A SUFFICIENT CONDITION FOR SUBELLIPTICITY OF THE \( \bar{\partial} \)-NEUMANN PROBLEM

DISSERTATION

Presented in Partial Fulfillment of the Requirement for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

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2004

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ABSTRACT

We give a sufficient condition for subelliptic estimates for the $d$-bar-Neumann problem on smoothly bounded, pseudoconvex domains. This condition is a quantified version of McNeal’s condition for compactness of the $d$-bar-Neumann operator, and it extends Catlin’s sufficiency condition for subellipticity as it is less stringent.
To my parents, Christine and Hans-Henry.
ACKNOWLEDGMENTS

It is a great pleasure to thank my advisor, Jeff McNeal, for introducing me to the subject of complex analysis of several variables, and for his guidance through it for the past years. I am deeply indebted to him for aiding me in my study of complex analysis, and it seems rather impossible to pay him back within a lifetime. I have greatly enjoyed and benefited from all our discussions filled with his contagious enthusiasm for mathematics. I sincerely appreciate all his support and patience over the years, and I feel immensely lucky to have had such an advisor.

I would like to thank my dear brother, Hans-Christian, for his persistent attempts to spark my interest in various branches of mathematics, and for all the distractions he provided by doing so.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
</tr>
<tr>
<td>Dedication</td>
</tr>
<tr>
<td>Acknowledgments</td>
</tr>
<tr>
<td>Vita</td>
</tr>
<tr>
<td>Chapters:</td>
</tr>
<tr>
<td>1. Introduction</td>
</tr>
<tr>
<td>2. Basics of the $\bar{\partial}$-Neumann problem</td>
</tr>
<tr>
<td>3. Basic estimates</td>
</tr>
<tr>
<td>4. Estimates for $\bar{\partial}^* N_q$</td>
</tr>
<tr>
<td>5. Estimates on $D^{0,q}(\Omega)$</td>
</tr>
<tr>
<td>6. Subelliptic estimate</td>
</tr>
<tr>
<td>Bibliography</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

In several complex variables, a fundamental and general problem is to construct holomorphic functions, which have certain specified properties, on domains or manifolds. One way to construct such functions is by studying solutions to the Cauchy-Riemann equations by partial differential equation methods. A starting difficulty is that, in more than one variable, the Cauchy-Riemann system is overdetermined. One method for dealing with this difficulty is by considering certain related, second order systems of partial differential equations. In this thesis, I study aspects of the $\bar{\partial}$-Neumann problem, which is a prototype for a second order elliptic problem with non-coercive boundary conditions. This second order problem handles the overdetermination perfectly, but at the cost of posing delicate boundary regularity questions.

Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain. We denote by $L^2_{p,q}(\Omega)$ the space of the $(p,q)$-forms with square-integrable coefficients. Let $\bar{\partial} : L^2_{p,q}(\Omega) \rightarrow L^2_{p,q+1}(\Omega)$ be the $L^2$-closure of $\partial$, and $\bar{\partial}^* : L^2_{p,q+1}(\Omega) \rightarrow L^2_{p,q}(\Omega)$ the Hilbert space adjoint of $\bar{\partial}$. Then the $\bar{\partial}$-Neumann problem can be stated as follows. For a given $f$ in $L^2_{p,q}(\Omega)$,
find \( u \) in \( L^2_{p,q}(\Omega) \), such that the following holds:

\[
\begin{aligned}
(\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \partial) u &= f \\
u \in \text{Dom}(\partial) \cap \text{Dom}(\partial^*) \\
\bar{\partial} u \in \text{Dom}(\partial^*), \bar{\partial}^* u \in \text{Dom}(\partial).
\end{aligned}
\]

Establishing a solution to the \( \bar{\partial} \)-Neumann problem does lead to a particular solution to the Cauchy-Riemann equations, but just in the \( L^2 \)-sense. Thus one is not just interested in the existence of such a \( L^2 \)-solution \( u \) for given data \( f \), but one is also interested in the kind of regularity statements that can be made about \( u \) when \( f \) is regular.

The complex Laplacian, \( \Box_{p,q} := \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \partial \), is itself elliptic, but the boundary conditions, which are implied by membership to \( \text{Dom}(\partial^*) \), are not. Thus Gårding’s inequality does hold in the interior of \( \Omega \), but not uniformly up to the boundary of \( \Omega \). So the usual results from elliptic boundary value problems can not be applied here, and indeed the usual methods do not work.

However, on domains with certain geometric conditions on the boundary, the question of existence of a solution to the \( \bar{\partial} \)-Neumann problem was settled through the works of Hörmander [Hör], Kohn [Koh1, Koh2] and Morrey [Mor]. In fact, Hörmander’s results in [Hör] imply that there exists a bounded operator \( N_{p,q} \) on \( L^2_{p,q}(\Omega) \), which inverts the complex Laplacian \( \Box_{p,q} \), under the assumption that \( \Omega \) is a bounded, pseudoconvex domain.

In the following, we will be concerned only with the local regularity question for the \( \bar{\partial} \)-Neumann problem, i.e. what conditions on \( \Omega \) are sufficient for \( u := N_{p,q} f \) being smooth wherever \( f \) is. A fundamental step concerning this question was done
by Kohn and Nirenberg. They showed in [Koh-Nir] that, if a so-called *subelliptic estimate of order* $\epsilon$ holds for the $\bar{\partial}$-Neumann problem on a neighborhood $V$ of a given point $p$ in $b\Omega$, then $f|_V \in H^{s}_{p,q}(V)$ implies $N_{p,q}f|_{V'} \in H^{s+2\epsilon}_{p,q}(V')$ for $V' \subset \subset V$; here $H^{s}_{p,q}$ denotes the $L^2$-Sobolev space of order $s$ on $(p,q)$-forms. Thus it is natural to inquire about subelliptic estimates for the $\bar{\partial}$-Neumann problem.

Denote by $\mathcal{D}^{p,q}(V \cap \bar{\Omega})$ the set of smooth $(p,q)$-forms $u$, which are supported in $V \cap \bar{\Omega}$, such that $u$ belongs to the domain of $\bar{\partial}^*$. A subelliptic estimate of order $\epsilon > 0$ near $p \in b\Omega$ is said to hold, if

$$\|u\|_\epsilon^2 \leq C(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2) \quad \text{for all } u \in \mathcal{D}^{p,q}(V \cap \bar{\Omega}),$$

(1.1)

where the norm on the left hand side of (1.1) is the $L^2$-Sobolev norm of order $\epsilon$.

The first major step towards subelliptic estimates for the $\bar{\partial}$-Neumann problem was achieved by Kohn [Koh1, Koh2]. In these papers he considered smoothly bounded, strongly pseudoconvex domains in $\mathbb{C}^n$ and showed that on such domains a subelliptic estimate of order $\frac{1}{2}$ holds, i.e.

$$\|u\|_{\frac{1}{2}}^2 \leq C(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2) \quad \text{for all } u \in \mathcal{D}^{p,q}(\Omega).$$

Combined with the above mentioned result in [Koh-Nir], this says that $N_{p,q}f$ gains one derivative over $f$ in the $L^2$-Sobolev scale. This gain is just half of what would occur in an ordinary elliptic boundary value problem, hence the name “subelliptic”.

The most general result concerning subelliptic estimates for the $\bar{\partial}$-Neumann problem was obtained by Catlin [Cat]. He showed that the existence of a certain family of functions on a pseudoconvex domain is sufficient for a subelliptic estimate to hold. Moreover, he proved that one can construct such a family of functions on any
smoothly bounded, pseudoconvex domain, which is of finite type. Catlin’s sufficiency result can be stated as follows:

**Theorem 1.2.** Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain. Let $p$ be a given point in $b\Omega$ and suppose that $b\Omega \cap U$ is pseudoconvex, where $U$ is a neighborhood of $p$. Denote by $S_\delta$ the set $\{z \in \Omega \mid -\delta < r(z) < 0\}$. Assume that for all $\delta > 0$ sufficiently small there exists a $\lambda_\delta \in C^2(\bar{\Omega} \cap U)$, such that

(i) $|\lambda_\delta(z)| \leq 1$ for all $z \in \bar{\Omega} \cap U$,

(ii) for all smooth $(p,q)$-forms $u$ and $z \in S_\delta \cap U$

$$
\sum' \sum_{|I|=p,|J|=q-1} \frac{\partial^2 \lambda_\delta}{\partial z_k \partial \bar{z}_l}(z) u_{I,k} \bar{u}_{I,l} \geq c \delta^{-2\epsilon} |u|^2,
$$

where the constant $c > 0$ does not depend on $\delta$ or $u$.

(iii) $\lambda_\delta$ is plurisubharmonic on $\bar{\Omega} \cap U$.

Then there exists a neighborhood $V \subset U$ of $p$ such that a subelliptic estimate of order $\epsilon$ holds.

We replace the boundedness condition on the weight functions $\lambda_\delta$ with that of *self-bounded complex gradient*, a weaker condition which allows unbounded families of functions. This idea was introduced by McNeal in [McN3].

**Definition 1.3.** Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain. A plurisubharmonic function $\phi \in C^2(\Omega)$ is said to have a self-bounded complex gradient, if there exists a constant $C > 0$ such that

$$
| \sum_{k=1}^n \frac{\partial \phi}{\partial z_k}(z) \xi_k |^2 \leq C \sum_{k,l=1}^n \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l}(z) \xi_k \xi_l
$$

(1.4)

holds for all $\xi \in \mathbb{C}^n$ and $z \in \Omega$. 

4
Notice that, if \( \lambda \in C^2(\Omega) \) is plurisubharmonic and bounded, then \( \phi = e^{\lambda} \) has a self-bounded complex gradient with \( C = \sup_{z \in \Omega} e^{\lambda(z)} \). Furthermore, notice the behavior of inequality (1.4) under scaling; replacing \( \phi \) by \( t\phi \) for \( t > 0 \), the left hand side of (1.4) is quadratic in \( t \), while the right hand side is linear in \( t \).

My result is the following:

**Theorem 1.5.** Let \( \Omega \subset \mathbb{C}^n \) be a smoothly bounded domain. Let \( p \) be a given point in \( b\Omega \) and suppose that \( b\Omega \cap U \) is pseudoconvex, where \( U \) is a neighborhood of \( p \). Denote by \( S_\delta \) the set \( \{ z \in \Omega \mid -\delta < r(z) < 0 \} \). Assume that for all \( \delta > 0 \) sufficiently small there exists a \( \phi_\delta \in C^2(\bar{\Omega} \cap U) \), such that

(i) \( \phi_\delta \) has a self-bounded complex gradient, with \( C \) in (1.4) independent of \( \delta \),

(ii) for all smooth \((p, q)\)-forms \( u \) and \( z \in S_\delta \cap U \)

\[
\sum_{|I|=p, |J|=q-1} \sum_{k,l=1}^n \frac{\partial^2 \phi_\delta}{\partial z_k \partial \bar{z}_l}(z) u_{I,k} \bar{u}_{I,l} \geq c_\delta^{-2}\epsilon|u|^2,
\]

where the constant \( c > 0 \) does not depend on \( \delta \) or \( u \).

(iii) \( \phi_\delta \) is plurisubharmonic on \( \bar{\Omega} \cap U \).

Then there exists a neighborhood \( V \subset U \) of \( p \) such that a subelliptic estimate of order \( \epsilon \) holds.

Observe that the existence of Catlin’s family of functions \( \{\lambda_\delta\} \) implies the existence of the above family \( \{\phi_\delta\} \) by setting \( \phi_\delta = e^{\lambda_\delta} \).

The main idea for proving Theorem 1.5 stems from a paper of McNeal, [McN3]. There he shows that the \( \bar{\partial} \)-Neumann operator, \( N^\Omega_q \), on a smoothly bounded, pseudoconvex domain \( \Omega \) is compact on \( L^2_{p,q}(\Omega) \), if \( \Omega \) satisfies condition \( (\bar{P}_q) \).
is said to satisfy condition \((\tilde{P}_q)\), if, for any \(M > 0\), there exists a plurisubharmonic function \(\phi_M \in C^2(\tilde{\Omega})\), which has a self-bounded complex gradient (with \(C = 1\)) such that

\[
\sum'_{|I| = p, |J| = q-1} \sum_{k,l=1}^n \frac{\partial^2 \phi_M}{\partial z_k \partial \bar{z}_l}(z) u_{I,k,J} \bar{u}_{I,l,J} \geq M|u|^2
\]

(1.6)

holds for all \(u \in \Lambda^{p,q}(\tilde{\Omega})\) and \(z \in b\Omega\). To prove compactness of \(N^\Omega_q\), McNeal considers the functional \(F : (\{e^{-\frac{\phi_M}{2}} \partial_{\phi_M}^* u \mid u \in D^{p,q}(\Omega)\}, \|\cdot\|_{\phi_M}) \rightarrow \mathbb{C}\) defined by

\[
F(e^{-\frac{\phi_M}{2}} \partial_{\phi_M}^* u) := (u, \alpha)_{\phi_M},
\]

where \(\alpha\) is a \(\bar{\partial}\)-closed \((p, q)\)-form. He then shows that

\[
|F| \leq C\left(\frac{1}{\sqrt{M}} \|\alpha\| + \sqrt{C(\phi_M)} \|\alpha\|_{-1}\right),
\]

where the factor \(\frac{1}{\sqrt{M}}\) comes from \(L^2\)-estimates on \(ue^{-\phi_M}\) near the boundary, while \(\sqrt{C(\phi_M)}\) comes from interior elliptic estimates on \(ue^{-\phi_M}\). By a duality argument, McNeal obtains a compactness estimate for \(\partial^* N^\Omega_q\), that is

\[
\|\partial^* N^\Omega_q \alpha\|^2 \leq C'(\frac{1}{M} \|\alpha\|^2 + C(\phi_M) \|\alpha\|^2_{-1}) \quad \text{for } \alpha \in \ker(\partial).
\]

(1.7)

Since condition (1.6) percolates up the \(\bar{\partial}\)-complex, one gets the analogous result for \(\partial^* N^\Omega_{q+1}\) by a similar proof. It follows then that \(N^\Omega_q\) is a compact operator.

The proof of Theorem 1.5 relies on the same duality argument. Since the weight functions \(\phi_\delta\) are just defined locally, we restrict our considerations to so-called approximating subdomains \(\Omega_a\) associated to \((\Omega, p, U)\). On such an approximating subdomain, we obtain estimates similar to (1.7) with the additional feature that we explicitly express \(C(\phi_\delta)\) in terms of \(\delta\). We then convert these inequalities to a family of \(L^2\)-estimates on \(D^{p,q}(\Omega)\). Having those at hand, we derive a subelliptic estimate by using the same method as Catlin in [Cat]. That is, we introduce a sequence of pseudodifferential operators which represent a partition of unity in the tangential Fourier
transform variables. Similar to [Cat], except that we do not go via the boundary, we then obtain that a subelliptic estimate of order $\epsilon$ holds near the boundary point $p$.

The thesis is structured as follows. In chapter 2 we review the setting of the $\bar{\partial}$-Neumann problem. In chapter 3 we derive several weighted $L^2$-inequalities, which are specific for weights having a self-bounded complex gradient. Using those inequalities we obtain two versions of compactness estimates on $\bar{\partial}^* N_q$ and $\bar{\partial}^* N_{q+1}$ in chapter 4. In chapter 5 we convert these compactness estimates to a family of $L^2$-estimates in terms of the Dirichlet form. With those estimates at hand we complete the proof of Theorem 1.5 in chapter 6.
CHAPTER 2

BASICS OF THE $\bar{\partial}$-NEUMANN PROBLEM

In this chapter, we give the general set-up of the $\bar{\partial}$-Neumann problem. We sketch how to obtain the existence of an $L^2$-solution to the $\bar{\partial}$-Neumann problem, and review the basic facts regarding subellipticity.

General Setting

Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain, i.e. $\Omega$ is bounded and there is a smooth function $r$ such that $\Omega = \{ z \in \mathbb{C}^n \mid r(z) < 0 \}$ and $\nabla r \neq 0$ whenever $r = 0$. We note that $b\Omega = \{ z \in \mathbb{C}^n \mid r(z) = 0 \}$. We assume that $r$ is normalized so that $|\nabla r| = 1$ holds on $b\Omega$.

Let $0 \leq p, q \leq n$. We can write an arbitrary $(p, q)$-form $u$ as

$$u = \sum_{|I|=p, |J|=q} u_{I,J} dz^I \wedge d\bar{z}^J,$$  \hspace{1cm} (2.1)

where $I = \{i_1, \ldots, i_p\}$, $J = \{j_1, \ldots, j_q\}$ and $dz^I = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, $d\bar{z}^J = d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$. Here $\sum'$ means that we only sum over strictly increasing index sets. We define the coefficients $u_{I,J}$ for arbitrary index sets $I$ and $J$, so that the $u_{I,J}$'s are antisymmetric functions of $I$ and $J$. 

8
Let $\Lambda^{p,q}(\bar{\Omega}), \Lambda^{p,q}_c(\Omega)$ and $L^2_{p,q}(\Omega)$ denote the $(p, q)$-forms with coefficients in $C^\infty(\bar{\Omega}), C^\infty_c(\Omega)$ and $L^2(\Omega)$, respectively. We use the pointwise inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle dz^k, dz^l \rangle = \delta^k_l = \langle d\bar{z}^k, d\bar{z}^l \rangle.$$ 

This inner product just differs by a constant from the standard euclidean inner product. By linearity we extend this inner product to $(p, q)$-forms.

The global $L^2$-inner product on $\Omega$ is defined by

$$(u, v)_\Omega = \int_\Omega \langle u, v \rangle dV,$$

where $dV$ is the euclidean volume form. The $L^2$-norm of a $(p, q)$-form $u$ on $\Omega$ is then given by $\|u\|^2_\Omega = (u, u)_\Omega$; we drop the subscript $\Omega$, when there is no reason for confusion.

If $\phi \in C^2(\bar{\Omega})$, we denote by $L^2_{p,q}(\Omega, \phi)$ the space of $(p, q)$-forms $u$ such that

$$\|u\|^2_{\phi, \Omega} = (u, u)_{\phi, \Omega} := \|ue^{-\phi}\|^2_\Omega = \int_\Omega \langle u, u \rangle e^{-\phi} dV < \infty.$$ 

Notice that the weighted $L^2$-space, $L^2_{p,q}(\Omega, \phi)$, equals $L^2_{p,q}(\Omega)$, since $\phi$ is continuous on $\bar{\Omega}$.

Let $u \in \Lambda^{p,q}(\bar{\Omega})$, then the $\bar{\partial}$-operator is defined as

$$\bar{\partial}_{p,q} u = \bar{\partial} u := \sum_{|I|=p, |J|=q} \sum_{k=1}^n \bar{\partial}_k u_{I,J} \, d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J,$$

where $\bar{\partial}_k := \frac{\partial}{\partial \bar{z}^k}$, and $u$ is expressed as in (2.1). Observe that $\bar{\partial}^2 = 0$. We can extend the differential operator $\bar{\partial}$, still denoted by $\bar{\partial}$, to act on non-smooth forms in the sense of distributions. Then, by restricting the domain of $\bar{\partial}$ to those forms $g \in L^2_{p,q}(\Omega)$, where $\bar{\partial}g$ in the distributional sense belongs to $L^2_{p,q+1}(\Omega)$, $\bar{\partial}$ becomes an operator.
on Hilbert spaces at each form level. Note that $\bar{\partial}$ is a densely defined operator on $L^2_{p,q}(\Omega)$, since the compactly supported forms $\Lambda^{p,q}_c(\Omega)$ are in Dom($\bar{\partial}$). Moreover, $\bar{\partial}$ is a closed operator, because differentiation is a continuous map in the distributional sense.

Now we define the Hilbert space adjoint, $\bar{\partial}^*$, to $\bar{\partial}$ with respect to the $L^2$-inner product on the appropriate form level in the following way:

we say that $u \in L^2_{p,q+1}(\Omega)$ belongs to the domain of $\bar{\partial}^*$, i.e. $u \in \text{Dom}(\bar{\partial}^*)$, if there exists a constant $C > 0$ so that

$$|(\bar{\partial} w, u)| \leq C \|w\| \text{ holds for all } w \in \text{Dom}(\bar{\partial}).$$

By the Riesz representation theorem it follows, that, if $u \in \text{Dom}(\bar{\partial}^*)$, there exists a unique $v \in L^2_{p,q}(\Omega)$, such that

$$(w, v) = (\bar{\partial} w, u)$$

holds for all $w \in \text{Dom}(\bar{\partial})$; we write $\bar{\partial}^* u$ for $v$. This characterization of $\text{Dom}(\bar{\partial}^*)$ reveals that certain boundary conditions must hold on any smooth $(p, q + 1)$-form, which belongs to $\text{Dom}(\bar{\partial}^*)$.

**Lemma 2.3.** Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain, $r$ a defining function of $\Omega$. Then $u \in D^{p,q+1}(\Omega) := \text{Dom}(\bar{\partial}^*) \cap \Lambda^{p,q+1}(\bar{\Omega})$ iff

$$\sum_{k=1}^{n} u_{I,k,j} \frac{\partial r}{\partial z_k} = 0 \text{ on } b\Omega,$$

for all $I$ and $J$ which are strictly increasing index sets of length $p$ and $q$, respectively.

**Proof.** Let $w \in \Lambda^{p,q}(\bar{\Omega})$ and $u \in D^{p,q+1}(\Omega)$. Using our notation (2.1) for $(p, q + 1)$-
forms, we can express \( u \) in the following manner:

\[
  u = (-1)^p \sum_{|I|, |J|} \sum_{k=1}^n u_{I,k,J} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J,
\]

where \(|I| = p\) and \(|J| = q\). By integration by parts it follows

\[
  (\bar{\partial}w, u) = (-1)^p \sum_{|I|, |J|} \sum_{k=1}^n (\bar{\partial}_k w_{I,J}, u_{I,k,J})
  = (-1)^{p+1} \sum_{|I|, |J|} \sum_{k=1}^n (w_{I,J}, \partial_k u_{I,k,J}) + (-1)^p \sum_{|I|, |J|} \sum_{k=1}^n \int_{b\Omega} w_{I,J} \bar{u}_{I,k,J} \frac{\partial r}{\partial \bar{z}_k} dS,
\]

where \(dS\) is the surface measure. Thus, if \(w\) has compact support in \(\Omega\), the boundary integral vanishes. This implies that

\[
  \sum_{|I|, |J|} \int_{b\Omega} w_{I,J} \sum_{k=1}^n \bar{u}_{I,k,J} \frac{\partial r}{\partial \bar{z}_k} dS = 0
\]

must hold for all \(w \in \Lambda^{p,q}(\Omega)\), since \(\Lambda^{p,q}_c(\Omega)\) is dense in \(L^2_{p,q}(\Omega)\). But that forces

\[
  \sum_{k=1}^n u_{I,k,J} \frac{\partial r}{\partial \bar{z}_k} = 0 \text{ on } b\Omega
\]

for all strictly increasing index sets \(I\) and \(J\) of length \(p\) and \(q\), respectively. Observe that we also have shown that for \(u \in D^{p,q+1}(\Omega)\) it holds that

\[
  \bar{\partial}^* u = (-1)^{p+1} \sum_{|I|, |J|} \sum_{k=1}^n \bar{\partial}_k u_{I,k,J} d\bar{z}^I \wedge dz^J.
\]

We can also inquire about the Hilbert space adjoint, \(\bar{\partial}_{\phi}^*\), to \(\bar{\partial}\) with respect to the \(L^2(\Omega, \phi)\)-inner product. In view of (2.2) it is easy to see that \(\text{Dom}(\bar{\partial}^*) = \text{Dom}(\bar{\partial}_{\phi}^*)\) holds, since \(\phi\) is continuous on \(\bar{\Omega}\). Let \(w \in \text{Dom}(\bar{\partial})\) and \(u \in \text{Dom}(\bar{\partial}^*)\), then

\[
  (\bar{\partial}w, u)_{\phi} = (\bar{\partial}w, e^{-\phi} u) = (w, \bar{\partial}^*(e^{-\phi} u)) = (w, e^\phi \bar{\partial}^*(e^{-\phi} u))_{\phi}.
\]
Hence the equation $\bar{\partial}^* = e^\phi \bar{\partial} e^{-\phi}$ defines $\bar{\partial}^*.$

Now we are ready to formulate the $\bar{\partial}$-Neumann problem. It is the following: given $f \in L^2_{p,q}(\Omega),$ find $u \in L^2_{p,q}(\Omega)$ such that the following holds

$$
\begin{cases}
(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})u = f \\
u \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \\
\bar{\partial}u \in \text{Dom}(\bar{\partial}^*), \, \bar{\partial}^*u \in \text{Dom}(\bar{\partial})
\end{cases}
$$

(2.4)

Let us remark that through solving the $\bar{\partial}$-Neumann problem we indeed obtain an $L^2$-solution to the Cauchy-Riemann equations. A necessary condition for $\bar{\partial}v = f$ to hold is that $f$ is annihilated by $\bar{\partial}.$ But then, if $u$ satisfies (2.4), it follows that $\bar{\partial}\bar{\partial}^*\bar{\partial}u = 0.$ Therefore

$$
0 = (\bar{\partial}\bar{\partial}^*\bar{\partial}u, \bar{\partial}u) = (\bar{\partial}^*\bar{\partial}u, \bar{\partial}^*\bar{\partial}u) = \|\bar{\partial}^*\bar{\partial}u\|^2,
$$

which implies that $\bar{\partial}\bar{\partial}^*u = f.$ Thus setting $v = \bar{\partial}^*u$ solves the $\bar{\partial}$-problem for $f.$ Moreover, $v$ is perpendicular to the kernel of $\bar{\partial},$ which says that among all solutions $v$ is the one of minimal $L^2$-norm.

The operator $\square_{p,q} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is called the complex Laplacian, and we say that a $(p,q)$-form belongs to the domain of the complex Laplacian, i.e. $u \in \text{Dom}(\square_{p,q}),$ if

$$
u \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*), \, \bar{\partial}u \in \text{Dom}(\bar{\partial}^*), \, \bar{\partial}^*u \in \text{Dom}(\bar{\partial}).
$$

A straightforward calculation shows that if $u \in \Lambda^{p,q}(\bar{\Omega}) \cap \text{Dom}(\square_{p,q}),$ then

$$
\square_{p,q}u = -\frac{1}{4} \sum'_{|I|=p, |J|=q} \Delta u_{I,J} dz^I \wedge d\bar{z}^J,
$$

12
where $\Delta = 4 \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \bar{z}_k}$ is just the usual Laplacian on functions. This not only justifies the name of $\square_{p,q}$, but more importantly it reveals that the complex Laplacian acting on forms, which are compactly supported in $\Omega$, is a strongly elliptic operator.

In view of Lemma 2.3, we notice that $u \in \text{Dom}(\bar{\partial}^*)$ and $\bar{\partial}u \in \text{Dom}(\bar{\partial}^*)$ are true boundary conditions. These boundary conditions are non-coercive and therefore the system (2.4) is not an elliptic boundary value problem (except when $n = q$, then (2.4) is just the classical Dirichlet problem). Furthermore, we remark that $\square_{p,q}$ is a closed, densely defined, selfadjoint operator; for a thorough treatment of this matter see e.g. chapter 4 in [Che-Sha].

From here on, we restrict our considerations to $(0, q)$-forms. The system (2.4) does not see the $dz$’s and the general case for $(p, q)$-forms can be derived easily. To simplify notation we shall write $\square_q$ for the complex Laplacian acting on $(0, q)$-forms. Also for notational ease we write $u_J$, instead of $u_{0,J}$, for the components of a $(0, q)$-form from $u$. We shall denote the Dirichlet form associated to $\square_q$ as usual by $Q(\ldots)$, i.e. $Q(u, v) := (\bar{\partial} u, \bar{\partial} v) + (\bar{\partial}^* u, \bar{\partial}^* v)$ for $u, v \in \mathcal{D}^{0,q}(\Omega)$.

For quantities $A$ and $B$ we use the notation $|A| \lesssim |B|$ to mean $|A| \leq C|B|$ for some constant $C > 0$, which is independent of relevant parameters. It will be specifically mentioned or clear from the context, what those parameters are.

Furthermore, we call the elementary inequality

$$|AB| \leq \epsilon A^2 + \frac{1}{4\epsilon} B^2 \quad \text{for } \epsilon > 0$$

the (sc)-(lc) inequality.
Existence of an $L^2$-solution

In the following we give a sketch on how to obtain existence of an $L^2$-solution to the system (2.4). From Hilbert space theory, we have the decomposition

$$L^2_{0,q}(\Omega) = \mathcal{R}(\Box_q) \oplus \ker(\Box_q),$$

where $\mathcal{R}(\Box_q)$ and $\ker(\Box_q)$ denote the range and the kernel of $\Box_q$, respectively. Since $\Box_q$ is a closed operator, it follows that $\ker(\Box_q)$ is also closed. In order to show that the complex Laplacian is invertible, we need that $\mathcal{R}(\Box_q)$ is closed as well as that $\ker(\Box_q) = \{0\}$. These two properties of the complex Laplacian are implied by the estimate

$$\|u\| \leq C\|\Box_q u\| \text{ for } u \in \text{Dom}(\Box_q),$$

(2.5)

where $C > 0$ is some constant independent of $u$. In the following outline of a proof of (2.5) we use an approach of Boas and Straube presented in [Boa-Str]. The starting point is the following basic equality:

**Proposition 2.6.** Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded domain and $u \in \mathcal{D}^{0,q}(\Omega)$, $1 \leq q \leq n$. Furthermore, assume that $\lambda, \psi \in C^2(\overline{\Omega})$ and $\psi \geq 0$. Then

$$\left\| \sqrt{\psi} \bar{\partial} u \right\|_\lambda^2 + \left\| \sqrt{\psi} \bar{\partial}^* u \right\|_\lambda^2$$

$$= \sum' \sum_{|I|=q-1}^n \int_{\partial \Omega} \psi \frac{\partial^2 u}{\partial z_k \partial \bar{z}_l} u_{kl} \bar{u}_{I} e^{-\lambda} dS + \sum' \sum_{|J|=q}^n \left\| \sqrt{\psi} \bar{\partial} u_{J} \right\|_\lambda^2$$

$$+ \sum' \sum_{|I|=q-1}^n \sum_{k,l=1}^n \int_{\Omega} \left( \psi \frac{\partial^2 \lambda}{\partial z_k \partial \bar{z}_l} - \frac{\partial^2 \psi}{\partial z_k \partial \bar{z}_l} \right) u_{kl} \bar{u}_{I} e^{-\lambda} dV$$

$$+ 2 \text{Re} \left( \sum' \sum_{|I|=q-1}^n u_{kl} \frac{\partial \psi}{\partial z_k} \bar{z}^I, \bar{\partial}^* u \right)_\lambda.$$
Versions of this equality, using different weight factors, were obtained by Berndtsson [Ber], McNeal [McN2, McN3], Ohsawa-Takegoshi [Ohs-Tak] and Siu [Siu]. The proof of (2.7) relies only on integration by parts and making use of the boundary conditions forced on \( u \in \mathcal{D}^{0,q}(\Omega) \); see [McN3]. Restricting our considerations to pseudoconvex domains, we get the boundary integral in (2.7) under control. That is:

**Definition 2.8.** Let \( \Omega \subset \mathbb{C}^n \) be a smoothly bounded domain, \( r \) a defining function of \( \Omega \). We say that \( \Omega \) is pseudoconvex at \( p \in \partial \Omega \) if

\[
\sum_{k,l=1}^{n} \frac{\partial^2 r}{\partial z_k \partial \bar{z}_l}(p) \xi_k \bar{\xi}_l \geq 0 \quad \text{for all} \quad \xi \in \mathbb{C}^n \quad \text{satisfying} \quad \sum_{k=1}^{n} \frac{\partial r}{\partial z_k}(p) \xi_k = 0. \tag{2.9}
\]

We say that \( \Omega \) is strictly pseudoconvex at \( p \in \partial \Omega \), if the inequality in (2.9) is strict. \( \Omega \) is said to be (strictly) pseudoconvex if it is (strictly) pseudoconvex at every point in the boundary of \( \Omega \).

We recall that \( u \in \mathcal{D}^{0,q}(\Omega) \) if and only if \( \sum_{k=1}^{n} u_{kI} \frac{\partial r}{\partial z_k} = 0 \) holds on \( \partial \Omega \). Comparing this boundary condition to the equation in (2.9) and assuming that \( \Omega \) is pseudoconvex, we see that the boundary integral in (2.7) is non-negative. Setting \( \lambda \equiv 0 \) and replacing \( \psi \) by \( 1 - e^{\phi} \), where \( \phi \in C^2(\bar{\Omega}) \) is an arbitrary non-positive function, (2.7) takes the following form

\[
\| \sqrt{1 - e^{\phi} \bar{\partial} u} \|^2 + \| \sqrt{1 - e^{\phi} \bar{\partial}^* u} \|^2 \\
\geq \sum_{|I|=q-1} \sum_{k,l=1}^{n} \int_{\Omega} \left( \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} + \frac{\partial \phi}{\partial z_k} \frac{\partial \phi}{\partial \bar{z}_l} \right) u_{kI} \bar{u}_{lI} e^{\phi} dV \\
+ 2 \text{Re} \left( \sum_{|I|=q-1} \sum_{k=1}^{n} u_{kI} e^{\phi} \frac{\partial \phi}{\partial z_k} d\bar{z}_I, \bar{\partial}^* u \right).
\]

Applying Cauchy-Schwarz inequality and (sc)-(lc) inequality to the last term in the
above estimate, it follows

\[ \| \sqrt{1 - e^\phi \bar{\partial} u} \|^2 + \| \sqrt{1 - e^\phi \tilde{\partial} u} \|^2 \]

\[ \geq \sum'_{|I|=q-1} \sum_{k,l=1}^n \int_\Omega \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} u_k I \bar{u}_l I e^\phi dV. \]

Since the factor \(1 - e^\phi\) is bounded from above by 1, we obtain

\[ \| \bar{\partial} u \|^2 + \| \tilde{\partial} u \|^2 \geq \sum'_{|I|=q-1} \sum_{k,l=1}^n \int_\Omega \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} u_k I \bar{u}_l I e^\phi dV. \]

Now we choose \(\phi = -1 + \frac{|z - p|^2}{D^2}\), where \(p\) is a fixed point in \(\Omega\) and \(D\) is the diameter of \(\Omega\). A straightforward computation gives

\[ \| u \|^2 \leq \frac{D^2 e}{q} (\| \bar{\partial} u \|^2 + \| \tilde{\partial} u \|^2) \]  

(2.10)

for all \(u \in D^{0,q}(\Omega)\). By approximation arguments, using Friedrich mollifiers, it holds that \(D^{0,q}(\Omega)\) is dense in \(\text{Dom}(\bar{\partial}) \cap \text{Dom}(\tilde{\partial}^*)\) with respect to the graph norm \((\| u \|^2 + \| \bar{\partial} u \|^2 + \| \tilde{\partial} u \|^2)^{1/2}\); see for instance [Hör]. This implies that for all \(u \in \text{Dom}(\Box_q)\) it holds that

\[ \| u \|^2 \leq \frac{D^2 e}{q} ((\tilde{\partial}^* \bar{\partial} + \bar{\partial} \tilde{\partial}^*) u, u) = \frac{D^2 e}{q} \| \Box_q u \| \cdot \| u \|. \]

Using (sc)-(lc) inequality again inequality (2.5) with \(C = \frac{D^2 e}{q}\) follows. To wit, the operator \(\Box_q\) is one-to-one and its range equals \(L^2_{0,q}(\Omega)\). Thus there exists an operator \(N_q : L^2_{0,q}(\Omega) \rightarrow \text{Dom}(\Box_q)\), which inverts the complex Laplacian. Moreover, inequality (2.5) says that \(N_q\) is bounded on \(L_{0,q}(\Omega)\). We summarize the basic properties of the \(\bar{\partial}\)-Neumann operator, \(N_q\), in the following theorem, which is due to Hörmander [Hör].
Theorem 2.11. Let $\Omega \subset \subset \mathbb{C}^n$ be a smoothly bounded, pseudoconvex domain. For $1 \leq q \leq n$, there exists a bounded operator $N_q : L^2_{0,q}(\Omega) \rightarrow \text{Dom}(\Box_q) \subset L^2_{0,q}(\Omega)$ such that the following holds

(1) $\Box_q N_q = N_q \Box_q = \text{id}$ on $\text{Dom}(\Box_q)$,

(2) $\bar{\partial} N_q = N_{q+1} \bar{\partial}$ on $\text{Dom}(\bar{\partial})$, $\bar{\partial}^* N_q = N_q \bar{\partial}^*$ on $\text{Dom}(\bar{\partial}^*)$,

(3) if $f \in L^2_{0,q}(\Omega)$, then $f = \bar{\partial} \bar{\partial}^* N_q f + \bar{\partial}^* \bar{\partial} N_q f$, and it holds

\[ \| N_q f \| \leq \frac{D^2 e}{q} \| f \|, \| \bar{\partial} N_q f \| \leq \left( \frac{D^2 e}{q} \right)^\frac{1}{2} \| f \|, \| \bar{\partial}^* N_q f \| \leq \left( \frac{D^2 e}{q} \right)^\frac{1}{2} \| f \| . \]

(4) If $f \in L^2_{0,q}(\Omega)$ is $\bar{\partial}$-closed, then the solution of $\bar{\partial} u = f$, which is orthogonal to the kernel of $\bar{\partial}$, is given by $u = \bar{\partial}^* N_q f$; if $f \in L^2_{0,q}(\Omega)$ is $\bar{\partial}^*$-closed, then the solution of $\bar{\partial}^* u = f$, which is orthogonal to the kernel of $\bar{\partial}^*$, is given by $u = \bar{\partial} N_q f$.

We use these basic properties (1) to (4) throughout the thesis without explicitly referring to Theorem 2.11.

Initial steps on the road to subellipticity

Having established the existence of an $L^2$-solution to the $\bar{\partial}$-Neumann problem, we now turn our attention to a couple of basic facts concerning regularity of such a solution. Again, a good starting point is the basic equality (2.7). Setting $\psi \equiv 1$ and $\lambda \equiv 0$ there, we obtain

\[ \| \bar{\partial} u \|^2 + \| \bar{\partial}^* u \|^2 = \sum_{|I|=q-1}^{n} \sum_{k,l=1}^{n} \int_{\Omega} \frac{\partial^2 r}{\partial z_k \partial \bar{z}_l} u_{k,l} \bar{u}_{i,l} dS + \sum_{|I|=q}^{n} \sum_{t=1}^{n} \| \bar{\partial}_t u \|^2 . \]
for $u \in \mathcal{D}^{0,q}(\Omega)$. Under the assumption that $\Omega$ is pseudoconvex, it follows that

$$
\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 \geq \sum'_{|J|=q} \sum_{l=1}^n \|\bar{\partial}_l u_J\|^2,
$$

(2.12)
i.e. all barred derivatives of the components of $u \in \mathcal{D}^{0,q}(\Omega)$ are under control. This enables us to derive Gårding’s inequality for the Dirichlet form of $\Box_q$ in the interior of $\Omega$. Suppose that $u$ is a smooth $(0,q)$-form compactly supported in $\Omega$. Then integrating by parts twice gives $\|\bar{\partial}_l u_J\|^2 = \|\partial_l u_J\|^2$. This equality combined with the estimates (2.10) and (2.12) implies that

$$
\|u\|_1^2 \lesssim \|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 \quad \text{for} \quad u \in \Lambda^{0,q}_c(\Omega),
$$

(2.13)
where $\|\cdot\|_1$ denotes the usual $L^2$-Sobolev 1-norm. Thus we have an elliptic estimate in the interior, which only reflects that the complex Laplacian is a strongly elliptic operator. We remark that such a uniform estimate does not hold for general $u \in \mathcal{D}^{0,q}(\Omega)$. In fact, an answer to the regularity question for forms in $\mathcal{D}^{0,q}(\Omega)$, which are non-vanishing near $\partial \Omega$, requires a different approach. Here again inequality (2.12) becomes useful, since it suggests that we only need to consider tangential $L^2$-Sobolev norms to get control over all derivatives. In detail: recall that the gradient of $r$ is non-vanishing on the boundary of $\Omega$. Since $r$ is a real-valued function, this implies that $|\bar{\partial}r| = |\sum_{j=1}^n \frac{\partial r}{\partial \bar{z}_j} d\bar{z}_j| \neq 0$ on the boundary of $\Omega$. This reveals that not all barred derivatives are tangential ones. By linear algebra it now follows that near the boundary we can write any first order derivative $D$ as a linear combination of the $\frac{\partial}{\partial \bar{z}_j}$’s and vector fields which are tangent to the boundary of $\Omega$. That is

$$
D = T + \sum_{j=1}^n a_j \frac{\partial}{\partial \bar{z}_j},
$$

(2.14)
where $T(r) = 0$ on $b\Omega$.

Recall that our ultimate goal is to show that under the assumptions of Theorem 1.5 a subelliptic estimate for the $\overline{\partial}$-Neumann problem holds. From the above paragraph it is clear that we need to focus only on a neighborhood of the boundary while trying to control fractional derivatives in tangential directions.

Let $p$ be a point in the boundary of $\Omega$. Then we can choose a neighborhood $U$ of $p$ and a local coordinate system $(x_1, \ldots, x_{2n-1}, r) \in \mathbb{R}^{2n-1} \times \mathbb{R}$, such that the last coordinate is a local defining function of the boundary. We call $(U, (x, r))$ a special boundary chart. We shall denote the dual variable of $x$ by $\xi$, and define $\langle x, \xi \rangle := \sum_{j=1}^{2n-1} x_j \xi_j$.

For $f \in C_0^\infty(U \cap \overline{\Omega})$ we define the tangential Fourier transform of $f$ by

$$\tilde{f}(\xi, r) := \int_{\mathbb{R}^{2n-1}} e^{-2\pi i \langle x, \xi \rangle} f(x, r) dx.$$ 

Via the tangential Bessel potential $\Lambda^s_t$ of order $s$,

$$(\Lambda^s_t f)(x, r) := \int_{\mathbb{R}^{2n-1}} e^{2\pi i \langle x, \xi \rangle} (1 + |\xi|^2)^{\frac{s}{2}} \tilde{f}(\xi, r) d\xi,$$

we can define the tangential $L^2$-Sobolev norm of $f$ of order $s$ by

$$\|f\|^2_s := \|\Lambda^s_t f\|^2 = \int_{-\infty}^{0} \int_{\mathbb{R}^{2n-1}} (1 + |\xi|^2)^s |\tilde{f}(\xi, r)|^2 d\xi dr.$$ 

Clearly, for $s > 0$, this norm is weaker than the full $L^2$-Sobolev norm of order $s$, since it just measures derivatives in the tangential directions.

Let $\omega_1, \ldots, \omega_n$ be an orthonormal basis of $\Lambda^{1,0}(U \cap \overline{\Omega})$ so that $\omega_n = \sqrt{2} \overline{\partial} r$. Then we can extend the definition of the tangential $L^2$-Sobolev norms to $(0, q)$-forms in the obvious way by setting

$$\|\phi\|^2_s := \sum_{|J|=q} \|\phi_J\|^2,$$

where $\phi = \sum_{|J|=q} \phi_J \omega_J \in \Lambda^0_{c}(U \cap \overline{\Omega})$. 

19
In this coordinate system the boundary condition for \( \phi \in D^{0,q}(U \cap \bar{\Omega}) \) translates to 
\( \phi_J \equiv 0 \) on \( \partial \Omega \) if \( n \in J \). That is, if \( n \in J \), then \( \phi_J(x,0) = 0 \), which implies that 
\( \Lambda^*_t \phi_J(x,0) = 0 \), i.e. the boundary condition is invariant under the action of \( \Lambda^*_t \).

In view of (2.14) it becomes even more clear that using the tangential \( L^2 \)-Sobolev norms, instead of the full \( L^2 \)-Sobolev norm, is naturally suited to our problem. If 
\( D_i = \frac{\partial}{\partial x_i} \) for \( i \in \{1, \ldots, 2n - 1\} \) and \( D_{2n} = \frac{\partial}{\partial r} \), we can write 
\[
\|\phi\|_\epsilon^2 \lesssim \sum_{|J|=q}' \left( \sum_{i=1}^{2n} \|D_i \phi_J\|_{\epsilon-1}^2 \right) + \|\phi\|^2 \lesssim \|\phi\|_\epsilon^2 + \sum_{|J|=q}' \|D_{2n} \phi_J\|_{\epsilon-1}^2 + \|\phi\|^2,
\]
since the \( D_i \)'s, for \( i \neq 2n \), are tangential derivatives. The term involving the normal derivative can be estimated with the help of (2.14). That is 
\[
\sum_{|J|=q}' \|D_{2n} \phi_J\|_{\epsilon-1}^2 \lesssim \sum_{|J|=q}' \left( \sum_{i=1}^{2n-1} \|D_i \phi_J\|_{\epsilon-1}^2 + \sum_{l=1}^n \|\bar{\partial} \phi_J\|_{\epsilon-1}^2 \right)
\lesssim \|\phi\|_\epsilon^2 + \sum_{|J|=q}' \sum_{l=1}^n \|\bar{\partial} \phi_J\|^2.
\]
The last term is dominated by \( Q(\phi, \phi) \), see (2.12), therefore it follows
\[
\|\phi\|_\epsilon^2 \lesssim \|\phi\|_\epsilon^2 + Q(\phi, \phi).
\]
Hence it suffices to show that \( \|\phi\|_\epsilon^2 \) is bounded by \( Q(\phi, \phi) \) in order to obtain a subelliptic estimate of order \( \epsilon \) for the \( \bar{\partial} \)-Neumann problem.
CHAPTER 3

BASIC ESTIMATES

In this chapter, we show some basic weighted inequalities for forms in \( \mathcal{D}^{0,q}(\Omega) \). These are derived from equality (2.7) by using plurisubharmonic weight functions, which have a self-bounded complex gradient. We will make extensive use of these inequalities in our proof of subelliptic estimates. We start out with a preliminary estimate, which follows quite easily from identity (2.7):

**Lemma 3.1.** Let \( \Omega \subset \mathbb{C}^n \) be a smoothly bounded, pseudoconvex domain. Let \( \psi, \lambda \in C^2(\overline{\Omega}) \) with \( \psi > 0 \). Then for all \( u \in \mathcal{D}^{0,q}(\Omega) \) it holds

\[
\left\| \sqrt{\psi} \partial u \right\|^2_\lambda + (1 + \frac{1}{\eta})\left\| \sqrt{\psi} \partial^* u \right\|^2_\lambda \geq \sum_{|I|=q-1} \int_{\Omega} \left( \psi \frac{\partial^2 \lambda}{\partial z_k \partial \bar{z}_l} - \frac{\partial^2 \psi}{\partial z_k \partial \bar{z}_l} \right) u_{kI} \bar{u}_{lI} \lambda e^{-\lambda} dV + \sum_{|I|=q-1} \eta \left\| \frac{1}{\sqrt{\psi}} \sum_{k=1}^n \frac{\partial \psi}{\partial z_k} u_{kI} \right\|^2_\lambda
\]

for any positive number \( \eta \).

**Proof.** First recall that \( \Omega \), with defining function \( r \), being pseudoconvex means that for any \( p \in \partial \Omega \) it holds that

\[
\sum_{k,l=1}^n \frac{\partial^2 r}{\partial z_k \partial \bar{z}_l} (p) \xi_k \xi_l \geq 0 \quad \text{whenever} \quad \sum_{k=1}^n \frac{\partial r}{\partial z_k} (p) \xi_k = 0.
\]
Also, recall that Lemma 2.3 says that \( u \in \mathcal{D}^{0,q}(\Omega) \) implies that \( \sum_{k=1}^{n} \frac{\partial r}{\partial z_k} u_k \equiv 0 \) on \( \partial \Omega \) holds for any strictly increasing index set \( I \) of length \( q - 1 \). Thus the boundary integral in (2.7) is non-negative. Therefore we just need to deal with the last term on the right hand side of (2.7). Using Cauchy-Schwarz inequality for that term we get

\[
2 \Re \left( \sum'_{|I|=q-1} \sum_{k=1}^{n} \frac{\partial \psi}{\partial z_k} u_{k|I} d\bar{z}^k, \bar{\partial}_\lambda^* u \right)_\lambda \\
\leq 2 \mathcal{O} \left( \sum'_{|I|=q-1} \frac{1}{\sqrt{\psi}} e^{-\frac{\lambda}{2}} \sum_{k=1}^{n} \frac{\partial \psi}{\partial z_k} u_{k|I} d\bar{z}^k, \sqrt{\psi} e^{-\frac{\lambda}{2}} \bar{\partial}_\lambda^* u \right)_\lambda \\
\leq 2 \| \sum'_{|I|=q-1} \frac{1}{\sqrt{\psi}} \sum_{k=1}^{n} \frac{\partial \psi}{\partial z_k} u_{k|I} d\bar{z}^k \|_\lambda \| \sqrt{\psi} \bar{\partial}_\lambda^* u \|_\lambda \\
\leq \sum'_{|I|=q-1} \eta \| \frac{1}{\sqrt{\psi}} \sum_{k=1}^{n} \frac{\partial \psi}{\partial z_k} u_{k|I} \|_\lambda^2 + \frac{1}{\eta} \| \sqrt{\psi} \bar{\partial}_\lambda^* u \|_\lambda^2,
\]

where the last step follows by the (sc)-(lc) inequality with \( \eta > 0 \). This proves our claim.

\[\square\]

We remark that Lemma 3.1 and the following Proposition 3.4 have been derived by McNeal in [McN3].

In the following, we shall consider only weight functions, which have a self-bounded complex gradient. Recall that a plurisubharmonic function \( \phi \in C^2(\overline{\Omega}) \) is said to have a self-bounded complex gradient, if there exists some \( C > 0 \) such that

\[
| \sum_{k=1}^{n} \frac{\partial \phi}{\partial z_k}(z) \xi_k |^2 \leq C \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l}(z) \xi_k \bar{\xi}_l
\]  

(3.3)

holds for all \( z \in \Omega \) and \( \xi \in \mathbb{C}^n \). Let us emphasize that the constant \( C \) in (3.3) is soft. That is, suppose \( \phi_1 \) satisfies (3.3) with \( C = C_1 > 0 \), let \( C_2 \) be some positive number.
Then $\phi_2 := \frac{C_2}{C_1} \phi_1$ satisfies (3.3) with $C = C_2$, since

\[
\left| \sum_{k=1}^{n} \frac{\partial \phi_2}{\partial z_j} \xi_j \right|^2 = \left( \frac{C_2}{C_1} \right)^2 \left| \sum_{k=1}^{n} \frac{\partial \phi_1}{\partial z_j} \xi_j \right|^2 \leq \frac{C_2}{C_1} \sum_{k,l=1}^{n} \frac{\partial^2 \phi_1}{\partial z_k \partial \bar{z}_l} \xi_k \bar{\xi}_l = C_2 \sum_{k,l=1}^{n} \frac{\partial^2 \phi_2}{\partial z_k \partial \bar{z}_l} \xi_k \bar{\xi}_l.
\]

We shall use the notation $|\partial \phi|_{i|\partial \phi} \leq \sqrt{C}$, when we mean (3.3). Also, we denote the set of plurisubharmonic functions on $\Omega$ by $PSH(\Omega)$.

**Proposition 3.4.** Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded, pseudoconvex domain, and suppose that $\phi \in C^2(\bar{\Omega}) \cap PSH(\Omega)$. If $|\partial \phi|_{i|\partial \phi} \leq 1$, then

\[
\frac{1}{2} \sum_{|I|=q-1}^{n} \int_{\Omega} \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} u_{kI} \bar{u}_{lI} e^{-2\phi} dV \leq \|\bar{\partial} u\|_{2\phi}^2 + 3\|\bar{\partial}^* u\|_{2\phi}^2 (3.5)
\]

holds for all $u \in D^{0,q}(\Omega)$.

**Proof.** To show inequality (3.5), set $\lambda = \phi$, $\psi = e^{-\phi}$ and $\eta = \frac{1}{2}$ in (3.2). Since $\phi$ has a self-bounded complex gradient, we have

\[
- \sum_{k,l=1}^{n} \frac{\partial^2 \psi}{\partial z_k \partial \bar{z}_l} u_{kI} \bar{u}_{lI} = e^{-\phi} \left( \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} u_{kI} \bar{u}_{lI} - \sum_{k=1}^{n} \frac{\partial \phi}{\partial z_k} u_{kI}^2 \right) \geq 0.
\]

Moreover, it holds

\[
-\eta \left| \frac{1}{\sqrt{\psi}} \sum_{k=1}^{n} \frac{\partial \psi}{\partial z_k} u_{kI} \right|^2 = - \frac{1}{2} \left\| \sum_{k=1}^{n} \frac{\partial \phi}{\partial z_k} u_{kI} \right\|_{2\phi}^2 \geq - \frac{1}{2} \int_{\Omega} \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} u_{kI} \bar{u}_{lI} e^{-2\phi} dV.
\]

Combining the last two estimates with inequality (3.2), we obtain

\[
\|\bar{\partial} u\|_{2\phi}^2 + 3\|\bar{\partial}^* u\|_{2\phi}^2 \geq \frac{1}{2} \sum_{|I|=q-1}^{n} \int_{\Omega} \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} u_{kI} \bar{u}_{lI} e^{-2\phi} dV.
\]

$\square$
We remark that inequality (3.5) is one of the key points leading to the subelliptic estimate. In fact, this inequality will be used in chapter 4 enabling us to obtain “good” estimates near the boundary. In the following, we derive a Gårding-like weighted inequality. This inequality is also crucial as it will give us “good” estimates in the interior.

**Proposition 3.6.** Let \( \Omega \subset \mathbb{C}^n \) be a smoothly bounded, pseudoconvex domain, and suppose that \( \phi \in C^2(\bar{\Omega}) \cap PSH(\Omega) \) satisfies \( |\partial \phi| \leq \frac{1}{\sqrt{2}} \). Then for all \( u \in \Lambda^0_{c}(\bar{\Omega}) \),

\[
\|ue^{-\phi}\|_1 \lesssim \|\bar{\partial}u\|_{2\phi} + \|\bar{\partial}^*u\|_{2\phi},
\]

(3.7)

where \( \| \cdot \|_1 \) denotes the \( L^2 \)-Sobolev 1-norm on \( \Omega \).

For the proof of Proposition 3.6 we need to introduce the Hodge-Star Operator \( \star \), that is the map

\[
\star : \Lambda^{p,q}(\bar{\Omega}) \longrightarrow \Lambda^{n-p,n-q}(\bar{\Omega})
\]

defined by \( \psi \wedge \star \varphi = \langle \psi, \varphi \rangle dV \) for \( \psi, \varphi \in \Lambda^{p,q}(\bar{\Omega}) \). The basic properties of the Hodge-Star Operator are summarized in the following lemma.

**Lemma 3.8.**  
(i) \( \star \star = (-1)^{p+q}id \) on \( \Lambda^{p,q}(\bar{\Omega}) \),

(ii) \( |\varphi| = |\star \varphi| \) for \( \varphi \in \Lambda^{p,q}(\bar{\Omega}) \),

(iii) \( \bar{\partial}^* = -\star \bar{\partial} \star \) on \( \Lambda^p_{c}(\bar{\Omega}) \).

A proof of Lemma 3.8 can be found in most textbooks introducing complex analysis in several variables, though the reader is advised to be cautious as there is a common typo through some literature defining \( \star \) by \( \psi \wedge \star \varphi = \langle \psi, \varphi \rangle dV \); however, a reliable source is for instance [Che-Sha], chapter 9.
Proof of Proposition 3.6. Let $u \in \Lambda_c^{0,q}(\bar{\Omega})$. By Gårding’s inequality (2.13), we have

$$
\|ue^{-\phi}\|_1^2 \lesssim \|\partial(ue^{-\phi})\|^2 + \|\partial^*(ue^{-\phi})\|^2 = \|\partial(ue^{-\phi})\|^2 + \|\partial^*_\phi u\|_{2\phi}^2.
$$

Thus we just need to consider the term $\|\partial(ue^{-\phi})\|^2$. For that define $v \in \Lambda_c^{n,n-q}(\bar{\Omega})$ by $v = \ast u$. Here we denote the coefficients of $v$ by $v_J$ for $|J| = n - q$. Then, by Lemma 3.8, it follows

$$
\|\partial(ue^{-\phi})\|^2 = \|\partial^*(ue^{-\phi})\|^2 \lesssim \|\partial^* v\|_{2\phi}^2 + \|[\partial^*, \phi]v\|_{2\phi}^2
$$

$$
= \|\ast \partial \ast v\|_{2\phi}^2 + \sum_{|J| = n - q - 1} \sum_{l=1}^{n} \partial^\phi v_{lJ} d_{\bar{z}J}^f \|_{2\phi}^2
$$

$$
= \|\partial u\|_{2\phi}^2 + \sum_{|J| = n - q - 1} \sum_{l=1}^{n} \partial^\phi v_{lJ} \|_{2\phi}^2
$$

$$
\leq \|\partial u\|_{2\phi}^2 + \sum_{|J| = n - q - 1} \sum_{l=1}^{n} \partial^\phi \frac{v_{k,l} \bar{v}_{lJ} e^{-2\phi}}{\bar{z}k \bar{z}l} dV,
$$

where the last step follows from $\phi$ having a self-bounded complex gradient. Note that $v \in \mathcal{D}^{n,n-q}(\Omega)$, since $v$ is identically zero on the boundary of $\Omega$. Hence we can apply inequality (3.5):

$$
\sum_{|J| = n - q - 1} \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial z_l} v_{k,l} \bar{v}_{lJ} e^{-2\phi} dV \leq 2\|\partial v\|_{2\phi}^2 + 6\|\partial^*_\phi v\|_{2\phi}^2.
$$

Since $|\partial \phi|_{i\partial \partial \phi} \leq \frac{1}{\sqrt{2\phi}}$, it follows that

$$
\|\partial^*_\phi v\|_{2\phi}^2 \leq 2\|\partial^* v\|_{2\phi}^2 + 2\|[\partial^*, \phi]v\|_{2\phi}^2
$$

$$
= 2\|\partial^* v\|_{2\phi}^2 + 2\sum_{|J| = n - q - 1} \sum_{l=1}^{n} \|\partial^\phi v_{lJ}\|_{2\phi}^2
$$

$$
\leq 2\|\partial^* v\|_{2\phi}^2 + \frac{1}{12} \sum_{|J| = n - q - 1} \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial z_l} v_{k,l} \bar{v}_{lJ} e^{-2\phi} dV.
$$

25
Thus we obtain
\[
\sum_{|J|=n-q-1}^{'} \int_{\Omega} \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} v_{k,l} \bar{u}_{l} e^{-2\phi} dV \leq 4 \| \bar{\partial} v \|_{2\phi}^2 + 24 \| \bar{\partial}^* v \|_{2\phi}^2 \\
= 4 \| \bar{\partial}^* u \|_{2\phi}^2 + 24 \| \bar{\partial} u \|_{2\phi}^2
\]
where the second line holds by Lemma 3.8. So we are left with estimating the term \( \| \bar{\partial}^* u \|_{2\phi}^2 \). As before, we just need to commute:
\[
\| \bar{\partial}^* u \|_{2\phi}^2 \lesssim \| \bar{\partial}^* u \|_{2\phi}^2 + \| [\bar{\partial}^*, \phi] u \|_{2\phi}^2 \\
\leq \| \bar{\partial}^* u \|_{2\phi}^2 + \sum_{|I|=q-1}^{'} \int_{\Omega} \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} u_{k,l} e^{-2\phi} dV,
\]
which, again, follows by the self-bounded complex gradient condition of \( \phi \). To finish we use inequality (3.5) again, that is
\[
\sum_{|I|=q-1}^{'} \int_{\Omega} \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} u_{k,l} e^{-2\phi} dV \lesssim \| \bar{\partial} u \|_{2\phi}^2 + \| \bar{\partial}^* u \|_{2\phi}^2.
\]
Collecting all our estimates, we obtain
\[
\| u e^{-\phi} \|_{1}^2 \lesssim \| \bar{\partial} u \|_{2\phi}^2 + \| \bar{\partial}^* u \|_{2\phi}^2
\]
for \( u \in \Lambda_{c}^{0,q}(\Omega) \).

Since the \( L^2 \)-Sobolev 1-norm dominates the \( L^2 \)-norm, (3.7) implies that
\[
\| u e^{-\phi} \|_{2}^2 \lesssim \| \bar{\partial} u \|_{2\phi}^2 + \| \bar{\partial}^* u \|_{2\phi}^2
\]
holds for all \( u \in \Lambda_{c}^{0,q}(\Omega) \). In the following, we show that this inequality is in fact true for all \( u \in D^{0,q}(\Omega) \).
Proposition 3.9. Let $\Omega \subset \subset \mathbb{C}^n$ be a smoothly bounded, pseudoconvex domain, and suppose that $\phi \in C^2(\bar{\Omega}) \cap PSH(\Omega)$ satisfies $|\partial \phi|_{\partial \partial \phi} \leq \frac{1}{\sqrt{2}}$. Then for $u \in \mathcal{D}^{0,q}(\Omega)$ it holds that

$$\|u\|_{2\phi}^2 \lesssim \|\bar{\partial} u\|_{2\phi}^2 + \|\bar{\partial}^* u\|_{2\phi}^2. \quad (3.10)$$

Proof. Set $\psi_t(z) = \phi(z) + t|z|^2$ for $t > 0$. Then $\psi_t$ is strictly plurisubharmonic, since for $\xi \in \mathbb{C}^n$, $z \in \Omega$ it holds

$$\sum_{k,l=1}^n \frac{\partial^2 \psi_t}{\partial z_k \partial \bar{z}_l}(z)\xi_k \bar{\xi}_l = \sum_{k,l=1}^n \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l}(z)\xi_k \bar{\xi}_l + t|\xi|^2.$$

Moreover, we observe that

$$|\sum_{k=1}^n \frac{\partial \psi_t}{\partial z_k}(z)\xi_k|^2 \leq 2|\sum_{k=1}^n \frac{\partial \phi}{\partial z_k}(z)\xi_k|^2 + 2t^2|\sum_{k=1}^n \bar{z}_k \xi_k|^2 \leq 2|\sum_{k=1}^n \frac{\partial \phi}{\partial z_k}(z)\xi_k|^2 + 2t^2|z|^2|\xi|^2.$$

Since $\Omega$ is a bounded domain, we can choose a $t > 0$, such that for all $z \in \Omega$ it holds

$$t^2|z|^2 \leq \frac{1}{24}.$$

Then $|\partial \psi_t|_{\partial \partial \psi_t} \leq 1$, and thus inequality (3.5) holds for $\psi_t$. That is

$$\frac{1}{2} \sum_{|l|=q-1} \int_{\Omega} \sum_{k,l=1}^n \frac{\partial^2 \psi_t}{\partial z_k \partial \bar{z}_l} u_{k \bar{l}} \bar{u}_l e^{-2\psi_t} dV \leq \|\bar{\partial} u\|_{2\psi_t}^2 + 3\|\bar{\partial}^* u\|_{2\psi_t}^2.$$

Note that $e^{-2t|z|^2}$ is bounded from above by 1 and $\phi$ is plurisubharmonic on $\Omega$. Hence it follows

$$\frac{1}{2} \int_{\Omega} t|u|^2 e^{-2\psi_t} dV \leq \|\bar{\partial} u\|_{2\phi}^2 + 3\|\bar{\partial}^* u\|_{2\psi_t}^2 \leq \|\bar{\partial} u\|_{2\phi}^2 + 6\|\bar{\partial}^* u\|_{2\phi}^2 + 6\|[\bar{\partial}^*, (t|z|^2)] u\|_{2\phi}^2.$$
By our choice of \( t \) we can estimate the last term

\[
6||[\bar{\partial}^*, (t|z|^2)]u||^2_{2\psi_t} = 6 \sum'_{|l|=q-1} \| \sum_k^n \frac{\partial (t|z|^2)}{\partial z_k} u_{k,l} \|^2_{2\psi_t} \leq \frac{1}{4} t ||u||^2_{2\psi_t}.
\]

Therefore it holds that

\[
\frac{1}{4} \int_{\Omega} t|u|^2 e^{-2\psi_t} \leq ||\bar{\partial}u||^2_{2\phi} + ||\bar{\partial}^* u||^2_{2\phi}.
\]

Since \( e^{-t|z|^2} \) is bounded from below on \( \Omega \), our claim follows, i.e.

\[
||u||^2_{2\phi} \lesssim ||\bar{\partial}u||^2_{2\phi} + ||\bar{\partial}^* u||^2_{2\phi}.
\]
CHAPTER 4

ESTIMATES FOR $\bar{\partial}^* N_q$

By a compactness estimate for $\bar{\partial}^* N_q$ we mean the following: for all $\eta > 0$ there exists a $C(\eta) > 0$ such that

$$\|\bar{\partial}^* N_q \alpha\| \lesssim \eta \|\alpha\| + C(\eta) \|\alpha\|_{-s}$$  (4.1)

for all $\alpha \in L^2_{0,q}(\Omega)$. Here $\|\cdot\|_{-s}$, $s > 0$, denotes the $L^2$-Sobolev norm of order $-s$, and the constant in $\lesssim$ does not depend on $\alpha$, $\eta$ or $C(\eta)$. This family of estimates (4.1) is indeed equivalent to $\bar{\partial}^* N_q$ being a compact operator from $L^2_{0,q}(\Omega)$ to $L^2_{0,q-1}(\Omega)$; for a proof see for instance [McN3]. We remark that for compactness of $\bar{\partial}^* N_q$ it is sufficient to establish (4.1) for $\bar{\partial}$-closed forms $\alpha \in L^2_{0,q}(\Omega)$. In fact, if $\gamma \in L^2_{0,q}(\Omega)$, then we can write $\gamma = \alpha + \beta$, where $\alpha \in \ker \bar{\partial}$ and $\beta \in (\ker \bar{\partial})^\perp$. However, $\bar{\partial}^* N_q \beta = 0$, since

$$\|\bar{\partial}^* N_q \beta\|^2 = (\bar{\partial}^* N_q \beta, \bar{\partial}^* N_q \beta) = (\bar{\partial} N_{q-1} \bar{\partial}^* N_q \beta, \beta) = 0.$$

Therefore compactness of $\bar{\partial}^* N_q|_{\ker \bar{\partial}}$ implies compactness of $\bar{\partial}^* N_q$.

In this chapter, we derive with the aid of our weighted estimates from chapter 3, two versions of compactness estimates for $\bar{\partial}^* N_q$. We start out with a quantified version of (4.1), i.e. we describe $C(\eta)$ for each $\eta$. 
Since the weight functions \( \{ \phi_\delta \} \) are just defined on \( \Omega \cap U \), where \( U \) is a neighborhood of a given \( p \in b\Omega \) (see hypotheses in Theorem 1.5), we need to restrict our considerations to a so-called approximating subdomain of \( \Omega \), which lies in \( U \).

**Proposition 4.2.** Suppose that \( \Omega \subset \mathbb{C}^n \) is a smoothly bounded domain. Let \( p \) be a point in \( b\Omega \) and suppose that \( b\Omega \cap U \) is pseudoconvex, where \( U \) is a neighborhood of \( p \). Then there exists a smoothly bounded, pseudoconvex domain \( \Omega_a \subset \Omega \cap U \) satisfying the following properties

1. \( b\Omega \cap b\Omega_a \) contains a neighborhood of \( p \) in \( b\Omega \),

2. all points in \( b\Omega_a \setminus b\Omega \) are strongly pseudoconvex.

A proof of Proposition 4.2 can be found in [McN1]. We call such a domain \( \Omega_a \) an approximating subdomain associated to \( (\Omega, p, U) \). The crucial feature, for our current purposes, of such an approximating subdomain \( \Omega_a \) is that \( \Omega_a \subset \subset U \cap \Omega \) is a smoothly bounded, pseudoconvex domain. Therefore we can apply our basic estimates, i.e. inequalities (3.5), (3.7) and (3.10), on \( \Omega_a \) using the \( \phi_\delta \)'s as weight functions. We remark that for using these inequalities a rescaling of the \( \phi_\delta \)'s might be necessary, so that \( |\partial \phi_\delta | \partial \bar{\partial} \phi_\delta \leq \frac{1}{\sqrt{24}} \) holds for all \( \delta > 0 \) sufficiently small.

**Theorem 4.3.** Assume the hypotheses of Theorem 1.5. Let \( \Omega_a \) be an approximating subdomain associated to \( (\Omega, p, U) \). Then there exists a neighborhood \( V \subset \subset U \) of \( p \), such that for \( \alpha \in L^2_{0,q}(\Omega_a) \), \( \bar{\partial} \)-closed and supported in \( V \cap \bar{\Omega}_a \), the following estimate holds:

\[
\| \bar{\partial}^* N_{q}^{\Omega_a} \alpha \|_{\Omega_a}^2 \lesssim \delta^{2q} \| \alpha \|_{\Omega_a}^2 + \delta^{-2q+2} \| \alpha \|_{L^2_{1,\Omega_a}}^2.
\]

(4.4)

The constant in \( \lesssim \) does neither depend on \( \alpha \) nor \( \delta \).
Proof. For notational simplification we write $\| \cdot \|$ for $\| \cdot \|_{\Omega_a}$ and $N_q$ for $N_{q_a}$. Let $W \subset \subset U$ be a neighborhood of $p$, such that $W \cap \Omega \subset \Omega_a$ and $\overline{W} \cap b\Omega_a \subset \subset b\Omega$. Also, let $V \subset \subset W$ be a neighborhood of $p$ and $\alpha \in L^2_{0,q}(\Omega_a)$ be a $\bar{\partial}$-closed form, which is supported in $V \cap \bar{\Omega}_a$. Define the functional $F : \{ e^{-\frac{\phi_s}{2}} \bar{\partial}_{\phi_s}^* u \mid u \in D^{0,q}(\Omega_a) \}, \| \cdot \|_{\phi_s} \rightarrow \mathbb{C}$ by

$$F(e^{-\frac{\phi_s}{2}} \bar{\partial}_{\phi_s}^* u) = (u, \alpha)_{\phi_s}.$$ 

We start out by showing that $F$ is bounded and satisfies the following estimate

$$|F(e^{-\frac{\phi_s}{2}} \bar{\partial}_{\phi_s}^* u)| \lesssim \| e^{-\frac{\phi_s}{2}} \bar{\partial}_{\phi_s}^* u \|_{\phi_s} (\delta^1 \| \alpha \| + \delta^{-1+\epsilon} \| \alpha \|^{q}_{-1}). \tag{4.5}$$

Denote $S_{\delta} = \{ z \in \Omega_a \mid -\delta < r(z) < 0 \}$, where $r$ is defining function of $\Omega$. Let $\chi \in C^\infty_c(W)$ such that $\chi \equiv 1$ on $V$ and $\chi \geq 0$. Recall that the support of $\alpha$ is in $V$.

Then, by the generalized Cauchy-Schwarz inequality, we obtain

$$|F(e^{-\frac{\phi_s}{2}} \bar{\partial}_{\phi_s}^* u)| = |(u, \alpha)_{\phi_s}|$$

$$\leq \| (u, \alpha)_{\phi_s} \|_{W \cap S_{\delta}} + |(u, \alpha)_{\phi_s} \cap \Omega_a \setminus S_{\delta}|$$

$$\leq \| ue^{-\phi_s} \|_{W \cap S_{\delta}} \| \alpha \| + |(\chi u, \alpha)_{\phi_s} \cap \Omega_a \setminus S_{\delta}|$$

$$\lesssim \| ue^{-\phi_s} \|_{W \cap S_{\delta}} \| \alpha \| + \| e^{-\phi_s} \chi u \|_{\Omega_a \setminus S_{\delta}} \| \alpha \|^{q}_{-1}.$$ 

In view of our claim (4.5) we need to estimate the terms $\| ue^{-\phi_s} \|_{W \cap S_{\delta}}$ and $\| e^{-\phi_s} \chi u \|_{\Omega_a \setminus S_{\delta}}$ appropriately.

1. Estimating $\| ue^{-\phi_s} \|_{W \cap S_{\delta}}$. Note that $\phi_s$ has a self-bounded complex gradient on $\Omega_a \subset U \cap \Omega$ by hypothesis (i). Hence inequality (3.5)

$$\sum'_{|I|=q-1} \int_{\Omega_a} \sum_{k,l=1}^{n} \frac{\partial^2 \phi_s}{\partial z_k \partial \bar{z}_l} u_{kl} \bar{u}_{kl} e^{-2\phi_s} dV \lesssim \| \bar{\partial} u \|_{2\phi_s}^2 + \| \bar{\partial}_{\phi_s}^* u \|_{2\phi_s}^2$$

31
holds uniformly for all \( \delta > 0 \) small. The plurisubharmonicity of \( \phi_\delta \) implies now, that

\[
\sum'_{|I|=q-1} \sum_{k,l=1}^{n} \frac{\partial^2 \phi_\delta}{\partial z_k \partial \bar{z}_l} u_{kl} \overline{u} e^{-2\phi_\delta} \, dV \lesssim \|\partial u\|^2_{2\phi_\delta} + \|\partial_{\phi_\delta}^* u\|^2_{2\phi_\delta}
\]

Invoking hypothesis (ii) and noting that inequalities (3.10) and (4.7), it follows

\[
\text{The last estimate holds since } \chi \text{ is supported in } W \text{ and } \partial_{\phi_\delta} \chi = 0 \text{ on } \Omega_a \setminus S_\delta. \text{ By the inequalities (3.10) and (4.7), it follows}
\]

\[
\|e^{-\phi_\delta} \zeta \chi u\|_{1}^2 \lesssim \|\partial u\|^2_{2\phi_\delta} + \|\partial_{\phi_\delta}^* u\|^2_{2\phi_\delta} + \delta^{-2}(\|u\|_{W^{2\phi_\delta}})^2
\]
for all $\delta > 0$ small enough. Using the estimate (4.6) for $\|u\|_{W^{\cap}S_\delta}$, we obtain

$$\|e^{-\phi_\delta \chi}u\|_1^2 \lesssim \delta^{-2+2\epsilon}(\|\bar{\partial}u\|_{2\phi_\delta}^2 + \|\bar{\partial}^*_\phi u\|_{2\phi_\delta}^2),$$

thus we can conclude

$$\left(\|e^{-\phi_\delta \chi}u\|_{1}^{\Omega_a \setminus S_\delta}\right)^2 \lesssim \delta^{-2+2\epsilon}(\|\bar{\partial}u\|_{2\phi_\delta}^2 + \|\bar{\partial}^*_\phi u\|_{2\phi_\delta}^2).$$

(4.8)

Write $u = u_1 + u_2$, where $u_1 \in \ker \bar{\partial}$ and $u_2 \perp \phi_\delta \ker \bar{\partial}$. Note that then $u_1 \in D^{0,q}(\Omega_a)$. Thus, since $\alpha \in \ker \bar{\partial}$, we get, using our estimates (4.6) and (4.8),

$$|(u, \alpha)_{\phi_\delta}| = |(u_1, \alpha)_{\phi_\delta}| \lesssim \|e^{-\phi_\delta \chi}u_1\|_{1}^{\Omega_a \setminus S_\delta}\|\alpha\| + \|e^{-\phi_\delta \chi}u_1\|_{1}^{\Omega_a \setminus S_\delta}\|\alpha\|_{-1} \lesssim \|\bar{\partial}^*_\phi u_1\|_{2\phi_\delta}(\delta^{\epsilon}\|\alpha\| + \delta^{-1+\epsilon}\|\alpha\|_{-1}).$$

However, $u_2 \perp \phi_\delta \ker \bar{\partial}$, therefore we get

$$\|\bar{\partial}^*_\phi u_1\|_{2\phi_\delta}^2 = (\bar{\partial}^*_\phi u_1 + \bar{\partial}^*_\phi u_2, \bar{\partial}^*_\phi u_1 + \bar{\partial}^*_\phi u_2)_{2\phi_\delta} = (\bar{\partial}^*_\phi u_1, \bar{\partial}^*_\phi u_1)_{2\phi_\delta} + 2\text{Re}(\bar{\partial}^*_\phi u_1, \bar{\partial}^*_\phi u_2)_{2\phi_\delta} + (\bar{\partial}^*_\phi u_2, \bar{\partial}^*_\phi u_2)_{2\phi_\delta} = \|\bar{\partial}^*_\phi u_1\|_{2\phi_\delta}^2 + 2\text{Re}(\bar{\partial}e^{-\phi_\delta \chi}u_1, u_2)_{\phi_\delta} + (\bar{\partial}e^{-\phi_\delta \chi}u_2, u_2)_{\phi_\delta} = \|\bar{\partial}^*_\phi u_1\|_{2\phi_\delta}^2.$$

Hence our claimed inequality (4.5) holds:

$$|F(e^{-\phi_\delta \chi}u)| = |(u, \alpha)_{\phi_\delta}| \lesssim \|e^{-\phi_\delta \chi}u\|_{\phi_\delta}(\delta^{\epsilon}\|\alpha\| + \delta^{-1+\epsilon}\|\alpha\|_{-1}).$$

That is, $F$ is bounded linear functional on $(\{e^{-\phi_\delta \chi}u \mid u \in D^{0,q}(\Omega_a)\}, \|\_\phi_\delta\|)$, which is a subset of $L^2_{0,q-1}(\Omega_a, \phi_\delta)$. By Hahn–Banach, $F$ extends to a bounded linear functional.
on $L^2_{0,q-1}(\Omega_a, \phi_\delta)$ with the same bound. The Riesz representation theorem yields, that there exists a unique $v \in L^2_{0,q-1}(\Omega_a, \phi_\delta)$ such that for all $g \in L^2_{0,q-1}(\Omega_a, \phi_\delta)$ holds

$$F(g) = (g, v)_{\phi_\delta},$$

$$\|v\|_{\phi_\delta}^2 \lesssim \delta^{2\epsilon} \|\alpha\|^2 + \delta^{-2+2\epsilon} \|\alpha\|^{-1}_\infty.$$

In particular, we get for all $u \in \mathcal{D}^{0,q}(\Omega_a)$

$$(u, \bar{\partial}(e^{-\frac{\phi_\delta}{2}}v))_{\phi_\delta} = (e^{-\frac{\phi_\delta}{2}}\partial_{\phi_\delta}^* u, v)_{\phi_\delta} = (u, \alpha)_{\phi_\delta}.$$

Note that $\mathcal{D}^{0,q}(\Omega_a)$ is dense in $L^2_{(0,q)}(\Omega_a, \phi_\delta)$. Hence, setting $s = e^{-\frac{\phi_\delta}{2}}v$, it follows that $\bar{\partial}s = \alpha$ holds in the distributional sense and

$$\|s\|^2 \lesssim \delta^{2\epsilon} \|\alpha\|^2 + \delta^{-2+2\epsilon} \|\alpha\|^{-1}_\infty.$$

But the minimal $L^2(\Omega_a)$-solution, $\bar{\partial}^*N_q\alpha$, to the $\bar{\partial}$-problem for $\alpha$ on $\Omega_a$ must also satisfy this estimate; that is

$$\|\bar{\partial}^*N_q\alpha\|^2 \lesssim \delta^{2\epsilon} \|\alpha\|^2 + \delta^{-2+2\epsilon} \|\alpha\|^{-1}_\infty.$$

(4.9)

\[\square\]

Remark. Observe that the only point where the form level $q$ of the $(0, q)$-forms comes into play, is in our hypothesis (ii), that is

$$\sum_{|J|=q-1}^{n} \sum_{k,l=1}^{n} \frac{\partial^2 \phi_\delta}{\partial z_k \partial z_l}(z)u_{k,l}\tilde{u}_{l,k} \geq C\delta^{-2\epsilon} |u|^2$$

(4.10)

holds for $z \in S_\delta$ and $u \in \Lambda^{0,q}(\Omega)$. Notice that this condition on the complex hessian of $\phi_\delta$ near the boundary percolates up the $\bar{\partial}$-complex, i.e. (4.10) implies that

$$\sum_{|J|=q}^{n} \sum_{k,l=1}^{n} \frac{\partial^2 \phi_\delta}{\partial z_k \partial z_l}(z)v_{k,l}\tilde{v}_{l,k} \geq C\delta^{-2\epsilon} |v|^2$$

34
holds for $z \in S_{\delta}$ and $v \in \Lambda^{0,q+1}(\Omega)$. Thus, assuming the hypotheses of Theorem 1.5, by an analogous proof as above, we obtain the following: there exists a neighborhood $V \subset U$ of $p$ such that for all $\beta \in L_{0,q+1}(\Omega_a)$, which are $\bar{\partial}$-closed and supported in $V \cap \bar{\Omega}_a$, the following estimate holds

$$
\|\bar{\partial}^* N_q^\Omega \beta\|_2^2 \lesssim \delta^{2\epsilon} \|\beta\|_{H^s_0(\Omega_a)}^2 + \delta^{-2+2\epsilon} \|\beta\|_{-1,\Omega_a}^2.
$$

These families of estimates, (4.9) and (4.11), are the heart of the matter for our proof of subellipticity. But to convert these estimates on $\bar{\partial}^* N_q^\Omega$ and $\bar{\partial}^* N_{q+1}^\Omega$ to usable estimates on $\mathcal{D}^{0,q}(\Omega)$, we need to have exact regularity of the operator $\bar{\partial}^* \bar{\partial} N_{q}^\Omega$. By exact regularity we mean that $\bar{\partial}^* \bar{\partial} N_{q}^\Omega$ preserves the $L^2$-Sobolev spaces, i.e. that $\bar{\partial}^* \bar{\partial} N_{q}^\Omega$ is a continuous operator from $H^s_{0,q}(\Omega_a)$, $s > 0$, to itself.

Kohn showed in [Koh3], that exact regularity of $\bar{\partial}^* \bar{\partial} N_q^\Omega$ follows from compactness of $N_q^\Omega$ on $L^2_{0,q}(\Omega)$, if $\Omega$ is a smoothly bounded, pseudoconvex domain. It is an easy consequence of the formula

$$
N_q = (\bar{\partial} N_{q-1}) (\bar{\partial}^* N_q) + (\bar{\partial}^* N_{q+1}) (\bar{\partial} N_q),
$$

that compactness of the operators $\bar{\partial}^* N_q$ and $\bar{\partial}^* N_{q+1}$ implies compactness of $N_q$.

In view of (4.1), it looks like we have just shown compactness of $\bar{\partial}^* N_{q}^\Omega$ and $\bar{\partial}^* N_{q+1}^\Omega$. But the estimates (4.9) and (4.11) do not even hold for all $\bar{\partial}$-closed forms in $L^2_{0,q}(\Omega_a)$ and $L^2_{0,q+1}(\Omega_a)$, respectively. This can be easily fixed. In fact, we can show that $N_q^\Omega$ is a compact operator on $L^2_{0,q}(\Omega_a)$ by using a proof very similar to the one of Theorem 4.3. The crucial property of the approximating subdomain $\Omega_a$ for this argument is that $\Omega_a$ is strongly pseudoconvex off the boundary of $\Omega$. In particular, we use Kohn’s result that near a point in the boundary of strong pseudoconvexity a subelliptic estimate of order $\frac{1}{2}$ holds.
Proposition 4.12. Assume that the hypotheses of Theorem 1.5 hold. Let $\Omega_a$ be an approximating subdomain associated to $(\Omega, p, U)$. Then the $\bar{\partial}$-Neumann operator $N_{q}^{\Omega_a}$ is a compact operator on $L_{0,q}^{2}(\Omega_a)$.

Proof. As before, we write $N_q$ for $N_{q}^{\Omega_a}$, and $\| \cdot \|$ for $\| \cdot \|_{\Omega_a}$. We start out with showing that $\bar{\partial}^* N_q$ is a compact operator. By the remark following (4.1) we obtain compactness of $\bar{\partial}^* N_q$, if we can show that for all $\eta > 0$ there exists a $C(\eta) > 0$ such that

$$
\| \bar{\partial}^* N_q \alpha \| \lesssim \eta \| \alpha \| + C(\eta) \| \alpha \|^{\frac{1}{2}}
$$

holds for all $\bar{\partial}$-closed $\alpha \in L_{0,q}^{0}(\Omega_a)$.

Let $\eta > 0$ be given. By our hypotheses there exists a function $\phi_{\eta} \in C^2(\bar{\Omega}_a) \cap PSH(\Omega_a)$ which has a self-bounded complex gradient and satisfies

$$
\sum_{|I|=q-1}^{n} \sum_{k,l=1}^{n} \frac{\partial^2 \phi_{\eta}}{\partial z_k \partial \bar{z}_l}(z) u_{kI} \bar{u}_{lI} \geq \eta^{-2} |u|^2
$$

for $u \in \Lambda^{0,q}(\Omega_a)$ on a strip $S_{\eta'} = \{ z \in \Omega_a \cap \Omega \mid -\eta' < r(z) < 0 \}$ for some $\eta' > 0$ chosen small enough, depending on $\eta$. Here $r$ is a defining function of $\Omega$.

Let $\alpha$ be a $\bar{\partial}$-closed $(0,q)$-form with coefficients in $L^2(\Omega_a)$. Define the linear functional $F : \{ e^{-\frac{\phi_{\eta}}{r}} \bar{\partial}_{\phi_{\eta}}^{*} u \mid u \in \mathcal{D}^{0,q}(\Omega_a) \}, \| \cdot \|_{\phi_{\eta}} \rightarrow \mathbb{C}$ by

$$
F(e^{-\frac{\phi_{\eta}}{r}} \bar{\partial}_{\phi_{\eta}}^{*} u) = (u, \alpha)_{\phi_{\eta}}.
$$

First, we will show that $F$ is a bounded functional satisfying

$$
|F(e^{-\frac{\phi_{\eta}}{r}} \bar{\partial}_{\phi_{\eta}}^{*} u)| \lesssim \| e^{-\frac{\phi_{\eta}}{r}} \bar{\partial}_{\phi_{\eta}}^{*} u \|_{\phi_{\eta}}(\eta \| \alpha \| + C(\eta) \| \alpha \|^{\frac{1}{2}})
$$

(4.13)

for some $C(\eta) > 0$. For that let $\chi \in C^\infty(\Omega_a)$ be a non-negative function such that
\( \chi = 1 \) on \( \Omega_a \setminus S' \) and \( \chi = 0 \) on \( S_{\frac{1}{2}} \). Then

\[
|F(e^{-\frac{\phi}{2}} \bar{\partial}_{\phi} u)| = |(u, \alpha)_{\phi}| = |(u, \alpha)^{S'}| + |(u, \alpha)^{\Omega_a \setminus S'}| 
\leq |(u, \alpha)^{S'}| + |(\chi u, \alpha)^{\Omega_a \setminus S'}| 
\leq \|ue^{-\phi}\|^{|S'|} + \|\chi ue^{-\phi}\|_{\frac{1}{2}} \|\alpha\|_{-\frac{1}{2}}.
\]

where the last line follows by the generalized Cauchy-Schwarz inequality. In view of our claimed inequality (4.13), we need to get some control of the terms \( \|ue^{-\phi}\|^{|S'|} \) and \( \|\chi ue^{-\phi}\|_{\frac{1}{2}} \).

Since \( \phi \in C^2(\Omega_a) \cap PSH(\Omega_a) \) has a self-bounded complex gradient and \( \Omega_a \) is pseudoconvex, we can use inequality (3.5) for estimating \( \|ue^{-\phi}\|^{|S'|} \). That is

\[
\sum' \int_{S_{\frac{1}{2}}} \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} u_{kli} \bar{u}_{kli} e^{-2\phi} dV \lesssim \|\bar{\partial} u\|_{2\phi}^2 + \|\bar{\partial}_{\phi} u\|_{2\phi}^2.
\]

Because \( \phi \) is plurisubharmonic on \( \Omega_a \), we obtain

\[
\sum' \int_{S_{\frac{1}{2}}} \sum_{k,l=1}^{n} \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l} u_{kli} \bar{u}_{kli} e^{-2\phi} dV \lesssim \|\bar{\partial} u\|_{2\phi}^2 + \|\bar{\partial}_{\phi} u\|_{2\phi}^2.
\]

By the blow-up condition on the complex hessian of \( \phi \) on \( S_{\frac{1}{2}} \), it follows

\[
(\|ue^{-\phi}\|^{|S'|})^2 \lesssim \|\bar{\partial} u\|_{2\phi}^2 + \|\bar{\partial}_{\phi} u\|_{2\phi}^2.
\]

In order to estimate \( \|\chi ue^{-\phi}\|_{\frac{1}{2}} \), note that \( \text{supp} \chi \cap b\Omega_a \subset b\Omega_a \setminus b\Omega \) and recall that, by our choice of \( \Omega_a \), we have that \( b\Omega_a \setminus b\Omega \) is strongly pseudoconvex. Thus an subelliptic estimate of order \( \frac{1}{2} \) holds for \( \chi ue^{-\phi} \), that is

\[
\|\chi ue^{-\phi}\|_{\frac{1}{2}}^2 \lesssim \|\bar{\partial}(\chi ue^{-\phi})\|^2 + \|\bar{\partial}_{\phi}(\chi ue^{-\phi})\|^2 
\lesssim C(\chi, \phi)(\|\bar{\partial} u\|_{2\phi}^2 + \|\bar{\partial}_{\phi} u\|_{2\phi}^2 + \|u\|_{2\phi}^2) 
\lesssim C(\chi, \phi)(\|\bar{\partial} u\|_{2\phi}^2 + \|\bar{\partial}_{\phi} u\|_{2\phi}^2),
\]

where \( C(\phi) \) is a constant depending on \( \phi \).
where the last line follows by inequality (3.10).

Now we are set up for proving inequality (4.13). Write \( u = u_1 + u_2 \), where \( u_1 \in \ker \bar{\partial} \) and \( u_2 \perp_{\phi_\eta} \ker \bar{\partial} \). Thus, since \( \alpha \in \ker \bar{\partial} \), we get, using our above estimates for the terms \( \|ue^{-\phi_\eta}\|_{S'} \) and \( \|\chi ue^{-\phi_\eta}\|_2 \),

\[
|F(e^{-\frac{\phi_\eta}{2}} \bar{\partial}_{\phi_\eta}^* u)| = |(u, \alpha)_{\phi_\eta}| = |(u_1, \alpha)_{\phi_\eta}|
\leq \|u_1 e^{-\phi_\eta}\|_{S'} \|\alpha\| + \|\chi u_1 e^{-\phi_\eta}\|_2 \|\alpha\|_{-\frac{1}{2}}
\leq \|\bar{\partial}_{\phi_\eta}^* u_1\|_{2\phi_\eta}(\eta \|\alpha\| + C(\eta) \|\alpha\|_{-\frac{1}{2}}).
\]

Recall that \( \|\bar{\partial}_{\phi_\eta}^* u\|_{2\phi_\eta} = \|\bar{\partial}_{\phi_\eta}^* u_1\|_{2\phi_\eta} \) holds, since \( u_2 \perp_{\phi_\eta} \ker \bar{\partial} \). This implies our claimed inequality (4.13), i.e. it holds

\[
|F(e^{-\frac{\phi_\eta}{2}} \bar{\partial}_{\phi_\eta}^* u)| \leq \|e^{-\frac{\phi_\eta}{2}} \bar{\partial}_{\phi_\eta}^* u\|_{\phi_\eta}(\eta \|\alpha\| + C(\eta) \|\alpha\|_{-\frac{1}{2}}).
\]

Thus \( F \) is a bounded, linear functional on \( \{e^{-\frac{\phi_\eta}{2}} \bar{\partial}_{\phi_\eta}^* u \mid u \in D^{0,q}(\Omega_a)\}, \| \cdot \|_{\phi_\eta} \) , which is a subset of \( L^2_{0,q-1}(\Omega_a, \phi_\eta) \). The Hahn-Banach theorem tells us now, that \( F \) extends to a bounded, linear functional on \( L^2_{0,q-1}(\Omega_a, \phi_\eta) \) with the same bound. By the Riesz representation theorem, there is a unique \( v \in L^2_{0,q-1}(\Omega_a, \phi_\eta) \) such that for all \( g \in L^2_{0,q-1}(\Omega_a, \phi_\eta) \) it holds

\[
F(g) = (g, v)_{\phi_\eta},
\]

\[
\|v\|_{\phi_\eta} \leq \eta \|\alpha\| + C(\eta) \|\alpha\|_{-\frac{1}{2}}.
\]

In particular, we get for all \( u \in D^{0,q}(\Omega_a) \)

\[
(u, \bar{\partial}(e^{-\frac{\phi_\eta}{2}} v))_{\phi_\eta} = (e^{-\frac{\phi_\eta}{2}} \bar{\partial}_{\phi_\eta}^* u, v)_{\phi_\eta} = (u, \alpha)_{\phi_\eta}.
\]

Thus, since \( D^{0,q}(\Omega_a) \) is dense in \( L^2_{0,q}(\Omega_a, \phi_\eta) \), we obtain by setting \( s = e^{-\frac{\phi_\eta}{2}} v \), that
$\partial s = \alpha$ holds in the distributional sense, and

$$\|s\| \lesssim \eta \|\alpha\| + C(\eta)\|\alpha\|_{-\frac{1}{2}}.$$  

The minimal $L^2(\Omega_a)$-solution, $\bar{\partial}^* N_q \alpha$, to the $\bar{\partial}$-problem for $\alpha$ must then also satisfy this estimate, that is

$$\|\bar{\partial}^* N_q \alpha\| \lesssim \eta \|\alpha\| + C(\eta)\|\alpha\|_{-\frac{1}{2}}.$$  

Thus $\bar{\partial}^* N_q$ is a compact operator from $L^2_{0,q}(\Omega_a)$ to $L^2_{0,q-1}(\Omega_a)$. An analogous proof yields the compactness of $\bar{\partial}^* N_{q+1}$. Therefore $N_q$, the $\bar{\partial}$-Neumann operator on $\Omega_a$, is a compact operator on $L^2_{0,q}(\Omega_a)$. 

$\square$
CHAPTER 5

ESTIMATES ON $\mathcal{D}^{0,q}(\Omega)$

In this chapter we convert the families of estimates, (4.9) and (4.11), obtained in chapter 4 to estimates for forms in $\mathcal{D}^{0,q}(\Omega)$. As already mentioned in chapter 4, we need exact regularity to yield for operators related to $N^{\Omega_a}$. We begin with a result of Kohn.

Proposition 5.1. Suppose $\Omega \subset \mathbb{C}^n$ is a smoothly bounded, pseudoconvex domain, such that its $\bar{\partial}$-Neumann operator, $N_q$, is compact on $L^2_{0,q}(\Omega)$. Let $s > 0$, then the following holds

1) if $\beta \in H^s_{0,q}(\Omega)$, then $\|\bar{\partial}^* \bar{\partial} N_q \beta\|_s \lesssim \|\beta\|_s$,

2) if $\beta \in H^s_{0,q-1}(\Omega)$, then $\|N_q \bar{\partial} \beta\|_s \lesssim \|\beta\|_s$.

Here, the constant in $\lesssim$ depends on $s$ but not on $\beta$.

A proof of Proposition 5.1 is contained in [Koh3]. An easy consequence of Proposition 5.1 is the exact regularity of the $L^2$-adjoint operators of $\bar{\partial}^* \bar{\partial} N_q$ and $N_q \bar{\partial}$ in the $L^2$-Sobolev spaces of negative order. In particular, the following holds.
Lemma 5.2. Suppose $\Omega \subset \subset \mathbb{C}^n$ is a smoothly bounded, pseudoconvex domain, such that its $\bar{\partial}$-Neumann operator, $N_q$, is compact on $L^2_{0,q}(\Omega)$. Then, if $\alpha \in \Lambda^{0,q}(\bar{\Omega})$, it follows

$$\|\bar{\partial}^*N_q\alpha\|_{-1} \lesssim \|\alpha\|_{-1},$$ (5.3)

$$\|ar{\partial}\bar{\partial}^*N_q\alpha\|_{-1} \lesssim \|\alpha\|_{-1}.$$ (5.4)

Proof. Let $\alpha \in \Lambda^{0,q}(\bar{\Omega})$. Then

$$\|\bar{\partial}^*N_q\alpha\|_{-1} = \sup\{(\bar{\partial}^*N_q\alpha, \beta) \mid \beta \in H^1_{0,q-1}(\Omega), \|\beta\|_1 \leq 1\}.$$ 

Since $\beta \in H^1_{0,q-1}(\Omega)$ is in $\text{Dom}(\bar{\partial})$, we obtain

$$(\bar{\partial}^*N_q\alpha, \beta) = (N_q\alpha, \bar{\partial}\beta) = (\alpha, N_q\bar{\partial}\beta) \lesssim \|\alpha\|_{-1}\|N_q\bar{\partial}\beta\|_1,$$

where the last step holds by the generalized Cauchy-Schwarz inequality. Now Proposition 5.1, part (2), yields exact regularity for $N_q\bar{\partial}$, in particular it holds

$$\|N_q\bar{\partial}\beta\|_1 \lesssim \|\beta\|_1$$

for all $\beta \in H^1_{0,q-1}(\Omega)$. Thus we have

$$\|\bar{\partial}^*N_q\alpha\|_{-1} \lesssim \sup\{\|\alpha\|_{-1}\|\beta\|_1 \mid \beta \in H^1_{0,q-1}(\Omega), \|\beta\|_1 \leq 1\}$$

$$\leq \|\alpha\|_{-1},$$

which proves (5.3).

The proof of (5.4) is very similar. Since $\alpha = (\bar{\partial}\bar{\partial}^*N_q + \bar{\partial}^*\bar{\partial}N_q)\alpha$, it holds that

$$\|\bar{\partial}\bar{\partial}^*N_q\alpha\|_{-1} = \|\alpha - \bar{\partial}^*\bar{\partial}N_q\alpha\|_{-1} \leq \|\alpha\|_{-1} + \|\bar{\partial}^*\bar{\partial}N_q\alpha\|_{-1},$$
where

\[ \| \bar{\partial}^* \bar{\partial} N_q \alpha \|_{-1} = \sup \{ (\bar{\partial}^* \bar{\partial} N_q \alpha, \beta) | \beta \in H^1_{0,q}(\Omega), \| \beta \|_1 \leq 1 \}. \]

As before, note that \( \beta \in H^1_{0,q}(\Omega) \) is in Dom(\( \bar{\partial} \)). Moreover, since \( \alpha \in \Lambda^{0,q}(\bar{\Omega}) \), it holds that \( \bar{\partial} N_q \alpha = N_{q+1} \bar{\partial} \alpha \). Thus we obtain

\[
(\bar{\partial}^* \bar{\partial} N_q \alpha, \beta) = (\bar{\partial} N_q \alpha, \bar{\partial} \beta) = (\alpha, \bar{\partial}^* N_{q+1} \bar{\partial} \beta) = (\alpha, \bar{\partial}^* \bar{\partial} N_q \beta) \\
\lesssim \| \alpha \|_{-1} \| \bar{\partial}^* \bar{\partial} N_q \beta \|_1.
\]

Now part (1) of Proposition 5.1 tells us that \( \bar{\partial}^* \bar{\partial} N_q \) is exactly regular, i.e.

\[ \| \bar{\partial}^* \bar{\partial} N_q \beta \|_1 \lesssim \| \beta \|_1 \]

holds for all \( \beta \in H^1_{0,q}(\Omega) \). Hence it follows

\[ \| \bar{\partial}^* \bar{\partial} N_q \alpha \|_{-1} \lesssim \sup \{ \| \alpha \|_{-1} \| \beta \|_1 | \beta \in H^1_{0,q}(\Omega), \| \beta \|_1 \leq 1 \} \]

\[ \leq \| \alpha \|_{-1}, \]

which proves (5.4).

Recall that we showed in Proposition 4.12 that the \( \bar{\partial} \)-Neumann operator, \( N^\Omega_{\Omega_a} \), associated to the approximating subdomain \( \Omega_a \) is compact. Therefore, the exact regularity results (5.3) and (5.4) hold for \( N^\Omega_{\Omega_a} \). Now we are ready to derive estimates for forms in \( D^{0,q}(\Omega) \).

**Proposition 5.5.** Assume the hypotheses of Theorem 1.5. Then there exists a neighborhood \( W \subset \subset U \) of \( p \), such that for all sufficiently small \( \eta > 0 \)

\[ \| u \|_{\Omega}^2 \lesssim \frac{\delta^{2\epsilon}}{\eta} (\| \bar{\partial} u \|_{\Omega}^2 + \| \bar{\partial}^* u \|_{\Omega}^2 + \delta^{-2} \| \bar{\partial} u \|_{-1, \Omega}^2) + \eta \delta^{-2} \| u \|_{-1, \Omega}^2 \]  

(5.6)
holds for \( u \in D^{0,q}(\Omega) \) supported in \( W \cap \bar{\Omega} \).

Proof. Recall that, under our hypotheses, Theorem 4.3 says that if \( \Omega_a \) is an approximating subdomain associated to \( (\Omega, p, U) \), then there exists a neighborhood \( \mathcal{U} \subset \subset U \) of \( p \) such that

\[
\| \bar{\partial}^* N^\Omega_{q^a} \alpha \|_{\Omega_a}^2 \lesssim \delta^{2e} \| \alpha \|_{\Omega_a}^2 + \delta^{-2+2e} \| \alpha \|_{-1, \Omega_a}^2, \tag{5.7}
\]

\[
\| \bar{\partial}^* N^\Omega_{q+1} \beta \|_{\Omega_a}^2 \lesssim \delta^{2e} \| \beta \|_{\Omega_a}^2 + \delta^{-2+2e} \| \beta \|_{-1, \Omega_a}^2. \tag{5.8}
\]

hold for all \( \alpha \in L_{0,q}(\Omega_a) \) and \( \beta \in L_{0,q+1}(\Omega_a) \), which are \( \bar{\partial} \)-closed and supported in \( V \cap \bar{\Omega}_a \). For notational ease we denote the \( L^2 \)-norm on \( \Omega_a \) by \( \| . \| \) and write \( N_q \) for the \( \bar{\partial} \)-Neumann operator on \( \Omega_a \).

Let \( W \subset \subset V \subset \subset U \) be a neighborhood of \( p \), and \( \zeta \in C^\infty_c(V), \zeta \geq 0 \) and \( \zeta \equiv 1 \) on \( W \). Let \( u \in D^{0,q}(\Omega) \) be supported in \( W \cap \bar{\Omega} \). Recall that \( \mathcal{U} \) in Theorem 4.3 was chosen such that \( \mathcal{U} \cap b\Omega_a \subset \subset b\Omega \). Thus it follows that \( u \in D^{0,q}(\Omega_a) \). We can write

\[
u = \zeta u = \zeta \bar{\partial} N_{q-1} \bar{\partial}^* u + \zeta \bar{\partial}^* N_{q+1} \bar{\partial} u,
\]

and we get

\[
\| u \|^2 \lesssim \| \zeta \bar{\partial} N_{q-1} \bar{\partial}^* u \|^2 + \| \bar{\partial}^* N_{q+1} \bar{\partial} u \|^2.
\]

Since \( \bar{\partial} u \) is a \( \bar{\partial} \)-closed \((0,q+1)\)-form supported in \( W \subset \subset V \), we can use (5.8) to estimate the last term in the above inequality, i.e.

\[
\| u \|^2 \lesssim \| \zeta \bar{\partial} N_{q-1} \bar{\partial}^* u \|^2 + \delta^{2e} \| \bar{\partial} u \|^2 + \delta^{-2+2e} \| \bar{\partial} u \|_{-1}^2, \tag{5.9}
\]

43
So we are left with estimating \( \| \zeta \bar{\partial} N_{q-1} \bar{\partial}^* u \|^2 \):

\[
\| \zeta \bar{\partial} N_{q-1} \bar{\partial}^* u \|^2 = (\zeta \bar{\partial} N_{q-1} \bar{\partial}^* u, \zeta \bar{\partial} N_{q-1} \bar{\partial}^* u)
= ([\zeta^2, \bar{\partial}] N_{q-1} \bar{\partial}^* u, \bar{\partial} N_{q-1} \bar{\partial}^* u) + (\bar{\partial} \zeta^2 N_{q-1} \bar{\partial}^* u, \bar{\partial} N_{q-1} \bar{\partial}^* u)
= ([\zeta^2, \bar{\partial}] N_{q-1} \bar{\partial}^* u, u - \bar{\partial}^* N_{q+1} \bar{\partial} u) + (\bar{\partial} \zeta^2 N_{q-1} \bar{\partial}^* u, N_q \bar{\partial} \bar{\partial}^* u),
\]

since \( \bar{\partial} N_{q-1} \bar{\partial}^* u = N_q \bar{\partial} \bar{\partial}^* u \) for \( u \in D^{0,q}(\Omega_a) \). By our choice of the cut-off function \( \zeta \) it follows, that the supports of \([\zeta^2, \bar{\partial}]\) and \( u \) are disjoint. Therefore

\[
\| \zeta \bar{\partial} N_{q-1} \bar{\partial}^* u \|^2 \\
\leq |([\zeta^2, \bar{\partial}] N_{q-1} \bar{\partial}^* u, \bar{\partial}^* N_{q+1} \bar{\partial} u)| + (\bar{\partial} N_q (\bar{\partial} \zeta^2 N_{q-1} \bar{\partial}^* u, \bar{\partial}^* u)
\leq \text{(A)} [\zeta^2, \bar{\partial}] \bar{\partial}^* N_q u \| \bar{\partial}^* N_{q+1} \bar{\partial} u \| + \text{(B)} [\bar{\partial} \zeta^2 N_q \bar{\partial}^* u] \| \bar{\partial}^* u \|.
\]

Using (sc)-(lc) inequality, we get

\[
(A) \lesssim \eta [\| \zeta^2, \bar{\partial} \| \bar{\partial}^* N_q u \|^2 + \frac{1}{\eta} \| \bar{\partial}^* N_{q+1} \bar{\partial} u \|^2
\]

for \( \eta > 0 \). Recall that \( \bar{\partial}^* N_{q+1} \) is a bounded map from \( L^2_{(0,q+1)}(\Omega_a) \) to \( L^2_{(0,q)}(\Omega_a) \), and also note that \([\zeta^2, \bar{\partial}]\) is a differential operator of order zero. Using (5.8) again, we obtain

\[
(A) \lesssim \eta [u \|^2 + \frac{1}{\eta} (\delta^2 \| \bar{\partial} u \|^2 + \delta^{-2+2\epsilon} \| \bar{\partial} u \|^2_{-1})
\]

To estimate (B) note that \( \bar{\partial} \zeta^2 N_{q-1} \bar{\partial}^* u \) is a \( \bar{\partial} \)-closed \((0,q)\)-form, which is supported in \( V \). Thus, by our estimate (5.7) on \( \bar{\partial}^* N_q \), it follows

\[
\| \bar{\partial}^* N_q (\bar{\partial} \zeta^2 N_{q-1} \bar{\partial}^* u) \| \lesssim \delta^\epsilon [\bar{\partial} \zeta^2 N_{q-1} \bar{\partial}^* u] + \delta^{-1+\epsilon} \| \bar{\partial} \zeta^2 N_{q-1} \bar{\partial}^* u \|_{-1}
\]

\(\text{(B1)}\)

\[
\| \bar{\partial}^* N_q (\bar{\partial} \zeta^2 N_{q-1} \bar{\partial}^* u) \| \lesssim \delta^{-1+\epsilon} \| \bar{\partial} \zeta^2 N_{q-1} \bar{\partial}^* u \|_{-1}
\]

\(\text{(B2)}\)

44
By commuting \( \bar{\partial} \) and \( \zeta^2 \), we obtain for \((B_1)\):

\[
(B_1) = \| \bar{\partial} \zeta^2 \bar{\partial}^* N_q u \| \leq \| \zeta^2 \bar{\partial} \bar{\partial}^* N_q u \| + \| [\zeta^2, \bar{\partial}] \bar{\partial}^* N_q u \| \\
\lesssim \| \bar{\partial} \bar{\partial}^* N_q u \| + \| \bar{\partial}^* N_q u \| \\
\lesssim \| u \|.
\]

The last line holds, since \( \bar{\partial} \bar{\partial}^* N_q \) is a bounded operator on \( L_{0,q}^2(\Omega_a) \) and \( \bar{\partial}^* N_q \) is a bounded operator from \( L_{0,q}^2(\Omega_a) \) to \( L_{0,q-1}^2(\Omega_a) \).

For estimating \((B_2)\) commute \( \bar{\partial} \) and \( \zeta^2 \) again, that is

\[
(B_2) = \| \bar{\partial} \zeta^2 \bar{\partial}^* N_q u \|_{-1} \leq \| \zeta^2 \bar{\partial} \bar{\partial}^* N_q u \|_{-1} + \| [\zeta^2, \bar{\partial}] \bar{\partial}^* N_q u \|_{-1} \\
\lesssim \| \bar{\partial} \bar{\partial}^* N_q u \|_{-1} + \| \bar{\partial}^* N_q u \|_{-1} \\
\lesssim \| u \|_{-1}.
\]

by (5.3) and (5.4). Combining our estimates for \((B_1)\) and \((B_2)\), we get

\[
(B) = \| \bar{\partial}^* N_q (\bar{\partial} \zeta^2 N_{q-1} \bar{\partial}^* u) \| \| \bar{\partial}^* u \| \\
\lesssim (\delta^4 \| u \| + \delta^{-1+\epsilon} \| u \|_{-1}) \| \bar{\partial}^* u \| \\
\lesssim \eta (\| u \|^2 + \delta^{-2} \| u \|_{-1}^2) + \frac{1}{\eta} \delta^{2\epsilon} \| \bar{\partial}^* u \|^2,
\]

where the last line, again, follows by the \((sc)-(lc)\) inequality, and \( \eta > 0 \).

Recall that we need above estimates on \((A)\) and \((B)\) to get control on the term

\[
\| \zeta \bar{\partial} N_{q-1} \bar{\partial}^* u \|. \quad \text{We now have}
\]

\[
\| \zeta \bar{\partial} N_{q-1} \bar{\partial}^* u \|^2 \leq \frac{\delta^{2\epsilon}}{\eta} (\| \bar{\partial} u \|^2 + \| \bar{\partial}^* u \|^2 + \delta^{-2} \| \bar{\partial} u \|_{-1}^2) + \eta (\| u \|^2 + \delta^{-2} \| u \|_{-1}^2)
\]

45
Combining this last estimate with inequality (5.9), it follows that

$$\|u\|^{2} \lesssim \frac{\delta^{2\epsilon}}{\eta} (\|\bar{\partial}u\|^{2} + \|\bar{\partial}^{*}u\|^{2} + \delta^{-2}\|\bar{\partial}u\|^{2}_{-1}) + \eta(\|u\|^{2} + \delta^{-2}\|u\|^{2}_{-1})$$

holds uniformly for all $\eta > 0$ small. Finally, there is an $\eta_0 > 0$ such that for all $0 < \eta < \eta_0$ we get

$$\|u\|^{2} \lesssim \frac{\delta^{2\epsilon}}{\eta} (\|\bar{\partial}u\|^{2} + \|\bar{\partial}^{*}u\|^{2} + \delta^{-2}\|\bar{\partial}u\|^{2}_{-1}) + \eta\delta^{-2}\|u\|^{2}_{-1}.$$  

Recall that here $\|\cdot\|$ denotes the $L^2$-norm on $\Omega_a$. However, $\Omega_a \subset \Omega$ and $u \in D^{0,q}(\Omega)$ is supported in $W \cap \Omega_a$. Thus we can conclude

$$\|u\|^{2}_{\Omega} \lesssim \frac{\delta^{2\epsilon}}{\eta} (\|\bar{\partial}u\|^{2}_{\Omega} + \|\bar{\partial}^{*}u\|^{2}_{\Omega} + \delta^{-2}\|\bar{\partial}u\|^{2}_{-1,\Omega}) + \eta\delta^{-2}\|u\|^{2}_{-1,\Omega}.$$  

for all $\eta > 0$ sufficiently small. \[\Box\]
CHAPTER 6

SUBELLIPTIC ESTIMATE

In this chapter we show how to derive subelliptic estimates from the family of estimates obtained in Proposition 5.5. We begin with stating the main result of this chapter.

Theorem 6.1. Let \( \Omega \subset \subset \mathbb{C}^n \) be a smoothly bounded domain, \( p \) a point on the boundary of \( \Omega \). Let \( V \) be a special boundary chart near \( p \) such that \( V \cap \partial \Omega \) is pseudoconvex. Suppose that

\[
\|u\|^2 \lesssim \frac{\delta^{2\epsilon}}{\eta} (Q(u, u) + \delta^{-2}\|\bar{\partial}u\|^2_{-1}) + \eta\delta^{-2}\|u\|^2_{-1}
\]  

(6.2)

holds for all \( u \in \mathcal{D}^{0,q}(\Omega) \) supported in \( V \cap \bar{\Omega} \), \( 0 < \eta < \eta_0 \), \( 0 < \delta < \delta_0 \). Let \( W \subset \subset V \) be a neighborhood of \( p \). Then

\[
\|u\|_{c}^2 \lesssim Q(u, u)
\]

holds for all \( u \in \mathcal{D}^{0,q}(\Omega) \) which are supported in \( W \cap \bar{\Omega} \).

For the proof of Theorem 6.1 we use a method which the author learned from Catlin in [Cat]. That is, we introduce a sequence of pseudo-differential operators, which represent a partition of unity in the tangential Fourier transform variables:

Let \( \{p_k(t)\}_{k=0}^\infty \) be a sequence of functions on \( \mathbb{R} \) satisfying the following conditions:
(1) \( \sum_{k=0}^{\infty} p_k^2(t) = 1 \) for all \( t \in \mathbb{R} \),

(2) \( p_0(t) = 0 \) for all \( t \geq 2 \), and \( p_k(t) = 0 \) for all \( t \notin (2^{k-1}, 2^{k+1}) \), \( k \geq 1 \).

We can choose the \( p_k \)'s such that \( |p_k'(t)| \leq C2^{-k} \) holds for all \( k \in \mathbb{N}_0 \), \( t \in \mathbb{R} \) for some \( C > 0 \). Let \( \mathcal{S}(\mathbb{R}^{2n}) \) be the class of Schwartz functions on \( \mathbb{R} \). Denote by \( \mathbb{R}^n_- \) the set \( \{(x_1, \ldots, x_{2n-1}, r) \mid r \leq 0\} \) and \( \mathcal{S}(\mathbb{R}^n_-) \) be the restriction of \( \mathcal{S}(\mathbb{R}^{2n}) \) to \( \mathbb{R}^n_- \).

For \( u \in \mathcal{S}(\mathbb{R}^n_-) \) define the operators \( P_k \) by

\[
\tilde{P}_k u(\xi, r) := p_k(|\xi|) \tilde{u}(\xi, r),
\]

where \( \tilde{u} \) is the tangential Fourier transform, that is

\[
\tilde{u}(\xi, r) = \int_{\mathbb{R}^{2n-1}} e^{-2\pi i (x, \xi)} u(x, r) dx.
\]

On \( (0, q) \)-forms we define the \( P_k \)'s to act componentwise.

One of the crucial features of such operators \( P_k \) is that it makes the tangential Sobolev \( s \)-norm of a function \( u \in \mathcal{S}(\mathbb{R}^n_-) \) comparable to a series involving \( L^2 \)-norms of \( P_k u \). In general, we have:

**Lemma 6.3.** For \( u \in \mathcal{S}(\mathbb{R}^n_-) \) and \( s = s_1 + s_2 \) it holds that

\[
\|u\|_s^2 \sim \sum_{k=0}^{\infty} 2^{2ks_1} \|P_k u\|_{s_2}^2.
\]

In particular, for the case \( s = s_1 \), it follows

\[
\|u\|_s^2 \sim \sum_{k=0}^{\infty} 2^{2ks} \|P_k u\|^2.
\]

**Proof.** Let \( u \in \mathcal{S}(\mathbb{R}^n_-) \), \( s = s_1 + s_2 \). By the definition of the tangential Sobolev \( s \)-norm we have

\[
\|u\|_s^2 = \int_{-\infty}^{0} \int_{\mathbb{R}^{2n-1}} (1 + |\xi|^2)^s |\tilde{u}(\xi, r)|^2 d\xi dr
\]

\[
= \int_{-\infty}^{0} \int_{\mathbb{R}^{2n-1}} (1 + |\xi|^2)^s (\sum_{k=0}^{\infty} p_k^2(|\xi|)) |\tilde{u}(\xi, r)|^2 d\xi dr,
\]

48
since $\sum_{k=0}^{\infty} p_k^2 = 1$. We notice that $(1 + |\xi|^2)^{s_1} \sim 2^{2ks_1}$ as long as $|\xi|$ is in the support of $p_k$. Thus, it follows

$$
\|u\|_{s_2}^2 = \sum_{k=0}^{\infty} \int_{-\infty}^{0} \int_{\mathbb{R}^{2n-1}} (1 + |\xi|^2)^{s_1+s_2} |p_k(|\xi|)\tilde{u}(\xi, r)|^2 d\xi dr \\
\sim \sum_{k=0}^{\infty} 2^{2ks_1} \int_{-\infty}^{0} \int_{\mathbb{R}^{2n-1}} (1 + |\xi|^2)^{s_2} |p_k(|\xi|)\tilde{u}(\xi, r)|^2 d\xi dr \\
= \sum_{k=0}^{\infty} 2^{2ks_1} \|P_k u\|_{s_2}^2.
$$

□

Another crucial property of such operators $P_k$ is that $P_k u \in \mathcal{D}^{0,q}(\Omega)$ whenever $u \in \mathcal{D}^{0,q}(\Omega)$. However, the $P_k$’s do not see the support of $u$, i.e. if $u$ is compactly supported, we can not conclude the same for $P_k u$. Thus inequality (6.2) does not hold for $P_k u$ in general. We shall introduce an appropriately chosen cut-off function $\chi$ and consider $\chi P_k u$. To be able to deal with certain error terms arising from inequality (6.2) applied to $\chi P_k u$, we collect a few facts in the following lemmata.

**Lemma 6.4.** If $f, u \in \mathcal{S}(\mathbb{R}^{2n})$ and $\sigma \in \mathbb{R}$

$$
\sum_{k=0}^{\infty} 2^{2k\sigma} \|[P_k, f]u\|^2 \lesssim \|u\|_{\sigma-1}^2,
$$

where the constant in $\lesssim$ does not depend on $u$.

The proof of Lemma 6.4 can be found in [Cat]. We remark that for $(0,q)$-forms results analog to Lemma 6.3 and Lemma 6.4 hold.

**Lemma 6.5.** Let $D$ be any differential operator of first order with coefficients in $C^\infty(\mathbb{R}^{2n})$ acting on smooth $(0,q)$-forms, let $\chi \in \mathcal{S}(\mathbb{R}^{2n})$ and $\sigma > 0$. Then

$$
\sum_{k=0}^{\infty} 2^{2k\sigma} \|D(\chi P_k u)\|_{-\sigma}^2 \lesssim \|Du\|^2 + \|u\|^2 + \sum_{|l|=q} \|\frac{\partial u}{\partial x_{2n}}\|_{-1}^2.
$$
holds for all \((0, q)\)-forms \(u\) with coefficients in \(\mathcal{S}(\mathbb{R}^{2n})\). Here, the constant in \(\lesssim\) does not depend on \(u\).

**Proof.** Recall that \(\Lambda_i^{-\sigma}\) denotes the tangential Bessel potential of order \(-\sigma\). We obtain by commuting

\[
\sum_{k=0}^{\infty} 2^{2k\sigma} \| D(\chi P_k u) \|_{-\sigma}^2 \leq \sum_{k=0}^{\infty} 2^{2k\sigma} \| D(\chi P_k u) \|_{-\sigma}^2 = \sum_{k=0}^{\infty} 2^{2k\sigma} \| \Lambda_i^{-\sigma} D(\chi P_k u) \|_{-\sigma}^2 \\
\lesssim \sum_{k=0}^{\infty} 2^{2k\sigma} \| \chi \Lambda_i^{-\sigma} D P_k u \|^2 + \sum_{k=0}^{\infty} 2^{2k\sigma} \| [\Lambda_i^{-\sigma} D, \chi] P_k u \|^2.
\]

We note that \([\Lambda_i^{-\sigma} D, \chi]\) is of tangential order \(-\sigma\) and of normal order 0. Therefore, invoking Lemma 6.3, we get

\[
\sum_{k=0}^{\infty} 2^{2k\sigma} \| [\Lambda_i^{-\sigma} D, \chi] P_k u \|^2 \lesssim \sum_{k=0}^{\infty} 2^{2k\sigma} \| P_k u \|_{-\sigma}^2 \sim \| u \|^2.
\]

Similarly, we obtain

\[
\sum_{k=0}^{\infty} 2^{2k\sigma} \| \chi \Lambda_i^{-\sigma} D P_k \|_{-\sigma}^2 \lesssim \sum_{k=0}^{\infty} 2^{2k\sigma} \| \chi D \Lambda_i^{-\sigma} P_k \|^2 + \sum_{k=0}^{\infty} 2^{2k\sigma} \| \chi [D, \Lambda_i^{-\sigma}] P_k \|^2 \\
\lesssim \sum_{k=0}^{\infty} 2^{2k\sigma} \| \chi D \Lambda_i^{-\sigma} P_k \|^2 + \sum_{k=0}^{\infty} 2^{2k\sigma} \| P_k \|_{-\sigma}^2 \\
\lesssim \sum_{k=0}^{\infty} 2^{2k\sigma} \| P_k (\chi D \Lambda_i^{-\sigma} u) \|^2 + \sum_{k=0}^{\infty} 2^{2k\sigma} \| \chi D \Lambda_i^{-\sigma}, u \|_{-\sigma}^2 + \| u \|^2 \\
\lesssim \| \chi D \Lambda_i^{-\sigma} u \|_{-\sigma}^2 + \sum_{k=0}^{\infty} 2^{2k\sigma} \| \chi D \Lambda_i^{-\sigma}, P_k u \|_{-\sigma}^2 + \| u \|^2,
\]

where the last line follows again by Lemma 6.3. To estimate term \((A)\) we write

\[
\chi D = \sum_{\mid l \mid = q} \sum_{j=1}^{2n} a_{ij} \frac{\partial}{\partial x_j},
\]

and commute again

\[
(A) = \| \chi D \Lambda_i^{-\sigma} u \|_{-\sigma}^2 \lesssim \| \Lambda_i^{-\sigma} \chi D u \|_{-\sigma}^2 + \| [\chi D, \Lambda_i^{-\sigma}] u \|_{-\sigma}^2 \\
\lesssim \| D u \|^2 + \sum_{\mid l \mid = q} \sum_{j=1}^{2n} \| a_{ij} \frac{\partial}{\partial x_j} \Lambda_i^{-\sigma} u \|_{-\sigma}^2.
\]
Since $\frac{\partial}{\partial x_j}$ and $\Lambda_t^{-\sigma}$ commute, it follows

$$\sum'_{I=q} \sum_{j=1}^{2n} \| [a_j^I, \Lambda_t^{-\sigma}] u_I \|_2^2 = \sum'_{I=q} \sum_{j=1}^{2n} \| [a_j^I, \Lambda_t^{-\sigma}] \frac{\partial u_I}{\partial x_j} \|_2^2 \lesssim \| u \|^2 + \sum'_{I=q} \| \frac{\partial u_I}{\partial x_{2n}} \|_{-1}^2.$$  

Here, the last estimate holds since $[a_j^I, \Lambda_t^{-\sigma}]$ is of tangential order $-\sigma - 1$ and $\frac{\partial}{\partial x_j}$ is a tangential derivative if $j \in \{1, \ldots, 2n - 1\}$. We are left with estimating the terms $(B_k)$. We first notice that

$$(B_k) = \| [\chi D \Lambda_t^{-\sigma}, P_k] u \|^2 \lesssim \sum'_{I=q} \sum_{j=1}^{2n} \| [a_j^I, \frac{\partial}{\partial x_j} \Lambda_t^{-\sigma}, P_k] u_I \|^2 = \sum'_{I=q} \sum_{j=1}^{2n} \| [a_j^I, P_k] \frac{\partial}{\partial x_j} \Lambda_t^{-\sigma} u_I \|^2.$$  

Lemma 6.4 implies now

$$\sum_{k=0}^{\infty} 2^{2k\sigma} (B_k) \lesssim \sum'_{I=q} \sum_{j=1}^{2n} \| \frac{\partial}{\partial x_j} \Lambda_t^{-\sigma} u_I \|_{\sigma-1}^2 \lesssim \| u \|^2 + \sum'_{I=q} \| \frac{\partial u_I}{\partial x_{2n}} \|_{-1}^2.$$  

Combining all our estimates we end up with the claimed inequality.

$$\sum_{k=0}^{\infty} 2^{2k\sigma} \| D(\chi P_k u) \|_{-\sigma}^2 \lesssim \| Du \|^2 + \| u \|^2 + \sum'_{I=q} \| \frac{\partial u_I}{\partial x_{2n}} \|_{-1}^2.$$  

\[\square\]

Having collected the basic facts concerning the $P_k$’s, we are ready to prove Theorem 6.1.

**Proof of Theorem 6.1.** Let $V$ be a special boundary chart near $p$ such that inequality (6.2) holds, that is

$$\| u \|^2 \lesssim \frac{\delta^{2\varepsilon}}{\eta} (Q(u, u) + \delta^{-2} \| \tilde{\partial} u \|^2_{-1}) + \eta \delta^{-2} \| u \|^2_{-1}$$
holds for all \( u \in \mathcal{D}^{0,q}(\Omega) \) supported in \( V \cap \bar{\Omega} \). Let \( W \subset \subset V \) be a neighborhood of \( p \), and \( u \in \mathcal{D}^{0,q}(\Omega) \) supported in \( W \cap \bar{\Omega} \). Moreover, let \( \chi \in C_c^\infty(V) \) such that \( \chi = 1 \) on \( W \). Then it follows by Lemma 6.3

\[
\| u \|_\epsilon^2 = \| \chi u \|_\epsilon^2 \sim \sum_{k=0}^{\infty} 2^{2k\epsilon} \| P_k (\chi u) \|_2^2 \\
\lesssim \sum_{k=0}^{\infty} 2^{2k\epsilon} \| \chi P_k u \|_2^2 + \sum_{k=0}^{\infty} 2^{2k\epsilon} \| [P_k, \chi] u \|_2^2 \\
\lesssim \sum_{k=0}^{\infty} 2^{2k\epsilon} \| \chi P_k u \|_2^2 + \| u \|_\epsilon^{-1},
\]

where the last line follows by Lemma 6.4. Recall that \( \epsilon \leq \frac{1}{2} \), therefore we have

\[
\| u \|_\epsilon^2 \lesssim \sum_{k=0}^{\infty} 2^{2k\epsilon} \| \chi P_k u \|_2^2 + \| u \|_2^2.
\]

Now inequality (6.2) comes into play. Since \( \chi P_k u \in \mathcal{D}^{0,q}(\Omega) \) is supported in \( V \cap \bar{\Omega} \), it follows that

\[
\| \chi P_k u \|_2^2 \lesssim \frac{\delta^2}{\eta} (Q(\chi P_k u, \chi P_k u) + \delta^{-2} \| \partial(\chi P_k u) \|_{-1}^2) + \eta \delta^{-2} \| \chi P_k u \|_{-1}^2
\]

holds uniformly for all \( k \in \mathbb{N}_0 \) and for all positive \( \delta < \delta_0 \). Let \( k_0 \in \mathbb{N} \) such that \( 2^{-k_0} \leq \delta_0 \). Then we obtain for all \( k \geq k_0 \)

\[
2^{2k\epsilon} \| \chi P_k u \|_2^2 \lesssim \frac{1}{\eta} (Q(\chi P_k u, \chi P_k u) + 2^{2k\epsilon} \| \partial(\chi P_k u) \|_{-1}^2) + \eta 2^{2k(1+\epsilon)} \| \chi P_k u \|_{-1}^2.
\]

Observe that

\[
\sum_{k=0}^{k_0-1} 2^{2k\epsilon} \| \chi P_k u \|_2^2 \leq \sum_{k=0}^{k_0-1} 2^{2k\epsilon} \| u \|_2^2 \lesssim \| u \|_2^2.
\]

Thus we can sum up over \( k \in \mathbb{N}_0 \), obtaining

\[
\sum_{k=0}^{\infty} 2^{2k\epsilon} \| \chi P_k u \|_2^2 \lesssim \frac{1}{\eta} \sum_{k=0}^{\infty} (Q(\chi P_k u, \chi P_k u)) + \frac{1}{\eta} \sum_{k=0}^{\infty} 2^{2k\epsilon} \| \partial(\chi P_k u) \|_{-1}^2 \\
+ \eta \sum_{k=0}^{\infty} 2^{2k(1+\epsilon)} \| \chi P_k u \|_{-1}^2 + \| u \|_2^2.
\]

52
Using Lemma 6.3, we have

\[ \sum_{k=0}^{\infty} 2^{2k(1+\epsilon)} \| \chi P_k u \|_{-1}^2 \lesssim \sum_{k=0}^{\infty} 2^{2k(1+\epsilon)} \| P_k u \|_{-1}^2 \sim \| u \|_{\epsilon}^2. \]

Furthermore, applying Lemma 6.5 with \( \sigma = 0 \) and \( \sigma = 1 \) resp., we get

\[ \sum_{k=0}^{\infty} Q(\chi P_k u, \chi P_k u) + \sum_{k=0}^{\infty} 2^{2k} \| \overline{\partial}(\chi P_k u) \|_{-1}^2 \lesssim Q(u, u) + \| u \|_2^2 + \sum_{|I|=q} \| \frac{\partial u_I}{\partial x_{2n}} \|_{-1}^2. \]

Recall that \( \frac{\partial}{\partial x_{2n}} \) can be expressed as a linear combination of the \( \frac{\partial}{\partial \bar{z}_j} \)'s and a tangential vector field \( T \). Then

\[
\| \frac{\partial u_I}{\partial x_{2n}} \|_{-1}^2 \lesssim \sum_{j=1}^{n} \| \frac{\partial u_I}{\partial \bar{z}_j} \|_{-1}^2 + \| Tu_I \|_{-1}^2 \lesssim \sum_{j=1}^{n} \| \frac{\partial u_I}{\partial \bar{z}_j} \|_2^2 + \| u \|_2^2 \lesssim \| \overline{\partial} u \|_2^2 + \| \overline{\partial}^* u \|_2^2 + \| u \|_2^2 \lesssim Q(u, u).
\]

Thus, by combining our estimates, we obtain

\[ \| u \|_{\epsilon}^2 \lesssim \sum_{k=0}^{\infty} 2^{2k\epsilon} \| \chi P_k u \|^2 + \| u \|_2^2 \lesssim \frac{1}{\eta} Q(u, u) + \eta \| u \|_{\epsilon}^2. \]

Choosing \( \eta > 0 \) small enough, we can absorb the term \( \eta \| u \|_{\epsilon}^2 \) into the right hand side, that is

\[ \| u \|_{\epsilon}^2 \lesssim Q(u, u). \]

\[ \square \]
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