Improvements On The Equity Indexed Annuity Market

DISSERTATION

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Graduate School of The Ohio State University

By

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ABSTRACT

Equity indexed annuities (EIA’s) were introduced to the market in the early 1995 and ever since then they have been producing a lot of interest and excitement in the option market. An EIA is an equity linked annuity whose return is based on the performance of an equity mutual fund or a stock index. These annuities have some really nice features such as: minimum guaranteed return, tax-deferral, and locking in of a credited interest rate, and also each EIA customer takes part in some way in the stock market. A typical example of an EIA is a “point-to-point” design. If S&P 500 has a realized 5-year return of 8% and the participation rate is set at 90%, then the actual interest rate credited to the policy will be \(8\% \times 90\% = 7.2\%\) if this is bigger than the minimum guaranteed rate (usually around 3%). There are also some other classic designs like “annual reset” and “continuous lookback” that we discuss in Chapter 2.

The problem the EIA market is facing today is the increased volatility of the market as well as an abrupt stop in the phenomenal growth of the stock market. Insurers aren’t able to offer the same participation rates as they used to in the late 90’s, and lower participation rates mean lower sale numbers. The need for some new designs or models of EIA’s is clear and that’s what this thesis will answer. We will show how to improve the classic EIA designs using a “multibarrier approach” and we will also introduce a few other new designs that perform much better then the classic
ones in today’s volatile market. These last designs are based on a “path-dependence approach” where we look at the value of the respective index throughout the period of the policy or just on an arbitrary period. So we introduced 5 new types of EIA’s and derived closed form formulae for the price of these new designs. We also provide tables and plots of the classic designs versus the new design which show that the new ones are better than the old ones and also show how much better they are in terms of actual numbers for the participation rates for different prices.
This is dedicated to the ones I love . . .
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First of all I want to thank my advisor, Professor Bostwick Wyman, for his constant help and guidance. His support and encouragement came always at just the right time.

Secondly, I want to thank Professor Elias Shiu for his constant help in keeping me up to date with all the news in this wonderful field of financial mathematics. Implicitly the whole group of financial mathematics we have here at Ohio State University benefitted greatly. I also want to thank Professor Shiu’s Ph. D. student - Mr. Hangsuck Lee for his kindness in sharing his work with all of us and for his research and insight in the equity indexed annuities market. I also have to thank Dr Serena Tiong for her research in pricing EIA’s with the help of the Esscher transform. This technique is one of fundamental tools used in this thesis.

And finally I want to thank my wife for her constant love, support and understanding and for all the help she’s given me through the years. I also want to thank my parents for all their efforts in bringing me where I am today.
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FIELDS OF STUDY

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CHAPTER 1

INTRODUCTION

1.1 Introduction and General Settings

In this chapter we discuss the basics for pricing contingent claims and we talk about some theorems useful in pricing the exotic options from Chapter 3 and the classic equity indexed annuities from Chapter 2.

The equity indexed annuity (EIA) market is a relatively new one: the first EIA’s were introduced in 1995. Even though this is a relatively new domain, there are already some very important results and ideas which started with the work of Professor Elias Shiu and Professor Hans Gerber. In their 1994 paper “Option Pricing by Esscher Transforms” (see [5]) they showed how formulae from financial mathematics (the very important Black-Scholes formula among others) and pricing certain financial derivatives can be derived using the Esscher transforms that we discuss later in this chapter in Section 1.3.

Later on (in 2000) Dr Serena Tiong used the Esscher transforms method to find closed form formulae for the most important designs of EIA known to that date. Pricing these EIA’s never seems like an easy task because of their embedded options but Dr Tiong, in her paper “Valuing Equity Indexed Annuities” (see [12]), provides
very nice and detailed proofs as well as tables with the different participation rate each design can offer for different market settings. Dr Tiong also remarks that even then (in 2000) the increased volatility of the market is driving down the participation rates and notes that these participation rates were a lot higher just a few years earlier.

The volatility problem and hence the low participation rate problem is one of the main problems the EIA market is facing today and obviously improvements are required in the form of new EIA designs and maybe new exotic options embedded in the already existent designs. Recently (in 2002), Dr. Hanneck Lee, one of Professor Shiu’s Ph. D. students, introduced a few new designs and showed in his Ph.D. thesis (see [10]) how these new designs can be better than the classic ones in certain market settings. Mr. Lee was kind enough to share his research and his insight with our group of financial mathematics here at Ohio State University and for that we are all very thankful. I want to add that his research was one of the starting points of my thesis and it challenged me to find new designs and new ideas that can improve the classic EIA models.

The rest of my thesis is organized as follows.

In Section 1.2 we will derive the fundamental theorem of asset pricing in a discrete-time model. We discuss the more difficult continuous-time model in Appendix A. We will not provide any proofs for this section because they can be easily found in literature. A very good source for all the proofs is [1] chapters 4 and 6 or [11] chapters 3, 5 and 10.

In Section 1.2 we will discuss the method of Esscher transforms which helps us find the equivalent martingale measure and is the fundamental tool in pricing all these EIA designs.
In Chapter 2 we will discuss the classic EIA models and discuss their designs, their strong points and their weak points.

In Chapter 3 we introduce the new EIA designs and we will show how they can outperform the classic design in different market settings. We will also provide tables and plots in which we compare the classic models versus the new models and we will provide actual numbers that show how the new models, in certain cases, have better prices and better participation rates.

Chapter 4 summarizes the results of the thesis and suggests paths for future research.

As mentioned before, Appendix A provides the continuous-time setting for the fundamental theorem of asset pricing, and Appendix B provides certain formulae that will be useful in pricing some of the new designs.

The general assumption of this thesis is that we are in the classic Black-Scholes model, which means that, according to the fundamental theorem of asset pricing, the price of any contingent claim can be calculated as the discounted expectation of the corresponding payoff with respect to the equivalent martingale measure.

1.2 The Fundamental Theorem of Asset Pricing

In this section we will consider the model of a finite market or a discrete-time model in which all relevant quantities take a finite number of values.

We consider a finite probability space \((\Omega, \mathcal{F}, P)\), with \(|\Omega|\) a finite number, and for any \(\omega \in \Omega\) \(P(\{\omega\}) > 0\). We have a time horizon \(T\), which is the terminal date for all economic activities considered. We use a filtration \(\mathcal{F}\) of \(\sigma\)-algebras \(\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_T\) and we take \(\mathcal{F}_0 = \{\emptyset, \Omega\}\) and \(\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)\) the power
set of $\Omega$. This financial market contains $d + 1$ financial assets. One is a risk-free asset (a bond or a bank account for example) labeled 0 and $d$ are risky assets (stocks) labeled 1 to $d$. The prices of these assets at time $t$: $S_0(t, \omega), S_1(t, \omega), \ldots, S_d(t, \omega)$ are non-negative and $\mathcal{F}_t$ - measurable. Let $S(t) = (S_0(t), \ldots, S_d(t))$ denote the vector of prices at time $t$. We assume $S_0(t)$ is strictly positive for all $t \in \{0, 1, \ldots, T\}$ and also assume that $S_0(0) = 1$. We define $\beta(t) = \frac{1}{S_0(t)}$ as a discount factor.

So, we constructed a market model $\mathcal{M}$ consisting in the probability space $(\Omega, \mathcal{F}, P)$, the set of trading dates, the price process $S$, and the information structure $F$.

**Definition 1.1** A trading strategy (or dynamic portfolio) $\varphi$ is a $\mathbb{R}^{d+1}$ vector stochastic process $\varphi = (\varphi_0(t, \omega), \varphi_1(t, \omega), \ldots, \varphi_d(t, \omega))$ which is predictable: each $\varphi_i(t)$ is $\mathcal{F}_{t-1}$ - measurable for $t \geq 1$.

Here, $\varphi_i(t)$ denotes the number of shares of asset $i$ held in the portfolio at time $t$ - to be determined on the basis of information available before time $t$. This means that the investor selects his time $t$ portfolio after observing the prices $S(t-1)$. However, the portfolio $\varphi(t)$ must be established before and held until after announcement of the prices $S(t)$.

**Definition 1.2** The value of the portfolio at time $t$ is the scalar product

$$V_\varphi(t) = \varphi(t) \cdot S(t) = \sum_{i=0}^{d} \varphi_i(t)S_i(t) \quad t = 1, 2, \ldots, T$$

and $V_\varphi(0) = \varphi(1)S(0)$

The process $V_\varphi(t, \omega)$ is called the wealth or value process of the trading strategy $\varphi$. We call $V_\varphi(0)$ the initial investment of the investor.

**Definition 1.3** The gains process $G_\varphi$ of a trading strategy $\varphi$ is given by:
\[ G_\varphi(t) = \sum_{x=1}^{t} \varphi(x) [S(x) - S(x-1)] , \quad t = 1, 2, \ldots, T \]

If we define \( \tilde{S}(t) = (1, \beta(t)S_1(t), \ldots, \beta(t)S_d(t)) \) the vector of discounted prices we also have the discounted value process \( \tilde{V}_\varphi(t) = \varphi(t)\tilde{S}(t) \) for \( t = 1, 2, \ldots, T \) and we can see that the discounted gains process \( \tilde{G}_\varphi(t) = \sum_{x=1}^{t} \varphi(x) [\tilde{S}(x) - \tilde{S}(x-1)] \)
reflects the gains from trading with assets 1 to \( d \) only.

**Definition 1.4** The strategy \( \varphi \) is **self-financing**, \( \varphi \in \Phi \), if
\[ \varphi(t)S(t) = \varphi(t+1)S(t) , \quad t = 1, 2, \ldots, T - 1 \]

This means that when new prices \( S(t) \) are quoted at time \( t \) the investor adjusts his portfolio from \( \varphi(t) \) to \( \varphi(t+1) \), without bringing in or consuming any wealth.

There are a few properties of the self financing strategies that are used in proving Theorem 1.1:

**Proposition 1.1** A trading strategy \( \varphi \) is self financing with respect to \( S(t) \) if and only if \( \varphi \) is self-financing with respect to \( \tilde{S}(t) \).

**Proposition 1.2** A trading strategy \( \varphi \) is self financing if and only if
\[ \tilde{V}(t) = V_\varphi(0) + \tilde{G}_\varphi(t) \]

The central principle in any market model is that of the absence of arbitrage opportunities, which means the absence of risk-free plans for making profit without any investment.

**Definition 1.5** Let \( \Phi_0 \subset \Phi \) be a set of self-financing strategies. A strategy \( \varphi \in \Phi_0 \) is called an **arbitrage opportunity** or arbitrage strategy with respect to \( \Phi_0 \) if \( P\{V_\varphi(0) = 0\} = 1 \) and the terminal wealth satisfies
\[ P\{V_\varphi(T) \geq 0\} = 1 \quad \text{and} \quad P\{V_\varphi(T) > 0\} > 0 \]
We say that a security market \( M \) is arbitrage-free if there are no arbitrage opportunities in the class \( \Phi \).

The next important thing we want to introduce is the notion of “risk-neutral probability.”

**Definition 1.6** A probability measure \( P^* \) on \( (\Omega, \mathcal{F}_T) \) equivalent to \( P \) is called a martingale measure for \( \tilde{S} \) if the process \( \tilde{S} \) follows a \( P^* \) - martingale with respect to the filtration \( F \). We denote \( \mathcal{P} (\tilde{S}) \) the class of equivalent martingale measures.

One proposition that follows quickly and is useful in proving Theorem 1.1 is:

**Proposition 1.3** Let \( P^* \in \mathcal{P} (\tilde{S}) \) and \( \varphi \) a self-financing strategy. Then the wealth process \( \tilde{V} (t) \) is a \( P^* \) martingale with respect to the filtration \( F \).

One of the central theorems of financial mathematics follows:

**Theorem 1.1** (No-Arbitrage Theorem) The market \( M \) is arbitrage-free if and only if there exists a probability measure \( P^* \) equivalent to \( P \) under which the discounted \( d \) - dimensional asset price process \( \tilde{S} \) is a \( P^* \) - martingale.

The question now is how we use this theorem to price contingent claims. We start with a definition:

**Definition 1.7** A contingent claim \( X \) with maturity date \( T \) is an arbitrary non-negative \( \mathcal{F}_T \) - measurable random variable.

We say that the claim is attainable if there exists a replicating strategy \( \varphi \in \Phi \) such that

\[
V_{\varphi}(T) = X
\]

Then, we have a second very important theorem:
Theorem 1.2  The arbitrage price process $\pi_X(t)$ of any attainable contingent claim $X$ is given by the risk-neutral valuation formula:

$$\pi_X(t) = \beta(t)^{-1} E^* \left( X \beta(T) \mid \mathcal{F}_t \right) \text{ for any } t = 0, 1, \ldots, T$$

where $E^*$ is the expectation taken with respect to an equivalent martingale measure $P^*$. Theorem 1.2 says that any attainable contingent claim can be priced using the equivalent martingale measure. So, clearly, “attainability” would be a very desirable property of any market. So the next definition follows naturally:

Definition 1.8  The market $\mathcal{M}$ is complete if every contingent claim is attainable.

The following theorem gives a nice characterization of a complete market:

Theorem 1.3 (Completeness Theorem)  An arbitrage-free market $\mathcal{M}$ is complete if and only if there exists a unique probability measure $P^*$ equivalent to $P$ under which discounted asset prices are martingales.

Let’s summarize what we have seen so far.

Theorem 1.1 tells us that if the market is arbitrage-free, equivalent martingale measures $P^*$ exist. Theorem 1.3 tells us that if the market is complete, equivalent martingale measures are unique. Putting them together we get:

Theorem 1.4 (Fundamental Theorem of Asset Pricing)  In an arbitrary-free complete market $\mathcal{M}$, there exists a unique equivalent martingale measure $P^*$.

And putting theorem 1.2 in the complete market setting we obtain:

Theorem 1.5 (Risk-Neutral Pricing Formula)  In an arbitrage-free complete market $\mathcal{M}$, arbitrage prices of contingent claims are their discounted expected values under the risk-neutral (equivalent martingale) measure $P^*$. 

7
1.3 Esscher Transform

We have seen in Section 1.2 that any attainable contingent claim can be priced using an equivalent martingale measure. We know that such a measure exists from Theorem 1.1 but we know nothing else about it. This is where the Esscher transform comes into play and proves itself very useful and efficient.

The option-pricing theory of Black and Scholes is probably the most important advance in the theory of financial economics in the past two decades. Their theory has been extended in many directions, usually by applying stochastic calculus and partial differential equations. A fundamental insight in the development of theory was provided by Cox and Ross when they pointed out the concept of risk-neutral valuation. Gerber and Shiu were the first to show that Esscher transforms can be used for pricing many contingent claims. The method they introduce in their 1994 paper (see [5]) is very efficient and elegant. The paper shows how the Esscher transform can be used to price derivative securities if the logarithms of the prices of the primitive securities are governed by certain stochastic processes with stationary and independent increments and continuous in probability and if the risk-free force of interest is assumed constant. This family of processes includes the Wiener process, the Poisson process, the gamma process, and the inverse Gaussian process.

Throughout this thesis we assume the Black-Scholes model. So, if we let $S(t)$ be the price of a non-dividend-paying stock at time $t \geq 0$, then

$$S(t) = S(0)e^{X(t)}$$

where $X(t)$ is a Brownian motion.

Let $F(x, t) = P[X(t) \leq x]$.

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and then the random variable $X(t)$ has a density

$$f(x, t) = \frac{d}{dx} F(x, t), \quad t > 0.$$ 

In general, the moment generating function of a random variable $X(t)$ is given by

$$M(z, t) = E\left[ e^{z X(t)} \right] = \int e^{z x} f(x, t) dx$$

and if we assume that $M(z, t)$ is continuous at $t = 0$, and that $X(t)$ has independent increments, it can be shown that

$$M(z, t) = \left[ M(z, 1) \right]^t$$

(see [4] vol.2, sec.9.5).

Let $h$ be a real number such that $M(h, t)$ exists. The new probability density function of $X(t)$ is

$$f(x, t; h) = \frac{e^{hx} f(x, t)}{\int e^{hu} f(u, t) du} = \frac{e^{hx} f(x, t)}{M(h, t)}$$

(1.2)

That is, the modified distribution of $X(t)$ is the Esscher transform of original distribution. It can be shown (see [10]) that under this new distribution $X(t)$ is still a Brownian motion. The corresponding moment generating function is:

$$M(z, t; h) = \int e^{zx} f(x, t; h) dx$$

$$= \int \frac{e^{zx} e^{hx} f(x, t)}{\int e^{hu} f(u, t) du} dx$$

$$= \int \frac{e^{(z+h)x} f(x, t)}{\int e^{hu} f(u, t) du} dx$$

$$= \frac{M(z + h, t)}{M(h, t)}$$

(1.3)
From (1.1) and (1.3) we get that

\[ M(z, t; h) = \left[ M(z, 1; h) \right]^t \]  \hspace{1cm} (1.4)

So, we considered the Esscher transform of a stochastic process, and hence the probability measure of the process has been modified. Since the exponential function is positive, the modified probability measure is equivalent to the original probability measure.

Now, we want to find an \( h \) (call it \( h^* \)) such that the discounted stock price process \( \tilde{S}(t) = e^{-rt}S(t) \) is a martingale with respect to the probability measure corresponding to \( h^* \). Here \( r \) is the constant risk-free force of interest. So basically we want to find an Esscher transform that gives us the equivalent martingale measure we discussed in Section 1.2.

If \( \tilde{S}(t) \) is a martingale with respect to the probability measure corresponding to \( h^* \) then:

\[ \tilde{S}(0) = E^*[\tilde{S}(t)] \]

which means

\[ S(0) = E^*[e^{-rt}S(t)] = e^{-rt}E^*[S(t)] \]

Then

\[ S(0) = e^{-rt}E^*[S(0)e^{X(t)}] = e^{-rt}S(0)E^*[e^{X(t)}] \]

Hence

\[ e^{rt} = E^*[e^{X(t)}] = M(1, t; h^*) \]

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Using (1.4) we get
\[ e^{rt} = \left[ M(1,1; h^*) \right]^t \]
and setting \( t=1 \) we obtain
\[ e^r = M(1,1; h^*) \] (1.5)

It is shown in [6] that the parameter \( h^* \) is unique.

Equation (1.5) is very important because for known original distributions of \( X(t) \) (Normal, Poisson) \( h^* \) can be computed explicitly. Therefore in certain cases we know exactly what the equivalent martingale measure is. And those certain cases include our general setting for this thesis (the Black-Scholes model).

We call the Esscher transform of parameter \( h^* \) the risk-neutral Esscher transform, and the corresponding equivalent martingale measure the risk-neutral Esscher measure.

We are now ready to state and prove the main theorem of Gerber and Shiu:

**Theorem 1.6 (Gerber-Shiu,[5])** Let \( g \) be a real-valued measurable function. Then, for each positive \( t \),

\[ E \left[ e^{-rt} S(t) g(S(t)) ; h^* \right] = S(0) E \left[ g(S(t)) ; h^* + 1 \right] \]

**Proof** The expectation on the left-hand side of the equation above is obtained by integrating
\[ e^{-rt} S(0) e^{x} g(S(0)e^{x}) f(x,t; h^*) \]
with respect to \( x \) over \( \mathbb{R} \). Since
\[ e^{x} f(x,t; h^*) = \frac{e^{(h^*+1)x} f(x,t)}{M(h^*,t)} = \frac{M(h^* + 1, t)}{M(h^*, t)} f(x,t; h^* + 1) \]
\[ M(1, t; h^*) f(x, t; h^* + 1) = e^{rt} f(x, t; h^* + 1) \]

the result follows immediately.

Comments

Let’s first note that in their 1994 paper Gerber and Shiu proved a more general theorem that deals with the prices of \( n \) non-dividend paying stocks or assets at time \( t \). But for our thesis we only need the 1-dimentional case.

We have to remark that the theorem is a statement about what happens at a certain time \( t \) with the price process \( S \). This is somehow restrictive because for those contingent claims that are defined with the help of more than just one moment in time this method might not work for the obvious reason that we might need more than one parameter \( h \). In the equity indexed annuity market the designs happen to be just right so that they can be priced with the help of this method.

The point-to-point designs can be dealt with using this method because the models are all based on what happens at certain moment in time \( T \) - the end term of the policies.

The annual reset designs can be dealt again with this method even though the models are based on the value of the stock index at different moment in time. What makes it work here is the fact that the time intervals have equal lengths and the fact that \( X(t) \) has independent increments and hence for every time interval we use the same \( h^* \).

Even though it looks like the continuous lookback designs and the new “path-dependent” designs from section 3.4 wouldn’t be able to be dealt with using this method, we can still apply it. The reason why it works is hidden in the distribution of the maximum of a Brownian motion over a certain interval (see equation (2.21)).
Basically the distribution of the maximum depends only on what happens with the Brownian motion at the end of the period being studied. Even the new designs from section 3.2 can be dealt with this method because the probability density function of those hitting times is actually related to the density of the maximum of the Brownian motion over the same interval (see equation (3.20)).

H. Gerber and E. Shiu proved in their paper (see [5]) that there are a lot of very important consequences of the method of Esscher transform. The Black-Scholes option-pricing formula is one of them as well as the binomial option-pricing formula.
CHAPTER 2

THE CLASSIC EQUITY INDEXED ANNUITIES

Equity indexed annuities (EIA’s) have been introduced to the market in early 1995 and ever since then they have generated a lot of interest on the option market. An EIA is an equity linked annuity whose return is based on the performance of an equity mutual fund or a family of mutual funds or a stock index. Usually, the Standard and Poor’s (S&P) 500 index is used. The insurers are presented with very difficult mathematical problems when it comes to the pricing and management of equity indexed annuities because of the embedded options in these products.

Let’s look at the features that make an EIA such an attractive and customer-friendly product. The most important feature is the existence of a minimum guaranteed return, typically around 3%. The second most important feature is the fact that the return is based on the performance of a stock index which allows the customers to participate in the exceptional growth of the stock market. Another very important feature is the fact that (for the annual reset types) interest rates, once credited, are locked in. Finally, equity indexed annuities are tax-deferred. The customers pay no taxes on the earnings, unless they make a withdrawal.

Different companies market EIA’s with many different designs. However, there are a few “classic” designs that are widely available. In this chapter we will discuss the
product design and pricing of these classic EIA policies. We will talk about their strengths and weaknesses in today’s economy. In the next chapter we will introduce new types of EIA designs that are better fitted for today’s volatile market and down economy.

The simplest type of equity indexed annuity is a “point-to-point” design. The policy earns the realized return on the index over a given period of time at a prescribed participation rate, but with a minimum guarantee. For example if the S&P 500 has a realized 10-year return of 8% and the participation rate is 90%, then the actual interest credited to the policy will be $8\% \times 90\% = 7.2\%$. Even when the market is down, the policy will earn a return at least equal to the guaranteed rate.

One type of equity indexed annuity product design that seems to be very favorable for customers is the “annual reset” design. Here, the interest rates are credited annually based on the higher of the index return over the year and a minimum guaranteed rate. This feature allows the resulting interest credits to be locked in, even if the index goes down in the subsequent periods.

Another popular type of design is the “continuous lookback”. This is also called a ”high-water mark” design or a “no regret” policy which earns the maximum between the highest return on the index attained during the life of the policy and a minimum guaranteed rate. These last policies are expensive by design, so usually they offer a lower participation rate to compensate for the cost.

Low risk-tolerant customers are attracted by the equity indexed annuities because of their potential to earn equity-like returns without undue risk. For the rest of the chapter we will take each one of these “classic” designs separately and discuss their pricings and their properties. This will give us a better understanding of the nature
of these products and help us tackle the more difficult designs introduced in the next chapter.

2.1 The Point-To-Point Design

Because of the way this design is set up, in the market, this product is also called “European” or “end of term.” Let

\[ S(t) = S(0) e^{X(t)}, t \geq 0 \]

be the value of a risky asset (the S&P 500 for example) at time \( t \). Under the Black-Scholes model \( X(t) \) is a Brownian motion with drift \( \mu \) and diffusion \( \sigma^2 \). This means that for any \( t \geq 0 \), \( X(t) \) follows a normal distribution with mean \( \mu t \) and variance \( \sigma^2 t \).

Let \( \alpha \) be the participation rate, \( \alpha > 0 \), and let \( g \) be the minimum guaranteed rate of return.

Suppose that at time \( T, T > 0 \), given an initial premium of $1, we have a policy that pays \( e^{\alpha X(T)} \) or \( e^{g T} \), whichever is higher. Therefore, at maturity, this policy earns a percentage of the realized return on the risky asset over \( T \) periods, with the provision of a minimum guaranteed return of \( g \) compounded continuously over time.

Under the risk-neutral Esscher parameter \( h^* \), we can write the value of this policy as (see Section 1.3 or [5]):

\[
P_{pp} = E \left[ e^{-r T} \max \left( e^{\alpha X(T)}, e^{g T} \right) ; h^* \right] \tag{2.1}
\]

where \( r \) is the constant risk-free force of interest.

We can write
\[
\max \left( e^{\alpha X(T)}, e^{gT} \right) = e^{\alpha X(T)} I (\alpha X(T) > gT) + e^{gT} I (\alpha X(T) \leq gT)
\]

Where
\[
I(A) = \begin{cases} 
1 & \text{if } A \text{ is true;} \\
0 & \text{if } A \text{ is false.}
\end{cases}
\]

Then (2.1) can be rewritten as
\[
P_{pp} = e^{-rT} E \left[ e^{\alpha X(T)} I (\alpha X(T) > gT) + e^{gT} I (\alpha X(T) \leq gT) ; h^* \right]
= e^{-rT} E \left[ e^{\alpha X(T)} I (\alpha X(t) > gT) ; h^* \right] + e^{(g-r)T} P \left( X(t) \leq \frac{gT}{\alpha} ; h^* \right)
\]

Since \( X(T) \) is normally distributed with mean \( \mu T \) and variance \( \sigma^2 T \), the moment generating function of \( X(T) \) is (see [2])
\[
M_X(z,T) = \exp \left[ \left( \mu z + \frac{1}{2} \sigma^2 z^2 \right) T \right]
\]

The moment generating function of \( X(T) \) under Esscher transform with respect to parameter \( h \) is (see Section 1.3)
\[
M_X (z, T, h) = \frac{M_X (z + h, T)}{M_X (h, T)} = \frac{\exp \left[ \left( \mu (z + h) + \frac{1}{2} \sigma^2 (z + h)^2 \right) T \right]}{\exp \left[ \left( \mu h + \frac{1}{2} \sigma^2 h^2 \right) T \right]}
= \exp \left[ \left( \mu z + \frac{1}{2} \sigma^2 z^2 + \frac{1}{2} 2zh \sigma^2 \right) T \right]
= \exp \left[ \left( \mu + h \sigma^2 \right) z + \frac{1}{2} \sigma^2 z^2 \right]
\]

The last expression in equation (2.4) is the moment generating function for a normal distribution with mean \( (\mu + h \sigma^2)T \) and variance \( \sigma^2 T \) (from equation (2.3)).

We know from Section 1.3 that the risk neutral Esscher parameter \( h^* \) is a constant such that
\[
e^r = M_X (1,1; h^*) = e^{\mu + h^* \sigma^2 + \frac{1}{2} \sigma^2}
\]
So \[ r = \mu + h^* \sigma^2 + \frac{1}{2} \sigma^2, \] or else
\[ \mu + h^* \sigma^2 = r - \frac{1}{2} \sigma^2 \] (2.5)

This means that \( X(T) \) is normally distributed with means \((r - \frac{1}{2} \sigma^2)T\), and \((r - \frac{1}{2} \sigma^2 + \alpha\sigma^2)T\) under the Esscher transform for parameters \( h^* \) and \( h^* + \alpha \) respectively, and the variance is \( \alpha^2 T \) in both cases.

Now let \( A \) be an event and \( \alpha \) an arbitrary real number. Then under the Esscher transform of parameter \( h \), we can write
\[
E \left[ e^{\alpha X(T)} I(A); h \right] = \frac{E \left[ e^{(\alpha+h) X(T)} I(A) \right]}{E \left[ e^{h X(T)} \right]}
\]
\[ = \frac{E \left[ e^{(\alpha+h) X(T)} I(A) \right]}{E \left[ e^{(\alpha+h) X(T)} \right]} \times \frac{E \left[ e^{(\alpha+h) X(T)} \right]}{E \left[ e^{h X(T)} \right]}
\]
\[ = P(A; h + \alpha) \times M_X(\alpha, T; h) \] (2.6)

Let us use equation (2.6) for \( A = \{ X(T) > \frac{gT}{\alpha} \} \) and \( h = h^* \). We obtain
\[
E \left[ e^{\alpha X(T)} I \left( X(T) > \frac{gT}{\alpha} \right); h^* \right] = P \left( X(T) > \frac{gT}{\alpha}; h^* + \alpha \right) M_X(\alpha, T; h^*)
\]
\[ = \Phi \left[ \frac{\left( r - \frac{1}{2} \sigma^2 + \alpha\sigma^2 \right) T - \frac{\alpha gT}{\sigma \sqrt{T}}}{\sigma \sqrt{T}} \right] e^{\left( r - \frac{1}{2} \sigma^2 + \frac{\alpha \sigma^2}{2} \right)T} \] (2.7)

Also from (2.5) we get
\[
P \left( X(T) \leq \frac{gT}{\alpha}; h^* \right) = \Phi \left[ \frac{\frac{gT}{\alpha} - \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right] \] (2.8)

From (2.2), (2.7), and (2.8) we can express the value of this policy as:
\[
P_{pp} = e^{\left( \alpha - 1 \right) T + \frac{1}{2} \alpha (\alpha - 1) \sigma^2} T \Phi \left[ \frac{\left( r - \frac{1}{2} \sigma^2 + \alpha \sigma^2 \right) T - \frac{\alpha gT}{\sigma \sqrt{T}}}{\sigma \sqrt{T}} \right] + e^{(g-r)T} \Phi \left[ \frac{\frac{gT}{\alpha} - \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right] \] (2.9)
where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable.

As one might expect, $P_{pp}$ is an increasing function of $\alpha$ for fixed $T$. In figure 2.1 we have the graphs of two different point-to-point designs. One design is for $T=10$, and the second one is for $T=20$. The plot represents the value of these policies as functions of the participation rate $z = \alpha$. The other market settings for figure 2.1 are: $g = 3\%$, $r = 5\%$, $\sigma = 20\%$. Also it is important to notice that for $\alpha < 1$ the value of a policy with a longer term is less expensive.

With the same settings as the ones for figure 2.1, we can see from figure 2.2 that for a fixed $\alpha < 1$, $P_{pp}$ is mostly a decreasing function of time with a maximum realized somewhere in the first 5 years. In figure 2.2 the most expensive design is the one for $\alpha = 1.1$, the next one is for $\alpha = .85$ and the cheapest one is for $\alpha = .6$.

### 2.2 The Annual Reset Design

The “annual reset design”, also called a “ratchet EIA” or a “cliquet design” (French for ratchet) is the most common type of EIA. This is a common type of policy because of its very attractive and customer driven features.

A ratchet annuity will earn, at the end of every period, the return on the asset $S(t)$ over that period with a participation rate $\alpha \geq 0$, or the minimum guaranteed return $g$, whichever is higher. Once interest is credited, the earnings are locked in and will never decrease, regardless of the future performance of the market. This gives the customer the very nice assurance that if the market is up one will profit, and if the market is down one still has the minimum guaranteed rate to rely on. This is clearly a big improvement over the simple point-to-point design. The market can
go up and down over a 10-year period, but a holder of a point-to-point EIA, can do nothing about it. A holder of an annual reset design EIA can take full advantage of the ups of the market without worries on a down year.

Using again the Esscher transform method, the value of this policy at time 0 is:

\[
P_{ar} = E \left[ e^{-rn} \prod_{i=1}^{n} \max \left( e^{g}, e^{\alpha X_i} \right) ; h^* \right] \tag{2.10}
\]

where \( X_i = X(i) - X(i-1), i = 1, 2, \ldots, n, \) denotes the rate of return on the asset over the \( i^{th} \) period of time. Since \( X(t) \) is a Brownian motion \( \{X_i\} \) is a sequence of independent and identically distributed random variables with \( X_i \sim N(\mu, \sigma^2) \) for any \( i = 1, 2, \ldots, n. \)

Then (2.10) becomes

\[
P_{ar} = e^{-rn} E \left[ \prod_{i=1}^{n} \max \left( e^{g}, e^{\alpha X_i} \right) ; h^* \right]
\]

\[
= e^{-rn} \prod_{i=1}^{n} E \left[ \max \left( e^{g}, e^{\alpha X_i} \right) ; h^* \right]
\]

\[
= e^{-rn} \left\{ E \left[ \max \left( e^{g}, e^{\alpha X_1} \right) ; h^* \right] \right\}^n
\]

\[
= \left\{ e^{-r} E \left[ \max \left( e^{g}, e^{\alpha X(1)} \right) ; h^* \right] \right\}^n \tag{2.11}
\]

The expression inside the parenthesis in the last equality of (2.11) is the value of a point-to-point policy design with \( T = 1. \) Then we can use equation (2.9) and obtain:

\[
P_{ar} = \left[ e^{(\alpha-1)r + \frac{1}{2} \alpha(\alpha-1)\sigma^2} \Phi \left( \frac{r - \frac{1}{2} \sigma^2 + \alpha \sigma^2 - \frac{g}{\alpha}}{\sigma} \right) \right] + e^{(g-r)} \Phi \left( \frac{\frac{g}{\alpha} - r + \frac{1}{2} \sigma^2}{\sigma} \right)^n \tag{2.12}
\]

which represents the value of an annual reset design policy.

The same settings from figure 2.1 apply to figure 2.3 and we can see that the policy costs more for bigger \( n \) (those are the plots for \( n = 5, n = 10, n = 15 \)) and
as expected $P_{ar}$ is an increasing function of $\alpha$. The big weakness of this design is definitely the cost. As we see from figure 2.4, for the same period of time - 10 years, and for the same market settings that we used for figure 2.1, the annual reset design is a lot more expensive than the point-to-point design.

In order to solve the cost problem and still keep the basic annual reset concept, two different types of annual reset designs were introduced in the late 90’s. The first one we discuss is the “Annual Reset with Cap” design which is somewhat similar to some of the new EIA’s introduced in the next chapter.

The “annual reset with cap” design is the same as a normal annual reset design with a fixed upper limit, or cap, on the periodic return.

Let $c$ be this cap, with $c > g$. This $c$ is the maximum rate of interest that the policy can earn in each period. The value of this policy is:

$$P_{arc} = \mathbb{E}\left[ e^{-rn} \prod_{i=1}^{n} \max\left(e^g, \min\left(e^c, e^{\alpha X_i}\right)\right) ; h^* \right]$$

by the independence of $X_i$’s.

We can write:

$$\max\left(e^g, \min\left(e^c, e^{\alpha X_i}\right)\right) = e^g I(\alpha X_i \leq g) + e^{\alpha X_i} I(g < \alpha X_i \leq c) + e^c I(\alpha X_i > c)$$

and then equation (2.13) becomes:

$$P_{arc} = e^{-rn} \prod_{i=1}^{n} \left[ e^g P\left(X_i \leq \frac{g}{\alpha} ; h^* \right) + e^{\alpha r + \frac{1}{2} \alpha (\alpha - 1) \sigma^2} \times P\left(\frac{g}{\alpha} < X_i \leq \frac{c}{\alpha} ; h^* + \alpha \right) + e^c P\left(X_i > \frac{c}{\alpha} ; h^* \right) \right]$$

(2.14)

where we used (2.7).
Since $X_i$’s are identically distributed normals with mean $\mu$ and variance $\sigma^2$, (2.14) becomes:

$$P_{arc} = \left[ e^{(g-r)} P \left( X_i \leq \frac{g}{\alpha} ; h^* \right) + e^{(\alpha-1)r+\frac{1}{2}\alpha(\alpha-1)\sigma^2} \times P \left( \frac{g}{\alpha} < X_i \leq \frac{c}{\alpha} + h^* \right) + e^{(c-r)} P \left( X_i > \frac{c}{\alpha} ; h^* \right) \right]^n$$

(2.15)

So the value of this annual reset with cap design is:

$$P_{arc} = \left\{ e^{(g-r)} \Phi \left( \frac{\frac{g}{\alpha} - r + \frac{\sigma^2}{2}}{\sigma} \right) + e^{(c-r)} \Phi \left( \frac{r - \frac{\sigma^2}{2} - \frac{c}{\alpha}}{\sigma} \right) + e^{(\alpha-1)r+\frac{1}{2}\alpha(\alpha-1)\sigma^2} \times \left[ \Phi \left( \frac{\frac{c}{\alpha} - r + \frac{\sigma^2}{2} - \alpha^2\sigma^2}{\sigma} \right) - \Phi \left( \frac{\frac{g}{\alpha} - r + \frac{\sigma^2}{2} - \alpha^2\sigma^2}{\sigma} \right) \right] \right\}^n$$

(2.16)

We notice that if we let $c = \infty$ in (2.16) we obtain the formula for a normal annual reset design given in (2.12).

In figure 2.5 we have the usual market settings as before with $n = 10$, $g = 3\%$, $r = 5\%$, $\sigma = 20\%$. As expected, as $c$ grows, so does the price of this particular design (in figure 2.5 we have the plots for $c = 30\%$, $c = 20\%$ and $c = 10\%$). The important cost problem has been somewhat resolved. As we can see in figure 2.6 (again with the same market settings as before) the price of the annual reset with cap design is still bigger than the price of the point-to-point design but less than the price of the normal annual reset design.

The second design introduced to solve the cost problem is called the “Annual Reset with Spread” design. This type of product usually earns a 100% participation rate in growth of the S&P500 in excess of a predetermined hurdle rate, which can be termed a “spread”, and of course there is a minimum guaranteed rate.

The design is clearly more attractive to the customers in a bull market (when S&P500 returns are high) than the design with cap, because of their potential to earn unlimited returns.
Let $s$ be the spread, $s \geq 0$, and let $g$ be the minimum guaranteed rate. In each period the policy will earn a return of

$$\max(g, \alpha(X_i - s))$$

Then the value of this policy is:

$$P_{ars} = E\left[e^{-rn} \prod_{i=1}^{n} \max\left(e^g, e^{\alpha(X_i - s)}\right); h^*\right]$$

$$= e^{-rn} \prod_{i=1}^{n} E\left[\max\left(e^g, e^{\alpha(X_i - s)}\right); h^*\right]$$

(2.17)

since $X_i$'s are independent.

Note that because

$$\max\left(e^g, e^{\alpha(X_i - s)}\right) = e^g I(\alpha(X_i - s) \leq g) + e^{\alpha(X_i - s)} I(\alpha(X_i - s) > g)$$

equation (2.17) becomes

$$P_{ars} = \prod_{i=1}^{n} \left[e^{(g-r)}P\left(X_i \leq s + \frac{g}{\alpha}; h^*\right) + e^{(\alpha-1)r-\alpha s + \frac{1}{2}\alpha(\alpha-1)\sigma^2} \times P\left(X_i > s + \frac{g}{\alpha}; h^* + \alpha\right)\right]$$

(2.18)

where we used equation (2.7) again.

Since $X_i$'s are identically distributed normals with mean $\mu$ and variance $\sigma^2$, using equation (2.5), we get from (2.18) that

$$P_{ars} = \left[e^{(g-r)}\Phi\left(\frac{s + \frac{g}{\alpha} - r + \frac{\sigma^2}{2}}{\sigma}\right) + e^{(\alpha-1)r-\alpha s + \frac{1}{2}\alpha(\alpha-1)\sigma^2} \times \Phi\left(\frac{r - \frac{\sigma^2}{2} + \alpha\sigma^2 - s - \frac{g}{\alpha}}{\sigma}\right)\right]^n$$

(2.19)

We should note that if we let $s = 0$ in (2.19) we obtain (2.12).

We can see from figure 2.7 (with the same settings as before) that $P_{ars}$ is an increasing function of $\alpha$ for fixed $n$ and, as expected, the smaller $s$ (we have two
models with $s = 5\%$ and $s = 10\%$ respectively) the bigger the value of the policy. It’s still more expensive than a point-to-point design, but less expensive than the normal annual reset design as shown in figure 2.8.

2.3 The Continuous Lookback Design

The last classic equity indexed annuity we discuss in this chapter is the continuous lookback design. The basic concept of this type of policy is that, at maturity, the interest earned will be based on the growth rate of the highest index value attained during the life of the policy over the index value at the start of the term.

In this section we consider the continuous lookback in order to obtain a closed-form formula for the price of this design. Let

$$M(T) = \max_{0 \leq t \leq T} X(t)$$

be the maximum rate of return on the index attained over the time interval $[0, T]$. At time $T$ the policy pays $e^{\alpha M(T)}$, $\alpha$ being the participation rate, or $e^{gT}$, whichever is higher. Here $g$ is again the minimum guaranteed rate. Then the value of this policy is:

$$P_{cl} = E\left[ e^{-rT} \max (e^{\alpha M(T)}, e^{gT}); h^* \right]$$

$$= E\left[ e^{-rT} e^{\alpha M(T)} I(\alpha M(T) > gT); h^* \right]$$

$$+ E\left[ e^{-rT} e^{gT} I(\alpha M(T) \leq gT); h^* \right]$$

$$= e^{-rT} E\left[ e^{\alpha M(T)} I(\alpha M(T) > gT); h^* \right]$$

$$+ e^{(g-r)T} P\left( M(T) \leq \frac{gT}{\alpha}; h^* \right)$$

(2.20)

We know that (see [8] or [9]) that

$$P( M(t) \leq x) = \Phi \left( \frac{x - \mu t}{\sigma \sqrt{t}} \right) - e^{\frac{2x \mu}{\sigma^2}} \Phi \left( \frac{-x - \mu t}{\sigma \sqrt{t}} \right)$$

(2.21)
for any \( t > 0, x \geq 0 \).

Then using equation (2.5) we obtain:

\[
P \left( M(T) \leq \frac{gT}{\alpha}; h^* \right) = \Phi \left[ \frac{\frac{gT}{\alpha} - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right]
\]

\[
- \exp \left[ 2 \frac{\left( r - \frac{\sigma^2}{2} \right) \frac{gT}{\alpha}}{\sigma^2} \right] \Phi \left[ \frac{-\frac{gT}{\alpha} - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] \tag{2.22}
\]

Here we computed the probability term from equation (2.20).

In order to get a closed-form formula for \( P_{cl} \) we need to compute the expectation term from (2.20).

We need the probability density function of \( M(t) \). For that we take the derivative with respect to \( x \) in equation (2.21). We get

\[
f_{M(t)}(x) = \frac{1}{\sigma \sqrt{t}} \varphi \left( \frac{x - \mu t}{\sigma \sqrt{t}} \right) + e^{\frac{2\mu x}{\sigma^2}} \frac{1}{\sigma \sqrt{t}} \varphi \left( \frac{-x - \mu t}{\sigma \sqrt{t}} \right)
\]

\[
- \frac{2\mu}{\sigma^2} e^{\frac{2\mu x}{\sigma^2}} \Phi \left( \frac{-x - \mu t}{\sigma \sqrt{t}} \right) \tag{2.23}
\]

where \( \varphi(\cdot) \) and \( \Phi(\cdot) \) denote the p.d.f and the c.d.f. of a standard normal variable, respectively. Using (2.23) and (2.5) we get

\[
E \left[ e^{\alpha M(T)} I (\alpha M(T) > gT); h^* \right]
\]

\[
= E \left[ e^{\alpha X_1} I \left( X_1 > \frac{gT}{\alpha} \right) \right] + E \left\{ \exp \left[ \frac{2}{\sigma^2} \frac{\left( r - \frac{\sigma^2}{2} \right) T}{\alpha} \right] X_2 I \left( X_2 > \frac{gT}{\alpha} \right) \right\}
\]

\[
- \frac{2 \left( r - \frac{\sigma^2}{2} \right)}{\sigma^2} \times \int_{\frac{gT}{\alpha}}^\infty e^{\frac{2}{\sigma^2} \frac{\left( r - \frac{\sigma^2}{2} \right) T}{\alpha}} \Phi \left[ \frac{-x - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] dx
\]

\[
= E \left[ e^{\alpha X_1} I \left( X_1 > \frac{gT}{\alpha} \right) \right] + E \left[ e^{\alpha(\alpha)X_2} I \left( X_2 > \frac{gT}{\alpha} \right) \right]
\]

25
\[-\left(\frac{2r}{\sigma^2} - 1\right) \int_0^\infty e^{a(x)x} \Phi \left[ -x - \frac{r - \frac{\sigma^2}{T}}{\sigma}\sqrt{T} \right] dx \] (2.24)

where \(a(\alpha) = \alpha + \frac{2r}{\sigma^2} - 1\) and \(X_1\) and \(X_2\) are normal random variables with means \(\left(r - \frac{1}{2}\sigma^2\right) T\) and \(-\left(r - \frac{1}{2}\sigma^2\right) T\) and common variance \(\sigma^2 T\).

By applying equation (2.7) with \(h^* = 0\) we obtain

\[E \left[ e^{\alpha X_1 I \left( X_1 > \frac{gT}{\alpha}\right)} \right] = e^{[\alpha r + \frac{1}{2} a(\alpha - 1) \sigma^2] T} \Phi \left[ \left(\frac{r - \frac{\sigma^2}{T} + a(\alpha) \sigma^2}{\sigma}\sqrt{T}\right) - \frac{gT}{\alpha} \right] \] (2.25)

and

\[E \left[ e^{a(\alpha) X_2 I \left( X_2 > \frac{gT}{\alpha}\right)} \right] = e^{[- a(\alpha) r + \frac{1}{2} a(\alpha)(a(\alpha) - 1) \sigma^2] T} \Phi \left\{ \left[ -r + \frac{\sigma^2}{T} + a(\alpha) \sigma^2 \right] T - \frac{gT}{\alpha} \sigma\sqrt{T} \right\} \] (2.26)

To compute the integral from (2.24) we will use equation (2.26) and integration by parts. We have

\[\int_0^\infty e^{a(x)x} \Phi \left[ -x - \frac{r - \frac{\sigma^2}{T}}{\sigma}\sqrt{T} \right] dx = \frac{1}{a(\alpha)} \left\{ e^{a(\alpha)x} \Phi \left[ -x - \frac{r - \frac{\sigma^2}{T}}{\sigma}\sqrt{T} \right] \right\}_{x = \frac{gT}{\alpha}} + \int_0^\infty \frac{d}{dx} e^{a(\alpha)x} \Phi \left[ -x - \frac{r - \frac{\sigma^2}{T}}{\sigma}\sqrt{T} \right] dx \] (2.27)
Using (2.20), (2.22), (2.24), (2.25), (2.26), and (2.27) we obtain the formula for the price of this “continuous lookback” policy:

\[
P_{cl} = e^{-rT} \left\{ \frac{\left(r - \frac{\sigma^2}{2} + \alpha\sigma^2\right) T - \frac{aT\alpha^2}{2}}{\sigma\sqrt{T}} \right\} \\
+ e^{-rT} \left( \frac{2r}{\sigma^2} - 1 \right) \frac{1}{a(\alpha)} \left\{ -e^{\frac{a(\alpha)gT}{\alpha}} \Phi \left[ \frac{-\frac{aT\alpha^2}{2} - (r - \frac{\sigma^2}{2}) T}{\sigma\sqrt{T}} \right] \right\} \\
- \frac{a(\alpha)}{\alpha} e^{-\left(\frac{a(\alpha)gT}{\alpha} - T\right)} \Phi \left[ \frac{\frac{aT\alpha^2}{2} - (r - \frac{\sigma^2}{2}) T}{\sigma\sqrt{T}} \right] \\
+ e^{(a(\alpha)gT - r)T} \frac{\frac{aT\alpha^2}{\alpha} - (r - \frac{\sigma^2}{2}) T}{\sigma\sqrt{T}} - e^{(a(\alpha)gT - r)T} \frac{\frac{aT\alpha^2}{\alpha} - (r - \frac{\sigma^2}{2}) T}{\sigma\sqrt{T}} \right\} 
\]

(2.28)

After further simplifications we get

\[
P_{cl} = e^{\left(\frac{(\alpha - 1)T}{\alpha} + \frac{1}{2}a(\alpha-1)\sigma^2\right)T} \Phi \left[ \frac{\left(r - \frac{\sigma^2}{2} + \alpha\sigma^2\right) T - \frac{gT\alpha^2}{\alpha}}{\sigma\sqrt{T}} \right] \\
+ \frac{\alpha}{a(\alpha)} e^{\left(-\left(\frac{a(\alpha)gT}{\alpha} + \frac{1}{2}a(\alpha)(a(\alpha)-1)\sigma^2\right)T\right)} \Phi \left[ \frac{-\frac{gT\alpha^2}{\alpha} - \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \right] \\
- \frac{\alpha}{a(\alpha)} e^{\left(\frac{a(\alpha)gT}{\alpha} - r\right)T} \Phi \left[ \frac{\frac{gT\alpha^2}{\alpha} - \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \right] \\
+ e^{(a(\alpha)gT - r)T} \frac{\frac{gT\alpha^2}{\alpha} - \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \right\} 
\]

(2.29)

On figures 2.9 and 2.10 we have the three “big” or “classic” designs. From the “annual reset” design we chose the least expensive one -the one with a cap- and even so it’s by far the most expensive among the three classics. Surprisingly, the “continuous lookback” design, even though it seems more customer-driven, it has almost the same price as a “point-to-point” design. In a more volatile market when
\( \sigma = 30\% \) (see figure 2.10 - the other settings remained the same as before) the point-to-point design is more expensive than the continuous lookback model, especially for practical values of \( \alpha \) (that is when \( \alpha < 1 \)). For a normal market, when \( \sigma = 20\% \), the “continuous lookback” design is a little bit more expensive than a “point-to-point” design (see figure 2.9 - same settings as before). We should also note that in a more volatile market all these designs are more costly. This is one of the main problems we will solve in the next chapter.
Figure 2.1: Ppp for different T’s

Figure 2.2: Ppp for different z’s
Figure 2.3: Par for different n’s

Figure 2.4: Par and Ppp for T=10
Figure 2.5: Parc for different c's

Figure 2.6: Par, Parc and Ppp for T=10
Figure 2.7: Pars for different $s$

Figure 2.8: Par, Pars and Ppp for $T=10$
Figure 2.9: Parc, Pcl and Ppp for sigma=20%

Figure 2.10: Parc, Ppp and Pcl for sigma=30%
CHAPTER 3

THE NEW EQUITY INDEXED ANNUITY DESIGNS

In this chapter we will introduce new EIA designs and we will show that this new designs can produce better participation rates in different market settings. And, in some cases, we will also see that this new designs can provide higher participation rates for lower prices. We will consider four different market settings in which we will compare the new designs we introduce in this chapter to the classic ones introduced in the previous chapter. These four market settings are as follows:

- **case 1**: $T = 10, g = 3\%$, $\sigma = 20\%$, $r = 4\%$;
- **case 2**: $T = 10, g = 3\%$, $\sigma = 20\%$, $r = 5\%$;
- **case 3**: $T = 10, g = 3\%$, $\sigma = 30\%$, $r = 4\%$;
- **case 4**: $T = 10, g = 3\%$, $\sigma = 30\%$, $r = 5\%$.

In the first three sections of this chapter we will improve each classic design discussed in chapter 2. In section 3.4 we will talk about some new designs that cannot be fitted in any of the three big categories of EIA we mentioned in chapter 2.
3.1 Improvements on The Point-To-Point Design

We start with a first improvement on the classic point-to-point design that we call a double barrier design. To put it in simple words, this policy will pay the realized return on the index, with a participation rate $\alpha_1$, as long as that is more than the minimum guaranteed rate $g$, and it does so only if the index reached a certain barrier $B_1$. It will pay a different participation rate $\alpha_2$ provided that the index reached a second (higher) barrier $B_2$.

In order for this policy to make sense a few conditions need to be satisfied:

$$B_2 \geq B_1 \geq g \quad \text{and} \quad \frac{B_2}{\alpha_2} > \frac{B_1}{\alpha_1}$$

(3.1)

So, this policy will pay:

$$\begin{align*}
&\left\{ \begin{array}{ll}
\max \left( e^{gT}, e^{\alpha_2 X(T)} \right) & \text{if } \alpha_2 X(T) \geq B_2 T; \\
\max \left( e^{gT}, e^{\alpha_1 X(T)} \right) & \text{if } \alpha_2 X(T) < B_2 T \quad \text{and} \quad \alpha_1 X(T) \geq B_1 T; \\
e^{gT} & \text{if } \alpha_1 X(T) < B_1 T.
\end{array} \right.
\end{align*}$$

$$= \begin{align*}
&\left\{ \begin{array}{ll}
e^{\alpha_2 X(T)} & \text{if } X(T) \geq \frac{B_2 T}{\alpha_2}; \\
e^{\alpha_1 X(T)} & \text{if } \frac{B_2 T}{\alpha_2} > X(T) \geq \frac{B_1 T}{\alpha_1}; \\
e^{gT} & \text{if } X(T) < \frac{B_1 T}{\alpha_1}.
\end{array} \right.
\end{align*}$$

$$= e^{\alpha_2 X(T)} I \left( X(T) \geq \frac{B_2 T}{\alpha_2} \right) + e^{\alpha_1 X(T)} I \left( \frac{B_2 T}{\alpha_2} > X(T) \geq \frac{B_1 T}{\alpha_1} \right)$$

$$+ e^{gT} I \left( X(T) < \frac{B_1 T}{\alpha_1} \right)$$

(3.2)

Using the Esscher transform method, the price of this policy is:

$$P_{bp} = e^{-rT} \left\{ E \left[ e^{\alpha_2 X(T)} I \left( X(T) \geq \frac{B_2 T}{\alpha_2} \right); h^* \right] + E \left[ e^{\alpha_1 X(T)} I \left( \frac{B_2 T}{\alpha_2} > X(T) \geq \frac{B_1 T}{\alpha_1} \right); h^* \right] \right\}.$$
Using equations (2.7) and (2.8) we obtain:

\[ P_{bp} = e^{-rT} e^{\left(\frac{1}{2} \sigma^2\right) T} \left[ \frac{\left(r - \frac{1}{2} \sigma^2 + \alpha_2 \sigma^2\right) T - B_2 T}{\sigma \sqrt{T}} \right] \Phi \left( \frac{r - \frac{1}{2} \sigma^2 + \alpha_1 \sigma^2}{\sigma \sqrt{T}} \right) \]

\[ + e^{-rT} e^{\left(\frac{1}{2} \sigma^2\right) T} \left\{ \Phi \left( \frac{r - \frac{1}{2} \sigma^2 + \alpha_1 \sigma^2}{\sigma \sqrt{T}} \right) \right\} - \Phi \left( \frac{r - \frac{1}{2} \sigma^2 + \alpha_1 \sigma^2}{\sigma \sqrt{T}} \right) \Phi \left( \frac{B_1 T - \left(r - \frac{1}{2} \sigma^2\right) T}{\sigma \sqrt{T}} \right) \]

so

\[ P_{bp} = e^{\left(\frac{1}{2} \sigma^2\right) T} \left[ \frac{\left(r - \frac{1}{2} \sigma^2 + \alpha_2 \sigma^2\right) T - B_2 T}{\sigma \sqrt{T}} \right] \Phi \left( \frac{r - \frac{1}{2} \sigma^2 + \alpha_1 \sigma^2}{\sigma \sqrt{T}} \right) \]

\[ + e^{\left(\frac{1}{2} \sigma^2\right) T} \left\{ \Phi \left( \frac{r - \frac{1}{2} \sigma^2 + \alpha_1 \sigma^2}{\sigma \sqrt{T}} \right) \right\} - \Phi \left( \frac{r - \frac{1}{2} \sigma^2 + \alpha_1 \sigma^2}{\sigma \sqrt{T}} \right) \Phi \left( \frac{B_1 T - \left(r - \frac{1}{2} \sigma^2\right) T}{\sigma \sqrt{T}} \right) \]

We can now generalize the double barrier design to a multiple barrier design. We take now \( n \) different barriers \( B_n > B_{n-1} > \ldots > B_1 > g \) with \( n \) different participation rates \( \alpha_n, \alpha_{n-1}, \ldots, \alpha_1 \) such that

\[ \frac{B_n}{\alpha_n} > \frac{B_{n-1}}{\alpha_{n-1}} > \ldots > \frac{B_1}{\alpha_1}. \]

This policy will pay:
\[
\begin{align*}
\begin{cases}
e^{\alpha_n X(T)} & \text{if } X(T) \geq \frac{B_n T}{\alpha_n}, \\
e^{\alpha_{n-1} X(T)} & \text{if } \frac{B_n T}{\alpha_n} > X(T) \geq \frac{B_{n-1} T}{\alpha_{n-1}}, \\
 \vdots \\
e^{\alpha_1 X(T)} & \text{if } \frac{B_2 T}{\alpha_2} > X(T) \geq \frac{B_1 T}{\alpha_1}, \\
e^{g T} & \text{if } X(T) < \frac{B_1 T}{\alpha_1}.
\end{cases}
\end{align*}
\]

(3.6)

and the price of this policy will be given by:

\[
P_{mp} = e^{[(\alpha_n-1)r + \frac{1}{2} \alpha_n (\alpha_n-1) \sigma^2]T} \Phi \left[ \frac{\left( r - \frac{1}{2} \sigma^2 + \alpha_n \sigma^2 \right) T - \frac{B_n T}{\alpha_n}}{\sigma \sqrt{T}} \right]
\]

\[
+ e^{[(\alpha_n-1)r + \frac{1}{2} \alpha_n (\alpha_n-1) \sigma^2]T} \Phi \left[ \frac{\left( r - \frac{1}{2} \sigma^2 + \alpha_{n-1} \sigma^2 \right) T - \frac{B_{n-1} T}{\alpha_{n-1}}}{\sigma \sqrt{T}} \right]
\]

\[
- \Phi \left[ \frac{\left( r - \frac{1}{2} \sigma^2 + \alpha_{n-1} \sigma^2 \right) T - \frac{B_n T}{\alpha_n}}{\sigma \sqrt{T}} \right]
\]

\[
+ \ldots + e^{[(\alpha_1-1)r + \frac{1}{2} \alpha_1 (\alpha_1-1) \sigma^2]T} \Phi \left[ \frac{\left( r - \frac{1}{2} \sigma^2 + \alpha_1 \sigma^2 \right) T - \frac{B_1 T}{\alpha_1}}{\sigma \sqrt{T}} \right]
\]

\[
- \Phi \left[ \frac{\left( r - \frac{1}{2} \sigma^2 + \alpha_1 \sigma^2 \right) T - \frac{B_2 T}{\alpha_2}}{\sigma \sqrt{T}} \right]
\]

\[
+ e^{(g-r)T} \Phi \left[ \frac{\frac{B_1 T}{\alpha_1} - \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right]
\]

(3.7)

where, we used again the Esscher transform method and the results from Chapter 2.

To see if these new designs are better than the classic point-to-point design we will compare the classic one with two new models - a double barrier design and a triple barrier one.

For model 1 we consider a design with $B_1 = 7\%$, $B_2 = 14\%$, and $\alpha_2 = 0.9\alpha_1$.

For model 2 we consider a design with $B_1 = 7\%$, $B_2 = 14\%$, $B_3 = 20\%$, $\alpha_3 = 0.9\alpha_1$, and $\alpha_2 = 1.1\alpha_1$. 

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We will compare these two models with the classic one in the four different market setting mentioned at the beginning of this chapter.

As we can see from tables 3.1-3.4, in each case the new design improves dramatically the classic design. Let’s look at the break-even price of $1 and notice that in each of the four cases, the participation rates of the new design are either similar ($\alpha_2$ in model 1 and $\alpha_3$ in model 2) or a lot higher than the participation rates of the classic designs. In fact, in each case the participation rates for the new designs for a price of $0.98 are still better than the participation rates for the classic design for a price of $1.

Looking at figures 3.1-3.4 we can also see that the new designs improve the classic one and clearly the new designs outperform the classic one for any price.

We should also mention that model 2 seems to be weaker than model 1 but that’s just because we set $\alpha_2 = 1.1\alpha_1$. If we actually set $\alpha_2 = a\alpha_1$ with $a < 1$, we have a model that is very similar to model 1.

We should note that the barriers we set for the two new models above are fairly low, considering that we are talking about a 10-year period. If we consider higher barriers, the new design is going to be even better. To see this, we will compare the classic point-to-point design with a double barrier model (say model 3) with $B_1 = 10\%$, $B_2 = 20\%$, and $\alpha_2 = 0.9\alpha_1$. We will compare these models in two different market settings - case 2 and case 4.

As we can see from tables 3.5 and 3.6 and from figures 3.5 and 3.6, the new design with higher barriers is a lot better than the classic one and improves the classic setting even further than models 1 and 2. Looking at the break-even price of all these policies for a market settings in case 2, we can see that we start with a participation rate.
of .7679 for the classic design, improve to a .8579 participation rate for model 1 and getting to a very high and impressive .8841 participation rate on model 3. In case 4 the same dramatic improvement can be noticed.

To conclude, depending on how aggressive the customer wants to get, the new design proposed in this section can improve the classic design by a lot, or just by a little bit. The key point is that we wanted to improve the classic model and we have done just that.

3.2 Improvements on The Annual Reset Design

To improve the classic annual reset design we are going to use the same multi-barrier approach that we used in the previous section. Let’s remind ourselves that the classic annual reset design is a policy that will earn, at the end of every period, the return on the index over that period with a participation rate $\alpha$ or the minimum guaranteed $g$, whichever is higher. Once the interest is credited, the earnings are locked in and will never decrease.

The new design that we propose in this section is described as follows: The policy will earn at the end of every period the realized return of the index, with a certain participation rate $\alpha_1$, as long as that return is more than the minimum guaranteed $g$, and it does so only if the index reached a certain barrier $B_1$. It will pay a different participation rate $\alpha_2$ if the index reached a second barrier $B_2$, and so on.

We will set again the conditions:

$$B_n > B_{n-1} > \ldots > B_1 > g$$

and

$$\frac{B_n}{\alpha_n} > \frac{B_{n-1}}{\alpha_{n-1}} > \ldots > \frac{B_1}{\alpha_1}$$
Let \( X_i = X(i) - X(i-1), i = 1, 2, \ldots, m \) be the rate of return on the asset over the \( i^{th} \) period of time. Since \( X(t) \) is a Brownian motion \( X_i \) is a sequence of independent and identically distributed random variables with
\[
X_i \sim N(\mu, \sigma^2) \quad \text{for any} \quad i = 1, 2, \ldots, m.
\]

Now let’s denote
\[
A_i = e^{\alpha_n X_i} I \left( X_i \geq \frac{B_n}{\alpha_n} \right) + e^{\alpha_{n-1} X_i} I \left( \frac{B_n}{\alpha_n} > X_i \geq \frac{B_{n-1}}{\alpha_{n-1}} \right) + \ldots + e^{\alpha_1 X_i} I \left( \frac{B_2}{\alpha_2} > X_i \geq \frac{B_1}{\alpha_1} \right) + e^g I \left( X_i < \frac{B_1}{\alpha_1} \right) \quad (3.8)
\]

Then the value of this new design at time 0 is given by:
\[
P_{mar} = E \left[ e^{-r m} A_1 A_2 \ldots A_m; h^* \right] \quad (3.9)
\]

Since \( X_i \)'s are independent and identically distributed, so are the \( A_i \)'s. Then (3.9) becomes:
\[
P_{mar} = e^{-r m} \prod_{i=1}^{m} E(A_i; h^*)
\]
\[
= \left[ e^{-r} E(A_1; h^*) \right]^m
\]
\[
= \left\{ e^{-r} E \left[ e^{\alpha_n X_1} I \left( X_1 \geq \frac{B_n}{\alpha_n} \right) + e^{\alpha_{n-1} X_1} I \left( \frac{B_n}{\alpha_n} > X_1 \geq \frac{B_{n-1}}{\alpha_{n-1}} \right) + \ldots + e^{\alpha_1 X_1} I \left( \frac{B_2}{\alpha_2} > X_1 \geq \frac{B_1}{\alpha_1} \right) + e^g I \left( X_1 < \frac{B_1}{\alpha_1} \right); h^* \right] \right\}^m \quad (3.10)
\]

The expression inside the parenthesis in the last equality of (3.10) is the value of the point-to-point new design (introduced in Section 3.1) with \( T = 1 \).
So using equation (3.7) we obtain:

\[
P_{mar} = \begin{cases} 
    e^{(\alpha_n-1)r + \frac{1}{2}\alpha_n(\alpha_n-1)\sigma^2} \Phi \left( \frac{r - \frac{1}{2}\sigma^2 + \alpha_n\sigma^2 - \frac{B_n}{\alpha_n}}{\sigma} \right) \\
    + e^{(\alpha_n-1)r + \frac{1}{2}\alpha_n(\alpha_n-1)\sigma^2} \left[ \Phi \left( \frac{r - \frac{1}{2}\sigma^2 + \alpha_n\sigma^2 - \frac{B_n}{\alpha_n}}{\sigma} \right) \right] \\
    \ldots + e^{(\alpha_1-1)r + \frac{1}{2}\alpha_1(\alpha_1-1)\sigma^2} \left[ \Phi \left( \frac{r - \frac{1}{2}\sigma^2 + \alpha_1\sigma^2 - \frac{B_1}{\alpha_1}}{\sigma} \right) \right] \\
    - \Phi \left( \frac{r - \frac{1}{2}\sigma^2 + \alpha_1\sigma^2 - \frac{B_2}{\alpha_2}}{\sigma} \right) \right] + \epsilon^{g-r} \Phi \left[ \frac{\frac{B_2}{\alpha_2} - \left( r - \frac{1}{2}\sigma^2 \right)}{\sigma} \right] \right] \right] m \tag{3.11}
\]

We note that the formulae for the classic designs can be obtained from the formula for the new design by setting \( \alpha_n = \alpha_{n-1} = \ldots = \alpha_1 = \alpha \) and \( B_n = B_{n-1} = \ldots = B_1 = g \).

To see if this new design is better than the classic annual reset design we will compare it (the classic one), as before, with two different models - a double barrier design and a triple barrier one.

For model 1 we consider a design with \( B_1 = 4\% \), and \( B_2 = 6\% \), and \( \alpha_2 = .9\alpha_1 \)

For model 2 we consider a design with \( B_1 = 4\% \), and \( B_2 = 6\% \), and \( B_3 = 8\% \), and \( \alpha_3 = .9\alpha_1 \), and \( \alpha_2 = 1.1\alpha_1 \)

We will study the same four cases we considered earlier based on the different assumptions on the market. We will consider \( n = 10 \) instead of \( T = 10 \).
We should note that since this policy resets every year the barriers we choose should be much smaller than the barriers we choose for a point-to-point design that goes over 10 years.

Looking at tables 3.7-3.10 we can see again the same pattern as in section 3.1: \( \alpha_2 \) in model 1 and \( \alpha_3 \) in model 2 are similar with the participation rate of the classic design but the other participation rates of the new design are higher. Also looking at the plots (see figures 3.7-3.10) of the prices of these policies as functions of the participation rate we can see again that for every price the new design provides better participation rates. Is to be mentioned again that if the customer is looking to be more aggressive (and hence being able to set higher barriers) the new design is going to improve the classic one by even more then what we see in these four cases studied. In all the cases studied here we notice that if the market is assumed to be more volatile (which means higher \( \sigma \) (case 3 and 4), the participation rates are lower than in a more stable market (case 1 and 2).

### 3.3 Improvements on The Continuous Lookback Design

The last classic design we look to improve is the continuous lookback design. In order to do this, we will use the same multibARRIER approach from the previous sections. As opposed to the first two classic designs we discussed, this one is “path related”. The point-to-point design only looks at the value of the index when the policy expires. The annual reset is looking at the value of the index at the end of every time period disregarding the way the index behaved during that period. That is what this last classic design does: Is looking at the highest value the index has
reached over the given period and if that rate is bigger than the minimum guaranteed
g, you will get that maximum with a certain participation rate.

The new design we propose here has essentially the same structure as the classic
one, only that we pay a certain participation rate \( \alpha \), if the maximum is above a
certain barrier \( B_1 \), and then a different participation rate \( \alpha_2 \), when the maximum has
reached a second barrier, and so on.

Let’s find a closed-form formula for the price of a new design with only two barriers.
Let

\[
M(T) = \max_{0 \leq t \leq T} X(t)
\]

The payoff of this new policy is given by:

\[
\begin{cases}
  e^{\alpha_2 M(T)} & \text{if } M(T) \geq \frac{B_2 T}{\alpha_2}; \\
  e^{\alpha_1 M(T)} & \text{if } \frac{B_2 T}{\alpha_2} > M(T) \geq \frac{B_1 T}{\alpha_1}; \\
  e^{g T} & \text{if } M(T) < \frac{B_1 T}{\alpha_1}.
\end{cases}
\]

\[
= e^{\alpha_2 M(T)} I \left( M(T) \geq \frac{B_2 T}{\alpha_2} \right) + e^{\alpha_1 M(T)} I \left( \frac{B_2 T}{\alpha_2} > M(T) \geq \frac{B_1 T}{\alpha_1} \right) \\
+ e^{g T} I \left( M(T) < \frac{B_1 T}{\alpha_1} \right)
\]

Then the value of this policy at time 0 is:

\[
P_{mcl} = E \left[ e^{-r T} e^{\alpha_2 M(T)} I \left( M(T) \geq \frac{B_2 T}{\alpha_2} \right); h^{*} \right] \\
+ E \left[ e^{-r T} e^{\alpha_1 M(T)} I \left( \frac{B_2 T}{\alpha_2} > M(T) \geq \frac{B_1 T}{\alpha_1} \right); h^{*} \right] \\
+ E \left[ e^{-r T} e^{g T} I \left( M(T) < \frac{B_1 T}{\alpha_1} \right); h^{*} \right]
\]

\[
= e^{-r T} E \left[ e^{\alpha_2 M(T)} I \left( M(T) \geq \frac{B_2 T}{\alpha_2} \right); h^{*} \right] \\
+ e^{-r T} E \left[ e^{\alpha_1 M(T)} I \left( \frac{B_2 T}{\alpha_2} > M(T) \geq \frac{B_1 T}{\alpha_1} \right); h^{*} \right]
\]

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\[ + e^{(g-r)T} P \left( M(T) < \frac{B_1 T}{\alpha_1}; h^* \right) \]  

(3.13)

Using equation (2.22) we can compute the last term of equation (3.13) above:

\[
P \left( M(T) < \frac{B_1 T}{\alpha_1}; h^* \right) = \Phi \left[ \frac{B_1 T}{\alpha_1} - \left( r - \frac{\sigma^2}{2} \right) T \sqrt{\frac{T}{\sigma^2}} \right] - \exp \left[ 2 \left( r - \frac{\sigma^2}{2} \right) \frac{B_1 T}{\alpha_1} \right] 
\times \Phi \left[ \frac{-B_1 T}{\alpha_1} - \left( r - \frac{\sigma^2}{2} \right) T \sqrt{\frac{T}{\sigma^2}} \right] \]  

(3.14)

The first term of equation (3.13) can be computed using equation (2.23) and (2.24):

\[
E \left[ e^{a_2 M(T)} I \left( M(T) \geq \frac{B_2 T}{\alpha_2} \right) ; h^* \right] 
= E \left[ e^{a_2 X_1} I \left( X_1 \geq \frac{B_2 T}{\alpha_2} \right) \right] + E \left[ e^{a_2 X_2} I \left( X_2 \geq \frac{B_2 T}{\alpha_2} \right) \right] 
- \left( \frac{2r}{\sigma^2} - 1 \right) \int_{\frac{B_2 T}{\alpha_2}}^{\infty} e^{a_2 x} \Phi \left( \frac{-x - \left( r - \frac{\sigma^2}{2} \right) T \sqrt{\frac{T}{\sigma^2}}}{\sigma \sqrt{T}} \right) dx \]  

(3.15)

where \( a(a_2) = \alpha_2 + \frac{2r}{\sigma^2} - 1 \), and \( X_1 \) and \( X_2 \) are normal random variables with means \( \left( r - \frac{1}{2} \sigma^2 \right) T \) and \( - \left( r - \frac{1}{2} \sigma^2 \right) T \) and common variance \( \sigma^2 T \).

Using equations (2.25), (2.26) and (2.27), equation (3.15) above becomes:

\[
E \left[ e^{a_2 M(T)} I \left( M(T) \geq \frac{B_2 T}{\alpha_2} \right) ; h^* \right] 
= e^{a_2 r + \frac{1}{2} \alpha_2 (a_2 - 1) \sigma^2 T} \Phi \left( \frac{\left( r - \frac{\sigma^2}{2} + \alpha_2 \sigma^2 \right) T - \frac{B_2 T}{\alpha_2}}{\sigma \sqrt{T}} \right) + e^{-a(a_2) r + \frac{1}{2} a(a_2) (a(a_2) - 1) \sigma^2 T} 
\times \Phi \left( \frac{-r + \sigma^2}{2 \sigma \sqrt{T}} \right) T - \frac{B_2 T}{\alpha_2} \right) 
\times \left\{ -e^{a(a_2) B_2 T \alpha_2} \Phi \left( \frac{-B_2 T}{\alpha_2} - \left( r - \frac{\sigma^2}{2} \right) T \sqrt{\frac{T}{\sigma^2}} \right) + e^{-a(a_2) r + \frac{1}{2} a(a_2) (a(a_2) - 1) \sigma^2 T} \right\} 
\times \Phi \left( \frac{\left( r - \frac{\sigma^2}{2} + \alpha_2 \sigma^2 \right) T - \frac{B_2 T}{\alpha_2}}{\sigma \sqrt{T}} \right) 
+ e^{-a(a_2) r + \frac{1}{2} a(a_2) (a(a_2) - 1) \sigma^2 T} \right\}
\]
and then using (3.16)

\[
\times \Phi \left\{ \frac{-r + \frac{\sigma^2}{2} + a(\alpha_2)\sigma^2}{\sigma\sqrt{T}} T - \frac{B_2 T}{\alpha_2} \right\}
\]

\[
= e^{[\alpha_2 r + \frac{1}{2} \alpha_2 (\alpha_2 - 1) \sigma^2] T} \Phi \left[ \frac{(r - \frac{\sigma^2}{2} + \alpha_2 \sigma^2) T - \frac{B_2 T}{\alpha_2}}{\sigma\sqrt{T}} \right]
\]

\[
+ \frac{\alpha_2}{a(\alpha_2)} e^{[-a(\alpha_2) r + \frac{1}{2} a(\alpha_2) (a(\alpha_2) - 1) \sigma^2] T} \Phi \left\{ \frac{-r + \frac{\sigma^2}{2} + a(\alpha_2)\sigma^2}{\sigma\sqrt{T}} T - \frac{B_2 T}{\alpha_2} \right\}
\]

\[
+ \frac{a(\alpha_2) - \alpha_2}{a(\alpha_2)} e^{[\sigma(\alpha_2) B_2 T - a(\alpha_2)] T} \Phi \left[ \frac{-B_2 T}{\alpha_2} - \left( r - \frac{\sigma^2}{2} \right) T \right]
\]

The middle term of equation (3.13) is a bit more complicated to get but, fortunately, most of the work is done when computing the first term (see equation (3.16)).

First we can write:

\[
E \left[ e^{\alpha_1 M(T)} I \left( \frac{B_2 T}{\alpha_2} > M(T) \geq \frac{B_1 T}{\alpha_1} \right) ; h^* \right]
\]

\[
= E \left[ e^{\alpha_1 M(T)} I \left( M(T) \geq \frac{B_1 T}{\alpha_1} \right) ; h^* \right] - E \left[ e^{\alpha_1 M(T)} I \left( M(T) \geq \frac{B_2 T}{\alpha_2} \right) ; h^* \right]
\]

and then using (3.16) we have:

\[
E \left[ e^{\alpha_1 M(T)} I \left( \frac{B_2 T}{\alpha_2} > M(T) \geq \frac{B_1 T}{\alpha_1} \right) ; h^* \right]
\]

\[
= e^{[\alpha_1 r + \frac{1}{2} \alpha_1 (\alpha_1 - 1) \sigma^2] T} \left\{ \Phi \left[ \frac{(r - \frac{\sigma^2}{2} + \alpha_1 \sigma^2) T - \frac{B_1 T}{\alpha_1}}{\sigma\sqrt{T}} \right] \right\}
\]

\[
- \left\{ \Phi \left[ \frac{(r - \frac{\sigma^2}{2} + \alpha_1 \sigma^2) T - \frac{B_2 T}{\alpha_2}}{\sigma\sqrt{T}} \right] \right\}
\]

\[
+ \frac{\alpha_1}{a(\alpha_1)} e^{[-a(\alpha_1) r + \frac{1}{2} a(\alpha_1) (a(\alpha_1) - 1) \sigma^2] T} \left\{ \Phi \left[ \frac{-r + \frac{\sigma^2}{2} + a(\alpha_1)\sigma^2}{\sigma\sqrt{T}} T - \frac{B_1 T}{\alpha_1} \right] \right\}
\]

\[
- \left\{ \Phi \left[ \frac{-r + \frac{\sigma^2}{2} + a(\alpha_1)\sigma^2}{\sigma\sqrt{T}} T - \frac{B_2 T}{\alpha_2} \right] \right\}
\]

\[
+ \frac{a(\alpha_1) - \alpha_1}{a(\alpha_1)} e^{[\sigma(\alpha_1) B_1 T / \alpha_1 - a(\alpha_1)] T} \Phi \left[ \frac{-B_1 T}{\alpha_1} - \left( r - \frac{\sigma^2}{2} \right) T \right]
\]

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To get the formula for the price $P_{mcl}$ of this new design we have to multiply equation (3.14) by $e^{(g-r)T}$ and equations (3.16) and (3.17) by $e^{-rT}$ and then add them up. We get:

\[
P_{mcl} = e^{[\alpha_2-1]r+\frac{1}{2}\alpha_2(\alpha_2-1)\sigma^2]T} \left\{ \Phi \left[ \frac{r - \frac{\sigma^2}{2} + \alpha_2 \sigma^2}{\sigma \sqrt{T}} T - \frac{B_2 T}{\alpha_2} \right] \right\} \\
+ \frac{\alpha_2}{a(\alpha_2)} e^{[-(a(\alpha_2)+1)r+\frac{1}{2}a(\alpha_2)(a(\alpha_2)-1)\sigma^2]T} \left\{ \Phi \left[ \frac{r - \frac{\sigma^2}{2} + \alpha_2 \sigma^2}{\sigma \sqrt{T}} T - \frac{B_2 T}{\alpha_2} \right] \right\} \\
+ \frac{a(\alpha_2) - \alpha_2}{a(\alpha_2)} e^{(a(\alpha_2)+1)r+\frac{1}{2}a(\alpha_2)(a(\alpha_2)-1)\sigma^2]T} \left\{ \Phi \left[ \frac{r - \frac{\sigma^2}{2} + \alpha_2 \sigma^2}{\sigma \sqrt{T}} T - \frac{B_2 T}{\alpha_2} \right] \right\} \\
+ e^{[\alpha_1-1]r+\frac{1}{2}\alpha_1(\alpha_1-1)\sigma^2]T} \left\{ \Phi \left[ \frac{r - \frac{\sigma^2}{2} + \alpha_1 \sigma^2}{\sigma \sqrt{T}} T - \frac{B_1 T}{\alpha_1} \right] \right\} \\
- \Phi \left[ \frac{r - \frac{\sigma^2}{2} + \alpha_1 \sigma^2}{\sigma \sqrt{T}} T - \frac{B_1 T}{\alpha_1} \right] \\
+ \frac{\alpha_1}{a(\alpha_1)} e^{[-(a(\alpha_1)+1)r+\frac{1}{2}a(\alpha_1)(a(\alpha_1)-1)\sigma^2]T} \left\{ \Phi \left[ \frac{r - \frac{\sigma^2}{2} + \alpha_1 \sigma^2}{\sigma \sqrt{T}} T - \frac{B_1 T}{\alpha_1} \right] \right\} \\
+ \frac{a(\alpha_1) - \alpha_1}{a(\alpha_1)} \left\{ e^{(a(\alpha_1)+1)r+\frac{1}{2}a(\alpha_1)(a(\alpha_1)-1)\sigma^2]T} \Phi \left[ \frac{r - \frac{\sigma^2}{2} + \alpha_1 \sigma^2}{\sigma \sqrt{T}} T = \frac{B_1 T}{\alpha_1} \right] \right\} \\
- e^{(a(\alpha_1)+1)r+\frac{1}{2}a(\alpha_1)(a(\alpha_1)-1)\sigma^2]T} \Phi \left[ \frac{r - \frac{\sigma^2}{2} + \alpha_1 \sigma^2}{\sigma \sqrt{T}} T = \frac{B_1 T}{\alpha_1} \right] \\
+ e^{(g-r)T} \left\{ \Phi \left[ \frac{B_1 T}{\alpha_1} - \frac{B_2 T}{\alpha_2} \right] \right\} \\
- \exp \left[ \frac{2 (r - \frac{\sigma^2}{2}) B_1 T}{\sigma^2} \right] \left\{ \Phi \left[ \frac{B_1 T}{\alpha_1} - \frac{B_2 T}{\alpha_2} \right] \right\} \right\} \\
(3.18)
\]
In a similar way we can find a closed form formula for a design with \( n \) barriers. But for the particular market cases we study next we will only use a design with two barriers.

We will compare a new design (say model 1) based on the multibarrier approach with the classic continuous lookback model on case 2 of market settings. Our model 1 will have the following settings: \( B_1 = 10\% \), \( B_2 = 20\% \), and \( \alpha_2 = .9\alpha_1 \). As we can see from figure 3.11 the new design is better than the classic one for every price. In table 3.11 we have the actual numbers and we can see that at the break even price the participation rates for the new design are \( \alpha_1 = .8931 \) and \( \alpha_2 = .8038 \), by far bigger than the regular participation rate for the classic design \( \alpha = .7567 \). Model 1 is pretty aggressive if we look at the barriers considered, but even if we take smaller barriers we would still be able to improve the classic design, but not as much.

For a more volatile market as in case 4 we will compare another new design (say model 2) with the classic one. This model has the following settings \( B_1 = 10\% \), \( B_2 = 20\% \), and \( \alpha_2 = .95\alpha_1 \). As we can see from table 3.12 and figure 3.12, the new model outperforms the classic one again, but this time the improvement is not as dramatic as in case 2 studied above.

### 3.4 New “Path-Dependent” Designs

In this section we will introduce a few other designs that can improve the classic ones. These new designs deserve a separate section because they can’t actually be fitted in any of the other sections. The first two sections of this chapter dealt with “path-independent” types of designs and the third section dealt with “path-dependent” designs. By “path-independent” we mean that the design only looks at
the end-value of the index and not at the way it got there. In section 3.3 we saw a “path-dependent” design where we took the maximum of the index over the given period of time.

Here we will try to combine the two ways of thinking into only one model. The key idea is to look at the first time the index hits a certain barrier and design models that takes this into consideration.

First let’s define:

$$\tau = \inf \{ t : X(t) = B \}$$

where $B$ is a positive number (the barrier rate).

A first model that takes only $\tau$ into consideration would have the following payoff:

\[
\begin{cases}
  e^{\alpha a BT} & \text{if } \tau \leq \frac{T}{2}; \\
  e^{\alpha b BT} & \text{if } \frac{T}{2} < \tau \leq T; \\
  e^{gT} & \text{if } \tau > T.
\end{cases}
\]

\[
= e^{\alpha a BT} I \left( \tau \leq \frac{T}{2} \right) + e^{\alpha b BT} I \left( \frac{T}{2} < \tau \leq T \right) + e^{gT} I (\tau > T)
\]

\[
= e^{\alpha a BT} \left[ 1 - I \left( \tau > \frac{T}{2} \right) \right] + e^{\alpha b BT} \left[ I \left( \tau > \frac{T}{2} \right) - I (\tau > T) \right] + e^{gT} I (\tau > T)
\]

\[
= e^{\alpha a BT} + \left( e^{\alpha b BT} - e^{\alpha a BT} \right) I \left( \tau > \frac{T}{2} \right) + \left( e^{gT} - e^{\alpha b BT} \right) I (\tau > T)
\]

where $\alpha$ is the participation rate, $g$ is the minimum guaranteed rate and $a$ and $b$ are two positive numbers such that $a < b < 1$. Let’s call this model a continuous lookback with barrier design.

Then, we can express the value of this policy, under the risk-neutral Esscher parameter $h^*$, as:

\[
P_{bc} = e^{(\alpha a B-r) T} + e^{-rT} E \left[ \left( e^{\alpha b BT} - e^{\alpha a BT} \right) I \left( \tau > \frac{T}{2} \right) ; h^* \right]
\]
\[ + e^{-rT} E \left[ \left( e^{gT} - e^{\alpha b B T} \right) I(\tau > T); h^* \right] \]
\[ = e^{(\alpha a B - r)T} + \left[ e^{(\alpha b B - r)T} - e^{(\alpha a B - r)T} \right] P(\tau > \frac{T}{2}; h^*) \]
\[ + \left[ e^{(g - r)T} - e^{(\alpha b B - r)T} \right] P(\tau > T; h^*) \]

(3.19)

Let’s denote \( p_0 = P(\tau > T; h^*) \)

To compute \( p_0 \) let’s note that

\[ P(\tau > T) = P(M(T) < B), \text{ where} \]
\[ M(T) = \max_{0 \leq t \leq T} X(t) \]

But we know from (2.21) that

\[ P(M(T) \leq x) = \Phi \left( \frac{x - \mu T}{\sigma \sqrt{T}} \right) - e^{\frac{2 \mu x}{\sigma^2}} \Phi \left( \frac{-x - \mu T}{\sigma \sqrt{T}} \right) \]

and then using (2.5) we have that:

\[
P(M(T) \leq x; h^*) = \Phi \left[ \frac{x - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] - \exp \left[ \frac{2 \left( r - \frac{\sigma^2}{2} \right) x}{\sigma^2} \right] \\
\times \Phi \left[ -x - \left( r - \frac{\sigma^2}{2} \right) T \right] \]

(3.21)

So, from (3.20) and (3.21) we get:

\[
p_0 = \Phi \left[ \frac{B - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] - \exp \left[ \frac{2 \left( r - \frac{\sigma^2}{2} \right) B}{\sigma^2} \right] \\
\times \Phi \left[ -B - \left( r - \frac{\sigma^2}{2} \right) T \right] \]

(3.22)

Then using (3.19) and (3.22) for \( T \) and for \( \frac{T}{2} \) we find the value of this policy to be:
\[ P_{bcl} = e^{(\alpha aB-r)T} + \left[ e^{(abB-r)T} - e^{(\alpha aB-r)T} \right] \left\{ \Phi \left[ \frac{B - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] \right\} - \exp \left[ \frac{2 \left( r - \frac{\sigma^2}{2} \right) B}{\sigma^2} \right] \Phi \left[ \frac{-B - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] \right\} + \left[ e^{(g-r)T} - e^{(abB-r)T} \right] \left\{ \Phi \left[ \frac{B - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] \right\} - \exp \left[ \frac{2 \left( r - \frac{\sigma^2}{2} \right) B}{\sigma^2} \right] \Phi \left[ \frac{-B - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] \right\} \] 

(3.23)

To see what kind of improvements this new design brings, we will compare it with all the classic ones. From the annual reset types we choose again the least expensive one - the annual reset with cap. Our model based on this new design will have the following settings: \( B = 10\% \), \( a = .6 \) and \( b = .8 \). We can see directly from figures 3.13 and 3.14 that the new design improves all of the classic ones regardless of the market settings (normal - as in case 2 or volatile - as in case 4). For the actual numbers, on a normal market we look on table 3.13 and note that for a price of only $.98, the new design produces a participation rate \( (\alpha = .8031) \) higher than the participation rates the classic design produce for a break even price of $1. And for a more volatile market we can see from table 3.14 that the new design improves the classic ones even more: this time the participation rate produced at a price of $.95 for the new design is still higher than all the other participation rates for the classic one at a break even price.

A second model that we call capped continuous lookback design takes \( \tau \) and \( X(T) \) in consideration. It has the following payoff: If the first time the index hits a certain barrier \( B \) occurs before time \( T \) then the payoff is \( e^{\alpha aBT} \), where \( \alpha \) is the participation
rate and \( aB \) represents a certain percentage of \( B \) (that is \( a \in (0, 1) \)). If the first time the index hits the same barrier \( B \) occurs after time \( T \), the policy pay \( e^{aX(T)} \) or \( e^{gT} \), whichever is higher. Here \( \alpha \) is the participation rate and \( g \) is the minimum guaranteed. We assume that \( a \) and \( B \) are chosen such that \( aB > g \). Then we can write the payoff of this design as:

\[
\begin{cases}
  e^{a\alpha BT} & \text{if } \tau \leq T; \\
  e^{aX(T)} & \text{if } \tau > T \text{ and } aX(T) \geq g; \\
  e^{gT} & \text{if } \tau > T \text{ and } aX(T) < g.
\end{cases}
\]

\[
e^{a\alpha BT} I(\tau \leq T) + e^{aX(T)} I(\tau > T, X(T) \geq \frac{g}{\alpha}) + e^{gT} I(\tau > T, X(T) < \frac{g}{\alpha})
\]

To express the value of this policy we will use again the risk-neutral Esscher measure of parameter \( h^* \). We have:

\[
P_{cel} = e^{-rT} \left\{ E\left[ e^{a\alpha BT} I(\tau \leq T); h^* \right] + E\left[ e^{aX(T)} I(\tau > T, X(T) \geq \frac{g}{\alpha}); h^* \right] \right. \\
+ \left. E\left[ e^{gT} I(\tau > T, X(T) < \frac{g}{\alpha}); h^* \right] \right\}
\]

(3.24)

Let’s compute each term of equation (3.24) separately:

\[
E\left[ e^{a\alpha BT} I(\tau \leq T); h^* \right] = e^{a\alpha BT} P(\tau \leq T; h^*) = e^{a\alpha BT} (1 - p_0)
\]

where \( p_0 \) is given by (3.22)
Then

\[
E \left[ e^{\alpha aB T} I (\tau \leq T); h^* \right] = e^{\alpha aB T} \left\{ 1 - \Phi \left[ \frac{B - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] \right. \\
+ \exp \left[ \frac{2 \left( r - \frac{\sigma^2}{2} \right) B}{\sigma^2} \right] \Phi \left[ \frac{-B - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] \right\}
\]

\[
= e^{\alpha aB T} \Phi \left[ \frac{-B + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] \\
+ \exp \left[ \alpha aB T + \frac{2 \left( r - \frac{\sigma^2}{2} \right) B}{\sigma^2} \right] \Phi \left[ \frac{-B - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right]
\]

(3.25)

since \( 1 - \Phi(x) = \Phi(-x) \)

From (2.6) we have that

\[
E \left[ e^{\alpha X(T)} I(A); h \right] = P(A; h + \alpha) E \left[ e^{\alpha X(T)}; h \right]
\]

Then

\[
E \left[ e^{\alpha X(T)} I (\tau > T, X(T) \geq \frac{g}{\alpha}); h^* \right] = P \left( \tau > T, X(T) \geq \frac{g}{\alpha}; h^* + \alpha \right) E \left[ e^{\alpha X(T)}; h^* \right]
\]

\[
= P \left( \tau > T, X(T) \geq \frac{g}{\alpha}; h^* + \alpha \right) e^\left[ (r - \frac{1}{2} \sigma^2) \alpha + \frac{1}{2} \sigma^2 \alpha^2 \right] T
\]

\[
= P \left( \tau > T, X(T) \geq \frac{g}{\alpha}; h^* + \alpha \right) e^\left[ \alpha + \frac{1}{2} \alpha (\alpha - 1) \sigma^2 \right] T
\]

(3.26)

But we proved in Appendix B (equation (B.1)) that

\[
P \left( \tau > T, X(T) \geq x \right) = \Phi \left( \frac{B - \mu T}{\sigma \sqrt{T}} \right) - \Phi \left( \frac{x - \mu T}{\sigma \sqrt{T}} \right)
\]

\[
- e^{2 \alpha \beta \sigma^2} \left[ \Phi \left( \frac{-B - \mu T}{\sigma \sqrt{T}} \right) - \Phi \left( \frac{x - 2B - \mu T}{\sigma \sqrt{T}} \right) \right]
\]

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Using equation (2.5) we obtain:

\[
E \left[ e^{\alpha X(T)} I \left( \tau > T, X(T) \geq \frac{g}{\alpha} \right) ; h^* \right] \\
= e^{[\alpha + \frac{1}{2} \alpha (\alpha - 1) \sigma^2] T} \left\{ \Phi \left[ \frac{B - \left( r - \frac{1}{2} \sigma^2 + \alpha \sigma^2 \right) T}{\sigma \sqrt{T}} \right] \\
- \Phi \left[ \frac{\frac{g}{\alpha} - \left( r - \frac{1}{2} \sigma^2 + \alpha \sigma^2 \right) T}{\sigma \sqrt{T}} \right] - exp \left[ \frac{2 \left( r - \frac{1}{2} \sigma^2 + \alpha \sigma^2 \right) B}{\sigma^2} \right] \\
\times \left\{ \frac{-B - \left( r - \frac{1}{2} \sigma^2 + \alpha \sigma^2 \right) T}{\sigma \sqrt{T}} \right\} \left\{ \frac{\frac{g}{\alpha} - 2B - \left( r - \frac{1}{2} \sigma^2 + \alpha \sigma^2 \right) T}{\sigma \sqrt{T}} \right\} \right\} 
\]

\[(3.27)\]

Now let’s compute the last term of equation (3.23):

\[
E \left[ e^{gT} I \left( \tau > T, X(T) < \frac{g}{\alpha} \right) ; h^* \right] \\
= e^{gT} P \left( \tau > T, X(T) < \frac{g}{\alpha} ; h^* \right) 
\]

\[(3.28)\]

And using the following formula (equation(B.2)) proved in Appendix B

\[
P \left( \tau > T, X(T) < x \right) = \Phi \left( \frac{x - \mu T}{\sigma \sqrt{T}} \right) - e^{\mu B} \Phi \left( \frac{x - 2B - \mu T}{\sigma \sqrt{T}} \right)
\]

we obtain from (3.28)

\[
E \left[ e^{gT} I \left( \tau > T, X(T) < \frac{g}{\alpha} \right) ; h^* \right] \\
= e^{gT} \left\{ \Phi \left[ \frac{\frac{g}{\alpha} - \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right] - exp \left[ \frac{2 \left( r - \frac{1}{2} \sigma^2 \right) B}{\sigma^2} \right] \\
\times \Phi \left[ \frac{\frac{g}{\alpha} - 2B - \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right] \right\} 
\]

\[(3.29)\]
Then from (3.24), (3.25), (3.27) and (3.29) we get the formula for the price of this new design

\[
P_{cdl} = e^{(\alpha a B - r)T} \Phi \left[ \frac{-B + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] \\
+ \exp \left[ (\alpha a B - r)T + \frac{2}{\sigma^2} \left( r - \frac{\sigma^2}{2} \right) B \right] \Phi \left[ \frac{-B - \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \right] \\
+ e^{[(\alpha - 1) r + \frac{1}{2} \alpha (\alpha - 1) \sigma^2]T} \left\{ \Phi \left[ \frac{B - \left( r - \frac{1}{2} \sigma^2 + \alpha \sigma^2 \right) T}{\sigma \sqrt{T}} \right] \\
- \Phi \left[ \frac{\frac{g}{\alpha} - \left( r - \frac{1}{2} \sigma^2 + \alpha \sigma^2 \right) T}{\sigma \sqrt{T}} \right] \right\} \\
\times \left\{ \Phi \left[ \frac{-B - \left( r - \frac{1}{2} \sigma^2 + \alpha \sigma^2 \right) T}{\sigma \sqrt{T}} \right] - \Phi \left[ \frac{\frac{g}{\alpha} - 2B - \left( r - \frac{1}{2} \sigma^2 + \alpha \sigma^2 \right) T}{\sigma \sqrt{T}} \right] \right\} \\
+ e^{(g - r)T} \left\{ \Phi \left[ \frac{\frac{g}{\alpha} - \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right] - \exp \left[ \frac{2}{\sigma^2} \left( r - \frac{1}{2} \sigma^2 \right) B \right] \right\} \\
\times \Phi \left[ \frac{\frac{g}{\alpha} - 2B - \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right] \right\} \right\} \\
(3.30)
\]

We will compare this new design with all the classic ones (the annual reset with cap is again chosen among the annual reset types). We take two models based on our new design. For model 1 we choose \( B = 10\% \) and \( a = .6 \), and for model 2 we choose \( B = 20\% \) and \( a = .3 \). From figures 3.15 and 3.16 we can see that both these models have lower prices than the classic ones for any participation rate regardless of the volatility of the market. For a normal setting like case 2 we can look on table 3.15 and notice that for a price of $.98, model 1 produces a participation rate of .8102 and model 2 produces a .8443 participation rate, both of them being higher than the participation rate the classic ones have at the break even price of $1. And in a more volatile market setting, like case 4, the improvements are even bigger, as we can see...
from table 3.16. It’s interesting to notice that between the two models presented above, model 2, which seems to be more “customer-friendly”, is cheaper than model 1.

3.5 Further Analysis

This thesis tried to answer the problem of low participation rates the equity indexed annuity market is faced with today. We solved that problem in the previous sections of this chapter through the five new designs we introduced. When we compared the new models with the classic ones we looked at what is better for the insurance companies that sell such products. For them, lower participation rates translate into lower sale numbers, and hence, the new designs with higher participation rates should be better from their point of view. On the other hand, from the point of view of customers this may or may not be important. In a bull market the higher participation rates is very important for customers. In general though, every customer is also interested in the actual return that these policies yield.

In this section we look at how some of the new designs fare against the classic ones in terms of actual returns to the customer. We can look at the actual return of these policies as functions of the return on the stock index. We can write these functions explicitly only for the point-to-point design. Using the same models from section 3.1 we have that in case 1 the return of the classic design is given by

\[ R_c = \max (e^{0.3}, e^{0.6184R}) \]

where R is the return of the stock index.

Model 1 has a return given by
\[ R_{bp} = \begin{cases} 
  e^3 & \text{if } R < .95; \\
  e^{7333R} & \text{if } .95 \leq R < 2.12; \\
  e^{6599R} & \text{if } 2.12 \leq R.
\end{cases} \]

Model 2 has a return given by

\[ R_{mp} = \begin{cases} 
  e^3 & \text{if } R < .97; \\
  e^{7228R} & \text{if } .97 \leq R < 1.76; \\
  e^{7951R} & \text{if } 1.76 \leq R < 3.07; \\
  e^{6505R} & \text{if } 3.07 \leq R.
\end{cases} \]

As we can see from these formulae and from Figures 3.17 and 3.19 the return of the classic design is better (higher) when the return of the index is in between the minimum guaranteed rate \( g \) and the first barrier of the new design. For any other returns of the index the new design provides a return at least equal (when the market is down) if not better (once the return of the index is past the first barrier) than the return of the classic design.

For a more volatile market as in case 3, the returns of all these designs are definitely lower and given by

\[ R_c = \max (e^{.3}, e^{.5344R}) \]

for the classic design;

\[ R_{bp} = \begin{cases} 
  e^3 & \text{if } R < 1.14; \\
  e^{6127R} & \text{if } 1.14 \leq R < 2.54; \\
  e^{5514R} & \text{if } 2.54 \leq R
\end{cases} \]

for model 1; and

\[ R_{mp} = \begin{cases} 
  e^3 & \text{if } R < 1.17; \\
  e^{5957R} & \text{if } 1.17 \leq R < 2.14; \\
  e^{6553R} & \text{if } 2.14 \leq R < 3.73; \\
  e^{5361R} & \text{if } 3.76 \leq R
\end{cases} \]
for model 2.

We can see again the same pattern as before. The return of the new designs are higher if the return of the index passed the first barrier. If it’s less than that but higher than \( g \) the classic ones yield higher returns to the customer, and for returns of the index lower than \( g \) the policies will give the customer the same return.

For the annual reset design is not possible to write an explicit formula of the actual return of the policy as a function of the return of the index. Therefore, we had to use computer simulation techniques for the models we discussed in section 3.2. We generated five different outcomes for the market over a period of 10 years and we looked at the end of every year for the return on the index to compute the actual return the classic annual reset design and the actual return for model 1 from section 3.2 had. As we can see from table 3.17 we have somewhat similar results with the ones we got for the point-to-point designs. Sometimes the new design yields higher returns than the classic one and sometimes is the classic one that is the better choice. We expect that to happen from the way these new designs are set up and a clear answer to the question of which design (classic vs new) is better in terms of actual returns can’t be given. We made certain statements based on the assumption the markets are going to behave a certain way but that’s about all we can say.

For the path-dependent designs simulation could be done to find out about their returns, but that would require a very powerful computer and even then we wouldn’t be able to come up with a definitive answer.

Another way to compare the new designs versus the classic ones is to look at the rate of change of the participation rates with respect to the volatility of the market. We would definitely prefer the design that has this rate of change smaller. For this
we are going to use the same models we discussed in the previous sections and we are
going to compare case 1 versus case 3 and case 2 versus case 4. They both provide a
change of 10% in the volatility of the market while all the other settings remain the
same.

In case 1 versus case 3 the point to point designs have the following rates of change:
the classic one - $\Delta \alpha = \alpha$.084, model 1 - $\Delta \alpha = \alpha$.1206 and model 2 - $\Delta \alpha = \alpha$.1271. In
case 2 versus case 4 we have: the classic one - $\Delta \alpha = \alpha$.0857, model 1 - $\Delta \alpha = \alpha$.106 and
model 2 - $\Delta \alpha = \alpha$.1192. So it seems that, from this point of view, the classic one is
the better one amongst the point-to-point designs.

For the annual reset designs similar conclusions can be drawn since in case 1 versus
case 3 we have the following rates of change: the classic one - $\Delta \alpha = \alpha$.0681, model 1
- $\Delta \alpha = \alpha$.074 and model 2 - $\Delta \alpha = \alpha$.0682. And for case 2 versus case 4 we have: the
classic one - $\Delta \alpha = \alpha$.0897, model 1 - $\Delta \alpha = \alpha$.0983 and model 2 - $\Delta \alpha = \alpha$.0923.

For the path dependent designs when we compare case 2 versus case 4 we have
the following rates of change: classic continuous lookback design - $\Delta \alpha = \alpha$.0239, mcl
design - $\Delta \alpha = \alpha$.1293, bcl design - $\Delta \alpha = \alpha$.0149, ccl design model 1 - $\Delta \alpha = \alpha$.0043,
ccl design model 2 - $\Delta \alpha = \alpha$.0064. So, clearly, the continuous lookback with barrier
design (bcl) and the capped continuous lookback designs (ccl) are the ones to choose
from the group of path dependent designs.
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Table 3.1: Ppp vs Pmp model 1 and 2 in case 1

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Table 3.2: Ppp vs Pmp model 1 and 2 in case 2

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Table 3.3: Ppp vs Pmp model 1 and 2 in case 3

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Table 3.4: Ppp vs Pmp model 1 and 2 in case 4
Table 3.5: Ppp vs Pmp model 3 in case 2

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Table 3.6: Ppp vs Pmp model 3 in case 4

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Table 3.7: Par vs Pmar model 1 and 2 in case 1

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Table 3.8: Par vs Pmar model 1 and 2 in case 2
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Table 3.9: Par vs Pmar model 1 and 2 in case 3

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Table 3.10: Par vs Pmar model 1 and 2 in case 4

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Table 3.11: Pcl vs Pmcl in case 2

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Table 3.12: Pcl vs Pmcl in case 4
### Table 3.13: Ppp, Pcl, Parc vs Pbcl in case 2

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### Table 3.14: Ppp, Pcl, Parc vs Pbcl in case 4

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### Table 3.15: Ppp, Pcl, Parc vs Pccl in case 2

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### Table 3.16: Ppp, Pcl, Parc vs Pccl in case 4

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Table 3.13: Ppp, Pcl, Parc vs Pbcl in case 2

Table 3.14: Ppp, Pcl, Parc vs Pbcl in case 4

Table 3.15: Ppp, Pcl, Parc vs Pccl in case 2

Table 3.16: Ppp, Pcl, Parc vs Pccl in case 4
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Table 3.17: Simulation of actual returns for the annual reset designs
Figure 3.1: Ppp, Pmp model 2, Pmp model 1 in case 1

Figure 3.2: Ppp, Pmp model 2, Pmp model 1 in case 2
Figure 3.3: Ppp, Pmp model 2, Pmp model 1 in case 3

Figure 3.4: Ppp, Pmp model 2, Pmp model 1 in case 4
Figure 3.5: Ppp, Pmp model 3 in case 2

Figure 3.6: Ppp, Pmp model 3 in case 4
Figure 3.7: Par, Pmar model 2, Pmar model 1 in case 1

Figure 3.8: Par, Pmar model 2, Pmar model 1 in case 2
Figure 3.9: Par, Pmar model 2, Pmar model 1 in case 3

Figure 3.10: Par, Pmar model 2, Pmar model 1 in case 4
Figure 3.11: Pcl, Pmcl model 1 in case 2

Figure 3.12: Pcl, Pmcl model 2 in case 4
Figure 3.13: Parc, Ppp, Pcl and Pbcl in case 2

Figure 3.14: Parc, Ppp, Pcl and Pbcl in case 4
Figure 3.15: Parc, Ppp, Pcl, Pcle model 1 and 2 in case 2

Figure 3.16: Parc, Ppp, Pcl, Pcle model 1 and 2 in case 4
Figure 3.17: Actual returns for the point-to-point Rbp and Rc in case 1

Figure 3.18: Actual returns for the point-to-point Rbp and Rc in case 3
Figure 3.19: Actual returns for the point-to-point $R_{mp}$ and $R_c$ in case 1

Figure 3.20: Actual returns for the point-to-point $R_{mp}$ and $R_c$ in case 3
CHAPTER 4

CONTRIBUTIONS AND FUTURE WORK

Due to the current volatile market the EIA’s have been forced to drop their participation rates and consequently the sales of the EIA products have been slowed down over the last few years. The EIA’s are still a very good option for those people looking to participate in the stock market without actually having to take the full risks that this might imply. In Chapter 2 we discussed the basic (classic) designs of EIA’s and we saw how the present volatile market is making for lower participation rates. In Chapter 3 we solved this problem by introducing five new designs and we provided detailed tables and plots that showed how these new designs are improving the old ones in different market settings.

We assumed throughout this thesis the perfect market setting with a constant risk-free force of interest, no lapse, no deaths, and no transaction costs and hence we were able to obtain close form formulae for the prices of all these designs. Some further research could take in consideration more realistic assumptions. Some areas of interest are stochastic interest rates that could provide a model where the interest rates are not constant over the period of the policies. Most of the research in pricing exotic options is done by monitoring the whole life of the policy and maybe some research needs to be done by looking at arbitrary periods inside the life of the respective policies.
Also a lapse model should be developed providing the customers with the possibility of withdrawing their money from an EIA that doesn’t produce much and to invest them in some other more profitable venues. Studies on the hedging strategies that the insurers should take to minimize the risks should be also developed. Especially difficult seems to be the problem of hedging those designs that are monitoring a stock index continuously over certain periods of time, because that would require continuous hedging which is not quite possible in real life.
APPENDIX A

THE CONTINUOUS BLACK-SCHOLES MODEL

We start with a general model of frictionless security market where investors are allowed to trade continuously up to some fixed finite planning horizon $T$. Uncertainty in this financial market is modeled by a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $F$ of $\sigma$-algebras $\mathcal{F}_t$, with $0 \leq t \leq T$, satisfying the usual conditions of right-continuity and completeness. We assume that $\mathcal{F}_0$ is trivial and that $\mathcal{F}_T = \mathcal{F}$.

There are $d+1$ primary traded assets, whose price processes are given by stochastic processes $S_0, \ldots, S_d$. We assume that $S = (S_0, \ldots, S_d)$ follows an adapted, right-continuous with left-limits (RCLL) and strictly positive semimartingale on $(\Omega, \mathcal{F}, P, F)$. We also assume that $S_0(t)$ is a non-dividend paying asset which is almost surely strictly positive and use it as a numeraire.

We denote by $M(P)$ the financial market described above.

**Definition A.1** A trading strategy (or dynamic portfolio) $\varphi$ is a $R^{d+1}$ vector stochastic process $\varphi(t) = (\varphi_0(t), \varphi_1(t), \ldots, \varphi_d(t))$, $0 \leq t \leq T$ which is predictable and locally bounded.

Here $\varphi_i(t)$ denotes the number of shares of asset $i$ held in the portfolio at time $t$ - determined on the basis of information available before time $t$. This means that the investor selects his time $t$ portfolio after observing the prices $S(t-)$.  


Definition A.2  The value of the portfolio $\varphi$ at time $t$ is given by the scalar product

$$V_\varphi(t) = \varphi(t) \cdot S(t) = \sum_{i=0}^{d} \varphi(t)S_i(t), \quad t \in [0, T]$$

Definition A.3  The gains process $G_\varphi(t)$ is defined by

$$G_\varphi(t) = \int_0^t \varphi(u)dS(u) = \sum_{i=0}^{d} \int_0^t \varphi_i(u)dS_i(u)$$

Definition A.4  A trading strategy is called self-financing if the value process satisfies

$$V_\varphi(t) = V_\varphi(0) + G_\varphi(t) \quad \text{for all} \quad t \in [0, T]$$

We can define the discounted price process, the discounted value process and discounted gains process with the help of the numeraire $S_0(t)$ the same way we did in Chapter 1.

Definition A.5  A self-financing strategy is called an arbitrage opportunity or arbitrage strategy if $V_\varphi(0) = 0$ and the terminal value satisfies

$$P\{V_\varphi(T) \geq 0\} = 1 \quad \text{and} \quad P\{V_\varphi(T) > 0\} > 0$$

Definition A.6  We say that a probability measure $Q$ defined on $(\Omega, \mathcal{F})$ is a (strong) equivalent martingale measure if $Q$ is equivalent to $P$ and the discounted process $\tilde{S}$ is a $Q$-local martingale (martingale).

We denote the set of martingale measures by $\mathcal{P}$

Definition A.7  A self financing strategy is called tame if $\tilde{V}_\varphi(t) \geq 0$ for all $t \in [0, T]$. We denote by $\Phi$ the set of tame trading strategies.
The following proposition assures us that the existence of an equivalent martingale measure implies the absence of arbitrage.

**Proposition A.1** Assume $\mathcal{P}$ is not empty. Then the market model contains no arbitrage opportunities in $\Phi$.

In order to get equivalence between the absence of arbitrage opportunities and the existence of an equivalent martingale measure we need some further definitions and requirements.

**Definition A.8** A simple predictable strategy is a predictable process which can be represented as a finite linear combination of stochastic processes of the form $\psi 1_{[\tau_1,\tau_2]}$ where $\tau_1$ and $\tau_2$ are stopping times and $\psi$ is an $\mathcal{F}_{\tau_1}$-measurable random variable.

**Definition A.9** We say that a simple predictable trading strategy is $\delta$-admissible if $V_\varphi(t) \geq -\delta$ for every $t \in [0, T]$.

**Definition A.10** A price process $S$ satisfies NFLVR (no free lunch with vanishing risk) if for any sequence $(\varphi_n)$ of simple trading strategies such that $\varphi_n$ is $\delta_n$-admissible and the sequence $\delta_n$ tends to zero, we have $V_{\varphi_n}(T) \to 0$ in probability as $n \to \infty$.

The following fundamental theorem of asset pricing is proved in [3]:

**Theorem A.1** (Fundamental Theorem of Asset Pricing) There exists an equivalent martingale measure for the financial market model $M(\mathcal{P})$ if and only if the condition NFLVR holds true.

For all the proofs we didn’t provide here, all the details omitted and a more complete look at the continuous setting please refer to [1].
APPENDIX B

PROOFS OF SOME IMPORTANT FORMULAE

First let’s define:

\[ \tau = \inf \{ t : X(t) = B \} \]

We assume \( x \leq B \)

\[
P(\tau > T, X(T) \geq x)
= P(M(T) < B, X(T) \geq x)
= P(M(T) < B) - P(M(T) < B, X(T) < x)
= \Phi \left( \frac{B - \mu_T}{\sigma \sqrt{T}} \right) - e^{2 \mu_B} \cdot \Phi \left( \frac{-B - \mu_T}{\sigma \sqrt{T}} \right)
- \left[ \Phi \left( \frac{x - \mu_T}{\sigma \sqrt{T}} \right) - e^{2 \mu_B} \cdot \Phi \left( \frac{x - 2B - \mu_T}{\sigma \sqrt{T}} \right) \right]
= \Phi \left( \frac{B - \mu_T}{\sigma \sqrt{T}} \right) - \Phi \left( \frac{x - \mu_T}{\sigma \sqrt{T}} \right)
- e^{2 \mu_B} \left[ \Phi \left( \frac{-B - \mu_T}{\sigma \sqrt{T}} \right) - \Phi \left( \frac{x - 2B - \mu_T}{\sigma \sqrt{T}} \right) \right]
\]

where we used the known distribution of \( M(t) \) (see [8] or [9]), equation (3.20) and equation (B.3) mentioned below.

A second formula that is used in Chapter 3 is the following
\( P(\tau > T, X(T) < x) \)

\[
P(\tau > T, X(T) < x) = P(M(T) < B, X(T) < x)
= \Phi \left( \frac{x - \mu T}{\sigma \sqrt{T}} \right) - e^{\frac{2 \mu B}{\sigma^2}} \Phi \left( \frac{x - 2B - \mu T}{\sigma \sqrt{T}} \right)
\] (B.2)

where we used again equation (3.20)

For both equations (B.1) and (B.2) we used the following formula proved in [7]:

\[
P(M(t) < B, X(t) < x) = \Phi \left( \frac{x - \mu t}{\sigma \sqrt{t}} \right) - e^{\frac{2 \mu B}{\sigma^2}} \Phi \left( \frac{x - 2B - \mu t}{\sigma \sqrt{t}} \right)
\] (B.3)
BIBLIOGRAPHY


