A Fundamental Investigation of Retention Phenomena in Snap-fit Features

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

By

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* * * * *

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ABSTRACT

This study explores issues related to modeling and analysis of insertion and retention phenomena in snap-fit features. An analytical methodology for developing detailed models of snap-fits is proposed and formulated. The strategy involves idealizing the feature as a rigid body supported on a flexible structure. A set of equations that comprehensively describes the system in its deformed configuration is formulated. The equation system is iteratively solved for several such configurations to obtain a model of insertion and retention processes for snap-fits. It has been implemented for the cantilever hook in this research. The model shows good agreement with experimental results for most snap-fit geometries. However, it still needs to be refined to provide better predictions for high-angle retention. Suggestions for possible improvements and future research directions are provided.

An experimental study of high-angle retention has been undertaken with a view to establishing a fundamental understanding of governing phenomena. Force-displacement curves are correlated to still frames from high speed video of the experiment, to identify the location of peaks in the insertion and retention force curves. Samples were fabricated from aluminum and steel to eliminate complexities due to material behavior. The maximum retention force and the shape of the insertion and retention force curves was found to be very sensitive to catch surface conditions. Large drops in insertion and retention force values were observed between successive tests on the same sample. Retention force curves exhibit evidence of stick-slip contact. Experiments were also conducted on injection
molded ABS samples. Once again, large differences in maximum insertion and retention force values were caused by change in contact edge conditions.

Initial steps for the development of predictive capability of time dependent effects in plastic parts were taken. A test material (polycarbonate) was characterized using a constitutive model capable of modeling long-term non-linear viscoelastic behavior. The model parameters were characterized and a master stress relaxation curve for the material was developed using time-temperature and time-strain superposition principles. The constitutive relationship was converted into an incremental form suitable for implementation in a commercial finite element analysis package.
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ACKNOWLEDGMENTS

I would like to thank my adviser, Dr. Anthony Luscher, for his unwavering support and guidance throughout my graduate education. His guidance and insight were instrumental in the success of the research presented in this dissertation. I would also like to acknowledge the support of my committee members, especially Prof. Harper, who provided invaluable advice for the viscoelastic characterization effort.

Thanks are also due to Keith Rogers for his help with the design and fabrication of samples and fixtures. Time and again, he made insightful suggestions and tips for improving upon current experimental designs. I would like to acknowledge the graciousness of Prof. Guezzenc for loaning high speed photography equipment for an extended period of time. Tieming was crucial in helping me out with debugging of my model on several occasions and providing a great sounding board for my research.

I would also like to express my appreciation for the moral and emotional support that Anju has provided over the last few years. Last but not the least, I would like to thank my parents for raising me to become capable of reaching this far in life and instilling values which guide me always.

This research was conducted as part of the NSF Center for Advanced Polymer and Composites Engineering. The financial and technical support of the sponsors of the Integral Attachment Program is highly appreciated.
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CHAPTER 1

INTRODUCTION

This thesis is part of the research being conducted in the Integral Attachment Program (IAP) at the Ohio State University. The program consists of a consortium of companies with an interest in joining and fastening of plastic parts. It is chartered with the mission of advancing snap-fit design from an art to an engineering activity. A variety of interdisciplinary research is conducted in the program to improve the state-of-the-art in snap-fit design. Traditionally, snap-fit design methodology has been disorganized and anecdotal in nature, relying greatly on the skill and the experience of the individual designer. The most popular sources of design information for snap-fit features have been the design guides published by resin suppliers as a service to their customers. The information disseminated by such guides is old and often obsolete and inaccurate. There has been tremendous progress in the design tools available to engineers, especially in the field of finite element analysis, design of experiments and symbolic mathematics. However, the snap-fit design data available in commercial literature has lagged far behind the progress in the supporting disciplines (mechanics, rheology etc.). Design engineers, working under the pressure of the reducing life-cycle times of their products, do not have the time or resources for generating generic design information for snap-fit design. Resin suppliers, on the other hand, faced with increasing competition and pressure to cut costs, cannot garner the resources to update
the design guides to reflect the improvement in the available design tools. In the current scenario, most snap-fit designs are one-of-a-kind efforts, with very little, if any, leveraging between different product families. Design knowledge generated in a particular project is leveraged for use in subsequent design only through the experience of the designer, or through project reports describing the design process and performance data. It will be of great value if design information for snap-fit features becomes available to the designers in an abstract form (as opposed to case-specific data) - design equations, response surfaces, design heuristics, evaluation metrics and the like. Identification and development of high-performance snap-fit topologies outside of the product life-cycle would be invaluable. Research conducted in the IAP attempts to fill this void between materials suppliers and the snap-fit design community.

This chapter presents a brief introduction to snap-fit features and some commonly used terminology, the motivation for this research and the scope of the goals of this thesis.

1.1 Snap-fit Features: Introduction

A few definitions of common snap-fit terms are in order.

1. Integral Attachment Features are defined as features formed into a part that provide mechanical attachment functionality in an assembly by establishing relative part location, alignment and orientation, transferring service loads, eliminating degrees of freedom, and/or absorbing the tolerance between these parts. Features that provide the final attachment between parts are known as snap-fit features.
2. **Insertion force** \((F_i)\) is the force that needs to be applied in the insertion direction of a snap-fit feature to engage it. Insertion force can be expressed as a single maximum value or as a graph of the force versus position relative to the snap-fit feature (Figure 1.1).

3. **Retention force** \((F_r)\) is the force that needs to be applied in the separation direction of a snap-fit feature to disassemble it (Figure 1.1). If designed to be a permanent assembly, disengagement occurs due to fracture, permanent deformation or loss of engagement between the two mating parts.

4. **Locking ratio** is defined to be the ratio of the maximum retention force to the maximum insertion force of a snap-fit feature.

Snap-fits are molded on parts to provide attachment functionality in assemblies. Unlike rivets, screws etc., which are discrete fasteners, snap-fit features are integral to the part. They thus help in reduction of part count and assembly time. The motions required for assembly are also usually simplified, which is beneficial from an ergonomic standpoint. The use of snap-fit features instead of discrete fasteners results in the reduction of the number of different materials in the assembly, which can be helpful for recycling purposes. Snap-fits have traditionally been used in toys and other consumer products. However, with an improvement in polymer technology and the rapid development of composite materials for structural applications, snap-fit features are now used even in demanding products. Underhood applications that snap-fits have found use in include air filter housings, throttlebodies, temperature, pressure sensors and manifolds in engines. These assemblies are expected to withstand harsh environmental conditions like high temperature and pressure and contaminants like oil and corrosive gases. Lens housings in big-screen televisions and high-end
projection systems, pager housings, single use cameras, compact disc players are other examples of consumer products that snap-fits have been successfully applied to. Quite often, snap-fits are used to provide secondary joining functionality while the primary fastening (adhesive, ultrasonic welding) joints form. The most commonly used snap-fit topologies are shown in Figure 1.2. The primary cause restricting the more widespread use of snap-fit features in a larger number of polymeric products is the lack of confidence in their design process and performance attributes. Often, criteria such as removability, cosmetic issues and packaging concerns drive product designs towards other fastening alternatives.

Figure 1.1: Typical experimental force curve for insertion and retention of snap-fit features.
Figure 1.2: Common snap-fit feature topologies.
1.2 Motivation for Research

The generic goal driving the research in this thesis is the need for more accurate and versatile models of the performance of snap-fit features. As mentioned earlier, information available in design guides is limited and often inaccurate. The most commonly used material models are not representative of polymer behavior, especially for modeling time dependent effects and loading rate dependency. The areas in which most significant contribution to the scientific underpinnings of snap-fit design can be made are systematically discussed below. The immediate need for an improvement in these areas forms the motivation behind the work done in this thesis.

1.2.1 Snap-fit Design Approaches

The development of equations for the detailed sizing of snap-fit features and for predicting their response can be accomplished using one of several techniques. In particular, two approaches have been popular in the past. The first, which will be referred to as the numerical design formulation, involves the development of response surface equations and main effect plots using statistical methods. Variables expected to influence the snap-fit’s performance (factors) and response variables are first identified. The portion of the design space that is of interest is then determined. An array of experimental designs spanning the region of interest is created using standard statistical techniques. Physical or computational experiments at each of the points in the design of experiments (DOE) array are conducted. Due to the cost associated with manufacturing physical test samples, the use of computational experiments has been more popular. Typically, finite element models of the snap-fit are created and queried for performance attributes. Response surface equations are fit to response variables values at experimental points. These equations are also used to predict
the performance of the snap-fit at other points in the design space. Possibility of satisfactorily extrapolating the equations to other regions of the design space also exists. Main effect plots depicting the sensitivity of the response variables to the factors are also created for quick determination of the most important design variables. Such an approach has been used for the cantilever hook [1], the bayonet & finger [2, 3], the compressive hook [4] and the post & dome [5] snap-fit features.

While the numerical design formulation is an effective and efficient method for generating design equations for snap-fits, it suffers from a number of limitations. Most of the drawbacks arise from it being a purely statistical technique rather than being based on a fundamental understanding of the phenomena that make a snap-fit work. The response surface equations are simply mathematical equations fit to the response variable values. Beyond this, their physical significance is very limited. They are a very useful alternative when it is difficult to model the response based on first principles and are most accurate when they span a region of the design space where only a single physical phenomena is active. If the design space is such that different mechanisms are active in different regions, use of a single response surface can introduce significant inaccuracies. Consider the example of the retention force of a snap-fit. Snap-fits can have multiple failure modes - for example, certain cantilever hook designs can disengage without any material failure (loss-of-engagement [6]). Other designs will fail only due to material damage or fracture. The particular mode of failure that will occur depends on the snap-fit dimensions. If a response surface is fit directly to the retention force data without considering the failure mode, the accuracy of the resulting regression model will be limited. However, it is difficult to plan a DOE array that will span a single failure mode only without any knowledge of a feature’s performance. Use of DOE approach as a black box for generating design equations can
thus lead to erroneous results. It is also not possible to identify the factors that cause the snap-fit to transition from one mode of behavior to another.

The other approach that is commonly used is the analytical design formulation. The snap-fit is idealized as a structure whose the stress, strain and deformation behavior can be modeled using available equations. Design guides from major resin suppliers use such an approach to develop simple equations for the detailed sizing of snap-fits. One of the most frequently used snap-fits is the cantilever hook. Its basic shape is shown in Figure 1.2(a). It is typically idealized as a cantilever beam with a rigid catch at the end. The transverse force required to deflect the end of the beam \( P \) by an amount equal to the catch offset is determined using Euler-Bernoulli beam theory. This is related to the forces acting on the catch in the engagement direction \( F_i \) to determine the insertion force for the feature (Figure 1.3). A similar approach is used to determine the retention force. Consideration of the equilibrium of the catch will show that the end of the beam is subject to an axial load and
a moment, in addition to the transverse load. Also, the deflections due to these forces and moments cannot be simply superposed. The bending moment in the beam due to the axial component depends on the deflection of the end, rather than being constant. This effect has not been considered in previous literature, although the need to incorporate an axial component and an overturning moment has been previously identified by Luscher [6]. The assumption that the maximum deflection of the end of the beam during insertion equals the offset of the catch also introduces some inaccuracy, because the end of the beam rotates, in addition to the transverse deflection. The Euler-Bernoulli beam theory is based on small deformation and long-beam assumptions, which are commonly violated in real-world designs.

The design guides also use properties such as Young’s modulus, secant modulus, Poisson’s ratio and yield strength which are not representative of the behavior of polymers. Though all the guides make mention of the phenomena of creep in plastics, no suggestions for including it in the design process are made. Creep data is typically presented as a set of curves and the designer is expected to work around possible creep (or stress relaxation) effects, rather than quantitatively including it in the design process. The issue of material characterization and incorporation of advanced material models is discussed in greater detail in the next section.

1.2.2 Polymeric Material Models

All design approaches, whether analytical or numerical, involve the use of a constitutive material model. Traditionally, material constants such as Young’s modulus, secant modulus, Poisson’s ratio etc. have been used in structural design. These quantities were first developed for metallic materials, which display distinct elastic and inelastic regimes
of response. Dependence of the response on loading history and observed creep effects are relatively insignificant, especially at low temperatures and strain values. Plasticity theories are used to model irrecoverable deformation of the material beyond the yield point. These constitutive models were very popular with engineers, since most structural applications made use of metallic materials. They were extended for use in polymers, as the scope of their application in structural tasks increased. The response of polymers to loading, however, is fundamentally different, consisting of an instantaneous elastic component and a time-dependent viscous component. Viscoelastic material models have been available in literature since the early 19th century. However, engineers adept at designing products with metals were (and still are) more comfortable with the use of elastic-plastic material models, even if not as accurate as viscoelastic models.

Plastic part designers have to contend with issues such as loss of preload, loss of sealing capacity, different failure modes at different loading rates - all of which are simple manifestations of the basic phenomena of viscoelasticity. Snap-fit features are usually assembled with some preload, so that a gap does not form between the parts. The preload causes an initial strain in the snap-fit, which will relax with time. A loss of preload will allow the formation of a gap between the parts, which will then rattle. Quite often, snap-fits are used in joints that need to form a seal between two parts. The sealing in such joints is usually created by compressing a sealing element such as gasket or an O-ring. The force required for this purpose is often provided by the snap-fit feature forming the joint. With time, the gasket and/or the snap-fit will undergo relaxation because of the viscoelastic nature of the elastomer forming the sealing element and/or the polymer forming the part, respectively. This effect is accelerated by temperature. Snap-fit features have been known to fail differently under different loading conditions. A feature that fails by material damage under low
loading rates may disengage without material failure in drop-test conditions. There is an immediate need to model these effects, irrespective of whether analytical design equations are being developed or predictive finite element analyses are being conducted.

Another issue is related to the recommended strain values in plastics. Resin suppliers usually specify a maximum allowable strain value for their plastics, which is used to form a “strength” criteria for the material. Their design guides do not always specify how allowable strain limits were determined. GE Plastics [7] does not give specific factors of safety for its materials. Allied Signals [8] suggests 70 % of the yield strength as a suitable value for allowable strain for a number of their polymers. Bayer [9] suggests the use of 1, 0.7 and 0.5 as factors of safety for semi-crystalline, amorphous and reinforced polymers respectively. A rational basis for the use of these values is not presented in any of the guides. The material properties of polymers are very dependent on loading rates and this can also affect allowable strain values. The failure modes of polymers should also be considered while estimating their load carrying capacity. Some polymers undergo brittle fracture via a phenomenon known as crazing, while others exhibit a more ductile mode of failure. Stress concentration effects are more critical in the former type of polymers. In view of the significant differences in the capacity of different polymers to sustain overload, custom factors of safety should be used, instead of applying generic values to broad material categories.

1.3 Research Objectives

A thorough discussion of the limitations of current modeling approaches has been presented. The overall objective of this research was to advance the accuracy and effectiveness of snap-fit modeling techniques. It also sought to establish an understanding of retention in high angle snap-fit features. The specific goals are discussed separately below.
1. **Development of improved analytical model for cantilever hook features.** The model would predict insertion and retention force curves and not just the maximum values. Effect of beam rotation and catch geometry would be included in the model equations. It would investigate the influence of axial force and moment components, in addition to transverse force, on the beam deflection relationships. Two separate beam formulations would be considered. Model predictions would be compared to experimental and numerical (finite element analysis) results. The model would attempt to quantify loss-of-engagement failures. The possible existence of a bifurcation between failure modes for the snap-fit would be investigated.

2. **Experimental study of high retention angle cantilever hook features.** The objective of this portion of the research was to establish an understanding of the physical phenomena governing retention, especially in high angle features (80° - 90°). This would be accomplished by fabricating specimens from aluminum and steel, to avoid complexity due to polymer viscoelasticity. The influence of factors such as snap-fit geometry, nature of friction and wear of the surfaces in contact and manufacturing variability would be studied. Using high speed photography, the geometry of deformation would be related to force curves for the feature. Available polymeric specimens would also be tested as benchmarks for the analytical model.

3. **Modeling creep and relaxation phenomena in snap-fit features** This study would develop the capability to model time-dependent effects in snap-fit features using finite element analysis or otherwise. The first step would be to perform a comprehensive characterization of the long-term non-linear viscoelastic behavior of an amorphous
test polymer (polycarbonate). Two-step stress relaxation and constant strain rate experiments would be used to validate the developed model. Incremental stress strain relationships would be formulated to implement the constitutive model in a finite element analysis package. Longer term, this model could be used for detailed analysis of snap-fit features to model creep and relaxation phenomena.

1.4 Summary and Thesis Organization

The state-of-the-art in snap-fit design has been presented and the drawbacks and limitations of current design and analysis approaches discussed. There is room for tremendous improvement in the accuracy of prediction models and validity of design equations. The particular areas that this work will focus on have been identified and the goals of this thesis clearly defined.

The remainder of this document contains a literature review and a description of the research conducted as part of this effort. Chapter 2 contains a summary of the literature reviewed from different areas relevant to this thesis. Chapter 3 details the development of analytical model for cantilever hook snap-fits, followed by a presentation of the results. The latter are compared to existing models and finite-element and experimental data. Chapter 4 describes the efforts for developing and implementing a constitutive model for the non-linear viscoelastic behavior of PC. Chapter 5 deals with experimental testing issues, fixture development and statistical analysis of test results for cantilever hook snap-fits. Chapter 6 summarizes the contributions of this thesis and lays the framework for future research in this area.
CHAPTER 2

LITERATURE SURVEY

2.1 Overview

Keeping in mind the goals of this thesis, a review of available literature in the following areas was conducted.

1. Numerical snap-fit design.

2. Analytical snap-fit design (advanced beam deflection formulations).

3. Viscoelasticity.

4. Viscoelastic finite element analysis.

The literature review for each area will be discussed separately. This is necessary to present a cohesive report, due to the broad interdisciplinary nature of this work. Past work in snap-fit design does not make effective use of advanced and accurate formulations available in the areas of beam analysis, viscoelasticity and viscoelastic finite element analysis. On the other hand, much of the literature in these three areas is purely theoretical in scope, and examples of applications to snap-fit design, or other practical problems are very limited.
2.2 Snap-fit Design

Snap-fit design has traditionally been classified into Attachment strategy design and Feature design.

Attachment strategy design deals with the issues of designing complete assemblies, rather than the detailed sizing of individual snap-fit features. This expands the scope of the field beyond snap-fits to auxiliary features like locators and compliant features. Issues such as the number of features to be used, their location and orientation to achieve optimal load bearing and constraint capacity are tackled. Guidelines and heuristics for the design of snap-fit assemblies are discussed [10, 11, 12, 13].

Luscher et. al. [14] classify snap-fits into three categories. Luscher and Bonenberger present guidelines for design of plastic parts utilizing such features [15]. A similar classification approach is proposed by Genc et. al. [16]. Qualitative criteria for selection of lock topologies are discussed by Genc et. al. [17, 18, 19]. Concepts for generation of alternative snap-fit assembly designs are provided by the same authors [20]. They suggest a maximum limit to the number of possible interfaces in snap-fit design in an attempt to encapsulate the design space for snap-fit features [21, 22]. A discussion of the motions required for the assembly of parts and their influence on the snap-fit design process is discussed by Luscher et. al. [23]. Design options for joining enclosures are discussed by Luscher, Boddmann et. al. [24] and Jenkins et. al. [25]. Although attachment strategy is an integral and important part of snap-fit design and considerable work has been done in this area, a detailed review of the literature will not be presented. This thesis primarily deals with the design of individual snap-fit features and enhancement of the accuracy and reliability with
which snap-fit performance can be predicted. Attachment strategy development is not one of the focus areas of this research.

Feature design deals with the development of design equations for the detailed sizing of snap-fit features. Models for accurately predicting the in-service performance of snap-fit features are also researched. The level of sophistication of the feature design approaches varies from simple application of engineering mechanics principles like beam theory and Castigliano’s theorem to the use of advanced finite element methods and design of experiments (DOE). In general, a model of an individual snap-fit feature is analyzed to predict performance attributes like insertion force, retention strength and locking ratio. Very often, these predictions are compared to experimental data. Most of the available literature investigates feature topologies that are currently used by designers. These include the bayonet & finger feature, the compressive hook feature, the cantilever hook and cylindrical and spherical snap-fit features. Some work deals with the development and design of high performance topologies, like the post & dome feature. Work done in the feature design area is discussed next.

### 2.2.1 Bayonet & Finger Feature

Wang et. al. [2] analyzed a bayonet & finger feature using a contact type finite element model in ABAQUS. High-impact polystyrene (HIPS) was modeled as an elastic-perfectly plastic material. Finite-element analysis predicted a snap-through failure mode. The simulation was compared to results from experimental tests. Reasonably accurate correlation between analysis and experiment was observed for insertion. Results for retention were not presented. Lewis et. al. [26] expanded upon the previous work. Instead of analyzing a
single instance of the feature, a DOE array was created. Two and three-dimensional finite-element models were used to generate approximate linear response surfaces for predicting the insertion and retention forces, based on feature geometry. A five factor, two level fractional factorial array was used for the DOE, requiring a total of sixteen experiments. Material was not considered as a variable. Design equations and sensitivity information for the feature is presented in the paper. However, predictions of the design equations do not correlate well with finite element analysis results for other feature sizes. No experimental verification of the results is presented. Shen [3] attempts to generate more accurate response surfaces for the bayonet & finger feature by using a quadratic DOE formulation. Additionally, the DOE is split into two design arrays, a catch array to optimize the detailed geometry of the catch, and a macro array to generate quadratic response surfaces for the feature. Issues such as sensitivity to base-part stiffness, over-engagement and base-part flexibility effects are also investigated. Novel modifications to improve the performance of the feature are suggested. No test data for the feature is, however, presented.

### 2.2.2 Compressive Hook Feature

The compressive hook feature is commonly used in automotive connectors. It derives its name from its retention mechanism which uses compression to provide the locking force. Hotra et. al. [27] use an approach similar to Wang et. al. [2] to model feature performance. The analysis is performed for 15% glass reinforced poly-butylene terephthalate (PBT). The material data is experimentally determined at 0.08%/sec strain rate. Bending and shearing failure modes are investigated separately. The simulation results are compared to experimental data and are shown to be in good agreement. The paper emphasizes the need for the use of advanced analysis techniques to model the performance of complex
snap-fit features. Hotra et. al. [28] extended the above work. Poly-phenylene oxide (PPO) was also considered, beside PBT. Bending and shear were again considered in separate analyses and the lower of the two taken as the actual failure mode. Tests were performed with a hard gage as a mating part in one case, and the actual terminal as the mating part in the other case. This was done to study the influence of mating part flexibility on the performance of the feature.

Roy [29] adopted an analytical approach for predicting the performance of the compressive hook feature. The hook was modeled as a cantilever beam with variable cross-section. In particular, it was divided into three different beam sections and Castigliano’s theorem used to determine the load-deflection relationships. A graphical solution method was also presented. Stiffness matrices for the feature were derived using an approach similar to the finite element method. Expressions for the insertion (assembly) force were derived based on equilibrium of the feature. For retention, buckling was recognized as the primary mode of failure. The Rayleigh-Ritz method was used to derive approximate expression for column buckling loads under different end conditions. The same method was also used to estimate eigen-frequencies for longitudinal and transverse vibration modes of the feature. All the results were compared to results from a finite element analysis conducted using ANSYS.

Lewis et. al. [4] used a DOE approach coupled with finite element analysis to generate approximate second order equations (response surfaces) that could calculate insertion and retention forces for initial design of compressive hook features. These equations were based on geometric and material properties of the feature and were compared to supplemental finite element and experimental results. Design guidelines for the feature are also presented.
2.2.3 Other Snap-fit Features

A DOE approach is also used by Luscher [1] and Matuschek and Michaeli [30] to model the performance of cylindrical and cantilever hook snap-fit features respectively. Matuschek and Michaeli [30] used a five factor, three level DOE array. A quadratic response surface was generated using the results of these analyses. The sensitivity of the maximum stress in the feature and the retention force to the geometric variables was presented graphically. An equidistant grid strategy was used to partition the design space of the feature and identify the optimal set of dimensions for a set of given design criteria.

Luscher [1] applied a combination of finite element analysis and design of experiments to the cantilever hook snap-fit feature. A four-factor, two level, orthogonal array was used to study insertion and a five-factor two-level array was used to study retention. The amount of engagement was also defined as a factor to incorporate the effect of warpage and shrinkage on feature performance. Although friction is not usually a design variable, it was included as a factor in the retention array. This was done to identify the sensitivity of the feature to changes in the value of coefficient of friction. The finite element results were compared to experimental data. An attempt to explain the discrepancies between the two was made. The sensitivity of the feature to geometric variables is discussed for insertion and retention separately. An optimal hook design formulation is also presented. Guidelines for the design of feature were presented based on the research.

Bader and Koch [31] were the first to use a viscoelastic material model in the finite element analysis of a ball type snap-fit feature. A “standard linear solid” viscoelastic model consisting of springs and dashpots was used. Plasticity was modeled through the introduction of an additional mechanical element that remains inactive below the yield point for the
material. The dependency of the spring and dashpot parameters on strain rate was modeled using a logarithmic curve fit. Plastic damage, recovery and stress relaxation behavior were experimentally observed and phenomenological models were fit to the data. Finite element analysis of the insertion and retention processes was performed using the commercial software MARC™ and MENTAT™. The results obtained using viscoelastic and elastic-plastic material models were compared to experimental test data. Though insertion force predictions for either of the material models were reasonably accurate, a significant improvement in the retention force and residual deformation predictions was seen.

2.2.4 Materials Issues in Snap-fit Design

There have been some efforts to promote the use of advanced material models for polymers in the field of snap-fit design. Most of these have, however, been limited to relatively simplistic elastic-plastic material formulations, in order to model residual strain in the polymer. This thesis will attempt to extend the state-of-the-art in snap-fit analysis by demonstrating improvement in the accuracy of finite element analysis predictions and the incorporation of time dependency into snap-fit design. A brief review of past work dealing with improved material models for snap-fit design is presented below, to emphasize the benefit of the proposed approach.

In order to evaluate the applicability and accuracy of experimental test data, Knapp II et. al. [32] compare three tensile test methods. It is well known that polymer behavior is dependent upon strain rate and that conventional tensile tests subject the specimen to variable strain rate during the test. In this work, tests were conducted at 0.002/s and 0.02/s strain rate. The experimental stress-strain data fit to an analytical curve and the value of tangent modulus ($E_t$) at each strain value was determined. This material model was used in a finite element
analysis and its results compared to experimental data. The authors propose that designers should use true stress-strain data gathered at strain rate values appropriate for their application. Trantina and Minnichelli [33] describe a software developed for automating snap-fit finite-element analysis. Deflection limited models are analyzed. Some elementary results describing the effect of dimensions on stress and strain on snap-fits are presented. The authors use a 0.1/s strain rate for determining the material properties. The work is also summarized, in more detail, in Trantina and Minnichelli [34, 35].

Sawyer et. al. [36] investigate the applicability of commonly reported coefficient of friction (µ) values for snap-fit design. Widely different values of µ have been reported in literature. For example, coefficients of friction ranging from µ = 0.28 to µ = 0.62 have been reported for unfilled nylon 6/6. In this study, friction measurements were made directly from polycarbonate cantilever hook lock pair contacts, and their dependence on load, sliding speed and contact geometry was investigated. An experimental setup for accomplishing this is detailed. A power law dependence of µ on normal load is suggested in the paper. Changes in the value of µ with sliding speed were found to be minimal. The contributing mechanisms to friction, viscoelastic deformation and adhesion are discussed briefly. The authors contend that adhesive contributions occur at a scale smaller than that modeled by finite elements and as such should be included in the value of µ. On the other hand, the effect of viscoelastic deformation should be captured by the finite element model of the snap-fit and as such need not be included in the value of µ.

2.3 Analytical Snap-fit Design (Advanced Beam Formulation)

The benefits and limitations of using a numerical approach (DOE, coupled with finite element analyses) for the design of cantilever hook features has been discussed in Chapter 1.
of this dissertation. An accurate analytical expression based on a fundamental understanding of insertion and retention phenomena will be of great value for snap-fit design.

Resin suppliers have provided rudimentary design equations for snap-fit features, especially the cantilever hook, as a service to the polymer design community. Some of the more popular “snap-fit design guides” are due to GE [7], Allied Signals [8] and Bayer [9]. These guides use a strength of materials approach to develop sizing equations for snap-fit features. Most of the design guides make the following assumptions:

1. The maximum displacement of the end of the cantilever beam is equal to the offset imposed by the catch of the snap-fit.

2. The only force acting at the end of the cantilever beam during insertion and retention is a transverse force.

3. Assumptions inherent in the use of the Euler-Bernoulli beam theory.

In view of the above assumptions, the maximum insertion force and retention force is estimated by transforming the transverse force required to deflect the end of the beam by an amount equal to the catch offset (Figure 2.1). The particular relationship that is most commonly used is shown below.

$$F_i = P \frac{\mu + \tan(\alpha)}{1 + \mu \tan(\alpha)}$$  \hspace{1cm} (2.1)

Data presented in popular design guides has been shown to vary widely and can be erroneous by orders of magnitude, especially for retention force [6]. Luscher [6] has pointed out that the forces acting on the end of the beam portion of a cantilever hook include an axial component and an overturning moment in addition to the transverse forces. He attempted to model the effect of these additional components on the insertion and retention
force for a cantilever hook by superposing the deflections due to the forces and moments. Though valid for small-deflection analysis, this approach fails for larger deflections, because the transverse and axial force components cause an additional moment as the beam deflects. This increases the effective moment acting on the end of the beam during insertion, and reduces it during retention. The limitations and inaccuracies of past efforts at analytically modeling the insertion and retention behavior of the cantilever hook has been discussed elsewhere. Several software programs which utilize these equations are also available. Design tools which incorporate automatic parametric mesh generation and finite element analysis [37, 38] have also been developed. The analytical formulation being pursued as part of this thesis is radically different from the “design guide” approach. The project envisions modeling the insertion and retention force curves instead of only the maximum insertion and retention force values. Very little work has been done in this area in the past. Meitinger [39] modeled the insertion behavior of the cantilever hook by considering
contact between the snap-fit and the mating part, force-deflection relationships for the beam portion of the cantilever hook, and the equilibrium of the catch. Sample results presented in the paper compare well with experimental data. However, the author Euler-Bernoulli beam theory with superposition, which is not strictly valid for large deformations. The effect of axial force component is also not considered in the analysis.

Effective modeling of the insertion and retention curves depends upon a formulation that accurately describes the force-deflection behavior the beam portion of the cantilever hook. A review of strength of materials literature was thus conducted to identify approaches that were more accurate in modeling this phenomena. A lot of the literature in this area was gleaned from textbook-type material, since it is not a “new” development. Timoshenko [40] presents a theory of elasticity type treatment of a cantilever beam loaded on end using the concept of Airy stress functions. Young [41] contains a comprehensive collection of formulas describing the strengths and stiffnesses of common structural shapes. Table 10 on pg. 16 of [41] lists formulas for shear, moment, slope and deflection for beams under simultaneous axial compressive and transverse loading. One of the results (Case 1b) from this table is similar to a formulation in Chapter 3 of this dissertation. Prescott [42] presents a solution for the deflection curve for a uniform cantilever beam under a given concentrated load and a given moment load at the free end. Frisch-Fay [43] presents a detailed treatment of non-linear deflections in beams using elliptic integrals.

2.4 Viscoelasticity

Most of the analytical efforts discussed in the previous sections make use of either elastic or elastic-plastic material models. Such material formulations are incapable of capturing any time-dependent effects in the material. It is common knowledge, however, that
polymers are inherently viscoelastic in nature. In fact, the response of a polymer is rarely elastic i.e. seldom does a polymer instantaneously recover its entire deformation upon load removal. There is usually a delay in the recovery phenomena. Also, the strain in a loaded plastic part under a given load will increase slowly with time due to molecular rearrangements induced by the stress. The phenomena of stress relaxation and creep are well known in polymers and have been the focus of an immense amount of research since the early nineteenth century. While a comprehensive review of the origin and literature of viscoelasticity is impossible, a thorough survey of research relevant to this thesis will be presented. We begin with a brief introduction to the basic concepts of viscoelastic models.

Viscoelastic response embodies aspects of both elastic solid and viscous fluid response. A popular approach for development of constitutive equations for polymers is to use combinations of mechanical analogs for linear elastic solid and linear viscous fluid response. The three most common mechanical analogs of viscoelastic response are the Maxwell model, the Kelvin-Voigt model and the Standard Linear Solid (SLS) model. These models, though attractive because of their simplicity, are limited in their capacity to represent realistic viscoelastic solid response. The Maxwell model represents fluid-like behavior and predicts a constant strain rate response to a step function stress input. For the Kelvin-Voigt model, the stress relaxation response is not gradual and the creep response is a poor approximation to the behavior of real polymeric materials.

Neither the Maxwell model nor the Kelvin-Voigt model provides a satisfactory simulation of observed viscoelastic response. The simplest model which does this for a solid consists of a Maxwell model and a linear spring in parallel or a Kelvin-Voigt model and a linear spring in series. However, the distribution of relaxation/retardation times observed in experimental work is broader than that predicted by these simple models [44].
This limitation of the *phenomenological approach* to viscoelasticity is easily overcome by a simple extension of the mechanical analog concept to form the *Generalized Maxwell model* which consists of N Maxwell models in parallel and the *Generalized Kelvin model* which consists of N Kelvin-Voigt elements in series. It is further found by experiment that the relaxation/retardation times for polymeric materials are so closely spaced that it is physically sensible, as well as mathematically convenient, to replace the discrete relaxation/retardation time constants corresponding to the generalized Maxwell or generalized Kelvin models with a continuous distribution of relaxation/retardation times. These are known as *relaxation/retardation spectra* and their use is quite popular in viscoelasticity literature.

All the modeling approaches discussed above are linear in nature i.e. they conform to the principles of proportionality and superposition. Linear viscoelasticity is valid only at small strains [44, 45] where the creep compliance or relaxation modulus is independent of applied stress or strain level respectively. At large strains or stresses, most materials exhibit non-linear behavior. The limit of linear behavior can be as low as one or two percent strain (or even lower). Constitutive equations for non-linear viscoelasticity are much more complex than linear theory. There has been considerable effort in past years to develop a generalized constitutive equation for non-linear viscoelastic materials. A theoretical survey of such work is presented by Lockett [46] using sophisticated mathematical methods. Findley [45] presents a review of some of the more practical approaches to characterizing nonlinear viscoelastic materials.

The approaches that have been adopted for nonlinear viscoelastic characterization range from purely empirical, in which experimental data is fit to common mathematical forms,
to purely theoretical, in which theories are constructed around physical axioms and the development is very rigorous. The first method can produce results for special materials and loading situations relatively quickly and inexpensively. However, such models are generally crude and cannot be applied to other loading situations with any degree of confidence. The theoretical approaches, though more sophisticated, have not reached a stage of practical development where they can replace the empirical methods. The modeling approach that is most popular in current literature is one that combines both of the above methods (semi-empirical approach). A detailed description of the nonlinear constitutive modeling theories will be presented in the chapter dealing with viscoelastic characterization of polycarbonate, only a brief mention of the most popular semi-empirical approaches will be made here.

The semi-empirical characterization approaches are also referred to as single-integral representations. The most well-known examples of single-integral approaches are the Leaderman’s modified superposition principle (MSP) [47], the Bernstein-Kearsely-Zapas (BKZ) theory and the Schapery thermodynamic theory (STT) [48]. Of the three, the Schapery theory has been found to be the most general and versatile. The Boltzmann superposition principle, the modified superposition principle and the BKZ theory are included in the Schapery representation as special cases. Smart and Williams [49] compare the accuracy of the above three single-integral representations in predicting the response of polypropylene (PP) and polyvinylchloride (PVC). They conclude that the Schapery theory models both loading and unloading behavior better than the other two theories.

Numerous examples of application of the Schapery theory to different classes of polymers are available in literature. Schapery [48] presents a graphical method of data reduction to estimate values of the non-linearizing parameters in his theory. The methodology
was illustrated using available data on nitrocellulose film, fiber-reinforced phenolic resin and polyisobutylene. Dillard et. al. [50] compare three approaches for modeling the non-linear viscoelastic response of glassy/epoxy composites. The time-temperature and the time-stress superposition techniques are compared to the Findley and Schapery approaches for modeling non-linear viscoelasticity. The Schapery approach was found to be the most general and convenient for adaptation into a numerical procedure. Popelar et. al. [51] modeled the mechanical response of medium and high density polyethylene using the Schapery approach. Master curves for stress relaxation were developed using horizontal and vertical shifting. The shift factors were also used to develop master curves for maximum stress vs strain rate and burst strength vs time to failure curves. The consistency of the shift data in the three cases was deemed to support the validity of the shift functions and master curves. Kenner et. al. [52] conducted stress relaxation tests on an epoxy-based molding compound, and used similar shifting procedures to create coherent master curves for different strain levels. Constant strain rate tests were used to augment and verify stress relaxation data. Crook and Letton [53] modeled the non-linear response of polycarbonate and a rubber toughened epoxy using the Schapery approach. The influence of damage on non-linear shift factors and creep compliance was investigated with a view towards modeling material response during repeated loading and unloading. They suggest that the use of non-linear factors is not necessary at low stress levels at which damage accumulation does not occur. Brinson et. al. [54] used the Schapery approach coupled with a power law model for the transient compliance component to characterize the behavior of Fiberite 934. A master creep curve was formed by horizontal and vertical shifting of test data from tests conducted at different temperatures. Some visco-plastic behavior was also observed. Bruller [55] presents a least squares formulation for obtaining a mathematical description of the long
term behavior of polymers in creep or stress relaxation. An exponential series identical
to the Kelvin model was assumed for linear viscoelasticity. Schapery’s model is used for
non-linear viscoelastic materials. Examples of the application of the approach to different
materials are presented. Material response to cyclic loading is also modeled and shown to
be reasonably accurate when compared to experimental data.

Several modifications of the original data reduction methods proposed by Schapery
have been presented in the literature. Zaoutsos, Papanicolaou et. al. [56] proposed an ana-
lytical methodology to estimate the non-linearizing parameters that enter into the Schapery
model. A visco-plastic term ($\epsilon_{vp}$) was also added to the model. Its value was estimated
as the residual strain in the plastic after long periods of time (168 h recovery test). The
approach was applied to the viscoelastic characterization of 90 degree carbon/epoxy com-
posite, using a power law model for the compliance component. The authors claim that this
approach eliminates the instability and dependence on the duration of the creep test of the
power law exponent. Papnicolaou, Zaoutsos et. al. [57] extended the above approach to
estimate the value of the last non-linearizing parameter ($g_2$) in the Schapery model analyt-
ically. This analytical procedure is an improvement over previously used mix of graphical
and numerical techniques. Gamby and Blugeon [58] suggested separating the instanta-
neous and delayed responses to determine the non-linear factors entering into the Schapery
model. Their contention was that such an approach would make the process of estimating
their values more accurate and reliable. The approach used was demonstrated for two mate-
rials, Polycarbonate and FM-73 adhesive. The results were not significantly more accurate
than previous work, however, the approach was more systematic, as it did not involve any
trial and error calculations.
Of particular interest to this work is studies related to the viscoelastic characterization of polycarbonate (PC). This has been chosen to be the material that the efforts of this thesis will be focussed on, being one of the most commonly used amorphous engineering polymers. A review of the literature revealed a number of different approaches and models that have been applied to PC in the past, with varying degrees of success. A limitation of the choice of a constitutive model for PC is the need for it to be suitable for use in a finite element analysis package. Some of the earliest work by Gamby and Blugeon [58] in this area has been discussed previously. The Schapery approach is used with some modifications in the data reduction method to improve the reproducibility of the results. Yee et. al. [59] used single- and double step stress relaxation experiments to characterize the behavior of polycarbonate. The Kohlrausch-Williams-Watts (KWW) form is used to model viscoelastic response. Since the KWW model could effectively describe only a single relaxation mechanism, a double-step relaxation experiment was used. After being held at a particular initial strain for a sufficiently long time, an additional small strain step was introduced. The response to this “tickle” step was considered as representative of the relaxation behavior of the structure at the magnitude of the initial strain. A master stress relaxation curve was formed by using simple horizontal shifting procedures. A data reduction procedure similar to that proposed by Matsuoka [60] was used. The material was found to be nonlinear at strains greater than 1% necessitating the use of strain dependent exponents in the KWW form. The KWW form with constant exponents was deemed unsuitable for modeling nonlinear viscoelastic response of PC, contrary to assertion by previous workers (Matsuoka [60] and others). A justification of modeling approach based on molecular theory and concepts of structural rearrangements is also presented. Malkin et. al. [61]
investigated the correctness and reliability of different methods of creep function calculations from relaxation curves, using computational methods. The Prony series approach was found to be the most accurate. Experiments were conducted with PC in the linear as well as non-linear stress range. Boyce et. al. [62] examined and compared the differences in the stress-strain response of PC in uniaxial compression, plane-strain compression, uniaxial tension and simple shear experiments. The behavior is related to the molecular orientation of the material. A constitutive model proposed by the same authors in earlier work was used to model the response. A finite-element implementation of the model was used to compare experimental results to predicted response and excellent agreement between the two was observed. Grzywinski and Woodford [63] demonstrated the feasibility of using stress relaxation test data for PC and polyphenylene oxide (PPO) to generate accurate and repeatable creep and secant modulus data for use in engineering design problems. Wing et. al. [64] compared the viscoelastic response of PC and microcellular PC and created a constitutive model for the two based on the Schapery theory. The onset of nonlinearity in the two cases was identified. Microcellular PC was found to more nonlinear and visco-plastic than standard PC. O’Connell and McKenna [65] discuss the applicability of time-temperature and time-aging superposition at temperatures below the glass-transition temperature ($T_g$) for PC. Experiments were conducted in torsion and the KWW function was used to model the relaxation curves. Limitations of the KWW forms discussed earlier in this section are identified. Colucci et. al. [66] attempted to relate differences in the tensile dilatation between two grades to PC to differences in their viscoelastic behavior. It was suggested that the concept can be extended to explain difference in yield and fracture behavior for the two polymers.
2.4.1 Viscoelastic Finite Element Analysis

With the rapid advances in the use of finite element analyses in practical design problems, there has been considerable research in the field of implementing different kinds of material models within the framework of commercially available finite element packages (ABAQUS™, MARC™ etc.). Initial work in this area focussed on manipulating the constitutive equations into a form suitable for use in finite element analysis. Examples of extension of the basic methodology to include more sophisticated and refined models and their application to design problems are also available in literature. The key issue in the implementation of any viscoelastic, or for that matter, any history dependent, material model is its expression in incremental form. This enables the expression of the material properties in a recursive equation, eliminating the expensive need of storing stress and strain data from all the previous increments. Some of the first work in the area of implementation of viscoelastic models in finite element analysis was done by Zienkiewicz et. al. [67]. Creep stress-strain relationships for the Kelvin (Voigt) viscoelastic model in incremental form were derived. These were necessary for implementation of viscoelasticity in finite element analysis. The approach was extended to cases where multiple Kelvin models are necessary to represent viscoelastic behavior and also to multi-axial stress states. The method was checked against some known solutions. Examples from the fields of propellant technology, concrete and rock behavior were included.

The exponential form of viscoelastic constitutive equations can be expressed in incremental form with relative ease, and has thus been one of the most commonly used forms in viscoelastic analysis. Henriksen [68] was one of the first researchers to implement the Schapery approach in a finite element setting. A stress operator that described uniaxial
strain as a function of current and past stress states was developed. Extension to a multi-
axial approach was achieved through the use of Poisson’s effects. Material properties were
assumed to be a function of the octahedral shear stress. The formulation was implemented
for plane strain, plane stress and axisymmetric formulations. Lai and Bakker [69] adopted
an alternative approach for implementation of the Schapery model. Instead of relating the
strain components to the stresses through linear operators analogous to linear elasticity,
the authors chose to partition the stresses and strains into deviatoric and hydrostatic com-
ponents. The deviatoric and hydrostatic behavior of the material is assumed to be fully
uncoupled. Skrypnyk et. al. [70] present an alternative approach for the implementation of
the Schapery model in MARC™, based on the Henriksen scheme of discretization. Poon
and Ahmad [71] extended the implementation to include anisotropic material behavior. Ha
and Schapery [72] applied the constitutive model to a particle filled rubber under three
dimensional stress states. A similar incremental approach coupled with a hygro-thermo-
rheologically simple material postulate has been used by Krishna et. al. [73].

The next chapter of this thesis describes the proposed analytical formulation in greater
detail.
CHAPTER 3

DEVELOPMENT OF ANALYTICAL MODEL OF CANTILEVER HOOK PERFORMANCE

In this chapter, the development of an analytical model for cantilever hook snap-fit features is described. A system of equations that model cantilever hook deformation and associated forces is developed from first principles. More specifically, these equations model deformation of the cantilever hook beam, equilibrium of the cantilever hook catch and contact between cantilever hook and mating part. Two reference frames are used to simplify the equations, and kinematic equations relating variables expressed relative to the two frames are also used. These equations are then iteratively solved for several contact positions during insertion and retention to obtain the respective force values. It is also possible to access other variables, like the displacement and rotation of the end of the beam, to gain greater insight into the physical phenomena. Separate equations are necessary for modeling the insertion and retention phenomena. The cantilever hook was chosen as the focus of this work because is one of the most commonly used snap-fits and data from previous efforts is available for comparison.

Section 3.1 establishes the nomenclature for cantilever hooks and describes the insertion and retention phenomena in detail.
3.1 Description and Nomenclature of Cantilever Hook

The basic shape of a cantilever hook feature is shown in Figure 3.1. The mating part, which engages with the cantilever hook is also shown. The terms *insertion face, retention face, dwell face, catch, mating part, insertion direction and retention direction*, which will be extensively used in this dissertation, are self-explanatory. It is important to note that the lower edge of the mating part is nominally at the same ordinate as the upper face of the cantilever beam (shown by dotted line). This will be referred to again later in this chapter. This condition is also referred to as 100 percent engagement in other literature in this area.

![Figure 3.1: Cantilever hook and mating part.](image)

Engagement, or insertion occurs via motion of the mating part towards the left (insertion direction). Once contact between the cantilever hook and the mating part has been established (on the insertion face), the beam deforms downwards as the mating parts moves
further along in the insertion direction. The point of contact slides along the insertion face, thereby bending the beam back further and increasing the force required to cause deformation. A sample deformed configuration of the cantilever hook and the mating part is shown in Figure 3.2. Contact along the dwell face is similar, with deformation of the beam end increasing, albeit at a much slower rate. As the trailing edge of the mating part crosses the end of the dwell face, the point of contact shifts to the retention face of the catch. The beam recovers its deformation, barring material effects. At the end of this “recovery phase”, when point E rests against the trailing edge of the mating part, the snap-fit is said to have engaged. **Disengagement, or retention** occurs as the mating part moves towards the right (retention direction). The underlying phenomena are similar to those in insertion, although the direction of the friction force is opposite to that during insertion.

Figure 3.2: Deformed configuration of cantilever hook and mating part during insertion.

Figure 3.3 illustrates the relevant geometric dimensions of cantilever hooks. Apart from sizing information, material properties such as the Young’s Modulus ($E$), Poisson’s
ratio ($\nu$), coefficient of friction ($\mu$) and allowable strain ($\epsilon_{all}$) are also used while designing these snap-fit features.

![Figure 3.3: Basic cantilever hook nomenclature.](image)

### 3.2 Current Cantilever Hook Design Methodology

The most commonly available sources of snap-fit design information are plastic design guides provided by resin manufacturers as a service to the design community [7, 8, 9]. The design approach commonly adopted in these guides is briefly mentioned in Chapter 1, but will be discussed in greater detail here to establish the motivation for developing a more accurate model. Luscher [1] also presents a comprehensive review of the design procedures adopted by commercial design guides. Calculation of insertion and retention forces is discussed separately in the following sections. It is worthwhile to note that the guides provide equations for estimating the maximum values of insertion and retention forces only, not the complete insertion and retention curves.
3.2.1 Calculation of Insertion Force

The insertion force \( F_i \) is related to bending force \( P \), Figure 3.3) required to bend the beam by considering equilibrium of the cantilever hook catch. The resulting equation is:

\[
F_i = P \frac{\mu + \tan(\alpha)}{1 + \mu \tan(\alpha)}
\]  

(3.1)

In Equation (3.1), \( \mu \) is the coefficient of dynamic friction and \( \alpha \) is the insertion face angle for the snap-fit (Figure 3.3). The above equation can be derived by identifying that the forces at the point of contact are the normal and friction forces \( F_n, F_\mu \) respectively. The forces can then be resolved along the directions of the insertion force \( F_i \) and transverse force \( P \) to obtain Equation 3.1. It is also important to realize that the Coulomb’s model of friction \( F_\mu = \mu F_n \) is also used in the derivation of the above equation. For more details, the reader is referred to one of the snap-fit design guides [9].

It is often desirable to eliminate \( P \) from Equation (3.1) since \( P \) is initially unknown in hook design. This is done by considering the geometry of the hook shown in Figure 3.3 and assuming that the entire deformation occurs exclusively in the beam portion of the hook (i.e. the catch is rigid). \( P \) can be expressed as a function of either the deflection \( y \) or the maximum allowable dynamic strain \( \epsilon_{all} \). For an untapered hook of rectangular cross-section, the relationship between \( P \) and \( y \) can be written as:

\[
y = \frac{PL^3}{3EI}
\]  

(3.2)

For a rectangular cross-section, the second moment of area \( I \) can be expressed as:

\[
I = \frac{bh^3}{12}
\]  

(3.3)

Equation (3.2) reduces to the following form:

\[
y = \frac{4PL^3}{bh^3E}
\]  

(3.4)
The relationship between strain ($\epsilon_{all}$) and beam deflection ($y$) can be expressed as follows:

$$y = \frac{2\epsilon_{all}L^2}{3h} \quad (3.5)$$

Eliminating $y$ between Equation (3.4) and Equations (3.5), $P$ can be expressed in terms of allowable strain yielding:

$$P = \frac{dh^2\epsilon_{all}E}{6L} \quad (3.6)$$

Equation (3.4) or Equation (3.6) can be used to eliminate bending force $P$ from Equation (3.1), thereby expressing the insertion force ($F_i$) in terms of $y$ or $\epsilon_{all}$ respectively.

Many thermoplastic resins behave non-linearly at the strain levels achieved during assembly. This nonlinearity is often incorporated in the design by using the secant modulus, $E_s$, instead of the Young’s modulus, $E$ in these equations.

3.2.2 Calculation of Retention Force

The design guides categorize cantilever hooks as being either removable or permanent. Removable snap-fits are designed to release under the application of a removal force to the assembly. Permanent snap-fits, on the other hand, are designed not to be removable unless the hook is mechanically deflected. If the hook is to be removable, the retention force ($F_r$) must be below some maximum value yet be high enough to retain engagement under normal service loads. The retention force can be calculated by using retention angle ($\beta$) instead of $\alpha$ in Equation (3.1), and considering that the friction force direction is opposite to that during insertion to yield:

$$F_i = P \frac{\mu + \tan (\beta)}{1 - \mu \tan (\beta)} \quad (3.7)$$

From the above equation, it is apparent that, for a removable cantilever hook, the retention angle must be below a critical angle of friction. The critical angle is a function only of
coefficient of friction ($\mu$):

$$\alpha_{crit} = \arctan \left( \frac{1}{\mu} \right)$$  \hspace{1cm} (3.8)

If $\beta$ is below this angle, the snap-fit will manually disengage but if $\beta$ is greater than this angle, disengagement can only occur via material failure of the beam, the catch or the mating part. For permanent cantilever hooks, simple tensile failure (of the beam) and shear failure (of the catch) are suggested for determining the retention force. While the above procedure for designing cantilever hooks is simple and efficient, the accuracy of its predictions is somewhat suspect, especially in retention. Several limitations of the design methodology are apparent:

- The maximum displacement of the end of the cantilever beam is implicitly assumed to be equal to the offset ($y$) of the cantilever hook. This assumption oversimplifies the deformation of the cantilever hook beam. Not all snap-fits are at such nominal tolerances.

- The end of the beam rotates in addition to deforming downwards. This can significantly influence the maximum deflection of the end of the beam during insertion. Any attempt to incorporate the rotation of the beam-end will also alter the force-deflection relationship of the beam (Equation (3.2)).

- The transverse force is assumed to be the only component of force responsible for the deformation of the cantilever beam. Consideration of the equilibrium of the catch reveals that the end of the beam is subject to additional force and moment components, due to the “offsetting” nature of the cantilever hook catch. The effect of these components (axial force $F_a$, moment $M$) should also be included in the design process to arrive at more accurate estimates of the forces.
3.3 Physical Description and Assumptions of Proposed Model

In order to describe the modeling strategy used in this work, we will refer to a particular deformed configuration of the cantilever hook and mating part during insertion, as shown in Figure 3.2. Contact forces \( F_N \) and \( F_\mu \) are generated at point C. These forces are responsible for the deformation of the cantilever hook. As the mating part moves along the insertion direction, the location of the contact point, the contact forces and the deformation of the cantilever hook all change. A two-step modeling strategy is used. A particular deformed configuration is first modeled using a set of equations and all the relevant variables solved for. The above solution procedure is then repeated for successive positions of the mating part (represented by coordinates of the contact point, C) to obtain a model of the insertion process. A similar approach can be used for modeling retention, with suitable modifications to the model equations.

The assumptions inherent in the modeling strategy are summarized below:

1. The mating part is rigid. This implies that all of the deformation occurs in the cantilever hook. This deformation is usually shared between the two parts being assembled in real life products. The extent of the inaccuracy introduced by this assumption depends on the relative stiffness of the two parts. This assumption is conservative in that it will lead to somewhat higher values for the insertion force, retention force and strain during insertion.

2. The cantilever hook can be idealized as a rigid catch supported on a cantilever beam. The forces developed during assembly and disassembly cause deformation only in the beam portion of the cantilever hook. The catch has the sole effect of offsetting
the contact point (C) forces from the end of the beam, resulting in a moment acting on the end of the beam, in addition to axial and transverse forces.

3. Insertion and retention processes are quasi-static. Dynamic effects are thus ignored, and static equilibrium of the catch can be assumed for model development. Additionally, each deformed configuration can be solved independently of the previous solutions (except for using it as an initial guess for iterative purposes).

4. The cantilever hook and mating part are always in perfect contact. Typically, generalized contact modeling approaches use some kind of distance tolerance to detect contact between bodies. If the distance between the bodies is less than that tolerance value, contact is assumed, otherwise the bodies are assumed to be out of contact. If the bodies are in contact, additional constraints are imposed on the equation system, otherwise one of the bodies is incrementally displaced until contact is detected. In our case, however, considerable simplification can be achieved by making use of the known cantilever hook geometry.

5. The mating part is in sliding contact with the catch. This assumption allows us to relate the normal force ($F_N$) and the friction force ($F_\mu$) via the dynamic coefficient of friction value ($\mu$).

As mentioned above, the cantilever hook snap-fit will be modeled as a rigid catch attached to the end of a cantilever beam. In fact, this is equivalent to considering the catch to be supported on a flexible foundation, with axial, transverse and rotational stiffnesses equivalent to those of the cantilever beam. The idealized model of the cantilever hook and the mating part is shown in Figure 3.4. For modeling purposes, two coordinate systems $O$ and
Figure 3.4: Idealized model of cantilever hook, showing the hook as a combination of a beam and a rigid catch.

$R$ will be used. Coordinate system $O$ is located at the base of beam with its $x$-axis along the neutral axis of the cantilever beam. This is a fixed coordinate system. Coordinate system $R$ is attached to the junction between the beam and the catch at the beam’s neutral axis. Therefore, it rotates with the end of the beam. The catch is stationary with respect to this coordinate system. The description of motion of the mating part with respect to the two reference coordinate systems is different. With respect to system $O$, its motion is along a straight line parallel to the $x$-axis, in the negative $x$ direction. With respect to $R$ however, its motion is along a straight line defined by the equation of the face of the catch that it is sliding along. This concept later proves useful while developing model equations. A deformed configuration of the idealized cantilever hook model is shown in Figure 3.5.

3.4 Formulation of Model Equations

The idealized model discussed above is used to develop a system of equations that describe the insertion and retention processes. As mentioned earlier, the basic step in the
modeling strategy is to solve for forces and displacements for a particular deformed configuration of the cantilever hook. Once a satisfactory solution has been achieved, the mating part (or the location of the contact point C on the catch) can be incrementally moved in the appropriate direction (depending on insertion or retention). For each such deformed configuration, the set of equations can be resolved and the forces obtained. This modeling approach is essentially quasi-static. Although the mating part position is incremented in time, no dynamic effects are modeled. The model equations can be divided into three distinct categories. Each set of equations models a particular aspect of cantilever hook insertion/retention. These sets are:

1. **Force-displacement relationships for the cantilever beam.** These relate the displacement and the rotation (θ) of the end of the cantilever beam (R) to the force and moment components (Fa, Ft, M, Figure 3.6). Two different forms are used in this thesis. A third form is suggested as future work to model large deformation behavior of beams more accurately.

![Figure 3.5: Idealized model of cantilever hook and mating part, shown in deformed configuration.](image)
Figure 3.6: Forces and displacement of cantilever beam, shown in deformed configuration.

Figure 3.7: Catch of cantilever hook showing forces acting on it (during insertion).

2. **Equilibrium equations for the catch.** These are simple equations derived by considering a free-body diagram of the catch (Figure 3.7). In particular, these relate the contact forces acting at the contact point (C) to the internal force and moment components at the junction between the beam and the catch (R).

3. **Geometric constraints.** These are additional equations derived by considering the specific geometry of the cantilever hook and mating part. These include kinematic transformation equations between the O and R coordinate systems, the equation of the face of the catch and equations arising due to the location of the contact point.

The detailed development of each of these sets of equations is presented next.
3.4.1 Force-Displacement Relationships for the Cantilever Beam

These equations can be represented in generic form as:

\[ O_{xR} = f_x(F_a, F_t, M, EI) \]  \hspace{1cm} (3.9)

\[ O_{yR} = f_y(F_a, F_t, M, EI) \]  \hspace{1cm} (3.10)

\[ O \left( \frac{dy}{dx} \right)_R = f_s(F_a, F_t, M, EI) \]  \hspace{1cm} (3.11)

The particular algebraic form of these equations depends upon the particular engineering mechanics formulation chosen. The two different formulations used in this thesis are:

1. **Euler-Bernoulli beam theory with superposition.** The individual deflections due to the force and moment components are superposed to determine the overall displacement of the end of the beam. These equations are popularly used in design guides, although only the effect of the transverse force \( F_t \) is commonly considered. For this case, the generic form shown in Equations 3.9 - 3.11 assumes the particular mathematical form shown below. The details of the derivation are shown in Appendix A.

\[ O_{yR} - \frac{ML^2}{2EI} - \frac{F_tL^3}{3EI} = 0 \]  \hspace{1cm} (3.12)

\[ O_{xR} - L = 0 \]  \hspace{1cm} (3.13)

\[ O \left( \frac{dy}{dx} \right)_R - \frac{M}{EI} x - \frac{F_tL}{EI} x + \frac{F_t}{2EI} x^2 = 0 \]  \hspace{1cm} (3.14)

Note that the axial force component \( F_a \) does not appear in the above equations. This is a consequence of the use of the superposition principle. \( F_a \) introduces an additional bending moment only when the beam deforms. This approach considers the undeformed configuration of the beam while calculating the bending moment. As
such, any coupling between the force and displacements is overlooked. This issue is addressed in the following beam formulation.

2. **Euler-Bernoulli beam theory without superposition.** As the cantilever beam deforms, an additional bending moment due to the axial component $F_a$ develops. This approach accounts for the increase in bending moment due to the axial component and is an improvement over the last formulation. The final form of the force-displacement relationship is reproduced below. The reader is referred to Appendix A for more details.

\[
O_y R - \frac{M}{F_a} \left[ \sec (\lambda L) - 1 \right] - \frac{F_t}{F_a} \left[ \frac{\tan (\lambda L)}{\lambda} - L \right] = 0 \quad (3.15)
\]

\[
O_x R - L = 0 \quad (3.16)
\]

\[
O \left( \frac{dy}{dx} \right)_R - \frac{M \tan (\lambda L)}{EI} - \frac{F_t}{F_a} \left[ \sec (\lambda L) - 1 \right] = 0 \quad (3.17)
\]

\[
\lambda^2 - \frac{F_a}{EI} = 0 \quad (3.18)
\]

It is important to note that the above equations have been derived with respect to the assumed force and moment directions shown in Figure 3.6. Note that the axial force is shown to be compressive, which corresponds to contact occurring on the insertion or dwell faces. For the simpler beam formulation, a change in the direction of axial force ($F_a$) does not require any additional consideration because the algebraic form of the solution does not change. However, for the second formulation, a change in the direction (sign) of the $F_a$ term, changes the algebraic form of the solution of the differential equation. In particular, the $\sec$ and $\tan$ terms change to their hyperbolic counterparts $sech$ and $tanh$ respectively, accompanied by some sign changes. More details are presented in the detailed derivations in Appendix A.
3.4.2 Equilibrium of Catch

As stated earlier, the catch is modeled as a rigid body. Conditions of equilibrium between the contact forces and the beam forces are developed in Appendix A and the final results, after considerable simplification, are shown here.

\[ F_a [1 - \mu \tan (\beta + \theta)] - F_t [\tan (\beta + \theta) + \mu] = 0 \]  \hspace{1cm} (3.19)

\[ M - R_{XC} (F_a \sin \theta + F_t \cos \theta) - R_{YC} (F_a \cos \theta - F_t \sin \theta) = 0 \]  \hspace{1cm} (3.20)

The above equations have been derived specifically for insertion (based on assumed force directions in Figure 3.7). The equilibrium conditions for retention will be different, because the direction of relative motion between the cantilever hook and mating part is opposite to that during insertion, with a similar change in the direction of friction force. These equations are derived in Appendix A.

3.4.3 Geometric Relationships

Geometric relationships are derived by considering the location of the contact point in the \( O \) and \( R \) coordinate systems. The transformation equations between the two systems are also used. Once again, only the final results are presented below. The reader is referred to the appendix for further details of the derivation. Figure 3.8 shows the cantilever hook and mating part in a deformed configuration.

\[ \begin{align*}
O_xR + O_{xRC} &= O_xC \hspace{1cm} (3.21) \\
O_yR + O_{yRC} &= O_yC \hspace{1cm} (3.22) \\
R_{XC} - O_{xRC} \cos(\theta) - O_{yRC} \sin(\theta) &= 0 \hspace{1cm} (3.23) \\
R_{YC} + O_{xRC} \sin(\theta) - O_{yRC} \cos(\theta) &= 0 \hspace{1cm} (3.24)
\end{align*} \]
The last equation in the above set (Equation 3.26) is simply the equation of the face of the catch on which contact occurs (Insertion face, AB in Figure 3.8). As the location of the contact point (C) changes from one face to another (dwell face, BD, or retention face DE), this equation will need to be updated to represent the current contact face. This essentially splits up the process of modeling insertion into three phases, corresponding to the three faces of the catch (insertion, dwell and retention face). The solution procedure (discussed in Section 3.4.4) is designed so that it tracks the location of the contact point (C) relative to the current face. If during an increment, it is detected that the contact point (C) falls outside the current face, then the system of equations is modified to reflect contact along the next face.

The equations developed in Section 3.3 form a set of non-linear equations that can be iteratively solved to obtain the deformed configuration and forces for any position of the mating part (or contact point). While solving non-linear equations, it is beneficial to
simplify the equation set as much as possible. With simple algebraic manipulations and substitutions, it is possible to reduce the number of equations. A typical simplified set of equations, for the Euler-Bernoulli beam formulation with superposition, is shown below. The other equation sets can be similarly simplified and reordered.

\[
6EI O_{y_{RC}} + 3ML^2 + 2F_t L^3 + 6EIt = 0 \quad (3.27)
\]

\[
2EI \tan \theta - 2ML + F_t L^3 = 0 \quad (3.28)
\]

\[
F_a [1 - \mu \tan (\beta + \theta)] - F_t [\tan (\beta + \theta) + \mu] = 0 \quad (3.29)
\]

\[
M - R_{x_C} (F_a \sin \theta + F_t \cos \theta) - R_{y_C} (F_a \cos \theta - F_t \sin \theta) = 0 \quad (3.30)
\]

\[
R_{x_C} - O_{x_{RC}} \cos \theta - O_{y_{RC}} \sin \theta = 0 \quad (3.31)
\]

\[
R_{y_C} + O_{x_{RC}} \sin \theta - O_{y_{RC}} \cos \theta = 0 \quad (3.32)
\]

\[
R_{y_C} - R_{x_C} \tan \beta - d = 0 \quad (3.33)
\]

### 3.4.4 Solution of Equation Set

A brief review of numerical analysis texts revealed a number of commonly used non-linear equation solvers. Some of them are listed below:

1. Newton’s method
2. Levenberg-Marquardt method
3. Powell’s method
4. Brent’s method
5. Brown’s method
6. Secant method with Broyden’s Jacobian update
Agrawal [74] presents a review of these solvers. In each case, the solution procedure is iterative and an initial starting point \( x^{(0)} \) must be provided. In most cases, subsequent estimates of \( x \) are provided by the iterative relationship:

\[
x^{i+1} = x^{(i)} + k^{(i)} \Delta x^{(i)}
\]

(3.34)

where \( \Delta x^{(i)} \) is the correction vector and \( k^{(i)} \) is a damping term selected such that either,

\[
|f_j (x^{(i+1)})| < |f_j (x^{(i)})| \quad \text{for } j = 1,2,\ldots,n
\]

(3.35)

or,

\[
\sum_{j=1}^{n} [f_j (x^{(i+1)})]^2 < \sum_{j=1}^{n} [f_j (x^{(i)})]^2
\]

(3.36)

In general, \( 0 < k^{(i)} \leq 1 \) is commonly chosen. The iterations are continued until a suitable stopping criteria is reached. The stopping criteria used in most cases is based on the Euclidean norm of the function residuals, i.e.

\[
F (x) = \|f (x)\| = \sum_{i=1}^{n} [f_i (x)]^2 \leq \epsilon_f
\]

(3.37)

where \( \epsilon_f \) is a convergence tolerance. An alternative termination criterion is based on the convergence of variables to a certain precision. This criterion is used to stop the iterations when the design variables are no longer progressing to the solution satisfying the design equations. It does not necessarily indicate convergence, and therefore, is not used to this research. The Newton’s method is based on the following iterative procedure:

Let the system of equations be represented by the following equation:

\[
f (x) = 0
\]

(3.38)

Start with \( x^{(1)}, f^{(1)}, [J^{(1)}] \) and \( k = 1 \). Then, iterate using the following equations:

\[
\Delta x^{(i)} = \left[J^{-1}\right]^{(i)} f^{(i)}
\]

(3.39)
\[ x^{(i+1)} = x^{(i)} + k \Delta x^{(i)} \] (3.40)

\([J]\) is the Jacobian matrix for the system of equations \( f(x) = 0 \) and is calculated as:

\[
[J] = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}
\] (3.41)

The terms of the Jacobian matrix can either be evaluated algebraically or approximated by using an appropriate difference formula. For our model, the Jacobian is evaluated algebraically, since MAPLE\textsuperscript{TM} allows the user to perform algebraic differentiation of functions. Newton’s method converts the problems of solving a non-linear set of equations to a linear system, albeit in incremental form. The convergence and efficiency of this and most other solution methods is very sensitive to a proper choice of the initial guess vector \( x^{(0)} \). Our problem naturally lends itself to an easy choice of initial variables, since incrementation is started from a configuration in which there is no deformation. Hence, at the very first increment, the exact solution of the equation set is known and is used as the starting guess for the next increment and so forth.

The symbolic mathematics package MAPLE\textsuperscript{TM} has been used to implement the Newton’s algorithm and also the incremental procedure for mating part position. Symbolic mathematics packages offer the additional advantage of allowing algebraic manipulations within the program. The program contains a symbolic definition of the equation set. Subroutines for computing the symbolic form of the Jacobian and determining the solution using Newton’s method have also been coded. Numerical forms of the Jacobian matrix and the equation sets are determined by substituting appropriate values for the variables into the symbolic forms. Due to the symbolic computation capability, re-implementation of the...
algorithm for a different beam deflection formulation, will therefore, be relatively simple. A flowchart of the solution algorithm is shown in Figure 3.9. During insertion, contact on the insertion, dwell and retention faces need to be solved for separately. Equation 3.33 contains the condition that constrains the contact point to lie on the surface of the catch. It is the equation of the particular face that the contact point is currently sliding along. Hence, the particular slope and intercept values for the three catch faces (insertion, retention and dwell) will be different. When the contact point reaches the junction between any pair of faces, these values need to be changed. The solution from the last increment is used as the initial variable set for the next face. The entire process is repeated for modeling retention. However, for the purpose of this work, modeling of retention phenomena is limited to contact along the retention face only. The maximum retention force value obtained during this phase of retention is typically the retention force at which disengagement or failure of the cantilever hook snap-fit occurs. Modeling retention processes on the dwell and insertion faces is thus of academic interest only, even though it will not require much additional effort.

Some results obtained using the script are presented and compared to experimentally obtained data in the following section.

3.5 Results and Discussion

The analytical model, implemented as a MAPLE script, can be used to simulate the insertion and retention behavior of most cantilever hook geometries. The script accepts input in the form of a text file containing the following design parameters: section modulus ($EI$), coefficient of friction ($\mu$), beam length ($L$), half beam thickness ($t$), insertion angle ($\alpha$), retention angle ($\beta$), catch offset ($y$) and length of dwell face ($d$). All these input
Figure 3.9: Algorithmic flowchart of analytical cantilever hook model.
variables are contained on a single line, separated by blank spaces. The script reads in these values and converts them into values more suitable for use in the model. Other variables that are defined by the user are the increment values, the error checking tolerances, the maximum number of iterations allowed per increment and the initial guess solution. While these values are currently hard-coded into the script, they can easily be transformed into user specified values via the input file. The script is divided into two parts, the subroutine library (library.maple) and the controlling program (insertion.maple or retention.maple). The former contains all the subroutines that form the complete model and post-processing utilities. The latter contains calls to the subroutines to generate model solutions for the different catch faces separately. The necessary convergence and control parameters are also defined in this portion of the program.

Once the script is loaded within MAPLE, it runs to completion without any further user intervention. There are two separate control scripts, one each for insertion and retention respectively. The insertion script calls the main solution subroutine thrice, once for the insertion, dwell and retention faces, in that order. The retention script solves the model for the retention face only. If at any point during the solution process, convergence is not achieved within the specified number of iterations, the script exits with an appropriate message. Two separate library scripts have been written, corresponding to the two beam deformation formulations adopted in this thesis. Typical insertion and retention results are shown in Figure 3.10 and Figure 3.11 respectively. The force curves show the different instances at which the model computes solutions of the model equations with each data point corresponding to one solution. Insertion proceeds from the right to the left and retention in the opposite direction in these figures. The different phases of insertion are apparent from the shape of the insertion force curve. Each discontinuity corresponds to a transition
between two adjacent catch faces, either from the insertion to the dwell face or from the dwell to retention face. This is not seen in the retention force curves since the retention model has only been implemented for the retention face.

It is relevant to point out here that, throughout this thesis, insertion force has been shown to be negative quantity while retention force is positive in magnitude. This is done to reflect the opposite directions of insertion and retention force and derives its motivation from the physical testing of cantilever hooks, during which the load cell remains in compression during insertion (hence the negative output) and in tension during retention. The term “maximum insertion force”, therefore refers to the maximum negative insertion force value.

![Graph of analytical cantilever hook model for insertion](image)

**Figure 3.10:** Typical results of analytical cantilever hook model for insertion, showing different phases of the insertion process.
Several quantities are output by the program. Numerical values of the axial force ($F_a$, insertion force), transverse force ($F_t$), overturning moment ($M$) and several geometric variables ($R_x C$, $R_y C$, $\theta$, $O_{xRC}$, $O_{yRC}$) are computed in the model. The program also contains subroutines for plotting the deformed shape of the cantilever hook and for creating an animation of the insertion or retention process being modeled.

The predicted influence of different design quantities on the performance of cantilever hooks can easily be observed by altering the appropriate numbers in the input file and re-running the script for each case. Some results obtained in this manner are shown in the following figures. The first set of figures (Figures 3.12 - 3.14) illustrate the influence of insertion angle, friction coefficient and catch offset respectively on insertion force. These curves were obtained using the Euler-Bernoulli beam formulation with superposition, described previously in Section 3.4. The next three figures (Figures 3.15 - 3.17) show the
effect of the same set of factors on insertion force values. However, these were derived using the second beam deformation formulation, the Euler-Bernoulli beam theory without superposition. The latter formulation also incorporates the influence of axial force on beam deformation and is expected to yield a more accurate description of the forces and deformation for the cantilever hook than the former.

![Graph showing typical results of analytical cantilever hook model for insertion, illustrating the influence of insertion angle on insertion force values. These results have been derived using the Euler-Bernoulli beam deformation formulation with superposition.](image)

Figure 3.12: Typical results of analytical cantilever hook model for insertion, illustrating the influence of insertion angle on insertion force values. These results have been derived using the Euler-Bernoulli beam deformation formulation with superposition.

The previous graphs indicate that the maximum insertion force value increases if the insertion angle, coefficient of friction or the catch offset is increased. The two beam deformation formulations qualitatively predict similar variation in the insertion force curve with changes in the different design quantities. The quantitative results from the two beam
Figure 3.13: Typical results of analytical cantilever hook model for insertion, illustrating the influence of coefficient of friction on insertion force values. These results have been derived using the Euler-Bernoulli beam deformation formulation with superposition.

Figure 3.14: Typical results of analytical cantilever hook model for insertion, illustrating the influence of catch offset on insertion force values. These results were derived using the Euler-Bernoulli beam deformation formulation with superposition.
Figure 3.15: Typical results of analytical cantilever hook model for insertion, illustrating the influence of insertion angle on insertion force values. These results were derived using the Euler-Bernoulli beam formulation without superposition and incorporate the influence of $F_a$.

Figure 3.16: Typical results of analytical cantilever hook model for insertion, illustrating the influence of coefficient of friction on insertion force values. These results were derived using the Euler-Bernoulli beam formulation without superposition and incorporate the influence of $F_a$. 
Figure 3.17: Typical results of analytical cantilever hook model for insertion, illustrating the influence of catch offset on insertion force values. These results were derived using the Euler-Bernoulli beam formulation without superposition and incorporate the influence of $F_a$.

Formulations are significantly different and this will be discussed in more detail in the following section.

(Figures 3.18 - 3.23) exhibit the effect of different input quantities on the retention force predictions of the model. Included are the effects of retention angle, catch offset and friction coefficient on retention force variation, as modeled using both beam formulations, in that order.

The maximum retention force values increase as any of the quantities, namely, retention angle, catch offset or friction coefficient are increased. This is expected from past experience and from results in previous work dealing with cantilever hook performance.

An interesting observation, however, is that the retention force curves for cantilever hooks with different catch offsets overlap, which is intuitively expected. A higher offset
Figure 3.18: Typical results of analytical cantilever hook model for retention, illustrating the influence of retention angle on retention force values. These results were derived using the Euler-Bernoulli beam formulation with superposition.

catch presents exactly the same geometry to a mating part as a catch with a lower offset with the same retention angle. The only difference is that the maximum retention force value occurs further along the retention face in the former case than in the latter. As for insertion, the predicted values for the maximum retention force are different for the two beam formulations adopted in this thesis.

Figure 3.24 graphically displays the various quantities output by the cantilever hook model script. Such plots allow for easy visualization of the phenomena during either insertion or retention.
Figure 3.19: Typical results of analytical cantilever hook model for retention, illustrating the influence of catch offset on retention force values. These results were derived using the Euler-Bernoulli beam formulation with superposition.

Figure 3.20: Typical results of analytical cantilever hook model for retention, illustrating the influence of friction coefficient on retention force values. These results were derived using the Euler-Bernoulli beam formulation with superposition.
Figure 3.21: Typical results of analytical cantilever hook model for retention, illustrating the influence of retention angle on retention force values. These results were derived using the Euler-Bernoulli beam formulation without superposition and incorporate the influence of $F_a$.

Figure 3.22: Typical results of analytical cantilever hook model for retention, illustrating the influence of catch offset on retention force values. These results were derived using the Euler-Bernoulli beam formulation without superposition and incorporate the influence of $F_a$. 

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3.5.1 Effect of Beam Formulation on Model Predictions

In this section, we shall compare model predictions for the two different beam formulations adopted in this thesis. These are (as discussed in Section 3.4) the following:

- Euler-Bernoulli beam theory with superposition.
- Euler-Bernoulli beam theory without superposition.

For convenience, we shall refer to the former as Formulation 1 and the latter as Formulation 2 in this section.

The case of insertion is compared first, the insertion force curves predicted using the two beam formulations are plotted together in Figure 3.25. The relevant catch dimensions are listed in the figure caption. For all the cases discussed here, the other cantilever hook
Figure 3.24: Typical results of analytical cantilever hook model for insertion, illustrating the variation in different quantities during the insertion process. These results were derived using the Euler-Bernoulli beam formulation with superposition and incorporate the influence of $F_a$. 
design inputs were \( L = 25.4 \) mm, \( EI = 25.95 e - 3 \) N-mm\(^2\) and \( d = 0.508 \) mm. It is clear that Formulation 2 predicts a consistently lower insertion force when contact occurs along the insertion face than Formulation 1. This can qualitatively be attributed to the fact that that the former includes the deformation due to \( F_a \) while the latter does not.

Figure 3.25: Comparison of typical insertion force curves obtained using the two different beam formulations adopted in this work. Relevant catch dimensions were \( y = 2.54 \) mm, \( \alpha = 45^\circ \), \( \beta = 87^\circ \) and \( \mu = 0.4 \).

In order to achieve a better understanding of the differences between the two approaches, it is necessary to consider the other model outputs (Figure 3.26). It is very apparent from these set of plots that the Formulation 1 consistently predicts forces of much higher magnitude as necessary for a given state of deformation than Formulation 2. This is true of \( F_a \), \( F_t \) and \( M \). Another interesting observation is the predicted variation of beam rotation (\( \theta \)) for the two cases. While Formulation 1 suggests that the beam starts to recover some of the angular rotation as the contact point moves up along the insertion face (around \( R_{x_C} = 1 \) mm
approximately), Formulation 2 exhibits no such behavior. The former prediction is contrary to common experimental observations, especially for low offset catch geometries, as is the case here. Such a reversal in the angle of rotation of the end of the beam has been experimentally observed only for cantilever hooks with very high offset values \((y = 7.112\text{mm})\), not at such low levels \((y = 2.54\text{mm})\).

A similar comparison of retention force curves is undertaken next. For retention, however, two separate catch geometries are compared. The first is a low retention angle cantilever hook \((\beta = 60^\circ)\) for which the axial force term is not expected to have a significant contribution. The second is a cantilever hook design with a high retention angle \((\beta = 89^\circ)\). The retention force curves for the two cases are shown in Figures 3.27 and 3.28 respectively. As expected, Formulation 1 predicts higher force magnitudes in comparison to Formulation 2. The magnitude of the discrepancy is lower for the former design \((\beta = 60^\circ)\) than for the latter \((\beta = 89^\circ)\). This is because \(F_a\) has a much larger contribution to the beam deformation for the high angle catch than for the low angle catch geometry, as can be seen from a consideration of catch equilibrium for both cases.

We again revert to plotting several output variables simultaneously to obtain additional insight into the retention phenomena. The matrix of variable plots similar to the one used for insertion is shown in Figure 3.29. The data simply reiterates that beam deformation formulation that utilizes the superposition assumption and ignores the effect of \(F_a\) on beam deflection overpredicts all the force and moment components. The axial retention force also appears to be tending towards an asymptotic value for Formulation 2. Earlier work dealing with the design of cantilever hooks has asserted that increasing catch offset aids retention only to a certain extent, beyond which it has little or no effect on maximum retention force of the feature. The observation that \(F_a\) tends towards an asymptotic value would lend some
Figure 3.26: Comparison of different output variables for insertion obtained using the two different beam formulations. Relevant catch dimensions were $y = 2.54\text{mm}$, $\alpha = 45^\circ$, $\beta = 87^\circ$ and $\mu = 0.4$. 

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Figure 3.27: Comparison of typical retention force curves for a cantilever hook with a low retention angle catch obtained using the two different beam formulations adopted in this work. Relevant catch dimensions were $y = 1.524\text{mm}$, $\alpha = 45^\circ$, $\beta = 60^\circ$ and $\mu = 0.1$.

Figure 3.28: Comparison of typical retention force curves for a cantilever hook with a high retention angle catch obtained using the two different beam formulations adopted in this work. Relevant catch dimensions were $y = 1.524\text{mm}$, $\alpha = 45^\circ$, $\beta = 89^\circ$ and $\mu = 0.1$. 
analytical justification to these design rules. No significant differences in the other script variables is observed in this case.

In the following section, we compare the insertion and retention force predictions obtained using the developed analytical model to experimental data from two cantilever hook testing efforts.
Figure 3.29: Comparison of different output variables for retention obtained using the two different beam formulations. Relevant catch dimensions were $y = 2.54\text{mm}, \alpha = 45^\circ, \beta = 89^\circ$ and $\mu = 0.1$. 
3.5.2 Comparison of Model Predictions to Experimental Cantilever Hook Test Results

The following graphs (Figures 3.30 - 3.35) evaluate the predictive capability and accuracy of the model developed in this effort by comparing the predicted insertion and retention force curves to physical test data obtained by experiments conducted as part of this thesis and by some previous researchers [75]. The sources of the presented results, respectively, are:

1. Experimental data from previous work by Rusli [75]. In these experiments, a small block of the same material as the cantilever hook was mounted on a fixture to simulate the mating part edge.

2. Experimental data presented later in this thesis. For these experiments, the mating part is a steel channel section, hence the contact conditions are different from the previous case.

3. Predictions of the analytical model developed as part of this work. These results utilize the Euler-Bernoulli beam deformation formulation with superposition, also referred to as Formulation 1 in the previous section, and

4. Predictions of the analytical model developed as part of this work. These results utilize the Euler-Bernoulli beam deformation formulation without superposition, also referred to as Formulation 2 in the previous section.

Comparisons are made for three cantilever hook designs with different catch geometries. The differences between the three designs are limited to catch dimensions, which are listed below, for reference purposes.
1. $y = 1.524 \text{mm}$, $\alpha = 25^\circ$, $\beta = 55^\circ$

2. $y = 2.54 \text{mm}$, $\alpha = 45^\circ$, $\beta = 90^\circ$

3. $y = 7.112 \text{mm}$, $\alpha = 45^\circ$, $\beta = 90^\circ$

Figures 3.30 - 3.32 present the insertion force data from various sources for the different cantilever hook geometries. For the first two cases, the predicted insertion force curve matches the shape of the experimentally obtained data very well. However, there is quite a significant difference between the location (x-displacement) at which contact shifts from the insertion to dwell face and the maximum insertion force. The model predictions lie between the two experimental values. The two experimental data sets represent experiments on similar cantilever hook samples, the only difference being the mating part. In one case, a small piece of the same polymer mounted on a steel fixture is used as the mating part, while in the other case, a steel channel section is used. The considerable difference between the two experiments can possibly be attributed to differences in the condition of the mating edge. The steel mating edge in the latter experiment is much sharper than the former polymer edge, which can affect the transition between faces. The friction condition between the two surfaces in contact also influences the shape and limit points of the insertion force curve. Similar observations and inferences can be made for the other phases of the insertion force curve.

The third cantilever hook geometry represents an extreme in the design space, having a very high offset ($y = 7.112 \text{mm}$) compared to the beam thickness ($t = 2.54 \text{mm}$). During the insertion experiment, the cantilever beam first bends back, as in the previous cases. However, as contact progresses along the insertion face, the end of the beam eventually begins to deform back towards the neutral axis, resulting in an S-shaped structure. Once
Figure 3.30: Comparison of model insertion force predictions to experimental test results for a low retention angle, low catch offset cantilever hook. Relevant catch dimensions and parameters were $y = 1.524\text{mm}, \alpha = 25^\circ, \beta = 55^\circ$ and $\mu = 0.1$.

Figure 3.31: Comparison of model insertion force predictions to experimental test results for a high retention angle, low catch offset cantilever hook. Relevant catch dimensions and parameters were $y = 2.54\text{mm}, \alpha = 45^\circ, \beta = 90^\circ$ and $\mu = 0.1$. 
again, there is considerable difference between data from the two experiments. The analytical model curves capture the initial curvature of the insertion force curve with reasonable accuracy. However, the two sets of data (model predictions and experimental) begin to diverge beyond a certain stage. Analytical Formulation 1 grossly overpredicts the maximum insertion force, compared to experimental data. The shape of the dwell region and snap-back portions of the insertion force curve show better agreement with experimental data.

Analytical Formulation 2 models the initial slope of the insertion force curve much better than the previous model. However, it fails to converge (within the specified number of increments) after the contact point progresses approximately three quarters of the distance to the tip of the insertion face. An inspection of the Jacobian matrix for the equation system reveals it to be nearly singular, which could be a possible reason for the difficulty in convergence.

The force curves for retention (two experimental data sets and two analytical model predictions) are shown in Figures 3.33 - 3.35 for the three geometries respectively. These curves show an agreement/disagreement pattern similar to the insertion force values. The analytical models predict retention force variation fairly well for low angle retention (Geometry 1). It is important to note the large difference in the two experimental retention force curves for this geometry. This points towards to the significance of surface phenomena (friction, wear) in retention. For Geometry 2, there is very little similarity between the two experimental curves or between the experimental data and model predictions.

Geometry 3 represents an even more extreme case, a high retention angle, coupled with a large offset. The failure mode of the snap-fit for the two experimental data sets is different. While the hook fails by tensile breakage in the experiments conducted elsewhere
Figure 3.32: Comparison of model insertion force predictions to experimental test results for a high retention angle, high catch offset cantilever hook. Relevant catch dimensions and parameters were $y = 7.112$ mm, $\alpha = 45^\circ$, $\beta = 90^\circ$ and $\mu = 0.1$.

In this thesis, Rusli’s samples undergo a loss-of-engagement failure. The initial slope of the experimental curves is very similar, however, the former displays a very smooth peak corresponding to the formation of a craze followed by beam fracture. The latter experimental curve exhibits more of a stick-slip phenomena as contact proceeds along the insertion face. The analytical model captures the initial slope fairly well. However, the models predict that the retention force tends towards an asymptotic value, which is quite different from the experimental data. There can be several reasons for the discrepancies between the experimental data and model predictions, which are discussed in detail next.
Figure 3.33: Comparison of model retention force predictions to experimental test results for a low retention angle, low catch offset cantilever hook. Relevant catch dimensions and parameters were $y = 1.524\,\text{mm}$, $\alpha = 25^\circ$, $\beta = 55^\circ$ and $\mu = 0.1$.

Figure 3.34: Comparison of model retention force predictions to experimental test results for a high retention angle, low catch offset cantilever hook. Relevant catch dimensions and parameters were $y = 2.54\,\text{mm}$, $\alpha = 45^\circ$, $\beta = 90^\circ$ and $\mu = 0.1$. 
Figure 3.35: Comparison of model retention force predictions to experimental test results for a high retention angle, high catch offset cantilever hook. Relevant catch dimensions and parameters were $y = 7.112\text{mm}$, $\alpha = 45^\circ$, $\beta = 90^\circ$ and $\mu = 0.1$.

### 3.6 Model Limitations and Possible Improvements

Section 3.5 provide us with a fair idea of the capabilities and limitations of the analytical model. While the predictions of insertion and retention force variation are quite accurate for low offset, low insertion and retention angle cases, the model is not as effective in modeling more extreme cases of insertion and retention. The test cases used in thesis were limited by the samples available from a past experimental effort by Rusli [75]. From the available samples, geometries were chosen to provide greatest representation of the overall design space for cantilever hooks. The influence of different design parameters on insertion and retention phenomena and the improvement in prediction accuracy by incorporating the influence of $F_a$ has been presented and discussed. This provides valuable insight into the
relative influence of physical phenomena governing cantilever hook performance and some
guiding principles for future work in this area.

The most important conclusion that can be drawn from all the different results presented
in this work is that surface condition and contact phenomena critically control cantilever
hook performance both during insertion as well as retention. The excellent agreement be-
tween the initial shapes of insertion and retention force curves in most cases suggests that
the model beam deformation formulation is fairly accurate. The fact that low offset and
low angle catches are modeled reliably indicates that the beam formulation is accurate for
small deformation problems. However, the discrepancies between maximum force values
suggests that contact phenomena are not very effectively modeled in this effort. The model
utilizes a relatively simple friction model, namely, Coulomb’s model ($F_\mu = \mu F_n$), which
is essential to formulating a determinate set of equations. It is well known, that in realistic
contact phenomena, the friction force varies from zero to a maximum value (determined by
the coefficient of static friction). Once motion between the two parts is initiated the value
of $\mu$ drops to a typically lower, dynamic friction coefficient value. As long as there is no
relative motion between the two parts in contact, the friction force remains indeterminate,
assuming whatever value is necessary to maintain equilibrium. Utilizing the Coulomb’s
model of friction in our work implies the assumption that the cantilever hook and mating
part are always in sliding contact. While such an assumption is acceptable for some can-
tilever hook designs, it places limitations on model accuracy in other cases, such as high
offset, high retention angle designs. For high offset, high retention angle snap-fits, it has
been observed in experiments that significant forces are developed in the system without
any appreciable motion between the two parts. In such cases, adopting the current friction
model has the effect of introducing a fictitious force in the system (almost like a “force generator, $F_\mu = F_n$”), the friction force. One of the problems with adopting a more advanced friction model is the elimination of one equation from the equation system developed in Section 3.4 while maintaining the number of unknowns. An additional equation would then be required to make the system determinate. Such an equation could possibly relate $\mu$ to some relative sliding velocity, akin to the approach used to model contact in finite element analysis. Another limitation in the current friction model is that contact in polymers is more sensitive to contact area and pressure, which is not captured by the current approach. Summarizing the ideas presented in this paragraph, contact phenomena are not realistically modeled by the current friction model or contact modeling approach. Improvements could be achieved by adopting an analytical contact formulation similar to that used in finite element analysis. Stick slip phenomena have been clearly observed in retention experiments on cantilever hooks and should be incorporated into the current model.

The discussion above points out what is deemed to be the single most important enhancement to the model. However, incremental gains in accuracy can be achieved by considering other aspects of the model. Currently, the material parameters that have been used in this work are typical values available in literature. Experiments to determine more accurate values of Young’s modulus and friction coefficient could provide data more appropriate for the current model. Previous work by Luscher [6] has suggested that the values of $\mu$ suggested in resin suppliers’ guides are not appropriate for use in snap-fit design. The reason that is cited that these values are obtained using experiments involving surface to surface contact, while data for edge on surface contact is more appropriate for use in snap-fit design. Polymers are well known to exhibit non-linear stress strain relationships - the material parameter used in this work models a linear isotropic elastic material. This results
in a stiffer than actual material at higher strain (deformation values). The use of a tangent modulus, secant modulus or a piecewise linear stress-strain relationship would be a more appropriate choice.

Deformation due to transverse shear is very dependent on the value of Poisson’s ratio ($\nu$). Polymers have high $\nu$ values and transverse shear is expected to have a significant contribution in the overall deformation of the beam. This effect is not modeled in the current work. Incorporation of axial deformation in addition to deformation due to transverse shear would also enhance the accuracy of the proposed model. The last geometry (Geometry 3) has a catch offset approximately three times the beam thickness. The small deformation assumption inherent in the beam formulations is no longer valid (catch offset controls the deflection of the end of the beam). A large deformation beam formulation would be more accurate in such cases. A possible enhancement to the model could be achieved by incorporating the ability to model tapered beams, which is not included in the current equation system. Such beams are commonly used in snap-fit designs and recommended by several of the resin manufacturer’s snap-fit design guides.

A more comprehensive validation approach would involve the physical testing of cantilever hook samples on a larger and more organized scale. This would allow us to identify the origin of deviation of model predictions from the experimental results. This can have a two-fold benefit - first, it would allow the definition of regimes of applicability of the model and second, provide greater insight into the causes of discrepancies.

The next chapter describes detailed experimental investigation of retention in snap-fits. Tests with steel, aluminum and polymeric specimens are conducted to establish a fundamental understanding of physical phenomena governing retention.
CHAPTER 4

EXPERIMENTAL INVESTIGATION OF INSERTION AND RETENTION PHENOMENA IN CANTILEVER HOOK FEATURES

This chapter describes the physical testing of cantilever hook features conducted to better understand the governing phenomena, especially in retention, for such snap-fits. The primary goal of this experimental effort was to identify the physical phenomena controlling the retention of high-angle cantilever hooks, and to compare and contrast them with low retention angle features. One of the anticipated outcomes of this work was an understanding of the reasons for so-called “permanent” cantilever hook snap-fits disengaging without failure, contrary to design equation predictions. This effect is also known as loss-of-engagement type failure. Another objective was to determine if there exists a “bifurcation” in the performance or failure mode of the snap-fit, and if so, discover its cause. This could later be incorporated into analytical models developed elsewhere in this thesis.

Fixture development, experimentation issues, test results and conclusions are discussed in this chapter.
4.1 Development of Fixture for Precise Positioning of Test Features

Rather than using an off-the-shelf fixture available from catalogs or reusing old designs, it was decided to develop a simple fixturing scheme that would eliminate much of the unnecessary features of standard fixtures, yet allow much more accurate alignment.

The X-Y table is mounted on the INSTRON T-slot table. The former has been fitted with a digital readout along one axis, which allows for accurate measurement of absolute or incremental displacement. It also eliminates the influence of backlash in the lead screw since it directly measures displacement of the top (movable) half of the table relative to the bottom (fixed) half. The displacement along that direction is no longer measured from the circular scale on the controlling crank. In order to mount the snap-fit on the X-Y table, a “breadboard” approach has been used. An array of alternate rows of threaded and straight thru holes has been machined on a stainless steel plate. This plate is bolted down on the X-Y table by four corner bolts and two bolts near its center. Correct positioning and alignment of the plate relative to the X-Y table is ensured by dowel pins. The steel plate has a 3/8” hole in the center. This is used to align the center of the plate with the main axis of the machine. A dowel pin is mounted in the actuator shaft and inserted into this hole. The position of the X-Y table is adjusted until the deflection of the dowel pin as it is inserted into the hole is reduced to a minimum.

A set of custom manufactured grip blocks can be mounted on the X-Y table. The blocks fixture the cantilever hook so that its face is perpendicular to one of the direction of motions of the X-Y table. In addition, the central axis of the INSTRON machine passes through the face of the cantilever hook beam. The above setup ensure the the gross linear alignment of the cantilever hook with the mating part. The steel grip blocks mentioned above are bolted
to each other to clamp the specimen. In the absence of any additional features, the gripping force would be developed only due to friction between the cantilever hook beam and the grip faces. In order to establish a “positive” grip, dowel pins that pass through holes in the grip blocks and cantilever beam ends are used.

The mating part is mounted on a threaded adapter. The other end of this adapter threads directly into the load cell mounted on the actuator shaft. It is necessary to ensure that the cantilever hook and mating part are parallel to each other. Angular alignment is achieved by using a dial indicator mounted on one of the grip blocks. The other end (indicator stylus) touches the mating part face. The angular position of the X-Y table is adjusted until the dial indicator readout is constant as the stylus is moved along the width of the cantilever hook face. This ensures that the mating part edge is parallel to the cantilever hook face. For more precise adjustment of the linear position, the X-Y table is moved perpendicular to the cantilever hook face until the two parts just touch. This is defined as the origin on the digital readout for future reference.

The assembly and disassembly of snap-fit features are very short duration events, almost instantaneous, events. A high speed video camera (capable of frame capture rates of 500 fps) was used to record the experiments as image sequences, which were then linked together to form a movie at much slower frame rates. This provides greater visual insight into retention. The large amount of data to be transferred and stored at such high frame rates limits the size of the images and duration of event that can be recorded. In fact, there is a direct trade-off between frame rate, event duration that can be recorded and image pixel size. For our experiments, a frame rate of 250 fps was used that allows an image size of
420X410 pixels to be captured. At this rate, a two second event could be recorded. Additional insight into the events was obtained by correlating the image sequence to the force displacement curve obtained from the INSTRON.

For insertion, the mating part mounted on the actuator shaft is moved downwards at a steady speed of 1mm/s. It is moved in the opposite direction for retention. The motion of the actuator is controlled via the WAVEMAKER software provided for control and data acquisition purposes. Force, time and position data is obtained as a comma delimited values text file (.csv) from the INSTRON data acquisition system. A number of different samples were tested. Machined steel and aluminum samples and injection molded ABS samples (available from a previous experimental effort, [75]) were tested. The mating part is machined out of steel.

4.2 Experimental Results

A number of different sets of samples were tested. Three to five samples of each kind were tested to estimate repeatability of the tests. As mentioned before, machined aluminum and steel and injection molded ABS samples were tested. The reason for using metallic snap-fits (even though metallic parts rarely use snap-fits) was that friction coefficients for metallic surfaces are known with much greater confidence than for plastics. Metals are also less susceptible to rate and time-dependent effects than plastics. The goal of testing metallic snap-fits was to isolate geometrical effects in the system, by considerably simplifying the friction and material effects on retention. However, sizing of metal snap-fits is more difficult because the maximum allowable strain is typically lower than for plastics, limiting feasible designs to lower offset values and/or longer beam lengths.
4.2.1 Test Results for Aluminum Samples

Only one particular snap-fit geometry were tested (Figure C.1). Two sets of four samples each were used, and each sample was tested four times to estimate the repeatability and reproducibility of the test procedure. Typical results obtained from the INSTRON machine are shown in Figure 4.1 and 4.2 for insertion and retention respectively. The shape of these curves is characteristic of the nature of the insertion and retention phenomena.

![Graph showing typical insertion force curve for cantilever hook snap-fits](image)

Figure 4.1: Typical insertion force curve for cantilever hook snap-fits

Multiple tests were run on each sample to determine the repeatability. The insertion test results for multiple runs on a single sample for insertion are shown in Figure 4.3 and those for retention are shown in Figure 4.4. The change in the shape of the insertion force curves and the peak insertion force values is much lower than that for retention. This seems to
Figure 4.2: Typical retention force curve for cantilever hook snap-fits

indicate that insertion does not have much effect on the surface condition of the cantilever hook or mating part. During retention, however, one can clearly observe the sharp edge of the mating part (steel) abrading the surface of the retention face. There is, thus, a definite change in the surface condition of the cantilever hook during retention. This has the effect of the maximum retention force being progressively lower in successive runs. The biggest drop, however, occurs between the first and second runs. Subsequent decreases in the force values are much lower.

The maximum insertion force and maximum retention force data is for all the samples and runs is summarized in Figures 4.5 and 4.6 respectively. Figure 4.5 shows the maximum insertion force values for different samples and multiple tests on each sample. No specific trend is observed for successive tests on the same sample. Samples 1-4 (Set I)
Figure 4.3: Insertion force curves for repeat tests on a particular machined aluminum sample.

Figure 4.4: Retention force curves for repeat tests on particular machined aluminum sample.
Figure 4.5: Maximum insertion force values for several identical machined aluminum samples and multiple tests on each sample.

Figure 4.6: Maximum retention force values for several identical machined aluminum samples and multiple tests on each sample.
were machined from one piece and Samples 5-8 (Set II) from another piece. The average values of insertion force for the two sets is distinctly different, that for the second set being lower than the first. This suggests some variation in machined dimensions or material properties (friction, elastic modulus) between the two sets. There is also some variation in the force values within each set. Summarizing, we observe some part-to-part and set-to-set variation in the maximum insertion force values. There is also a change in the maximum insertion force values for successive tests on the same sample, however, no specific trend is observed and this variation is much smaller than either the part-to-part or the set-to-set variation. Making any statistical inferences about the cause of such variation would require more samples to be tested in a planned manner, including effects such as blocking and randomization.

The maximum retention force values shown in Figure 4.6 show similar trends. There is quite a large variation in the force values for different samples, when corresponding runs are compared. This is especially true for the first couple of runs, when the maximum abrasion of the cantilever hook face occurs. This “wear” effect reduces with later runs and the part-to-part variation is much lower for Runs 3, 4 and 5. Unlike insertion, there is a definite decrease in the retention force values with successive tests. The retention force values (∼500N) are typically much larger than the insertion force values (∼30N), hence the wear during retention is much larger than during insertion. The wear causes a reduction in the friction coefficient, and hence, the retention force. From the above discussion, one can conclude that the surface properties play an important role in determining the retention force for the snap-fit.

Another set of experiments was conducted after sandblasting the surface of the cantilever hooks and machining the mating part to recreate a sharp contact edge. The purpose
of sandblasting was to recreate a uniform high friction contact surface on all the samples and then determine if the maximum retention force values reverted to their original high values. The maximum retention force values for these samples are presented in Table 4.1. Sandblasting the contact surfaces does have the desired effect of “rejuvenating” the sample. The retention force values are considerably higher than the previous uses of the same samples. Even in this case, the retention force values for the second test one each sample are much lower than those for the first test. These observations reinforce the assertion that the phenomena of retention in snap-fits is very sensitive to the surface conditions of the mating parts and the friction characteristics of the mating materials. Additionally, a steel mating part when sliding along an aluminum cantilever hook abrades the latter’s surface, resulting in a change of the tribological characteristics and a reduction in the retention force. Insertion force data has not been presented here because no meaningful conclusions can be drawn from the maximum insertion force values.

<table>
<thead>
<tr>
<th>Sample #</th>
<th>Max. Retention Force (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Test 1</td>
</tr>
<tr>
<td>1</td>
<td>2289.1</td>
</tr>
<tr>
<td>2</td>
<td>1342.7</td>
</tr>
<tr>
<td>3</td>
<td>1988.5</td>
</tr>
<tr>
<td>4</td>
<td>1938.7</td>
</tr>
<tr>
<td>5</td>
<td>1618.3</td>
</tr>
<tr>
<td>6</td>
<td>1873.5</td>
</tr>
<tr>
<td>7</td>
<td>2112.2</td>
</tr>
</tbody>
</table>

Table 4.1: Maximum retention force values for seven sandblasted aluminum cantilever hook samples.
4.2.2 Test Results for Steel Samples

While testing aluminum cantilever hooks against steel mating parts, it was observed that the sharp contact edge of the mating part would shave off material from the snap-fit face, often producing chip-like shavings. This effect would diminish with successive tests on the same sample, and the retention force in the first try would be much higher than the later tests. This has been discussed in the previous section. Steel samples were tested next, in an attempt to reduce the influence of these wear-related effects on the test results. Two geometries were used, and three samples of each geometry were tested. The only difference between the two geometries was the retention angle ($\beta$). The first geometry had a retention angle of 90° and the latter 93°. Five tests were successively run on each sample. The shape of the force curves were similar to the ones shown earlier in Figure 4.1 and Figure 4.4. For the 90° geometry, the maximum insertion force values are shown graphically in Figure 4.7 and the maximum retention force values are in Figure 4.8. The results are very similar in trend to those for aluminum samples. The insertion force values show a definite decrease in value with successive tests (reduction in absolute value), the biggest drop occurring between the first and second use tests. Similar observations can be made about the retention force. Overall, the consistency in the results from different samples is higher than that for aluminum samples. The maximum insertion and retention force values for the first test on the samples are tabulated in Table 4.2 and Table 4.3 respectively.

The difference in insertion force and retention force values for the two different geometries tested does not appear to be significant. The 93° samples were tested to investigate whether the common design practice of providing a relief on the retention face results in an improvement in performance of the cantilever hook. When the shape of the retention force
Table 4.2: Maximum insertion force values for the first test on two machined steel cantilever hook geometries tested.

<table>
<thead>
<tr>
<th>Sample #</th>
<th>Max. Insertion Force (N)</th>
<th>Retention Angle = 90°</th>
<th>Retention Angle = 93°</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-9.3</td>
<td>-19.2</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-11.6</td>
<td>-10.0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-14.7</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.3: Maximum retention force values for the two machined steel cantilever hook geometries tested.

<table>
<thead>
<tr>
<th>Sample #</th>
<th>Max. Retention Force (N)</th>
<th>Retention Angle = 90°</th>
<th>Retention Angle = 93°</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1277.9</td>
<td>1412.7</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1102.6</td>
<td>1189.2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1073.6</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>
curves is compared for the two geometries, no significant differences are observed which would point towards a fundamentally different retention mode in the two cases.

![Graph showing maximum insertion force values for several identical machined steel samples (Retention Angle = 90°) and multiple tests on each sample.](image)

Figure 4.7: Maximum insertion force values for several identical machined steel samples (Retention Angle = 90°) and multiple tests on each sample.
Figure 4.8: Maximum retention force values for several identical machined steel samples (Retention Angle = 90°) and multiple tests on each sample.

4.2.3 Test Results for ABS Samples

Five different cantilever hook geometries available from a previous effort by Rusli [75] were tested. The detailed dimensions of the geometry are shown in Appendix C. All the geometries are identical except for the catch dimensions and the depth taper on the beam. The depth taper is defined as the ratio $w_t/w_b$ (Figure 4.9) and is expressed in percentage. The differences are highlighted in Figure 4.9 and Table 4.4. In particular, geometries 1 and 4 represent low retention angle (retention angle $< \text{critical angle}$) snap-fit features. The difference between the two is that the cantilever beam has a 2:1 depth taper on the latter but not on the former. Similarly, the pair of Geometries 2 and 5 are cantilever hooks with high retention angle (retention angle $= 90^\circ$), but the latter has a depth taper on the beam, as in
the previous pair. Geometry 3 has a high offset, high retention angle catch (retention angle = 90°). These geometries represent different regimes of the design space for cantilever hooks fairly well. More specifically, these include low offset, high offset, low retention angle (semi-permanent) and high retention angle (permanent) cantilever hooks. We were limited to using samples already available in the lab, because molding additional geometries and creating the mold inserts required for the same for was beyond the scope of this work.

![Diagram of cantilever hook nomenclature](image)

Figure 4.9: Basic cantilever hook nomenclature.

Five samples of each geometry were tested to capture the reproducibility of the data. A set of insertion and retention force data for each geometry was obtained. The maximum insertion and retention force values for each of the geometries are tabulated in Tables 4.5 and Table 4.6 respectively. It is important to note here that not only are the maximum insertion and retention force values relevant to understanding the performance of snap-fit features, the shape and form of the insertion and retention force curves are relevant too.
Figure 4.10: Insertion force curves for injection molded ABS cantilever hooks with Geometry 1.

Figure 4.11: Insertion force curves for injection molded ABS cantilever hook samples, Geometry 3.
<table>
<thead>
<tr>
<th>Geometry</th>
<th>Depth taper ( % )</th>
<th>Offset (mm)</th>
<th>Insertion angle $\alpha$ (°)</th>
<th>Retention Angle $\beta$ (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>1.524</td>
<td>25</td>
<td>55</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>1.524</td>
<td>45</td>
<td>90</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>7.112</td>
<td>25</td>
<td>55</td>
</tr>
<tr>
<td>4</td>
<td>50</td>
<td>1.524</td>
<td>25</td>
<td>55</td>
</tr>
<tr>
<td>5</td>
<td>50</td>
<td>1.524</td>
<td>45</td>
<td>90</td>
</tr>
</tbody>
</table>

Table 4.4: Dimensional differences between the five injection molded ABS sample geometries tested.

<table>
<thead>
<tr>
<th>Sample #</th>
<th>Max. Insertion Force (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Geometry 1</td>
</tr>
<tr>
<td>1</td>
<td>-5.4</td>
</tr>
<tr>
<td>2</td>
<td>-4.9</td>
</tr>
<tr>
<td>3</td>
<td>-5.5</td>
</tr>
<tr>
<td>4</td>
<td>-5.4</td>
</tr>
<tr>
<td>5</td>
<td>-5.0</td>
</tr>
</tbody>
</table>

Table 4.5: Summary of maximum insertion force values for injection molded ABS cantilever hooks, five samples were tested for each geometry.

In fact, it can be argued that the latter provide greater insight into the physical phenomena governing cantilever hook performance than the former. Figure 4.10 and Figure 4.12 show typical insertion and retention force curves respectively for the five different Geometry 1 samples tested. These are quite similar in shape to the force curves from the steel and aluminum samples and are characteristic of cantilever hook type snap-fits. The curves for
<table>
<thead>
<tr>
<th>Sample #</th>
<th>Max. Retention Force (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Geometry 1</td>
</tr>
<tr>
<td>1</td>
<td>34.4</td>
</tr>
<tr>
<td>2</td>
<td>38.0</td>
</tr>
<tr>
<td>3</td>
<td>34.2</td>
</tr>
<tr>
<td>4</td>
<td>35.8</td>
</tr>
<tr>
<td>5</td>
<td>41.7</td>
</tr>
</tbody>
</table>

Table 4.6: Summary of maximum retention force values for injection molded ABS cantilever hooks, five samples were tested for each geometry.

Geometries 2, 4 and 5 are very similar in nature and are not reproduced here. However, the insertion and retention behavior of the high offset, high retention angle geometry (Geometry 3) are quite different and are shown in Figure 4.11 (insertion) and Figure 4.13 (retention). Comparing the insertion curves for Geometry 1 (Figure 4.10) and Geometry 3 (Figure 4.11), one can clearly observe that process insertion for the latter occurs over a longer displacement of the mating part than the former. This is fairly obvious if we compare the two geometries. The latter has a much higher offset, resulting in a larger catch and a longer insertion face. It can also be seen from the graphs that the insertion force curve for Geometry 3 displays much more non-linearity than that for Geometry 1. The larger offset (7.112 mm) on the former implies that the beam has to deform much more than for the geometry with the smaller catch offset (1.524 mm).

The retention force curves for the two geometries are compared next. These are shown in Figures 4.12 and 4.13 for Geometries 1 and 3 respectively. Once again, the magnitude of the retention force for Geometry 3 is much higher than that for Geometry 1, ostensibly
Figure 4.12: Retention force curve for injection molded ABS cantilever hooks with Geometry 1.

due to the higher offset. However, it is important to point out here that Geometry 3 disassembles by material failure at the junction between the beam and catch, while Geometry 1 cantilever hook disengages without any apparent material failure, a “loss-of-engagement” type failure. Also, the retention angle for Geometry 3 is $90^\circ$ while that for Geometry 1 is $55^\circ$.

Another relevant observation is that the footprint of the retention force curve on the displacement axis for Geometry 3 is much larger than that for Geometry 1. The longer duration of the upward portion of the retention curve of the former can be attributed to the higher offset value, as previously. This results in the mating part having to displace a larger amount before it completely disengages, if at all. The gentler downward slope of the retention force curve for Geometry 3 compared to Geometry 1 is due to the fact that
the former occurs due to the catch being gradually separated from the beam as the material fails. This is in contrast to the former case, it is actually the mating part that disengages resulting in the beam recovering its deformation almost instantaneously.

The abovementioned effects become more apparent when the force displacement curves are correlated to the pictures of the deformed configuration of the beam and mating part recorded using the high speed video equipment.

4.3 High Speed Photography of Cantilever Hook Tests and Correlation to Force Curves

Retention in snap-fit features is a very short duration event and as such, detailed observation of the occurring phenomena is not possible without the aid of some recording
device. A normal video camera is capable of recording events at a rate of 30 fps. However, video of retention events recorded using such devices did not provide any additional insight over the naked eye. A high speed CCD camera capable of recording upto 500 fps was used to record the experiments to achieve a greater understanding of the physical nature of retention. For all of our experiments, a fixed frame rate of 250 fps was used. There is a tradeoff between frame rate, image size and duration of event that can be captured, and this frame rate (250 fps) was high enough to capture events of interest during retention. At this recording rate, an event of approximately 2 s duration could be captured at an image size of 420 X 410 pixels.

The output from the camera is in the form of individual still frames, a total of 512 frames for each test. These frames are images in .tif format and can be combined to form a movie using commonly available commercial software (Quicktime Pro). The camera is controlled using a personal computer and custom software supplied by the manufacturer. The images supplied by the CCD camera are stored in a cyclic buffer, hence only the 512 most recent images are stored. The actual time interval that is recorded thus controlled by the instant when the “stop recording” button is hit on the software - the previous 512 frames are then available for viewing or storage. The camera can also be triggered automatically and has a number of trigger modes. However, triggering for our experiments was performed manually, since the goal was to record the entire retention process.

In order to achieve a fundamental understanding of the governing phenomena during retention, an attempt to correlate the retention force curve to the photographic information gathered during the experiment. For this purpose, it was necessary to accurately identify a common time instant on the series of photographs (from the CCD camera) and the force data (from the INSTRON data acquisition system). We will refer to this as the time origin.
Once this was accomplished, the image frames were related to data points on the force-displacement curve by tracking the elapsed time from the time origin. The identification of the time origin can be tricky. Two alternate approaches were used. The first approach was to identify the first instant where deformation of the cantilever was observed in the photographs as the first non-zero force point on the INSTRON data. Alternately, the last point on the retention force curve peak was assumed to be the transition between the retention and dwell face, corresponding to the “snap-back” of the cantilever hook. By using these two methods, the visual deformation evidence could be related to the force data with a fair degree of uncertainty. However, since the rate of image capture was different from the rate of INSTRON data acquisition, some margin of error was unavoidable in this process. The estimated error in terms of the number of frames is included in the caption for each figure.

Figure 4.14 shows the correlation between geometric deformation and variation of retention force for injection molded ABS samples with Geometry 1. Its relevant catch characteristics are a low retention angle and a low offset value. The initial increase in retention force with upward motion of the mating part is quite typical. It occurs due an increase in the lateral deformation of the beam. The increase continues until the limit of static friction is reached, beyond which the mating part edge slips along the retention face and there is a sudden drop in the force value. The stick-slip event occurs again corresponding to the second peak of the retention force-displacement curve. The surprising observation in this case is that the final maxima in the retention force curve does not occur at the transition of the contact edge between the retention and dwell faces. Typically, one would expect that the retention force to increase sharply as the mating edge rides up the retention face. Once the end of the face is reached, the direction of the forces at the contact edge change. The angle between the normal force at the point of contact and retention direction increases and
the majority of the retention force is primarily due to friction, as opposed to normal forces in the previous case. It is reasonable to thus expect a sharp drop in retention force as the contact point transitions from the retention to the dwell face. Thereafter, only a small or no decrease in the force would be expected, corresponding to the increase in effective beam length.

In this case, however, it is clearly seen that the maximum retention force does not occur at the transition discussed above. Instead, it appears to happen at some location on the dwell face. This is very surprising observation and requires further explanation. The geometry of the catch plays an important role here. The low retention angle and low offset of the catch ensures that there are no sudden transitions in the deformation of the cantilever hook. This fact, coupled with the viscoelastic nature of ABS, can explain the apparent lag in the force values with respect to the deformation of the cantilever hook. Since there is no instantaneous recovery of deformation (which is typical of high retention angle snap-fits, and since the retention face ramp is rather gradual) the viscous portion of the material response dominates over the elastic response. A stick-slip type contact phenomena is, however, observed between the mating part edge and retention face.

Figure 4.15 displays the results of a test on another geometry of the injection molded ABS samples (Geometry 2). This is a low offset, high retention angle cantilever hook snap-fit. The force variation observed here is similar to the previous case. However, the detailed shape of the curve deserves some attention. For such a snap-fit, once contact between the mating edge and retention face is established, further upward motion of the mating part does not cause the cantilever hook beam to bend backwards. Initially, the beam stretches a small amount, since the force is primarily axial. As the axial force increases, the overturning moment acting on the beam becomes larger, causing the beam to bend
Figure 4.14: Correlation of shape of retention force-displacement curve to high speed photographs (250 fps) for injection molded ABS sample with Geometry 1. Numbers in pictures indicate frame number.
Figure 4.15: Correlation of shape of retention force-displacement curve to high speed photographs (250 fps) for injection molded ABS sample with Geometry 2. Numbers in pictures indicate frame number.
backwards. The forces increase without any relative motion between the two parts until the mating parts slips. This stick-slip phenomena repeats itself, resulting in 2-3 peaks until the final maximum is reached at the top edge of the retention face. Subsequently, there is a sharp drop in the retention force. This is due to the change in direction of contact forces corresponding to transition of contact from the retention to dwell face.

Figure 4.16 shows the same principles applied to an ABS specimen with Geometry 4. Noteworthy here is the fact that disengagement here occurs due to material failure (fracture at the junction of the beam and catch). The maximum force corresponds to initiation of the fracture beyond which progressively lesser force is required to pull the mating part. The duration of the retention force peak is longer than two seconds, hence the time origin could not be located accurately on the insertion force curve. The frames are placed on the retention force graph to provide a qualitative idea of the failure mode.

The next two figures (Figures 4.17 and 4.18) illustrate the deformation-force correlation for successive tests on a sandblasted aluminum sample, results for which have been presented earlier. The initial shape and nature of the curve is similar to that in the previous case. However, the maximum value of retention force is much larger (due to higher material stiffness). Only a single peak is seen in the force-displacement curve for the snap-fit. Initially, the cantilever hook hinges about the mating part edge, bending backwards in the process. It is important to note that at this stage, there is no relative motion at the contact edge. Once the static coefficient of friction is exceeded (corresponding to the maximum retention force value), the mating part slips right through. The maximum retention force does not occur at the transition between the retention and dwell surfaces. Instead, it happens near the base of the retention face. This can be explained by considering the static and dynamic coefficients of friction for sandblasted aluminum surfaces. If the latter is much
Figure 4.16: Correlation of shape of retention force-displacement curve to high speed photographs (250 fps) for injection molded ABS sample with Geometry 4. Numbers in pictures indicate frame number.
lower than the former, then such an effect is to be expected. This is because once sliding
between the mating part edge and retention face is initiated, the dynamic friction values
are too low to cause another stick-slip transition (as was the case in previously discussed
geometries). Figure 4.18 shows the results for the second test on the same sandblasted alu-
minum sample as before. Changes, if any, are expected to be caused solely by the change
in the surface condition of the retention face. The sharp mating part edge of the steel mat-
ing part abrades the softer surface of the aluminum cantilever hook. The biggest change
from the last case is the appearance of multiple local maxima peaks in the retention force
curve. The maximum still occurs very near the base of the retention face, however, another
peak of almost the same magnitude occurs closer to the face’s top edge. This suggests a
change in friction coefficient values for the surface, with the static coefficient most likely
being lower than for the original sandblasted surface.
Figure 4.17: Correlation of shape of retention force-displacement curve to high speed photographs (250 fps) for first test on sandblasted aluminum sample. Numbers in pictures indicate frame number.
Figure 4.18: Correlation of shape of retention force-displacement curve to high speed photographs (250 fps) for second test on sandblasted aluminum sample. Numbers in pictures indicate frame number.
CHAPTER 5

DEVELOPMENT OF A LONG-TERM CONSTITUTIVE MODEL FOR POLYCARBONATE

In previous chapters, it has been discussed that current analytical and numerical snap-fit models lack predictive capability with respect to time-dependent effects and loading rate dependence. These phenomena are a consequence of the viscoelastic nature of polymers. It is also common knowledge that most polymers exhibit non-linear stress-strain properties, limiting the applicability of linear viscoelastic models to very low strain values. The range of linear behavior depends on the particular polymer being evaluated. This chapter describes the characterization of long-term properties of Polycarbonate (PC) using a non-linear viscoelastic model proposed by Schapery [48]. Short-term stress relaxation experiments (typical duration - 1 hr.) are used in conjunction with time-temperature and time-strain superposition principles to predict the relaxation modulus at longer durations. The validity of these superposition principles is evaluated, Two-step stress relaxation and constant strain rate experiments are used to verify and refine the model parameters. The reasons for the choice of the Schapery constitutive model over other single integral superposition principles are discussed.
5.1 Constitutive Model for Polycarbonate

The test polymer for this thesis has been chosen to be PC. It is amorphous in nature and finds extensive use in engineering applications. It is known to exhibit non-linear behavior at strain levels as low as 1%. Several different constitutive models have been applied to PC in the past. The choice of an appropriate model for this work is driven by several factors.

1. The model should be capable of capturing non-linear effects.

2. It should accurately model the behavior of PC over several decades, and not just over a restricted window of time.

3. It should lend itself to relatively simple and effective implementation in a finite element analysis program.

4. The experimentation necessary to completely characterize the model variables should not be prohibitive in terms of cost or time.

Based on these considerations, and a review of relevant literature, the Schapery approach was chosen as the most suitable model for use in this thesis. It is a semi-empirical constitutive equation, drawing on the fundamental strengths of purely theoretical approaches and the effectiveness of empirical models. It has been shown to be versatile, and has found extensive application to several different classes of polymers. Some initial literature dealing with the use of this model for viscoelastic behavior of PC exists. With appropriate choice of kernel terms, its implementation in a finite element package is a simple extension of the approach used to model linear viscoelastic materials.
5.2 The Schapery Model for Non-linear Viscoelasticity

The Schapery representation, also known as the Schapery thermodynamic theory was derived from thermodynamic considerations. The constitutive equation expresses the stress response to an arbitrary strain history in terms of a modified convolution integral, as shown below.

\[
\sigma = h_e E_e \epsilon + h_1 \int_0^t \Delta E (\rho - \rho') \frac{dh_2}{d\tau} d\tau 
\]  

(5.1)

with the reduced time, \(\rho\), defined as

\[
\rho \equiv \int_0^t \frac{dt'}{a_\epsilon [\epsilon (t')]} 
\]  

(5.2)

and

\[
\rho' \equiv \rho (\tau) = \int_0^\tau \frac{dt'}{a_\epsilon [\epsilon (t')]} 
\]  

(5.3)

The only strain dependent quantities in the above equations are \(h_e, h_1, h_2\) and \(a_\epsilon\). The term \(h_e E_e \epsilon\) represents the equilibrium (long-term) stress response to the strain input. \(\Delta E\) corresponds to the linear transient relaxation modulus. No restriction on the particular algebraic form of the linear relaxation modulus terms \(E_e\) and \(\Delta E\) is specified. The two forms that have been most commonly used are the power law form and the exponential series form. A similar set of equations can be written to express the strain response to an arbitrary stress history.

The Schapery theory, in the form shown, does not represent the time-temperature correspondence (thermo-rheological simplicity) that is commonly observed in polymers. The mathematical representation of stress response of a thermo-rheologically simple material (TSM) to an arbitrary strain history is:

\[
\sigma = E_e \epsilon + \int_{t'}^t \Delta E (\rho - \rho') \frac{d\epsilon}{d\tau} d\tau 
\]  

(5.4)
where $\rho$ is known as reduced time and is mathematically defined as:

$$\rho \equiv \int_0^t \frac{dt'}{a_T[T(t')]}$$

and

$$\rho' \equiv \rho(\tau) = \int_0^\tau \frac{dt'}{a_T[T(t')]}$$

The ability to model long-term properties (up to several decades of time) is of great interest in snap-fit design. It is usually not possible to conduct experiments for that long a period of time. Hence, extensive use of the concept of thermorheological simplicity (TSM) is made in polymer material characterization. This allows the use of short-run experimental data to construct a master stress relaxation curve which predicts material behavior to long times.

It is important to note that Equations 5.4-5.6 are very similar to Equations 5.1-5.3 in their representation of temperature and stress effects respectively on the stress relaxation behavior of a polymer. In fact, Schapery [48] originally suggested that the Equation 5.1 can be extended to incorporate TSM behavior by introducing temperature as an argument of the coefficient $a_\epsilon$ i.e.

$$a_\epsilon = a_\epsilon(\epsilon, T)$$

Other effects such as moisture and physical ageing have been recognized as changing the relaxation behavior of polymers in a similar manner to temperature and can also dealt with by the Schapery representation. The Schapery thermodynamic theory is the most generalized among single integral representations, as discussed in Chapter 2. The Boltzmann superposition principle, the modified superposition principle and the BKZ theory can be represented as special cases of the Schapery representation.
5.3 Determination of Model Parameters: Theoretical Basis of Data Reduction Procedure

The data reduction procedure suggested by Schapery [48] will be used to determine the model parameters via a combination of single step and two-step stress relaxation experiments. In order to establish the theoretical basis of the data reduction procedure, the response of the Schapery model to single step and two step experiments will be determined following the steps in the original work. The basic form of Schapery’s constitutive model can be written as:

\[ \sigma = h_e E \epsilon + h_1 \int_0^t \Delta E (\rho - \rho') \frac{dh_2}{d\tau} d\tau \]  

(5.8)

with the reduced time, \( \rho \), defined as

\[ \rho \equiv \int_0^t \frac{dt'}{a_{\epsilon T} \left[ \epsilon (t') \right]} \]  

(5.9)

and

\[ \rho' \equiv \rho (\tau) = \int_0^\tau \frac{dt'}{a_{\epsilon T} \left[ \epsilon (t') \right]} \]  

(5.10)

It is important to note that the shift factor \( a_{\epsilon T} \) is a function of both strain (\( \epsilon \)) and time (\( T \)) i.e.

\[ a_{\epsilon T} = a_{\epsilon T} (\epsilon, T) \]  

(5.11)

Hence, our implementation of the Schapery model will result in coupling of the time-temperature and time-strain correspondence phenomena. The change in time scale due to strain and temperature will be modeled simultaneously. Experimental results will be used to separate the temperature and strain dependence of the shift factors. This issue is discussed in more detail later in this chapter. For the sake of notational simplicity we shall use the symbol \( a \) instead of \( a_{\epsilon T} \) for the shift factors without any change in its significance.
5.3.1 Schapery Response to Single Step Stress Relaxation Experiments

The stress response of the Schapery model a step strain input at temperature \( T \) can be determined by substituting \( \epsilon = \epsilon_0 H(t) \) in Equation (5.1). It can be written as:

\[
\sigma = h e^0 \epsilon_0 + h e^0 h e^0 \epsilon_0 \Delta E \left( \frac{t}{a(\epsilon_0, T)} \right)
\]

(5.12)

The non-linear relaxation modulus \( (E_n) \) can be defined as:

\[
E_n \approx \frac{\sigma}{\epsilon_0} = h e^0 E_e + h e^0 h e^0 \Delta E \left( \frac{t}{a(\epsilon_0, T)} \right)
\]

(5.13)

Rearranging terms, we can write,

\[
E_n - h e^0 E_e = h e^0 h e^0 \Delta E \left( \frac{t}{a(\epsilon_0, T)} \right)
\]

(5.14)

Taking log of both sides,

\[
\log (E_n - h e^0 E_e) = \log (h e^0 h e^0) + \log \Delta E \left( \frac{t}{a(\epsilon_0, T)} \right)
\]

(5.15)

From the above equation, it can be inferred that the values of \( \log (h_1 h_2) \) and \( \log (a(\epsilon, T)) \) are the amounts of vertical and horizontal shifts respectively necessary to overlap a \( \log \Delta E_n \) versus \( \log t \) curve from a single step stress relaxation experiment to some reference \( \log \Delta E_n \) versus \( \log t \) curve. The transient non-linear relaxation modulus \( \Delta E_n \) is defined as \( \Delta E_n = E_n - h_e E_e \).

Let \( \epsilon_R, T_R \) be the reference strain and temperature for a set of experiments. Consider an isothermal single step stress relaxation experiment conducted under the conditions \( \epsilon_1, T_1 \). The parameters \( \log (h_1 h_2) \) and \( \log (a(\epsilon, T)) \), are, according to the Schapery model, the vertical and horizontal shifts necessary to make the stress relaxation curve from the \( \epsilon_1, T_1 \) experiment overlap that at \( \epsilon_R, T_R \). However, since the quantities \( E_e \), the equilibrium relaxation modulus, and \( h_e \), the associated non-linearizing factor, are not known apriori, this
becomes a trial and error procedure, with values of $E_e$ repeatedly chosen until acceptable overlap of the two curves becomes possible. The shift factors $a(\epsilon, T)$ determined in this manner are functions of both temperature and strain. Hence by running a series of experiments at different temperatures and strains, an attempt to separate the influence of the two variables can be made. The non-linearizing factors $h_1, h_2$ are not expected to be functions of temperature, only of strain.

Considerable simplification in the data reduction procedure can be achieved if it is possible to superpose the relaxation modulus ($E_n(t)$) curves directly (instead of $\Delta E_n(t)$) to form the master curve. It can then be shown that $h_e$ has the same strain dependence as the product $h_1h_2$ i.e.

$$h_e = h_1h_2$$ (5.16)

Some of the common forms that have been used to express the temperature dependence of $a$ below the glass transition temperature of the material ($T_g$) are:

$$\log (a_T) = -A \left( \frac{T - T_R}{T} \right)$$ (5.17)
$$\log (a_T) = -A \left( \frac{T - T_R}{T_R} \right)$$ (5.18)
$$\log (a_T) = -A \left( \frac{T - T_R}{T_R} \right)^a$$ (5.19)

Of the above, the first equation is a theoretical result known as the Arrhenius equation and the other relationships are empirical in nature. Similar relationships can be used to relate $a$ to strain. The particular form that will be used will be discussed later, after presenting the experimental results.
5.3.2 Schapery Response to Two Step Stress Relaxation Experiments

From the single step experiments, it is only possible to obtain the non-linearizing factors in product form. Two-step experiments can aid in separating $h_1$ and $h_2$ as first suggested by Schapery. Consider a strain input of the form shown in Figure 5.1. The response of the Schapery model to such a strain input can be determined by substituting the following input function into the convolution integral:

$$h_2 \epsilon = h_2^\alpha \epsilon_a H(t) + (h_2^\alpha \epsilon_b - h_2^\alpha \epsilon_a) H(t - t_a)$$  \hspace{1cm} (5.20)

Making the above substitution and simplifying the resulting equation, we get the response equation.

Figure 5.1: Strain input for two step stress relaxation experiment.
For $0 < t < t_a$: 

$$\sigma = E^\epsilon_a \epsilon_a \tag{5.21}$$

and for $t_a < t < t_b$

$$\sigma(t) = E^\epsilon_b \epsilon_b - \frac{h_1^\epsilon_b}{h_1^\epsilon_a} \left[ \Delta E \left( \frac{t - t_a}{a(\epsilon_a, T)} \right) - \Delta E \left( \frac{t_a}{a(\epsilon_a, T)} \right) + \frac{t - t_a}{a(\epsilon_b, T)} \right] \epsilon_a \tag{5.22}$$

Assuming that single step relaxation data is available at strain level $\epsilon_b$, all quantities in the stress response equation (Equation (5.22)) will be known except for $\frac{h_1^\epsilon_b}{h_1^\epsilon_a}$. This ratio can then be found by matching Equation (5.22) to stress data at a convenient point of time. If $\epsilon_b$ is sufficiently small, $h_1^\epsilon_b \simeq 1$, and all properties will thereby be determined at (arbitrary) strain level $\epsilon_a$.

### 5.4 Experimental Setup, Sample Preparation and Test Procedure

The experimental setup is shown in Figure 5.2. The testing machine is a servo-hydraulic INSTRON™ tensile test machine capable of crosshead speeds of up to 50 mm/s. The strain in the specimen is directly controlled using a commercial dynamic extensometer from INSTRON™, instead of using cross-head displacement to estimate the strain in the specimen. The PID (proportional-integral-derivative) strain channel controller on the machine was tuned to attain the desired strain value rapidly with minimal overshoot. This is beneficial in attempting to estimate the instantaneous elastic response of the material, since data for a length of time of the order of five times the initial ramp-up period is usually discarded. The machine is equipped with a specialized load cell designed to be insensitive to off-axis moment loads, which can be considerable in snap-fits. A PC equipped with a National Instruments GPIB board and commercial INSTRON™ control software is used for test control and data acquisition. The data acquired through the GPIB board is stored
in a comma delimited text file. The temperature chamber is equipped with a temperature controller with built-in PID control. An unsheathed thermocouple is used for temperature measurement to reduce the lag in the system and achieve more sensitive temperature control. The temperature controller was tuned for fastest ascent to the set-point temperature with minimal overshoot. The temperature fluctuation inside the chamber has been measured to be less than 0.5°C. The grips are light-weight to reduce inertial effects. One of the grips is activated by a pneumatic cylinder and was designed and manufactured in-house. This enables the specimen to be gripped after the desired temperature has been reached inside the controller. This allows the nullification of thermal strain effects from the measured mechanical strain values and results in an improvement in accuracy. The grips have serrated faces for firmly gripping the polymer specimens, to eliminate slippage during the tests. Dogbone shaped tensile test specimens (ASTM D639) were injection molded from a general purpose grade of Polycarbonate manufactured by GE Plastics (LEXAN 141) using a standard ASTM mold in the Department of Industrial, Welding and Systems Engineering (IWSE). All samples were annealed at 159°C for 1.5 hours, which was approximately 10°C above the $T_g$ of the material. This was done to reduce the internal residual stresses generated during injection molding and also to refresh the aging history of the material. The specimens were then stored at room temperature in airtight plastic bags with small desiccant packs ($\sim 25°C$) for 45 days prior to testing.

Stress relaxation experiments were conducted at strain levels between 0.1% and 4.5% and temperatures between 25°C and 105°C. Tests were carried out at intervals of 0.5% strain and 20°C, except in cases where some additional data at intermediate strain or temperature levels was deemed necessary. Numerous replicate tests were conducted to establish the repeatability of the experimental results. The duration of most tests was 1.5 hours,
Figure 5.2: Experimental setup for viscoelastic characterization of PC showing uniaxial test machine (INSTRON), temperature chamber, controller and lightweight grips
except for some tests which were run overnight (14-18 hours). Table 5.1 displays the exact set of tests that were conducted.

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Table 5.1: Matrix of stress relaxation tests showing the number of tests run at each temperature-strain combination.

In a typical test, the sample was attached to the top grip which was then threaded into the actuator inside the chamber. The extensometer was then mounted. After closing the door of the chamber, the temperature controller was switched on and temperature inside the chamber allowed to reach the desired value. Sufficient time was allowed for the temperature to fully stabilize inside the chamber. The other end of the specimen was then clamped using the pneumatic grip (which is controlled from outside the chamber). The strain and load values were balanced on the machine control interface.
Generating a perfect step strain input is not possible, so a waveform of the type shown in Figure 5.3 was used. Care was taken to make the duration of the ramp as small as possible to most accurately approximate the step function. As mentioned previously, the PID controller of the strain channel on the INSTRON was tuned to reduce ramp-up times ($\Delta t_a$) without any overshoot. The duration of the ramp-up time governs the amount by which the initial stress relaxation response measured deviates from the real stress relaxation response. It has been shown that the discrepancy between the two responses is negligible when the elapsed time is greater than five times the ramp-up time ($5\Delta t_a$). Hence data for times less than that duration ($< 5\Delta t_a$) is discarded before reduction. It is for this reason that the ramp-up times should be as low as possible so that the instantaneous elastic response of the material is modeled more accurately. It is necessary to find an optimum between the amount of overshoot and the ramp-up time.
5.5 Reduction of Experimental Data for Estimation of Shift Factors and Schapery Parameters

The raw stress relaxation data, obtained in comma delimited format from the data acquisition setup on the INSTRON testing machine, contains time, displacement, force and strain information. Each data set was manipulated to obtain $\log(t)$, $\log(E(t))$ and $E(t)$ using the statistical analysis program SAS. The values from all tests conducted at a particular strain and different temperatures were then collected in a single data file for analysis and graphing purposes. Graphs of the raw stress relaxation data at different strain levels and temperatures are shown in Appendix B.

The next step in the data reduction procedure was to graphically shift the individual curves ($\log E(t)$ versus $\log t$) using a combination of horizontal and vertical shifts to estimate the values of $\log a(\epsilon, T)$ and $\log h_1 h_2$ respectively. Several different shifting schemes were evaluated to obtain the best possible master-curve. Initially, horizontal shifting only was used to form a master-curve. It was found that vertical shifts were also necessary to form a contiguous master curve. The next step was to perform shifting with respect to temperature and strain separately. Stress relaxation curves from experiments at a particular strain (say $\epsilon_R$) and a range of temperatures ($25^\circ C$ - $105^\circ C$) were shifted to form a master curve at $\epsilon_R$ and $25^\circ C$. Time-temperature superposition was used in this manner to form several master curves at different $\epsilon_R$s and $T_R = 25^\circ C$.

An analogous shifting methodology was used with respect to strain instead of temperature (time-strain superposition) to generate master curves at several $T_R$s and $\epsilon_R = 0.005$. Another set of master curves was thus obtained with a different set of reference strain and
temperature values. Theoretically, it should be possible to coalesce all these curves (time-temperature and time-strain) into a unified master stress relaxation curve with respect to a particular $\epsilon_R, T_R$ by further shifting. Stress relaxation behavior at any desired temperature and strain could then be determined by using this master curve coupled with appropriate values of the horizontal and vertical shift factors.

An alternative shifting procedure would be to shift the individual stress relaxation curves, obtained using the tests depicted in Table 5.1, to a common $\epsilon_R, T_R$ and directly form a single master-curve instead of following the two step procedure discussed above. This would also have the added benefit of inherently averaging out the experimental variation in the stress relaxation data (Appendix B). This procedure was finally adopted after attempting the other approaches described in the previous paragraph. The resulting master-curve ($\epsilon_R = 0.005, T_R = 25^\circ C$) is shown in Figure 5.4. The 0.1% and 0.3% data was not included in the master curve because the relaxation modulus curves (Figure B.1 and Figure B.2 respectively) were not representative and the repeatability was poor at such low strain levels. The experimental variation is large enough to cause significant error in the data at these strain levels. The horizontal shift factors associated with the above master curve are shown in Figures 5.5 and 5.6. In Figure 5.5 the quantity $\log a$ is plotted against temperature for different strain levels. The horizontal shift factors exhibit a linear variation with temperature. Linear least-squares trendlines fit to the data are also shown in the figure.

Figure 5.6 shows the variation of $\log a$ with $\epsilon - \epsilon_R$ for different temperatures. This figure contains the same data as the previous figure, except that it is plotted versus strain instead of temperature. Again, one can observe a linear variation of the horizontal shift factors with strain for each temperature level. It is significant that, in both plots, the trendlines depicting
Figure 5.4: Master relaxation modulus curves for polycarbonate $\epsilon_R = 0.5\%$ and $T_R = 25^\circ C$. 
Figure 5.5: Shift factors $a(T, \epsilon)$ as a Function of temperature for different strain levels for the master curve of Figure B.12 and least-squares fit curves.
the variation of $\log a$ with temperature and strain are almost parallel. This implies that, in addition to the linear variation of $\log a$ against both temperature and strain, the effect of the two can be separated out (recall Equation 5.7) according to the following equation:

$$a (T, \epsilon) = a_T (T) a_\epsilon (\epsilon)$$ (5.23)

or,

$$\log (T, \epsilon) = \log a_T (T) + \log a_\epsilon (\epsilon)$$ (5.24)

The above equation suggests that if the multiplicative decomposition assumption of $\log a$ (Equation 5.23) is valid, then plots of $\log a$ versus temperature or strain at different strains or temperatures respectively will be parallel curves. This is supported by Figures 5.5 and 5.6. Additionally, since the variation is observed to be linear, one can write,

$$\log a_T (T) = -A \left( \frac{T - T_R}{T_R} \right)$$ (5.25)

and

$$\log a_\epsilon (\epsilon) = -B \left( \frac{\epsilon - \epsilon_R}{\epsilon_R} \right)$$ (5.26)

Together, the variation of the horizontal shift factor with strain and temperature can be represented as:

$$\log a (T, \epsilon) = -A \left( \frac{T - T_R}{T_R} \right) - B \left( \frac{\epsilon - \epsilon_R}{\epsilon_R} \right)$$ (5.27)

Essentially, Equation 5.27 implies that the variation of the horizontal shift factors can be represented as a surface when $\log a$ is plotted versus temperature and strain. This is shown in Figure 5.7 along with the trend surface fit to the data using least squares approach. The equation of the fit surface was determined to be:

$$\log a (T, \epsilon) = -20.9661 \left( \frac{T - T_R}{T_R} \right) - 0.479043 \left( \frac{\epsilon - \epsilon_R}{\epsilon_R} \right)$$ (5.28)
Figure 5.6: Variation of horizontal shift factors $a(T, \varepsilon)$ with temperature for different strains and least-squares fit lines.
where $T_R = 298K$ and $\epsilon_R = 5e-3$. This approach for modeling the horizontal shift factors has the added advantage of averaging out the scatter in the shift factor data. Figure 5.8 shows the variation of the quantity $\log h_1 h_2$ with temperature for different strain levels. The least square fit trendlines are also shown for each case. A closer look at the graph reveals that the values vertical shifts corresponding to $T - T_0 = 80^\circ C$ significantly deviate from the trend shown by the data points at all other $T - T_0$ values. This is true at all strain levels. This data set is ignored to plot the trendlines shown in the figure. The considerable difference in the vertical shifts required at $T = 105^\circ C$ from that at other values seems to suggest a breakdown in the validity of the superposition assumptions and data reduction scheme being used in this thesis. The sudden change in the vertical shifts required for master curve formation at higher temperatures may indicate some additional relaxation phenomena or structural changes that do not occur at lower temperatures. A more comprehensive experimental effort at temperatures close to the glass transition temperature of the material would be required to firmly establish the reasons for such behavior. For the purposes of master curve formation however, this effect is not expected to introduce any inaccuracy because the isothermal stress relaxation curves at these temperatures get shifted several decades on the time scale (approx. 13-15) for a reference strain of 0.5% and a reference temperature of $25^\circ C$. For our work, room temperature data is desired only to 6-7 decades of time, corresponding to 1-2 years of service. Hence, later regions of the master curve are not very critical to this research.

No significant or consistent trends are observed in the variation of the vertical shifts with temperature. Most of the trendlines are almost horizontal, although, the 3.5% and 4.0% lines do show a slight increase in the magnitude of vertical shifts required at higher temperatures. However, the trend is not considerable enough to require inclusion in the
Figure 5.7: Variation of horizontal shift factors $a(T, \epsilon)$ with strain for different temperatures and least-squares fit surface.
model. This implies that the quantity $\log h_1 h_2$ is a function of strain only, which confirms one of the basic assertions of the Schapery model.

Figure 5.9 shows the vertical shift factors plotted against $\epsilon - \epsilon_R$ for different temperatures. The data shows a strong trend, the vertical shifts being larger in magnitude for higher strain levels. In this figure, the drop in the vertical shifts at temperature $T = 105^\circ C$ discussed previously is more apparent. The remainder of the data points show very consistent trends. No clear effect of temperature on the values of $\log h_1 h_2$ can be seen in this figure either. A second order curve fit using least squares approximation is fit to all the data points clubbed together (excluding the data at $T = 105^\circ C$). The equation of the curve is given below:

$$\log (h_1 h_2) = 0.0225943 \left( \frac{\epsilon - \epsilon_R}{\epsilon_R} \right) - 0.00485444 \left( \frac{\epsilon - \epsilon_R}{\epsilon_R} \right)^2$$

(5.29)

Since it was possible to shift the relaxation modulus curves directly to form the master curve, we recall Equation 5.16, which is valid for the current model.

$$h_e = h_1 h_2$$

(5.30)

Summarizing the above discussion, the temperature and strain dependence of the horizontal and vertical shift factors can be modeled using the following equations:

$$\log a (T, \epsilon) = -20.9661 \left( \frac{T - T_R}{T_R} \right) - 0.479043 \left( \frac{\epsilon - \epsilon_R}{\epsilon_R} \right)$$

(5.31)

$$\log (h_1 h_2) = +0.00225943 \left( \frac{\epsilon - \epsilon_R}{\epsilon_R} \right) - 0.00485444 \left( \frac{\epsilon - \epsilon_R}{\epsilon_R} \right)^2$$

(5.32)

$$h_e = h_1 h_2$$

(5.33)
Figure 5.8: Variation of vertical shift factors $h_1, h_2$ with temperature for different strain levels and least-squares fit lines.
Figure 5.9: Variation of vertical shift factors $h_1h_2$ with strain for different temperatures and least-squares fit curve.
These equations coupled with a Prony series representation of the relaxation modulus master curve (discussed next) will be used to model the overall response of the material to arbitrary strain and temperature history.

### 5.5.1 Prony Series Representation of Master Curve Data

The relaxation modulus master curve obtained by combined horizontal and vertical shifting of individual isothermal stress relaxation curves is shown in Figure 5.4. The next step in the data reduction procedure is to convert the raw data into a form suitable for use in analytical models. The most commonly used analytical representation utilizes a summation of exponential terms known as the Prony series, which can be algebraically expressed as:

\[
E(t) = E_\infty + \sum_{i=1}^{N} E_i \exp \left( -\frac{t}{\tau_i} \right)
\]  

(5.35)

The collocation method (insert citation) is used to determine the Prony series parameters \(E_i, \tau_i, i = 1 \ldots N\). The relaxation times and Prony constants are listed in Table 5.2 below. The MATLAB script used to generate these values is provided in the Appendix. The master curve spans 15 decades of time \((\sim 10^6 \text{ years})\), the full duration of which is of academic interest only.

The experimental master curve and its Prony series representation is shown in Figure 5.10. It is important to note that the value of the equilibrium relaxation modulus \(E_e\) does not have any physical significance. It is chosen to achieve the closest fit to experimental data using the collocation method. No evidence of polycarbonate approaching an equilibrium long term modulus is exhibited during the experiments. The use of an apparent \(E_e\) value can further be justified by considering the fact that \(10^{15}\) seconds is much larger than the practical time window of interest (typically 1-2 years or \(10^7\) seconds). Furthermore, our goal is primarily to predict the properties of the material at room temperature.
<table>
<thead>
<tr>
<th>n</th>
<th>$t_i$ (s)</th>
<th>log($E_i$) (MPa)</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>2.00e+01</td>
<td>36.7566</td>
</tr>
<tr>
<td>2</td>
<td>2.00e+02</td>
<td>37.7439</td>
</tr>
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<td>3</td>
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<td>2.00e+06</td>
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<td>2.00e+07</td>
<td>160.6603</td>
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<td>8</td>
<td>2.00e+08</td>
<td>137.4210</td>
</tr>
<tr>
<td>9</td>
<td>2.00e+09</td>
<td>189.8020</td>
</tr>
<tr>
<td>10</td>
<td>2.00e+10</td>
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<td>2.00e+13</td>
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<td>2.00e+14</td>
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</tr>
<tr>
<td>15</td>
<td>2.00e+15</td>
<td>3.9233</td>
</tr>
<tr>
<td>16</td>
<td>$\infty(E_e)$</td>
<td>2.84</td>
</tr>
</tbody>
</table>

Table 5.2: Prony series constants for polycarbonate stress relaxation master-curve.
conditions, so the later portions of the master curve will rarely, if ever, be used in any calculations or simulations.

5.5.2 Two-Step Stress Relaxation Experiments

As discussed in Section 5.5, the original work by Schapery [48] suggests that two step stress relaxation experiments can be used to separate the non-linearizing parameters $h_1$ and $h_2$. This can be a cumbersome procedure. Additionally, changes in the individual expressions of $h_1$ and $h_2$ while maintaining their product is not expected to have a significant effect on the overall shape of the two-step stress relaxation response. With this justification in mind, an alternative procedure was adopted to establish separate functional forms of $h_1$ and $h_2$. Two assumptions would be considered as separate cases: $h_1 = 1$ and $h_2$. The
predicted two step stress relaxation and constant strain rate response for each case would then be compared to corresponding experimental results to determine the more appropriate assumption. The value of the appropriate factor, \( h_1 \) or \( h_2 \) would then be identically set to one in the constitutive model.

In order to obtain the predicted response using the model developed in this thesis, two separate assumptions were considered, \( h_1 = 1 \) and \( h_2 = 1 \) respectively. The model predictions would then be compared to experimental results for these experiments (and constant strain rate experiments, if necessary). The appropriate factor would then be identically set to one in the model based on the assumption that yielded better agreement between to experiments.

The following figures (Figure 5.11 - Figure 5.18) present experimental results and model prediction of stress response to a two-step strain input. In all cases, \( \epsilon_a = 0.005 \) and \( T = 25^\circ C \). Some figures contain multiple experimental curves, corresponding to repeated experiments. The model response was obtained using a MAPLE script, attached in Appendix B. Overall, the model predictions and experimental results show excellent agreement, within the range of observed variation. However, the two assumptions being tested appear to have no influence on the response. In fact, it is not possible to distinguish the two predicted responses in any of the experiments. Hence, a decision regarding the more appropriate assumption (\( h_1 = 1 \) v/s \( h_2 = 1 \)) cannot be made on the basis of these two step stress relaxation experiments alone.

The next section presents a similar comparison of experimental results and model predictions for constant strain rate experiments.
Figure 5.11: Comparison of two step stress relaxation ($\epsilon_a = 0.005, \epsilon_b = 0.01$) experimental results and model predictions using $h_1 = 1$ and $h_2 = 1$ respectively.

Figure 5.12: Comparison of two step stress relaxation ($\epsilon_a = 0.005, \epsilon_b = 0.015$) experimental results and model predictions using $h_1 = 1$ and $h_2 = 1$ respectively.
Figure 5.13: Comparison of two step stress relaxation \((\epsilon_a = 0.005, \epsilon_b = 0.02)\) experimental results and model predictions using \(h_1 = 1\) and \(h_2 = 1\) respectively.

Figure 5.14: Comparison of two step stress relaxation \((\epsilon_a = 0.005, \epsilon_b = 0.025)\) experimental results and model predictions using \(h_1 = 1\) and \(h_2 = 1\) respectively.
Figure 5.15: Comparison of two step stress relaxation ($\varepsilon_a = 0.005, \varepsilon_b = 0.03$) experimental results and model predictions using $h_1 = 1$ and $h_2 = 1$ respectively.

Figure 5.16: Comparison of two step stress relaxation ($\varepsilon_a = 0.005, \varepsilon_b = 0.035$) experimental results and model predictions using $h_1 = 1$ and $h_2 = 1$ respectively.
Figure 5.17: Comparison of two step stress relaxation ($\epsilon_a = 0.005, \epsilon_b = 0.04$) experimental results and model predictions using $h_1 = 1$ and $h_2 = 1$ respectively.

Figure 5.18: Comparison of two step stress relaxation ($\epsilon_a = 0.005, \epsilon_b = 0.045$) experimental results and model predictions using $h_1 = 1$ and $h_2 = 1$ respectively.
5.5.3 Constant Strain Rate Experiments

Constant strain rate experiments were conducted at room temperature \((T = 25^\circ C)\). Polycarbonate does not exhibit any significant dependence of the stress-strain response on strain rate (Figure 5.19). Once again, results from repeated experiments are also shown in the figure. It is not straightforward to obtain an analytical expression of the constrain strain rate input response, hence a numerical approach was adopted in this case. In particular, the convolution integral is represented as a summation:

\[
\int_0^{t_j} \Delta E (\rho - \rho') \frac{d}{d\tau} (h_2 \epsilon) d\tau = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta E (\rho - \rho') \frac{\Delta h_2 \epsilon}{\Delta \tau_i} \Delta \tau_i \quad (5.36)
\]

where,

\[
\rho - \rho' = \int_{\tau_i}^{t_j} \frac{dt'}{a(\epsilon, T)} \quad (5.37)
\]

\[
\tau_i = \frac{it_j}{n} \quad (5.38)
\]

\[
\Delta \tau_i = \frac{t_j}{n} \quad (5.39)
\]

Equation 5.36 can be further simplified to the following form:

\[
\int_0^{t_j} \Delta E (\rho - \rho') \frac{d}{d\tau} (h_2 \epsilon) d\tau = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta E (\rho - \rho') \left[ (h_2 \epsilon)_{\tau_i} - (h_2 \epsilon)_{\tau_{i-1}} \right] \quad (5.40)
\]

where, once again,

\[
\rho - \rho' = \int_{\tau_i}^{t_j} \frac{dt'}{a(\epsilon, T)} \quad (5.41)
\]

\[
\tau_i = \frac{it_j}{n} \quad (5.42)
\]

These relationships have been implemented using MAPLE (the source code of the script is attached in Appendix B). Figure 5.19 displays the predicted stress-strain trends along with experimental results. The agreement is excellent at strains of up to 3.5 %. The model
Figure 5.19: Comparison of model predictions and experimental results for constant strain rate experiments, strain rates $\dot{\varepsilon} = 0.01\%/s, 0.1\%/s$ and $1\%/s$ respectively.

It does not predict any significant dependence of the stress-strain curve on strain rate, for the $\dot{\varepsilon}$ levels considered here. Also, there is very little difference in the predictions obtained using two assumptions, namely $h_1 = 1$, and $h_2 = 1$.

Beyond 3.5% strain levels, the model predicts a decrease in stress with increasing strain, a phenomena not exhibited by the experimental results. Upon closer examination of numerical results, this apparent reduction in stress values occurs because the rate of decrease of $h_2$ is greater than the increase in $\varepsilon$, resulting in a reduction of the magnitude of the product $h_2\varepsilon$ with time. This results in a negative contribution due to the term $[(h_2\varepsilon)_{\tau_i} - (h_2\varepsilon)_{\tau_{i-1}}]$ to the overall stress. There is no physical justification of this effect and this is clearly a fallacious prediction.
Referring back to Figure 5.9, there appears to be greater scatter in the values of vertical shift factors at $\epsilon - \epsilon_R = 0.040$ strain level than at lower strains. It has also been discussed previously that the vertical shift factor values at temperature levels corresponding to $T - T_R = 80^\circ$C do not follow the general trend. The vertical shift factor data at these temperatures was excluded from the curve fit in Equation 5.29. These observations suggest that the vertical shifts at higher temperatures and strains require further experimentation and investigation.

In this chapter, we have described a comprehensive experimental effort to characterize the non-linear viscoelastic behavior of a test polymer, Polycarbonate. Time-temperature and time-strain superposition principles are used to form a master stress relaxation curve for the material. Equations for modeling the corresponding horizontal and vertical shift factors have been obtained. Finally, the proposed material model is verified and refined using two-step stress relaxation and constant strain rate experiments.

The next chapter describes the efforts related to finite-element implementation of the constitutive material model developed here.
CHAPTER 6

FINITE ELEMENT IMPLEMENTATION OF VISCO-ELASTIC CONSTITUTIVE MODEL

The previous chapter describes the experimental efforts to characterize the non linear viscoelastic behavior of Polycarbonate. The goal of developing such a model was to predict the time-dependent and loading rate dependent behavior of snap-fit features. The majority of snap-fit modeling efforts involve the use of finite element analysis (FEA). Hence, the next step in the research was to implement the constitutive model for use in a commercial finite element package. The theory behind the FEA implementation of the constitutive model is described in this chapter. The basic steps are listed below:

- Extension of uniaxial constitutive model to multiaxial stress states.
- Conversion of hereditary integral form of constitutive equation into incremental form.
- Implementation of incremental constitutive model for FEA via user subroutines.
- Testing and verification of above model using simple test cases.

Each of the above steps are discussed separately in this chapter.
6.1 Extension of Uniaxial Model to Multiaxial Stress States

The Schapery model, in its most basic form, represents the stress response to arbitrary strain input according to the following equation.

\[ \sigma = h_e E_e \epsilon + h_1 \int_0^t \Delta E (\rho - \rho') \frac{dh_2 \epsilon}{dT} dT \]  

(6.1)

with the reduced time, \( \rho \), defined as

\[ \rho \equiv \int_0^t \frac{dt'}{a_T [\epsilon(t')]} \]  

(6.2)

and

\[ \rho' \equiv \rho (\tau) = \int_0^\tau \frac{dt'}{a_T [\epsilon(t')]} \]  

(6.3)

This is the uniaxial form of the Schapery model. It can be extended to multiaxial form in manner analogous to the generalized Hooke’s Law. The Schapery model for multiaxial deformation is presented in the following section.

6.2 Multiaxial Deformation

6.2.1 Linear Elasticity

In linear elasticity, the stress-strain relationship for an isotropic solid is described by the generalized Hooke’s Law.

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & (1-2\nu) & 0 & 0 \\
0 & 0 & 0 & 0 & (1-2\nu) & 0 \\
0 & 0 & 0 & 0 & 0 & (1-2\nu)
\end{bmatrix} \begin{bmatrix}
\epsilon_{xx} \\
\epsilon_{yy} \\
\epsilon_{zz} \\
\epsilon_{xy} \\
\epsilon_{yz} \\
\epsilon_{zx}
\end{bmatrix}
\]  

(6.4)
where $E$ is the Young’s Modulus and $\nu$ is the Poisson’s ratio. In index notation the above equation can be written as:

\[
\sigma_{ij} = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \epsilon_{kk} \delta_{ij} + \frac{E}{1 + \nu} \epsilon_{ij} \tag{6.5}
\]

In terms of the Lame’s constants $\lambda$ and $G$, the above equation becomes:

\[
\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2G \epsilon_{ij} \tag{6.6}
\]

where the Lame’s constants are defined as follows:

\[
\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad G = \frac{E}{2(1 + \nu)} \tag{6.7}
\]

The summation convention is used in the above equation i.e.

\[
\epsilon_{kk} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \tag{6.8}
\]

Some authors recommend decomposition of the stress and strain into hydrostatic and deviatoric components for description of multiaxial constitutive laws. This approach is based on the assertion that the deformation mechanisms in materials are different for the deviatoric and hydrostatic components of stress and should therefore be considered independent of each other. The deviatoric stress tensor $s_{ij}$ and strain tensor $d_{ij}$ are defined as follows:

\[
s_{ij} = \sigma_{ij} - \delta_{ij} \sigma_m \tag{6.9}
\]

\[
d_{ij} = \epsilon_{ij} - \delta_{ij} \epsilon_m \tag{6.10}
\]

where $\delta_{ij}$ is the Kronecker delta and $\sigma_m$ and $\epsilon_m$ are hydrostatic stress and strain respectively, given by:

\[
\sigma_m = \frac{1}{3} \sigma_{ii} \tag{6.11}
\]

\[
\epsilon_m = \frac{1}{3} \epsilon_{ii} \tag{6.12}
\]
The stress-strain law (Equation 6.4) for a Hookean solid can then be rewritten as:

$$\sigma_{ij} = 2Gd_{ij} + 3K\delta_{ij}\epsilon_m$$

(6.13)

where $G$ and $K$ are shear and bulk modulus respectively, and related to tensile modulus and Poisson’s ratio as follows:

$$K = \frac{E}{3(1-2\nu)}$$

(6.14)

$$G = \frac{E}{2(1+\nu)}$$

(6.15)

For the purpose of our research however, the generalized form of the stress-strain relationship, shown in Equation 6.5, will be used. A similar approach was followed by Henriksen [68] with good results. Another reason for the adoption of this form of the constitutive model over the hydrostatic-deviatoric form is that the experiments used for material characterization are all uniaxial stress-state experiments. The relaxation modulus derived from the experimental effort can be used directly to obtain the multi-axial stress formulation. Adoption of the alternate approach would simply involve the conversion of $E$ into $K$ and $G$, which, though not a complicated procedure, is not expected to yield any further insight into the deformation mechanisms of the material. It would be more appropriate to decompose stress into deviatoric and hydrostatic components if experiments conducted to characterize the material directly provided $K$ and $G$ values (i.e. pure shear and pure compression tests were used) instead of using $E$ and $\nu$.

### 6.2.2 Linear Viscoelasticity

For an isotropic, linear viscoelastic solid, the Young’s modulus $E$ and Poisson’s ratio $\nu$ are expected to be functions of time. Consider a stress relaxation test in which a strain $\epsilon_{ij}$
is applied at time $\tau = 0$. In view of Equation 6.5, the stress response at time $\tau > 0$ is given by,

$$\sigma_{ij}(t) = \frac{\nu(t)}{1 + \nu(t)} \frac{E(t)}{(1 - 2\nu(t))} \epsilon_{kk} \delta_{ij} + \frac{E(t)}{1 + \nu(t)} \epsilon_{ij}$$

(6.16)

In terms of Lame’s constants, the above equation can be rewritten as:

$$\sigma_{ij}(t) = \lambda(t) \epsilon_{kk} \delta_{ij} + 2G(t) \epsilon_{ij}$$

(6.17)

where the two sets of constants are related as follows at time $t$:

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}$$

(6.18)

$$G = \frac{E}{2(1 + \nu)}$$

(6.19)

In order to describe the linear viscoelastic response under arbitrary loading history, one can readily generalize the 1-D Boltzmann’s superposition principle to the case of multiaxial deformation as follows:

$$\sigma_{ij}(t) = \lambda_c \delta_{ij} \epsilon_{kk}(t) + \delta_{ij} \int_0^t \Delta \lambda(t - \tau) \frac{d}{d\tau} \epsilon_{kk}(\tau) d\tau + 2G_c \epsilon_{ij}(t)$$

$$+ 2 \int_0^t \Delta G(t - \tau) \frac{d}{d\tau} \epsilon_{ij}(\tau) d\tau$$

(6.20)

where $\lambda_c$ and $G_c$ are the long-term equilibrium values of the Lame’s constants respectively, $\Delta \lambda(t)$ and $\Delta G(t)$ are the transient portions respectively. These quantities can be related back to the original set of constants $E, \nu$ by relationships similar to Equation 6.18 and Equations 6.19. However, two separate Poisson’s ratios would have to be considered for the time dependent response and the equilibrium response. For most polymers, considerable experimental evidence exists that suggests that the value of Poisson’s ratio is time-independent. Additionally, the two values mentioned above can be assumed to be identical.
This results in simplification of the multiaxial stress-strain formulation, as shown below:

\[
\sigma_{ij}(t) = \frac{\nu E_e}{(1 + \nu)(1 - 2\nu)} \delta_{ij} \epsilon_{kk}(t) + \delta_{ij} \int_0^t \frac{\nu \Delta E(t - \tau)}{(1 + \nu)(1 - 2\nu)} \frac{d}{d\tau} (\epsilon_{kk}) d\tau \\
+ \frac{E_e}{(1 + \nu)} \epsilon_{ij}(t) + \int_0^t \frac{\Delta E(t - \tau)}{(1 + \nu)} \frac{d}{d\tau} (\epsilon_{ij}) d\tau
\] (6.21)

Here, we have reverted back to the use of the constants \( E, \nu \) instead of the Lamé’s constants \( \lambda, G \). Simplifying the above equation, we get:

\[
\sigma_{ij} = E_e \left[ \frac{\nu \delta_{ij} \epsilon_{kk}}{(1 + \nu)(1 - 2\nu)} + \frac{\epsilon_{ij}}{(1 + \nu)} \right] + \frac{\nu \delta_{ij}}{(1 + \nu)(1 - 2\nu)} \int_0^t \Delta E(t - \tau) \frac{d}{d\tau} (\epsilon_{kk}) d\tau \\
+ \frac{1}{(1 + \nu)} \int_0^t \Delta E(t - \tau) \frac{d}{d\tau} (\epsilon_{ij}) d\tau
\] (6.22)

**6.2.3 Non-linear Viscoelasticity**

For an isotropic non-linear viscoelastic solid, the 3-D Schapery representation can be written by analogy with Equation 6.21 as

\[
\sigma_{ij} = h_e E_e \left[ \frac{\nu \delta_{ij} \epsilon_{kk}}{(1 + \nu)(1 - 2\nu)} + \frac{\epsilon_{ij}}{(1 + \nu)} \right] + \frac{h_1 \nu \delta_{ij}}{(1 + \nu)(1 - 2\nu)} \int_0^t \Delta E(t - \tau) \frac{d}{d\tau} (h_2 \epsilon_{kk}) d\tau \\
+ \frac{h_1}{(1 + \nu)} \int_0^t \Delta E(t - \tau) \frac{d}{d\tau} (h_2 \epsilon_{ij}) d\tau
\] (6.23)

where,

\[
\rho \equiv \int_0^t \frac{dt'}{a_{eT} [\epsilon, T]} \quad (6.24)
\]

and

\[
\rho' \equiv \rho(\tau) = \int_0^\tau \frac{dt'}{a_{eT} [\epsilon, T]} \quad (6.25)
\]

As discussed in the previous chapter, the effect of temperature and strain is to influence mechanical behavior by changing the time scale of the relaxation mechanisms. The quantity \( a_{eT} \) captures this effect via the reduced time \( \rho \). Furthermore, it has also been shown
that the dependence of the shift factor $a_{\epsilon T}$ on temperature and strain can be separated out in a multiplicative fashion, i.e.

$$a_{\epsilon T}(\epsilon, T) = a_\epsilon(\epsilon)a_T(T)$$  \hfill (6.26)\

The exact relationships for the uniaxial case have also been derived in the previous chapter of this thesis.

In the uniaxial form of the Schapery model, the quantities $h_0, h_1, h_2$ are functions of strain. This dependence also needs to be extended to the multiaxial case. It is assumed that the material properties are functions of the effective strain $\hat{\epsilon}$, i.e.

$$h_0 = h_0(\hat{\epsilon}), \quad h_1 = h_1(\hat{\epsilon}), \quad h_2 = h_2(\hat{\epsilon})$$  \hfill (6.27)\

The effective strain $\hat{\epsilon}$ is proportional to the octahedral shear strain $\epsilon_{oct}$ and is defined as:

$$\hat{\epsilon} = \sqrt{\frac{2}{3}}d_{ij}d_{ij}$$  \hfill (6.28)\

Uniaxial strain relaxation experiments have been used for material characterization in this work. For such a case, effective strain is given by: and therefore, the effective strain is,

$$\hat{\epsilon} = \epsilon$$  \hfill (6.29)\

Lai [69] suggests that a direct effect of the hydrostatic component of strain is that it changes the volume of a polymer and hence also the relaxation times. A positive hydrostatic pressure decreases the free volume and shifts the relaxation times to longer times while a negative pressure has the opposite effect. Therefore, it is more reasonable to assume that the time shift factor $a_\epsilon$ is a function of $\epsilon_{kk}$ rather than $\hat{\epsilon}$, namely,

$$a_\epsilon = a_\epsilon(\epsilon_{kk})$$  \hfill (6.30)
and
\[ a_e (−\epsilon_{kk}) = \frac{1}{a_e (\epsilon_{kk})} \]  

(6.31)

### 6.3 3-D Incremental Constitutive Equations

The hereditary integral form of the Schapery constitutive theory for multiaxial stress states has been developed in the previous section. The determination of the current stress and strain state of a polymer requires a knowledge of the stress and strain history due to the integration from time = 0 to time = \( t \). Storing the entire history information for large real world problems can be prohibitively expensive and impractical, hence the motivation for developing an incremental form of the constitutive model. In order to have the hereditary integration computed recursively, the transient relaxation modulus \( \Delta E(t) \) is expressed in the form of the Prony series, as suggested by Henriksen [68].

\[ \Delta E(\rho) = \sum_{n=1}^{N} E_n \exp \left( \frac{-\rho}{\tau_n} \right) \]  

(6.32)

In order to derive the incremental form, we first consider the following term from Equation 6.23, defining the quantity \( \Phi_1(t) \):

\[ \Phi_1(t) = \int_0^t \Delta E(\rho - \rho') \frac{d}{d\tau} (h_2 \epsilon_{kk}) d\tau \]  

(6.33)

Substituting Equation 6.32 into the above equation, we get:

\[ \Phi_1(t) = \sum_{n=1}^{N} \int_0^t E_n \exp \left( \frac{-\rho - \rho'}{\tau_n} \right) \frac{d}{d\tau} (h_2 \epsilon_{kk}) d\tau \]  

(6.34)

Consider only the \( n \)th term of the above summation (denoted by \( \Phi_n^1 \)). Then,

\[ \Phi_n^1(t) \]  

(6.35)
\[ \Phi_n^1(t) = \exp \left( -\frac{\Delta \rho}{\tau_n} \right) \Phi_n^1(t - \Delta t) + \frac{E_n \tau_n h_2 \Delta \epsilon_{kk}}{\Delta \rho} \left[ 1 - \exp \left( -\frac{\Delta \rho}{\tau_n} \right) \right] \]

(6.39)

Defining \( \alpha_n \) as

\[ \alpha_n = \exp \left( -\frac{\Delta \rho}{\tau_n} \right) \]

(6.40)

and \( \beta_n \) as

\[ \beta_n = \left[ 1 - \exp \left( -\frac{\Delta \rho}{\tau_n} \right) \right] = 1 - \alpha_n \]

(6.41)

Equation 6.39 can be written more simply as:

\[ \Phi_n^1(t) = \alpha_n \Phi_n^1(t - \Delta t) + \frac{E_n \tau_n h_2 \Delta \epsilon_{kk}}{\Delta \rho} \beta_n \]

(6.42)
Having obtained the above result, we will now derive a quantity that will be used later in
the incremental formulation.

\[ \Phi^n_1(t) - \Phi^n_1(t - \Delta t) = - (1 - \alpha_n) \Phi^n_1(t - \Delta t) + \frac{E_n\tau_nh2\Delta\varepsilon_{kk}}{\Delta\rho} \beta_n \] (6.43)

Using the definition of \( \beta_n \) in the above equation, we get:

\[ \Phi^n_1(t) - \Phi^n_1(t - \Delta t) = \beta_n \left[ \frac{E_n\tau_nh2\Delta\varepsilon_{kk}}{\Delta\rho} - \Phi^n_1(t - \Delta t) \right] \] (6.44)

Returning to Equation 6.23, the second integral on the RHS can similarly be evaluated.

Defining \( \Phi_2 \) as

\[ \Phi_2(t) = \int_0^t \Delta E(\rho - \rho') \frac{d}{d\tau} (h2\varepsilon_{ij}) d\tau \] (6.45)

By comparison of the form of the above equation with Equation 6.33, and defining \( \Phi^n_2 \) as

\[ \Phi^n_2(t) = \int_0^t E_n \exp \left( \frac{\rho - \rho'}{\tau_n} \right) \frac{d}{d\tau} (\varepsilon_{ij}) (h2\varepsilon_{ij}) d\tau \] (6.46)

we can directly write the final simplified form as:

\[ \Phi^n_2(t) = \alpha_n \Phi^n_2(t - \Delta t) + \frac{E_n\tau_nh2\Delta\varepsilon_{ij}}{\Delta\rho} \beta_n \] (6.47)

Similar to Equation 6.44, we can write

\[ \Phi^n_2(t) - \Phi^n_2(t - \Delta t) = \beta_n \left[ \frac{E_n\tau_nh2\Delta\varepsilon_{ij}}{\Delta\rho} - \Phi^n_2(t - \Delta t) \right] \] (6.48)

Recall Equation 6.23, repeated below, for purposes of clarity:

\[ \sigma_{ij}(t) = h_{c}E_{c} \left[ \frac{\nu\delta_{ij}\varepsilon_{kk}}{(1 + \nu)(1 - 2\nu)} + \frac{\varepsilon_{ij}}{(1 + \nu)} \right] + \frac{h_{1}\nu\delta_{ij}}{(1 + \nu)(1 - 2\nu)} \int_0^t \Delta E(t - \tau) \frac{d}{d\tau} (\varepsilon_{kk}) d\tau \]

\[ + \frac{h_{1}}{(1 + \nu)} \int_0^t \Delta E(t - \tau) \frac{d}{d\tau} (\varepsilon_{ij}) d\tau \] (6.49)
The above equation, combined with the definitions of the quantities \( \Phi_1 \) (Equation 6.33) and \( \Phi_2 \) (Equation 6.45), can be much more simply written as:

\[
\sigma_{ij}(t) = h_e E_e \left[ \frac{\nu \delta_{ij} \epsilon_{kk}}{(1 + \nu)(1 - 2\nu)} + \frac{\epsilon_{ij}}{(1 + \nu)} \right] + \frac{\delta_{ij} \nu h_1}{(1 + \nu)(1 - 2\nu)} \Phi_1(t) + \frac{h_1}{(1 + \nu)} \Phi_2(t)
\]  
(6.50)

Using Equations 6.37 and 6.46 in the above equation, we get:

\[
\sigma_{ij}(t) = h_e E_e \left[ \frac{\nu \delta_{ij}}{(1 + \nu)(1 - 2\nu)} \epsilon_{kk}(t) + \frac{1}{(1 + \nu)} \epsilon_{ij}(t) \right] 
+ \sum_{n=1}^{N} \left[ \frac{\nu \delta_{ij} h_1}{(1 + \nu)(1 - 2\nu)} \Phi^n_1(t) + \frac{h_1}{(1 + \nu)} \Phi^n_2(t) \right]
\]  
(6.51)

The above equation can be written for time \( t = t - \Delta t \) instead of time \( t \) to get:

\[
\sigma_{ij}(t - \Delta t) = h_e E_e \left[ \frac{\nu \delta_{ij}}{(1 + \nu)(1 - 2\nu)} \epsilon_{kk}(t - \Delta t) + \frac{1}{(1 + \nu)} \epsilon_{ij}(t - \Delta t) \right] 
+ \sum_{n=1}^{N} \left[ \frac{\nu \delta_{ij} h_1}{(1 + \nu)(1 - 2\nu)} \Phi^n_1(t - \Delta t) + \frac{h_1}{(1 + \nu)} \Phi^n_2(t - \Delta t) \right]
\]  
(6.52)

Subtracting Equation 6.52 from Equation 6.51, we get:

\[
\Delta \sigma_{ij}(t) = h_e E_e \left[ \frac{\nu \delta_{ij}}{(1 + \nu)(1 - 2\nu)} \Delta \epsilon_{kk}(t) + \frac{1}{(1 + \nu)} \Delta \epsilon_{ij}(t) \right] 
+ \sum_{n=1}^{N} \left[ \frac{\nu \delta_{ij} h_1}{(1 + \nu)(1 - 2\nu)} \left( \Phi^n_1(t) - \Phi^n_1(t - \Delta t) \right) 
+ \frac{h_1}{(1 + \nu)} \left( \Phi^n_2(t) - \Phi^n_2(t - \Delta t) \right) \right]
\]  
(6.53)

Recalling Equation 6.44 and Equation 6.48, repeated below:

\[
\Phi^n_1(t) - \Phi^n_1(t - \Delta t) = \beta_n \left[ \frac{E_n \tau_n h_2 \Delta \epsilon_{kk}}{\Delta \rho} - \Phi^n_1(t - \Delta t) \right]
\]  
(6.54)

and,

\[
\Phi^n_2(t) - \Phi^n_2(t - \Delta t) = \beta_n \left[ \frac{E_n \tau_n h_2 \Delta \epsilon_{ij}}{\Delta \rho} - \Phi^n_2(t - \Delta t) \right]
\]  
(6.55)
Using Equations 6.54 and 6.55 in Equation 6.53, we get the following form of the incremental constitutive relationships:

\[
\Delta \sigma_{ij}(t) = h_e E_x \left[ \frac{\nu \delta_{ij}}{(1 + \nu)(1 - 2\nu)} \Delta \epsilon_{kk}(t) + \frac{1}{1 + \nu} \Delta \epsilon_{ij}(t) \right] \\
+ \sum_{n=1}^{N} \frac{\delta_{ij} \nu h_1 \beta_n}{(1 + \nu)(1 - 2\nu)} \left( \frac{E_n \tau_n \Delta \epsilon_{kk}}{\Delta \rho} - \Phi_1^n(t - \Delta t) \right) \\
+ \frac{h_1 \beta_n}{1 + \nu} \left( \frac{E_n \tau_n \Delta \epsilon_{ij}}{\Delta \rho} - \Phi_2^n(t - \Delta t) \right)
\] (6.56)

Rearranging and collecting terms, we get:

\[
\Delta \sigma_{ij}(t) = h_e E_x \left[ \frac{\nu \delta_{ij}}{(1 + \nu)(1 - 2\nu)} \Delta \epsilon_{kk}(t) + \frac{1}{1 + \nu} \Delta \epsilon_{ij}(t) \right] \\
+ \sum_{n=1}^{N} \frac{h_1 h_2 \beta_n E_n \tau_n}{\Delta \rho} \left[ \frac{\nu \delta_{ij}}{(1 + \nu)(1 - 2\nu)} \Delta \epsilon_{kk}(t) + \frac{1}{1 + \nu} \Delta \epsilon_{ij}(t) \right] \\
- \sum_{n=1}^{N} \frac{\nu \delta_{ij} h_1 \beta_n}{(1 + \nu)(1 - 2\nu)} \Phi_1^n(t - \Delta t) + \frac{h_1 \beta_n}{1 + \nu} \Phi_2^n(t - \Delta t)
\] (6.57)

which can be simplified a step further to get:

\[
\Delta \sigma_{ij}(t) = \left[ h_e E_x + \sum_{n=1}^{N} \frac{h_1 h_2 \beta_n E_n \tau_n}{\Delta \rho} \right] \left[ \frac{\nu \delta_{ij} \Delta \epsilon_{kk}}{(1 + \nu)(1 - 2\nu)} + \frac{\Delta \epsilon_{ij}}{(1 + \nu)} \right] \\
- \sum_{n=1}^{N} \frac{\nu \delta_{ij} h_1 \beta_n}{(1 + \nu)(1 - 2\nu)} \Phi_1^n(t - \Delta t) + \frac{h_1 \beta_n}{1 + \nu} \Phi_2^n(t - \Delta t)
\] (6.58)

A slight rearrangement of terms in the above equation gives us:

\[
\Delta \sigma_{ij}(t) = \left[ h_e E_x + \sum_{n=1}^{N} \frac{h_1 h_2 \beta_n E_n \tau_n}{\Delta \rho} \right] \left[ \frac{\nu \delta_{ij} \Delta \epsilon_{kk}}{(1 + \nu)(1 - 2\nu)} + \frac{1}{(1 + \nu)} \Delta \epsilon_{ij} \right] \\
- \sum_{n=1}^{N} \frac{\nu \delta_{ij} h_1 \beta_n}{(1 + \nu)(1 - 2\nu)} \Phi_1^n(t - \Delta t) + \frac{h_1 \beta_n}{1 + \nu} \Phi_2^n(t - \Delta t)
\] (6.59)

The above equation gives us the relationship between the stress and strain tensor. With some manipulation, it can be written to provide the stress vector update in terms of the strain vector and other state variables \( \Phi_1^n, \Phi_2^n \). Defining new quantities \( \hat{\nu}, \bar{\nu} \) as,

\[
\hat{\nu} = \frac{\nu}{(1 + \nu)(1 - 2\nu)}
\]

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\[ \bar{\nu} = \frac{1}{1 + \nu} \quad (6.60) \]

Equation 6.59 can be written as:

\[
\Delta \sigma_{ij}(t) = \left[ h_e E_e + \sum_{n=1}^{N} \frac{h_1 h_2 \beta_n E_n \tau_n}{\Delta \rho} \right] \left[ \bar{\nu} \delta_{ij} \Delta \epsilon_{kk} + \bar{\nu} \Delta \epsilon_{ij} \right] - \sum_{n=1}^{N} h_1 \beta_n \left[ \bar{\nu} \delta_{ij} \Phi_1^n(t - \Delta t) + \bar{\nu} \Phi_2^n(t - \Delta t) \right] \quad (6.61)
\]

Converting the tensor stress and strain components into physical components, we can write:

\[
\Delta \sigma(t) = \left[ h_e E_e + \frac{h_1 h_2}{\Delta \rho} \sum_{n=1}^{N} \beta_n E_n \tau_n \right] M \Delta \epsilon - \sum_{n=1}^{N} h_1 \beta_n M \Psi^n(t - \Delta t) \quad (6.62)
\]

where,

\[
\Delta \sigma = \{ \Delta \sigma_{xx}, \Delta \sigma_{yy}, \Delta \sigma_{zz}, \Delta \sigma_{xy}, \Delta \sigma_{yz}, \Delta \sigma_{zx} \}^T
\]

\[
\Delta \epsilon = \{ \Delta \epsilon_{xx}, \Delta \epsilon_{yy}, \Delta \epsilon_{zz}, \Delta \epsilon_{xy}, \Delta \epsilon_{yz}, \Delta \epsilon_{zx} \}^T
\]

\[
\Psi^n = \{ \psi_{xx}, \psi_{yy}, \psi_{zz}, \psi_{xy}, \psi_{yz}, \psi_{zx} \}
\]

\[
\Psi_{xx}(t) = \int_0^t \Delta E(\rho - \rho') \frac{d}{d\tau} \left( h_2 \epsilon_{xx} \right) d\tau
\]

\[
\Psi_{yy}(t) = \int_0^t \Delta E(\rho - \rho') \frac{d}{d\tau} \left( h_2 \epsilon_{yy} \right) d\tau
\]

\[
\Psi_{zz}(t) = \int_0^t \Delta E(\rho - \rho') \frac{d}{d\tau} \left( h_2 \epsilon_{zz} \right) d\tau
\]

\[
\Psi_{xy}(t) = \int_0^t \Delta E(\rho - \rho') \frac{d}{d\tau} \left( h_2 \epsilon_{xy} \right) d\tau
\]

\[
\Psi_{yz}(t) = \int_0^t \Delta E(\rho - \rho') \frac{d}{d\tau} \left( h_2 \epsilon_{yz} \right) d\tau
\]

\[
\Psi_{zx}(t) = \int_0^t \Delta E(\rho - \rho') \frac{d}{d\tau} \left( h_2 \epsilon_{zx} \right) d\tau
\]

\[
\beta_n = \left[ 1 - \exp \left( -\frac{\Delta \rho}{\tau_n} \right) \right] \quad (6.63)
\]
and the matrix $[M]$ is defined as:

$$M = \begin{bmatrix}
\dot{\nu} + \bar{\nu} & \dot{\nu} & \dot{\nu} & 0 & 0 & 0 \\
\dot{\nu} & \dot{\nu} + \bar{\nu} & \dot{\nu} & 0 & 0 & 0 \\
\dot{\nu} & \dot{\nu} & \dot{\nu} + \bar{\nu} & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{\nu} & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{\nu} & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{\nu}
\end{bmatrix}$$  \hspace{1cm} (6.64)

Recalling the definitions of $\bar{\nu}$ and $\hat{\nu}$ (Equation 6.60), the $[M]$ matrix can be rewritten as:

$$M = \frac{1}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 - \nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1 - \nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1 - \nu & 0 & 0 & 0 \\
0 & 0 & 0 & 1 - 2\nu & 0 & 0 \\
0 & 0 & 0 & 0 & 1 - 2\nu & 0 \\
0 & 0 & 0 & 0 & 0 & 1 - 2\nu
\end{bmatrix}$$ \hspace{1cm} (6.65)

We also need update equations for the quantity $\Psi^n$ in Equation 6.62. In order to derive this relationship, we write:

$$M\Psi^n(t) \equiv \hat{\nu}\delta_{ij}\Phi^n_1(t) + \dot{\nu}\Phi^n_2(t)$$ \hspace{1cm} (6.66)

By substituting the update equations for $\Phi^n_1(t)$ (Equation 6.44) and $\Phi^n_2(t)$ (Equation 6.48) into the above equations performing some minor algebraic manipulations, we derive the update formula in vector form as,

$$\Psi^n(t) = \frac{E_n h_3 \tau n \beta_n}{\Delta \rho} \Delta \epsilon + \alpha_n \Psi^n(t - \Delta t)$$ \hspace{1cm} (6.67)
6.4 Summary of 3-D Incremental Constitutive Equations

We have described the detailed development of the constitutive equation in incremental form. Here, we present only the final result, for the purpose of clarity and brevity.

\[ \Delta \sigma(t) = \left[ h_e E_e + \frac{h_1 h_2}{\Delta \rho} \sum_{n=1}^{N} \beta_n E_n \tau_n \right] M \Delta \epsilon - \sum_{n=1}^{N} h_1 \beta_n M \Psi^n(t - \Delta t) \] (6.68)

\[ \Psi^n(t) = \frac{E_n h_2 \tau_n \beta_n}{\Delta \rho} \Delta \epsilon + \alpha_n \Psi^n(t - \Delta t) \] (6.69)

\[ \alpha_n = \exp \left( \frac{-\Delta \rho}{\tau_n} \right) \] (6.70)

\[ \beta_n = 1 - \alpha_n \] (6.71)

\[ \Delta \rho = \frac{\Delta t}{a_{cT}(t)} \] (6.72)

\[ M = \frac{1}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix} 1 - \nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1 - \nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1 - \nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 - 2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - 2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 - 2\nu \end{bmatrix} \] (6.73)
RELATIONSHIPS FOR DEVELOPMENT OF ANALYTICAL CANTILEVER HOOK MODEL

A.1 Force-Displacement Relationships for Cantilever Beam Under Concentrated Axial & Transverse End Loads and Concentrated End Moment

As discussed in Chapter 3, two beam deflection formulations have been used in this thesis. In the first formulation, the Euler Bernoulli beam theory coupled with superposition is used. The second formulation incorporates the effect of the axial force component in addition to the transverse force and moment. The derivation of expressions for the deflection and slope at the end of the beam is presented in the following sections. The steps in the development of the catch equilibrium equations and the geometric relationships are also provided here.

A.1.1 Euler-Bernoulli beam theory with superposition

This model assumes superposition to be valid. The total deflection of the end of the beam is calculated as the sum of the individual deflections due to the transverse end load and the end moment. The axial force does not appear either in the the final expression for the deflection curve or its slope.
The resultant bending moment at any cross-section of the beam is obtained by summing the moment due to the transverse force and applied moment. (Figure A.1). The bending moment at a distance $x$ from the fixed end of the beam is thus:

\[
M' = -M - F_t(L - x)
\]  \hspace{1cm} (A.1)

Using the Euler-Bernoulli beam theory, we can write:

\[
EI \frac{d^2y}{dx^2} = -M'
\]  \hspace{1cm} (A.2)

Substituting for bending moment $M'$ and integrating twice, we get:

\[
EI \frac{d^2y}{dx^2} = M + F_t(L - x)
\]  \hspace{1cm} (A.3)
\[ EI \frac{dy}{dx} = Mx + F_t L x - F_t \frac{x^2}{2} + C \quad \text{(A.4)} \]
\[ EI y = \frac{M}{2}x^2 + F_t L \frac{x^2}{2} - F_t \frac{x^3}{6} + Cx + D \quad \text{(A.5)} \]

The boundary conditions for this case are:

\[ @ \quad x = 0 \quad y = 0 \quad \text{(A.6)} \]
\[ @ \quad x = 0 \quad \frac{dy}{dx} = 0 \quad \text{(A.7)} \]

Substituting the boundary conditions into the expression for displacement, we find \( C = 0 \) and \( D = 0 \). Hence the displacement along the length of the beam can be expressed in the form:

\[ y = \frac{M}{2EI}x^2 + \frac{F_t x^2}{2EI} (L - \frac{x}{3}) \quad \text{(A.8)} \]

We are interested in the displacement at the end of the beam that attaches to the catch. It can be obtained by substituting \( x = L \) into Equation A.8. The displacement of the end of beam can be expressed as:

\[ Oy_R = ML^2 \frac{2EI}{2EI} + F_t L^3 \frac{3EI}{3EI} \quad \text{(A.9)} \]

This formulation does not capture any effects due to the shortening of the beam, hence the \( x \)-coordinate of the end of the beam does not change. Hence:

\[ Ox_R = L \quad \text{(A.10)} \]

The angle of rotation of the catch also enters into the formulation. This is controlled by the slope of the beam at the free end. Differentiating Equation A.8, we get an expression for the slope of the beam deflection curve along its length,

\[ \frac{dy}{dx} = \frac{M}{EI} x + \frac{F_t L}{EI} x - \frac{F_t}{2EI} x^2 \quad \text{(A.11)} \]

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The slope of the beam at its end is obtained by substituting \( x = L \) into Equation A.11.

\[
\Omega \left( \frac{dy}{dx} \right)_R = \frac{ML}{EI} + \frac{F_t L^2}{2EI}
\]  

(A.12)

It should be noted here that the Equations A.9 through A.12 have been derived under the assumed force directions shown in Figure A.1. The direction of the axial force component will change directions as the location of the contact point changes from the dwell to the retention face during retention - it will become tensile. The axial force will also be tensile during retention. The expressions for end deflection and end slope can be had by simply changing the sign of the \( F_a \) term. Hence, no special treatment of the two cases when \( F_a \) becomes tensile is necessary. The derivation of these relationships when the assumption of superposition is no longer made is presented next.

A.1.2 Euler Bernoulli Beam Theory Without Superposition Assumption

This formulation incorporates the additional bending moment due to the axial end load into the Euler-Bernoulli beam equation. The free body diagram of the beam for this case is shown in Figure A.2. Note that the forces are shown in the deformed configuration. By summing the forces and moments in this configuration, the axial force \( (F_a) \) contributes to the bending moment at any cross-section along the length of the beam, unlike the previous case, when it had no effect on the bending moment.

The bending moment at a section a distance \( x \) from the fixed end is given by the equation:

\[
M' = -M - F_a(y_l - y) - F_t(L - x)
\]  

(A.13)

Rearranging, we get,

\[
M' = (-M - F_a y_l - F_t L) + F_a y + F_t x
\]  

(A.14)
The beam equation can then be written as

\[ EI \frac{d^2 y}{dx^2} = - (-M - F_a y - F_t l) - F_a y - F_t x \]  \hspace{1cm} (A.15)

The above equation contains the term \( y_L \) which is the value of the variable \( y \) at \( x = L \). To simplify the equation, we differentiate it once more to get

\[ EI \frac{d^3 y}{dx^3} = -F_a \frac{dy}{dx} - F_t \]  \hspace{1cm} (A.16)

Rewriting

\[ \frac{d^3 y}{dx^3} + \frac{F_a}{EI} \frac{dy}{dx} = -\frac{F_t}{EI} \]  \hspace{1cm} (A.17)

The boundary conditions for this case are:

\[ @ \hspace{0.2cm} x = 0 \hspace{0.2cm} \quad y = 0 \]  \hspace{1cm} (A.18)
\[ \begin{align*}
@ & \quad x = 0 \quad \frac{dy}{dx} = 0 \quad \text{(A.19)} \\
@ & \quad x = L \quad \frac{d^2 y}{dx^2} = \frac{M}{EI} \quad \text{(A.20)}
\end{align*} \]

We can convert the above equation into a more generic form by introducing the following substitutions:

\[ \begin{align*}
\lambda^2 &= \frac{F_a}{EI} \quad \text{(A.21)} \\
\beta^2 &= \frac{F_t}{EI} \quad \text{(A.22)}
\end{align*} \]

Equation A.17 then becomes:

\[ \frac{d^3 y}{dx^3} + \lambda^2 \frac{dy}{dx} = -\beta^2 \quad \text{(A.23)} \]

The homogenous solution of the above equation is obtained by substituting \( y = \exp (\gamma x) \), to get the following characteristic equation:

\[ \gamma^3 + \lambda^2 \gamma = 0 \quad \text{(A.24)} \]

whose roots are \( \gamma = 0, \pm i\lambda \). Hence the homogenous solution is given as:

\[ y_H = c_1 + c_2 \exp (i\lambda x) + c_3 \exp (-i\lambda x) \quad \text{(A.25)} \]

The particular solution is obtained by substituting \( y = c_4 x \) in Equation A.23, to get

\[ y_p = -\frac{\beta^2}{\lambda^2} x \quad \text{(A.26)} \]

The total solution is therefore,

\[ y = c_1 + c_2 \exp (i\lambda x) + c_3 \exp (-i\lambda x) - \frac{\beta^2}{\lambda^2} x \quad \text{(A.27)} \]

We can solve for \( c_1, c_2 \) and \( c_3 \) by substituting the boundary conditions into Equation A.27 and solving a set of three simultaneous equations. The final expression for the deflection.
curve of the beam is obtained as:

\[ y = c_1 + c_2 e^{i\lambda x} + c_3 e^{-i\lambda x} - \frac{\beta^2}{\lambda^2} x \]  (A.28)

\[ c_1 = -\frac{1}{\lambda^2 (e^{i\lambda L} + e^{-i\lambda L})} \left[ -2\xi^2 + \frac{i\beta^2}{\lambda} \left( e^{i\lambda L} - e^{-i\lambda L} \right) \right] \]  (A.29)

\[ c_2 = \frac{1}{\lambda^2 (e^{i\lambda L} + e^{-i\lambda L})} \left[ -\xi^2 - \frac{i\beta^2 e^{-i\lambda L}}{\lambda} \right] \]  (A.30)

\[ c_3 = \frac{1}{\lambda^2 (e^{i\lambda L} + e^{-i\lambda L})} \left[ -\xi^2 + \frac{i\beta^2 e^{i\lambda L}}{\lambda} \right] \]  (A.31)

\[ c_2 = \frac{1}{\lambda^2 (e^{i\lambda L} + e^{-i\lambda L})} \left[ -\xi^2 - \frac{i\beta^2 e^{-i\lambda L}}{\lambda} \right] \]  (A.32)

The slope of the deflection curve is obtained to be:

\[ \frac{dy}{dx} = i\lambda c_2 e^{i\lambda x} - i\lambda c_3 e^{-i\lambda x} - \frac{\beta^2}{\lambda^2} \]  (A.33)

In order to obtain the deflection and slope at the end of the beam, we substitute \( x = L \) in Equation A.32 and Equation A.33 to get:

\[ O_y R = \frac{M}{F_a} \left[ \sec (\lambda L) - 1 \right] + \frac{F_t}{F_a} \left[ \tan (\lambda L) - \frac{\lambda}{\lambda} \right] \]  (A.34)

and,

\[ O \left( \frac{dy}{dx} \right)_R = \frac{M}{EI} \tan (\lambda L) + \frac{F_t}{F_a} \left[ \sec (\lambda L) - 1 \right] \]  (A.35)

The above equations have been derived under the assumption that the axial force component \( (F_a) \) is compressive (Figure A.2). Similar relationships can be derived for a tensile \( F_a \) following an identical procedure. Only the final results are presented here:

\[ y = c_1 + c_2 e^{\lambda x} + c_3 e^{-\lambda x} + \frac{\beta^2}{\lambda^2} x \]  (A.36)

\[ c_1 = -\frac{1}{\lambda^2 (e^{\lambda L} + e^{-\lambda L})} \left[ 2\xi^2 + \frac{\beta^2}{\lambda} \left( e^{\lambda L} - e^{-\lambda L} \right) \right] \]  (A.37)

\[ c_2 = \frac{1}{\lambda^2 (e^{\lambda L} + e^{-\lambda L})} \left[ \xi^2 - \frac{i\beta^2 e^{-\lambda L}}{\lambda} \right] \]  (A.38)
\[ c_3 = \frac{1}{\lambda^2 (e^{\lambda L} + e^{-\lambda L})} \left[ \xi^2 + \frac{\beta^2 e^{\lambda L}}{\lambda} \right] \quad \text{(A.39)} \]

\[ \frac{dy}{dx} = \lambda c_2 e^{\lambda x} - \lambda c_3 e^{\lambda x} + \frac{\beta^2}{\lambda^2} \quad \text{(A.40)} \]

The expressions for end-deflection and slope at end of beam are:

\[ O_y_R = -\frac{M}{F_a} \left[ \text{sech} (\lambda L) - 1 \right] + \frac{F_t}{F_a} \left[ \frac{\tanh (\lambda L)}{\lambda} - L \right] \quad \text{(A.41)} \]

\[ O \left( \frac{dy}{dx} \right)_R = \frac{M \tanh (\lambda L)}{EI} \frac{1}{\lambda} + \frac{F_t}{F_a} \left[ \sec (\lambda L) - 1 \right] \quad \text{(A.42)} \]

Although Equations A.41 & A.42 appear to have a different algebraic form than Equations A.34 & A.35 respectively, the two can be interconverted by replacing \( F_a \) by \(-F_a\) in these equations and in the expression for \( \lambda \) (Equation A.21). The latter is equivalent to replacing \( \lambda \) by \( i\lambda \) in the equations for end-deflection and slope at end of beam. In view of the above comments, no special treatment of the case when the axial force \( (F_a) \) is necessary in the solution algorithm.

### A.2 Derivation of Catch Equilibrium Equations

The equilibrium equations for insertion and retention processes need to be developed separately. This is due to the presence of the friction force. The relative motion between the cantilever hook and mating part is opposite during insertion and retention and so is the direction of the friction force.

#### A.2.1 Catch Equilibrium Equations for Insertion

The free body diagram of the catch is shown in Figure A.3. The forces acting on the catch are the contact forces from the mating part and the forces form the supporting beam of the cantilever hook snap-fit. From force equilibrium, we get:
Figure A.3: Free body diagram of cantilever hook catch during insertion.

\[ F_t + F_\mu \sin (\beta + \theta) - F_n \cos (\beta + \theta) = 0 \]  \hspace{0.5cm} (A.43)

\[ F_a - F_\mu \cos (\beta + \theta) - F_n \sin (\beta + \theta) = 0 \]  \hspace{0.5cm} (A.44)

Under the assumption that the two bodies are always on the verge of sliding, the relationship between the normal force \((F_n)\) and the friction force \((F_\mu)\) can be written as:

\[ F_\mu = \mu F_n \]  \hspace{0.5cm} (A.45)

Substituting this relationship between normal force and friction force into Equations A.43 & A.44, we get,

\[ \frac{F_a}{F_t} = \frac{\tan (\beta + \theta) + \mu}{1 - \mu \tan (\beta + \theta)} \]  \hspace{0.5cm} (A.46)

The above equation can be rewritten as:

\[ F_a [1 - \mu \tan (\beta + \theta)] - F_t [\tan (\beta + \theta) + \mu] = 0 \]  \hspace{0.5cm} (A.47)

Next, we establish the moment equilibrium equation for the cantilever hook catch. Considering moment equilibrium about the point C, we get:

\[ ^{R}M + {}^{R}{r_{CR}} \times ^{R}F_R = 0 \]  \hspace{0.5cm} (A.48)
\[-M^Rk - \left( R_{xC}^Ri + R_{yC}^Rj \right) \times \left[ (F_a \cos \theta - F_t \sin \theta)^Ri - (F_a \sin \theta + F_t \cos \theta)^Rj \right] = 0 \]  
(A.49)

Rearranging and collecting terms, we get:

\[-M^Rk + R_{xC}^R (F_a \sin \theta + F_t \cos \theta)^Rk + R_{yC}^R (F_a \cos \theta - F_t \sin \theta)^Rk = 0 \]  
(A.50)

In scalar form, the above equation can thus be written as:

\[ M - R_{xC}^R (F_a \sin \theta + F_t \cos \theta) - R_{yC}^R (F_a \cos \theta - F_t \sin \theta) = 0 \]  
(A.51)

The equilibrium equations Equations A.47 & A.51 are valid for contact along the insertion and dwell faces during the insertion process. When the contact point during insertion shifts to the retention face care should be taken to replace the insertion face angle ($\beta$) with the angle that the retention face makes from the positive direction of the $^Rx$ axis. Figure A.3 shows that angle to be ($\pi - \alpha$) and not just $\alpha$. Hence, the equilibrium equations for insertion when contact occurs on the retention face are obtained by replacing the quantity $\beta$ by ($\pi - \alpha$) and not by $\alpha$ alone.

### A.2.2 Catch Equilibrium Equations for Retention

The free body diagram of the catch during retention is shown in Figure A.4. The forces acting on the catch are similar to those during insertion, except the direction of the friction force ($F_\mu$) is reversed. From force equilibrium, we get:

\[ F_t + F_\mu \sin (\alpha - \theta) - F_n \cos (\alpha - \theta) = 0 \]  
(A.52)

\[ F_a + F_\mu \cos (\alpha - \theta) + F_n \sin (\alpha - \theta) = 0 \]  
(A.53)

Under the assumption that the two bodies are always in sliding contact, the relationship between the normal force ($F_n$) and the friction force ($F_\mu$) can be written as:

\[ F_\mu = \mu F_n \]  
(A.54)
Figure A.4: Free body diagram of cantilever hook catch during insertion.

Substituting this relationship between normal force and friction force into Equations A.43 & A.44, we get,

\[
\frac{F_a}{F_t} = - \frac{[\tan (\alpha - \theta) + \mu]}{1 + \mu \tan (\alpha - \theta)} \quad (A.55)
\]

The above equation can be rewritten as:

\[
F_a [1 + \mu \tan (\alpha - \theta)] + F_t [\tan (\alpha - \theta) + \mu] = 0 \quad (A.56)
\]

Next, we establish the moment equilibrium equation for the cantilever hook catch. Considering moment equilibrium about the point C, we get:

\[
R M + R_{CR} \times R F_R = 0 \quad (A.57)
\]

\[
-M R k - (R_{xC} R i + R_{yC} R j) \times \left[ (F_a \cos \theta - F_t \sin \theta) R i - (F_a \sin \theta + F_t \cos \theta) R j \right] = 0 \quad (A.58)
\]

Rearranging and collecting terms, we get:

\[
-M R k + R_{xC} (F_a \sin \theta + F_t \cos \theta) R k + R_{yC} (F_a \cos \theta - F_t \sin \theta) R k = 0 \quad (A.59)
\]
In scalar form, the above equation can thus be written as:

\[ M - R_x C (F_a \sin \theta + F_t \cos \theta) - R_y C (F_a \cos \theta - F_t \sin \theta) = 0 \quad (A.60) \]

The next section describes the development of kinematic relationships that help relate variables in the two coordinate systems and define the incremental motion of the contact point.

### A.3 Geometric Relationships

The goal of establishing geometric relationships is to completely locate the point of contact \( (C) \) between the mating part and the catch with respect to the \( O \) and the \( R \) coordinate systems. This is dependent upon the coordinates of the end of the beam, which defines the location of the origin of the coordinate system \( R \). Figure 3.8 shows the cantilever hook and mating part in a deformed configuration. We can write the following relationships directly:

\[ O \overrightarrow{OR} + O \overrightarrow{RC} = O \overrightarrow{OC} \quad (A.61) \]

\[ R \overrightarrow{RC} = [T] O \overrightarrow{RC} \quad (A.62) \]

\( [T] \) is a standard coordinate transformation matrix. In component form, the above equations become:

\[
\begin{pmatrix}
O_x R \\
O_y R
\end{pmatrix}
+ 
\begin{pmatrix}
O_x RC \\
O_y RC
\end{pmatrix}
= 
\begin{pmatrix}
O_x C \\
O_y C
\end{pmatrix}
\quad (A.63)
\]

\[
\begin{pmatrix}
R_x C \\
R_y C
\end{pmatrix}
= 
\begin{bmatrix}
\cos \theta & \sin \theta \\
- \sin \theta & \cos \theta
\end{bmatrix}
\begin{pmatrix}
O_x RC \\
O_y RC
\end{pmatrix}
\quad (A.64)
\]

For our model, the mating part moves horizontally towards or away from the cantilever hook. Its position is represented the location of the contact point \( C \). The \( x \)-coordinate of
the contact point is the quantity that is incremented in the algorithm to represent the motion of the mating part. The variables $O_x$ and $O_y$ are known. This can be formally expressed as:

$$ O_x = \delta $$

$$ O_y = -t $$

Another relationship is derived from the fact that the contact point lies on the insertion face of the catch. Hence its coordinates should satisfy the equation of the line representing the current contact face. Mathematically, this can be written as:

$$ R_y - R_x \tan \beta = d $$

The development of equations that enter into the analytical model discussed in Chapter 3 have been described in this part of the thesis. It is relevant to reiterate that the nature of the algorithm requires that some of the equations for insertion and retention be derived separately. The equations that are expected to be different in insertion and retention have been derived separately, and common equations have been presented for everything else. Also,
since the equation of the current contact face enters into the model, it becomes necessary to keep track of contact point coordinates and check if its incremented value falls outside the current face limits. When the end of the current face is reached, the face equation needs to be changed to represent the next contact face.
APPENDIX B

RESULTS OF STRESS RELAXATION EXPERIMENTS AND DATA REDUCTION
Figure B.1: Relaxation Modulus Curves for Polycarbonate at a Strain of 0.1%.
Figure B.2: Relaxation Modulus Curves for Polycarbonate at a Strain of 0.3%.
Figure B.3: Relaxation Modulus Curves for Polycarbonate at a Strain of 0.5%.
Figure B.4: Relaxation Modulus Curves for Polycarbonate at a Strain of 1.0%.
Figure B.5: Relaxation Modulus Curves for Polycarbonate at a Strain of 1.5%.
Figure B.6: Relaxation Modulus Curves for Polycarbonate at a Strain of 2.0%.
Figure B.7: Relaxation Modulus Curves for Polycarbonate at a Strain of 2.5%.
Figure B.8: Relaxation Modulus Curves for Polycarbonate at a Strain of 3.0%.
Figure B.9: Relaxation Modulus Curves for Polycarbonate at a Strain of 3.5%.
Figure B.10: Relaxation Modulus Curves for Polycarbonate at a Strain of 4.0%.
Figure B.11: Relaxation Modulus Curves for Polycarbonate at a Strain of 4.5%.
Figure B.12: Master Stress Relaxation Modulus Curves for Polycarbonate at a Reference Strain of 0.5% and Reference Temperature of 25°C.
Figure B.13: Shift factors $a(T,\varepsilon)$ as a Function of Temperature for Different Strain Levels for the Master Curve of Figure B.12 and Corresponding Least-Square Approximation.
Figure B.14: Shift factors $\alpha(T,\varepsilon)$ as a Function of Strain for Different Temperatures for the Master Curve of Figure B.12 and Corresponding Least-Squares Approximation.
Figure B.15: Vertical Shift factors $h_1 h_2$ as a Function of Temperature for Different Strain Levels for the Master Curve of Figure B.12 and Corresponding Least-Squares Approximation.
Figure B.16: Vertical shift factors $h_1h_2$ as a Function of Strain for Different Temperatures for the Master Curve of Figure B.12 and Corresponding Least-Squares Approximation.
Figure B.17: Variation of Horizontal Shift factors $a(T, \epsilon)$ with Strain and Temperature and Least-squares Fit Surface Plot.
Figure B.18: Variation of Vertical Shift factors $h_1 h_2$ with Strain and Temperature and Least-squares Fit Surface Plot.
APPENDIX C

DETAILED GEOMETRY OF CANTILEVER HOOK SPECIMENS
Figure C.1: Detailed dimensions of machined aluminum cantilever hook specimens.
Figure C.2: Detailed dimensions of machined steel cantilever hook specimens with 90° retention angle.
Figure C.3: Detailed dimensions of machined steel cantilever hook specimens with 93° retention angle.
APPENDIX D

LISTING OF PROGRAMS USED IN THIS DISSERTATION

D.1 MAPLE Script Containing Subroutines for Beam Formulation 1
(library_w0.maple)

restart;
with(linalg):
# Initialize some variables
nEqs := 7:
SymbolicJacobian := array(1..nEqs,1..nEqs):
SymbolicEquations := array(1..nEqs):
v := array(1..nEqs):
InitialVarArr := array(1..nEqs):
initialVar := array(1..nEqs):
FinalSolutionList := list():

# Procedure for plotting certain columns from a list of data.
The output from the solution subroutine is a list of lists.
The subroutine takes the complete solution and the indices of
of the columns to be plotted as input and generates a plot.
PlotColumns := proc(VarList,XAxisIndex,YAxisIndex,DisplayOptions)
local opts, PlotList, NumOfTerms, i;
opts := \[args[4..nargs]\];
PlotList := list():
NumOfTerms := nops(VarList):
for i from 1 to NumOfTerms do
PlotList :=
[op(PlotList),[VarList[i,XAxisIndex],VarList[i,YAxisIndex]]]:
od:
plot(PlotList,seq(opts[i],i=1..nargs-3)):
end proc:

########################################################################
writeData := proc(VarList,filename)
local varList, printList, printLst, fd, insAnglea, retAnglea:
insertAnglea := evalf(insertAngle*180/\Pi);
retAnglea := evalf(retAngle*180/\Pi);
fd := fopen(filename, WRITE, TEXT):
fprintf(fd, \"EI \t nu \t L \t t/2 \t alpha \t beta \t y \t d \n\"):
fprintf(fd,\"%f %f %f %f %f %f %f %f %f \n", sectionModulus, frictionCoeff, beamLength,

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halfbeamThick, insAnglea, retAnglea, catchOffset, catchDwell);
writeData(fd, VarList):
fclose(fd):
end proc:

#******************************************************************************

matrixPlot := proc()
local pp,
pp := array(1..2,1..3,[[]]):
pp[1,1] := PlotColumns(FinalSolutionList, 1, 8, labels = ["Abs_x","Axial_force"], style = point, axesfont = [HELVETICA,4], axes= boxed, labelfont = [HELVETICA,4]):
pp[1,2] := PlotColumns(FinalSolutionList, 2, 8, style = point, labels = ["Abs_x","Transverse_force"], axesfont = [HELVETICA,2], axes= boxed, labelfont = [HELVETICA,4]):
pp[1,3] := PlotColumns(FinalSolutionList, 3, 8, style = point, labels = ["Abs_x","Moment"], axesfont = [HELVETICA,2], axes= boxed, labelfont = [HELVETICA,4]):
pp[2,1] := PlotColumns(FinalSolutionList, 4, 5, style = point, labels = ["Rel_x","Rel_y"], axesfont = [HELVETICA,2], axes= boxed, labelfont = [HELVETICA,4]):
pp[2,2] := PlotColumns(FinalSolutionList, 4, 6, style = point, labels = ["Rel_x","Beam_Rotation"], axesfont = [HELVETICA,2], axes= boxed, labelfont = [HELVETICA,4]):
pp[2,3] := PlotColumns(FinalSolutionList, 1, 7, style = point, labels = ["Abs_x","Abs_y"], axesfont = [HELVETICA,2], axes= boxed, labelfont = [HELVETICA,4]):
display(pp):
end proc:

#******************************************************************************

cREATEPlot := proc(var1, var2, var3, var4, var5, var6)
# var1 -> Axial Force
# var2 -> Transverse Force
# var3 -> Moment
# var4 -> Local x
# var5 -> Rotation of beam
local abspointH, abspointI, abspointJ, abspointK, abspointL, abspointM, currPlot, origin, pointR, lambda, xi, beta, c1, c2, c3:
global beamPlot, catchPlot, matingPlot:
origin := [0,0]:
pointR := [beamLength, var2*beamLength^3/(3*sectionModulus) + var3*beamLength^2/(2*sectionModulus)]:
pointC := [var4, var5]:
abspointH := convertCoord(pointH, pointR, var6):
abspointI := convertCoord(pointI, pointR, var6):
abspointJ := convertCoord(pointJ, pointR, var6):
abspointK := convertCoord(pointK, pointR, var6):
abspointL := convertCoord(pointL, pointR, var6):
abspointM := convertCoord(pointM, pointR, var6):
abspointC := convertCoord(pointC, pointR, var6):
abspointCC := [abspointC[1], abspointC[2] - catchOffset]:
plotList := [pointR, abspointH, abspointI, abspointJ, abspointK, abspointL, abspointM, pointR]:
beamPlot := plot(var3*x^2/(2*sectionModulus) +

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var2*x^2/(2*sectionModulus)*(beamLength - x/3),
x=0..beamLength, view = [0..2, -1..1],
axes = none, scaling = constrained):
catchPlot := plot(plotList , view = [0..2, -0.2..0.25], axes = none,
scaling = constrained, style = line):
matingPlot := plot([abspointC, abspointC], view = [0..2, -0.2..0.25],
axes = none, scaling = constrained, style = line,
color = green);
totalPlot := display(beamPlot, catchPlot, matingPlot):
return totalPlot:
end proc:
#########################################################################
animatePlot := proc(varList)
# This subroutine will read the number of lines of results in
# FinalSolutionList and then execute
# a for loop that repeatedly draws a line sketch of the
# cantilever hook feature for the different
# solution positions.
# varList -> variable list containing the solutions
# numof Terms -> number of terms in varList
local numofTerms, i, currPlot:
global animPlot;
animPlot := list():
numofTerms := nops(varList):
for i from 1 to numofTerms do
# Recall that varList[i,8] contains the absolute axial
# displacement.
currPlot := createPlot(varList[i,8], varList[i,2], varList[i,3],
varList[i,4], varList[i,5], varList[i,6]):
animPlot := [op(animPlot), currPlot]:
end od:
end proc:
#########################################################################
convertCoord := proc(point1, pointR, v6)
# point1 -> Local coordinates of point that need to be transformed into
# global coordinates
# pointR -> Global coordinates of the origin of the catch-fixed coordinate
# system
# abspoint1 -> Global coordinates of the point passed to this subroutine
# v2 -> current value of solution variable #2, corresponding to F_t
# v3 -> current value of solution variable #3, corresponding to M
# v6 -> current value of solution variable #6, corresponding to theta
local abspoint1:
abspoint1 :=[pointR[1] + point1[1]*cos(v6) - point1[2]*sin(v6),
pointR[2] + point1[1]*sin(v6) + point1[2]*cos(v6)]:
return abspoint1:
end proc:
#########################################################################
readDesigndata := proc(filename)
local file, fn;
global sectionModulus, frictionCoeff, beamLength, halfbeamThick,
insAngle, retAngle, catchOffset, catchDwell, designData, lst;
file := filename:fn := fopen(file,READ):
while (not feof(fn)) do
lst := fscanf(fn, '%f %f %f %f %f %f %f %f'): od:close(fn):

sectionModulus := lst[1];
frictionCoeff := lst[2];
beamLength := lst[3];
halfbeamThick := lst[4];
insAngle := evalf(lst[5]*Pi/180);
retAngle := evalf(lst[6]*Pi/180);
catchOffset := lst[7];
catchDwell := lst[8];
end proc:

#########################################################################
convertDesignData := proc()
# Design Parameters -> Script Parameters
# The input file contains design parameters, variables that the designer
# would know values of. However,
# the script needs the parameters in slightly different format. The
# following code performs the conversions.
# Three points H,I,J have been identified.
# H -> End point of retention face (where it meets the beam)
# I -> Transition between dwell and retention faces
# J -> Transition between insertion and dwell faces
# The coordinates of these points are defined first, on the basis of the
# design inputs and then other variables
# are defined based on the coordinates of these points.

global pointH, pointI, pointJ, pointK, pointL, pointM, insIntercept,
retIntercept, dwellIntercept, dwellAngle:
pointH := array(1..2 , [0 , -halfbeamThick]):
pointI := array(1..2 , [catchOffset*cot(retAngle) ,
-(halfbeamThick+catchOffset)]):
pointJ := array(1..2 , [catchOffset*cot(retAngle)+catchDwell ,
-(halfbeamThick+catchOffset)]):
pointK := array(1..2 , [pointJ[1] + 1.25*catchOffset*cot(insAngle) ,
pointJ[2] + 1.25*catchOffset]):
pointL := array(1..2 , [pointK[1] , halfbeamThick]):
pointM := array(1..2 , [0 , halfbeamThick]):
retIntercept := -halfbeamThick:
dwellAngle := 0:
end proc:

#########################################################################
# Procedure for Jacobian evaluation
findJacobian := proc(SymEquations, Variables, constantList)
local SymJacobian, i, j, numJacobian:
SymJacobian := array(1..nEqs,1..nEqs):
numJacobian := array(1..nEqs,1..nEqs):
for i from 1 to nEqs do
  for j from 1 to nEqs do
    SymJacobian[i,j] := diff(SymEquations[i],Variables[j]):
  od:
numJacobian := subs(constantList, evalm(SymJacobian)):
end proc:

#########################################################################
getEquations := proc(whichFace, constantList)
local SymbolicEquations, numEquations:
SymbolicEquations := array(1..nEqs):
numEquations := array(1..nEqs):
if (whichFace = 1) then
  # Insertion face equations
  SymbolicEquations[3] := v[1]*(1-mu*tan(beta)) - v[2]*(tan(beta)+mu);
elif (whichFace = 2) then
  # Dwell face equations
elif (whichFace = 3) then
  # Retention face equations
elif (whichFace = 4) then
  # Note here than v[2] and v[8] have been interchanged from the previous
  # sets of equation. This is to assist incrementation - we will increment
  # axial force instead of x-displacement.
fi:
numEquations := subs(constantList, evalm(SymbolicEquations));
end proc:

solutionLoop := proc(iniSoln, iniVar, symEqs, symJac, faceStart, faceEnd, inc)
  ## iniSoln -> Initial solution array, excluding the variable to be incremented
  ## iniVar -> Initial value of variable to be incremented
  ## incVar -> Amount the incremental variable should be incremented by
  numEquations := subs(constantList, evalm(SymbolicEquations));
end proc:
local pointonFaceflag, guessSoln, convSoln, incVar, k, numVar, eps:

# Prepare variables for passing to the solution subroutine
eps := 1e-7:
numVar := 7:
k := 100:
guessSoln := array(1..nEqs):
guessSoln := iniSoln:
incVar := iniVar:

# symEqs is passed to this routine
# symJac is passed to this routine

# Prepare flags
pointonFaceflag := 1:
convFlag := 1:
while (pointonFaceflag = 1) do
    # Call a subroutine to find the converged solution
    # The subroutine returns convSoln
    results := newtonRaph(eps, numVar, k, guessSoln, incVar,
                      symEqs, symJac):
    convSoln := results[1]:
    convFlag := results[2]:
    # If convFlag = 2 then the limit on the number of iterations has
    # been exceeded. Return!
    if (convFlag = 2) then return fi:
    # Check if converged solution is still within the limits of the current
    # face. The subroutine checkPoint returns 1 if the point is still on
    # the face and 0 otherwise
    # convSoln[3], convSoln[4] -> relative x and y coordinates of contact
    # point
    pointonFaceflag := checkPoint(convSoln[4], convSoln[5],
                                   faceStart, faceEnd):
    if (pointonFaceflag = 1) then
        # Store converged solution
        FinalSolutionList := [op(FinalSolutionList),
                               [seq(convSoln[i],i=1..7), incVar]]:
        # Increment incVar
        incVar := incVar + inc:
        # Reset guessSoln to convSoln
        guessSoln := convSoln:
    else
        print("Converged Solution is outside the face"): print(convSoln):
    fi:
end proc:

########################################################################
newtonRaph := proc(eps, n, k, iniSoln, incVar, symEq, symJac)
# eps -> Convergence criteria
# n -> Number of equations and variables
# k -> Maximum number of iterations

# Prepare variables for passing to the solution subroutine
eps := 1e-7:
umVar := 7:
k := 100:
iniSoln := array(1..nEqs):
motherLine := [seq(iniSoln[i],i=1..nEqs),
              incVar, k, numVar, eps]:

# Prepare flags
pointonFaceflag := 1:
convFlag := 1:
while (pointonFaceflag = 1) do
    # Call a subroutine to find the converged solution
    # The subroutine returns convSoln
    results := newtonRaph(eps, numVar, k, motherLine,
                       symEq, symJac):
    convSoln := results[1]:
    convFlag := results[2]:
    # If convFlag = 2 then the limit on the number of iterations has
    # been exceeded. Return!
    if (convFlag = 2) then return fi:
    # Check if converged solution is still within the limits of the current
    # face. The subroutine checkPoint returns 1 if the point is still on
    # the face and 0 otherwise
    # convSoln[3], convSoln[4] -> relative x and y coordinates of contact
    # point
    pointonFaceflag := checkPoint(convSoln[4], convSoln[5],
                                   faceStart, faceEnd):
    if (pointonFaceflag = 1) then
        # Store converged solution
        FinalSolutionList := [op(FinalSolutionList),
                               [seq(convSoln[i],i=1..7), incVar]]:
        # Increment incVar
        incVar := incVar + inc:
        # Reset guessSoln to convSoln
        guessSoln := convSoln:
    else
        print("Converged Solution is outside the face"): print(convSoln):
    fi:
end proc:
initSoln -> Initial (guess solution)
incVar -> Value of variable being incremented
symEq -> Equations, with constants substituted in already
symJac -> Jacobian matrix, with constants substituted in already
convFlag -> Convergence flag, returned by convergence checking routine?
trySoln -> Trial solution (array)
totTryList -> Total trial solution list, also includes variable being
incremented. Used for finding numerical values of the Jacobian
and Equations. (8 elements)
incVarList -> Incremental variable in list form, [v8=xx]
convSoln -> Converged solution
v -> Symbolic array of variables
numJac -> Numerical Jacobian matrix
numEq -> Numerical value of equations
suggInc -> Suggested increment in variable values

local trySoln, trySolnList, numJac, numEq, job, newTrySolnList, newtrySoln,
totTryList, suggInc, incVarList, totTryList, convSoln, newtrySolnList:

convSoln := array(1..nEqs):
newtrySoln := array(1..nEqs):
job := 0:
convFlag := 0:
trySoln := iniSoln:
incVarList := [v8=incVar]:
while (job <= k) do
job := job + 1:
# Form a list of trial variables, so that they can be substituted into the
# symbolic Jacobian and equations. Note that the incremental variable (v8)
# is also included, in the 2nd statement.
trySolnList := [seq(v[i] = trySoln[i], i=1..n)]:
totTryList := [op(trySolnList), op(incVarList)]:
# Substitute above list into symbolic forms of the Jacobian and Equations
numJac := subs(totTryList, evalm(symJac)): numEq := subs(totTryList, evalm(symEq)):
# Get suggested variable increment by solving numJac*suggInc=numEq
suggInc := -1*linsolve(numJac, numEq):
newtrySoln := matadd(trySoln, suggInc):
# Convert the new trial solution into list form
newtrySolnList := [seq(v[i] = newtrySoln[i], i=1..n)]:
# Append the value of the variable being incremented to
# the above list, to send to the checkConvergence routine
# later.
newtotTryList := [op(newtrySolnList), incVarList]:
# Check if the new solution is a converged solution or not.
# If not, check if it has iterated too many times - exit out and print a
# message. Otherwise, set the new solution to be the trial solution for
# the next increment.
convFlag := checkConvergence(newtotTryList, symEq, eps):
if (convFlag = 1) then
norm1 := computeNorm(newtotTryList, symEq, eps):
condition := evalf(cond(numJac)) :
printf("%10.7f",incVar, trySoln[5],
job,condition, norm1):
return newtrySoln, convFlag:
else
trySoln := newtrySoln:
fi:
od:

print("Too many iteration without convergence"): convFlag := 2:
return newtrySoln, convFlag:
end proc:

*****************************************************************************
checkPoint := proc(pointX, pointY, faceStart, faceEnd)
# pointX -> x coordinate of point to be checked
# pointY -> y coordinate of point to be checked
# faceStart -> array[1..2] relative coordinates of starting point of face
# faceEnd -> array[1..2] relative coordinates of ending point of face
# midPoint -> array[1..2] relative coordinates of midpoint of face
# distMid -> distance of current contact point from face midpoint
# flag -> flag for whether point is on face (1 = yes).
local midPoint, faceLength, distMid, flag:
midPoint := array(1..2):
faceLength := sqrt((faceStart[1]-faceEnd[1])^2 +
    (faceStart[2]-faceEnd[2])^2):
midPoint := evalm(0.5*matadd(faceStart, faceEnd)):
distMid := sqrt((pointX-midPoint[1])^2 + (pointY-midPoint[2])^2):
if (2*distMid < faceLength + 0.001) then
    flag := 1:
else
    flag := 0:
fi:
flag:
end proc:

checkConvergence := proc(solutionList, symbolicEquations, epsilon)
local numericEquations, convergenceFlag:
# substitute values of variables into symbolic form of equations
numericEquations := evalf(subs(op(solutionList), evalm(symbolicEquations))):
# check for convergence and assign flag accordingly
if norm(numericEquations, 1) >= epsilon then
    convergenceFlag := 0:
else
    convergenceFlag := 1:
end if:
end proc:

computeNorm := proc(solutionList, symbolicEquations, epsilon)
local numericEquations, convergenceFlag:
numericEquations := evalf(subs(op(solutionList), evalm(symbolicEquations))):
norm(numericEquations, 1):
end proc:
restart:

with(linalg):

# Initialize some variables
nEqs := 7:
SymbolicJacobian := array(1..nEqs,1..nEqs):
SymbolicEquations := array(1..nEqs):
v := array(1..nEqs):
InitialVarArr := array(1..nEqs):
initialVar := array(1..nEqs):
FinalSolutionList := list():

#Procedure for plotting certain columns from a list of data.
The output from the solution subroutine is a list of lists.
The subroutine takes the complete solution and the indices of
of the columns to be plotted as input and generates a plot.

PlotColumns := proc(VarList,XAxisIndex,YAxisIndex,DisplayOptions)
local opts, PlotList, NumOfTerms, i;
opts := [args[4..nargs]];
PlotList := list();
NumOfTerms := nops(VarList);
for i from 1 to NumOfTerms do
PlotList := [op(PlotList), [VarList[i,XAxisIndex],VarList[i,YAxisIndex]]]:
od: plot(PlotList,seq(opts[i],i=1..nargs-3)):
end proc:

writeData := proc(VarList,filename)
#Dump the solution list into a text file.
local varList, printList, printLst, fd:
fd := fopen(filename, WRITE, TEXT):
fprintf(fd, "# EI 		 nu 		 L 		 t/2 		 alpha 		 beta 		 y 		 d 
"
sectionModulus, frictionCoeff, beamLength,
halfbeamThick, insAngle, retAngle, catchOffset, catchDwell);
writedata(fd, VarList):
fclose(fd):
end proc:

matrixPlot := proc()
local pp:
pp := array(1..2,1..3,[]):
pp[1,1] := PlotColumns(FinalSolutionList, 1, 8, labels = [
"Abs_x","Axial_force"], style = point,
axesfont = [HELVETICA,4], axes=boxed,
labelfont = [HELVETICA,4]):
pp[1,2] := PlotColumns(FinalSolutionList, 2, 8, style = point,
labels = ["Abs_x","Transverse_force"],
axesfont = [HELVETICA,2], axes=boxed,
labelfont = [HELVETICA,4]):
pp[1,3] := PlotColumns(FinalSolutionList, 3, 8, style = point,
labels = ["Abs_x","Moment"],
axesfont = [HELVETICA,2], axes=boxed,
labelfont = [HELVETICA,4]):
pp[2,1] := PlotColumns(FinalSolutionList, 4, 5, style = point,
labels = ["Rel_x","Rel_y"],
axesfont = [HELVETICA,2], axes=boxed,
labelfont = [HELVETICA,4]):

pp[2,2] := PlotColumns(FinalSolutionList, 4, 6, style = point,
labels = ["Rel_x","Beam_Rotation"],
axesfont = [HELVETICA,2], axes= boxed,
labelfont = [HELVETICA,4]):

pp[2,3] := PlotColumns(FinalSolutionList, 1, 7, style = point,
labels = ["Abs_x","Abs_y"],
axesfont = [HELVETICA,2], axes= boxed,
labelfont = [HELVETICA,4]):

display(pp):
end proc:

#########################################################################
createPlot := proc(var1, var2, var3, var4, var5, var6 )
# var1 -> Axial Force
# var2 -> Transverse Force
# var3 -> Moment
# var4 -> Local x
# var5 -> Local y
# var6 -> Rotation of beam
local abspointH,abspointI,abspointJ,abspointK,abspointL,abspointM,
currPlot, origin, pointR, lambda, xi, beta, c1, c2, c3:
global beamPlot, catchPlot, matingPlot:
lambda := sqrt(var1/sectionModulus):
beta := sqrt(var2/sectionModulus):
xi := sqrt(var3/sectionModulus):
c1 := -1/(lambda*lambda*(exp(I*lambda*beamLength)+
exp(-I*lambda*beamLength)))*(-2*xi*xi+I*beta*beta/lambda*
(exp(I*lambda*beamLength)-exp(-I*lambda*beamLength))):
c2 := 1/(lambda*lambda*(exp(I*lambda*beamLength)+exp(-I*lambda*beamLength)))*
(-xi*xi-I*beta*beta/lambda*exp(-I*lambda*beamLength)):
c3 := 1/(lambda*lambda*(exp(I*lambda*beamLength)+exp(-I*lambda*beamLength)))*
(-xi*xi+I*beta*beta/lambda*exp(I*lambda*beamLength)):
beamFunction := c1 + c2*exp(I*lambda*x) + c3*exp(-I*lambda*x) -
beta^2/lambda^2*x:
origin := [0,0]:
pointR := [beamLength, c1 + c2*exp(I*lambda*beamLength) +
c3*exp(-I*lambda*beamLength) - beta^2/lambda^2*beamLength]:
pointC := [var4, var5]:
abspointH := convertCoord(pointH, pointR, var6):
abspointI := convertCoord(pointI, pointR, var6):
abspointJ := convertCoord(pointJ, pointR, var6):
abspointK := convertCoord(pointK, pointR, var6):
abspointL := convertCoord(pointL, pointR, var6):
abspointM := convertCoord(pointM, pointR, var6):
abspointC := convertCoord(pointC, pointR, var6):
abspointCC := [abspointC[1], abspointC[2] - catchOffset]:
plotList := [pointR, abspointH, abspointI, abspointJ,abspointK,
abspointL, abspointM, pointR]:
beamPlot := plot(beamFunction, x=0..beamLength, view = [0..1.5, -0.2..0.25],
axes = none, scaling = constrained):
catchPlot := plot(plotList , view = [0..2, -0.2..0.25], axes = none,
scaling = constrained, style = line):
matingPlot := plot([abspointC, abspointCC], view = [0..2, -0.2..0.25],
axes = none, scaling = constrained, style = line,
color = green):
totalPlot := display(beamPlot, catchPlot, matingPlot):
return totalPlot:
end proc:

#########################################################################
animatePlot := proc(varList)
# This subroutine will read the number of lines of results in
# FinalSolutionList and then execute
# a for loop that repeatedly draws a line sketch of the
# cantilever hook feature for the different
# solution positions.
# varList -> variable list containing the solutions
# numofTerms -> number of terms in varList

local numofTerms, i, currPlot:

numofTerms := nops(varList):
for i from 1 to numofTerms do
    currPlot := createPlot(varList[i,8], varList[i,2], varList[i,3], varList[i,4],
                           varList[i,5], varList[i,6]):
    animPlot := [op(animPlot), currPlot]:
od:
end proc:

convertCoord := proc(point1, pointR, v6)
# point1 -> Local coordinates of point that need to be transformed into
# global coordinates
# pointR -> Global coordinates of the origin of the catch-fixed coordinate
# system
# absPoint1 -> Global coordinates of the point passed to this subroutine
# v2 -> current value of solution variable #2, corresponding to F_t
# v3 -> current value of solution variable #3, corresponding to M
# v6 -> current value of solution variable #6, corresponding to theta

local absPoint1:
    absPoint1 :=[pointR[1] + point1[1]*cos(v6) - point1[2]*sin(v6),
                 pointR[2] + point1[1]*sin(v6) + point1[2]*cos(v6)]:
    return absPoint1:
end proc:

readDesignData := proc(filename)
local file, fn;

global sectionModulus, frictionCoeff, beamLength, halfBeamThick,
      insAngle, retAngle, catchOffset, catchDwell, designData, lst;

file := filename:fn := fopen(file,READ):
while (not feof(fn)) do
    lst := fscanf(fn, '%f %f %f %f %f %f %f %f'):
od:
fclose(fn):

sectionModulus := lst[1]:
frictionCoeff := lst[2]:
beamLength := lst[3]:
halfBeamThick := lst[4]:
insAngle := evalf(lst[5]*Pi/180):
retAngle := evalf(lst[6]*Pi/180):
catchOffset := lst[7]:
catchDwell := lst[8]:
end proc:

convertDesignData := proc()
# Design Parameters -> Script Parameters
# The input file contains design parameters, variables that the designer
# would know values of. However,  
# the script needs the parameters in slightly different format. The  
# following code performs the conversions.  
# Three points H,I,J have been identified.  
# H -> End point of retention face (where it meets the beam)  
# I -> Transition between dwell and retention faces  
# J -> Transition between insertion and dwell faces  
# The coordinates of these points are defined first, on the basis of the  
# design inputs and then other variables  
# are defined based on the coordinates of these points.

global pointH, pointI, pointJ, pointK, pointL, pointM, insIntercept,  
retIntercept, dwellIntercept, dwellAngle:

    pointH := array(1..2 , [0 , -halfbeamThick]);  
    pointI := array(1..2 , [catchOffset*cot(retAngle) ,  
                        -(halfbeamThick+catchOffset)]);  
    pointJ := array(1..2 , [catchOffset*cot(retAngle)+catchDwell ,  
                        -(halfbeamThick+catchOffset)]);  
    pointK := array(1..2 , [pointJ[1] + 1.25*catchOffset*cot(insAngle) ,  
                        pointJ[2] + 1.25*catchOffset]);  
    pointL := array(1..2 , [pointK[1] , halfbeamThick]);  
    pointM := array(1..2 , [0 , halfbeamThick]);  
    retIntercept := -halfbeamThick;  
    dwellAngle := 0;  

end proc:

elif (whichFace = 2) then
  # Dwell face equations
  SymbolicEquations[2] := v[1]*v[1]*(sqrt(EI)*tan(v[6]) - v[3])*sqrt(v[1])^3*tan(sqrt(v[1]/EI)*L) - v[1]*v[2]*sqrt(EI)*(sec(sqrt(v[1]/EI)*L) - 1);

else (whichFace = 3) then
  # Retention face equations
  SymbolicEquations[2] := v[1]*v[1]*sqrt(EI)*tan(v[6]) - v[3]*sqrt(v[1])^3*tan(sqrt(v[1]/EI)*L) - v[1]*v[2]*sqrt(EI)*(sec(sqrt(v[1]/EI)*L) - 1);

elif (whichFace = 4) then
  # Note here than v[1] and v[8] have been interchanged from the previous
  # sets of equation. This is to assist incrementation - we will increment
  # axial force instead of x- displacement.
  SymbolicEquations[2] := v[8]*v[8]*sqrt(EI)*tan(v[6]) - v[3]*sqrt(v[8])^3*tan(sqrt(v[8]/EI)*L) - v[1]*v[2]*sqrt(EI)*(sec(sqrt(v[8]/EI)*L) - 1);
fi:

# Substitute the constant list into the symbolic form and return value.
numEquations := subs(constantList, evalm(SymbolicEquations));
end proc:

solutionLoop := proc(iniSoln, iniVar, symEqs, symJac, faceStart, faceEnd, inc)
  # iniSoln -> Initial solution array, excluding the variable to be incremented
  # iniVar -> Initial value of variable to be incremented
  # increment -> Amount the incremental variable should be incremented by
  # incVar -> Value of variable to be incremented
  # symEqs -> Symbolic equations, with constants substituted already
  # symJac -> Symbolic Jacobian matrix, with constants substituted already
  # guessSoln -> Guess solution for the solution subroutine
  # convSoln -> Converged solution returned by the solution subroutine
  # pointonFaceflag -> Flag for whether contact point is still on
end proc:
# convFlag - Flag returned by solution subroutine, indicating convergence condition
# faceStart - array[1..2] relative coordinates of face starting point
# faceEnd - array[1..2] relative coordinates of face end point
# inc - Iteration increment
# eps - Convergence check
# k - Max number of iterations

local pointonFaceflag, guessSoln, convSoln, incVar, k, numVar, eps:
global FinalSolutionList:

# Prepare variables for passing to the solution subroutine
eps := 1e-7:
umVar := 7:
k := 100:
guessSoln := array(1..nEqs):
guessSoln := iniSoln:
incVar := iniVar:

# symEqs is passed to this routine
# symJac is passed to this routine

# Prepare flags
pointonFaceflag := 1:
convFlag := 1:

while (pointonFaceflag = 1) do

## Call a subroutine to find the converged solution
# The subroutine returns convSoln
# The subroutine returns convSoln

results := newtonRaph(eps, numVar, k, guessSoln, incVar, symEq, symJac):

convSoln := results[1]:
convFlag := results[2]:

# If convFlag = 2 then the limit on the number of iterations has been exceeded. Return.
if (convFlag = 2) then
fi:

# Check if converged solution is still within the limits of the current face. The subroutine checkPoint returns 1 if the point is still on the face and 0 otherwise.

pointonFaceflag := checkPoint(convSoln[4], convSoln[5], faceStart, faceEnd):

if (pointonFaceflag = 1) then

# Store converged solution
FinalSolutionList := [op(FinalSolutionList),
[seq(convSoln[i],i=1..7), incVar]]:

# Increment incVar
incVar := incVar + inc:

# Reset guessSoln to convSoln

else
print("Converged Solution is outside the face"): print(convSoln):
fi:

od:
end proc:

########################################################################

newtonRaph := proc(eps, n, k, iniSoln, incVar, symEq, symJac)
# eps - Convergence criteria
# n - Number of equations and variables
# k - Maximum number of iterations
# iniSoln - Initial (guess solution)
# incVar - Value of variable being incremented
# symEq - Equations, with constants substituted in already
# symJac - Jacobian matrix, with constants substituted in already
# convFlag - Convergence flag, returned by convergence checking routine?

```
local trySoln, trySolnList, numJac, numEq, job, newTrySolnList, newtrySoln, newtotTryList, suggInc, incVarList, totTryList, convSoln, newtrySolnList:

global v, convFlag:

convSoln := array(1..nEqs):
newtrySoln := array(1..nEqs):
job := 0:
# convFlag := 0:
trySoln := iniSoln:
incVarList := [v8=incVar]:

while (job <= k) do
  job := job + 1:
  # Form a list of trial variables, so that they can be substituted into the
  # symbolic Jacobian and equations. Note that the incremental variable (v8)
  # is also included, in the 2nd statement.
  trySolnList := [seq(v[i] = trySoln[i], i=1..n)]:
totTryList := [op(trySolnList), op(incVarList)]:
  # Substitute above list into symbolic forms of the Jacobian and Equations
  numJac := subs(totTryList, evalm(symJac)):
  numEq := subs(totTryList, evalm(symEq)):
  # Get suggested variable increment by solving numJac*suggInc=numEq
  suggInc := -1*linsolve(numJac, numEq):
  newtrySoln := matadd(trySoln, suggInc):
  # Convert the new trial solution into list form
  newtrySolnList := [seq(v[i] = newtrySoln[i], i=1..n)]:
  # Append the value of the variable being incremented to
  # the above list, to send to the checkConvergence routine
  # later.
  newtotTryList := [op(newtrySolnList), incVarList]:
  # Check if the new solution is a converged solution or not.
  # If not, check if it has iterated too many times - exit out and print a
  # message. Otherwise, set the new solution to be the trial solution for
  # the next increment.
  convFlag := checkConvergence(newtotTryList, symEq, eps):
  if (convFlag = 1) then
    norm1 := computeNorm(newtotTryList, symEq, eps):
    condition := evalf(cond(numJac)) :
    printf("%g	%10.7f	%d	%10.3e	%10.3e
", incVar, trySoln[5], job, condition, norm1):
    print(evalf(evalm(numJac)),condition):
    print(numEq):
    return newtrySoln, convFlag:
  else
    trySoln := newtrySoln:
  fi:
  # if (convFlag = 0) then
  # if (job >= k) then
  #   convFlag := 2:
  #   print("Too many iterations without convergence"):
  # else
  #   trySoln := newtrySoln:
  # fi:
  od:

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print("Too many iteration without convergence"):  
convFlag := 2:  
return newtrySoln, convFlag:
end proc:

########################################################################
checkPoint := proc(pointX, pointY, faceStart, faceEnd)
# pointX -> x coordinate of point to be checked
# pointY -> y coordinate of point to be checked
# faceStart -> array[1..2] relative coordinates of starting point of face
# faceEnd -> array[1..2] relative coordinates of ending point of face
# midPoint -> array[1..2] relative coordinates of midpoint of face
# distMid -> distance of current contact point from face midpoint
# flag -> flag for whether point is on face (1 = yes).
local midPoint, faceLength, distMid, flag:
midPoint := array(1..2):
faceLength := sqrt((faceStart[1]-faceEnd[1])^2 +
(faceStart[2]-faceEnd[2])^2):
midPoint := evalm(0.5*matadd(faceStart, faceEnd)):
distMid := sqrt((pointX-midPoint[1])^2 + (pointY-midPoint[2])^2):
if (2*distMid < faceLength + 0.001) then
  flag := 1:
else flag := 0:
fi:
flag:
end proc:

########################################################################
checkConvergence := proc(solutionList, symbolicEquations, epsilon)
local numericEquations, convergenceFlag:
# substitute values of variables into symbolic form of equations
numericEquations := evalf(subs(op(solutionList),
evalm(symbolicEquations))):
# check for convergence and assign flag accordingly
if norm(numericEquations, 1) >= epsilon then
  convergenceFlag := 0:
else convergenceFlag := 1:
fi:
convergenceFlag:
end proc:

########################################################################
computeNorm := proc(solutionList, symbolicEquations, epsilon)
local numericEquations, convergenceFlag:
numericEquations := evalf(subs(op(solutionList),
evalm(symbolicEquations))):
norm(numericEquations,1):
end proc:

########################################################################
computeOtherVars := proc(VarList)
# Use this subroutine to calculate the values of Normal contact
# force, friction force and strain in the beam. Can also add
# variables later.
local normalForce, fricForce, maxStrain, numTerms, maxBM, beamEndy, i:
global auxList:
auxList := list():
umTerms := nops(VarList):
for i from 1 to numTerms do
  normalForce := -1*VarList[i,8]/
    (frictionCoeff*cos(retAngle)+sin(retAngle)):
  fricForce := frictionCoeff*normalForce:
  beamEndy := VarList[i,7]-halfbeamThick:
end proc:
maxBM := -1*VarList[i,3] - VarList[i,2]*beamLength - VarList[i,8]*beamEndy:
maxStrain := maxBM*halfbeamThick/sectionModulus:
auxList := [op(auxList), [normalForce, fricForce, maxStrain, VarList[i,1], VarList[i,8]]]:
o:
ed proc:
### D.3 MAPLE Script for Insertion Model (insertion.maple)

```maple
readDesigndata("design.dat");
convertDesigndata();

initialVar[5] := -halfbeamThick;
initialVar[1] := 1e-3;
initialVar[2] := 0;
initialVar[3] := 0;
initialVar[6] := 0;
initialVar[7] := -halfbeamThick;

insertion_face_solution

print("Starting Insertion Face Solution");
ConstantList := [EI=sectionModulus, mu=frictionCoeff, L=beamLength, beta=insAngle, t=halfbeamThick, d=insIntercept];
Equations := getEquations(1, ConstantList):
Jacobian := findJacobian(Equations, v, ConstantList):
InitialArr := initialVar:
InsLowerLimit := initialVar[4]-0.001:
insfaceStart := array(1..2,
                       [initialVar[4], initialVar[5]]);
insfaceEnd := array(1..2,
                    [pointJ[1], pointJ[2]]);
InsIncrement := -1e-3;
lastSolution := solutionLoop(InitialArr, InsLowerLimit, Equations, Jacobian,
                            insfaceStart, insfaceEnd, InsIncrement);

dwell_face_solution

if (convFlag<>2) then
print("Starting Dwell Face Solution");
ConstantList := [EI=sectionModulus, mu=frictionCoeff, L=beamLength, beta=dwellAngle, t=halfbeamThick, d=dwellIntercept];
Equations := getEquations(2, ConstantList):
Jacobian := findJacobian(Equations, v, ConstantList):
InitialArr := lastSolution;
DwellLowerLimit := lastSolution[-1, -1] + InsIncrement;
dwellfaceStart := array(1..2,
                        [lastSolution[-1,1], lastSolution[-1,2]]);
dwellfaceEnd := array(1..2,
                      [pointI[1], pointI[2]]);
dwellIncrement := -1e-3;
lastSolution := solutionLoop(InitialArr, DwellLowerLimit, Equations, Jacobian,
                             dwellfaceStart, dwellfaceEnd, dwellIncrement);
```

---

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if (convFlag <> 2) then
   print("Starting Retention Face Solution");
   ConstantList := [EI=sectionModulus, mu=frictionCoeff, L=beamLength,
                    beta=retAngle, t=halfbeamThick, d=retIntercept]:
   Equations := getEquations(3, ConstantList);
   Jacobian := findJacobian(Equations, v, ConstantList);
   # Set the guess solution as the last solution from the previous face
   InitialArr := FinalSolutionList[-1,1..7];
   # Initial value of the variable being incremented - set to the value
   # from the last increment of the insertion face plus a small increment.
   RetLowerLimit := FinalSolutionList[-1, -1] + dwellIncrement;
   # Starting and ending point of current face (insertion) in relative
   # coordinates.
   # Artificially make the retention face twice the length to investigate
   # what is going on at the start of the retention face.
   retfaceStart := array(1..2, [artpointI[1], artpointI[2]]);
   retfaceEnd := array(1..2, [pointH[1], pointH[2]]);
   retIncrement := -1e-3;
   lastSolution := solutionLoop(InitialArr, RetLowerLimit, Equations,
                                Jacobian,retfaceStart, retfaceEnd,
                                retIncrement);
fi:

if (convFlag = 2) then
   print("The script terminated because convergence was not");
   print(" achieved in the specified number of iterations");
fi:
D.4 MAPLE Script for Retention Model (*retention.maple*)

```maple
readDesigndata("design.dat"): convertDesigndata():

# Move the contact point in the retention direction and try to model retention.

ConstantList := [EI=sectionModulus, mu=frictionCoeff, L=beamLength,
                 beta=retAngle, t=halfbeamThick, d=retIntercept]: Equations := getEquations(4, ConstantList):
Jacobian := findJacobian(Equations, v, ConstantList):

# Set the guess solution based on no deformation. Introduce small values for
# some variable to aid
InitialArr := array(1..7,[-0.001,0,0,0,-halfbeamThick,0,-halfbeamThick]):
RetLowerLimit := -1e-4:

# Starting and ending point of current face (insertion) in relative
# coordinates.
retfaceStart := array(1..2, [pointH[1], pointH[2]]):
retfaceEnd := array(1..2, [pointI[1], pointI[2]]):
retIncrement := -1e-1:
lastSolution := solutionLoop(InitialArr, RetLowerLimit, Equations,
                             Jacobian, retfaceStart, retfaceEnd, retIncrement):
```

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D.5 MAPLE Script to Model Two-step Stress Relaxation Response of Schapery Model

restart;
with(linalg):
# Define reference temperature and strain
Tr:=298; epsilon_r:=5e-3;
# Define the expression for horizontal shift factor
a := epsilon,T) -> 10^(-(20.9661*(T-Tr)/Tr+.479043*
(epsilon-epsilon_r)/epsilon_r));
# Define vertical shift factors
h1 := epsilon -> 1;
# need to be a little careful about this.
# Below the reference strain value, we want the value of
# h_2 to be 1.
h2 := proc(epsilon)
if (epsilon < epsilon_r) then
  h2 := 1:
else
  h2 := 10^(-(-0.00225943*(epsilon - epsilon_r)/epsilon_r +
    0.00485444*(epsilon - epsilon_r)/epsilon_r*
    (epsilon - epsilon_r)/epsilon_r));
fi:
return h2:
end proc:
# he has to be same as the product h1h2 since we were able
# to form master curve without any subtraction
# same approach for he as h2
he := proc(epsilon)
if (epsilon < epsilon_r) then
  he := 1:
else
  he := 10^(-(-0.00225943*(epsilon - epsilon_r)/epsilon_r +
    0.00485444*(epsilon - epsilon_r)/epsilon_r*
    (epsilon - epsilon_r)/epsilon_r));
fi:
return he:
end proc:
# Define the prony series constants.
Prony_Constants :=
array(1..15,[36.7566,37.7439,62.5874,76.8598,68.6610,110.6774,
  192.6603,137.4210,189.8020,327.3961,284.8248,
  279.7057,159.6305,165.8243,3.9233]);
# Equilibrium modulus from Prony series fit
Ee := 10^2.48;
# Set of assumex relaxation times for Prony series fit
Relaxation_Times := [seq(20*10^(i-1),i=1..15)];
# Form master curve equation
# En -> transient portion of master curve, be careful
# about notation w.r.t Schapery’s paper.
for i from 1 to 15 do E||i := Prony_Constants[i] od;
for i from 1 to 15 do t||i := Relaxation_Times[i] od;
En := (x) -> add(E||i*exp(-x/t||i), i=1..15) ;
Etotal := x-> En(x) + Ee; evalf(Ee); evalf(Etotal(10));
# Single step Schapery Response
# Single step Schapery response. Note that sigmasinglestep needs to be
# redefined as a function of t, epsilon and temperature
# otherwise, the function is not evaluated properly. For a single step
# stress relaxation experiment, the value of epsilon is fixed
and so is the value of temperature (we are only considering isothermal experiments), so they are not really variables in that sense. However, if they are not included in the argument list, MAPLE will not evaluate the value of stress completely.

\[
sigma_{\text{singlestep}} := (t, \epsilon, T) \rightarrow \text{he}(\epsilon) \times E_e \times \epsilon + \text{h1}(\epsilon) \times \text{h2}(\epsilon) \times \epsilon \times \text{En}(t/a(\epsilon, T));
\]

Two step Schapery Response
The equation for two step response has been derived in the thesis. Just the final equation is reproduced below:

\[
\text{sigma}_{\text{twostep}} := (t, t_a, \epsilon_a, \epsilon_b, T) \rightarrow \text{he}(\epsilon_b) \times E_e + \text{h1}(\epsilon_b) \times \text{h2}(\epsilon_b) \times \text{En}((t-t_a)/a(\epsilon_b, T)) \times \epsilon_b - \text{h1}(\epsilon_b) \times \text{h2}(\epsilon_a) \times \epsilon_a \times \text{En}((t-t_a)/a(\epsilon_b, T)) - \text{En}(t_a/a(\epsilon_a, T) + (t-t_a)/a(\epsilon_b, T));
\]

Output the Two step Response to file
Naming convention: (h2=1 \epsilon_a = 0.5\% \epsilon_b = 1\% \Rightarrow h2_10_51_0.dat
Just have to write the output to file.

\[
\text{writeTwoStepData} := \text{proc}(t_a, t_{\text{end}}, \epsilon_a, \epsilon_b, T, \text{filename}, n1, n2)}
local tstep, currtime, currstress, i, fd:
fd := fopen(filename, WRITE, TEXT):
for i from 0 to n1 do
    tstep := t_a/n1:
    currtime := tstep*i:
    currstress := \sigma_{\text{singlestep}}(currtime, \epsilon_a, 298):
    fprintf(fd, "%f \t %f \n", currtime, currstress):
od:
for i from 0 to n2 do
    # for a uniform time scale
    tstep := (t_{\text{end}} - t_a)/n2:
    currtime := t_a + tstep*i:
    currstress := \sigma_{\text{twostep}}(currtime, t_a, \epsilon_a, \epsilon_b, 298):
    fprintf(fd, "%f \t %f \n", currtime, currstress):
od:
fclose(fd):
end proc:
D.6 MAPLE Script to Model Constant Strain Rate Input Response of Schapery Model

```maple
eps := t-> c*t;

# Define a function for horizontal shift factor
a := (t,temp) -> 10^(-1*(20.9661*(temp-temp_r)/temp_r + 0.479043*(eps(t) - eps_r)/eps_r));

# Define a function for a vertical shift factor
# Two scenarios arise
# need to be a little careful about this.
# Below the reference strain value, we want the value of h2 to be 1.

h1 := 1;

h2 := proc(t)
if eps(t) < eps_r then
  h2 := 1:
else
  h2 := 10^(-(-0.00225943*(eps(t) - eps_r)/eps_r + 0.00485444*(eps(t)- eps_r)/eps_r* (eps(t) - eps_r)/eps_r));
fi:
return h2:
end proc:

# The other scenario is:
# h2 := 1;
# h1 := proc(t) -> 10^(-(-0.00225943*(eps(t) - eps_r)/eps_r + 0.00485444*(eps(t) - eps_r)/eps_r* (eps(t) - eps_r)/eps_r));
# In both scenarios above he remains the same:
he := proc(t):
if eps(t) < eps_r then
  he := 1:
else
  he := 10^(-(-0.00225943*(eps(t) - eps_r)/eps_r + 0.00485444*(eps(t)- eps_r)/eps_r* (eps(t) - eps_r)/eps_r));
fi:
return he:
end proc:

# Define the rho-rho' quantity in the convolution integral
# Define it as a function, since we will use it in the convolution
# integral
red_time := (t,tau) -> int(a(tt,298),tt=tau..t);

# Define the second term in the convolution integral
eps_term := tau -> diff(h2(tau)*eps(tau),tau);

# Define the transient modulus term in the convolution integral (Delta_E)
# Equilibrium modulus from Prony series fit
Ee := 10^2.48;

# Define the prony series constants.
Prony_Constants := array(1..15,[36.7566,37.7439,62.5874,76.8598,68.6610,110.6774,160.6603,137.4210,189.8020,327.3961,184.8248,279.7057,159.6305,165.8243,3.9233]);

# Set of assumex relaxation times for Prony series fit
Relaxation_Times := [seq(20*10^(-1-i),i=1..15)];

# Form master curve equation
# dE -> transient portion of master curve, be careful
```

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# about notation w.r.t Schapery's paper.
for i from 1 to 15 do E||i := Prony_Constants[i] od;
for i from 1 to 15 do eta||i := Relaxation_Times[i] od;
dE := (t) -> add(E||i*exp(-t/eta||i), i=1..15) ;
# Now we have all the quantities in the integrand defined. In particular, #
# these are:
# dE(t) -> transient relaxation modulus
# red_time(t,tau) -> rho - rho'
# eps_term(tau) -> d(h2*eps)/dtau
# Theoretically, we should be ready to integrate now:
#
#integrand(t,tau) := dE(red_time(t,tau))*eps_term(tau);
# The other option is to work the problem numerically
# See the notes for derivation of the equation
#
# This determines sigma(tj)
# Procedure to calculate the quantity within the integral at #
yany moment of time.
trans_stress := proc(curr_time,n_int)
local taui,sigma,i,taulast:
taulast := 0.0;
sigma := 0.0;
for i from 1 to n_int do
  taui := evalf(i*curr_time/n_int);
sigma:=sigma + dE(red_time(curr_time,taui))*(h2(taui)*eps(taui)
  -h2(taulast)*eps(taulast));
tmpqty:=h2(taui)*eps(taui)-h2(taulast)*eps(taulast);
  # print(red_time(curr_time,taui),h2(taui),h2(taulast),eps(taui),
  # eps(taulast),tmpqty):
taulast:=taui;
  od:
end proc:
#sigma:=sigma + dE(red_time(curr_time,taui))*(h2(taui)*eps(taui) - #
#If we want to find the variation of stress in a constant strain rate #
#experiment, # then we will have to call the above subroutine numerous #
#number of times, changing # the value of curr_time at each instant. #
#This is a bit of repetition, since the # way the program is set up #
#right now, it sums everything from zero every time which # is not #
#really necessary. More efficient way of doing this would be to just #
#find the # increment in stress after the last time instant. # Maybe #
#we can implement that later. #

nsteps := 10;
#the number of data points we want in the curve
results := array(1..100,1..2):
temp_r:=298;
eps_r:=0.005;
c:=0.0001;
final_time := 450:
#trans_comp:= trans_stress(.45,100):
fd := fopen("resultsh10_01.dat",WRITE,TEXT):
for j from 1 to nsteps do
  curr_time := j*final_time/nsteps;
trans_comp := trans_stress(curr_time,100):
  # call the subroutine to find the transient #
  # stress component
eqm_comp := Ee*eps(curr_time)*he(curr_time):
total_stress := trans_comp + eqm_comp:
curr_strain := eps(curr_time):
results[j,1] := curr_time;
results[j,2] := total_stress;
print(curr_time,total_stress):
fprintf(fd,"%f \t %f \t %f \n",curr_time,c curr_strain,total_stress):
od:
fclose(fd):

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D.7 MATLAB Script to Estimate Prony Series Parameters

% Number of terms in Prony Series
N = 15
% Define the constant a
a = 2.0
% Definition of collocation times
logcolloctimes = [1:1:N]
colloctimes = 10.^logcolloctimes
tautimes = a*logcolloctimes
logtautimes = log10(tautimes)
% Glassy Modulus and Equilibrium Modulus.
loge_infinity = 2.48;
e_infinity = 10^loge_infinity;
% Our equilibrium modulus is not zero
% e_infinity = 100.0
% We are not enforcing fit of the glassy modulus
loge_g = 2.3;
e_g = 10^loge_g;
% Experimental Data at the collocation times.
loge_colloc = [3.378
3.362
3.349
3.334
3.316
3.288
3.252
3.213
3.14
3.03
2.9
2.75
2.61
2.485]
e_colloc = 10.^loge_colloc
% Setting up the [B] matrix
for j=1:N
    for i=1:N
        B(j,i)=exp(-10^((j-i)/a));
    end
end
% Setting up the a vector and finding the values of
% the collocation constants.
AA=e_colloc-e_infinity;
e_constants=B\AA
logsampletimes = [1:0.2:15]
sampletimes = 10.^logsamp	model = e_infinity
for i = 1:N
    model = model + e_constants(i)*exp(-sampletimes/tautimes(i))
end
logmodel=log10(model);
plot(logsampletimes,logmodel,'gx-',logcolloctimes,loge_colloc,'r*');
legend('Collocation model','Experimental Data');
xlabel('log(t) - s');
ylabel('log(E) - MPa');
grid on;
%errorpercent = (logmodel-lograwdata)./logmodel*100;
results = [tautimes' e_constants]
 fid=fopen('PCresults.dat','w');
 fprintf(fid,'tau(i)				E(i)
(i)s		(MPa)
',
 fprintf(fid,'%4.2e		%6.4f
',results');
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fclose(fid);
BIBLIOGRAPHY


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