NUMERICAL FORMULATION FOR A DYNAMIC ANALYSIS OF THE PLASTIC BEHAVIOR IN SATURATED GRANULAR SOILS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

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ABSTRACT

The behavior of saturated soils is dominated by the interaction between solid phase and fluid phase. This interaction is particularly strong in dynamic problems and causes the catastrophic loss of strength known as liquefaction. This study presents a numerical framework developed for the dynamic analysis of wave propagation characteristics in saturated granular soils. The description of dynamic behaviors in saturated soils proposed by Hiremath (1987) is formulated in the numerical procedure with the variational formulation by Sandhu (1975a, 1975b, 1976). Singh’s elastic-plastic Cap model (1972) is incorporated as a constitutive relation for the plastic behavior of solid phase. The constitutive model is extended for the prediction of nonlinear elastic behavior of a granular soil. A predictor-corrector form of Newmark method performs direct time domain integration. For the verifications of the numerical procedure, numerical predictions are compared with analytical solutions for the case of an elastic solid and laboratory experimental equipped with a shock tube apparatus for the case of an elastic-plastic solid. Comparison results show that the numerical predictions are in good agreement with analytical solutions. The comparison between numerical predictions and experimental test results shows that numerical predictions agree with experimental observations in the cases for the sand
solid, but with the lead shot solid, numerical predictions experience difficulties by oscillatory responses in predicted pore pressures. To solve oscillatory response problem, solid damping equivalent to the mass coupling is evaluated. Predictions of non-linear elastic-plastic responses with the equivalent solid damping are agree well with experimental observations. The numerical procedure presented in this study can be used for the study of dynamic problems in geotechnical engineering particularly related to liquefaction that is typically caused by the loss of effective stress due to the build-up of pore pressure.
Dedicated to my parents
I wish to express my sincere gratitude to my advisor, Professor William E. Wolfe, for his guidance, support, and encouragement throughout this research. His patient and careful evaluation of the manuscript was of great assistance in the preparation of this dissertation.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

In 1964, Alaska and Niigata earthquakes were milestone events and initiated a keen interest in the study of earthquake effects on soil properties. The liquefaction of saturated granular soils has been recognized as one of the major causes of ground failure during earthquakes. Therefore a considerable effort has been made in understanding liquefaction hazards and the associated circumstances that cause liquefaction. A review of the extent of physical and economical damage from liquefaction during recent earthquakes (Loma Prieta in 1989, Northridge in 1994, Kobe in 1995, and Turkey and Taiwan in 1999) shows that disastrous earthquakes still occur on a regular basis and provides the necessary motivation to continue studies to improve our understanding of liquefaction.

Initial efforts to explain this phenomenon focused on laboratory cyclic triaxial testing of reconstituted sand specimens to make a prediction of liquefaction susceptibility or resistance (Seed and Idriss (1971) and Seed (1979)). A procedure, termed the "simplified procedure," evolved for evaluating liquefaction resistance of
soils and has become the standard of engineering practice (National Center for Earthquake Engineering Research (1996)). This procedure has been successfully adapted in many practical engineering designs and the buildup of pore water pressure and the triggering of soil liquefaction have been reasonably well characterized. However, the prediction of soil behavior during liquefaction and post-liquefaction has been difficult since only the initiation of liquefaction could be predicted by this procedure.

To address concerns with soil deformation and ground failure caused by earthquake motion, dynamics of fluid saturated porous media have been employed to analyze the liquefaction of saturated soils. Based on Biot’s pioneering work (1941, 1956a, 1956b, and 1962), various theories have been proposed to explain the mechanical behavior of saturated soils under dynamic loads and methodologies have been suggested for the analysis of liquefaction. However, due to the complexity of problem in dynamics of fluid saturated soils, practical solutions are possible only through numerical approaches at the present time. Therefore, the need for further research on the development of computational procedures to produce numerical solutions incorporating dynamic theory for fluid saturated soils is clear for in-depth understanding of liquefaction related issues and for providing the prediction of ground behaviors.

To provide a numerical framework developed for modeling wave propagation characteristics of a two-phase system subjected to dynamic ground events, Hiremath’s approach (1987) to describe the dynamic behavior of saturated soils is chosen for the
formulation of governing field equations. This approach is based on the use of a convected coordinate system to combine both the Eulerian and the Lagrangian descriptions. Adopting the Lagrangian description, soil particles move from the original to final configuration within the fixed reference volume. Using the Eulerian description, the flow of fluid is regarded to be relative to a reference volume of the soil so i.e. the flow of fluid is focused on the motion through a stationary control volume.

In the system described, the attenuation of the propagating wave is associated with both the relative dynamic motion between the fluid and the solid and the inelastic deformation of the soil matrix. For the inelastic behavior of soil, a number of constitutive models are available, but the complexities of many of these models can be a major obstacle incorporating them into numerical procedures for wave propagation in a two-phase system. Thus, the focus described in this thesis is on a comprehensive and yet simple constitutive relation with relatively few material parameters. The model chosen is based on an elastic-plastic Cap model proposed by Singh (1972). The numerical procedure is verified by comparing the results with existing analytical solutions and with experimental observations.

1.2 Organization

Chapter 2 presents a review of the literature on various existing dynamic theories for describing fluid-saturated soils. The scope of the literature review also includes constitutive relations for granular soils to describe the inelastic behavior. The description of dynamic behavior of saturated soils proposed by Hiremath (1987) is
presented in Chapter 3. Field equations for dynamics of fluid saturated soils are described in Chapter 4. Variational formulation for self adjoint equations in operator form is carried out in Chapter 5. The formulation of finite element and its spatial and temporal discretization is given in Chapter 6. Chapter 7 includes the description of non-linear plastic constitutive relations. Comparisons of numerical predictions against the results of analytical solutions and laboratory experiments are presented in Chapter 8. Chapter 9 reports the conclusions as well as recommendations for future work.
CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The importance of an accurate description for analyzing soil dynamics problems resulting from transient ground motion has been widely recognized. The theory of porous media has provided a good solution to the dynamic problems of saturated granular soils. A porous medium is an assemblage of solid particles forming a skeleton whose voids are filled with fluid. The overall behavior of a porous medium is controlled by the various interactions between the solid and the fluid. The variations in effective stress and pore pressure due to dynamic excitations are well described in the porous media theory. In recent years, the main efforts to complete the theory have been concerned with the development of constitutive relations for solids in the elastic, plastic, and elastic-plastic ranges as well as with the calculation of initial- and boundary-value problems. This review is focused on the development and the current state of the dynamic theory for porous media and its application in the constitutive relation for granular soils.
2.2 Description of Motion

Various approaches to describe the constituents and their mixture as well as their motion, the formulation of equations for the balance of mass, momentum, and rate of energy, and constitutive relations have been proposed.

Biot (1941, 1956a, 1956b, and 1962) considered harmonic wave propagation through a fluid saturated porous medium with different levels of interaction. Initially, to generalize Terzaghi’s theory of consolidation by extending it to the three-dimensional case, Biot (1941) expressed the change in water content per unit volume of soil mass in terms of intrinsic fluid pressure. Using the volumetric strain, equations of motion for a two-phase system were expressed as a general theory of wave propagation through a saturated porous medium (Biot (1956a, 1956b, and 1962)). For the description of the two-phase system with different states of motion, the saturated porous medium was idealized as two continuous and superimposed phases, the solid skeleton and the pore fluid, which are coupled at different levels.

In Biot’s theory, the equations of motion were derived using the assumption that the kinetic energy of the system per unit volume and a dissipation function were dependent on the relative motion between the fluid and the solid. In the dynamic equations, the independent variables were time and position and the dependent variables were the displacements of solid and fluid. The kinetic energy of the two-phase system per unit volume was assumed to be

\[
K = \frac{1}{2} \left( \rho^{11} \dot{u}_i \dot{u}_i + 2 \rho^{12} \dot{u}_i \dot{w}_i + \rho^{22} \dot{w}_i \dot{w}_i \right)
\]  

(2.1)
where superposed dots denote time derivatives, \( u_i \) and \( w_i \) are the components of the displacement vectors associated with the solid and the fluid respectively. The density terms \( \rho^{11} \) and \( \rho^{22} \) are related to the density of the solid (\( \rho_s \)) and the fluid (\( \rho_f \)) respectively by means of the porosity \( n \)

\[
\rho^{11} = (1-n)\rho_s
\]  
(2.2)

\[
\rho^{22} = n\rho_f
\]  
(2.3)

\( \rho^{12} \) is the added mass or mass coupling between the fluid and solid and \( n \) is the porosity.

Biot also assumed a dissipation function, given as

\[
D = \frac{1}{2} b(u_i - \dot{w}_i)(u_i - \dot{w}_i)
\]  
(2.4)

where the coefficient \( b \) is a viscous coupling term. The Lagrange equations concerning the dissipation function and the kinetic energy can be written as

\[
\frac{\partial}{\partial t} \frac{\partial K}{\partial \dot{u}} + \frac{\partial D}{\partial \dot{w}} = f^{(1)}
\]  
(2.5)

\[
\frac{\partial}{\partial t} \frac{\partial K}{\partial \dot{w}} + \frac{\partial D}{\partial \dot{u}} = f^{(2)}
\]  
(2.6)

where \( f^{(1)} \) and \( f^{(2)} \) denote the generalized forces per unit volume acting on the solid and the fluid, respectively. Biot expressed these forces by the gradients of the solid stresses and of the fluid pressure as

\[
f^{(1)} = \text{div} \tau + \rho_s g
\]  
(2.7)
\[ f^{(2)} = \text{div} \pi + \rho_f g \]  

(2.8)

where \( \tau \) and \( \pi \) are the stresses of the solid and the fluid respectively, and \( g \) is the body force term. Finally inserting the relations (2.1), (2.4), (2.5) and (2.6) into (2.7) and (2.8), the Biot’s equations of motion are given as

\[ \rho^{11} \ddot{u}_i + \rho^{12} \ddot{w}_i + b(\dot{u}_i - \dot{w}_i) = \text{div} \tau + \rho_s g \]  

(2.9)

\[ \rho^{12} \ddot{u}_i + \rho^{22} \ddot{w}_i + b(\dot{u}_i - \dot{w}_i) = \text{grad} \pi + \rho_f g \]  

(2.10)

where \( \pi \) is the pressure in the pore fluid.

There are two interaction terms, viscous and mass interactions, between two phases in Biot’s theory. Biot (1956a) expressed the viscous coupling which was related to Darcy’s coefficient of permeability (\( k \)) as:

\[ b = \frac{\mu(n)^2}{k} \]  

(2.11)

where \( \mu \) is the dynamic fluid viscosity.

The mass interaction or the dynamic interaction is dependent on the relative acceleration of the two phases. When a rigid body moves through a fluid, a force must be applied to accelerate the mass. This force must include additional force required to accelerate the coupled fluid. This additional force works as a resistance to the body’s motion. In equations (2.9) and (2.10), the added mass or apparent mass introduced by Biot (1956a) was an attempt to account for this effect. The relation of apparent mass is given as

\[ \rho^{11} + 2\rho^{12} + \rho^{22} = \rho \]  

(2.12)
The apparent mass increases the mass of the solid in the dynamic equation of motion of the solid if the fluid does not move and it increases the mass of the fluid in the dynamic equation of motion of the fluid if the solid does not move.

Biot (1956a, 1956b and 1962) stated the stress strain relationships for a linear elastic material as

\[\tau_{ij} = E_{ijkl} e_{kl}^{(i)} + \alpha M [\alpha e_{kl}^{(i)} + \zeta] \delta_{ij}\]  

(2.13)

\[\pi = M [\alpha e_{kl}^{(i)} + \zeta]\]  

(2.14)

in which,

\[e_{kl}^{(i)} = \frac{1}{2} [u_{i,j}^{(0)} + u_{j,i}^{(0)}]\]  

(2.15)

where \(\alpha\) and \(M\) are compressibilities of solid and fluid, respectively, \(E_{ijkl}\) denotes elasticity tensor and \(\zeta\) is the relative strain of the fluid.

Truesdell (1960, 1965, and 1969) postulated balance equations for a rational theory of mixtures. Further contributions to the theory of mixture were due to Green (1965, 1967, 1969, and 1970) who used different notions for the balance equations. To apply the principles of continuum mechanics, Truesdell (1960) regarded the mixture as a superposed continuum. To ensure the resulting equations for the motion of mixtures have the same form as the equations of motion of a single constituent, several quantities are introduced such as specific internal energy, heat flux vector, specific energy supply, and energy production density. He assumed that all properties of the mixture were mathematical consequences from the properties of the constituents. As a
result, the sum of the specific internal energy and the kinetic energy per unit volume of the mixture was taken to be equal to the sum of the corresponding quantities associated with the constituents contained in the unit volume. Sandhu (1985) noted that it was difficult to assign a physical meaning to some of these quantities since the mixture could not be regarded as a continuum consisting of a set of non-penetrating particles in motion.

Green (1967) however established that the internal energy per unit mass of the mixture need not equal the sum of the internal energies of the constituents. Rather, he proposed that the total stress and the total heat flux for the mixture should equal the sum of the corresponding quantities for the constituents. Due to difficulty accepting the interpretations associated with some of the quantities which occurred in Truesdell's theory, Green and Naghdi (1969) derived the mass and the momentum balance equations from the material frame invariance of the rate of energy equality. The form of the balance equations in Green’s theory and Truesdell’s theory appears to be the same but the terms have a quite different meaning. Bowen (1976) regarded Green’s theory to be a special case of Truesdell’s theory. However, Sandhu (1985) noted “Bowen’s opinion was incorrect and asserted that the definitions of total stress, heat flux and specific energy supply were natural in Green’s theory but those in Truesdell’s theory were artificial and unnecessary”.

Based on the work of Truesdell (1960, 1965) and Green (1965), Garg (1971, 1974) and Morland (1972) proposed theories to explain the transient response and wave propagation in fluid saturated granular media. Garg et al. (1974) appears to have
been the first to propose an analytical solution to the problem of wave propagation in a
one-dimensional soil column subjected to step loading. Based on Green’s work (1965, 1967), the balance equations are written in the form as

$$\tau_{ij,i} = \rho^{(1)} \dot{u}_i - b[(\dot{u}_j - \dot{w}_j)]$$  \hspace{1cm} (2.16)

$$\pi_{,j} = \rho^{(2)} \ddot{u}_i + b[(\dot{u}_i - \dot{w}_i)]$$  \hspace{1cm} (2.17)

Here $b$ is the same form as Biot’s viscous coupling term in equation (2.11).

Garg et al. (1974) used following constitutive relations as

$$\tau_{ij} = \bar{\alpha} e_{kk}^{(1)} \delta_{ij} + ce_{kk}^{(1)} \delta_{ij} + 2\mu [e_{ij}^{(1)} - \frac{1}{3} e_{kk}^{(1)} \delta_{ij}]$$

$$\pi = ce_{kk}^{(1)} + de_{kk}^{(2)}$$  \hspace{1cm} (2.18)

where $\mu$ is the bulk shear modulus of the solid and $\bar{\alpha}, c, d$ are functions of the volume fractions of constituents. Sandhu (1985) showed that Garg’s constitutive relations were same as Biot (1962) with relations as,

$$\bar{\alpha} = \lambda + \frac{2}{3} \mu + M(\alpha - n)^2$$

$$b = M(n)^2$$

$$c = M(\alpha - n)n$$  \hspace{1cm} (2.19)

Morland (1972) used similar equations of motion as Garg but he expressed the partial density and partial stress tensor decomposed by the volume and surface fractions to introduce the concept of “realistic densities” and “realistic stress tensors”.

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Boer (1996) noted “the decomposition of the partial density could be physically founded however, there is no physical foundation of decomposition of the partial stress tensor.” Based on the Garg’s approach (1974), Morland (1987) provided an analytical solution to wave equations. This analytical solution was extended to include impermeable rigid boundary conditions. However, to solve the equations of motion analytically with boundary conditions, the solution was limited for a simple step loading input in a velocity field. Garg et al. (1974) used a numerical inversion in the Laplace transform solution but Morland et al. (1987) used a decomposition procedure for the inversion.

Gibson (1967) removed the limitation of small strains in Biot’s theory for one-dimensional nonlinear consolidation. In Gibson’s theory, the solid has a reference configuration and the motion of fluid is considered relative to the current configuration of the solid. Gibson assumed the effective stress of the solid depended upon the deformation or the deformation rate and history of deformation of the solid’s skeleton. The fluid pressure was assumed to depend on the fluid density in an isothermal system. Hiremath (1987) extended the one-dimensional quasi-static deformation concepts presented by Gibson (1967) for the case of soils, to three dimensions and to include inertia. According to Hiremath, the momentum balance equations could be obtained in terms of the bulk stress and partial pressure as

\[
\tau_{ij,j} + \rho g = \rho^{(1)} \ddot{u}_i + \rho^{(2)} \ddot{w}_i \\
\pi_{j,j} + \rho^{(2)} g = \rho^{(2)} \ddot{w}_i + \frac{\mu}{k} \bar{n}^{(2)} (\ddot{w}_i - \ddot{u}_i)
\]  

\[ (2.20) \]

\[ (2.21) \]
Comparing with Biot’s equations in (2.9) and (2.10), these balance equations do not have mass coupling terms.

Several approaches have been used to describe the motion of the constituents of a multi-component mixture. Among them, the convected coordinate frame that combines both the Eulerian and the Lagrangian descriptions as described by Hiremath (1987) appears to be an appropriate system of coordinates for the description of the motion of a two-phase system. The motion of soil particles is characterized by a Lagrangian description and the flow of water by an Eulerian description. In the theory, the reference state of the solid is known and the fluid has no specific identification. The coordinate system then moves along with the material volume that contains the same solid particles whose initial unstrained state is known. The fluid is referred with respect to the current position of the solid. The stability problem related to unstable oscillatory responses was one of major difficulties in numerical solutions to the dynamic problems of saturated soils.

2.3 Numerical Solutions

2.3.1 Previous Numerical solutions

Garg (1971) obtained a numerical solution for wave propagation in a fluid saturated porous solid considering relative motions between the fluid and the solid. The solutions were limited to low fluid pressures due to instability in numerical procedures. Ghaboussi and Wilson (1972) also presented numerical solutions to
Biot’s equations of wave propagation for a porous saturated medium without the limitation on pore pressures. Later, Ghaboussi and Wilson (1973) an arbitrary Rayleigh’s type viscous damping term to reflect the intrinsic damping in the saturated soil in addition to the dissipation of relative motion based on permeability and porosity introduced. Simon et al. (1984) presented an analytical solution for the transient response of one-dimensional fluid saturated porous elastic solids. Exact solutions to evaluate the accuracy of finite element models for analyses of porous media were suggested. Later, Simon et al. (1986) evaluated three types of finite element procedures for dynamic analysis of saturated porous media. Extending their earlier work, Simon et al. (1986) formulated higher order, mixed and Hermitean finite elements.

Sandhu et al. (1987) developed a computational program to solve Biot’s equations of motion for linear elastic saturated soils. Sandhu et al. (1988) extended that program to include non-linear dynamic response of saturated soils. Kuo (1990) and Dreger (1995) verified Sandhu’s numerical program (1987) with test results obtained in the laboratory using a shock tube. Song (1999) also verified the Sandhu’s (1988) non-linear numerical program using the laboratory experiment results provided by Dreger (1995). However, during the numerical the verification, unstable oscillatory pore pressure response was experienced. Song (1999) showed that applying the solid damping improved the stability problem. The solid damping was obtained from the proportional damping (Bath (1996)) during time domain integration procedure.
2.3.2 Stability in Numerical Solutions

As Zienkiewicz et al. (1994) pointed out, many computational difficulties prevented numerical solutions to the dynamic problems for saturated soils. For instance, numerical solutions may display unstable oscillatory responses leading to unsatisfactory results. Murad and Loula (1992, 1994) attempted to estimate the numerical error in a finite element analysis of Biot’s theory. They showed that the effect of the time interval used in the time domain integration had greater effect on the convergence rate than the element size. Bardet and Shiv (1995) found that the instability of a two-phase system was more likely to occur in contractive hardening material descriptions than dilative descriptions. In dilative descriptions, the instability also took place but was less catastrophic than in contractive materials. Miga et. al. (1998) performed the stability analysis for a finite element formulation of Biot's equations on two-dimensional bilinear elements. The results showed that the presence and persistence of stable spurious oscillations in the pore pressure were influenced by the ratio of time-step size to the square of the space-step for fixed time-integration weightings and physical property selections. The analysis also revealed that explicitly dominated time integration schemes were not stable for porous saturated media and only became possible through a decoupling of the equilibrium and continuity equations. Zhang and Schrefler (2001) investigated the loss of solution uniqueness of coupled problems in elastic-plastic saturated porous media by means of positiveness of
the second-order work density. They showed that there was a domain of permeability values where the waves could still propagate but stability was lost.

2.3.3 Time Domain Integration

In the numerical analysis of a two phase system, the dynamic system equations are solved in a step-by-step manner through direct integration. Unconditionally stable procedures widely used in the literature to integrate the equations of motion are the Newmark (1959) and the Wilson methods (Bathe, 1996). Many time steps are required in these procedures. Any errors introduced in each time step are carried over to the subsequent time steps. Consequently, even if the errors introduced in each time step are small, the end results may deviate significantly from the actual behavior (Zienkiewicz et al. (1994)).

Improvements have been made by introducing a mixed time integration method (Belytschko and Mullen (1976)) which was intended mainly for non-uniform finite element meshes because different time steps can be used in two or more mesh domains. Because of such a mesh subdivision, it was unnecessary to update nodal values within the entire mesh with the time step controlled by the smallest element. Similarly, Hughes and Liu (1978) introduced an explicit-implicit method that applied different integration schemes within separate mesh partitions. These integration methods were mainly focused on the improvement of effectiveness in numerical computations.
Chang (1997) and Chang et. al. (1998) introduced an integral form of the Newmark method to minimize errors in each time step. This integral algorithm is similar to the so called $\alpha$ method (Bathe, 1996) except that the parameters are no longer constants. Algaard et. al. (2001) modified Chang’s integral algorithm by including an additional term in the estimate for the restoring force to eliminate numerical damping. Because the parameters in the integral algorithms can be varied during computations to minimize the period of elongation and to preserve a spectral radius close to unity for small time steps, more stable and accurate time integration is possible. However, this numerical algorithm is limited to explicit computations.

The Newmark family of methods is second-order accurate. Fung (1997) and Jamal et. al. (2002) attempted third-order or high-order algorithms for the stable computations based on the Newmark method. These higher-order algorithms provide accurate results and control stability characteristics by introducing weighting factors. However, the number of equations to be solved simultaneously increases about six to eight times when compared with the ordinary Newmark method. Moreover, due to the doubled bandwidth, four times the computational effort is necessary (Fung (1997)).

2.4 Constitutive Relations

When the soil is subjected to external loads, strains may not be completely recovered. The portion of un-recovered strain is called plastic strain. Depending upon the particular loading history or condition, the plastic strain can be dominant in the
behavior of soils. Thus, an ability to analyze the behavior of soils in a plastic range is important for providing accurate solutions to wave propagation problems in saturated soils. This section provides a review of constitutive relations for an elastic-plastic behavior of granular soils. However, the review does not aim to cover the whole field of soil plasticity or plastic constitutive models for soils.

2.4.1 Basic Concept of Soil Plasticity Theory

The soil plasticity theory is concerned with the analysis of stresses and strains in the plastic range. In 1773, Coulomb proposed a yield criterion for soils represented by a function or an equation as

$$f(\sigma_y) = k$$

where \( k \) is a condition of yield. In general, the yield surface is dependent on the stress or strain history and the properties of the material. Any point on the yield surface represents a point where yielding can start. The equation \( f = k \) means that the stress state is on the yield surface. If \( f < k \), the stress state is in the elastic domain and \( f > k \) indicates that the stress state is in the plastic domain and plastic deformation can take place. The coefficient \( k \) can be a constant or dependent on stress and strain histories by a hardening rule such as a strain (work) hardening.

The strain-hardening is that when a material is beyond the yield point, an additional stress is required to produce further plastic deformation and the material apparently becomes stronger and more difficult to deform. Because, for an isotropic hardening condition, the strain-hardening is expressed with plastic work done by the
plastic strain, the strain-hardening is commonly referred as a work-hardening in the traditional soil plasticity. For a work-hardening material, the yield surface must change for continued straining beyond the initial yield surface. After initial yielding has started, the condition \( k \) assumes a new value depending on the hardening properties of the material. If the material is unloaded and reloaded, additional yielding will not take place until the new value of \( k \) is reached. Thus the function \( f \) can be looked upon as a loading function.

In order for plastic deformation to take place, the condition for the yield surface is given as

\[
\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} > 0
\]

(2.23)

For work-hardening materials, the condition is given from Drucker’s stability postulate (1951):

\[
d\sigma_{ij} d\varepsilon_{ij}^p > 0 \quad \text{or} \quad d\sigma_{ij} (\varepsilon_{ij}^p - \varepsilon_{ij}^e) > 0
\]

(2.24)

where \( d\sigma_{ij} \), \( d\varepsilon_{ij} \), \( d\varepsilon_{ij}^p \), and \( d\varepsilon_{ij}^e \) are the increments of stress, total strain, plastic strain, and elastic strain, respectively. The plastic strain is obtained from the stress relation as

\[
d\varepsilon_{ij}^p = C_{ijkl} d\sigma_{kl}
\]

(2.25)

where \( C_{ijkl} \) is an elastic-plastic compliance. The plastic strain is given from the definition of an equal plastic strain locus which is called as a plastic potential function \((g)\):
\[ d\varepsilon_{ij}^p = d\lambda \frac{\partial g}{\partial \sigma_{ij}} \]  

(2.26)

where \( \lambda \) is scalar constant. If the yield function is same as the plastic potential function, the plastic strain follows the associated flow rule and the direction of the plastic strain is normal to the yield surface. In this case, the plastic strain is given as

\[ d\varepsilon_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \]  

(2.27)

2.4.2 Constitutive Relations for Granular Soils

The description of the plastic behavior of granular soils was introduced by Drucker and Prager (1952). Drucker and Prager extended the von Mises criterion to a generalized failure function for granular materials. Von Mises’s criterion can be written as:

\[ J_2 = \kappa^2 \]  

(2.28)

where \( \kappa \) is a material constant and \( J_2 \) is the second invariant of stress deviation tensor. By adding the hydrostatic pressure, \( P \), to von Mises criterion in (2.28), Drucker and Prager gave the failure condition as:

\[ \alpha P + J_2^{1/2} = \kappa \]  

(2.29)

where \( \alpha \) is a material parameter. Figure 2.1 shows a schematic representation of the Drucker and Prager failure surface on the stress space with the hydrostatic pressure, \( P \), the negative of the mean stress, which is one-third of the first invariant of the stress
tensor, and the square root of the second invariant of the deviatoric stress tensor, commonly referred as a shear stress term.

Roscoe and Schofield (1963) and Roscoe and Burland (1968) developed a constitutive relation referred to as the Cam-clay model from extensive research on the basic properties of soils. Practical expressions describing volumetric hardening and the concept of residual states were provided. The plastic volumetric strain was evaluated from the compression and rebound indices obtained from an isotropic compression test.

DiMaggio and Sandler (1971) developed an elastic-plastic model for granular materials. Figure 2.2 shows a two-dimensional simplified representation of the DiMaggio and Sandler’s Model. The yield surface is composed of a curved failure surface together with a movable cap that intersects the pressure axis. In this model, three different behavior modes are possible. If, the point representing the stress state lies below the failure surface and to the left of the cap in Figure 2.2, elastic behavior occurs within the yield surface. Ideally plastic behavior takes a place when the stress point lies on the failure surface. The role of the failure surface is to limit the level of the shear stress that a matrial can support without failure. The cap mode behavior occurs when the stress point lies on the movable cap and moves the cap outward or backward. The motion of the cap is related to the plastic volumetric strain through the use of a hardening rule. The cap does not move during purely elastic deformation. When the stress point lies on the failure surface alone, the movement of the cap depends on the resultant plastic volumetric strain. In this case, the cap can contract.
Singh (1972) simplified the DiMaggio and Sandler’s model by specifying that the hardening yield surface does not contract when stresses are reduced even when it has moved out during the loading. Because the yield surface does not contract upon unloading, the deformations during unloading and reloading are purely elastic until stresses reaches to previous cap location. The failure surface is a straight line as a Drucker and Prager’s failure condition. Figure 2.3 shows a two-dimensional representation of Singh’s simplification. The plastic volumetric strain is obtained by compression and rebound indices similar to those used by Roscoe et al. (1963).

The improvement and extension of these models are provided by a number of researchers. To avoid an excessive amount of dilatancy on the Drucker and Prager’s failure surface, Desai et al. (1982) proposed a curved failure surface similar to DiMaggio and Sandler’s model (1971). However, to define the curved failure surface, many additional material parameters are required. To extend DiMaggio and Sandler’s model for a brittle material, such as a rock or a concrete, Bathe et al. (1980) proposed the plane cap model. Similar to the elliptical cap models, this model consists of the Drucker and Prager’s failure surface, a compression plane cap surface, and a tension cut-off limit surface. Kim (1988) attempted to improve Singh’s model based on a series of laboratory tests. He found that the ratio of the major to minor axes of elliptical cap decreased as relative density increased and the difference between compression and rebound indexes is dependent on density.

Other types of constitutive relations for granular soils are also proposed. Lade and Duncan (1975, 1976) and Lade (1977) developed a yield condition for granular
soils which is expressed in terms of the first and third invariants of the stresses. The isotropic hardening behavior was formulated by curved meridians of the yield surface with smooth functions in the principle stress space. Mroz (1967) proposed a multi-surface hardening model for granular soils. A set of loading surfaces allowed to keep track of loading events such as the maximum stress level was postulated. Later, Mroz et al. (1981) extended previous work to an infinite number of nested loading surfaces making the hardening modulus dependent on the ratio of active loading sizes and outer loading surfaces. Desai and his co-workers (Desai and et al, (1989, 1990)) developed a hierarchical single surface model by describing isotropic and kinematical hardening. This model contains all three stress invariants and requires many strength properties of granular materials.

While a large number of constitutive models for granular soils has been developed over the years, many of them are very complex and it is often difficult to relate the model’s variables, parameters, and constants to basic soil properties. Moreover, in many models, little consideration is given to how these models are incorporated to the computational analysis. In addition to the capacity to represent observed stress-strain behaviors, constitutive models for numerical solutions to the dynamic problems of saturated soils require that the constitutive model has to be properly posed with basic physical laws, such as conservation of mass, momentum and energy, in the initial and boundary value problems. Simplicity is also important to capture the rapid variation of a dynamic loading condition in the wave propagation problems. For these reasons described above, Singh’s elastic-plastic Cap model is
chosen for a constitutive model to be incorporated into numerical solutions for wave propagation problems. This model requires few material parameters and is simple and comprehensive. Within a small strain range, this model provides satisfied prediction results (Kim (1988)). More details of this model and its implementation will be discussed in Chapter 7.

2.5 Summary

This chapter provided a review of various theories to describe the dynamic behavior of saturated soils and numerical solutions to the dynamic problems with difficulties confronted in the numerical solutions. For the behavior of soils in the plastic range, a review of elastic plastic constitutive models for granular soils was also included. Among several descriptions of a porous media theory, Hiremath’s method (1987) appears to be an appropriate approach to describe the motion of a two-phase system. To minimize the stability problem related to unstable oscillatory responses in numerical solutions to the dynamic problems of saturated soils, an unconditionally stable implicit method for the time domain integration should be chosen. Limited amount of work for a non-linear plastic response of saturated soils has been done in most numerical solutions. Singh’s elastic-plastic Cap model (1972) is chosen to extend the study to a plastic range of soil behavior.
Figure 2.1 Drucker and Prager’s Failure Surface
Figure 2.2 DiMaggio and Sandler’s Cap Model
Figure 2.3 Singh’s Elastic Plastic Cap Model
CHAPTER 3

DYNAMIC THEORY OF SATURATED SOILS

3.1 Introduction

This Chapter presents the description of dynamic behavior of saturated soils proposed by Hiremath (1987). This description uses a convected coordinate system. The motion of the solid is determined with respect to a fixed reference volume which the motion of the fluid is characterized in relation to the solid. Therefore the fluid itself does not have a reference state. The definitions, relations and formulae for the use of convected coordinates in the mechanics of continua can be referred in Hiremath (1987).

3.2 Definitions

The total mass, \( M \) of the mixture is given by the sum of the constituents masses

\[
M = M^{(1)} + M^{(2)} \quad (3.1)
\]

The superscripts 1 and 2 refer to the solid and fluid, respectively. The total volume, \( V \) of mixture is occupied by the individual volumes \( V^{(1)} \) and \( V^{(2)} \) at any instant of time as

\[
V = V^{(1)} + V^{(2)} \quad (3.2)
\]
which leads to

\[ n^{(1)} + n^{(2)} = 1 \]  \hspace{1cm} (3.3)

with

\[ n^{(1)} = \frac{V^{(1)}}{V} \]
\[ n^{(2)} = \frac{V^{(2)}}{V} \]  \hspace{1cm} (3.4)

The intrinsic densities are defined as

\[ \rho^{(1)} = \frac{M^{(1)}}{V^{(1)}} \]
\[ \rho^{(2)} = \frac{M^{(2)}}{V^{(2)}} \]  \hspace{1cm} (3.5)

The mixture density, \( \rho \) is then defined by

\[ \rho = \frac{M}{V} = \frac{M^{(1)}}{V} + \frac{M^{(2)}}{V} \]
\[ = \frac{M^{(1)}}{V^{(1)}} + \frac{M^{(2)}}{V^{(2)}} = n^{(1)} \rho^{(1)} + n^{(2)} \rho^{(2)} \]
\[ = \rho^{(1)} + \rho^{(2)} \]  \hspace{1cm} (3.6)

Total stress can be expressed as the sum of the partial stresses

\[ t_{ij} = t_{ij}^{(1)} + t_{ij}^{(2)} \]  \hspace{1cm} (3.7)

If the fluid is isotropic, then the fluid partial stress is

\[ t_{ij}^{(2)} = \pi \delta_{ij} \]  \hspace{1cm} (3.8)
where \( \pi \) is the fluid pressure. The intrinsic stresses are defined by their relationship with the partial stresses as

\[
t_{ij}^{(1)} = n^{(1)} t_{ij}^{(1)*} \tag{3.9}
\]

\[
t_{ij}^{(2)} = n^{(2)} t_{ij}^{(2)*} = \pi \delta_{ij} = n^{(2)} \pi^* \delta_{ij} \tag{3.10}
\]

where \( t_{ij}^{(1)*} \) is the intrinsic partial stress for the solid and \( \pi^* \) is the intrinsic fluid pressure.

### 3.3 Kinematics

A material volume \( V_o \) in the reference state \( C_o \), upon motion and deformation, occupies a volume \( V \) in deformed state \( C \). The material volume is followed through its motion so that it encompasses the same set of solid particles. The convected coordinate frame is assumed to be rectangular Cartesian in the reference configuration \( C_o \) and to be curvilinear in any other state.

The strain in the solid, \( \gamma_{ij}^{(1)} \), is defined as

\[
\gamma_{ij}^{(1)} = \frac{1}{2} \left[ u_{i,j}^{(1)} + u_{j,i}^{(1)} + u_{m,i}^{(1)} u_{m,j}^{(1)} \right] \tag{3.11}
\]

Here \( u^{(1)} \) are components of solid displacement referred to the Cartesian system in \( C_o \). This is the same form of the Green’s strain tensor. The fluid is assumed to be isotropic and only infinitesimal strains are considered in fluid deformations. Components of the strain tensor for the fluid, \( \gamma_{kk}^{(2)} \), referred to a Cartesian system in \( C_o \), are

\[
\gamma_{kk}^{(2)} = u_{k,k}^{(2)} \tag{3.12}
\]
3.4 Balance Law

$\rho^{(1)}_0$ and $\rho^{(1)}$ denote the mass densities in the configurations $C_0$ and $C$, respectively, in a volume element containing the same set of solid particles,

$$\rho^{(1)}_0 = n^{(1)}_0 \rho^{(1)*}_0$$  \hspace{1cm} (3.13)

and

$$\rho^{(1)} = n^{(1)} \rho^{(1)*}$$  \hspace{1cm} (3.14)

in which $n^{(1)}_0$ and $n^{(1)}$ are the solid volume fractions in the initial and the current configurations, respectively and a superposed asterisk denotes an intrinsic quantity. The volume element embraces the same solid particles in the two configurations.

This leads to an equation of mass balance in the form,

$$\int_{V_0} \rho^{(1)*}_0 n^{(1)}_0 dV_0 = \int_{V} \rho^{(1)*} n^{(1)} dV$$  \hspace{1cm} (3.15)

Using the convected coordinates (Hiremath (1987)),

$$dV = \sqrt{G} dV_0$$  \hspace{1cm} (3.16)

Substituting in the mass balance equation (3.15), yields

$$\int_{V_0} \left[ \rho^{(1)*}_0 n^{(1)}_0 - \sqrt{G} \rho^{(1)*} n^{(1)} \right] dV_0 = 0$$  \hspace{1cm} (3.17)

Since this hold for arbitrary initial volume $V_0$, the continuity of the solid phase is defined as

$$\rho^{(1)*}_0 n^{(1)}_0 = \sqrt{G} \rho^{(1)*} n^{(1)}$$  \hspace{1cm} (3.18)
In Hiremath’s description, the motion of the fluid is relative to the solid and the fluid itself does not have a reference state. Mass continuity of the fluid constituent is described for solid volume in the current configuration. The fluid mass continuity is given as

\[ \frac{\partial}{\partial t} \left[ \rho \varepsilon \sqrt{G} \right] + \frac{\partial}{\partial x^i} \left[ \rho \varepsilon \sqrt{G} \frac{\partial \varepsilon}{\partial x^i} \left[ \dot{u}_i - \dot{u}_i \right] - \dot{\varepsilon} \right] = 0 \]  

(3.19)

The overall equilibrium instead of the equilibrium of the solid phase is considered. The balance of momentum for the mixture is given as

\[ [S]_m^i + \sqrt{G} \rho \dot{F}^i_m = \sqrt{G} \rho f_1 - \sqrt{G} \rho f_2 \]  

(3.20)

where \( S_m^i \) is stress tensor in the form of First Piola-Kirchoff tensor. The balance of momentum for the fluid is given as

\[ \pi_m + \sqrt{G} \rho f_2 \dot{F}_m^{(2)} = \sqrt{G} \rho f_2 \dot{w}_m + \sqrt{G} D \dot{w}_m - \dot{u}_m \]  

(3.21)

3.5 Specialization on One-dimensional Problem

3.5.1 Kinematical Quantities

For one dimensional problem, the direction \( x_1 \) is simply referred to as \( x \) and associated quantities are denoted by a superscript and subscript \( x \). Defining \( z \) as a single valued and continuously differentiable as many times as required,

\[ z = z(x, t) \]  

(3.22)

with
\[ x(t = 0) = a \]  \hspace{1cm} (3.23)

For motion to be possible,

\[ \left| \frac{\partial z}{\partial x} \right| > 0 \]  \hspace{1cm} (3.24)

The position vectors, \( \mathbf{r} \) of the point \( P_0 \) in the reference state, \( C_0 \) and \( \mathbf{R} \) of the point \( P \) in the deformed state, \( C \) are, respectively,

\[ \mathbf{r} = x \mathbf{e} = a \mathbf{e} \]  \hspace{1cm} (3.25)

\[ \mathbf{R} = z \mathbf{e} \]  \hspace{1cm} (3.26)

The displacement vector, \( \mathbf{u} \) is

\[ \mathbf{u} = \mathbf{R} - \mathbf{r} = u \mathbf{e} \]  \hspace{1cm} (3.27)

\[ u = z - x = z - a \]  \hspace{1cm} (3.28)

The strain tensor is,

\[ \gamma_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} \]  \hspace{1cm} (3.29)

The changes in volume are obtained as,

\[ dV_0 = dx \]  \hspace{1cm} (3.30)

\[ dV = \sqrt{G} dV_0 \]  \hspace{1cm} (3.31)

where,

\[ G = \left( 1 + \frac{\partial u}{\partial x} \right)^2 = \left( 1 + \frac{\partial u}{\partial x} \right)^2 = \left( \frac{\partial z}{\partial x} \right)^2 \]  \hspace{1cm} (3.32)

(3.31) and (3.32) give

\[ dV = \left( 1 + \frac{\partial u}{\partial x} \right) dx \]  \hspace{1cm} (3.33)
3.5.2 Balance Laws

From the continuity equation for the solid in (3.18), the mass continuity for solid in one-dimension is

\[ \rho_0^{(1)r} n_0^{(1)} = \frac{\partial}{\partial x} \rho^{(1)r} n^{(1)} \]  

(3.34)

The continuity for the fluid in (3.19) and the coordinate relation (3.32) give the mass continuity for fluid in one-dimension as

\[ \frac{\partial}{\partial t} \left[ n^{(2)} \rho^{(2)r} \frac{\partial z}{\partial x} \right] + \frac{\partial}{\partial x} \left[ \rho^{(2)r} \left[ w - \ddot{u} \right] \right] = 0 \]  

(3.35)

For one-dimensional analysis, the momentum Balance of the fluid is,

\[ \frac{\partial \pi^*}{\partial x} + \frac{\partial z}{\partial x} \rho^{(2)} \hat{F}^{(2)}_x = \frac{\partial z}{\partial x} \rho^{(2)} \ddot{w} + \frac{\partial z}{\partial x} D \left[ \dot{w} - \ddot{u} \right] \]  

(3.36)

and the momentum Balance of the fluid-saturated solid is

\[ \left[ S^x \right]_x + \frac{\partial z}{\partial x} \rho \hat{F}^x = \frac{\partial z}{\partial x} \rho^{(1)} \ddot{u} + \frac{\partial z}{\partial x} \rho^{(2)} \ddot{w} \]  

(3.37)

3.6 Specialization on Biot’s Theory

Assuming small strain, explicit forms of continuity equations are not required as the changes in density are small. All quantities are referred to the initial state with rectangular Cartesian system as a frame of reference. In that case, the distinction between the contravariant and covariant components disappears. The kinematical relations in (3.11) and (3.12) reduce to
\[
\gamma^{(1)}_{ij} = \frac{1}{2} \left[ u_{i,j} + u_{j,i} \right]
\]

\[
\gamma^{(2)}_{kk} = w_{k,k}
\]  

(3.38)

The momentum balance equations in terms of the bulk stress, \( t_{ij} \) and the partial pressure, \( \pi \) are obtained from (3.20) and (3.21), respectively, as

\[
\tau_{ij,j} + \rho g_{ij} = \rho^{(1)} \ddot{u}_i + \rho^{(2)} \ddot{\omega}_i
\]  

(3.39)

\[
\pi_{,i} + \rho^{(2)} g_i = \rho^{(2)} \ddot{u}_i^{(2)} + D [\ddot{\omega}_i - \dot{u}_i]
\]  

(3.40)

which is often also in the form;

\[
\pi_{,i} + \rho^{(2)} g_i = \rho^{(2)} \ddot{u}_i^{(2)} + \frac{\mu}{k} n^2 [\ddot{\omega}_i - \dot{u}_i]
\]  

(3.41)

The equations (3.39) and (3.40) are the same in the formulation of Biot’s theory. Subtracting (3.40) from (3.39), an equilibrium equation in terms of the partial solid stress is obtained as

\[
t^{(1)}_{ij,j} + \rho^{(1)} g_{ij} = \rho^{(1)} \ddot{u}_i - D [\ddot{\omega}_i - \dot{u}_i]
\]  

(3.42)

Comparing with Biot’s equations, (3.41) and (3.42) do not have the mass coupling terms. For the constitutive relations, Hiremath adopted Biot’s relationships (1956a, 1956b and 1962) for the linear elastic material. These are given as

\[
\tau_{ij} = E_{ijkl} e_{kl}^{(i)} + \alpha M [\alpha e_{kl}^{(i)} + \zeta \delta_{ij}]
\]  

(3.43)

\[
\pi = M [\alpha e_{kl}^{(i)} + \zeta]
\]  

(3.44)
CHAPTER 4

FIELD EQUATIONS FOR DYNAMICS OF SATURATED SOILS

In the chapter, field equations for dynamics of fluid saturated soils are described. The domain of all functions is the Cartesian product, $R \times [0, \infty)$, where $R$ is the closure of the spatial region and $[0, \infty)$ is the positive time interval. To derive field equations, the soil skeleton is assumed to be linear elastic but, later, non-linear plastic behavior is considered during the formulation process. Integral form of the field equations is obtained by Laplace transformation of the equilibrium equations followed by inversion.

4.1 Differential Form of Field Equations

4.1.1 Equilibrium Equations

From (3.39) and (3.40), the equilibrium equations are given in the Hiremath’s description as

\[
\tau_{i,j} + \rho g_i = \rho^{(1)} \ddot{u}_i + \rho^{(2)} \ddot{\hat{w}}_i
\]

(4.1)

\[
\gamma_{i,j} + \rho \gamma^{(2)} g_i = \rho^{(2)} \ddot{\hat{w}}_i + D[\dot{\hat{w}}_i - \dot{u}_i]
\]

(4.2)
where $u$ and $w$ are displacements of the solid and the fluid, respectively.

### 4.1.2 Kinematics

For small deformation from (3.38), the kinematical relations are given as

$$e_{ij} = \frac{1}{2} [u_{i,j} + u_{j,i}] \quad (4.3)$$

$$\xi = w_{i,j} \quad (4.4)$$

where $e_{ij}$ and $\xi$ are components of the symmetric strain tensor of solid and fluid, respectively.

### 4.1.3 Constitutive relations

From (3.43) and (3.44), the constitutive equations for linear elastic fluid saturated soil are given as

$$\tau_{ij} = E_{kij} e_{kl} + \alpha M [\alpha \delta_{kl} e_{kl} + \xi] \delta_{ij} \quad (4.5)$$

$$\pi = M [\alpha e_{kl} + \xi]$$

The inverse relationships are

$$e_{kl} = C_{ijkl} (\delta_{kl} + \alpha \pi \delta_{kl})$$

$$\xi = \pi \left( \frac{1}{M} + \alpha^2 C_{ijkl} \delta_{jkl} \right) - \alpha C_{ijkl} \delta_{kl} \tau_{ij} \quad (4.6)$$
Here $E_{ijkl}$ and $C_{ijkl}$ are components of the elasticity and compliance tensor of the elastic solid, respectively. $\alpha$ is the compressibility of the solid and $M$ is that of the fluid.

4.1.4 Boundary Conditions

The displacement boundary conditions are

$$u_i(x,t) = \hat{u}_i(x,t) \quad \text{on} \quad S_1 \times [0, \infty)$$

$$w_i(x,t) = \hat{w}_i(x,t) \quad \text{on} \quad S_2 \times [0, \infty) \quad (4.7)$$

and the traction boundary conditions are

$$\pi_i(x,t)n_i = \hat{\pi}_i(x,t) \quad \text{on} \quad S_3 \times [0, \infty)$$

$$\tau_{0i}(x,t)n_i = T_i(x,t) = \hat{T}_i(x,t) \quad \text{on} \quad S_4 \times [0, \infty) \quad (4.8)$$

4.1.5 Initial Conditions

The initial conditions for the problems are

$$u(x,0) = u_0(x)$$

$$\dot{u}(x,0) = \dot{u}_0(x)$$

$$w(x,0) = w_0(x)$$
The equations (4.1) through (4.9) completely define the initial-boundary value problem of small deformation of fluid saturated soil.

4.2 Integral Form of the Field Equations

For development of variational principles, the field equations need to be rewritten in the form of convolution product so that the time derivatives are avoided. This can be done through applying Laplace transform and taking inverse after appropriate rearrangement.

4.2.1 Dynamic Equilibrium

Laplace transformation of (4.1) and (42) followed by inversion gives

\[
\begin{align*}
    t^* \tau_{ij,j} + F_i - \rho^{(1)} u_i - \rho^{(2)} w_i &= 0 \\
    t^* \pi_{,j} + G_j - \rho^{(2)} w_i - D[w_i - u_i] &= 0
\end{align*}
\]

where

\[
\begin{align*}
    F_i &= t^* \rho b_i + \rho^{(1)} [u_i (0) - t \cdot \dot{u}_i (0)] \\
    &\quad + \rho^{(2)} [w_i (0) - t \cdot \dot{w}_i (0)] \\
    G_j &= t^* \rho b_j + \rho^{(2)} [w_j (0) + t \cdot \dot{w}_j (0)] \\
    &\quad + D[t \cdot w_i (0) - t \cdot u_i (0)]
\end{align*}
\]

The symbol “*” denotes the convolution product defined as
\[ f^* g = \int_0^t f(\tau) g(t-\tau) d\tau \]  

(4.14)

4.2.2 Kinematics

Equations (4.3) and (4.4) need to be restated in the form

\[ t^* e_{ij} = \frac{1}{2} t^*(u_{i,j} + u_{j,i}) \]  

(4.15)

\[ t^* \xi = t^* w_{ij} \]  

(4.16)

4.2.3 Constitutive Equations

Equations (4.4) to (4.5) must be restated so that the constitutive relations show the dependence of quantities appearing in the equilibrium equations upon corresponding kinematical quantities in them.

\[ t^* \tau_y = t^* E_{ijkl} e_{kl} + t^* M \delta_{ij} (\alpha \delta_{ijkl} e_{kl} + \xi) \]

\[ t^* \pi = t^* M (\alpha \delta_{ij} e_{ij} + \xi) \]

\[ t^* e_{ij} = t^* C_{ijkl} (\tau_{kl} - \alpha \pi \delta_{ij}) \]

\[ t^* \xi = t^* \pi \left( \frac{1}{M} + \alpha^2 C_{ijkl} \delta_{kl} \delta_{ij} \right) - t^* \alpha C_{ijkl} \delta_{kl} \tau_{ij} \]  

(4.17)
CHAPTER 5

VARIATIONAL PRINCIPLES

For the development of a numerical procedure, the variational formulation is carried out in the chapter. The formulation is based on the principles of variational theory developed by Sandhu and Saalam (1975a) and Sandhu (1975b, 1976). To transform the problem of wave equations for the fluid-saturated soils into an equivalent variational problem, a set of displacement variables is regarded as multiple in the admissible space satisfying the field equations, the initial and boundary conditions to the problem.

5.1 Preliminaries

5.1.1 Boundary Value Problem

The linear vector space $W$ consisting of all admissible states is referred to as the product space

$$W = W_1 \times W_2 \times \cdots \times W_n$$

(5.1)

where $W_i$ is a subspace whose elements represent the admissible state for a specific field variable, $u$. Consider the boundary value problem given as
\[ A(u) = f \text{ on } R \times [0, \infty) \]
\[ C(u) = g \text{ on } \partial R \times [0, \infty) \]  \hspace{1cm} (5.2)

where \( R \) is an open connected region of interest, \( \partial R \) is the boundary of \( R \), and \( A, C \) are linear operator matrices. The field operator \( A \) and the boundary operator \( C \) are bounded and defined such that

\[ A : W_R \to V_R \]
\[ C : W_{\partial R} \to V_{\partial R} \]  \hspace{1cm} (5.3)

\( V_R, V_{\partial R} \) are linear vector spaces defined on the regions indicated by the subscripts and \( W_R, W_{\partial R} \) are subsets in \( V_R, V_{\partial R} \), respectively. Throughout, \( A \) and \( C \) are assumed to be linear so that

\[ A(\alpha u + \beta v) = \alpha A(u) + \beta A(v) \quad u, v \in W_R \]
\[ C(\alpha u + \beta v) = \alpha C(u) + \beta C(v) \quad u, v \in W_{\partial R} \]  \hspace{1cm} (5.4)

where \( \alpha, \beta \) are arbitrary scalars. Solution of boundary value problem implies determination of \( u \in W_R \) for given \( f \in V_R \) and \( g \in V_{\partial R} \) subject to the satisfaction of equation (5.1) – (5.4).

5.1.2 Bilinear Mapping

A bilinear mapping \( B : W \times V \to S \), where \( W, V, S \) are linear vector spaces, for given \( w \in W, v \in V \), is defined as a function to assign an element in \( S \) corresponding to an ordered pair \((w, v)\). \( B \) is said to be bilinear if
\[ B(\alpha w_1 + \beta w_2, v) = \alpha B(w_1, v) + \beta B(w_2, v) \]
\[ B(w, \alpha v_1 + \beta v_2) = \alpha B(w, v_1) + \beta B(w, v_2) \]

where \( \alpha, \beta \) are scalars. The notation can be used as

\[ B_R(w, v) = \langle w, v \rangle_R \]

\( B \) is said to be non-degenerate if

\[ \langle w, v \rangle_R = 0 \quad w \in W \text{ if and only if } v = 0 \]

For \( W = V \), \( B \) is symmetric if

\[ \langle w, v \rangle_R = \langle v, w \rangle_R \]

5.1.3 Self-Adjoint Operator

Let \( A : V \to W \) be an operator on the linear vector space \( V \) defined on spatial region \( R \). Operator \( A^* \) is said to be adjoint of \( A \) with respect to a bilinear mapping \( \langle \cdot, \cdot \rangle_R : W \times W \to S \) if

\[ \langle w, Av \rangle_R = \langle v, A^*w \rangle_R + D_{cr}(v, w) \]

for all \( w \in W \) and \( v \in V \). Here, \( D_{cr}(v, w) \) represents quantities associated with the boundary \( \partial R \) of \( R \). If \( A = A^* \), then \( A \) is said to be self-adjoint. In particular, a self-adjoint operator \( A \) on \( V \) is symmetric with respect to the bilinear mapping if \( V = W \) and

\[ \langle w, Av \rangle_R = \langle v, Aw \rangle_R \]
5.1.4 Gateaux Differential of a Function

Considering $V$ and $S$ as linear vector spaces, the Gateaux differential of a continuous function $F : V \rightarrow S$ is defined as

$$\Delta_F F(u) = \lim_{\lambda \to 0} \frac{1}{\lambda} \left[ F(u + \lambda v) - F(u) \right]$$

(5.12)

provided the limit exists. $v$ is referred to as the ‘path’ and $\lambda$ is a scalar. For $u, v \in V$, $u + \lambda v \in V$. Equation (5.12) can be equivalently written as

$$\Delta_F F(u) = \frac{d}{d\lambda} F(u + \lambda v)|_{\lambda=0}$$

(5.13)

5.1.5 Basic Variational Principles

For the boundary value problem given by (5.1) with homogeneous boundary condition, Mikhlin (1965) showed the functional, $\Omega(u)$ to be a minimum value for the unique solution $u_0$ with self-adjoint, positive definite operator $A$,

$$\Omega(u) = \langle Au, u \rangle_R - 2\langle u, f \rangle_R$$

(5.14)

where $\langle \cdot, \cdot \rangle_R$ denotes inner product over the separable space of square functions. The $u_0$ that minimizes the functional (5.14) is the solution of the problem (5.1). Taking Gateaux differential of (5.14),

$$\Delta_F \Omega(u) = \lim_{\lambda \to 0} \frac{1}{\lambda} \left[ \langle A(u + \lambda v, u + \lambda v) - 2\langle u + \lambda v, f \rangle - \langle Au, u \rangle + 2\langle u, f \rangle \right]$$

$$= \langle Au, v \rangle + \langle Av, u \rangle - 2\langle v, f \rangle$$

$$= 2\langle v, Au - f \rangle = 0$$

(5.15)
In (5.15), the linearity and self-adjointness of $A$ with respect to the bilinear mapping and the symmetry of the bilinear mapping are assumed. The Gateaux differential vanishes at the solution $u_0$ such that $Au_0 - f = 0$. For the vanishing of the Gateaux differential at $u = u_0$ to imply $Au_0 - f = 0$, the bilinear mapping has to be into the real line and the operator must be positive. However, in general, it is only necessary to use vanishing of the Gateaux differential as equivalent to (5.1) being satisfied. The governing function for the operator equation (5.2) can be defined as

$$
\Omega = \langle u, A_j u_j - 2f_j \rangle_E + \langle u, C_j u_j - 2g_i \rangle_E \quad (5.16)
$$

5.1.6 Consistent Boundary and Initial Discontinuity

Sandhu (1976) pointed out that appropriate boundary terms should be included in the governing function even if they are homogeneous. This is important for approximation procedures such as the finite element method, where the functions of limited smoothness are used. The boundary operators must be in a form consistent with the field operator. Considering the boundary value problem of multi-variables given (5.5) and (5.6), Sandhu (1976) defined consistency of boundary operators with the field operators to be the property;

$$
D_{\partial R}(u_i, u_j) = \left\langle v_j, \sum_{j} C_j u_j \right\rangle_{\partial R} - \sum_{j} \left\langle u_j, C_j v_i \right\rangle_{\partial R}, \quad i = 1, 2, \ldots, n \quad (5.17)
$$

To find an approximation to the exact solution by the finite element method, the function space with limited smoothness over the entire domain is sometimes used. In
order to properly handle this limited smoothness problem in the variational formulation, Sandhu (1976) introduced internal discontinuity conditions in the form;

\[ C(u) = g \text{ on } \partial R, \quad (5.18) \]

where a prime denotes the internal jump discontinuity along element boundary \( \partial R_i \) embedded in the region \( R \). Sandhu and Salaam (1975a) and Sandhu (1975b) showed that this condition can be included explicitly in the governing function.

**5.2 Variational Principles for Dynamics of Fluid Saturated Soils**

**5.2.1 Field Equations**

The integral form of field equations (4.10) through (4.17) is can be written in a self-adjoint matrix form;

\[ A(u) = f \text{ on } R \times [0, \infty) \quad (5.19) \]

Here,

\[
A = \begin{bmatrix}
\rho & \rho_2 & 0 & -L & 0 & 0 \\
\rho_2 & \rho_2 / f + (1/k) & -t^* \frac{\partial}{\partial m} & 0 & 0 & 0 \\
0 & t^* \frac{\partial}{\partial m} & 0 & 0 & 0 & -t^* \\
L & 0 & 0 & -t^* & 0 & 0 \\
0 & 0 & 0 & -t^* & P & t^* \alpha M \delta_{ij} \\
0 & 0 & -t^* & 0 & t^* \alpha M \delta_{kl} & t^* M
\end{bmatrix}
\]

\[ (5.20) \]
where

\[ L = \left( \frac{1}{2} \right) t^* \left( \delta_{lm} \frac{\partial}{\partial k} + \delta_{km} \frac{\partial}{\partial l} \right) \]  

(5.21)

\[ P = t^* (E_{ijkl} + \alpha^2 M \delta_{ij} \delta_{kl}) \]

(5.22)

\[ u = \begin{bmatrix} u_m \\ w_m \\ \pi \\ \tau_{ij} \\ e_{ij} \\ \xi \end{bmatrix}, \quad \text{and} \quad f = \begin{bmatrix} F_m \\ G_m \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]  

(5.23)

Elements of \( A \) satisfy self-adjointness in the sense of equation (5.10). The operators on the diagonal are symmetric and the off-diagonal operators constitute adjoint pairs with respect to the bilinear mapping (5.7). Consistent boundary conditions for the equations (5.19) are

\[-t^* u_i n_j = -t^* \hat{u}^*_i n_j \quad \text{on} \quad S_1 \times [0, \infty)\]

\[-t^* w_i n_j = -t^* \hat{w}^*_i n_i \quad \text{on} \quad S_2 \times [0, \infty)\]

\[ t^* m_i = t^* \hat{\pi}^*_i n_i \quad \text{on} \quad S_3 \times [0, \infty)\]

\[ t^* \tau_{ij} n_j = t^* \hat{\tau}^*_i \quad \text{on} \quad S_4 \times [0, \infty) \]

(5.24)
Consistent form of the internal jump discontinuities is

\[-t \ast (u_i n_j)' = -t \ast (g_i) n_j \quad \text{on} \quad S_u \times (0, \infty)\]

\[-t \ast (w_i n_j)' = -t \ast g_2 \quad \text{on} \quad S_{2i} \times (0, \infty)\]

\[t \ast (\tau i n_j)' = t \ast g_3 n_j \quad \text{on} \quad S_{3i} \times (0, \infty)\]

\[t \ast (\sigma i n_j)' = t \ast g_4 n_j \quad \text{on} \quad S_{4i} \times (0, \infty)\] (5.25)

Here, surface \(S_u, S_{2i}, S_{3i}\) and \(S_{4i}\) are embedded in the interior of \(R\). Operators in the self-adjoint operator matrix equation (5.19) have the following relationships:

\[\left\langle t \ast u_i, j, \tau ij \right\rangle_R = -\left\langle t \ast u_i, j, \tau ij \right\rangle_R\]

\[+ \left\langle t \ast u_i n_j, \tau ij \right\rangle_{S_1} + \left\langle t \ast u_i, \tau ij n_j \right\rangle_{S_4}\]

\[+ \left\langle t \ast (u_i n_j)' \tau ij \right\rangle_{S_u} + \left\langle t \ast u_i, (\tau ij n_j)' \right\rangle_{S_{4i}}\] (5.26)

\[\left\langle t \ast w_i, \pi i \right\rangle_R = -\left\langle t \ast w_i, \pi i \right\rangle_R\]

\[+ \left\langle t \ast w_i n_i, \pi i \right\rangle_{S_2} + \left\langle t \ast w_i, \pi n_i \right\rangle_{S_3}\]

\[+ \left\langle t \ast (w_i n_i)' \pi i \right\rangle_{S_{2i}} + \left\langle t \ast w_i, (\pi n_i)' \right\rangle_{S_{3i}}\] (5.27)

In (5.26) and (5.27), the \(\left\langle \cdot , \cdot \right\rangle_R\) can be evaluated as the sum of quantities evaluated over subregions of \(R\) such that all the surfaces \(S_u, S_{2i}, S_{3i}, S_{4i}\) are contained in the union of the boundaries of these subregions.
5.2.2 A General Variational Principle

For the operator equation (5.19), the governing function following (5.16) is defined as;

\[
\Omega(u) = \left\langle \rho u_i, u_i \right\rangle_R + 2\left\langle \rho_2 w_i, u_i \right\rangle_R
\]

\[
- \left\langle t^* \tau_{ij}, u_i \right\rangle_R + \left\langle \frac{\rho_2}{f} + 1 - \frac{1}{k} \right\rangle w_i, w_i \right\rangle_R
\]

\[
- \left\langle t^* \pi_j, w_i \right\rangle_R + \left\langle t^* w_i, \pi \right\rangle_R
\]

\[
- 2\left\langle t^* \xi, \pi \right\rangle_R + \left\langle t^* u_i, \tau_{ij} \right\rangle_R
\]

\[
- 2\left\langle t^* e_g, \tau_{ij} \right\rangle_R + \left\langle t^* (E_{ijkl} + \alpha^2 M \delta_{ij} e_{\omega}), e_{ij} \right\rangle_R
\]

\[
+ 2\left\langle t^* \alpha M \delta_{ij} e_{ij}, \xi \right\rangle_R + \left\langle t^* M \xi, \xi \right\rangle_R
\]

\[
- 2\left\langle u_i, F_i \right\rangle_R - 2\left\langle w_i, G_i \right\rangle_R
\]

\[
- \left\langle \tau_{ij}, t^* (u_i - 2u_i), n_j \right\rangle_{S_i} - \left\langle \pi, t^* (w_i - 2w_i), n_i \right\rangle_{S_i}
\]

\[
+ \left\langle w_i, t^* (\pi - 2\pi), n_i \right\rangle_{S_i} + \left\langle u_i, t^* (\tau_{ij} - 2T_i), n_j \right\rangle_{S_i}
\]

\[
- \left\langle \tau_{ij}, t^* ((u_i, n_j) - 2(g_1), n_j) \right\rangle_{S_i}
\]

\[
- \left\langle \pi, t^* ((w_i n_i) - 2g_2) \right\rangle_{S_i}
\]

\[
+ \left\langle w_i, t^* ((m_i) - 2g_3 n_i) \right\rangle_{S_i}
\]
\[ + \langle u_i, t^* ((\tau_j n_j) - 2g_\lambda n_i) \rangle_{S_{\lambda, i}} \quad (5.28) \]

The Gateaux differential of this function along \( v = \{ \overline{u}_i, \overline{w}_i, \pi, \tau_{ij}, \overline{\tau}_{ij}, \overline{\tau}_{ij} \} \) is:

\[
\Delta_{v} \Omega(u) = \left\{ \begin{array}{l}
\langle \overline{u}_i, \rho u_i + \rho_2 \overline{w}_j - t^* \tau_{ij,j} - 2F_i \rangle_R \\
+ \langle u_i, \rho \overline{u}_i + \rho_2 \overline{w}_i - t^* \overline{\tau}_{ij,j} \rangle_R \\
+ \langle \overline{w}_i, \rho_2 u_i + (\frac{\rho_2}{f} + 1) \overline{w}_i - t^* \pi_j - 2G_i \rangle_R \\
+ \langle \overline{w}_i, \rho_2 \overline{u}_i + (\frac{\rho_2}{f} + 1) \overline{w}_i - t^* \overline{\pi}_j \rangle_R \\
- \langle \overline{\pi}_j, t^* w_{ij,j} - t^* \hat{\xi} \rangle_R + \langle \pi, t^* \overline{w}_{ij,j} - t^* \overline{\xi} \rangle_R \\
- \langle \overline{\tau}_{ij,j} - t^* \epsilon_{ij} \rangle_R + \langle \tau_{ij,j} - t^* \overline{\epsilon}_{ij} \rangle_R \\
- \langle \overline{\epsilon}_{ij}, t^* \tau_{ij} + t^* (E_{ij} + \alpha^2 M \delta_{ij} \delta_{kl}) \epsilon_{kl} + \alpha M \delta_{ij} \hat{\xi} \rangle_R \\
- \langle \epsilon_{ij}, t^* \overline{\tau}_{ij} + t^* (E_{ij} + \alpha^2 M \delta_{ij} \delta_{kl}) \overline{\epsilon}_{kl} + \alpha M \delta_{ij} \overline{\xi} \rangle_R \\
+ \langle \overline{\xi}, t^* \pi + t^* \alpha M \delta_{ij} \epsilon_{kl} + t^* M \hat{\xi} \rangle_R \\
+ \langle \overline{\xi}, t^* \overline{\pi} + t^* \alpha M \delta_{ij} \overline{\epsilon}_{kl} + t^* M \overline{\xi} \rangle_R \\
- \langle \overline{\tau}_{ij,j}, t^* (u_i n_j - 2 \hat{u}_i n_j) \rangle_{S_{\lambda, i}} - \langle \tau_{ij,j}, t^* \overline{u}_i n_j \rangle_{S_{\lambda, i}} \end{array} \right. \]

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\[- \langle \pi, t^* (w_i n_i - 2 \dot{w}_i n_i) \rangle_{S_2} - \langle \pi, t^* \overline{w}_i n_i \rangle_{S_2} \]

\[- \langle \overline{w}_i, t^* (\pi n_i - 2 \pi n_i) \rangle_{S_3} - \langle w_i, t^* \overline{\pi}_i n_i \rangle_{S_3} \]

\[- \langle u_j, t^* (\tau_{ij} n_j - 2 \dot{T}_i) \rangle_{S_4} - \langle u_i, t^* \tau_{ij} n_j \rangle_{S_4} \]

\[- \langle \tau_{ij}, t^* (u_i n_j)^\prime - 2(g_1) n_j \rangle_{S_{5i}} - \langle \tau_{ij}, t^* (\dot{u}_i n_j)^\prime \rangle_{S_{5i}} \]

\[- \langle \pi, t^* ((w_i n_i)^\prime - 2(g_2)) \rangle_{S_{5i}} - \langle \pi, t^* (\overline{w}_i n_i) \rangle_{S_{5i}} \]

\[+ \langle \overline{w}_i, t^* (\pi n_i)^\prime - 2 g_3 n_i \rangle_{S_{5i}} + \langle w_i, t^* (\overline{\pi}_i n_i)^\prime \rangle_{S_{5i}} \]

\[- \langle u_i, t^* ((\tau_{ij} n_j)^\prime - 2(g_4) n_j) \rangle_{S_{5i}} + \langle u_i, t^* (\dot{\tau}_{ij} n_j)^\prime \rangle_{S_{5i}} \]

(5.29)

Using equation (5.26) and (5.27), the gateaux differential can be rewritten as;

\[\Delta \Omega(u) = 2 \langle \dot{\pi}, \rho u_i + \rho_2 w_i - t^* \tau_{ij}, - F_i \rangle_R \]

\[- 2 \langle \overline{\pi}, t^* \dot{w}_i - t^* \dot{\pi}_i \rangle_R + 2 \langle \tau_{ij}, t^* u_{ij} - t^* e_{ij} \rangle_R \]

\[- 2 \langle \varepsilon_{ij} - t^* \tau_{ij} + t^* (E_{ijkl} + \alpha^2 M \delta_{ji} \delta_{kl}) e_{kl} + \alpha M \delta_{ji} \dot{\varepsilon}_j \rangle_R \]

\[+ 2 \langle \overline{\varepsilon}, - t^* \pi + t^* \alpha M \delta_{k} e_{kl} + t^* M \dot{\varepsilon}_j \rangle_R \]

\[- 2 \langle \overline{\tau}_{ij}, t^* (u_{ij} n_j - u_{ij} n_j) \rangle_{S_{4i}} \]

\[- 2 \langle \overline{\pi}, t^* (w_i n_i - w_i n_i) \rangle_{S_2} \]

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\[-2 \langle \overline{w}_i, t^*, (\pi n_i - \pi i n_i) \rangle_{S_3} \]
\[-2 \langle \overline{u}_i, t^*, (\tau_{ij} n_j - T_i) \rangle_{S_4} \]
\[-2 \langle \overline{r}_{ij}, t^* (u_i n_j)' - (g_1) n_j \rangle_{S_{ij}} \]
\[-2 \langle \overline{\pi}, t^* ((w_i n_i)' - (g_2)) \rangle_{S_{ij}} \]
\[+2 \langle \overline{w}_i, t^* ((\overline{m}_i)' - g_3 n_i) \rangle_{S_{ij}} \]
\[+2 \langle \overline{u}_i, t^* ((\tau_{ij} n_j)' - (g_4) n_j) \rangle_{S_{ij}} \] (5.30)

The Gateaux differential vanishes if and only if all the field equations along with the boundary conditions (5.24) and the jump conditions (5.25) are satisfied because of linearity and non-degeneracy of bilinear mapping. Hence, vanishing of $\Delta_{i} \Omega(u)$ for all $v \in w$ implies satisfaction of (5.19), (5.24), and (5.25).

5.2.3 Extended Variational Principles

Equation (5.26) and (5.27) relate pairs of operators in the operator matrix (5.19). These relations may be used to eliminate either of the elements in each pair of the functions $\Omega(u)$ in (5.28). Eight alternative forms can be obtained by using either or both relations. Elimination of an operator $A_{ij}$ from the function implies that state of variable $u_j$ needs not be in the domain $M_{ij}$ of $A_{ij}$. Where $A_{ij}$ are differential operators, this result in relaxing the requirement of smoothness in $u_j$ thereby
extending the space of admissible states. In the context of finite element method, it is clear that the extension of the admissible space provides greater freedom in selection of approximation function. In the following, the possible extensions are explicitly stated.

Using (5.26) to eliminate \( \tau_{y,j} \) from (5.28),

\[
\Omega_i = \left( \rho u_i, u_i \right)_R + 2 \left( \rho w_i, u_i \right)_R + \left( \frac{D}{f} + \frac{1}{k} \right) w_i, w_i \right)_R \\
- \left( t^* \pi_j, w_i \right)_R + \left( t^* w_{ij}, \pi \right)_R \\
- 2 \left( t^* \xi, \pi \right)_R + 2 \left( t^* u_{ij}, \tau \right)_R \\
- 2 \left( t^* \xi, \tau \right)_R + \left( t^* (E_{\xi \xi} + \alpha^2 M \delta \delta_{\xi \xi} e_{\xi \xi}, e_{\xi \xi} \right)_R \\
+ 2 \left( t^* M \delta e_{\xi \xi} \delta \xi \xi \right)_R + \left( t^* M \xi \xi \right)_R \\
- 2 \left( \pi, t^* (u_j - \bar{u}_j) n_j \right)_S_i \\
- \left( \pi, t^* (w_i - 2 \bar{w}_i) n_i \right)_S_2 \\
+ \left( w_i, t^* (\pi - 2 \bar{\pi}) n_i \right)_S_3 + 2 \left( \bar{u}_i, t^* \bar{T}_i \right)_S_4 \\
- 2 \left( \tau_{y,j}, t^* ((u_i n_j)' - (g_i)_j) n_j \right)_S_{ii}
\]
\[-\langle \pi, t^* (w_i n_i) \rangle_{S_{2i}} - 2g_{2i} \rangle_{S_{2i}}

\[\langle w_i, t^*(\delta n_i) - 2g_{3i} n_i \rangle_{S_{3i}} + 2\langle u_i, t^* g_{4i} n_i \rangle_{S_{4i}} \tag{5.31}\]

In (5.31) the stress components need not be differentiable. Using (5.26) to eliminate, alternatively, \(u_{i,j} \) (5.28) gives

\[
\Omega_2 = \langle \rho \mu_j, u_j \rangle_R + 2\langle \rho_j w_j, u_j \rangle_R - 2\langle t^* \tau_{y,j}, u_j \rangle_R \\
+ \left\langle \left( \frac{\rho_j}{j} + 1 \right) w_j, w_j \right\rangle_R \\
- \langle t^* \pi_j, w_j \rangle_R + \langle t^* w_{y,j}, \pi \rangle_R - 2\langle t^* \pi_{y,j}, \pi \rangle_R \\
- 2\langle t^* e_{y,j}, \tau_{y,j} \rangle_R + \left\langle \left( E_{yji} + \alpha^2 M \delta_{y} \delta_{ji} \right) e_{y,j}, e_{y,j} \right\rangle_R \\
+ 2\left\langle t^* \alpha M \delta_{y} e_{y,j}, \xi \right\rangle_R + \left\langle t^* M \xi, \xi \right\rangle_R \\
- 2\langle u_i, F_i \rangle_R - 2\langle w_i, G_i \rangle_R \\
- 2\langle \tau_y, t^* \hat{u}_j n_i \rangle_{S_y} - \langle \pi, t^*(w_i - 2\hat{u}_i) n_i \rangle_{S_y} \\
+ \langle w_i, t^*(\pi - 2\hat{\pi}) n_i \rangle_{S_y} + 2\langle u_i, t^*(\tau_y n_j - \hat{T}_i) \rangle_{S_y} \\
+ 2\langle \tau_y, t^*(g_1) n_j \rangle_{S_y} - \langle \pi, t^*((w_i n_i) - 2g_{2i}) \rangle_{S_{2i}} \\
+ \langle w_i, t^*((\delta n_i) - 2g_{3i} n_i) \rangle_{S_{3i}}
\]

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\[ + 2\left(\mathbf{u}_i, t^* \left(\left((\mathbf{\tau}_{ij} n_j\right))^2 \mathbf{g}_4 n_i\right)\right)_{S_i} \]  

(5.32)

in which the displacement \( \mathbf{u}_i \) need not be differentiable. Elimination of \( \mathbf{w}_{ij} \) by (5.27) form (5.28) gives

\[
\Omega_3 = \left\langle \rho \mathbf{u}_i, \mathbf{u}_i \right\rangle_R + 2\left\langle \rho_2 \mathbf{w}_i, \mathbf{u}_i \right\rangle_R - \left\langle t^* \mathbf{\tau}_{ij}, \mathbf{u}_i \right\rangle_R
\]

\[
+ \left\langle \left( \frac{\rho_2}{f} + \frac{1}{k} \right) \mathbf{w}_i, \mathbf{w}_i \right\rangle_R
\]

\[
- 2\left\langle t^* \mathbf{\pi}_j, \mathbf{w}_i \right\rangle_R - 2\left\langle t^* \mathbf{\xi}, \mathbf{\pi} \right\rangle_R + \left\langle t^* \mathbf{u}_i, \mathbf{\tau}_{ij} \right\rangle_R
\]

\[
- 2\left\langle t^* \mathbf{e}_j, \mathbf{\tau}_{ij} \right\rangle_R + \left\langle t^* \left( \mathbf{E}_{ijkl} + \alpha^2 M \mathbf{\delta}_{ij} \mathbf{\delta}_{kl} \right) \mathbf{e}_{ij}, \mathbf{e}_{ij} \right\rangle_R
\]

\[
+ 2\left\langle t^* \mathbf{G}, \mathbf{\xi} \right\rangle_R + \left\langle t^* \mathbf{M} \mathbf{\xi}, \mathbf{\xi} \right\rangle_R
\]

\[
- 2\left\langle \mathbf{u}_i, \mathbf{F}_i \right\rangle_R - 2\left\langle \mathbf{w}_i, \mathbf{G}_i \right\rangle_R
\]

\[
- \left\langle \mathbf{\tau}_{ij}, t^* \left( \mathbf{u}_i - 2\mathbf{\hat{u}}_i \right) n_j \right\rangle_{S_i} + 2\left\langle \mathbf{\pi}, t^* \mathbf{\hat{w}}_i n_i \right\rangle_{S_i}
\]

\[
+ 2\left\langle \mathbf{w}_i, t^* \left( \mathbf{\pi} - \mathbf{\hat{\pi}} \right) n_i \right\rangle_{S_i} + \left\langle \mathbf{u}_i, t^* \left( \mathbf{\tau}_{ij} n_j - 2\mathbf{\hat{\tau}}_i \right) \right\rangle_{S_i}
\]

\[
- \left\langle \mathbf{\tau}_{ij}, t^* \left( \mathbf{u}_i n_j \right) - 2\left( g_{1j} \right) n_j \right\rangle_{S_i} - 2\left\langle \mathbf{\pi}, t^* \mathbf{g}_2 \right\rangle_{S_i}
\]

\[
+ 2\left\langle \mathbf{w}_i, t^* \left( \left( \mathbf{\varpi}_i \right) n_i \right) \right\rangle_{S_i}
\]

\[
+ \left\langle \mathbf{u}_i, t^* \left( \mathbf{\tau}_{ij} n_j \right) - 2\mathbf{g}_3 n_i \right\rangle_{S_i}
\]

\[
+ \left\langle \mathbf{u}_i, t^* \left( \mathbf{\tau}_{ij} n_j \right) - 2\mathbf{g}_4 n_i \right\rangle_{S_i}
\]

(5.33)
Here, $w_i$ need not be differentiable. In the same way $\pi_i$ can be dropped out by using (5.27), yielding

$$
\Omega_4 = \left\langle \rho u_i, u_i \right\rangle_R + 2\left\langle \rho_2 w_i, u_i \right\rangle_R - \left\langle t^* \pi_{i,j}, u_i \right\rangle_R
$$

$$
+ \left\langle \frac{\rho_2}{f} + 1 \cdot \frac{1}{k} w_i, w_i \right\rangle_R
$$

$$
+ 2\left\langle t^* w_{i,j}, \pi \right\rangle_R - 2\left\langle t^* \xi, \pi \right\rangle_R + \left\langle t^* u_{i,j}, \tau_{i,j} \right\rangle_R
$$

$$
- 2\left\langle t^* e_{\gamma}, \tau_{i,j} \right\rangle_R + \left\langle t^* (E_{ijkl} + \alpha^2 M\delta_{i,j} \delta_{k,l}) e_{i,j}, e_{i,j} \right\rangle_R
$$

$$
+ 2\left\langle t^* \alpha M \delta_{i,j} e_{i,j}, \xi \right\rangle_R + \left\langle t^* M \xi, \xi \right\rangle_R
$$

$$
- 2\left\langle u_i, F_j \right\rangle_R - 2\left\langle w_i, G_i \right\rangle_R
$$

$$
- \left\langle \tau_{i,j}, t^*(u_i - 2\hat{u}_i)n_j \right\rangle_{S_1} - 2\left\langle \pi, t^*(w_i - \hat{w}_i)n_j \right\rangle_{S_2}
$$

$$
+ 2\left\langle w_i, t^* \hat{m}_i \right\rangle_{S_3} + \left\langle u_i, t^* (\tau_{i,j}n_j - 2\hat{T}_i) \right\rangle_{S_4}
$$

$$
= \left\langle \tau_{i,j}, t^*((u_i n_j) - 2\hat{g}_1)n_j \right\rangle_{S_{1i}}
$$

$$
- 2\left\langle \pi, t^*((w_i n_j) - 2g_1)n_j \right\rangle_{S_{2i}}
$$

$$
- 2\left\langle w_i, t^* g_2 n_i \right\rangle_{S_{3i}} + \left\langle u_i, t^* ((\tau_{i,j}n_j) - 2g_4 n_i) \right\rangle_{S_{4i}}
$$

(5.34)
In (5.34) the fluid pressure is not required to be differentiable. As can be seen from (5.31) to (5.34), both the differential operators in an adjoint pair cannot be removed at the same time. Use of (5.26) and (5.27), however, eliminates the differentiability of the two field variables form (5.30). Eliminating \( \tau_{y,j} \) and \( \pi_j \) from (5.30),

\[
\Omega_s = \left\langle \rho u_i, u_i \right\rangle_R + 2\left\langle \rho_2 w_i, u_i \right\rangle_R + \left\langle \left( \frac{\rho_2}{f} + 1 \times \frac{1}{k} \right) w_j, w_j \right\rangle_R
\]

\[
+ 2\left\langle t * w_{i,j}, \pi \right\rangle_R - 2\left\langle t * \xi, \pi \right\rangle_R + \left\langle t * u_{i,j}, \tau \right\rangle_R
\]

\[
- 2\left\langle t * e_{y,j}, \tau \right\rangle_R + \left\langle t * (E_{ijkl} + \alpha^2 M\delta_{ij}\delta_{kl}) e_{ij}, e_{ij} \right\rangle_R
\]

\[
+ 2\left\langle t * \alpha M \delta_{ij} e_{y,j}, \xi \right\rangle_R + \left\langle t * M e_{ij}, \xi \right\rangle_R
\]

\[
- 2\left\langle u_i, F_i \right\rangle_R - 2\left\langle w_i, G_i \right\rangle_R
\]

\[
- 2\left\langle \tau_{y,j}, t * (u_i - \bar{u}_i)n_j \right\rangle_{S_i} - 2\left\langle \pi, t * (w_i - \bar{w}_i)n_i \right\rangle_{S_i}
\]

\[
- 2\left\langle e_{y,j}, t * e_{ij} \right\rangle_{S_i} - 2\left\langle u_i, t * \tilde{F}_i \right\rangle_{S_i}
\]

\[
- 2\left\langle \tau_{y,j}, t * ((u_i n_j) - (g_1)) n_j \right\rangle_{S_i}
\]

\[
- 2\left\langle \pi, t * ((w_i n_i) - g_2) \right\rangle_{S_i}
\]

\[
- 2\left\langle w_i, t * g_{2j} n_j \right\rangle_{S_i} - 2\left\langle u_i, t * g_{2i} n_i \right\rangle_{S_i}
\]

\[(5.35)\]
In (5.35), the total stress field and the fluid pressure need not be differentiable. Elimination of \( u_{i,j} \) and \( \pi_j \) form (5.30) gives

\[
\Omega_6 = \langle \rho u_i, u_i \rangle_R + 2 \langle \rho_2 w_i, u_i \rangle_R - \langle t^* \tau_{i,j}, u_i \rangle_R \\
+ \left( \frac{\rho_2}{f} + 1 \cdot \frac{1}{k} \right) w_i, w_i \rangle_R \\
+ \langle t^* w_{i,i}, \pi \rangle_R - 2 \langle t^* \xi, \pi \rangle_R - 2 \langle t^* e_{ij}, \tau_{i,j} \rangle_R \\
+ \langle t^* (E_{gij} + \alpha^2 M \delta_{ij} \delta_{ij}) e_{ij}, e_{ij} \rangle_R \\
+ 2 \langle t^* \alpha M \delta_{ij} e_{ij}, \xi \rangle_R + \langle t^* M \xi, \xi \rangle_R \\
- 2 \langle u_i, F_i \rangle_R - 2 \langle w_i, G_i \rangle_R \\
+ 2 \langle \tau_{y,i}, t^* \tilde{u}_j \rangle_{S_i} - 2 \langle \pi, t^* (w_i - \tilde{w}_j) n_j \rangle_{S_i} \\
- 2 \langle w_i, t^* \tilde{n}_j \rangle_{S_i} + 2 \langle u_i, t^* (\tau_{y,j} n_j - \tilde{T}_j) \rangle_{S_i} \\
+ 2 \langle \tau_{y,j}, t^* (g_1)_j, n_j \rangle_{S_i} - 2 \langle \pi, t^* ((w_j n_j) - g_2) \rangle_{S_j} \\
- 2 \langle w_i, t^* g_j n_j \rangle_{S_i} + 2 \langle u_i, t^* ((\tau_{y,j})' - g_3 n_j) \rangle_{S_i} \]

(5.36)

where \( u_i \) and \( \pi \) need not be differentiable. Eliminating \( \tau_{y,j} \) and \( w_{ij} \) from (5.30)

\[
\Omega_7 = \langle \rho u_i, u_i \rangle_R + 2 \langle \rho_2 w_i, u_i \rangle_R + \left( \frac{\rho_2}{f} + 1 \cdot \frac{1}{k} \right) w_i, w_i \rangle_R 
\]

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\[-2\langle t^* \pi_i, w_i \rangle_R - 2\langle t^* \xi, \pi \rangle_R + 2\langle t^* u_{i,j}, \tau_{y} \rangle_R\]

\[-2\langle t^* e_y, \tau_{y} \rangle_R + \langle t^* (E_{ijkl} + \alpha^2 M \delta_{y} \delta_{kl}) e_{kl}, e_{y} \rangle_R\]

\[+ 2\langle t^* \alpha M \delta_{y} e_y, \xi_e \rangle_R + \langle t^* M \xi_e, \xi_e \rangle_R\]

\[-2\langle u_i, F_j \rangle_R - 2\langle w_i, G_j \rangle_R\]

\[-2\langle \tau_{y}, t^* (u_i - \bar{u}_i) n_j \rangle_{S_1} + 2\langle \pi, t^* \bar{w}_i n_i \rangle_{S_2}\]

\[+ 2\langle w_i, t^* (\pi - \bar{\pi}) n_i \rangle_{S_1} - 2\langle u_i, t^* \bar{T}_i \rangle_{S_1}\]

\[-2\langle \tau_{y}, t^* ((u_i n_j)^* - (g_i), n_j) \rangle_{S_{su}} - 2\langle \pi, t^* g_2 \rangle_{S_2}\]

\[+ 2\langle w_i, t^* ((\pi n_i)^* - g_3 n_i) \rangle_{S_{su}} + 2\langle u_i, t^* g_4 n_i \rangle_{S_{su}}\]

(5.37)

which does not require the total stress and relative displacement of fluid to be differentiable. Using (5.26) and (5.27) to eliminate $u_{i,j}$ and $w_{i,j}$, (5.30) is

$$
\Omega_8 = \langle \rho u_i, u_i \rangle_R + 2\langle \rho w_i, u_i \rangle_R - \langle t^* \tau_{y,i}, u_i \rangle_R
$$

$$
+ \left( \frac{\rho^*_2}{f} + 1 \frac{1}{k} \right) w_i, w_j \rangle_R
$$

$$
- \langle t^* \pi_y, w_i \rangle_R - 2\langle t^* \xi, \pi \rangle_R - 2\langle t^* e_y, \tau_{y} \rangle_{RR}
$$

$$
+ \langle t^* (E_{ijkl} + \alpha^2 M \delta_{y} \delta_{kl}) e_{kl}, e_{y} \rangle_R
$$
Here, the solid displacement and relative displacement of fluid need not be differentiable.

5.2.4 Specializations

If the admissible state is constrained to satisfy some field equations and the boundary conditions, certain specialized forms of the variational principle are possible. This procedure is used to reduce the number of free variables in the governing function. Also, certain assumptions in the spatial or temporal variation of some of the variables lead to approximate theories. In the context of direct methods of approximation, the constraints assumed in the specialization must be satisfied by admissible states. If it is difficult to satisfy the constraints, such specialization of the variational formulation will not be useful in practice. Some specializations of the extended variational stated in the previous section are presented below.
For the functional $\Omega_5$ in equation (5.35), in which the soil stress and fluid pressure need not be differentiable, specialization to satisfy (5.19), i.e. satisfying identically the kinematics relationships gives

$$\Omega_5 = \langle p u_i, u_i \rangle_R + 2\langle \rho_2 w_i, u_i \rangle_R + \left( \frac{\rho_2}{f} + 1 \cdot \frac{1}{k} \right) w_i, w_i \rangle_R$$

$$+ \left\{ t^* (E_{\mu \mu} + 2M \delta_\mu \delta_\mu) e_{\mu i}, e_{\mu j} \right\}_R$$

$$+ 2 \left\{ t^* \alpha M \delta_\mu \varepsilon_{\alpha i} \right\}_R + \left\{ t^* M \xi_{ij}, \xi_{ij} \right\}_R$$

$$- 2 \langle u_i, F_i \rangle_R - 2 \langle w_i, G_i \rangle_R$$

$$- 2 \langle \tau_{ij}, t^* (u_i - \dot{u}_i) n_j \rangle_{S_i} - 2 \langle \pi, t^* (w_i - \dot{w}_i) n_i \rangle_{S_i}$$

$$- 2 \langle w_i, t^* \dot{m}_i \rangle_{S_i} - 2 \langle u_i, t^* \dot{T}_i \rangle_{S_i}$$

$$- 2 \langle \tau_{ij}, t^* ((u_i n_j) - (g_{ij}) n_j) \rangle_{S_i}$$

$$- 2 \langle \pi, t^* ((w_i n_i) - g_{2i}) \rangle_{S_i}$$

$$- 2 \langle w_i, t^* g_{ni} \rangle_{S_i} - 2 \langle u_i, t^* g_{ni} \rangle_{S_i}$$

(5.39)

If the field variables over the domain are continuous, the jump discontinuity terms drop out giving the specialization;

$$\Omega_{io} = \langle p u_i, u_i \rangle_R + 2\langle \rho_2 w_i, u_i \rangle_R + \left( \frac{\rho_2}{f} + 1 \cdot \frac{1}{k} \right) w_i, w_i \rangle_R$$
Further specialization of (5.40) to the case where displacement boundary conditions are identically satisfied yields the function governing the two field formulation proposed by Ghaboussi and Wilson (1972) except that in the present formulation the boundary terms are consistent. Alternatively, specializing equation (5.37) to satisfy the (5.19),

$$\Omega_{11} = \left\langle \rho u_i, u_i \right\rangle_R + 2\left\langle \rho_z, u_i \right\rangle_R + \left\langle \left( \frac{\rho_z}{\varepsilon} + \frac{1}{k} \right) w_i, w_i \right\rangle_R$$

$$- \left\langle t^* \pi^*_j, w_i \right\rangle_R - 2\left\langle t^* \xi, \pi \right\rangle_R$$

$$+ \left\langle t^* (E_{ijkl} + \alpha^2 M_{ijkl} \delta_{kl}) e_{ii}, e_{ij} \right\rangle_R$$

$$+ 2\left\langle t^* \alpha M_{ij} \varepsilon_{ij} \right\rangle_R + \left\langle t^* \xi, \xi \right\rangle_R$$

$$- 2\left\langle u_i, F_i \right\rangle_R - 2\left\langle w_i, G_i \right\rangle_R$$

$$- 2\left\langle \tau_{ij}, t^* (u_i - \hat{u}_i) n_j \right\rangle_{S_i} + 2\left\langle \pi, t^* \hat{w}_i n_i \right\rangle_{S_i}$$
Furthermore, assuming that the internal discontinuities and the boundary conditions on \( S_1 \) and \( S_3 \) are identically satisfied and eliminating \( \xi \) by using (5.19), \( \Omega_{11} \) gives

\[
\Omega_{12} = \langle \rho u, u \rangle_R + 2\langle \rho_2 w, u \rangle_R + \left( \frac{\rho_2}{f} + 1 \right) \langle \frac{1}{k} w, w \rangle_R
\]

\[-2\langle t^*(\pi - \hat{\pi}) n_i \rangle_{S_i} - 2\langle u_i, t^* \hat{T}_i \rangle_{S_i} \]

\[-2\langle \pi, t^* ((u, n_j)^{(j)}, (g, n_j)) \rangle_{S_{ij}} + 2\langle \pi, t^* g_2 \rangle_{S_{ij}} \]

\[+ 2\langle w, t^* ((u, n_j)^{(j)}, (g, n_j)) \rangle_{S_{ij}} - 2\langle u_i, t^* g_2 n_i \rangle_{S_{ij}} \]

\[(5.41)\]

\( \Omega_{12} \), which is three field formulations, has important meaning in the context of finite analysis, since this mixed formulation of \( u - w - \pi \) can produce the continuity of the pore pressure which is the physically important quantity in the analysis of dynamic response of the fluid saturated solid. Similar three field formulation can be obtained by specializing \( \Omega_i \) to satisfy (5.19).
6.1 Introduction

In this Chapter, the finite element discretization of the governing functional equation is described. The equilibrium equations of saturated soils are thus set up in matrix form. A predictor-corrector form of Newmark method to perform time domain integration scheme is used for temporal discretization.

6.2 Finite Element Method

In the finite element method, the domain of interest $R$, is divided into a finite number of non-overlapping sub regions or elements $R_e$ such that

$$\overline{R} = \lim_{m \to \infty} \bigcup_{e=1}^{m} \overline{R_e} \quad \text{where } e = 1, 2, 3, \ldots, m \quad (6.1)$$

Here $m$ represents the number of elements. Elements, $R_e$ are connected with each other through a finite number of nodal points. $\overline{R}$ and $\overline{R_e}$ represent the closure of $R$ and $R_e$, respective.
For an arbitrary element, the matrix form of approximation for an unknown field variable can be expressed as,

\[
\bar{\nu}_\alpha = \left[\psi\right]^T_e \{a_\nu\}_e
\]  

(6.2)

where \(\left[\psi\right]^T_e\) represents the set of base functions (i.e. column vector) and \(\{a_\nu\}_e\) represents the column vector of unknown coefficients. Evaluating the function and its derivatives up to the required order of differentiability at the nodal points gives:

\[
\{\bar{\nu}_\alpha\}_e = \left[\psi\right]^T_e \{a_\nu\}_e
\]  

(6.3)

where \(\{\bar{\nu}_\alpha\}_e\) is the vector of nodal point values of the unknown field variable and its derivatives up to the order of differentiability selected and \(\left[\psi\right]^T_e\) is the matrix of base functions and their higher order base derivatives. The rows and columns of \(\left[\psi\right]^T_e\) are linearly independent. Further, if \(\left[\psi\right]^T_e\) is invertible, equation (9.3) gives:

\[
\{a_\nu\}_e = \left[\left[\psi\right]^T_e\right]^{-1} \{\bar{\nu}_\alpha\}_e
\]  

(6.4)

where \(\Psi = \left[\psi\right]^T_e\). Substituting equation (9.4) into (9.2) yields:

\[
\bar{\nu}_\alpha = \left[\psi\right]^T_{\nu\alpha} \left[\Psi\right]^{-1} \{\bar{\nu}_\alpha\}_e
\]  

\[
= \left[\psi_{\nu\alpha}\right]^T_e \{\bar{\nu}_\alpha\}_e
\]  

(6.5)

in which

\[
\left[\psi_{\nu\alpha}\right]^T_e = \left[\psi\right]^T_e \left[\Psi\right]^{-1}
\]  

(6.6)
where \( \psi_{\nu} \) is the set of interpolating functions relating the nodal point function values and their derivatives to the value of the function \( \psi_{\alpha} \) at any arbitrary point within an arbitrary element “e”. Similarly, for the other field variables:

\[
\begin{align*}
\bar{u}_m &= \left[ \psi_{u_{\nu}} \right]_e^T \{ \bar{u}_m \}_e \\
\bar{w}_m &= \left[ \psi_{w_{\nu}} \right]_e^T \{ \bar{w}_m \}_e \\
\bar{\pi}_m &= \left[ \psi_{\pi_{\nu}} \right]_e^T \{ \bar{\pi}_m \}_e \\
\bar{\tau}_m &= \left[ \psi_{\tau_{\nu}} \right]_e^T \{ \bar{\tau}_m \}_e
\end{align*}
\]  

(6.7)

The governing function \( \Omega \) was developed for the finite element solution of the given problem. The governing function \( \Omega \), (5.42) can be expressed as a summation of all the non-overlapping sub-regions or elements in the form:

\[
\Omega = \sum_{e=1}^{m} \Omega_e
\]  

(6.8)

where

\[
\begin{align*}
\Omega_e = \Omega_{12} &= \left\langle \rho u_{i}, u_{i} \right\rangle_R + 2\left\langle \rho_{2} w_{i}, u_{i} \right\rangle_R + \left\langle \frac{\rho_{2}}{f} + 1 \frac{1}{k} \right\rangle w_{i}, w_{i} \right\rangle_R \\
&- 2\left\langle t^{*} \pi_{j}, w_{i} \right\rangle_R + \left\langle t^{*} (E_{ijkl} e_{kl}, e_{ij}) \right\rangle_R \\
&+ 2\left\langle t^{*} \delta_{ij} e_{ij}, \pi \right\rangle_R - \left\langle t^{*} \pi / M, \pi \right\rangle_R \\
&- 2\left\langle u_{i}, F_{i} \right\rangle_R - 2\left\langle w_{i}, G_{i} \right\rangle_R
\end{align*}
\]  

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\[ + 2\langle \pi, \dot{t} \cdot \dot{w}, n_i \rangle_{S_2} - 2\langle u_i, \dot{t} \cdot \dot{T}_i \rangle_{S_4} \]  \quad (6.9)

Here the inner product is,

\[ \langle u, v \rangle_{R} = \int_{R} (u \ast v) dR \]  \quad (6.10)

The equation (6.9) is the mixed formulation involving \( u - w - \pi \). The governing functional, \( \Omega \) involving the pore pressure field can be ensure continuity of pore pressure in analysis of dynamic response of saturated soils. The Gateaux differential of the governing functional vanishes if and only the field equation, the boundary conditions, and initial conditions are all satisfied.

The governing function (6.9) with (6.10) is explicitly,

\[ \Omega = \int_{R} \left( \rho u_i \ast u_i \right) dR + 2\int_{R} \left( \rho_2 w_i \ast u_i \right) dR \]

\[ + \int_{R} \left( \frac{\rho_2}{f} + g'w \frac{1}{k} \right) w_i \ast w_i dR - 2\int_{R} \left( g \ast \pi_j \ast w_i \right) dR \]

\[ - \int_{R} \left( g \ast E_{ijkl} \epsilon_{kl} \ast e_{ij} \right) dR + 2\int_{R} \left( g \ast \alpha \delta_{ij} \epsilon_{ij} \ast \pi \right) dR \]

\[ - \int_{R} \left( \frac{g \ast \pi}{M} \ast \pi \right) dR - 2\int_{R} \left( u_i \ast F_i \right) dR - 2\int_{R} \left( w_i \ast G_i \right) dR \]

\[ + 2\int_{S_2} \left( \pi \ast g \ast \dot{w}, n_i \right) dS - 2\int_{S_2} \left( u_i \ast t \ast \dot{T}_i \right) dS \]
\begin{align*}
&= \int_V (\rho u_i u_i) dV + 2\int_V (\rho_2 w_i u_i) dV \\
&\quad + \int_V \left( \frac{\rho_2}{f} + \frac{g}{k} \right) w_i w_i dV - 2\int_V (g \pi_j w_i) dV \\
&\quad - \int_V (g E_{ijkl} e_{kl} e_{ij}) dV + 2\int_V (g \alpha \delta_{ij} e_{ij} \pi) dV \\
&\quad - \int_V \left( g \pi M \pi \right) dV - 2\int_V (u_i F_j) dV - 2\int_V (w_i G_i) dV \\
&\quad + 2\int_{S_2} (\pi g \dot{w}_i n_i) dS - 2\int_{S_4} (u_i t \dot{T}_i) dS
\end{align*}

(6.11)

where \( g \) is a function of time variable, with the following properties

\[ g(t) = t \quad \text{and} \quad g'(t) = \frac{dg(t)}{dt} = 1 \quad (6.12) \]

### 6.3 Discretization of Governing Function

The spatial domain is divided into a finite number of non-trivial sub-regions or elements so that their geometry is completely defined by the location in space of a set of nodal points. The displacements and the pore-pressure field within each element are defined in terms of the nodal point values through interpolating functions. Thus, the variables are

\[ u^n (x, t) = \left\{ \phi^n(x) \right\}^T \{ \bar{u}(t) \} \]
\[ w^n(\mathbf{x},t) = \left\{ \psi^n(\mathbf{x}) \right\}^T \left\{ \mathbf{w}(t) \right\} \]

\[ \pi^n(\mathbf{x},t) = \left\{ \eta^n(\mathbf{x}) \right\}^T \left\{ \pi(t) \right\} \quad (6.13) \]

Here \( u^n(\mathbf{x},t) \) represents the solid displacement at \((\mathbf{x},t)\) in the element of the mixture. \( \left\{ \psi^n(\mathbf{x}) \right\} \) is the set of solid displacement interpolation functions, \( \left\{ \mathbf{w}(t) \right\} \) is the set of nodal point solid displacement. \( w^n(\mathbf{x},t) \) represents displacement of fluid relative to the solid. \( \left\{ \eta^n(\mathbf{x}) \right\} \) is the set of relative displacement interpolation functions and \( \left\{ \pi(t) \right\} \) is the set of nodal point values of the fluid displacements relative to the solid. \( \pi^n(\mathbf{x},t), \left\{ \eta^n(\mathbf{x}) \right\}, \) and \( \left\{ \pi(t) \right\} \) are respectively, the scalar pressure field at \((\mathbf{x},t)\) in the element and the set of nodal point values of the pore pressure. The symbol \( \mathbf{x} \) and \( t \) in the parentheses indicate the space and time variables.

The strain-displacement relationship can be written by differentiation of \( u(\mathbf{x},t) \) as

\[ e^n(\mathbf{x},t) = \left\{ \phi^n(\mathbf{x}) \right\}^T \left\{ \mathbf{u}(t) \right\} \quad (6.14) \]

in which \( e^n(\mathbf{x},t) \) is the reduced strain tensor,

\[ e^n(\mathbf{x},t) = \begin{bmatrix} e_x \\ e_y \\ e_{xy} \end{bmatrix} \quad (6.15) \]
at point \( x \) at the time \( t \), and \( \{ \phi^n \} \) is the transformation matrix derived from the
displacement interpolation functions, \( \{ \phi^n(x) \} \) by suitable differentiation and
reorganization of terms.

6.3.1 Matrix From of Governing Functional

For a total of \( N \) elements, substituting equation (6.13) into equation (6.11), the
governing equations can be written in terms of nodal point values of the soil
displacements, the relative fluid displacements, and the pore pressures. The process to
find the matrix form of the governing functional is following as term by term.

\[
\int_V (\rho u_i * u_j) dV = \sum_{n=1}^N \int_{V^n} \left( \rho u^T \phi^n \phi^n^T \right) u^n
\]

\[
= u^T \left( \sum_{n=1}^N \int_{V^n} \left( \phi^n \rho \psi \phi^n^T \right) u^n \right) * u
\]

\[
= u^T M_s * u
\]

(6.16)

where

\[
M_s = \sum_{n=1}^N \int_{V^n} \left( \phi^n \rho \psi \phi^n^T \right) u^n
\]

\[
2 \int_V (\rho w_i * u_i) dV = 2 \sum_{n=1}^N \int_{V^n} \left( \rho w^T \psi \phi^n \phi^n^T \right) u^n
\]
\[
2 \sum_{n=1}^{N} \int_{v} \left( w^{T} \psi^{n} \rho^{n} \phi^{nT} u \right) dv^{n} = w^{T} \left( \sum_{n=1}^{N} \int_{v} \left( \psi^{n} \rho^{n} \phi^{nT} \right) dv^{n} \right) u
\]

\[
= w^{T} \left( \sum_{n=1}^{N} \int_{v} \left( \phi^{n} \psi^{n} \right) dv^{n} \right) w
\]

\[
+ u^{T} \left( \sum_{n=1}^{N} \int_{v} \left( \phi^{n} \psi^{n} \right) dv^{n} \right) w
\]

\[
= w^{T} M_{c} u + u^{T} M_{c}^{T} w
\]

where \( M_{c} = \sum_{n=1}^{N} \int_{v} \left( \psi^{n} \rho^{n} \phi^{nT} \right) dv^{n} \)

\[
\int_{V} \left( \frac{\rho^{2}}{f} + g^{*} \frac{1}{k} \right) w^{*} w_{i} dV = \sum_{n=1}^{N} \int_{v} \left( \frac{\rho^{2}}{f} + g^{*} \frac{1}{k} \right) \psi^{n} w_{i} \psi^{nT} w dv^{n}
\]

\[
= w^{T} \left( \sum_{n=1}^{N} \int_{v} \psi^{n} \left( \frac{\rho^{2}}{f} + g^{*} \frac{1}{k} \right) \psi^{nT} dv^{n} \right) w
\]

\[
= w^{T} \left( M_{f} + g^{*} C \right) w
\]

where \( M_{f} = \sum_{n=1}^{N} \int_{v} \psi^{n} \left( \frac{\rho^{2}}{f} \right) \psi^{nT} dv^{n} \)

and \( C = \sum_{n=1}^{N} \int_{v} \psi^{n} \left( \frac{1}{k} \right) \psi^{nT} dv^{n} \)
- 2 \int V (g \pi^j w_i) dV = -2 \sum_{n=1}^N \int_{\nu^v} (g \eta_i^T \pi^\nu w) dv^n

= -\pi^T \left( g \sum_{n=1}^N \int_{\nu^v} (\eta^\nu \psi^\nu) dv^n \right) * w

= -w^T \left( g \sum_{n=1}^N \int_{\nu^v} (\psi^\nu \eta^\nu) dv^n \right) * \pi

= -\left( w^T * gL * w + w^T * gL^T * gL * \pi \right) \quad (6.19)

where  

L = \sum_{n=1}^N \int_{\nu^v} (\eta^\nu \psi^\nu) dv^n

\int V (g * E_{ijkl} e_{kl} e_{ij}) dV = \sum_{n=1}^N \int_{\nu^v} (g * E_{ijkl} \phi_e^T u * \phi_e u) dv^n

= g \left( \sum_{n=1}^N \int_{\nu^v} (u^T \phi_e^T E_{ijkl} \phi_e^T) dv^n \right) * u

= u^T * g \left( \sum_{n=1}^N \int_{\nu^v} (\phi_e^T E_{ijkl} \phi_e^T) dv^n \right) * u

= u^T * gK * u \quad (6.20)

where  

K = \sum_{n=1}^N \int_{\nu^v} (\phi_e^T E_{ijkl} \phi_e^T) dv^n

and  

\phi_e^T = \frac{1}{2} \left( \phi_i^T_{i,j} + \phi_j^T_{j,i} \right)
\[ 2 \int \left( g^* \alpha \delta_{ij} e_{ij} \pi \right) dV = 2 \sum_{n=1}^{N} \int_{\Omega} \left( g^* (\alpha_n^*) \phi_{n^T u} \eta_{n^T} \pi \right) dV^* \]

\[ = 2 g^* \sum_{n=1}^{N} \int_{\Omega} u^{T} \phi_{n^T} (\alpha_n^*) \eta_{n^T} dV^* \pi \]

\[ = u^{T} * g \sum_{n=1}^{N} \int_{\Omega} \left( \phi_{n^T} (\alpha_n^*) \eta_{n^T} \right) dV^* \pi \]

\[ + \pi^T * g \sum_{n=1}^{N} \int_{\Omega} \left( \eta_n (\alpha_n^*) \phi_{n^T} \right) dV^* \pi \]

\[ = u^{T} * g K_T \pi^* + \pi^T * g K_T u \]

(6.21)

where \[ K_p = \sum_{n=1}^{N} \int_{\Omega} \left( \phi_{n^T} (\alpha_n^*) \eta_{n^T} \right) dV^* \]

and \[ \phi_{n^T} = \phi_{n^T} \]

\[ - \int \left( g^* \frac{\pi}{M} \pi \right) dV = - \sum_{n=1}^{N} \int_{\Omega} \left( g^* \eta_{n^T} \left( \frac{1}{M} \right) \pi^* \eta_{n^T} \pi \right) dV^* \]

\[ = - g^* \sum_{n=1}^{N} \int_{\Omega} \left( \pi^T \eta_n \left( \frac{1}{M} \right) \eta_{n^T} \right) dV^* \pi \]

\[ = - g^* \pi^T \sum_{n=1}^{N} \int_{\Omega} \left( \eta_n \left( \frac{1}{M} \right) \eta_{n^T} \right) dV^* \pi \]

\[ = - \pi^T g K_{\pi^T} \pi \]

(6.22)
where

$$K_{n} = \sum_{n=1}^{N} \int_{\omega_{n}} \left( \eta^{n} \left( \frac{1}{M} \right) \eta^{nT} \right) dv^{n}$$

$$-2 \int_{V} (u_{i} \ast F_{i}) dV = -2 \sum_{n=1}^{N} \int_{\omega_{n}} \left( \phi^{nT} u \ast F_{i} \right) dv^{n}$$

$$= -2 u^{T} \sum_{n=1}^{N} \int_{\omega_{n}} \left( \phi^{n} F_{i} \right) dv^{n}$$

$$= -2 u^{T} \ast P_{1} \quad (6.23)$$

where

$$P_{1} = \sum_{n=1}^{N} \int_{\omega_{n}} \left( \phi^{n} F_{i} \right) dv^{n}$$

$$-2 \int_{V} (w_{i} \ast G_{i}) dV = -2 \sum_{n=1}^{N} \int_{\omega_{n}} \left( \psi^{nT} w \ast G_{i} \right) dv^{n}$$

$$= -2 w^{T} \sum_{n=1}^{N} \int_{\omega_{n}} \left( \psi^{n} G_{i} \right) dv^{n}$$

$$= -2 w^{T} \ast P_{2} \quad (6.24)$$

where

$$P_{2} = \sum_{n=1}^{N} \int_{\omega_{n}} \left( \psi^{n} G_{i} \right) dv^{n}$$

$$2 \int_{S_{2}} (\pi \ast g \ast \tilde{w_{i}} n_{i}) dS = 2 \sum_{n=1}^{N} \int_{S_{2}} \left( g \ast \eta^{nT} \pi \ast \tilde{w_{i}} n_{i} \right) ds^{n}$$
\[ = 2g \sum_{n=1}^{N} \int_{S_n} (\pi^T \eta^n \cdot \hat{w}_i n_i) ds_n \]

\[ = 2\pi^T g \sum_{n=1}^{N} \int_{S_n} (\eta^n \hat{w}_i n_i) ds_n \]

\[ = 2\pi^T g R_i \]  \hspace{1cm} (6.25)

where

\[ R_i = \sum_{n=1}^{N} \int_{S_n} (\eta^n \hat{w}_i n_i) ds_n \]

\[-2 \int_{S_i} (u_i \cdot t_i \cdot \hat{T}_i) ds = -2 \sum_{n=1}^{N} \int_{S_n} (g^* \phi^n \cdot u_i \cdot \hat{T}_i) ds_n \]

\[ = -2g^* \sum_{n=1}^{N} \int_{S_n} (u^T \phi^n \cdot \hat{T}_i) ds_n \]

\[ = -2u^T g R_2 \]  \hspace{1cm} (6.26)

where

\[ R_2 = \sum_{n=1}^{N} \int_{S_n} (\phi^n \hat{T}_i) ds_n \]

Combining equations (6.16) through (6.26), the governing function can be written in matrix form as

\[
\Omega(u_2, w_2, \pi) = \begin{bmatrix}
    u^T & M_s \cdot (g^* K^*) & M_c^T \cdot (g^* K_p^*) \\
    w^T & M_c & (g^* M_f^*) + (g^* H^*) & -g^* L^T \\
    \pi^T & (g^* K_p^T) & -(g^* L^*) & -(g^* K_P^*)
\end{bmatrix}
\]

\[
\begin{bmatrix}
    u \\
    w \\
    \pi
\end{bmatrix}
\]

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6.3.2 Variation of the Governing Function

The governing function after finite element discretization, can be written in the form

$$\Omega(U) = \{U\}^T [S][U] + \{U\}^T [X]$$

(6.28)

where

$$\{U\} = \begin{bmatrix} u \\ w \\ \pi \end{bmatrix}$$

$$[S] = \begin{bmatrix} (M_s)+(g*K) & M_c^T & (g*K_p) \\ M_c & (M_f)+(g*H) & -(g*L) \\ (g*K_p^T) & -(g*L) & -(g*K_p) \end{bmatrix}$$

$$\{X\} = \begin{bmatrix} -(2P_1) - 2(g*R_2) \\ -(2P_2) \\ (2g*R_1) \end{bmatrix}$$

(6.29)

Vanishing of variation, $\Omega$ of with respect to the nodal values, $u, w, \pi$ implies

$$\frac{\partial \Omega(U)}{\partial U_i} = 0 \quad \text{for all } i.$$
This gives

$$2[S][U] + \{X\} = 0$$  \hspace{1cm} (6.30)$$

Therefore corresponding to equation (6.14), the following set of matrix equations is possible as the Euler equations,

$$
\begin{pmatrix}
(M_x) + (g^* K) & M_c^T & (g^* K_p) \\
M_c & (M_f) + (g^* H) & -(g^* L^T) \\
(g^* K_p^T) & -(g^* L) & -(g^* K_x)
\end{pmatrix} \begin{pmatrix} u \\ w \\ \pi \end{pmatrix} = \begin{pmatrix} P_1 + (g^* R_2) \\ P_2 \\ -g^* R_1 \end{pmatrix}
$$

(6.30)

or

$$
\begin{pmatrix}
M_x & M_c^T & 0 \\
M_c & M_f & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix} u \\ w \\ \pi \end{pmatrix} = g^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \\ \pi \end{pmatrix}
$$

$$+ g^* \begin{pmatrix} K & 0 & K_p \\
M_c & 0 & -L^T \\
K_p^T & L & -K_x \end{pmatrix} \begin{pmatrix} u \\ w \\ \pi \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ 0 \end{pmatrix} + g^* \begin{pmatrix} R_2 \\ 0 \\ -R_1 \end{pmatrix}
$$

(6.31)

Equation (6.31) after differentiation twice with respect to time is

$$
\begin{pmatrix}
M_x & M_c^T & 0 \\
M_c & M_f & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix} \ddot{u} \\ \dot{w} \\ \ddot{\pi} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{u} \\ \dot{w} \\ \ddot{\pi} \end{pmatrix} = \begin{pmatrix} \dddot{u} \\ \ddot{w} \\ \dddot{\pi} \end{pmatrix}
$$
\begin{align*}
\frac{\partial^2 P_1}{\partial t^2} + \begin{bmatrix} K & 0 & K_p \\ M_c & 0 & -L^T \\ K_p^T & -L & -K_p \end{bmatrix} \begin{bmatrix} u \\ w \\ \pi \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 P_1}{\partial t^2} \\ \frac{\partial^2 P_2}{\partial t^2} + 0 \\ 0 \end{bmatrix} + \begin{bmatrix} R_2 \\ 0 \\ -R_1 \end{bmatrix} = (6.32)
\end{align*}

From equations (4.10), (4.11) and (6.23), (6.24),

\begin{align*}
\frac{\partial^2 P_1}{\partial t^2} = \sum_{n=1}^{N} \int_{x} (\phi^n \rho_{g_1}) h v^n = F
\end{align*}

which represents volumetric body force on the mixture, and

\begin{align*}
\frac{\partial^2 P_2}{\partial t^2} = \sum_{n=1}^{N} \int_{x} (\psi^n \rho_{g_2} s_1) h v^n = G
\end{align*}

which represents surface traction applied to the fluid. With equations (6.33) and (6.34), equation (6.32) can be rewritten as

\begin{align*}
\begin{bmatrix} M_s & M_c^T \\ M_c & M_f \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{w} \\ \ddot{\pi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{\pi} \end{bmatrix} + \begin{bmatrix} K & 0 & K_p \\ M_c & 0 & -L^T \\ K_p^T & -L & -K_p \end{bmatrix} \begin{bmatrix} u \\ w \\ \pi \end{bmatrix} = \begin{bmatrix} F + R_2 \\ G \\ -R_1 \end{bmatrix} = (6.35)
\end{align*}
6.3.3 Solution Procedures

In equation (6.35) the third equation does not have mass and damping terms. Therefore static equilibrium equation can be used to eliminate one of the variables from analysis. The third equation is

\[ K_p^T u - L w - K_\pi \pi = -R_1 \]

or

\[ K_\pi \pi = K_p^T u - L w + R_1 \] (6.36)

Thus, the pore pressures can be calculated in terms of \( u \), and \( w \) as:

\[ \pi = K_\pi^{-1} K_p^T u - K_\pi^{-1} L w + K_\pi^{-1} R_1 \] (6.37)

The other two equations are:

\[ M_s \dddot{u} + M_c \dddot{w} + K u + K_p \pi = \bar{F} + R_2 \] (6.38)

and

\[ M_c \dddot{u} + M_f \dddot{w} + C \ddot{w} - L^T \pi = \bar{G} \] (6.39)

The following expressions are made by substituting (6.37) into (6.38) and (6.39),

\[ M_s \dddot{u} + M_c \dddot{w} + K u + (K_p K_\pi^{-1} K_p^T) u - (K_p K_\pi^{-1} L) \dot{w} \]

\[ = \bar{F} + R_2 - K_p K_\pi^{-1} R_1 \] (6.40)
and

\[ M_c \dddot{u} + M_f \dddot{w} + C \dddot{w} - L^T K \pi^{-1} K_p^T u + L^T K \pi^{-1} L \dot{w} = G - L^T K \pi^{-1} R_1 \]  

(6.41)

Combining equations (6.40) and (6.41), the equations of motion in matrix form are:

\[
\begin{bmatrix}
M_x & M_c^T \\
M_c & M_f
\end{bmatrix}
\begin{bmatrix}
\dddot{u} \\
\dddot{w}
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & C
\end{bmatrix}
\begin{bmatrix}
\dddot{u} \\
\dddot{w}
\end{bmatrix}
+ \begin{bmatrix}
K + K_p K \pi^{-1} K_p^T & -K_p K \pi^{-1} L \\
-L^T K \pi^{-1} K_p^T & L^T K \pi L
\end{bmatrix}
\begin{bmatrix}
u \\
w
\end{bmatrix}
= \begin{bmatrix}
F + R_2 - K_p K \pi^{-1} R_1 \\
G - L^T K \pi^{-1} R_1
\end{bmatrix}
\]

(6.42)

The third equation in Equation (6.42) is implemented in a numerical program. After solving for the unknown \( u \) and \( w \), pore pressures is obtained using (6.37).

### 6.4 Incremental Form

Rewriting (6.42) as

\[
\begin{bmatrix}
M_x & M_c^T \\
M_c & M_f
\end{bmatrix}
\begin{bmatrix}
\dddot{u} \\
\dddot{w}
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & C
\end{bmatrix}
\begin{bmatrix}
\dddot{u} \\
\dddot{w}
\end{bmatrix}
+ \begin{bmatrix}
K_p K \pi^{-1} K_p^T & -K_p K \pi^{-1} L \\
-L^T K \pi^{-1} K_p^T & L^T K \pi L
\end{bmatrix}
\begin{bmatrix}
u \\
w
\end{bmatrix}
= \begin{bmatrix}
F + R_2 - K_p K \pi^{-1} R_1 \\
G - L^T K \pi^{-1} R_1
\end{bmatrix}
\]

(6.43)
Equation (6.43) can be expressed compactly as;

\[
[M] \ddot{\{U\}} + [C] \dot{\{U\}} + [K] \{U\} + [K]^{**} \{U\} = R
\]  

(6.44)

At any instant time \(t\), the equilibrium forces acting on the idealized system can be represented by

\[
F_I(t) + F_D(t) + F_S^*(t) + F_S^{**}(t) = P(t)
\]  

(6.45)

where \(F_I\), \(F_D\), \(F_S^*\), \(F_S^{**}\), and \(P\) can be identified with the corresponding terms in the equation (6.44). \(F_I\) and \(F_D\) are respectively the inertial and the damping force vector. \(F_S^*\) represents the internal resisting force related to the solid deformation only and \(F_S^{**}\) is the internal resisting force arising out of deformation of the fluid and coupling between the two phases. \(P\) denotes the applied load vector. A short time later, the equation would be

\[
F_I(t + \Delta t) + F_D(t + \Delta t) + F_S^*(t + \Delta t) + F_S^{**}(t + \Delta t) = P(t + \Delta t)
\]  

(6.46)

Subtracting (6.45) from (6.46) results

\[
F_I(t + \Delta t) - F_I(t) + F_D(t + \Delta t) - F_D(t) + F_S^*(t + \Delta t) - F_S^*(t)
\]

\[
+ F_S^{**}(t + \Delta t) - F_S^{**}(t) = P(t + \Delta t) - P(t)
\]  

(6.47)

From the definition,

\[
F_I(t + \Delta t) = M(t + \Delta t)\ddot{u}(t + \Delta t)
\]  

(6.48)
Expanding $M(t + \Delta t)$ and $\ddot{u}(t + \Delta t)$ in terms of Taylor series and retaining only the first order terms in $\Delta t$

$$F_j(t + \Delta t) = [M(t) + M'(t)\Delta t][\ddot{u}(t) + \dddot{u}(t)\Delta t]$$

$$= M(t)\ddot{u}(t) + M(t)\dddot{u}(t)\Delta t + M'(t)\dddot{u}(t)\Delta t$$  \hspace{1cm} (6.49)

Writing

$$\dddot{u}(t)\Delta t = \Delta \dddot{u}(t)$$  \hspace{1cm} (6.50)

and

$$M'(t)\Delta t = \Delta M(t)$$  \hspace{1cm} (6.51)

gives

$$F_j(t + \Delta t) = M(t)\ddot{u}(t) + M(t)\Delta \dddot{u}(t) + \Delta M(t)\dddot{u}(t)$$  \hspace{1cm} (6.52)

Similarly,

$$F_B(t + \Delta t) = C(t)\dot{u}(t) + C(t)\Delta \ddot{u}(t) + \Delta C(t)\dddot{u}(t)$$

$$F_S^*(t + \Delta t) = K^*(t)u(t) + K^*\Delta u(t) + \Delta K^*(t)u(t)$$

$$F_S^{**}(t + \Delta t) = K^{**}(t)u(t) + K^{**}\Delta u(t) + \Delta K^{**}(t)u(t)$$  \hspace{1cm} (6.53)

Use of (6.51) and (6.52) into (6.47) gives

$$M(t)\Delta \dddot{u}(t) + \Delta M(t)\dddot{u}(t) + C(t)\Delta \dddot{u}(t) + \Delta C(t)\dddot{u}(t)$$

$$+ K^*(t)\Delta u(t) + \Delta K^*(t)u(t) + K^{**}(t)\Delta u(t) + \Delta K^{**}(t)u(t)$$
This represents a general form of incremental equations. If all the mass, damping and stiffness quantities at the time \( t \) are known, the equation (6.54) can be solved for \( \Delta u(t) \) by step-forward integration scheme, which also yields \( \Delta u(t) \) and \( u(\Delta t) \). The quantities themselves are dependent on the solution \( \Delta u(t) \) and hence iterative scheme to reduce the cumulative accumulation of error at each step becomes inevitable. The representation of the form (6.49) is only an approximation and the residual force is given by

\[
R(t) = P(t + \Delta t) - F_i(t + \Delta t) - F_D(t + \Delta t) - F_S^*(t + \Delta t) - F_S^{**}(t + \Delta t)
\]

(6.55)

### 6.5 Material Non-linearity

This section gives the general system of non-linear behavior in which any or all of the mass, damping and stiffness properties may be path and time dependent. For small deformations, the incremental changes in these properties may be excluded. Further, if the changes in densities, permeability, compressibilities and porosity are negligible, the only time dependent property in the general form (6.54) is the stiffness matrix \( K^*(t) \). Rewriting (6.54) without the negligible terms gives

\[
M(t)\Delta \ddot{u}(t) + C(t)\Delta \dot{u}(t) + K^*(t)\Delta u(t) + K^{**}(t)\Delta u(t)
\]
\[ R(t + \Delta t) - M(t) \ddot{u}(t) - C(t) \dot{u}(t) - K^* (t) u(t) - F^*_S(t) u(t) \]  

(6.56)

Here \( F^*_S(t) \) represents the internal resisting force at \( t \). The incremental force \( \Delta F^*_S(t) \) is expressed as \( K^P(t) \Delta u(t) \). \( K^P(t) \) is the stiffness matrix dependent on the non-linear solid stress-strain relations. It is both path and time dependent and may be elastic or plastic unlike \( K^*(t) \), which is related to only elastic deformations in solid.

Now stress is

\[ \Delta \sigma = D \Delta \varepsilon = DB \Delta u \]  

(6.57)

where \( D \) is the plasticity matrix and \( B \) is the strain-displacement matrix. Recalling (6.43) and (6.56),

\[
K^P = \begin{bmatrix}
\int_{R} B^T DB \, dR & 0 \\
0 & 0
\end{bmatrix}
\]  

(6.58)

The force, \( F^*_S(t) \) is obtained from the stiffness at time \( t \). For non-linear stress-strain relations, (6.58) is evaluated iteratively at elemental level in the numerical program by using local iterative scheme. For the iteration at the global level, the convergence is checked by

\[
\frac{\| \Delta u_i - \Delta u_{i-1} \|}{\| \Delta u_i \|} < T
\]  

(6.59)

where \( \Delta u_i \) and \( \Delta u_{i-1} \) are, respectively, the solutions at \( i^{th} \) and \( (i-1)^{th} \) iterations. \( T \) is the tolerance and \( \| \Delta u \| \) represents the norm given by \( \sum |\Delta u_i| \). If the convergence is
not reached at the global level, the time step $\Delta t$ may required to be changed and analysis repeated all over again. However, in the wave propagation analysis, the time step cannot be arbitrarily reduced. The temporal refinement has to be accompanied by spatial refinement, which means reordering of elements and nodes. This would increase the computational effort. This is overcome to a large extent by appropriate choice of time step so as to eliminate the necessity of spatial and temporal refinement. For this, the time step is so chosen that the faster elastic wave would travel no more than one element length in a single time step.

### 6.6 Finite Elements Used

Linear, quadratic one-dimensional and two-dimensional elements are implemented. For the two-dimensional analysis, two types of elements, 4-4 and 8-8 isoparametric quadrilaterals, are used. In 4-4 element, the solid displacement and the relative fluid displacement are interpolated using bilinear Lagrange polynomials while in 8-8 element, biquadratic Lagrange polynomials are used. Linear interpolation functions corresponding to the nodal quantities for one-dimensional element are

$$
\psi_1 = \phi_1 = 1 - \frac{x}{l} \quad \psi_2 = \phi_2 = \frac{x}{l}
$$

(6.60)

where $l$ is the length of element. The shape functions for 4 node and 8 node isoparametric elements are well documented in literature (Zienkiewicz, 1979 and Bath (1996)). Each nodal point has 4 degrees of freedom. Thus the element matrices are 16
x 16 for the 4-4 element and 32 x 32 for the 8-8 element. The interpolation functions for the 4-4 element are

\[ \psi_i = \phi_i = \frac{1}{4}(1+s_0)(1+t_0) \]  

(6.61)

where the coordinate system, \( s_0 = s_i, t_0 = t_i \) and \( i=1,2,3,4 \). For 8-8 element, the shape functions are

\[ \psi_i = \phi_i = \frac{1}{4}(1+s_0)(1+t_0)(s_0 + t_0 - 1) \]  

for \( i=1,2,3,4 \)

\[ \psi_i = \phi_i = \frac{1}{4}(1-s_0^2)(1+t_0) \]  

for \( i=5,7 \)

\[ \psi_i = \phi_i = \frac{1}{4}(1+s_0)(1-t_0^2) \]  

for \( i=6,8 \)  

(6.62)

6.7 Time Domain Integration Procedure

For a time domain integration procedure of dynamic systems, the predictor-corrector form of Newmark method (Newmark, 1959) is choosen. In this method, using small time intervals, a step forward integration sequence is established. It is assumed that the displacements of solid and fluid in the mixture, as well as pore pressure, have been determined at the beginning of a small time interval before the time for which the analysis is required. Denoting \( u, \dot{u}, \) and \( \ddot{u} \) as the displacements, velocities, and accelerations respectively, Newmark method assumes

\[ \ddot{u}(t + \Delta t) = \frac{\ddot{u}(t)}{\beta} + (1 - \beta)\ddot{u}(t + \Delta t) + \gamma \Delta t \dot{u}(t + \Delta t) \]  

(6.63)
where $\beta$ and $\gamma$ are Newmark’s coefficients (Newmark, 1959) and can be determined to obtain integration accuracy and stability. The equation of motion at time $(t + \Delta t)$ is used to evaluate the acceleration, $\dddot{u}(t + \Delta t)$ as

$$\dddot{u}(t + \Delta t) = [M \dddot{u}(t + \Delta t) + [C \dddot{u}(t + \Delta t) + [K]u(t + \Delta t) = \{R(t + \Delta t)\} \tag{6.65}$$

From Equation (5.63), the acceleration, $\dddot{u}(t + \Delta t)$ is

$$\dddot{u}(t + \Delta t) = \frac{1}{\gamma \Delta t^2}[(u(t + \Delta t) - u(t)) - \frac{1}{2}(\gamma - 1)\dddot{u}(t + \Delta t)\tag{6.66}$$

Substituting Equation (6.66) into Equation (6.64), an expression for the velocity at time $(t + \Delta t)$ is obtained in terms of $u(t + \Delta t)$ as

$$\dot{u}(t + \Delta t) = [1 - \frac{\beta}{\gamma}]\dot{u}(t) + [(2 - \frac{\beta}{\gamma})\frac{\Delta t}{2}]\ddot{u}(t) + \frac{\beta}{\gamma}\{u(t + \Delta t) - u(t)\} \tag{6.67}$$

The Equations (6.66) and (6.67) cab be used in Equation (6.65) to get

$$u(t + \Delta t)[K + M(\frac{1}{\gamma \Delta t^2}) + C(\frac{\beta}{\gamma \Delta t})]$$

$$= R(t + \Delta t) + M[(\frac{1}{\gamma \Delta t^2})u(t) + (\frac{1}{\gamma \Delta t})\ddot{u}(t) + (\frac{1}{2})\gamma]$$

$$+ C[(\frac{\beta}{\gamma \Delta t})u(t) + (\frac{\beta}{\gamma})\ddot{u}(t) + (\frac{\Delta t}{2})]\dddot{u}(t)\tag{6.68}$$
An elastic-plastic Cap model proposed by Singh (1972) is chosen for modeling the plastic behavior of the saturated soil. This constitutive relation consists of the failure surface for the ideal plastic condition and the strain hardening yield surface which is a family of ellipsoids defined by the cumulative plastic volumetric strain. In this chapter, the implementation of the constitutive relation is provided.

7.1 Constitutive Model

This model is a specialization of the model introduced by DiMaggio and Sandler (1971). It consists of a failure cone of an ideal plastic condition defined by the Drucker-Prager yield condition (1952) and a strain hardening yield surface, which is a family of ellipsoids defined by the cumulative plastic volumetric strain. The ellipsoids are assumed to be of constant eccentricity with an axis parallel to hydrostatic axis and centers on the hydrostatic axis as shown in Figure 7.1.
Figure 7.1 Elastic Plastic Cap Model
7.1.1 Yield Surfaces

The failure surface of Singh’s model is expressed as

\[ f_1 = \alpha J_1 + J_2^{1/2} - \kappa = 0 \]  
(7.1)

The equation for the hardening yield surface is

\[ f_2 = \left( \frac{(J_1 - P_0)}{a} \right)^2 + \left( \frac{\sqrt{J_2}}{b} \right)^2 - 1 = 0 \]  
(7.2)

where \( \alpha \) and \( \kappa \) are material constants, \( P_0 \) is the center of ellipse, \( a \) and \( b \) are representing the radius of the ellipsoid. \( J_1 \) is the first invariant of the stress tensor and \( J_2 \) is the second invariant of the stress deviation tensor,

\[ J_1 = \sigma_{ii} \]  
(7.3)

\[ J_2 = \left( \frac{1}{2} \right) \sigma_{ij} \sigma_{ij} - \left( \frac{1}{6} \right) \sigma_{kk}^2 = \left( \frac{1}{2} \right) S_{ij} S_{ij} \]  
(7.4)

where

\[ S_{ij} = \sigma_{ij} - \left( \frac{1}{3} \right) \sigma_{kk} \delta_{ij} \]  
(7.5)

is the stress deviation tensor.

Because of the existence of intermediate principal stresses, the material constants \( \alpha \) and \( \kappa \) are not uniquely related to Mohr-Coulomb parameters, cohesion \( (c) \) and friction angle \( (\phi) \). However, for the case of conventional triaxial compression, \( \alpha \) and \( \kappa \) can be written in terms of \( c \) and \( \phi \) as
\[ \alpha = \frac{2 \sin \phi}{\sqrt{3(3 - \sin \phi)}} \]

\[ \kappa = \frac{6c \cos \phi}{\sqrt{3(3 - \sin \phi)}} \quad (7.6) \]

7.1.2 Parameters for Yield Surfaces

All points on a yield surface have the same plastic volumetric strain. The center of the ellipse is assumed to be on the hydrostatic axis and the tangent to the ellipse at its point of intersection with the failure surface is assumed to be horizontal. Referring to Figure 7.1, it is clear these requirements ensure that the center of the ellipse is on the x-axis directly below the point of intersection. All ellipses are assumed to have the same ratio of major to minor axis.

The point \((P_0, b)\) in Figure 7.1 lies on the failure surface \((f_1=0)\) as well as on the yield surface \((f_2=0)\). Substituting \(P_0\) for \(J_1\) and \(b\) for \(J_2\) in (7.1) gives

\[ \alpha P_0 + b = \kappa \quad (7.7) \]

or

\[ b = (\kappa - \alpha P_0) \quad (7.8) \]

From the constant ratio \((R=a/b)\),

\[ a = R(\kappa - \alpha P_0) \quad (7.9) \]

The yield condition with multiplication by \(a^2\) is

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\[(J_1 - P_0)^2 + R^2 J_2 - a^2 = 0 \quad (7.10)\]

From (7.9), (7.10) gives

\[P_0 \left[1 - \alpha^2 R^2 \right] - P_0 \left(2J_1 - 2R^2 \kappa \alpha \right) - \left( R^2 k - R^2 J_2 - J_1^2 \right) = 0 \quad (7.11)\]

Defining

\[A = \left(1 - \alpha^2 R^2 \right) \quad (7.12)\]

\[B = \left(2J_1 - 2R^2 \kappa \alpha \right) \quad (7.13)\]

\[C = - \left( - R^2 k + R^2 J_2 + J_1^2 \right) \quad (7.14)\]

(7.11) can be written as

\[AP_0^2 - BP_0 - C = 0 \quad (7.15)\]

When \( A \neq 0 \), the solution of (7.14) yields

\[P_0 = \left[ B + \left( B^2 + 4AC \right)^{1/2} \right] / 2A = 0 \quad (7.16)\]

Only the positive sign for the square root term is taken as the interest is only in the higher root. When \( A = 0 \),

\[P_0 = - \frac{C}{B} \quad (7.17)\]

Once the center of the ellipse is known, the other dimensions can be evaluated as
\[ a = R(k - \alpha P_0) \]

\[ b = (k - \alpha P_0) \]

\[ P_\varepsilon = P - a \] (7.18)

7.1.3 Plastic Volumetric Strain

The plastic volumetric strain was determined from the compression (\( \lambda^* \)) and rebound (\( \kappa^* \)) indices respectively obtained from an isotropic compression test. Singh (1972) gave the relationship between the dimension \( P_C \) and \( \varepsilon_{kk}^P \) as,

\[ \varepsilon_{kk}^P = -A \ln(-P_C) + B \] (7.19)

where

\[ A = \left( \frac{1}{2.3} \right) \left| \frac{\lambda^* - \kappa^*}{1 + \epsilon_0} \right| \] (7.20)

\[ B = A \ln(-3P_0) \] (7.21)

From (7.19),

\[ P_C = -\exp\left( \frac{B - \varepsilon_{kk}^P}{A} \right) \] (7.22)

Using Equation (7.22), the major and minor axes of the ellipse in terms of \( \varepsilon_{kk}^P \) are
$$P_0 = \left[ \frac{1}{1 + \alpha R} \right] - \exp \left( \frac{B - \varepsilon_{kk}^p}{A} \right)$$  \hfill (7.23)$$

$$a = P_C - P_0 = \left[ \frac{1}{1 + \alpha R} \right] \left( Rk - \exp \left( \frac{B - \varepsilon_{kk}^p}{A} \right) \right) + \exp \left( \frac{B - \varepsilon_{kk}^p}{A} \right)$$  \hfill (7.24)$$

$$b = \frac{a}{R} = \left[ \frac{1}{R(1 + \alpha R)} \right] \left( Rk - \exp \left( \frac{B - \varepsilon_{kk}^p}{A} \right) \right) + \frac{1}{R} \exp \left( \frac{B - \varepsilon_{kk}^p}{A} \right)$$  \hfill (7.25)$$

For the plastic volumetric strain,

$$\frac{\partial P_0}{\partial \varepsilon_{ij}^p} = \left[ \frac{1}{A(1 + \alpha R)} \right] \left( \exp \left( \frac{B - \varepsilon_{kk}^p}{A} \right) \right) \delta_{ij}$$  \hfill (7.26)$$

$$\frac{\partial a}{\partial \varepsilon_{ij}^p} = \left[ \frac{1}{A(1 + \alpha R)} \right] \left( \exp \left( \frac{B - \varepsilon_{kk}^p}{A} \right) \right) \delta_{ij} - \frac{1}{A} \exp \left( \frac{B - \varepsilon_{kk}^p}{A} \right) \delta_{ij}$$  \hfill (7.27)$$

$$\frac{\partial b}{\partial \varepsilon_{ij}^p} = -\left[ \frac{1}{AR(1 + \alpha R)} \right] \left( \exp \left( \frac{B - \varepsilon_{kk}^p}{A} \right) \right) \delta_{ij} + \frac{1}{A} \exp \left( \frac{B - \varepsilon_{kk}^p}{A} \right) \delta_{ij}$$  \hfill (7.28)$$

Differenciating the yield function with respect to
\[ \frac{\partial f_2}{\partial \sigma_{ij}} = 2b^2(J_1 - P_0)\delta_{ij} + a^2 S_{ij} \quad (7.29) \]

and

\[ \frac{\partial f_2}{\partial e_{ij}^p} = -2b^2(J_1 - P_0)\left( \frac{\partial P_0}{\partial e_{ij}^p} \right) + 2b(J_1 - P_0) \left( \frac{\partial b}{\partial e_{ij}^p} \right) + 2J_2a \left( \frac{\partial a}{\partial e_{ij}^p} \right) - 2ab^2 \left( \frac{\partial a}{\partial e_{ij}^p} \right) - 2a^2b \left( \frac{\partial b}{\partial e_{ij}^p} \right) \quad (7.30) \]

Substituting (7.20) through (7.23) into (7.24) and (7.25),

\[ \frac{\partial f_2}{\partial \sigma_{ij}} = 2b^2(J_1 - P_0)\delta_{ij} + a^2(\sigma_{ij} - \sigma_{kk}\delta_{ij}) \quad (7.31) \]

and

\[ \frac{\partial f_2}{\partial e_{ij}^p} = 2bP_c\delta_{ij} \left[ J_1 - P_0 \right] \left[ b + \alpha(J_1 - P_0) \right] + aR^2 \left( \frac{J_2 - 2b^2}{A(1 + \alpha R)} \right) \quad (7.32) \]

7.1.4 Elastic Stress-Strain Relations

The total strain rate, \( \dot{\varepsilon}_{ij} \), can be decomposed into the elastic strain rate, \( \dot{\varepsilon}_{ij}^e \) and the plastic strain rate, \( \dot{\varepsilon}_{ij}^p \), as:

\[ \dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p \quad (7.33) \]
The elastic strain components, $\dot{\varepsilon}_{ij}$, are uniquely related to the components of the effective stress rate, $\dot{\sigma}_{ij}$, by the fourth order elasticity tensor, $E_{ijkl}$, as

$$\dot{\sigma}_{ij} = E_{ijkl} \dot{\varepsilon}_{ij}$$

or

$$\dot{\sigma} = E \dot{\varepsilon}$$

(7.34)

7.1.5 Stress-Strain Relations for Plasticity

From the equation (7.33), stress-strain relations for work-hardening plasticity are given by

$$\dot{\sigma}_{ij} = E_{ijkl} \left( \dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^p \right)$$

(7.35)

The flow rule associated with the yield surface $f$ is,

$$\dot{\varepsilon}_{ij}^p = \lambda \left( \frac{\partial f}{\partial \sigma_{ij}} \right)$$

(7.36)

Substitution into (7.35) gives

$$\dot{\sigma}_{ij} = E_{ijkl} \left[ \dot{\varepsilon}_{kl} - \lambda \left( \frac{\partial f}{\partial \sigma_{kl}} \right) \right]$$

(7.37)

The consistency condition, $\dot{f} = 0$, implies

$$\dot{\sigma}_{ij} \left( \frac{\partial f}{\partial \sigma_{ij}} \right) + \dot{\varepsilon}_{ij}^p \left( \frac{\partial f}{\partial \varepsilon_{ij}^p} \right) = 0$$

(7.38)
Multiplying throughout by $\frac{\partial f}{\partial \sigma_{ij}}$ in the equation (7.37) and substituting for $\frac{\partial f}{\partial \sigma_{ij}}\sigma_{ij}$

give

$$\left(\frac{\partial f}{\partial \sigma_{ij}}\right)E_{ijkl}\dot{\varepsilon}_{kl} = \lambda \left[ -\left(\frac{\partial f}{\partial \sigma_{ij}}\right)\left(\frac{\partial f}{\partial \varepsilon_{ij}^p}\right) + \left(\frac{\partial f}{\partial \sigma_{ij}}\right)E_{ijkl}\left(\frac{\partial f}{\partial \sigma_{kl}}\right)\right]$$

(7.39)

which yields at

$$\lambda = \left(\frac{1}{\beta}\right)\left(\frac{\partial f}{\partial \sigma_{ij}}\right)E_{ijkl}\dot{\varepsilon}_{kl}$$

(7.40)

Denoting $\frac{\partial f}{\partial \sigma_{ij}} = q_{ij}$ and $\frac{\partial f}{\partial \varepsilon_{ij}^p} = p_{ij}$, $\beta$ can be expressed as

$$\beta = -p_{ij}q_{ij} + q_{ij}E_{ijkl}q_{kl}$$

(7.41)

From the equations (7.36), (7.40) and (7.41),

$$\dot{\varepsilon}_{ij}^p = L_{ijkl}\dot{\varepsilon}_{kl}$$

(7.42)

where

$$L_{ijkl} = \beta^{-1} q_{rs} E_{rsij} q_{kl}$$

(7.43)

The stress rate can be written in terms of elastic strain rate as:

$$\dot{\sigma}_{ij} = E_{ijkl}\dot{\varepsilon}_{kl}^e$$

$$= E_{ijkl} \left(\dot{\varepsilon}_{kl} - \dot{\varepsilon}_{kl}^p\right)$$

(7.44)

Use of (7.42) and (7.44) gives,
\[ \dot{\sigma}_{ij} = E_{ijkl} [ \delta_{km} \delta_{ln} - L_{klmn} ] \dot{\varepsilon}_{mn} \]  

(7.45)

Here \( L_{klmn} \) is a fourth order tensor given by the equation (7.43) and \( \beta \) is a scalar. Also \( p_{ij} \) and \( q_{ij} \) are evaluated for the known yield surface. Symbolically, the equation (7.45) is represented by

\[ \dot{\sigma} = D \dot{\varepsilon} \]

(7.46)

7.1.6 Stress-Strain Relations for Yield Surface \( f_1 = 0 \)

In case when yield surface \( f_1 \) alone governs the stress-strain relations, there is no strain hardening and the yield surface \( f_1 \)

\[ f_1 = \alpha J_1 + J_2^{1/2} - k = 0 \]  

(7.47)

For this surface

\[ q_{ij} = \frac{\partial f_1}{\partial \sigma_{ij}} = \left( \alpha \delta_{ij} + \frac{S_{ij}}{2 J_2^{1/2}} \right) \]  

(7.48)

and

\[ \dot{\sigma} = D_1 \dot{\varepsilon} \]

(7.49)

7.1.7 Stress-Strain Relations for Yield Surface \( f_2 = 0 \)

For the expanding cap, the yield surface is given by
\[ f_2(\sigma_{ij}, \varepsilon_{ij}^p) = 0 \]  \hspace{1cm} (7.50)

\( p_{ij} \) and \( q_{ij} \) are given as

\[ q_{ij} = \frac{\partial f_2}{\partial \sigma_{ij}} = 2b^2(J_1 - P_0)\delta_{ij} + a^2(\sigma_{ij} - \sigma_{kk}\delta_{ij}) \]  \hspace{1cm} (7.51)

\[ P_{ij} = \frac{\partial f_2}{\partial \varepsilon_{ij}^p} \]

\[ = \left[ 2bP_c\delta_{ij}(J_1 - P_0)\left[ \frac{b + \alpha(J_1 - P_0) + \alpha R^2(J_2 - 2b^2)}{A(1 + \alpha R)} \right] \right] \]  \hspace{1cm} (7.52)

The equation (7.33) and (7.34) give the stress-strain in the symbolic form

\[ \dot{\sigma} = D_2\dot{\varepsilon} \]  \hspace{1cm} (7.53)

7.1.8 Stress-Strain Relations for Singular Yield Surface (\( f_1 = 0 \) and \( f_2 = 0 \))

In Singh’s model, when the stress state is a point of intersection of the Drucker and Prager line with the cap where \( f_1 = 0 = f_2 \), the stress-strain relations are obtained by

\[ \dot{\varepsilon}_{ij}^p = \frac{1}{2} \left[ \lambda_{ij} \frac{\partial f_1}{\partial \sigma_{ij}} + \lambda_{ij} \frac{\partial f_2}{\partial \sigma_{ij}} \right] \]  \hspace{1cm} (7.54)

From (7.49) and (7.53),

\[ \dot{\varepsilon} = \frac{1}{2} \left[ D_1^{-1} + D_2^{-1} \right] \dot{\sigma} \]  \hspace{1cm} (7.55)

which gives
\[ \dot{\sigma} = D_3 \dot{\varepsilon} \quad \text{(7.56)} \]

with

\[ D_3 = 2 \left[ D_1^{-1} + D_2^{-1} \right]^{-1} \quad \text{(7.57)} \]

Equations (7.49), (7.53) and (7.56) completely define the stress-strain relations in the Singh’s model. In finite element analysis, depending the state of stress in any element at a particular instant of time the appropriate relationship is used.
8.1 Introduction

It is important to verify the proposed numerical procedure by comparing predictions against known solutions or available analytical solutions. Two analytical solutions proposed Garg et al. (1974) and Morland et al. (1987) for problems of wave propagations through saturated soils will be used to validate the proposed procedures. Garg’s method (1974) determines a Laplace transform solution which is inverted numerically for the case of equal step-function solid and fluid velocities imposed at the surface. Morland’s method (1987) uses a decomposition procedure to separate the singular field which incorporates all discontinuities of the velocity field from twice differential fields and considers successive reflections when the base is rigid and impervious. Both procedures are assumed a linear elastic solid and ideal fluid and are focused on one-dimensional compression waves induced by a surface velocity loading. Neither analytical method was extended to non-linear or plastic solids and the author is not aware of an analytical solution for this more general cases. Thus in this case, the numerical procedure is compared to experimental results presented by Dreger
To compare the proposed method to the solutions of Garg and Morland, the solution procedure described in the Chapter 6 needed to be modified to allow for specified velocity on the boundary. The modified time integration scheme for the velocity boundary conditions is described in Appendix A.

8.2 Garg’s Analytical Solution

In order to solve the field equations (4.1) and (4.2) analytically, Garg et. al. (1974) observed that these relations hold when the displacement are replaced by velocities. Restricting the case of homogeneous initial conditions, the boundary conditions can be written as:

\[ \dot{u}(0,t) = \dot{\psi}(0,t) = H(t) \]  

where \( H(t) \) is Heaviside function. Figure 8.1 shows the half space restricted to vertical direction and the dynamic event applied at the top surface.

Applying the Laplace transform of the field equations gives the coupled ordinary differential equations

\[
\begin{align*}
\hat{C}_{11} \frac{\partial^2 \hat{W}^s}{\partial Z^2} + C_{12} \frac{\partial^2 \hat{W}^f}{\partial Z^2} - r_s^2 \hat{W}^s + D \cdot s \left( \hat{W}^f - \hat{W}^s \right) &= 0 \\
C_{21} \frac{\partial^2 \hat{W}^s}{\partial Z^2} + C_{22} \frac{\partial^2 \hat{W}^f}{\partial Z^2} - r_s^2 \hat{W}^f - D \cdot s \left( \hat{W}^f - \hat{W}^s \right) &= 0
\end{align*}
\]

where

\[ L[\hat{u}(Z,t), \hat{\psi}(Z,t)] = \left[ \hat{W}^s(Z,s), \hat{W}^f(Z,s) \right] \]
and $s$ is the transform parameter. The material constants used by Garg et al. (1974) can be converted as

$$C_0^2 = \frac{\lambda + 2\mu + \alpha^2 M}{\rho}$$

$$C_1^2 = \frac{\lambda + 2\mu + M(\alpha - n)}{\rho^2}$$

$$C_2^2 = \frac{M(n)^2}{\rho^2}$$

$$C_{12}^2 = \frac{M(n)(\alpha - n)}{\rho^1}$$

$$C_{21}^2 = \frac{M(n)(\alpha - n)}{\rho^2}$$

$$C_\pm^2 = \left( C_1^2 + C_2^2 \pm \left[ (C_1^2 + C_2^2)^2 + 4C_{12}^2C_{21}^2 \right]^{1/2} \right)$$

$$\nu = D \frac{\rho}{\rho^s \rho^f}$$

$$r_s = \frac{\rho^s}{\rho}$$

$$r_f = \frac{\rho^f}{\rho}$$

$$r_s + r_f = 1 \quad (8.4)$$

A wave speed magnitude, $c_\sigma$, is given by (Garg et al. (1974))

$$\rho c_\sigma^2 = \tilde{c}_{11} + c_{12} + c_{21} + c_{22} \quad (8.5)$$

The normalized elastic moduli, $C_{ln}$ defined by

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\[ c_{lm} = \rho c_0^2 C_{lm} \quad (\ell, m = 1, 2) \]

\[ \hat{c}_{11} = \rho c_0^2 \hat{C}_{11} \quad (8.6) \]

The characteristic equation can be rewritten as (Garg et. al. (1974)):

\[ \nu(C_0^2 \theta^2 - s^2) - (C_+^2 \eta_+^2 - s^2)(C_-^2 \eta_-^2 - s^2) = 0 \quad (8.7) \]

It has been shown by Garg et. al. (1974) that \( C_0 \) is the wave velocity when the saturated soil acts a single material and \( C_\pm \) are the wave velocities when no viscous coupling exists. The characteristic equation has four roots of \( \theta \). Since the real part of the complex roots being negative implies amplification of the wave without bound, \( \theta \) is required to have positive real part. For the case of unit velocity at the top, the solution is given as:

\[
\begin{bmatrix}
\hat{u} \\
\hat{w}
\end{bmatrix} = \begin{bmatrix}
A^*_1 \\
B^*_1
\end{bmatrix} \exp\left(-\frac{\eta_1 Z}{C_+} \right) + \frac{\eta_1 Z}{C_+} \int_{Z/C_+}^{t} \exp(-\eta_1 \tau) f_1(\tau) d\tau \\
\begin{bmatrix}
A^*_2 \\
B^*_2
\end{bmatrix} \exp\left(-\frac{\eta_2 Z}{C_-} \right) - \frac{\eta_2 Z}{C_-} \int_{Z/C_-}^{t} \exp(-\eta_2 \tau) f_2(\tau) d\tau
\end{bmatrix}
\times \left[ H\left(t - \frac{Z}{C_+}\right) - H\left(t - t_1 - \frac{Z}{C_+}\right) \right]
\]

\[ + \begin{bmatrix}
A^*_2 \\
B^*_2
\end{bmatrix} \exp\left(-\frac{\eta_2 Z}{C_-} \right) - \frac{\eta_2 Z}{C_-} \int_{Z/C_-}^{t} \exp(-\eta_2 \tau) f_2(\tau) d\tau \\
\begin{bmatrix}
A^*_1 \\
B^*_1
\end{bmatrix} \exp\left(-\frac{\eta_1 Z}{C_+} \right) + \frac{\eta_1 Z}{C_+} \int_{Z/C_+}^{t} \exp(-\eta_1 \tau) f_1(\tau) d\tau
\end{bmatrix}
\times \left[ H\left(t - \frac{Z}{C_-}\right) - H\left(t - t_1 - \frac{Z}{C_-}\right) \right] \quad (8.8) \]

where
\[ A_1^* = \frac{C_1^2 - C^2 + C_{12}^2}{C_+^2 - C_-^2} \]

\[ B_1^* = \frac{-C_1^2 + C_+^2 + C_{21}^2}{C_+^2 - C_-^2} \]

\[ A_2^* = 1 - A_1^* \]

\[ B_2^* = 1 - B_1^* \]

\[ \eta_1 = \frac{\nu}{2} \frac{C_+^2 - C_0^2}{C_+^2 - C_-^2} \]

\[ \eta_2 = \frac{\nu}{2} \frac{C_0^2 - C_-^2}{C_+^2 - C_-^2} \] (8.9)

and

\[ f_1 = \frac{I_1 \left[ \eta_1 \left( \frac{\tau^2 - \frac{Z^2}{C_+}}{C_+} \right)^{\frac{1}{2}} \right]}{\left( \frac{\tau^2 - \frac{Z^2}{C_+}}{C_+} \right)^{\frac{1}{2}}} \]

\[ f_2 = \frac{I_1 \left[ \eta_2 \left( \frac{\tau^2 - \frac{Z^2}{C_-}}{C_-} \right)^{\frac{1}{2}} \right]}{\left( \frac{\tau^2 - \frac{Z^2}{C_-}}{C_-} \right)^{\frac{1}{2}}} \] (8.10)
Here $H(t)$ is Heaviside step function and $I_1$ is the modified Bessel function of first kind with order one.
Figure 8.1 One-Dimensional Idealization of Half Space
8.3 Morland’s Analytical Solution

To verify successive reflections at rigid boundary, Morland’s analytical solution is used. In addition to Garg’s case of applied solid and fluid velocities on surface:

\[ z = 0: \quad \dot{u} = w_s(t) \]
\[ \dot{w} = w_f(t) \quad (8.11) \]

the base \( z = L \) is supposed rigid and impervious (Figure 8.2), so that

\[ \text{for } z = L: \quad u = w = 0 \text{ and } \dot{u} = \dot{w} = 0 \quad (8.12) \]

The solution to the transform velocity equations (8.1) and (8.2) satisfying the zero velocity condition (8.12) on the rigid and impervious base is given by Morland et al. (1987) as

\[
\overline{W}^s = A_1(s)\left[\exp\left[-\theta_1(s)Z\right] - \exp\left[-\theta_1(s)Z(2L - L)\right]\right]
\]
\[
+ A_2(s)\left[\exp\left[-\theta_2(s)Z\right] - \exp\left[-\theta_2(s)Z(2L - L)\right]\right] \quad (8.13)
\]
\[
\overline{W}^f = B(\theta_1^2)A_1(s)\left[\exp\left[-\theta_1(s)Z\right] - \exp\left[-\theta_1(s)Z(2L - L)\right]\right]
\]
\[
+ B(\theta_2^2)A_2(s)\left[\exp\left[-\theta_2(s)Z\right] - \exp\left[-\theta_2(s)Z(2L - L)\right]\right] \quad (8.14)
\]

where

\[
B(\theta^2) = \frac{D \cdot s + r_s s^2 - \hat{C}_1 \theta^2}{D \cdot s + C_{12} \theta^2} \quad (8.15)
\]

and \( \theta_1^2, \theta_2^2 \) are the roots of

\[
\Delta \lambda^2 - (as^2 + bD \cdot s) \lambda + r_s + r_f s^4 + D \cdot s^3 = 0 \quad (8.16)
\]

where
\[ a = r_f \hat{C}_{11} + r_s C_{22} \]
\[ b = \hat{C}_{11} + C_{22} + C_{12} + C_{21} \]  
(8.17)

\( \theta_1 \) and \( \theta_2 \) are the roots with positive real part as \( |s| \to \infty \) in \( \text{Re}(s) > 0 \). Setting

\[ B(\theta_r^2) = B_r \quad (r = 1, 2) \]  
(8.18)

and writing \( A_r \) for \( A_r(s) \) the surface velocity conditions (8.11) give

\[
A_1 = \frac{B_2 W_s - W_f}{X_1(B_2 - B_1)}
\]
\[
A_2 = \frac{W_f - B_1 W_s}{X_2(B_2 - B_1)}
\]  
(8.19)

where

\[ X_r(s) = 1 - e^{-2L\theta_r} r(s) \quad (r = 1, 2) \]  
(8.20)

Similarly, setting

\[ Y_r(s) = 1 + e^{-2L\theta_r} r(s) \quad (r = 1, 2) \]  
(8.21)

Transforms of the form (8.13) and (8.14) have no analytic inversion. Since

\[ |e^{-2L\theta_r}| < 1 \quad \text{when} \quad \text{Re}(\theta_r) > 0, \]

restrictions that \( \theta_r^2 \) has no zeros in \( \text{Re}(s) > 0 \), and \( \text{Re}(\theta_r) \) is positive in \( \text{Re}(s) > 0 \) as \( |s| \to \infty \), are imposed, then

\[ |e^{-2L\theta_r}| < 1 \quad \text{in} \quad \text{Re}(s) > 0 \quad (r=1,2). \]  
(8.22)

Hence

\[ X_r^{-1} = \sum_{n=0}^{\infty} \exp(-2Ln\theta_r) \]
\[
Y_r^{-1} = \sum_{n=0}^{\infty} (-1)^n \exp(-2Ln\theta_r) \quad (r=1,2) \quad (8.23)
\]

are convergent in Re(s) > 0 on the inversion contour, and can be substituted directly in the coefficients (8.19).

In the velocity transform (8.13) and (8.14), each exponential term gives rise to the combination

\[
X_r^{-1}e^{-\theta r} = \sum_{n=0}^{\infty} \exp\{- (k + 2nL)\theta_r\} \quad (8.24)
\]

Define

\[
\alpha_n = Z + 2Ln \quad (n=1,2) \quad (8.25)
\]

\[
\beta_n = 2L(n+1) - Z
\]

then

\[
\bar{W}^s = \sum_{r=1}^{2} \sum_{n=0}^{\infty} \left\{ A_r \left[ e^{-\alpha n^r} - e^{-\beta n^r} \right] \right\} \quad (8.26)
\]

\[
\bar{W}^f = \sum_{r=1}^{2} \sum_{n=0}^{\infty} \left\{ B_r \hat{A}_r \left[ e^{-\alpha n^r} - e^{-\beta n^r} \right] \right\} \quad (8.27)
\]

where

\[
\hat{A}_r = X_r A_r \quad (r=1, 2) \quad (8.28)
\]

Note that \( \alpha_0 > 0, \beta_0 > 0 \) and \( \alpha_n, \beta_n > 0 \) (n=1,2,…). Since \( \theta_r \) is analytic in Re(s) > 0 (r=1, 2):

\[
\theta_r = \frac{s}{q_r} \left[ 1 + \frac{\ell_r}{s} + \frac{m_r}{s^2} + \frac{n_r}{s^3} + 0 \left( \frac{1}{s^4} \right) \right]
\]
for $|s| > R > 0$ in $\text{Re}(s) > 0$ \hfill (8.29)

and the inversion

$$L^{-1}[\hat{f}] = f(T) = \frac{1}{2} \int_{c-i\infty}^{c+i\infty} f(s) e^{st} \, ds \quad (c > 0). \hfill (8.30)$$

with $c > R$ can be chosen. The coefficients $\ell_r, m_r, n_r$ are given later. Thus

$$- \exp(-k_n \theta_r) = \exp(-k_n s / q_r) \cdot \exp(-k_n \ell_r / q_r) \cdot \varphi_r(k_n, s),$$

for $|s| > R \hfill (8.31)$

where $k_n$ denotes $\alpha_n$ or $\beta_n$ in (8.26), (8.27), and

$$w_r(k_n, s) = 1 - \frac{k_n m_r}{q_r} \frac{n_r}{s} + \left( n_r - \frac{k_n m_r}{2q_r} \frac{1}{s^2} + 0 \left( \frac{1}{s^3} \right) \right)$$

for $|s| > R \hfill (8.32)$

Also, in $\text{Re}(s) > 0$,

$$B_r = b_{r1} + \frac{b_{r2}}{s} + 0 \left( \frac{1}{s^3} \right)$$

for $|s| > R \hfill (8.33)$

Allowing step function surface conditions, in $\text{Re}(s) > 0$,

$$\bar{W}_s = \frac{1}{s} \left[ g_0 + \frac{g_1}{s} + \frac{g_2}{s^2} + 0 \left( \frac{1}{s^3} \right) \right] \quad |s| > R \hfill (8.34)$$

and hence

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\[
\hat{A}_r = \frac{1}{s} \left[ a_{r0} + \frac{a_{r1}}{s} + \frac{a_{r2}}{s^2} + 0\left(\frac{1}{s^3}\right)\right] \quad |s| > R
\]

\[
B_r \hat{A}_r = \frac{1}{s} \left[ e_{r0} + \frac{e_{r1}}{s} + \frac{e_{r2}}{s^2} + 0\left(\frac{1}{s^3}\right)\right]
\]  (8.35)

where \(a_{r0}, e_{r0}\) are non-zero if \(g_0\) or \(f_0\) is non-zero; that is, a step function surface condition.

Incorporating the expansions (8.31), (8.32), and (8.35) in (8.26), (8.27) shows that in \(|s| > R\):

\[
\mathcal{W}^s = \sum_{r=1}^{\infty} \sum_{n=0}^{\infty} \exp\left(\frac{-\alpha_n s}{q_r}\right) \exp\left(\frac{-\alpha_n \ell r}{q_r}\right) \left[ G_r^* (\beta_n, s) + O\left(\frac{1}{s^4}\right)\right] - \exp\left(\frac{-\beta_n s}{q_r}\right) \exp\left(\frac{-\beta_n \ell r}{q_r}\right) \left[ G_r^* (\beta_n, s) + O\left(\frac{1}{s^4}\right)\right]
\]  (8.36)

\[
\mathcal{W}^f = \sum_{r=1}^{\infty} \sum_{n=0}^{\infty} \exp\left(\frac{-\alpha_n s}{q_r}\right) \exp\left(\frac{-\alpha_n \ell r}{q_r}\right) \left[ F_r^* (\alpha_n, s) + O\left(\frac{1}{s^4}\right)\right] - \exp\left(\frac{-\beta_n s}{q_r}\right) \exp\left(\frac{-\beta_n \ell r}{q_r}\right) \left[ F_r^* (\beta_n, s) + O\left(\frac{1}{s^4}\right)\right]
\]  (8.37)

where

\[
G_r^* (k_n, s) = \frac{G_{r0}}{s} + \frac{G_{r1}(k_n)}{s^2} + \frac{G_{r2}(k_n)}{s^3}
\]

\[
F_r^* (k_n, s) = \frac{F_{r0}}{s} + \frac{F_{r1}(k_n)}{s^2} + \frac{F_{r2}(k_n)}{s^3}
\]  (8.38)

with \(G_{r0}, F_{r0}\), (independent of \(k_n\)) non-zero for a step-function surface condition (Morland et al. (1987)). Invoking the shift theorem
\[ L^{-1}\left[e^{-rs} \hat{f}(s)\right] = f(T - r)H(T - r) \quad r > 0 \] (8.39)

Each * term in (8.49), (8.50), and (8.51) has an inverse form

\[ L^{-1}\left[ \exp\left(-\frac{k_n s}{q_r}\right) \exp\left(-\frac{k_n \ell_r}{q_r}\right) G^*_r(k_n, s)\right] = \exp\left(-\frac{k_n \ell_r}{q_r}\right) G^*_r(k_n, T - \frac{k_n}{q_r})H(T - \frac{k_n}{q_r}) \] (6.40)

and the remainder \(0(s^{-4})\) terms contribute terms \(0\left(T - \frac{k_n}{q_r}\right)^3\)\(H(T - \frac{k_n}{q_r})\) as \(T \rightarrow \frac{k_n}{q_r}\).

From (8.38),

\[ G^*_r(k_n, T - \frac{k_n}{q_r}) = G_{r0} + G_{r1}(k_n) \left[T - \frac{k_n}{q_r}\right] + \frac{1}{2} G_{r2}(k_n) \left[T - \frac{k_n}{q_r}\right]^2 \]

\[ F^*_r(k_n, T - \frac{k_n}{q_r}) = F_{r0} + F_{r1}(k_n) \left[T - \frac{k_n}{q_r}\right] + \frac{1}{2} F_{r2}(k_n) \left[T - \frac{k_n}{q_r}\right]^2 \] (8.41)

and the terms \(G_{r0}e^{-\alpha_{n, \ell}/q_r}, F_{r0}e^{-\alpha_{n, \ell}/q_r}\) are the amplitudes of the velocity discontinuities on \(k_n = q_r T\), non-zero when the prescribed surface velocities have a discontinuity \(g_0\) or \(f_0\). Define

\[ W^*_s(Z, T) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \exp\left(-\frac{\alpha_{n, \ell}}{q_r}\right) G^*_r \left(\alpha_{n, T} - \alpha_n\right) H\left(\frac{T - \alpha_n}{q_r}\right) \]

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\[-\exp\left(-\frac{\beta_n \ell}{q_r}\right) G_r\left(\frac{\beta_n , T - \beta_n}{q_r}\right) H\left(\frac{T - \beta_n}{q_r}\right) \quad (8.42)\]

\[W_s f (Z,T) = \sum_{r=0}^{2} \sum_{n=0}^{\infty} \exp\left(-\frac{\alpha_n \ell}{q_r}\right) F_r\left(\frac{\alpha_n , T - \alpha_n}{q_r}\right) H\left(\frac{T - \alpha_n}{q_r}\right)\]

\[-\exp\left(-\frac{\beta_n \ell}{q_r}\right) F_r\left(\frac{\beta_n , T - \beta_n}{q_r}\right) H\left(\frac{T - \beta_n}{q_r}\right) \quad (8.43)\]

and make the decompositions

\[W_s = W_s^s + W_c^s\]

\[W_f = W_s^f + W_c^f \quad (8.44)\]

then \(W_s^s, W_c^s\) are twice continuously differentiable since the remainder terms in (8.36), (8.37) are zero and have zero first and second derivatives at each \(T = k_n / q_r\) (Morland et al. (1987)). Since \(W_s^s, W_c^f\) are zero at \(T = 0, 0 < Z < L\), then \(W_c^s, W_c^f\) are zero at \(T = 0, 0 < Z < L\), while boundary conditions give

\[Z = L:\]

\[W_c^s = -W_s^s(L,T)\]

\[W_c^f = -W_s^f(L,T) \quad (8.45)\]

\[Z = 0:\]

\[W_c^s = W_s(T) - W_s^s(0,T)\]

\[W_c^f = W_f(T) - W_s^f(0,T) \quad (8.46)\]

or

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\[ Z = 0 : \]

\[ W_0^f + W_0^s (0, T) = W_e^f + W_e^s (0, T) \]  \hspace{1cm} (8.47)

Prescribed step-function velocities on \( Z = 0 \) at \( T = 0 \) are given by non-zero \( g_0, f_0 \) in the expansions (8.34), and hence non-zero \( a_{r0}, e_{r0} \) in the expansions (8.35) given by the relation (8.19) for general prescribed velocity conditions.

In the analytic decomposition (8.44) with singular parts \( W_e^s, W_e^f \) depicted by (8.42) and (8.43), the construction involves expansion of terms of the form \( e^{-k_0r} \) in (8.31), for large \(|s|\), hinging on the expansion (8.30) for \( q_0 \theta_0 / s \) in powers of \( 1/s \) with coefficients \( \ell_r, m_r, n_r \), which in turn enter the coefficients in the expansion of \( w_r(k_n, s) \), in (8.34) and \( \beta_r \) in (8.35). The coefficients \( \ell_r, m_r, n_r \) are given by Morland et al. (1987) as

\[ \ell_r = \frac{1}{2} \hat{d} \ell_r \]

\[ m_r = \frac{1}{2} \hat{d}^2 (\hat{m}_r - \frac{1}{4} \hat{\ell}_r) \]

\[ n_r = \frac{1}{2} \hat{d}^3 (\hat{n}_r - \frac{1}{2} \hat{\ell}_r \hat{m}_r + \frac{1}{8} \hat{\ell}_r^3) \]  \hspace{1cm} (8.48)

where

\[ \hat{\ell}_1 = \frac{q_1^2 - q_\infty^2}{q_1^2 - q_2^2} \]

\[ \hat{\ell}_2 = \frac{q_\infty^2 - q_2^2}{q_1^2 - q_2^2} \]
\[ \hat{m}_1 = -\frac{q_1^2 (q_1^2 - q_2^2)(q_2^2 - q_2^2)}{(q_1^2 - q_2^2)^3} \]

\[ \hat{m}_2 = -\frac{q_2^2}{q_1^2} \hat{m}_1 \]

\[ \hat{n}_1 = \frac{q_2^2 (q_1^2 - q_2^2) - q_1^2 (q_1^2 - q_2^2)}{(q_1^2 - q_2^2)^2} \]

\[ \hat{n}_2 = -\frac{q_2^2}{q_1^2} \hat{n}_1 \] (8.49)
Figure 8.2 Representative Soil Column with Rigid Base
8.4 Experimental Program

Dreger (1995) conducted an experimental program in which a one-dimensional wave was propagated through a fluid-saturated porous solid. To obtain a range of solid-fluid coupling effects, sand and lead shot were used as the solids and different viscosities of fluid were used in a shock tube test. The sand was a fine to medium grained silica sand commonly referred to as Ottawa sand. Lead shot with an average diameter of 2.3 mm was used to conduct highly dense samples for the tests. The material properties of tested Ottawa sand and lead shot (Wolfe et al. (1986) and Dreger (1995)) are summarized in Table 8.1. While the modulus of elasticity of sand and shot were similar, the specific gravity of lead shot was more than four times that of sand. The absolute permeability of sand was lower than that of lead shot by two orders of magnitude. Dow Corning 200 Fluid is a water-clear silicon fluid available in viscosities ranging from 0.65 to 100,000 centistokes. Dow Corning 200 Fluid was used as a test fluid to construct saturated samples for varying viscosity. For the experiments, Kinematic viscosities of 1.0, 5.0, and 200.0 centistokes were used. The properties of the silicone fluid used in the laboratory tests are summarized in Table 8.2. A typical configuration of the shock tube is shown in Figure 8.3. Table 8.3 shows the combinations and arrangement of solid and fluid materials used for the samples tested. More details of the experimental setup and the test method can be found in Dreger (1995).
<table>
<thead>
<tr>
<th>Property</th>
<th>Ottawa Sand</th>
<th>Lead Shot</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific Gravity</td>
<td>2.65</td>
<td>11.35</td>
</tr>
<tr>
<td>Total Mass Density (g/cm³)</td>
<td>1.62</td>
<td>6.70</td>
</tr>
<tr>
<td>Modulus of Elasticity (kg/cm²)</td>
<td>483</td>
<td>477</td>
</tr>
<tr>
<td>Poisson’s Ratio</td>
<td>0.16</td>
<td>0.3</td>
</tr>
<tr>
<td>Bulk Modulus (Grains) (kg/cm²)</td>
<td>370000</td>
<td>457000</td>
</tr>
<tr>
<td>Bulk Modulus (Skeleton) (kg/cm²)</td>
<td>240</td>
<td>397</td>
</tr>
<tr>
<td>Porosity</td>
<td>0.61</td>
<td>0.59</td>
</tr>
<tr>
<td>Absolute Permeability (cm²)</td>
<td>3.10x10⁻⁷</td>
<td>7.19x10⁻⁵</td>
</tr>
</tbody>
</table>

Table 8.1 Properties of Tested Solids
<table>
<thead>
<tr>
<th>Kinematic Viscosity (centistokes)</th>
<th>Specific Gravity (Gs)</th>
<th>Specific Weight (g/cm³)</th>
<th>Bulk Modulus (kg/cm²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.816</td>
<td>9.816</td>
<td>7,547</td>
</tr>
<tr>
<td>5.0</td>
<td>0.913</td>
<td>0.913</td>
<td>8,444</td>
</tr>
<tr>
<td>200.0</td>
<td>0.967</td>
<td>0.967</td>
<td>8,943</td>
</tr>
</tbody>
</table>

Table 8.2 Properties of Dow Corning 200 Fluid
<table>
<thead>
<tr>
<th>Cases</th>
<th>Fluid Viscosity (cSt)</th>
<th>Solid</th>
<th>Relative Density (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>1.0</td>
<td>Sand</td>
<td>95.6</td>
</tr>
<tr>
<td>Case 2</td>
<td>5.0</td>
<td>Sand</td>
<td>58.2</td>
</tr>
<tr>
<td>Case 3</td>
<td>200.0</td>
<td>Sand</td>
<td>53.3</td>
</tr>
<tr>
<td>Case 4</td>
<td>1.0</td>
<td>Lead Shot</td>
<td>105.4</td>
</tr>
<tr>
<td>Case 5</td>
<td>5.0</td>
<td>Lead Shot</td>
<td>66.5</td>
</tr>
<tr>
<td>Case 6</td>
<td>200.0</td>
<td>Lead Shot</td>
<td>66.5</td>
</tr>
</tbody>
</table>

Table 8.3 Combinations of Solid and Fluid Materials for Shock Tube Tests from Dreger (1995)
8.5 Results

8.5.1 Verification with Analytical Solutions

With compression wave propagation in a homogeneous linear elastic isotropic porous material with velocity boundary conditions, the verification by comparisons with Garg’s analytical solution is made to represent the one-dimensional wave propagation phenomenon. The effect of time step size on accuracy of the numerical solution is investigated. By comparisons with Morland’s analytical solution, successive reflections at the rigid and impervious base are verified.

Garg et al. (1974) originally provided solutions to two extreme cases which he called strong and weak viscous couplings. In reality, the weak coupling case that allows relative motion between the solid and the fluid is of more interest because in the strong coupling case, the solid and the fluid act as single material. Thus, the case of weak coupling is studied. The physical properties used by Garg et al. (1974) are given in Table 8.4.

The analytical and numerical analyses are performed for a time history of velocities. Velocity profiles at time stages of 0.5T and 1.0T are also plotted. Here, T is the time required for the wave to travel from top surface to a depth of 0.5m distance. Figure B.1 through B.4 in Appendix B show the results of velocity histories at time stage 0.5T and 1.0T. The numerical results are in good agreement with Garg’s solutions. Selected results representing velocity histories of the fluid and the solid at 0.1m, 0.2m, 0.3m, and 0.4m from the top are plotted in Figure B.5 through B.12. The results are also analyzed for the effect of time step size on the solution for velocities.
The choice of time step is an important factor in the quality of numerical results in transient response studies. Three time steps are employed and the influence of the variation on the histories of the solid and the fluid velocities at different locations is observed. Figure B.13 through B.20 show the result of comparisons. It is clear that in general, smaller time step gives good results. However, the comparison results also show that reducing the step size provides oscillations at the change of velocities in some cases.

The same physical properties used by Garg et al. (1974) are applied by Morland. To investigate the effect of reflection, both numerical and analytical solutions are performed up to seven reflections (7.0T) and compared. The velocity profiles for the solid and the fluid at 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, and 7.0T are plotted and compared in Figure C.1 through C.28 at Appendix C. Velocity histories of the solid and the fluid at four different locations (0.2, 0.4, 0.6, and 0.8m from the top) are also plotted. Figure C.29 through C.36 shows comparisons of velocity histories between analytical and numerical solutions. Numerical solution shows very good agreement with Morland’s analytical solutions.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>0.18</td>
</tr>
<tr>
<td>$\rho(kgm^{-3})$</td>
<td>2316.2</td>
</tr>
<tr>
<td>$\rho'(kgm^{-3})$</td>
<td>180</td>
</tr>
<tr>
<td>$\kappa (N^{-1}m^2)$</td>
<td>0.8475x10^{-10}</td>
</tr>
<tr>
<td>$\kappa_1 (N^{-1}m^2)$</td>
<td>0.2778x10^{-10}</td>
</tr>
<tr>
<td>$\kappa_f (N^{-1}m^2)$</td>
<td>4.5455x10^{-10}</td>
</tr>
<tr>
<td>$\dot{C}_{11} (Nm^{-2})$</td>
<td>2.7537 x10^{10}</td>
</tr>
<tr>
<td>$C_{12} (Nm^{-2})$</td>
<td>0.0928x10^{10}</td>
</tr>
<tr>
<td>$C_{22} (Nm^{-2})$</td>
<td>0.0339x10^{10}</td>
</tr>
<tr>
<td>$c_0 (ms^{-1})$</td>
<td>3548.5</td>
</tr>
</tbody>
</table>

Table 8.4 Material Properties for Garg’s and Morland’s Solutions
8.5.2 Verification with Experimental Results

8.5.2.1 Input Parameters

The elastic properties of materials are given in the Tables 8.1 and 8.2. However, Hardin and Richart (1963) found that the shear modulus of a granular soil varied with its density or void ratio, \( e \) and mean effective stress, \( p \) and many empirical relations were suggested for a variety of soils. In present study, Wan and Guo’s empirical relation (1998) is used as

\[
G = G_o \frac{(2.17 - e)^2}{(1 + e)} \sqrt{p}
\]

(8.50)

where \( G_o \) is a material constant which is the shear modulus at very small amplitudes of shear strains. Because the Poisson’s ratio is constant, the Young’s modulus is non-linear by the relation.

The input parameters for plastic behavior are Drucker and Prager parameters (\( \alpha \) and \( \kappa \)), the compression (\( \lambda^* \)) and rebound (\( \kappa^* \)) indices, and the shape aspect ratio, \( R \). The compression (\( \lambda^* \)) and rebound (\( \kappa^* \)) indices can be obtained from basic soil property tests such as an isotropic compression test. The material parameters \( \alpha \) and \( \kappa \) can be obtained from the relations of Mohr- Coulomb parameters, friction angle (\( \phi \)) and cohesion (\( c \)) as following;

\[
\alpha = \frac{2 \sin \phi}{\sqrt{3(3 - \sin \phi)}}
\]

\[
k = \frac{6c \cos \phi}{\sqrt{3(3 - \sin \phi)}}
\]

(7.6)
In order to define the elliptical cap, it is necessary to choose an initial volume at an initial stress that represents $\varepsilon^P_v = 0$ and a change of volume at a different stress. Wu et al. (1983) suggested a simple relationship between $R$ and $\alpha$. Using the approximation of Wu et al. the shape ratio $R$ of elliptical cap is chosen as:

$$ R = \frac{1}{\alpha} $$  \hspace{1cm} (8.51)

However, Kim (1988) found that this simple relationship showed good agreement between predicted and measured results only within small range of strains. He suggested the shape ratio of the ellipse is a function of relative density. For numerical predictions, the shape ratio, $R$ for the sand solid is obtained from Kim’s (1988) results. For the lead shot solid, the shape ratio is obtained from Wu’s simple relation (1983) due to the difficulty of obtaining the triaxial test results for lead shot.

8.5.2.2 Results

The measured pore pressure histories from the results of experiments are used to verify the non-linear plastic portion of the numerical program. The test results of pore pressure time histories for sand and lead shot solids are presented in Appendix D. Predicted pore pressure time histories with comparisons of experimental results are presented in Figures E.1 through E.12. In Figures E.1 through E.6, comparisons of the pore pressure time histories for sand as a solid with fluid viscosities of 1.0 cSt., 5.0 cSt., and 200.0 cSt., are given. In each case, the numerical predictions are in good agreement with experimental results. The differences between the numerical predictions and experimental results are less than 10% through a measured time period
of nine milliseconds. This is enough time for more than three reflections of the waves off the rigid bottom of the tube. The predicted peak responses of the first cycle in the pore pressure time history at the locations of transducers Pa and Pb are slightly over-predicted while subsequent predicted peak responses are slightly under-predicted. These results indicate that the numerical program is capable of analyzing the dynamic behavior of saturated sand with a wide range of the fluid viscosities. The comparisons of the measured and predicted pore pressure time histories for the cases of lead shot solid are shown in Figure E.7 through E.12. Although the maximum values compare very well and the trend of pore pressure time history for the numerical predictions is similar to one measured in the experiments, the calculated values predicted a high frequency oscillation not measured in the actual pore pressure time history.

8.5.2.3 Numerical Consideration of Equivalent Mass Coupling

The reason that the predictions for sand solid cases show good agreement with experimental results, but in the case of the lead shot solid there are significant oscillations in the predicted pore pressure time histories, may be found in the major difference between two solid properties given in Table 8.1. Although the bulk modulus and porosity are similar to each other, the mass density of lead shot is more than four times that of sand. To address the oscillation problem, it may be assumed that the oscillation problem is related to the solid mass.

The added mass term in Biot’s original theory (1956a and 1956b) does not appear in Hiremath’s formulation. Biot stated that “when the solid is accelerated
(\(\rho^{11}\ddot{u}\)), a force, \(\rho^{12} \dddot{w}\), must be exerted on the fluid to prevent an average displacement of the fluid. This effect is measured by the coupling effect, \(\rho^{12}\).

According to the interpretation of de Boer (1996), Dreski (1978) proposed similar effects with a velocity force. Dreski divided the fluid density into the density of the free fluid, \(\rho^f\), moving with the velocity of the fluid (\(v_f\)) and the density of the trapped fluid, \(\rho^g\), moving with the velocity of the skeleton (\(v_s\)). The mass-center velocity of the whole fluid, \(\dot{w}\), is given as

\[
\rho^f \dot{w} = \rho^f v_f + \rho^g v_s
\]

However, both Biot’s added mass and Dreski’s trapped mass did not provide how there terms could be evaluated.

In fluid dynamics, the conservation of mass in saturated soils can be given as (Ingebritsen and Sanford (1998))

\[
-\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = \frac{1}{\rho^f} \frac{\partial (\rho^f n)}{\partial t}
\]

For isotropic conditions, Darcy’s Law gives

\[
\left(\frac{\partial}{\partial x} \left( K \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y} \left( K \frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z} \left( K \frac{\partial h}{\partial z}\right)\right) = \frac{1}{\rho^f} \frac{\partial (\rho^f n)}{\partial t}
\]

\[
K \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right) = \frac{1}{\rho^f} \frac{\partial (\rho^f n)}{\partial t} = \frac{n}{\rho^f} \frac{\partial \rho^f}{\partial t} + \frac{\dot{c}}{\partial t}
\]

where \(h\) is the hydraulic head. In static conditions, the right side of this equation is zero. In dynamic conditions, these terms play an important role. The first term on the right side of this equation can be expressed as a function of pore pressure (\(p\)):
\[
\frac{n \frac{\partial \rho^f}{\partial t}}{\rho^f} = \frac{n \frac{\partial \rho^f}{\partial p}}{\rho^f} \frac{\partial p}{\partial t} = \frac{\partial}{\partial t} \left( n \frac{\partial \rho^f}{\partial p} \frac{\partial p}{\partial t} \right)
\] (8.56)

A fixed fluid mass \( M_f \) with a volume \( V_f \) is given as

\[ M_f = \rho^f V_f \] (8.57)

When the pore pressure is changed, the change of mass is fixed so that

\[ \Delta M_f = \rho^f \Delta V_f + \Delta \rho^f V_f = 0 \]

\[ \Rightarrow \quad \frac{\Delta \rho^f}{\rho^f} = -\frac{\Delta V_f}{V_f} \] (8.58)

Thus

\[ \frac{n \frac{\partial \rho^f}{\partial t}}{\rho^f} = \frac{n \frac{\partial \rho^f}{\partial p}}{\rho^f} \frac{\partial p}{\partial t} = -n \frac{\partial V_f}{V_f} \frac{\partial p}{\partial t} \] (8.59)

Since the compressibility of fluid is given as

\[ \beta_f = -\frac{1}{V_f} \frac{\partial V_f}{\partial p} \] (8.60)

The equation (8.59) can be expressed with the compressibility of fluid

\[ \frac{n \frac{\partial \rho^f}{\partial t}}{\rho^f} = \beta_f \frac{\partial p}{\partial t} \] (8.61)

For the second term on the right side of this equation, it is assumed that the change of the mixture volume is primarily from the pore volume change and the pore is filled with the fluid (perfectly saturated condition) so that

\[ \Delta V_f \approx \Delta V_f \Rightarrow \Delta n = \frac{\Delta V_f}{V_f} \approx \frac{\Delta V_T}{V_T} \] (8.62)

The second term is then
\[ \frac{\partial n}{\partial t} \approx \frac{1}{V_T} \frac{\partial V_T}{\partial t} \] (8.63)

Applying Terzaghi’s effective stress principle, the volume change of the mixture is a function of the effective stress, equation can be expressed as

\[ \frac{\partial n}{\partial t} = \frac{\partial n}{\partial \sigma_e} \frac{\partial \sigma_e}{\partial t} = \frac{1}{V_T} \frac{\partial V_T}{\partial \sigma_e} \frac{\partial \sigma_e}{\partial t} \] (8.64)

Since the compressibility of the porous solid is given as

\[ \alpha_s = -\frac{1}{V_T} \frac{\partial V_T}{\partial \sigma_e} \] (8.65)

The equation (8.64) can be expressed with the compressibility of the solid

\[ \frac{\partial n}{\partial t} = -\alpha_s \frac{\partial \sigma_e}{\partial t} \] (8.66)

If the mixture is deformed under drained condition with the pore pressure maintained constant then

\[ \Delta \sigma_e = \Delta \sigma_T \] (8.67)

However, in transient flow where the total stress is relatively constant but the pore pressure varies significantly, the change of the effective stress is by Terzaghi’s effective stress principle

\[ \Delta \sigma_e = -\Delta p \] (8.68)

In this case, the equation (8.66) is

\[ \frac{\partial n}{\partial t} = \alpha_s \frac{\partial p}{\partial t} \] (8.69)

The right side of the equation (8.55) is expressed with a function of pore pressure as
\[ \frac{n}{\rho'} \frac{\partial \rho'}{\partial t} + \frac{\partial n}{\partial t} = n \beta \frac{\partial p}{\partial t} + \alpha \frac{\partial p}{\partial t} = (n \beta + \alpha) \frac{\partial p}{\partial t} \]  

(8.70)

This expression is purely the effect of transient flow in the saturated soils. Thus, this relation should be considered in the theoretical description for the dynamic behavior of saturated soils. Although, the theoretical development of the description is out of present study’s scope, this effect can be examined by the numerical consideration presented in Appendix F, but this consideration is limited on the numerical predictions of Dreger’s tested results.

Figure E.13 through E.18 show comparisons of predicted pore pressure time histories with and without the equivalent mass damping for the cases of the lead shot solid. As can be seen, the magnitude of the oscillations is greatly reduced when the mass coupling is considered. However, the results show not only a reduction in the magnitude of the oscillations, but better agreement in predictions.

8.5.2.4 Prediction Comparisons between Elastic and Plastic

The difference between elastic and plastic constitutive relationships is investigated. Figure E.19 through E.30 show comparisons of pore pressure histories between elastic and plastic predictions. The average trend of the pore pressure time history with the plastic constitutive model follows the path of experimental pore pressure responses. Although deviations are only shown at high pressure parts and predictions with non-linear plastic constitutive model are better in agreement with
experimental results, it is appeared that the stress level in the experimental results does not reach to a plasticade range.
Figure 8.3 Sketch of Shock Tube

High Pressure Section
Mylar Membrane
Low Pressure Section

Silicone Fluid
Saturated Sample

Pr, Pa, Pb : Pressure Transducers
Q : Quick Connect
CHAPTER 9

SUMMARY AND RECOMMENDATIONS

9.1 Summary

A numerical framework to produce computational solutions incorporating Hiremath’s method to describe the dynamic behavior of saturated soils is developed. In this method, the motion of the solid is described with respect to its reference configuration but the motion of the fluid is described relative to the solid. The coupled initial-boundary value problem of wave equations is transformed into an equivalent variational problem. The field variables are expressed in the admissible space whose elements are defined in the spatial region. A solution of the mixed problem is an admissible state of the field variables, which satisfies the field equations, the initial conditions and the boundary conditions to the problem. To achieve more stable numerical solutions, an unconditionally stable time domain integration scheme is chosen. The predictor-corrector form of the Newmark method performs the time domain integration procedure for the dynamic analysis. For the elastic-plastic behavior of soils, Singh’s Cap model (1972) is incorporated as a constitutive relation. This model is extended to predict non-linear elastic behavior of a granular soil. The
validity of the numerical procedure is confirmed. For the case of an elastic solid, numerical predictions are compared with Garg’s and Morland’s analytical solutions. The predicted responses are in good agreement with both analytical solutions. For the case of non-linear elastic-plastic solids, the numerical procedure is verified against laboratory experimental observations. To improve stability, Biot’s mass coupling is numerically considered in the form of equivalent solid damping. The predictions of non-linear elastic-plastic response with equivalent solid damping are in good agreement with experimental observations.

9.2 Recommendations for Future Work

The applied solid damping is achieved numerically. However, the mass coupling has to be considered theoretically so that the mass coupling can be formulated in the field equation during the numerical procedure. Thus development of the dynamic theory to include the mass coupling term is recommended.

The speed of a measurement system in the experimental results used for verifications may not have been fast enough to capture some high frequency responses. It is also appeared that the tested stress level did not consistently reach to a plastic range. An experimental setup, which is capable of performing experiments over a wide range of stress levels, with high speed measurement system is recommended for better verification for the theoretical and numerical formulations. The observation points for pore pressures in the experimental setup were limited near the top of tested
samples. Locating additional pore pressure transducers near the bottom of a sample will provide useful data for reflection responses at the boundary.

Extensions of this study are applicable for analyzing multi-phase systems such as coupled problems with the simultaneous presence of water and air in which air pressure plays important role or that of water and oil for the treatment of oil reservoirs. The extensions can be done by allowing the field equations of motion to contain two different fluids.
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APPENDIX A

Modified Scheme for Velocity Boundary Conditions

To compare the proposed method to the solutions of Garg and Morland, the solution procedure described in the Chapter 6 needed to be modified to allow for specified velocity on the boundary. Recalling the equations of motion in symbolic notation,

\[ [M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = R \quad (A.1) \]

In a given problem, when velocities are prescribed at certain points, the corresponding accelerations and displacements are also known. In developing the modified scheme, all features of time domain method were sought to be retained. This enabled application of the scheme directly without elimination of known degrees of freedom which is quite cumbersome in dynamics problems. Letting \( u_a, \dot{u}_a, \) and \( \ddot{u}_a \) be the unknown displacements, velocities, and accelerations and \( u_b, \dot{u}_b, \) and \( \ddot{u}_b \) be the specified displacements, velocities, and accelerations, for the stage \((n+1)\) at time \( t_n + \Delta t \), the equations of motion can be written in partitioned form as
where the subscript 'b' indicates the quantities corresponding to specified boundary conditions. Rearrangement of terms in (A.2) gives

\[
\begin{bmatrix}
K_{ab} & K_{bb}
\end{bmatrix}
\begin{bmatrix}
\dot{u}_a
\end{bmatrix}
\begin{bmatrix}
R_a
\end{bmatrix}
- \begin{bmatrix}
C_{ab} & C_{bb}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_a
\end{bmatrix}
- \begin{bmatrix}
0
\end{bmatrix}
- \begin{bmatrix}
M_{ab} & M_{bb}
\end{bmatrix}
\begin{bmatrix}
\dot{w}_b
\end{bmatrix}
\begin{bmatrix}
R_b
\end{bmatrix}
- \begin{bmatrix}
C_{ab} & C_{bb}
\end{bmatrix}
\begin{bmatrix}
\ddot{w}_b
\end{bmatrix}
\begin{bmatrix}
0
\end{bmatrix}
- \begin{bmatrix}
M_{ab} & M_{bb}
\end{bmatrix}
\begin{bmatrix}
\dot{w}_b
\end{bmatrix}
\begin{bmatrix}
0
\end{bmatrix}
\]

(A.3)

From (6.63)

\[
\dot{u}_{n+1} = \frac{1}{\beta(\Delta t)^2}
\begin{bmatrix}
u_{n+1} - u_n - \Delta t \dot{u}_n - \left(\frac{1}{2} - \beta \right)(\Delta t)^2 \ddot{u}_n
\end{bmatrix}
\]

(A.4)

Substituting (A.4) in (6.64) gives

\[
\dot{u}_{n+1} = \frac{\gamma}{\beta(\Delta t)^2}
\begin{bmatrix}
u_{n+1} - u_n - \left(1 - \frac{\beta}{\gamma}\right)(\Delta t) \dot{u}_n - \left(\frac{1}{2} - \frac{\beta}{\gamma}\right)(\Delta t)^2 \ddot{u}_n
\end{bmatrix}
\]

(A.5)

Using these relationships for the unknown quantities \(\dot{u}_a\) and \(\ddot{u}_a\)

\[
\begin{bmatrix}
\dot{u}_a
\end{bmatrix}
\begin{bmatrix}
0
\end{bmatrix}
\begin{bmatrix}
\dot{u}_a
\end{bmatrix}
\begin{bmatrix}
0
\end{bmatrix}
\]

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\begin{equation}
-\Delta t \left\{ \begin{array}{c}
\dot{u}_a \\
0
\end{array} \right\}_{(n)} - \left( \frac{1}{2} - \beta \right) (\Delta t)^2 \left\{ \begin{array}{c}
\dot{u}_a \\
0
\end{array} \right\}_{(n)} \right)
\end{equation}

\begin{equation}
\left\{ \begin{array}{c}
\ddot{u}_a \\
0
\end{array} \right\}_{(n+1)} = \frac{\gamma}{\beta(\Delta t)} \left[ \begin{array}{c}
u_a \\
0
\end{array} \right]_{(n+1)} - \left[ \begin{array}{c}
u_a \\
0
\end{array} \right]_{(n)}
\end{equation}

\begin{equation}
- \left( 1 - \frac{\beta}{\gamma} \right) (\Delta t) \left\{ \begin{array}{c}
\dot{u}_a \\
0
\end{array} \right\}_{(n)} - \left( \frac{1}{2} - \frac{\beta}{\gamma} \right) (\Delta t)^2 \left\{ \begin{array}{c}
\ddot{u}_a \\
0
\end{array} \right\}_{(n)} \right)
\end{equation}

Substituting (A.6) and (A.7) in (A.3) and adding a term

\begin{equation}
[M] \frac{1}{\beta(\Delta t)^2} \left\{ \begin{array}{c}
u_a \\
0
\end{array} \right\}_{(n+1)} + [C] \frac{\gamma}{\beta(\Delta t)} \left\{ \begin{array}{c}
u_a \\
0
\end{array} \right\}_{(n+1)}
\end{equation}

on both sides of the equation, yields

\begin{equation}
\left[ \begin{array}{c}
[K] + [M] \frac{1}{\beta(\Delta t)^2} \\
[C] \frac{\gamma}{\beta(\Delta t)}
\end{array} \right] \left\{ \begin{array}{c}
u_a \\
\nu_a
\end{array} \right\}_{(n+1)}
\end{equation}

\begin{equation}
= \left\{ \begin{array}{c}
R_a \\
R_b
\end{array} \right\}_{(n+1)}
\end{equation}

\begin{equation}
+ [M] \frac{1}{\beta(\Delta t)^2} \left[ \begin{array}{c}
u_a \\
0
\end{array} \right]_{(n)} + \left\{ \begin{array}{c}
u_a \\
0
\end{array} \right\}_{(n+1)}
\end{equation}

\begin{equation}
+ \Delta t \left[ \begin{array}{c}
\ddot{u}_a \\
0
\end{array} \right]_{(n)} + \left( \frac{1}{2} - \beta \right) (\Delta t)^2 \left\{ \begin{array}{c}
\dot{u}_a \\
0
\end{array} \right\}_{(n)}
\end{equation}

\begin{equation}
+ [C] \frac{\gamma}{\beta(\Delta t)} \left[ \begin{array}{c}
u_a \\
0
\end{array} \right]_{(n)} + \left\{ \begin{array}{c}
u_a \\
0
\end{array} \right\}_{(n+1)}
\end{equation}

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\[ + \left( 1 - \frac{\beta}{\gamma} \right) (\Delta t) \begin{bmatrix} \dot{u}_a \\ 0 \end{bmatrix}_{(n)} + \left( \frac{1}{2} - \frac{\beta}{\gamma} \right) (\Delta t)^2 \begin{bmatrix} \ddot{u}_a \\ 0 \end{bmatrix}_{(n)} \]

\[-[M] \begin{bmatrix} 0 \\ \ddot{u}_b \end{bmatrix}_{(n+1)} - [C] \begin{bmatrix} 0 \\ \ddot{u}_b \end{bmatrix}_{(n+1)} \] (A.9)

Symbolically,

\[
\begin{bmatrix} K_{aa}^* & K_{ab}^* \\
K_{ba}^* & K_{bb}^* \end{bmatrix} \begin{bmatrix} \ddot{u}_a \\ \ddot{u}_b \end{bmatrix}_{(n+1)} = \begin{bmatrix} R_a^* \\
R_b^* \end{bmatrix} \] (A.10)

where

\[ K_{aa}^* = \left[ K_{aa} + \left[ M_{aa} \right] \frac{1}{\beta (\Delta t)^2} + \left[ C_{aa} \right] \frac{\gamma}{\beta (\Delta t)} \right] \]

\[ K_{ab}^* = \left[ K_{ab} + \left[ M_{ab} \right] \frac{1}{\beta (\Delta t)^2} + \left[ C_{ab} \right] \frac{\gamma}{\beta (\Delta t)} \right] \]

\[ K_{ba}^* = \left[ K_{ba} + \left[ M_{ba} \right] \frac{1}{\beta (\Delta t)^2} + \left[ C_{ba} \right] \frac{\gamma}{\beta (\Delta t)} \right] \]

\[ K_{bb}^* = \left[ K_{bb} + \left[ M_{bb} \right] \frac{1}{\beta (\Delta t)^2} + \left[ C_{bb} \right] \frac{\gamma}{\beta (\Delta t)} \right] \]

\[ \begin{bmatrix} R_a^* \\
R_b^* \end{bmatrix} = \begin{bmatrix} R_a \\
R_b \end{bmatrix}_{(n+1)} \]

\[ + [M] \frac{1}{\beta (\Delta t)^2} \begin{bmatrix} \ddot{u}_a \\ 0 \end{bmatrix}_{(n)} + \begin{bmatrix} 0 \\ \ddot{u}_b \end{bmatrix}_{(n+1)} \]

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Newmark’s method allowing for prescribed displacement boundary conditions can be used to solve (A.10).

\[
\begin{bmatrix}
K_{aa} & 0 \\
0 & I
\end{bmatrix} \begin{bmatrix}
\dot{u}_a \\
\dot{u}_b
\end{bmatrix}_{(n)} + \begin{bmatrix}
\frac{1}{2} - \beta \\
\frac{1}{2} - \gamma
\end{bmatrix} \Delta t \begin{bmatrix}
\ddot{u}_a \\
\ddot{u}_b
\end{bmatrix}_{(n)} = \begin{bmatrix}
R_a^{**} \\
R_b^{**}
\end{bmatrix}
\]  

(A.12)

where

\[
R_a^{**} = R_a^* - K_{ab}^* \dot{u}_{b(n+1)}
\]  

(A.13)
APPENDIX B

Comparisons of Velocity Histories with Garg’s Analytical Solution
Figure B.1 Comparison of Solid Velocity History with Garg’s Analytical Solution at Period T=0.5
Figure B.2 Comparison of Fluid Velocity History with Garg’s Analytical Solution at Period T=0.5
Solid Velocity History at T=1.0

Figure B.3 Comparison of Solid Velocity History with Garg’s Analytical Solution at Period T=1.0
Figure B.4 Comparison of Fluid Velocity History with Garg’s Analytical Solution at Period T=1.0
Figure B.5 Comparison of Solid Velocity History with Garg’s Analytical Solution at Depth Z=0.1
Figure B.6 Comparison of Fluid Velocity History with Garg’s Analytical Solution at Depth Z=0.1
Figure B.7 Comparison of Solid Velocity History with Garg's Analytical Solution at Depth Z=0.2
Figure B.8 Comparison of Fluid Velocity History with Garg's Analytical Solution at Depth Z=0.2
Figure B.9 Comparison of Solid Velocity History with Garg’s Analytical Solution at Depth Z=0.3
Figure B.10 Comparison of Fluid Velocity History with Garg’s Analytical Solution at Depth Z=0.3
Figure B.11 Comparison of Solid Velocity History with Garg’s Analytical Solution at Depth Z=0.4
Figure B.12 Comparison of Fluid Velocity History with Garg’s Analytical Solution at Depth Z=0.4
Figure B.13 Solid Velocity History Comparison for Different Time Step Sizes at Depth Z=0.1
Figure B.14 Fluid Velocity History Comparison for Different Time Step Sizes at Depth Z=0.1
Figure B.15 Solid Velocity History Comparison for Different Time Step Sizes at Depth $Z=0.2$
Figure B.16 Fluid Velocity History Comparison for Different Time Step Sizes at Depth Z=0.2
Figure B.17 Solid Velocity History Comparison for Different Time Step Sizes at Depth Z=0.3
Figure B.18 Fluid Velocity History Comparison for Different Time Step Sizes at Depth Z=0.3
Figure B.19 Solid Velocity History Comparison for Different Time Step Sizes at Depth Z=0.4
Figure B.20 Fluid Velocity History Comparison for Different Time Step Sizes at Depth Z=0.4
APPENDIX C

Comparisons of Velocity Histories with Morland’s Analytical Solution
Solid Velocity History at T=0.5

Figure C.1 Comparison of Solid Velocity History with Morland’s Solution at Period T=0.5
Figure C.2 Comparison of Fluid Velocity History with Morland’s Solution at Period T=0.5
Figure C.3 Comparison of Solid Velocity History with Morland’s Solution at Period $T=1.0$
Figure C.4 Comparison of Fluid Velocity History with Morland’s Solution at Period $T=1.0$
Figure C.5 Comparison of Solid Velocity History with Morland’s Solution at Period T=1.5
Figure C.6 Comparison of Fluid Velocity History with Morland’s Solution at Period T=1.5
Figure C.7 Comparison of Solid Velocity History with Morland’s Solution at Period $T=2.0$
Fluid Velocity History at T=2.0

Figure C.8 Comparison of Fluid Velocity History with Morland’s Solution at Period T=2.0
Figure C.9 Comparison of Solid Velocity History with Morland’s Solution at Period T=2.5
Figure C.10 Comparison of Fluid Velocity History with Morland’s Solution at Period T=2.5
Figure C.11 Comparison of Solid Velocity History with Morland’s Solution at Period T=3.0
Figure C.12 Comparison of Fluid Velocity History with Morland’s Solution at Period T=3.0
Figure C.13 Comparison of Solid Velocity History with Morland’s Solution at Period T=3.5
Figure C.14 Comparison of Fluid Velocity History with Morland’s Solution at Period T=3.5
Figure C.15 Comparison of Solid Velocity History with Morland’s Solution at Period T=4.0
Fluid Velocity History at $T=4.0$

Figure C.16 Comparison of Fluid Velocity History with Morland’s Solution at Period $T=4.0$
Figure C.17 Comparison of Solid Velocity History with Morland’s Solution at Period T=4.5
Figure C.18 Comparison of Fluid Velocity History with Morland’s Solution at Period $T=4.5$
Figure C.19 Comparison of Solid Velocity History with Morland’s Solution at Period $T=5.0$
Figure C.20 Comparison of Fluid Velocity History with Morland’s Solution at Period T=5.0
Figure C.21 Comparison of Solid Velocity History with Morland’s Solution at Period T=5.5
Figure C.22 Comparison of Fluid Velocity History with Morland’s Solution at Period T=5.5
Figure C.23 Comparison of Solid Velocity History with Morland’s Solution at Period T=6.0
Figure C.24 Comparison of Fluid Velocity History with Morland’s Solution at Period T=6.0
Solid Velocity History at T=6.5

Figure C.25 Comparison of Solid Velocity History with Morland’s Solution at Period T=6.5
Figure C.26 Comparison of Fluid Velocity History with Morland’s Solution at Period T=6.5
Solid Velocity History at $T=7.0$

Figure C.27 Comparison of Solid Velocity History with Morland’s Solution at Period $T=7.0$
Figure C.28 Comparison of Fluid Velocity History with Morland’s Solution at Period T=7.0
Figure C.29 Comparison of Solid Velocity History with Morland’s Solution at Depth 0.2m
Figure C.30 Comparison of Fluid Velocity History with Morland’s Solution at Depth 0.2m
Figure C.31 Comparison of Solid Velocity History with Morland’s Solution at Depth 0.4m
Figure C.32 Comparison of Fluid Velocity History with Morland’s Solution at Depth 0.4m
Figure C.33 Comparison of Solid Velocity History with Morland’s Solution at Depth 0.6m
Figure C.34 Comparison of Fluid Velocity History with Morland’s Solution at Depth 0.6m
Solid Velocity History at Z=0.8

Figure C.35 Comparison of Solid Velocity History with Morland’s Solution at Depth 0.8m
Fluid Velocity History at Z=0.8

Figure C.36 Comparison of Fluid Velocity History with Morland’s Solution at Depth 0.8m
APPENDIX D

Test Results of Pore Pressure Time History
Figure D.1 Experimental Pore Pressure Time Histories for Sand Solid and 1.0 cSt. Fluid
Figure D.2 Experimental Pore Pressure Time Histories for Sand Solid and 5.0 cSt. Fluid
Figure D.3 Experimental Pore Pressure Time Histories for Sand Solid and 200.0 cSt. Fluid
Figure D.4 Experimental Pore Pressure Time Histories for Lead Shot Solid and 1.0 cSt. Fluid
Figure D.5 Experimental Pore Pressure Time Histories for Lead Shot Solid and 5.0 cSt. Fluid
Figure D.6 Experimental Pore Pressure Time Histories for Lead Shot Solid and 200.0 cSt. Fluid
APPENDIX E

Comparisons of Pore Pressure Time Histories with Experimental Results
Figure E.1 Comparison of Pore Pressure Time Histories for Sand Solid & 1.0cSt Fluid at Transducer Pa
Figure E.2 Comparison of Pore Pressure Time Histories for Sand Solid & 1.0cSt Fluid at Transducer Pb
Figure E.3 Comparison of Pore Pressure Time Histories for Sand Solid & 5.0cSt Fluid at Transducer Pa
Figure E.4 Comparison of Pore Pressure Time Histories for Sand Solid & 5.0cSt Fluid at Transducer Pb.
Figure E.5 Comparison of Pore Pressure Time Histories for Sand Solid & 200.0cSt Fluid at Transducer Pa
Figure E.6 Comparison of Pore Pressure Time Histories for Sand Solid & 200.0cSt Fluid at Transducer Pb
Figure E.7 Comparison of Pore Pressure Time Histories for Lead Shot Solid & 1.0cSt Fluid at Transducer Pa
Figure E.8 Comparison of Pore Pressure Time Histories for Lead Shot Solid & 1.0cSt Fluid at Transducer Pb
Figure E.9 Comparison of Pore Pressure Time Histories for Lead Shot Solid & 5.0cSt Fluid at Transducer Pa
Figure E.10 Comparison of Pore Pressure Time Histories for Lead Shot Solid & 5.0cSt Fluid at Transducer Pb
Figure E.11 Comparison of Pore Pressure Time Histories for Lead Shot Solid & 200.0cSt Fluid at Transducer Pa
Figure E.12 Comparison of Pore Pressure Time Histories for Lead Shot Solid & 200.0cSt Fluid at Transducer Pb
Figure E.13 Comparison of Pore Pressure Time Histories for Lead Shot Solid & 1.0cSt Fluid at Transducer Pa with Mass Damping
Figure E.14 Comparison of Pore Pressure Time Histories for Lead Shot Solid & 1.0cSt Fluid at Transducer Pb with Mass Damping
Figure E.15 Comparison of Pore Pressure Time Histories for Lead Shot Solid & 5.0cSt Fluid at Transducer Pa with Mass Damping
Figure E.16 Comparison of Pore Pressure Time Histories for Lead Shot Solid & 5.0cSt Fluid at Transducer Pb with Mass Damping
Figure E.17 Comparison of Pore Pressure Time Histories for Lead Shot Solid & 200cSt Fluid at Transducer Pa with Mass Damping
Figure E.18 Comparison of Pore Pressure Time Histories for Lead Shot Solid & 200cSt Fluid at Transducer Pb with Mass Damping
Figure E.19 Comparison of Pore Pressure Time Histories between Elastic and Plastic for Sand & 1.0 cSt Fluid at Transducer Pa
Figure E.20 Comparison of Pore Pressure Time Histories between Elastic and Plastic for Sand & 1.0 cSt Fluid at Transducer Pb
Figure E.21 Comparison of Pore Pressure Time Histories between Elastic and Plastic for Sand & 5.0 cSt Fluid at Transducer Pa
Figure E.22 Comparison of Pore Pressure Time Histories between Elastic and Plastic for Sand & 5.0 cSt Fluid at Transducer Pb
Figure E.23 Comparison of Pore Pressure Time Histories between Elastic and Plastic for Sand & 200 cSt Fluid at Transducer Pa
Figure E.24 Comparison of Pore Pressure Time Histories between Elastic and Plastic for Sand & 200 cSt Fluid at Transducer Pb
Figure E.25 Comparison of Pore Pressure Time Histories between Elastic and Plastic for Lead Shot & 1.0 cSt Fluid at Transducer Pa
Figure E.26 Comparison of Pore Pressure Time Histories between Elastic and Plastic for Lead Shot & 1.0 cSt Fluid at Transducer Pb
Figure E.27 Comparison of Pore Pressure Time Histories between Elastic and Plastic for Lead Shot & 5.0 cSt Fluid at Transducer Pa
Figure E.28 Comparison of Pore Pressure Time Histories between Elastic and Plastic for Lead Shot & 5.0 cSt Fluid at Transducer Pb
Figure E.29 Comparison of Pore Pressure Time Histories between Elastic and Plastic for Lead Shot & 200 cSt Fluid at Transducer Pa
Figure E.30 Comparison of Pore Pressure Time Histories between Elastic and Plastic for Lead Shot & 200 cSt Fluid at Transducer Pb
The material properties of the solid and the fluid in Table 8.1 and 8.2 show that the bulk modulus of the fluid is more than 20 times than that of the solid. Section 8.5.2.3 shows that the deformation of the solid causes most of system volume changes. Because the change of volume is mostly affected by the deformation of the solid, it is assumed that the effect of the added mass or trapped mass is only relevant to the solid motion to address present oscillation problems. To take into account the effect of mass change in numerical solutions, Dreski’s velocity force term (1978) is adapted.

The amount of the mass change is limited to the solid term as

\[ H = \rho^{12} \cdot \ddot{u} = \Delta \rho^{11} \cdot \ddot{u} \quad (F.1) \]

\[ \Delta \rho^{11} \cdot \ddot{u} = \left( \frac{\rho^{11}}{\rho^{11}_{t=0}} - \frac{\rho^{(1)}}{\rho^{(1)}_{t=0}} \right) \cdot \ddot{u} \quad (F.2) \]

Applying the relation (3.6) to (F.2)

\[ \left( \frac{\rho^{11}}{\rho^{11}_{t=0}} - \frac{\rho^{(1)}}{\rho^{(1)}_{t=0}} \right) \cdot \ddot{u} = \left( \frac{M^{(1)}}{V} \right)_{t=0} - \left( \frac{M^{(1)}}{V} \right)_{t} \cdot \ddot{u} \quad (F.3) \]

From the deformed state at time \((t + \Delta t)\) with reference state at \((t)\),
Using the convected coordinates

\[ dV = \sqrt{G} dV_0 \]  

(F.4) becomes

\[
\left( \begin{array}{c}
\left( \frac{M^{(i)}}{V} \right) - \left( \frac{M^{(f)}}{V} \right)
\end{array} \right) \cdot \dot{u} = \left( \begin{array}{c}
\left( \frac{M^{(i)}}{V} \right) - \left( \frac{M^{(f)}}{V_0} \right)
\end{array} \right) \cdot \dot{u}
\]

\[ = \frac{M^{(i)}}{V_0} \left[ \left( \frac{1}{\sqrt{G}} \right) - (1) \right] \cdot \dot{u} \]  

(F.5)

Inserting (3.32) into (F.5) gives

\[
\frac{M^{(i)}}{V_0} \left[ \frac{1}{1 + \frac{\partial u}{\partial x}} - 1 \right] \cdot \dot{u} = \frac{M^{(i)}}{V_0} \left[ -\frac{\partial u}{\partial x} \right] \cdot \dot{u} = \frac{M^{(i)}}{V_0} \left[ \frac{\partial u}{1 + \epsilon} \right] \cdot \dot{u} \]  

(F.6)

Letting the strain,

\[ \epsilon = \frac{\partial u}{\partial x} \]  

(F.7)

\[
\frac{M^{(i)}}{V_0} \left[ \frac{\partial u}{1 + \frac{\partial u}{\partial x}} \right] = \frac{M^{(i)}}{V_0} \left[ \frac{-\epsilon}{1 + \epsilon} \right] \cdot \dot{u} \equiv -\frac{M^{(i)}}{V_0} \epsilon \cdot \dot{u} = -\rho_{11}^1 \epsilon \cdot \dot{u} \]  

(F.8)

The negative sign means opposite direction to the movement of \( u \). This term, numerically equivalent to the mass change, can be applied as solid damping in the damping matrix in (6.42) as:
\[ \begin{bmatrix} M_s & M_c^T \\ M_c & M_f \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} H & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \end{bmatrix} \]

\[ + \begin{bmatrix} K + K_p K_{\pi}^{-1} K_p^T & -K_p K_{\pi}^{-1} L \\ -L^T K_{\pi}^{-1} K_p^T & L^T K_{\pi} L \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} F + R_2 - K_p K_{\pi}^{-1} R_1 \\ G - L^T K_{\pi}^{-1} R_1 \end{bmatrix} \]

(F.9)

where

\[ [H] = e[M_s] \]  

(F.10)