DYNAMIC ANALYSIS OF DRY FRICITION PATH IN A TORSIONAL SYSTEM

DISSERTATION

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By

Chengwu Duan, B.S., M.S.

* * * * *

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Dissertation Committee:               Approved by
Dr. Rajendra Singh, Adviser
Dr. Ahmet Selamet
Dr. Marcelo Dapino               Adviser
Dr. Ahmet Kahraman               Department of Mechanical Engineering
Traditionally, dry friction non-linear elements have been treated as vibration dampers, in parallel with elastic elements. However, in some practical systems (including automotive drivelines) the dry friction element (with high saturation torque) exists by itself and it functions as a key power transmission path for both mean and dynamic loads. First, to address such path issues, we examine multi degree of freedom torsional systems with time-invariant normal load. A procedure to predict pure stick to stick-slip boundaries, based on a linear system theory, is developed when the torsional system is excited by harmonic excitation. For non-linear studies, both discontinuous and smoothened friction formulations are examined. The effects of a secondary inertia are analytically and numerically investigated. Results show that the secondary mass significantly affects the quasi-discontinuous nature of the system response. Next, in order to fully understand the non-linear frequency characteristics generated by stick-slip vibration, a new analytical method is developed based on assumed torque and velocity profiles. Super-harmonics are efficiently calculated and effect of the mean torque is qualitatively identified. Further, a refined multi-term harmonic balance proposed and associated computational issues are addressed. Studies show that the
mean load significantly affects the response as it could induce asymmetric stick-slip motions.

Second, the effect of time varying actuation pressure (or the normal load) on the dry friction element on transient and steady state responses has been studied. Analytical solution for pure slip motion is obtained based on an approximate linear system model. Effects of time-varying parameters, such as phase, frequency and amplitude of the actuation pressure are observed over several frequency regimes. The negative friction slope is found to be the major cause of judder-induced phenomena such as bifurcations and quasi-periodic or chaotic responses. Some instabilities such as abrupt jumps in the amplitude-frequency maps of relative velocity are also seen around the super-harmonic peak frequencies.
DEDICATION

Dedicated to
my parents and wife
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VITA

November 13, 1973 ..........................................  Born - Hunan, China

1997 ..........................................................  B.S. Mechanical Engineering
Tongji University, Shanghai, China

1997-1999 .................................................... Product Development Engineer
Shanghai Hella Automotive Lighting Ltd.

2001 ............................................................ M.S. Engineering Mechanics
The University of Alabama, AL

2001 - present .............................................. Graduate Research Associate
The Ohio State University

PUBLICATIONS


FIELDS OF STUDY

Major Fields: Mechanical Engineering

Powertrain Dynamics and Non-linear vibration

Vibro-acoustics and Noise Control
# TABLE OF CONTENTS

ABSTRACT ................................................................................................................... ii

DEDICATION .............................................................................................................. iv

ACKNOWLEDGMENTS .............................................................................................. v

VITA .............................................................................................................................. vi

LIST OF TABLES ......................................................................................................... x

LIST OF FIGURES ....................................................................................................... xi

LIST OF SYMBOLS ..................................................................................................... xix

CHAPTER 1 INTRODUCTION .................................................................................. 1

1.1 Dry Friction Path .............................................................................................. 1

1.2 Literature Review ............................................................................................ 4

1.3 Problem Formulation ...................................................................................... 9

References for Chapter 1 ........................................................................................ 15

CHAPTER 2 DYNAMICS OF A 3DOF TORSIONAL SYSTEM WITH A DRY FRICTION CONTROLLED PATH ............................................................................. 19

2.1 Introduction .................................................................................................. 19

2.2 Literature Review and Research Issues ........................................................... 21

2.3 Problem Formulation .................................................................................... 27

2.4 Smoothened and Discontinuous Friction Torque Models ......................... 31

2.5 Computation of Stick to Slip Boundaries Based on Linear System Theory .. 33

2.6 Solution to Non-linear Path Formulation ...................................................... 37
2.7 Effect of Friction Controlled Path Parameters .....................................................47
2.8 Comparison with Benchmark Studies .................................................................64
2.9 Negative Slope in Friction Formulation .............................................................71
2.10 Conclusion .........................................................................................................72
References for Chapter 2 .......................................................................................74

CHAPTER 3  SUPER-HARMONICS IN A TORSIONAL SYSTEM WITH DRY FRICTION PATH .................................................................78
3.1 Introduction .......................................................................................................78
3.2 Problem Formulation .......................................................................................81
3.3 Linear System Analysis ....................................................................................86
3.4 Analytical Solution Using One-Term Harmonic Balance Method ..................88
3.5 Refined Multi-term Harmonic Balance Method .............................................102
3.6 Non-linear Response and Super-Harmonics .................................................110
3.7 Effect of Negative Slope in Friction Formulation ........................................120
3.8 Comparison with Conventional Friction Damper Problem (Den Hartog’s System) ..............................................................123
3.9 Conclusion .......................................................................................................127
References for Chapter 3 ....................................................................................128

CHAPTER 4  EFFECT OF TIME VARYING DRY FRICTION ON TRANSIENT AND STEADY STATE RESPONSES .........................................................131
4.1 Introduction ......................................................................................................131
4.2 Problem Formulation ......................................................................................133
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 2.1 Values of parameter and excitation amplitude used for simulating the system of Figure 2.3</td>
<td>28</td>
</tr>
<tr>
<td>Table 2.2 State to state transition for the non-linear formulation of Figure 2.3 with discontinuous Coulomb friction model</td>
<td>42</td>
</tr>
<tr>
<td>Table 3.1 Parameters (in the dimensionless form) used to study an automotive driveline system corresponding to Figure 3.1</td>
<td>109</td>
</tr>
<tr>
<td>Table 4.1 Parameters and excitation amplitude used for simulating the system of Figure 4.2</td>
<td>146</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 1.1</td>
<td>Torque converter clutch sub-system a) physical system; b) simplified torsional model; c) operational ranges as a function of vehicle speed.</td>
</tr>
<tr>
<td>Figure 1.2</td>
<td>Power distribution between mechanical and fluid paths in a Toyota A541 automatic transmission [1.14].</td>
</tr>
<tr>
<td>Figure 1.3</td>
<td>Selected dry friction formulations. (a) Classical (discontinuous) Coulomb model; (b) Karnopp Model [1.17]; (c) Menq et. al.’s models [1.19]: --- , elastic bar on rigid base; , elastic bar with an elastoplastic shear layer on a rigid base; (d) Model by Imamura et. al. [1.20];</td>
</tr>
<tr>
<td>Figure 1.4</td>
<td>Reduced order driveline torsional systems with dry friction non-linearity. a) 3DOF semi-definite model with $C_{13}$ and $C_{23}$ elements; b) simplified 3DOF system with only $C_{13}$ element; c) 2DOF definite system with a dominant mechanical path, non-negligible $C_{23}$ and significant $I_3$.</td>
</tr>
<tr>
<td>Figure 1.5</td>
<td>Smoothed friction formulation $T_f(\delta)$ as employed in our study with a variable conditioning factor $\sigma$ given saturation torque $T_{sf} = 350$ N-m and $\mu_k = \mu_s$: Key for (e): _ _ , $\sigma = 50$; - - - , $\sigma = 10^2$; . . . , $\sigma = 10^3$; ___ , $\sigma = 10^4$</td>
</tr>
<tr>
<td>Figure 2.1</td>
<td>Schematic of a typical automotive torque converter and dry friction clutch (TCC).</td>
</tr>
<tr>
<td>Figure 2.2</td>
<td>Selected dry friction formulations. (a) Classical (discontinuous) Coulomb model; (b) Karnopp Model [4]; (c) Menq et. al.’s models [6]: - - - , elastic bar on rigid base; , elastic bar with an elastoplastic shear layer on a rigid base; (d) Model by Imamura et. al. [7]; (e) Smoothed friction formulation $T_f(\delta)$ of our study with a variable conditioning factor $\sigma$ and with saturation torque $T_{sf} = 350$ N-m: Key for (e): _ _ , $\sigma = 50$; - - - , $\sigma = 10^2$; . . . , $\sigma = 10^3$; ___ , $\sigma = 10^4$.</td>
</tr>
<tr>
<td>Figure 2.3</td>
<td>Physical Example: 3DOF semi-definite torsional model of an automotive driveline system. Here, $T_f(\delta)$ represents a dry friction path.</td>
</tr>
</tbody>
</table>
Figure 2.4 Resulting torsional systems under different conditions. (a) Pure stick condition; (b) Free body diagram for $I_1$ under pure stick condition; (c) Positive slip condition

Figure 2.5 Stick-slip boundaries based on linear system analysis. (a) Variation with respect to $I_2$; (b) Variation with respect to $T_{sf}$; (c) Variation with respect to excitation amplitude of $T_p$

Figure 2.6 Comparison of two friction path models in terms of time histories. , discontinuous friction model; - - - , smoothened model with $\sigma = 0.5$; _ _ _, smoothened model with $\sigma = 10^2$

Figure 2.7 Comparison of two friction path models in terms of time histories. , discontinuous model; - - - , smoothened model with $\sigma = 10^3$; _ _ _, smoothened model with $\sigma = 2 \times 10^3$

Figure 2.8 Schematic of the bi-linear torsional system (with $I_2 \to 0$). Here, $T_f$ represents a dry friction path.

Figure 2.9 Effect of $I_2$ on time histories. a) numerical solutions: , $I_2/I_1 = 0$; - - - , $I_2/I_1 = 0.005$. b) comparison between numerical and analytical solution during the positive slip state: , numerical solution given $I_2/I_1 = 0$; - - - , analytical solution given $I_2/I_1 = 0$; _ _ _, numerical solution given $I_2/I_1 = 0.005$; _ _ _, analytical solution given $I_2/I_1 = 0.005$

Figure 2.10 Effect of $I_2$ on time histories. a) numerical solutions: , $I_2/I_1 = 0$; - - - , $I_2/I_1 = 0.01$. b) comparison between numerical and analytical solution during the positive slip state: , numerical solution given $I_2/I_1 = 0$; - - - , analytical solution given $I_2/I_1 = 0$; _ _ _, numerical solution given $I_2/I_1 = 0.01$; _ _ _, analytical solution given $I_2/I_1 = 0.01$

Figure 2.11 Effect of $I_2$ on phase-plane plots. (a) and (c) are with $I_2/I_1 = 0$; (b) and (d) are with $I_2/I_1 = 0.01$

Figure 2.12 Effect of $I_2$ on time histories: , $I_2/I_1 = 0$; - - - , $I_2/I_1 = 0.1$; _ _ _, $I_2/I_1 = 0.4$.

Figure 2.13 Effect of $I_2$ on responses over a range of excitation frequencies. (a) Maximum and minimum responses: $\circ$, $I_2/I_1 = 0$; - - - , $I_2/I_1 = 0.01$; $\times$, $I_2/I_1 = 0.1$; _ _ _, $I_2/I_1 = 0.2$; _ _ _, $I_2/I_1 = 0.4$; (b) rms responses: $\circ$, $I_2/I_1 = 0$; - - - , $I_2/I_1 = 0.01$; $\times$, $I_2/I_1 = 0.1$; _ _ _, $I_2/I_1 = 0.2$; _ _ _, $I_2/I_1 = 0.4$

Figure 2.14 Effect of $T_{sf}$ on responses over a range of excitation frequencies. (a) Maximum and minimum responses: $\varnothing$, $T_f = 250$ Nm; - - - , $T_f = 350$ Nm; $\times$, $T_f =$
450 Nm; ___, Tf = 550 Nm; (b) rms responses: ◊, Tf = 250 Nm; - - -, Tf = 350 Nm; ×, Tf = 450 Nm; ___, Tf = 550 Nm ..............................................................62

Figure 2.15 Effect of Tsf on time histories. (a) Excitation frequency is 10 Hz: - - -, Tf = 250 Nm; . . . , Tf = 350 Nm; _ . _, Tf = 450 Nm; ___, Tf = 550 Nm; (b) Excitation frequency is 8 Hz: - - -, Tf = 250 Nm; . . . , Tf = 350 Nm; _ . _, Tf = 450 Nm; ___, Tf = 550 Nm..........................................................................................................63

Figure 2.16 Benchmark models with dry friction element. (a) Ferri and Heck’s turbine blade friction damper model [2.15]; (b) Harung et. al.'s passive vibration absorber model [2.26]...........................................................................................65

Figure 2.17 Comparison with Ferri and Heck’s results. (a) Frequency response curves with f = 2, p = 0.5 and r = 0.2. ×, pure stick solution; ◊, discontinuous friction model; ___, smoothened friction model with σ = 50; - - -, smoothened friction model with σ = 10; □, Ferri and Heck’s results. (b) Time histories at ω = 0.94: ___, discontinuous friction model; - - -, smoothened friction model with σ = 10; _ . _, smoothened friction model with σ = 50. ......................................................66

Figure 2.18 Frequency response curves with f = 2, p = 0.2 and r = 0.05. ×, pure stick solution; ◊, discontinuous friction model; ___, smoothened friction model with σ = 50; - - -, smoothened friction model with σ = 10; □, Ferri and Heck’s results. 67

Figure 2.19 Measured frequency response curves (Hartung et. al.’s experiment scanned from [2.26]) ............................................................................................69

Figure 2.20 Simulated frequency response curves. ×, pure stick solution; ◊, discontinuous model (stick-slip) solution; ___, smoothened friction model with σ = 50...............................70

Figure 3.1 Schematic of the 2DOF Torsional Automotive Dry Friction Clutch System. a) Non-linear model. b) Linear system with viscous damper. All parameters and variables have dimensions....................................................................................80

Figure 3.2 Conventional Friction Damper System. a) 2DOF semi-definite system. b) SDOF definite system (Den Hartog’s Model [1]). All parameters and variables have dimensions. ..........................................................................................84

Figure 3.3 Linear System Frequency Response for I1 = 0.01, ζ1 = 0.4, ζ = 0.001, Tm = 0.5, Tp = 4.5. a) Max-mean-min frequency responses of δ1; b) Max-min frequency response of δ2. ............................................................................................................87

Figure 3.4 Simplifications to Figure 3.1a Yield Two De-coupled Sub-systems. a) Non-Linear sub-system with dry friction; b) Linear sub-system. Dimensionless parameters and variables are shown here. ........................................................................90
Figure 3.5 Assumed Time Domain Profiles for Relative Velocity and Friction Torque.
..............................................................................................................................91

Figure 3.6 Time Histories for System of Figure 3.4 Given $I_1 = 0.01$, $\zeta = 0.001$, $T_m = 0.5$, $T_p = 4.5$. a) $\dot{\delta}_i$ and $T_f$ at $\Omega = 0.5$: ____ analytical solution; - - -, numerical solution. b) $\dot{\delta}_i$ and $T_f$ at $\Omega = 0.9$: ____ analytical solution; - - -, numerical solution. ................................................................................................................95

Figure 3.7 Frequency Responses for System of Fig. 4 Given $I_1 = 0.01$, $\zeta = 0.001$, $T_m = 0.5$, $T_p = 4.5$. a) Max-mean-min frequency responses of $\dot{\delta}_i$: --○--, analytical solution; --x--, numerical solution; b) Max-min frequency responses of $\dot{\delta}_2$: --o--, analytical solution; --x--, numerical solution. .......................................................96

Figure 3.8 Frequency Responses for System of Fig. 4 Given $I_1 = 0.01$, $\zeta = 0.001$, $T_m = 0.5$, $T_p = 4.5$. a) Max-mean-min responses of $\dot{\delta}_i$: --o--, $T_m = 0$; ++--, $T_m = 0.2$; b) Max-min responses of $\dot{\delta}_2$: --o--, $T_m = 0$; ++--, $T_m = 0.2$. ....................................................97

Figure 3.9 Frequency Responses for System of Figure 3.4 Given $I_1 = 0.04$, $\zeta = 0.001$, $T_m = 0.5$, $T_p = 1.0$. a) Max-min frequency responses of $\dot{\delta}_i$: --o--, analytical solution; --x--, numerical solution; b) Max-min frequency responses of $\dot{\delta}_2$: --o--, analytical solution; --x--, numerical solution. .......................................................99

Figure 3.10 Time Histories for System of Figure 3.4 Given $I_1 = 0.04$, $\zeta = 0.001$, $T_m = 0.5$, $T_p = 1.0$. a) $\dot{\delta}_i$ and $T_f$ at $\Omega = 0.5$: ____ analytical solution; - - -, numerical solution. b) $\dot{\delta}_i$ and $T_f$ at $\Omega = 0.9$: ____ analytical solution; - - -, numerical solution. ..............................................................................................................100

Figure 3.11 A Generic Non-Linear Frequency Response Map of $\delta_2$ under Significant Stick-Slip Motions..............................................................................................112

Figure 3.12 Frequency Response for System of Figure 1a Given $I_1 = 10$, $\zeta = 0.02$, $T_m = 0.5$, $T_p = 1.5$. a) Max-min frequency responses of $\dot{\delta}_i$: ooo, semi-analytical solution with 12 harmonic terms; ---, semi-analytical solution with 3 harmonic terms; xxx, numerical solution; b) rms frequency response of $\dot{\delta}_2$: ooo, semi-analytical solution with 12 harmonic terms; ---, semi-analytical solution with 3 harmonic terms; xxx, numerical solution. ..........................................................113

Figure 3.13 Time Histories at $\Omega = 0.23$ for System of Figure 3.1a Given $I_1 = 10$, $\zeta = 0.02$, $T_m = 0.5$, $T_p = 1.5$. a) harmonic excitation torque; b) time history of $\dot{\delta}_i$: ____,
semi-analytical solution; ____, numerical solution; c) time history of $T_f$: ____,
semi-analytical solution; ____, numerical solution; d) time history of $\delta_2$: ____,
semi-analytical solution; ____, numerical solution.................................114

Figure 3.14 Frequency Responses for System of Figure 1a Given. $I_1 = 10, \zeta = 0.02, T_m$
= 0, $T_p = 1.0$. a) Max-min frequency responses of $\delta_1$; b) rms frequency response
of $\delta_2$. ..................................................................................................................115

Figure 3.15 Time Histories and Fast Fourier Transform of $\delta_2$ Given $I_1= 10, \zeta = 0.02,$
$T_m = 0.5, T_p = 1.5$. a) $\Omega = 0.32$; b) $\Omega = 0.23$. ........................................116

Figure 3.16 3-D Response Map for System of Figure 3.1a Given $I_1= 10, \zeta = 0.02, T_m$
= 0.5, $T_p = 1.5$........................................................................................................118

Figure 3.17 Frequency Response for System of Figure 1a Given $I_1 = 10, \zeta = 0.02, T_m$
= 0.5, $T_p = 1.5$. a) Max-min frequency responses of $\delta_1$: ____, $\mu_k = 1.0$; ooo,
$\mu_k = 0.95$; xxx, $\mu_k = 0.9$; b) rms frequency response of $\delta_2$: ____, $\mu_k = 1.0$; ooo,
$\mu_k = 0.95$; xxx, $\mu_k = 0.9$. ..................................................................................121

Figure 3.18 Comparison between Semi-analytical Solution and Den Hartog’s
Analytical Solution Given $\varphi = 0$ and $T_p = 1.25$ For System of Figure 3.2. ooo,
semi-analytical solution with 12 harmonics; xxx, semi-analytical solution with 24
harmonics; -□-, Den Hartog’s analytical solution (from Figure 4 of reference
[3.1]). ........................................................................................................................124

Figure 3.19 3-D Response Map for System in Figure 3.2 Given $\varphi = 0$ and $T_p = 1.25$.
Semi-Analytical Solutions used to Construct this Map.................................125

Figure 4.1 Torsional systems with dry friction element. a) Classical dry friction
damper system; b) Dry friction path in an automotive system.........................132

Figure 4.2 Driveline System with Time-Varying Friction Torque. a) 3DOF model; b)
2DOF model ........................................................................................................135

Figure 4.3 Three kinetic friction coefficient formulations for the clutch liner: a)
$\mu_k = \mu_c$; b) $\mu_k < \mu_c$; c) $\mu_k > \mu_c$. key: ____, discontinuous formulation; ____,
smoothened formulation....................................................................................139

Figure 4.4 Typical time histories for 3DOF driveline system given $\omega = 100$ rad/s, $T_m$
= 300, $T_p = 250, T_{sm} = 350, T_{sp} = 0, \mu_s = \mu_k$ and $\Omega_e = 100$ rad/s. a) responses up
to 2.5 s using RK45; b) zoomed response from 1.5 to 1.6s using, ____, RK45, …,
Gear, ____, RK4 methods. .................................................................................144
Figure 4.5 Typical time histories for 3DOF driveline system given \( \omega = 150 \text{ rad/s}, T_m = 300; T_p = 250, T_{sm} = 350, T_{sp} = 1/6T_{sm}, \omega_r = \omega, \psi = 0, \mu_s = 1.2\mu_k \) and \( \Omega_e = 150 \text{ rad/s}. \) a) responses up to 2s using RK45; b) zoomed response from 1.22 to 1.3s using: ____, RK45, ____ Gear, ___, RK4 methods. ...............................................145

Figure 4.6 Effect of 2DOF system parameters on pure slip transients. a) effect of \( T_{sm}: \) ____ \( T_m/T_{sm} = 0.75, T_{sp} = 0; \ldots, T_m/T_{sm} = 0.75, T_{sp}/T_{sm} = 0.167; \ldots \) \( T_m/T_{sm} = 0.9, T_{sp}/T_{sm} = 0.167. \) b) effect of \( I_1: \) ____ \( I_1 = 0.2; \ldots. I_1 = 0.4. \) ........................................151

Figure 4.7 Effect of \( T_{sm} \) on the clutch engagement in Figure 4.2b given \( T_m/T_{sm} = 1.05 \) and \( T_{sp}/T_{sm} = 0.5. \) ..................................................................................................152

Figure 4.8 Effect of \( \mu_k \) on transient stick-slip responses of 2DOF system. a) \( \omega = 70 \text{ rad/s and } T_{sm} = 550 \text{ Nm; } \omega = 50 \text{ rad/s and } T_{sm} = 550 \text{ Nm. key: } ____ \( \mu_k = \mu_s; \ldots \) \( \mu_k = 0.75 \mu_s; \ldots \) \( \mu_k = 1.25 \mu_s. \) ........................................155

Figure 4.9 Effect of reduced \( T_{sm} \) and negative slope in friction torque on response of a 2DOF system when excited at \( \omega = 50 \text{ rad/s. a) } ____ \( T_m = 550, \mu_k = \mu_s; \ldots, T_m = 412.5, \mu_k = \mu_s; \) b) ____ \( T_m = 550, \mu_k = \mu_s, \ldots, T_m = 412.5, \mu_k = 0.75 \mu_s. \) 156

Figure 4.10 Effect of time-varying friction on stick-slip transients of a 2DOF system. a) effect of phase lag \( \psi \) given \( \omega = 60 \text{ rad/s and } \omega_r = \omega: ____ \( \psi = 0, \ldots, \psi = \pi/2, \ldots. \) \( \psi = \pi. \) b) effect of mismatched frequencies given \( \omega = 80 \text{ rad/s and } \psi = 0: ____ \( \omega_r = \omega; \ldots, \omega_r = 2\omega; \ldots. \) \( \omega_r = 0.5\omega. \) ..................................................157

Figure 4.11 Physical explanation on the effect of slip motion attenuation. All shaded areas represent the case for enhanced slip motions. a) effect of phase lag \( \omega_r = \omega): ____ T_e(t); \ldots, T_s(t) \) in phase; ____ \( T_e(t) \) not in phase. b) effect of mismatched frequency \( \psi = 0): ____ T_e(t); \ldots, \omega_r = \omega; \ldots, \omega_r > \omega. \) c) effect of mismatched frequency \( \psi = 0): ____ T_e(t); \ldots, \omega_r = \omega; \ldots, \omega_r < \omega. \) 158

Figure 4.12 Effect of time-varying friction \( T_s(t) \) on clutch judder. Key: ____ \( \mu_k = \mu_s \) and \( T_{sp} = 0; \ldots, \mu_k = 0.75 \mu_s \) and \( T_{sp} = 0; ____ \mu_k = 0.75 \mu_s \) and \( T_{sp} = 1/3 T_{sm} \) 159

Figure 4.13 Effect of smoothening friction law on maximum friction coefficient a) smoothened friction law; b) zoomed part. Key: ____ \( \mu_k = \mu_s; \ldots, \mu_k = 0.8 \mu_s; \ldots. \) \( \mu_k = 0.75 \mu_s; \ldots. \) \( \mu_k = 0.677 \mu_s. \) ..................................................162

Figure 4.14 Effect of \( \mu_k \) given a time-invariant friction torque.a) \( \delta_{\text{max}} \) maps. B) bifurcation diagrams with \( \omega/\omega_n \) as a bifurcation parameter. ..........................166

Figure 4.15 Bifurcation diagram employing \( \mu_k / \mu_s \) as a bifurcation parameter given \( \omega/\omega_n = 0.36. \) ..........................................................167
Figure 4.16 Sample time histories and Poincare sections given \( \omega/\omega_n = 0.36 \). a) \( \mu_k/\mu_s = 0.94 \); b) \( \mu_k/\mu_s = 0.74 \); c) \( \mu_k/\mu_s = 0.64 \) .......................................................... 168

Figure 4.17 Effect of phase \( \psi \) on steady-state response given \( \omega_f = \omega \) and \( T_{sp}/T_{sm} = 0.25 \). a) maximum frequency response map for \( \hat{\delta}_1 = \ldots x \ldots, \psi = 0 \); \( \ldots o \ldots, \psi = \pi/2 \); \( \ldots + \ldots, \psi = \pi \). b) bifurcation diagram .................................................. 171

Figure 4.18 Time delay in positive slip motion with respect to excitation. a) \( \omega/\omega_n = 0.15 \); b) \( \omega/\omega_n = 0.18 \); c) \( \omega/\omega_n = 0.37 \); d) \( \omega/\omega_n = 0.40 \) .............................................. 172

Figure 4.19 Effect of \( \omega_f \) on steady-state response given \( \psi = 0 \) and \( T_{sp}/T_{sm} = 0.25 \). a) maximum frequency response map for \( \hat{\delta}_1 = \ldots + \ldots, \omega_f = 0.5\omega \); \( \ldots x \ldots, \omega_f = \omega \); \( \ldots + \ldots, \omega_f = 2.0\omega \); b) bifurcation diagram .................................................. 173

Figure 4.20 Effect of \( \omega_f \) on steady-state response given \( \psi = 0 \) and \( T_{sp}/T_{sm} = 0.25 \). a) maximum frequency response map for \( \hat{\delta}_1 = \ldots + \ldots, \omega_f = 0.667\omega \); \( \ldots x \ldots, \omega_f = \omega \); \( \ldots o \ldots \), \( \omega_f = 1.33 \); b) bifurcation diagram .................................................. 174

Figure 4.21 Sample time histories for different \( \omega_f \) given \( \omega = 150 \), \( T_{sp}/T_{sm} = 1/4 \). a) \( \omega_f = 0.5\omega \); b) \( \omega_f = 0.67\omega \); c) \( \omega_f = 1.33 \omega \) .................................................. 175

Figure 4.22 Effect of \( \omega_f \) on steady-state response given \( \psi = 0 \) and \( \omega_f = \omega \) a) maximum frequency response map for \( \hat{\delta}_1 = \ldots x \ldots, T_{sp}/T_{sm} = 0 \); \( \ldots o \ldots, T_{sp}/T_{sm} = 0.125 \); \( \ldots x \ldots, T_{sp}/T_{sm} = 0.25 \); \( \ldots o \ldots, T_{sp}/T_{sm} = 0.50 \); b) bifurcation diagram .................................................. 176

Figure 4.23 Interaction between \( \mu_k(\hat{\delta}_1) \) and \( \psi \) given \( T_{sp}/T_{sm} = 0.25 \), \( \omega_f = \omega \) and \( \mu_k = 0.75\mu_s \). a) maximum frequency response maps of \( \hat{\delta}_{\max} = \ldots x \ldots, \psi = 0 \); \( \ldots o \ldots, \psi = \pi/2 \); \( \ldots + \ldots, \psi = \pi \). b) bifurcation diagram .................................................. 179

Figure 4.24 Interaction between \( \mu_k(\hat{\delta}_1) \) and \( \omega_f \) given \( T_{sp}/T_{sm} = 0.25 \), \( \omega_f = \omega \) and \( \mu_k = 0.67\mu_s \). a) maximum frequency response maps of \( \hat{\delta}_{\max} = \ldots x \ldots, \psi = 0 \); \( \ldots o \ldots, \psi = \pi/2 \); \( \ldots + \ldots, \psi = \pi \). b) bifurcation diagram .................................................. 180

Figure 4.25 Interaction between \( \mu_k(\hat{\delta}_1) \) and \( \omega_f \) given \( T_{sp}/T_{sm} = 0.25 \), \( \psi = 0 \) and \( \mu_k = 0.75\mu_s \). a) maximum frequency response maps of \( \hat{\delta}_{\max} = \ldots x \ldots, \omega_f = \omega \); \( \ldots + \ldots, \omega_f = 2\omega \); \( \ldots o \ldots, \omega_f = 0.5\omega \). b) bifurcation diagram .................................................. 181
Figure 4.26 Interaction between $\mu_k(\delta_i)$ and $\omega_f$ given $T_{sp}/T_{sm} = 0.25$, $\psi = 0$ and 
$\mu_k = 0.75 \mu_s$. a) maximum frequency response maps of $\delta_{i_{max}}$: .. x .., $\omega_f = \omega$; .. + 
.., $\omega_f = 0.67 \omega$; .. o .., $\omega_f = 1.33 \omega$. b) bifurcation diagram. .................................182

Figure 4.27 Interaction between $\mu_k(\delta_i)$ and $T_{sp}$ given $\omega_f = \omega$, $\psi = 0$ and 
$\mu_k = 0.75 \mu_s$. a) maximum frequency response maps of $\delta_{i_{max}}$: ___,$T_{sp}/T_{sm} = 0$; .. o .., $T_{sp}/T_{sm} = 0.125$; .. x .., $T_{sp}/T_{sm} = 0.25$; _ _ $T_{sp}/T_{sm} = 0.5$. b) bifurcation 
diagram..................................................................................................................................................183
LIST OF SYMBOLS

LIST OF SYMBOLS FOR CHAPTER 1

\[ C \] torsional viscous damping coefficient (N-m-s/rad)
\[ I \] torsional inertia (kg-m\(^2\))
\[ K \] torsional stiffness (N-m/rad)
\[ T \] torque (N-m)
\[ t \] time (s)
\[ V \] vehicle speed (km/h)
\[ \delta \] relative angular displacement (radian)
\[ \theta \] absolute angular displacement (radian)
\[ \mu \] friction coefficient
\[ \phi \] phase (radian)
\[ \Omega \] angular speed (rad/s)
\[ \omega \] circular frequency (rad/s)

Subscripts
\[ 1, 2, \ldots \] inertia element index
\[ c \] critical
\[ D \] drag
\[ e \] engine
\[ FS \] friction shoe
\[ f \] friction
\[ IMP \] impeller
\[ k \] kinetic
\[ m \] mean
\[ p \] pulsating
\[ s \] static
\[ sf \] saturation
\[ T \] transmission
\[ TB \] turbine

Superscript
\[ . \] first derivative with time

Abbreviations
2 or 3DOF two or three degree of freedom
MDOF multi-degree of freedom
SDOF  single degree of freedom
TCC  torque converter clutch

LIST OF SYMBOLS FOR CHAPTER 2

C  torsional viscous damping coefficient (N-m-s/rad)
F  force (N)
f  dimensionless force
I  torsional inertia (kg-m²)
K  torsional stiffness (N-m/rad)
k  dimensionless stiffness
m  dimensionless mass
P  period (s)
p  dimensionless stiffness
r  dimensionless mass
t  time (s)
T  torque (N-m)
V  relative velocity
Δ  small quantity
δ  relative angular displacement (radian)
ζ  viscous damping ratio
θ  absolute angular displacement (rad)
μ  friction coefficient
ξ  normalized absolute displacement
ρ  dimensionless friction force amplitude
σ  conditioning factor
τ  integration time (s)
ϕ  phase (rad)
ψ  phase lag (rad)
Ω  angular speed (rad/s)
ω  excitation frequency (rad/s)
Γ  test function (N-m)
Ξ  test function ((rad/s)²)
ℜ  decision function (N-m)
γ  normalized absolute displacement
φ  phase lag (rad)

Subscripts

1,2,3  inertial element indices
c  critical or transition
D  drag load
e  engine or equivalent
LIST OF SYMBOLS FOR CHAPTER 3

C  torsional viscous damping coefficient
D  differential operator matrix
f  function
I  torsional inertia
H  characteristic matrix
J  Jacobian matrix
K  torsional stiffness
P  period
R  residue vector
T  torque
t  time (dimensional)
Δ  discrete Fourier transform matrix
δ  relative angular displacement
ε  tolerance
ζ  viscous damping ratio
θ  absolute angular displacement
μ  friction coefficient
σ  conditioning factor
τ  dimensionless time
ϕ  phase angle
ψ  phase lag
Ω  excitation frequency (dimensionless)
ω  excitation frequency (rad/s)

Subscripts

1,2,3  inertial element indices
e  engine
f  friction
k  kinetic
m  mean
max  maximum
min  minimum
n  natural frequency
p  fluctuating component or perturbation
s  static
sf  saturation

Superscripts

_  dimensional value
.
..  first derivative with respect to time
..  second derivative with respect to time
'  first derivative with respect to dimensionless time
"  second derivative with respect to dimensionless time
-1  inverse
+  pseudo inverse
T  transpose

Operators

||  absolute value
\|\|  Euclidean or L₂ norm
< >ₜ  time-average operator
Abbreviations

2DOF 2 degree of freedom system
DFT discrete Fourier transform
HBM one term harmonic balance method
MHBM multi-term harmonic balance method
max maximum value
min minimum value
rms root-mean-square value
SDOF single degree of freedom system
TCC torque converter clutch

LIST OF SYMBOLS FOR CHAPTER 4

A area of actuation pressure
C torsional viscous damping coefficient
I torsional inertia
K torsional stiffness
M mass matrix
N normal force
P pressure
R moment arm
T torque
t time
α exponentially decaying factor
δ relative angular displacement
ζ viscous damping ratio
θ absolute angular displacement
µ friction coefficient
ψ phase lag
Ω angular speed
ω angular frequency

Subscripts

1, 2, 3 inertial element indices
c constraint
d damped
e engine
f friction
k kinetic
m mean
max maximum
n natural frequency
p  fluctuating component or perturbation
s  static or saturation
ss  steady-state
t  transmission

Superscripts

_  normalized value
.  first derivative with respect to time
..  second derivative with respect to time
T  transpose

Operators

||  absolute value
< >_t  time-average operator

Abbreviations

2DOF  2 degree of freedom system
HBM  one term harmonic balance method
MDOF  multi-degree of freedom
MHBM  multi-term harmonic balance method
max  maximum value
NLTV  non-linear time-varying
SDOF  single degree of freedom system
TCC  torque converter clutch

LIST OF SYMBOLS FOR CHAPTER 5

T  torque (N-m)

Subscript
m  mean

Abbreviations

3DOF  three degree of freedom
MHMB  multi-term harmonic balance method
CHAPTER 1

INTRODUCTION

1.1 Dry Friction Path

Dry friction elements are commonly found in many mechanical and structural systems [1.1-1.10]. For example, consider the automotive torque converter clutch (TCC) sub-system that consists of a fluid torque converter and in parallel a mechanical dry friction clutch as shown in Figure 1.1a. When the vehicle speed \( V \) is low (say in the first gear), the dry friction clutch is fully disengaged and only the fluid torque converter path is operational, as indicated in Figure 1.1c. The pump drives the turbine with a torque generated by a change in the momentum of the fluid. Torque amplification is allowed and a smooth shift or transition is allowed [1.11]. Improved vehicle ride quality is achieved through the fluid coupling. However, fuel efficiency is reduced as a result of significant difference between the speed of impeller and turbine. At a higher speed (say in the fourth gear), the mechanical clutch is fully engaged and the fluid path is no longer functional. Under this condition, the transmission is directly driven by the engine. The energy dissipated within the torque converter is minimized to enhance the fuel efficiency. Nonetheless, the engine torque fluctuations are also transmitted to the downstream driveline system that would deteriorate the ride quality.
Figure 1.1 Torque converter clutch sub-system a) physical system; b) simplified torsional model; c) operational ranges as a function of vehicle speed.
Over the mid-speed range (2\textsuperscript{nd} or 3\textsuperscript{rd} gear), the TCC is partially engaged and both the dry friction clutch and the fluid torque converter transmit torque [1.12-1.16]. The bypass mechanical clutch is intentionally made to slip since a slipping clutch is expected to achieve a best compromise between fuel efficiency and riding quality. Although sparse information is available in the open literature on the power distribution among the mechanical and fluid paths as well their control, the TCC is typically designed to transmit very high torque loads and to suppress a large slip ($\Omega_e - \Omega_t$) between the engine and transmission speeds to avoid overheating [1.13].

For instance, Kono et. al. [1.14] have illustrated the power distribution of a Toyota automatic transmission (A541) as a function of slip speed $\Delta\Omega = \Omega_e - \Omega_t$ between the torque converter input shaft (impeller) and output shaft (turbine), as shown in Figure 1.2. At low slip speeds, more power is carried by the mechanical path. For this reason, our study will focus on the mechanical dry friction path. As a consequence of significant torque pulsations from the engine, the stick-slip phenomenon often takes place within the TCC. The resulting stick-slip could generate several vibration problems in the driveline system. To fully understand the dynamic effects of stick-slip and related phenomena within a driveline system with TCC, we must analytically examine the non-linear dynamic characteristics of a dry friction path in a multi-degree-of-freedom torsional system. It is the main focus of this dissertation.
Figure 1.2 Power distribution between mechanical and fluid paths in a Toyota A541 automatic transmission [1.14].

1.2 Literature Review

1.2.1 Typical Dry Friction Models

Several dry friction formulations have been proposed based on the classical (discontinuous) Coulomb model that is illustrated in Figure 1.3a. For instance, Karnopp [1.17] developed a stick-slip friction law in which the friction force or torque $T_f$ is defined as a function of the relative velocity $\dot{\delta}$. A small region of velocity $\pm \Delta \dot{\delta}$ is defined as the stick stage in Figure 1.3b. Beyond this, the friction interface is considered to be in the slip stage. Following Iwan’s bi-linear hysteresis formulation [1.18], Menq et. al. [1.19] proposed a modified micro-slip model by modeling the friction interface in terms of an elasto-plastic shear layer. Three different states are defined in Figure 1.3c: purely elastic, partial slip and total slip. When the relative
Figure 1.3 Selected dry friction formulations. (a) Classical (discontinuous) Coulomb model; (b) Karnopp Model [1.17]; (c) Menq et. al.’s models [1.19]: --- , elastic bar on rigid base; ___ , elastic bar with an elastoplastic shear layer on a rigid base; (d) Model by Imamura et. al. [1.20];
displacement reaches a certain value $\delta_c$, the deformation between interfaces becomes plastic and gross slip takes place. Consequently, the friction force reaches a saturation value $T_{sf}$. This model could further consider unloading and reloading cases, in which the material memory effect is taken into account. Imamura et. al. [1.20] developed a new friction torque model as shown in Figure 1.3d with an application to automotive clutch system. Unlike the bi-linear hysteresis model in which $\delta_c$ is defined at the kinetic transition point, a critical velocity $\dot{\delta}_c$ is defined here. The friction torque varies linearly with $\dot{\delta}$ within the critical bounds $\pm \dot{\delta}_c$. Finally, other models could be derived from the aforementioned formulations by selecting the static friction coefficient $\mu_s$ or kinetic friction coefficient $\mu_k$, or by assuming a specific $\mu_k(\dot{\delta})$ relation.

1.2.2 Solution Methods

Several analytical or computational methods have been applied to solve the dry friction problems. Den Hartog [1.1] initiated research in this area by analytically solving the forced vibration problem of a single degree of freedom (SDOF) system with Coulomb friction. He determined an equivalent viscous damping value under pure slipping case and then found the time history with a limit of no more than 2 stops. With the assumption of periodic and aperiodic motions, Pratt and Williams [1.2] extended Den Hartog’s work and calculated the system response with multi-lockups by using a shooting method. Also, they calculated the equivalent viscous damping on a
time-averaged basis. Shaw [1.3] extended the previous work by including different static $\mu_s$ and kinetic $\mu_k$ coefficients; he also conducted bifurcation and stability analyses. Menq et. al. [1.4, 1.21] and Wang et. al. [1.5, 1.22] used a multi-term harmonic balance method to find the dynamic response of a bi-linear hysteresis problem. Ferri and Heck [1.23] and Ferri [1.6] solved a two degree of freedom (2DOF) system by using a modified singular perturbation method. In their method, the system order is reduced to eliminate the numerical stiffness problem. Van De Vrande et. al. [1.7] solved a dry friction induced stick-slip problem by smoothening the discontinuous dry friction force with an arc-tangent function and then studied the autonomous systems from the phase plane viewpoint. Leine et. al. [1.8] proposed an alternate friction model using a concept similar to Karnopp’s formulation [1.17]. Berger et. al. [1.24] proposed a mixed differential-algebraic equation approach that uses differential equations to describe the slipping dynamics and algebraic equations to model the interfacial sticking. In their method, a zeroth-order optimization algorithm is proposed to detect the transition from stick to slip.

1.2.3 Unresolved Research Issues

Three unresolved issues in this area emerge. First, many investigators have focused on SDOF systems with an application to the dry friction damper. In such systems, the saturation friction force is relatively small compared to the excitation force amplitude and the dry friction damper is used mainly to absorb or dampen forced vibrations [1.1-1.10]. Further, much research has focused on the description of friction interfacial regimes [1.17-1.20]. Nonetheless, some dry friction elements in real-life
mechanical systems, rather than dissipating energy, act as key energy transmitting paths. For instance, the friction capacity of the dry friction clutch (TCC) could be as high as 450 N-m, which is comparable to the peak value of the dynamic torque generated by the engine [1.20]. In this case, TCC cannot be treated as a purely frictional vibration damper. Instead, the friction torque from TCC becomes a dominant excitation to the subsequent torsional driveline sub-system (downstream of TCC) consisting of gearbox, propeller shaft, differential, axle and wheels. Consequently, the dynamics of the driveline system is significantly determined by the localized stick-slip motions within TCC. However, the scientific literature on this effect of dry friction path on system dynamics is very limited. For example, some researchers have focused on the bi-linear hysteresis problem by assuming a massless link between the spring and Coulomb friction elements [1.4, 1.5, 1.9, 1.21, 1.22]. However, that is not always true for some physical systems since a small secondary inertial element could exist. Effect of such a secondary inertia (or mass) on system dynamics is still not well understood though Ferri and Heck [1.21] briefly mentioned its significance in the context of a dry friction damper. Recently Berger and Krousgrill [1.9] examined the role of a secondary mass in dissipating energy and evaluated its influence on the kinetic state of the damper. They found that the non-zero secondary mass damper would substantially attenuate the resonant response when compared with a massless bi-linear system.

Second, most of previous work on dry friction damper system has focused on primary harmonic resonance given harmonic excitation [1.1, 1.2, 1.4-1.6]. However, for a system in which dry friction element is a dominant path, super-harmonics could
appear and may actually dictate the responses in time or frequency domain. The appearance of such super-harmonics needs to be identified via analytical or semi-analytical methods. Also, a non-zero mean torque is typical for many real-life systems and its effects on system dynamics are yet to be fully investigated.

Third, the effects of a time varying dry friction formulation on system dynamics are unknown. This is because literature has focused on time-invariant friction torque or force, especially when the saturation forces or torques are small [1.1-1.10]. One may find physical processes where the contact loads may change periodically, such as in gear pairs [1.25-1.26] or in automotive system where the actuation pressure varies with time [1.13-1.16]. Thus, to obtain a complete understanding of system, the effects of a time varying friction formulation on both transient and steady state responses must be determined.

1.3 Problem Formulation

1.3.1 Torsional System with Dry Friction Controlled Path

To investigate the dynamics of a multi-degree-of-freedom (MDOF) system subject to stick-slip motion, the propulsion system can be reduced to a simplified 3 DOF semi-definite torsional model as in Figure 1.4a, which is conceptually similar to the manual transmission formulation employed by Padmanabhan and Singh [1.27] to study gear rattle and to the automatic transmission model utilized by Yamada and Ando [1.28] to examine clutch judder. The slipping clutch is represented by a pure dry friction element; also two viscous damping elements are added to this system.
Figure 1.4. Reduced order driveline torsional systems with dry friction non-linearity. 
a) 3DOF semi-definite model with $C_{13}$ and $C_{23}$ elements; b) simplified 3DOF system with only $C_{13}$ element; c) 2DOF definite system with a dominant mechanical path, non-negligible $C_{23}$ and significant $I_3$. 

\[ T_e(t) = T_m + T_p(t) \]

\[ \theta_1 \quad \theta_2 \quad \theta_3 \]

\[ I_1 \quad I_2 \quad I_3 \]

\[ T_{\text{fr}2} \quad K_{23} \quad C_{23} \]

\[ C_{13}(\dot{\theta}_1) \]

\[ T_D \rightarrow \infty \]
In Figure 1.4a and b, $I_1$ represents the flywheel and impeller, $I_2$ is the friction shoe and $I_3$ is the reflected inertia of the rest of the driveline system. The absolute angular displacements are $\theta_1$, $\theta_2$ and $\theta_3$. Further, $K_{23}$ is the equivalent driveline stiffness and $C_{23}$ is the equivalent viscous damping. An engine speed-dependent damping represents the fluid path, i.e. $C_{13} = C_{13}(\dot{\theta}_1)$ and, $T_{f12}$ is the friction torque transmitted by the dry clutch and it is a function of the relative velocity $\dot{\delta}_1 = \dot{\theta}_1 - \dot{\theta}_2$ across the dry friction interface. Finally, $T_e(t)$ represents the dynamic torque excitation, $T_m$ is the mean torque, and $T_D$ is the drag torque.

Depending on the objective of a particular study, the torsional system of Figure 1.4a could be reduced to two sub-sets. First, to focus on the non-linear dynamics of a 3DOF torsional system and to examine the effect of a secondary inertia, $C_{13}(\dot{\theta}_1)$ is assumed to be a linear viscous damping term and $C_{23}$ is excluded for the sake of simplicity. The resulting simplified model is shown in Figure 1.4b. Second, to examine the non-linear effect of a dominant dry friction path, the damping term $C_{13}$ representing the fluid path is neglected. Nonetheless, a small damping element $C_{23}$ is included to represent the effective damping in the downstream driveline sub-system. Further, the inertia of downstream sub-system ($I_3$) employed to study is very large compared with $I_1$ and $I_2$ in a realistic automotive system. Consequently, the 3DOF semi-definite system in Figure 1.4a is further modified to a 2DOF definite system as in Figure 1.4c. Again, $I_1$ represents the combined torsional inertia of engine, flywheel, front cover and impeller and $I_2$ is the inertia of friction shoe assembly.
In all models, the engine torque $T_e(t)$ is composed of mean $T_m = \langle T_e \rangle$, and pulsating (dynamic) $T_p(t)$ components, where $\langle \rangle$ is the time-average operator. Using the Fourier series expansion, define $T_e(t) = T_m + \sum_n T_{pn} \cos(\omega_{pn} t + \phi_{pn})$, where $n$ is the harmonic order of the firing sequence, $\omega_{pn} = (N_e / 2)n\Omega_e$, $\Omega_e$ is the engine speed, $N_e$ is the number of engine cylinders \[1.29\], $T_{pn}$ is the amplitude for the $n^{th}$ harmonic and $\phi_{pn}$ is the associated phase lag. The drag load $T_d(t)$ consists of losses with the gearbox, wheel rolling resistance and aerodynamic drag. It is further assumed that the vehicle speed is constant and the vehicle drag and the mean engine load are balanced under steady state condition, $T_m = T_d$.

1.3.4 Scope and Assumptions

Primary scope and assumptions of the proposed work are as follows:

1. Only 2DOF and 3DOF torsional systems of Figure 1.4 are studied. Although many degrees of freedoms are necessary to represent a realistic automotive driveline system, a reduced order system can efficiently capture the key phenomena over the frequency range of interest.

2. This study focuses on the mechanical path and the fluid path is modeled as a linear viscous damper or is neglected. Further, only one dry friction non-linear element (path) is included. Although many friction models have been proposed in literature \[1.17-1.20\], the following non-linear friction law used in our study \[1.30\].
Further, a hyper-tangent function is used to smoothen the discontinuous friction law when necessary, where $\sigma$ is the conditioning factor.

\[
\mu(\dot{\delta}_1) = \left[ \mu_k + (\mu_s - \mu_k)e^{-\psi|\dot{\delta}|} \right] \text{sgn}(\dot{\delta}_1) \\
\leq \mu_s \quad \dot{\delta}_1 > 0
\]

(1.1)

Figure 1.5 shows typical smoothened and discontinuous friction relationship given $T_{sf} = 350$ N-m and $\mu_k = \mu_s$.

Figure 1.5 Smoothened friction formulation $T_f(\dot{\delta}_1)$ as employed in our study with a variable conditioning factor $\sigma$ given saturation torque $T_{sf} = 350$ N-m and $\mu_k = \mu_s$. Key for (e): _ _ _ _ , $\sigma=50$; -- -- -- , $\sigma=10^2$; . . . , $\sigma=10^3$; _____ , $\sigma = 10^4$
3. Although the engine torque is composed of multiple harmonic terms as discussed in previous section, only the mean and dominant pulsating term is used for the sake of simplicity, i.e. \( T_e(t) = T_m + T_p \sin(\omega_p t) \).

4. When applied, the time varying dry friction (or actuation pressure) is assumed as a combination of sinusoidal and mean terms. Positive definite saturation friction force is assumed to ensure that no separation across the frictional interface occurs.

5. Static and kinetic friction coefficients are assumed to be different only when the judder problem is considered.

1.3.5 Objectives

Chief objectives of this study are as follows:

1. Calculate non-linear time and frequency domain response of a torsional system using numerical methods. A procedure to predict pure stick to stick-slip boundaries will be first developed based on linear system theory. Both smoothened and discontinuous friction formulations will be examined and effect of smoothening factor will be investigated. Further, effect of the secondary inertia will be studied and compared with the conventional bi-linear system.

2. Develop new or refined semi-analytical methods to construct non-linear frequency responses and investigate effect of mean torque. An analytical method will be first developed based on assumed torque profile given
continuous slipping motion. Then a refined multi-term harmonic balance method will be proposed. Associated computational issues will be addressed and a procedure for selecting an initial guess for Newton Raphson scheme will be developed. Super-harmonics induced by significant stick-slip motions will be identified and effects of mean torque will be studied.

3. Investigate the effect of time varying dry friction on transient and steady state responses of a torsional system with dry friction path. Three response stages, namely pure slip, transient stick-slip and steady state stick-slip, will be identified. Analytical solution will be obtained for pure slipping motion. The effect of time varying dry friction on transient stick-slip and steady state responses will be examined. Further, interaction of friction characteristics and time varying dry friction will be studied.

Objective 1, 2 and 3 are addressed in Chapter 2, 3 and 4 respectively. Each chapter is self-sufficient in terms of problem formulation, results and conclusion.

References for Chapter 1


CHAPTER 2

DYNAMICS OF A 3DOF TORSIONAL SYSTEM WITH A DRY FRICTION CONTROLLED PATH

2.1 Introduction

Dry friction elements are commonly found in many mechanical and structural systems. For example, consider the automotive torque converter clutch (TCC) subsystem that consists of a fluid torque converter and in parallel a mechanical dry friction clutch as shown in Figure 2.1. When the engine speed $\Omega_e$ is low, the dry friction clutch is fully disengaged and only the fluid torque converter path is operational. The pump drives the turbine with a torque generated by a change in the momentum of the fluid. Torque amplification is allowed and a smooth shift or transition is allowed [2.1]. At a higher speed, the mechanical clutch is fully engaged and the fluid path is no longer in effect. Under this condition, the transmission is directly driven by the engine. The energy dissipated within the torque converter is minimized to enhance the fuel efficiency. Over the mid-speed range, the TCC is partially engaged and both the dry friction clutch and the fluid torque converter transmit torque [2.2-2.3]. The TCC is designed to transmit very high torque loads and to suppress a large slip between engine and transmission to avoid overheating. However, the stick-slip phenomenon often takes place within the TCC as a
Figure 2.1 Schematic of a typical automotive torque converter and dry friction clutch (TCC)
consequence of significant torque pulsations from the engine. The resulting stick-slip could excite several vibration problems in the driveline system, thereby reducing the vehicle ride quality. To study the dynamic effects of stick-slip within a driveline system with TCC, we will study the non-linear dynamic characteristics of a three degree of freedom (3DOF) semi-definite torsional system with a dry friction controlled path.

2.2 Literature Review and Research Issues

2.2.1 Typical Dry Friction Models

Several dry friction formulations have been proposed based on the classical (discontinuous) Coulomb model that is illustrated in Figure 2.2a. For instance, Karnopp [2.4] developed a stick-slip friction law in which the friction force or torque \( T_f \) is defined as a function of the relative velocity \( \dot{\delta} \). A small region of velocity \( \pm \Delta \dot{\delta} \) is defined as the stick stage in Figure 2.2b. Beyond this, the friction interface is considered to be in the slip stage. Following Iwan’s bi-linear hysteresis formulation [2.5], Menq et. al. [2.6] proposed a modified micro-slip model by modeling the friction interface in terms of an elasto-plastic shear layer. Three different states are defined in Figure 2.2c: purely elastic, partial slip and total slip. When the relative displacement reaches a certain value \( \delta_c \), the deformation between interfaces becomes plastic and the gross slip takes place. Consequently, the friction force reaches a saturation value \( T_{sf} \). This model could further consider unloading and reloading cases,
Figure 2.2 Selected dry friction formulations. (a) Classical (discontinuous) Coulomb model; (b) Karnopp Model [4]; (c) Menq et. al.'s models [6]: \ldots, elastic bar on rigid base; \ldots, elastic bar with an elastoplastic shear layer on a rigid base; (d) Model by Imamura et. al. [7]; (e) Smoothened friction formulation $T_f(\dot{\delta})$ of our study with a variable conditioning factor $\sigma$ and with saturation torque $T_{sf} = 350$ N-m: Key for (e): \ldots, $\sigma = 50$; \ldots, $\sigma = 10^2$; \ldots, $\sigma = 10^3$; \ldots, $\sigma = 10^4$. 
22
in which the material memory effect is taken into account. Imamura et. al. [2.7] developed a new friction torque model as shown in Figure 2.2d with an application to automotive clutch system. Unlike the bi-linear hysteresis model in which $\delta_c$ is defined at the kinetic transition point, a critical velocity $\dot{\delta}_c$ is defined here. The friction torque varies linearly with $\dot{\delta}$ within the critical bounds $\pm \dot{\delta}_c$. Finally, other models can be derived from the aforementioned formulations by selecting the static friction coefficient $\mu_s$ or kinetic friction coefficient $\mu_k$, or by assuming a specific $\mu_k(\dot{\delta})$ relation.

2.2.2 Solution Methods

Several analytical or computational methods have been applied to solve the dry friction problems. Den Hartog [2.8] initiated research in this area by analytically solving the forced vibration problem of a single degree of freedom (SDOF) system with Coulomb friction. He determined an equivalent viscous damping value under pure slipping case and then found the time history with a limit of no more than 2 stops. With the assumption of periodic and aperiodic motions, Pratt and Williams [2.9] extended Den Hartog’s work and calculated the system response with multi-lockups by using a shooting method. Also, they calculated the equivalent viscous damping on a time-averaged basis. Shaw [2.10] extended the previous work by including different static $\mu_s$ and kinetic $\mu_k$ coefficients; he also conducted bifurcation and stability
analyses. Menq et. al. [2.11-2.12] and Wang et. al. [2.13-2.14] used a multi-term harmonic balance method to find the dynamic response of a bi-linear hysteresis problem. Ferri and Heck [2.15] and Ferri [2.16] solved a two degree of freedom (2DOF) system by using a modified singular perturbation method. In their method, the system order is reduced to eliminate the numerical stiffness problem. Van De Vrande et. al. [2.17] solved a dry friction induced stick-slip problem by smoothening the discontinuous dry friction force with an arc-tangent function and then studied the autonomous systems from the phase plane viewpoint. Leine et. al. [2.18] proposed an alternate friction model using a concept similar to Karnopp’s formulation. Berger et. al. [2.19] proposed a mixed differential-algebraic equation approach that uses differential equations to describe the slipping dynamics and algebraic equations to model the interfacial sticking. In their method, a zeroth-order optimization algorithm is proposed to detect the transition from stick to slip.

2.2.3 Some Unresolved Problems

Many investigators have focused on SDOF systems with application to the dry friction damper. In such systems, the friction force is relatively small compared to the excitation force amplitude and the dry friction damper is used mainly to absorb or dampen forced vibration [2.8-2.16, 2.20]. Further, much research has focused on the description of friction interfacial regimes [2.4-2.6]. Nonetheless, dry friction elements invariably exist in many real-life mechanical systems and rather than dissipating energy, the dry friction path acts as a key energy transmitting element. For instance,
the friction capacity of the dry friction clutch (TCC) could be as high as 450 N-m, which is comparable to the peak value of the dynamic torque generated by the engine [2.7]. In this case, TCC cannot be considered as a purely frictional vibration damper. Instead, the friction torque from TCC becomes a dominant excitation to the subsequent torsional driveline sub-system (downstream of TCC) consisting of gearbox, propeller shaft, differential, axle and wheels. Consequently, the dynamics of the driveline system is significantly determined by the localized stick-slip motions within TCC. However, the scientific literature on this effect of dry friction on system dynamics is very limited. For example, some researchers have focused on a bi-linear hysteresis problem by assuming a massless link between the spring and Coulomb friction elements [2.5-2.6, 2.11-2.14]. However, that is not always true for some physical systems since a small secondary lumped inertial element could exist. Effect of such a secondary inertia on the system dynamics is still not well understood though Ferri and Heck [2.15] briefly mentioned the significance on a secondary mass in the context of a dry friction damper. Recently Berger and Krousgrill [2.20] examined the role of a secondary mass in dissipating energy and evaluated its influence on the kinetic state of the damper. They found that the non-zero secondary mass damper would substantially attenuate the resonant response when compared with a massless bi-linear system.
Figure 2.3 Physical Example: 3DOF semi-definite torsional model of an automotive driveline system. Here, $T_f(\dot{\delta})$ represents a dry friction path.

$$T_e(t) = T_m + T_p(t)$$
2.3 Problem Formulation

2.3.1 Torsional System with Dry Friction Controlled Path

The driveline system can be reasonably represented by a 3DOF semi-definite system with focus on the TCC sub-system. This is conceptually similar to the manual transmission formulation employed by Padmanabhan and Singh [2.21] to study gear rattle and to the automatic transmission model utilized by Yamada and Ando [2.22] to examine clutch judder. As shown in Figure 2.3, I₁ represents the combined torsional inertia of flywheel, front cover and impeller, I₂ is the inertia of friction shoe assembly and I₃ is the reflected torsional inertia of the rest of the driveline system. The governing equations for this 3DOF semi-definite system with a non-linear dry friction path as given by \( T_f \) are:

\[
I_1 \ddot{\theta}_1 + C(\dot{\theta}_1)(\dot{\theta}_1 - \dot{\theta}_2) + T_f(\delta_1, \dot{\delta}_1) = T_e(t)
\]  \hspace{1cm} (2.1)

\[
I_2 \ddot{\theta}_2 + K(\theta_2 - \theta_3) - T_f(\delta_1, \dot{\delta}_1) = 0
\]  \hspace{1cm} (2.2)

\[
I_3 \ddot{\theta}_3 - C(\dot{\theta}_1)(\dot{\theta}_1 - \dot{\theta}_2) - K(\theta_2 - \theta_3) = -T_D(t)
\]  \hspace{1cm} (2.3)

Here, \( \theta_1, \theta_2 \) and \( \theta_3 \) are absolute angular displacements, \( C(\dot{\theta}_1) \) is the engine speed-dependent viscous damping term which represents the fluid path, \( K \) is the linear torsional stiffness, \( T_e(t) \) is the engine torque (including mean and dynamic terms) and \( T_D(t) \) is the drag load as experienced by the driveline. Further, in Equation (2.1), \( T_f(\delta_1, \dot{\delta}_1) \) is a function of the relative displacement \( \delta = \theta_1 - \theta_2 \) and relative
<table>
<thead>
<tr>
<th>Parameters and Excitation</th>
<th>Value(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsional Inertias (kg-m²)</td>
<td>$I_1 = 0.20$, $I_2 = 0.02$, $I_3 = 6.6$</td>
</tr>
<tr>
<td>Torsional Viscous Damping (Nm-rad/s)</td>
<td>$C = 1.0$</td>
</tr>
<tr>
<td>Torsional Stiffness (Nm/rad)</td>
<td>$K = 1010$</td>
</tr>
<tr>
<td>Saturation Friction Torque (Nm)</td>
<td>$T_{sf} = 350$</td>
</tr>
<tr>
<td>Torque Excitation Amplitude (Nm)</td>
<td>$T_m = 100$, $T_p = 250$</td>
</tr>
</tbody>
</table>

Table 2.1 Values of parameter and excitation amplitude used for simulating the system of Figure 2.3
velocity $\delta_1 = \dot{\theta}_1 - \dot{\theta}_2$ across the dry friction interface. When the relative motions are of interest, the system can be further reduced to the following 2DOF definite system, where $\delta_2 = \theta_2 - \theta_3$ and $\dot{\delta}_2 = \dot{\theta}_2 - \dot{\theta}_3$:

$$I_2 \ddot{\delta}_1 + C(\dot{\theta}_1) \frac{I_2}{I_1} (\dot{\delta}_1 + \dot{\delta}_2) - K \delta_2 + \frac{I_1 + I_2}{I_1} T_f (\delta_1, \dot{\delta}_1) = \frac{I_2}{I_1} T_e(t) \quad (2.4)$$

$$I_2 \ddot{\delta}_2 + C(\dot{\theta}_1) \frac{I_2}{I_3} (\dot{\delta}_1 + \dot{\delta}_2) + \frac{I_2 + I_3}{I_3} K \delta_2 - T_f (\delta_1, \dot{\delta}_1) = \frac{I_2}{I_3} T_D(t) \quad (2.5)$$

Table 2.1 lists typical values of parameters and excitation used for simulation studies.

2.3.2 Scope, Assumptions and Objectives

The engine torque $T_e(t)$ is composed of mean $T_m = <T_e>$, and pulsating $T_p(t)$ components, where $<$, $>$, is the time-average operator. Using the Fourier series expansion, express it as $T_e(t) = T_m + \sum_n T_{pn} \cos(\omega_{pn} t + \phi_{pn})$, where $n$ is the harmonic order of the firing sequence, $\omega_{pn} = (N_e / 2)n\Omega_e$, $N_e$ is the number of engine cylinders [2.23, 2.24], $T_{pn}$ is the amplitude for the $n^{th}$ harmonic and $\phi_{pn}$ is the associated phase lag. In this chapter, only the fundamental term ($n = 1$) is considered for the sake of simplicity and the phase angle is assumed to be zero. The drag load $T_d(t)$ consists of wheel rolling resistance and aerodynamic drag. It is further assumed that the vehicle
speed is constant and the vehicle drag and the mean engine load are balanced, \( T_m = T_D \).

Even when the TCC is partially engaged, most of the torque is transmitted by the mechanical path [2.3]. Thus the driveline system dynamics is assumed to be mostly affected by the mechanical stick-slip motions. In this case, the fluid path term \( C(\dot{\theta}) \) is further approximated by a linear viscous damping \( C \). In a real system, \( C \) could be around 1.0 N-m-s/rad [2.1]. Further, under a high normal load, the shear stiffness of the friction interface is assumed to be very large. Consequently, the discontinuous Coulomb model is used and \( \mu_s = \mu_k \) is assumed. The normal force on the friction interface remains unchanged and the friction torque \( T_f \) capacity is fixed accordingly.

Given a non-linear dry friction problem, time domain integration methods are usually employed [2.4, 2.8, 2.9, 2.18]. However, the solution process consumes significant time and yet the resulting time history at a certain frequency (under a given mean load) does not provide an overall characterization of the non-linear system from the design standpoint. Further, actual time histories may not be important for some engineering applications. Consequently, steady state frequency response characteristics must be constructed to provide better and easy-to-follow dynamic design guidelines [2.25]. This chapter will accordingly also examine the non-linear frequency response of the driveline system since it is rarely discussed in the dry friction literature [2.21, 2.25]. Finally, for any non-linear system, super or sub-harmonic responses, multi-valued equilibrium points, quasi-periodic or chaotic
responses may be present. However, such non-linear issues will be addressed in a future chapter.

Chief objective of this study is to investigate the non-linear dynamics of the 3DOF system representing the torsional driveline system subject to the localized stick-slip motions of the dry friction path. In particular, the effect of the secondary inertia \( I_z \) that represents the friction shoe assembly here is studied and compared with the conventional bi-linear system. Two solution methods, namely the smoothened friction model and discontinuous Coulomb model, are employed. The effect of the conditioning factor \( \sigma \) (as defined later) on system dynamics will be studied. Approximate analytical solutions, based on assumed states (such as pure stick, positive or negative slip) will also be developed and compared with computational solutions. Time histories are presented to assist the frequency domain analyses. The stick to slip boundaries will be first determined based on the linear system theory and then examined using the non-linear models. The effect of the friction torque amplitude (a path parameter) is of particular interest. Further, the effect of secondary inertia will be carefully examined using both numerical and approximate analytical solutions. Finally, our models are applied to two benchmark analytical and experimental studies [2.15, 2.26] for the sake of validation.

### 2.4 Smoothened and Discontinuous Friction Torque Models

The classical Coulomb model is \( T_f(\dot{\delta}) \) a non-smooth function since it is discontinuous with respect to \( \dot{\delta} \) and a singularity exists at \( \dot{\delta} = 0 \). For this reason, the
direct numerical integration scheme cannot be applied. To overcome this difficulty, a 
smoothening procedure could be used to condition the abrupt transitions. Ways to 
osmoothen a discontinuous function using arc-tangent, hyperbolic tangent, hyperbolic-
cosine or quintic spline functions have been proposed by Kim et. al. [2.27] with an 
application to the clearance non-linearity. In their study, hyperbolic tangent or arc-
tangent function were preferable because of their applicability to both direct time 
domain integration and semi-analytical (such as multi-term harmonic balance) 
methods. The smoothened friction model can be described by a continuous but still a 
strongly non-linear function as shown below in terms of the hyperbolic tangent 
function. The resulting non-linear friction torque is given by:

\[ T_f = T_{sf} \tanh(\sigma \dot{s}) \] (2.6)

where \( T_{sf} \) is the saturation torque (torque at the kinetic state) and \( \sigma \) is the 
conditioning factor that control the abruptness of the transition as illustrated in Figure 
2.2e. The higher the value of \( \sigma \) is, the more abrupt the transition is. An extremely 
high value of \( \sigma \) would yield a profile that would resemble the classical Coulomb 
model. Further the smoothened friction model of equation (2.6) yields a unique \( T_f \) 
corresponding to a certain value of \( \dot{s} \). Thus the friction torque would be viewed as an 
“active torque” with \( \dot{s} \). The smoothening procedure is used mainly to facilitate the 
direct numerical integration scheme as \( T_f \) becomes a smooth function of \( \dot{s} \) over the 
entire vector space. Conversely, the discontinuous friction torque model, as given
below, treats $T_f$ as a “passive torque” that is determined by external excitation and system response. Here, $T_f$ can assume any value between $-T_{sf}$ and $T_{sf}$ at $\dot{\delta}_1 = 0$:

$$T_f = \begin{cases} T_{sf} & \dot{\delta}_1 > 0 \\ [-T_{sf} \ T_{sf}] & \dot{\delta}_1 = 0 \\ -T_{sf} & \dot{\delta}_1 < 0 \end{cases} \quad (2.7)$$

### 2.5 Computation of Stick to Slip Boundaries Based on Linear System Theory

In the case of a high friction torque $T_{sf}$, a closer observation would find that the friction interface is under purely the stick condition over some frequencies. A simple linear system analysis is conducted to find this frequency regime. The schematic of a system under the pure stick condition is shown in Figure 2.4a. In this case, $I_1$ and $I_2$ move together as a single rigid body and the 3DOF semi-definite system is reduced to a 2DOF semi-definite system with equations as:

$$ (I_1 + I_2) \ddot{\theta}_1 + C(\dot{\theta}_1 - \dot{\theta}_3) + K(\theta_1 - \theta_3) = T_e \quad (2.8) $$

$$ I_4 \ddot{\theta}_3 - C(\dot{\theta}_1 - \dot{\theta}_3) - K(\theta_1 - \theta_3) = -T_D \quad (2.9) $$

Reducing this linear system further into SDOF definite system, the governing equation is given as follows where the sinusoidal excitation (at $\omega$ with amplitude $T_{eq}$) under a mean load $T_m$ is applied:

$$ I_{eq} \ddot{\delta} + C \dot{\delta} + K \delta = T_m + T_{eq} \sin(\omega t) \quad (2.10) $$
Where, $\delta = \theta_1 - \theta_3 = \theta_2 - \theta_3$, $I_{eq} = I_3(I_1 + I_2)/(I_1 + I_2 + I_3)$ and $T_{eq} = T_p I_3/(I_1 + I_2 + I_3)$.

The analytical solution for the steady state oscillatory motions ($\delta$ and $\dot{\delta}$) is as follows where $\psi = \tan^{-1}[C\omega/(K - I_{eq}\omega^2)]$ is the phase lag.

$$\delta(t) = \frac{T_m}{K} + \frac{T_{eq}}{\sqrt{(K - I_{eq}\omega^2)^2 + (C\omega)^2}} \sin(\omega t - \psi) \tag{2.11}$$

$$\dot{\delta}(t) = \frac{T_{eq}\omega}{\sqrt{(K - I_{eq}\omega^2)^2 + (C\omega)^2}} \cos(\omega t - \psi) \tag{2.12}$$

As seen from the free body diagram of Figure 2.4b, the following relationship holds under the pure stick condition:

$$T_e(t) - I_1\dot{\theta}_1(t) - C\dot{\delta}(t) = T_f(t) \tag{2.13}$$

Then the transition condition from stick to slip is:

$$\left| T_e(t) - I_1\dot{\theta}_1(t) - C\dot{\delta}(t) \right| > T_{sf} \tag{2.14}$$

From equation (2.9), substitute $\dot{\theta}_1(t) = [T_e(t) - C\dot{\delta}(t) - K\dot{\delta}(t)]/(I_1 + I_2)$ into equation (2.14) to find the breakaway condition as:

$$\left| \frac{I_2}{I_1 + I_2} T_e(t) - \frac{I_2}{I_1 + I_2} C\dot{\delta}(t) + \frac{I_1}{I_1 + I_2} K\dot{\delta}(t) \right| > T_{sf} \tag{2.15}$$

As evident from this analysis, the stick to slip boundary is determined by system parameters that are summarized in Table 2.1. As shown in Figure 2.5a, the upper threshold frequency drops as the secondary inertia $I_2$ is increased. In the case of a very small $I_2$, the pure stick regime is not found at the lower frequency threshold. Depending on the friction saturation torque, the pure stick regime exists at lower
Figure 2.4 Resulting torsional systems under different conditions. (a) Pure stick condition; (b) Free body diagram for $I_1$ under pure stick condition; (c) Positive slip condition
Figure 2.5 Stick-slip boundaries based on linear system analysis. (a) Variation with respect to $I_2$; (b) Variation with respect to $T_{sf}$; (c) Variation with respect to excitation amplitude of $T_p$
frequencies as shown in Figure 2.5b. The excitation amplitude yields an opposite trend in Figure 2.5c where the pure stick regime occurs at a lower excitation torque. This analysis can also be applied to other physical systems where a dry friction element is present. The stick-slip transition could be quickly located and the entire simulation process would become more time efficient.

2.6 Solution to Non-linear Path Formulation

2.6.1 Smoothened Friction Torque Model

For the case when the smoothened friction model is used, the non-linear friction torque becomes an explicit non-linear function of $\dot{\delta}$. Using (2.4-2.6), the governing equations can be rewritten as:

\[ I_2 \ddot{\delta}_1 + C \frac{I_2}{I_1} (\dot{\delta}_1 + \dot{\delta}_2) - K \delta_2 + \frac{I_1 + I_2}{I_1} T_{sf} \tanh(\sigma \dot{\delta}_1) = \frac{I_2}{I_1} T_c(t) = \frac{I_2}{I_1} (T_m + T_p \sin \omega t) \]  \hspace{1cm} (2.16)

\[ I_2 \ddot{\delta}_2 + C \frac{I_2}{I_3} (\dot{\delta}_1 + \dot{\delta}_2) + \frac{I_2 + I_3}{I_3} K \delta_2 - T_{sf} \tanh(\sigma \dot{\delta}_1) = \frac{I_2}{I_3} T_D \]  \hspace{1cm} (2.17)

Since there is no spring in parallel with the dry friction element, $\delta_1$ does not appear in the system equations. An explicit Runge-Kutta method of order 5(4) with a step size control due to Dormand and Prince [2.28] can be applied directly to this system to find the resulting response.
2.6.2 Analytical Solution for the Positive Slip State

Let us assume that the friction torque behaves in a piecewise linear manner. Three different dynamic states (positive slip, negative slip and pure stick) are defined depending on the frictional interfacial condition. In each state, an effective linear model is considered. First, examine the positive slip state, \( \dot{\delta}_1 = \dot{\theta}_1 - \dot{\theta}_2 > 0 \). The friction torque is constant and equal to \( T_{sf} \) as shown in Figure 2.4c. The governing equations for this state are:

\[
I_2 \ddot{\delta}_1 + C \frac{I_2}{I_1} (\dot{\delta}_1 + \dot{\delta}_2) - K \delta_2 + \frac{I_1 + I_2}{I_1} T_{sf} = \frac{I_2}{I_1} T_s(t) \quad (2.18)
\]

\[
I_2 \ddot{\delta}_2 + C \frac{I_2}{I_3} (\dot{\delta}_1 + \dot{\delta}_2) + \frac{I_2 + I_3}{I_3} K \delta_2 - T_{sf} = \frac{I_2}{I_3} T_D \quad (2.19)
\]

As noted in Table 2.1, \( I_3 \gg I_2 \) for a typical automotive driveline system. For this reason, equation (2.19) can be approximated by discarding the viscous damping torque associated with \( \dot{\delta}_1 \).

\[
I_2 \ddot{\delta}_2 + C \frac{I_2}{I_3} \delta_2 + \frac{I_2 + I_3}{I_3} K \delta_2 - T_{sf} = \frac{I_2}{I_3} T_D \quad (2.20)
\]

The solution for \( \delta_2(t) \) is first obtained where \( \beta = I_2 / I_3 \) and \( t_c \) is the transition time from the pure stick to the positive slip.

\[
\delta_2(t) = \frac{\beta T_D + T_{sf}}{(1 + \beta)K} e^{-\lambda(t-t_c)} \{p_1 \cos(\omega_d(t-t_c)) + p_2 \sin(\omega_d(t-t_c))\} \quad (2.21a)
\]

\[
\lambda = \frac{C \beta}{2I_2}, \quad \omega_d = \frac{\sqrt{4I_2(1 + \beta)K - C^2 \beta^2}}{2I_2} \quad (2.21b-c)
\]

Assuming the initial values:
The coefficients $p_1$ and $p_2$ are found as shown below.

\[
p_1 = \Lambda_2 - \frac{\beta T_D + T_{df}}{(1 + \beta)K}, \quad p_2 = \frac{V_2 + \lambda P_1}{\omega_d}
\]

(2.23a-b)

Rewrite (2.18) in the following form where $\alpha = I_2 / I_1$,

\[
I_2 \ddot{\delta}_1 + C\alpha \dot{\delta}_1 = \alpha T_e(t) - C\alpha \dot{\delta}_2 + K \delta_2 - (1 + \alpha)T_{df}
\]

(2.24)

Note that the right hand side of (2.24) can be viewed as an equivalent excitation to a first order linear system with a time constant $I_2 / C\alpha - 1 / \xi$. We obtain the solution for \( \dot{\delta}_1(t) \) in the following torque functional form.

\[
\dot{\delta}_1(t) = A_0 + A_4 e^{-\xi(t-t_c)} + \left\{ A_{21} \cos(\omega(t-t_c)) + A_{22} \sin(\omega(t-t_c)) \right\} + \left\{ A_{31} \cos(\omega_d(t-t_c)) + A_{32} \sin(\omega_d(t-t_c)) \right\} + A_4(t-t_c)
\]

(2.25)

The coefficients ($A_0$, $A_4$, $A_{21}$, $A_{22}$, $A_{31}$, $A_{32}$ and $A_4$) can be obtained by satisfying the initial conditions $\delta_1|_{t=m_2} = \Lambda_1$ and $\dot{\delta}_1|_{t=m_2} = 0$. Although the exact expressions for the coefficients are not displayed here, (2.25) shows the oscillations occur with $\omega$ and $\omega_d$ in addition to a bias term ($A_0$), an exponentially decaying term, and a linear ramp.

2.6.3 Analytical Solution for the Negative Slip State

Next, the equations for the negative slip state in which $\dot{\delta}_1 = \dot{\theta}_1 - \dot{\theta}_2 < 0$ and $T_f = -T_{sf}$ are given as follows. In Figure 2.4c, the direction of $T_{sf}$ must be reversed to illustrate this case.
The analytical solutions for this state can be obtained in a manner similar to those reported for the positive slip state in section 2.6.2.

2.6.4 Analytical Solution for the Pure Stick State

Finally, consider the pure stick condition. Now, I₁ and I₂ move as a single inertial body and the 3DOF system is essentially reduced to a 2DOF semi-definite system or a SDOF definite system as shown in Figure 2.4a. The governing equation is given by (2.10) as defined in section 2.5. In this state, \( \dot{\delta}_1 = 0 \) and \( \delta_1 \) remains constant and equal to the one from the end of previous state. The solution for \( \delta_2(t) \) can be obtained by solving (2.10) as follows where \( \gamma = C / 2I_{eq} \). Assuming that the transition time from the positive or negative slip to the pure stick state is \( t_d \), we again express it in a functional form where \( B_{11} \) and \( B_{12} \) are coefficients that are determined by the initial conditions. Like (2.25), observe a bias term \( (T_m / K) \) and an exponentially decaying term. But oscillations occur only with frequency \( \omega \).

\[
\delta_2(t) = e^{-\gamma(t-t_d)} \left\{ B_{11} \cos(\omega_d(t-t_d)) \right\} + \frac{T_m}{K} + \frac{T_{eq}}{\sqrt{(K-I_{eq}\omega^2)^2+(C\omega)^2}} \sin(\omega(t-t_d) - \psi) 
\]

(2.28)
2.6.5 Numerical Solution of the Discontinuous Friction Torque Model

As presented in the previous sections, each state can be analytically solved and then the entire solution could be assembled. Alternatively, the equations can be solved by a 5th (4th) order Runge-Kutta integration algorithm. Test conditions and functions, however, must be defined at each state. At each integration step, the test function is checked. For instance when system undergoes positive or negative slip, the test function is defined as $\mathcal{I}(\tau_i) = \delta_i(\tau_i)\delta_i(\tau_{i+1})$, which is the product of the relative velocities of two successive integration steps. Whenever $\mathcal{I}(\tau_i)$ becomes negative, a possible transition to the stick state is detected and the program automatically finds the exact transition point within a pre-specified precision. A bi-section root finding algorithm is used; although this method may not be the most efficient algorithm, yet it is found very reliable by Gear and Osterby [2.29]. With a specified tolerance $10^{-6}$, our code usually finds the exact point within 5 to 10 iterations. A test function for the stick state is defined as $\Gamma(\tau)$, which is actually the instantaneous interfacial friction torque. From (2.15), $\Gamma(\tau)$ is given by:

$$\Gamma(\tau) = \frac{I_2}{I_1 + I_2} T_e(\tau) - \frac{I_2}{I_1 + I_2} C\dot{\delta}_2(\tau) + \frac{I_1}{I_1 + I_2} K\dot{\delta}_2(\tau)$$ (2.29)

Whenever $|\Gamma(\tau)|$ is greater than $T_{sf}$, a possible transition from stick to slip is detected. Once the exact transition points are calculated, the subsequent state is determined by a decision function $\mathcal{R}(\tau)$ that is numerically identical to $\Gamma(\tau)$. The detailed state to state transition process is illustrated in Table 2.2. The $\text{sgn}$ function used here is the
<table>
<thead>
<tr>
<th>Previous State</th>
<th>Transition Condition</th>
<th>Test Function</th>
<th>Future State</th>
<th>$\delta_i$ and $\dot{\delta}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive Slip</td>
<td>$\Im(\tau) &lt; 0$</td>
<td>$</td>
<td>\Re(\tau)</td>
<td>\leq T_{sf}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Negative Slip</td>
<td>$\sgn(\dot{\delta}_i(\tau + 0)) = \sgn(f(\tau - 0))$</td>
</tr>
<tr>
<td>Pure Stick</td>
<td>$</td>
<td>\Gamma(\tau)</td>
<td>&gt; T_{sf}$</td>
<td>$\Re(\tau) &gt; T_{sf}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Negative Slip</td>
<td>$\sgn(\dot{\delta}_i(\tau + 0)) = \sgn(f(\tau - 0))$</td>
</tr>
<tr>
<td>Negative Slip</td>
<td>$\Im(\tau) &lt; 0$</td>
<td>$</td>
<td>\Re(\tau)</td>
<td>\leq T_{sf}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Pure Stick</td>
<td>$\delta_i = \delta_i^*$; $\dot{\delta}_i = 0$</td>
</tr>
</tbody>
</table>

Table 2.2 State to state transition for the non-linear formulation of Figure 2.3 with discontinuous Coulomb friction model
conventional triple-valued signum function:

\[
\text{sgn}(\dot{\delta}_1) = \begin{cases} 
1 & \dot{\delta}_1 > 0 \\
0 & \dot{\delta}_1 = 0 \\
-1 & \dot{\delta}_1 < 0 
\end{cases} \quad (2.30)
\]

Note that \(\delta^*_1\) in Table 2.2 is the value of \(\delta_1\) at the end of previous positive or negative slip state and as discussed earlier, \(\delta_1\) remains this value during the subsequent pure stick state. In our simulation, the values of \(\delta_1, \dot{\delta}_1, \delta_2, \) and \(\dot{\delta}_2\) at the end of a certain state provide the initial conditions for the next state. Finally, the solutions for all states are assembled.

### 2.6.6 Effect on Conditioning Factor \(\sigma\)

In the smoothened friction model of equation (2.6), the singularity at \(\dot{\delta} = 0\) of the classical Coulomb model is eliminated. However, this smoothening process brings an artificial uncertainty to the system since it is virtually impossible to determine the precise value of \(\sigma\) without a significant knowledge of its dynamics. From the standpoint of the rate of change of the friction torque at \(\dot{\delta} = 0\), the proposed procedure replaces the impulse function (with the discontinuous model) with the following finite derivative:

\[
\frac{dT_f}{d\delta} = T_{sf} \sigma [1 - \tanh^2(\sigma \dot{\delta})] \quad (2.31)
\]

The maximum amplitude, \(T_{sf} \sigma\), can now be taken as an indicator of its abruptness. As seen from Equation (2.31), even for the same \(\sigma\) value, the abruptness of \(T_f(\dot{\delta})\)
profile would vary from system to system because of the saturation friction torque $T_s$. Moreover, $\sigma$ controls the integration speed. For example, a lower $\sigma$ value gives a smoother torque curve and faster convergence is achieved in numerical integration. Conversely, a higher $\sigma$ value can produce a Coulomb-like $T_f(\dot{\delta})$ curve. But too abrupt transition(s) would ill-condition the Jacobian matrix \[2.30\], and thereby introducing numerical stiffness issues. Illustrative results are given in Figures 2.6 and 2.7. When $\sigma$ is 100, the calculated system response is almost identical to the one found with the discontinuous model. When $\sigma$ is low such as 0.5, no pure stick regime is found in $\dot{\delta}_1(t)$ of Figure 2.6. The friction torque undergoes a relatively smooth transition but the calculated response of $\delta_2(t)$ shows some differences. On the other hand, when $\sigma$ is 1000, the time domain responses of Figure 2.7 are incorrect. Therefore, the dynamic response is very sensitive to the judicious choice of $\sigma$.

Meanwhile, some almost periodic solutions may appear. These could be very misleading because the periodic solution is being sought, and a relative higher value of $\sigma$ provides results that resemble the ones given by the discontinuous models. A comparison between the computational times clearly shows the advantage of a smoothened model. For instance, the calculations for $\dot{\delta}_1(t)$ etc. at one particular excitation frequency takes 15 seconds with the smoothened friction model with $\sigma=100$ on a Pentium 4 1.7GHz processor, while the discontinuous friction model consumes 50 seconds. However, as discussed earlier, the best value of $\sigma$ usually is
Figure 2.6 Comparison of two friction path models in terms of time histories. Dashed line, discontinuous friction model; solid line, smoothened model with $\sigma = 0.5$; dotted line, smoothened model with $\sigma = 10^2$. 
Figure 2.7 Comparison of two friction path models in terms of time histories. , discontinuous model; , , smoothened model with \( \sigma = 10^3 \); , , smoothened model with \( \sigma = 2 \times 10^3 \)
not known \textit{a priori} and it could depend on system parameters. Consequently, the discontinuous friction model should be used as benchmark if successfully implemented. Further, our solution algorithm for discontinuous friction model could also be extended to other piecewise linear systems like the clearance non-linearity. However, test or decision functions would need to be modified.

2.7 Effect of Friction Controlled Path Parameters

2.7.1 Bi-Linear Friction System Analysis

Many researchers [2.2, 2.3, 2.5, 2.6, 2.8, 2.11] have studied the conventional bi-linear friction system that assumes a massless link between the spring and the dry friction elements as shown in Figure 2.8. Similar to the physical system of Figure 2.3, equations for Figure 2.8 can be given on a state by state basis. First, for the positive slip state, the equation is:

\[
\frac{I_1I_3}{I_1 + I_3} \ddot{\delta}_1 + C\dot{\delta}_1 + T_{sf} = \frac{I_3}{I_1 + I_3} T_e + \frac{I_1}{I_1 + I_3} T_D
\]  

(2.32)

Since no inertial body exists between the dry friction and spring elements, the torque acting on the torsional spring is constant \((T_{sf})\). Consequently, the value of \(\delta_2\) remains \(T_{sf} / K\) and \(\dot{\delta}_2\) is zero. Second, for the case of negative slip, the equation is rewritten as follows where \(\delta_2 = -T_{sf} / K\) and \(\dot{\delta}_2 = 0\) :

\[
\frac{I_1I_3}{I_1 + I_3} \ddot{\delta}_1 + C\dot{\delta}_1 - T_{sf} = \frac{I_3}{I_1 + I_3} T_e + \frac{I_1}{I_1 + I_3} T_D
\]  

(2.33)
Figure 2.8 Schematic of the bi-linear torsional system (with $I_z \rightarrow 0$). Here, $T_f$ represents a dry friction path.
Third, the equation for the pure stick state is given by:

\[
\frac{I_1 I_3}{I_1 + I_3} \ddot{\delta}_2 + C \dot{\delta}_2 + K \delta_2 = T_m + \frac{I_3}{I_1 + I_3} T_p(t)
\]  \hspace{1cm} (2.34)

As depicted in Figures 2.4a and 2.4b, \( \delta_1 \) remains constant and \( \dot{\delta}_1 \) is equal to zero. The integration procedure used here with the discontinuous friction model can now be applied to this system with a redefinition of the test \( \Gamma(\tau) \) or decision \( \Re(\tau) \) function:

\[
\Gamma(\tau) = \Re(\tau) = K \delta_2
\]  \hspace{1cm} (2.35)

The systems of Figures 2.3 and 2.8 can now be compared. Both show the second order system behavior under the pure stick condition. However, under the positive or negative slip state condition, the bi-linear friction system (of Figure 2.8) exhibits a first order system behavior while the model of Figure 2.3 follows a second order system. For this reason, some key differences between these two systems are expected, as explored in the subsequent section.

2.7.2 Effect of the Secondary Inertia \( I_2 \)

A comparison of equations (2.10) and (2.34) shows that the effect of the secondary \( (I_2) \) inertia could be negligible under the pure stick condition in the presence of a very small \( I_2 \). For this reason, we examine the effect of \( I_2 \) in the positive or negative slip state. First, consider a conventional bi-linear friction system undergoes transition from pure stick to positive slip state. Equation (2.32) describes the governing equation for the system in the pure positive slipping motion. A phase term \( \phi_e \) is intentionally included in the excitation torque expression since the absolute
transition time \( t_c \) may not be integer multiplier of the excitation period \( P \).

Consequently, given \( T_p = T_m \), equation (2.32) has to be rewritten as follows where

\[
I_{le} = I_l I_3 / (I_1 + I_3) \quad \text{and} \quad T_{pe} = T_p I_3 / (I_1 + I_3) .
\]

\[
I_{le} \dot{\delta}_l + C \dot{\delta}_l = (T_m - T_{sf}) + T_{pe} \sin(\omega t + \varphi_c) .
\]  

(2.36)

\[
P = \frac{2\pi}{\omega} , \quad \varphi_c = \left[ t_c - \text{int}(\frac{t_c}{P}) \right] \omega .
\]  

(2.37a-b)

The operator \( \text{int}(\cdot) \) yields the integer portion of a fraction number. Now, the general solution of \( \dot{\delta}_l(t) \) is a combination of homogeneous and particular solutions as follows:

\[
\dot{\delta}_l(t) = A e^{\frac{C}{I_{le}} t} + \left[ a_0 + a_1 \cos(\omega t) + a_2 \sin(\omega t) \right]
\]  

(2.38)

The coefficients \( a_0, a_1 \) and \( a_2 \) are obtained by matching the particular solutions on both sides of (2.36) as follows:

\[
a_0 = \frac{T_m - T_{sf}}{C} , \quad a_1 = T_{pe} \frac{C \sin \varphi_c - I_{le} \omega \cos \varphi_c}{I_{le}^2 \omega^2 + C^2} , \quad a_2 = T_{pe} \frac{C \cos \varphi_c + I_{le} \omega \sin \varphi_c}{I_{le}^2 \omega^2 + C^2} .
\]  

(2.39a-c)

The constant \( A \) needs to be determined by the initial condition. As noted from the previous section, when the frictional interface experiences a transition from pure stick to positive slip motion, \( \delta_2 \) reaches the maximum amplitude \( T_{sf} / K \) and retains this value during the entire subsequent pure slip state. To satisfy this dynamic condition, \( \dot{\delta}_2 \) experiences a finite jump from a certain value \( V \) at the end of pure stick state to 0 at the start of pure slip state. For this reason, a corresponding finite jump from 0 to \( V \) must occur in \( \dot{\delta}_1 \) to satisfy the following continuity condition when we reset the transition time as 0.
\[
(\dot{\theta}_1 - \dot{\theta}_2)\Big|_{t=0} = (\dot{\theta}_1 - \dot{\theta}_2)\Big|_{t=0}
\]  
(2.40)

Thus, the initial condition of \( \dot{\delta}_1(t) \) is determined, \( \dot{\delta}_1(0) = V \). Use this to find \( A \) of (2.38) and write the complete response of \( \dot{\delta}_1(t) \) as follows:

\[
\dot{\delta}_1(t) = (V - a_0 - a_1)e^{\frac{-c}{I_2\omega}} + a_0 + a_1\cos(\omega t) + a_2\sin(\omega t) \tag{2.41}
\]

Observe that the oscillatory terms associated with \( a_1 \) and \( a_2 \) only contains only one frequency (\( \omega \)).

Next, the 3DOF system \( (I_2 \neq 0) \) is considered. Assume a very small \( I_2 \), say \( I_2/I_1 = 0.005 \sim 0.01 \). Equations (2.18) and (2.19) are approximated as follows:

\[
I_2\ddot{\delta}_1 + C\frac{I_2}{I_1}(\ddot{\delta}_1 + \ddot{\delta}_2) - K\delta_2 + T_{sf} = 0 \tag{2.42}
\]

\[
I_2\ddot{\delta}_2 + C\frac{I_2}{I_3}(\ddot{\delta}_1 + \ddot{\delta}_2) + K\delta_2 - T_{sf} = 0 \tag{2.43}
\]

Since the excitation torque is neglected due to \( I_1 \gg I_2 \), the phase lag \( \varphi_c \) term that accurately represent the excitation as discussed in the previous section is no longer an issue here. Further, equation (2.43) is further simplified given \( I_3 \gg I_2 \); refer to Table 2.1 for typical parameters.

\[
I_2\ddot{\delta}_2 + C\frac{I_2}{I_3}\ddot{\delta}_2 + K\delta_2 - T_{sf} = 0 \tag{2.44}
\]

Equation (2.44) can be conveniently solved and the approximated solution for \( \delta_2 \) is obtained where \( \omega_n \cong \sqrt{K/I_2} \).

\[
\delta_2(t) = b_0 + b_1e^{-\frac{c}{I_3\omega}}\cos(\omega_n t) + b_2e^{-\frac{c}{I_3\omega}}\sin(\omega_n t) \tag{2.45}
\]
The initial conditions for $\delta_2(t)$ at the transition time are $\delta_2(0) = T_{sf} / K$ and $\dot{\delta}_2(0) = V$. Since $I_2$ has a very small but non-zero value, its absolute velocity or momentum can only be changed by an infinite large impulsive force within an infinitesimal time span that is physically impossible. Consequently, no jump could take place in $\dot{\delta}_2(t)$ unlike the one exhibited by bi-linear friction system. Instead, $\dot{\delta}_2(t)$ is a smooth function of time.

$$\dot{\delta}_2(t) = \frac{T_{sf}}{K} + \frac{V}{\omega_n} e^{-\frac{C}{2nI_1}} \sin(\omega_n t)$$ (2.46)

As noted, $\delta_2(t)$ oscillates around the mean value $T_{sf} / K$ unlike the bi-linear friction system. Further, since $I_2$ is very small, the oscillation frequency ($\omega_n$) is very high. To conveniently obtain an approximate analytical solution for $\dot{\delta}_1(t)$, we neglect the damping term in equation (2.44) and substitute the relation in equation (2.42), and obtain the following governing equation.

$$I_2 (\ddot{\delta}_1 + \ddot{\delta}_2) + \frac{C}{I_1} I_2 (\ddot{\delta}_1 + \ddot{\delta}_2) = 0$$ (2.47)

Where the initial condition is defined as $(\dot{\delta}_1 + \dot{\delta}_2)|_{t=0} = \dot{\delta}_2|_{t=0} = V$. Finally, the approximate analytical solution of $\dot{\delta}_1(t)$ is obtained.

$$\dot{\delta}_1(t) = V \left[ e^{-\frac{C}{I_1}} - e^{-\frac{C}{2nI_1}} \cos(\omega_n t) \right]$$ (2.48)
Similar to $\delta_2(t)$, a very high frequency oscillatory term is found in $\dot{\delta}_1(t)$ along with an exponentially decaying term. Further, it is noted that the above analyses could be applied to the negative slip state in a similar manner.

Figures 2.9 and 2.10 compare the results corresponding to $I_2/I_1 \neq 0$ (system of Figure 2.3) and 0 (the conventional bi-linear system of Figure 2.8) cases. As expected, the motion differences under the pure stick condition ($\dot{\delta}_1 = 0$) are minimal since $I_2$ is very small compared to $I_1$ as shown in Figures 2.9a and 2.10a. Under the slip condition ($\dot{\delta}_1 > 0$), the difference is however noticeable. As response makes a transition from stick to slip, $\dot{\delta}_1$ of the bi-linear system (with $I_2 = 0$) shows a finite jump from zero to the value of $\dot{\delta}_2$ at the end of previous stick state and then it goes back to zero (stick) gradually. The values of $\delta_2$ are bounded within $\pm T_{sf} / K$. Overall, the response resembles the “relaxation oscillation” [2.31], i.e. the potential energy is incrementally stored in the spring during the pure stick state and then suddenly released during the stick to slip transition. However, our response does not quite follow the classical “relaxation oscillation” behavior since the underlying mechanism is essentially different. According to Den Hartog [2.32] and Andronov et. al. [2.33], the “relaxation oscillation” is self-excited due to the existence of negative damping, such as the dry friction element with a negative slope in an autonomous system. In contrast, the response displayed by our bi-linear friction system is a result of the
Figure 2.9 Effect of $I_2$ on time histories. a) numerical solutions: $\cdots$, $I_2/I_1 = 0$; $\cdots$, $I_2/I_1 = 0.005$. b) comparison between numerical and analytical solution during the positive slip state: $\cdots$, numerical solution given $I_2/I_1 = 0$; $\cdots$, analytical solution given $I_2/I_1 = 0$; $\cdots$, numerical solution given $I_2/I_1 = 0.005$; $\cdots$, analytical solution given $I_2/I_1 = 0.005$. 

54
Figure 2.10 Effect of $I_2$ on time histories. a) numerical solutions: $\cdots$, $I_2/I_1 = 0$; $\cdots\cdots$, $I_2/I_1 = 0.01$. b) comparison between numerical and analytical solution during the positive slip state: $\cdots\cdots$, numerical solution given $I_2/I_1 = 0$; $\cdots\cdots\cdots$, analytical solution given $I_2/I_1 = 0$; $\cdots\cdots\cdots$, numerical solution given $I_2/I_1 = 0.01$; $\cdots\cdots\cdots$, analytical solution given $I_2/I_1 = 0.01$. 
external torque excitation in the presence of a friction term and as such no negative
damping element is present. In fact, Minorsky [2.34] has suggested that generic quasi-
discontinuous motions may describe two kinds of oscillations: (i) relaxation
oscillations and (ii) impulse-excited oscillations that are induced by an external
impulsive cause. Although our response cannot be classified as impulse-excited
oscillations either, we would still categorize as a quasi-discontinuous oscillations.
Further research is needed to explore this issue.

On the contrary, the system response with a non-zero $I_2$ is, however, quite
different. With the existence of a secondary inertia, an abrupt change or finite jump in
$\dot{\delta}_1$ at the transition point does not occur. But the slip velocity is much higher than the
one in the bi-linear friction case. With the initial conditions and absolute transition
time provided by numerical solution, approximate analytical solutions for $\dot{\delta}_1$ and $\dot{\delta}_2$
are also obtained by using equations (2.41) and (2.46), and (2.48) respectively.
Comparative results in Figures 2.9b and 2.10b show a good agreement between the
approximate analytical solutions and exact numerical solutions. Some minor
difference for the $I_2 \neq 0$ case could be due to inadequate damping in the approximate
model. The reason that $\dot{\delta}_1(t)$ just decreases gradually with time without active
oscillations with bi-linear friction system is that the oscillatory term in equation (2.41)
is at excitation frequency $\omega$ and this frequency is very low compared to $\omega_n = \sqrt{K/I_2}$
since a very small value of $I_2$ is used. Also, only period-one motions are observed.

Further, the response is investigated using phase-plane plots. As evident from
Figures 2.11a and 2.11c, the friction interface in the bi-linear hysteresis case always
experiences two stops per cycle and the corresponding $\delta_{2}(t)$ shows a noticeable boundary at $\pm T_{sf}/K$. Compare the results of Figure 2.11a and 2.11c ($I_2 = 0$) with Figures 2.11b and 2.11d ($I_2 \neq 0$). More than 2 stops are experienced and $\delta_{2}(t)$ shows an oscillatory motion around $\pm T_{sf}/K$. The cusps formed in Figures 2.11b and 2.11d are due to the stick-slip transitions.

More time histories plots are presented in Figure 2.12. It is seen that an increase in $I_2$ decreases the severity of the interfacial stick-slip. At the excitation frequency of 7 Hz, the negative slip disappears when $I_2 = 0.4I_1$. As stated previously in the stick to slip boundary analysis, there exists a specific value of $I_2$ that would induce a purely stick regime and thus $\delta_{2}(t)$ is governed strictly by a linear system.

A more comprehensive understanding on the effect of $I_2$ could be achieved by constructing the non-linear frequency response characteristics. Dynamic responses may be quantified by taking the maximum (max) and minimum (min) values from the calculated time history at each excitation frequency. Figure 2.13a shows the response maps in terms of $\delta_{2\text{max}}$ and $\delta_{2\text{min}}$ values. Stick-slip boundaries as found earlier by the linear system theory are also plotted. Since there is no spring in parallel with the dry friction element, the relative displacement $\delta_{1}(t)$ may grow up to a very large value under the influence of a mean load $T_m$. Thus, no physical meaning could be associated with max or min values of the steady state $\delta_{1}(t)$. Instead, the non-linear frequency response of relative displacement $\delta_{2}$ is of chief interest here. Also, we find the root-mean-square (rms) values from the calculated time histories, again at each excitation frequency as shown in Figure 2.13b.
Figure 2.11 Effect of $I_2$ on phase-plane plots. (a) and (c) are with $I_2/I_1 = 0$; (b) and (d) are with $I_2/I_1 = 0.01$. 
Figure 2.12 Effect of $I_2$ on time histories: \( I_2/I_1 = 0 \); \( I_2/I_1 = 0.1 \); \( I_2/I_1 = 0.2 \); \( I_2/I_1 = 0.4 \).
Figure 2.13 Effect of $I_2$ on responses over a range of excitation frequencies. (a) Maximum and minimum responses: $\circ$, $I_2/I_1 = 0$; $\ldots$, $I_2/I_1 = 0.01$; $\times$, $I_2/I_1 = 0.1$; $\ldots$, $I_2/I_1 = 0.2$; $\ldots$, $I_2/I_1 = 0.4$; (b) rms responses: $\circ$, $I_2/I_1 = 0$; $\ldots$, $I_2/I_1 = 0.01$; $\times$, $I_2/I_1 = 0.1$; $\ldots$, $I_2/I_1 = 0.2$; $\ldots$, $I_2/I_1 = 0.4$. 
As expected from the stick to slip boundary analysis, an increase in the value of \( I_2 \) narrows the stick-slip regime. It is also clearly observed in Figure 2.13a that as \( I_2 \) increases, the max to min value of \( \delta_2 \) over the pure stick (linear) regime is lowered. However, the values over the stick-slip regime go up. Unlike the bi-linear hysteresis case where the peak amplitude is constant (\( \delta_2 = \frac{T_{sf}}{K} \)), the amplitudes with non-zero \( I_2 \) exhibit “resonance-like” curves in Figure 2.13. Such resonances are dictated by a combination of 2 states: 2DOF and 3DOF system responses. A comparison of the rms maps in Figure 2.13b shows similar effects of \( I_2 \).

### 2.7.3 Effect on the Saturation Friction Torque \( T_{sf} \)

Figure 2.14 shows the effect of \( T_{sf} \) on non-linear frequency responses. As \( T_{sf} \) increases, the stick-slip regime narrows but the peak amplitude increases. Such a narrowing trend of the stick-slip regime is consistent with the stick to slip boundary analysis. Higher amplitudes of \( \delta_2 \) can be explained by the fact that the sub-system of \( I_2 \) and \( I_3 \) now receives more excitation which is equal to \( T_{sf} \) under the slip condition. The severity of the stick-slip transition also decreases as evident from the time histories of Figures 2.15a (at 10 Hz) and 2.15b (at 8 Hz). Figure 2.15a shows that when \( T_{sf} \) is small such as 250 N-m, 2 slips are found in each cycle. But when \( T_{sf} \) increases to 450
Figure 2.14 Effect of $T_{sf}$ on responses over a range of excitation frequencies. (a) Maximum and minimum responses: $\bigcirc$, $T_f = 250$ Nm; $\ldots\ldots\ldots$, $T_f = 350$ Nm; $\times$, $T_f = 450$ Nm; $\bigcirc\ldots\bigcirc$, $T_f = 550$ Nm; (b) rms responses: $\bigcirc$, $T_f = 250$ Nm; $\ldots\ldots\ldots$, $T_f = 350$ Nm; $\times$, $T_f = 450$ Nm; $\bigcirc\ldots\bigcirc$, $T_f = 550$ Nm
Figure 2.15 Effect of $T_{sf}$ on time histories. (a) Excitation frequency is 10 Hz: \ldots, $T_f$ = 250 Nm; \ldots, $T_f$ = 350 Nm; \ldots, $T_f$ = 450 Nm; \ldots, $T_f$ = 550 Nm; (b) Excitation frequency is 8 Hz: \ldots, $T_f$ = 250 Nm; \ldots, $T_f$ = 350 Nm; \ldots, $T_f$ = 450 Nm; \ldots, $T_f$ = 550 Nm
N-m, the number of slips per cycle reduces to one. One could predict the trend of stick-slip regime as follows: $T_{sf}$ first reduces the slip velocity and its duration. Then the number of slips is reduced as $T_{sf}$ is increased. Ultimately, this reduction would introduce the pure stick condition, i.e. non-slip or zero slip velocity.

### 2.8 Comparison with Benchmark Studies

Next we compare our method with two benchmark examples as available in the literature. These include Ferri and Heck’s turbine blade damper study [2.15] and Hartung et. al.’s passive vibration absorber analyses [2.26]. Physical models of both studies could be conceptually described by the sub-sets of Figure 2.3 since the dry friction is the only non-linear element in two degree of freedom systems.

The model of Ferri and Heck’s turbine blade damper is presented in Figure 2.16a. The governing equations with dimensionless parameters (using the nomenclature of [2.15]) are as follows where $y_1$ and $y_2$ are the absolute displacements; $g(y_1)$ is the classical Coulomb friction force.

\[
\begin{align*}
    r\ddot{y}_1 + p(y_1 - y_2) + g(y_1) &= 0 \quad (2.49) \\
    \ddot{y}_2 + 2\zeta\dot{y}_2 + y_2 + p(y_2 - y_1) &= f\cos(\omega t) \quad (2.50)
\end{align*}
\]

Two example cases are selected to demonstrate the significant dynamic effect of the secondary mass ($r$). Our methods are applied to this system to first calculate the stick to slip boundaries, and then we conduct non-linear simulations. Using both
Figure 2.16 Benchmark models with dry friction element. (a) Ferri and Heck’s turbine blade friction damper model [2.15]; (b) Harung et. al.’s passive vibration absorber model [2.26]
Figure 2.17 Comparison with Ferri and Heck’s results. (a) Frequency response curves with \( f = 2, p = 0.5 \) and \( r = 0.2 \). \( \times \), pure stick solution; \( \circ \), discontinuous friction model; \( \cdash \), smoothened friction model with \( \sigma = 50 \); \( \cdash \cdash \), smoothened friction model with \( \sigma = 10 \); \( \Box \), Ferri and Heck’s results. (b) Time histories at \( \omega = 0.94 \): \( \cdash \), discontinuous friction model; \( \cdash \cdash \), smoothened friction model with \( \sigma = 10 \); \( \cdash \cdash \cdash \), smoothened friction model with \( \sigma = 50 \).
Figure 2.18 Frequency response curves with \( f = 2, \ p = 0.2 \) and \( r = 0.05 \). ×, pure stick solution; ○, discontinuous friction model; —, smoothened friction model with \( \sigma = 50 \); ——, smoothened friction model with \( \sigma = 10 \); □, Ferri and Heck’s results.
discontinuous and smoothened friction torque models of earlier sections, we predict frequency responses that match quite well with Ferri and Heck’s results in Figures 2.17 and 2.18. A minor difference is seen around $\omega = 0.94$ in Figure 2.17a when $\sigma$ is changed from 50 to 10. However, as shown by the time history in Figure 2.17b, this difference is within 5%. Further, our results generate more insight into the system by analytically pinpointing the pure stick (linear) and stick-slip (non-linear) frequency regimes.

Hartung et. al. [2.26] studied the dynamics of a 2DOF system (vibration absorber) as shown in Figure 2.16b. The governing equations are as follows using their nomenclature.

$$m\ddot{x}_1 + c\dot{x}_1 - c\dot{x}_2 + (1 + k)x_1 - x_2 = k\cos(\omega t)$$  \hspace{1cm} (2.51)

$$\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 - \dot{x}_1 + \dot{x}_2 = -\rho_s g(\dot{x}_2)$$  \hspace{1cm} (2.52)

Aside from numerical simulation based on the discontinuous friction model, they also conducted an experimental study and investigated the effect of friction force amplitude $\rho_s$. Selected measured frequency response curves under three friction forces are extracted from [2.26] and illustrated in Figure 2.19. First we predict the stick to slip boundaries using the linear system procedure proposed earlier. Second, we employ the discontinuous and smoothened friction models. Our simulation results, as shown in Figure 2.20, show a very good match with measured curve of Figure 2.19 even when the smoothened friction model is chosen. As the friction force increases, the stick-slip motion is suppressed in Figure 2.20. The stick-slip motion virtually disappears in
Figure 2.19 Measured frequency response curves (Hartung et. al.’s experiment scanned from [2.26])
Figure 2.20 Simulated frequency response curves. ×, pure stick solution; ○, discontinuous model (stick-slip) solution; ——, smoothened friction model with $\sigma = 50$
Figure 2.20c when the friction force increases to a certain value and finally the system is degenerated into a linear single degree of freedom system. Although Hartung et. al.’s experiment [2.26] showed that the friction-induced characteristics depend on normal or friction force amplitude, our comparisons reveal that the even simplest friction model could still be used to qualitatively assess the non-linear responses or quantitatively estimate the stick-slip boundaries. Thus, we validate our linear and non-linear methods.

2.9 Negative Slope in Friction Formulation

Although only the case \( \mu_s = \mu_k \) has been considered in this chapter, several previous researchers, including Karnopp [2.4] and Shaw [2.10], have suggested a general formulation where \( \mu_k < \mu_s \). For example, consider the following formulation [2.35] where \( \mu_s \) has been normalized with respect to \( \mu_k (=1.0) \) and \( \alpha \) is a factor that controls the exponentially decaying gradient.

\[
\mu(\dot{\delta}_1) = \begin{cases} 
1.0 + (\mu_s - 1.0)e^{-\alpha|\dot{\delta}_1|} & \text{sgn}(\dot{\delta}_1) \quad |\dot{\delta}_1| > 0 \\
[0 \quad \mu_s] & \dot{\delta}_1 = 0
\end{cases}
\]

Further, we can condition the discontinuous formulation (2.53) by employing the hyperbolic-tangent function (2.6).

\[
\mu(\dot{\delta}_1) = [1.0 + (\mu_s - 1.0)e^{-\alpha|\dot{\delta}_1|}] \tanh(\sigma\dot{\delta}_1)
\] (2.54a)
\[
\frac{\partial \mu}{\partial \dot{\delta}} = \sigma[1.0 + (\mu_s - 1.0)e^{-a|\dot{\delta}|}][1.0 - \tanh^2(\sigma\dot{\delta})] - \alpha(\mu_s - 1.0)e^{-a|\dot{\delta}|}\text{sgn}(\dot{\delta})\tanh(\sigma\dot{\delta})
\]  
(2.54b)

Note that when \( \mu_s > 1.0 \), a negative slope (\( \partial \mu / \partial \dot{\delta} < 0 \)) is found in the friction law. Although (2.54) is different from (2.6) that implicitly assumes \( \mu_s = \mu_k \), the transition frequency from the pure stick state to the stick-slip regime can still be analytically determined via the procedure of section 2.5. For example, an increase in \( \mu_s \) will enhance the value of \( T_{sf} \) and hence yield a narrower the stick-slip frequency regime as shown previously in Figure 2.5b. Further, some previous researchers have shown that the resulting “negative damping” could induce dynamic instabilities into some physical systems in where the dry friction element acts as a passive damper [2.10]. This issue will be studied in subsequent chapter [2.36] in the context of a significant dry friction path in torsional systems.

2.10 Conclusion

Unlike previous studies that focused on the frictional interface models, we have examined the non-linear time and frequency domain responses of a 3DOF system subjected to localized stick-slip motions. Three contributions emerge. First, a procedure to calculate the stick to slip boundaries has been developed based on the linear system theory. This procedure yields reliable prediction of the thresholds and thus it can quickly identify linear or non-linear frequency regimes. Second, both smoothened and discontinuous friction models are studied and compared with each
other. A direct time domain numerical integration can be employed to solve the smoothened friction model and it speeds up the calculation. Though a judicious choice of the conditioning factor can generate reasonably close solutions, one must exercise caution in using the smoothening procedure without some prior knowledge of the system. Third, the effect of the secondary inertia is investigated in depth using both time and frequency domain calculations. Both approximate analytical and numerical solutions clearly illustrate that even a very small secondary inertia has a significant influence on the behavior and thus the bi-linear friction system (such as Figure 2.8) must be used with caution. During the slip states, \( \dot{\delta}_1(t) \) follows a first order behavior when \( I_2 = 0 \) and a second order system when \( I_2 \neq 0 \). As evident from time histories and frequency responses, the amplitudes of \( \delta_2(t) \) are bounded for a bi-linear system and its frequency responses form a flat top. But when \( I_2 \neq 0 \), no specific bounds can be defined for \( \delta_2(t) \) and its frequency responses show resonance-like curves. These resonances are essentially determined by a combination of 2 states: 2DOF and 3DOF non-linear system responses.

The frictional interface dynamics and the response of the overall system is dictated by the secondary inertia \( I_2 \). At a higher value of \( I_2 \), the stick-slip phenomena are suppressed. However, the peak amplitudes of the response could still go up. Similar effect is observed by a change in the saturation friction torque. Although the system response is strongly non-linear, the suppression of stick-slip phenomena seems to follow the following pattern: first reduce the slip velocity and then reduce the number of slips.
The advantage of our smoothened and discontinuous friction models is limited to a system of low dimension. As the system dimension increases, time domain simulations could become very cumbersome since many non-linear functions must be evaluated. Conceptually, semi-analytical techniques need to be developed. Effects along this direction are reported in a subsequent chapter [2.36].

References for Chapter 2


[2.3] Personal discussion with DaimlerChrysler powertrain engineers in April 2002.


76

CHAPTER 3

SUPER-HARMONICS IN A TORSIONAL SYSTEM WITH DRY FRICTION PATH

3.1 Introduction

Dry friction elements are commonly found in mechanical systems and yet much of prior research has focused on the dry friction damper and its characterization \[3.1-3.6\]. Den Hartog [3.1] initiated work in this area by analytically determining the forced harmonic response of a single degree of freedom (SDOF) system with combined Coulomb and viscous friction elements. But his solution was limited to no more than two stops. Pratt and Williams [3.2] extended Den Hartog’s work and calculated the system response with multiple lock-ups by using a numerical shooting method. Wang [3.3] developed an analytical solution for the periodic response of a bi-linear hysteresis friction system. However, numerical iterations were still needed to match the solutions obtained from stick and slip states. Further, Menq and Yang [3.4], and Wang and Chen. [3.5] have used multi-term harmonic balance methods (MHBM) to find the dynamic response of a bi-linear hysteresis problem. Overall, three common features exist among the formulations that have been employed by many researchers \[3.1-3.6\]. First, only the primary harmonic resonances are examined under sinusoidal excitation. Second, a spring path is placed in parallel with the dry friction element,
thus having two parallel paths for force transmission. Third, the saturation friction force is assumed to be small; this is obviously valid from the friction damper standpoint.

In our study we examine the dry friction element in those situations where it is a key path in transmitting mechanical power in real-life torsional systems rather than acting as a pure friction damper. For example, the controlled slip clutch technology is now being widely used in automotive drive train systems to increase fuel efficiency and to improve ride quality [3.7-3.10]. Clutch systems such as the automotive torque converter clutch (TCC) [3.7], smart clutch [3.8] and dual clutch transmission [3.9, 3.10] employ the dry friction element as the sole or dominant power transmission path. The spring element is usually in series (but not in parallel) with the dry friction element. The scientific literature on such torsional problems is sparse. Recently, Duan and Singh [3.11] studied the TCC sub-system using numerical methods and constructed the non-linear frequency responses based on the cyclic time histories under sinusoidal excitations in the presence of mean torques. However, a thorough understanding of the non-linear characteristics is yet to be achieved. For instance, the existence of super-harmonics has not been demonstrated, especially when a mean torque load is also applied. This chapter will focus on the above mentioned issues and propose semi-analytical methods for a two-degree of freedom (2DOF) torsional system with a single dry friction element of relatively high saturation torque.
Figure 3.1 Schematic of the 2DOF Torsional Automotive Dry Friction Clutch System.
a) Non-linear model. b) Linear system with viscous damper. All parameters and variables have dimensions.
3.2 Problem Formulation

3.2.1 Physical System and Governing Equations

The 2DOF definite torsional system of Figure 3.1 represents, in a generic sense, key features of the automotive clutch systems [3.7, 3.8, 3.11]. The dry friction $T_{f12}$ is the sole path that transmits torque from the flywheel to the downstream driveline system. Here, $\overline{I}_1$ represents the combined torsional inertia of flywheel, front cover and impeller, $\overline{I}_2$ is the inertia of friction shoe, and $\overline{I}_3$ is the lumped inertia of transmission, differential and vehicle. Further, $\overline{I}_3$ is reasonably approximated as a grounded inertia since it is substantial compared to $\overline{I}_1$ and $\overline{I}_2$ during the typical TCC operation, i.e. in a high gear position [3.7, 3.12]. The governing equations for this 2DOF torsional system are:

$$ \overline{T}_1 \ddot{\theta}_1 + \overline{T}_{f12} (t) = \overline{T}_e (\overline{T}) = \overline{T}_m + \overline{T}_p \sin(\omega T), \quad (3.1a) $$

$$ \overline{T}_2 \ddot{\theta}_2 + \overline{C}_{23} \dot{\theta}_2 + \overline{K}_{23} \theta_2 = \overline{T}_{f12} (t). \quad (3.1b) $$

Here, $\dot{\theta}_1$ and $\dot{\theta}_2$ are the absolute angular displacements, $\overline{C}_{23}$ is the lumped viscous damping between the friction shoe and the rest of the driveline, $\overline{K}_{23}$ is the linear torsional stiffness, and $\overline{T}_e (\overline{T})$ is the engine torque excitation composed of mean torque $\overline{T}_m = < \overline{T}_e >_T$ and pulsating $\overline{T}_p (\overline{T})$ components; here $< >_T$ is the time-average operator. The pulsating torque generally contains many harmonic (or torque order) components. However, in this study we will only consider one term, i.e. $\overline{T}_e (\overline{T}) = \overline{T}_m + \overline{T}_p \sin(\omega \overline{T})$ where $\omega = (N_e / 2) \overline{\Omega}_e$ is the dominant frequency for a multi-
cylinder engine [3.13]; here, $N_e$ is the number of engine cylinders and $\Omega_e$ is the engine speed.

In Equation (1), $\bar{T}_{f12}(t) = \bar{T}_{sf} f(\dot{\theta}_1 - \dot{\theta}_2)$ is the non-linear friction torque, which is a function ($f$) of the relative velocity $\dot{\theta}_1 - \dot{\theta}_2$ across the friction interface, and $\bar{T}_{sf}$ represents the saturation friction torque. Further, the classic Coulomb model is used and static and kinetic friction coefficients are assumed to be the same. The normal force on the friction interface remains unchanged and the friction torque $\bar{T}_{sf}$ is accordingly constant during the slip state.

The governing equations (3.1a, b) are then non-dimensionalized for three reasons. First, this process would reduce the number of system parameters and would permit more efficient parametric studies. Second, the numerical integration (or iteration) procedure is easier with a dimensionless formulation. Finally, the resulting dimensionless frequency would help in mapping the non-linear frequency response characteristics. The dimensionless parameters are given as follows; also refer to the list of symbols for identification.

$$\bar{\omega}_n = \sqrt{\frac{K_{23}}{T_2}}, \quad I_1 = \frac{\bar{I}_1}{\bar{T}_2}, \quad I_2 = \frac{\bar{I}_2}{\bar{T}_2} = 1.0, \quad \zeta = \frac{\bar{C}_{23}}{2\sqrt{K_{23}\bar{T}_2}},$$

$$T_m = \frac{\bar{T}_m}{\bar{T}_{sf}}, \quad T_p = \frac{\bar{T}_p}{\bar{T}_{sf}}, \quad \theta_1 = \frac{\bar{\theta}_1\bar{K}_{23}}{\bar{T}_{sf}}, \quad \theta_2 = \frac{\bar{\theta}_2\bar{K}_{23}}{\bar{T}_{sf}},$$

$$\Omega = \frac{\bar{\omega}}{\bar{\omega}_n}, \quad \tau = \bar{\omega}_n \bar{T}.$$
Thus, the governing equations in dimensionless form are as follows where derivatives (superscripts ' and ") are with respect to dimensionless time $\tau$.

\[ I_i \dot{\theta}_1 + f(\dot{\theta}_1 - \dot{\theta}_2) = T_m + T_p \sin(\Omega \tau), \]  
\[ (3.3a) \]

\[ \theta_2'' + 2\zeta \theta_2' + \theta_2 = f(\dot{\theta}_1 - \dot{\theta}_2). \]  
\[ (3.3b) \]

For the case where relative motions are of interest, define $\delta_1(\tau) = \theta_1 - \theta_2$ and $\delta_2(\tau) = \theta_2 - \theta_3 = \theta_2$ since $\theta_3 = 0$ and rewrite the governing equations as:

\[ I_i \delta_1'' - 2\zeta I_i \delta_1' - I_i \delta_2 + (1 + I_1) f(\dot{\delta}_1) = T_m + T_p \sin(\Omega \tau), \]  
\[ (3.4a) \]

\[ \delta_2'' + 2\zeta \delta_2' + \delta_2 = f(\dot{\delta}_1). \]  
\[ (3.4b) \]

### 3.2.2 Objectives

The first major objective is to develop semi-analytical methods and determine the harmonic and super-harmonic responses of the torsional system of Figure 3.1 and as described by (4). For the sake of illustration and validation, we will also apply our method to the single degree of freedom (SDOF) of Figure 3.2 that has been used for many researchers [3.1, 3.2, 3.14]. For instance, Den Hartog obtained a closed form solution under sinusoidal excitation by assuming non-stop frictional oscillations [3.1]. In this chapter, we will extend his study to a system with a single dry friction path and subject it to a sinusoidal excitation in the presence of a mean load. By obtaining a
\[ T_e = T_m + T_p(t) \]

Figure 3.2 Conventional Friction Damper System. a) 2DOF semi-definite system. b) SDOF definite system (Den Hartog’s Model [1]). All parameters and variables have dimensions.
closed form solution to this simplified system, better qualitative understanding of the
non-linear characteristics could be obtained. Further, in those cases where multiple-
stops take place, time domain integration methods are usually employed [3.2, 3.6].
However, the solution process is time-intensive since the numerical integration step
has to be very small to capture the stick-slip transitions. Also, it takes significant time
to obtain a steady-state response especially for a lightly damped system. Some
researchers [3.5, 3.14] also utilized the incremental harmonic balance methods
(IHBM) to examine the friction damper that was placed in parallel with spring
elements. However, their methods are generally limited to three harmonics. This is
not sufficient for our system since significant stick-slip motions could take place.
Accordingly, in this chapter, a refined multi-term harmonic balance method (MHBM)
that can accommodate up to 12 harmonics is proposed to predict the non-linear
characteristics in a more efficient way.

The second major objective is to generate the non-linear frequency response
characteristics of a torsional system with a dry friction controlled path and in
particular to demonstrate the existence of super-harmonic resonances. Much of the
prior work on dry friction damper system (as in Figure 3.2) is limited to an
examination of response in the vicinity of primary harmonic resonance [3.4, 3.5, 3.14].
However, we intend to show that significant super-harmonic resonances could exist in
the system of Figure 3.1. They are also controlled by the mean torque load and they
could even dominate the time domain responses. To clearly show the super-harmonic
resonances, the non-linear frequency response maps will be constructed from the
calculated steady-state cyclic time histories. Two kinds of frequency domain maps are
presented in the subsequent sections. The first is the max-min map that is generated by
picking the maximum (max) and minimum (min) response amplitudes at the excitation
frequency ($\Omega$) of interest; the second is the rms map by calculating the root-mean-
square (rms) values of time history at each frequency. Also, the mean (dc term) values
are plotted over the $\Omega$ range.

3.3 Linear System Analysis

For an automotive driveline coupled with a clutch, the whole system as in
Figure 3.1a may be modeled as a linear system under two premises: (i) assume the
pure stick condition or (ii) replace the dry friction element by a linear viscous damper.
However, under the first premise, with a change in the inertia (for example, from $I_2$ to
$I_1 + I_2$) and in the degree of freedom, the essential system properties such as natural
frequencies and modes of the downstream sub-system will significantly change. Thus
it is not suitable for benchmark studies. Consequently, in this section, we will analyze
a linear system coupled by a viscous damper as shown in Figure 3.1b using the second
premise. The governing equations are obtained by replacing $f(\dot{\theta}_1 - \dot{\theta}_2)$ in equation
(3.4) with $2\zeta_1(\dot{\theta}_1 - \dot{\theta}_2)$ where $\zeta_1 = \bar{C}_{12} / (2\sqrt{K_{12}})$:

$$I_1 \ddot{\delta}_1 + (1 + I_1)2\zeta_1 \dot{\delta}_1 - 2\zeta I_1 \dot{\delta}_2 - I_1 \ddot{\delta}_2 = T_m + T_p \sin(\Omega \tau), \tag{3.5a}$$

$$\ddot{\delta}_2 + 2\zeta \dot{\delta}_2 + \delta_2 = 2\zeta_1 \dot{\delta}_1. \tag{3.5b}$$

A closer look at the above equations reveals that there has to be a mean velocity for $\dot{\delta}_1$.
Figure 3.3 Linear System Frequency Response for $I_1 = 0.01$, $\zeta_1 = 0.4$, $\zeta = 0.001$, $T_m = 0.5$, $T_p = 4.5$. a) Max-mean-min frequency responses of $\delta_1$; b) Max-min frequency response of $\delta_2$. 
to balance the mean torque \( T_m \) under the dynamic condition. Further, \( T_m \) will be carried on to \( \delta_2 \) through the viscous coupling. By assuming harmonic solutions, the mean part can be very easily obtained where \( < >_\tau \) implies time-average.

\[
<\delta'_1(\tau)>_\tau = \frac{T_m}{2\zeta_1}, \quad <\delta'_2(\tau)>_\tau = T_m. \quad (3.6a, b)
\]

Note that these values remain constant as \( \Omega \) changes. One could obtain complete analytical solutions for \( \delta'_1(\tau) \) and but the resulting expressions would be tediously long. Thus these are not included here. Rather, solutions are conveniently obtained by using a conventional numerical integration scheme such as Runge-Kutta. Sample frequency responses are presented in Figure 3.3. It is seen that the relative velocity maintains an almost constant level except around \( \Omega = 1.0 \). As evident from Figure 3.3b, only the primary harmonic resonance appears in \( \delta_2 \) as it should for a linear system. Numerical solutions confirm the analytical mean velocity \( <\delta'_1(\tau)>_\tau \) and mean displacement \( <\delta'_2(\tau)>_\tau \) as observed in Figure 3.3.

### 3.4. Analytical Solution Using One-Term Harmonic Balance Method

#### 3.4.1 Closed Form Solution to a Simplified Torsional System

First, an approximate analytical solution based on one-term harmonic balance method (HBM) is constructed for a simplified system to qualitatively establish the nature of non-linear frequency responses and super-harmonic resonances. In this simplified system (of Figure 3.1) we assume that \( I_1 << 1.0 \) and continuous slipping motion takes place across the friction interface. Further, \( \delta_2 << \delta_1 \) and \( \delta'_2 << \delta'_1 \) are
assumed. These assumptions are reasonable because the displacement and velocity of $I_2$ are constrained by the torsional spring and the viscous damper respectively. Conversely, the dry frictional path, unlike the viscous damper which can at least constrain the velocity, can not limit motions of $I_1$ especially in the presence of $T_m$. Thus, we further approximate (3.4) as:

$$I_1\ddot{\delta}_1 + f(\dot{\delta}_1) = T_m + T_p \sin(\Omega \tau). \quad (3.7a)$$

$$\ddot{\delta}_2 + 2\zeta \dot{\delta}_2 + \ddot{\delta}_2 = f(\dot{\delta}_1). \quad (3.7b)$$

The above formulation essentially de-couples the 2DOF definite system into two SDOF sub-systems as shown in Figures 3.4a and 3.4b respectively. Next, consider the system of Figure 3.4a and equation (3.7a). The relative velocity and acceleration across the frictional interface, under harmonic excitation, are assumed as follows where the mean velocity $A$ is a consequence of $T_m$, $B$ is the amplitude of dynamic velocity and $\phi$ is the phase lag.

$$\dot{\delta}_1 = A + B \sin(\Omega \tau + \phi), \hspace{1cm} \dot{\delta}_1 = B \Omega \cos(\Omega \tau + \phi). \quad (3.8a, b)$$

Assume profiles for relative velocity and corresponding friction torque as illustrated in Figure 3.5. Note that an asymmetric slip motion occurs due to the mean velocity and the corresponding friction torque. i.e. $\dot{\delta}(\tau) \neq -\dot{\delta}(\tau + P/2)$ and $T_f(\tau) \neq -T_f(\tau + P/2)$ where $P = 2\pi / \Omega$ is the period. For continuous slipping motions, different amplitudes corresponding to positive or negative slip result due to the bias
$T_e = T_m + T_p \sin(\Omega \tau)$

Figure 3.4 Simplifications to Figure 3.1a Yield Two De-coupled Sub-systems. a) Non-Linear sub-system with dry friction; b) Linear sub-system. Dimensionless parameters and variables are shown here.
Figure 3.5 Assumed Time Domain Profiles for Relative Velocity and Friction Torque.
term $A$. However, in the case of stick-slip as explored in the subsequent section, the asymmetry could be introduced by either a bias term or even order harmonics. The transition times $\tau_1$ and $\tau_2$ (when $T_f$ undergoes an abrupt change) can be determined by setting $\delta' = 0$ as:

$$\Omega \tau_1 + \phi = -\sin^{-1} \frac{A}{B} , \quad \tau_1 = \frac{-\sin^{-1} \frac{A}{B} - \phi}{\Omega} . \quad (3.9a, b)$$

$$\Omega \tau_2 + \phi = \pi + \sin^{-1} \frac{A}{B} , \quad \tau_2 = \frac{\pi + \sin^{-1} \frac{A}{B} - \phi}{\Omega} . \quad (3.10a, b)$$

The corresponding friction torque is seen as follows.

$$T_f(\tau) = \begin{cases} 1.0 & \tau_1 \leq \tau < \tau_2 \\ -1.0 & \tau_2 \leq \tau < P + \tau_1 \end{cases} . \quad (3.11)$$

Approximate the above expression using a truncated Fourier series as:

$$T_f(\tau) = \frac{T_{f0}}{2} + \sum_{n=1}^{N} T_{cn} \cos(n\Omega \tau) + T_{sn} \sin(n\Omega \tau) , \quad (3.12a)$$

$$T_{f0} = \frac{2}{P} \int_{\tau_1}^{P + \tau_1} T_f(\tau) d\tau = \frac{\Omega}{\pi} (\tau_2 - \tau_1 - \frac{\pi}{\Omega}) , \quad (3.12b)$$

$$T_{cn} = \frac{2}{P} \int_{\tau_1}^{P + \tau_1} T_f(\tau) \cos(n\Omega \tau) d\tau = \frac{2}{n\pi} [\sin(n\Omega \tau_2) - \sin(n\Omega \tau_1)] , \quad (3.12c)$$

$$T_{sn} = \frac{2}{P} \int_{\tau_1}^{P + \tau_1} T_f(\tau) \sin(n\Omega \tau) d\tau = \frac{2}{n\pi} [\cos(n\Omega \tau_2) - \cos(n\Omega \tau_1)] . \quad (3.12d)$$

Substituting $T_{f0}$, $T_{c1}$ and $T_{s1}$ into equation (3.7a) and applying the harmonic balance to both sides, we find the following non-linear algebraic equations by sorting the like terms:
Further, insert equations (3.9b) and (3.10b) into (3.13a) to find:

\[
\sin^{-1} A = \frac{T_m \pi}{B},
\]

(3.14)

Since \( \sin^{-1} (A/B) \) is bounded in \([0.5, 0.5] \pi \), it is noted that \(-1 \leq T_m \leq 1\). Analytically, if \( T_m \) is beyond such bounds, then either pure positive slip (\( T_m > 1 \)) or pure negative slip (\( T_m < -1 \)) motion takes place. In either case, the transmitted frictional torque \( f(\delta') \) will always assume a constant value (\( \pm 1 \)) accordingly. That would result in a constant relative displacement \( \delta_2 \). Use trigonometric relations to observe the following:

\[
\sin(\Omega \tau_2) - \sin(\Omega \tau_1) = 2 \cos\left(\frac{T_m \pi}{2}\right) \sin(\phi),
\]

(3.15a)

\[
\cos(\Omega \tau_2) - \cos(\Omega \tau_1) = -2 \cos\left(\frac{T_m \pi}{2}\right) \cos(\phi).
\]

(3.15b)

Substitute equations (3.15) into (3.13b) and (3.13c) respectively, and further simplify the non-linear algebraic equations as:

\[
I_1 B \Omega \cos(\phi) + \frac{4}{\pi} \cos\left(\frac{T_m \pi}{2}\right) \sin(\phi) = 0,
\]

(3.16a)

\[
-I_1 B \Omega \sin(\phi) + \frac{4}{\pi} \cos\left(\frac{T_m \pi}{2}\right) \cos(\phi) = T_p.
\]

(3.16b)

Finally, the closed form solutions of A, B and \( \phi \) of equations (3.8), corresponding to Figure 3.4a, can be obtained as:
\[
A = B \sin \frac{mT \pi}{2}, \quad B = \frac{\sqrt{T^2_m \pi^2 - 16 \cos^2 \left(\frac{T_m \pi}{2}\right)}}{I \pi \Omega}, \quad (3.17a, b)
\]

\[
\phi = -\sin^{-1} \sqrt{\frac{T^2_m \pi^2 - 16 \cos^2 \left(\frac{T_m \pi}{2}\right)}} {T_m \pi}, \quad (3.17c)
\]

Although \(A\) and \(B\) change with \(\Omega\), \(\phi\) retains a constant value that is solely determined by the \(T_m\) and \(T_p\). This is different from the first order linear time-invariant system for which the \(\phi\) varies with \(\Omega\) \cite{3.15}. Also, it differs from the classical dry friction damper system where a discontinuous jump in \(\phi\) is seen at the primary resonance \cite{3.1}.

Once \(\delta'_1(\tau)\) is obtained, the corresponding \(T_f(\tau)\) is also determined, which now acts as an exciter for the sub-system 2 of Figure 3.4b. Feeding the solution of \(T_f(\tau)\) into equation (3.7b), an analytical solution for \(\delta_2(\tau)\) is obtained:

\[
\delta_2(\tau) = T_m + \sum_{n=1}^{N} \left\{ \frac{T_m}{\Lambda_n} \cos(n\Omega \tau + \psi_n) + \frac{T_m}{\Lambda_n} \sin(n\Omega \tau + \psi_n) \right\}, \quad (3.18a)
\]

\[\Lambda_n = \sqrt{(1 - n^2 \Omega^2)^2 + (2n \zeta \Omega)^2}, \quad \psi_n = -\tan^{-1} \frac{2n \zeta \Omega}{1 - n^2 \Omega^2}. \quad (3.18b, c)\]
Figure 3.6 Time Histories for System of Figure 3.4 Given $I_1 = 0.01$, $\zeta = 0.001$, $T_m = 0.5$, $T_p = 4.5$. a) $\delta'$ and $T_f$ at $\Omega = 0.5$: ___ analytical solution; - - -, numerical solution. b) $\delta'$ and $T_f$ at $\Omega = 0.9$: ___ analytical solution; - - -, numerical solution.
Figure 3.7 Frequency Responses for System of Fig. 4 Given $I_1 = 0.01$, $\zeta = 0.001$, $T_m = 0.5$, $T_p = 4.5$. a) Max-mean-min frequency responses of $\delta_1$: --o--, analytical solution; --x--, numerical solution; b) Max-min frequency responses of $\delta_2$: --o--, analytical solution; --x--, numerical solution.
Figure 3.8 Frequency Responses for System of Fig. 4 Given $I_1 = 0.01$, $\zeta = 0.001$, $T_p = 4.5$. a) Max-mean-min responses of $\delta_1'$: --o--, $T_m = 0$; --+--, $T_m = 0.2$; b) Max-min responses of $\delta_2'$: --o--, $T_m = 0$; --+--, $T_m = 0.2$. 

97
3.4.2 Analytical and Numerical Solutions

The analytical solutions obtained in previous section clearly reveal multiple harmonics in steady-state response of $\delta_2(\tau)$ and thereby raise the possibility of super-harmonic resonances. Typical analytical results, in both time and frequency domains are shown in Figures 3.6 and 3.7. These are compared with numerical solutions from a standard Runge-Kutta $5^{th}(4^{th})$ order integration scheme with step size adaptation due to Dormand and Prince [3.16]. Excellent match between analytical and numerical solutions is achieved. Minor differences in the frequency response of velocity $\delta_1^i$ are noticed at $\Omega=1.0$ and $\Omega=0.5$. This is because the analytical solution does not account for a coupling between the sub-systems that is maximized at primary and super-harmonic resonances. And, differences in time domain velocity $\delta_1^i$ result seem to produce minimal effects in the frequency response maps of $\delta_2$ as seen in Figure 3.7b. Unlike the response for a linear system as observed in Figure 3.3b, super-harmonic resonances are generated in Figure 3.7b by the dry friction non-linearity at $\Omega=1/3$ and $1/2$ respectively.

Although the proposed analytical solution is limited to a simplified system with pure (or almost pure) slipping motion, the influence of $T_m$ can be easily identified. From equation (3.17b), it is seen that the mean velocity $A$ vanishes as $T_m$ goes to zero. In this case, the friction interface would experience symmetric positive and negative slipping motions. These in turn would generate a symmetric square
Figure 3.9 Frequency Responses for System of Figure 3.4 Given $I_1 = 0.04$, $\zeta = 0.001$, $T_m = 0.5$, $T_p = 1.0$. a) Max-min frequency responses of $\delta_1$: --o--, analytical solution; --x--, numerical solution; b) Max-min frequency responses of $\delta_2$: --o--, analytical solution; --x--, numerical solution.
Figure 3.10 Time Histories for System of Figure 3.4 Given $I_1 = 0.04$, $\zeta = 0.001$, $T_m = 0.5$, $T_p = 1.0$. a) $\delta_i$ and $T_f$ at $\Omega = 0.5$: ____ analytical solution; - - -, numerical solution. b) $\delta_i$ and $T_f$ at $\Omega = 0.9$: ____ analytical solution; - - -, numerical solution.
profile of the friction torque in Figure 3.5. Thus, only the odd orders (1/3, 1/5, ...) of $T_f$ are generated and consequently only the odd order super-harmonic resonances would occur in the response $\delta_2$. Additionally, Figure 3.8 shows that even a very small $T_m$ will generate even orders (such as 1/2 and 1/4). For example, the 1/2 super-harmonic resonance is clearly observed in Figure 3.8 at $\Omega = 0.5$ when $T_m = 0.2$.

3.4.3 Limitations

The one-term harmonic balance method has been successful in qualitatively establishing the nature of super-harmonic resonances. Although the approximate analytical solution quickly predicts responses, it is valid only for a very simplified problem. In a more realistic driveline system, significant stick-slip motions occur and the assumption of continuous slipping motions is no longer valid. Thus, the analytical solution cannot provide accurate results for the system of Figure 3.1a, as evident from Figure 3.9 for frequency responses and from Figure 3.10 for corresponding time histories. For this reason, a refined multi-term harmonic balance method (MHBM) is proposed in the next section to reasonably approximate the steady-state stick-slip motions under harmonic torque excitation given a mean load.
3.5 Refined Multi-term Harmonic Balance Method

3.5.1 Formulation

Consider the torsional system of Figure 3.1 again and observe that no spring element exists in parallel with the dry friction element. Consequently, the relative displacement $\delta_1(\tau)$ would grow monotonically with a mean velocity and thus it cannot be defined as a periodic function under the effect of $T_m$. Instead, the relative velocity $\dot{\delta}_1(\tau)$ is assumed to be periodic and is expressed as a truncated Fourier series as:

$$\dot{\delta}_1(\tau) = a_o + \sum_{n=1}^{nh} a_{2n-1} \sin(n\Omega \tau) + a_{2n} \cos(n\Omega \tau). \quad (3.19a)$$

Similarly, the relative displacement $\delta_2(\tau)$ and the non-linear friction torque $f(\delta_1)$ are expanded in the form of Fourier series:

$$\delta_2(\tau) = b_o + \sum_{n=1}^{nb} b_{2n-1} \sin(n\Omega \tau) + b_{2n} \cos(n\Omega \tau), \quad (3.19b)$$

$$f(\delta_1) = c_o + \sum_{n=1}^{nh} c_{2n-1} \sin(n\Omega \tau) + c_{2n} \cos(n\Omega \tau). \quad (3.19c)$$

First, substitute (3.19b) and (3.19c) into (3.7b) and sort out the like terms on both sides to yield:

$$b_o = c_o, \quad (3.20a)$$

$$b_{2n-1}(1-n^2\Omega^2) - b_{2n}2\zeta_2 n\Omega = c_{2n-1}, \quad (3.20b)$$

$$b_{2n-1}2\zeta_2 n\Omega + b_{2n}(1-n^2\Omega^2) = c_{2n}. \quad (3.20c)$$

Represent the above in matrix form to yield the following where $H$ is a characteristic matrix of the linear sub-system of Figure 3.4b.

$$Hb = c, \quad b = H^{-1}c, \quad (3.21a, b)$$

102
\[ b = (b_0 \ b_1 \ b_2 \ \cdots \ b_{2nh})^T, \quad \mathbf{c} = (c_0 \ c_1 \ c_2 \ \cdots \ c_{2nh})^T, \quad (3.21c,d) \]

\[
H = \begin{bmatrix}
1 & \cdots & \cdots & \cdots \\
& 1-n^2\Omega^2 & -2\zeta n\Omega \\
& & 2\zeta n\Omega & 1-n^2\Omega^2 \\
& & & \ddots \\
\end{bmatrix}.
\] (3.21e)

The introduction of \( H \) would efficiently reduce the original 2DOF system problem into SDOF system problem. Further, the characteristic matrix concept can be extended to a linear sub-system of very large dimension, thus allowing a non-linear path synthesis concept similar to Rook’s formulation \([3.17]\).

The time domain response \( \delta_i(\tau) \) can be written in terms of a discrete vector \( \mathbf{\delta}_i \) by utilizing a discrete Fourier transform (DFT) matrix \( \Delta \) where \( a \) is the corresponding Fourier coefficients of \( \delta_i(\tau) \)[3.18]. Generally, the number of discrete points to represent a steady-state response cycle is a multiple of 2, i.e. \( N = 2^p \). This is consistent with the requirement of a fast Fourier transform routine.

\[
\mathbf{\delta}_1(\tau) = \left( \delta_1'(\tau_0) \ \delta_1'(\tau_1) \ \cdots \ \delta_1'(\tau_{N-1}) \right)^T = \Delta \mathbf{a}.
\] (3.22a)

Similarly:

\[
\mathbf{\delta}_2(\tau) = \Delta \mathbf{b}.
\] (3.22b)

Introduce a differential operator \( D \) as:

\[
D = \begin{bmatrix}
0 & & & \\
& \ddots & & \\
& & 0 & -n \\
& & n & 0 \\
& & & \ddots \\
\end{bmatrix}.
\] (3.23)
Thus, we have:

\[ \delta_1'(\tau) = \Omega \Delta Da \], \hfill (3.24a) \\
\[ \delta_2'(\tau) = \Omega \Delta Db \], \quad \delta_3'(\tau) = \Omega^2 \Delta^2 b \]. \hfill (3.24b, c)

The non-linear torque is also written as \( f(\delta) = \Delta c \), which implies \( c = \Delta^+ f \) where \( \Delta^+ = (\Delta^T \Delta)^{-1} \Delta^T \). The torque excitation is also defined in the form \( T_e(\tau) = \Delta Q \), where \( Q \) is a known vector. Substitute (3.19-3.24) into equation (3.4) and define the residue \( \Delta R \) in time domain as:

\[ \Delta R = I_i \Omega \Delta Da + I_i \Omega^2 \Delta^2 b + \Delta c - \Delta Q \]. \hfill (3.25)

Further, substituting \( b = H^{-1} c = H^{-1} \Delta^+ f \) and pre-multiplying both sides by \( \Delta^+ \), the residue in frequency domain \( \Delta R \) is obtained:

\[ R = I_i \Omega Da + [I_i \Omega^2 D^2 H^{-1} \Delta^+ + \Delta^+] f - Q \]. \hfill (3.26)

Essentially, our MHBM minimizes \( R \) in the frequency domain by using an iterative approach. For instance, the Newton-Raphson iteration has been widely used [3.14, 3.17, 3.18]. In this process, a Jacobian Matrix \( J \) is first defined as follows:

\[ J = \frac{\partial R}{\partial a} = I_i \Omega D + [I_i \Omega^2 D^2 H^{-1} \Delta^+ + \Delta^+] \frac{\partial f}{\partial a} \]. \hfill (3.27)

Here, \( \frac{\partial f}{\partial a} \) can be calculated by the chain rule as:

\[ \frac{\partial f}{\partial a} = \frac{\partial f}{\partial \delta} \frac{\partial \delta}{\partial a} = \frac{\partial f}{\partial \delta} \Delta \]. \hfill (3.28)

At each iterative step, the value of \( a \) is updated as:

\[ a_{k+1} = a_k - J_k^{-1} R_k \]. \hfill (3.29)
3.5.2 Computational Issues and Choice of Initial Conditions

The following error criteria are usually followed [3.5, 3.14, 3.18, 3.21] where $\varepsilon$ is a pre-defined numerical tolerance:

$$\|a_{k+1} - a_k\| \leq \varepsilon, \quad \|R_{k+1} - R_k\| \leq \varepsilon.$$  \hfill (3.30a, b)

Here, $\|$ represents Euclidean or $L_2$ norm. When the Jacobian matrix is ill-conditioned, the first criterion (3.30a) is more reliable [3.19]. The chief benefit of the Newton-Raphson technique is its quadratic convergence feature, i.e.

$$\|R_{k+1} - R_k\| = O\|R_k - R_{k-1}\|^2$$

where $O$ represents order of magnitude. To take advantage of the quick convergence ability, the partial derivative of the residue has to be evaluated as in equation (3.27). This implies the non-linear function has to be continuous as indicated in equation (3.28). This is however not the case for the classical Coulomb friction formulation in which a discontinuity exists at zero velocity. Therefore, a smoothening or conditioning procedure, using the hyperbolic or arctangent function, has been usually employed by some researchers [3.11, 3.18, 3.20, 3.21]. In our study, a hyperbolic tangent function is used to approximate the classic Coulomb friction function:

$$T_f = f(\delta_i) = \tanh(\sigma\delta_i).$$  \hfill (3.31a)

$$\frac{\partial f}{\partial \delta_i} = \sigma[1 - \tanh^2(\sigma\delta_i)]$$  \hfill (3.31b)

Duan and Singh [3.11] have shown that the conditioning factor $\sigma$ should be very carefully chosen to ensure an appropriate representation of the theoretical
discontinuous Coulomb friction when a direct Runge-Kutta 4\textsuperscript{th}(5\textsuperscript{th}) numerical integration scheme is employed. Kim et al. discussed the effect of the “smoothening factor” on non-linear frequency responses with application of clearance type non-linearity [3.21]. Our harmonic balance method is also sensitive to the choice of σ. Chief reason lies in the calculation of $\frac{\partial f}{\partial \delta_i}$ that plays an important role in the Jacobian matrix.

$$
\frac{\partial f}{\partial \delta_i} = \text{diag} \left[ \frac{\partial f}{\partial \delta_i} \bigg|_{r=r_0}, \frac{\partial f}{\partial \delta_i} \bigg|_{r=r_1}, \ldots, \frac{\partial f}{\partial \delta_i} \bigg|_{r=r_{N+1}} \right] \tag{3.32}
$$

As evident from equation (3.31a), a lower value of σ would make the stick to slip transition more smooth and thus it is desirable in terms of numerical convergence. However, a more smooth transition would indicate fewer harmonics are contained in the approximated friction torque as explained by the Gibbs phenomena. Consequently, the calculated response may not be sufficiently accurate especially when the super-harmonic components significantly contribute to the overall response. On the other hand, a high value of σ is intuitively preferred. Mathematically, as the value of σ increases, the approximated $T_f = f(\delta_i)$ asymptotically converges to the discontinuous Coulomb friction. However, the values of $\frac{\partial f}{\partial \delta_i}$ vary from relatively large numbers (corresponding to the stick state) to almost zeros (corresponding to the slip state) as obtained by equation (3.31b). The order of magnitude difference in such numerical values would ultimately contribute to the numerical stiffness of the Jacobian matrix that could be defined by the ratio of the largest to the smallest non-zero local eigenvalues [3.22]. The widely separated eigenvalues in turn would indicate the
coexistence of slowly-varying and rapidly-varying responses when the solution is slightly perturbed. This would require unreasonably small calculation steps to warrant numerical stability. Thus it would hinder convergence especially when significant stick-slip motions take place. We selected $\sigma = O(50)$ when 256 discrete points are used to represent the continuous time history within an excitation cycle. The resulting responses have been validated by using a discontinuous numerical integration scheme that is already documented in an earlier paper [3.11].

Further, the initial guess of solution $a_0$ is very important for the predictor-corrector type exercises [3.23]. If $a_0$ is far away from true solution, the convergence speed of Newton-Raphson is limited because the quadratic convergence seems to occur only during last steps. In the worst possible case, convergence may not be achieved at all and the solution could diverge. But much of previous work on this topic [3.4, 3.5, 3.14, 3.18] does not address this issue in sufficient detail. One approach is to just make a random guess [3.18]. Further, Wang et. al. have proposed that one could use the first order (one-term) harmonic balance solution as the initial guess for a bi-linear hysteresis problem [3.5]. But as seen in Figure 3.9, our one-term harmonic balance solution is still sufficiently far from the true solution. Consequently, we propose the following scheme. First, determine the stick to slip boundaries prior to a non-linear analysis, using a similar procedure introduced by Duan and Singh [3.11]. For the sake of clarity, this procedure is briefly introduced here. When the frictional element is under the pure stick condition, $\bar{T}_1$ and $\bar{T}_2$ stick together to form a single
rigid body and the system in Figure 3.1 degenerates into a SDOF system.

Consequently, the governing equation, in the dimensionless form, is as:

\[
(I_1 + 1)\dot{\delta}^2 + 2\zeta\dot{\delta} + \delta = T_m + T_p \sin(\Omega\tau). 
\] (3.33)

The corresponding steady-state forced harmonic response is:

\[
\begin{align*}
\delta(\tau) &= \delta_m + \delta_p(\tau) = T_m + \frac{T_p}{\sqrt{1 - (I_1 + 1)\Omega^2}^2 + (2\zeta\Omega)^2}} \sin(\Omega\tau + \varphi), \\
\varphi &= -\tan^{-1} \frac{2\zeta\Omega}{1 - (I_1 + 1)\Omega^2}. 
\end{align*}
\] (3.34a)
(3.34b)

Under the pure stick condition, the frictional torque in the interface is the difference between the excitation torque and the inertial torque \( I_1\dot{\delta} \) where \( \dot{\delta} = -\Omega^2\delta_p \) is obtained from equation (3.34a):

\[
T_f(\tau) = [T_m + T_p \sin(\Omega\tau)] - [I_1(-\Omega^2\delta_p(\tau))], 
\] (3.35)

\[
T_f(\tau) = T_m + T_p \sin(\Omega\tau) + I_1 \frac{T_p\Omega^2}{\sqrt{1 - (I_1 + 1)\Omega^2}^2 + (2\zeta\Omega)^2}} \sin(\Omega\tau + \varphi). 
\] (3.36)

Thus, the criterion to determine the stick to slip transition is defined as follows where \( \| \| \) represents the absolute value.

\[
\|T_f\| > T_{sf}. 
\] (3.37)

Using this criterion, the frequency regime(s) over which the pure stick condition takes place can be found by numerically sweeping the excitation frequency in either downward or upward direction. Additionally, the determination of stick to slip boundaries would not only bound the frequency regime over which the non-linear
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Dimensionless Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1$</td>
<td>10~20</td>
</tr>
<tr>
<td>$I_2$</td>
<td>1.0</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>0.02</td>
</tr>
<tr>
<td>$K$</td>
<td>1.0</td>
</tr>
<tr>
<td>$T_{sf}$</td>
<td>1.0</td>
</tr>
<tr>
<td>$T_m$</td>
<td>0~0.8</td>
</tr>
<tr>
<td>$T_p$</td>
<td>0.5~2.0</td>
</tr>
</tbody>
</table>

Table 3.1 Parameters (in the dimensionless form) used to study an automotive driveline system corresponding to Figure 3.1
analysis is needed, but it could also provide a good clue regarding the initial frequency for a sweep up or down; Note that the zeros are natural choices in the initial guess of \(a_0\). This way, the solution at the initial frequency is easily obtained. Given a high frequency resolution, the response at subsequent frequencies can be conveniently determined by assuming the solution at the previous frequency as the initial guess.

3.6 Non-linear Response and Super-Harmonics

3.6.1 Typical Non-linear Responses and Effect of Mean Load

Table 3.1 lists typical parameters (in the dimensionless form) of an automotive driveline system. As noted before, the saturated dry friction torque is generally high for a realistic TCC, in contrast with the dry frictional damper system, for the following reasons: to increase the fuel efficiency by allowing more power transmitted to the downstream system and to avoid thermal issues induced by excessive slipping motions [3.7, 3.24]. For example, the friction torque capacity of a typical dry friction torque converter clutch is typically of the same order of magnitude as the peak dynamic torque generated by a nominal multi-cylinder engine [3.24, 3.25]. Further, unlike the simplified system that was studied in section 3.3, \(I_1\) representing flywheel, front cover and impeller is much higher than \(I_2\) of the friction shoe assembly. Nonetheless, Duan and Singh [3.11] have shown that \(I_2\) can still significantly affect the system dynamics.

The appearance of super-harmonic peaks is obviously related to the number of harmonics \((n)\) that must be included by the multi-term harmonic balance method. In our study, 12 harmonics are used to construct the stick-slip motions and accordingly,
12 terms are included in $\delta_z(t)$ as these should be enough to predict real-life periodic motions. Also, as more harmonics are included, the super-harmonic resonances like $1/12$, $1/13$ and $1/14$, etc. will be squeezed into a smaller frequency region and this would pose some difficulty in distinguishing them. Thus, in our study, we will only show the frequency range starting from $\Omega = 0.12$ and assume minimal coupling effect between the super-harmonics below $\Omega = 0.12$. Under the condition of significant stick-slip, a generic non-linear frequency response map of $\delta_z$ is illustrated in Figure 3.11. As shown, two types of peaks could be present. The first type is the transitional peak which occurs as the frictional interface undergoes sudden transition from a pure stick state to the stick-slip state. The second type is the resonant peak which occurs at $\Omega = 1/n$ due to the system resonance effect.

Figure 3.12 represents a typical non-linear frequency response when significant stick-slip motions occur. Figure 3.13 shows a sample time history at $\Omega = 0.23$. The semi-analytical solutions match quite well with numerical solutions. The difference in the $T_f(\tau)$ plot of Figure 3.13c clearly illustrates the Gibbs phenomena. Further, super-harmonic peaks in Figure 3.12b can be seen at $\Omega = 1/3, 1/4, 1/5, 1/7$ etc. However, it should be noted that the peak response at $\Omega = 1/2$ cannot be presumed to be a super-harmonic peak. In fact, it is a transitional peak, as evident from Figure 3.12a. When the system parameters and excitation change, the transition frequency will also change. This transition frequency can be determined by the procedure introduced in section 3.2.3, ahead of the non-linear calculation. Again, the calculated
Figure 3.11 A Generic Non-Linear Frequency Response Map of $\delta_2$ under Significant Stick-Slip Motions.
Figure 3.12 Frequency Response for System of Figure 1a Given $I_1 = 10$, $\zeta = 0.02$, $T_m = 0.5$, $T_p = 1.5$. a) Max-min frequency responses of $\delta_1$: ooo, semi-analytical solution with 12 harmonic terms; ---, semi-analytical solution with 3 harmonic terms; xxx, numerical solution; b) rms frequency response of $\delta_2$: ooo, semi-analytical solution with 12 harmonic terms; ---, semi-analytical solution with 3 harmonic terms; xxx, numerical solution.
Figure 3.13 Time Histories at $\Omega = 0.23$ for System of Figure 3.1a Given $I_1 = 10$, $\zeta = 0.02$, $T_m = 0.5$, $T_p = 1.5$. a) harmonic excitation torque; b) time history of $\delta_1'$: ___, semi-analytical solution; ___, numerical solution; c) time history of $T_f$: ___, semi-analytical solution; ___, numerical solution; d) time history of $\delta_2'$: ___, semi-analytical solution; ___, numerical solution.
Figure 3.14 Frequency Responses for System of Figure 1a Given. $I_1 = 10, \zeta = 0.02, T_m = 0, T_p = 1.0$. a) Max-min frequency responses of $\delta_1';$ b) rms frequency response of $\delta_2'$. 
Figure 3.15 Time Histories and Fast Fourier Transform of $\delta_2$ Given $I_1 = 10$, $\zeta = 0.02$, $T_m = 0.5$, $T_p = 1.5$. a) $\Omega = 0.32$; b) $\Omega = 0.23$. 
non-linear frequency response of $\delta_2$ is much different from the anticipated linear system response for which a primary harmonic resonance occurs at $\Omega \approx 1.0$ as in Figure 3.3b. Instead, the super-harmonic peaks dominate the response level at low frequencies and the primary harmonic resonance is not excited at all. Further, it is noted that 3 harmonic terms are not sufficient to represent significant stick-slip motions as shown in Figure 3.12.

Again, the effect of $T_m$ is investigated. Figure 3.14 presents the non-linear frequency responses for $T_m = 0$. Compare these with Figure 3.12 and observe that the existence of non-zero $T_m$ has an effect on the frequency responses similar to the one discussed in section 3.4 for a simplified system. When $T_m = 0$, symmetric stick-slip motions are strictly followed as seen in Figure 3.14a. But asymmetric stick-slip motions take place at $T_m = 0.5$ as observed in Figure 3.12a. Further, more super-harmonic peaks are excited by the friction torque that is generated by the asymmetric stick-slip. For example in Figure 3.14, when $T_m = 0$, super-harmonic resonances only occur around $\Omega \approx 1/4$ and $1/6$; but when $T_m = 0.5$, resonances take place around $\Omega \approx 1/3, 1/4, 1/5, 1/7$ etc. as shown in Figure 3.12b. However, under significant stick-slip motions, the generated friction torque as in Figure 3.13 is no longer a pulse excitation as in Figures 3.6 and 3.10. It is hard to analytically predict which super-harmonic peaks will appear unlike the simple case studied in section 3.4. Nevertheless, semi-analytical methods such as MHBM or numerical integration can be utilized to find the resonant peaks.
Figure 3.16 3-D Response Map for System of Figure 3.1a Given $I_1 = 10$, $\zeta = 0.02$, $T_m = 0.5$, $T_p = 1.5$. 
3.6.2 Effect of the Number of Harmonics on Resonant Peaks

As discussed in section 3.4, the super-harmonic contents in $T_f(\tau)$ induced by pure slip or stick-slip motions will generate super-harmonic response in $\delta_2(\tau)$. Accordingly, resonant peaks at lower $\Omega$ as in Figures 3.12b and 3.14b occur. However, in the case of significant stick-slip motions, it does not mean the $n^{th}$ super-harmonic component of $\delta_2$ would always dominate the time history when excited at $\Omega = 1/n$. For instance, consider the time history corresponding to the peak around $\Omega = 0.32$ (1/3 super-harmonic peak) as shown in Figure 3.15a. The response is dictated by the first 4 harmonics and the mean ($n = 0$) part. At the peak around $\Omega = 0.23$ (1/4 super-harmonic peak) in Figure 3.15b, the response is dictated by the first 5 harmonics and the mean part. The dominance of $n = 1$ and 2 components is due to the sticking phase which prevails in the stick-slip response. The difference in rms value responses as shown in Figure 3.11b is due to the involvement of more period-motions (or more harmonics) in the time history.

The mechanism of generating super-harmonic peaks discussed above can be further explained by a 3-D response map that is constructed in Figure 3.16. The x coordinate indicates the harmonic order $n$ (including the dc part) of the response of $\delta_2$; the y coordinate is the excitation frequency $\Omega$ and the z coordinate is the response amplitude $\delta_2$. As seen in Figure 3.16, $n = 0, 1$ and 2 components contribute much to the overall response over the entire frequency regime, which is consistent with our observation at $\Omega = 0.32$ and $\Omega = 0.23$. As $\Omega$ moves down, more super-
harmonic components get involved. Accordingly, various resonant peaks are formulated.

### 3.7 Effect of Negative Slope in Friction Formulation

Finally, we study a more general friction formulation, with different static ($\mu_s$) and kinetic ($\mu_k$) friction coefficients. In the following formulation [3.26], $\mu_k$ has been normalized with respect to $\mu_s$ (=1.0), $\alpha$ is a factor that controls the exponentially decaying gradient and sgn is the signum function.

$$
T_f(\delta'_1) = f(\delta'_1) = \begin{cases} 
\mu_k + (1.0 - \mu_k)e^{-\alpha|\delta'_1|} \text{sgn}(\delta'_1) & |\delta'_1| > 0 \\
[0 \ 1.0] & \delta'_1 = 0
\end{cases} 
$$

(3.38)

$$
\text{sgn}(\delta'_1) = \begin{cases} 
\delta'_1 & |\delta'_1| > 0 \\
0 & \delta'_1 = 0
\end{cases}
$$

(3.39)

Further, we can further condition the discontinuous formulation (3.38) by a hyperbolic-tangent function.

$$
f(\delta'_1) = [\mu_k + (1.0 - \mu_k)e^{-\alpha|\delta'_1|}] \tanh(\sigma \delta'_1)
$$

(3.40a)

$$
\frac{\partial f}{\partial \delta'_1} = \sigma[\mu_k + (1.0 - \mu_k)e^{-\alpha|\delta'_1|}][1.0 - \tanh^2(\sigma \delta'_1)] - \alpha(1.0 - \mu_k)e^{-\alpha|\delta'_1|} \text{sgn}(\delta'_1) \tanh(\sigma \delta'_1)
$$

(3.40b)

Note that when $\mu_k < 1.0$, a negative slope ($\partial f / \partial \delta'_1 < 0$) is found in the friction law.
Figure 3.17 Frequency Response for System of Figure 1a Given $I_1 = 10$, $\zeta = 0.02$, $T_m = 0.5$, $T_p = 1.5$. a) Max-min frequency responses of $\delta_1'$: _____, $\mu_k = 1.0$; ooo, $\mu_k = 0.95$; xxx, $\mu_k = 0.9$; b) rms frequency response of $\delta_2'$: _____, $\mu_k = 1.0$; ooo, $\mu_k = 0.95$; xxx, $\mu_k = 0.9$. 

121
Insert \( f(\delta_i) \) and \( \frac{\partial f}{\partial \delta_i} \) into (3.26) and (3.28) and apply the MHBM formulation of section 3.5 with the provision that the system is still dynamically stable and the response is periodic. Consequently, only a minor variation in \( \mu_k \) is permitted.

Figure 3.17 compares the results for 3 values of \( \mu_k \) (1.0, 0.95 and 0.9), given \( \alpha = 2 \). First, Figure 3.17a shows that the stick to slip transition frequencies for all cases are almost identical as it should be since the friction capacity that is determined by \( \mu_* \) (=1.0) remains unchanged. Differences in \( \delta_i^\prime \) values are also seen at lower frequencies. As shown in Figure 3.17b, it appears that the downstream system response could be sensitive to \( \mu_k \). A minor change in \( \mu_k \) here induces relatively big difference in \( \delta_{2rms} \) especially at the resonant frequencies. When \( \mu_k \) is reduced, the peak values of \( \delta_{2rms} \) are lower. This is because of the reduced friction torque during the slip state that is determined by the value of \( \mu_k \); this torque constitutes an equivalent excitation to the downstream sub-system. Similar to the variation in \( \delta_i^\prime \), the differences in \( \delta_{2rms} \) between three cases are more visible at lower frequencies where significant stick-slip motions tend to occur. Of course, a further decrease of \( \mu_k \) will introduce numerical instabilities and chaotic responses. Those factors will pose difficulties on the application of MHBM that assumes periodic responses. A subsequent chapter will address this particular issue.
3.8 Comparison with Conventional Friction Damper Problem (Den Hartog’s System)

Finally, we examine the conventional SDOF friction damper system of Figure 3.2 that has been studied by many researchers [3.1, 3.2, 3.14, 3.20]. The governing equation is as follows:

\[ T_e \ddot{\delta} + C \dot{\delta} + K \delta + T_f(\dot{\delta}) = T_{me} + T_{pe} \sin(\delta \tau), \quad T_f(\delta) = T_{sf} f(\dot{\delta}) . \]  

(3.41a, b)

Again, it is non-dimensionalized by introducing the following parameters:

\[ \omega_n = \sqrt{\frac{K}{I_e}}, \quad \zeta = \frac{C}{2 \sqrt{KI_e}}, \quad \delta = \frac{K \delta}{T_{sf}}, \]  

(3.42a-c)

\[ T_m = \frac{T_{me}}{T_{sf}}, \quad T_p = \frac{T_{pe}}{T_{sf}} , \]  

(3.42d-e)

\[ \Omega = \frac{\ddot{\delta}}{\omega_n^2}, \quad \tau = \frac{\dot{\delta}}{\omega_n}. \]  

(3.42f-g)

The following dimensionless governing equation is obtained where derivatives are with respect to \( \tau \):

\[ \dddot{\delta} + 2\zeta \ddot{\delta} + \dot{\delta} + f(\dot{\delta}) = T_m + T_p \sin(\Omega \tau) \]  

(3.43)

First, note that \( T_m \) can be balanced out by the spring with a mean or static displacement \( \langle \delta \rangle = T_m \). Thus the existence of non-zero \( T_m \) has no effect on the relative slip or stick-slip velocities. This is quite different from our 2DOF system of Figure 3.1. This implies the friction interface in Figure 3.2 would always experience symmetric stick-slip motion, i.e. \( \delta(\tau) = -\delta(\tau + P/2) \). Indeed, this was the basis of
Figure 3.18 Comparison between Semi-analytical Solution and Den Hartog’s Analytical Solution Given $\varsigma = 0$ and $T_p = 1.25$ For System of Figure 3.2. ooo, semi-analytical solution with 12 harmonics; xxx, semi-analytical solution with 24 harmonics; -□-, Den Hartog’s analytical solution (from Figure 4 of reference [3.1]).
Figure 3.19 3-D Response Map for System in Figure 3.2 Given $\zeta = 0$ and $T_p = 1.25$. Semi-Analytical Solutions used to Construct this Map.
solution as originally proposed by Den Hartog [3.1]. Further, equation (3.41) can be simplified by excluding $T_m$ and by re-setting $\delta(\tau)$:

$$\ddot{\delta} + 2\xi\dot{\delta} + \delta + f(\dot{\delta}) = T_p \sin(\Omega \tau)$$  \hspace{1cm} (3.44)

Den Hartog developed the analytical solutions for (3.44) for two cases: pure slipping motion and two-stop motion. He also obtained a boundary (dashed line in Figure 3.18) between the motions without any stop and with two stops. Based on Den Hartog’s boundary, stick-slip motion tends to occur in the low frequency range when the friction force or torque is generally high. This is consistent with our system of Figure 3.1 as discussed in section 3.6. Here, we employ the multi-term harmonic balance method (proposed in section 3.5) to find the stick-slip responses. For the sake of brevity, only the case of $\varsigma = 0$ and $T_p = 1.25$ is compared. Figure 3.18 compares the calculated responses and Den Hartog’s analytical solution. Good match is observed in terms of the magnification factors ($\delta_{0,\text{peak}}/(T_p/K)$) and peak frequency ($f_{\text{peak}}$). Minor differences are found around the $f_{\text{peak}}$ regime. But these results could be improved by increasing $n$ from 12 to 24. Further, as seen from the 3-D response map in Figure 3.19, only the first harmonic dominates the response over the entire frequency regime. Although a third harmonic component is involved at lower $\Omega$ values, it is too weak to generate an active super-harmonic peak. This explains the absence of super-harmonic peaks in the SDOF frequency responses unlike the observations for the torsional system of Figure 3.1 we studied in this chapter.
3.9 Conclusion

The non-linear frequency response characteristics of a torsional system with dry friction controlled path have been studied. Three key contributions emerge. First, an analytical solution based on one-term harmonic balance is developed for a simplified torsional system subjected to continuous slipping motions. The nature of super-harmonic peaks as generated by the dry friction non-linearity is efficiently found. The effect of a non-zero mean load is also determined and qualitatively understood. Second, a refined multi-harmonic balance method (MHBM) is proposed to study an automotive drive train system that experiences significant stick-slip motions. Our method includes up to 12 harmonics and yet yields responses in an efficient manner. Associated computational issues with the conditioning factor are addressed in detail. Moreover, a procedure to properly select the initial conditions is developed based on the linear system theory. Two types of peaks are seen in non-linear frequency responses: relaxation peaks and super-harmonic resonant peaks. The non-zero mean load in the MDOF system has an effect similar to the simplified system and it generates asymmetric stick-slip motions. Further, studies have shown that the occurrence of super-harmonic resonant peaks is also related to the number of harmonic terms included in assumed solutions especially over the lower frequency regime. This is well explained by the 3-D response maps. Third, the conventional SDOF dry friction damper system (Den Hartog’s problem) in which a spring path is in parallel with a dry friction path is revisited. Our results show that this conventional system differs, in terms of the dynamic behavior, from our torsional system with a sole dry friction path (with a high saturation torque). In particular, the mean load in our
system dictates the nature of nonlinear system responses. The SDOF damper system response (Den Hartog’s problem) is controlled by the primary harmonic resonance, unlike our torsional system (with a sole dry friction path) where many super-harmonic resonant peaks are present. Future work will deal with periodic and transient response of the torsional system (with a dry friction path), as well as applications to real-life automotive problems.

References for Chapter 3


CHAPTER 4

EFFECT OF TIME VARYING DRY FRICTION ON TRANSIENT AND STEADY STATE RESPONSES

4.1 Introduction

Dry friction elements are encountered in many mechanical and structural systems, under a variety of operational conditions [4.1-4.10]. First, consider the classical friction damper example of Figure 4.1a where a single degree of freedom (SODF) torsional system is shown. Assuming a time-invariant normal load $N$, the equation of motion is

$$I \ddot{\theta} + T_c(\theta, \dot{\theta}) + T_f(N, \dot{\theta}) = T_e(t)$$

where $T_c(\theta, \dot{\theta})$ is the constraint torque, $T_e(t)$ is the externally applied torque excitation, $I$ is the inertia, $\theta$ is the angular displacement and $T_f$ is the friction torque given time-invariant $N$, a moment arm $R$ and a coefficient of friction $\mu$. Further, one could employ one of the several $T_f(\dot{\theta})$ relationships that are available in the literature [4.3, 4.8, 4.11-4.15]. There is a substantial body of literature on time-invariant friction torque or force, especially when the saturation forces or torques are small [4.1-4.6]. Second, assume that $N$
Figure 4.1 Torsional systems with dry friction element. a) Classical dry friction damper system; b) Dry friction path in an automotive system
varies with time, say intentionally through an actively controlled actuation mechanism that applies time-varying pressure $P(t)$ on an area $A$. Thus the non-linear friction torque $T_f(N(t), \dot{\theta})$ is given by $\mu N(t) = \mu P(t) A$. Of course, one may find yet physical processes where the contact loads may change periodically anyway, such as in gear pairs [4.16-4.17]. To the best of our knowledge, no prior researcher has addressed the time-varying non-linear friction force or torque issue, with the exception of sliding friction in gears [4.16-4.17].

4.2 Problem Formulation

4.2.1 Physical System

In this chapter, we investigate the effect of time-varying $T_f$ on the dynamics of a multi-degree of freedom (MDOF) torsional system. The immediate application of this study is the slipping torque converter clutch (TCC) that is employed in automotive driveline system as conceptually illustrated in Figure 4.1b. Other examples may include smart clutch [4.18] and dual clutch transmission [4.19-4.20] concepts. Unlike a pure dry friction damper (such as in Figure 4.1a), the dry friction element in Figure 4.1b is used as a key path to transmit the mechanical power. For example, in a vehicle with automatic transmission, the fuel economy can be improved by applying a slipping TCC to avoid the power loss within the fluid torque converter without reducing the ride quality [4.21-4.22]. As shown in Fig. 1b, the non-linear friction torque $T_f$ transmitted within the TCC is applied by a hydraulic actuation pressure $P(t)$ that is
controlled by a pulse-width modulated solenoid valve [4.23], i.e. \( T_f(t) = \mu P(t) AR \). By controlling \( P \), a target value of slip speed \((\Omega_e - \Omega_s)\) is achieved, ensuring a best compromise between the fuel efficiency and ride quality [4.23-4.24]. Several researchers have studied the effect of the feedback control system and hydraulic actuation system dynamics [4.23, 4.25]. However, the essential dry friction non-linearity has been either ignored or linearized around the operating point assuming small motions.

Recently, Duan and Singh studied a torsional system with a dry friction controlled path and investigated its dynamics [4.26-4.27]. Significant stick-slip motions are found under a harmonic excitation but these are strongly influenced by a mean torque load \( T_m \). Non-linear frequency response characteristics have been studied using analytical or semi-analytical methods. However, the earlier studies assumed a constant \( N \). Further, \( \mu_s \) was assumed to be equal to \( \mu_k \) for the sake of simplicity. In this chapter, we assume the actuation pressure \( P \) to be harmonic along with a bias term, as a result of the oscillations within the hydraulic control circuit [4.23]. Additionally, we consider the possibility that the \( T_f(\dot{\delta}) \) slope is negative (the Strubeck effect of the friction liner). Both of these effects could induce quasi-periodic or chaotic responses and consequently pose difficulty for the system control and introduce objectionable noise and vibration problems.
Figure 4.2 Driveline System with Time-Varying Friction Torque. a) 3DOF model; b) 2DOF model
4.2.2 Governing Equations of 3DOF System

The vehicle driveline system can be represented by a simplified 3 degree of freedom (3DOF) torsional model as in Figure 4.2a. Here, \( I_1 \) represents the combined torsional inertia of flywheel, \( I_2 \) is the inertia of friction shoe and pressure plate, \( I_3 \) is the transmission and differential, and the wheel and vehicle sub-system is assumed to be rigid. The governing equations for this system are:

\[
I_1 \ddot{\theta}_1 + T_f (\dot{\theta}_1 - \dot{\theta}_1, t) = T_e (t) \\
(4.1a)
\]

\[
I_2 \ddot{\theta}_2 + C_{23} (\dot{\theta}_2 - \dot{\theta}_3) + K_{23} (\theta_2 - \theta_3) = T_f (\dot{\theta}_1, t) \\
(4.1b)
\]

\[
I_3 \ddot{\theta}_3 + C_{34} \dot{\theta}_3 + K_{34} \theta_3 = C_{23} (\dot{\theta}_2 - \dot{\theta}_3) + K_{23} (\theta_2 - \theta_3) \\
(4.1c)
\]

Here, \( \theta_1, \theta_2 \) and \( \theta_3 \) are absolute angular displacements; \( C_{23} \) and \( K_{23} \) are the lumped viscous damping and stiffness associated the torsional clutch damper; and \( C_{34} \) and \( K_{34} \) represent damping and stiffness of the axle shaft(s). When relative motions are of interest, rewrite equations where \( \delta_1 = \theta_1 - \theta_2, \delta_2 = \theta_2 - \theta_3 \) and \( \delta_3 = \theta_3 \).

\[
I_1 \ddot{\delta}_1 - \frac{I_1}{I_2} C_{23} \dot{\delta}_2 - \frac{I_1}{I_2} K_{23} \delta_2 + (1 + \frac{I_1}{I_2}) T_f (\dot{\delta}_1, t) = T_e (t) \\
(4.2a)
\]

\[
I_2 \ddot{\delta}_2 + (1 + \frac{I_3}{I_2}) C_{23} \dot{\delta}_2 + (1 + \frac{I_3}{I_2}) K_{23} \delta_2 - C_{34} \dot{\delta}_3 - K_{34} \delta_3 = \frac{I_3}{I_2} T_f (\dot{\delta}_1, t) \\
(4.2b)
\]

\[
I_3 \ddot{\delta}_3 + C_{34} \dot{\delta}_3 + K_{34} \delta_3 = C_{23} \dot{\delta}_2 + K_{23} \delta_2 \\
(4.2c)
\]

The engine torque excitation \( T_e (t) \) is composed of mean \( T_m = < T_e > \), and pulsating \( T_p (t) \) components, where \( < > \) is the time-average operator. Using the Fourier series expansion, express it as \( T_e (t) = T_m + \sum_n T_p (t) \sin (\omega_p n t + \phi_p n) \), where \( n \) is the harmonic.
order of the firing torque sequence, \( \omega_{pn} = (N_e/2)n\Omega_e \), \( N_e \) is the number of engine cylinders [4.28], \( T_{pn} \) is the amplitude for the \( n^{th} \) harmonic and \( \phi_n \) is the associated phase lag. In this study, only the dominant harmonic component \( (\omega_{p1} = \omega) \) is considered for the sake of simplicity, i.e. \( T_{e}(t) = T_m + T_p \sin(\omega t) \).

### 4.2.3 Friction Torque Formulation

The non-linear friction torque \( T_f(\dot{\delta}, t) \) is carried by the clutch and then it acts as an equivalent torque excitation to the downstream system. In a realistic automotive system, a pulse-width modulated solenoid valve would generate a time-varying \( P \) by changing the command value or duty ratio [4.23]. That results in a non-linear time-varying (NLTV) friction torque formulation, \( T_f(\dot{\delta}, t) = \mu(\dot{\delta})P(t)AR \). Note that \( A \) and \( R \) are assumed to be time-invariant. Express \( P(t) \) profile in the form of a sinusoidal signal with mean pressure \( P_m \), amplitude \( P_p \), starting time \( t_0 \) and the actuation pressure frequency \( \omega_f \).

\[
P(t) = \begin{cases} 
0 & t < t_{0-} \\
P_m + P_p \sin(\omega_f t + \psi) & t \geq t_{0+} 
\end{cases}
\]  

(4.3)

Here, \( \psi \) is the phase lag between the actuation pressure and \( T_e(t) \). Further approximate (4.3) with the following:

\[
P(t) = \frac{\text{sgn}(t-t_0) + \text{sgn}(t+t_0)}{2}[P_m + P_p \sin(\omega_f t + \psi)]
\]  

(4.4a)

Where \( \text{sgn} \) represents a triple-valued signum function defined as:
\[ \text{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \] (4.4b)

In this study, we set the initial engagement time at \( t_{0+} = 0 \) without a loss of generality and \( P(t) \) is assumed to be positive-definite to ensure that no separation occurs across the frictional interface, i.e. \( P^p / P^m \in [0, 1) \). The following friction formulation \( \mu(\dot{\delta}_1) \) is employed since the aim of this study is to examine the phenomenological dynamic behavior [4.14].

\[
\mu(\dot{\delta}_1) = \begin{cases} 
\mu_k + (\mu_s - \mu_k)e^{-\alpha|\dot{\delta}_1|} & |\dot{\delta}_1| > 0 \\
0 & |\dot{\delta}_1| = 0 
\end{cases}
\] (4.5)

Here, \( \alpha \) is a positive constant that controls the gradient of \( \mu \) with respect to \( \dot{\delta}_1 \). In our study, \( \alpha = 2 \) is chosen for the sake of illustration. Figure 4.3 considers three cases of \( \mu_k \): (i) \( \mu_k = \mu_s \); (ii) \( \mu_k < \mu_s \); (iii) \( \mu_k > \mu_s \). Previous researchers have extensively studied the friction characteristics applying to a SDOF dry friction damper system [4.1-4.5, 4.7-4.8, 4.10, 4.15]. For example, Den Hartog initiated the research by analytically solving a SDOF system with dry friction damping subjected to harmonic excitation assuming \( \mu_k = \mu_s \) [4.1]. Shaw [4.3] extended the previous work by including static \( \mu_s \) and kinetic \( \mu_k \) coefficients; he also conducted bifurcation and stability analyses. Note that when \( \mu_s = \mu_k \), the above friction law reduces to the form used before by Duan and Singh [4.26-4.27]. In addition, a multi-valued regime exists
Figure 4.3 Three kinetic friction coefficient formulations for the clutch liner:  
a) $\mu_k = \mu_s$;  
b) $\mu_k < \mu_s$;  
c) $\mu_k > \mu_s$.  
key: ____ discontinuous formulation; ____ smoothened formulation.
at $\hat{\delta}_1 = 0$. To further reduce the system parameters, we could incorporate some parameters such as $\mu_k$, $A$ and $R$ within $P(t)$ to yield the non-linear time-varying (NLTV) friction torque $T_f(\dot{\delta}, t)$ as:

$$T_f(\dot{\delta}, t) = \bar{\mu}(\dot{\delta}_1) T_s(t)$$  \hspace{1cm} (4.6a)

$$T_s(t) = \mu_k A R P(t) = T_{sm} + T_{sp} \sin(\omega_f t + \psi)$$  \hspace{1cm} (4.6b)

Here, $\bar{\mu}(\dot{\delta}_1)$ is a normalized friction coefficient function:

$$\bar{\mu}(\dot{\delta}_1) = \begin{cases} 
1.0 + \left( \frac{\mu_s}{\mu_k} - 1.0 \right) e^{-10|\dot{\delta}_1|} \text{sgn}(\dot{\delta}_1) & |\dot{\delta}_1| > 0 \\
0 & |\dot{\delta}_1| = 0
\end{cases}$$  \hspace{1cm} (4.7)

### 4.2.4 Objectives

The first objective is to investigate the transient response during the clutch engagement process that is controlled by the dry friction. An analytical solution will be sought for the pure slip regime and then a general trend for the engagement process will be obtained. In the transient stick-slip regime, self-excited vibrations (clutch judder) that are induced by the Stribeck effect will be investigated. The second objective is to examine the steady-state responses under the effect of time-varying $T_f(t)$ and compare with the time-invariant case. Both periodic and chaotic responses will be identified via results in time history and frequency domain, bifurcation diagram and poincare maps. In our study, the bifurcation diagram and poincare
sections will be constructed by cascading the steady-state time domain responses. In addition to the negative slope characteristics, effects of clutch actuation parameters such as $\omega_f$, $\psi$ and $T_{sp}$ will be identified. Finally, our numerical solution method will be validated by comparing some predictions with those yielded by the *Xppaut* software [4.29].

### 4.3 Solution Methodology

For a piece-wise linear or non-linear system as defined in the previous section, two numerical schemes, namely the discontinuous and continuous solutions, can be employed. The first one finds the solutions for different transition states and then assembles them [4.3, 4.8, 4.10, 4.26]. This method can give an “exact” solution of the non-analytical system but enormous time is often required in the matching process since bi-section or secant methods are used. Recently, Leine successfully employed the Hénon’s scheme but it is still not convenient for a MDOF system [4.8]. The second method employs a stiff ODE solver but one must first smoothen or condition the discontinuous non-linearity. Typical smoothening functions include arc-tangent, hyper-tangent and the like [4.7, 4.26, 4.30]. This method is computationally efficient than the first one [4.26] but an artificial uncertainty is introduced by the smoothening factor $\sigma$. Duan and Singh have studied the effect of $\sigma$ with an application to the dry friction non-linearity [4.26]. Based on their study, the user must exercise caution and use the discontinuous solution as a benchmark.
Several researchers have recently used the harmonic balance method (HBM) to study the non-linear systems. For example, Blankenship and Kahraman developed a harmonic balance method to study the geared system under parametric excitation [4.31]. However, their method can not be conveniently implemented especially when many harmonics must be included in the expansion. Kim et. al. [4.32] and Duan and Singh [4.27] proposed a refined multi-term HBM (MHBM) but both formulations are not suitable for a system with time-varying friction torque excitation because the procedure that converts the residue from time domain to frequency domain by factoring out the discrete Fourier transform (DFT) matrix is on longer feasible. In addition, Padmanabhan and Singh [4.33] used the shooting’s method with a path following technique to systematically identify key non-linear phenomena such as sub-harmonic, quasi-periodic and chaotic responses. However, it is limited to a low dimensional system since too many variational equations would need to be evaluated. Further, analytical methods such as HBM, MHBM and shooting always look for periodic steady-state solutions. Transient responses, quasi-period or chaotic responses that are very important from the vehicle driveline performance standpoint cannot be obtained. For example, Kim et. al. has reported that the MHBM can only detect a possible chaotic motion by determining the stability of the periodic steady-state trajectory but it is incapable of accurately obtaining time histories or its peak to peak (pp) values [4.32].

To efficiently obtain a complete map of the system behavior including both transient and steady-state responses, an explicit Runge-Kutta 4th(5th) order numerical integration (designated as RK45) routine with adaptive step size control due to
Dormand and Prince is employed [4.34] in our study. To facilitate the numerical integration, the discontinuous friction law is replaced by the following smoothened one with a hyper-tangent function.

\[
\mu(\dot{\delta}_1) = [1.0 + \left(\frac{\mu_s}{\mu_k} - 1.0\right)e^{-\alpha|\dot{\delta}|}]\tanh(\sigma\dot{\delta}_1)
\] (4.8)

A value of 50 is chosen for the smoothening factor \(\sigma\). This choice has been justified by Duan and Singh [4.26-4.27] who applied to a similar automotive drive train torsional system with the following friction formulation. To validate our numerical method, a popular ordinary differential equation solver \(Xppaut\) that is placed in the public domain by Ermentrout [4.29, 4.35] will be utilized.

### 4.4 Typical responses for a 3DOF Torsional Driveline System

Table 4.1 lists typical system parameters of the driveline system of Figure 4.2a. First, consider the initial state of clutch engagement. Assume that the flywheel rotates with the engine speed \(\Omega_e\) but the rest of driveline system is stationary \((\Omega_i = 0)\). Thus, the following initial conditions are defined:

\[
\begin{align*}
\delta_1(0) &= 0, & \dot{\delta}_1(0) &= \Omega_e, \\
\delta_2(0) &= \delta_3(0) = 0, & \dot{\delta}_2(0) &= \dot{\delta}_3(0) = 0.
\end{align*}
\] (4.9a-b)

Figure 4.4 presents typical time domain responses for \(\mu_s = \mu_k\), in the absence of time-varying actuation pressure \((T_{sp} = 0)\). In addition to our RK45 with adaptive step size
Figure 4.4 Typical time histories for 3DOF driveline system given $\omega = 100$ rad/s, $T_m = 300$, $T_p = 250$, $T_{sm} = 350$, $T_{sp} = 0$, $\mu_s = \mu_k$ and $\Omega_e = 100$ rad/s. a) responses up to 2.5 s using RK45; b) zoomed response from 1.5 to 1.6s using, ____, RK45, …, Gear, _ _ _ _, RK4 methods.
Figure 4.5 Typical time histories for 3DOF driveline system given $\omega = 150$ rad/s, $T_m = 300$; $T_p = 250$, $T_{sm} = 350$, $T_{sp} = 1/6T_{sm}$, $\omega_f = \omega$, $\psi = 0$, $\mu_k = 1.2\mu_k$ and $\Omega_e = 150$ rad/s. a) responses up to 2s using RK45; b) zoomed response from 1.22 to 1.3s using: ____ , RK45, ..., Gear, ____ , RK4 methods.
### Parameters and Excitation Value(s)

<table>
<thead>
<tr>
<th>Parameters and Excitation</th>
<th>Value(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torsional Inertias (kg-m²)</td>
<td>$I_1 = 0.20$, $I_2 = 0.02$, $I_3 = 0.25$</td>
</tr>
<tr>
<td>Torsional Viscous Damping (Nm-rad/s)</td>
<td>$C_{23} = 0.6$, $C_{34} = 0.3$</td>
</tr>
<tr>
<td>Torsional Stiffness (Nm/rad)</td>
<td>$K_{23} = 3000$, $K_{34} = 58000$</td>
</tr>
<tr>
<td>Torque Excitation Amplitude (Nm)</td>
<td>$T_m = 300$, $T_p = 250$</td>
</tr>
<tr>
<td>Friction coefficients</td>
<td>$\mu_s = 0.3$, $\mu_k = 0.15 \sim 0.36$</td>
</tr>
<tr>
<td>Pressure Area (m²)</td>
<td>$A = 0.08$</td>
</tr>
<tr>
<td>Moment Arm (m)</td>
<td>$R = 0.1$</td>
</tr>
<tr>
<td>Actuation Pressure (kPa)</td>
<td>$P_m = 200\sim400$, $P_p = 0 \sim P_m$</td>
</tr>
<tr>
<td>Phase Lag (radian)</td>
<td>$\psi = 0$, $\pi/2$, $\pi$</td>
</tr>
</tbody>
</table>

Table 4.1 Parameters and excitation amplitude used for simulating the system of Figure 4.2
control, two numerical schemes including the Gear’s method and classical 4th order Runge-Kutta method (RK4) are selected from *Xppaut* [4.29, 4.35]. As seen in Figure 4.4, excellent agreement is achieved between RK45 and Gear’s methods. Differences with the RK4 routine especially during the pure stick state could be due to the numerical instability of that method. Figure 4.5 shows time histories for time-varying friction with $\mu_s=1.2\mu_k$ and $T_{sp}/T_{sm}=0.167$. Again, RK45 and Gear’s methods yield almost identical results. When the computational problem becomes even more stiff by the introduction of a negative slope in the friction formulation, the results of RK4 deviate further away from the solutions of RK45 and Gear’s methods. Both transient and steady-state regimes can be clearly observed in Figures 4.4 and 4.5. The transient solution includes pure slip and stick-slip regimes, and the steady-state regime contains mostly stick-slip (and possibly some pure stick) motions.

Further, results in Figures 4.4 and 4.5 show that $\delta_3$ is an order of magnitude lower than $\delta_2$. Consequently, a linear model of the downstream sub-system (including $I_2$ and $I_3$) is developed.

\[
M \ddot{X} + K X = 0, \quad (4.10a)
\]

\[
M = \begin{bmatrix} I_2 & 0 \\ 0 & I_3 \end{bmatrix}, \quad K = \begin{bmatrix} K_{23} & -K_{23} \\ -K_{23} & K_{23} + K_{34} \end{bmatrix}, \quad X = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}. \quad (4.10b-d)
\]

Employing the parameters of Table 4.1, two normalized eigenvectors are found $[1 \ 0.11 \ 0]^T$ at 365.6 rad/s and $[1 \ -0.73 \ 0]^T$ at 510.2 rad/s. Due to the fact that $K_{23}$ is much smaller than $K_{34}$, the relative motion between $I_2$ and $I_3$ is much higher than that between $I_3$ and vehicle at $\omega_1 = 365.6 \ rad/s$. At $\omega_2 = 510.2 \ rad/s$, the motion
between $I_3$ and vehicle increases. Nevertheless, previous studies by Den Hartog [4.1] and Duan and Singh [4.27] have shown that significant stick-slip motions tend to occur at the lower frequencies. Consequently, the 3DOF definite driveline system of Figure 4.2a is further reduced to a 2DOF definite system (of Figure 4.2b) by incorporating $I_3$ into vehicle and wheels. The non-linear governing equations for the reduced 2DOF system are given as:

$$I_1 \ddot{\theta}_1 + T_f (\dot{\theta}_1 - \dot{\theta}_2, t) = T_m(t) + T_p \sin(\omega t), \tag{4.11a}$$

$$I_2 \ddot{\theta}_2 + C_{23} \dot{\theta}_2 + K_{23} \dot{\theta}_2 = T_f (\dot{\theta}_1 - \dot{\theta}_2, t). \tag{4.11b}$$

For the case when relative motions are of interest, rewrite the governing equations as:

$$I_1 \ddot{\delta}_1 - \frac{I_1}{I_2} C_{23} \ddot{\delta}_2 - \frac{I_1}{I_2} K_{23} \dot{\delta}_2 + (1.0 + \frac{I_1}{I_2}) T_f \dot{\delta}_1(t) = T_m + T_p \sin(\omega t), \tag{4.12a}$$

$$I_2 \ddot{\delta}_2 + C_{23} \ddot{\delta}_2 + K_{23} \dot{\delta}_2 = T_f (\dot{\delta}_1, t). \tag{4.12b}$$

### 4.5 Transient Responses of Reduced 2DOF Torsional System

#### 4.5.1 Pure Slip Transient Response

As a result of speed difference between $I_1$ and $I_2$ during the initial engagement, a pure slip transient is seen in Figures 4.4 and 4.5. If we assume pure positive slip motion ($\dot{\delta}_1 > 0$) and assume that $\mu_k = \mu_s$ since the negative gradient dominates only when $\dot{\delta}_1$ approaches zero, equations (4.12a-b) can be approximated as:
\[
I_1 \ddot{\delta}_1 - \frac{I_1}{I_2} C_{23} \dot{\delta}_2 - \frac{I_1}{I_2} K_{23} \delta_2 + (1.0 + \frac{I_1}{I_2})(T_{sp} \sin(\omega_f t + \psi)) = T_m + T_p \sin(\omega t) ,
\] (4.13a)

\[
I_2 \ddot{\delta}_2 + C_{23} \dot{\delta}_2 + K_{23} \delta_2 = T_{sp} \sin(\omega_f t + \psi) .
\] (4.13b)

Assume the following initial conditions:
\[
\delta_i(0) = 0 , \quad \dot{\delta}_i(0) = \Omega_e , \quad \delta_2(0) = 0 , \quad \dot{\delta}_2(0) = 0 . \tag{4.14a-d}
\]

Analytical solution of equation (4.13b) is given by a sum of complementary and particular solutions:
\[
\delta_2(t) = \left\{ e^{-\zeta \omega_d t} \left( a \cos(\omega_d t) + b \sin(\omega_d t) \right) \right\} + \left\{ \frac{T_{sm}}{K_{23}} + \frac{T_{sp}}{\sqrt{\Lambda}} \sin(\omega_f t + \psi + \phi) \right\}.
\] (4.15a)

\[
\omega_n = \sqrt{K_{23} / I_2} , \quad \zeta = C_{23} / (2 \sqrt{K_{23} I_2}) , \quad \omega_d = \omega_n \sqrt{1 - \zeta^2} , \tag{4.15b-d}
\]

\[
\Lambda = (K_{23} - I_2 \omega^2) + (C_{23} \omega)^2 , \quad \phi = -\tan^{-1}(C_{23} \omega / (K_{23} - I_2 \omega^2)) . \tag{4.15e-f}
\]

The constants \( a \) and \( b \) of (4.15a) are determined by applying the initial conditions (4.14c-d):
\[
a = -\frac{T_{sm}}{K_{23}} - \frac{T_{sp}}{\sqrt{\Lambda}} \sin(\psi + \phi) , \quad b = \frac{1}{\omega_d} \left\{ a \zeta \omega_n - \frac{T_{sp}}{\sqrt{\Lambda}} \omega_f \cos(\psi + \phi) \right\} . \tag{4.16b}
\]

Substitute (4.15) into (4.13) to yield the solution for \( \ddot{\delta}_1 \) and \( \dot{\delta}_1 \):
\[
\dot{\delta}_1(t) = \left( \frac{T_m - T_{sm}}{I_1} \right) + \frac{T_p}{I_1} \sin(\omega t) - \frac{T_{sp}}{I_1} \sin(\omega_f t + \psi)
\]
\[
+ \frac{T_{sp}}{\sqrt{\Lambda}} \omega_d^2 \sin(\omega_f t + \psi + \phi) + e^{-\zeta \omega_d t} \left\{ (-a \omega_d^2 - 2\zeta b \omega_n \omega_d + \zeta^2 \omega_n^2 a) \cos(\omega_d t) \right\}
\]
\[
+ (-b \omega_d^2 + 2\zeta a \omega_n \omega_d + \zeta^2 \omega_n^2 b) \sin(\omega_d t) \right\} . \tag{4.17a}
\]
\[
\dot{\delta}(t) = V_m + \frac{(T_m - T_{sm})}{I_1} t \\
+ \int \left\{ \frac{T_p}{I_1} \sin(\omega t) - \frac{T_{sp}}{I_1} \sin(\omega_j t + \psi) + \frac{T_{sp}}{\sqrt{\Lambda}} \omega_j \sin(\omega_j t + \psi + \phi) \right\} \, dt \\
+ \int e^{-\zeta\omega_n t} \left\{ (-a\omega_n^2 - 2\xi b\omega_n \omega_d + \xi^2 \omega_n^2 a) \cos(\omega_d t) \right\} \, dt \\
+ \left\{ \frac{a_1}{I_1} \sin(\omega t + \phi_1) + \frac{a_2}{I_1} \sin(\omega_j t + \phi_2) \right\} + a_3 e^{-\zeta\omega_n t} \sin(\omega_n t + \phi_3)
\]

(4.17b)

The above solutions lead to some interesting results. As evident from (4.17b), the solution of \(\dot{\delta}(t)\) is given in the following functional form where the coefficients \(a_0\), \(a_1\), \(a_{21}\), \(a_{22}\) and \(a_3\) are given by (4.17b):

\[
\dot{\delta}(t) = a_0 + a_1 t + \left\{ \frac{a_{21}}{I_1} \sin(\omega t + \phi_{21}) + \frac{a_{22}}{I_1} \sin(\omega_j t + \phi_{22}) \right\} + a_3 e^{-\zeta\omega_n t} \sin(\omega_n t + \phi_3)
\]

(4.18)

Under the situation when the amplitude of oscillatory part is small and the decaying component dies out quickly, the clutch engagement rate is controlled by a ramp of gradient \(a_1 = (T_m - T_{sm})/I_1\). The time-varying friction torque \(T_{sp}\) contributes to the oscillatory motion during the ramp. Numerical results of Figures 4.6a confirm the analytical functional forms. As shown in Figure 4.6a, when \(T_m/T_{sm}\) is increased from 0.75 to 0.9, the ramp gradient decreases. Further, Figure 4.6b shows the effects of \(I_1\) from 0.2 to 0.4. Analytically, an increase in \(I_1\) indicates more kinetic energy and thus the dissipation process should take more time. Conversely, when the oscillatory and decaying parts dominate the pure slip motion, it is difficult to determine the ramp
Figure 4.6 Effect of 2DOF system parameters on pure slip transients. a) effect of $T_{sm}$:
- $T_m/T_{sm} = 0.75$, $T_{sp} = 0$; 
- $T_m/T_{sm} = 0.75$, $T_{sp}/T_{sm} = 0.167$; 
- $T_m/T_{sm} = 0.9$, $T_{sp}/T_{sm} = 0.167$. b) effect of $I_1$:
- $I_1 = 0.2$; 
- $I_1 = 0.4$
Figure 4.7 Effect of $T_{sm}$ on the clutch engagement in Figure 4.2b given $T_m / T_{sm} = 1.05$ and $T_{sp} / T_{sm} = 0.5$. 
gradient rate. Nonetheless, our analysis still gives a guideline regarding \((T_m - T_{sm})\). As shown in Figure 4.7, when \(T_m\) is higher than \(T_{sm}\), no final engagement can be realized because since \(a_t\) term in (4.17b) monotonically grows with time and ultimately it would dominate the response.

### 4.5.2 Stick-Slip Transient Response (Judder)

When \(\dot{\theta}_2\) approaches \(\dot{\theta}_1\) following the pure slip motion as discussed in the previous section, stick-slip transient motions would take place. In addition to introducing objectionable noise problem, significant stick-slip motions could be transferred by the differential to vehicle in the form of fore-after jerk. This phenomenon is usually referred to the clutch judder problem that typically occurs at low frequencies [4.36]. Yamada and Ando [4.37] and Centea et. al. [4.36] have called this as the “negative damping” problem, introduced by the negative slope of \(\mu_k(\delta_t)\). Similar to prior research, the negative slope of \(\mu_k\) will be first investigated under a time-invariant friction torque \(T_{sm}\). Then the effect of \(T_s(t)\) will be examined under the \(\mu_k = \mu_s\) assumption.

From the friction formulation of (4.6) and (4.7), a decrease in \(\mu_k\) with \(\dot{\delta}_t\) affects the system in two ways. First, a negative slope regime is formed. Second, the saturation friction torque \(T_{sm} = \mu_k ARP_m\) is reduced and thus more slip motions are allowed. Figure 4.8 shows results for three values of \(\mu_k\). As expected, the stick-slip
motions become more serious when $\mu_k < \mu_s$ because the negative damping enhances the slip motions. Conversely, when $\mu_k > \mu_s$, the stick-slip motions are attenuated as a result of the positive damping as well as a higher value of $T_{sm}$ as seen in Figures 4.8a and b. To clearly illustrate the effect of the negative slope more clearly, Figure 4.9 compares results for two cases of $T_{sm}$ and $\mu_k$. Note that although a reduction in $T_{sm}$ would enhance the stick-slip motions, the negative slope characteristics ($\mu_k < \mu_s$) could further amplify them.

To examine the effect of time-varying friction torque on judder, first apply $T_s(t)$ at $\omega_j = \omega$, where $\omega$ is the frequency of engine torque $T_e(t)$ but with a phase lag of $\psi$. Results of Figure 4.10a show that in-phase $T_s(t)$ tends to attenuate the stick-slip motions. A physical explanation can be found by analyzing the relationship between $T_e(t)$ and $T_s(t)$ in a quasi-static manner as shown in Figure 4.11a. Note that only positive stick-slip motions ($\dot{\delta} \geq 0$) are excited under significant positive mean torque $T_{ms}$. When the engine torque $T_e(t)$ is higher in the first half cycle ($\omega t \in [n2\pi, n2\pi + \pi/2]$, $n = \text{integer}$), the friction interface tends to initiate positive slipping motions. Quasi-statically a higher value of $T_s$ should suppress this tendency. But when there is a phase lag between $T_s$ and $T_e$, such a suppression should be reduced. Figure 4.10b illustrates the case of mismatch between $\omega_j$ and $\omega$. Observe
Figure 4.8 Effect of $\mu_k$ on transient stick-slip responses of 2DOF system. a) $\omega = 70$ rad/s and $T_{sm} = 550$ Nm; b) $\omega = 50$ rad/s and $T_{sm} = 550$ Nm. key: _____, $\mu_k = \mu_s$; . . ., $\mu_k = 0.75 \mu_s$; _ _ _, $\mu_k = 1.25 \mu_s$. 

155
Figure 4.9 Effect of reduced $T_{sm}$ and negative slope in friction torque on response of a 2DOF system when excited at $\omega = 50$ rad/s. a) $\ldots$, $T_{sm} = 550$, $\mu_k = \mu_s$; $\ldots$, $T_{sm} = 412.5$, $\mu_k = \mu_s$; b) $\ldots$, $T_{sm} = 550$, $\mu_k = \mu_s$; $\ldots$, $T_{sm} = 412.5$, $\mu_k = 0.75 \mu_s$. 

156
Figure 4.10 Effect of time-varying friction on stick-slip transients of a 2DOF system. a) effect of phase lag $\psi$ given $\omega = 60$ rad/s and $\omega_f = \omega$: $\ldots$, $\psi = 0$, $\ldots$, $\psi = \pi/2$, $\ldots$, $\psi = \pi$. b) effect of mismatched frequencies given $\omega = 80$ rad/s and $\psi = 0$: $\ldots$, $\omega_f = \omega$; $\ldots$, $\omega_f = 2\omega$; $\ldots$, $\omega_f = 0.5\omega$. 
Figure 4.11 Physical explanation on the effect of slip motion attenuation. All shaded areas represent the case for enhanced slip motions. a) effect of phase lag ($\omega_f = \omega$): ___, $T_e(t)$; ___, $T_s(t)$ in phase; ___, $T_s(t)$ not in phase. b) effect of mismatched frequency ($\psi = 0$): ___, $T_e(t)$; ___, $\omega_f = \omega$; ___, $\omega_f > \omega$. c) effect of mismatched frequency ($\psi = 0$): ___, $T_e(t)$; ___, $\omega_f = \omega$; ___, $\omega_f < \omega$. 
Figure 4.12 Effect of time-varying friction $T_s(t)$ on clutch judder. Key: ___, $\mu_k = \mu_s$ and $T_{sp} = 0$; . . ., $\mu_k = 0.75 \mu_s$ and $T_{sp} = 0$; _ _ _ , $\mu_k = 0.75 \mu_s$ and $T_{sp} = 1/3 T_{sm}$.
that $T_s(t)$ with a mismatched frequency produces more slip motions. Similar physical explanation is presented in Figure 4.11b. Since $\omega_f \neq \omega$ and $\psi \neq 0$, some “leakage” as indicated by the shaded areas occurs and consequently the slip motions are enhanced. Further, a time varying $T_s(t)$ with $\omega_f = \omega$ and $\psi = 0$ is applied to a clutch with negative damping. Results in Figure 4.12 show that $T_s(t)$ could efficiently reduce the judder problem in this case. Nonetheless, explanation of Figure 4.11 only applies at lower frequencies due to its quasi-static nature. As $\omega$ increases, phase delay between excitation and response may become important and the quasi-static explanation is no longer valid.

4.6 Nature of Steady State Responses

For a physical system of Figure 4.2b, three kinds of steady-state responses across the frictional interface are possible following the initial transients: pure stick, pure slip and stick-slip. While stick-slip is essentially controlled by the strong non-linearity of dry friction element, pure stick or pure slip can be determined ahead of numerical simulation by analytical solutions. For example, Duan and Singh [4.26] have proposed a procedure to determine the pure stick response regime for a time-invariant dry friction torque, this still holds for a time-varying dry friction formulation. Assume pure stick motion ($\delta_i = 0$) that would consolidate $I_1$ and $I_2$ into a single inertial body. The non-linear system of Figure 4.2b now behaves as a linear system.
The internal friction force $T_{fi}$ across the interface is determined by $(T_e(t) - I_i\ddot{\theta}_i)$, where $\ddot{\theta}_i = \ddot{\theta}$ under the condition $\dot{\theta}_i = 0$.

$$\delta_2 = \frac{T_m + T_p}{K_{23}} \frac{T_p}{\sqrt{(K_{23} - (I_1 + I_2)\omega^2)^2 + (C_{23}\omega)^2}} \sin(\omega t + \varphi)$$  \hspace{1cm} (4.19a)

$$\varphi = -\tan^{-1} \frac{C_{23}\omega}{K_{23} - (I_1 + I_2)\omega^2}$$  \hspace{1cm} (4.19b)

Thus,

$$T_{fi}(t) = T_m + T_p \sin(\omega t) + \frac{\omega^2 T_p}{\sqrt{(K_{23} - (I_1 + I_2)\omega^2)^2}} \sin(\omega t + \varphi)$$  \hspace{1cm} (4.20)

When the internal friction torque is asymptotically less than the time-varying dry friction, pure stick motion is satisfied.

$$T_{fi}(t) \leq \mu P(t) AR \quad \forall t \in [0, \infty)$$  \hspace{1cm} (4.21)

Thus we define a difference variable $\Delta(t) = T_{fi}(t) - \mu P(t) AR$ and given its periodicity, the following criterion is defined to satisfy the pure stick condition:

$$\Delta(t) \leq 0 \quad \forall t \in [0, P_{\text{period}}), \quad P_{\text{period}} = \frac{2\pi}{\omega}. \hspace{1cm} (4.22a-b)$$

As noted, when the amplitudes of excitation and friction torques are fixed, the lower and higher transition frequencies from pure stick to stick-slip can be determined via upward and downward frequency sweeps respectively. However, it should be pointed out that this process is based on the piece-wise non-linear friction law of (4.7). In contrast when that smoothened friction law of (4.8) is used, the following two effects
Figure 4.13 Effect of smoothening friction law on maximum friction coefficient a) smoothened friction law; b) Zoomed part. Key: \( _{-} \), \( \mu_k = \mu_s \); \( \cdots \), \( \mu_k = 0.8\mu_s \); \( \_\_\_\_\_\_ \), \( \mu_k = 0.75\mu_s \); \( \_\_\_\_ \), \( \mu_k = 0.677\mu_s \).
are brought into the system dynamics. First, the assumption of only pure stick motions may not be valid. Second, the maximum friction coefficient or torque decreases with $\mu_k$ given the same $\mu_s$ value as a result of the smoothening process as shown in Figure 4.13. Thus the resulting “transition” frequency would be affected by the value of $\mu_k$, as explored in the subsequent section.

Further, pure slip steady-state motions are possible as observed from equation (4.18). As noted in Figure 4.7, when $a_i = (T_m - T_{sm})/I_i$ is greater than zero, the motion across the frictional interface is pure slip although no steady-state responses can be defined since the relative velocity keeps on increasing. However, when $a_i = 0$ ($T_m = T_{sm}$), steady-state pure slip motions could take place and the resulting response can be analytically obtained by eliminating the exponentially decaying term.

$$\dot{\delta}_{1s}(t) = a_0 + a_{21} \sin(\omega t + \phi_{21}) + a_{22} \sin(\omega_f t + \phi_{22})$$

(4.23)

However, the above mentioned condition alone cannot warrant pure slip motions. Assuming pure positive slip motions, another condition, as stated below, has to be added to avoid a crossing of zero velocity.

$$|a_0| > |a_{21}| + |a_{22}|$$

(4.24)

Although a detailed description of $a_0$, $a_{21}$ and $a_{22}$ expression would be lengthy for the time-varying dry friction case, analytical results for the time-invariant dry friction are discussed below. In the case of $T_{sp} = 0$ and $T_m = T_{sm}$, equation (4.17b) can be conveniently reduced to the following expression:
\[
\dot{\delta}_1(t) = V_m - \frac{T_p}{I_1 \omega} \cos(\omega t) + \frac{T_{sm} \ 2\xi \ e^{-\omega_d t}}{K_{23} \omega_n} \cos(\omega_d t) + \frac{T_{sm} \ 2\xi^2 - 1 \ e^{-\omega_d t}}{\omega_d} \sin(\omega_d t) \quad (4.25)
\]

Substitute the initial condition \( \dot{\delta}_1(0) = \Omega_e \) to obtain \( V_m \), the steady-state response of \( \dot{\delta}_{ss}(t) \) is found as:

\[
\dot{\delta}_1(t) = (\Omega_e - \frac{T_{sm} \ 2\xi}{K_{23} \omega_n}) + \frac{T_p}{I_1 \omega} (1 - \cos(\omega t)) \quad (4.26)
\]

The typical parameters of Table 4.1 suggest that the term \( (T_{sm} \ 2\xi)/(K_{23} \omega_n) \) is negligible compared to the engine speed \( \Omega_e \), as thus \( \dot{\delta}_1(t) = \Omega_e + (T_p/I_1 \omega)(1 - \cos(\omega t)) \). Thus the positive definite condition of \( \dot{\delta}_1(t) \) \( (\dot{\delta}_1(t) > 0 \ \forall t \in [0, \infty]) \) is also satisfied and correspondingly pure positive slip motions are guaranteed. Finally, note that the dynamic amplitude will be dictated by \( T_p \), \( I_1 \) and \( \omega \).

4.7 Effect of Time Invariant Dry Friction on Steady State Response

Assuming \( \mu_s = \mu_k \) and \( T_s \neq T_s(t) \), Duan and Singh had earlier studied the non-linear frequency characteristics of a similar dry friction path problem [4.27]. In this chapter, we focus more on the steady-state response when \( \mu_k \leq \mu_s \). Two spectral maps are constructed for the use of examining the effect of different values of \( \mu_k \) in frequency domain. The first is the maximum response versus frequency ratio map by picking the maximum (max) response amplitude of \( \dot{\delta}_1 \) at each excitation frequency \( \omega \).
Further, to illustrate the non-linear frequency maps, $\omega$ is normalized with respect to the linear sub-system natural frequency $\omega_n = \sqrt{K_{23}/I_2}$. The second is the bifurcation diagram (with $\omega/\omega_n$ or $\mu_k/\mu_s$ as the parameter) by picking the values of $\delta_2$ corresponding to starting time of each excitation cycle (i.e. $t = 2\pi n/\omega$, $n = \text{integer}$).

The reason to use $\delta_2$ in constructing the bifurcation diagram is that the variation in $\delta_2$ is relatively smaller than $\dot{\delta}_1$ at the fixed phase point. Since there is a deterministic relationship between $\delta_2$ and $\dot{\delta}_1$, a bifurcation in $\delta_2$ would also indicate one in $\dot{\delta}_1$. To reach steady-state response, the first 50 transient cycles is discarded.

Figure 4.14a shows the effect of $\mu_k$ on $\delta_{\text{max}}$. When $\mu_k = \mu_s$, the map of $\delta_{\text{max}}$ is a smooth curve and no active super-harmonic resonant peak is visible. As $\mu_k$ is decreased, more super-harmonic resonant peaks around $\omega/\omega_n = 0.167, 0.20, 0.25$ and 0.33 appear along with jumps. The upward shift in peak frequencies with a decrease in $\mu_k$ can be attributed to “negative damping” effect. Further, the “transition” frequency from pure stick to stick-slip change with respect to the value of $\mu_k$; this is due to the smoothened friction law as discussed in the previous section. As noted in Figure 4.14b, a jump in $\delta_{\text{max}}$ yields a corresponding bifurcation point. For example, a jump at $\omega/\omega_n = 0.2$ with $\mu_k = 0.83\mu_s$ changes the period-one motions to period-two type. As $\mu_k$ is decreased further, the period-two motions become quasi-periodic or even
Figure 4.14 Effect of $\mu_k$ given a time-invariant friction torque. a) $\delta_1_{\text{max}}$ maps. B) bifurcation diagrams with $\omega/\omega_n$ as a bifurcation parameter.
Figure 4.15 Bifurcation diagram employing $\mu_k / \mu_s$ as a bifurcation parameter given $\omega/\omega_n=0.36$. 
Figure 4.16 Sample time histories and poincare sections given $\omega/\omega_n = 0.36$. a) $\mu_k/\mu_s = 0.94$; b) $\mu_k/\mu_s = 0.74$; c) $\mu_k/\mu_s = 0.64$
chaotic responses. Figure 4.15 illustrates the tendency by employing $\mu_k / \mu_s$ as bifurcation parameter. As observed, when $\mu_k / \mu_s$ is closer to unity, only period-one motions are seen. As $\mu_k / \mu_s$ is decreased to around 0.92, a sudden bifurcation yields period-two motions. Then as $\mu_k / \mu_s$ is decreased further down to 0.7, the period-two motions become quasi-periodic or chaotic. Figures 4.16 shows the sample time histories of $\dot{\delta}_1$, $\dot{\delta}_2$ and poincare sections corresponding to three $\mu_k / \mu_s$ values.

4.8 Effect of Time Varying Dry Friction on Steady State Response

4.8.1 Effect of Actuation Parameters $\psi$, $\omega_f$ and $T_{sp}$ with $\mu_k = \mu_s$

Under the condition of $\mu_k = \mu_s$, the effect of actuation pressure phase lag $\psi$ is first studied with $\omega_f = \omega$. Figure 4.17 shows the max map of $\dot{\delta}_1$ and the corresponding bifurcation diagram with $\omega / \omega_n$ as the bifurcation parameter. At lower frequencies ($\omega / \omega_n < 0.3$ here) the $\psi = 0$ value yields the best attenuation of stick-slip motions compared with $\psi = \pi / 2$ and $\pi$. This is consistent with our analysis of the transient stick-slip motion as reported in the previous section. But when the frequency is increased, the attenuation effect is reversed. This can be attributed to an increased time-delay of the positive slip motion. Figure 4.18 shows the coupled excitation torque and $\dot{\delta}_1$ response at several frequencies with $T_{sp} = 0$. As shown in Figures 4.18a and 4.18b, when the excitation frequency is low ($\omega / \omega_n = 0.15$ and 0.18) the positive slip starts and persists during the first half cycle of engine torque. But when the excitation
frequency is higher (\(\omega / \omega_n = 0.37\) and 0.40), the positive slip is initiated and continues during the second half cycle of the engine torque as shown in Figures 4.18c and 4.18d. In a similar manner to the quasi-static analysis discussed in the previous section, differences in the attenuation effects of lower and higher frequencies can be clearly explained. Further, all curves in Figure 4.17a are relatively smooth for the case of \(\mu_k = \mu_s\) and no jumps are seen. Accordingly, no bifurcation is observed in Figure 4.17b.

Second, the effect of \(\omega_f\) is shown in Figures 4.19 and 4.20. As seen in Figure 4.19a and 4.20a, best attenuation of the slip motion occurs at lower frequencies occurs as \(\omega_f = \omega\). This effect is however reversed as \(\omega\) is increased. Also, no bifurcations seem to take place in Figures 4.19b and 4.20b. Assume \(\omega_f = M \omega / N\) where both \(M\) and \(N\) are integers and \(\omega\) is the engine excitation frequency. When \(M\) and \(N\) are commensurable, the resulting motions will possess a period identical to the lower frequency component of \(\omega\) and \(\omega_f\). For example, when \(M / N = 1/2\), then the motions are period-two; when \(M / N = 2\), the motions are period-one type as shown in Figure 4.19b. In the case when \(M\) and \(N\) are incommensurable, the resulting motions are period-\(N\) type. For example, when \(M / N = 2/3\), the motions are period-three type; when \(M / N = 4/3\), the motions are also period-three as shown in Figure 4.20b. Time histories in Figure 4.21 confirm such results.
Figure 4.17 Effect of phase $\psi$ on steady-state response given $\omega_f = \omega$ and $T_{sp}/T_{sm}=0.25$. a) maximum frequency response map for $\delta_1$: ..x.., $\psi = 0$; ..o.., $\psi = \pi/2$; ..+. .., $\psi = \pi$. b) bifurcation diagram.
Figure 4.18 Time delay in positive slip motion with respect to excitation. a) $\omega/\omega_n=0.15$; b) $\omega/\omega_n=0.18$; c) $\omega/\omega_n=0.37$; d) $\omega/\omega_n=0.40$
Figure 4.19 Effect of $\omega_f$ on steady-state response given $\psi = 0$ and $\frac{T_{sp}}{T_{sm}} = 0.25$. a) maximum frequency response map for $\delta_{i}$. .. $\omega_f = 0.5\omega$; .. $\omega_f = \omega$; .. $\omega_f = 2.0\omega$; b) bifurcation diagram
Figure 4.20 Effect of $\omega_f$ on steady-state response given $\psi = 0$ and $T_{sp}/T_{sm} = 0.25$. a) maximum frequency response map for $\delta_1$: + , $\omega_f = 0.667\omega$; x , $\omega_f = \omega$; o , $\omega_f = 1.33\omega$ b) bifurcation diagram.
Figure 4.21 Sample time histories for different $\omega_r$ given $\omega =150$, $T_{sp}/T_{sm}=1/4$. a) $\omega_r=0.5\ \omega$; b) $\omega_r=0.67\ \omega$; c) $\omega_r=1.33\ \omega$. 

175
Figure 4.22 Effect of \( \omega_f \) on steady-state response given \( \psi = 0 \) and \( \omega_r = \omega \). a) maximum frequency response map for \( \dot{\delta}_1 \): , \( T_{sp}/T_{sm} = 0 \); ..o.., \( T_{sp}/T_{sm} = 0.125 \); .. x .., \( T_{sp}/T_{sm} = 0.25 \); _ . _, \( T_{sp}/T_{sm} = 0.50 \); b) bifurcation diagram

176
Finally, the effect of $T_{sp}$ is studied given $\omega_f = \omega$ and $\psi = 0$. Figure 4.22 shows the corresponding max maps and bifurcation diagram. Consistent with the previous analyses, a higher value of $T_{sp}$ provides the best attenuation of the slip motion at lower frequencies and as $\omega$ goes up, the results are reversed. Also, no bifurcation or possible quasi-periodic or chaotic motions are generated.

4.8.2 Interaction between Friction Characteristics $\mu(\dot{\delta})$ and $T_f(t)$

First, the interaction between $\mu(\dot{\delta})$ and $\psi$ is investigated under the condition $\omega_f = \omega$ and $T_{sp}/T_{sm} = 0.25$. Figure 4.23 shows the resulting max maps and bifurcation diagram with $\psi = 0$, $\pi / 2$ and $\pi$, given $\mu_k = 0.75 \mu_s$. The combined effects of negative damping and varying phases are clearly observed in Figure 4.23a. Similar to Figure 4.14a, the negative damping introduces active super-harmonic peaks. Further, three phase $\psi$ values produce the same attenuation in the slip motion as the case $\mu_k = \mu_s$ that shown in Figure 4.17a. The $\psi = 0$ phase lag has the best attenuation at lower $\omega$ but its effect is reversed at higher $\omega$. When $\psi = 0$, only the pure stick motions exist when $\omega / \omega_n < 0.15$. Consequently, only period-one motions occur in this regime. Note that the transition occurs at a lower value of $\omega / \omega_n$ than that observed (around 0.18) in Figure 4.17a. This could be attributed to the amplification by the negative damping. As $\omega$ is increased further, super-harmonic peaks appear with jumps. Such jumps certainly indicate a loss in stability of the non-linear system and thus bifurcations take place as seen in Figure 4.23b. It also appears that the
severity of bifurcations is more dependent on the abruptness of jumps than the amplitudes of slip motions. As noted in Figure 4.23a, a significant jump occur around $\omega/\omega_n = 0.35$ given $\psi = 0$; this makes the periodic motions into chaotic. In contrast the slip motions for $\psi = \pi$ are higher at lower $\omega$, but the occurrence of peaks are relatively smooth and accordingly no bifurcations are seen. Similar phenomena are seen in Figure 4.24 except that more bifurcations or quasi-periodic or chaotic motions occurs as a result of further decrease in $\mu_k$.

Second, the effect of $\mu_k$ with several $\omega_f$ values is studied given $\psi = 0$ and $T_{sp}/T_{sm} = 0.25$. Resulting max maps and bifurcation diagrams are shown in Figures 4.25 and 4.26. Again, the combined effects of negative damping and mismatched frequencies $\omega_f$ can be clearly observed. At lower $\omega$, a friction torque with $\omega_f = \omega$ yields the best suppression of the slip motion. As shown in Figure 4.25, numerous jumps take place in the particular case of $\omega_f = 0.5\omega$ and quasi-periodic or chaotic motions seem to prevail over the entire frequency range of interest.

Figure 4.27 presents results with several values of $T_{sp}$ under the conditions $\omega_f = \omega$ and $\psi = 0$. As a result of increased attenuation with $T_{sp}$ in Figure 4.27a, the initial quasi-periodic or chaotic motions at lower $\omega$ becomes period-one type at the transition from stick-slip to pure stick motion as shown in Figure 4.27b. However, enhanced slip motions and enlarged jumps convert the periodic motions back into quasi-periodic or chaotic ones at higher $\omega$ around 0.25 and 0.35.
Figure 4.23 Interaction between $\mu_k(\delta)$ and $\psi$ given $T_{sp}/T_{sm} = 0.25$, $\omega_f = \omega$ and $\mu_k = 0.75\mu_c$. a) maximum frequency response maps of $\dot{\delta}_{\text{max}}$: .. x .., $\psi = 0$; .. o .., $\psi = \pi/2$; .. + .., $\psi = \pi$. b) bifurcation diagram.
Figure 4.24 Interaction between $\mu_k(\dot{\delta})$ and $\psi$ given $T_{sp}/T_{sm} = 0.25$, $\omega_f = \omega$ and $\mu_k = 0.67 \mu_s$. a) maximum frequency response maps of $\delta_{\text{max}}$: $\psi = 0$; $\psi = \pi/2$; $\psi = \pi$. b) bifurcation diagram.
Figure 4.25 Interaction between $\mu_k(\delta_p)$ and $\omega_f$ given $T_{sp}/T_{sm} = 0.25$, $\psi = 0$ and $\mu_k = 0.75\mu_s$. a) maximum frequency response maps of $\dot{\delta}_{\text{max}}$: .. x .., $\omega_f = \omega$; .. + .., $\omega_f = 2\omega$; .. o .., $\omega_f = 0.5\omega$. b) bifurcation diagram.
Figure 4.26 Interaction between $\mu_k(\dot{\delta})$ and $\omega_f$ given $T_{sp}/T_{sm} = 0.25$, $\psi = 0$ and $\mu_k = 0.75\mu_s$. a) maximum frequency response maps of $\dot{\delta}_{1max}$: .. $x$ .., $\omega_f = \omega$; .. $+$ .., $\omega_f = 0.67\omega$; .. $o$ .., $\omega_f = 1.33\omega$. b) bifurcation diagram.
Figure 4.27 Interaction between \( \mu_k(\delta_1') \) and \( T_{sp} \) given \( \omega_f = \omega \), \( \psi = 0 \) and \( \mu_k = 0.75\mu_s \). a) maximum frequency response maps of \( \delta_{1\max} \): ___, / 0 \( T_{sp}/T_{sm} = 0 \); .. o .., \( T_{sp}/T_{sm} = 0.125 \); .. x .., \( T_{sp}/T_{sm} = 0.25 \); _. _ \( T_{sp}/T_{sm} = 0.5 \). b) bifurcation diagram
4.9 Conclusion

Specific effects of the dry friction path on both transient and steady-state responses of a torsional system have been studied. To the best of our knowledge, no prior researcher has addressed the NLTV friction force or torque issue, with the exception of sliding friction issues in gears in which the time-varying friction is in the off-line of action while mesh force excitation is in the line of action [4.16-4.17]. Two major contributions of our study emerge. First, key response regimes, namely pure slip, transient stick-slip and steady-state stick-slip, have been identified. Analytical solution for the pure slip motion is obtained based on a simplified linear system analysis. Analyses have shown that three key parameters \( T_m, T_{sm} \) and \( I_1 \) control the engagement rate within the friction interface. The \( T_m \leq T_{sm} \) guideline has to be strictly followed to ensure the final engagement. The friction characteristics with a negative slope \( \mu_k < \mu_s \) are shown to significantly affect the system dynamics in two ways: the “negative” damping effect and a reduction in the saturation friction torque. Although the negative damping may induce judder (substantial stick-slip), a well-tuned time-varying friction torque could possibly quench the stick-slip behavior. Our numerical solutions have been validated by comparing prediction with a popular non-linear solver \textit{Xppauto} [4.29, 4.35]. Second, the effects of time-varying friction on steady-state responses have been numerical studied. Results have shown the importance of actuation system parameters \( \psi, \omega \) and \( T_{sp} \) in attenuating the system
response. Bifurcation diagram are constructed to detect the qualitative changes in the dynamic behavior. It is found that the negative slope (in $T_f$) is the major cause of bifurcations and quasi-periodic or chaotic responses. Around the super-harmonic peak frequencies, the non-linear system tends to lose stability in the form of an abrupt jump in the max maps of $\dot{\delta}_i$ and consequently bifurcations take place. Further, the severity of bifurcations appears to be more dependent on the abruptness of jumps rather than on the amplitudes of slip motions.

Finally, several limitations of our work are evident. First, a more efficient computational solution method for the time-varying problem has to be developed since numerical stiffness issues are observed. Second, the path following or parametric continuation study would be needed to systematically identify the nature of the bifurcations. As well, the threshold of stability of the periodic solution has to be determined. Research along these directions is planned.

References for Chapter 4


CHAPTER 5

CONCLUSION

5.1 Summary

This study has resulted in new or improved insights dry friction path of a torsional system. Non-linear time and frequency domain responses have been calculated using numerical methods. Smoothened and discontinuous friction formulations are examined and effect of smoothening factor is investigated. In addition, the effect of a secondary inertia on system dynamics is studied. A new analytical method based on torque profile is developed and successfully applied to continuous slipping system. This method differs from the classical analytical solution [5.1] since we must consider asymmetric stick-slip motions subjected to a non-zero mean torque. Further, a refined multi-term harmonic balance method (MHBM) is proposed to accurately capture significant stick-slip motions in a realistic torsional system. This method can be efficiently applied to a system consisting of two large dimensional linear sub-systems connected by a non-linear dry friction path. Super-harmonics generated by stick-slip motions are identified and in particular the effect of a mean torque is studied. Finally, the effect of time varying dry friction on system dynamics is investigated. Both transient and steady state responses are studied under
the influence of time varying actuation pressure. Interactions of frictional characteristics and time varying parameters are studied.

In Chapter 2, a 3DOF semi-definite torsional system representing an automotive driveline is studied in presence of a torque converter clutch that manifests itself as a dry friction path. An analytical procedure based on the linear system theory is proposed first to establish the stick to slip boundaries. Smoothened and discontinuous Coulomb friction formulations are then applied to the non-linear system, and the differential governing equations are numerically solved given harmonic torque excitation and a mean load. Time domain histories illustrating dry friction induced stick-slip motions are predicted for different saturation torques and system parameters. Analysis shows that the conditioning factor associated with the smoothened friction model (hyperbolic tangent) must be carefully selected. Then non-linear frequency responses are constructed from cyclic time histories and the stick-slip boundaries predictions (as yielded by the linear system theory) are confirmed. In particular, the effect of secondary inertia is investigated since it has a significant influence on the dynamic response. Results for our system are also compared with the conventional bi-linear hysteresis model in which the secondary inertia is ignored. Finally, our methods are successfully compared with two benchmark analytical and experimental studies, as available in the literature on two degree-of-freedom translational systems.

In Chapter 3, the non-linear frequency response characteristics of a two degree of freedom torsional system with a significant dry friction controlled path are studied, when excited by sinusoidal torque under a mean load. An analytical solution
is first developed for a simplified system subjected to continuous slipping motions. The nature of super-harmonic peaks as generated by the dry friction non-linearity is efficiently found. The effect of a non-zero mean load is also determined and qualitatively understood. Further, a refined multi-harmonic balance method (MHBM) is proposed that includes up to 12 terms. It is used to study an automotive drive train system that experiences significant stick-slip motions. Associated computational issues including the selection of initial conditions are addressed. Studies show that the mean load could induce asymmetric stick-slip motions and accordingly it has significant effect on time and frequency domain responses. Reasons for the occurrence of super-harmonic resonant peaks and relaxational peaks are investigated. Finally, our MHBM method is applied to the conventional single degree of freedom system where the spring path exists in parallel with a dry friction damper (Den Hartog's problem). Our predictions match well with Den Hartog's analytical solution. Den Hartog's system differs, in terms of the dynamic behavior, from our torsional system (with a sole dry friction path).

In Chapter 4, the effect of a time-varying dry friction (or actuation pressure) on transient and steady-state responses of a torsional system with dry friction path has been studied. First, three different response stages, namely pure slip, transient stick-slip and steady-state stick-slip, have been identified via numerical simulations. Analytical solution for pure slipping motion is obtained based on an approximated linear system model. Detailed analyses show that the mean torque load, mean saturation friction torque (or actuation pressure) and flywheel-impeller inertia control the engagement rate at the friction interface. Second, the friction characteristics with a
negative slope are found to affect the system dynamics in two ways: through the "negative" damping effect and a reduced (or enhanced) saturation friction torque. Third, the effects of time-varying friction on steady-state responses have been numerically studied. Results reveal the varying attenuation effects of time-varying parameters, such as phase, frequency and amplitude of the actuation pressure, in different frequency regimes. The negative friction slope is found to be the major cause of judder-induced phenomena such as bifurcations and quasi-periodic or chaotic responses. The non-linear system with time-varying dry-friction shows several instabilities as abrupt jumps in the amplitude-frequency maps of relative velocity are seen around the super-harmonic peak frequencies. Further, the severity of bifurcations appears to be more dependent on the abruptness of jump than the amplitude of slipping motion.

5.2 Contributions

This study has significantly advanced the subject of dry friction non-linearity in the context of two or three degree freedom torsional systems when subjected to harmonic excitation with a mean torque. In particular, research has been initiated on the dry friction path problems. Unlike previous work, we have examined the following issues: torque transmission through the dry friction only, high saturation friction torque and time varying normal load. Specifically, the following major contributions emerge.
1. Non-linear time and frequency domain responses of a torsional system with a dry friction controlled path have been calculated using numerical methods. A procedure to predict pure stick to stick-slip boundaries has been developed based on linear system theory. Both smoothened and discontinuous friction formulations have been examined and the effect of smoothening factor is investigated. Analytical and numerical results reveal that the secondary inertia could significantly affect the system dynamics, especially the quasi-discontinuous responses.

2. A new analytical method based on assumed torque profile is developed and successfully applied to the continuous slipping system. It differs from Den Hartog’s analytical solution [5.1] since we must consider asymmetric stick-slip motions. Super-harmonics are efficiently established and the effect of mean torque is qualitatively identified. Further, a refined multi-term harmonic balance method (MHBM) is proposed. Associated computational issues are addressed and a procedure for selecting proper initial values for Newton-Raphson scheme is developed. The proposed method has been validated by numerical solutions and by comparing with Den Hartog’s analytical solution [5.1].

3. The effect of time varying dry friction on transient and steady state responses of a multi-degree of freedom torsional system has been investigated. An approximate linear system solution is also obtained and successfully applied to pure slipping transients. The effect of time varying parameters leading on transient stick-slip (judder) has been studied. Steady state responses including
periodic and quasi-periodic or chaotic motions are predicted and identified via bifurcation diagrams. Finally, interaction between frictional characteristics and time varying parameters is examined.

5.3 Future Work

The proposed numerical and semi-analytical methods can be easily extended to analyze more complex dry friction problem as well as piece-wise non-linear stiffness problems. The following research issues are suggested for future studies.

1. Investigate interactions between periodic torque excitation and periodic time varying dry friction.

2. Examine transient responses such those induced by tip-in (a sudden increase in throttle angle or $T_m$) and tip-out (a sudden decrease in $T_m$) with a slipping torque converter clutch [5.2-5.5].

3. Examine the effects of several concurrent non-linear elements. For example, frictional hysteresis and multi-staged stiffness element associated with a torsional clutch damper and gear backlashes could be included to model a realistic driveline system [5.2, 5.7, 5.8].

4. Apply parametric continuation or path following technique [5.6] to identify the nature of bifurcations. Also, determine the stability of periodic responses.
References for Chapter 5


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