], and Duenez and Miller [6] studied the distribution of low-lying zeros for various families of \( L \)-functions and they gave further evidence to the truth of the density conjecture. All the above results except the functional field case that Katz and Sarnak considered are partially proved for a restricted class of test functions. The main method in studying the distribution of zeros is to use an analogue of Riemann’s explicit formula which relates sums over zeros to multiple sums over primes.

Our aim in this thesis is to provide further evidence for the density conjecture under the generalized Riemann Hypothesis by examining the one-level densities for the following families of \( L \)-functions:

1. Symmetric power \( L \)-functions associated with cusp forms on \( \text{GL}_2 \),

2. Hecke \( L \)-functions associated with cubic characters.

Below we state the main results of this thesis. Chapter 2 will present the basic facts and prove auxiliary results while in Chapter 3 and Chapter 4 we shall prove the main theorems of this chapter.

1.3 Thesis Results

1.3.1 Symmetric Power \( L \)-functions

We first describe the family in more detail. Let \( k \geq 2 \) be an even integer and \( H_k \) denote the set of arithmetically normalized cusp forms of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \) which are eigenfunctions of all the Hecke operators. Any \( f \in H_k \) has a Fourier expansion at infinity given by

\[
f(z) = \sum_{n \geq 1} \lambda_f(n)n^{(k-1)/2} e(nz)
\]  

(1.11)
where $\lambda_f(1) = 1$ and $\lambda_f(n) \in \mathbb{R}$ is the $n$-th Hecke eigenvalue. For each $f$ in $H_k$, there is an associated cuspidal automorphic representation $\pi_f = \otimes_p \pi_{f,p}$ of $GL_2(\mathbb{A}_\mathbb{Q})$ with trivial central character such that

- $\pi_{f,\infty} \simeq D_k$, the discrete series representation of $GL_2(\mathbb{R})$ of weight $k$

- $\pi_{f,p}$ is an unramified principal series representation with an associated semisimple conjugacy class of $SL_2(\mathbb{C})$ given by

\[
\begin{bmatrix}
\alpha_f(p) & 0 \\
0 & \alpha_f(p)^{-1}
\end{bmatrix}
\]

such that for any $r \geq 0$,

\[
\lambda_f(p^r) = \sum_{i=0}^{r} \alpha_f(p)^{2i-r} = \text{tr} \left( \text{Sym}^r(t_f(p)) \right)
\]

where $\text{Sym}^r$ is the symmetric $r$-th power representation of the standard representation of $GL_2(\mathbb{C})$, an irreducible representation of dimension $r + 1$,

- $\pi_{f,p}$ is tempered, as proved by Deligne [4, 5], i.e. $|\alpha_f(p)| = 1$.

For any $r \geq 1$, symmetric $r$-th power $L$-function associated with $f$ in $H_k$ is given by the Euler product of degree $r + 1$

\[
L(s, \text{Sym}^r f) = \prod_p L_p(s, \text{Sym}^r f),
\]

with local factors

\[
L_p(s, \text{Sym}^r f) = \det \left( I_{r+1} - p^{-s} \text{Sym}^r t_f(p) \right)^{-1} = \prod_{i=0}^{r} \left( 1 - \alpha_f(p)^{r-2i} p^{-s} \right)^{-1}
\]

absolutely convergent and non-vanishing for $\text{Re}(s) > 1$. As a consequence of Langlands Functoriality, we have
**Conjecture (Sym r).** There exists an automorphic self-dual representation $\operatorname{Sym}^r \pi_f = \bigotimes_p \operatorname{Sym}^r \pi_{f,p}$ of $\operatorname{GL}_{r+1}(\mathbb{A}_{\mathbb{Q}})$ whose local $L$-factors $L(s, \operatorname{Sym}^r \pi_{f,p})$ agree with the local factors $L_p(s, \operatorname{Sym}^r f)$ defined above; more precisely,

- **At the infinite place, the local $L$-factor of $\operatorname{Sym}^r \pi_{f,\infty}$ is given by**

  
  
  $$L(s, \operatorname{Sym}^r \pi_{f,\infty}) = \begin{cases} 
    \prod_{0 \leq j \leq \frac{r-1}{2}} \Gamma_C\left(s + \frac{r-2j}{2}\left(k-\frac{1}{2}\right)\right) & \text{if } r \text{ is odd} \\
    \Gamma_R(s + \kappa) \prod_{1 \leq j \leq \frac{r}{2}} \Gamma_C(s + j(k-1)) & \text{if } r \text{ is even} \end{cases} \quad (1.14)$$

  
  where $\Gamma_R(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$, $\Gamma_C(s) = \Gamma_R(s)\Gamma_R(s + 1)$ and $\kappa = 0$ if $\frac{r}{2}$ is even and $1$ otherwise.

- **$\operatorname{Sym}^r \pi_{f,p}$ is an unramified principal series representation associated to the conjugacy class given by $\operatorname{Sym}^r t_f(p)$**.

Furthermore, the completed function

$$\Lambda(s, \operatorname{Sym}^r f) = L_\infty(s, \operatorname{Sym}^r f) L(s, \operatorname{Sym}^r f) \quad (1.15)$$

satisfies the functional equation

$$\Lambda(s, \operatorname{Sym}^r f) = \epsilon_{\operatorname{Sym}^r f} \Lambda(1 - s, \operatorname{Sym}^r f) \quad (1.16)$$

with

$$\epsilon_{\operatorname{Sym}^r f} = \begin{cases} 
    i^k, & r \equiv 1 \mod 8 \\
    -1, & r \equiv 3 \mod 8 \\
    -i^k, & r \equiv 5 \mod 8 \\
    1, & \text{otherwise} \end{cases} \quad (1.17)$$

This conjecture has already been established by Gelbart and Jacquet [9] for $r = 2$, Kim and Shahidi [23] for $r = 3$ and Kim [20] for $r = 4$. Since for our family, $\operatorname{Sym}^r \pi_f$ is defined over $\mathbb{Q}$, unramified at all finite places and $\operatorname{Sym}^r \pi_f \not\cong \operatorname{Sym}^r \pi_f \otimes \chi$ for
any non-trivial primitive Dirichlet character $\chi$, we expect $\text{Sym}^r \pi_f$ to be cuspidal as proved for $r \leq 4$ by Gelbart and Jacquet [9] for $r = 2$ and Kim and Shahidi for $r = 3, 4$ [20, 23, 24]. Also Kim and Shahidi established the functional equation and meromorphic continuation of $L(s, \text{Sym}^r f)$ to $\mathbb{C}$ for $5 \leq r \leq 9$ and the holomorphy and non-vanishing of $L(s, \text{Sym}^r f)$ in the half plane $\text{Re}(s) \geq 1$ for $5 \leq r \leq 8$. For the discussion of the local $L$-factors $L(s, \text{Sym}^r \pi_{f, \infty})$ as well as the computation of the root numbers, we refer to the recent work of Cogdell and Michel [1].

As we assume the generalized Riemann Hypothesis, we shall denote the non-trivial zeros of $L(s, \text{Sym}^r f)$ for fixed $r \geq 1$ and $f \in H_k$ by

$$\rho_f = \frac{1}{2} + i \gamma_f. \quad (1.18)$$

These zeros appear in complex conjugate pairs by (1.16) and by a well-known argument of Riemann the number of zeros with $|\gamma_f|$ bounded by an absolute large constant is of order $\log c_{\text{Sym}^r f}$ where $c_{\text{Sym}^r f}$ is the analytic conductor of $\text{Sym}^r f$ given by

$$c_{\text{Sym}^r f} = \begin{cases} \frac{k}{r} & \text{if } r \text{ is even} \\ \frac{k+1}{r} & \text{if } r \text{ is odd.} \end{cases} \quad (1.19)$$

For any even Schwartz function $\phi(x)$ whose Fourier transform

$$\hat{\phi}(y) = \int_{-\infty}^{\infty} \phi(x) e(-xy) \, dx \quad (1.20)$$

has compact support in $(-v, v)$, we define

$$D(\text{Sym}^r f; \phi) = \sum_{\rho_f} \phi \left( \frac{\gamma_f}{2\pi} \log R \right). \quad (1.21)$$

As mentioned in the previous section we shall consider various averages over the family $H_k$. Hence we first evaluate

$$\lim_{k \to \infty} \frac{1}{|H_k|} \sum_{f \in H_k} D(\text{Sym}^r f; \phi). \quad (1.22)$$
In Lemma 3.1 we express $D(\text{Sym}^r f; \phi)$ using the explicit formula as multiple sums over primes involving Hecke eigenvalues with an additional term that reveals the underlying symmetry group. After estimating the resulting sums over primes using the Riemann Hypothesis, we see that the only significant contribution comes from

$$\sum_p \lambda_f(p^r) \hat{\phi}\left(\frac{\log p}{\log R}\right) \frac{2 \log p}{\sqrt{p}}.$$  \hfill (1.23)

We handle this sum by applying Petersson’s formula to

$$\sum_{f \in \mathcal{H}_k} \lambda_f(p^r).$$  \hfill (1.24)

which appear after interchanging the order of summation over primes with that over cusp forms. In Lemma 3.4 this term yields sums involving Kloosterman sums and the Bessel function. By the well-known estimates for these terms we obtain our first result:

\textbf{Theorem 1.1.} For any even Schwartz function $\phi(x)$ whose Fourier transform $\hat{\phi}(y)$ has support in $(-v, v)$ with

$$v = \begin{cases} 
\frac{2}{r^2} & \text{if } r \text{ even}, \\
\frac{2}{r^2 + r} & \text{if } r \text{ odd},
\end{cases} \quad (1.25)$$

we have

$$\lim_{k \to \infty} \frac{1}{|\mathcal{H}_k|} \sum_{f \in \mathcal{H}_k} D(\text{Sym}^r f; \phi) = \int_{-\infty}^{\infty} \phi(x) W(G)(x) \, dx$$ \hfill (1.26)

where $\text{Sym}^r \pi_f$ is assumed to be cuspidal for $r > 4$, $W(G)$ is given by (1.6) and the symmetry group $G$ equals

$$G = \begin{cases} 
\text{Sp} & \text{if } r \text{ even,} \\
\text{O} & \text{if } r \equiv 1, 5 \text{ mod } 8, \\
\text{SO(odd)} & \text{if } r \equiv 3 \text{ mod } 8, \\
\text{SO(even)} & \text{if } r \equiv 7 \text{ mod } 8.
\end{cases} \quad (1.27)$$
Remark 1.1. Note that by Plancherel theorem we have

$$
\int_{-\infty}^{\infty} \phi(x) W(G)(x) \, dx = \int_{-\infty}^{\infty} \hat{\phi}(y) \hat{W}(G)(y) \, dy.
$$

(1.28)

The Fourier transforms of the densities for the groups in (1.6) are

\[
\hat{W}(G)(y) = \begin{cases} 
\delta_0(y) & \text{if } G = U \text{ or SU} \\
\delta_0(y) - \frac{1}{2} \eta(y) & \text{if } G = \text{Sp} \\
\delta_0(y) + \frac{1}{2} \eta(y) & \text{if } G = \text{SO(even)} \\
\delta_0(y) - \frac{1}{2} \eta(y) + 1 & \text{if } G = \text{SO(odd)} \\
\delta_0(y) + \frac{1}{2} & \text{if } G = O,
\end{cases}
\]

(1.29)

where \( \eta(y) = 1, \frac{1}{2}, 0 \) for \(|y| < 1, y = \pm 1, |y| > 1 \). As the last three densities which correspond to the orthogonal symmetry groups agree in \(-1 < y < 1\), it is impossible to distinguish these types unless the support of \( \hat{\phi} \) is larger than \([-1, 1]\). Hence Theorem 1.1 cannot distinguish the orthogonal symmetry types O, SO(odd) and SO(even) as \( v \leq 1 \). However the expected symmetry for a family with even (odd) functional equation, ie. with root number +1 (-1), is SO(even), (SO(odd)). Moreover, two-level scaling densities can also be used as in [25] to distinguish between SO(odd) and SO(even) for arbitrarily small support. Note that for \( r \equiv 1, 5 \mod 8 \), the root number \( \epsilon_{\text{Sym}^r f} \) in (1.17) changes sign as \( k \) varies modulo 4. Hence we expect a similar result for the subsets \( H_k^{\pm} \) of the forms \( f \) for which \( \epsilon_{\text{Sym}^r f} = \pm 1 \). This case will be handled separately.

Next, we shall average over the weight as well. More precisely, we fix \( h \in C_0^\infty(\mathbb{R}^+) \), a smooth real valued function compactly supported on \( \mathbb{R}^+ \). We evaluate as \( K \to \infty \)

\[
\mathcal{A}_r(K) = \sum_{k \equiv 0(2)} \frac{24}{k - 1} h \left( \frac{k - 1}{K} \right) \sum_{f \in H_k} D(\text{Sym}^r f; \phi)
\]

(1.30)

normalized by

\[
A_r(K) = \sum_{k \equiv 0(2)} \frac{24}{k - 1} h \left( \frac{k - 1}{K} \right) |H_k|.
\]

(1.31)
As mentioned above the main contribution comes from (1.23). This time we carry out the summation over $k$ first which gives the following sum of Bessel functions

$$I(x) = \sum_{k \equiv 0 \mod 2} 2^k h \left( \frac{k - 1}{K} \right) J_{k-1}(x). \quad (1.32)$$

By Lemma 3.5 the main term of $I(x)$ behaves like $e^{ix}/\sqrt{x}$. For even $r$, this exponential factor can be attached to the sums of Kloosterman sums which in turn yields an exponential sum of the form

$$\sum_{t \mod c} \exp \left( \frac{2\pi i A t^{r/2}}{c} \right). \quad (1.33)$$

This sum is handled in [16] when $r = 2$ while for $r = 4$, it is the Gauss sum and we only need to know its size when $c$ is prime. The case for odd $r$, however, poses a different and more difficult situation as one needs a strong estimate of the following exponential sum

$$\sum_{p \equiv a \mod c} \exp \left( \frac{4\pi i p^{r/2}}{c} \right). \quad (1.34)$$

In this case we trivially estimate these terms without making further assumptions.

Our next result is the following:

**Theorem 1.2.** For any even Schwartz function $\phi(x)$ whose Fourier transform has support in $(-v, v)$ with

$$v = \begin{cases} 
2 & \text{if } r = 1 \\
\frac{8}{r(r+2)} & \text{if } r \text{ even} \\
\frac{8}{(r+1)(r+2)} & \text{if } r \text{ odd},
\end{cases} \quad (1.35)$$

we have

$$\lim_{K \to \infty} \frac{\mathcal{A}_r(K)}{\mathcal{A}_r(K)} = \int_{-\infty}^{\infty} \phi(x) W(G)(x) \, dx \quad (1.36)$$

where the cuspidality of $\text{Sym}^r \pi_f$ is assumed for $r > 4$ and $G$ is given by (1.27).
Remark 1.2. This theorem extends the support of \( \hat{\phi}(y) \) by a multiple of \( \frac{4r}{r + 2} \) for \( r > 1 \). However this is still weaker for \( r = 2 \) than \( v = \frac{3}{2} \) obtained in [16] by the explicit computation of sums of Kloosterman sums. We shall only do this computation for \( r = 4 \).

Next, we turn to the case \( r \equiv 1, 5 \mod 8 \) in which the sign of the functional equation changes modulo 4. We shall consider the sums

\[
\mathcal{A}_r^\pm(K) = \sum_{k \leq K} \frac{24}{k - 1} h \left( \frac{k - 1}{K} \right) \sum_{f \in H_k^r} D(Sym^r f; \phi).
\]  

(1.37)

In this case we obtain the following:

Theorem 1.3. Given a test function \( \phi(x) \) with the support of \( \hat{\phi} \) in \((-v, v)\) where \( v = \frac{6}{(r+1)(r+2)} \), we have

\[
\lim_{K \to \infty} \frac{\mathcal{A}_r^\pm(K)}{A_r^\pm(K)} = \int_{-\infty}^{\infty} \phi(x) W(G^\pm)(x) \, dx
\]

(1.38)

where \( G^- = \text{SO(odd)} \), \( G^+ = \text{SO(even)} \) and the cuspidality of \( \text{Sym}^r \pi_f \) is assumed for any \( r > 4 \) such that \( r \equiv 1, 5 \mod 8 \).

Finally we look at the case \( r = 4 \). Note that Theorem 1.1 gives \( v = \frac{1}{8} \) for the support while Theorem 1.2 improves it to \( v = \frac{1}{3} \). Our last result yields a slight improvement which actually triples the first as in [16] by an explicit local computation of sums of Kloosterman sums at prime arguments.

Theorem 1.4. For any even Schwartz function \( \phi(x) \) whose Fourier transform has support in \((-\frac{3}{8}, \frac{3}{8})\), we have

\[
\lim_{K \to \infty} \frac{\mathcal{A}_4(K)}{A_4(K)} = \int_{-\infty}^{\infty} \phi(x) W(\text{Sp})(x) \, dx.
\]

(1.39)
1.3.2 Hecke L-functions Associated With Cubic Characters

Let \( K = \mathbb{Q}(\omega) \), where \( \omega = e^{2\pi i/3} \) is the third root of unity. For any square-free integer \( c \neq 1 \) of \( K \) such that \( c \equiv 1 \) mod 9, the cubic residue character \( \chi_c = (\cdot)^3 \) is trivial on the units of \( K \), hence it can be regarded as a primitive Dirichlet character modulo \( (c) \), i.e., a character of the ray class group \( J^{(c)}/P^{(c)} \) where \( J^{(c)} \) is the group of all ideals in \( K \) relatively prime to \( (c) \) and \( P^{(c)} \) is the subgroup of all principal ideals \((a)\) with \( a \equiv 1 \) mod \( (c) \).

The Hecke \( L \)-function associated with \( \chi_c \) is defined by

\[
L(s, \chi_c) = \sum_{a \subset \mathcal{O}_K} \chi_c(a) N a^{-s}
\]

where \( a \) runs over the integral ideals of \( K \) and we put \( \chi_c(a) = 0 \) whenever \((a, (c)) \neq 1\).

It converges absolutely for \( \text{Re}(s) > 1 \) and has the Euler product

\[
L(s, \chi_c) = \prod_p \left( 1 - \chi_c(p)Np^{-s} \right)^{-1}
\]

where \( p \) varies over prime ideals of \( K \). Hecke showed that the completed \( L \)-function

\[
\Lambda(s, \chi_c) = \left( |d_K|N(c) \right)^{s/2}(2\pi)^{-s} \Gamma(s) L(s, \chi_c),
\]

where \( d_K \) is the discriminant of \( K \), is entire and satisfies

\[
\Lambda(s, \chi_c) = W(\chi_c) N(c)^{-1/2} \Lambda(1 - s, \overline{\chi_c}).
\]

Here \( W(\chi_c) \) is the Gauss sum

\[
W(\chi_c) = \sum_{a \in \mathfrak{D}_{\mathcal{O}_K}/(c)} \chi_c(a) \exp \left( 2\pi i \text{Tr} \left( \frac{a}{\delta(c)} \right) \right)
\]

where \( \mathfrak{D}_{\mathcal{O}_K/Z} = (\delta) \) is the different of \( K \). \( (\delta = 2\omega + 1) \).

As in the previous section, we shall consider the sum

\[
\sum_{\rho_c} \phi \left( \frac{\gamma_c}{2\pi} \log y \right)
\]
where $\rho_c = \frac{1}{2} + \gamma_c$ runs over the non-trivial zeros of $L(s, \chi_c)$. This time we average over the characters $\chi_c$ with $c \equiv 1 \mod 9$ and $c \neq 1$. In other words, we evaluate as $y \to \infty$

$$
\frac{1}{\mathcal{A}(y)} \sum_{c \equiv 1 \mod 9}^* e^{-N(c)/y} \sum_{\rho_c} \phi \left( \frac{\gamma_c}{2\pi} \log y \right)
$$

(1.46)

where $\mathcal{A}(y)$ is the normalizing factor

$$
\mathcal{A}(y) = \sum_{c \equiv 1 \mod 9}^* e^{-N(c)/y}.
$$

(1.47)

Once again the explicit formula followed by an estimate using the Riemann Hypothesis reveals the term with the main contribution to be

$$
\sum_{c \equiv 1 \mod 9}^* \sum_{p} (\chi_c(p) + \chi_c(p^2)) \hat{\phi} \left( \frac{\log Np}{\log y} \right) \frac{\log Np}{\sqrt{Np \log y}}.
$$

(1.48)

We shall handle this sum in two ways. First, we use estimates on character sums using the Riemann Hypothesis and an analogue of Polya-Vinogradov inequality due to Heath-Brown and Patterson [14] and this allows a support within [-1,1]. Next, we use two dimensional Poisson Summation for the sum over $c \equiv 1 \mod 9$ and use the crucial bound of Patterson on the distribution of cubic Gauss sums at prime arguments to get a slightly larger support. In fact, Patterson shows in [28] that

$$
\sum_{\pi \equiv 1 \mod 3, N(\pi) \leq z} \hat{G}(\pi) \mu(\pi) \Lambda(\pi) \ll z^E
$$

(1.49)

holds for $E = \frac{30}{31}$ where $\hat{G}$ is the normalized cubic Gauss sum (see Chap. 4) and $\mu$ can be thought of as the cubic root of a Grössencharakter. He also conjectures

$$
\sum_{N(c) \leq X, c \equiv 1 \mod 3} \hat{G}(c) \mu(c) \Lambda(c) = b(r, \omega) X^{5/6} + O \left( X^{1/2 + \epsilon} \right)
$$

(1.50)

where $b(r, \omega)$ is an explicit number and $b(r, \omega) = 0$ unless $\omega^3 = 1$. In Chapter 4 we prove the following:
Theorem 1.5. Assuming
\[ \sum_{\pi \equiv 1 \mod 3, N(\pi) \leq z} \hat{G}(\pi) \mu(\pi) \Lambda(\pi) \ll z^E, \]  
we have for any even Schwartz function $\phi(x)$ whose Fourier transform has compact support in $(-\frac{1}{E}, \frac{1}{E})$
\[ \lim_{y \to \infty} \frac{1}{\mathcal{A}(y)} \sum_{c \equiv 1 \mod 9}^* e^{-N(c)/y} \sum_{\rho_c} \phi \left( \frac{\gamma c}{2\pi} \log y \right) = \int_{-\infty}^{\infty} \phi(x) W(U)(x) \, dx \]
where the implied constant depends only on the test function.

Remark 1.3. Note that this theorem allows the support of $\hat{\phi}(y)$ to go beyond $[-1, 1]$ as (1.51) holds for $E < 1$ by Patterson’s result mentioned above. In case Patterson’s Conjecture (1.50) is also true, we can get the support up to $(-2, 2)$.
In this chapter we state the basic facts and prove auxiliary results that are needed for the main theorems.

We begin with Perron's formula which will be used to estimate certain sums over primes that appear after the explicit formula is derived in Chapters 3 and 4. We refer to [3, p. 105] for the proof.

**Lemma 2.1 (Perron's Formula).** For \( T > 0, \ c > 0, \ x > 0 \), we have
\[
\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \, ds - \delta(x) \right| < \begin{cases} 
  x^c \min \left\{ 1, \frac{1}{T|\log x|} \right\} & \text{if } x \neq 1, \\
  \frac{c}{T} & \text{if } x = 1 
\end{cases} 
\tag{2.1}
\]

where
\[
\delta(x) = \begin{cases} 
  1 & \text{if } x > 1, \\
  \frac{1}{2} & \text{if } x = 1, \\
  0 & \text{if } 0 < x < 1. 
\end{cases} \tag{2.2}
\]

Note that one of the incidents that the generalized Riemann Hypothesis is utilized will be when we apply Perron's formula to the logarithmic derivatives of the \( L \)-functions in Lemma 3.2. Hence our next step is to look at the properties of the \( L \)-functions.
2.1 Symmetric Power L-functions

For our computations, we need $\Lambda(s, \text{Sym}^r f)$ in (1.15) to be entire. Therefore throughout this thesis we assume for $r > 4$ that $\text{Sym}^r \pi_f$ is a self-dual cuspidal automorphic representation of $\text{GL}_{r+1}(\mathbb{A}_\mathbb{Q})$. As mentioned in introduction, this is already known for $r \leq 4$.

Recall that $H_k$ is the set of Hecke eigenforms of weight $k$ for the full modular group $\text{SL}_2(\mathbb{Z})$. The dimension of $H_k$ is known explicitly but for our purposes we use

**Lemma 2.2.** For any even $k \geq 2$, we have

$$|H_k| = \frac{k - 1}{12} + O(1). \quad (2.3)$$

Any $f \in H_k$ has the Fourier expansion at infinity given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz) \quad (2.4)$$

where $e(z) = e^{2\pi i z}$ and $\lambda_f(n) \in \mathbb{R}$ is the $n$th Hecke eigenvalue. The Hecke L-function associated with $f(z)$

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s} \quad (2.5)$$

is absolutely convergent for $\text{Re}(s) > 1$ and has the Euler product

$$L(s, f) = \prod_p (1 - \lambda_f(p)p^{-s} + p^{-2s})^{-1} \quad (2.6)$$

$$= \prod_p (1 - \alpha_f(p)p^{-s})^{-1}(1 - \alpha_f(p)^{-1}p^{-s})^{-1}$$

where the Hecke eigenvalues satisfy

$$\lambda_f(n) \lambda_f(m) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right) \quad (2.7)$$

$$\lambda_f(p^n) = \sum_{i=0}^{n} \alpha_f(p)^{n-2i}. \quad (2.8)$$
The completed $L$-function

$$\Lambda(s, f) = \Gamma_C \left( s + \frac{k - 1}{2} \right) L(s, f)$$  \hspace{1cm} (2.9)

is entire and satisfies the functional equation

$$\Lambda(s, f) = i^k \Lambda(1-s, f).$$  \hspace{1cm} (2.10)

For any $r \geq 1$, we have introduced in (1.12) the symmetric $r$-th power $L$-function associated with $f$ in $H_k$. Using the identity

$$\Gamma_C(s) = \pi^{-s-1/2} \Gamma \left( \frac{s}{2} \right) \Gamma \left( \frac{s+1}{2} \right)$$  \hspace{1cm} (2.11)

for $L_\infty(s, \text{Sym}^r f)$ in (1.14), we rewrite the completed function (1.15) as

$$\Lambda(s, \text{Sym}^r f) = Q \pi^{-r+1/2} \prod_{j=0}^r \Gamma \left( \frac{s}{2} + \mu_j \right) \prod_p (1 - \alpha_f(p)^{r-2j} p^{-s})^{-1}$$ \hspace{1cm} (2.12)

where $Q > 0$ and $\mu_j$’s depend on $k$. By logarithmic differentiation we obtain

$$\frac{\Lambda'}{\Lambda} (s, \text{Sym}^r f) = \log \pi^{-r+1/2} + \frac{1}{2} \sum_{j=0}^r \Psi \left( \frac{s}{2} + \mu_j \right) - \sum_{n=1}^\infty \lambda_f^{(r)}(n) \Lambda(n) n^{-s}$$ \hspace{1cm} (2.13)

where $\Psi(s) = \frac{\Gamma'}{\Gamma}(s)$, $\Lambda(n)$ is the von-Mangoldt’s function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (2.14)

and

$$\lambda_f^{(r)}(p^m) = \sum_{j=0}^r \alpha_f(p)^{m(r-2j)}.$$  \hspace{1cm} (2.15)

**Lemma 2.3.** For any $r \geq 1$, we have

$$\lambda_f^{(r)}(p^2) = \sum_{n=0}^r (-1)^{r-n} \lambda_f(p^{2n}).$$  \hspace{1cm} (2.16)
Proof. For \( r = 1 \), we find from (2.8) that

\[
\lambda_f^{(1)}(p^2) = \alpha f(p)^2 + \alpha f(p)^{-2} = \lambda f(p^2) - 1.
\]

Assuming (2.16) holds for \( r > 1 \) we see that

\[
\lambda_f^{(r+1)}(p^2) = \sum_{j=0}^{r+1} \alpha f(p)^{2(r+1)-4j}
= \sum_{j=0}^{2(r+1)} \alpha f(p)^{2(r+1)-2j} - \sum_{j=0}^{r} \alpha f(p)^{2r-4j}.
\]

Hence the result follows from equations (2.8) and (2.15) and the induction hypothesis.

 Lemma 2.3 will be used in the proof of the explicit formula in Chapter 3 to express \( \lambda_f^{(r)}(p^2) \) in terms of the Hecke eigenvalues \( \lambda_f(p^{2n}) \). The sums over primes that appear after the explicit formula which involve these eigenvalues will then be estimated in Lemma 3.2 applying Perron’s formula to the logarithmic derivatives of \( L(s, \text{Sym}^{2n} f) \) for \( 1 \leq n \leq r \). It is when we estimate these derivatives below that the cuspidality of \( \text{Sym}^r \pi_f \) is used. Before we do this, however, we need an auxiliary lemma to make sure we get our final Theorems for \( r = 3 \) and \( r = 4 \) unconditionally, that is assuming only the generalized Riemann Hypothesis.

**Lemma 2.4.** For \( m \geq n \geq 1 \), we have

\[
L(s, \text{Sym}^m f \times \text{Sym}^n f) = \prod_{0 \leq j \leq n} L(s, \text{Sym}^{m+n-2j} f) \tag{2.17}
\]

where the \( L \)-function on the left is the Rankin-Selberg \( L \)-function and the last factor on the right is the Riemann zeta function \( \zeta(s) \) if \( m = n \).

Proof. Comparing the local factors on both sides we get for \( m \geq n = 1 \)

\[
L(s, \text{Sym}^m f \times f) = L(s, \text{Sym}^{m+1} f) L(s, \text{Sym}^{m-1} f) \tag{2.18}
\]
where the last factor is $\zeta(s)$ for $m = 1$, while for $m \geq n \geq 2$, we have

$$L(s, \text{Sym}^m f \times \text{Sym}^n f) = L(s, \text{Sym}^{m+n} f) L(s, \text{Sym}^{m+n-2} f) \left( L(s, \text{Sym}^{m-2} f \times \text{Sym}^{n-2} f) \right)$$

where $\text{Sym}^0 f = 1$, $\text{Sym}^1 f = f$, and the last factor is $\zeta(s)$ if $m = n = 2$. 

**Lemma 2.5.** Assuming the cuspidality of $\text{Sym}^r \pi_f$ for $r > 4$, we have for $\Re(s) = \sigma > 1/2$ and for any $n \geq 1$,

$$\frac{L'(s, \text{Sym}^{2n} f)}{L(s, \text{Sym}^{2n} f)} \ll \frac{\log(k + |t|)}{\sigma - \frac{1}{2}}. \quad (2.20)$$

**Proof.** By (2.13) we get for any $n \geq 1$,

$$-\frac{L'(s, \text{Sym}^{2n} f)}{L(s, \text{Sym}^{2n} f)} = \sum_{j=0}^{2n} \frac{1}{2} \Psi \left( \frac{s}{2} + \mu_j \right) - \frac{N'}{N} (s, \text{Sym}^{2n} f) + O(1). \quad (2.21)$$

From the cuspidality of $\text{Sym}^r \pi_f$ for $r > 4$ and similar arguments to those in [3, p. 79] we see that $\Lambda(s, \text{Sym}^{2n} f)$ is entire of order 1 and hence it can be written as

$$\Lambda(s, \text{Sym}^{2n} f) = e^{A+Bs} \prod_{\rho} \left( 1 - \frac{1}{s/\rho} \right) e^{s/\rho} \quad (2.22)$$

where $\rho$ runs through the nontrivial zeros of $L(s, \text{Sym}^{2n} f)$ and

$$\sum_{\rho} |\rho|^{-1-\epsilon} \quad (2.23)$$

converges for any $\epsilon > 0$.

Note that by Lemma 2.3 we get

$$\lambda_f^{(3)}(p^2) = \lambda_f(p^6) - \lambda_f(p^4) + \lambda_f(p^2) - 1,$$

$$\lambda_f^{(4)}(p^2) = \lambda_f(p^8) - \lambda_f(p^6) + \lambda_f(p^4) - \lambda_f(p^2) + 1.$$
Hence we should find substitutes for the entire functions $\Lambda(s, \text{Sym}^{2n}f)$ in order to use the above argument without assuming the cuspidality for $n = 3$ and $n = 4$. For this reason, we use Lemma 2.4 to obtain

$$
\frac{L'}{L}(s, \text{Sym}^6f) = \frac{L'}{L}(s, \text{Sym}^2f \times \text{Sym}^4f) - \frac{L'}{L}(s, \text{Sym}^4f)
$$

(2.24)

$$
-\frac{L'}{L}(s, \text{Sym}^2f),
$$

In this case we know by [19] that $L(s, \text{Sym}^4f \times \text{Sym}^2f)$ extends to an entire function since $\text{Sym}^4\pi_f$ and $\text{Sym}^2\pi_f$ are both cuspidal by [9, 20]. Hence we use (2.21) for the $L$-functions on the right to express $(L'/L)(s, \text{Sym}^6f)$ in a similar way. For $n = 4$, on the other hand, we obtain

$$
\frac{L'}{L}(s, \text{Sym}^8f) = \frac{L'}{L}(s, \text{Sym}^4f \times \text{Sym}^4f) - \zeta'(s)
$$

(2.25)

$$
-\frac{L'}{L}(s, \text{Sym}^4f \times \text{Sym}^2f).
$$

By [19] $L(s, \text{Sym}^4f \times \text{Sym}^4f)$ extends to an entire function with the exception of a simple pole at $s = 1$ and admits a functional equation which takes $s$ to $1 - s$. As the simple pole at $s = 1$ will be cancelled by that of the zeta function in (2.25), we can work in this case with the entire functions

$$
\tilde{\Lambda}(s, \text{Sym}^4f \times \text{Sym}^4f) = s(s - 1)\Lambda(s, \text{Sym}^4f \times \text{Sym}^4f),
$$

$$
\xi(s) = \frac{1}{2} s(s - 1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).
$$

Hence by the cuspidality assumption for $n > 4$ and the above arguments for $n = 3$ and $n = 4$ we can assume from now on that for any $n \geq 1$,

$$
-\frac{L'}{L}(s, \text{Sym}^{2n}f) = \sum_j \Psi\left(\frac{s}{2} + \mu_j\right) - \sum_\rho \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right) + O(1)
$$

(2.26)

where (2.23) holds. Note that for $s = 2 + it$,

$$
\sum_{|\gamma - t| \leq 1} 1 \ll \text{Re}\left(\sum_\rho \frac{1}{s - \rho} + \frac{1}{\rho}\right).
$$

(2.27)
By the Laurent expansion of $\Psi(s)$

$$
\Psi(s) = -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty} \frac{s}{n(n+s)},
$$

(2.28)

where $\gamma$ is the Euler constant, we see that for $\text{Re}(s) > 1/4$,

$$
\Psi(s) \ll \log(1 + |s|).
$$

(2.29)

Recalling that each $\mu_j$ in (2.26) depends on $k$ linearly and using (2.29) we obtain

$$
\sum_{|\gamma-t| \leq 1} 1 \ll \log(k + |t|).
$$

(2.30)

Replacing $s = \sigma + it$ by $2+it$ in (2.26) and subtracting we see that

$$
\frac{L'}{L}(s, \text{Sym}^2f) = \sum_{|\gamma-t| \leq 1} \frac{1}{s - \rho} + O(\log(k + |t|))
$$

(2.31)

which finishes the proof.

2.1.1 Petersson’s Formula

In this section we introduce the essential tools to handle the main term that appears after the explicit formula. For the discussion of the results below, we refer to [16].

We first introduce for any cusp form $f$ of weight $k$ for $\text{SL}_2(\mathbb{Z})$, the normalized coefficients

$$
\Psi_f(n) = \left( \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \right)^{1/2} \|f\|^{-1} \lambda_f(n)
$$

(2.32)

where $\|f\|$ is the norm of $f$ with respect to Petersson inner product. We define

$$
\Delta_k(m, n) = \sum_{f \in \mathcal{B}_k} \overline{\Psi_f(m)} \Psi_f(n)
$$

(2.33)

where $\mathcal{B}_k$ is any orthogonal basis for the finite dimensional space of cusp forms of weight $k$ for $\text{SL}_2(\mathbb{Z})$. This sum is basis independent, being the $n$-th Fourier coefficient of the $m$-th Poincaré series up to some normalizing factors.

Below we give have two equivalent descriptions for $\Delta_k(m, n)$, the first of which is due to Petersson [29]:
Proposition 2.6. For any \( m, n \geq 1 \) we have
\[
\Delta_k(m, n) = \delta(m, n) + 2\pi i^k \sum_{c=1}^{\infty} c^{-1} S(m, n; c) J_{k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right)
\]
where \( \delta(m, n) \) is the diagonal symbol, \( J_k(x) \) is the \( k \)th Bessel function and
\[
S(m, n; c) = \sum_{d \mod c}^* e\left( \frac{md + nd}{c} \right)
\]
is the classical Kloosterman sum. The symbol \( \sum^* \) implies the summation is over the primitive residue classes mod \( c \).

By [16, Lemma 2.7] we can also express \( \Delta_k(m, n) \) in terms of Hecke eigenvalues as follows:

Lemma 2.7. For any \( m, n \geq 1 \) we have
\[
\Delta_k(m, n) = \frac{12}{k-1} \sum_{f \in H_k} \lambda_f(n) \lambda_f(m) Z(1, f)
\]
where
\[
Z(s, f) = \sum_{m=1}^{\infty} \frac{\lambda_f(m^2)}{m^s} = \zeta(2s)^{-1} L(s, \text{Sym}^2 f).
\]

Lemma 2.8. For any \( x > 1 \) we have,
\[
Z(1, f) = \sum_{m \leq x} \frac{\lambda_f(m^2)}{m} + O \left( x^{-1/2} (kx)^{\epsilon} \right),
\]
where the implied constant depends only on \( \epsilon \).

Proof. By Cauchy's integral formula,
\[
Z(1, f) = \text{Res}_{s=0} \left\{ Z(s+1, f) \frac{x^s}{s} \right\} = \frac{1}{2\pi i} \int_{\gamma} Z(s+1, f) \frac{x^s}{s} ds,
\]
where \( \gamma \) is the rectangular contour with vertices at \( 1 \pm iT, a \pm iT \) where \( T > 1 \) and \( a = -1/2 + (\log x)^{-1} \). By Perron's formula with \( x \in (1/2) \mathbb{Z}^+ \) we have
\[
\frac{1}{2\pi i} \int_{1-iT}^{1+iT} Z(s+1, f) \frac{x^s}{s} ds = \sum_{m \leq x} m^{-1} \lambda_f(m^2)
\]
\[
+ O \left( \frac{x}{T} \sum_{m=1}^{\infty} \frac{1}{m^2 |\log \frac{x}{m}|} \right)
\]

where the error term is $O(x/T)$. By the Lindelöf hypothesis we find

$$
\frac{1}{2\pi i} \int_{1\pm iT}^{a\pm iT} Z(s+1,f) \frac{x^s}{s} \, ds \ll \frac{x(kT)^\epsilon}{T},
$$

$$
\frac{1}{2\pi i} \int_{a-iT}^{a+iT} Z(s+1,f) \frac{x^s}{s} \, ds \ll x^{-1/2}(kT)^\epsilon \log T.
$$

Finally choosing $T = x^{3/2}$ yields the desired result. 

**Corollary 2.9.** For $Y \geq 1$ we have

$$r_f(Y) = \frac{1}{Z(1,f)} \sum_{m > Y} \lambda_f(m^2) \frac{m}{m} \ll Y^{-1/2}(kY)^\epsilon. \quad (2.41)$$

**Proof.** By [12] we have

$$Z(1,f)^{-1} = \zeta(2) L(1, \text{Sym}^2 f)^{-1} \ll \log^2 k.$$

Hence the result follows by the above lemma.

### 2.1.2 Sums of Kloosterman Sums

This section is concerned with the computation of sums of Kloosterman sums that appear after Petersson’s formula and is essentially relevant only to the family of symmetric fourth power $L$-functions.

By the Riemann hypothesis for Dirichlet $L$-functions we have

$$\sum_{p \leq x} \chi(p) \log p = \delta_\chi x + O(x^{1/2} \log^2 cx) \quad (2.42)$$

where $\chi$ is any Dirichlet character to modulus $c$, and $\delta_\chi$ is the indicator of the principal character, the implied constant being absolute.

**Lemma 2.10.** For any $n \geq 1$, $x \geq 2$ and any integer $m$, we have

$$\sum_{p \leq x} S(m^2, p^{2n}; c) e\left(\frac{2mp^n}{c}\right) \log p = s_n(m; c)x + O\left(cx^{1/2}(cx)^\epsilon\right) \quad (2.43)$$

where

$$s_n(m; c) = \frac{1}{\varphi(c)} \sum_{a \mod c}^* S(m^2, a^{2n}; c) e\left(\frac{2ma^n}{c}\right). \quad (2.44)$$
Proof. For any integer \( m \) and any Dirichlet character \( \chi \) to modulus \( c \), we let
\[
T_m(\chi) = \sum_{a \mod c} \chi(a) S(m^2, a^{2n}; c) e\left(\frac{2ma^n}{c}\right).
\] 
(2.45)

In particular, \( T_m(\chi_0) = \varphi(c)s_n(m; c) \) for the principal character \( \chi_0 \). By (2.42) we have
\[
\sum_{p \leq x \atop p \nmid c} S(m^2, p^{2n}; c) e\left(\frac{2mp^n}{c}\right) \log p = \frac{1}{\varphi(c)} \sum_{\chi \mod c} T_m(\chi) \sum_{p \leq x} \chi(p) \log p
\]
\[
= s_n(m; c) x + O\left(\frac{x^{1/2}(cx)^{\epsilon}}{\varphi(c)} \sum_{\chi \mod c} |T_m(\chi)|\right).
\]

Note that
\[
\sum_{\chi \mod c} |T_m(\chi)|^2 = \sum_{a, b \mod c}^* S(m^2, a^{2n}; c) S(m^2, b^{2n}; c)
\cdot e\left(\frac{2m(a^n - b^n)}{c}\right) \sum_{\chi \mod c} \chi(ab^{-1})
\]
\[
= \varphi(c) \sum_{a \mod c}^* S(m^2, a^{2n}; c)^2 \leq \varphi(c)^2 c \tau(c)^2,
\]
where \( \tau(c) \) is the number of divisors of \( c \) and \( \sum^* \) denotes the summation over primitive residue classes modulo \( c \). By the Cauchy-Schwartz inequality,
\[
\sum_{\chi \mod c} |T_m(\chi)| \leq \varphi(c)^{1/2} \left(\sum_{\chi} |T_m(\chi)|^2\right)^{1/2} \ll \varphi(c) c \tau(c).
\] 
(2.46)

Finally, the contribution from prime divisors of \( c \) is \( O(c\tau(c)^2 \log x) \). Hence the proof follows.

Next we shall compute \( s_n(m; c) \) locally for \( n = 2 \). By (2.35) we have
\[
s_n(m; c) = \frac{1}{\varphi(c)} \sum_{a, d \mod c}^* e_c \left(m^2d + a^{2n}d + 2ma^n\right)
\] 
(2.47)

where \( e_c(x) = e^{2\pi ix/c} \). By the change of variable \( d \to d\overline{a}^n \) we get
\[
s_n(m; c) = \frac{1}{\varphi(c)} \sum_{a, d \mod c}^* e_c \left(a^n\overline{d}(m + d)^2\right)
\] 
(2.48)

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where \( \overline{d} \) is the inverse of \( d \) modulo \( c \). In particular for \( n = 2 \), we have

\[
\begin{align*}
\sum_{x \equiv d \pmod{c}} y \equiv -d \pmod{c} \text{ for } (c,d) = 1
\end{align*}
\]

**Lemma 2.11.** For \((c,d) = 1\), we have

\[
s(m; cd) = s(md; c) s(mc; d).
\]

**Proof.** By (2.44) we have

\[
\begin{align*}
s(m; cd) &= \frac{1}{\varphi(cd)} \sum_{x \mod c}^{*} S\left(m, (xd + yc)^{2}; cd\right) e_{cd}(2m(xd + ye)^{2})
\end{align*}
\]

where the second equality follows for \((c,d) = 1\) by (see [18, p.59])

\[
S(m, n; cd) = S(md, n\overline{d}; c) S(m\overline{c}, n\overline{c}; d)
\]

\[
S(dm, n; c) = S(m, dn; c).
\]

\[\square\]

**Proposition 2.12.** If \((c, 2n) = 1\) then

\[
\sum_{t \mod c} e_{c}(nt^{2}) = \left(\frac{n}{c}\right) \epsilon_{c} \sqrt{c}
\]

where for any positive integer \( c \),

\[
\epsilon_{c} = \frac{1 + e_{4}(-c)}{1 - i}.
\]

We refer to [18, p.66] for the proof of the above proposition.

**Lemma 2.13.** For any prime \( p \), we have

\[
s(m; p) = \begin{cases} 
\varphi(p)^{-1} \left( 1 - \epsilon_{p} \sqrt{p} \left( \frac{-m}{p} \right) \right) & \text{if } p \nmid m, \\
-1 & \text{if } p \mid m,
\end{cases}
\]

where \((\cdot/\cdot)\) is the Legendre-Jacobi symbol.
Proof. Note that \( s(m; 2) = \mu((m, 2)) \). For an odd prime \( p \nmid m \), we have

\[
\varphi(p) s(m; p) = \sum_{a \mod p}^* \left( 1 + \sum_{d \mod p \atop p \nmid d + m} e_p \left( a^2 d(d + m)^2 \right) \right)
\]

\[
= \varphi(p) + \sum_{d \mod p \atop p \nmid d + m}^* \left( \left( \frac{d}{p} \right) e_p \sqrt{p} - 1 \right)
\]

\[
= 1 - e_p \sqrt{p} \left( -\frac{m}{p} \right)
\]

which follows by proposition (2.12). Similarly for \( p \mid m \), we obtain

\[
\varphi(p) s(m; p) = \sum_{d \mod p}^* \left( \sum_{a \mod p}^* e_p \left( a^2 d^3 - 1 \right) \right)
\]

\[
= e_p \sqrt{p} \sum_{d \mod p}^* \left( \frac{d}{p} \right) - \varphi(p).
\]

which finishes the proof. \( \square \)

**Corollary 2.14.** For square-free \( c \) with \((c, m) = 1\), we have

\[
s(m; c) \leq 2^{\nu(c)} \frac{\sqrt[c]{c}}{\varphi(c)}
\]

(2.56)

where \( \nu(c) \) is the number of distinct prime factors of \( c \).

**Proof.** By Lemma 2.11 we have

\[
s(m; c) = \prod_{p \mid c} s \left( \frac{mc}{p}; p \right) \leq \prod_{p \mid c} 2^{\sqrt[p]{p}} \varphi(p).
\]

(2.57)

where the inequality follows by the above lemma. \( \square \)

**Lemma 2.15.** We have

\[
s(m; c) = \mu(c)
\]

(2.58)

for those \( c \) such that \( c \mid m \) where \( \tilde{c} = \prod_{p \mid c} p \) is the square-free part of \( c \).
Proof. For \( c = p^n \), \( n \geq 2 \) and \( p \mid m \) we have
\[
\varphi(c) \ s(m; c) = \sum_{d \mod c} \sum_{x \mod p^{n-1}}^{*} \ e_{p^n} \left( x^2 d(d + m)^2 \right) \\
\quad \cdot \sum_{y \mod p} e_p \left( 2xyd(d + m)^2 \right) \\
= 0.
\] (2.59)

Similarly for \( c = 2^n \), \( n \geq 4 \) and \( 2 \mid m \) we obtain
\[
\varphi(c) \ s(m; c) = \sum_{d \mod c} \sum_{x \mod 2^{n-2}}^{*} \ e_{2^n} \left( x^2 d(d + m)^2 \right) \\
\quad \cdot \sum_{y \mod 4} e_2 \left( xyd(d + m)^2 \right) \\
= 0.
\] (2.60)

Finally for \( 2 \mid m \), we have
\[
s(m; 4) = e_4 \left( (m + 1)^2 \right) + e_4 \left( 3(m - 1)^2 \right) = 0,
\]
\[
s(m; 8) = \sum_{d \mod 8}^{*} e_8(d) = 0.
\]

Hence the result follows from Lemma 2.11 and Lemma 2.13. \( \square \)

Remark 2.1. We shall consider only those \( c \) for which \( \gcd\left( \frac{c}{(m,c)}, m \right) = 1 \) since otherwise \( s(m; c) = 0 \) by the above lemma.

2.2 Hecke L-functions Associated With Cubic Characters

Recall that our base field is the imaginary quadratic field \( K = \mathbb{Q}(\omega) = \mathbb{Q}(\sqrt{-3}) \) where \( \omega = e^{2\pi i/3} \) is the third root of unity. It is well-known that the the ring of integers of \( K \) is \( \mathcal{O}_K = \mathbb{Z}[\omega] \) and it is a Euclidean domain with the norm given by
\[
N(a + b\omega) = a^2 - ab + b^2
\] (2.61)
for any integer \( a + b\omega \in \mathcal{O}_K \). The units of \( K \) are \( \pm \omega^i \) for \( i = 1, 2, 3 \). The discriminant of \( K \) is \( d_K = -3 \). Hence 3 is the only rational prime that ramifies in \( K \). In fact,
3 = -\omega^2(1 - \omega)^2 and if \( p \equiv 1 \mod 3 \) is a rational prime, then \( p = \pi \pi' \) where \( \pi \) is a prime in \( \mathcal{O}_K \), while if \( p \equiv 2 \mod 3 \) is a rational prime, then it is prime in \( \mathcal{O}_K \) as well.

One can easily show using the units of \( K \) that every ideal coprime to 3 has unique generator congruent to 1 modulo 3.

Given \( \alpha \in \mathcal{O}_K \), for any prime \( \pi \) with \( N\pi \neq 3 \), the cubic residue character of \( \alpha \) modulo \( \pi \) is determined by

\[
\left( \frac{\alpha}{\pi} \right)_3 \equiv \alpha^{(N\pi-1)/3} \mod (\pi)
\]

if \( \pi \nmid \alpha \) and it equals zero otherwise. We can then extend this definition to any ideal \( a \neq (1) \), coprime to 3, by

\[
\left( \frac{\alpha}{a} \right)_3 = \prod_{i=1}^{n} \left( \frac{\alpha}{\pi_i} \right)_{3}^{e_i}
\]

where \( a = \prod_{i=1}^{n} (\pi_i)^{e_i} \). From now on we denote this character by \( \chi_a \) for \( a = (a) \).

We call an integer \( \pi \in \mathcal{O}_K \) primary if \( \pi \) is prime and \( \pi \equiv 1 \mod 3 \). Below we state the Cubic Reciprocity Theorem and Eisenstein’s Supplement Theorem to the cubic reciprocity law for whose proofs we refer to [17].

**Theorem 2.16.** Let \( \pi_1 \) and \( \pi_2 \) be primary and \( N\pi_1 \neq N\pi_2 \). Then

\[
\chi_{\pi_1}(\pi_2) = \chi_{\pi_2}(\pi_1)
\]

**Theorem 2.17.** Let \( \pi \) be a prime, \( N(\pi) \neq 3 \). If \( \pi \) is rational, write \( \pi = 3m - 1 \). If \( \pi = a + b\omega \) is a primary complex prime, write \( -a = 3m - 1 \). Then

\[
\chi_{\pi}(1 - \omega) = \omega^{2m}.
\]

For any square-free \( c \neq 1 \) such that \( c \equiv 1 \mod 9 \), \( \chi_c \) is trivial on the units of \( K \). Hence it can be regarded as a primitive Dirichlet character modulo \( (c) \), a character of the ray class group \( J^{(c)}/P^{(c)} \) where \( J^{(c)} \) is the group of all ideals relatively prime to \( (c) \) and \( P^{(c)} \) is the subgroup of all principal ideals \( (a) \) with \( a \equiv 1 \mod (c) \).
As in the previous section we shall first estimate the logarithmic derivative of \( L(s, \chi_c) \) given by (1.40) and then use Perron’s formula together with the generalized Riemann Hypothesis to estimate the sum

\[
\sum_{Np \leq x} \chi_c(p) \log Np \tag{2.66}
\]

which will appear after the explicit formula in Chapter 4.

**Lemma 2.18.** For \( \text{Re}(s) = \sigma > 1/2 \), we have

\[
\frac{L'(s, \chi_c)}{L(s, \chi_c)} \ll \frac{\log(N(c)(3 + |t|))}{\sigma - \frac{1}{2}}. \tag{2.67}
\]

**Proof.** We know \( \Lambda(s, \chi_c) \) is entire of order 1 and hence it can be written as

\[
\Lambda(s, \chi_c) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho} + \frac{1}{\rho} \right) e^{s/\rho} \tag{2.68}
\]

where \( \rho \) runs through the non-trivial zeros of \( L(s, \chi_c) \). By logarithmic differentiation of (1.42) we obtain

\[
-\frac{L'(s, \chi_c)}{L(s, \chi_c)} = \frac{\Gamma'(s)}{\Gamma(s)} + \log \sqrt{N(c)} - \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) + O(1). \tag{2.69}
\]

Using the same idea as in the proof of Lemma 2.5 we find

\[
\frac{L'(s, \chi_c)}{L(s, \chi_c)} = \sum_{|\gamma - t| \leq 1} \frac{1}{s - \rho} + O\left( \log N(c)(3 + |t|) \right). \tag{2.70}
\]

\[\square\]

**Lemma 2.19.** Under the generalized Riemann Hypothesis we have for \( x \in (1/2)\mathbb{Z}^+ \) and \( x > 1 \),

\[
\sum_{Np \leq x} \chi_c(p) \log Np \ll x^{1/2} \log^3 x \log N(c) \tag{2.71}
\]

**Proof.** By Perron’s formula we have for \( a = 1 + (\log x)^{-1} \),

\[
\frac{1}{2\pi i} \int_{a-iT}^{a+iT} -\frac{L'(s, \chi_c)}{L(s, \chi_c)} \frac{x^s}{s} ds = \sum_{Np \leq x} \chi_c(p) \log Np + O(x^{1/2} \log x) \tag{2.72}
\]

\[+ O\left( \frac{x}{T} \sum_a \frac{\Lambda(a)}{(Na)^{x/\log Na}} \right) \]

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where
\[
\Lambda(a) = \begin{cases} 
\log Np & \text{if } a = p^k, \\
0 & \text{if otherwise.}
\end{cases}
\] (2.73)

For \(Na \leq x/2\), we see that
\[
\sum_{Na \leq x/2} \frac{\Lambda(a)}{(Na)^a|\log \frac{x}{Na}|} \ll \log x,
\] (2.74)

while for \(x/2 < Na \leq x\), we use
\[
\left|\log \frac{x}{Na}\right| = -\log(1 - \frac{x - Na}{x}) \geq \frac{x - Na}{x}
\] (2.75)

which yields
\[
\sum_{\frac{x}{2} < Na \leq x} \frac{\Lambda(a)}{(Na)^a|\log \frac{x}{Na}|} \ll \sum_{\frac{x}{2} < Na \leq x} \frac{\Lambda(a)}{(Na)^{a-1}(x - Na)} \ll \log^2 x.
\] (2.76)

We get the same bounds similarly for \(Na \geq 3x/2\) and for \(x < Na < 3x/2\), respectively.

We shift the contour for the integral in (2.72) to \(b = 1/2 + (\log x)^{-1}\) and then use (2.67) to get
\[
\int_{b+iT}^{a+iT} \frac{L'(s, \chi_c) x^s}{s} ds \ll \frac{x \log x \log(TN(c))}{T},
\]
\[
\int_{b-iT}^{b+iT} \frac{L'(s, \chi_c) x^s}{s} ds \ll x^{1/2} \log x \log^2 T \log N(c).
\]

Finally, we obtain the result by choosing \(T = x^{1/2}\).

2.2.1 Twisted Poisson Summation

We shall prove an analog of Poisson Summation Formula on \(K\) with a twist by a primitive character to allow larger support for the test functions. For our computations, we shall only be interested in
\[
g(x, y) = e^{-A(x^2 - xy + y^2)}
\] (2.77)

where \(A > 0\) will be chosen later.
Lemma 2.20. The Fourier transform of \( g(x,y) \) is

\[
\hat{g}(a,b) = \frac{2\pi}{\sqrt{3A}} e^{-\frac{4\pi^2}{3A} (a^2 + ab + b^2)}.
\]

(2.78)

Proof. The Fourier transform of \( h(x) = e^{-tx^2} \) equals

\[
\hat{h}(y) = \sqrt{\frac{\pi}{t}} h\left(\frac{\pi y}{t}\right).
\]

(2.79)

Hence using this equation we see that

\[
\hat{g}(a,b) = \int_{\mathbb{R}^2} e^{-A(x^2 - xy + y^2)} e^{-2\pi i (x,y) \cdot (a,b)} \, dx \, dy
\]

which yields the claimed result. \( \square \)

Lemma 2.21. For any primitive character \( \chi \) of conductor \((f_\chi)\), we have

\[
\sum_{c \in \mathcal{O}_K} \chi(c) e^{-AN(c)} = 2\pi \chi(-\omega) \frac{\sqrt{3AW(\chi)}}{A} \sum_{c \in \mathcal{O}_K} \overline{\chi(c)} e^{-\frac{4\pi^2 N(c)}{\sqrt{3AN(\chi)}}} (2.80)
\]

where \( W(\chi) \) is the Gauss sum

\[
W(\chi) = \sum_{a \mod (f_\chi)} \chi(a) e^{\left(\frac{\text{Tr}(a)}{\delta f_\chi}\right)}.
\]

(2.81)

Proof. We let for any \( z \in \mathbb{C} \),

\[
e_T(z) = e^{2\pi i \text{Tr}(z)}.
\]

(2.82)

As \( \chi \) is primitive, we can replace \( \chi(c) \) in (2.80) by

\[
\chi(c) = \frac{1}{W(\chi)} \sum_{d \mod (f_\chi)} \overline{\chi(d)} e_T\left(\frac{dc}{f_\chi \delta}\right)
\]

(2.83)
and obtain
\[
\sum_{c \in \mathcal{O}_K} \chi(c) e^{-AN(c)} = \frac{1}{W(\chi)} \sum_{d \mod (f\chi)} \sum_{c \in \mathcal{O}_K} \chi(d) e_{T} \left( \frac{dc}{f\chi\delta} \right) e^{-AN(c)}
\] (2.84)
where the innermost sum over \(c\) can be written as
\[
\sum_{c \in \mathcal{O}_K} e_{T} \left( \frac{dc}{f\chi\delta} \right) e^{-AN(c)} = \sum_{n,m \in \mathbb{Z}} h(n,m).
\] (2.85)

Here for any \(x, y\) in \(\mathbb{R}\),
\[
h(x, y) = e_{T} \left( \frac{d}{f\chi\delta} (x + y \omega) \right) g(x, y).
\] (2.86)

Choosing \(r\) and \(s\) in \(\mathbb{Q}\) such that
\[
d \frac{f}{\chi} = r + s(1 + \omega)
\] (2.87)
we see that
\[
h(x, y) = e^{2\pi i (x,y) \cdot (s,r)} g(x, y).
\] (2.88)

By the two dimensional Poisson Summation formula we get
\[
\sum_{n,m \in \mathbb{Z}} h(n,m) = \sum_{a,b \in \mathbb{Z}} \hat{h}(-a, -b)
\] (2.89)
where by (2.86)
\[
\hat{h}(-a, -b) = \int_{\mathbb{R}^2} h(x, y) e^{2\pi i (x,y) \cdot (a,b)} dx dy
\] (2.90)
\[
= \int_{\mathbb{R}^2} g(x, y) e^{2\pi i (x,y) \cdot (a+s, b+r)} dx dy
\]
\[
= \hat{g}(a + s, b + r).
\]

Hence from Lemma 2.20 and equation (2.87) we find
\[
\sum_{n,m \in \mathbb{Z}} h(n,m) = \sum_{a,b \in \mathbb{Z}} \hat{g}(a + s, b + r)
\] (2.91)
\[
= \frac{2\pi}{\sqrt{3}A} \sum_{c \in \mathcal{O}_K} e^{-\frac{4\pi^2}{3A} N(c - \omega(r+s(1+\omega)))}
\]
\[
= \frac{2\pi}{\sqrt{3}A} \sum_{c \in \mathcal{O}_K} e^{-\frac{4\pi^2}{3A} N\left(\frac{cf\chi - \omega d}{\chi}\right)}.
\]
Finally, inserting (2.91) into (2.84) we obtain

\[
\sum_{c \in \mathcal{O}_K} \chi(c) e^{-AN(c)} = \frac{2 \pi \chi(-\omega)}{\sqrt{3} AW(x)} \sum_{d \mod (f_x)} \chi(d) \sum_{c \in \mathcal{O}_K} e^{-\frac{4 \pi^2 N(c_f c - d)}{3A N(f_x)}}
\]

\[
= \frac{2 \pi \chi(-\omega)}{\sqrt{3} AW(x)} \sum_{c \in \mathcal{O}_K} \chi(c) e^{-\frac{4 \pi^2 N(c)}{3A N(f_x)}}, \tag{2.92}
\]

which proves the lemma. \[\square\]
CHAPTER 3

SYMMETRIC POWER L-FUNCTIONS ASSOCIATED WITH CUSP FORMS ON GL(2)

3.1 Explicit Formula

In this chapter we shall prove the main theorems stated in introduction. Following the same outline as in [16] we first derive the explicit formula to express

$$D(\text{Sym}^r f; \phi) = \sum_{\rho_f} \phi\left(\frac{\gamma_f}{2\pi} \log R\right)$$

(3.1)

which was defined in (1.21) as multiple sums over primes.

Lemma 3.1. Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}(y)$ has compact support. Then for any $f \in H_k$, we have

$$D(\text{Sym}^r f; \phi) = \hat{\phi}(0) \frac{\log c_{\text{Sym}^r f}}{\log R} + (-1)^{r+1} \frac{\phi(0)}{2} + O\left(\frac{1}{\log R}\right)$$

(3.2)

$$- \sum_p \lambda_f(p^r) \hat{\phi}\left(\frac{\log p}{\log R}\right) \frac{2 \log p}{\sqrt{p} \log R}$$

$$- \sum_p \left(\sum_{n=0}^{r-1} (-1)^n \lambda_f(p^{2r-2n})\right) \hat{\phi}\left(\frac{2 \log p}{\log R}\right) \frac{2 \log p}{p \log R}$$

where the implied constant depends only on the test function $\phi$.

Proof. We define

$$G(s) := \phi\left(\left(s - \frac{1}{2}\right) \frac{\log R}{2\pi i}\right).$$

(3.3)
It is entire in the strip \(-1 \leq \text{Re}(s) \leq 2\) and satisfies

\[
G(s) = G(1 - s),\quad (3.4)
\]

\[
G(s) \ll s^{-2}.
\]

Let \(\rho_f = \frac{1}{2} + i\gamma_f\) run over the non-trivial zeros of \(\Lambda(s, \text{Sym}^r f)\) counted with multiplicities. By Cauchy’s theorem for integrals and the functional equations (1.16) and (3.4),

\[
\sum_{\rho} G(\rho_f) = \frac{1}{2\pi i} \int_{(2)} 2G(s) \frac{\Lambda'(s, \text{Sym}^r f)}{\Lambda(s, \text{Sym}^r f)} ds.
\]

(3.5)

Using (2.13) we get

\[
D(\text{Sym}^r f; \phi) = \sum_{j=0}^{r} F_j - 2 \sum_{n=1}^{\infty} \lambda_f^{(r)}(n) F(n) \Lambda(n) + O\left(\frac{1}{\log R}\right)
\]

(3.6)

where \(F(y)\) is the inverse Mellin transform of \(G(s)\) given by

\[
F(y) = \frac{1}{2\pi i} \int_{(\frac{1}{2})} G(s) y^{-s} ds = \frac{1}{\sqrt{y} \log R} \hat{\phi}\left(\frac{\log y}{\log R}\right)
\]

(3.7)

and

\[
F_j = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \Psi\left(\frac{s}{2} + \mu_j\right) G(s) ds
\]

(3.8)

\[
= \frac{1}{\log R} \int_{-\infty}^{\infty} \Psi\left(\frac{1}{4} + \mu_j + \frac{i\pi y}{\log R}\right) \phi(y) dy.
\]

(3.9)

By the approximate formula (see (8.363.3) of [GR])

\[
\Psi(a + ib) + \Psi(a - ib) = 2\Psi(a) + O\left(\frac{b^2}{a^2}\right)
\]

we obtain

\[
F_j = \hat{\phi}(0) \Psi(1/4 + \mu_j) + O\left(\frac{1}{\log^3 R}\right).
\]

(3.11)

We insert this back in (3.6) and then use

\[
\Psi(\alpha + 1/4) = \log \alpha + O(1), \quad \text{for } \alpha \geq 1/4
\]
to get
\[
\sum_{j=0}^{r} F_j = \hat{\phi}(0) \frac{\log c_{\text{Sym}}}{\log R} + O\left(\frac{1}{\log R}\right)
\]  
(3.12)

where \(c_{\text{Sym}}\) is given by (1.19). The second term in (3.6) yields
\[
\sum_{n=1}^{\infty} \lambda_f(r)(n)F(n)\Lambda(n) = \sum_{n=1}^{\infty} \sum_{p} \lambda_f(r)(p^n) \hat{\phi}\left(\frac{n \log p}{\log R}\right) \frac{\log p}{p^{n/2} \log R}
\]
\[
= \sum_{p} \lambda_f(r)(p) \hat{\phi}\left(\frac{\log p}{\log R}\right) \frac{\log p}{\sqrt{p} \log R} + O\left(\frac{1}{\log R}\right)
\]
\[
+ \sum_{p} \lambda_f(r)(p^2) \hat{\phi}\left(\frac{2 \log p}{\log R}\right) \frac{\log p}{p \log R}
\]

where \(\lambda_f(r)(p^n)\) is given by (2.15).

Note that \(\lambda_f(r)(p) = \lambda_f(p^r)\). Moreover, by Lemma 2.3 we have
\[
\lambda_f(r)(p^2) = \sum_{n=0}^{r-1} (-1)^n \lambda_f(p^{2r-2n}) + (-1)^r.
\]  
(3.13)

Finally, we use the prime number theorem for
\[
\sum_{p} \hat{\phi}\left(\frac{2 \log p}{\log R}\right) \frac{2 \log p}{p \log R} = \int_{2}^{\infty} \hat{\phi}\left(\frac{2 \log t}{\log R}\right) \frac{2 \log t}{t \log R} d\pi(t) + O\left(\frac{1}{\log R}\right). 
\]  
(3.14)

By the Riemann Hypothesis,
\[
\pi(t) = \int_{2}^{t} \frac{1}{\log v} dv + O\left(t^{1/2} \log t\right) 
\]  
(3.15)

for \(t \geq 2\). Hence we get
\[
\sum_{p} \hat{\phi}\left(\frac{2 \log p}{\log R}\right) \frac{2 \log p}{p \log R} = \frac{1}{2} \hat{\phi}(0) + O\left(\frac{1}{\log R}\right)
\]  
(3.16)

which completes the proof. \(\square\)

Next, we estimate the last term in (3.2), namely the sum
\[
\frac{2}{\log R} \sum_{n=0}^{r-1} (-1)^n. \mathcal{S}(2r-2n)
\]  
(3.17)

where
\[
\mathcal{S}(n) = \sum_{p} \lambda_f(p^n) \hat{\phi}\left(\frac{2 \log p}{\log R}\right) \frac{\log p}{p}.
\]  
(3.18)
Assuming the cuspidality of $\text{Sym}^r \pi_f$ for $r > 4$ we shall prove that for any $n \geq 1$,
\begin{equation}
S(n) \ll \log \log k. \tag{3.19}
\end{equation}

**Lemma 3.2.** Under the generalized Riemann Hypothesis, we have for $x > 1$ and $k \geq 2$,
\begin{equation}
\sum_{p \leq x} \lambda_f(p^r) \log p \ll x^{1/2+\epsilon} \log k. \tag{3.20}
\end{equation}

**Proof.** We apply Perron’s formula (Lemma 2.1) to
\begin{equation}
\frac{L'}{L}(s, \text{Sym}^r f) = -\sum_{n=1}^{\infty} \Lambda_f^{(r)}(n) \Lambda(n) n^{-s} \tag{3.21}
\end{equation}
with $x \in (1/2)\mathbb{Z}^+$ to get
\begin{equation}
\sum_{p \leq x} \lambda_f(p^r) \log p = \int_{c-iT}^{c+iT} -\frac{L'}{L}(s, \text{Sym}^r f) \frac{x^s}{s} \, ds + O(x^{1/2} \log x) \tag{3.22}
\end{equation}
\begin{equation*}
+ O \left( \sum_{n=1}^{\infty} \left( \frac{x}{n} \right)^c \Lambda(n) \frac{T}{\log \frac{x}{n}} \right).
\end{equation*}

For $c = 1 + (\log x)^{-1}$, we obtain by [3, p. 107]
\begin{equation}
\sum_{n=1}^{\infty} \left( \frac{x}{n} \right)^c \Lambda(n) \frac{T}{\log \frac{x}{n}} \ll \frac{x \log^2 x}{T}. \tag{3.23}
\end{equation}

Shifting the contour to $b = 1/2 + (\log x)^{-1}$ and using (2.20) we get
\begin{equation}
\sum_{p \leq x} \lambda_f(p^r) \log p \ll x^{1/2} \log x + \frac{x \log^2 x}{T} + \frac{x \log x}{T} \log(k + T) \tag{3.24}
\end{equation}
\begin{equation*}
+ x^{1/2} \log x \log(k + T) \log T.
\end{equation*}

Finally, choosing $T = x^{1/2}$ yields the result. \qed

**Corollary 3.3.** For any $n \in \mathbb{Z}^+$, (3.19) holds.

**Proof.** Using integration by parts we get
\begin{equation}
S(n) = -\int_1^\infty \sum_{p \leq t} \lambda_f(p^n) \log p \, d \left\{ \delta \left( 2 \log \frac{t}{\log R} \right) \frac{1}{t} \right\}
\end{equation}
\begin{equation*}
\ll \int_1^{\log^2 k} t^{-1} dt + \int_{\log^2 k}^{\infty} \left( \sum_{p \leq t} \lambda_f(p^n) \log p \right) t^{-2} dt.
\end{equation*}

Hence the result follows from Lemma 3.2. \qed
3.2 Averaging Over The Cusp Forms

We have shown so far that for any even test function $\phi(x)$ whose Fourier transform has compact support, $R \asymp c_{\text{Sym}} f$ and any $f \in H_k$, we have

$$D(\text{Sym}^r f; \phi) = \hat{\phi}(0) + (-1)^{r+1} \frac{\phi(0)}{2} + O\left(\frac{\log \log k}{\log k}\right) \quad (3.25)$$

$$- \sum_p \lambda_f(p^r) \hat{\phi}\left(\frac{\log p}{\log R}\right) \frac{2 \log p}{\sqrt{p} \log R}.$$

Next, we average over the cusp forms. In fact, by (3.25) and Lemma 2.2 we obtain

$$\sum_{f \in H_k} D(\text{Sym}^r f; \phi) = \left(\hat{\phi}(0) + (-1)^{r+1} \frac{\phi(0)}{2}\right)|H_k| - \mathcal{P}_R(\phi) \quad (3.26)$$

$$+ O\left(k \frac{\log \log k}{\log k}\right)$$

where

$$\mathcal{P}_R(\phi) = \sum_p \left(\sum_{f \in H_k} \lambda_f(p^r)\right) \hat{\phi}\left(\frac{\log p}{\log R}\right) \frac{2 \log p}{\sqrt{p} \log R}. \quad (3.27)$$

As $\hat{\phi}(y)$ has support in $(-v, v)$, one can always find a $v'$ with $0 < v' < v$ such that only the primes with $p \leq P = R^{v'}$ contribute to the above sum. Next, we use Petersson formula to handle $\mathcal{P}_R(\phi)$.

Lemma 3.4. For any $Y \geq 1$, we have

$$\mathcal{P}_R(\phi) = \frac{k-1}{12} \sum_{m \leq Y} m^{-1} \sum_c c^{-1} Q_k(m; c) + O\left(\frac{k}{\log k}\right) \quad (3.28)$$

$$+ O\left(\frac{k(kY)^\epsilon}{\sqrt{Y \log k}}\right)$$

where

$$Q_k(m; c) = \frac{4\pi i^k}{\log R} \sum_p S(m^2, p^r; c) J_{k-1}\left(\frac{4\pi m p^r/2}{c}\right) \hat{\phi}\left(\frac{\log p}{\log R}\right) \frac{\log p}{\sqrt{p}} \sqrt{p}. \quad (3.29)$$
Proof. We have

\[
\sum_{f \in H_k} \lambda_f(p^r) = \sum_{f \in H_k} \frac{\lambda_f(p^r)}{Z(1, f)} \sum_{m \leq Y} \lambda_f(m^2) m^{-1} + \sum_{f \in H_k} \lambda_f(p^r) r_f(Y)
\]

(3.30)

where \(r_f(Y)\) is given by (2.41). We substitute the second term on the right back into (3.27) and then use Corollary 2.9 and Lemma 2.2 to get for sufficiently small \(\epsilon\),

\[
\frac{1}{\log R} \sum_{f \in H_k} r_f(Y) \sum_{p \leq P} \lambda_f(p^r) \hat{\phi}\left(\frac{\log p}{\log R}\right) \frac{\log p}{\sqrt{p}} \ll \frac{k(kY)^{\epsilon}}{\sqrt{Y} \log k}.
\]

(3.31)

As for the first term on the right of (3.30), we apply Lemma 2.7 and Petersson formula 2.6 and we obtain

\[
\sum_{m \leq Y} m^{-1} \sum_{f \in H_k} \lambda_f(p^r) \lambda_f(m^2) Z(1, f) = \frac{k - 1}{12} \sum_{m \leq Y} m^{-1} \Delta_k(m^2, p^r)
\]

(3.32)

\[
= \frac{k - 1}{12} \Delta_{p^{r/2}} + \frac{k - 1}{12} \sum_{m \leq Y} m^{-1} 2\pi i^k
\]

\[
\cdot \sum_{c=1}^{\infty} c^{-1} S(m^2, p^r; c) J_{k-1}\left(\frac{4\pi mp^{r/2}}{c}\right)
\]

where the first term is present only if \(p^r = m^2\) with \(m \leq Y\) and it contributes \(O\left(\frac{k}{\log R}\right)\) to (3.27) for any \(r > 1\). Hence the result follows. \(\square\)

We are finally ready to prove Theorem 1.1. We first estimate (3.29) using

\[
|S(m, n; c)| \ll (m, n, c)^{1/2} \tau^{1/2}(c)
\]

(3.33)

for the Kloosterman sum, which is essentially due to A. Weil (see [18]), and

\[
J_{k-1}(x) \ll 2^{-k} x \quad \text{if } k \geq 2, \quad 0 < x \leq \frac{k}{3},
\]

(3.34)

for the Bessel function (see [35]). Using these two estimates we obtain

\[
Q_k^r(m; c) \ll 2^{-k} P^{(r+1)/2} \epsilon^{-1/2} m^2
\]

(3.35)

which in turn yields

\[
\mathcal{R}_k^r(\phi) \ll 2^{-k} kY^2 P^{(r+1)/2} + \frac{k(kY)^{\epsilon}}{\sqrt{Y} \log k} + \frac{k}{\log k}.
\]

(3.36)
The condition on $x = 4\pi mp^{r/2}/c$ in (3.34) is satisfied if we assume

$$12\pi Y p^{r/2} \leq k.$$  \hfill (3.37)

Choosing $Y = k^{3\epsilon}$ and $\epsilon$ sufficiently small we get

$$\mathcal{R}_k(\phi) \ll \frac{k}{\log k}$$  \hfill (3.38)

and (3.37) translates into the claimed restriction on the support of $\hat{\phi}$ which proves Theorem 1.1.

### 3.3 Averaging Over The Weight

In this section, we average over the weight as well to allow a larger support for $\hat{\phi}$. More precisely, we fix a smooth real valued function $h$ compactly supported on $\mathbb{R}^+$ and evaluate as $K \to \infty$

$$\mathcal{A}_r(K) = \sum_{2|k} \frac{24}{k-1} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} D(Sym^r f; \phi)$$  \hfill (3.39)

normalized by

$$A_r(K) = \sum_{2|k} \frac{24}{k-1} h\left(\frac{k-1}{K}\right) |H_k|.$$  \hfill (3.40)

By Lemma 2.2 we obtain

$$A_r(K) = K \hat{h}(0) + O(K^{2/3}).$$  \hfill (3.41)

Inserting (3.26) into (3.39) gives

$$\mathcal{A}_r(K) = A_r(K) \left(\hat{\phi}(0) + (-1)^{r+1} \frac{\phi(0)}{2}\right) - \mathcal{R}_r(\phi)$$  \hfill (3.42)

$$+ O\left(K \frac{\log \log K}{\log K}\right)$$

where

$$\mathcal{R}_r(\phi) = \sum_{2|k} \frac{24}{k-1} h\left(\frac{k-1}{K}\right) \mathcal{R}_k(\phi).$$  \hfill (3.43)
Choosing \( Y = K^{3\epsilon} \) and \( \epsilon \) sufficiently small in Lemma 3.4 we obtain

\[
\mathcal{P}_r(\phi) = \frac{4\pi}{\log R} \sum_{m \leq Y} m^{-1} \sum_p \hat{\phi}(\log p) \frac{\log p}{\sqrt{p}} \cdot \sum_c c^{-1} S(m^2, p^r; c) I\left(\frac{4\pi m p^{r/2}}{c}\right)
\]

where \( I(x) \) is the sum of Bessel functions

\[
I(x) = \sum_{k=1}^{2|k|} 2i^k h\left(\frac{k-1}{K}\right) J_{k-1}(x).
\]

**Lemma 3.5.** For \( K \geq 1 \) and \( x > 0 \), we have

\[
I(x) = -\frac{K}{\sqrt{x}} \text{Im} \left\{ \xi_8 e^{ix} h\left(\frac{K^2}{2x}\right) \right\} + O\left(\frac{x}{K^4}\right)
\]

where \( \xi_8 = e^{2\pi i/8} \) and

\[
h(v) = \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} e^{\text{iv}uv} dv.
\]

**Proof.** By [10] we have

\[
J_n(x) = \int_{-1/2}^{1/2} e^{ix\sin \pi t} e(-nt) dt,
\]

which gives the Fourier series representation for

\[
e^{ix\sin \pi t} = \sum_{n \in \mathbb{Z}} J_n(x) e(nt).
\]

Then for any \( u \in \mathbb{R} \) and any \( g \in C_0^\infty(\mathbb{R}^+) \), we have

\[
\sum_{n \in \mathbb{Z}} g(n) e(nu) J_n(x) = \int_{-\infty}^{\infty} \hat{g}(t) \sum_{n \in \mathbb{Z}} J_n(x) e(n(t+u)) dt
\]

\[
= \int_{-\infty}^{\infty} \hat{g}(t) e^{ix\sin 2\pi(t+u)} dt.
\]

In particular for \( u = \pm 1/4 \),

\[
\sum_{2|k} 2i^k g(k-1) J_{k-1}(x) = i \sum_{n \in \mathbb{Z}} g(n) \left( e\left(\frac{n}{4}\right) - e\left(-\frac{n}{4}\right) \right) J_n(x)
\]

\[
= -2 \int_{-\infty}^{\infty} \hat{g}(t) \sin(x \cos 2\pi t) dt.
\]
By Taylor expansion of \( \sin x \) and \( \cos t \) we obtain

\[
\sin(x \cos 2\pi t) = \sin(x + xf_n(t)) + O(x^{2n+2}) \tag{3.52}
\]

where

\[
f_n(t) = \sum_{j=1}^{n} (-1)^j \frac{(2\pi t)^{2j}}{(2j)!}. \tag{3.53}
\]

Hence for \( g(x) = h \left( \frac{x}{K} \right) \), we have

\[
I(x) = -2K \text{ Im} \left\{ e^{ix} \int_{0}^{\infty} h(u) \int_{-\infty}^{\infty} e^{i(xf_n(t) - 2\pi K t)u} \, dt \, du \right\} + O(xc_{2n+2}(g)) \tag{3.54}
\]

where

\[
c_n(g) = \int_{-\infty}^{\infty} |\hat{g}(t) t^n| \, dt. \tag{3.55}
\]

By using \((2\pi it)^{n} \hat{g}(t) = \hat{g^{(n)}(t)}\) we get

\[
c_n(g) \leq \int_{-\infty}^{\infty} K^{-n} |\hat{h}^{(n)}(t)| \, dt \ll K^{-n}. \tag{3.56}
\]

In particular for \( n = 1 \), we get by [18, p. 87]

\[
2 \int_{-\infty}^{\infty} \hat{g}(t) \sin(x + xf_1(t)) \, dt = \int_{0}^{\infty} g(\sqrt{\frac{2xy}{\pi y}}) \sin\left(x + y - \frac{\pi}{4}\right) \, dy \tag{3.57}
\]

which yields the desired result.

We are now ready to prove our second result. We shall determine the support of \( \hat{\phi} \) so that (3.44) is absorbed by the error term in (3.42). First using integration by parts for (3.47) we see that \( h(v) \ll v^{-A} \) for any \( A \geq 0 \). Hence by Lemma 3.5

\[
I(x) \ll \frac{K}{\sqrt{x}} \left( \frac{x}{K^2} \right)^A + xK^{-4}. \tag{3.58}
\]

For \( x \ll K^{2-\epsilon} \), \( I(x) \ll xK^{-4} \) which in turn gives

\[
\mathcal{P}_r(\phi) \ll Y^{2}K^{-4}P^{(r+1)/2}. \tag{3.59}
\]
As for $x \gg K^{2-\epsilon}$, we take $A = 0$ in (3.58) so that

$$I(x) \ll \frac{K}{\sqrt{x}} + xK^{-4}. \quad (3.60)$$

In this case we obtain

$$\mathcal{P}_r(\phi) \ll Y^{3/2+\epsilon} K^{-1} P^{(r+2)/4+\epsilon} + Y^2 K^{-4} P^{(r+1)/2}. \quad (3.61)$$

Combining the last two estimates we conclude that $\mathcal{P}_r(\phi)$ is smaller than the error term in (3.42) if $v$ is given by (1.35) which proves Theorem 1.2 except for the case $r = 1$. This case is handled in [16]. In fact, as $x = 4\pi m p^{1/2}/c$, $p \leq P$ and $m \leq Y$, we get $x \ll K^{2-\epsilon}$ if we assume $YP^{1/2} \ll K^{2-\epsilon}$ which gives $v = 2$. As $x \ll K^{2-\epsilon}$, we then obtain (3.59) and this finishes the proof of Theorem 1.2.

### 3.4 The case $r \equiv 1, 5 \pmod{8}$

Recall $H^\pm_k$ stands for the set of forms in $H_k$ for which the sign of the functional equation is $\pm 1$. We see from (1.17) that $H^+_k$ is empty unless $k \equiv 0, 2 \pmod{4}$ for $r \equiv 1, 5 \pmod{8}$, respectively. Similarly, $H^-_k$ is empty unless $k \equiv 0, 2 \pmod{4}$ for $r \equiv 5, 1 \pmod{8}$, respectively.

Below we evaluate as $K \to \infty$

$$\mathcal{A}_r^\pm(K) = \sum_{k \leq K} \frac{24}{k-1} h \left( \frac{k-1}{K} \right) \sum_{f \in H_k^\pm} D(\text{Sym}^r f; \phi) \quad (3.62)$$

normalized by

$$A_r^\pm(K) = \sum_{k \leq K} \frac{24}{k-1} h \left( \frac{k-1}{K} \right) |H_k^\pm|. \quad (3.63)$$

As before we obtain

$$\mathcal{A}_r^\pm(K) = A_r^\pm(K) \left( \hat{\phi}(0) + (-1)^{r+1} \frac{\phi(0)}{2} \right) - (-1)^{\frac{r-1}{2}} \mathcal{P}_r(\phi) \quad (3.64)$$

$$+ O \left( K \frac{\log \log K}{\log K} \right)$$
where

\[ 2 \mathcal{P}_r^\pm(\phi) = \mathcal{R}_r(\phi) \pm \mathcal{P}_r(\phi). \]  

(3.65)

Here \( \mathcal{P}_r(\phi) \) is given by (3.44) and

\[ \mathcal{R}_r(\phi) = 4\pi \sum_{m \leq Y} m^{-1} \sum_p \hat{\phi}\left(\frac{\log p}{\log R}\right) \frac{\log p}{\sqrt{p} \log R} \cdot \sum_c c^{-1} S(m^2, p^r; c) J\left(\frac{4\pi mp^{r/2}}{c}\right) \]  

(3.66)

where

\[ J(x) = \sum_{k \equiv 0(2)} 2h\left(\frac{k-1}{K}\right) J_{k-1}(x). \]  

(3.67)

By [16, p.102] we get

\[ J(x) = h\left(\frac{x}{K}\right) + O\left(xK^{-3}\right). \]  

(3.68)

Inserting (3.68) into (3.66) we obtain

\[ \mathcal{R}_r(\phi) = 4\pi \sum_{m \leq Y} m^{-1} \sum_p \hat{\phi}\left(\frac{\log p}{\log R}\right) \frac{\log p}{\sqrt{p} \log R} \cdot \sum_c c^{-1} S(m^2, p^r; c) h\left(\frac{4\pi mp^{r/2}}{cK}\right) + O\left(\frac{p(r+1/2)Y^2}{K}\right) \]  

\[ \ll p(r+2)/4+\epsilon Y^{3/2+\epsilon} K^{-1/2} + p(r+1/2)Y^2 K^{3-3}. \]

We see that for sufficiently small \( \epsilon \),

\[ \mathcal{R}_r(\phi) \ll \frac{K}{\log K} \]  

(3.70)

when \( v = 6/((r + 1)(r + 2)) \) which concludes the proof of Theorem 1.3.

### 3.5 Symmetric Fourth Power

By Lemma 3.5, (3.44) becomes

\[ \mathcal{P}_4(\phi) = -\frac{2\sqrt{\pi} K}{\log R} \sum_{m \leq Y} m^{-3/2} \sum_c c^{-1/2} Q_4(m; c) + O(Y^2 K^{-4} P^{5/2}) \]  

(3.71)
where

\[ Q_4(m; c) = \text{Im} \left\{ \bar{\zeta}_8 \sum_p S(m^2, p^4; c) \hat{\phi} \left( \frac{\log p}{\log R} \log p \frac{p^{3/2} e^{\left( 2mp^2 - \frac{2m^2}{c} \right) h(cK^2)}}{8\pi mp^2} \right) \right\}. \]

As \( h(v) \ll v^{-A} \) for any \( A \geq 0 \), we see that the contribution to \( \mathcal{P}_4(\phi) \) from primes \( p \leq P_0 = \left( \frac{K^{2-\epsilon}}{m} \right)^{1/2} \) is negligible. In the remaining range \( P_0 < p \leq P \), we use integration by parts to rewrite \( Q_4(m; c) \) as the product of \( \bar{\zeta}_8 \) and the imaginary part of

\[ \int_{P_0}^{P} h(cK^2) \hat{\phi} \left( \frac{\log y}{\log R} \right) y^{-3/2} d \left( \sum_{p \leq x} S(m^2, p^4; c) e^{\left( 2mp^2 - \frac{2m^2}{c} \right) h\left( \frac{cK^2}{8\pi mp^2} \right)} \right) \tag{3.72} \]

which by Lemma 2.10 becomes

\[ s(m; c) \int_{P_0}^{P} h(cK^2) \hat{\phi} \left( \frac{\log y}{\log R} \right) y^{-3/2} dy + O \left( m^{3/2 + \epsilon} P^{2 + 3\epsilon} K^{-A} \right) \tag{3.73} \]

where \( s(m; c) \) is given by (2.44) for \( n = 2 \). Hence we obtain

\[ \mathcal{P}_4(\phi) = -\frac{2\sqrt{\pi}K}{\log R} \sum_{m \leq Y} \frac{1}{m^{3/2}} \text{Im} \left( \bar{\zeta}_8 \int_{P_0}^{P} S(y) \hat{\phi} \left( \frac{\log y}{\log R} \right) y^{-3/2} dy \right) \tag{3.74} \]

\[ + O \left( K^{-2} Y^{1+\epsilon} P^{2 + 3\epsilon} K^{-A} \right) \]

where

\[ S(y) = \sum_c c^{-1/2} s(m; c) h \left( \frac{cK^2}{8\pi my^2} \right). \tag{3.75} \]

**Lemma 3.6.** For any \( y > 1 \) and \( A > 0 \), we have

\[ S(y) \ll K^{-1} (\sqrt{my})^{1+\epsilon} + (my^2)^{1+\epsilon} K^{-A}. \tag{3.76} \]

**Proof.** By Remark 2.1 we consider only those \( c \) such that \( \gcd \left( \frac{c}{(c, m)} \right) \) = 1. Hence \( c \) can be written as \( c = nab^2d^2 \) where \( n = (c, m) \) coprime to \( abd \), \( (a, bd) = 1 \), \( a \) and \( b \) square-free. As \( h(v) \ll v^{-A} \) for any \( A \geq 0 \), we have

\[ S(y) = \sum_{c \leq C} c^{-1/2} s(m; c) h \left( \frac{cK^2}{8\pi my^2} \right) + O \left( (my^2)^{1+\epsilon} K^{-A} \right). \tag{3.77} \]
where \( C = my^2 K^{\epsilon-2} \). Moreover, by Lemma 2.11 and Corollary 2.14 we get
\[
c^{-1/2} s(m; c) \ll \frac{c\epsilon}{n^{1/2+\epsilon} \varphi(a)}.
\] (3.78)

Hence we obtain
\[
\sum_{c \ll C} \frac{s(m; c)}{\sqrt{c}} h\left(\frac{cK^2}{8\pi my^2}\right) \ll \sum_{n|n,m,a,b} \sum_{d^2 \ll \frac{c}{\sqrt{n} \varphi(a)}} \frac{(ab^3d^2)^\epsilon}{\sqrt{n} \varphi(a)} \ll C^{1/2+\epsilon} \sum_{n|n,m,a,b} \frac{1}{n^{1+\epsilon} (ab)^{3/2}} \ll \frac{(\sqrt{my})^{1+\epsilon}}{K}.
\] (3.79)

which completes the proof. \( \square \)

Finally, choosing \( Y = K^{3\epsilon} \) yields
\[
\mathcal{P}_4(\phi) \ll K^{-2+4\epsilon} p^{2+3\epsilon} + p^{1/2+\epsilon} K^{2\epsilon} + K^{6\epsilon-4} p^{5/2}.
\] (3.80)

Hence we finish the proof of Theorem 1.4 by choosing \( v = 3/8 \) and \( \epsilon \) sufficiently small, in which case \( \mathcal{P}_4(\phi) \) is smaller than the error term which is \( O(K/\log K) \).
CHAPTER 4
HECKE L-FUNCTIONS ASSOCIATED WITH CUBIC CHARACTERS

4.1 Explicit Formula

In this chapter we shall prove Theorem 1.5. We start as in the previous chapter by deriving the explicit formula.

Lemma 4.1. Let \( \phi(x) \) be an even Schwartz function whose Fourier transform has compact support. Then we have

\[
\sum_{\rho \in \mathcal{C}} \phi \left( \frac{\gamma_{c} \log y}{2\pi} \right) = \hat{\phi}(0) \log N(c) + O \left( \frac{\log \log N(c)}{\log y} \right) \quad (4.1)
\]

\[
- \sum_{\rho \in \mathcal{C}} (\chi_{c}(p) + \chi_{c}(p^2)) \hat{\phi} \left( \frac{\log Np}{\log y} \right) \frac{\log Np}{\sqrt{Np} \log y}
\]

where the implied constant depends only on the test function.

Proof. As in the previous chapter, we define

\[
G(s) = \phi \left( \left( s - \frac{1}{2} \right) \frac{\log y}{2\pi i} \right). \quad (4.2)
\]

It is holomorphic in \(-1 \leq \text{Re}(s) \leq 2\) and satisfies

\[
G(s) = G(1 - s) \quad (4.3)
\]

\[
G(s) \ll s^{-2}.
\]
By Cauchy’s Integral formula and the functional equations (4.3) and (1.43)
\[
\sum_{\rho_c} G(\rho_c) = \frac{1}{2\pi i} \int_{(2)} G(s) \left( \frac{N'}{\Lambda}(s, \chi_c) + \frac{N'}{\Lambda}(s, \overline{\chi}_c) \right) ds. \tag{4.4}
\]
By (1.42) we obtain the logarithmic derivative of \(\Lambda(s, \chi_c)\)
\[
\frac{N'}{\Lambda}(s, \chi_c) = \frac{1}{2} \log \frac{3N(c)}{4\pi^2} + \Gamma'(s) - \sum_{p} \sum_{n=1}^{\infty} \chi_c(p)(Np)^{-ns} \log Np. \tag{4.5}
\]
Using this expression we get
\[
\sum_{\rho_c} \phi \left( \frac{\gamma_c \log y}{2\pi} \right) = F(1) \log \frac{3N(c)}{4\pi^2} - \sum_{p} \sum_{n \geq 1} H_n(p) \tag{4.6}
\]
\[+ \frac{1}{2\pi i} \int_{(1/2)} G(s) \frac{\Gamma'}{\Gamma}(s) ds \]
where
\[
F(t) = \frac{1}{2\pi i} \int_{(1/2)} G(s) t^{-s} ds = \frac{1}{\sqrt{t \log y}} \hat{\phi} \left( \frac{\log t}{\log y} \right). \tag{4.7}
\]
is the inverse Mellin transform of \(G(s)\) and
\[
H_n(p) = \left( \chi_c(p^n) + \overline{\chi}_c(p^n) \right) F(\log Np) \log Np
\]
\[= \left( \chi_c(p^n) + \chi_c(p^{2n}) \right) \hat{\phi} \left( \frac{n \log Np}{\log y} \right) \log Np (Np)^{n/2} \log y. \tag{4.8}
\]
For the finite primes \(p\), we see that
\[
\sum_{p,n} H_n(p) \ll \frac{1}{\log y} \tag{4.9}
\]
while for \(n=2\), partial integration yields
\[
\sum_{Np \leq x} H_2(p) = \int_{1}^{x} \hat{\phi} \left( \frac{\log t^2}{\log y} \right) \frac{1}{t \log y} dS(t) \tag{4.10}
\]
\[= - \int_{1}^{x} 2S(t) \frac{\log t^2}{(t \log y)^2} \hat{\phi}' \left( \frac{\log t^2}{\log y} \right) - S(t) \frac{\log t^2}{t^2 \log y} \hat{\phi} \left( \frac{\log t^2}{\log R} \right) dt \]
where
\[
S(t) = \sum_{Np \leq t} \chi_c(p^2) \log Np. \tag{4.11}
\]
By Lemma 2.19 we find that
\[
\sum_{Np \le x} H_2(p) \ll \int_1^{\log^3 N(c)} \frac{1}{t \log y} \, dt + \frac{\log N(c)}{\log y} \int_{\log^3 N(c)}^x t^{-3/2+\epsilon} \, dt
\]
\[
\ll \frac{\log \log N(c)}{\log y}.
\]

Finally, by the approximate formula (see (8.363.3) of [10])
\[
\frac{\Gamma'}{\Gamma} (a + ib) + \frac{\Gamma'}{\Gamma} (a - ib) = 2 \frac{\Gamma'}{\Gamma} (a) + O \left( \frac{b^2}{a^2} \right)
\]
we obtain
\[
\frac{1}{\pi i} \int_{(1/2)} G(s) \frac{\Gamma'}{\Gamma} (s) \, ds = \frac{2}{\log R} \int_{-\infty}^{\infty} \phi(t) \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + i \frac{2\pi t}{\log y} \right) \, dt
\]
\[
= 2 \hat{\phi}(0) \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} \right) + O \left( \frac{1}{\log^4 y} \right),
\]
which finishes the proof. \qed

### 4.2 Averaging Over The Characters

In this section we average over the characters $\chi_c$ for square-free $c \neq 1$ in $\mathcal{O}_K$ such that $c \equiv 1 \mod 9$. More precisely, we evaluate as $y \to \infty$
\[
\frac{1}{\mathcal{A}(y)} \sum_{\chi_c \equiv 1 \mod 9}^* e^{-N(c)/y} \sum_{\rho_c} \phi \left( \frac{\gamma_c \log y}{2\pi} \right)
\]
where $\sum^*$ denotes summation over square-free integers $c \neq 1$ and
\[
\mathcal{A}(y) = \sum_{c \equiv 1 \mod 9}^* e^{-N(c)/y}.
\]

**Lemma 4.2.** We have
\[
\mathcal{A}(y) = y \frac{\text{Res}_{s=1} \zeta_K(s)}{h(9) \zeta_K(2)} \prod_{\mathfrak{p} \mid (9)} \left( 1 - \frac{1}{N\mathfrak{p}} \right)^{-1} + O \left( y^{1/2+\epsilon} \right).
\]
where $\zeta_K(s)$ is the Dedekind zeta function and $h(9)$ is the size of the ray class group $J_{(9)}/P_{(9)}$. 

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Proof. Using ray class characters modulo (9) we get
\[
\mathcal{A}(y) = \frac{1}{h(9)} \sum_{\chi \mod 9} \frac{1}{2\pi i} \int_{(2)} \Gamma(s) y^s L_\chi(s) \, ds
\]
(4.18)
where \( h(9) = |J^{(9)}/P^{(9)}| \) and
\[
L_\chi(s) = \sum_a |\mu(a)| \chi(a) (Na)^{-s}
\]
(4.19)

We shift the contour in (4.18) to \( \frac{1}{2} + \epsilon \) and pick up the residue at \( s = 1 \) of \( L_\chi(s) y^s \Gamma(s) \) when \( \chi = \chi_0 \), which gives the desired result.

**Lemma 4.3.** We have
\[
\sum_{c \equiv 1 \mod 9} \frac{\log N(c)}{\log y} e^{-N(c)/y} = \mathcal{A}(y) + O\left( \frac{y}{\log y} \right) + O\left( y^{1/2+\epsilon} \right).
\]
(4.21)

**Proof.** As in Lemma 4.2 we use ray class characters modulo (9) to get
\[
\sum_{c \equiv 1 \mod 9} \frac{\log N(c)}{\log y} e^{-N(c)/y} = -\frac{1}{h(9)} \sum_{\chi \mod 9} \frac{1}{2\pi i} \int_{(2)} \Gamma(s) y^s L'_\chi(s) \, ds
\]
(4.22)
where \( L_\chi(s) \) is given by (4.19). Once again moving the line of integration to \( \frac{1}{2} + \epsilon \) we pick up the residue at \( s = 1 \) of \( L'_\chi(s) y^s \Gamma(s) \) when \( \chi = \chi_0 \). We see that
\[
\text{Res}_{s=1} (L'_\chi(s) y^s \Gamma(s)) = \log y \text{ Res}_{s=1} (L_\chi(s) y^s \Gamma(s)) + Cy
\]
(4.23)
for some constant \( C \). Hence the result follows from the previous lemma. \qed
Inserting (4.1) into (4.15) and using Lemmas 4.2 and 4.3 we obtain

\[ \sum_{c \equiv 1 \mod 9}^* e^{-N(c)/y} \sum_{\rho_c} \phi \left( \frac{\gamma_c \log y}{2\pi} \right) = \mathcal{I}(y) \widehat{\phi}(0) - \mathcal{I}(y) + O \left( y \frac{\log \log y}{\log y} \right) \]

where

\[ \mathcal{I}(y) = \sum_{c \equiv 1 \mod 9}^* \sum_p \left( \chi_c(p) + \chi_c(p^2) \right) \widehat{\phi} \left( \frac{\log Np}{\log y} \right) \frac{\log Np}{\sqrt{Np} \log y}. \quad (4.24) \]

### 4.3 Estimate of \( \mathcal{I}(y) \)

We see that the main contribution comes from (4.24). Hence for the rest of this chapter, we shall work on estimating \( \mathcal{I}(y) \). To this end, we define

\[ \mathcal{I}_1(y) = \sum_{c \equiv 1 \mod 9}^* e^{-N(c)/y} \sum_{p \mid 3 \mod Np \leq t} \chi_c(p) e^{-N(c)/y} \quad (4.25) \]

where \( x \) is determined by the support of \( \widehat{\phi} \). We then have

\[ \mathcal{I}(y) = \frac{1}{\log y} (\mathcal{I}_1(y) + \mathcal{I}_2(y)) + O \left( \frac{\mathcal{I}(y)}{\log y} \right). \quad (4.26) \]

We shall only work with \( \mathcal{I}_1(y) \) as one can get the same results for \( \mathcal{I}_2(y) \). Using integration by parts we obtain

\[ \mathcal{I}_1(y) = \int_1^x \widehat{\phi} \left( \frac{\log t}{\log y} \right) \frac{\log t}{\sqrt{t}} dG_y(t) \quad (4.27) \]

\[ = - \int_1^x \frac{G_y(t)}{t^{3/2}} \left\{ \widehat{\phi}' \left( \frac{\log t}{\log y} \right) \frac{\log t}{\log y} + \widehat{\phi} \left( \frac{\log t}{\log y} \right) (1 - \log \sqrt{t}) \right\} dt \]

where

\[ G_y(t) = \sum_{p \mid 3 \mod Np \leq t} \sum_{c \equiv 1 \mod 9}^* \chi_c(p) e^{-N(c)/y}. \quad (4.28) \]

Assuming that

\[ G_y(t) \ll t^\alpha y^{1/2+\epsilon} \quad (4.29) \]
holds for some \( \alpha > \frac{1}{2} \), we immediately see that
\[
\mathcal{A}_1(y) \ll y^{1/2+\epsilon} x^{\alpha-1/2}. \tag{4.30}
\]
Similarly, we get the same bound for \( \mathcal{A}_2(y) \). Combining these two estimates we get

**Theorem 4.4.** Let \( \phi(x) \) be an even Schwartz function whose Fourier transform has compact support in \( (-v, v) \) with \( v = \frac{1}{2^{\alpha-1}} \). Then, we have
\[
\frac{1}{\mathcal{A}(y)} \sum_{c \equiv 1 \pmod{9}} \sum_{p \equiv c \pmod{9}} \phi \left( \frac{\gamma_p \log y}{2\pi} \right) = \hat{\phi}(0) + O \left( \frac{\log \log y}{\log y} \right) \tag{4.31}
\]
where the implied constant depends only on the test function.

**Remark 4.1.** Note that \( \hat{\phi}(0) \) can be written as
\[
\hat{\phi}(0) = \int_{-\infty}^{\infty} \phi(x) W(U)(x) \, dx \tag{4.32}
\]
where \( W(U)(x) = 1 \) is the density function determined by Katz and Sarnak [21] for the unitary symmetry group \( U \).

Hence in the next section we shall deal with the estimate in (4.29) in order to prove Theorem 4.4.

### 4.3.1 Estimate of \( G_y(t) \)

We first remove the square-free condition on \( c \) and rewrite (4.28) as
\[
G_y(t) = \sum_{p \equiv 3 \pmod{3}} \sum_{d \equiv 1 \pmod{3}} \mu(d) \sum_{c \equiv d^2 \pmod{9}} \chi_{cd^2}(p) e^{-N(cd^2)/y}. \tag{4.33}
\]
We then separate \( G_y(t) \) into two sums for which we shall use different estimates, one depending on the Riemann Hypothesis and the other using an analog of Polya-Vinogradov inequality. Hence we write
\[
G_y(t) = R_y(t) + S_y(t) \tag{4.34}
\]
where

\[
R_y(t) = \sum_{\substack{p \nmid 3 \\ Np \leq t}} \sum_{d \equiv 1 \mod 3} \mu(d) \sum_{c \equiv d^2 \mod 9} \chi_{cd^2}(p) e^{-N(cd^2)/y}
\]

and

\[
S_y(t) = \sum_{\substack{d \equiv 1 \mod 3 \\ N(d) > B}} \mu(d) \sum_{c \equiv d^2 \mod 9} e^{-N(cd^2)/y} \sum_{\substack{p \nmid 3 \\ Np \leq t}} \chi_{cd^2}(p).
\]

where \(B < \sqrt{y}\) will be chosen optimally. First, we consider \(S_y(t)\). By Lemma 2.19 the innermost sum over primes will be \(O(t^{1/2+\epsilon} \log N(cd^2))\) and hence we obtain

\[
S_y(t) \ll t^{1/2+\epsilon} \sum_{\substack{d \equiv 1 \mod 3 \\ N(d) > B}} \sum_{c \equiv d^2 \mod 9} \log N(cd^2) e^{-N(cd^2)/y}
\]

\[
\ll t^{1/2+\epsilon} y^{1+\epsilon} B^{-1}.
\]

In the above estimate the essential contribution comes from \(B < N(d) \leq y^{1/2+\epsilon}\) while in the remaining range the sum will be smaller for sufficiently large \(y\).

Next, we shall estimate \(R_y(t)\) using two different methods. First we use the following result due to Heath-Brown and Patterson [14, Lemma 2] :

**Lemma 4.5.** Let \(\chi\) be a character of modulus \(f \neq 1\), not necessarily primitive. Then, if \(w \leq 1\), \(\epsilon > 0\),

\[
\sum_{c \equiv 1 \mod 3 \atop (c,f)=1} \chi(c) e^{-2\pi N(c)w} \ll E(\chi)w^{-1} + N(f)^{1/2+\epsilon}
\]

where \(E(\chi) = 1\) if \(\chi\) is principal, 0 otherwise and the implied constant depends only on \(\epsilon\).

Note that any prime ideal \(p \nmid 3\) has unique prime generator \(\pi \equiv 1 \mod 3\). By the Cubic Reciprocity Theorem 2.16 we obtain

\[
\chi_c(\pi) = \chi_{\pi}(c)
\]
for \( c \equiv 1 \mod 9 \). Therefore

\[
R_y(t) = \sum_{\pi \equiv 1 \mod 3} \sum_{d \equiv 1 \mod 3} \frac{\mathcal{N}(\pi)}{N(\pi) \leq t} \mu(d) \sum_{c \equiv \pm d^2 \mod 9} \mathcal{N}(\pi) e^{-N(cd^2)/y}.
\]

(4.40)

Using ray class characters modulo (9) and Lemma 4.5 with \( w = N(d^2)/2\pi y \) we obtain

\[
R_y(t) \ll t^{3/2+\epsilon} B.
\]

(4.41)

Combining (4.41) and (4.37) we see that

\[
G_y(t) \ll t^{1/2+\epsilon} y^{1+\epsilon} B^{-1} + t^{3/2+\epsilon} B.
\]

(4.42)

Hence choosing \( B = \sqrt{\frac{y}{t}} \) yields Theorem 4.4 with \( \alpha = 1 \).

Next, we shall improve our result by using the crucial bound of Patterson for which we have to use characters \( \chi \) such that if \( u \) is a unit of \( K \), then \( \chi(u) \) is a third root of unity. This already holds for any \( \chi_c \) while for the characters modulo (9) that we use below, we need to make sure they are even. Therefore, we rewrite (4.35) as

\[
R_y(t) = \frac{1}{2} \sum_{\pi \equiv 1 \mod 3} \sum_{d \equiv 1 \mod 3} \frac{\mathcal{N}(\pi)}{N(\pi) \leq t} \mu(d) \sum_{c \equiv \pm d^2 \mod 9} \mathcal{N}(\pi) e^{-N(cd^2)/y}.
\]

(4.43)

Then using even ray class characters modulo (9) the innermost sum over \( c \) becomes

\[
\sum_{c \equiv \pm d^2 \mod 9} \mathcal{N}(\pi) e^{-N(cd^2)/y} = \frac{2}{|h(9)|} \sum_{\chi \equiv \chi \text{ even}} \mathcal{N}(\pi) \sum_{c \in \mathcal{D}_K} \chi(c) e^{-N(c\pi)/y}.
\]

(4.44)

When \( \chi = \chi_0 \) is the principal character modulo (9) we get

\[
\sum_{c \in \mathcal{D}_K} \mathcal{N}(\pi) e^{-N(c\pi)/y} = \sum_{c \in \mathcal{D}_K} \mathcal{N}(\pi) e^{-N(c\pi)/y} - \sum_{r \geq 1 \text{ unit}} \mathcal{N}(u(1 - \omega)^r) e^{-3^r N\pi}/y.
\]

(4.45)

We see that the contribution of \( c \) which are not relatively prime to 3 is \( O(tB\gamma^r) \) and we get

\[
R_y(t) = \frac{1}{|h(9)|} \sum_{\pi \equiv 1 \mod 3} \sum_{d \equiv 1 \mod 3} \frac{\mathcal{N}(\pi)}{N(\pi) \leq t} \mu(d) \sum_{\chi \mod (9) \chi \text{ even}} \mathcal{N}(\pi) e^{-N(c\pi)/y} + O(tB\gamma^r)
\]

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\[ \tau = \begin{cases} 
\chi_1 \chi \pi & \text{if } \chi \neq \chi_0, \\
\chi \pi & \text{if } \chi = \chi_0.
\end{cases} \] (4.47)

Here \( \chi_1 \) is the primitive ray class character of conductor \((f_\chi)\) which induces \( \chi \). Note that \( \tau \) is primitive of conductor \((\pi f \chi)\) for \( \chi \neq \chi_0 \) and of conductor \((\pi)\) for \( \chi_0 \). We shall apply twisted poisson summation formula to the sum over \( c \) above in \( R_y(t) \). In fact, by Lemma 2.21 we obtain

\[
\sum_{c \in \mathcal{O}_K} \tau(c) e^{-N(cd^2)/y} = \frac{2\pi y \tau(-\omega)}{\sqrt{3N(d^2)W(\tau)}} \sum_{c \in \mathcal{O}_K} \tau(c) e^{-\frac{4\pi^2 y}{3N(\chi \pi^2 d^2)} N(c)}
\] (4.48)

where \( W(\chi) \) is the Gauss sum associated to \( \chi \) given by (1.44). For \( \chi \neq \chi_0 \), we have

\[
\frac{1}{W(\tau)} = \frac{W(\chi_1 \chi \pi)}{N(f_\chi \pi)} = \frac{W(\chi_1) W(\chi \pi) (f_\chi \chi_1 \chi \pi)}{N(f_\chi \pi)}.
\] (4.49)

Substituting this back in the expression for \( R_y(t) \) and rearranging the sums we get

\[
R_y(t) = \frac{2\pi y}{\sqrt{3|h(9)|}} \sum_{d \equiv 1 \bmod 3} \frac{\mu(d)}{N(d^2)} \sum_{\chi \bmod 3} \chi(d^2) \chi(\omega \chi \pi) W(\chi_1) N(f_\chi \pi) \sum_{c \in \mathcal{O}_K} \chi_1(c) \sum_{\pi \equiv 1 \bmod 3} \sum_{N(\pi) \leq t} \frac{W(\pi) \chi_\pi(\omega^2 cd^2)}{N(\pi)} \chi_1(\pi) e^{-D/N(\pi)}
\] (4.50)

where \( D = \frac{4\pi^2 y N(c)}{3N(f_\chi d^2)} \) and \( \chi_1 = 1 \) if \( \chi \) is principal. Note that

\[
W(\chi \pi) = \chi_\pi(\delta) \sum_{a \bmod (\pi)} \chi_\pi(a) e^{\left(\frac{Tr\left(\frac{a}{\pi}\right)}{\pi}\right)} := \chi_\pi(\delta) G(\pi).
\] (4.51)

We use Theorem 2.16 and Theorem 2.17 for \( \chi_\pi(\delta \omega^2 cd \chi) \chi_1(\pi) \) in (4.50) so that the innermost sum over primes can be written as

\[
\frac{1}{|h(9)|} \sum_{a \equiv 1 \bmod 3} \omega^{Tr(a)} \chi_1(\pi) \sum_{\psi \bmod 9} \psi(a) \sum_{\pi \equiv 1 \bmod 3} \frac{G(\pi) \omega(\pi)}{N(\pi)} e^{-D/N(\pi)}
\] (4.52)
where $\omega(\pi) = \psi_{\chi_{d}}(\pi)$ and $\tilde{c}$ is the divisor of $c$ coprime to 3. We apply integration by parts to the last term over primes to get

$$
\sum_{\pi \equiv 1 \mod 3 \atop N(\pi) \leq t} \frac{G(\pi) \omega(\pi)}{N(\pi)} e^{-D/N(\pi)} = \int_2^t \frac{e^{-D/u}}{\sqrt{u} \log u} \, dP(u) \tag{4.53}
$$

$$
= \frac{P(t) e^{-D/t}}{\sqrt{t} \log t} - \int_2^t P(u) \left( \frac{e^{-D/u}}{\sqrt{u} \log u} \right)' \, du
$$

where

$$
P(u) = \sum_{\pi \equiv 1 \mod 3 \atop N(\pi) \leq u} \frac{G(\pi) \omega(\pi)}{\sqrt{N(\pi)}} \Lambda(\pi). \tag{4.54}
$$

Note that $P(u) \ll u^E$ holds trivially for $E = 1$ and as mentioned in introduction Patterson’s estimate [28] gives $E = 30/31$. Hence we obtain

$$
\sum_{\pi \equiv 1 \mod 3 \atop N(\pi) \leq t} \frac{G(\pi) \omega(\pi)}{N(\pi)} e^{-D/N(\pi)} \ll t^{E-1/2} e^{-D/t}. \tag{4.55}
$$

In (4.50) this estimate yields

$$
R_y(t) \ll t^{E+1/2} y^f B. \tag{4.56}
$$

Hence choosing $B = \frac{\sqrt{y}}{t^A}$ in (4.37) and (4.56) we get

$$
G_y(t) \ll \left( t^{E+1/2-A} + t^{1/2+A+\epsilon} \right) y^{1/2+\epsilon}. \tag{4.57}
$$

We see that if $A = E/2$, then (4.29) holds with $\alpha = (E + 1)/2$. Applying Theorem 4.4 then yields our final result, Theorem 1.5, with $v = 1/E$. 

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