On Lagrangian Algebras in Braided Fusion Categories

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This dissertation titled
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Abstract

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This work is part of an ongoing study of a generalization of the notion of symmetry. In mathematics, the notion of symmetry is formalized into the concept of a group. Examples of groups arise naturally throughout mathematics, chemistry, physics, and other fields. However, recent developments in high-energy and condensed matter physics show that not all kinds of symmetry that arise in nature can be captured by groups. A versatile, more general tool that does this job is a certain class of tensor categories known as fusion categories. We add to the study of these categories by constructing certain analogues to the classical algebraic notion of an associative algebra over a field inside them, and then expounding the properties of these generalized algebras.
For my parents, Al and Jennifer, without whose unwavering love and support none of this would have been possible.
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0 INTRODUCTION

0.1 General Background and Motivation

The concept of a group underlies the entirety of abstract algebra. Virtually all of the algebraic structures that are studied—groups (obviously), rings, modules, fields, vector spaces, et cetera—can be thought of as groups with some additional structure imposed thereon. Consequently, the theory of groups is the foundation upon which is built all of modern algebra, so by studying groups, we study algebraic structures in general.

Groups arise naturally throughout mathematics, chemistry\(^1\), physics, and other fields of study as abstract generalizations of the notion of symmetry. Some are quite concrete—e.g., the group of geometric symmetries of a polygon or polyhedron—while others are far more abstract—e.g., the Galois group of admissible permutations of the roots of a polynomial. That latter example, of course, dates back to Galois and was one of the crucial factors behind the historical development of the theory of groups. Galois himself was only interested in understanding why the general polynomial equations of degree 5 or higher were not solvable by radicals, but the theory that he developed\(^2\) to answer this question was the first step towards the abstract study of groups for their own sake.

The primary motivation behind this work is the study of “generalized groups” in some sense. Specifically, we study braided fusion categories, which are the “generalized groups” to which we referred. One might ask, “Why study categories if your real interest is groups?” One (admittedly complicated) answer is as follows: there is a limitation to the utility of groups as models of symmetry as it exists in the real world, at least at the quantum level. As it was noted in [30], certain topological states of matter\(^3\) are classified by categories of a certain type. In particular, particle-like excitations correspond to objects

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\(^1\) E.g., the VSEPR theory of molecular bonding.

\(^2\) Which now bears his name.

\(^3\) These are called “2+1D topological orders”. See [30].
of a nondegenerate unitary braided fusion categories, with the vacuum corresponding to
the monoidal unit. This gives one possible answer to the natural question of “Why
categories?”—just as groups arise naturally in the study of symmetry in the real world, so
braided fusion categories arise in the study of symmetry in the quantum world.

In modern mathematics, we often study groups via their actions on other objects.
This is the idea underlying classical representation theory, wherein a group is studied by
allowing it to act on a vector space by linear transformations. Just as the class of all
finite-dimensional vector spaces over a given field $k$, together with linear transformations,
forms a category, so too does the collection of all finite-dimensional $k$-linear
representations of a given finite group $G$. This category, which we call $\text{Rep}(G)$, can be
interpreted as the category of finite-dimensional vector spaces with $k$-linear $G$-action, with
$G$-invariant linear transformations as its morphisms. In this interpretation, $\text{Rep}(G)$ is
naturally a subcategory of $k\text{-}\text{Vect}$. A rich theory has been developed that describes the
category $\text{Rep}(G)$. E.g., character theory arises from the notion of the trace of a linear
operator from linear algebra as applied to the linear action of a group on a
finite-dimensional vector space.

The category of finite-dimensional vector spaces is naturally endowed with a “formal
addition”: the direct sum. On the level of dimension, the direct sum is indeed additive.
The direct sum also has an identity object: the trivial vector space $\{0\}$. Thus the collection
of (isomorphism classes of) vector spaces forms a monoid with respect to the direct sum.
Moreover, every vector space is a direct sum of finitely many copies of the simple vector
space $k$. Thus $k\text{-}\text{Vect}$ is a semisimple category—an Abelian category for which all objects
are direct sums of simple objects. It follows from Maschke’s lemma$^4$ that the category
$\text{Rep}(G)$ is semisimple precisely when the characteristic of $k$ does not divide the order of $G$.

$^4$ Lemma 2.1.1.1 (q.v.).
$k\cdot\text{Vect}$ is also endowed with a “formal multiplication”: the *tensor product*, which is also multiplicative on the level of dimension. The tensor product also has an identity: the field $k$ itself, *i.e.*, the isomorphism class of the one-dimensional space. Since the tensor product is also associative, the tensor product of vector spaces makes $k\cdot\text{Vect}$ into the simplest example of what we call a *monoidal category*. This is a category equipped with a tensor product that has a unit object and is associative up to a natural isomorphism called the *associator*. For $k\cdot\text{Vect}$, the associator is trivial, since the tensor product of vector spaces is associative on the nose.

The diagonal action allows us to make the tensor product of representations into a representation in its own right. This turns $\text{Rep}(G)$ into a monoidal category, with the monoidal unit being the trivial representation. The tensor product in $\text{Rep}(G)$ is also associative on the nose, and so the associator of $\text{Rep}(G)$ is also trivial.

The category of finite-dimensional vector spaces also has *dual objects*. To each vector space $V$, there is associated the vector space $V^*$ of linear functionals on $V$. The dual vector space comes equipped with two natural maps: the *evaluation map*—which takes a simple tensor $\ell \otimes v \in V^* \otimes V$ and evaluates the functional $\ell$ at the vector $v$—and the *coevaluation map*—which takes a scalar $\lambda \in k$ and returns the sum

$$\sum_{i=1}^{\dim_k(V)} \lambda e_i \otimes e_i^*,$$

where $\{e_i | i = 1, 2, \ldots, \dim_k(V)\}$ is any basis for $V$ and $\{e_i^* | j = 1, 2, \ldots, \dim_k(V)\}$ is the corresponding dual basis. The existence of these dual objects makes $k\cdot\text{Vect}$ a *rigid category*.

Colloquially, a monoidal category is called *fusion* if it has (left) duals, every object is a direct sum of simple objects, and the collection of (isomorphism classes of) simple objects is finite. Fusion categories are generalized groups in the sense that the collection

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5. Hence the name.

6. Surprisingly enough, the value of the coevaluation map does not depend upon the choice of basis.
of simple objects of a fusion category forms a group under the tensor product. The simplest possible example of a fusion category is not \( \mathcal{Rep}(G) \), but rather something else: the category of \( G \)-graded vector spaces. The collection of \( G \)-graded vector spaces, along with linear maps that preserve the grading, forms a category that we denote by \( \mathcal{V}(G) \). The simple objects of \( \mathcal{V}(G) \) are one-dimensional vector spaces concentrated in a single degree. That degree is, in turn, labeled by a single group element. Consequently, there is a one-to-one correspondence between simple objects of \( \mathcal{V}(G) \) and elements of \( G \). In particular, when \( G \) is finite, so too is the set of (isomorphism classes of) simple objects of \( \mathcal{V}(G) \).

The tensor product of vector spaces extends\(^7\) to \( G \)-graded vector spaces in a natural way in terms of the group operation on \( G \). The unit object is a one-dimensional vector space concentrated in the degree labeled by the identity element \( e \in G \). Applied to simple objects \( I(g) \) and \( I(h) \) of \( \mathcal{V}(G) \), where \( g, h \in G \), the tensor product \( I(g) \otimes I(h) \) is isomorphic to \( I(gh) \). Moreover, the notion of the dual of a vector space passes\(^8\) without change to the category of \( G \)-graded vector spaces. It follows from this discussion that the category of \( G \)-graded vector spaces is a fusion category. Indeed, on the level of simple objects, \( \mathcal{V}(G) \) is just \( G \). In other words, \( \mathcal{V}(G) \) is a proper-class-sized model of a finite object. \( \mathcal{V}(G) \) also has a rather unusual property: despite the fact that \( \mathcal{V}(G) \) is a subcategory of \( k\cdot\text{Vect} \), the dimension of \( \mathcal{V}(G) \) is \( |G| \), while the dimension of \( k\cdot\text{Vect} \) is 1.

The theory of fusion categories has applications in representation theory, theoretical physics, and quantum computing. Consequently, it is, as of the time of this writing, experiencing a period of rapid development. Our interest in fusion categories stems from the fact that the class of fusion categories forms a nice generalization of the notion of groups. They almost—but not quite—fit the mold for this generalized symmetry that we

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\(^7\) See § 1.4.1.

\(^8\) See § 2.2.3.
wish to describe. We are interested in a special class of fusion categories that come equipped with an extra structure—a braiding. This additional structure can be seen as an answer to a natural question about the tensor product in a monoidal category. That question is, “Given two objects $X$ and $Y$, is $X \otimes Y$ isomorphic to $Y \otimes X$?” Put more simply, the question amounts to “Is the tensor product commutative or not?” The answer, in general, is that it can be, but it does not have to be. Moreover, there are two possible interpretations of the meaning of the word “commutative” in the context of a tensor category. One is called symmetry, and the other, nondegeneracy. Colloquially, a symmetry is a swap that, when done twice, is akin to doing nothing. Also colloquially, a swap is nondegenerate if it is as far from being symmetric as it can possibly be. The former is what happens in the category of vector spaces\(^9\). The latter is what we want the objects of our study to satisfy. We call such swaps braidings. The terms “symmetry” and “braiding” come from the notions of the symmetric and braid groups, respectively, and the actions of these groups on tensor factors in a monoidal category. In short, symmetries are symmetric because the square of a generator of the symmetric group $S_n$ is the identity, whereas braidings are nondegenerate because the square of a generator of the braid group $B_n$ is not the identity.

One method of producing examples of nondegenerate categories is by the so-called monoidal center construction—the categorification of a classical algebraic construction. This is an analogue, at the level of categories, of the notion of the center of a monoid, group, or ring. Given a monoidal category $C$, we look at the collection of all those objects $Z \in C$ for which $Z \otimes X$ is isomorphic to $X \otimes Z$ for all $X \in C$. This collection forms a subcategory $Z(C)$ of $C$ called the monoidal center. The monoidal center $Z(C)$ has the property that, for any two objects $X, Y \in Z(C)$, $X \otimes Y$ is isomorphic to $Y \otimes X$. $Z(C)$ is our archetype for a braided monoidal category: a monoidal category with a braiding.

\(^9\) This is because isomorphism classes of vector spaces correspond to cardinal numbers, multiplication of cardinal numbers is commutative, and the tensor product is multiplicative on the level of dimension.
When we apply the monoidal center construction to the category of $G$-graded vector spaces, the result is a braided fusion category which we call the Drinfeld center of $G$. The Drinfeld center $\mathcal{Z}(G)$ also has a rather odd property that is at odds with what usually happens in algebra when we compute the center of something. Namely, the dimension of the Drinfeld center $\mathcal{Z}(G)$ is $|G|^2$, while the dimension of $\mathcal{V}(G)$ is $|G|$. It is for this reason that the Drinfeld center is sometimes called the Drinfeld double. Unusually among algebraic objects, the category $\mathcal{V}(G)$ gets larger (in some sense) when you compute its monoidal center, rather than smaller. Another nice consequence is that the braiding in the Drinfeld center $\mathcal{Z}(G)$ is always nondegenerate (and hence not symmetric) so long as $G$ is not the trivial group.

The braiding in $\mathcal{Z}(G)$ amounts to the structure of a $G$-action on the objects of $\mathcal{Z}(G)$. Thus we can realize the category $\mathcal{Z}(G)$ as the category whose objects are $G$-graded vector spaces with compatible $G$-action, and whose morphisms are linear maps that preserve both the grading and the $G$-action. We use this interpretation as our definition of $\mathcal{Z}(G)$, and prove later that $\mathcal{Z}(G)$ is equivalent to the monoidal center of the category $\mathcal{V}(G)$ of graded vector spaces.

The tensor product of vector spaces is naturally associative on the nose, but for graded vector spaces, it need not be. Instead, a sort of “skew associativity” can be imposed via a function $\alpha : G \times G \times G \to k^*$ that appears as a constraint:

$$u \otimes (v \otimes w) = \alpha(f, g, h)(u \otimes v) \otimes w, \quad \forall u \in U_f, \ \forall v \in V_g, \ \forall w \in W_h$$

for $G$-graded vector spaces $U$, $V$, and $W$. It can be shown that such a function $\alpha$ must satisfy the 3-cocycle condition

$$\alpha(y, z, w)\alpha(x, yz, w)\alpha(x, y, z) = \alpha(xy, z, w)\alpha(x, y, zw), \quad \forall x, y, z, w \in G.$$

---

10 See § 1.4.3 for the definition of the braiding.
11 See § 1.4.3.
12 In chapter 2, proposition 2.2.3.3.
13 See § 1.3.1.
If we apply the monoidal center construction to the category $\mathcal{V}(G, \alpha)$ of $G$-graded vector spaces with skew associativity controlled by $\alpha$, then we obtain the **twisted Drinfeld center** $\mathcal{Z}(G, \alpha)$, whose objects are $G$-graded vector spaces with projective $G$-action, and whose morphisms are $k$-linear maps preserving the grading and projective action.

The notion of an **associative unital algebra** also passes$^{14}$ from the category $k$-$\mathbf{Vect}$ to arbitrary monoidal categories $C$. In particular, this notion passes to the categories $\mathcal{R}ep(G)$, $\mathcal{V}(G, \alpha)$, and $\mathcal{Z}(G, \alpha)$. The twisted associativity in the latter two categories makes these algebras skew associative in an absolute sense, but associative within their indicated categories$^{15}$.

The braiding in a braided category $C$ also allows us to consider **commutative algebras** in $C$—in particular, in the braided category $\mathcal{Z}(G, \alpha)$. As with algebras in the classical sense, we can also consider the categories of **modules over these algebras**, as well as **local modules**. Local modules$^{16}$ are those modules over a commutative algebra in a braided category that are unaffected by a double braiding swap$^{17}$ prior to multiplication. There is no classical algebraic analogue for this notion. Rather, it is a purely quantum$^{18}$ phenomenon. Those commutative algebras whose categories of local modules are trivial in some sense$^{19}$ we call **Lagrangian algebras**.

### 0.2 Main Results

#### 0.2.1 Executive Summary

In this work, we study Lagrangian algebras in braided fusion categories. One reason for which we choose to study these algebras in this environment is that Lagrangian

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14 See § 1.2.1.
15 See §§ 1.2.1 and 1.4.3.
16 This is unrelated to the usual algebraic notion of a local module.
17 See the diagram (1.1) in § 1.2.1.
18 This is explained to some degree in the introduction of [30].
19 Namely, in the sense that they are equivalent to $k$-$\mathbf{Vect}$. See § 1.2.1.
algebras are (labels for) modular invariants for rational conformal field theories\textsuperscript{20}. Another reason is that they allow us to identify a given nondegenerate braided fusion category with the monoidal center of a finite group. In this sense, Lagrangian algebras can be thought of as being analogous to a choice of coordinates in linear algebra.

In chapters 1 and 2, we build up the necessary background to discuss the main objects of our study. In chapter 3, we give a classification of Lagrangian algebras in group-theoretical braided fusion categories: Lagrangian algebras in the Drinfeld center $\mathcal{Z}(G, \alpha)$ correspond to pairs $(H, \gamma)$, where $H \leq G$ is a subgroup, and $\gamma \in C^2(G, k^*)$ is a coboundary for the restriction of $\alpha$ to $H$. The aforementioned classification is the main result of chapter 3. We then use the classification to compute the characters of Lagrangian algebras in general, and for a tractable concrete example.

It is known that Lagrangian algebras in pointed braided fusion categories corresponded to Lagrangian subgroups. It is also not difficult to compute the automorphism groups of these Lagrangian algebras. Another quantum phenomenon equips these Lagrangian algebras with associators—classes in the third cohomology of the grading group of the category\textsuperscript{21}. However, there was no known formula for the associator of the group of symmetries of such a Lagrangian algebra. The main result of chapter 4 is the formula for the associator that we derive in § 4.5. As a byproduct of describing the associator, we also obtain a description of the third cohomology group of an arbitrary finite Abelian group $B$ in terms of Lagrangian extensions of the character group $\hat{B}$, the group of antisymmetric 3-linear forms on $B$, the 2-torsion subgroup $B_2$ of $B$, Abelian group extensions of $B$ by $\hat{B}$, and the quotient group $B/2B$.

\textsuperscript{20} See, e.g., [5].
\textsuperscript{21} See § 1.4.5.
0.2.2 More Details — Main Results of Chapter 3

The classification of Lagrangian algebras from chapter 3, in particular, gives a method for explicitly computing physical modular invariants\(^{22}\) for group-theoretical modular data. It should be mentioned that the set of labels for physical modular invariants was obtained in [39] using the language of module categories. By establishing a correspondence between Lagrangian algebras and module categories, and by computing the characters of Lagrangian algebras, we give a method for determining modular invariants corresponding to module categories. Indeed, computing modular invariants becomes nothing more or less than knowing Lagrangian algebras in a group-theoretical braided fusion category.

We classify Lagrangian and the more general etale algebras in the Drinfeld center \(\mathcal{Z}(G, \alpha)\) in two steps. First, we describe etale algebras with the trivial grading. These are nothing but indecomposable commutative separable algebras with a \(G\)-action, and hence\(^{23}\) they are just function algebras on transitive \(G\)-sets. Up to isomorphism, they are labeled by conjugacy classes of subgroups \(H \leq G\). Then we identify the category of local modules \(\mathcal{Z}(G, \alpha)_{\text{loc}}^{k(G/H)}\) with the group-theoretical modular category \(\mathcal{Z}(H, \alpha|_H)\). A general etale algebra in \(\mathcal{Z}(G, \alpha)\) is an extension of its trivial degree component, and hence is an etale algebra in one of \(\mathcal{Z}(G, \alpha)_{\text{loc}}^{k(G/H)}\). Considered as an algebra in \(\mathcal{Z}(H, \alpha|_H)\), its trivial-degree component is one-dimensional. Our second step is to classify such algebras and their categories of local modules. Then we combine the results obtaining the description of all etale algebras in \(\mathcal{Z}(G, \alpha)\) and their local modules. As a corollary, we get a classification of Lagrangian algebras in \(\mathcal{Z}(G, \alpha)\): they are parametrized by conjugacy classes of subgroups \(H \leq G\) together with a coboundary \(\partial \gamma = \alpha|_H\), matching with the answer from [39]. We explain this matching by identifying Lagrangian algebras with full centers of

\(^{22}\) This problem was raised in [5].
\(^{23}\) We work over an algebraically closed field \(k\). See lemma 3.2.2.1.
indecomposable separable algebras in $\mathcal{V}(G, \alpha)$. Finally, after recalling the character theory for objects of $\mathcal{Z}(G, \alpha)$, we compute characters of Lagrangian algebras. We treat the case $G = D_3$—the dihedral group of order 6—as an example.

### 0.2.3 More Details — Main Results of Chapter 4

The third Abelian cohomology classifies\(^{24}\) braided pointed fusion categories. Such categories are labeled by triples $(A, \alpha, c)$, where $A$ is an Abelian group, $\alpha$ is a 3-cocycle of $A$ with coefficients in $k^*$, and $c$ is a 2-cochain of $A$ with coefficients in $k^*$ such that the pair $(\alpha, c)$ belongs to the third Abelian cohomology group\(^{25}\) $H^3_{ab}(A, k^*)$. Moreover, the Eilenberg–Mac Lane interpretation of the third Abelian cohomology\(^{26}\) as the group of quadratic forms has a natural manifestation on level of categories: braided pointed fusion categories can instead be labeled by pairs $(A, q)$, where $A$ is as above and $q : A \to k^*$ is the (unique) quadratic function corresponding to the Abelian 3-cocycle $(\alpha, c)$.

Associators for a pointed fusion category are classified by the third cohomology group $H^3(G, k^*)$. These are associativity constraints for the tensor product of $G$-graded vector spaces. Recently, the group structure of the third cohomology was interpreted\(^{27}\) as the group of modular extensions of the category $\mathcal{R}ep(G)$ of finite-dimensional $k$-linear representations of $G$.

Lagrangian algebras in modular categories seem to play a very special role. For example, a Lagrangian algebra $L \in C$ allows us to identify $C$ with the monoidal center $\mathcal{Z}(C_L)$ of the category $C_L$ of $L$-modules in $C$. The group of invertible modules over a Lagrangian algebra comes equipped with a natural 3-cocycle (the associator). The group of invertible modules over a Lagrangian algebra contains a great deal of information about

---

\(^{24}\) See [26, Proposition 3.1].
\(^{25}\) See § 1.3.2. More details can also be found in [17] and [26].
\(^{26}\) See [17].
\(^{27}\) In [30].
the ambient category. For example, when all simple modules over a Lagrangian algebra are invertible, the ambient category can be identified with a twisted Drinfeld center\(^{28}\).

In a modular pointed category, all Lagrangian algebras correspond to Lagrangian subgroups of the grading group. Moreover, all simple modules over Lagrangian algebras are invertible and the group of invertible modules is the quotient by the corresponding Lagrangian subgroup. Thus a Lagrangian subgroup\(^{29}\) \(L \leq A\) gives a braided equivalence

\[
C(A, q) \cong \mathbb{Z}(A/L, \beta)
\]

for some \(\beta \in H^3(A/L, k^*)\). The main purpose of chapter 4 is to describe the associator \(\beta \in H^3(A/L, k^*)\) as a function of \(A, q\) and \(L\). We derive a formula for a cocycle representing it\(^{30}\), but the explicit expression is not very easy to use. E.g., the formula does not reveal the cohomology class of the associator. In order to find a more useful description of the associator, we look at the correspondence between triples \((A, q, L)\) (a nondegenerate quadratic group and a Lagrangian subgroup) and the cohomology classes from \(H^3(A/L, k^*)\) from a different perspective. By fixing \(L\) and varying \((A, q)\), we turn this correspondence into a homomorphism of groups. We define a natural group structure on the set of isomorphism classes of Lagrangian extensions of \(L\). Assigning the associator to the triple \((A, q, L)\) becomes a homomorphism from the group \(\text{Lex}(L)\) of Lagrangian extensions to the third cohomology:

\[
\text{Lex}(L) \longrightarrow H^3(\hat{L}, k^*) .
\]

Here, we use the natural identification of \(A/L\) with the character group \(\hat{L}\) induced by the quadratic function on \(A\). The homomorphism \((\dagger)\) is a compact way of expressing the associator \(\beta \in H^3(A/L, k^*)\) as a function of \(A, q\) and \(L\).

\(^{28}\) See chapter 3 or [12] for more.

\(^{29}\) Here, we abuse notation by referring to the Lagrangian subgroup as \(L\) also.

\(^{30}\) See § 4.5.
Furthermore, the homomorphism (†) has a nice interpretation in terms of low-dimensional cohomology. It can be used to unpack the internal structure of the third cohomology of an Abelian group. To simplify the exposition, write \( \hat{L} = B \). The homomorphism (†) is an embedding whose image is the kernel of the alternation map \( \text{Alt}_3 : H^3(B, k^\ast) \to \text{Hom}_\mathbb{Z}(\Lambda^3 B, k^\ast) \). Here we denote by \( \text{Hom}_\mathbb{Z}(\Lambda^n B, M) \) the group of antisymmetric \( n \)-linear forms on \( B \) with values in an Abelian group \( M \). Our ground field \( k \) is algebraically closed of characteristic zero; consequently, its multiplicative group \( k^\ast \) is a divisible group. The torsion subgroup of \( k^\ast \) is isomorphic to \( \mathbb{Q}/\mathbb{Z} \). Thus for finite \( B \), we can trade \( k^\ast \) for the universal torsion group \( \mathbb{Q}/\mathbb{Z} \) as coefficient groups for cohomology without changing the isomorphism classes of the cohomology groups.

The functor \( B \mapsto H^n(B, \mathbb{Q}/\mathbb{Z}) \) is polynomial of degree \( n \). As a polynomial functor, it has a filtration whose associated graded components are homogeneous polynomial functors. For degrees one and two, the cohomology is already homogeneous:

\[
H^1(B, \mathbb{Q}/\mathbb{Z}) = \hat{B}, \quad H^2(B, \mathbb{Q}/\mathbb{Z}) = \text{Hom}_\mathbb{Z}(\Lambda^2 B, \mathbb{Q}/\mathbb{Z}).
\]

The degree-three cohomology is a combination of homogeneous polynomial functors of degrees three and two. When \( B \) is 2-torsion-free, the third cohomology group fits in the split exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ext}_\mathbb{Z}^1(B, \hat{B})^\ast & \longrightarrow & H^3(B, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Hom}_\mathbb{Z}(\Lambda^3 B, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.
\end{array}
\]

with the map \( \text{Ext}_\mathbb{Z}^1(B, \hat{B})^\ast \to H^3(B, \mathbb{Q}/\mathbb{Z}) \) given by factoring through the embedding \( \text{Ext}_\mathbb{Z}^1(B, \hat{B}) \hookrightarrow H^2(B, \hat{B}) \) and then assigning \( \gamma \mapsto \gamma(x, y)(z) \) for \( x, y, z \in B \).
For a general finite Abelian group $B$, the situation is a bit more complicated. We have a diagram with exact rows and columns:

\[
\begin{array}{c}
\{0\} \\
\downarrow \\
\hat{B}_2 \\
\downarrow \psi \\
{0} \rightarrow \text{Lex} (\hat{B}) \rightarrow H^3 (B, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}_\mathbb{Z} (\Lambda^3 B, \mathbb{Q}/\mathbb{Z}) \rightarrow \{0\} \\
\downarrow \phi \\
\text{Ext}_\mathbb{Z}^1 (B, \hat{B})^\tau \\
\downarrow \\
B/2B \\
\downarrow \\
\{0\}
\end{array}
\]

In order to build the vertical sequence in the diagram (‡), we interpret both the collection of Lagrangian extensions of $\hat{B}$ and the collection of extensions of $B$ by $\hat{B}$ as the categorical groups $\text{Lex} (\hat{B})$ and $\text{Ext} (B, \hat{B})$, respectively. The zeroth homotopy groups of $\text{Lex} (\hat{B})$ and $\text{Ext} (B, \hat{B})$ are $\text{Lex} (\hat{B})$ and $\text{Ext}_\mathbb{Z}^1 (B, \hat{B})$, respectively. We construct a functor $\text{Lex} (\hat{B}) \rightarrow \text{Ext} (B, \hat{B})$. This functor induces the indicated homomorphism $\phi : \text{Lex} (\hat{B}) \rightarrow \text{Ext}_\mathbb{Z}^1 (B, \hat{B})$ from the diagram (‡). Note also that in the case where $B$ is 2-torsion-free, all terms of the vertical sequence in (‡) except $\text{Lex} (\hat{B})$ and $\text{Ext}_\mathbb{Z}^1 (B, \hat{B})$ vanish.\(^{32}\)

The map $\psi : \hat{B}_2 \rightarrow \text{Lex} (\hat{B})$ sends $\chi$ into a Lagrangian extension $(B \times \hat{B}, q_{\text{std}} + \tilde{q}, \iota)$, where $\tilde{q}$ is a quadratic function $B \rightarrow \mathbb{Q}/\mathbb{Z}$ such that $2\tilde{q}(x) = \chi (x)$ for all $x \in \hat{B}_2$. The map $\phi : \text{Lex} (\hat{B}) \rightarrow \text{Ext}_\mathbb{Z}^1 (B, \hat{B})^\tau$ assigns to a Lagrangian extension $(A, q, \iota)$ an extension of

\[^{31}\text{See, e.g., [26, § 3.1].}
\[^{32}\text{In that case, exactness of the vertical sequence gives an isomorphism } \text{Lex} (\hat{B}) \cong \text{Ext}_\mathbb{Z}^1 (B, \hat{B})^\tau.\]
Abelian groups $B \xrightarrow{i} A \xrightarrow{\pi} \hat{B}$, where the projection $\pi$ sends $a \in A$ into 

$$(\pi(a))(x) = q(a + \iota(x)) - q(a)$$

for $x \in B$. See § 4.4 for more.
1 Definitions

Here, we review a number of preliminary concepts.

1.1 Categories, Part I, or “An Introduction to Abstract Nonsense”

1.1.1 Categories and Morphisms

A category $C$ is a pair of classes $C = \langle \text{Obj}(C), \text{Map}(C) \rangle$. Elements of $\text{Obj}(C)$ are called objects and elements of $\text{Map}(C)$ are called morphisms or arrows. We will follow the standard abuse of notation and write $A \in C$ to denote that $A$ is an object of the category $C$. For each morphism $f \in \text{Map}(C)$, there exist uniquely determined $A, B \in C$, called the source object and target object, respectively. We write

$$f : A \rightarrow B$$

and say that $f$ is a morphism from $A$ to $B$. We denote by $\text{Hom}_C(A, B)$ the class of morphisms with source object $A$ and target object $B$. For any three objects $A, B, C \in C$, there is a binary operation

$$\circ : \text{Hom}_C(B, C) \times \text{Hom}_C(A, B) \rightarrow \text{Hom}_C(A, C)$$

called composition of morphisms. The composition of $f : A \rightarrow B$ and $g : B \rightarrow C$ is denoted by $g \circ f$.

For any category $C$, the following axioms hold:

- (Associativity): If $f : A \rightarrow B, g : B \rightarrow C$, and $h : C \rightarrow D$ are morphisms, then $h \circ (g \circ f) = (h \circ g) \circ f$.

- (Identity): For every $X \in C$, there exists a morphism $\text{Id}_X : X \rightarrow X$ such that, for any morphisms $f : A \rightarrow X, g : X \rightarrow B$, we have $\text{Id}_X \circ f = f$ and $g \circ \text{Id}_X = g$, respectively.

\[33\] By $f$. 

The morphism $\text{Id}_X$ is called the *identity morphism*. For any object $X$, it can be shown that the identity morphism $\text{Id}_X$ is unique.

A morphism $f : X \rightarrow Y$ is a *monomorphism* if it is left cancellative, *i.e.*, for all morphisms $g_1, g_2 : Z \rightarrow X$, we have

$$ (f \circ g_1 = f \circ g_2) \implies g_1 = g_2. $$

A monomorphism is *normal* if it is the kernel of some morphism. Dually, $f : X \rightarrow Y$ is an *epimorphism* if it is right cancellative, *i.e.*, for all morphisms $g_1, g_2 : Y \rightarrow Z$, we have

$$ (g_1 \circ f = g_2 \circ f) \implies g_1 = g_2. $$

An epimorphism is *conormal* if it is the cokernel of some morphism.

If $C$ is a category, then the *opposite category to $C$*, denoted by $C^{\text{op}}$, is the category whose objects are the objects of $C$, and whose morphisms are the morphisms of $C$ with all arrows reversed.

An object $I \in C$ is an *initial object* if, for every object $Y \in C$, there is a unique morphism $f : I \rightarrow Y$. Dually, an object $T \in C$ is a *terminal object* if, for every object $X \in C$, there is a unique morphism $g : X \rightarrow T$. An object $0 \in C$ that is both initial and terminal is called a *zero object*.

A category $C$ is *Abelian* if it has a zero object, it has all pullbacks and pushouts, all of its monomorphisms are normal, and all of its epimorphisms are conormal. An object $X$ of an Abelian category $C$ is *simple* if $X$ is not isomorphic to the zero object $0 \in C$ and the only subobjects of $X$ are $0$ and $X$. The set of isomorphism classes of simple objects of $C$ is denoted by $\text{Irr}(C)$. An Abelian category $C$ is *semisimple* if every object $X \in C$ is a direct sum of simple objects.
1.1.2 Functors

Let \( C \) and \( D \) be categories. A functor \( F : C \to D \) is a mapping such that each of the following hold:

- For each \( X \in C \), \( F(X) \in D \).
- For each \( f \in \text{Hom}_C(X,Y) \), \( F(f) \in \text{Hom}_D(F(X),F(Y)) \), and
  - \( F(\text{Id}_X) = \text{Id}_{F(X)} \) for all \( X \in C \).
  - \( F(g \circ f) = F(g) \circ F(f) \) for all morphisms \( f : X \to Y, g : Y \to Z \).

1.1.3 Monoidal Categories and Braiding

A category \( C \) is called monoidal or tensor if there is a bifunctor

\[ \otimes : \text{Obj}(C) \times \text{Obj}(C) \to \text{Obj}(C) \]

that is associative up to a natural isomorphism \( \alpha : A \otimes (B \otimes C) \sim (A \otimes B) \otimes C \), called the associator, and for which there is an object \( I \in C \), called the monoidal unit, such that, for all \( A \in C \), there are isomorphisms \( \lambda_A : I \otimes A \sim A \) and \( \rho_A : A \otimes I \sim A \), called the left and right unit maps, respectively. The monoidal unit \( I \) can be shown to be unique up to natural isomorphism. To say that \( \otimes \) is a bifunctor means that, for any \( A, B \in C \), each of \( A \otimes - : C \to C \) and \( - \otimes B : C \to C \) are functors. A monoidal category \( C \) is strict if each of \( \alpha, \lambda, \) and \( \rho \) are identities.

A monoidal category \( C \) is called braided if, for each pair of objects \( A, B \in C \), there is an isomorphism

\[ c_{A,B} : A \otimes B \to B \otimes A, \]

natural in \( A \) and \( B \), that satisfies the hexagon axioms. By “satisfies the hexagon axioms”, we mean that the collection of braiding maps \( c_{(\cdot),()} \) makes both of the following diagrams
Let \( \langle C, \otimes, I \rangle \) be a monoidal category. The collection

\[
\mathcal{Z}(C) = \left\{ Z \in C \mid A \otimes Z \cong Z \otimes A \ \forall A \in C \right\}
\]

forms a full braided subcategory of \( C \) that we call the monoidal center or Drinfeld center of \( C \). For \( Z \in \mathcal{Z}(C) \) and \( A \in C \), there is a natural isomorphism \( z_A : A \otimes Z \to Z \otimes A \) that we call the half-braiding.

### 1.1.4 Invertible Objects and Pointed Categories

An object \( X \) of a monoidal category \( \langle C, \otimes, I \rangle \) is called invertible if there is an object \( Y \in C \) and isomorphisms \( X \otimes Y \sim I \), \( Y \otimes X \sim I \). The set of isomorphism classes of invertible objects of a monoidal category \( C \) forms a group under \( \otimes \) that is called the Picard group of \( C \) and is denoted by Pic(\( C \)). A monoidal category \( C \) is called pointed if every simple object of \( C \) is invertible.

The Picard subcategory Pic(\( C \)) of a monoidal category \( C \) is the full subcategory of \( C \) generated by all invertible objects.
1.1.5 Rigid, Fusion, and Ribbon Categories

A monoidal category \( \langle C, \otimes, I \rangle \) is rigid if, for every object \( X \in C \), there is an object \( X^* \in C \), called the (left) dual of \( X \), and morphisms \( \text{ev}_X : X^* \otimes X \to I \) (the evaluation map) and \( \text{coev}_X : I \to X \otimes X^* \) (the coevaluation map) such that each of the following compositions is the identity:

\[
\begin{align*}
X & \xrightarrow{\text{coev} \otimes \text{Id}} (X \otimes X^*) \otimes X \xrightarrow{\alpha^{-1}} X \otimes (X^* \otimes X) \xrightarrow{\text{Id} \otimes \text{ev}} X \\
X^* & \xrightarrow{\text{Id} \otimes \text{coev}} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha} (X^* \otimes X) \otimes X^* \xrightarrow{\text{ev} \otimes \text{Id}} X^*
\end{align*}
\]

A \( k \)-linear monoidal category \( C \) is fusion if \( C \) is semisimple, \( C \) is rigid, and the collection \( \text{Irr}(C) \) of isomorphism classes of simple objects of \( C \) is finite.

A rigid monoidal category \( C \) is a ribbon category if, for each object \( X \in C \), there is a morphism \( \theta_X : X \to X \), called the ribbon twist, such that \( \theta_X \otimes \theta_Y = c_{Y,X} \circ c_{X,Y} (\theta_X \otimes \theta_Y) \).

1.1.6 Pivotal and Spherical Categories

Following [2], we say that a \( k \)-linear monoidal category \( C \) is a category with dual data if there is a functor \( (\cdot)^* : C^{\text{op}} \to C \), natural transformations \( \tau : \text{Id}_C \to (\cdot)^* \circ (\cdot)^* \) and \( \gamma : ((\cdot)^* \times (\cdot)^*) \circ \otimes \to \otimes^{\text{op}} \circ (\cdot)^* \), and an isomorphism \( \nu : I \to I^* \). Each component of the natural transformations \( \tau \) and \( \gamma \) is required to be an isomorphism. If the dual data are coherent, then the category \( C \) is called a category with duals. A category with duals is called a pivotal category if, for each object \( X \in C \), there is a morphism \( \varepsilon_X : I \to X \otimes X^* \) that satisfies the following conditions:

- For all morphisms \( f : X \to Y \), the following diagram commutes:

\[
\begin{array}{ccc}
I & \xrightarrow{\varepsilon_X} & X \otimes X^* \\
\downarrow{\varepsilon_Y} & & \downarrow{f \otimes \text{Id}_{X^*}} \\
Y \otimes Y^* & \xrightarrow{\text{Id}_{Y} \otimes f} & Y \otimes X^*
\end{array}
\]
• For all $X \in C$, the following composition is $\text{Id}_X$:

\[
\begin{array}{c}
X^* \\
\downarrow \alpha^{-1} \\
\downarrow \epsilon_{X^*} \otimes \text{Id}_{X^*} \\
I \otimes X^* \\
\downarrow \text{Id}_{X^*} \otimes \lambda^{-1} \\
X^* \otimes I \\
\downarrow \text{Id}_{X^*} \otimes \rho^{-1} \\
(X^* \otimes X^{**}) \otimes X^* \\
\downarrow \tau^{-1} \\
X^* \otimes (X^{**} \otimes X^*) \\
\downarrow \text{Id}_{X^*} \otimes \gamma_{X^{**}X} \\
X^* \otimes (X \otimes X^*)^* \\
\end{array}
\]

• For all $X, Y \in C$, the following composition is $\epsilon_{X \otimes Y}$:

\[
\begin{array}{c}
I \\
\downarrow \epsilon_X \\
X \otimes X^* \\
\downarrow \text{Id}_{X^*} \otimes \lambda^{-1} \\
X \otimes (I \otimes X^*) \\
\downarrow \text{Id}_{X} \otimes \epsilon_{X^*} \otimes \text{Id}_{X^*} \\
X \otimes ((Y \otimes Y^*) \otimes X^*) \\
\end{array}
\]

Let $C$ be a pivotal category, and let $X \in C$. The monoid $\text{End}_C(X) = \text{Hom}_C(X, X)$ has two trace maps, $\text{Tr}_L, \text{Tr}_R : \text{End}_C(X) \to \text{End}_C(I)$, given by the following compositions:

\[
\begin{array}{c}
I \\
\downarrow \epsilon_X \\
X^* \otimes X^{**} \\
\downarrow \text{Id}_{X^*} \otimes \tau^{-1} \\
X^* \otimes X \\
\downarrow \text{Id}_{X} \otimes f \\
X^* \otimes X \\
\downarrow \tau^{-1} \otimes \tau_X \\
X^{***} \otimes X^{**} \\
\end{array}
\]

\[
\begin{array}{c}
I \\
\downarrow \nu^{-1} \\
I^* \\
\downarrow \epsilon(X^*) \\
X^* \otimes X \\
\downarrow \text{Id}_{X} \otimes f \\
X^* \otimes X \\
\downarrow \tau^{-1} \otimes \tau_X \\
X^{***} \otimes X^{**} \\
\end{array}
\]
In the case where $C$ is strict, these compositions simplify to

$$\text{Tr}_L(f) = \varepsilon_{X^*}(\text{Id}_{X^*} \otimes f)\varepsilon_X^*, \quad \text{Tr}_R(f) = \varepsilon_X(f \otimes \text{Id}_{X^*})\varepsilon_{X^*}^*.$$  

A pivotal category $C$ is called a spherical category if, for all objects $X \in C$ and all morphisms $f : X \to X$, we have $\text{Tr}_L(f) = \text{Tr}_R(f)$. We denote the common value of the left and right traces simply by $\text{Tr}(f)$.

1.2 All About Algebras, or “So You Want to Multiply”

1.2.1 Algebras and Modules

Let $k$ be a field, and let $C$ be a $k$-linear monoidal category. An associative unital algebra in $C$ is an object $A \in C$ together with two morphisms $\mu : A \otimes A \to A$ and $\iota : I \to A$ such that

$$\mu(\mu \otimes \text{Id}_A) = \mu(\text{Id}_A \otimes \mu),$$

and

$$\mu(\iota \otimes \text{Id}_A) = \text{Id}_A = \mu(\text{Id}_A \otimes \iota).$$

Henceforth, when we refer to an “algebra” in a monoidal category, we will tacitly assume that the algebra is an associative unital algebra unless we explicitly state otherwise. Let $A, B \in C$ be algebras. A right $A$-module is an object $M \in C$ together with a
morphism $\nu : M \otimes A \to M$ such that

$$\nu(\nu \otimes \text{Id}_A) = \nu(\text{Id}_M \otimes \mu).$$

Left $A$-modules are defined similarly. If $M \in C$ is both a left $A$-module and a right $B$-module such that the respective actions are compatible, then $M$ is said to be an $(A, B)$-bimodule. Compatibility of the actions amounts to commutativity of the following diagram:

$$
\begin{array}{ccc}
A \otimes M \otimes B & \overset{\text{Id}_A \otimes \rho}{\longrightarrow} & A \otimes M \\
\downarrow \rho \otimes \text{Id}_B & & \downarrow \lambda \\
M \otimes B & \overset{\rho}{\longrightarrow} & M
\end{array}
$$

Here, $\rho$ (resp. $\lambda$) denotes right (resp. left) action by $B$ (resp. $A$) on $M$. We denote the category of left $A$-modules in $C$ by $\mathcal{A}C$, that of right $A$-modules by $\mathcal{C}A$, and that of $(A, B)$-bimodules by $\mathcal{A}C_B$.

An algebra $A$ in a monoidal category $C$ is said to be simple if any nonzero algebra homomorphism $A \to B$ is a monomorphism.

An algebra $A$ is said to be separable if it is equipped with a map $\varepsilon : A \to I$ such that the following composition is a nondegenerate pairing (denoted $e : A \otimes A \to I$):

$$A \otimes A \overset{\mu}{\longrightarrow} A \overset{\varepsilon}{\longrightarrow} I.$$ 

Nondegeneracy of $e$ means that there is a morphism $\kappa : I \to A \otimes A$ such that the composition

$$A \overset{\text{Id}_A \circ \kappa}{\longrightarrow} A \otimes A \otimes A \overset{e \otimes \text{Id}_A}{\longrightarrow} A$$

is the identity. It also implies that the similar composition

$$A \overset{\kappa \otimes \text{Id}_A}{\longrightarrow} A \otimes A \otimes A \overset{\text{Id}_A \otimes e}{\longrightarrow} A$$

is also the identity.
For a separable algebra $A$, the adjunction

$$C \rightleftharpoons C_A$$

splits. Indeed, the splitting $M \to M \otimes A$ of the adjunction map $M \otimes A \to M$ is given by the composition

$$M \xrightarrow{\text{Id}_M \otimes \varepsilon} M \otimes A \otimes A \xrightarrow{\mu \otimes \text{Id}_A} M \otimes A.$$

An algebra $A = \langle A, \mu, \iota \rangle$ in a braided monoidal category is \textit{commutative} if

$$\mu = \mu \circ c_{A,A}.$$ 

An indecomposable algebra $A$ is an \textit{etale algebra} if it is both commutative and separable.

A right $A$-module $M = \langle M, \nu \rangle$ is said to be \textit{local} if the following diagram commutes:

$$
\begin{array}{ccc}
M \otimes A & \xrightarrow{\nu} & M \\
\downarrow{c_{M,A}} & & \nu \\
A \otimes M & \xrightarrow{c_{A,M}} & M \otimes A
\end{array}
$$

(1.1)

Locality in this sense is unrelated to the notion of $M$ having a unique maximal submodule.

We denote by $C_A^{\text{loc}}$ the full subcategory of local right $A$-modules. An algebra $L$ in a $k$-linear braided monoidal category $C$ is said to be a \textit{Lagrangian algebra} if the category $C_L^{\text{loc}}$ of local right $L$-modules is equivalent to the category $k$-$\text{Vect}$ of $k$-vector spaces.

\subsection{1.2.2 The Full Center of an Algebra}

The \textit{full center} \footnote{Cf. [9] and [12].} $Z(A)$ of an algebra $A$ in a monoidal category $C$ is an object of the monoidal center $Z(C)$ together with a morphism $Z(A) \to A$ in $C$, terminal among pairs $(Z, \zeta)$, where $Z \in Z(C)$ and $\zeta : Z \to A$ is a morphism in $C$ such that the following diagram
commutes:

\[
\begin{array}{c}
\xymatrix{A \otimes Z \ar[r]^{\text{Id}_A \otimes \zeta} \ar[d]^{\zeta_A} & A \otimes A \
Z \otimes A \ar[r]_{\zeta \otimes \text{Id}_A} & A \otimes A \ar[ru]_{\mu} & \mu}
\end{array}
\] (1.2)

Here, \( \zeta_A \) is the half-braiding of \( Z \) as an object of \( \mathcal{Z}(C) \). The terminality condition means that for any such pair \((Z, \zeta)\), there is a unique morphism \( \phi : Z \to Z(A) \) in the monoidal center \( \mathcal{Z}(C) \), which makes the following diagram commute:

\[
\begin{array}{c}
\xymatrix{Z \ar[r]^{\phi} & Z(A) \
A \ar[ru]_{\iota} &}
\end{array}
\]

Here, \( \iota : Z(A) \to A \) denotes the canonical inclusion.

1.3 Group Cohomology

1.3.1 \textit{G-modules and the Standard Normalized Complex}

Let \( G \) be a group, and let \( M \) be an Abelian group. We say that \( M \) is a \textit{G-module} if \( G \) acts \( \mathbb{Z} \)-linearly on \( M \), \textit{i.e.}, \( G \) acts on \( M \) such that

\[ g.(m + n) = g.m + g.n \quad \forall g \in G, m, n \in M. \]

Let \( M \) be a \( G \)-module. For an integer \( n \geq 0 \), we denote by

\[ C^n(G, M) = \left\{ c : G \times \cdots \times G \to M \middle| c(g_1, \ldots, g_n) = 0 \text{ if } \exists j \text{ s.t. } g_j = e \right\} \]

the group of \textit{normalized} \( n \)-cochains. This is the Abelian group of set-theoretic functions from the \( n \)-fold Cartesian power of \( G \) into \( M \) that vanish whenever at least one of their arguments is the identity element of \( G \), with group operation given by pointwise addition.
For an integer $n \geq 0$, the \textit{standard differential or coboundary homomorphism}
\[ \partial^n : C^n(G, M) \rightarrow C^{n+1}(G, M) \]
is given by the formula
\[ [\partial^n(c)](g_1, \ldots, g_{n+1}) = g_1 \cdot c(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} c(g_1, \ldots, g_n). \] (1.3)

Through an heroic piece of elementary algebra, it can be shown that
\[ \partial^{n+1} \circ \partial^n = 0 \quad \forall n \geq 0. \]

We define the group of \textit{normalized $n$-cocycles} to be
\[ Z^n(G, M) = \ker(\partial^n), \]
and the subgroup of \textit{normalized $n$-coboundaries} to be
\[ B^n(G, M) = \begin{cases} 
\{0\}, & n = 0 \\
\partial^{n-1}[C^{n-1}(G, M)], & n > 0 
\end{cases}. \]

The $n^{\text{th}}$ \textit{cohomology group of $G$ with coefficients in $M$} is the quotient group
\[ H^n(G, M) = Z^n(G, M) / B^n(G, M). \]

Here, the condition that $\partial^{n+1} \circ \partial^n = 0$ is used to guarantee that $B^n(G, M) \subseteq Z^n(G, M)$, so that this quotient makes sense.
### 1.3.2 Abelian Cohomology

Let $G$ be an Abelian group and $M$ a trivial $G$-module. Under these assumptions, the standard differential (1.3) simplifies to

$$[\partial^n(c)](g_1, \ldots, g_{n+1}) = c(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i c(g_1, \ldots, g_ig_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} c(g_1, \ldots, g_n).$$

Let $C_*(G)$ denote the graded Abelian group for which, for $n > 0$, $C_n(G) = \mathbb{Z}[G^{	imes n}]$, i.e., $C_n(G)$ is the free Abelian group with basis $\{(g_1, g_2, \ldots, g_n) | g_i \in G\}$. Define $\mathbb{Z}$-linear maps $d_{n+1} : C_{n+1}(G) \to C_n(G)$ by

$$d_{n+1}(g_1, \ldots, g_{n+1}) = (g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (g_1, \ldots, g_ig_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} (g_1, \ldots, g_n).$$

It can be shown that the maps $d_n$ are chain differentials.

An $(n, n')$-shuffle is a permutation $\sigma \in S_{n+n'}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$ and $\sigma(n + 1) < \sigma(n + 2) < \cdots < \sigma(n + n')$. Denote by $\text{Sh}(n, n')$ the set of all $(n, n')$-shuffles. Define the shuffle product $\star : C_n(G) \otimes_{\mathbb{Z}} C_{n'}(G) \to C_{n+n'}(G)$ on generators by

$$(g_1, \ldots, g_n) \star (g_{n+1}, \ldots, g_{n+n'}) = \sum_{\sigma \in \text{Sh}(n, n')} (-1)^{\text{sgn}(\sigma)} (g_{\sigma(1)}, \ldots, g_{\sigma(n)}, g_{\sigma(n+1)}, \ldots, g_{\sigma(n+n')}).$$

It can be shown that the shuffle product is associative and commutative, and satisfies the derivation identity

$$d_*(x \star y) = d_*(x) \star y + (-1)^{|y|} x \star d_*(y), \quad x, y \in C_*(G).$$

The shuffle product endows $C_*(G)$ with the structure of a differential graded algebra.

Denote by $B_*(G) = B(C_*(G))$ the bar complex of the differential graded algebra $C_*(G)$, i.e., for $n > 0$, $B_n(G)$ is generated by symbols $[x_1| \cdots |x_r]$, where $x_i \in C_n(G)$ such that $n_1 > 0$ and $n_1 + n_2 + \cdots + n_r + (r - 1) = n$. The differential $d : B_{n+1}(G) \to B_n(G)$ has the form

$$d[x_1| \cdots |x_r] = \sum_{i=1}^{r} (-1)^{n_{i-1}} [x_1| \cdots |d(x_i)| \cdots |x_r] + \sum_{j=i}^{r-1} (-1)^{n_j} [x_1| \cdots |x_i \star x_j| \cdots |x_r].$$
where \( a_i = |x_1| + |x_2| + \cdots + |x_i| + i \).

The cohomology \( H^*_{ab}(G, M) \) of the cochain complex \( \text{Hom}_\mathbb{Z}(B_\ast(G), M) \) is called the \textit{Abelian cohomology}. The first Abelian cohomology group has the form
\[
H^1_{ab}(G, M) \simeq \{ \phi : G \to M \mid \phi(fg) = \phi(f) + \phi(g) \} = \text{Hom}_\mathbb{Z}(G, M),
\]
and so it coincides with the ordinary first cohomology group \( H^1(G, M) \). The second Abelian cohomology group has the form
\[
H^2_{ab}(G, M) = \left\{ \gamma \in Z^2(G, M) \mid \gamma(f, g) = \gamma(g, f) \forall f, g \in G \right\} / B^2(G, M),
\]
and hence is a subgroup of the ordinary second cohomology group \( H^2(G, M) \) consisting of cohomology classes of symmetric 2-cocycles, \( i.e., H^2_{ab}(G, M) = \text{Ext}^1_\mathbb{Z}(G, M) \). The third Abelian cohomology group has the form
\[
H^3_{ab}(G, M) = Z^3_{ab}(G, M) / B^3_{ab}(G, M),
\]
where an Abelian 3-cocycle is a pair \((\alpha, c) \in Z^3(G, M) \times C^2(G, M)\) such that
- \( c(f, g) + c(g, h) + \alpha(f, h, g) = c(fg, h) + \alpha(f, g, h) + \alpha(h, f, g) \), and
- \( c(f, h) + c(f, g) + \alpha(f, g, h) = c(f, gh) + \alpha(g, f, h) + \alpha(g, h, f) \)
for all \( f, g, h \in G \). An Abelian 3-cocycle \((\alpha, c)\) is an Abelian 3-coboundary if is of the form
\[
(\alpha, c) = (\partial u(f, g, h), u(f, g) - u(g, f))
\]
for some \( u \in C^2(G, M) \).

1.4 Categories, Part II, or “Generalized Abstract Nonsense”

1.4.1 Graded Vector Spaces

Let \( X \) be a set. A \( k \)-vector space \( V \) is \( X \)-\textit{graded} if there exists a direct sum decomposition
\[
V = \bigoplus_{x \in X} V_x
\]
of $V$ into $k$-vector subspaces indexed by the set $X$.

A $G$-graded vector space

$$A = \bigoplus_{g \in G} A_g$$

is said to be a $G$-graded algebra if it is equipped with a multiplication that preserves the $G$-grading:

$$A_f A_g \subseteq A_{fg}, \quad f, g \in G.$$  

For a 3-cocycle $\alpha \in Z^3(G, k^*)$, a $G$-graded algebra $A$ is $\alpha$-associative if

$$a(bc) = \alpha(f, g, h)(ab)c, \quad \forall a \in A_f, b \in A_g, c \in A_h.$$  

A $G$-algebra is an algebra $A \in k$-Vect together with an action of $G$ on $A$ by algebra homomorphisms.

### 1.4.2 The Category $\mathcal{V}(G, \alpha)$

For a group $G$, a field $k$, and a 3-cocycle $\alpha \in Z^3(G, k^*)$, we define the monoidal category $\mathcal{V}(G, \alpha)$ as follows. Objects of $\mathcal{V}(G, \alpha)$ are $G$-graded $k$-vector spaces. Morphisms in $\mathcal{V}(G, \alpha)$ are $k$-linear maps $T : X \to Y$ that preserve the $G$-grading in the sense that, for each $g \in G$, we have $T[X_g] \subseteq Y_g$. The tensor product in $\mathcal{V}(G, \alpha)$ is the usual tensor product of graded vector spaces, \(\text{viz.}\),

$$ (V \otimes W)_f = \bigoplus_{g, h : gh = f} \left( V_g \otimes W_h \right), \quad f \in G, \quad (1.4) $$

so that

$$ V \otimes W = \bigoplus_{f \in G} \left[ \bigoplus_{g, h : gh = f} \left( V_g \otimes W_h \right) \right]. $$

Associativity of the tensor product in $\mathcal{V}(G, \alpha)$ is twisted by $\alpha$ in the following way. Let $U, V, W \in \mathcal{V}(G, \alpha)$, and assume that $u \in U_f, v \in V_g$, and $w \in W_h$ for $f, g, h \in G$. Then

$$ u \otimes (v \otimes w) = \alpha(f, g, h)(u \otimes v) \otimes w. \quad (1.5) $$
The monoidal unit in $V(G, \alpha)$ is $I = I_e = k$.

1.4.3 The Category $\mathcal{Z}(G, \alpha)$

Let $G$ be a group, and let $\alpha \in Z^3(G, k^*)$ be a normalized 3-cocycle of $G$ with coefficients in the trivial $G$-module $k^*$. An $\alpha$-projective $G$-action on a $G$-graded vector space $V$ is a collection of automorphisms $f : V \to V$ for each $f \in G$ such that

$$f[V_g] = V_{fg^{-1}},$$

and

$$(fg).v = \alpha(f, g|h)f.(g.v), \quad \forall v \in V_h. \quad (1.6)$$

Here,

$$\alpha(f, g|h) = \frac{\alpha(f, gh^{-1}, g)}{\alpha(f, g, h)\alpha(fghg^{-1}f^{-1}, f, g)}. \quad (1.7)$$

Similarly, define

$$\alpha(f|g, h) = \frac{\alpha(f, g, h)\alpha(fgf^{-1}, fhf^{-1}, f)}{\alpha(fgf^{-1}, f, h)}. \quad (1.8)$$

The following identities follow directly from the 3-cocycle condition for normalized $\alpha$:

$$\alpha(f, gh|h)\alpha(g, h|u) = \alpha(fg, h|u)\alpha(f, g|huh^{-1}), \quad (1.9)$$

$$\alpha(fg|u, v)\alpha(f, g|u)\alpha(f, g|v) = \alpha(f, g|uv)\alpha(g|u, v)\alpha(f|gug^{-1}, gvg^{-1}),$$

$$\alpha(g, h, u)\alpha(f|gh, u)\alpha(f|g, h) = \alpha(f|g, hu)\alpha(f|h, u)\alpha(fgf^{-1}, fhf^{-1}, fuf^{-1}).$$

The twisted Drinfeld center $\mathcal{Z}(G, \alpha)$ is the braided monoidal category defined as follows. Objects of $\mathcal{Z}(G, \alpha)$ are $G$-graded vector spaces

$$V = \bigoplus_{g \in G} V_g$$

together with a given $\alpha$-projective $G$-action. We will denote the action of an element $g \in G$ on a vector $v \in V$ by $g.v$. Morphisms of $\mathcal{Z}(G, \alpha)$ are graded and action-preserving homomorphisms of vector spaces. The tensor product in $\mathcal{Z}(G, \alpha)$ is the tensor product of
$G$-graded vector spaces as in (1.4), with associativity twisted by $\alpha$ as in (1.5), and $\alpha$-projective $G$-action defined by

$$f.(x \otimes y) = \alpha(f|g,h)\left((f.x) \otimes (f.y)\right), \quad x \in X_g, y \in Y_h. \quad (1.10)$$

The monoidal unit is $I = I_e = k$ with trivial $G$-action. The braiding is given by

$$c_{X,Y}(x \otimes y) = (f.y) \otimes x, \quad x \in X_f, y \in Y. \quad (1.11)$$

The category $Z(G,\alpha)$ is rigid\textsuperscript{35}, with dual objects

$$X^* = \bigoplus_{f \in G} (X^*)_f$$

given by

$$(X^*)_f = \left(X_{f^{-1}}\right)^* = \text{Hom}_k(X_{f^{-1}}, k), \quad f \in G,$$

with the $\alpha$-projective action

$$(g.\ell)(x) = \frac{1}{\alpha(g|f^{-1}f,g^{-1}f^{-1})}f(g^{-1}.x), \quad \ell \in \text{Hom}_k(X_{f^{-1}}, k), \quad x \in X_{g^{-1}f^{-1}g}.$$

The category $Z(G,\alpha)$ is unitarizable with the ribbon twist

$$\theta_X(x) = \alpha(f, f^{-1}, f) f^{-1}.x, \quad x \in X_f.$$

The (unitary) trace of an endomorphism $a : X \to X$ can be written in terms of ordinary traces on vector spaces $X_g$:

$$\text{Tr}(a) = \sum_{g \in G} \text{Tr}_X(a_g),$$

and the (unitary) dimension of an object $X \in Z(G, \alpha)$ is the dimension of its underlying (graded) vector space

$$\text{dim}(X) = \sum_{g \in G} \text{dim}(X_g).$$

It will be shown that the Drinfeld center $Z(G,\alpha)$ is precisely the monoidal center of the category $\mathcal{V}(G,\alpha)$ of graded vector spaces.

\textsuperscript{35} See lemma 2.2.2.1.
1.4.4 The Category $C(G, H, \alpha)$

For a (finite) group $G$, a subgroup $H \leq G$, and a 3-cocycle $\alpha \in Z^3(G, k^\times)$, we define the monoidal category $C(G, H, \alpha)$ as follows: Objects of $C(G, H, \alpha)$ are $G$-graded $k$-vector spaces with a given $\alpha|_H$-projective $H$-action. Morphisms of $C(G, H, \alpha)$ are $k$-linear maps preserving both the $G$-grading and $H$-action. The tensor product in $C(G, H, \alpha)$ is the tensor product of $G$-graded $k$-vector spaces, with $\alpha|_H$-projective $H$-action given by

$$h.(x \otimes y) = \alpha(h|_f, g)(h.x) \otimes (h.y), \quad x \in X_f, \ y \in Y_g, \ h \in H, \ f, g \in G.$$ 

The monoidal unit in $C(G, H, \alpha)$ is $I = I_e = k$ with trivial $H$-action. Note that the category $C(G, H, \alpha)$ defined above contains, as a full subcategory, the braided category $Z(H, \alpha|_H)$, and that the category $C(G, G, \alpha)$ is $Z(G, \alpha)$.

1.4.5 The Category $C(A, q)$ and Quadratic Functions

Let $A$ be an Abelian group, let $\alpha \in Z^3(A, k^\times)$ be normalized, and let $c \in C^2(A, k^\times)$ be a normalized 2-cochain such that the pair $(\alpha, c) \in H^3_{ab}(A, k^\times)$ is an Abelian 3-cocycle. Define the pointed category $C(A, \alpha, c)$ as follows. Objects of $C(A, \alpha, c)$ are $A$-graded $k$-vector spaces. Morphisms of $C(A, \alpha, c)$ are linear maps preserving the $A$-grading. The category $C(A, \alpha, c)$ is braided monoidal with respect to the tensor product of graded vector spaces, with associativity given by

$$\alpha_{V,U,W}(v \otimes (u \otimes w)) = \alpha(a, b, c)((v \otimes u) \otimes w), \quad v \in V_a, \ u \in U_b, \ w \in W_c, \ a, b, c \in A,$$

and braiding given by

$$c_{V,W}(v \otimes w) = c(a, b)(w \otimes v), \quad v \in V_a, \ w \in W_b.$$

The category $C(A, \alpha, c)$ is fusion, with its simple objects being one-dimensional vector spaces $I(a)$ concentrated in degree $a \in A$. 

\[ (1.12) \]
Let $A$ be an Abelian group. A function $q : A \to M$ is a quadratic function if the following hold: For all $n \in \mathbb{Z}$ and $x \in A$, we have $q(nx) = n^2 (q(x))$, and the function $\sigma : A \times A \to M$ defined by $\sigma(x, y) = q(x + y) - q(x) - q(y)$ is a symmetric $\mathbb{Z}$-bilinear form. We call $\sigma$ the polarization of $q$. The set of such quadratic functions forms a group that we denote by $Q(A, M)$. Via the assignment $(\alpha, c) \mapsto q : G \to M$, where $q(x) = c(x, x)$, the Abelian cohomology group $H^3_{ab}(G, M)$ is isomorphic to $Q(G, M)$. We apply the correspondence $(\alpha, c) \mapsto q$ to write $C(A, \alpha, c) = C(A, q)$ for short.

When the quadratic function $q$ is trivial, we will write $C(A) = C(A, q)$. The category $C(A)$ is symmetric and coincides with $\text{Rep}(\hat{A})$. Here, $\hat{A} = \text{Hom}_\mathbb{Z}(A, \mathbb{Q}/\mathbb{Z})$.

A quadratic function $q : A \to M$ is nondegenerate if its polarization $\sigma$ is nondegenerate. In the case where $A$ is finite, a quadratic function $q : A \to M$ is nondegenerate if and only if $\ker(\sigma) = \{0\}$.

For a subgroup $B$ of a quadratic Abelian group $(A, q)$, the orthogonal complement of $B$ is the subgroup $B^\perp \leq A$ given by

$$B^\perp = \left\{ a \in A \middle| q(a + b) = q(a) + q(b) \ \forall b \in B \right\}.$$

A subgroup $B \leq A$ is isotropic (with respect to $q$) if $q|_B \equiv 0$. If $B$ is an isotropic subgroup of $A$, then $B \subseteq B^\perp$. In this case, we call the quotient group $B^\perp / B$ the isotropic contraction along $B$. The quadratic function $q$ descends to the isotropic contraction $B^\perp / B$.

We call an isotropic subgroup $L$ of a nondegenerate quadratic group $(A, q)$ Lagrangian if $L = L^\perp$. If $A$ is finite, then $L$ is Lagrangian if and only if $|L|^2 = |A|$.

### 1.4.6 Categorical Groups and Their Standard Invariants

Following [26, § 3.1], we define a categorical group to be a monoidal category $(\mathcal{G}, \otimes, I)$ in which all morphisms are invertible and, for every object $X \in \mathcal{G}$, there is an object $X^* \in \mathcal{G}$ and a morphism $e_X : X^* \otimes X \to I$. 
Let $\mathcal{G}$ be a categorical group. The *standard invariants* of $\mathcal{G}$ are the zeroth and first homotopy groups $\pi_0(\mathcal{G})$ and $\pi_1(\mathcal{G})$, respectively. $\pi_0(\mathcal{G})$ is the group of isomorphism classes of objects of $\mathcal{G}$, and $\pi_1(\mathcal{G})$ is the group of automorphisms of the monoidal unit $I \in \mathcal{G}$. The associativity constraint $\alpha$ for the tensor product in $\mathcal{G}$ is a class in $H^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G}))$. Thus up to equivalence, categorical groups are labeled by triples $(\pi_0(\mathcal{G}), \pi_1(\mathcal{G}), [\alpha])$. Here, $[\alpha] \in H^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G}))$ is the cohomology class of the associator of $\mathcal{G}$.

### 1.4.7 Module Categories

Let $C$ be a monoidal category with associator $\alpha$. A *module category* over $C$ is a category $\mathcal{M}$ equipped with an action $\ast : C \times \mathcal{M} \to \mathcal{M}$, a unit isomorphism $\ell_M : \text{Id} \otimes M \to M$, and a natural isomorphism $a_{X,Y,M} : (X \otimes Y) \ast M \to X \ast (Y \ast M)$ for all $X, Y \in C$ and $M \in \mathcal{M}$ such that the pentagon diagram

\[
\begin{align*}
(X \otimes (Y \otimes Z)) \ast M &\cong (X \otimes Y) \ast (Z \ast M) \\
&\cong (X \ast ((Y \otimes Z) \ast M)) \\
&\cong (X \ast ((Y \ast (Z \ast M)))
\end{align*}
\]

commutes for all $X, Y, Z \in C$ and $M \in \mathcal{M}$. An action of $C$ on $\mathcal{M}$ is nothing but a monoidal functor $C \to \mathcal{End}(\mathcal{M})$.

### 1.4.8 Module Functors

Let $\mathcal{M}$ and $\mathcal{N}$ be module categories with associativity constraints $m$ and $n$, respectively. Then a *module functor* $F : \mathcal{M} \to \mathcal{N}$ is an isomorphism $F_{X,M} : F(X \ast M) \to X \ast F(M)$ for all $X \in C$ and $M \in \mathcal{M}$ such that the following diagrams
commute:

\[ F(X \ast (Y \ast M)) \quad (1.14) \]

\[
\begin{array}{ccc}
F((X \otimes Y) \ast M) & \xrightarrow{F_{(X \otimes Y) \ast M}} & X \ast F(Y \ast M) \\
F_{X \otimes Y} & \downarrow & \downarrow \text{Id}_X \ast F_{Y \ast M} \\
(X \otimes Y) \ast F(M) & \rightarrow & X \ast (Y \ast F(M)) \\
\end{array}
\]

and

\[
\begin{array}{ccc}
F(\text{Id} \otimes M) & \xrightarrow{F_{(\text{Id} \otimes M)}} & \text{Id} \otimes F(M) \\
F_{(\text{Id} \otimes M)} & \downarrow & \downarrow F_{(\text{Id} \otimes M)} \\
F(M) & \rightarrow & F(M) \\
\end{array}
\]

for all \( X, Y \in C \) and \( M \in M \). For module functors \( F : M \rightarrow N \) and \( G : N \rightarrow P \), the composition \( G \circ F \) has the following module structure:

\[ G(F(V \ast M)) \xrightarrow{G(F_{V \ast M})} G(V \ast F(M)) \xrightarrow{G_{V \ast F(M)}} V \ast G(F(M)) \]

1.4.9 Module Natural Transformations

Now let \( F \) and \( G \) be functors of \( C \)-module categories \( M \rightarrow N \). Let \( \alpha \) be a natural transformation of \( F \) into \( G \). Then \( \alpha \) is given by a collection of isomorphisms \( a_M : F(M) \rightarrow G(M) \), \( M \in M \), such that the following diagram commutes:

\[ F(X \ast M) \xrightarrow{F_{X \ast M}} X \ast F(M) \]

\[ \xrightarrow{a_{X \ast M}} \]

\[ G(X \ast M) \xrightarrow{G_{X \ast M}} X \ast G(M) \]

1.4.10 Simple \( V(G, \alpha) \)-module Categories

Consider a pointed fusion category of the form \( C = V(G, \alpha) \), where \( G \) is a finite group. Let \( X \) be a \( G \)-set. Define an action of \( V(G, \alpha) \) on the category \( V(X) \) of \( X \)-graded
vector spaces by

\[(V \ast M)_x = \bigoplus_{g, y = x} V_g \otimes M_y, \quad V \in \mathcal{V}(G, \alpha), \ M \in \mathcal{V}(X).\]

For \(\gamma \in C^2(G, \text{Map}(X, k))\), define the action associativity by

\[a_{U,V,M}((u \otimes v) \ast m) = \gamma(f, g, x)u \ast (v \ast m), \quad u \in U_f, \ v \in V_g, \ m \in M_x.\]

The pentagon axiom (1.13) is equivalent to the condition that \(\partial \gamma = \alpha\). We will denote the resulting \(\mathcal{V}(G, \alpha)\)-module category by \(M(X, \gamma)\).

For this example, we can generate full module functors

\[F = F(f, \varphi) : M(X, \gamma) \to M(Y, \eta)\]

using the following choices. First, choose a map of \(G\)-sets \(f : X \to Y\) and \(\varphi : G \times X \to k^\ast\) with \(\varphi \in C^1(G, \text{Map}(X, k^\ast))\). Coherence of (1.14) implies \(\gamma = \partial(\varphi)f^\ast(\eta)\) while the definition gives for \(M \in \mathcal{M}\),

\[F(f, \varphi)(M) = F(M)_y = \bigoplus_{f(x) = y} M_x.\]

The composition of module functors \(F(f, \varphi)\) and \(F'(f', \varphi')\) is

\[F(f, \varphi) \circ F'(f', \varphi') = F(f'f, f^\ast(\varphi')\varphi).\]

For \(g \in V_g\) and \(x \in X\), the composition has the form

\[F'(F(g, x)) = F'(f(g), \varphi(g, x)) = (f^\ast(f(g)), \varphi'(g, f(x))).\]

In the specific case where \(\mathcal{M} = \mathcal{N} = M(X, \gamma)\), and \(F = F(f, \varphi)\) and \(G = F(f', \varphi')\), a natural transformation \(\alpha\) is an element of \(\text{Map}(X, k^\ast) = C^0(G, \text{Map}(X, k^\ast))\), and the commutativity of (1.15) amounts to the equality \(f(\alpha) \cdot \varphi = \varphi'\).

A \(\mathcal{V}(G, \alpha)\)-module category \(\mathcal{V}(X, \gamma)\) is simple if and only if \(X\) is a transitive \(G\)-set. Write \(X = G/H\) for some subgroup \(H \leq G\). The associativity constraint \(\gamma\) is a class in \(C^2(G, \text{Map}(G/H, k^\ast))\). Simple \(\mathcal{V}(G, \alpha)\)-module categories correspond to pairs \((H, \gamma)\), where \(H \leq G\) is a subgroup and \(\gamma \in C^2(H, k^\ast)\) is a coboundary \(\partial \gamma = \alpha|_H\).
2 Preliminary Results

Here, we establish a number of general preliminary results that will be of use later.

2.1 Algebras in Monoidal Categories

2.1.1 Maschke’s Lemma

Here, we state and prove a well-known result of classical representation theory.

Lemma 2.1.1.1 ([42], Proposition 9; [24], Proposition IX.5.8). Let $k$ be a field, and let $G$ be a finite group. The group algebra $k[G]$ is semisimple if and only if $\text{char}(k) \nmid |G|$. In particular, if $\text{char}(k) = 0$, then the group algebra $k[G]$ is semisimple.

Proof. ($\Leftarrow$) Suppose that $\text{char}(k) \not| |G|$. Let $M$ be a left $k[G]$-module, and let $N \subseteq M$ be a submodule. Then in particular, $N$ is a $k$-vector subspace of $M$, so there is a complementary subspace $W$ such that $M = N \oplus W$. Hence every element $m \in M$ can be written uniquely as $m = n + w$ for some $n \in N$, $w \in W$. Define $\pi : M \to N$ by $\pi(n + w) = n$. Observe that $\pi$ is a $k$-linear projection of $M$ onto $N$, and that $\ker(\pi) = W$.

Since $\text{char}(k) \mid |G|$, either $\text{char}(k) = 0$ or $|G|$ is a unit modulo $\text{char}(k)$. In view of this fact, define $\hat{\pi} : M \to N$ by

$$\hat{\pi}(m) = \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}m).$$

Note that $\hat{\pi}$ is $k$-linear, $\ker(\hat{\pi}) = W$, and $\hat{\pi}[M] \subseteq N$. We claim that $\hat{\pi}$ is $k[G]$-linear. Given $m \in M$ and $x \in G$, we have

$$\hat{\pi}(xm) = \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}xm).$$

As $g$ runs over $G$, so does $h := x^{-1}g$, whence we have

$$\hat{\pi}(xm) = \frac{1}{|G|} \sum_{h \in G} xh\pi(h^{-1}m) = x \left[ \frac{1}{|G|} \sum_{h \in G} h\pi(h^{-1}m) \right] = x\hat{\pi}(m).$$

Therefore $\hat{\pi}$ is $k[G]$-linear, as claimed.
To show that $\hat{\pi}$ is a projection onto $N$, it suffices to show that $\hat{\pi}^2 = \hat{\pi}$. Observe that, given $n \in N$ and $g \in G$, we have $(gn) \in N$, whence $\pi(gn) = gn$. Therefore

$$\hat{\pi}(n) = \frac{1}{|G|} \sum_{g \in G} g\pi(g^{-1}n) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1}n) = \frac{1}{|G|} \sum_{g \in G} (gg^{-1})n = \frac{1}{|G|} \sum_{g \in G} n = \frac{1}{|G|} \cdot |G| \cdot n = n.$$ 

It follows that $\hat{\pi}|_N = \text{Id}_N$, and therefore $\hat{\pi}^2 = \hat{\pi}$, so that $\hat{\pi}$ is a projection with image $N$. It follows that $M = N \oplus W$ is a direct sum of $k[G]$-modules, whence $k[G]$ is semisimple.

$(\Rightarrow)$ By contrapositive. Suppose that $\text{char}(k) \mid |G|$. Note that in this case, we must have $\text{char}(k) > 0$. Let $w := \sum_{g \in G} g$. Note that $w \neq 0$. Moreover, for $h \in G$, we have

$$hwh^{-1} = h \left( \sum_{g \in G} g \right) h^{-1} = \sum_{g \in G} hgh^{-1},$$

which is equal to $w$ because conjugation is an automorphism of $G$. It follows that $hw = wh$ for all $h \in G$, whence $w \in Z(k[G])$. Let $n := |G|$, and enumerate $G$ as $G = \{g_1, g_2, \ldots, g_n\}$.

Then

$$w^2 = \left( \sum_{i=1}^n g_i \right) \left( \sum_{j=1}^n g_j \right) = \sum_{i=1}^n \left[ g_i \left( \sum_{j=1}^n g_j \right) \right].$$

Since for each $i = 1, 2, \ldots, n$ we have $g_i \sum_{j=1}^n g_j = g_i w = w$, it follows that

$$w^2 = \sum_{i=1}^n w = nw.$$

Since $\text{char}(k) \mid n$, it follows that $w^2 = 0$. Let $I := k[G]w$. In view of the previous argument, $I^2 = \{0\}$, and hence $I$ is nilpotent, so it follows that $I \subseteq J(k[G])$. Moreover, $w \in I$ and $w \neq 0$, so $I \neq \{0\}$. Therefore $J(k[G]) \neq \{0\}$, whence $k[G]$ is not semisimple.

Remark 2.1.1.2. In what follows, we will be more concerned with the case where the regular group algebra $k[G]$ is replaced by a twisted group algebra of the form $k[G, \beta]$ for some $\beta \in Z^2(G, k^*)$. It is still true in this case that, if $\text{char}(k) \nmid |G|$, then the twisted group algebra $k[G, \beta]$ is semisimple.
2.1.2 The Simple Things

**Lemma 2.1.2.1** ([12], Lemma 2.1). *Let $A$ be an indecomposable separable algebra in a spherical fusion category $C$. Then $A$ is simple.*

**Proof.** If a separable algebra $A$ is not simple, then there is a surjective, but not bijective, algebra homomorphism $A \twoheadrightarrow B$. Via the inverse image functor, the category of right $B$-modules $C_B$ becomes a full subcategory of $C_A$. Moreover, $C_B$ is a full left $C$-module subcategory of $C_A$. Recall from [18, Proposition 3.9] that a semisimple module category over a fusion category is a direct sum of its simple module subcategories. In particular, $C_B$ is a direct summand of $C_A$ as a $C$-module category. Hence the algebra $\text{Hom}_C(I, A)$ in $k\text{-Vect}$, which coincides with the algebra $\mathcal{E}nd_C(\text{Id}_{C_A})$ of $C$-module endomorphisms of the identity functor of $C_A$, is a nontrivial direct sum. Thus $A$ is decomposable. ■

2.1.3 The Full Center of an Algebra

Here, we enumerate several properties of the full center$^{36}$ of an algebra.

**Proposition 2.1.3.1** ([12], Proposition 2.3). *The full center $Z(A)$ of an indecomposable separable algebra $A$ in a fusion category $C$ coincides with the action internal end $[A, A] \in Z(C)$ of the trivial bimodule $A \in AC_A$ with respect to the $Z(C)$-action on $AC_A$ given by $\alpha$-induction. Moreover, the category of modules $Z(C)_{Z(A)}$ is equivalent, as a fusion category, to the category $AC_A$ of $(A, A)$-bimodules.*

**Proof.** The universal property of the action internal end says that $[A, A]$ is the terminal object among pairs $(Z, z)$ where $Z \in Z(C)$ and $\xi : \alpha(Z) \rightarrow A$ is a morphism of $(A, A)$-bimodules. The right $A$-module map $\xi : Z \otimes A \rightarrow A$ is completely determined by the morphism $\zeta = \xi(\text{Id}_{\otimes A}) : Z \rightarrow A$ (which is still a left $A$-module map). The left $A$-module property for $\zeta$ is precisely (1.2).

$^{36}$ *Cf.* [9], [12], and the diagram (1.2).
According to [38], the functors
\[ Z(C)_{[A,A]} \xrightarrow{\sim} \mathcal{A} \]
\[ [A, -] \]
are quasi-inverse equivalences. The tensor structure
\[ [A, M] \otimes_{[A,A]} [A,N] \rightarrow [A, M \otimes_A N] \quad (2.1) \]
for the functor \([A, -]\) comes from the universal property of the action internal hom.

Indeed, the composition
\[ \alpha([A, M] \otimes [A, N]) \otimes_A A \simeq \alpha([A, M]) \otimes_A A \otimes_A \alpha([A, N]) \otimes_A A \rightarrow M \otimes_A N \]
induces the map \([A, M] \otimes [A, N] \rightarrow [A, M \otimes_A N]\), which is naturally \([A, A]\)-balanced, viz., factors through \([A, M] \otimes_{[A,A]} [A, N]\), giving rise to (2.1).

\[ \square \]

**Remark 2.1.3.2.** Here is a slightly different proof of the second statement of proposition 2.1.3.1, due to A. Davydov. The canonical braided equivalence (Morita invariance of the monoidal center) \( Z(C) \rightarrow Z(A \mathcal{C} A) \) sends the full center \( Z(A) \) to the full center \( Z(I) \) of the monoidal unit \( I \in \mathcal{A} \mathcal{C} A \) (which is really the \((A, A)\)-bimodule \( A \)).

For a fusion category \( D (= A \mathcal{C} A) \), the full center \( Z(I) \in Z(D) \) coincides with \( L(I) \), where \( L : D \rightarrow Z(D) \) is the adjoint to the forgetful functor \( F : Z(D) \rightarrow D \). The adjunction is monadic. Moreover, the monad \( T = L \circ F \) on \( Z(D) \) is a \( Z(D) \)-module functor. Thus \( T \)-algebras are the same as \( T(I) \)-modules. Finally, \( T(I) = L(I) = Z(I) \), and the forgetful functor factorizes:

\[ \xymatrix{ \mathcal{Z}(D) \\ \mathcal{Z}(D)_{Z(I)} \ar[u]^{F} \\
 D } \]

**Theorem 2.1.3.3** ([12], Theorem 2.5). The full center \( Z(A) \) of an indecomposable separable algebra \( A \) in a fusion category \( C \) is a Lagrangian algebra in \( Z(C) \).
Proof. The tensor equivalence $\mathcal{Z}(\mathcal{C})_{\mathcal{A}} \rightarrow \mathcal{A}C_A$ from proposition 2.1.3.1 induces a braided tensor equivalence $\mathcal{Z}(\mathcal{Z}(\mathcal{C})_{\mathcal{A}}) \rightarrow \mathcal{Z}(A C_A)$. By Morita invariance of the monoidal center, $\mathcal{Z}(A C_A) \simeq \mathcal{Z}(\mathcal{C})$.

By [16, Proposition 3.7], we have the decomposition into the Deligne product

$$\mathcal{Z}(\mathcal{C}) \boxtimes \mathcal{Z}(\mathcal{C})_{\mathcal{A}} \simeq \mathcal{Z}(\mathcal{Z}(\mathcal{C})_{\mathcal{A}}).$$

Combining the above, we get $\mathcal{Z}(\mathcal{C}) \boxtimes \mathcal{Z}(\mathcal{C})_{\mathcal{A}} \simeq \mathcal{Z}(\mathcal{C})$, which means that $\mathcal{Z}(\mathcal{C})_{\mathcal{A}} \simeq k\text{-}\text{Vect}$, i.e., $\mathcal{Z}(\mathcal{A})$ is Lagrangian. ■

2.2 Commutative Algebras in Group-theoretical Categories

2.2.1 Group-theoretical Braided Fusion Categories

In what follows, we will frequently make use of the following result, which was established in [19]. We state it here without proof.

Lemma 2.2.1.1 ([19]). Let $(A, m, i)$ be a commutative algebra in a braided category $\mathcal{C}$. Let $(B, \mu, \iota)$ be an algebra in $\mathcal{C}_A$. Define $\mu$ and $\iota$ as compositions

$$B \otimes B \rightarrow B \otimes_A B \xrightarrow{\mu} B, \quad I \rightarrow A \xrightarrow{\iota} B.$$

Then $(B, \mu, \iota)$ is an algebra in $\mathcal{C}$. Moreover, the map $\iota : A \rightarrow B$ is a homomorphism of algebras in $\mathcal{C}$. Furthermore, the algebra $(B, \mu, \iota)$ in $\mathcal{C}$ is separable or commutative if and only if the algebra $(B, \mu, \iota)$ in $\mathcal{C}_A$ is such. Finally, the functor $\left(C^\text{loc}_A\right)_B \rightarrow C^\text{loc}_B$ given by

$$(M, m : B \otimes_A M \rightarrow M) \mapsto \left(M, \overline{m} : B \otimes M \rightarrow B \otimes_A M \xrightarrow{m} M\right)$$

(2.2)

is a braided monoidal equivalence.

2.2.2 Ribbon Structure on Group-theoretical Braided Fusion Categories

Lemma 2.2.2.1 ([12], Lemma 3.2). The category $\mathcal{Z}(\mathcal{G}, \alpha)$ is rigid, with dual objects

$$X^* = \bigoplus_{f \in G} (X^*)_f$$
given by

\[(X^*)_f = (X_{f^{-1}})^* = \text{Hom}_k(X_{f^{-1}}, k),\]

with the $\alpha$-projective action

\[(g.\ell)(x) = \frac{1}{\alpha(g|f, f^{-1})\alpha(g^{-1}, g|f^{-1})}\ell(g^{-1}.x), \quad \ell \in \text{Hom}_k(X_{f^{-1}}, k), x \in X_{gf^{-1}g^{-1}}.\]

**Proof.** Using the identities (1.9), it can be shown that the formula given above does indeed define an $\alpha$-projective action on the associated graded vector space. The evaluation morphism $\text{ev}_X : X^* \otimes X \to I$ is given by the canonical pairing

\[(X^*)_f \otimes X_{f^{-1}} \to k.\]

Similarly, the coevaluation morphism $\text{coev}_X : I \to X \otimes X^*$ is given by the canonical elements

\[k \to X_{f^{-1}} \otimes (X^*)_f.\]

The category $\mathcal{Z}(G, \alpha)$ is ribbon, with the ribbon twist

\[\theta_X(x) = \alpha(f, f^{-1}, f)f^{-1}.x, \quad x \in X_f.\]

The (unitary) trace of an endomorphism $a : X \to X$ can be written in terms of ordinary traces on vector spaces $X_g$:

\[\text{Tr}(a) = \sum_{g \in G} \text{Tr}_{X_g}(a_g),\]

and the (unitary) dimension of an object $X \in \mathcal{Z}(G, \alpha)$ is the dimension of its underlying (graded) vector space

\[\text{dim}(X) = \sum_{g \in G} \text{dim}(X_g).\]
2.2.3 The Monoidal Center of $\mathcal{V}(G, \alpha)$

Here, we describe the monoidal center $Z(\mathcal{V}(G, \alpha))$ of the category of graded vector spaces. We begin with a simple lemma.

**Lemma 2.2.3.1.** Let $\alpha \in Z^3(G, k^*)$ be a normalized 3-cocycle. Then we have the following identities:

$$\alpha(f, g, h)\gamma(g, h)\gamma(f, gh) = \alpha(h^{-1}, g^{-1}, f^{-1})\gamma(f, g)\gamma(fg, h), \quad (2.3)$$

$$\tau(fg)\tau(f)\gamma(g^{-1}) = \gamma(g^{-1}, f^{-1})\gamma(f, g^{-1}), \quad (2.4)$$

where $\gamma(f, g) = \alpha(f, g^{-1}, f^{-1})^{-1}\alpha(f, g, g^{-1})$ and $\tau(f) = \alpha(f^{-1}, f, f^{-1}) = \alpha(f, f^{-1}, f)^{-1}$.

**Proof.** Indeed, for a normalized 3-cocycle $\alpha$, the ratio of left- and right-hand sides of (2.3) is equal to

$$\frac{\partial \alpha(fgh, h^{-1}, g^{-1}, f^{-1})\partial \alpha(fgh, h, h^{-1}, g^{-1})}{\partial \alpha(fh, g, h, h^{-1}g^{-1})\partial \alpha(g, h, h^{-1}, g^{-1})},$$

while the ratio of left- and right-hand sides of (2.4) coincides with

$$\frac{\partial \alpha(f, g, f^{-1}g^{-1})\partial \alpha(f, g, (fg)^{-1}, fg)\partial \alpha(f^{-1}, f, g)}{\partial \alpha(g^{-1}, f^{-1}, f, g)\partial \alpha(g, g^{-1}, f^{-1}, fg)}.$$

$$\blacksquare$$

It is well-known\(^{37}\) that associativity constraints for the category of $G$-graded vector spaces with the ordinary tensor product (1.4) correspond to 3-cocycles

$$\phi_{V,U,W} : V \otimes (U \otimes W) \rightarrow (V \otimes U) \otimes W,$$

$$\phi_{V,U,W}(v \otimes (u \otimes w)) = \alpha(f, g, h)(v \otimes u) \otimes w, \quad \forall v \in V_f, u \in U_g, w \in W_h.$$  

Denote by $\mathcal{V}(G, \alpha)$ such a monoidal category. Clearly, $\mathcal{V}(G, \alpha)$ is a fusion category with the set of simple objects $\text{Irr}(\mathcal{V}(G, \alpha)) = G$.

It is quite straightforward to verify that the category of finite-dimensional vector spaces is spherical, with dual objects $V^* = \text{Hom}_k(V, k)$ given by spaces of linear maps.

---

\(^{37}\) See, e.g., [26].
(algebraic dual spaces), with the inverse monoidal structure \( \gamma_{V,U}^{-1} : U^* \otimes V^* \to (U \otimes V)^* \) for the functor \((\cdot)^*\) defined as

\[
\gamma_{V,U}^{-1}(m \otimes \ell) = \overline{(m \otimes \ell)}, \quad \ell \in V^*, \ m \in U^*,
\]

where \((m \otimes \ell)(x \otimes y) = \ell(x)m(y)\), and with the involution isomorphism \(\tau_V : V \to V^{**}\) given by

\[
\tau_V(v) = \overline{x}, \quad \overline{x}(\ell) = \ell(x), \quad \ell \in V^*.
\]

This structure passes without change to the category of graded vector spaces with trivial associativity. The introduction of nontrivial coefficients in associativity constraints perturbs this structure in the following way.

**Proposition 2.2.3.2.** The fusion category \(\mathcal{V}(G, \alpha)\) is spherical, with dual objects

\[
X^* = \bigoplus_{f \in G} (X^*)_f
\]

given by

\[
(X^*)_f = \left((X_f)^{-1}\right)^* = \text{Hom}_k(X_{f^{-1}}, k),
\]

with the inverse monoidal structure \(\gamma_{X,Y}^{-1} : Y^* \otimes Y^* \to (X \otimes Y)^*\) for the functor \((\cdot)^*\) defined as

\[
\gamma_{X,Y}^{-1}(m \otimes \ell) = \frac{1}{\gamma(f, g)} \cdot \overline{(m \otimes \ell)}, \quad \ell \in X^*_f, \ m \in Y^*_g,
\]

and with the involution isomorphism \(\tau_X : X \to X^{**}\), given by

\[
\tau_X(x) = \tau(f)\overline{x}, \quad x \in X_f.
\]

Here, \(\gamma(f, g)\) and \(\tau(f)\) are the same as in lemma 2.2.3.1.

**Proof.** Coherence for the monoidal structure \(\gamma\)

\[
\begin{align*}
(X \otimes (Y \otimes Z))^* &\xrightarrow{\gamma_{X,Y} \gamma_{Y,Z}} (Y \otimes Z)^* \otimes X^* \xrightarrow{\gamma_{X,Y} \text{id}_X} (Z^* \otimes Y^*) \otimes X^* \\
((X \otimes Y) \otimes Z)^* &\xrightarrow{\gamma_{X,Y} \gamma_{X,Z}} Z^* \otimes (X \otimes Y)^* \xrightarrow{\text{id}_Z \gamma_{X,Y}} Z^* \otimes (Y^* \otimes X^*)
\end{align*}
\]
is equivalent to the equation (2.3), while monoidality of the isomorphism $\tau$

\[
\begin{array}{ccc}
(X \otimes Y)^{**} & \xrightarrow{\gamma_{X,Y}} & (Y^* \otimes X^*)^* \\
\tau_{X,Y} & \downarrow & \gamma_{Y^*,X^*} \\
X \otimes Y & \xrightarrow{\tau_{X,Y}} & X^{**} \otimes Y^{**}
\end{array}
\]

is equivalent to the equation (2.4).

\[\blacksquare\]

**Proposition 2.2.3.3** ([12], Proposition 4.1). The monoidal center $\mathcal{Z}(\mathcal{V}(G, \alpha))$ is isomorphic, as braided monoidal category, to the category $\mathcal{Z}(G, \alpha)$. The canonical forgetful functor $\mathcal{Z}(\mathcal{V}(G, \alpha)) \to \mathcal{V}(G, \alpha)$ corresponds to the functor $\mathcal{Z}(G, \alpha) \to \mathcal{V}(G, \alpha)$ forgetting the $G$-action.

**Proof.** For an object $(X, x)$ of the monoidal center $\mathcal{Z}(\mathcal{V}(G, \alpha))$, the natural isomorphism

\[x_V : V \otimes X \to X \otimes V, \quad V \in \mathcal{V}(G, \alpha)\]

is defined by its evaluations on one-dimensional graded vector spaces. The isomorphism $x_{I(f)}$ can be seen as an automorphism $f : X \to X$. The fact that $x_{I(f)}$ preserves grading, amounts to the condition $f[X_g] = X_{f^g f^{-1}}$:

\[X_g = (I(f) \otimes X)_{f g} \xrightarrow{x_{I(f)}} (X \otimes I(f))_{f g} = X_{f^g f^{-1}}.\]

The coherence condition for $x$ is equivalent to the the equation (1.6), with the associativity constraints giving rise to $\alpha(h|f, g)$. The diagram, defining the second component $x|y$ of the tensor product $(X, x) \otimes (Y, y) = (X \otimes Y, x|y)$, is equivalent to the tensor product of projective actions (1.10), with the associativity constraints giving rise to $\alpha(g, h|f)$.

The description of the monoidal unit in a monoidal center corresponds to the answer for the monoidal unit in $\mathcal{Z}(G, \alpha)$. Clearly, the braiding $c_{(X,x), (Y,y)} = y_X$ of the monoidal center $\mathcal{Z}(\mathcal{V}(G, \alpha))$ corresponds to the braiding (1.11) of $\mathcal{Z}(G, \alpha)$.

\[\blacksquare\]
3 Lagrangian Algebras in Group-theoretical Braided Fusion Categories

In this chapter, we study Lagrangian algebras in the Drinfeld center \( \mathcal{Z}(G, \alpha) \) of a finite group \( G \). We show that such algebras in \( \mathcal{Z}(G, \alpha) \) are parametrized by pairs \( (H, \gamma) \), where \( H \) is a subgroup of \( G \) and \( \gamma \) is a 2-cochain of \( H \) with values in \( k^* \) that trivializes \( \alpha \) in the sense that the differential of \( \gamma \) is \( \alpha \) on \( H \).

3.1 Description of the Category \( \mathcal{Z}(G, \alpha) \)

3.1.1 Simple Objects of \( \mathcal{Z}(G, \alpha) \)

Lemma 3.1.1.1. The category \( \mathcal{Z}(G, \alpha) \) admits the following decomposition into a direct sum of \( k \)-linear subcategories:

\[
\mathcal{Z}(G, \alpha) = \bigoplus_{f \in \text{cl}(G)} \mathcal{Z}_f(G, \alpha),
\]

where the sum is taken over a set \( \text{cl}(G) \) of representatives of conjugacy classes of elements of \( G \), and for \( f \in G \), the subcategory \( \mathcal{Z}_f(G, \alpha) \) is given by

\[
\mathcal{Z}_f(G, \alpha) = \left\{ Z \in \mathcal{Z}(G, \alpha) \mid \text{supp}(Z) = \text{cl}(f) \right\}.
\]

Here, \( \text{cl}(f) = \{ gfg^{-1} \mid g \in G \} \) denotes the conjugacy class of \( f \) in \( G \).

Proof. Clearly, the support of an object of \( \mathcal{Z}(G, \alpha) \) is a union of conjugacy classes of \( G \). It is also straightforward that, for \( Z \in \mathcal{Z}(G, \alpha) \), we have

\[
Z = \bigoplus_{c \in \text{cl}(G)} Z_c,
\]

where

\[
Z_c = \bigoplus_{f \in c} Z_f,
\]

giving a decomposition of \( Z \) into a direct sum of objects of \( \mathcal{Z}(G, \alpha) \). The result follows.
Lemma 3.1.1.2. For $f \in G$, the category $\mathcal{Z}_f(G, \alpha)$ is equivalent, as a $k$-linear category, to the category $k[C_f(G), \alpha(\cdot, |f|^{-1})]\text{-Mod}$ of left modules over the twisted group algebra $k[C_f(G), \alpha(\cdot, |f|^{-1})]$ in the category $k\text{-Vect}$ of vector spaces.

Proof. Consider the assignment $F : \mathcal{Z}_f(G, \alpha) \to k[C_f(G), \alpha(\cdot, |f|^{-1})]\text{-Mod}$ given by $F(Z) = Z_f$. We make $F(Z)$ into a left $k[C_f(G), \alpha(\cdot, |f|^{-1})]\text{-module}$ by defining, for $h \in C_f(G)$ and $z \in Z_f$, $e_h \cdot z = h \cdot z$. It can be easily checked that this definition gives rise to a left $k[C_f(G), \alpha(\cdot, |f|^{-1})]\text{-module}$ structure on $F(Z)$.

Now consider the assignment $E : k[C_f(G), \alpha(\cdot, |f|^{-1})]\text{-Mod} \to \mathcal{Z}_f(G, \alpha)$ given by

$$E(M) = \left\{ a : G \to M \mid a(xy) = \alpha(y^{-1}, x^{-1} |xf x^{-1}|) y^{-1} . a(x) \ \forall x \in G, y \in C_f(G) \right\}.$$

Define $G$-grading on $E(M)$ as follows: a function $a \in E(M)$ is homogeneous if and only if $\text{supp}(a)$ is a single coset modulo $C_f(G)$, i.e.,

$$|a| = xf x^{-1} \iff \text{supp}(a) = xC_f(G).$$

In particular,

$$|a| = f \iff \text{supp}(a) = C_f(G). \quad (3.2)$$

Define $\alpha$-projective $G$-action on $E(M)$ by $(g.a)(x) = \alpha(x^{-1}, g |xf x^{-1}|) a(g^{-1}x)$. It can be checked that these definitions make $E(M)$ into an object of $\mathcal{Z}_f(G, \alpha)$.

Now we proceed to defining the adjunction isomorphisms. For $Z \in \mathcal{Z}_f(G, \alpha)$, given a homogeneous $z \in Z_{xf^{-1}}$ for some $x \in G$, define $\tilde{z} : G \to Z_f$ by

$$\tilde{z}(y) = \begin{cases} y^{-1} . z, & y \in xC_f(G) \\ 0, & \text{otherwise} \end{cases}.$$

Observe that, for $g \in C_f(G)$, we have

$$\tilde{z}(gx) = (g^{-1}x^{-1}) . z = \alpha(g^{-1}, x^{-1} |xf x^{-1}|) g^{-1} (x^{-1} . z) = \alpha(g^{-1}, x^{-1} |xf x^{-1}|) g^{-1} (\tilde{z}(x)).$$
so that $\tilde{z} \in E(F(Z))$. Now define $\phi_Z : Z \to E(F(Z))$ by $\phi_Z(z) = \tilde{z}$. We first check that $\phi_Z$ preserves the $\alpha$-projective action. We have

$$\phi_Z(g,z)(x) = g\tilde{z}(x) = x^{-1}.(g.z) = \frac{1}{\alpha(x^{-1}, g|xfx^{-1})}(x^{-1}g).z = \frac{1}{\alpha(x^{-1}, g|xfx^{-1})}(g^{-1}x) = (g, \tilde{z})(x) = (g, \phi_Z(z))(x),$$

and so $\phi_Z$ preserves the $\alpha$-projective action. Now we check that $\phi_Z$ preserves the grading.

Let $|z| = bfb^{-1}$ for some $b \in G$. Recall from equation (3.10) that

$$|z| = bfb^{-1} \iff y|\tilde{z}(y)|y^{-1} = bfb^{-1} \forall y \in G.$$ 

Since supp($\tilde{z}$) = $xC_G(f)$, it suffices to check that this holds when $y = xg$ for some $g \in C_G(f)$. We obtain

$$(yg)|\tilde{z}(yg)|(yg)^{-1} = (yg)(yg)^{-1}.z|(yg)^{-1} = (yg)(yg)^{-1}|z|(yg)(yg)^{-1} = e|z|e = |z| = bfb^{-1},$$

as claimed. Therefore $\phi_Z$ preserves the grading, whence $\phi_Z$ is a morphism in the category $\mathcal{Z}_f(G, \alpha)$. To see that $\phi_Z$ is an isomorphism, we exhibit an inverse. Define a map

$$\eta_Z : E(F(Z)) \to Z \text{ by } \eta_Z(z) = \frac{1}{|C_G(f)|} \sum_{g \in G} \alpha\left(g, g^{-1} | \tilde{z}\right) g. (a(g)). \quad (3.3)$$

We claim that $\eta_Z = \phi_Z^{-1}$. Let $z \in Z_{xf^{-1}}$ for some $x \in G$, so that $|z| = xfx^{-1}$ as well. We have

$$(\eta_Z \circ \phi_Z)(z) = \eta_Z(\tilde{z}) = \frac{1}{|C_G(f)|} \sum_{g \in G} \alpha\left(g, g^{-1} | \tilde{z}\right) g. (\tilde{z}(g)) = \frac{1}{|C_G(f)|} \sum_{g \in xC_G(f)} \alpha\left(g, g^{-1} | \tilde{z}\right) g. (\tilde{z}(g)) = \frac{1}{|C_G(f)|} \sum_{g \in xC_G(f)} \alpha\left(g, g^{-1} | \tilde{z}\right) z = \frac{1}{|C_G(f)|} \sum_{g \in xC_G(f)} z = \frac{1}{|C_G(f)|} (|xC_G(f)| \cdot z) = \frac{1}{|C_G(f)|} (|C_G(f)| \cdot z) = z,$$
so that $\eta_Z \circ \phi_Z = \text{Id}_Z$.

Now let $a \in E(F(Z))$ be homogeneous of degree $|a| = yf^{-1}$, so that $\text{supp}(a) = yC_G(f)$. For $x \in \text{supp}(a)$, we have

\[
((\phi_Z \circ \eta_Z)(a))(x) = \phi_Z \left( \frac{1}{|C_G(f)|} \sum_{g \in gC_G(f)} \alpha \left( g, g^{-1} |a| \right) g \cdot (a(g)) \right)(x) = \frac{1}{|C_G(f)|} \sum_{g \in gC_G(f)} \alpha \left( g, g^{-1} |a| \right) (g \cdot (a(g))(x)) = \frac{1}{|C_G(f)|} \sum_{g \in gC_G(f)} \alpha \left( g, g^{-1} |a| \right) (x^{-1} \cdot (g \cdot (a(g)))) = \frac{1}{|C_G(f)|} \sum_{g \in gC_G(f)} \alpha \left( g, g^{-1} |a| \right) (x^{-1} \cdot (a(g))) = \frac{1}{|C_G(f)|} \sum_{g \in gC_G(f)} \alpha \left( x^{-1} g, g^{-1} |a| \right) (g^{-1} x)^{-1} \cdot (a(g)). \tag{3.4}
\]

Set $h = g^{-1}x$, so that $x = gh$. Then we have that (3.4) coincides with

\[
= \frac{1}{|C_G(f)|} \sum_{g \in gC_G(f)} \alpha \left( h^{-1}, g^{-1} |a| \right) h^{-1} \cdot (a(g)). \tag{3.5}
\]

Since $x \in \text{supp}(a) = yC_G(f)$ and $g \in yC_G(f)$, it follows that $gC_G(f) = xC_G(f)$, and therefore $h = g^{-1}x \in C_G(f)$. We apply $C_G(f)$-equivariance of $a$ to rewrite (3.4) as

\[
\frac{1}{|C_G(f)|} \sum_{g \in gC_G(f)} a(x) = \frac{1}{|C_G(f)|} \left( |yC_G(f)| \right) \cdot a(x) = a(x), \tag{3.6}
\]

which shows that $\phi_Z \circ \eta_Z = \text{Id}_{E(F(Z))}$, whence $\eta_Z = \phi_Z^{-1}$.

For an object $V \in C((f), C_G(f), \alpha)$, we have $F(E(V)) = (E(V))_f$, so by equation (3.2), a function $a : G \to V$ belongs to $F(E(V))$ if and only if $\text{supp}(a) = C_G(f)$.

Moreover, the equivariance condition can be rewritten as follows: for $x \in C_G(f)$, we have

\[
a(x) = a(ex) = \alpha \left( e^{-1}, x^{-1} \big| f \right) x^{-1} \cdot (a(e)) = x^{-1} \cdot (a(e)), \tag{3.7}
\]
where the last equality follows from the fact that \( \alpha \) is normalized. Effectively, therefore, any such function is completely determined by its value at the identity element \( e \in G \). For an object \( V \in \mathcal{C}([f], C_G(f), \alpha) \), define \( \psi_V : F(E(V)) \to V \) by

\[
\psi_V(a) = a(e),
\]

and define \( \xi_V : V \to F(E(V)) \) by

\[
\xi_V(v) = \tilde{v},
\]

where \( \tilde{v} : G \to V \) is defined by

\[
\tilde{v}(x) = x^{-1}v.
\]

We first check that \( \psi_V \) is a morphism in \( \mathcal{C}([f], C_G(f), \alpha) \). Being an evaluation map, \( \psi_V \) is \( k \)-linear. For \( a \in F(E(V)) \), we have \( |a| = f \), i.e.,

\[
|a(x)| = x^{-1}fx \quad \forall x \in G.
\]

Therefore \( |\psi_V(a)| = |a(e)| = e^{-1}|a|e = |a| = f \), so that \( \psi_V \) preserves the \( \{f\}\)-grading. Next we check that \( \psi_V \) preserves the \( \alpha \)-projective \( C_G(f) \)-action. Let \( g \in C_G(f) \). Then

\[
\psi_V(g.a) = (g.a)(e) = a(g^{-1}) = g.(a(e)) \text{ by } C_G(f)\text{-equivariance, which coincides with } g.(\psi_V(a)).
\]

Therefore \( \psi_V \) preserves the \( \alpha \)-projective \( C_G(f) \)-action, whence \( \psi_V \) is indeed a morphism in \( \mathcal{C}([f], C_G(f), \alpha) \). Finally, we claim that \( \xi_V = \psi_V^{-1} \). We have

\[
(\psi_V \circ \xi_V)(v) = \psi_{F(E(V)}(\tilde{v}) = \tilde{v}(e) = e.v = v.
\]

On the other hand, for \( x \in C_G(f) \), we have

\[
((\xi_V \circ \psi_V)(a))(x) = (\xi_V(a(e)))(x) = \overline{a(e)}(x) = x^{-1}.(a(e)) = a(x)
\]

in view of (3.7), and our claim is proven.

\[\blacksquare\]

**Corollary 3.1.1.3.** Simple objects \( Z \in \mathcal{Z}(G, \alpha) \) are parametrized by conjugacy classes of pairs \((f, U)\), where \( f \in G \) and \( U \) is a simple module over the twisted group algebra \( k[C_G(f), \alpha(\cdot, |f|^1)] \).
Proof. Clearly, the support of a simple object $Z \in \mathcal{Z}(G, \alpha)$ must be an indecomposable $G$-subset in $G$ with respect to the adjoint action, i.e., a conjugacy class of $G$. Let $g \in \text{supp}(Z)$. The axioms of the $\alpha$-projective action imply that $Z$ is induced from the $k[C_G(g), \alpha(\cdot, \cdot|g)^{-1}]$-module $Z_g$. For $Z$ to be simple, the module $Z_g$ must be simple as well. Setting $f = g$ and $U = Z_g$ gives the result. 

**Corollary 3.1.1.4.** If $k$ is an algebraically closed field of characteristic zero, then the category $\mathcal{Z}(G, \alpha)$ is fusion.

Proof. Follows from corollary 3.1.1.3 and lemma 2.1.1.1.

### 3.2 Algebras in Group-theoretical Modular Categories

#### 3.2.1 Algebras in the Category $\mathcal{Z}(G, \alpha)$

We start with expanding the structure of an algebra in the category $\mathcal{Z}(G, \alpha)$ in plain algebraic terms.

**Proposition 3.2.1.1** ([12], Proposition 3.3). An algebra $A$ in the category $\mathcal{Z}(G, \alpha)$ is a $G$-graded $\alpha$-associative algebra together with an $\alpha$-projective $G$-action such that

$$f.(ab) = \alpha(f|g, h) (f.a) (f.b), \quad a \in A_g, b \in A_h. \quad (3.8)$$

An algebra $A$ in the category $\mathcal{Z}(G, \alpha)$ is commutative if and only if

$$ab = (f.b) a, \quad \forall a \in A_f, b \in A. \quad (3.9)$$

Proof. Being a morphism in the category $\mathcal{Z}(G, \alpha)$, the multiplication of an algebra in $\mathcal{Z}(G, \alpha)$ preserves grading and $\alpha$-projective $G$-action, whence the property (3.8) holds. Associativity of multiplication in $\mathcal{Z}(G, \alpha)$ is equivalent to $\alpha$-associativity. The formula (1.11) for the braiding in $\mathcal{Z}(G, \alpha)$ implies that commutativity for an algebra $A$ in the category $\mathcal{Z}(G, \alpha)$ is equivalent to the condition (3.9).
Proposition 3.2.1.2 ([12], Corollary 3.4). The degree $e$ component $A_e$ of an algebra $A$ in the category $\mathcal{Z}(G, \alpha)$ is an associative $G$-algebra and $A$ is an $(A_e, A_e)$-bimodule. The algebra $A_e$ is commutative if $A$ is a commutative algebra in the category $\mathcal{Z}(G, \alpha)$.

Proof. The normalization condition for the cocycle $\alpha$ together with $\alpha$-associativity of $A$ implies that $A_e$ is an associative algebra and $A$ is an $(A_e, A_e)$-bimodule. The same normalization condition together with $\alpha$-projectivity of the $G$-action and the property (3.8) implies that the action of $G$ on $A_e$ is genuine and $G$ acts on $A_e$ by algebra homomorphisms. Commutativity of $A_e$ for a commutative algebra $A \in \mathcal{Z}(G, \alpha)$ follows directly from the identity (3.9).

Proposition 3.2.1.3 ([12], Proposition 3.5). An algebra $A$ in the category $\mathcal{Z}(G, \alpha)$ is separable if and only if the composition

$$A_f \otimes A_{f^{-1}} \xrightarrow{\mu} A_e \xrightarrow{\text{Tr}} k$$

defines a nondegenerate bilinear pairing for any $f \in G$. In particular, the algebra $A_e$ is separable if $A$ is a separable algebra in the category $\mathcal{Z}(G, \alpha)$.

Proof. Being a homomorphism of graded vector spaces, the standard trace map $\text{Tr} : A \to I$ is zero on $A_f$ for $f \neq e$. Hence the standard bilinear form is zero on $A_f \otimes A_g$ unless $fg = e$. In particular, the restriction of $\text{Tr}$ to $A_e$ makes $A_e$ into a separable algebra in the category of vector spaces.

3.2.2 Commutative Separable Algebras in Trivial Degree and Their Modules

We start with a well-known\textsuperscript{38} description of etale $G$-algebras. We give a sketch of the proof for completeness.

\textsuperscript{38} See, e.g., [29].
Lemma 3.2.2.1 ([29], Theorems 2.1 and 2.2; [12], Lemma 3.6). Commutative separable $G$-algebras are function algebras on $G$-sets. Indecomposable $G$-algebras correspond to transitive $G$-sets.

Proof. An etale algebra over an algebraically closed field $k$ is a function algebra $k(X)$ on a finite set $X$, with elements of $X$ corresponding to minimal idempotents of the algebra. The $G$-action on the algebra amounts to a $G$-action on the set $X$. Clearly, the algebra of functions $k(X \sqcup Y)$ on the disjoint union of $G$-sets is the direct sum of $G$-algebras $k(X) \oplus k(Y)$, and any direct sum decomposition of $G$-algebras appears in that way.

Let $k(X)$ be an indecomposable $G$-algebra. By choosing a minimal idempotent $p \in X$, we can identify the $G$-set $X$ with the set $G/H$ of left cosets modulo the stabilizer subgroup $H = \mathrm{stab}_G(p)$.

This brings us to our first major result. A sketch of the proof of this result was given in [12]; we provide a complete proof here.

Theorem 3.2.2.2 ([12], Theorem 3.7). Let $G$ be a finite group, and let $H \leq G$ be a subgroup. Then the category $\mathcal{Z}(G, \alpha)_{k(G/H)}$ of right modules over the function algebra $k(G/H)$ in the Drinfeld center $\mathcal{Z}(G, \alpha)$ is equivalent, as a monoidal category, to the category $\mathcal{C}(G, H, \alpha)$. Moreover, the full subcategory $\mathcal{Z}(G, \alpha)_{k(G/H)}^{\text{loc}}$ of local modules is equivalent, as a braided monoidal category, to the Drinfeld center $\mathcal{Z}(H, \alpha|_H)$.

Proof. We shall exhibit the claimed equivalence of categories by constructing quasi-inverse functors $D$ and $E$ fitting in the following diagram:

$$
\begin{array}{ccc}
\mathcal{Z}(G, \alpha)_{k(G/H)} & \xrightarrow{D} & \mathcal{C}(G, H, \alpha) \\
\searrow & & \nearrow \\
\mathcal{Z}(H, \alpha|_H) & & \\
\end{array}
$$

Let $p : G/H \to k$ be defined by

$$
p(xH) = \begin{cases} 
1, & x \in H \\
0, & \text{otherwise}
\end{cases}
$$
so that \( p \) is, effectively, the indicator function on \( H \). Then \( p \) is a minimal idempotent in the function algebra \( k(G/H) \), and \( H = \text{stab}_G(p) \). Let \( M \) be a right \( k(G/H) \)-module, and consider the assignment \( D : \mathcal{Z}(G, \alpha)_{k(G/H)} \to C(G, H, \alpha) \) given by \( D(M) = Mp \). Since \( p \) is of trivial degree, \( Mp \) is a \( G \)-graded vector space in a natural way: \((Mp)_g = \left(M_s\right)_g \). The \( G \)-action on \( M \) reduces to an \( H \)-action on \( Mp \). This makes \( D(M) \) an object of \( C(G, H, \alpha) \).

Clearly, \( D \) is a functor. If \( M \) and \( N \) are objects of \( \mathcal{Z}(G, \alpha)_{k(G/H)} \), then

\[
D(M \otimes_{k(G/H)} N) = (M \otimes_{k(G/H)} N)p = (M \otimes_{k(G/H)} N)p^2 = (Mp) \otimes_{k(G/H)} (Np) = D(M) \otimes D(N),
\]

whence \( D \) is a tensor functor.

The functor \( E \) requires a lengthier setup. For \( V \in C(G, H, \alpha) \), let \( V^G \) be the vector space of set-theoretic functions \( a : G \to V \). Define \( G \)-grading on \( V^G \) by

\[
|a| = f \in G \iff |a(x)| = x^{-1}f x \quad \forall x \in G. \tag{3.10}
\]

Define an \( \alpha \)-projective \( G \)-action on \( V^G \) as follows. For a homogeneous \( a \in V^G \) of degree \( |a| = f \), define \( g.a : G \to V \) by

\[
(g.a)(x) = \alpha\left(x^{-1}, g \big| f\right)^{-1}a(\left.g^{-1}x\right).
\]

The action property follows from equation (1.9):

\[
\alpha(u, v|f)(u.(v.a))(x) = \frac{\alpha(u, v|f)}{\alpha(x^{-1}, u|vf^{-1})}((v.a)(u^{-1}x)) = \frac{\alpha(u, v|f)}{\alpha(x^{-1}, u|vf^{-1})}\alpha(x^{-1}u, v|f)\alpha^{-1}(v^{-1}u^{-1}x),
\]

which coincides with

\[
(\alpha uv).a(x) = \frac{1}{\alpha(x^{-1}, uv|f)}a((uv)^{-1}x).
\]

Observe that

\[
|g.a(x)| = \left|a(g^{-1}x)\right| = g|a(x)|g^{-1}
\]
So $g.a \in V^G_{gf^{-1}}$ for $a \in V^G_f$. All this makes $V^G$ an object of $Z(G, \alpha)$. Now consider a subspace of $V^G$ given by

$$E(V) = \left\{ a : G \to V \mid a(xh) = a(h^{-1}, x^{-1}|f)h^{-1}.a(x), \ h \in H, x \in G, |a| = f \right\} ,$$

for $V \in C(G, H, \alpha)$. Since the defining condition is homogeneous, $E(V)$ is a graded subspace of $V^G$. Moreover, $E(V)$ is invariant under the $\alpha$-projective $G$-action on $V^G$:

$$(g.a)(xh) = \frac{1}{\alpha(h^{-1}, x^{-1}, g|f)}a(g^{-1}xh) = \frac{\alpha(h^{-1}, x^{-1}g|f)}{\alpha(h^{-1}, x^{-1}, g|f)}h^{-1} \cdot \left( a(g^{-1}x) \right) =$$

$$= \frac{\alpha(h^{-1}, x^{-1}gfg^{-1})}{\alpha(x^{-1}, g|f)}h^{-1} \cdot \left( a(g^{-1}x) \right) = \alpha(h^{-1}, x^{-1}|gfg^{-1})h^{-1} \cdot \left( (g.a)(x) \right).$$

Thus $E(V)$ is also an object of $Z(G, \alpha)$. To make $E(V)$ into a right $k(G/H)$-module, we must make sense of a morphism $\nu : E(V) \otimes k(G/H) \to E(V)$ in $Z(G, \alpha)$ satisfying the usual module multiplication properties. To do this, we first identify $k(G/H)$ with $E(k)$. We can then define $\nu : E(V) \otimes E(k) \to E(V)$ by $\nu(a \otimes b) = ab$, where $(ab)(x) = a(x) \cdot b(x)$, rescaling the vector $a(x) \in V$ by the scalar $b(x) \in k$. Therefore $E$ is a functor $C(G, H, \alpha) \to Z(G, \alpha)_{k(G/H)}$. For $V, W \in C(G, H, \alpha)$, the universal property of the tensor product gives an isomorphism

$$E(V) \otimes_{k(G/H)} E(W) = E(V) \otimes_{E(k)} E(W) \simeq E(V \otimes W),$$

which shows that $E$ is a monoidal functor.

We now proceed to defining the adjunction isomorphisms. Given $V \in C(G, H, \alpha)$, define a map $\phi_V : E(V)p \to V$ by $\phi_V (ap) = a(e)$, where $e \in G$ is the identity element. Observe that, since $e \in H$, we have $p(e) = 1$, and so $a(e) = a(e)p(e)$. We must show that $\phi_V$ is a morphism in the category $C(G, H, \alpha)$, i.e., $\phi_V$ is a $k$-linear map that preserves the $G$-grading and $H$-action. Clearly, $\phi_V$ is $k$-linear. Suppose that $a \in E(V)$ satisfies $|a| = g$. Then $a(e) \in V_{e^1 ge} = V_g$, and hence $|a(e)| = g$. Since $a(e) = \phi_V (ap)$, it follows that $\phi_V$ preserves the $G$-grading. Let $h \in H$, and consider

$$\phi_V (h.(ap)) = (h.(ap))(e) = (ap)(h^{-1}e) = (ap)(h^{-1}) = a(h^{-1})p(h^{-1}) = a(h^{-1}) =$$
\[= h. (a(e)) = h. (\phi_V (ap)) \].

Therefore \(\phi_V\) preserves the \(H\)-action, and hence \(\phi_V\) is indeed a morphism in the category \(C(G, H, \alpha)\). It remains to show that \(\phi_V\) is bijective. We claim that the map \(\eta_V : V \to E(V)p\) defined by

\[\eta_V(v) = \tilde{v},\]

where

\[\tilde{v}(x) = \begin{cases} x^{-1}.v, & x \in H \\ 0, & \text{otherwise} \end{cases},\]

is the inverse of \(\phi_V\). Observe that \(\tilde{v} = \tilde{v}p\). To see that \(\eta_V\) is indeed the inverse of \(\phi_V\), let \(ap \in E(V)p\) for some \(a \in E(V)\). Then for \(x \in G\), we have

\[(ap)(x) = \begin{cases} a(x), & x \in H \\ 0, & \text{otherwise} \end{cases}.\]

Then

\[ap \mapsto a(e) \mapsto \overline{a(e)},\]

where

\[\overline{a(e)}(x) = \begin{cases} x^{-1}.(a(e)), & x \in H \\ 0, & \text{otherwise} \end{cases}.\]

By definition of the action, this is equal to

\[\begin{cases} a(x), & x \in H \\ 0, & \text{otherwise} \end{cases},\]

which is clearly equal to \((ap)(x)\). This shows that \(\eta_V \circ \phi_V = \text{Id}_{E(V)p}\). On the other hand, given \(v \in V\), we obtain

\[v \mapsto \tilde{v} \mapsto \tilde{v}(e).\]

By definition, \(\tilde{v}(e) = e^{-1}.v = e.v = v\), and so \(\phi_V \circ \eta_V = \text{Id}_V\). Therefore \(\phi_V\) is indeed invertible, and \(\eta_V = \phi^{-1}_V\), as claimed.
Now let $M \in \mathcal{Z}(G, \alpha)_{k(G/H)}$. Define a map $\psi_M : E(mp) \rightarrow M$ by

$$
\psi_M(a) = \frac{1}{|H|} \sum_{g \in G} \alpha(g, g^{-1}|a|g(a(g)).
$$

We show that $\psi_M$ is an isomorphism by exhibiting an inverse, and showing that that inverse is a morphism in $\mathcal{Z}(G, \alpha)_{k(G/H)}$. Define a map $\xi_M : M \rightarrow E(mp)$ by

$$
\xi_M(m)(x) = (x^{-1}.m)p, \ x \in G.
$$

We first claim that $\xi_M$ is a $G$-graded $k$-linear map that preserves the given $G$-action and is a homomorphism of right $k(G/H)$-modules. $k$-linearity of $\xi_M$ is clear. Let $m \in M$. For $x \in G$, we have

$$
|\xi_M(m)| = |(x^{-1}.m)p| = |x^{-1}.m| \cdot |p| = (x^{-1}|m| x) \cdot e = x^{-1}|m| x.
$$

According to the $G$-grading on $E(-)$, this means that $|\xi_M(m)| = |m|$, i.e., $\xi_M$ preserves the $G$-grading.

Next, we show that $\xi_M$ preserves the $G$-action. Suppose that $|m| = f$ for some $f \in G$. For $g, x \in G$ we have

$$
(\xi_M(g.m))(x) = (x^{-1}(g.m))p = \frac{1}{\alpha(x^{-1}, g|f)}((x^{-1}g).m)p =
$$

$$
= \frac{1}{\alpha(x^{-1}, g|f)}((g^{-1}x)^{-1}.m)p = \frac{1}{\alpha(x^{-1}, g|f)}(\xi_M(m)(g^{-1}x)) = (g.(\xi_M(m)))(x),
$$

i.e., $\xi_M$ preserves the $G$-action.

Next, we show that $\xi_M$ is a homomorphism of right $k(G/H)$-modules. Let $m$ be as above, and let $s \in k(G/H)$. Then for $x \in G$,

$$
\xi_M(m \cdot s)(x) = (x^{-1}.(m \cdot s))p = (x^{-1}.m)p(x^{-1}.s)(H) = (x^{-1}.m)p \cdot s(xH).
$$

On the other hand,

$$
(\xi_M(m) \cdot s)(x) = \xi_M(m)(x) \cdot s(xH) = (x^{-1}.m)p \cdot s(xH),
$$
so that \( \xi_M(m \cdot s) = \xi_M(m) \cdot s \). Thus \( \xi_M \) is a morphism in \( \mathcal{Z}(G, \alpha)_{k(G/H)} \).

Finally, we claim that \( \xi_M \) is the inverse of \( \psi_M \). For \( m \in M_f \), consider

\[
(\psi_M \circ \xi_M)(m) = \frac{1}{|H|} \sum_{g \in G} \alpha(g, g^{-1}f)g.(\xi_M(m)(g)) = \frac{1}{|H|} \sum_{g \in G} \alpha(g, g^{-1}f)g.(g^{-1}(m)p) =
\]

\[
= \frac{1}{|H|} \sum_{g \in G} \frac{\alpha(g, g^{-1}f)}{\alpha(g^{-1}f)}(g(g^{-1})m)g.p = \frac{1}{|H|} \sum_{g \in G} m \cdot p(g^{-1}H) = \frac{1}{|H|} \sum_{h \in H} m =
\]

\[
= \frac{1}{|H|}(|H| \cdot m) = m,
\]

and therefore \( \psi_M \circ \xi_M = \text{Id}_M \).

Now let \( a \in E(Mp)_f \) and \( x \in G \), and consider

\[
(\xi_M \circ \psi_M)(a)(x) = \xi_M \left( \frac{1}{|H|} \sum_{g \in G} \alpha(g, g^{-1}f)g.(a(x)) \right) =
\]

\[
= \frac{1}{|H|} \sum_{g \in G} \alpha(g, g^{-1}f)(g.(\xi_M(a(g))))(x) =
\]

\[
= \frac{1}{|H|} \sum_{g \in G} \frac{\alpha(g, g^{-1}f)}{\alpha(x^{-1}, g^{-1}f)} \xi_M(a(g))(g^{-1}x) =
\]

\[
= \frac{1}{|H|} \sum_{g \in G} \frac{\alpha(g, g^{-1}f)}{\alpha(x^{-1}, g^{-1}f)}((g^{-1}x)^{-1}.(a(g))p). \tag{3.12}
\]

Now, by cocycle identities for \( \alpha \), we have

\[
\frac{\alpha(g, g^{-1}f)}{\alpha(x^{-1}, g^{-1}f)} = \alpha(x^{-1}g, g^{-1}f).
\]

Also, for any \( g \in G \), we have \( a(g) = a(g) \cdot p \), so that

\[
\left((g^{-1}x)^{-1}.(a(g))p \right) = \left((g^{-1}x)^{-1}.(a(g) \cdot p) \right) =
\]

\[
= \left((g^{-1}x)^{-1}.(a(g))\cdot p \right) = \delta_{g^{-1}, xH}(g^{-1}x)^{-1}.(a(g))p.
\]

We can therefore rewrite (3.12) as

\[
\frac{1}{|H|} \sum_{g \in H} \alpha(x^{-1}g, g^{-1}f)((g^{-1}x)^{-1}.(a(g))p). \tag{3.13}
\]
Now set \( h = g^{-1}x \in H \), and note that

\[
a\left(x^{-1}g, g^{-1}f \right) \left((g^{-1}x)^{-1}.(a(g)) \right) = a\left(h^{-1}, g^{-1}f \right) h^{-1}.(a(g)).
\]

Observe that we have

\[
a(x) = a\left(gg^{-1}x \right) = a(gh),
\]

so by \( H \)-equivariance of \( a \),

\[
a(x) = a(gh) = a\left(h^{-1}, g^{-1}f \right) h^{-1}.(a(g)).
\]

Using this, we may rewrite (3.13) as

\[
\frac{1}{|H|} \sum_{g \in xH} a(x) \cdot p. \tag{3.14}
\]

Finally, (3.14) coincides with

\[
\frac{1}{|H|} \sum_{g \in xH} a(x) = \frac{1}{|H|} (|xH| \cdot a(x)) = \frac{1}{|H|} (|H| \cdot a(x)) = a(x),
\]

as desired. This shows that \( \xi_M \circ \psi_M = \text{Id}_{E(Mp)} \). Therefore \( \psi_M \) is indeed an isomorphism with inverse \( \xi_M = \psi_M^{-1} \), as claimed. Therefore the categories \( Z(G, \alpha)_{k(G/H)} \) and \( C(G, H, \alpha) \) are tensor equivalent.

At the very end of the proof, we address the locality statement. For an object \( V \in Z(G, \alpha) \), we denote by \( \text{supp}(V) \) the \textit{support} of \( V \): the set

\[
\text{supp}(V) = \left\{ g \in G \mid V_g \neq \{0\} \right\}.
\]

Let \( M \) be a right \( k(G/H) \)-module, and \( p \) be as above. If \( m \in M_g \) for some \( g \in G \), then by (1.1) and (1.11), \( M \) is local if and only if we have

\[
mp = m(g.p) \quad \text{and} \quad mp = mp^2 = m(p \cdot (g.p)).
\]

By definition of the action, \( g.p = p(g^{-1}H) \), and hence \( g.p \neq 0 \) if and only if \( g \in H \). It follows that \( \text{supp}(Mp) \subseteq H \). Moreover, if \( mp \neq 0 \), then necessarily \( p \cdot (g.p) \neq 0 \), which
can happen if and only if \( g.p = p \), i.e., \( g \in \text{stab}_G(p) = H \). As we have already shown, \( D(M) = M.p \) is a \( G \)-graded vector space with \( H \)-action. Combining these two facts, we have that \( M \) is local if and only if \( D(M) = M.p \) is, in fact, \( H \)-graded with \( H \)-action; i.e., \( M \) is local if and only if \( D(M) \) is an object of the category \( \mathcal{Z}(H, \alpha|_H) \) of \( H \)-graded \( k \)-vector spaces with \( H \)-action. This shows that \( \mathcal{Z}(G, \alpha)_{k(G/H)}^{\text{loc}} \) is tensor equivalent to \( \mathcal{Z}(H, \alpha|_H) \). It is straightforward to see that this tensor equivalence is braided, as claimed, which completes the proof.

We mention an immediate consequence of theorem 3.2.2.2.

**Corollary 3.2.2.3.** For a finite group \( G \) and a subgroup \( H \leq G \), the function algebra \( A = k(G/H) \) in the category \( \mathcal{Z}(G, \alpha) \) is Lagrangian if and only if \( H = \{e\} \).

**Proof.** An algebra \( A \) in \( \mathcal{Z}(G, \alpha) \) is Lagrangian if and only if \( \mathcal{Z}(G, \alpha)_{A}^{\text{loc}} \cong k\text{-Vect} \). By theorem 3.2.2.2, \( \mathcal{Z}(G, \alpha)_{k(G/H)}^{\text{loc}} \cong \mathcal{Z}(H, \alpha|_H) \), and clearly \( \mathcal{Z}(H, \alpha|_H) \cong k\text{-Vect} \) if and only if \( H = \{e\} \).

**Remark 3.2.2.4.** Theorem 3.2.2.2 in combination with lemma 2.2.1.1 gives an interpretation of the transfer, defined in [43]. The transfer turns an algebra from \( \mathcal{Z}(H, \alpha|_H) \) into an algebra from \( \mathcal{Z}(G, \alpha) \). Indeed, by theorem 3.2.2.2, an algebra from \( \mathcal{Z}(H, \alpha|_H) \) is an algebra in \( \mathcal{Z}(G, \alpha)_{k(G/H)}^{\text{loc}} \), which by lemma 2.2.1.1 gives an algebra in \( \mathcal{Z}(G, \alpha) \).

**Corollary 3.2.2.5 ([12], Corollary 3.9).** For a simple separable algebra \( A \) in \( \mathcal{Z}(G, \alpha) \) there is a subgroup \( H \leq G \) such that \( A \) is the transfer of a simple separable algebra \( B \) in \( \mathcal{Z}(H, \alpha|_H) \) with \( B_e = k \).

**Proof.** The subalgebra \( A_e \) is an indecomposable commutative \( G \)-algebra. By lemma 3.2.2.1, it is isomorphic to \( k(X) \) for some transitive \( G \)-set \( X \). By lemma 2.2.1.1, \( A \) is a commutative algebra in \( \mathcal{Z}(G, \alpha)_{A_e}^{\text{loc}} \). Thus, by theorem 3.2.2.2, \( A \) is the transfer of the etale algebra \( B = Ap \) from \( \mathcal{Z}(H, \alpha|_H) \). Here, \( p \) is the minimal idempotent of \( A_e \), corresponding.
to an element of $X$, with the stabilizer $H = \text{stab}_G(p)$. Finally, $B_e = A_e p = k$ by minimality of $p$.

### 3.2.3 Commutative Separable Algebras Trivial in Trivial Degree and Their Local Modules

Here we describe simple commutative separable algebras $B$ in $\mathbb{Z}(H, \beta)$ with $B_e = k$.

**Proposition 3.2.3.1** ([12], Lemma 3.10 and Proposition 3.11). Let $B$ be a separable algebra in $\mathbb{Z}(H, \alpha|_H)$ such that $B_e = k$. Then $B$ is a twisted group algebra of the form $k[F, \gamma]$, where $F = \text{supp}(B) \trianglelefteq H$ and $\gamma \in C^2(F, k^*)$ is a 2-cochain with $\partial(\gamma) = \alpha|_F$.

$H$-action on $k[F, \gamma]$ is given by

$$h. e_f = \varepsilon_h(f) e_{hf^{-1}},$$

where $\varepsilon : H \times F \rightarrow k^*$ satisfies each of the following identities:

$$\varepsilon_{gh}(f) = \varepsilon_g(hf^{-1})\varepsilon_h(f)\alpha(g, h|f), \quad g, h \in H, f \in F$$ (3.15)

$$\gamma(f, g)\varepsilon_h(fg) = \alpha(f, gh)\varepsilon_h(f)\varepsilon_h(g)\gamma(hfh^{-1}, hgh^{-1}), \quad h \in H, f, g \in F$$ (3.16)

If, in addition, $B$ is commutative, then $\varepsilon$ must also satisfy

$$\gamma(f, g) = \varepsilon_f(g)\gamma(fgf^{-1}, f), \quad f, g \in F.$$ (3.17)

Moreover,

$$\dim_k(B_h) \leq 1 \quad \forall h \in H.$$

**Proof.** An $H$-graded algebra $B$ such that $B_e = k$ is separable if and only if the multiplication map $\mu$ defines a nondegenerate pairing $m : B_g \otimes B_{g^{-1}} \rightarrow B_e = k$. Thus, $\alpha$-associativity of multiplication implies that, for any $a, c \in B_g$ and $b \in B_{g^{-1}}$, we have

$$\alpha(g, g^{-1}, g)m(a, b)c = am(b, c).$$
Given nonzero $a$ and $c$, by choosing $b$ such that $m(a, b), m(b, c) \neq 0$, we obtain that $a$ and $c$ are proportional. It follows from the nondegeneracy of $m : B_g \otimes B_{g^{-1}} \to B_c = k$ that a generator of a nonzero $B_f$ is invertible, and hence $F$ is closed under inversion. Thus, for nonzero components $B_f$ and $B_g$, the product $B_f B_g \subseteq B_{fg}$ is also nonzero, which shows that $F$ is closed under multiplication. Therefore $F \leq H$. Suppose that $f \in F$ and $h \in H$.

Consider $B_{hfh^{-1}}$. Since $B$ is $H$-graded, $H$ acts on the set of graded components of $B$ via

$$h.(B_{h'}) = B_{hh'h^{-1}}, \ h, h' \in H.$$ 

Hence $B_{hfh^{-1}} = h.(B_f)$. Since $f \in F = \text{supp}(B)$, we must have $B_f \neq \{0\}$. Since $h$ is invertible, we must have that $h.(B_f) = B_{hfh^{-1}} \neq \{0\}$ as well. Therefore $F$ is closed under conjugation, so that $F \leq H$, as claimed. Since the action of $H$ is by assumption $\alpha$-projective, we must have that $(gh).e_f = \alpha(g, h|f)g.(h.e_f)$. By definition of the action, this is equal to $\varepsilon_{gh}(f)e_{ghfh^{-1}g^{-1}}$. We therefore obtain

$$\varepsilon_{gh}(f)e_{ghfh^{-1}g^{-1}} = \alpha(g, h|f)g.(h.e_f) = \alpha(g, h|f)g.(\varepsilon_h(f)e_{hfh^{-1}}) = \alpha(g, h|f)\varepsilon_h(f)g.e_{hfh^{-1}} = \alpha(g, h|f)\varepsilon_h(f)e_{ghfh^{-1}g^{-1}},$$

which proves the first identity. Multiplicativity of the action amounts to the equality

$$h(e_f e_g) = \gamma(f, g)e_{hfgf^{-1}}$$

and

$$\alpha(f, gh)h(e_f)h(e_g) = \alpha(f, gh)e_h(f)e_h(g)\gamma(hfh^{-1}, gh^{-1})e_{hfgf^{-1}},$$

which gives the second identity. Finally, commutativity implies that $e_f e_g = \gamma(f, g)e_{fg}$ is equal to

$$f(e_g)e_f = e_f(g)e_{fgf^{-1}}e_f = e_f(g)\gamma(fgf^{-1}, f)e_{fg},$$ \hspace{1cm} (3.18)
Denote by \( k[F, \gamma, \varepsilon] \) an etale algebra in \( \mathcal{Z}(H, \alpha|_H) \), as defined in proposition 3.2.3.1.

**Lemma 3.2.3.2** ([12], Lemma 3.12). Two algebras \( k[F, \gamma, \varepsilon] \) and \( k[F', \gamma', \varepsilon'] \) in the category \( \mathcal{Z}(H, \alpha|_H) \) are isomorphic if and only if there is a cochain \( c : F \to k^* \) such that

\[
c(fg)\gamma(f, g) = \gamma'(f, g)c(f)c(g), \quad \varepsilon(h)f(hf^{-1}) = c(f)e'_h(g).
\]

**Proof.** Indeed, isomorphic algebras in \( \mathcal{Z}(H, \alpha|_H) \) have to have the same supports. Thus \( F = F' \). Since the components of both \( k[F, \gamma, \varepsilon] \) and \( k[F', \gamma', \varepsilon'] \) are all one-dimensional, an isomorphism \( k[F, \gamma, \varepsilon] \to k[F', \gamma', \varepsilon'] \) has the form \( e_f \mapsto c(f)e_f \) for some \( c(f) \in k^* \).

Finally, multiplicativity of this mapping is equivalent to the first condition, while \( H \)-equivariance is equivalent to the second. \( \square \)

Note that \( \varepsilon \) can be considered as an element of \( C^1(H, C^1(F, k^*)) \), while \( \gamma \) lies naturally in \( C^2(F, k^*) = C^0(H, C^2(F, k^*)) \). Thus, in the terminology of the Appendix of [8], \( (\varepsilon, \gamma) \) is a 2-cochain of \( \tilde{C}^\ast(H, F, k^*) \). The equations (3.15),(3.16), together with the condition \( \partial\gamma = \alpha|_F \), are equivalent to the coboundary condition \( \partial(\varepsilon, \gamma) = (\alpha_2, \alpha_1, \alpha_0) = \tau(\alpha) \) in \( \tilde{C}^\ast(H, F, k^*) \). The equations (3.18) say that \( (\varepsilon, \gamma) = \partial(c)(\varepsilon', \gamma') \) for \( c \in C^1(F, k^*) = \tilde{C}^1(H, F, k^*) \).

Before we describe local modules over the algebras, we need to explain how the cochains \( \gamma \) and \( \varepsilon \) associated to them allow us to reduce the cocycle \( \alpha \in Z^3(H, k^*) \) to \( \tilde{\alpha} \in Z^3(H/F, k^*) \). It will be shown, during the proof of theorem 3.2.3.3, that \( \tilde{\alpha}(x, y, z) \) defined by

\[
\alpha(s(x), s(y), s(y)^{-1}s(x)^{-1}s(zy))\gamma(\tau(y, z), \tau(x, yz))\gamma(\tau(y, z)\tau(x, yz), \tau'(x, y, z))\times
\]

\[
\times \varepsilon_{s(zy)^{-1}s(x)}(\tau(x, y))\gamma(\tau'(x, y, z)^{-1}, \tau'(x, y, z))
\]

(3.19)
is a 3-cocycle of \( H/F \). Here \( s : H/F \to H \) is a section of the canonical projection \( H \to H/F \), \( \tau(y, z) = s(z)^{-1}s(y)^{-1}s(yz) \in F \) and

\[
\tau'(x, y, z) = s(zy)^{-1}s(x)s(y)\tau(x, y)^{-1}s(y)^{-1}s(x)^{-1}s(zy).
\]
Our strategy in the following proof will differ from that which we used in the proof of theorem 3.2.2.2 above (q.v.). In what follows, we will compute local modules over etale algebras first, and then use the result to single out those etale algebras which are Lagrangian. We then compute all modules for these Lagrangian algebras only.

**Theorem 3.2.3.3** ([8], Theorem 3.5.4; [12], Theorem 3.13). Let $k[F, \gamma, \varepsilon]$ be the algebra in $\mathcal{Z}(H, \alpha|_H)$, as defined in proposition 3.2.3.1. Then the category $\mathcal{Z}(H, \alpha|_H)^{\text{loc}}_{[F, \gamma, \varepsilon]}$ of local right $k[F, \gamma, \varepsilon]$-modules in $\mathcal{Z}(H, \alpha|_H)$, is equivalent, as a ribbon category, to $\mathcal{Z}(H/F, \overline{\alpha})$, where $\overline{\alpha}$ is as defined above.

**Proof.** The structure of a right $k[F, \gamma, \varepsilon]$-module on an object $M = \bigoplus_{h \in H} M_h$ of $\mathcal{Z}(H, \alpha|_H)$ amounts to a collection of isomorphisms $e_f : M_h \to M_{hf}$ (right multiplication by $e_f \in k[F, \gamma, \varepsilon]$) such that

$$e_e = I, \quad e_f e_{f'} = \gamma(f, f') e_{f'f}, \quad he_fh^{-1} = \varepsilon_h(f) e_{hfh^{-1}}, \quad f, f' \in F, h \in H.$$ 

Here $h : M_{hf} \to M_{hfh^{-1}}$ is the $\alpha$-projective $H$-action on $M$. The $k[F, \gamma, \varepsilon]$-module $M$ is local iff $e_f = \varepsilon_h(f) hfh^{-1} e_{hfh^{-1}}$ on $M_h$.

Indeed, the double braiding in $\mathcal{Z}(H, \alpha|_H)$ transforms an element $m \otimes e_f \in M \otimes A$ (with $m \in M_h$) as follows:

$$m \otimes e_f \mapsto h(e_f) \otimes m = \varepsilon_h(f) e_{hfh^{-1}} \otimes m \mapsto \varepsilon_h(f) hfh^{-1}(m) \otimes e_{hfh^{-1}}.$$ 

An equivalent way of expressing the locality condition is the following:

$$f = \varepsilon_h(h^{-1}fh^{-1}) \gamma(h^{-1}fh, f^{-1}) \gamma(f, f^{-1})^{-1} e_{h^{-1}fh^{-1}} = \varepsilon_h(f)e_{[h^{-1}, f]}.$$ 

Now let $s : H/F \to H$ be a section of the quotient map $H \to H/F$. For a $k[F, \gamma, \varepsilon]$-module $M$ define an $H/F$-graded vector space $V$ by $V_x = M_{s(x)}$, where $x \in H/F$.

For local $M$ the vector space $V$ can be equipped with a projective $H/F$-action: for $y \in H/F$ define $y : V_x \to V_{yxy^{-1}}$ as the composition

$$V_x = M_{s(x)} \xrightarrow{\delta(y)} M_{s(y)s(x)y^{-1}} \xrightarrow{e_{f(xy)}} M_{s(yxy^{-1})} = V_{yxy^{-1}},$$
where
\[ f(x, y) = s(y)s(x)^{-1}s(y)^{-1}s(yxy^{-1}) = s(y)s(x)^{-1}s(yx) \in F. \]

To compute the projective multiplier one would need to compare \( zy \) on \( V_x \) with the composition of \( y \) and \( z \). This can be done with the help of the following diagram:

Here, \( \sigma(z,y) = s(z)s(y)s(zy)^{-1} \in F \) and
\[ g(x, y, z) = s(z)f(x, y)f(y, z) = s(z)s(y)s(x)^{-1}s(zy) \]
so that
\[ [s(z)s(x)^{-1}, \sigma(z,y)]s(y)s(x)^{-1}s(zy) \]
coincides with
\[ s(zy)s(x)^{-1}s(zy) = f(x, zy). \]

The cells of the diagram commute up to multiplication by a scalar (except two top and one bottom cells, which commute on the nose). Carefully gathering the scalars one can write down the multiplier for the projective \( H/F \)-action on \( V \). Fortunately, we do not have to do it. In view of the proposition proved in the appendix of [8], it is enough to know that such a multiplier exists.
The $H/F$-graded vector space $V \otimes U$, corresponding to the tensor product $M \otimes_{k[F, \gamma, \varepsilon]} N$ of local $B = k[F, \gamma, \varepsilon]$-modules, can be identified with the graded tensor product of $V$ and $U$. Indeed, the composition

$$\bigoplus_{y=x} M_{y(x)} \otimes N_{s(x)} \rightarrow (M \otimes_B N)_{s(x)}$$

defines an isomorphism $\bigoplus_{y=x} V_y \otimes U_z \rightarrow (V \otimes U)_x$. Here, as before, $\tau(y, z) = s(z)^{-1}s(y)^{-1}s(yz) \in F$. Again, we will use a diagrammatic language to prove the compatibility (up to a scalar) of the $H/F$-action on $V \otimes U$ with the $H/F$-actions on $V$ and $U$:

\[
\begin{array}{cccccccc}
(V \otimes U)_x & \xrightarrow{y} & (V \otimes U)_{xy^{-1}} \\
(M \otimes_B N)_{s(x)} & \xrightarrow{pr} & (M \otimes_B N)_{s(y)s(x)s(y^{-1})} & \xrightarrow{pr} & (M \otimes_B N)_{s(y)xy^{-1}} \\
(M \otimes N)_{s(x)} & \xrightarrow{pr} & (M \otimes N)_{s(y)s(x)s(y)} & \xrightarrow{pr} & (M \otimes N)_{s(y)xy^{-1}} \\
\bigoplus_{f_g = s(x)} M_f \otimes N_g & \xrightarrow{\tau(f, g)} & \bigoplus_{f_g = s(x)} M_{s(y)f_s(x)^{-1}} \otimes N_{s(y)g_s(y^{-1})} & \xrightarrow{pr} & M_{f'} \otimes N_{g'} \\
\bigoplus_{y = x} M_{s(y)} \otimes N_{s(y)} & \xrightarrow{\tau(y)} & \bigoplus_{y = x} M_{s(y)s(y^{-1})} \otimes N_{s(y)s(y^{-1})} & \xrightarrow{pr} & \bigoplus_{y = x} M_{s(y)y^{-1}} \otimes N_{s(y)y^{-1}} \\
\bigoplus_{y = x} V_y \otimes U_u & \xrightarrow{y \otimes} & \bigoplus_{y = x} M_{s(y)y^{-1}} \otimes N_{s(y)y^{-1}} & \xrightarrow{pr} & \bigoplus_{y = x} V_{yy^{-1}} \otimes U_{yy^{-1}} \\
\end{array}
\]
Here \( h(v, y, g) = s(y)g s(y)^{-1} f(v, y)^{-1} s(y)g^{-1} s(y)^{-1} \). Again the cells of the diagram commute up to scalars (one for each \( v \) and \( u \)), resulting in an overall factor, rescaling \( y \otimes y \) into \( y \) on \( V \otimes U \). Note that the six vertex cell in the right half of the diagram commutes by the following property of the projection map: for any \( u \in F \) the diagram

\[
\begin{array}{ccc}
(M \otimes_B N) & \xrightarrow{pr} & (M \otimes N) \\
\downarrow \cong & & \downarrow \cong \\
\bigoplus_{f, g} M_f \otimes N_g & \xrightarrow{\oplus \phi \otimes e -1_u -1_h} & \bigoplus_{f', g'} M_{f'} \otimes N_{g'}
\end{array}
\]

commutes up to multiplication by scalars (one for each \( f \) and \( g \)). So far their actual form was not important, but it will become crucial in what we are going to do later. To calculate this factor we start with the identity

\[
me_u \otimes n = \varepsilon_{h^{-1}}(u)(m \otimes ne^{h^{-1}uh}),
\]

which follows from the definition of the tensor product over \( B \). Multiplying this identity with \( e^{h^{-1}u^{-1}h} \) (from the right) we get

\[
me_u \otimes ne^{h^{-1}u^{-1}h} = \varepsilon_{h^{-1}}(u)\gamma(h^{-1}uh, h^{-1}u^{-1}h)(m \otimes n).
\]
The last step we need to make is to calculate the associator on $V \otimes (U \otimes W)$ and to prove that it is scalar on $V_x \otimes (U_y \otimes W_z)$. Once again we apply diagrammatic technique:

\[
\begin{align*}
&V_x \otimes (U_y \otimes W_z) \xrightarrow{\overline{\alpha}_{(x,y,z)}} (V_x \otimes U_y) \otimes W_z \\
&\downarrow \\
&M_{s(x)} \otimes (N_{s(y)} \otimes L_{s(z)}) \xrightarrow{\text{Id}\otimes \text{Id} \otimes \tau_{(y,z)}} (M_{s(x)} \otimes N_{s(y)}) \otimes L_{s(z)} \\
&\downarrow \\
&M_{s(x)} \otimes (N_{s(y)} \otimes L_{s(y)^{-1}s(z)}) \xrightarrow{\text{Id}\otimes \text{Id} \otimes \text{Id}} (M_{s(x)} \otimes N_{s(y)}^{-1}) \otimes L_{s(z)} \\
&\downarrow \\
&M_{s(x)} \otimes (N_{s(y)} \otimes L_{s(y)^{-1}s(x)y}) \xrightarrow{\text{Id} \otimes \text{Id} \otimes \tau_{(y,z)}} (M_{s(x)} \otimes N_{s(y)}^{-1}^{-1}) \otimes L_{s(y)^{-1}s(x)y} \\
&\downarrow \\
&M \otimes (N \otimes L) \xrightarrow{\alpha_{M,N,L}} (M \otimes N) \otimes L \\
&\downarrow \\
&M \otimes_B (N \otimes_B L) \xrightarrow{\alpha_{M,N,L}} (M \otimes_B N) \otimes_B L \\
\end{align*}
\]

Here $\alpha$ stands for the multiplication by $\alpha(s(x), s(y), s(y)^{-1}s(x)^{-1}s(xyz))$,

\[
* = e_{s(xyz)^{-1}^{s(x)}s(x)}(\tau(x,y))\gamma(\tau'(x, y, z)^{-1}, \tau'(x, y, z))(\text{Id} \otimes e_{\tau(x,y)} \otimes e_{\tau'(x,y,z)}),
\]

and again

\[
\tau'(x, y, z) = s(xyz)^{-1}s(x)s(y)\tau(x, y)^{-1}s(y)^{-1}s(x)^{-1}s(xyz).
\]

Now composing the arrows of the top cell and comparing the coefficients gives the formula (3.19). Finally, by the proposition from the appendix of [8], the category $\mathcal{Z}(H, \alpha|_H)^{\text{loc}}_{k[F,\gamma,e]}$ is ribbon equivalent to $\mathcal{Z}(H/F, \overline{\alpha})$.

We close this section by mentioning an immediate consequence of theorem 3.2.3.3.

**Corollary 3.2.3.4** ([12], Corollary 3.14). An etale algebra $B = k[F, \gamma, e]$ in the category $\mathcal{Z}(H, \alpha|_H)$ is Lagrangian if and only if $F = H$. 
Proof. By theorem 3.2.3.3, an etale algebra $B = k[F, \gamma, \varepsilon]$ in $\mathcal{Z}(H, \alpha|_H)$ is Lagrangian if and only if $\mathcal{Z}(H/F, \alpha) \simeq k^\gamma$-Vect, i.e., if and only if the quotient group $H/F$ is trivial, which happens if and only if $F = H$.  

3.2.4 Etale Algebras and Their Local Modules

In this section, we combine the previous results on commutative separable algebras in group-theoretical modular categories and on their local modules. Define

$$A(H, F, \gamma, \varepsilon) = E(k[F, \gamma, \varepsilon]),$$

where $E$ is the functor from the proof of theorem 3.2.2.2 above.

**Theorem 3.2.4.1** ([12], Theorem 3.15). *Etale algebras in $\mathcal{Z}(G, \alpha)$ correspond to quadruples $(H, F, \gamma, \varepsilon)$, where $H \leq G$ is a subgroup, $F \trianglelefteq H$ is a normal subgroup, $\gamma \in C^2(F, k^\ast)$ is a coboundary $\partial \gamma = \alpha|_F$ and $\varepsilon : H \times F \to k^\ast$ satisfies the conditions (3.15), (3.16), and (3.17).*

**Proof.** Follows from corollary 3.2.2.5 and lemma 2.2.1.1. 

**Theorem 3.2.4.2** ([8], Theorem 3.6.3; [12], Theorem 3.16). *The category $\mathcal{Z}(G, \alpha)^{\text{loc}}_{A(H, F, \gamma, \varepsilon)}$ of local right $A(H, F, \gamma, \varepsilon)$-modules in $\mathcal{Z}(G, \alpha)$, is equivalent, as a ribbon category, to $\mathcal{Z}(H/F, \alpha)$.***

**Proof.** Follows from lemma 2.2.1.1 and theorems 3.2.2.2 and 3.2.3.3. 

**Remark 3.2.4.3.** When $F = H$, the function $\varepsilon$ is completely determined by $\gamma$. Indeed, by the commutativity condition (3.17), we have

$$\varepsilon_f(g) = \frac{\gamma(f, g)}{\gamma(fgf^{-1}, f)}, \quad f, g \in H.$$

Assume now that $F = H$. Write $L(H, \gamma)$ in place of $A(H, H, \gamma, \varepsilon)$. Theorems 3.2.4.1 and 3.2.4.2 allow us to describe Lagrangian algebras in group-theoretical modular categories in purely group-theoretical terms. This is the next major result:
Corollary 3.2.4.4 ([12], Corollary 3.17). Lagrangian algebras $L \in \mathcal{Z}(G, \alpha)$ correspond to pairs $(H, \gamma)$, where $H \leq G$ is a subgroup and $\gamma \in C^2(H, k^*)$ is a coboundary $\partial \gamma = \alpha|_H$.

Proof. Follows from corollary 3.2.3.4 and theorems 3.2.4.1 and 3.2.4.2. ■

Remark 3.2.4.5. A Lagrangian algebra $L = L(H, \gamma)$ is completely characterized by the following conditions: the trivial-degree component $L_e$ is isomorphic to the function algebra $k(G/H)$, and the image $D(L)$ under the functor $D$ from the proof of theorem 3.2.2.2 is isomorphic to the twisted group algebra $k[H, \gamma] \in C(G, H, \alpha)$.

3.3 The Full Center of an Algebra Redux

3.3.1 The Full Center of the Twisted Group Algebra

Theorem 3.3.1.1 ([12], Theorem 4.2). Let $k[H, \gamma]$ be the twisted group algebra, viewed as an algebra in $\mathcal{V}(G, \alpha)$. Then the full center $Z(k[H, \gamma])$ coincides with $L(H, \gamma) = E(k[H, \gamma])$.

Proof. It suffices to construct a homomorphism of algebras $\zeta : L(H, \gamma) \to k[H, \gamma]$ in $\mathcal{V}(G, \alpha)$ fitting in the diagram (1.2). Indeed, such a homomorphism induces a homomorphism of algebras $\tilde{\zeta} : L(H, \gamma) \to Z(k[H, \gamma])$ in $\mathcal{Z}(G, \alpha)$. Since $L(H, \gamma)$ is a separable indecomposable algebra in $\mathcal{Z}(G, \alpha)$, $\tilde{\zeta}$ is a monomorphism by lemma 2.1.2.1. Finally, the dimension of $\mathcal{Z}(G, \alpha)$ is $|G|$, which coincides with the dimension of the full center $Z(k[H, \gamma])$.

Let us define a map $\zeta : E(k[H, \gamma]) \to k[H, \gamma]$ by $\zeta(a) = a(e)$. Observe that this definition, effectively, implies that $\zeta(a) = 0$ if $|a| \in G \setminus H$. We claim that $\zeta$ is a homomorphism of algebras in the category $\mathcal{V}(G, \alpha)$. As $\zeta$ is an evaluation map, it is automatically additive and multiplicative. It remains to check that $\zeta$ is $G$-graded. Let $a \in E(k[H, \gamma])$ such that $|a| = f \in G$. Recall from the proof of theorem 3.2.2.2 that $|a| = f \in G \iff |a(x)| = x^{-1}f x \quad \forall x \in G,$
or equivalently,

\[ |a| = x |a(x)| x^{-1} \quad \forall x \in G. \]

From this, we see that

\[ |\zeta(a)| = |a(e)| = e^{-1} |a| e = e^{-1} f e = f = |a|, \]

and therefore \( \zeta \) does indeed preserve the \( G \)-grading, so that \( \zeta \) is a morphism in \( \mathcal{V}(G, \alpha) \), as claimed.

Now we consider the diagram (1.2). Here, \( A = k[H, \gamma] \) and \( Z = L(H, \gamma) \), so that the diagram becomes

\[
\begin{array}{ccc}
k[H, \gamma] \otimes L(H, \gamma) & \xrightarrow{\text{Id} \otimes \zeta} & k[H, \gamma] \\
\downarrow{\epsilon} & & \downarrow{\mu} \\
L(H, \gamma) \otimes k[H, \gamma] & \xrightarrow{\zeta \otimes \text{Id}} & k[H, \gamma] \otimes k[H, \gamma]
\end{array}
\] (3.20)

Let \( a \in L(H, \gamma) \), and let \( e_h \in k[H, \gamma] \) for some \( h \in H \). Going the short way, we obtain

\[ e_h \otimes a \mapsto e_h \otimes a(e) \mapsto e_h a(e). \] (3.21)

Going the long way, we obtain

\[ e_h \otimes a \mapsto (h.a) \otimes e_h \mapsto ((h.a)(e)) \otimes e_h \mapsto ((h.a)(e)) e_h. \] (3.22)

We can use \( H \)-equivariance of \( a \) to rewrite the rightmost side of (3.22) as

\[ ((h.a)(e)) e_h = a \left( e_h^{-1} \right) e_h = \left( h. (a(e)) \right) e_h \] (3.23)

The rightmost sides of (3.21) and (3.23) coincide because the \( H \)-action on \( k[H, \gamma] \) is inner in the sense that, for \( y \in k[H, \gamma] \) and \( h \in H \), we have

\[ h.y = e_h y (e_h)^{-1}. \] (3.24)
It therefore follows that $\zeta$ makes the diagram (3.20) commute. This completes the proof. ■

Remark 3.3.1.2. It follows from theorem 3.2.4.4 that Lagrangian algebras in the Deligne product $\mathcal{Z}(G, \alpha) \boxtimes \mathcal{Z}(G, \alpha^{-1}) \simeq \mathcal{Z}(G \times G, \alpha \times \alpha^{-1})$ correspond to pairs $(U, \gamma)$, where $U \subseteq G \times G$ is a subgroup and $\gamma \in C^2(U, k^*)$ is a coboundary $\partial \gamma = \left(\alpha \times \alpha^{-1}\right)_{|U}$. This coincides with the parametrization of module categories obtained in [38], which illustrates the fact\textsuperscript{39} that the full center defines a bijection between equivalence classes of indecomposable module categories over $\mathcal{Z}(G, \alpha)$ and Lagrangian algebras in $\mathcal{Z}(G, \alpha) \boxtimes \mathcal{Z}(G, \alpha^{-1})$.

3.4 Characters of Lagrangian Algebras

3.4.1 Irreducible Characters

For an object $Z \in \mathcal{Z}(G, \alpha)$, define its character as the following function on pairs of commuting elements of $G$:

$$
\chi_Z(f, g) = \text{Tr}_Z(f(g)).
$$

Lemma 3.4.1.1 ([12], Lemma 5.1). Let $Z \in \mathcal{Z}(G, \alpha)$. Then the character $\chi_Z$ satisfies

$$
\chi_Z(x f x^{-1}, x g x^{-1}) = \frac{\alpha(x g x^{-1}, x) f}{\alpha(x, g) f} \cdot \chi_Z(f, g) \quad (3.25)
$$

Proof. Let $\rho : G \to \text{Aut}(Z)$ be the representation homomorphism corresponding to the action of $G$ on the graded vector space $Z$. For $f, g, x \in G$, we have, by definition,

$$
\chi_Z(f, g) = \text{Tr}_Z(\rho(g)) = \text{Tr}_{Z_{f^{-1}g^{-1}}}((\rho(x) \rho(g) \rho(x^{-1})).
$$

Note that by (1.6), we can write

$$
\rho(x) \rho(g) \rho(x^{-1}) = \frac{\alpha(x g x^{-1}, x) f}{\alpha(x, g) f} \rho(x g x^{-1}).
$$

\textsuperscript{39} As formulated in [8, § 2.5].
Indeed,
\[ \rho(x)\rho(g) = \frac{1}{\alpha(x, g|f)}\rho(xg). \]

Similarly,
\[ \rho(xgx^{-1})\rho(x) = \frac{1}{\alpha(xgx^{-1}, x|f)}\rho(xg). \]

Combining these, we see that
\[ \text{Tr}_{Z_{sf}}(\rho(x)\rho(g)\rho(x)^{-1}) = \frac{\alpha(xgx^{-1}, x|f)}{\alpha(x, g|f)}\text{Tr}_{Z_{sf}}(\rho(xgx^{-1})) = \]
\[ = \frac{\alpha(xgx^{-1}, x|f)}{\alpha(x, g|f)}\chi_Z(xfx^{-1}, xgx). \]

The result follows.

\[ \textbf{Remark 3.4.1.2.} \] Equation (3.25) implies that a character of \( Z(G, \alpha) \) can be nonzero only on those commuting pairs \((f, g) \in G \times G\) for which
\[ \frac{\alpha(x, f, g)\alpha(g, x, f)\alpha(f, g, x)}{\alpha(x, g, f)\alpha(f, x, g)\alpha(g, f, x)} = 1 \]
for all \( x \in C_G(f) \cap C_G(g)\).

By a character of \( Z(G, \alpha) \), we mean a function on pairs of commuting elements of \( G \) satisfying the projective class function property (3.25).

For characters \( \chi \) and \( \chi' \), define the product \( \chi \chi' \) by
\[ (\chi \chi')(f, g) = \sum_{f_1f_2=f, f_g=g, f_{i_1}} \alpha(g|f_1, f_2)\chi(f_1, g)\chi'(f_2, g). \]  \hspace{1cm} (3.26)

\[ \textbf{Lemma 3.4.1.3 (}[12], Lemma 5.3). \] Let \( Z, W \in Z(G, \alpha) \), and let \( \chi_Z, \chi_W \) denote their respective characters. Then \( \chi_Z \chi_W = \chi_{Z \otimes W} \).

\[ \textbf{Proof.} \] We have
\[ \chi_{Z \otimes W}(f, g) = \text{Tr}_{Z \otimes W}(g) = \sum_{f_1f_2=f} \text{Tr}_{Z_{f_1} \otimes W_{f_2}}(g). \]
If \( f_i \notin C_G(g) \), then the corresponding summand contributes zero to the trace. Otherwise,

\[
\sum_{f_1, f_2 = f} \text{Tr}_{Z_{f_1} \otimes W_{f_2}}(g) = \sum_{f_1, f_2 = f} \alpha(g|f_1, f_2) \text{Tr}_{Z_{f_1}}(g) \text{Tr}_{W_{f_2}}(g) = \\
= \sum_{f_1, f_2 = f \atop f_k = g f_i} \alpha(g|f_1, f_2) \chi_Z(f, g) \chi_W(f, g) = \chi_Z \chi_W,
\]
as desired. \( \blacksquare \)

**Remark 3.4.1.4.** The character of the dual object (the *dual character*) has the form

\[
\chi' = \frac{1}{\alpha(g^{-1}|f^{-1})} \chi(f^{-1}, g^{-1}).
\]

As in the ordinary character theory, the space of characters of \( \mathcal{Z}(G, \alpha) \) comes equipped with a *scalar product* (see [1]):

\[
(\chi, \chi') = \frac{1}{|G|} \sum_{f, g \in G \atop f \cdot g = g \cdot f} \alpha(g^{-1}|f^{-1}) \chi(f, g^{-1}) \chi'(f, g).
\]

The scalar product calculates dimensions of corresponding Hom-spaces in \( \mathcal{Z}(G, \alpha) \), as the next lemma shows.

**Lemma 3.4.1.5** ([12], Lemma 5.5). Let \( Z, W \in \mathcal{Z}(G, \alpha) \). Then

\[
(\chi_Z, \chi_W) = \dim(\text{Hom}_{\mathcal{Z}(G, \alpha)}(Z, W)).
\]

In particular, for simple \( Z, W \in \mathcal{Z}(G, \alpha) \), the scalar product \( (\chi_Z, \chi_W) = 1 \) if and only if \( Z \cong W \), and zero otherwise.

**Proof.** First, observe that the Hom-space \( \text{Hom}_{\mathcal{Z}(G, \alpha)}(I, W) \) coincides with the vector space of \( G \)-invariants \( W_e^G \), so that

\[
\dim(\text{Hom}_{\mathcal{Z}(G, \alpha)}(I, W)) = \dim_k(W_e^G) = \frac{1}{|G|} \sum_{g \in G} \chi_W(e, g).
\]
More generally, \( \text{Hom}_{Z(G, \alpha)}(Z, W) \approx \text{Hom}_{Z(G, \alpha)}(I, Z^* \otimes W) \), and hence

\[
\dim(\text{Hom}_{Z(G, \alpha)}(Z, W)) = \dim(\text{Hom}_{Z(G, \alpha)}(I, Z^* \otimes W)) = \frac{1}{|G|} \sum_{g \in G} \chi_{Z^* \otimes W}(e, g) =
\]

\[
= \frac{1}{|G|} \sum_{f, g \in G \atop fg = gf} \alpha(g|f^{-1}, f) \chi_Z(f^{-1}, g) \chi_W(f, g) =
\]

\[
= \frac{1}{|G|} \sum_{f, g \in G \atop fg = gf} \alpha(g|f^{-1}, f) \frac{\alpha(g^{-1}, g|f^{-1})}{\alpha(g|f, f^{-1})} \chi_Z(f, g) \chi_W(f, g) =
\]

\[
= \frac{1}{|G|} \sum_{f, g \in G \atop fg = gf} \alpha(g^{-1}, g|f) \chi_Z(f, g^{-1}) \chi_W(f, g) = (\chi_Z, \chi_W).
\]

\[\blacksquare\]

### 3.4.2 Characters of Lagrangian Algebras in \( Z(G, \alpha) \)

**Lemma 3.4.2.1.** Let \( G \) be a group, let \( k \) be a field, and let \( \beta \in Z^2(G, k^*) \). Assume that \( \beta \) is a coboundary

\[
\beta(x, y) = (\partial \delta)(x, y) = \frac{\delta(x) \delta(y)}{\delta(xy)}
\]

(3.27)

for some \( \delta \in C^1(G, k^*) \). Then the map \( \phi : k[G, \beta] \to k[G] \) given by

\[
\phi(e_x) = \delta(x)e_x, \quad x \in G,
\]

is an isomorphism of algebras in \( k^{-}\text{Vect} \). Moreover, for \( M \in k[G]\)-Mod, the character \( \chi_M \) is given by the formula

\[
\chi_{\phi^{-1}M}(x) = \delta(x)\chi_M(x), \quad x \in G.
\]

(3.28)

**Proof.** The map \( \phi \) as defined is clearly an isomorphism of \( k \)-vector spaces. It suffices to show that \( \phi \) preserves the multiplication:

\[
\phi(e_x e_y) = \phi(\beta(x, y)e_{xy}) = \beta(x, y)\delta(xy)e_{xy}.
\]
On the other hand,

\[ \phi(e_x)\phi(e_y) = (\delta(x)e_x)(\delta(y)e_y) = \delta(x)\delta(y)e_xe_y = \delta(x)\delta(y)e_{xy}. \]

These coincide by the coboundary equation (3.27). The equation (3.28) follows as an immediate consequence.  

**Lemma 3.4.2.2 ([12], Lemma 5.6).** For \( V \in \mathcal{Z}(H, \alpha|_H) \), the character \( \chi_{E(V)} \) has the form

\[ \chi_{E(V)}(f, g) = \sum_{y \in Y} \frac{\alpha(y^{-1}g y, y^{-1}|f)}{\alpha(y^{-1}, g|f)} \chi_V(y^{-1}f y, y^{-1}gy), \quad f, g \in G, \]  

(3.29)

where \( Y \) is a set of representatives in \( G \) of the cosets of

\[ \left\{ y \in G \mid y^{-1}f y, y^{-1}gy \in H \right\} / H \subseteq G/H. \]

**Proof.** Let \( V \in \mathcal{Z}(H, \alpha|_H) \). It follows from the defining condition that functions from \( E(V) \) are supported by unions of right \( H \)-cosets of \( G \). Clearly, any function from \( E(V) \) is a unique sum of functions, each of which is supported on a single coset. For a coset \( yH \), a function \( a \in E(V) \) with support \( \text{supp}(a) = yH \) is uniquely determined by its value \( a(y) = v \in V \) at \( y \). Denote such a function by \( a_{y,v} \). The space \( E(V) \) is spanned by the \( a_{y,v} \)s, where \( y \in G \) and \( v \in V \). By (3.10), the degree of \( a_{y,v} \) is \( f \in G \) if and only if \( v \in V_{y^{-1}fy} \). Note that for \( v \in V_{h^{-1}y^{-1}f y h} \), we have

\[ \alpha(h^{-1}, y^{-1}|f)\alpha(h, h^{-1}|y^{-1}f y)a_{y,h,v} = a_{y,h,v}, \quad h \in H. \]

Indeed,

\[ v = a_{y,h,v}(yh) = \alpha(h^{-1}, y^{-1}|f)h^{-1}.\left(a_{y,h,v}(y)\right), \]

and thus

\[ a_{y,h,v}(y) = h.v = \alpha(h^{-1}, y^{-1}|f)\alpha(h, h^{-1}|y^{-1}f y)a_{y,h,v}(y). \]

We can therefore write

\[ E(V)_f = \bigoplus_y \left\langle a_{y,v} \mid v \in V_{y^{-1}fy} \right\rangle \simeq \bigoplus_y V_{y^{-1}fy}, \]  

(3.30)
where the sum is taken over a set of representatives of the cosets $G/H$.

For $g \in C_G(f)$, consider the $k$-linear operator $g : E(V)_f \to E(V)_f$. Recall from the proof of theorem 3.2.2.2 that the action of $g$ on $E(V)$ is given by

$$(g.a)(x) = \alpha(x^{-1}, g|f)a(g^{-1}x), \quad x \in G, a \in E(V)_f.$$ 

In particular, $\text{supp}(g.a) = g\text{supp}(a)$. Therefore $g.a_v$ is supported by the coset $gyH$ and hence can be written as $a_{y^iv'}$ for some $v' \in V$. The action of $g$ on $E(V)_f$ permutes the direct summands of (3.30) according to the left action of $g$ on the set of cosets $G/H$. Thus the trace $\text{Tr}_{E(V)_f}(g)$ has the form

$$\text{Tr}_{E(V)_f}(g) = \sum_y \text{Tr}_{(a_{y^iv}) \in V_{y^{-1}f}}(g), \quad (3.31)$$

where the sum runs over representatives of those cosets $yH$ for which $gyH = yH$. Now let $y \in G$ be such that $gyH = yH$, or, equivalently, $y^{-1}gy \in H$. To compute the trace

$$\text{Tr}_{(a_{y^iv}) \in V_{y^{-1}f}}(g),$$

note that for $v \in V_{y^{-1}f}$, $g.a_v$ can be written as $a_{y^iv'}$ for some $v' \in V$. More explicitly,

$$v' = g.a_v(y) = \alpha(y^{-1}, g|f)a_{y^iv}(g^{-1}y) = \alpha(y^{-1}, g|f)a_{y^iv}(yy^{-1}g^{-1}y) =$$

$$= \frac{\alpha(y^{-1}gy, y^{-1}f)}{\alpha(y^{-1}, g|f)}(y^{-1}g)_va_{y^iv}(y) = \frac{\alpha(y^{-1}gy, y^{-1}f)}{\alpha(y^{-1}, g|f)}(y^{-1}g)_v.$$ 

Thus the trace $\text{Tr}_{(a_{y^iv}) \in V_{y^{-1}f}}(g)$ coincides with $\frac{\alpha(y^{-1}gy, y^{-1}f)}{\alpha(y^{-1}, g|f)}\text{Tr}_{V_{y^{-1}f}}(y^{-1}g)_v$. Finally,

$$\chi_{E(V)}(f, g) = \text{Tr}_{E(V)_f}(g) = \sum_{y \in Y} \frac{\alpha(y^{-1}gy, y^{-1}f)}{\alpha(y^{-1}, g|f)}\text{Tr}_{V_{y^{-1}f}}(y^{-1}g)_v =$$

$$= \sum_{y \in Y} \frac{\alpha(y^{-1}gy, y^{-1}f)}{\alpha(y^{-1}, g|f)}\chi_V(y^{-1}fy, y^{-1}gy),$$

which gives the desired result.
Lemma 3.4.2.3 ([12], Lemma 5.7). Let $k[H, \gamma]$ be the twisted group algebra, viewed as an algebra in the category $\mathcal{Z}(H, \alpha|_H)$. Then the character $\chi_{k[H, \gamma]}$ of $k[H, \gamma]$ is given by

$$\chi_{k[H, \gamma]}(h, u) = \frac{\gamma(u, h)}{\gamma(h, u)}, \quad hu = uh. \quad (3.32)$$

Proof. Let $h \in H$. The degree-$h$ component $k[H, \gamma]_h$ is one-dimensional and has the form

$$k[H, \gamma]_h = ke_h,$$

where $e_h$ is the standard basis vector of degree $h$ in $k[H, \gamma]$. Now taking $u \in H$, we have

$$u.e_h = \frac{\gamma(u, h)}{\gamma(uh^{-1}, u)}e_{uh^{-1}}.$$

If $u \in C_H(h)$, then this formula becomes

$$u.e_h = \frac{\gamma(u, h)}{\gamma(h, u)}e_h.$$

If $u \not\in C_H(h)$, then $u.e_h \not\in k[H, \gamma]_h$, so the degree-$h$ component is not a fixed point of the action and therefore contributes zero to the character. Hence by lemma 3.4.2.2, the character $\chi_{k[H, \gamma]}$ has the form given by equation (3.32), as claimed. \hfill \blacksquare

Combining the last two results proves the main result of this section:

Theorem 3.4.2.4 ([12], Theorem 5.8). Let $L = L(H, \gamma)$ be a Lagrangian algebra in the Drinfeld center $\mathcal{Z}(G, \alpha)$. Then the character $\chi_L$ has the form

$$\chi_L(f, g) = \sum_{y \in Y} \frac{\alpha(y^{-1}gy, y^{-1}f)\gamma(y^{-1}fy, y^{-1}gy)}{\alpha(y^{-1}, g|f)\gamma(y^{-1}gy, y^{-1}fy)}, \quad f, g \in G,$$

where $Y$ is a set of representatives in $G$ of the cosets of

$$\left\{ y \in G \mid y^{-1}fy, y^{-1}gy \in H \right\} \cap H \subseteq G/H.$$

Proof. Follows from lemmata 3.4.2.2 and 3.4.2.3. \hfill \blacksquare
3.5 Example: Characters of $\mathcal{Z}(D_3, \alpha)$

Here, we compute the characters for a relatively tractable example.

### 3.5.1 Brief Description of $H^3(D_3, \mathbb{C}^*)$

We choose the following presentation of $D_3$:

$$D_3 = \left\langle r, s \mid r^3 = s^2 = e; srs = r^{-1} \right\rangle.$$  

According to [5, § 6.3], the third cohomology group $H^3(D_3, \mathbb{C}^*)$ is cyclic of order 6. It is generated by the cohomology class of the function $\varpi : D_3 \times D_3 \times D_3 \to \mathbb{C}^*$ given by

$$\varpi(s^{m_1}r^{p_1}, s^{m_2}r^{p_2}, s^{m_3}r^{p_3}) = \exp\left[\frac{2\pi i}{9} \left((-1)^{m_2+m_3}n_1((-1)^{m_1}n_2 + n_3 - [(-1)^{m_1}n_2 + n_3]_3) + \frac{9}{2}m_1m_2m_3\right)\right],$$

where $[\cdot]_3$ denotes taking the congruence class modulo 3.

Recall that the second cohomology group $H^2(D_3, \mathbb{C}^*)$ is trivial. This implies that a coboundary for the restriction $\alpha|_H$ of a 3-cocycle $\alpha \in Z^3(D_3, \mathbb{C}^*)$ to a subgroup $H \leq D_3$ is unique if it exists. It follows that Lagrangian algebras in $\mathcal{Z}(D_3, \varpi)$ are labeled only by conjugacy classes of subgroups on which the restriction of $\alpha$ is cohomologically trivial. We call such subgroups *admissible subgroups*.

In terms of the cocycle $\alpha$, there are four distinct cases, depending upon the order of the cohomology class of $\alpha$ in $H^3(D_3, \mathbb{C}^*)$. In what follows, $\omega$, $\epsilon$, and $\eta$ denote primitive cube, fourth, and ninth roots of unity, respectively.

### 3.5.2 Wherein $|\alpha| = 1$

Taking $\alpha = \varpi^0 \equiv 1$ as our associator turns the twisted Drinfeld center $\mathcal{Z}(D_3, \alpha)$ into the untwisted version $\mathcal{Z}(D_3)$. The character table for $\mathcal{Z}(D_3)$ is as follows:
Table 3.1: Characters of $\mathbb{Z}(D_3)$

<table>
<thead>
<tr>
<th></th>
<th>$(e, e)$</th>
<th>$(e, r)$</th>
<th>$(e, s)$</th>
<th>$(r, e)$</th>
<th>$(r, r)$</th>
<th>$(r, r^2)$</th>
<th>$(s, e)$</th>
<th>$(s, s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\omega$</td>
<td>$\omega^{-1}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\omega^{-1}$</td>
<td>$\omega$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>

All subgroups are admissible in this case. Up to conjugacy, there are four subgroups:

\[
\{e\}, \quad \langle s \rangle, \quad \langle r \rangle, \quad D_3.
\]

The characters of the corresponding Lagrangian algebras are as follows:

Table 3.2: Characters of Lagrangian Algebras in $\mathbb{Z}(D_3)$

<table>
<thead>
<tr>
<th></th>
<th>$(e, e)$</th>
<th>$(e, r)$</th>
<th>$(e, s)$</th>
<th>$(r, e)$</th>
<th>$(r, r)$</th>
<th>$(r, r^2)$</th>
<th>$(s, e)$</th>
<th>$(s, s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L({e})$</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L(\langle s \rangle)$</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$L(\langle r \rangle)$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$L(D_3)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

In view of the character table for $\mathbb{Z}(D_3)$, the above table gives rise to the following decompositions into sums of irreducible characters:
Table 3.3: Decomposition into Irreducible Characters

<table>
<thead>
<tr>
<th></th>
<th>$L(\langle e \rangle)$</th>
<th>$L(\langle s \rangle)$</th>
<th>$L(\langle r \rangle)$</th>
<th>$L(D_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0 + \chi_1 + 2\chi_2$</td>
<td>$\chi_0 + \chi_2 + \chi_6$</td>
<td>$\chi_0 + \chi_1 + 2\chi_3$</td>
<td>$\chi_0 + \chi_3 + \chi_6$</td>
<td></td>
</tr>
</tbody>
</table>

3.5.3 Wherein $|a| = 2$

In this case, $a = \omega^3$. The corresponding character table is as follows:

Table 3.4: Characters of $Z(D_3, a); |a| = 2$

<table>
<thead>
<tr>
<th></th>
<th>$(e, e)$</th>
<th>$(e, r)$</th>
<th>$(e, s)$</th>
<th>$(r, e)$</th>
<th>$(r, r)$</th>
<th>$(s, e)$</th>
<th>$(s, s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\omega$</td>
<td>$\omega^{-1}$</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\omega^{-1}$</td>
<td>$\omega$</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\varepsilon$</td>
</tr>
<tr>
<td>$\chi_7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$-\varepsilon$</td>
</tr>
</tbody>
</table>

The admissible subgroups are $\{e\}$ and $\langle r \rangle$. The characters of the corresponding Lagrangian algebras are as follows:
Table 3.5: Characters of Lagrangian Algebras in $\mathbb{Z}(D_3, \alpha)$

<table>
<thead>
<tr>
<th></th>
<th>$(e, e)$</th>
<th>$(e, r)$</th>
<th>$(e, s)$</th>
<th>$(r, e)$</th>
<th>$(r, r)$</th>
<th>$(r, r^2)$</th>
<th>$(s, e)$</th>
<th>$(s, s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L(e)$</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L(r)$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>1 + $\omega^2$</td>
<td>1 + $\omega$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In view of the character table for $\mathbb{Z}(D_3, \alpha)$, the above table gives rise to the following decompositions into sums of irreducible characters:

Table 3.6: Decomposition into Irreducible Characters

<table>
<thead>
<tr>
<th></th>
<th>$L({e})$</th>
<th>$L(\langle r \rangle)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0 + \chi_1 + 2\chi_2$</td>
<td>$\chi_0 + \chi_1 + \chi_3 + \chi_5$</td>
<td></td>
</tr>
</tbody>
</table>

3.5.4 Wherein $|\alpha| = 3$

In this case, $\alpha = \omega^2$. The character table for $\mathbb{Z}(D_3, \alpha)$ is as follows:

Table 3.7: Characters of $\mathbb{Z}(D_3, \alpha); |\alpha| = 3$

<table>
<thead>
<tr>
<th></th>
<th>$(e, e)$</th>
<th>$(e, r)$</th>
<th>$(e, s)$</th>
<th>$(r, e)$</th>
<th>$(r, r)$</th>
<th>$(r, r^2)$</th>
<th>$(s, e)$</th>
<th>$(s, s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\eta$</td>
<td>$\eta^2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\eta^4$</td>
<td>$\eta^8$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\eta^7$</td>
<td>$\eta^3$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>
The admissible subgroups are \{e\} and \langle s \rangle. The characters of the corresponding Lagrangian algebras are as follows:

Table 3.8: Characters of Lagrangian Algebras in $\mathbb{Z}(D_3, \alpha)$

<table>
<thead>
<tr>
<th></th>
<th>$(e, e)$</th>
<th>$(e, r)$</th>
<th>$(e, s)$</th>
<th>$(r, e)$</th>
<th>$(r, r)$</th>
<th>$(r, r^2)$</th>
<th>$(s, e)$</th>
<th>$(s, s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L({e})$</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L(\langle s \rangle)$</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

In view of the character table for $\mathbb{Z}(D_3, \alpha)$, we have the following decomposition into irreducibles:

Table 3.9: Decomposition into Irreducible Characters

<table>
<thead>
<tr>
<th></th>
<th>$L({e})$</th>
<th>$L(\langle s \rangle)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0 + \chi_1 + 2\chi_2$</td>
<td>$\chi_0 + \chi_2 + \chi_6$</td>
<td></td>
</tr>
</tbody>
</table>

3.5.5 Wherein $|\alpha| = 6$

In this case, $\alpha = \varpi$. The character table for $\mathbb{Z}(D_3, \alpha)$ is as follows:
Table 3.10: Characters of $\mathcal{Z}(D_3, \alpha); |\alpha| = 6$

<table>
<thead>
<tr>
<th></th>
<th>$(e,e)$</th>
<th>$(e,r)$</th>
<th>$(e,s)$</th>
<th>$(r,e)$</th>
<th>$(r,r)$</th>
<th>$(r, r^2)$</th>
<th>$(s,e)$</th>
<th>$(s,s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\eta$</td>
<td>$\eta^2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\eta^4$</td>
<td>$\eta^8$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\eta^7$</td>
<td>$\eta^3$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\varepsilon$</td>
</tr>
<tr>
<td>$\chi_7$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$-\varepsilon$</td>
</tr>
</tbody>
</table>

Only the trivial subgroup $\{e\}$ is admissible in this case. Its character is identical to the untwisted case:

Table 3.11: Characters of Lagrangian Algebras in $\mathcal{Z}(D_3, \alpha)$

<table>
<thead>
<tr>
<th></th>
<th>$(e,e)$</th>
<th>$(e,r)$</th>
<th>$(e,s)$</th>
<th>$(r,e)$</th>
<th>$(r,r)$</th>
<th>$(r, r^2)$</th>
<th>$(s,e)$</th>
<th>$(s,s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L({e})$</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Finally, the decomposition into irreducible characters is, also, identical to the untwisted case:

Table 3.12: Decomposition into Irreducible Characters

<table>
<thead>
<tr>
<th></th>
<th>$L({e})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_0 + \chi_1 + 2\chi_2$</td>
<td></td>
</tr>
</tbody>
</table>
3.6 Automorphisms of Lagrangian Algebras in Group-theoretical Braided Fusion Categories

Here, we make a few remarks about the automorphism groups of Lagrangian algebras in $\mathcal{Z}(G, \alpha)$.

3.6.1 Lagrangian Algebras in Drinfeld Centers of Finite Groups

The following classification of Lagrangian algebras in $\mathcal{Z}(G, \alpha)$ was proved in [12] and can be found in § 3.2.4 of this document as corollary 3.2.4.4. We restate it here.

**Theorem 3.6.1.1.** Lagrangian algebras in the Drinfeld center $\mathcal{Z}(G, \alpha)$ correspond to pairs $(H, \gamma)$, where $H \leq G$ is a subgroup and $\gamma \in C^2(H, k^*)$ is a coboundary $\partial \gamma = \alpha|_H$ of the restriction of the associator to $H$.

Explicitly, let $\gamma \in C^2(H, k^*)$ be such that $\partial \gamma = \alpha|_H$. Denote by $k[H, \gamma]$ the twisted group algebra of $H$, viewed as an algebra in $\mathcal{Z}(H, \alpha|_H)$. Then

$$L(H, \gamma) = \left\{ a : G \to k[H, \gamma] \big| a(xh) = \alpha(h^{-1}, x^{-1}| f)h^{-1}(a(x)), \ h \in H, \ x \in G, \ |a| = f \right\}.$$

Here, $|a| = f \iff |a(x)| = x^{-1}f x \ \forall x \in G$. The $G$-action on $L(H, \gamma)$ is as follows: given a homogeneous $a \in L(H, \gamma)$ of degree $f \in G$ and an arbitrary $g \in G$, the function $g.a : G \to k[H, \gamma]$ is given by

$$(g.a)(x) = \frac{1}{\alpha(x^{-1}, g| f)}a(g^{-1}x), \ x \in G.$$

3.6.2 Automorphisms of Lagrangian Algebras

Let $H \leq G$ be a subgroup, and let $\gamma \in C^2(H, k^*)$ be such that $\partial \gamma = \alpha|_H$. The automorphism group $\text{Aut}_{\mathcal{Z}(G, \alpha)}(L(H, \gamma))$ of the corresponding Lagrangian algebra fits in the short exact sequence of groups

$$\text{Aut}_{\mathcal{Z}(G, \alpha)_{L(H, \gamma)}}(L(H, \gamma)) \longrightarrow \text{Aut}_{\mathcal{Z}(G, \alpha)}(L(H, \gamma)) \longrightarrow \text{Aut}_{\text{Rep}(G)}(L(H, \gamma)_e) \quad (3.33)$$

\[40\] Cf. the proof of theorem 3.2.2.2 in chapter 3.
The trivial-degree component $L(H, \gamma)_e$ is the algebra $k(G/H)$ of functions on cosets. Consequently,

$$\text{Aut}_{\text{Rep}(G)}(L(H, \gamma)_e) = \text{Aut}_{\text{Rep}(G)}(k(G/H)) \simeq \text{Aut}_G(G/H) \simeq N_G(H)/H.$$ (3.34)

Since the category $\mathcal{Z}(G, \alpha)_{L(H, \gamma)_e}$ is equivalent to $\mathcal{Z}(H, \alpha|_H)$, and since $L(H, \gamma)$ is $k[H, \gamma]$ when viewed as an algebra in $\mathcal{Z}(H, \alpha|_H)$, it follows that

$$\text{Aut}_{\mathcal{Z}(G, \alpha)_{L(H, \gamma)_e}}(L(H, \gamma)) \simeq \text{Aut}_{\mathcal{Z}(H, \alpha|_H)}(k[H, \gamma]),$$

which is, in turn, isomorphic to the character group $\hat{H}$. Thus the sequence (3.33) becomes

$$\hat{H} \to \text{Aut}_{\mathcal{Z}(G, \alpha)}(L(H, \gamma)) \to N_G(H)/H$$ (3.35)

Recall the category $C(G, H, \alpha)$, whose objects are $G$-graded vector spaces with $\alpha|_H$-projective $H$-action and associativity twisted by $\alpha$, and whose morphisms are $k$-linear maps preserving the $G$ grading and $\alpha|_H$-projective $H$ action. Recall the functor

$E : C(G, H, \alpha) \to \mathcal{Z}(G, \alpha)_{k(G/H)}$ given by

$$E(V) = \left\{ a : G \to V \left| a(xh) = a(h^{-1}, x^{-1}|f)h^{-1}, (a(x)) \forall h \in H \forall x \in G; |a| = f \right. \right\},$$

where the grading on $E(V)$ is $|a| = f \iff |a(x)| = x^{-1}f x \forall x \in G$, and $G$-action on $E(V)$ is given by $(g.a)(x) = a\left(x^{-1}, g|f\right)^{-1}a\left(g^{-1} x\right)$. It was proven in § 3.2.2 that the functor $E$ is a tensor equivalence; moreover, when restricted to the full subcategory

$\mathcal{Z}(H, \alpha|_H) \subseteq C(G, H, \alpha)$, $E$ induces a braided equivalence $\mathcal{Z}(H, \alpha|_H) \simeq \mathcal{Z}(G, \alpha)_{k(G/H)}^{\text{loc}}$.

Let now $\zeta \in \text{Aut}_{\mathcal{Z}(G, \alpha)}(L(H, \gamma))$, and let $\zeta_e = \zeta|_{L(H, \gamma)_e}$ be the restriction to the trivial-degree component. The associated inverse image functor

---

41 See theorem 3.2.2.2.
42 This is an abuse of language. The image of $L(H, \gamma)$ under the functor $D$ from the proof of theorem 3.2.2.2 is isomorphic to $k[H, \gamma]$. See also remark 3.2.4.5.
43 E.g., from “Definitions” or [12, Appendix A].
44 See the proof of theorem 3.2.2.2 in § 3.2.2.
45 See also [12, Theorem 3.7].
\( \zeta_e : \mathcal{Z}(G, \alpha)_{k(G/H)} \to \mathcal{Z}(G, \alpha)_{k(G/H)} \) gives a tensor autoequivalence of the category of modules. Moreover, since \( \zeta_e \) preserves the locality condition (1.1), its restriction to the subcategory \( \mathcal{Z}(G, \alpha)_{k(G/H)}^{\text{loc}} \) of local modules is a braided autoequivalence. The diagram of functors

\[
\begin{array}{ccc}
\mathcal{Z}(H) & \xrightarrow{E} & \mathcal{Z}(G, \alpha)_{k(G/H)}^{\text{loc}} \\
\downarrow_{F_{\zeta_e}} & & \downarrow_{\zeta_e} \\
\mathcal{Z}(H) & \xrightarrow{E} & \mathcal{Z}(G, \alpha)_{k(G/H)}^{\text{loc}}
\end{array}
\]

commutes up to a natural isomorphism. Note that \( \zeta_e \) can be considered as an element of \( N_G(H)/H \leq \text{Out}(H) \) using (3.34), and \( F_{\zeta_e} \) is a braided autoequivalence of \( \mathcal{Z}(H) \) induced by \( \zeta_e \in \text{Out}(H) \).
4 LAGRANGIAN ALGEBRAS IN POINTED CATEGORIES

In this chapter, we consider the case of pointed fusion categories in detail.

Lagrangian algebras in a pointed category of the form $C(A, q)$, where $A$ is a finite Abelian group and $q$ is a quadratic form on $A$, correspond to Lagrangian subgroups $L$ of $A$. These are subgroups onto which $q$ restricts trivially\(^{46}\) of the largest possible order\(^ {47}\). We derive a formula for the associator $\beta$ of the group of symmetries of the Lagrangian algebra corresponding to a Lagrangian subgroup $L$ of $A$.

4.1 Modular Extensions of Finite Groups

In this section, we interpret the third cohomology group $H^3(G, k^*)$ as the group of modular extensions of $\text{Rep}(G)$, and then consider pointed modular extensions of $\text{Rep}(G)$.

4.1.1 The Categorical Group $\text{Mex}(S)$

Recall\(^ {48}\) that a braided fusion category $\mathcal{D}$ is nondegenerate if its symmetric center $Z_{\text{sym}}(\mathcal{D}) = \left\{ X \in \mathcal{D} \left| c_{X,Y} \circ c_{Y,X} = \text{Id}_{Y \otimes X}, \ \forall Y \in \mathcal{D} \right. \right\}$ coincides with $k$-$\text{Vect}$.

Let $S$ be a symmetric fusion category. Following [30], we say that a nondegenerate braided category $\mathcal{D}$ containing $S$ as a full subcategory is a modular extension\(^ {49}\) of $S$ if the symmetric centralizer $C_\mathcal{D}(S) = \left\{ X \in \mathcal{D} \left| c_{X,Y} \circ c_{Y,X} = \text{Id}_{Y \otimes X}, \ \forall Y \in \mathcal{S} \right. \right\}$ coincides with $S$.

\(^{46}\)I.e., isotropic subgroups of $A$.
\(^{47}\)I.e., the order of $L$ squares to the order of $A$.
\(^{48}\)E.g., from [11].
\(^{49}\)We are abusing language here; what we call a “modular extension” would more properly be termed a “nondegenerate braided extension”. See [30, Remark 4.2].
**Remark 4.1.1.1.** A nondegenerate braided category $\mathcal{D}$ containing $\mathcal{S}$ as a full subcategory is a modular extension if and only if $\dim(\mathcal{D}) = (\dim(\mathcal{S}))^2$. Indeed, $\mathcal{S}$ is always a subcategory of its symmetric centralizer $C_{\mathcal{D}}(\mathcal{S})$, and $\dim(\mathcal{D}) = \dim(\mathcal{S}) \cdot \dim(C_{\mathcal{D}}(\mathcal{S}))$ makes $\dim(C_{\mathcal{D}}(\mathcal{S})) = \dim(\mathcal{S})$ equivalent to $\mathcal{S} = C_{\mathcal{D}}(\mathcal{S})$.

Define the category $\text{Mex}(\mathcal{S})$ as follows. Objects of $\text{Mex}(\mathcal{S})$ are modular extensions of $\mathcal{S}$. Morphisms are isomorphism classes of braided equivalences preserving the subcategory $\mathcal{S}$, i.e., making the diagram

$$
\begin{array}{ccc}
C & \rightarrow & \mathcal{D} \\
\downarrow & & \downarrow \\
\mathcal{S} & \rightarrow & \mathcal{D}
\end{array}
$$

commute on the nose.

**Remark 4.1.1.2.** If one considers isomorphisms between braided equivalences of modular extensions as 2-cells, one naturally ends up with a 2-category $\text{Mex}(\mathcal{S})$.

For $\mathcal{D}, \mathcal{D}' \in \text{Mex}(\mathcal{S})$, the Deligne product $\mathcal{S} \boxtimes \mathcal{S}$ is a full subcategory of $\mathcal{D} \boxtimes \mathcal{D}'$. The tensor product functor $\otimes : \mathcal{S} \boxtimes \mathcal{S} \rightarrow \mathcal{S}$ has a two-sided adjoint $F : \mathcal{S} \rightarrow \mathcal{S} \boxtimes \mathcal{S}$. This gives rise to a commutative algebra $A = F(I) \in \mathcal{S} \boxtimes \mathcal{S}$, where $I \in \mathcal{S}$ is the monoidal unit. This algebra is Lagrangian. Indeed, $\dim(A) = \dim(\mathcal{S})$, and so

$$\dim(\mathcal{D}) = \dim(\mathcal{S}) \cdot \dim(C_{\mathcal{D}}(\mathcal{S})) = (\dim(\mathcal{S}))^2 = (\dim(A))^2.$$

Define an operation $\circ_A : \text{Mex}(\mathcal{S}) \times \text{Mex}(\mathcal{S}) \rightarrow \text{Mex}(\mathcal{S})$ by $\mathcal{D} \circ_A \mathcal{D}' = (\mathcal{D} \boxtimes \mathcal{D}')_{\text{loc}}^A$. The category $\text{Mex}(\mathcal{S})$ is monoidal with respect to $\circ_A$. The monoidal unit is the monoidal (or Drinfeld) center $Z(\mathcal{S})$ of $\mathcal{S}$ with the natural embedding $\mathcal{S} \rightarrow Z(\mathcal{S})$.

Moreover, $\text{Mex}(\mathcal{S})$ is a categorical group. Indeed, the quasi-inverse of a modular extension $\mathcal{C}$ with respect to $\circ_A$ is $\overline{\mathcal{C}}$, the category with the inverse braiding. Denote by $\text{Mex}(\mathcal{S})$ the zeroth homotopy group $\pi_0(\text{Mex}(\mathcal{S}))$. By the definition, the first homotopy
group $\pi_1(Mex(S))$ coincides with the group $\text{Aut}_{br}(\mathcal{Z}(S)/S)$ of isomorphism classes of braided tensor autoequivalences of $\mathcal{Z}(S)$ fixing objects of $S$ on the nose.

**Remark 4.1.1.3.** The 2-category $\text{M}ex(S)$ from remark 4.1.1.2 is clearly a 2-categorical group. The standard invariants $\pi_0$ and $\pi_1$ of $\text{M}ex(S)$ are the same as for $\text{M}ex(S)$. It follows from the definition that $\pi_2(\text{M}ex(S)) = \ker(\text{Aut}_{\otimes}(Id_{\mathcal{Z}(S)}) \to \text{Aut}_{\otimes}(Id_S))$.

We will be interested in the case when the base symmetric category $S$ is the category $\text{Rep}(G)$ of finite-dimensional representations of a finite group $G$. Modular extensions of $\text{Rep}(G)$ turn out to be twisted Drinfeld centers\(^{50}\) of $G$. For a 3-cocycle $\alpha \in Z^3(G, k^*)$ denote by $\mathcal{Z}(G, \alpha)$ the category whose objects are $G$-graded $k$-vector spaces with $\alpha$-projective $G$-action and whose morphisms are $k$-linear maps preserving the grading and $\alpha$-projective action. Note that $\text{Rep}(G)$ is a full symmetric subcategory of $\mathcal{Z}(G, \alpha)$ consisting of trivially-graded objects. Moreover, $\dim(\mathcal{Z}(G, \alpha)) = |G|^2 = (\dim(\text{Rep}(G)))^2$, so $\mathcal{Z}(G, \alpha)$ is indeed a modular extension of $\text{Rep}(G)$ by remark 4.1.1.1.

The following result was proved in [30, Theorem 4.22]. We add a sketch of the proof for the sake of completeness.

**Proposition 4.1.1.4 ([13], Proposition 2.4).** The assignment $H^3(G, k^*) \to \text{Mex}(\text{Rep}(G))$ given by $\alpha \mapsto \mathcal{Z}(G, \alpha)$ is an isomorphism.

**Proof.** The homomorphism property of the assignment is established by a braided equivalence $\mathcal{Z}(G, \alpha) \otimes_{\text{Rep}(G)} \mathcal{Z}(G, \beta) \to \mathcal{Z}(G, \alpha\beta)$. Note that $\mathcal{Z}(G, \alpha) \otimes \mathcal{Z}(G, \beta) \simeq \mathcal{Z}(G \times G, \alpha \times \beta)$ as braided fusion categories. Let $\delta : G \to G \times G$ be the diagonal embedding $g \mapsto (g, g)$. This induces a functor $\delta^* : \text{Rep}(G \times G) \to \text{Rep}(G)$ making the diagram

$$
\begin{array}{ccc}
\text{Rep}(G) \otimes \text{Rep}(G) & \xrightarrow{\otimes} & \text{Rep}(G) \\
\downarrow & & \\
\text{Rep}(G \times G) & \xrightarrow{\delta^*} & \text{Rep}(G)
\end{array}
$$

\(^{50}\) See chapter 3 or [12, § 2] for details.
commute. This functor has a two-sided adjoint $\delta_*$. By Frobenius reciprocity, the adjoint $\delta_*$ is just the induction with respect to $\delta$, and so $\delta_*(I) = k$. This implies that $\delta_*(k)$ is the function algebra $^{51} k((G \times G)/\Delta(G)) \in \mathbb{Z}(G \times G, \alpha \times \beta)$. By theorem 3.2.2.2, its category of local modules is equivalent to $\mathbb{Z}(G, \alpha \beta)$.

Now let $\mathcal{D}$ be a modular extension of $\text{Rep}(G)$. The function algebra $A = k(G) \in \text{Rep}(G)$ with the regular $G$-action is etale in $\text{Rep}(G)$ and in $\mathcal{D}$ as well. Moreover, the group of algebra automorphisms is $\text{Aut}_{\text{alg}}(A) = G$. Since $\dim(A) = \dim(\text{Rep}(G))$, $A$ is Lagrangian. Hence the category of modules $\mathcal{D}_A$ admits a decomposition

$$\mathcal{D}_A = \bigoplus_{g \in G} \mathcal{D}_A^{g,\text{loc}}$$

into $g$-local subcategories for each $g \in G$. The result is tensor equivalent $^{52}$ to $\mathcal{V}(G, \alpha)$ for some $\alpha \in Z^3(G, k^*)$. Thus we have $\mathcal{D} = \mathcal{Z}(\mathcal{D}_A) \simeq \mathcal{Z}(\mathcal{V}(G, \alpha)) \simeq \mathcal{Z}(G, \alpha)$ which shows that the assignment is bijective.

The first homotopy group $\pi_1(\text{Mex}(\text{Rep}(G)))$ also has a cohomological description.

**Lemma 4.1.1.5** ([13], Lemma 2.5). $\pi_1(\text{Mex}(\text{Rep}(G))) \simeq H^2(G, k^*)$.

**Proof.** By definition, $\pi_1(\text{Mex}(\text{Rep}(G)))$ is the group $\text{Aut}_{\text{br}}(\mathcal{Z}(\text{Rep}(G))/\text{Rep}(G))$ of braided autoequivalences of the monoidal center $\mathcal{Z}(\text{Rep}(G))$ that fix the full subcategory $\text{Rep}(G) \subseteq \mathcal{Z}(\text{Rep}(G))$ on the nose. The monoidal center $\mathcal{Z}(\text{Rep}(G))$ is equivalent, as a braided monoidal category, to the untwisted Drinfeld center $\mathcal{Z}(G)$. The algebra of functions $k(G) \in \text{Rep}(G)$ is an etale algebra in $\mathcal{Z}(G)$; its category of right modules $\mathcal{Z}(G)_{k(G)}$ is equivalent to the category $\mathcal{V}(G)$ of $G$-graded vector spaces. This equivalence gives rise to a homomorphism $\text{Aut}_{\text{br}}(\mathcal{Z}(\text{Rep}(G))/\text{Rep}(G)) \rightarrow \text{Aut}_{\otimes}(\mathcal{V}(G))$. It follows from [37, Corollary 6.9] that it is an isomorphism with the group $\text{Aut}_{\otimes}(\mathcal{V}(G))$ of isomorphism

---

$^{51}$ Here, $\Delta(G)$ denotes the diagonal subgroup $\{(g, g) | g \in G\} < G \times G$.

$^{52}$ See [30, § 4.3] for details.
classes of tensor structures on the identity functor (the so-called soft autoequivalences). The group $\text{Aut}_0^1(\mathcal{V}(G))$ is, in turn, isomorphic to $H^2(G, k^*)$ by [6, Proposition 2.5].

**Remark 4.1.1.6.** The group $\pi_2(\text{Mex}(\text{Rep}(G)))$ is isomorphic to the first cohomology group $H^1(G, k^*)$. Indeed, according to remark 4.1.1.3, the group $\pi_2(\text{Mex}(\text{Rep}(G)))$ coincides with $\ker\left(\text{Aut}_0(\text{Id}_{\text{Rep}(G)}) \to \text{Aut}_0(\text{Rep}(G))\right)$.

Finally, we make a few remarks about the associator of the categorical group $\text{Mex}(\text{Rep}(G))$. This associator is a class in $H^3\left(H^3(G, k^*), H^2(G, k^*)\right)$. The last result of this section shows that it is the class of a certain crossed module.

**Lemma 4.1.1.7** ([13], Lemma 2.7). The categorical group $\mathcal{G}$ associated with the crossed module

$$Z^3(G, k^*) \leftarrow C^2(G, k^*)/B^2(G, k^*) \quad (4.1)$$

is equivalent to $\text{Mex}(\text{Rep}(G))$.

**Proof.** Objects of the categorical group $\mathcal{G}$ are elements of $Z^3(G, k^*)$. A morphism between $\alpha, \beta \in Z^3(G, k^*)$ is a 2-cochain $c \in C^2(G, k^*)/B^2(G, k^*)$ such that $\partial c = \alpha \cdot \beta^{-1}$. Define a functor $F : \mathcal{G} \to \text{Mex}(\text{Rep}(G))$ by $F(\alpha) = Z(G, \alpha)$. For a morphism $c : \alpha \to \beta$ define a braided tensor functor $\text{Id}(c) : \mathcal{V}(G, \alpha) \to \mathcal{V}(G, \beta)$, which is the identity functor with the tensor structure given by the 2-cochain $c$. Define $F(c) : Z(G, \alpha) \to Z(G, \beta)$ to be the functor $Z(\text{Id}(c)) : Z(\mathcal{V}(G, \alpha)) \to Z(\mathcal{V}(G, \beta))$ induced by $\text{Id}(c)$.

By proposition 4.1.1.4 and lemma 4.1.1.5, the effects of $F$ on both $\pi_0$ and $\pi_1$ are bijective. Hence $F$ is an equivalence of categorical groups. ■

**Remark 4.1.1.8.** Similarly, $\text{Mex}(\text{Rep}(G))$ corresponds to the crossed-complex

$$Z^3(G, k^*) \leftarrow C^2(G, k^*) \leftarrow C^1(G, k^*)$$.
4.1.2 Invertible Objects of $\mathcal{Z}(G, \alpha)$

Let

$$P = \left\{ (z, c) \in \mathcal{Z}(G) \times C^1(G, k^*) \mid c(x)c(y) = \alpha(x, y\vert z)c(xy), \quad x, y \in G \right\}.$$  

It is straightforward to see that the operation

$$(z, c)(w, d) = \left( zw, cda(-\vert z, w)^{-1} \right).$$  

(4.2)

makes $P$ a group with the identity element\(^{53}\) $(e, 1)$ and the inverses

$$(z, c)^{-1} = \left( z^{-1}, c^{-1}a(-\vert z, z^{-1}) \right).$$

Denote by

$$Z_{a}(G) = \left\{ z \in \mathcal{Z}(G) \vert \exists c \in C^1(G, k^*) \text{ such that } c(x)c(y) = \alpha(x, y\vert z)c(xy) \quad \forall x, y \in G \right\}$$

a subgroup of central elements of $G$.

**Remark 4.1.2.1.** Note that $P$ and $Z_{a}(G)$ fit in the short exact sequence

$$\mathcal{G} \overset{\iota}{\rightarrow} P \overset{\pi}{\rightarrow} Z_{a}(G),$$  

(4.3)

where $\iota$ is the map $\chi \mapsto (e, \chi)$ and $\pi$ is the canonical projection $(z, c) \mapsto z$.

Recall that the Picard group $\text{Pic}(C)$ of a monoidal category $C$ is the group of isomorphism classes of invertible objects of $C$.

**Theorem 4.1.2.2** ([13], Theorem 2.8). The Picard group $\text{Pic}(\mathcal{Z}(G, \alpha))$ is isomorphic to $P$.

**Proof.** The natural forgetful functor $\mathcal{Z}(G, \alpha) \rightarrow \mathcal{V}(G, \alpha)$ (forgetting about the braiding) induces a group homomorphism $\text{Pic}(\mathcal{Z}(G, \alpha)) \rightarrow \text{Pic}(\mathcal{V}(G, \alpha)) = G$. However, because $\mathcal{Z}(G, \alpha)$ is braided, this homomorphism factors through the center $\mathcal{Z}(G)$. Hence the

\(^{53}\) Here, $1$ denotes the constant function $1(x) = 1 \in k$. 

support of an invertible object $I \in \text{Pic}(\mathcal{Z}(G, \alpha))$ is a single central element $z \in Z(G)$. The $\alpha$-projective $G$-action\(^{54}\) on an invertible object $I = I(z)$ is given by multiplication by $c$, which shows the correspondence on the level of sets. The group operation on \(\text{Pic}(\mathcal{Z}(G, \alpha))\) is the tensor product in $\mathcal{Z}(G, \alpha)$. The $\alpha$-projective action on the tensor product $I(z) \otimes I(w)$ is diagonal and contributes a factor of $\alpha(-|z, w)$, which shows that the correspondence of sets is an isomorphism of groups. \(\blacksquare\)

The function $q : P \to \mathbb{Q}/\mathbb{Z}$ given by

$$q(z, c) = c(z)$$

(4.4)

is induced by the braiding on $\mathcal{Z}(G, \alpha)$. It is a quadratic form on $P$.

A 3-cocycle $\alpha \in Z^3(G, k^*)$ is said to be soft if for any $g \in G$ there is a $c \in C^1(G, k^*)$ such that $c(x)c(y) = \alpha(x, y|g)c(xy)$ for all $x, y \in G$.

**Corollary 4.1.2.3** ([13], Corollary 2.9). The category $\mathcal{Z}(G, \alpha)$ is pointed if and only if $G$ is Abelian and $\alpha$ is soft.

**Proof.** The category $\mathcal{Z}(G, \alpha)$ is pointed if and only if its group of invertible objects is of order $|G|^2$. By theorem 4.1.2.2, we have

$$|\hat{G}| = |G| = |Z_\alpha(G)|.$$ 

The first equality is equivalent to $G$ being Abelian, and the second is equivalent to $\alpha$ being soft. \(\blacksquare\)

**Remark 4.1.2.4.** For a pointed category $\mathcal{Z}(G, \alpha)$, the group of objects $P$ is a Lagrangian extension\(^{55}\) of $\hat{G}$ with respect to $\iota : \hat{G} \to P$ and the quadratic function (4.4).

\(^{54}\) See § 1.4.3.

\(^{55}\) See § 4.2.
To pursue our interest in pointed Drinfeld centers $Z(G, \alpha)$, we now assume that $G = B$ is Abelian. Denote by $Z_3^{\text{soft}}(B, k^*)$ the group of soft 3-cocycles. It is straightforward to see that all 3-coboundaries are soft. Write $H_3^{\text{soft}}(B, k^*) = Z_3^{\text{soft}}(B, k^*)/B_3(B, k^*)$.

**Lemma 4.1.2.5** ([13], Lemma 2.10). Let $B$ be an Abelian group. Then the third soft cohomology group $H_3^{\text{soft}}(B, k^*)$ is the kernel of the alternation map
\[
\text{Alt}_3 : H^3(B, k^*) \to \text{Hom}_\mathbb{Z}(\Lambda^3 B, k^*).
\]

**Proof.** The assignment $\alpha \mapsto \alpha(-, -| -)$ gives a homomorphism
\[
H^3(B, k^*) \to H^2(B, \text{Map}(B, k^*))
\]
into the second cohomology of $B$ with coefficients in the trivial $B$-module $\text{Map}(B, k^*)$ of set-theoretic functions $B \to k^*$. The soft cohomology $H_3^{\text{soft}}(B, k^*)$ is the kernel of this homomorphism.

Since $k^*$ and $\text{Map}(B, k^*)$ are divisible, the universal coefficient formula implies that the alternation map $\text{Alt}_2$ gives an isomorphism
\[
H^2(B, \text{Map}(B, k^*)) \to \text{Hom}(\Lambda^2 B, \text{Map}(B, k^*)).\]
The following commutative diagram
\[
\begin{array}{ccc}
0 & \to & H_3^{\text{soft}}(B, k^*) \\
& & \downarrow \text{Alt}_3 \\
& & H^3(B, k^*) \\
& & \downarrow \text{Alt}_2 \\
& & \text{Hom}(\Lambda^3 B, k^*) \\
& & \to \text{Hom}(\Lambda^2 B, \text{Map}(B, k^*))
\end{array}
\]
proves the result. $\blacksquare$

For an Abelian group $B$, define the category $\text{Mex}_p(\text{Rep}(B))$ to be the full categorical subgroup of $\text{Mex}(\text{Rep}(B))$ consisting of those modular extensions $\mathcal{D}$ of $\text{Rep}(B)$ that are pointed categories. The computations above prove the following:

**Theorem 4.1.2.6.** The group of connected components
\[
\pi_0\left(\text{Mex}_p(\text{Rep}(B))\right) = \text{Mex}_p(\text{Rep}(B)) \text{ fits into the short exact sequence}
\]
\[
\{0\} \to \text{Mex}_p(\text{Rep}(B)) \to H^3(B, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_\mathbb{Z}(\Lambda^3 B, \mathbb{Q}/\mathbb{Z}) \to \{0\}
\]
4.2 Lagrangian Extensions of Abelian Groups

4.2.1 The Categorical Group $\mathcal{L}ex(B)$

Let $B$ and $D$ be Abelian groups. An extension of $B$ by $D$ is a short exact sequence of Abelian groups

$$D \longrightarrow A \longrightarrow B.$$  

Denote by $\boxplus$ the Baer sum of extensions:

$$\left( D \overset{i}{\longrightarrow} A \overset{\pi}{\longrightarrow} B \right) \boxplus \left( D \overset{i'}{\longrightarrow} A' \overset{\pi'}{\longrightarrow} B \right) = \left( D \overset{i}{\longrightarrow} A \times_B A' / \tilde{\Delta}(D) \overset{p}{\longrightarrow} B \right),  \quad (4.5)$$

where $A \times_B A'$ is the fibered product, $\tilde{\Delta}(D)$ is the antidiagonal subgroup, $i(d) = (i(d), 0) = (0, i'(d))$, and $p(a, a') = \pi(a) = \pi'(a')$. The Baer sum is both associative and commutative, and the Baer sum of extensions is again an extension$^{56}$.

Let $B$ be an Abelian group. A Lagrangian extension of $B$ is a triple $(A, q, \iota)$, where $A$ is an Abelian group, $\iota : B \hookrightarrow A$ is an embedding of groups, and $q : A \to \mathbb{Q}/\mathbb{Z}$ is a nondegenerate quadratic function such that $\iota[B] \trianglelefteq A$ is a Lagrangian subgroup with respect to $q$. The collection of Lagrangian extensions of a given Abelian group $B$ forms a category which we denote by $\mathcal{L}ex(B)$. A morphism $(A, q, \iota) \to (A', q', \iota')$ in $\mathcal{L}ex(L)$ is a homomorphism $\omega : A \to A'$ such that $q' \circ \omega = q$ and $\omega \circ \iota = \iota'$.

Let $(A, q, \iota), (A', q', \iota') \in \mathcal{L}ex(B)$. Define the sum $(A, q, \iota) \boxplus (A', q', \iota')$ by

$$(A, q, \iota) \boxplus (A', q', \iota') = \left( A \times_{\tilde{\Delta}} A' / \tilde{\Delta}(B), q \boxplus q', \delta \right),$$

where

$$A \times_{\tilde{\Delta}} A' = \{(a, a') \in A \times A' \mid q(a + \iota(x)) + q'(a' - \iota'(x)) = q(a) + q'(a') \quad \forall x \in B\}$$

is the fibered product, and

$$\tilde{\Delta}(B) = \{(\iota(b), -\iota'(b)) \mid b \in B\}$$

$^{56}$ See § 4.3 for more information.
is the antidiagonal subgroup. Define a quadratic function $\tilde{q} : A \times A' \to \mathbb{Q}/\mathbb{Z}$ by $\tilde{q}(a, a') = q(a) + q'(a')$. The kernel of $\tilde{q}|_{A \times \hat{B} A'}$ is precisely $\tilde{\Delta}(B)$, and hence the restriction descends to the quotient $A \times \hat{B} A'/\tilde{\Delta}(B)$ and gives a nondegenerate function $q \boxplus q'$ on the quotient. Indeed, the orthogonal complement $\tilde{\Delta}(B)\perp$ in $A \times A'$ (with respect to $\tilde{q}$) is the fibered product $A \times \hat{B} A'$: for $(a, a') \in A \times A'$, the orthogonality condition $\tilde{\sigma}((a, a), (\iota(x), -\iota'(x))) = 0$ with $(\iota(x), -\iota'(x))$ is equivalent to $\sigma(a, \iota(x)) = \sigma'(a', \iota'(x))$, or simply $\pi(a) = \pi'(a')$.\footnote{Here, $\sigma$ and $\sigma'$ are the symmetric bilinear forms associated to $q$ and $q'$, respectively.} Finally, the embedding $\delta : B \to A \times \hat{B} A'/\tilde{\Delta}(B)$ is given by $\delta(b) = (\iota(b), 0) = (0, \iota'(b))$.

Symmetry of the operation $\boxplus$ is clear. Associativity of $\boxplus$ follows from associativity of the Baer sum (4.5). This works because the Baer sum is compatible with the quadratic structure on the extensions $A$ and $A'$.

By the trivial Lagrangian extension, we mean $(B \times \hat{B}, q_{\text{std}}, \iota)$ with $q_{\text{std}}(b, \chi) = \chi(b)$ and $\iota(b) = (b, 0)$. The trivial Lagrangian extension is the monoidal unit object in $\mathcal{L}ex(B)$. That is, $\mathcal{L}ex(B)$ is a symmetric monoidal groupoid.

Define the conjugate of the Lagrangian extension $(A, q, \iota)$ by $(A, q, \iota) = (A, -q, \iota)$. The conjugate is the dual object in $\mathcal{L}ex(B)$. This makes $\mathcal{L}ex(B)$ a symmetric categorical group with respect to $\boxplus$.

Now we shall discuss the functoriality of $\mathcal{L}ex(B)$. Namely, for a homomorphism of Abelian groups $f : B \to D$, we will define a functor between symmetric categorical groups $\mathcal{L}ex(B) \to \mathcal{L}ex(D)$. For a Lagrangian extension $(A, q, \iota)$ of $B$, set $A' = \frac{(A \times \hat{B} \hat{D}) \times D}{\tilde{\Delta}(B)}$.\footnote{Here, $\sigma$ and $\sigma'$ are the symmetric bilinear forms associated to $q$ and $q'$, respectively.}
Note that $A'$ fits into the commutative diagram with exact rows and columns

\[
\begin{array}{c}
D & \xrightarrow{\iota'} & (A \times \widehat{\bar{B}D}) \times D/\Delta(B) & \xrightarrow{\pi'} & \widehat{\bar{D}} \\
& & \downarrow & & \\
B & \xrightarrow{i} & A \times \widehat{\bar{B}D} & \xrightarrow{pr_2} & \widehat{\bar{D}} \\
& & \downarrow & & \\
& & & \pi & B
\end{array}
\]

Now define a Lagrangian extension of $D$

\[
\mathcal{L}ex(f)(A, q, \iota) = \left( \frac{(A \times \widehat{\bar{B}D}) \times D}{\Delta(B)}, q', \iota' \right),
\]

(4.6)

where $q' : A' \to \mathbb{Q}/\mathbb{Z}$ is defined as follows. The formula $q'(a, \chi, d) = q(a) + \chi(d)$ define a degenerate quadratic function on $(A \times \widehat{\bar{B}D}) \times D$ with kernel $\Delta(B)$.

**Remark 4.2.1.1.** Suppose now that $f$ is surjective, and let $K = \ker(f)$. It can be verified that in this case $A \times \widehat{\bar{B}D}$ coincides with the orthogonal complement $K^\perp$ of $K$ in $A$.

Moreover, the quotient $\frac{(A \times \widehat{\bar{B}D}) \times D}{\Delta(B)}$ coincides with the isotropic contraction $K^\perp/K$ along $K$. This contains $B/K$, which is isomorphic to $D$. Thus (4.6) reduces to

\[
\mathcal{L}ex(f)(A, q, \iota) = (K^\perp/K, q|_{K^\perp}, \iota')
\]

where $\iota' : D \cong B/K \to K^\perp/K$ is induced by $\iota : B \to K^\perp$.

Define a functor $\mathcal{L}ex(B) \to \mathcal{M}ex_{pl}(\mathcal{C}(B))$ by

\[
(A, q, \iota) \mapsto C(A, q).
\]

(4.7)

**Proposition 4.2.1.2** ([13], Proposition 3.3). The functor (4.7) is an equivalence of groupoids.

**Proof.** Indeed, any pointed modular extension of $\mathcal{C}(B)$ has the form $C(A, q)$ for some Lagrangian extension $(A, q)$ of $B$. Any equivalence of extensions $C(A, q) \to C(A', q')$ correponds to an isomorphism of Lagrangian extensions $(A, q) \to (A', \iota')$.  ■
The equivalence (4.7) allows us to transfer the monoidal structure $\odot_{C(B)}$ from $\text{Mex}_{pr}(C(B))$ to $\text{Lex}(B)$. Note first that the etale algebra $R = F(I) \in C(B) \boxtimes C(B)$ corresponds, under the standard identification $C(B) \boxtimes C(B) \simeq C(B \oplus B)$, to the antidiagonal subgroup $\bar{\delta}(B) = \{(b, -b) | b \in B\} \leq B \oplus B$. Indeed, the tensor product functor $C(B) \boxtimes C(B) \to C(B)$ corresponds to the direct image of the addition homomorphism $a : B \times B \to B$, $(x, y) \mapsto x + y$. Thus $R$ is the inverse image $a^*(I) = R(\bar{\delta}(B))$.

Now, the chain of braided equivalences

$$C(A, q) \odot_{C(B)} C(A', q') = (C(A, q) \boxtimes C(A', q'))_{\text{loc}}^{R} \simeq C(A \times A', q \times q')_{\text{loc}}^{R(\bar{\delta}(B))} \simeq C(\bar{\delta}(B)^+ / \bar{\delta}(B), q \times q')$$

identifies the product $C(A, q) \odot_{C(B)} C(A', q')$ with the sum $(A, q, i) \boxplus (A', q', i')$.

The equivalence (4.7) implies that the group $\text{Lex}(B)$ of Lagrangian extensions is isomorphic to $\text{Mex}_{pr}(C(B))$. Since $C(B) \simeq \mathcal{R}ep(\hat{B})$ as symmetric fusion categories, we have:

**Corollary 4.2.1.3.** The group $\text{Lex}(B)$ of Lagrangian extensions of $B$ is isomorphic to the group $\text{Mex}_{pr}(\mathcal{R}ep(\hat{B}))$.

**Proof.** Follows immediately from theorem 4.1.2.6 and proposition 4.2.1.2. □

**Corollary 4.2.1.4.** The group $\text{Lex}(\hat{B})$ fits in the short exact sequence

$$\{0\} \to \text{Lex}(\hat{B}) \to H^3(B, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_{\mathbb{Z}}(\Lambda^3 B, \mathbb{Q}/\mathbb{Z}) \to \{0\}$$

**Proof.** Follows immediately from theorem 4.1.2.6, proposition 4.2.1.2, and corollary 4.2.1.3. □

**Remark 4.2.1.5.** Here is an alternative description of the group $\text{Lex}(\hat{B})$. In [3], it was proved that for a finite Abelian group $B$, there is a short exact sequence

$$\{0\} \to \Lambda^3 B \to H_3(B, \mathbb{Z}) \to \text{Tor}^\mathbb{Z}_1(B, B)^{\mathbb{Z}} \to \{0\}$$
Here, \( \text{Tor}_1^\mathbb{Z}(B, B)^{S_2} \) is the subgroup of \( \text{Tor}_1^\mathbb{Z}(B, B) \) consisting of \( S_2 \)-anti-invariants.

Applying the contravariant functor \( \text{Hom}_\mathbb{Z}(-, \mathbb{Q}/\mathbb{Z}) \) to this sequence yields

\[
\{0\} \to \text{Hom}_\mathbb{Z}\left(\text{Tor}_1^\mathbb{Z}(B, B)^{S_2}, \mathbb{Q}/\mathbb{Z}\right) \to \text{Hom}_\mathbb{Z}(H_3(B, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_\mathbb{Z}(\Lambda^3 B, \mathbb{Q}/\mathbb{Z}) \to \{0\}
\]

By the Universal Coefficient Theorem\(^{59}\), there is a short exact sequence

\[
\{0\} \to \text{Ext}^1_\mathbb{Z}(H_2(B, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \to H^3(B, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_\mathbb{Z}(H_3(B, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \to \{0\}
\]

Since \( \mathbb{Q}/\mathbb{Z} \) is a divisible group, it is an injective\(^{60}\) \( \mathbb{Z} \)-module, whence \( \text{Ext}^1_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) \) is trivial for any Abelian group \( M \). Thus the above short exact sequence gives an isomorphism \( H^3(B, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}_\mathbb{Z}(H_3(B, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \). The commutative diagram with short exact rows

\[
\begin{array}{cccccc}
\text{Lex}(\hat{B}) & \longrightarrow & H^3(B, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Hom}_\mathbb{Z}(\Lambda^3 B, \mathbb{Q}/\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
\text{Hom}_\mathbb{Z}\left(\text{Tor}_1^\mathbb{Z}(B, B)^{S_2}, \mathbb{Q}/\mathbb{Z}\right) & \longrightarrow & \text{Hom}_\mathbb{Z}(H_3(B, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Hom}_\mathbb{Z}(\Lambda^3 B, \mathbb{Q}/\mathbb{Z})
\end{array}
\]

shows that \( \text{Lex}(\hat{B}) \) is isomorphic to the dual of the group \( \text{Tor}_1^\mathbb{Z}(B, B)^{S_2} \).

**Remark 4.2.1.6.** For any positive integer \( n \), the third cohomology group

\( H^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \). Since \( \mathbb{Z}/n\mathbb{Z} \) is cyclic, \( \Lambda^3(\mathbb{Z}/n\mathbb{Z}) = \{0\} \). Therefore the short exact sequence

\[
\{0\} \to \text{Lex}(\mathbb{Z}/n\mathbb{Z}) \to H^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_\mathbb{Z}(\Lambda^3(\mathbb{Z}/n\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \to \{0\}
\]

gives an isomorphism \( \text{Lex}(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \).

**Example 4.2.1.7.** Here, we explicitly compute the group of Lagrangian extensions

\( \text{Lex}(\mathbb{Z}/2\mathbb{Z}) \). Since a Lagrangian extension \( A \supseteq \mathbb{Z}/2\mathbb{Z} \) is an Abelian group extension of \( \mathbb{Z}/2\mathbb{Z} \) by \( \mathbb{Z}/2\mathbb{Z} \), \( A \) is of order 4, and either \( A \cong (\mathbb{Z}/2\mathbb{Z})^2 \) or \( A \cong \mathbb{Z}/4\mathbb{Z} \).

\(^{58}\) Here, we implicitly use the fact that \( \mathbb{Q}/\mathbb{Z} \) is an injective \( \mathbb{Z} \)-module, and so its contravariant Hom functor preserves short exact sequences.

\(^{59}\) See, e.g., [22].

\(^{60}\) For the basic facts on injective modules, see, e.g., [22, § IV.3].
Let \( A = (\mathbb{Z}/2\mathbb{Z})^2 \). The matrices of possible nondegenerate symmetric bilinear forms \( \sigma : A \times A \to \mathbb{Q}/\mathbb{Z} \) such that the first summand \( \mathbb{Z}/2\mathbb{Z} \) is Lagrangian are
\[
\begin{bmatrix}
0 & \frac{1}{2} \\
\frac{1}{2} & 1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & \frac{1}{2} \\
\frac{1}{2} & 0
\end{bmatrix}.
\]
Here, entries are in \( \mathbb{Q}/\mathbb{Z} \). For the first one, a compatible quadratic function \( q_\pm : A \to \mathbb{Q}/\mathbb{Z} \) has the form
\[
q_\pm(e_1) = 0, \quad q_\pm(e_2) = \pm \frac{1}{4}, \quad q_\pm(e_1 + e_2) = \mp \frac{1}{4}.
\]
The corresponding classes in \( \text{Lex}(\mathbb{Z}/2\mathbb{Z}) \) are isomorphic and nontrivial. The second bilinear form corresponds to the trivial Lagrangian extension
\[
(\mathbb{Z}/2\mathbb{Z} \times \hat{\mathbb{Z}}/2\mathbb{Z}, q_{\text{std}}, \text{canonical inclusion}).
\]
It follows from the following example 4.2.1.8 that the group \( A = \mathbb{Z}/4\mathbb{Z} \) is not realizable as a Lagrangian extension of \( \mathbb{Z}/2\mathbb{Z} \).

**Example 4.2.1.8.** More generally, let \( B = \mathbb{Z}/2^\ell\mathbb{Z} \) for some positive integer \( \ell \). As in example 4.2.1.7 above, a Lagrangian extension \( A \supseteq \mathbb{Z}/2^\ell\mathbb{Z} \) is an Abelian group extension of \( \mathbb{Z}/2^\ell\mathbb{Z} \) by \( \mathbb{Z}/2^\ell\mathbb{Z} \), whence \( |A| = 2^{2\ell} \). Suppose that \( A = \mathbb{Z}/2^{2\ell}\mathbb{Z} \), and consider the short exact sequence
\[
\begin{array}{cccc}
0 & \to & \mathbb{Z}/2^\ell\mathbb{Z} & \xrightarrow{2^\ell} \mathbb{Z}/2^{2\ell}\mathbb{Z} & \xrightarrow{1_{2\ell}} & \mathbb{Z}/2^\ell\mathbb{Z} & \to & 0
\end{array}
\]
Because subgroups of finite cyclic groups of a given order are unique, a Lagrangian subgroup of the middle term is unique if it exists. A quadratic function \( q : \mathbb{Z}/2^{2\ell}\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \) is determined by its value at 1, and this must be of the form
\[
q(1) = \frac{k}{2^{2\ell+1}}
\]
for some integer \( k \). Since the image of \( \mathbb{Z}/2^\ell\mathbb{Z} \) must be Lagrangian with respect to such \( q \), we have \( q(2^\ell) \equiv 0 \pmod{\mathbb{Z}} \), which implies that \( 2^{2\ell} q(1) \equiv 0 \pmod{\mathbb{Z}} \). Thus
\[ \frac{2^\ell k}{2^{\ell+1}} \equiv 0 \pmod{\mathbb{Z}}, \text{ and so } \frac{k}{2} \equiv 0 \pmod{\mathbb{Z}}. \] Therefore \( k \) is even. Write \( k = 2m \) for some integer \( m \). We have

\[ q(1) = \frac{m}{2^\ell}. \]

Now

\[ \sigma(1, 1) = q(2) - 2q(1) = 4q(1) - 2q(1) = 2q(1) = \frac{m}{2^{\ell-1}}. \]

But then

\[ \sigma(2^{\ell-1}, 1) = 2^{\ell-1} \sigma(1, 1) = 2^{\ell-1} \frac{m}{2^{\ell-1}} = m \equiv 0 \pmod{\mathbb{Z}}, \]

so \( 0 \neq 2^{\ell-1} \in \ker(\sigma) \) and hence \( \sigma \) is degenerate. Therefore \( \mathbb{Z}/2^\ell\mathbb{Z} \) is not realizable as a Lagrangian extension of \( \mathbb{Z}/2^\ell\mathbb{Z} \).

**Example 4.2.1.9.** Let now \( A = \mathbb{Z}/2^{\ell-1}\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). Set \( e_1 = (1, 0), \ e_2 = (0, 1) \), so that \( A = \langle e_1, e_2 \rangle \), and set \( L = \langle 2^{\ell-1} e_1 + e_2 \rangle \). We wish to determine possible quadratic functions \( q : A \to \mathbb{Q}/\mathbb{Z} \) with respect to which the subgroup \( L \subseteq A \) is Lagrangian. Since any element \( \vec{x} \in A \) has the form \( \vec{x} = ae_1 + be_2 \) for some \( a, b \in \mathbb{Z} \), such a function \( q \) has the form

\[ q(a, b) = a^2 q(e_1) + b^2 q(e_2) + ab \sigma(e_1, e_2), \quad (4.8) \]

where \( \sigma : A \times A \to \mathbb{Q}/\mathbb{Z} \) is the symmetric bilinear form associated to \( q \). In order that \( q \) be well-defined, we must have \( q(a + 2^{\ell-1} b) = q(a, b) \) and \( q(a, b + 2) = q(a, b) \) for all \( (a, b) \in A \). Therefore

\[ 2^{\ell} q(e_1) = 0, \ 4q(e_2) = 0, \ 2\sigma(e_1, e_2) = 0. \quad (4.9) \]

The quadratic condition on such \( q \) requires that

\[ q((a, b) + (c, d)) = q(a, b) + q(c, d) + \sigma((a, b), (c, d)) \quad \forall \ (a, b), (c, d) \in A. \]

Therefore

\[ 2q(e_i) = \sigma(e_i, e_i), \quad i \in \{1, 2\}. \quad (4.10) \]
Combining the conditions (4.9) and (4.10), we may set
\[ q(e_1) = \frac{1}{2^{2\ell}}, \quad q(e_2) = \pm \frac{1}{4}. \]

To remove the indeterminacy in \( q(e_2) \), we apply the Lagrangian condition. Since the subgroup \( L \leq A \) has order \( 2^{\ell}, |L|^2 = |A| \). Therefore \( L \) is Lagrangian with respect to \( q \) if and only if \( L \) is isotropic with respect to \( q \). Isotropy of \( L \) amounts to the condition
\[ q\left(2^{2\ell-1}e_1 + e_2\right) = 0. \]

By (4.8),
\[ q\left(2^{\ell-1}e_1 + e_2\right) = 2^{2\ell-2}q(e_1) + q(e_2) \]
\[ \implies \frac{2^{2\ell-2}}{2^{2\ell}} \pm \frac{1}{4} = \frac{1}{4} \pm \frac{1}{4} = 0 \iff q(e_2) = -\frac{1}{4}. \]

Therefore (4.8) becomes
\[ q(a, b) = \frac{a^2}{2^{2\ell}} - \frac{b^2}{4} + ab\sigma(e_1, e_2). \quad (4.11) \]

It remains to determine \( \sigma(e_1, e_2) \). By (4.9), \( \sigma(e_1, e_2) \in \left\{ 0, \frac{1}{2} \right\} \). We consider two cases.

**Case I:** \( \sigma(e_1, e_2) = 0 \). Then \( \sigma \) has the matrix representation
\[
\begin{bmatrix}
\frac{1}{2^{\ell-1}} & 0 \\
0 & \frac{1}{2}
\end{bmatrix}.
\]

Let \((a, b) \in \ker(\sigma)\). Then \( \sigma((a, b), (x, y)) = 0 \) for all \((x, y) \in A\). By bilinearity of \( \sigma \), this condition amounts to
\[ ax\sigma(e_1, e_1) + by\sigma(e_2, e_2) = 0 \quad \forall (x, y) \in A, \]

viz.,
\[ \frac{ax}{2^{2\ell-1}} + \frac{by}{2} \equiv 0 \pmod{\mathbb{Z}} \quad \forall (x, y) \in A. \]
In particular, for \((x, y) = (0, 1)\), we have \(\frac{b}{2} \equiv 0 \pmod{\mathbb{Z}}\), i.e., \(b \equiv 0 \pmod{2}\). Similarly, setting \((x, y) = (1, 0)\), we get that \(a \equiv 0 \pmod{2^{\ell - 1}}\), and therefore \((a, b) = (0, 0)\). Therefore \(\sigma\) is nondegenerate.

In this case, we can define \(q : A \to \mathbb{Q}/\mathbb{Z}\) according to the formula (4.11):

\[
q(a, b) = \frac{a^2}{2^\ell} - \frac{b^2}{4}.
\]

**Case II:** \(\sigma(e_1, e_2) = \frac{1}{2}\). Then \(\sigma\) has the matrix representation

\[
\begin{pmatrix}
1 & 1 \\
2^{\ell-1} & 2 \\
1 & 1 \\
\end{pmatrix}.
\]

Let \((a, b) \in \text{ker}(\sigma)\). Then by bilinearity of \(\sigma\), we have

\[
\frac{ax}{2^{2\ell - 1}} + \frac{ay}{2} + \frac{bx}{2} + \frac{by}{2} \equiv 0 \pmod{\mathbb{Z}} \quad \forall (x, y) \in A.
\]

In particular, for \((x, y) = (0, 1)\), we have

\[
\frac{a + b}{2} \equiv 0 \pmod{\mathbb{Z}},
\]

so that either both \(a\) and \(b\) are even or both \(a\) and \(b\) are odd. Setting \((x, y) = (1, 0)\) yields

\[
\frac{a}{2^{2\ell - 1}} + \frac{b}{2} \equiv 0 \pmod{\mathbb{Z}}.
\]

(4.12)

Assume by way of contradiction that \(a\) and \(b\) are both odd. Write \(a = 1 + 2a', b = 1 + 2b'\) for some \(a', b' \in \mathbb{Z}\). Substituting these values into (4.12) yields

\[
\frac{1 + 2a'}{2^{2\ell - 1}} + \frac{1 + 2b'}{2} \equiv 0 \pmod{\mathbb{Z}}.
\]

(4.13)

Rewriting (4.13) in terms of the common denominator \(2^{2\ell - 1}\) yields

\[
\frac{1 + 2a' + 2^{2\ell - 2} + 2^{2\ell - 1}b'}{2^{2\ell - 1}} \equiv 0 \pmod{\mathbb{Z}}.
\]

(4.14)

Since the LHS of (4.14) is an integer, its numerator must be even, yet its numerator is clearly odd\(^{61}\) \((\Rightarrow \Leftarrow\). Therefore we cannot have that \(a\) and \(b\) are odd, so we must have that

\(^{61}\) Here, we assume without loss of generality that \(\ell \neq 1\), since example 4.2.1.7 (q.v.) already disposed of that case.
a and b are even. But b even means that $b \equiv 0 \pmod{2}$, i.e., $b = 0$, so by (4.12),

$$\frac{a}{2^{2\ell-1}} \equiv 0 \pmod{\mathbb{Z}},$$

i.e., $a \equiv 0 \pmod{2^{2\ell-1}}$. Therefore $a = 0$ as well, and therefore $\sigma$ is nondegenerate.

In this case, we can define $q : A \to \mathbb{Q}/\mathbb{Z}$ according the the formula (4.11):

$$q(a, b) = \frac{a^2}{2^{2\ell}} - \frac{b^2}{4} + \frac{ab}{2}.$$

In both cases, the subgroup $L \leq A$ generated by the pair $(2^{\ell-1}, 1)$ is Lagrangian with respect to $q$ and is isomorphic to $\mathbb{Z}/2^\ell\mathbb{Z}$.

### 4.3 The Categorical Group $\mathcal{E}xt(B, D)$

Let $B$ and $D$ be Abelian groups. Denote by $\mathcal{E}xt(B, D)$ the category whose objects are Abelian group extensions $D \hookrightarrow A \xrightarrow{\pi} B$ and whose morphisms are group homomorphisms $\xi : A \to A'$ making the diagram commute. Note that such $\xi$ is necessarily an isomorphism, i.e., $\mathcal{E}xt(B, D)$ is a groupoid.

The category $\mathcal{E}xt(B, D)$ is monoidal with respect to the Baer sum (4.5). The unit object in $\mathcal{E}xt(B, D)$ is the trivial extension

$$D \to D \times B \to B,$$
with the canonical embedding and projection maps. Associativity of the operation \( \boxplus \) is given by the following diagram:

\[
\begin{array}{ccc}
D & \xrightarrow{i_1} & \left( A \times_B \left( A' \times_B A'' / \tilde{\Delta}(D) \right) \right) / \tilde{\Delta}(D) & \xrightarrow{p_1} & B \\
\downarrow \alpha & & \downarrow \alpha & & \\
D & \xrightarrow{i_2} & \left( \left( A \times_B A' / \tilde{\Delta}(D) \right) \times_B A'' \right) / \tilde{\Delta}(D) & \xrightarrow{p_2} & B
\end{array}
\]

Consider the triple fibered product

\[ A \times_B A' \times_B A'' = \left\{ (a, a', a'') \in A \times A' \times A'' \mid \pi(a) = \pi'(a') = \pi''(a'') \right\}. \]

Define a subgroup \( \nabla(D) \leq A \times_B A' \times_B A'' \) by

\[ \nabla(D) = \left\{ (0, \iota'(x), -\iota''(x)) + (\iota(y), \iota'(y), 0) \mid x, y \in D \right\}. \]

There are natural maps \( A \times_B A' \times_B A'' / \nabla(D) \to \left( A \times_B \left( A' \times_B A'' / \tilde{\Delta}(D) \right) \right) / \tilde{\Delta}(D) \) and

\[ A \times_B A' \times_B A'' / \nabla(D) \to \left( \left( A \times_B A' / \tilde{\Delta}(D) \right) \times_B A'' \right) / \tilde{\Delta}(D) \]

that commute with \( \alpha \). We equip the product \( A \times A' \times A'' \) with the quadratic function \( \bar{q} = q \oplus q' \oplus q'' \) and its associated bilinear form \( \bar{\sigma} = \sigma \oplus \sigma' \oplus \sigma'' \), and consider the orthogonal complement \( \nabla(D)^\perp \) in \( A \times A' \times A'' \). We claim that \( \nabla(D)^\perp = A \times_B A' \times_B A'' \). Let \( (a, a', a'') \in \nabla(D)^\perp \). Then for any \( x, y \in D \), we have

\[ \bar{\sigma}((a, a', a''), (\iota(x), \iota'(x), 0)) = 0 = \bar{\sigma}((a, a', a''), (0, \iota'(y), \iota''(y))). \]

The first equality implies that \( \sigma(a, \iota(x)) = \sigma'(a', \iota'(x)) \) for any \( x \in D \), i.e., \( \pi(a) = \pi'(a') \).

The second equality implies that \( \sigma'(a', \iota'(y)) = \sigma''(a'', \iota''(y)) \) for any \( y \in D \), i.e.,

\( \pi'(a') = \pi''(a'') \). It follows that \( (a, a', a'') \in A \times_B A' \times_B A'' \). It follows that the operation \( \boxplus \) is associative up to isomorphism. The \textit{opposite extension} to \( D \xrightarrow{i} A \xrightarrow{\pi} B \) is the extension \( D \xrightarrow{i} A \xrightarrow{\pi} B \). Note that the Baer sum

\[ \left( D \xrightarrow{i} A \xrightarrow{\pi} B \right) \boxplus \left( D \xrightarrow{i} A \xrightarrow{-\pi} B \right) \]
is isomorphic to the trivial extension. Altogether, this makes the category $\mathcal{E}xt(B, D)$ into a categorical group with respect to $\boxplus$. The standard invariants of the categorical group $\mathcal{E}xt(B, D)$ are $\pi_0(\mathcal{E}xt(B, D)) = \text{Ext}_2^Z(B, D)$ and $\pi_1(\mathcal{E}xt(B, D)) = \text{Hom}_2^Z(B, D)$.

$\mathcal{E}xt$ is functorial in each of its arguments; it is contravariant in its first argument and covariant in its second argument.

### 4.3.1 Wherein the Arguments of $\mathcal{E}xt$ are Dual to One Another

In the special case when the arguments $B$ and $D$ are dual to each other, the categorical group $\mathcal{E}xt(B, D)$ has a natural symmetry. Namely, define the contravariant functor $T : \mathcal{E}xt(\widehat{B}, B) \to \mathcal{E}xt(\widehat{B}, B)$ by

$$T \left( B \xrightarrow{i} A \xrightarrow{\pi} \widehat{B} \right) = \left( B \xrightarrow{\widehat{\pi} \circ \text{ev}_B} \widehat{A} \xrightarrow{\widehat{i}} \widehat{B} \right),$$

where $\text{ev}_B : B \to \widehat{B}$ is the canonical evaluation isomorphism. The functor $T$ is an involutive autoequivalence. More precisely, we have a natural isomorphism $i : \text{Id} \to T^2$ (given by $\text{ev}_A : A \to \widehat{A}$) such that $i_{T(X)} = T(i_X)^{-1}$. Namely, let $X \in \mathcal{E}xt(\widehat{B}, B)$ be the Lagrangian extension $B \xrightarrow{i} A \xrightarrow{\pi} \widehat{B}$, so that $T(X)$ is the extension

$$\widehat{B} \xrightarrow{\widehat{\pi}} \widehat{A} \xrightarrow{\widehat{i}} \widehat{B}.$$

Define a map $X \to T^2(X)$ by the following diagram:

$$
\begin{array}{ccc}
B & \xrightarrow{i} & A \xrightarrow{\pi} \widehat{B} \\
\downarrow{\text{ev}_B} & & \downarrow{\text{ev}_A} \\
\widehat{B} & \xrightarrow{\widehat{i}} & \widehat{A} \xrightarrow{\widehat{\pi}} \widehat{B}
\end{array}
$$

The left-hand square of this diagram commutes due to naturality of the evaluation map $\text{ev}_{(\cdot)} : \text{Id} \to \widehat{\cdot}$. Note that $\widehat{\text{ev}_B} = \text{ev}_B^{-1}$. Indeed, $\widehat{\text{ev}_B} : \chi \mapsto (\text{ev}(a) \mapsto \text{ev}(a)(\chi) = \chi(a))$ for $a \in B$, whence $\widehat{\text{ev}_B} \circ \text{ev}_B : \chi \mapsto (a \mapsto \chi(a))$ for $a \in B$. This implies that the right-hand square of the diagram also commutes, and therefore the extensions $X$ and $T^2(X)$ are equivalent.
More generally, let $\mathcal{G}$ be a categorical group. For any $X \in \mathcal{G}$, there is a map $\text{Aut}_{\mathcal{G}}(I) \to \text{Aut}_{\mathcal{G}}(X)$ defined as follows. For $\alpha \in \text{Aut}_{\mathcal{G}}(I)$, define $a \in \text{Aut}_{\mathcal{G}}(X)$ by the diagram

\[
\begin{array}{c}
I \otimes X \xrightarrow{\lambda_X} X \\
\alpha \otimes I \downarrow \downarrow a \\
I \otimes X \xrightarrow{\lambda_X} X
\end{array}
\]

Here, $\lambda$ denotes the left unit isomorphism in $\mathcal{G}$. If $\mathcal{G}$ is a categorical group, then the above map is an isomorphism. Let $g_X : \text{Aut}_{\mathcal{G}}(X) \to \text{Aut}_{\mathcal{G}}(I)$ denote the inverse of this map.

Let $T : \mathcal{G} \to \mathcal{G}$ be a contravariant monoidal autoequivalence with $i : \text{Id} \to T^2$ a natural isomorphism such that $i_{T(X)} = T(i_X)^{-1}$ for any $X \in \mathcal{G}$. We say that an object $X \in \mathcal{G}$ is $T$-equivariant if there is an isomorphism $x : X \to T(X)$ making the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{i_X} & T^2(X) \\
\downarrow x & & \downarrow T(x) \\
T(X) & & \end{array}
\]

(4.15)

Let $\tau$ be the effect of $T$ on $\pi_0(\mathcal{G})$, and denote by $\pi_0(\mathcal{G})^{\tau}$ the subgroup of classes of $T$-equivariant objects.

**Proposition 4.3.1.1** ([13], Proposition 3.8). The sequence

\[
\begin{array}{c}
0 \longrightarrow \pi_0(\mathcal{G})^\tau_0 \longrightarrow \pi_0(\mathcal{G})^\tau \longrightarrow H^1(\mathbb{Z}/2\mathbb{Z}, \pi_1(\mathcal{G})) \longrightarrow 0
\end{array}
\]

is exact.

**Proof.** We say that the class of an object $X \in \mathcal{G}$ is $\tau$-invariant if there is a morphism $x : X \to T(X)$. Define $a = a(x) \in \text{Aut}_{\mathcal{G}}(X)$ as the counterclockwise composition of the diagram (4.15). Let $\alpha(x) \in \text{Aut}_{\mathcal{G}}(I) = \pi_1(\mathcal{G})$ be the corresponding automorphism of the monoidal unit $I$. Then we have $\tau(\alpha) = T(\alpha) = T(a)$. Moreover, for any $Y \in \mathcal{G}$ and any morphism $x : X \to Y$, we have $g_Y(a) = g_1(xax^{-1})$. 
Applying the functor $T$ to the diagram (4.15) gives

\[
\begin{array}{c}
X \\
\downarrow \scriptstyle i_X \\
T^2(X) \\
\downarrow \scriptstyle T(x) \\
T^3(X) \\
\downarrow \scriptstyle T(i_X) \\
T(X)
\end{array}
\]

(4.16)

Note that $T(a) \in \text{Aut}_G(T(X))$ is the clockwise composition of the diagram (4.16). We claim that

\[
T(a) = x a^{-1} x^{-1}
\]

(4.17)

In terms of elements of $\text{Aut}_G(I)$ we have $\tau(a) = \alpha^{-1}$.

Consider the diagram

\[
\begin{array}{c}
X \\
\downarrow \scriptstyle i_X \\
T^2(X) \\
\downarrow \scriptstyle T(x) \\
T^3(X) \\
\downarrow \scriptstyle T(i_X) \\
T(X)
\end{array}
\]

(4.18)

The rightmost cell of (4.18) commutes by definition of $T(a)$. The bottom left cell commutes by definition of $a$. The top center cell commutes by naturality of $i$, and so the diagram commutes overall. (4.17) follows from commutativity of (4.18).

Now let $X \in G$ be $\tau$-invariant with an isomorphism $x : X \to T(X)$. Let $x' : X \to T(X)$ be another isomorphism. Define $b \in \text{Aut}(X)$ by the diagram

\[
\begin{array}{c}
X \\
\downarrow \scriptstyle b \\
X
\end{array}
\]

\[
\begin{array}{c}
X \xrightarrow{x} T(X) \\
\downarrow \scriptstyle b \\
X \xrightarrow{x'} T(X)
\end{array}
\]

\[
\begin{array}{c}
X \\
\downarrow \scriptstyle x' \\
X
\end{array}
\]
It follows from the diagram

![Diagram](image-url)

that \( a(x') = a(x)xT(b)^{-1}x^{-1}b \). This translates to \( \alpha(x') = \alpha(x)\tau(\beta)^{-1}\beta \), where \( \beta \in \text{Aut}_G(I) \) is the automorphism corresponding to \( b \in \text{Aut}_G(X) \).

Thus we have a map

\[
\pi_0(G)^\tau \to H^1(\mathbb{Z}/2\mathbb{Z}, \pi_1(G))
\]

sending \( X \in \pi_0(G)^\tau \) to the class of \( \alpha(x) \) in

\[
H^1(\mathbb{Z}/2\mathbb{Z}, \pi_1(G)) = \{ \alpha \in \pi_1(G) | \tau(\alpha) = \alpha^{-1}/[\beta \tau(\beta)^{-1} | \beta \in \pi_1(G)] \}.
\]

Now we check that this map is a group homomorphism. Given \( X, Y \in \pi_0(G)^\tau \), choose \( x : X \to T(X) \) and \( y : Y \to T(Y) \). Then

\[
X \otimes Y \xrightarrow{x \otimes y} T(X) \otimes T(Y) \xrightarrow{T_{xy}} T(X \otimes Y)
\]
is a weak equivariance structure for $X \otimes Y$. Compute $a(x \otimes y) = a(x|y)$ according to the following diagram:

![Diagram](image-url)

It follows from the above diagram that $a(x|y) = a(x) \otimes a(y)$. Finally, by the construction, the class of $\alpha(x)$ is trivial if and only if a $\tau$-invariant structure on $X$ can be promoted to a $T$-equivariant one.

The effect of $T$ on $\pi_0(\text{Ext}(\widehat{B}, B)) = \text{Ext}_{\mathbb{Z}}^1(\widehat{B}, B)$ is an involution

$\tau = \pi_0(T) : \text{Ext}_{\mathbb{Z}}^1(\widehat{B}, B) \to \text{Ext}_{\mathbb{Z}}^1(\widehat{B}, B)$, which sends an extension of $\widehat{B}$ by $B$ to its dual.

Upon identification of the double dual with $B$, the dual extension becomes another extension of $\widehat{B}$ by $B$.

If we identify\textsuperscript{62} $\text{Hom}_{\mathbb{Z}}(\widehat{B}, B)$ with the tensor square $B \otimes B$, then the effect of $T$ on $\pi_1(\text{Ext}(\widehat{B}, B))$ becomes the permutation involution.

**Remark 4.3.1.2.** The above proposition gives an exact sequence

\[ 0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(\widehat{B}, B)_0^\tau \rightarrow \text{Ext}_{\mathbb{Z}}^1(\widehat{B}, B)^\tau \rightarrow H^1(\mathbb{Z}/2\mathbb{Z}, B \otimes B) \]

\textsuperscript{62} Using the fact that $B$ is finite.
The last group is

\[ H^1(\mathbb{Z}/2\mathbb{Z}, B \otimes B) = \{ \alpha \in B \otimes B \mid \tau(\alpha) = -\alpha \} / \{ \beta - \tau(\beta) \mid \beta \in B \otimes B \}, \]

with \( \tau \) acting as the transposition of tensor factors. Define a map\(^63\) \( B_2 \to H^1(\mathbb{Z}/2\mathbb{Z}, B \otimes B) \) by \( b \mapsto b \otimes b \). This map is an isomorphism. To see that, first note that the functor \( B \mapsto H^1(\mathbb{Z}/2\mathbb{Z}, B \otimes B) \) is linear. By linearity, it suffices to show that \( B_2 \to H^1(\mathbb{Z}/2\mathbb{Z}, B \otimes B) \) is an isomorphism for cyclic \( B \). This is obviously true when \( B \) is of odd order, since both \( B_2 \) and \( H^1(\mathbb{Z}/2\mathbb{Z}, B \otimes B) \) are equal to \( \{0\} \) in that case. For \( B = \mathbb{Z}/2^r\mathbb{Z} \), we have

\[ Z^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2^r\mathbb{Z} \otimes \mathbb{Z}/2^r\mathbb{Z}) = (\mathbb{Z}/2^r\mathbb{Z})^2 \quad \text{and} \quad B^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2^r\mathbb{Z} \otimes \mathbb{Z}/2^r\mathbb{Z}) = \{0\}, \]

whence the map \( B_2 \to H^1(\mathbb{Z}/2\mathbb{Z}, B \otimes B) \) is indeed an isomorphism.

Altogether, this gives an exact sequence

\[ \{0\} \longrightarrow \mathcal{E}xt^1_{\mathbb{Z}}(\hat{B}, B) \longrightarrow \mathcal{E}xt^1_{\mathbb{Z}}(\hat{B}, B) \longrightarrow B_2 \]

4.4 Comparing \( \mathcal{L}ex(B) \) with \( \mathcal{E}xt(\hat{B}, B) \)

4.4.1 \( F \)'n' \( \phi \)

Define a functor \( F : \mathcal{L}ex(B) \to \mathcal{E}xt(\hat{B}, B) \) by

\[ F(A, q, \iota) = \left( B \xrightarrow{\iota} A \xrightarrow{\pi} \hat{B} \right), \]

where \( (\pi(a))(x) = q(a + \iota(x)) - q(a) \forall a \in A, x \in B \). Note that \( F \) is monoidal with the obvious monoidal structure.

The functor \( F \) induces a homomorphism of Abelian groups \( \phi : \mathcal{L}ex(B) \to \mathcal{E}xt^1_{\mathbb{Z}}(\hat{B}, B) \).

First, we examine the image of \( \phi \).

**Proposition 4.4.1.1** ([13], Proposition 3.10). The extension \( F(A, q, \iota) \) is \( T \)-equivariant.

\(^{63}\) Note that the condition \( 2b = 0 \) implies that \( \tau(b \otimes b) = -b \otimes b \).
Proof. Define $f : A \to \hat{A}$ by $f(a) = \sigma(a, -)$ for $a \in A$. Here, $\sigma$ is the polarization of $q$.

Such $f$ makes the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{i} & A \\
\downarrow{e} & & \downarrow{\pi} \\
\hat{B} & \xrightarrow{f} & \hat{A} \\
\end{array}
\]

commute. Indeed, for $x \in B$ and $a \in A$, we have

\[
(f(\iota(x)))(a) = \sigma(\iota(x), a) = (\pi(a))(x) = e(x)(\pi(a)) = \hat{\pi}(e(x))(a),
\]

and

\[
(\pi(a))(x) = \sigma(a, x) = (f(a))(x) = (\hat{\iota}(f(a)))(x).
\]

It is straightforward to check that $f$ is a $T$-equivariant structure on $F(A, q, \iota)$. \qed

In other words, proposition 4.4.1.1 shows that the homomorphism $\phi$ lands in $\text{Ext}^1_Z(\hat{B}, B)^\tau$, with the image of $\phi$ being precisely $\text{Ext}^1_Z(\hat{B}, B)_0$. Denote by $K(B)$ and $C(B)$ the kernel and the cokernel, respectively, of the homomorphism $\phi : \text{Lex}(B) \to \text{Ext}^1_Z(\hat{B}, B)^\tau$. Clearly, $K$ and $C$ are functors from the category of finite Abelian groups to itself that fit into the exact sequence

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K(B) & \longrightarrow & \text{Lex}(B) & \phi & \longrightarrow & \text{Ext}^1_Z(\hat{B}, B)^\tau & \longrightarrow & C(B) & \longrightarrow & 0
\end{array}
\]

Next, we compute the polarization of the functor $\text{Lex}$. Note that for an extension

\[
\begin{array}{ccc}
0 & \longrightarrow & B \\
\downarrow{\iota} & & \downarrow{\pi} \\
& \longrightarrow & \hat{D} \\
\end{array}
\]

the embedding\footnote{The direct sum of $\iota$ and $\hat{\pi} \circ \text{ev}^{-1}$.} $B \oplus D \hookrightarrow E \oplus \hat{E}$ is a Lagrangian subgroup of the quadratic group $(E \oplus \hat{E}, q_{\text{std}})$. In other words, $(E \oplus \hat{E}, q_{\text{std}})$ becomes a Lagrangian extension of $B \oplus D$. This allows us to define a map $\nu : \text{Ext}^1_Z(\hat{D}, B) \to \text{Lex}(B \oplus D)$ by

\[
\nu \left( B \xrightarrow{i} E \xrightarrow{\pi} \hat{D} \right) = \left( E \oplus \hat{E}, q_{\text{std}}, \iota \right).
\]
Proposition 4.4.1.2 ([13], Proposition 3.11). The sequence

\[ 0 \longrightarrow \text{Ext}^1_\mathcal{Z} (\hat{D}, B) \stackrel{\nu}{\longrightarrow} \text{Lex}(B \oplus D) \stackrel{\kappa}{\longrightarrow} \text{Lex}(B) \oplus \text{Lex}(D) \longrightarrow 0, \]

where \( \kappa : \text{Lex}(B \oplus D) \rightarrow \text{Lex}(B) \oplus \text{Lex}(D) \) is the canonical mapping, is exact.

Proof. By remark 4.2.1.1, \( \kappa \) maps the class of a Lagrangian extension \((A, q, \iota)\) of \(B \oplus D\) into the pair of Lagrangian extensions

\[ B \subset D^\perp / D, \quad D \subset B^\perp / B. \]

For \((A, q, \iota)\) in the kernel of \(\kappa\), we have

\[ B \subset D^\perp / D \simeq B \oplus \hat{B}, \quad D \subset B^\perp / B \simeq D \oplus \hat{D}, \]

where \(B\) (respectively \(D\)) is embedded as the first summand and is Lagrangian with respect to the standard quadratic function on \(B \oplus \hat{B}\) (respectively \(D \oplus \hat{D}\)).

Now we want show that the Lagrangian extension \((E \oplus \hat{E}, q_{\text{std}})\) of \(B \oplus D\) (the effect of the map \(\nu\) on an extension \(B \rightarrow E \rightarrow \hat{D}\)) is always in the kernel of \(\kappa\). Indeed, the orthogonal complement of \(D\) in \(E \oplus \hat{E}\) has the form

\[ D^\perp = \left\{ (e, e) \in E \oplus \hat{E} \, | \, \sigma_{\text{std}} ((e, e), (0, \pi(d))) = 0 \, \forall \, d \in D \right\} = B \oplus \hat{E}. \]

Furthermore,

\[ D^\perp / D \simeq \left( B \oplus \hat{E} \right) / (0 \oplus \pi[D]) \simeq B \oplus \hat{B}. \]

Analogous results hold for \(B^\perp / B\). This shows that \(\text{im}(\nu) \subseteq \ker(\kappa)\).

To see that \(\text{im}(\nu) = \ker(\kappa)\), we show that, for \((A, q, \iota) \in \ker(\kappa)\), there exists an extension \(B \longrightarrow E \longrightarrow \hat{D}\) of \(\hat{D}\) such that

\[ (A, q, \iota) \simeq \left( E \oplus \hat{E}, q_{\text{std}}, \iota \oplus \hat{\pi} \right). \]

We start by extracting such \(E\) out of \(A\). Since \((A, q, \iota)\) is in the kernel of \(\kappa\), the isotropic contraction along \(B\) is trivial as a Lagrangian extension of \(D\): \(B^\perp / B \simeq D \oplus \hat{D}\). By lifting
the corresponding projections \( p_1 : B^\perp / B \to D, \ p_2 : B^\perp / B \to \widehat{D} \), we obtain surjections 
\( \overline{p}_1 : B^\perp \to D \) and \( \overline{p}_2 : B^\perp \to \widehat{D} \). Now define \( E \) to be \( \ker(\overline{p}_1) \). Since \( \ker(\overline{p}_1) \cap \ker(\overline{p}_2) = B \), the sequence

\[
B \overset{i}{\to} E \overset{\overline{p}_2}{\to} \widehat{D} \tag{4.19}
\]

is short exact. Similarly, \( D^\perp / D \cong B \oplus \widehat{B} \), and so we have projections \( w_1 : D^\perp / D \to B, \ w_2 : D^\perp / D \to \widehat{B} \) and corresponding surjections \( \overline{w}_1 : D^\perp \to B, \overline{w}_2 : D^\perp \to \widehat{B} \). Define \( E' = \ker(\overline{w}_1) \). We then have the analogous short exact sequence

\[
D \overset{i'}{\to} E' \overset{\overline{w}_2}{\to} \widehat{B}. \tag{4.20}
\]

By their definitions, \( E \) and \( E' \) are subgroups of \( A \). We show that \( A \) is their direct sum.

Since the orders of \( E \) and \( E' \) square to the order of \( A \), all we need to show is that
\[
E \cap E' = 0.
\]
Let \( x \in E \cap E' \). Then \( x \in B^\perp \cap D^\perp = (B \oplus D)^\perp = B \oplus D \) since \( B \oplus D \) is Lagrangian in \( A \). Since \( \overline{p}_1|_{B \oplus D} \) is the second projection and \( \overline{w}_1|_{B \oplus D} \) is the first,
\[
\overline{p}_1(x) = \overline{w}_1(x) = 0 \implies x = 0.
\]

Now we show that \( E \) and \( E' \) are Lagrangian subgroups of the quadratic group \( A \).

Together with the decomposition \( E \oplus E' = A \), that will give us a nondegenerate pairing between \( E \) and \( E' \), identifying \( E' \) with \( \widehat{E} \). What we are actually going to show is that \( E \) is isotropic; this is sufficient, since \( |E|^2 = |A| \). Recall that the kernel of the restriction of \( q \) to \( B^\perp \) is \( B \), so for any \( x \in B^\perp \) we have \( q(x) = q_{B^\perp / B}(\bar{x}) \), where \( \bar{x} \) is the coset of \( x \) modulo \( B \).

Now for \( x \in E = \ker(\overline{p}_1) \), the coset \( \bar{x} \) lies in \( \ker(p_1) \), which is an isotropic subgroup of \( B^\perp / B \). Hence \( q(x) = q_{B^\perp / B}(\bar{x}) = 0 \). An analogous argument shows that \( E' \) is also Lagrangian. Therefore since \( E \oplus E' = A \), we must have that \( E \) and \( E' \) are dual to one another. More precisely, the pairing \( E' \otimes E \to \mathbb{Q} / \mathbb{Z} \) given by \( (y, x) \mapsto q(y + x) \) is nondegenerate and gives an isomorphism of quadratic groups \( E' \sim \widehat{E} \).
The last thing we need to check is that, upon identification between \( E' \) and \( \hat{E} \), the extension (4.20) is the dual of (4.19). In other words, we need \( \hat{\tau} = \hat{w}_2 \) and \( \hat{\nu} = \hat{p}_2 \), which follows immediately from the definitions.

\[ \boxdot \]

**Corollary 4.4.1.3** ([13], Corollary 3.12). *The functors \( K \) and \( C \) are additive.*

**Proof.** The polarization of the functor \( B \mapsto \text{Ext}^1_{\mathbb{Z}}(\hat{B}, B)^\tau \) is \( \text{Ext}^1_{\mathbb{Z}}(\hat{D}, B) \). Moreover, the natural transformation \( \phi \) induces an isomorphism of polarizations. Finally, note that \( K([0]) = C([0]) = \{0\} \).

\[ \boxdot \]

Next, we describe the kernel functor \( K \).

**Lemma 4.4.1.4** ([13], Lemma 3.13). *The functor \( K \) fits into the exact sequence*

\[ \{0\} \longrightarrow \wedge^2 B 
\longrightarrow B \otimes B 
\longrightarrow Q(\hat{B}, \mathbb{Q}/\mathbb{Z}) 
\longrightarrow K(B) 
\longrightarrow \{0\}, \]

*where \( \wedge^2 B \) is the subgroup of antisymmetric elements of \( B \otimes B \).*

**Proof.** Let \( (A, q, \iota) \in K(B) \). Then, as an Abelian group, \( A \) can be identified with \( B \times \hat{B} \). Moreover, under this identification, \( \iota \) becomes the canonical inclusion. Furthermore, the quadratic function \( q \) is such that

\[ q(b + b', \chi) - q(b', \chi) = \pi(b', \chi)(b) = \chi(b). \]

Denote by \( \tilde{q} \) the restriction of \( q \) to \( \hat{B} \). Then for \( b \in B, \chi \in \hat{B} \), we have

\[ q(b, \chi) = \chi(b) + \tilde{q}(\chi). \]

Noting that \( \chi(b) = q_{std}(b, \chi) \), one can write

\[ (A, q, \iota) \simeq (B \times \hat{B}, q_{std} + \tilde{q}, \iota) \]

as Lagrangian extensions of \( B \). In other words, the map \( Q(\hat{B}, \mathbb{Q}/\mathbb{Z}) \to K(B) \) given by \( \tilde{q} \mapsto (B \times \hat{B}, q_{std} + \tilde{q}, \iota) \) is an epimorphism of groups.
Note that the kernel of the map \( Q(\widehat{B}, \mathbb{Q}/\mathbb{Z}) \to K(B) \) is the image of the map
\[ B \otimes B \to Q(\widehat{B}, \mathbb{Q}/\mathbb{Z}) \] sending \( r \) to the quadratic function \( q(\chi) = (\chi \otimes \chi)(r) \). Clearly such \( q \) is zero if and only if \( \tau(r) = -r \), where \( \tau: B \otimes B \to B \otimes B \) is the transposition automorphism.

**Corollary 4.4.1.5** ([13], Corollary 3.14). The functor \( K \) is isomorphic to the functor \( B \mapsto B/2B \).

**Proof.** Define a map \( Q(\widehat{B}, \mathbb{Q}/\mathbb{Z}) \to B/2B \) by assigning to \( q \) an element \( x \in B/2B \) such that \( 2q(\chi) = \chi(x) \) for all \( \chi \in \widehat{B}/2B \). It is straightforward that this map is zero on the image of \( B \otimes B \to Q(\widehat{B}, \mathbb{Q}/\mathbb{Z}) \), thus giving the map \( K(B) \to B/2B \). Finally, by looking at the effect on a cyclic 2-group \( B \), we can see that the map \( K(B) \to B/2B \) is an isomorphism.

In the rest of the section, we examine the cokernel functor \( C \).

**Remark 4.4.1.6.** Note that \( C(\mathbb{Z}/2^t \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \). Indeed, a generator of \( \text{Ext}^1_\mathbb{Z}(\mathbb{Z}/2^t \mathbb{Z}, \mathbb{Z}/2^t \mathbb{Z}) \simeq \mathbb{Z}/2^t \mathbb{Z} \) has the form
\[ \mathbb{Z}/2^t \mathbb{Z} \to \mathbb{Z}/2^{t+1} \mathbb{Z} \to \mathbb{Z}/2^t \mathbb{Z} \]

Example 4.2.1.8 shows that \( \mathbb{Z}/2^{t+1} \mathbb{Z} \) is not realizable as a Lagrangian extension of \( \mathbb{Z}/2^t \mathbb{Z} \). Thus the map \( \text{Lex}(\mathbb{Z}/2^t \mathbb{Z}) \to \text{Ext}^1_\mathbb{Z}(\mathbb{Z}/2^t \mathbb{Z}, \mathbb{Z}/2^t \mathbb{Z}) \) is not surjective. Note also that any extension of the form
\[ \mathbb{Z}/2^t \mathbb{Z} \to \mathbb{Z}/2^{t-1} \mathbb{Z} \times \mathbb{Z}/2 \mathbb{Z} \to \mathbb{Z}/2^t \mathbb{Z} \]

is twice a generator of \( \text{Ext}^1_\mathbb{Z}(\mathbb{Z}/2^t \mathbb{Z}, \mathbb{Z}/2^t \mathbb{Z}) \simeq \mathbb{Z}/2^t \mathbb{Z} \). Now, example 4.2.1.8 shows that the middle term of (4.21) is realizable as a Lagrangian extension of \( \mathbb{Z}/2^t \mathbb{Z} \), and moreover, that the image of the map \( \text{Lex}(\mathbb{Z}/2^t \mathbb{Z}) \to \text{Ext}^1_\mathbb{Z}(\mathbb{Z}/2^t \mathbb{Z}, \mathbb{Z}/2^t \mathbb{Z}) \) is \( 2\mathbb{Z}/2^t \mathbb{Z} \). Thus the cokernel of the map \( \text{Lex}(\mathbb{Z}/2^t \mathbb{Z}) \to \text{Ext}^1_\mathbb{Z}(\mathbb{Z}/2^t \mathbb{Z}, \mathbb{Z}/2^t \mathbb{Z}) \) must be isomorphic to \( \mathbb{Z}/2\mathbb{Z} \).
Lemma 4.4.1.7 ([13], Lemma 3.16). The functor $C$ is isomorphic to the functor 
$B \mapsto B_2 = \{ b \in B \mid 2b = 0 \}$.

Proof. Since the image of $\phi$ coincides with $\text{Ext}^1_{\mathbb{Z}}(\widehat{B}, B)\tau$, we have a short exact sequence

$$
\begin{array}{c}
\{0\} \longrightarrow \text{Ext}^1_{\mathbb{Z}}(\widehat{B}, B)\tau \longrightarrow \text{Ext}^1_{\mathbb{Z}}(\widehat{B}, B) \longrightarrow C(B) \longrightarrow \{0\}
\end{array}
$$

Thus the homomorphism $\text{Ext}^1_{\mathbb{Z}}(\widehat{B}, B)\tau \rightarrow H^1(\mathbb{Z}/2\mathbb{Z}, B \otimes B)$ defined in § 4.3 factors through an embedding $C(B) \rightarrow H^1(\mathbb{Z}/2\mathbb{Z}, B \otimes B)$. By remark 4.3.1.2, this is a natural transformation of linear functors in $B$. To show that it is an isomorphism, it suffices to check it for cyclic $B$. It is obvious for $B$ of odd order. Remark 4.3.1.2 says that $H^1(\mathbb{Z}/2\mathbb{Z}, B \otimes B)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Finally, remark 4.4.1.6 shows that $C(\mathbb{Z}/2\mathbb{Z})$ is also isomorphic to $\mathbb{Z}/2\mathbb{Z}$, making the map $C(B) \rightarrow H^1(\mathbb{Z}/2\mathbb{Z}, B \otimes B)$ an isomorphism. ■

Finally, we summarize the results of this section as a theorem.

Theorem 4.4.1.8 ([13], Theorem 3.17). For any finite Abelian group $B$, the following sequence is exact:

$$
\begin{array}{c}
\{0\} \longrightarrow B/2B \longrightarrow \text{Lex}(B) \longrightarrow C(B) \longrightarrow \{0\}
\end{array}
$$

Proof. Follows directly from corollary 4.4.1.5 and lemma 4.4.1.7. ■

4.5 Explicit Computation of the Homomorphism $\text{Lex}(L) \rightarrow H^3(\widehat{L}, k^*)$

Here, we describe the homomorphism $\text{Lex}(L) \rightarrow H^3(A/L, k^*)$ explicitly by giving a 3-cocycle $\beta \in Z^3(A/L, k^*)$ representing the Lagrangian extension $(A, q, \iota)$. Upon identification of $A/L$ with $\widehat{L}$, this gives a formula for the associator $\beta$ of the Lagrangian algebra corresponding to $L$. 
4.5.1 Lagrangian Algebras in Pointed Categories

Let now \( C \) be a braided monoidal category with the braiding \( c_{X,Y} : X \otimes Y \to Y \otimes X \).

Recall from [40] that the category \( C_R \) of right \( R \)-modules is monoidal with respect to the relative tensor product \( \otimes_R \), and that the category \( C_R^{\text{loc}} \) of local modules is braided.

Computations will be greatly simplified by taking a strict model of the category \( C(A, \alpha, c) \). Simple objects of \( C(A, \alpha, c) \) are labeled by elements of \( A \). Passing to a strict model gives rise to fusion isomorphisms \( \iota_{a,b} : I(a) \otimes I(b) \to I(a + b) \) for \( a, b \in A \).

Let \( a, b, c \in A \). By going around the diagram

\[
\begin{array}{ccc}
I(a) \otimes I(b) \otimes I(c) & \xrightarrow{\iota_{a,b} \otimes \text{Id}} & I(a + b) \otimes I(c) \\
\text{Id} \otimes I(c) & \xrightarrow{\iota_{a+b,c}} & I(a + b + c)
\end{array}
\]

clockwise, we obtain an automorphism of the simple object \( I(a + b + c) \in C(A, \alpha, c) \) that is precisely \( \alpha(a, b, c) \cdot \text{Id}_{I(a+b+c)} \). The braiding in \( C(A, \alpha, c) \) is given by (1.12).

Recall that the category \( C(A, \alpha, c) \) can also be written as \( C(A, q) \), where \( q \in Q(A, k^*) \) is the unique quadratic form corresponding to the class of the pair \( (\alpha, c) \in H^3_{ab}(A, k^*) \). Now let \( B \leq A \) be isotropic with respect to \( q \), and let \( R(B) \in C(A, q) \) be the object given by

\[
R(B) = \bigoplus_{b \in B} I(b). \tag{4.22}
\]

Let \( \eta \in C^2(B, k^*) \) be such that \( \partial \eta = \alpha|_B \), and define the multiplication map \( \mu : R(B) \otimes R(B) \to R(B) \) by

\[
\mu(b, b') = \eta(b, b') \cdot \iota_{b,b'} \tag{4.23}
\]

The coboundary condition on \( \eta \) makes this multiplication associative. This makes \( R(B) \) into an associative algebra in \( C(A, q) \). \( R(B) \) is commutative if and only if

\[
\eta(b, b') \cdot \iota_{b,b'} = c(b, b') \eta(b', b) \cdot \iota_{b',b} \text{ for all } b, b' \in B.
\]

\( ^{65} \text{In the case where } C \text{ has coequalizers.} \)
The natural forgetful functor \( C(A, q) \to \mathcal{V}(A, \alpha) \) (forgetting the quadratic structure) is a tensor equivalence. This induces a tensor equivalence \( C(A, q)_{R(B)} \to \mathcal{V}(A, \alpha)_{R(B)} \). Let \( p : A \to A/B \) be the canonical projection. The associated direct image functor \( p_* \) takes \( \mathcal{V}(A, \alpha) \) to \( \mathcal{V}(A/B, \beta) \) for some \( \beta \in H^3(A/B, k^*) \), and its restriction to \( \mathcal{V}(A, \alpha)_{R(B)} \) is a tensor equivalence \( \mathcal{V}(A, \alpha)_{R(B)} \simeq \mathcal{V}(A/B, \beta) \). Thus the composition

\[
C(A, q)_{R(B)} \xrightarrow{\text{forget}} \mathcal{V}(A, \alpha)_{R(B)} \xrightarrow{p_*} \mathcal{V}(A/B, \beta)
\]

exhibits a tensor equivalence between \( C(A, q)_{R(B)} \) and \( \mathcal{V}(A/B, \beta) \).

Recall that Lagrangian algebras in a pointed nondegenerate braided fusion category \( C(A, q) \) correspond to Lagrangian subgroups of \( A \). Let \( L \leq A \) be a Lagrangian subgroup, and let \( R(L) \in C(A, q) \) be the corresponding Lagrangian algebra. Let \( J : C(A, q) \to C(A, q)_{R(L)} \) be the free module functor defined by \( J(X) = X \otimes R(L) \), with corresponding right \( R(L) \)-multiplication given by \( \nu_X : J(X) \otimes R(L) \to J(X) \). The functor \( J \) gives rise to a natural collection of isomorphisms \( J_{X,Y} : J(X) \otimes_{R(L)} J(Y) \to J(X \otimes Y) \) induced from the composition

\[
X \otimes R(L) \otimes Y \otimes R(L) \xrightarrow{\text{Id}_X \otimes_{R(L)} \text{Id}_Y \otimes_{R(L)}} X \otimes Y \otimes R(L) \otimes R(L) \xrightarrow{\text{Id}_X \otimes \text{Id}_Y \otimes \mu} X \otimes Y \otimes R(L) .
\]

The maps \( J_{X,Y} \) are \( R(L) \)-balanced in the sense that they make the diagram

\[
\begin{array}{cccc}
J(X) \otimes_{R(L)} (R(L) \otimes_{R(L)} J(Y)) & \xrightarrow{\text{Id}_{J(X) \otimes_{R(L)} J(Y)}} & J(X) \otimes_{R(L)} (J(Y) \otimes_{R(L)} R(L)) & \xrightarrow{\text{Id}_{J(X) \otimes_{R(L)} J(Y)}} & J(X) \otimes_{R(L)} J(Y) \\
& \downarrow{}^{\alpha} & & & \downarrow{}^{\text{Id}_{J(X) \otimes_{R(L)} J(Y)}} \\
(J(X) \otimes_{R(L)} R(L)) \otimes_{R(L)} J(Y) & \xrightarrow{\nu_X \otimes_{R(L)} J(Y)} & J(X) \otimes_{R(L)} J(Y) & \xrightarrow{J_{X,Y}} & J(X \otimes Y)
\end{array}
\]
commute for all \(X, Y \in C(A, q)\), and coherent in the sense that they make the diagram

\[
\begin{array}{ccc}
J(X) \otimes_{R(L)} J(Y) \otimes_{R(L)} J(Z) & \xrightarrow{J_{X,Y} \otimes \text{Id}_{J(Z)}} & J(X \otimes Y) \otimes_{R(L)} J(Z) \\
\text{Id}_{J(X) \otimes J(Z)} & & J_{X \otimes Y, Z} \\
J(X) \otimes_{R(L)} J(Y) \otimes_Z J(Z) & \xrightarrow{J_{X,Y, Z}} & J(X \otimes Y \otimes Z)
\end{array}
\]

commute for all \(X, Y, Z \in C(A, q)\). They also give rise to an isomorphism \(J(I) \sim R(L)\) according to the diagram

\[
\begin{array}{ccc}
J(I) & \xrightarrow{\sim} & R(L) \\
\downarrow & \uparrow \rho_{R(L)} & \\
I \otimes R(L) & &
\end{array}
\]

Here, \(I \in C(A, q)\) denotes the monoidal unit and \(\rho_{R(L)}\) denotes the right unit isomorphism in \(C(A, q)\).

We will also take a strict model of the category of modules \(C(A, q)_{R(L)}\). All simple \(R(L)\)-modules are induced from simple objects of \(C(A, q)\). For \(a \in A\), denote \(J(I(a))\) by \(J(a)\), and define maps of \(R(L)\)-modules

\[
\phi_{a,b} : J(a) \otimes_{R(L)} J(b) \to J(a + b)
\]

by the following diagram:

\[
\begin{array}{ccc}
J(a) \otimes_{R(L)} J(b) & \xrightarrow{\phi_{a,b}} & J(a + b) \\
\downarrow & \downarrow & \downarrow \\
J(I(a)) \otimes_{R(L)} J(I(b)) & & J(I(a + b)) \\
J(I(a) \otimes I(b)) & \xrightarrow{J_{I(a), I(b)}} & J(I(a) \otimes I(b))
\end{array}
\]
For $a, b, c \in A$, the clockwise composition of the diagram

$$
\begin{array}{c}
J(a) \otimes_{R(L)} J(b) \otimes_{R(L)} J(c) \\
\downarrow \text{Id}_{R(L)} \otimes \phi_{b,c} \\
J(a) \otimes_{R(L)} J(b) \\
\downarrow \phi_{a,b+c} \\
J(a) \otimes_{R(L)} J(c)
\end{array}
\xrightarrow{\phi_{a,b} \otimes \text{Id}_{R(L)}}
\begin{array}{c}
J(a + b) \otimes_{R(L)} J(c) \\
\downarrow \phi_{a+b,c} \\
J(a + b + c)
\end{array}
\xrightarrow{\phi_{a,b,c} \otimes \text{Id}_{R(L)}}
\begin{array}{c}
J(a + b + c)
\end{array}
$$

(4.24)

is precisely $\alpha(a, b, c) \cdot \text{Id}_{R(a+b+c)}$.

For $\ell \in L$, define $\theta_\ell : I(\ell) \otimes R(L) \to R(L)$ by

$$
\theta_\ell = \bigoplus_{\ell' \in L} \eta(\ell, \ell') \cdot \iota_{\ell, \ell'}.
$$

This gives an isomorphism $\theta_\ell : J(\ell) \to J(0)$ of right $R(L)$-modules in $C(A, q)$. The collection of isomorphisms $\theta_\ell$ allows one to define, for any $a \in A$ and any $\ell \in L$, isomorphisms of $R(L)$-modules $\vartheta_{a,\ell} : J(a + \ell) \to J(a)$ by the diagram

$$
\begin{array}{c}
J(a + \ell) \\
\downarrow \phi_{a,\ell} \\
J(a) \otimes_{R(L)} J(\ell)
\end{array}
\xrightarrow{\theta_{a,\ell} \otimes \text{Id}_{R(L)}}
\begin{array}{c}
J(a) \otimes_{R(L)} J(\ell)
\end{array}
\xrightarrow{\text{Id}_{R(L)} \otimes \phi_{a,\ell}}
\begin{array}{c}
J(a)
\end{array}
$$

Now define $\vartheta'_{\ell, a} : J(\ell + a) \to J(a)$ by the diagram

$$
\begin{array}{c}
J(\ell + a) \\
\downarrow \phi_{\ell, a} \\
J(\ell) \otimes_{R(L)} J(a)
\end{array}
\xrightarrow{\vartheta'_{\ell, a} \otimes \text{Id}_{R(L)}}
\begin{array}{c}
J(\ell) \otimes_{R(L)} J(a)
\end{array}
\xrightarrow{\text{Id}_{R(L)} \otimes \phi_{\ell, a}}
\begin{array}{c}
R(L) \otimes_{R(L)} J(a)
\end{array}
$$

Commutativity of addition implies that the maps $\vartheta_{a,\ell}$ and $\vartheta'_{\ell, a}$ differ by a nonzero scalar, viz., $\vartheta'_{\ell, a} = c(a, \ell) \cdot \vartheta_{a,\ell}$. The collections $\vartheta$ establishes that up to isomorphism, simple $R(L)$-modules are labeled by elements of the quotient group $A/L$. 
To finally define a strict model of the category $C(A, q)_{R(L)}$, choose a set-theoretic section $s : A/L \to A$ of the canonical projection. Define $\gamma : A/L \times A/L \to L$ by $\gamma(x, y) = s(x) + s(y) - s(x + y)$, and define $J_{x,y} : J(s(x)) \otimes_{R(L)} J(s(y)) \to J(s(x + y))$ as the composition

$$J(s(x)) \otimes_{R(L)} J(s(y)) \xrightarrow{\phi_{s(x),s(y)}} J(s(x) + s(y)) \xrightarrow{\delta_{s(x+y),s(x+y)}} J(s(x + y)) .$$

The tensor category $C(A, q)_{R(L)}$ of right $R(L)$-modules is equivalent to $\mathcal{V}(A/L, \beta)$ for some $\beta \in Z^3(A/L, k^*)$. The clockwise composition of the arrows of the diagram

$$J(s(x)) \otimes_{R(L)} J(s(y)) \otimes_{R(L)} J(s(z)) \xrightarrow{J_{x,y} \otimes \text{Id}_{J(s(z))}} J(s(x) + s(y)) \otimes_{R(L)} J(z) \quad (4.25)$$

is precisely $\beta(x, y, z) \cdot \text{Id}_{J(s(x+y+z))}$, and so we need only compute this composition to see the value of the associator $\beta(x, y, z)$.

We now expand the diagram (4.25). In doing so, we will suppress the tensor product symbols for the sake of compactness of notation. We will also suppress labels on identity morphisms. Juxtaposition of objects in the following diagram should be understood to denote their tensor product over $R(L)$. Moreover, we will write $R$ instead of $R(L)$ where applicable.
It is straightforward to see that the clockwise reading of the upper left cell of the diagram (4.26) contributes a factor of $a(s(x), s(y), s(z))$.

**Lemma 4.5.1.1** ([13], Lemma A.1). The lower left cell of the diagram (4.26) contributes a factor of $a(s(x), s(y + z), \gamma(y, z))^{-1}$.

**Proof.** Let $a = s(x), b = s(y + z), \ell = \gamma(y, z)$. Consider the diagram

$$\begin{array}{c}
J(a)J(b + \ell) \\
\downarrow \phi_{a+b+\ell} \\
J(a)(b + \ell) \\
\downarrow \phi_{a+b+\ell} \\
J(a)J(b)J(\ell) \\
\downarrow \phi_{a+b+\ell} \\
J(a)J(b) \\
\downarrow \phi_{a,b} \\
J(a) \\
\end{array}$$

The right cell has the form

$$\begin{array}{c}
J(a + b)J(\ell) \\
\downarrow \phi_{a+b+\ell} \\
J(a + b + \ell) \\
\downarrow \phi_{a+b+\ell} \\
J(a + b) \\
\end{array}$$
and commutes on the nose, as it is the definition of $\theta_{a+b,\ell}$. The bottom cell has the form

$$J(a)J(b) \xrightarrow{\phi_{a,b} \otimes \text{Id}} J(a + b)R$$

and commutes for tautological reasons. The left cell has the form

$$J(a)J(b) \xrightarrow{\phi_{a,b}} J(a + b)$$

$J(a)$ plays no part. This cell is the definition of $\text{Id} \otimes \vartheta_{b,\ell}$ with orientation reversed. Hence it defines $\vartheta_{b,\ell}^{-1}$ and thus commutes on the nose. The center cell has the form

$$J(a)J(b)J(\ell) \xrightarrow{\phi_{a,b} \otimes \text{Id}} J(a + b)J(\ell)$$

This cell commutes because its composition gives the scalar $\phi_{a,b}^{-1} \phi_{a+b,\ell}^{-1}$, and scalars commute pairwise. The top cell is the only cell that does not commute on the nose. It has the form

$$J(a)J(b + \ell) \xrightarrow{\text{Id} \otimes \vartheta_{b,\ell}} J(a)J(b)J(\ell)$$

By (4.24), its clockwise composition gives $\alpha(a, b, \ell)^{-1}$. The result follows.

**Lemma 4.5.1.2** ([13], Lemma A.2). The upper right cell of the diagram (4.26) contributes a factor of

$$\frac{\alpha(\gamma(x, y), s(x + y), s(z))c(s(x + y) + s(z), \gamma(x, y))}{c(s(x + y), \gamma(x, y))}.$$
Proof. Let now $\ell = \gamma(x, y)$, $a = s(x + y)$, and $b = s(z)$, and consider the diagram

The rightmost cell has the form

$$RJ(a)J(b) \xrightarrow{\phi_{a,b}} J(a)J(b)$$

and commutes on the nose for tautological reasons. The center cell has the form

$$J(\ell)J(a)J(b) \xrightarrow{\theta_{a,b} \otimes \text{Id}} RJ(a)J(b)$$

and also commutes on the nose. The leftmost cell has the form

$$J(\ell + a)J(b) \xrightarrow{\phi_{a,b} \otimes \text{Id}} J(\ell)J(a)J(b)$$

and contributes a factor of $\alpha(\ell, a, b)$. The bottom cell has the form

$$J(\ell)J(a + b) \xrightarrow{\phi_{a,b} \otimes \text{Id}} RJ(a + b)$$
and is the definition of $\vartheta'_{\ell,a+b}$. Similarly, the top cell has the form

$$J(\ell + a)J(b) \xrightarrow{\vartheta'_{\ell,a} \otimes \text{Id}} J(a)J(b)$$

and is the definition of $\vartheta'_{\ell,a}$.  Now, in the diagram (4.26), $\vartheta'$ do not appear. Instead, $\vartheta$s appear. Comparing $\vartheta'$ with $\vartheta$ gives factors of $c(a + b, \ell)$ from the bottom cell and $c(a, \ell)^{-1}$ from the top cell. The inverse appears in the second factor due to the orientation being reversed relative to the definition of $\vartheta_{a,\ell}$. The result follows. ■

**Lemma 4.5.1.3** ([13], Lemma A.3). *The lower right cell of the diagram (4.26) contributes a factor of*

$$\eta(\gamma(x, y + z), \gamma(y, z))$$

$$\eta(\gamma(x + y, z), \gamma(x, y)).$$

**Proof.** We split the lower right cell of the diagram (4.26) into two triangles as follows:

Let us now consider the general form of such triangles:

$$J(a + \ell + \ell') \xrightarrow{\vartheta_{a,\ell} + \ell'} J(a + \ell) \xrightarrow{\vartheta_{a,\ell}} J(a)$$

(4.28)
Expanded, this becomes

\[
J(a + \ell + \ell') \xrightarrow{\phi_{a,\ell} \Id} J(a) J(\ell) J(\ell') \xrightarrow{\phi_{a,\ell} \Id} J(a) J(\ell) \xrightarrow{\Id \phi_{\ell',\ell}} J(a) \xrightarrow{\Id \phi_{\ell',\ell}} J(a) R
\]

All of the quadrilaterals in this diagram commute on the nose, leaving only the right-center cell, which collapses to

\[
J(a) J(\ell + \ell') \xrightarrow{\Id \phi_{\ell',\ell}} J(a) R \quad (4.29)
\]

Now, since \(J(a)\) plays no part in the cell (4.29), we may omit it and rewrite (4.29) as

\[
J(\ell + \ell') \xrightarrow{\phi_{\ell',\ell}} R \quad (4.29)
\]

Going clockwise through this diagram from the bottom left gives us a factor of \(\eta(\ell, \ell')\).

Therefore, the lower triangle of (4.27) contributes a factor of \(\eta(\gamma(x, y + z), \gamma(y, z))\), and the upper triangle contributes a factor of \(\eta(\gamma(x + y, z), \gamma(x, y))^{-1}\). The inverse appears in the second factor because the orientation of the upper triangle is reversed relative to (4.28).
Finally, we summarize the results of this section as a theorem.

**Theorem 4.5.1.4** ([13], Theorem A.4). The category $C(A, q)_{R(L)}$ of right modules over a Lagrangian algebra $R(L) \in C(A, q)$ is equivalent, as a monoidal category, to the category $\text{V}(A/L, \beta)$ of $A/L$-graded vector spaces, where $\beta \in \mathbb{Z}^3(A/L, k^*)$ is given by the formula

$$\beta(x, y, z) =$$

$$= \frac{\alpha(s(x), s(y), s(z))\alpha(\gamma(x, y), s(x + y), s(z))c(s(x + y) + s(z), \gamma(x, y))\eta(\gamma(x, y + z), \gamma(y, z))}{\alpha(s(x), s(y + z), \gamma(y, z))c(s(x + y), \gamma(x, y))\eta(\gamma(x + y, z), \gamma(x, y))}$$

(4.30)

**Proof.** Follows from lemmata 4.5.1.1, 4.5.1.2, and 4.5.1.3. ■

### 4.6 Examples

**Example 4.6.1.1.** Let $A = B \oplus \hat{B}$ for an Abelian group $B$, and let $q_{\text{std}} : B \oplus \hat{B} \to k^*$ be a quadratic function given by $q_{\text{std}}(b, \chi) = \chi(b)$. The subgroup $L = B \oplus \{1\}$ is Lagrangian with respect to $q$. The character group $\hat{L}$ can be identified with $\hat{B}$. We have a braided equivalence

$$C(B \oplus \hat{B}, q_{\text{std}}) \simeq \mathbb{Z}(\hat{B}).$$

**Example 4.6.1.2.** Let $A = B \oplus B$ for a nondegenerate quadratic Abelian group $(B, q)$. Equip $A$ with a quadratic function $q \oplus q^{-1}$. The diagonal subgroup $\Delta = \{(b, b) | b \in B\}$ is Lagrangian. The character group $\hat{L}$ can be identified with $\hat{B}$. We have a braided equivalence

$$C(B \oplus B, q \oplus q^{-1}) \simeq \mathbb{Z}(\hat{B}, \alpha_q).$$

**Example 4.6.1.3.** Let $A = \mathbb{Z}/m^2\mathbb{Z}$ be a cyclic group of order $m^2$ for an odd $m$ and let $q_{\varepsilon} : \mathbb{Z}/m^2\mathbb{Z} \to k^*$ be a quadratic function given by $q(s) = \varepsilon s^2$, where $\varepsilon \in k^*$ is a primitive root of unity of order $m^2$. The subgroup $L = m\mathbb{Z}/m^2\mathbb{Z}$ is Lagrangian with respect to $q_{\varepsilon}$. The character group $\hat{L}$ can be identified with $\mathbb{Z}/m\mathbb{Z}$ via

$$\mathbb{Z}/m\mathbb{Z} \to \hat{L}, \quad j \mapsto \chi_j, \quad \chi_j(ml) = \varepsilon^{iml}.$$
For \( \nu \in k^* \) such that \( \nu^{m^2} = 1 \), denote by
\[
\alpha_{\nu}(x, y, z) = \nu^{(y+z-[y+z]_m)},
\]
where \([x]_m\) is the residue of \( x \in \mathbb{Z} \) modulo \( m \) (taking values between 0 and \( m - 1 \)). Note that \( \alpha_{\nu} \) is a normalised 3-cocycle of \( \mathbb{Z}/m\mathbb{Z} \) and that the assignment \( \nu \mapsto \alpha_{\nu} \) gives an isomorphism between \( H^3(\mathbb{Z}/m\mathbb{Z}, k^*) \) and the group \( \mu_m(k) \) of \( m \)th roots of 1 in \( k \). We have a braided equivalence
\[
C(\mathbb{Z}/m^2\mathbb{Z}, q_{e^m}) \cong \mathbb{Z}(\mathbb{Z}/m\mathbb{Z}, \alpha_{e^m}).
\]
In order that the braided category \( \mathbb{Z}(\mathbb{Z}/m\mathbb{Z}, \alpha) \) be pointed, it is necessary that the cocycle \( \alpha \) be a coboundary. Given \( z \in \mathbb{Z}/m\mathbb{Z} \), let \( c_z \in C^1(\mathbb{Z}/m\mathbb{Z}, k^*) \) be such that
\[
(\partial c_z)(x, y) = \alpha(x, y|z)
\]
for all \( x, y \in \mathbb{Z}/m\mathbb{Z} \). Computing \( \alpha(x, y|z) \) for the cocycle \( \alpha \) from (4.31) gives
\[
\alpha(x, y|z) = \nu^{-(x+y-[x+y]_m)}.
\]
Used inductively, the coboundary equation allows one to describe \( c_z \). Namely \( c_z \) is completely determined by its value at \( 1 \in \mathbb{Z}/m\mathbb{Z} \):
\[
c_z(x) = \frac{(c_z(1))^x}{\prod_{i=1}^{x-1} \alpha(i, 1|z)}.
\]
Indeed, for \( x \in \mathbb{Z}/m\mathbb{Z} \), we have
\[
c_z(x) = c_z((x - 1) + 1) = \frac{c_z(x - 1) c_z(1)}{\alpha(x - 1, 1|z)}.
\]
Moreover, the formula (4.33) yields a normalized 1-cochain of \( \mathbb{Z}/m\mathbb{Z} \) with coefficients in \( k^* \) only if \( c_z(m) = 1 \). This requires that
\[
(c_z(1))^m = \prod_{i=1}^{m-1} \alpha(i, 1|z).
\]
It follows from (4.32) that the only nontrivial value of \( \alpha(i, 1|z) \) is \( \alpha(m - 1, 1|z) = \nu^{-zm} \). That gives
\[
(c_z(1))^m = \nu^{-zm}.
\]
Making a choice \( c_z(1) = \nu^{-\sigma(z)} \) gives

\[
c_z(x) = \nu^{-\sigma(z)(x)}, \quad x \in \mathbb{Z}/m\mathbb{Z}.
\]

Using \( z \mapsto (z, c_z) \) as a section of \( \text{Pic}(\mathbb{Z}(\mathbb{Z}/m\mathbb{Z}, \alpha)) \to \mathbb{Z}/m\mathbb{Z} \), we can write down the extension class as an element of \( \mathbb{Z}^2(\mathbb{Z}/m\mathbb{Z}, \hat{\mathbb{Z}}/m\mathbb{Z}) \). Namely, for \( z, w \in \mathbb{Z}/m\mathbb{Z} \), we can define the character \( \chi_{z,w} \in \hat{\mathbb{Z}}/m\mathbb{Z} \) by

\[
\chi_{z,w}(g) = \frac{\alpha (g|z, w) c_{z+w}(g)}{c_z(g)c_w(g)}.
\]

(4.35)

Note that the isomorphism

\[
\hat{\mathbb{Z}}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}, \quad \chi \mapsto k, \text{ such that } \chi(1) = (\nu^m)^k
\]

identifies the extension cocycle (4.35) with the canonical extension cocycle \( z, w \mapsto z + w - [z + w]_m \) of the extension \( \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m^2\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \).
REFERENCES


Appendix: Homotopy Action on the Standard Normalized Complex

It is well known that the standard differential (1.3) comes from a cosimplicial structure on the standard complex \( \partial = \sum_{i=0}^{n+1} (-1)^i \partial_i \). Here, \( \partial_i : C^n(G, M) \to C^{n+1}(G, M), \ i = 0, ..., n + 1 \) are defined by

\[
\partial_i(c)(g_1, ..., g_{n+1}) = \begin{cases} 
  c(g_2, ..., g_{n+1}), & i = 0 \\
  c(g_1, ..., g_i, g_{i+1}, g_{i+2}, ..., g_{n+1}), & 1 \leq i \leq n \\
  c(g_1, ..., g_n), & i = n + 1
\end{cases}
\]

and satisfy the cosimplicial identities

\[
\partial_j \partial_i = \partial_i \partial_j, \quad i < j.
\]

The group of automorphisms \( \text{Aut}(G) \) of the group \( G \) acts on the complex \( C^*(G, M) \) via

\[
\phi(c)(g_1, ..., g_n) = c(\phi^{-1}(g_1), ..., \phi^{-1}(g_n)), \quad \phi \in \text{Aut}(G), \ c \in C^n(G, M).
\]

For \( g \in G \) denote by \( \phi_g \) the corresponding inner automorphism \( \phi_g(x) = gxg^{-1} \).

Proposition A.1.1.1 ([15]). The action of inner automorphisms on the complex \( C^*(G, M) \) is homotopically trivial. More precisely, for any \( c \in C^n(G, M) \) and \( g \in G \), we have

\[
\phi_g(c) = c + \partial(h^g(c)) - h^g(\partial(c)),
\]

where

\[
h^g(c)(g_1, ..., g_{n-1}) = c(g^{-1}, g_1, ..., g_{n-1}) + \sum_{i=1}^{n-1} (-1)^{i} c(g^{-1}g_1g, ..., g^{-1}g_1g, g^{-1}g_{i+1}, g_{i+1}, ..., g_{n-1}). \quad (A.1)
\]

See, e.g., [20].
Proof. For an $n$-cochain $c$, write $h^g(c) = \sum_{i=0}^{n-1} (-1)^i h^g_i(c)$, where

$$h^g_i(c)(g_1, \ldots, g_{n-1}) = \begin{cases} c(g^{-1}, g_1, \ldots, g_{n-1}), & i = 0 \\ c(g^{-1}g_1g, \ldots, g^{-1}g_i, g^{-1}, g_{i+1}, \ldots, g_{n-1}), & 1 \leq i \leq n-1 \end{cases}$$

The following identities

$$h^g_0 \partial_0 = \text{Id}, \quad h^g_n \partial_{n+1} = \phi^g,$$

$$h^g_i \partial_i = \begin{cases} \partial_i h^g_{i-1}, & i < j \\ h^g_{i-1} \partial_i, & i = j \neq 0 \\ \partial_{i-1} h^g_i, & i > j + 1 \end{cases}$$

show that $h^g_i$ is a cosimplicial homotopy between $\phi^g$ and the identity $\text{Id}$. We now prove the identities.

Assume that $i = j = 0$. Then $\left[ \partial_0(c) \right](g_1, \ldots, g_{n+1}) = c(g_2, \ldots, g_{n+1})$. Call this function $b$. Then $\left[ h^g_0(b) \right](g_1, \ldots, g_n) = b(g^{-1}, g_1, \ldots, g_n) = c(g_1, \ldots, g_n)$. The first identity follows.

Assume that $i = n + 1$ and $j = n$. Note that this value of $j$ makes sense, as it entails acting upon the $(n + 1)$-cochain $\partial_{n+1}(c)$. We obtain

$$\left[ \partial_{n+1}(c) \right](g_1, \ldots, g_{n+1}) = c(g_1, \ldots, g_n).$$

Call this function $b$. Then $\left[ h^g_n(b) \right](g_1, \ldots, g_n) = b(g^{-1}g_1g, g^{-1}g_2g, \ldots, g^{-1}g_ng, g^{-1}) = c(g^{-1}g_1g, g^{-1}g_2g, \ldots, g^{-1}g_ng) = \phi^g(c)$ by definition of the action of $\text{Aut}(G)$ on the complex $C^*(G, M)$. The second identity follows. The proof of the third identity must be split into several cases. We first assume that $i < j$. If $i = 0$ and $j = 1$, then we have $\left[ \partial^g_0(c) \right](g_1, \ldots, g_{n+1}) = c(g_2, \ldots, g_{n+1})$. Here, the superscript on $\partial$ denotes the degree of the differential map under consideration. Call this function $b$. Then $\left[ h^g_0(b) \right](g_1, \ldots, g_n) = b(g^{-1}g_1g, g^{-1}g_2g, \ldots, g^{-1}g_ng) = c(g^{-1}, g_2, \ldots, g_n)$. On the other hand, $\left[ h^g_0(c) \right](g_1, \ldots, g_{n-1}) = c(g^{-1}, g_1, \ldots, g_{n-1})$. Call this function $b$. Then $\left[ \partial^g_0^{-1}(b) \right](g_1, \ldots, g_n) = b(g_2, \ldots, g_n) = c(g^{-1}, g_2, \ldots, g_n)$, and the desired equality follows.
Now assume that $i = 0$ and $j \geq 2$. As above, $\left[ \partial_0^g (c) \right] (g_1, \ldots, g_{n+1}) = c (g_2, \ldots, g_{n+1})$.

Call this function $b$. Then
\[
\left[ h_j^g (b) \right] (g_1, \ldots, g_n) = b \left( g^{-1} g_1 g, g^{-1} g_2 g, \ldots, g^{-1} g_j g, g^{-1}, g_{j+1}, \ldots, g_n \right) = c \left( g^{-1} g_2 g, \ldots, g^{-1} g_j g, g^{-1}, g_{j+1}, \ldots, g_n \right). \quad \text{On the other hand,}
\]
\[
\left[ h_{j-1}^g (c) \right] (g_1, \ldots, g_{n-1}) = c \left( g^{-1} g_1 g, \ldots, g^{-1} g_{j-1} g, g^{-1}, g_j, \ldots, g_{n-1} \right). \quad \text{Call this function $b$.}
\]

Then $\left[ \partial_0^{g-1} (b) \right] (g_1, \ldots, g_n) = g_1 b (g_2, \ldots, g_n) =
\\quad = \left( g^{-1} g_1 g \right) c \left( g^{-1} g_2 g, \ldots, g^{-1} g_j g, g^{-1}, g_{j+1}, \ldots, g_n \right)$, and the desired equality follows.

Finally, assume that $i \geq 1$ and $j \geq 2$. Note that in this case, it is not possible to have $i = n + 1$. We have $\left[ \partial_i^g (c) \right] (g_1, \ldots, g_{n+1}) = c (g_1, \ldots, g_i g, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1})$. Call this function $b$. Then we have $\left[ h_j^g (b) \right] (g_1, \ldots, g_n) =
\\quad = b \left( g^{-1} g_1 g, g^{-1} g_2 g, \ldots, g^{-1} g_i g, g^{-1}, g_{j-1}, \ldots, g_n \right) =
\\quad = c \left( g^{-1} g_2 g, \ldots, g^{-1} g_i g, g^{-1}, g_{j-1}, \ldots, g_n \right). \quad \text{On the other hand,}
\]
\[
\left[ h_{j-1}^g (c) \right] (g_1, \ldots, g_{n-1}) = c \left( g^{-1} g_1 g, \ldots, g^{-1} g_{j-1} g, g^{-1}, g_j, \ldots, g_n \right). \quad \text{Call this function $b$.}
\]

Then $\left[ \partial_i^{-1} (b) \right] (g_1, \ldots, g_n) = b (g_1, \ldots, g_i, g_i g_{i+1}, g_{i+2}, \ldots, g_n) =
\\quad = c \left( g^{-1} g_1 g, \ldots, g^{-1} g_i g, g^{-1}, g_{i+1}, \ldots, g_n \right)$, and the desired equality follows.

We next assume that $i = j \neq 0$. We first consider the case where $i = j = 1$.

$\left[ \partial_1^g (c) \right] (g_1, \ldots, g_{n+1}) = c (g_1 g_2, g_3, \ldots, g_{n+1})$. Call this function $b$. Then
\[
\left[ h_1^g (b) \right] (g_1, \ldots, g_n) = b \left( g^{-1} g_1 g, g^{-1} g_2, \ldots, g_n \right) = c \left( g^{-1} g_1 g g^{-1}, g_2, \ldots, g_n \right) =
\\quad = c \left( g^{-1} g_1 g, g_2, \ldots, g_n \right). \quad \text{On the other hand,}
\]
\[
\left[ h_0^g (b) \right] (g_1, \ldots, g_n) = b \left( g^{-1}, g_1, \ldots, g_n \right) = c \left( g^{-1} g_1, g_2, \ldots, g_n \right), \quad \text{and the desired equality follows.}
\]

We now consider the case where $i = j \geq 2$. We have
\[
\left[ \partial_i^g (c) \right] (g_1, \ldots, g_{n+1}) = c (g_1, \ldots, g_i, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1})$. Call this function $b$. Then
\[
\left[ h_i^g (b) \right] (g_1, \ldots, g_n) = b \left( g^{-1} g_1 g, \ldots, g^{-1} g_i g, g^{-1}, g_{i+1}, \ldots, g_n \right) =
\\quad = c \left( g^{-1} g_1 g, \ldots, g^{-1} g_i g, g^{-1}, g_{i+1}, \ldots, g_n \right). \quad \text{On the other hand,}
\]
On the other hand, \[ [h^g_{i-1}(b)](g_1, \ldots, g_n) = b \left( g^{-1}_1g, \ldots, g^{-1}_{i-1}g, g^{-1}_i, g_i, \ldots, g_n \right) = c \left( g^{-1}_1g, \ldots, g^{-1}_{i-1}g, g^{-1}_i, g_i, \ldots, g_n \right), \]
and the desired equality follows.

Finally, we assume that \( i > j + 1 \). We first assume that \( j = 0 \) and \( i \geq 2 \). As in the immediately preceding case, \[ [\partial^i_j(c)](g_1, \ldots, g_{n+1}) = c \left( g_1, \ldots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \ldots, g_{n+1} \right). \]
Call this function \( b \). Then \[ [h^g_0(b)](g_1, \ldots, g_n) = b \left( g^{-1}_1g, \ldots, g^{-1}_n \right) = c \left( g^{-1}_1g, \ldots, g^{-1}_n \right). \]
On the other hand, \[ [h^g_0(c)](g_1, \ldots, g_{n-1}) = c \left( g^{-1}_1g, \ldots, g^{-1}_{n-1} \right). \] Call this function \( b \). Then \[ [\partial^{i-1}_j(b)](g_1, \ldots, g_n) = b \left( g_1, \ldots, g_{i-2}, g_{i-1}g_i, g_{i+1}, \ldots, g_n \right) = c \left( g^{-1}_1g, \ldots, g^{-1}_{i-2}, g_{i-1}g_i, g_{i+1}, \ldots, g_n \right), \]
and the desired equality follows.

Lastly, we consider the case where \( j > 0 \). We have \[ \left[ \partial_i(c) \right](g_1, \ldots, g_{n+1}) = c \left( g_1, \ldots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \ldots, g_n \right). \] Call this function \( b \). Then \[ \left[ h^g_i(b) \right](g_1, \ldots, g_n) = b \left( g^{-1}_1g, \ldots, g^{-1}_i, g_i, g_{i+1}, \ldots, g_n \right) = c \left( g^{-1}_1g, \ldots, g^{-1}_i, g_i, g_{i+1}, \ldots, g_n \right). \]
On the other hand, \[ \left[ h^g_i(c) \right](g_1, \ldots, g_{n-1}) = c \left( g^{-1}_1g, \ldots, g^{-1}_{i-1}, g_i, g_{i+1}, \ldots, g_n \right). \] Call this function \( b \). Then \[ \left[ \partial^{i-1}_j(b) \right](g_1, \ldots, g_n) = b \left( g_1, \ldots, g_{i-2}, g_{i-1}g_i, g_{i+1}, \ldots, g_n \right) = c \left( g^{-1}_1g, \ldots, g^{-1}_{i-2}, g_{i-1}g_i, g_{i+1}, \ldots, g_n \right), \] and the proof of the identities is completed.

With the identities proven, we now turn our attention to the proof of the result. We begin by writing \[ h^g = \sum_{i=0}^{n-1} (-1)^i h^g_i, \quad \partial = \sum_{j=0}^{n+1} (-1)^j \partial_j. \]
We claim that \( \phi_g - \text{Id} = \partial h^g - h^g \partial \). Indeed, we have

\[
\begin{align*}
\partial h^g - h^g \partial &= \left[ \sum_{j=0}^{n} (-1)^j \partial_j \right] \left[ \sum_{i=0}^{n-1} (-1)^i h^g_i \right] - \left[ \sum_{i=0}^{n} (-1)^i h^g_i \right] \left[ \sum_{j=0}^{n+1} (-1)^j \partial_j \right] \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} \partial_j h^g_i - \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} h^g_i \partial_j = \end{align*}
\]
\[
= \sum_{0 < i \leq j \leq n-1} (-1)^{i+j} \partial_i h_j^g + \sum_{0 \leq j < i \leq n} (-1)^{i+j} \partial_j h_i^g - \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} h_j^g \partial_i - \sum_{1 \leq i \leq n} (-1)^{i+j} h_j^g \partial_i +
\]
\[
+ \sum_{\substack{1 \leq i \leq n \cr j = i-1}} (-1)^{i+j} h_j^g \partial_i - \sum_{0 \leq j < i-1 \leq n} (-1)^{i+j} h_j^g \partial_i - h_0^g \partial_0 + h_n^g \partial_{n+1} \]
\[
= \left[ \sum_{0 < i \leq j \leq n-1} (-1)^{i+j} \partial_i h_j^g - \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} h_j^g \partial_i \right] +
\]
\[
+ \left[ \sum_{0 \leq j < i \leq n} (-1)^{i+j} \partial_j h_i^g - \sum_{0 \leq j < i-1 \leq n} (-1)^{i+j} h_j^g \partial_i \right] +
\]
\[
+ \left[ \sum_{1 \leq i \leq n} (-1)^{i+j} h_j^g \partial_i + \sum_{1 \leq i \leq n} (-1)^{i+j} h_j^g \partial_i \right] - h_0^g \partial_0 + h_n^g \partial_{n+1} \]
\[
= 0 + 0 + 0 - h_0^g \partial_0 + h_n^g \partial_{n+1} = h_n^g \partial_{n+1} - h_0^g \partial_0 = \phi_g - \text{Id},
\]
and the proof is completed. \[\blacksquare\]

**Remark A.1.1.2.** It follows from proposition A.1.1.1 that the group of inner automorphisms \(\text{Inn}(G) = \{\phi_g \mid g \in G\}\) acts trivially on the cohomology \(H^\ast(G, M)\). Therefore the action of \(\text{Aut}(G)\) on \(C^\ast(G, M)\) gives rise to an action of the outer automorphism group \(\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)\) on \(H^\ast(G, M)\).