A Bayesian Method for Planning Reliability Demonstration Tests
for Multi-Component Systems

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ABSTRACT

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The objective of this research is to develop a methodology to plan a zero-failure reliability demonstration test (RDT) for multi-component systems. A fully Bayesian hierarchical approach is applied to model multi-level systems, with independent components. The system failure time distribution is expressed according to the component failure time distributions based on the system structure determined by the reliability block diagram. Component-level prior information is incorporated into a joint prior distribution. Markov chain Monte Carlo methods, such as Gibbs sampling, are developed to find the minimum sample size for planning the system RDT.
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1 INTRODUCTION

This chapter presents the motivation and objective of this thesis research, and then it provides a brief introduction to reliability, reliability tests, and reliability data analysis.

1.1 Objective and Motivation

A reliability demonstration test (RDT) is a type of life test that is used to demonstrate whether a product has met a certain reliability criterion [1]. Based upon such criterion decision on acceptance or rejection of the product can be made. RDT provides the confidence on the reliability of the product. In a RDT, the test is either stopped at the failure of \( r \) number of samples out of \( n \) testing samples, or at the predefined testing period \( t_c \). After the test, if the number of samples failed are less than or equal to the predefined quantity \( r \), the demonstration is successful. In a zero-failure RDT, \( r = 0 \). RDT needs to be carefully planned to minimize the testing sample size \( n \) and/or testing period \( t_c \).

Traditional maximum likelihood estimation (MLE) based zero-failure RDT planning methods usually work for simple cases such as binomial, exponential, and Weibull with fixed shape parameters; the MLE plans usually require testing of a large sample size. Applying Bayesian approach in planning reliability tests may result in a reduced sample size due to the integration of available prior knowledge governing the product’s reliability. For a complex system, system-level prior knowledge may be limited due to the complexity of the system and the scarcity of system-level historical data.

Testing the whole system for reliability estimation and optimization is a challenging and complicated task. This task is restricted with parameters such as cost,
longer testing times, and limited resource availability. Many complex systems such as nuclear reactors, space satellites, water turbines, and one-time use military missiles are very costly and time consuming to produce [2]. For such complex systems, a large number of entire systems cannot be tested. A prior knowledge concerning the component reliability is usually available. For example, the components may be supplied by different vendors, who have performed extensive component-level reliability tests. It is possible to reduce a sample size for a system RDT if component-level prior knowledge can be effectively used.

Our goal in this research is to plan a system level RDT for zero failure test using prior information at the component level.

1.2 Background

This section provides a brief introduction of reliability. It will discuss basic reliability elements, system reliability, reliability tests, and reliability data analysis methods.

1.2.1 Reliability

Reliability is a product’s probability to perform satisfactorily, under designed working conditions, for the intended period of time \( t \). Durability of the product and its resistance to failure affects its reliability. The reliability of a product is usually described by some lifetime distribution functions, such as reliability function and cumulative distribution function, and is characterized by some reliability measures, such as the mean
time to failure (MTTF).

1.2.1.1 Reliability Function

Reliability function shows the probability of a product’s functionality over the intended period of time $t$. Let us define a continuous random variable $T$ as the time to failure of a product, with $T \geq 0$. For a given value of $t$, $R(t)$ is the probability that the time to failure is greater than or equal to $t$ [1]; that is

$$R(t)=P\{T \geq t\}$$

where $0 \leq R(t) \leq 1$, $R(0)=1$, and $\lim_{t \to \infty} R(t)=0$ [1].

1.2.1.2 Cumulative Distribution Function (CDF)

CDF for the failure distribution is given by

$$F(t)=1-R(t)=P\{T \leq t\}$$

where $F(0)=0$, $\lim_{t \to \infty} F(t)=1$, and $0 \leq F(t) \leq 1$. $R(t)$ is used when survival probabilities are computed, while CDF, $F(t)$, is used in the calculation of failure probabilities.

1.2.1.3 Probability Density Function (PDF)

PDF of the failure distribution $f(t)$ indicates the probability density of the random variable, falling within certain bounded limits inside the distribution [3]. The PDF is defined by

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt} \quad \text{where} \quad \int_0^\infty f(t)dt = 1 \quad \text{and} \quad f(t) \geq 0.$$ 

Thus, $F(t)=\int_0^t f(u)du$, and $R(t)=\int_t^\infty f(u)du$. 
1.2.1.4 Hazard Rate Function

Hazard rate function provides an instantaneous failure rate at time \( t \). The hazard rate function is defined as the conditional probability that the product will fail in a unit interval of time, after \( t \), given it was working at time \( t \) \[4]\; i.e.,

\[
h(t) = \lim_{\Delta t \to 0} \frac{P\{t \leq T \leq t + \Delta t|T \geq t\}}{\Delta t} = \frac{R(t) - R(t + \Delta t)}{\Delta t R(t)} = \frac{f(t)}{R(t)}.
\]

Hazard rate is closely related to the failure mechanism of a product. It may be distinguished into constant, decreasing, and increasing hazard rate. For a constant hazard rate model, such as exponential distribution, the conditional probability of failure, which is conditioned on a previous time period, is independent of the previous time period. Failure due to completely random or chance events will follow this distribution. This property of exponential distribution is also known as “memory-less” property. It assumes that the products under the exponential distribution, even after a working period of a certain time period \( t \), are as “good as new,” and thus their hazard rate is constant; i.e., it is neither increasing nor decreasing. An increase in a hazard rate indicates wear and tear of parts \[5\]; however, \( h(t) \) becomes a decreasing function when early life failures of a product are eliminated \[6\]. For a normal distribution, \( h(t) \) is an increasing function; whereas, for a Weibull distribution, \( h(t) \) can be either an increasing or a decreasing function.
1.2.1.5 Mean Time to Failure (MTTF)

MTTF is the expected time to failure and it is defined by

\[
MTTF = E(T) = \int_0^\infty t \cdot f(t)\,dt = \int_0^\infty R(t)\,dt.
\]

MTTF is one of the measures of central tendencies of the failure distribution. The other commonly used measure is the median time to failure.

1.2.1.6 Median Time to Failure

Median time to failure divides the distribution in two halves. It may be the preferred choice to consider the middle value when the distribution is highly skewed, for which the MTTF does not yield satisfactory results. It’s also known as the time required for 50% of the test samples to fail [7]. Median time to failure, \( t_{med} \) is defined as,

\[
R(t_{med}) = 0.5.
\]

1.2.1.7 Quantile Life \( (t_p) \)

The \( p \)-quantile life of a lifetime distribution is defined as \( F(t_p) = p \); i.e.,

\[
t_p = F^{-1}(p), \quad \text{where, } t_p \text{ corresponds to the time at which } p \text{ proportion of the total sample experiences failure [8]. A reliability requirement is usually specified for a quantile life. For example, a customer may require that the product } p\text{-quantile life exceeds a certain requirement } t^*_p. \text{ A RDT can be used to demonstrate that this reliability requirement may be satisfied.} \]
1.2.2 System Reliability

While analyzing the reliability of a system, one can apply two main approaches. The first approach considers the reliability model for the entire system; while the second approach derives the system reliability from the components’ reliability functions. A reliability block diagram (RBD) is a schematic representation of a system with individual components, connected to each other as blocks. A RBD represents a string of blocks connected in a parallel or a series configuration, with each individual block representing a component of the system with a failure rate. A RBD does not necessarily indicate the physical positioning of the components in the system; however, it indicates how the components are functionally connected in the system and how their functionality is going to affect the system. Serial and parallel formations and their combinations are typical component configurations used for system reliability in operational systems.

1.2.2.1 Serial System

Consider in Figure 1.1, components 1, 2, ..., m be individual mutually independent components connected in series. Thus, to make this serial system work, all these components must work.
Let $R_0(t)$ be the system reliability for the period of time $t$ and, $R_1(t), R_2(t), ..., R_m(t)$ be the individual reliabilities of the components that are connected in serial configuration. Then, $R_0(t)$ is given by

$$R_0(t) = \prod_{i=1}^{m} R_i(t) \leq \min \{R_1(t), R_2(t), ..., R_m(t)\}$$

and

$$h_0(t) = \sum_{i=1}^{m} h_i(t)$$

where $h_0(t)$ is the system hazard rate function and $h_i(t)$ is the individual component hazard rate, for $i = 1, 2, ..., m$.

1.2.2.2 Parallel System

When components are connected in parallel, as long as one of them is working, the system is working. Consider a system of $m$ mutually independent components placed in the parallel combination as shown in Figure 1.2.
Here, $R_0(t)$ is given by

$$R_0(t) = 1 - \prod_{i=1}^{m} [1 - R_i(t)] \geq \max \{R_1(t), R_2(t), \ldots, R_m(t)\}.$$ 

Many systems consist of both parallel and serial combinations, and they are usually referred to as combined series-parallel systems.

### 1.2.2.3 Combined System

A combined system is the one made up of serial and parallel combinations of components. Consider the example of a combined system as shown in Figure 1.3. According to the series and parallel configuration in the system, $R_0(t)$ is stated as

$$R_0(t) = [1 - (1 - R_2(t))(1 - R_1(t))]R_1(t).$$
1.2.3 Reliability Tests

There are many types of reliability tests such as life tests and accelerated life tests (ALT) that are being used to evaluate a product’s reliability. In an ALT, the product is placed under elevated stress conditions of operations to speed up its failure process in order to collect its failure information in a short period of time. RDT is a type of life test that demonstrates if a product has met a certain required reliability at a specific confidence [9]. RDT has been widely explored in many research papers since 1970. A RDT has been known for its estimation of the minimum sample size for testing of a product, which has cost and quantity restrictions and is mainly associated with the automotive and aerospace industry. Multiple types of demonstration tests which are used worldwide include [9] reliability, maintainability, and availability tests. Other types of classifications are based on whether it is required to estimate reliability parameters for comparison with the known value during planning, or whether the specific parameter value is better at a specified level with the stated confidence interval [10]. Such types of tests are classified under RDTs. RDT goes through the following three stages [11]:

![Figure 1.3: Example of combined series-parallel system’s RBD](image)
1. Planning of a test

2. Testing period

3. Evaluation

In the planning stage, a reliability goal and a RDT model are decided. Prior parameters that affect the decision matrix, for example, survival period, customer risk, producer risk and their desired reliability are noted. A producer’s risk is defined as the probability of rejecting the product where the test parameter is inside an acceptable range; whereas, consumer’s risk is the probability of accepting the product when the test parameter lies outside the acceptable range [12]. Length of test $t$, test quantity $n$, sample availability and confidence level are determined prior to the test and specific to the product. A planning approach (Bayesian/ MLE) is chosen and distribution is decided for the product survival data. Acceptance and rejection criterion are created and plotted with the number of accepted failures and sample survival duration. For example, in Bayesian approach, let the product hazard rate be a function of $r$ and $n$, and let $\lambda^*_h$ be the maximum allowed hazard rate desired by the customer. Then, posterior confidence is given by

$$P(\lambda_n \leq \lambda^*_h | Accept) \geq 1 - \gamma,$$

where $\gamma$ indicates posterior risk and $1 - \gamma$ indicates customer assurance. Posterior confidence indicates the certainty based on posterior risk that the product meets the demonstrated reliability.

Once the testing begins, it is monitored with the set up time span for its status as estimated in the planning phase. Test units are checked for functionality periodically. Progress graphs are updated with testing. In the final analysis phase, the number of survived samples is compared with the primary goal $r$. This phase concludes the RDTs.
with model and the result reports for acceptance or rejection.

### 1.2.4 Maximum Likelihood Estimation (MLE)

MLE is a very popular method of estimating reliability parameters with the help of statistical model from a finite sample [4]. When reliability data is available, it can be incorporated to form a joint likelihood function. The joint likelihood function is given by

\[
L(\theta_1, \theta_2, \ldots, \theta_k) = \prod_{i=1}^{n} L_i(t_i|\theta_1, \theta_2, \ldots, \theta_k)
\]

where \( t = t_1, t_2, \ldots, t_n \) denotes the component-level reliability data and \( L_i(t_i|\theta_1, \theta_2, \ldots, \theta_k) \) is the likelihood contribution of \( t_i \). When the first partial derivative of the log-likelihood function with respect to unknown parameters \( \theta_1, \theta_2, \ldots, \theta_k \) is equated to zero, MLE is obtained [4] as follows

\[
\frac{\partial \ln L(\theta_1, \theta_2, \ldots, \theta_k)}{\partial (\theta_j)} = 0, \quad j = 1, \ldots, k.
\]

### 1.2.5 Bayesian Approach, Priors and Gibbs Sampling

This section provides brief information about the Bayesian approach and provides a Markov chain Monte Carlo (MCMC) [13] simulation algorithm through Gibbs sampling. The maximum likelihood estimation method assumes the parameters to be unknown, but having a fixed quantity [9]. The Bayesian approach considers that the parameters can be random and measures the degree of belief in the form of subjective distribution. The probability distribution that provides this basis is called “prior
distribution”. This prior distribution along with the sample information conglomerates the “posterior distribution”, which is a joint distribution and a compilation of assumed uncertainties with historical data [14]. Parameters of the prior distribution are called hyperparameters. Different priors have different hyperparameters for the distribution-specific parameters. The capability of accommodating prior knowledge in the analysis makes Bayesian a favorite choice, considering the limited availability of data sets [15] [16]. Based upon the observed data, inferences to the model parameters are made with probability statements. Prior distribution is updated with Bayes’ theorem once this observed sample data gets collected. Bayes’ theorem is given by

\[ f(\theta|x) = \frac{f(x|\theta)f(\theta)}{f(x)} \]

where \( \theta \) denotes the parameter vector, while \( x \) denotes the sample data. \( f(x|\theta) \) is a likelihood function. \( f(\theta|x) \) and \( f(\theta) \) denote joint posterior and prior distributions respectively. The marginal density function is given by

\[ f(x) = \int f(x|\theta)f(\theta)d\theta. \]

An equivalent form which omits the factor \( f(x) \) where \( x \) is fixed and \( f(x) \) is independent of \( \theta \) is given by

\[ f(\theta|x) \propto f(x|\theta)f(\theta). \]

Consider an example of a model with three parameters \( \theta = (\theta_1, \theta_2, \theta_3) \). Their joint posterior distribution is given by \( f(\theta_1, \theta_2, \theta_3|x) \). Inferences on each of the parameters depend on their marginal posterior distributions. For example, marginal posterior
distribution of $\theta_i$ is given by $f(\theta_i|x) = \int \int f(\theta_1, \theta_2, \theta_3|x) d\theta_2 d\theta_3$, which requires multiple integrations. Often, for complicated models these integrations are difficult to get through. Under such cases, MCMC simulations provide the easiest way to compute the results without solving the integrals [13]. Gibbs sampler is a MCMC algorithm which is applied to generate approximated values from a joint probability distribution of random variables from high dimensional problems. Each iteration draws a sample of one parameter conditioned upon the most recent values of the other parameters. Consider for example, a Gibbs sampling simulation, where each parameter is simulated in the ‘$i+1$th’ iteration from the following parameters using the conditional posterior distributions as

\[
\begin{align*}
\theta_1^{(i+1)} &\sim f(\theta_1|\theta_2^{(i)}, \theta_3^{(i)}, x) \\
\theta_2^{(i+1)} &\sim f(\theta_2|\theta_1^{(i+1)}, \theta_3^{(i)}, x) \\
\theta_3^{(i+1)} &\sim f(\theta_3|\theta_1^{(i+1)}, \theta_2^{(i+1)}, x)
\end{align*}
\]

where ‘$i$’ and ‘$i+1$’ represent their respective iterations. For larger iterations, samples taken from one parameter can be considered as its simulated value from its marginal posterior distribution, which can then be used to formulate a posterior inference on the model parameters [17].

1.3 Thesis Overview

This thesis is organized as follows. Chapter 2 reviews the related research work in the literature, Chapter 3 explains the research and applied methodologies, Chapter 4 focuses on the case study and results, and Chapter 5 summarizes the results with a conclusion and provides the future scope.
2 LITERATURE REVIEW

In the literature review of this document, research done and methods followed by various authors for different types of multi-level information integration methods, approaches used for planning RDT at component and system level, and Bayesian approach for planning other types of reliability tests are discussed.

2.1 Multi-level Information Integration

A system is made up of different components, and each component has different reliability, which in turn affects system reliability. System-level data are generally not available, because most of the time, system tests are limited due to low resources and feasibility constraints. Component-level historical data are available with vendors at various conditions. These data can definitely be used to determine system level reliability estimation, which signifies the importance of multi-level information integration. System reliability estimation has been proposed with three approaches to date: top-down approach, bottom-up approach and Bayesian hierarchical approach.

2.1.1 Top-down Approach

Assuming prior information is available for system reliability, Mastran and Singpurwala [18] and Mastran [19] proposed a top-down approach. The top-down approach was explained by Mastran [19] with the following steps:
1. Using Bayes’ theorem, system-level prior distribution is combined with system-level reliability data for obtaining a revised prior distribution for a system’s reliability.

2. Derivation of the prior distributions of component reliabilities consistent with the revised prior of the system’s reliability with the known component configuration from the system.

3. Using Bayes’ theorem for combining the derived component-level prior with the component-level test data to obtain posterior distributions of component reliabilities.

4. Ultimately obtaining posterior distributions of system reliability consistent with the posterior distributions of component reliabilities.

Top-down approach is applied when only system level prior information is available which then can be used to derive component level priors for obtaining the desired posterior distribution.

### 2.1.2 Bottom-up Approach

Martz et al. [20] and Martz and Waller [21] used bottom-up approach for the integration of multi-level binary data. In this approach, lower level posterior distributions were obtained by using prior information and integrating the data at the same level. With the next upper level, “induced” higher level posterior distribution was obtained by taking the lower-level posterior distribution up through the system RBD and combining it with data and “native” prior distribution at a higher level to obtain a posterior distribution.
This approach goes up the process until it reaches the system level. Some of the studies on multi-level systems with proportional hazard modeling for hierarchical systems [22], interdependent subsystems and components [23], and failure time and repair data [24] have opted for this approach.

### 2.1.3 Bayesian Hierarchical Approach

Approximations are used in both top down and bottom up approaches. Approximations lead to errors, especially when the sample size is significantly small. Different data integration methods lead to different posterior results, which in turn lead to aggregation errors [25]. A fully Bayesian hierarchical approach, which can naturally eliminate approximation and aggregation errors, has been chosen by various recent papers [26] [27] [28] [29] [30] [31] [32]. In this approach, system and sub-system failure time distributions are reiterated using deterministic relations derived from RBDs, Bayesian networks, or fault tree analyses. Multi-level data are incorporated in a joint likelihood function and, therefore, combined and analyzed simultaneously. This approach has been applied in the past for analyzing failure time, binomial data (go/no-go), and degradation data [33]. In this paper, the same approach proposes a method to plan a system reliability demonstration test.
2.2 Planning Reliability Demonstration Tests

Reliability demonstration tests are mainly planned with two approaches: Bayesian and MLE. Each of these approaches has its own merits and is used according to the historical data availability, testing level, and sample size.

2.2.1 Planning RDT Using MLE

The maximum likelihood (ML) method assumes the parameters to be unknown, but having a fixed quantity [9]. MLE works on the principle of choosing the parameters that maximize the likelihood function. Use of MLE is prominent in RDT for higher sample sizes and when there is no prior data available. It works on the test data, and obtained values are used to estimate the reliability of “to be tested” products.

Using the MLE approach, McKane et al. [34] applied log-location scale distributions and failure censoring approach to design RDT. Likelihood was derived from a log-location scale random variable $T$, for a failure censored sample of $r$ failures among $n$ independent observations. ML estimates of moment and standard deviation were obtained from statistical softwares such as SAS and Matlab and were further used to derive point estimators. Quantile and reliability approaches were used to define lower confidence bound for determining the sample size and the number of failures for a RDT.

For demonstrating equipment reliability to a customer, Eryilmaz [35] planned a startup demonstration test following a geometric distribution using the MLE approach. $n$ identical units were tested with a total success total failure (TSTF) procedure for $x$ number of trials in order to estimate the test length. The invariance property of MLE was
used to obtain an approximate confidence interval for the probability of success and failure for initial startup with the help of the fisher information matrix. Fernández [12] assumed the MLE method to design optimal RDTs at the component level. A mixed integer nonlinear programming problem was formulated considering the producer’s and consumer’s risk. A likelihood ratio obtained from lifetime data was used to derive an acceptance criterion for the sampling plans.

2.2.2 Planning RDT Using Bayesian

The Bayesian approach considers that the failure rate can be random, and hence, this method requires a previous judgmental approach to give a subjective distribution [36]. Prior distribution along with the sample information amalgamates in the posterior distribution, which is a joint distribution and a compilation of assumed uncertainties with historical data. This capability of accommodating prior knowledge in the analysis makes Bayesian a favorite choice, considering the limited availability of data sets [15]. Bayesian methods can be further classified into ‘fixed time - sample size variation’ and ‘fixed sample size - testing time variation’ methods. BAZE [37] (Bayesian zero-failure reliability demonstration test) was developed by Martz and Waller [37] with a gamma prior and exponential failure time model for fixed-time demonstration tests. The designed procedure was to test $n$ identical components for a specified time $t$, where zero failure occurrences determined the success of the test. The sample size could be reduced for longer test durations, and so the test duration could be reduced with a bigger sample size.
Similar to Martz and Waller [37], Fan and Chang [2] also conducted BAZE for demonstrating the reliability of high quality electro-explosive devices; they designed an accelerated test assuming an exponential failure distribution. The Bayesian approach was applied here to incorporate a prior distribution with the goal to reduce test length and sample size.

Applying the Bayesian approach, Higgins and Tsokos [38] used uniform distributions and fitted inverted gamma as priors to compare posterior producer and consumer risks. Higgins and Tsokos [38] and Coolen and Coolen-Schrijner [39] pointed out that applying two different priors for mean time between failures (MTBF) may lead to a significant difference in the posterior risk of Bayesian RDTs, even if both of the priors fit the data poorly or equally well. Coolen and Coolen-Schrijner [39] described Bayesian zero failure RDTs with the brief discussion on deterministic and non-deterministic tests, where a conservative number of tests can be obtained based on the expected value of number of tasks required from the RDTs.

Ten and Xie’s [40] RDT plan for series-systems and Mann’s [41] “Approximately Optimum Lower Confidence Bound Model” considered the Bayesian approach to incorporate substantial binomial subsystem test data.

2.3 Planning of Reliability Demonstration Tests for Multi-Level Systems

Multi-level systems use prior knowledge about component data to integrate into system. RDT planning is classified into system-level RDT planning and component-level RDT planning.
2.3.1 Planning RDTs for System-Level

Planning of the system-level RDT consists of testing the entire system for failure. In this case, even if a system fails due to failure of an individual component, it is considered a failure of the system. Two types of system-level RDTs have been researched: one system-level RDT considers component-level prior information along with system test data to calculate posterior distribution at the system level, while the other considers only the system-level prior and the system test data. A system-level RDT is never completely dependent on just the component data because the main purpose is to test the system in its operational condition when all of its components are working together. Yadav et al. [42] studied system-level RDT with three-dimensional considerations of physical, functional, and time. Here, prior reliability targets were allocated for each of the dimensions from failure analysis and warranty information. Each function was then disaggregated into sub functions, and their individual contribution was considered for the effect on functional reliability of the system. With a serial configuration system, a RDT was classified under zero-failure RDT and zero or one failure RDT. Using a hydraulic power rack and pinion steering system, the system reliability was demonstrated for different failure modes.

Ten and Xie [40] applied objective subsystem test data to demonstrate the reliability test for the series-systems. High prior confidence from substantial subsystem level data was incorporated with the test data, inclusive of a consumer requirements parameter, to obtain the posterior distribution. Similar to Yadav et al. [42], Ten and Xie [40] also considered the serial system configuration for binomially distributed test data.
for attribute systems. Lower confidence bounds from the subsystem data were derived to fit them into the system prior with Mann’s [41] “Approximately Optimum Lower Confidence Bound Model Using Bayesian Approach.” Mastran [19] gathered the data both at the component and system levels to demonstrate a system-level RDT. The Bayesian approach was used to incorporate consistent moments of both the component-level and the system-level data where each individual component was assumed to have an exponential failure distribution.

2.3.2 Planning RDTs for Component-Level

A system may fail because of the failure of one of the components of the system. This makes it important to consider component-level RDT planning. In planning a RDT for component level, each individual component is tested to obtain its posterior distribution inclusive of its prior historical data and the likelihood of failure from the current testing. These data are then incorporated to form a system-level RDT. Component-level RDT planning, thus, has the advantage of considering all the components which may become part of the system failure; but at the same time, it fails to contain the probability of failure when all the components work together. Gal [43] tested components to demonstrate system reliability under the assumption that a system will work properly for a unit time; whereas components might fail during the same period. Gal [43] assumed the components of the system to be having a very high reliability with an expectation of zero failure during the test. Independent testing of each component of the system was done by using serial and parallel configurations. Gal [43] demonstrated
the reliability for the components with assumptions of two cases of failure rates: one with an increasing hazard rate and the other with a constant hazard rate.

Mazumdar [44] described both acceptable and unacceptable reliability levels for limiting the number of component failures \( k \) as an extension for Gal’s [43] case of serial connection of redundant units. Mazumdar [45] derived an optimum component-testing procedure for the independent subsystems connected in a serial configuration. He considered the case where the serial formation of subsystems with redundant components with the same limiting assumption of failures \( k \) as stated by Mazumdar [44]. Rajgopal and Mazumdar [46] designed an individual component test plan for a serial system in which they considered exponentially distributed failure times, which followed gamma distribution. With the cases of availability and unavailability of prior information on failure rates, Rajgopal and Mazumdar [46] created a linear programming model for limiting the total number of observed failures \( m \) as an acceptance criterion.

2.4 Bayesian Approach for Planning Other Reliability Tests

The Bayesian approach provides the advantage of incorporating prior knowledge in the reliability analysis. Most of the research done using the Bayesian approach applied it for its inherent advantage of using the least sample size. Martz and Waller [21] applied a two-staged Bayesian model with \( \beta \) prior distribution for estimating reliability of a heat seeking missile system. Liu [16] planned a second and third level constant stress ALT with a Bayesian approach for repairable systems having multiple failure modes, where minimizing the pre-posterior variance of a test quantity at regular stress levels was the
optimization criterion. A surface-smoothing technique was applied for obtaining the optimal plan. Erkanli and Soyer [47] used the same technique to plan a constant-stress ALT using a Bayesian approach. Yuan et al. [15] used the Bayesian approach in finding the optimal stress changing time in a simple step-stressed accelerated life testing (SSALT).
3 PROPOSED RESEARCH TASK AND METHODOLOGIES

This chapter illustrates the proposed methodologies for the research. These include the modeling of a multi-component system, the integration of component-level prior information, the design of a Bayesian RDT, the development of a simulation algorithm, the effect of a prior on optimal plan, and the effect of test duration and customer requirement on the optimal sample size of two example systems.

3.1 Parametric Modeling of a Multi-Component System

Consider a reliability block diagram as mentioned in section 1.2.2. Now, let the components be labeled as $i$, where $i = 1, 2, \ldots, m$. Let $T_i$ be a random variable for the failure time of the considered components $i$, with pdf $f_i(t|\theta_i)$, cdf $F_i(t|\theta_i)$, and the reliability function $R_i(t|\theta_i)$, where $\theta_i$ denotes the parameter vector. The reliability function for the system was derived with deterministic system structure as

$$R_0(t|\theta_1, \theta_2, \ldots, \theta_m) = \varphi_0(R_1(t|\theta_1), R_2(t|\theta_2), \ldots, R_m(t|\theta_m))$$

where the structure function $\varphi_0$ depends upon the system configuration.

For example, system reliability for the series-parallel system shown in Figure 1.3 is expressed as

$$R_0(t|\theta_1, \theta_2, \theta_3) = \varphi_0(R_1(t|\theta_1), R_2(t|\theta_2), R_3(t|\theta_3))$$

$$= [1 - (1 - R_2(t|\theta_3))(1 - R_3(t|\theta_3))]R_1(t|\theta_1)$$

Here, the system-level reliability function has been stated as a parametric function of the parameters of the component-level reliability functions.
3.2 Component-Level Prior Information Integration

An advantage of Bayesian methodology is that with the information from expert opinions, historical data can be integrated in terms of priors [31] [40]. In case component-level prior information is available, it can be combined to form a prior distribution to obtain the posterior distribution with the help of Bayesian methodology.

Let \( f(\theta_1, \theta_2, \ldots, \theta_m) \) be a joint prior distribution for the model parameters. If the component parameter vectors \( \theta_i \)'s are assumed to be independent, then

\[
f(\theta_1, \theta_2, \ldots, \theta_m) = f(\theta_1)f(\theta_2)\ldots f(\theta_m)
\]

where \( f(\theta_1)f(\theta_2)\ldots f(\theta_m) \) are prior distributions of the individual components’ parameters. Bayes’ theorem combines joint prior distribution \( f(\theta_1, \theta_2, \ldots, \theta_m) \) with the likelihood function to obtain posterior distribution.

3.3 Design of Bayesian System Reliability Demonstration Plans

A customer can specify any requirements such as MTTF, hazard rate etc. In this thesis \( t_p \) is considered as the customer requirement criterion and customer specified \( t_p \) is denoted by \( t_p^* \). Thus, probabilities in this thesis are specified in terms of \( t_p \). Prior probability in this thesis is given by \( \Pr(t_p \geq t_p^*) \); it is the probability obtained based on the prior information without testing any samples when \( t_p \geq t_p^* \). Posterior probability is stated as \( \Pr(t_p \geq t_p^*|Accept) \) when all the test samples have already survived the testing period \( t_c \) for the condition \( t_p \geq t_p^* \).
Joint prior distribution \( f(\theta_1, \theta_2, \ldots, \theta_m) \) as mentioned in section 3.2 can be used for planning a system-level reliability demonstration test with the Bayesian approach to obtain posterior probability. The test is accepted by the customer when all the test samples survive the testing period \( t_c \). To have confidence over the decision based on the test, the customer needs an assurance in the form of a probability. A Bayesian reliability plan for the system can be generated by considering specified customer assurance \( 1 - \gamma \), where \( \gamma \) indicates customer risk. Once the customer accepts the test, the customer requirement is met according to the desired customer assurance as

\[
\Pr(t_p \geq t^*_p | \text{Accept}) \geq 1 - \gamma.
\]

Note that \( \Pr(t_p \geq t^*_p | \text{Accept}) \) has been denoted by \( \Gamma_n \) and \( \Pr(t_p \geq t^*_p) \) has been denoted by \( \Gamma_0 \) in this thesis frequently.

Then, according to Bayes’ theorem,

\[
f(\theta_1, \theta_2, \ldots, \theta_m | \text{Accept}) \propto L(\theta_1, \theta_2, \ldots, \theta_m | \text{Accept}) f(\theta_1, \theta_2, \ldots, \theta_m)
\]

\[
\propto \Pr(\text{Zero failure} | \theta_1, \theta_2, \ldots, \theta_m) f(\theta_1, \theta_2, \ldots, \theta_m)
\]

\[
\propto R_0(t_c | \theta_1, \theta_2, \ldots, \theta_m) f(\theta_1, \theta_2, \ldots, \theta_m).
\]

Here, \( f(\theta_1, \theta_2, \ldots, \theta_m | \text{Accept}) \) is the joint posterior distribution, \( L(\theta_1, \theta_2, \ldots, \theta_m | \text{Accept}) \) stands for likelihood based upon the experiment which is again stated by

\[
\Pr(\text{Zero failure} | \theta_1, \theta_2, \ldots, \theta_m)
\]

for zero failure RDT, and \( f(\theta_1, \theta_2, \ldots, \theta_m) \) is the prior distribution of independent component parameters \( \theta_1, \theta_2, \ldots, \theta_m \). For independent components, \( \theta_0 = (\theta_1, \theta_2, \ldots, \theta_m) \).
Thus, for a given sample size \( n \),

\[
f(\theta_0|\text{Accept}) \propto \Pr(\text{Zero failure}|\theta_0) f(\theta_0) \propto R_0(t_c|\theta_0)^n f(\theta_0).
\] (3.1)

By the definition of quantile life \( t_p \) in section 1.2.1.7 for the given parameter \( \theta_0 \),

\[
R_0(t_p|\theta_0) = 1 - p
\]
i.e.;
\[
t_p = R_0^{-1}(1 - p|\theta_0).
\] (3.2)

Joint posterior distribution \( f(t_p|\text{Accept}) \) can be derived from \( f(\theta_0|\text{Accept}) \) by random variable transformation and marginization. Then, the posterior probability can be evaluated by

\[
\Pr(t_p \geq t_p^*|\text{Accept}) = \int_{t_p}^{*} f(t_p|\text{Accept}) dt_p.
\] (3.3)

This does not have a closed form solution. Hence, MCMC methods such as Gibbs sampling can be used to draw a random sample from the joint posterior distribution \( f(\theta_0|\text{Accept}) \), namely \( \theta_0^{(k)}, k = 1,2,\ldots,K \), where \( K \) is the number of Gibbs sampling iterations and \( \theta_0^{(k)} \) is the parameter value drawn at the \( k \)th Gibbs sampling iteration. When \( \theta_0^{(k)} \) obtained through Gibbs sampling is used in equation (3.2), we get \( t_p^{(k)} \).

Then \( \Pr(t_p \geq t_p^*|\text{Accept}) \) can be estimated through

\[
\Pr(t_p \geq t_p^*|\text{Accept}) \approx \frac{1}{K} \sum_{k=1}^{K} I(t_p^{(k)} \geq t_p^*) = \frac{1}{K} \sum_{k=1}^{K} I(R_0^{-1}(1 - p|\theta_0^{(k)}) \geq t_p^*)
\] (3.4)

where \( I(.) \) indicates the indicator function which provides either 1 or 0 as the output; when \( t_p^{(k)} \geq t_p^* \), \( I(.) = 1 \) and when \( t_p^{(k)} < t_p^* \), \( I(.) = 0 \). From the obtained multiple \( t_p \) values,
the value of $t_p$ that passes the condition $t_p^{(k)} \geq t_p^*$ will be used to obtain $\Pr(t_p \geq t_p^*|\text{Accept})$ using equation (3.4).

### 3.4 Development of Simulation Algorithm

The design of Bayesian reliability demonstration tests as shown by equation (3.3) requires multiple levels of integration to obtain posterior distribution which has no closed form solution. A MCMC algorithm has been developed in this study for the same purpose.

“The BUGS (Bayesian inference Using Gibbs Sampling) project” is specially made for the Bayesian approach with Gibbs sampling [48]. A Winbugs code has been applied to perform the Monte Carlo simulation algorithm and Gibbs sampling. A Matlab program code is used that summons this Winbugs code to calculate the $t_p$ value and verify the condition $\Pr(t_p \geq t_p^*|\text{Accept}) \geq 1 - \gamma$. The fzero function used in the Matlab code tries to find the root of the nonlinear function in equation (3.2) for the assumed example systems.

In this analysis, $\Pr(t_p \geq t_p^*|\text{Accept}) \geq 1 - \gamma$ is the customer requirement as mentioned in section 3.3 and $\Pr(t_p \geq t_p^*|\text{Accept})$ can be obtained with the equation (3.3). As equation (3.3) does not have a close form solution, equation (3.4) shows the MCMC method to obtain the value of $\Pr(t_p \geq t_p^*|\text{Accept})$. $\Pr(t_p \geq t_p^*|\text{Accept})$ is approximated by the fractions of simulated values of $t_p$ that are greater than or equal to $t_p^*$. For a given $\epsilon$, $\Pr(t_p \geq t_p^*|\text{Accept})$ can be evaluated for different values of sample sizes ($n$), and the
minimum sample size satisfying the condition \( \Pr(t_p \geq t_p^* | Accept) \geq 1 - \gamma \) is selected as the sample size for an optimal plan of RDT.

3.5 Effect of a Prior on an Optimal Plan

Example systems assumed in this thesis are of components with Weibull and exponential failure time distributions. Gamma prior has been assumed in this thesis being a two-parameter distribution [4]. Assumed gamma priors will have 2 hyperparameters for each parameter \((a, b)\) and will be chosen such that they will have the same mean \((a/b)\) but different variance \((a/b^2)\) values for evaluating the optimal sample size trend across each parameter.

3.6 Example Systems

Two example systems have been used in this thesis for case study. The systems under study include Example I, a two-component series system configuration, and Example II, a three-component combined system configuration. For each of these systems, different lifetime distributions for the components with RBDs are as shown in Figure 3.1 (a) and (b) respectively.
The reliability function for each of them is found by its specific component configuration.

For system (a) the system reliability $R_0(\theta_0)$ is given by

$$R_0(\theta_0) = R_1(\theta_1) \times R_2(\theta_2)$$

where $R_1(\theta_1)$ and $R_2(\theta_2)$ are individual component reliability functions of components 1 and 2. Similarly, for system (b), the system reliability $R_0(\theta_0)$ is given by

$$R_0(\theta_0) = R_1(\theta_1) \times (1 - (1 - R_2(\theta_2))(1 - R_3(\theta_3)))$$

where $R_1(\theta_1)$, $R_2(\theta_2)$, and $R_3(\theta_3)$ are individual component reliability functions of components 1, 2 and 3. As mentioned earlier, different distributions for each of these individual components have been assumed. Example I system consists of two components connected in series, the first with a Weibull distribution and the second with
an exponential distribution. Example II has a Weibull distribution component 1 connected in series with exponential distribution components 2 and 3, which are connected in a parallel configuration to each other as shown in Figure 3.1 (b).

3.7 Investigating Test Duration and Reliability Requirement Effects on RDT

Plan of the Example Systems

The overall effect of example systems on the RDT plan can be summarized through variations in $Pr(t_p \geq t^*_p)$, $Pr(t_p \geq t^*_p|Accept)$, $t_c$, $t^*_p$, $n$ and priors.

For the systems under consideration, the customer specifies the desired quantile life $t^*_p$ at a certain customer assurance $1-\gamma$. $Pr(t_p \geq t^*_p|Accept)$ is obtained by solving the equation (3.3). When $Pr(t_p \geq t^*_p|Accept) \geq 1-\gamma$, the minimum sample size obtained is the optimal sample size $n^*$ for the given priors. To check the effect of different customer requirements at specified $1-\gamma$, $t^*_p$ can be varied to see the variation in the obtained optimal sample size $n^*$ with the numerical analysis.

Similarly, for $n = 0$, i.e., when only prior probability $Pr(t_p \geq t^*_p)$ is desired without the actual test, numerical analysis yields $Pr(t_p \geq t^*_p)$ values at current priors and $t^*_p$ values. If the $Pr(t_p \geq t^*_p)$ value already exceeds the customer requirement $1-\gamma$, then as an optimal plan for RDT, no test is recommended, which may save both time and resources.
4 CASE STUDY

The numerical study that has been performed in this thesis consists of two example systems. Through these two example systems, the effect of various prior distributions and different values of $t_p^*$ on the optimal sample size $n^*$ will be discussed for the specified customer assurance $1-\gamma$ and at given $t_c, T_0$ will be obtained for multiple values of $t_p^*$ for zero failure RDT.

4.1 Priors for the Example Systems

The two example systems considered for the case study had RBD as shown in Figure 3.1 (a) and (b). For each parameter, assumed priors had the same mean but different variance values. In Example I, $\text{Gamma}(a_{\beta}, b_{\beta}), \text{Gamma}(a_{\theta}, b_{\theta})$ and $\text{Gamma}(a_{\lambda_1}, b_{\lambda_1})$ were the priors for $\beta$, $\theta$, and $\lambda_1$ respectively. Priors assumed for Example I were the same as for Example II except instead of $\text{Gamma}(a_{\lambda_1}, b_{\lambda_1})$ additional prior parameters were added for the second and third exponential distribution components as $\text{Gamma}(a_{\lambda_2}, b_{\lambda_2})$ and $\text{Gamma}(a_{\lambda_3}, b_{\lambda_3})$ with the same values as $\text{Gamma}(a_{\lambda_1}, b_{\lambda_1})$. Behavior of increase or decrease in variances at the same mean value of priors to obtain their effect on optimal test sample size will be discussed in this chapter. Table 4.1 shows three sets of hyperparameters differentiated for each parameter.
When Gamma\((a, b)\) prior distribution is considered for the random variable \(X\), its probability density function (PDF) can be stated by

\[
f(x|a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \quad \text{where, } a > 0, b > 0, x > 0.
\]

Table 4.1: Three hyperparameter sets of prior distributions

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Hyperparameters</th>
<th>Variance (a_\beta / b_\beta^2)</th>
<th>Mean (a_\beta / b_\beta)</th>
<th>Priors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beta ((\beta))</td>
<td>(a_\beta) = 900, (b_\beta) = 600</td>
<td>0.0025</td>
<td>1.5</td>
<td>I</td>
</tr>
<tr>
<td></td>
<td>(a_\beta) = 450, (b_\beta) = 300</td>
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<td>1.5</td>
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<tr>
<td></td>
<td>(a_\beta) = 225, (b_\beta) = 150</td>
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<td>1.5</td>
<td>III</td>
</tr>
<tr>
<td>Theta ((\theta))</td>
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<td>10000</td>
<td>1000</td>
<td>I</td>
</tr>
<tr>
<td></td>
<td>(a_\theta) = 50, (b_\theta) = 0.05</td>
<td>20000</td>
<td>1000</td>
<td>II</td>
</tr>
<tr>
<td></td>
<td>(a_\theta) = 25, (b_\theta) = 0.025</td>
<td>40000</td>
<td>1000</td>
<td>III</td>
</tr>
<tr>
<td>Lambda ((\lambda))</td>
<td>(a_\lambda) = 25, (b_\lambda) = 25,000</td>
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<td>0.001</td>
<td>I</td>
</tr>
<tr>
<td></td>
<td>(a_\lambda) = 15, (b_\lambda) = 15,000</td>
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<td>III</td>
</tr>
</tbody>
</table>

These prior distributions are also shown in Figures 4.1, 4.2 and 4.3 with the help of PDF plots. The difference in the priors by their variance values can be identified through these plots.
Figure 4.1: Plot of PDF of Gamma priors of $\beta$

Figure 4.2: Plot of PDF of Gamma priors of $\theta$

Figure 4.2: Plot of PDF of Gamma priors of $\theta$
4.2 Example I: A Two-Component System

The first example of a two-component system is composed of a Weibull component and an exponential component connected in a series configuration. As shown in Figure 3.1(a), components 1 and 2 are the individual components connected in the series configuration having respective reliability functions of $R_1(t)$ and $R_2(t)$.

Component 1 is assumed to have an exponential lifetime distribution with a reliability function $R_1(t) = e^{-\lambda_1 t}$, where $\lambda_1$ is a rate parameter of exponential distribution. Component 2 has been considered to follow the Weibull distribution with a reliability function $R_2(t) = e^{-(t/\theta)^\beta}$, where $\theta$ is a scale parameter and $\beta$ is a shape parameter of Weibull distribution.
The system reliability of this two-component series system $R_0(t)$ is derived to be

$$R_0(t) = R_1(t) \times R_2(t) = e^{-1} \times e^{\frac{-t}{\theta}}.$$ 

Comparing this $R_0(t)$ with the $R_0(t_p | \theta_0)$ in equation (3.2) we get

$$R_0(t_p | \theta_0) = R_0(t) = R_1(t) \times R_2(t) = e^{-\lambda t} \times e^{\frac{-t}{\theta}} = 1 - p$$

and therefore

$$t_p = R_0^{-1}(1 - p | \lambda, \theta, \beta).$$

(4.1)

The Matlab `fzero` function as mentioned in APPENDIX B solves this equation (4.1), where the Gibbs sampling simulated parameter values were obtained from the Winbugs code mentioned in APPENDIX A. $p$ with a value 0.1 has been assumed in both Example I and Example II systems. Assuming three different priors with specified $t_c$ and $t_p^*$ values for $1 - \gamma = 0.95$, numerical trials were conducted to determine the optimal sample size, i.e. $n^*$. The obtained result is $\Gamma_n$ for different $n$ values, where at $\Gamma_n \geq 1 - \gamma$, $n^*$ is obtained.

For example, graphical representation of one of the trials is shown in Figure 4.4 where the acceptable customer assurance $1 - \gamma$ is 0.95 and thus the minimal sample size $n = 29$ that achieved this $\Gamma_n (0.9509)$ at customer specified $t_p^* = 70$ and $t_c = 100$ for the given priors is the optimal sample size $n^*$. Results of this trial are as shown in Table 4.2.
Table 4.2: Optimal plan of RDT for Example I at $1 - \gamma = 0.95$

<table>
<thead>
<tr>
<th>Trial #</th>
<th>$t_c$</th>
<th>$t_p^*$</th>
<th>$n^*$</th>
<th>Hyperparameters</th>
</tr>
</thead>
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<td>900, 600</td>
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<td>15</td>
<td>150</td>
<td>80</td>
<td>29</td>
<td>225, 150</td>
</tr>
<tr>
<td>16</td>
<td>150</td>
<td>70</td>
<td>16</td>
<td>900, 600</td>
</tr>
<tr>
<td>17</td>
<td>150</td>
<td>70</td>
<td>19</td>
<td>450, 300</td>
</tr>
<tr>
<td>18</td>
<td>150</td>
<td>70</td>
<td>18</td>
<td>225, 150</td>
</tr>
<tr>
<td>19</td>
<td>150</td>
<td>70</td>
<td>130</td>
<td>900, 600</td>
</tr>
<tr>
<td>20</td>
<td>100</td>
<td>90</td>
<td>96</td>
<td>450, 300</td>
</tr>
<tr>
<td>21</td>
<td>100</td>
<td>90</td>
<td>60</td>
<td>225, 150</td>
</tr>
<tr>
<td>22</td>
<td>100</td>
<td>80</td>
<td>76</td>
<td>900, 600</td>
</tr>
<tr>
<td>23</td>
<td>100</td>
<td>80</td>
<td>62</td>
<td>450, 300</td>
</tr>
<tr>
<td>24</td>
<td>100</td>
<td>80</td>
<td>43</td>
<td>225, 150</td>
</tr>
<tr>
<td>25</td>
<td>100</td>
<td>70</td>
<td>25</td>
<td>900, 600</td>
</tr>
<tr>
<td>26</td>
<td>100</td>
<td>70</td>
<td>29</td>
<td>450, 300</td>
</tr>
<tr>
<td>27</td>
<td>100</td>
<td>70</td>
<td>27</td>
<td>225, 150</td>
</tr>
</tbody>
</table>
The same analysis was conducted at $n = 0$, where prior probabilities were obtained based on the same priors. If the $\Gamma_0$ value obtained for a specified $t_p^*$ meets the customer assurance $1 - \gamma$, for the given set of priors, then there is no need to perform a RDT for any sample size because the purpose of planning a RDT is already fulfilled, i.e. $\Gamma_0 \geq 1 - \gamma$. For example, if the $1 - \gamma$ requirement for the test is 0.85 and at $t_p^* = 70$ and if the obtained $\Gamma_0 = 0.867$, then there is no need to perform a RDT for any sample size. $\Gamma_0$ values ranged from 0.266 to 0.867 as shown in Table 4.3.
Table 4.3: $\Pr(t_p \geq t_p^*)$ for Example I at $n = 0$

<table>
<thead>
<tr>
<th>Trial #</th>
<th>$t_p^*$</th>
<th>$I_0$</th>
<th>Hyperparameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Beta ($\beta$)</td>
</tr>
<tr>
<td>1</td>
<td>90</td>
<td>0.266</td>
<td>900, 600</td>
</tr>
<tr>
<td>2</td>
<td>90</td>
<td>0.318</td>
<td>450, 300</td>
</tr>
<tr>
<td>3</td>
<td>90</td>
<td>0.39</td>
<td>225, 150</td>
</tr>
<tr>
<td>4</td>
<td>80</td>
<td>0.57</td>
<td>900, 600</td>
</tr>
<tr>
<td>5</td>
<td>80</td>
<td>0.56</td>
<td>450, 300</td>
</tr>
<tr>
<td>6</td>
<td>80</td>
<td>0.552</td>
<td>225, 150</td>
</tr>
<tr>
<td>7</td>
<td>70</td>
<td>0.867</td>
<td>900, 600</td>
</tr>
<tr>
<td>8</td>
<td>70</td>
<td>0.805</td>
<td>450, 300</td>
</tr>
<tr>
<td>9</td>
<td>70</td>
<td>0.725</td>
<td>225, 150</td>
</tr>
</tbody>
</table>

Results of the trials shown in Table 4.2 and Table 4.3 indicated the following:

a. For the same priors as the $t_p^*$ requirement increased, the sample size required to achieve the necessary $1 - \gamma$, i.e. 0.95 also increased as shown in Table 4.2.

b. $t_p^*$ is defined by the customer requirement. If a higher testing time ($t_c$) is provided for a specific test and the required customer quantile life $t_p^*$ of the product is comparatively lower, then the sample size $n$ that must be tested to achieve this lower $t_p^*$ is definitely lower when compared to the lower testing time and higher customer specified quantile life $t_p^*$. Thus, in order to satisfy the accuracy of the prediction, it can be concluded that prior should be strengthened by increasing the number of tests ($n$) for the same test time ($t_c$).

c. For the same prior and $t_p^*$, $n^*$ increased when $t_c$ decreased and when $t_c$ increased, $n^*$ decreased.
d. For the same analysis conducted at $n = 0$ and across the same values of $t_p^*$ from Table 4.2, the obtained results show that for the same prior, $\Gamma_0$ decreased as $t_p^*$ increased.

e. As seen in Table 4.2, when $t_p^* = 80$ or 90, for the same $t_c$, $n_0^*$ decreased when the prior uncertainty increased (i.e., from Prior I to Prior III). As shown in Table 4.3, when $t_p^* = 80$ or 90, the prior probability $\Gamma_0$’s were very low, therefore large sample sizes will be needed planning the test.

f. When $t_p^* = 70$, for the same $t_c$, $n_0^*$ may not be sensitive to the priors. When $t_p^* = 70$, the prior probability $\Gamma_0$’s were close to 0.95, therefore, small sample size may be sufficient for the test.

g. Thus, it may be concluded from both of these tables that as $t_p^*$ decreases, prior probability $\Gamma_0$ increases and may reduce $n^*$ for RDT.

4.3 Example II: A Three-Component System

Similar to Example I, the same numerical analysis was conducted for Example II on a three-component system.

As discussed in section 3.6 for Example II, components 2 and 3 are connected in parallel to each other while their parallel configuration is connected in a series configuration with component 1 as shown in Figure 3.1 (b). Components 1, 2, and 3 have individual reliabilities of $R_1(t)$, $R_2(t)$ and $R_3(t)$ respectively. Considering the respective
components’ lifetime distributions as mentioned in section 3.5, \( R_1(t) = e^{-\lambda t} \), \( R_2(t) = e^{-\lambda t} \), and \( R_3(t) = e^{-\lambda t} \). The system reliability \( R_0(t) \) of this system has been derived to be

\[
R_0(t) = (1 - (1 - R_2(t))(1 - R_3(t))) \times (R_1(t)).
\]

Substituting the individual reliabilities of components, we obtain

\[
R_0(t) = \left(1 - \left(1 - e^{-\lambda t}\right)\left(1 - e^{-\lambda t}\right)\right) \times e^{-\left(\frac{t}{\theta}\right)^\beta}.
\]

Comparing this \( R_0(t) \) with the \( R_0(t|\theta_0) \) in equation (3.2) we get

\[
R_0(t|\theta_0) = R_0(t) = \left(1 - \left(1 - e^{-\lambda t}\right)\left(1 - e^{-\lambda t}\right)\right) \times e^{-\left(\frac{t}{\theta}\right)^\beta} = 1 - p,
\]

and, therefore, \( t_p = R_0^{-1}(1 - p|\lambda_2, \lambda_3, \theta, \beta) \). (4.2)

This equation for \( t_p \) was solved with the help of the Matlab \texttt{fzero} function as mentioned in APPENDIX D, with the parameter values obtained from the Winbugs code mentioned in APPENDIX C.

Assuming three different priors with specified \( t_c \) and \( t_p^* \) values for \( 1 - \gamma = 0.95 \), numerical trials were conducted to determine the optimal sample size i.e. \( n^* \). Similar to Example I, the obtained result is \( \Pr(t_p \geq t_p^*|Accept) \) for different \( n \) values. As the mean and variance remain the same as in Example I, the system configuration of the individual components decides the \( I_n \) values for various \( t_p^* \) and \( t_c \). An optimal plan for RDT for Example II system is shown in Table 4.4.
Results of the Example II analysis indicated similar observations as Example I. For example, a graphical representation of one of the trials is shown in Figure 4.5, where the acceptable customer assurance $1 - \gamma$ is 0.95. Thus, the minimum sample size $n = 39$
that achieved this $1 - \gamma$ at specified $t_p^* = 210$ and $t_c = 300$ by $\Gamma_n(0.9527)$ for the given priors is the optimal sample size $n^*$. 

![Graphical example plot of $\Gamma_n$ vs. $n$ in Example II](image)

**Figure 4.5: Graphical example plot of $\Gamma_n$ vs. $n$ in Example II**

As in Example I, similar analysis was conducted on Example II system at $n = 0$, where, in the Example II system, different $\Gamma_0$ values were obtained for the same priors and same $t_p^*$ values and are shown in Table 4.5.
Table 4.5: Pr($t_p \geq t_p^*$) for Example II at $n = 0$

<table>
<thead>
<tr>
<th>Trial #</th>
<th>$t_p^*$</th>
<th>$\Gamma_0$</th>
<th>Hyperparameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.53</td>
<td>Beta ($\beta$)</td>
</tr>
<tr>
<td>1</td>
<td>220</td>
<td>900, 600</td>
<td>100, 0.1</td>
</tr>
<tr>
<td>2</td>
<td>220</td>
<td>450, 300</td>
<td>50, 0.05</td>
</tr>
<tr>
<td>3</td>
<td>220</td>
<td>225, 150</td>
<td>25, 0.025</td>
</tr>
<tr>
<td>4</td>
<td>210</td>
<td>900, 600</td>
<td>100, 0.1</td>
</tr>
<tr>
<td>5</td>
<td>210</td>
<td>450, 300</td>
<td>50, 0.05</td>
</tr>
<tr>
<td>6</td>
<td>210</td>
<td>225, 150</td>
<td>25, 0.025</td>
</tr>
<tr>
<td>7</td>
<td>200</td>
<td>900, 600</td>
<td>100, 0.1</td>
</tr>
<tr>
<td>8</td>
<td>200</td>
<td>450, 300</td>
<td>50, 0.05</td>
</tr>
<tr>
<td>9</td>
<td>200</td>
<td>225, 150</td>
<td>25, 0.025</td>
</tr>
</tbody>
</table>

Observations from the trial on Example II from Table 4.4 and Table 4.5 are as stated below:

a. As in Example I system, for the same prior parameters, as the $t_p^*$ requirement is increased, the sample size $n^*$ required to achieve this required $t_p^*$, at specified $1 - \gamma$, i.e. 0.95, also increased.

b. $t_p^*$ behavior with $n^*$ is the same as observed in Example I. If the value of $t_p^*$ was dropped compared to the specified $t_c$, sample size required to test at specified $1 - \gamma$ also was dropped and when the $t_p^*$ value was increased, the required sample size to test also increased.

c. $n^*$ increased when $t_c$ decreased, and $t_c$ increased when $n^*$ decreased, similar to Example I for the same prior.
d. For \( n=0 \), for each \( t^*_p \), as the variance was decreased, the \( \Gamma_0 \) value has been observed to be increasing as shown in Table 4.5.

e. Similar to Example I, as seen in Table 4.4, when \( t^*_p = 210 \) or 220, for the same \( t_c \), \( n^* \) decreased when the prior uncertainty was increased from Prior I to Prior III. As shown in Table 4.3, when \( t^*_p = 210 \) or 220, the prior probability \( \Gamma_0 \)'s were very low, therefore large sample sizes will be needed in the test. For \( t^*_p = 200 \), for the same \( t_c \), \( n^* \) may not be sensitive to the priors and as the \( \Gamma_0 \) value is closer to 0.95 small sample size might be sufficient for the test.

f. Similar to the results of Table 4.2 and Table 4.3, it can be seen that as the \( t^*_p \) value was dropped in Table 4.4, required \( n^* \) to achieve the specified \( 1 - \gamma \) also dropped, and for \( n = 0 \) in Table 4.5, when the \( t^*_p \) value was dropped, \( \Gamma_0 \) value obtained increased across the priors. Thus, similar to Example I, it may be concluded from both of these cases that for decreased \( t^*_p \), prior probability \( \Gamma_0 \) may increase and may decrease the value of \( n^* \) during RDT.
5 CONCLUSION, CONTRIBUTION, AND FUTURE RESEARCH

With this thesis, the planning of a reliability demonstration test for multiple-component systems with the help of Bayesian methodology has been explained. Component level prior information from each component has been utilized with a joint prior distribution in order to obtain the optimal sample size for various test duration as well as customer specified quantile life. An objective achieved from this thesis is the planning of a zero failure system-level RDT from the component-level prior, for a customer specified quantile life, with desired customer assurance, at a minimum possible sample size.

5.1 Conclusion

As per the objective and tasks described in section 1.1 for this thesis, the Bayesian approach has been applied to achieve a RDT that can yield the desired results of customer assurance at a minimum sample size. Though the results obtained show a specific trend for sample size prediction, these results are restricted for assumed distributions and component configurations for certain priors. The major outcomes that can be derived from the numerical analysis are as follows:

- $t^*_p$ is a very important factor while making a decision on optimal sample size $n^*$ for a specified customer assurance $1 - \gamma$. For $n = 0$, here $t^*_p$ decided the trend of $\Gamma_0$ because for the specified $t^*_p$ only prior information will be utilized to calculate $t_p$, because the likelihood will not be affected by $n$ and $t_c$. Here, for the same prior and $t_c, n^*$
increased when $t^*_p$ was increased, which may be explained by $\Gamma_0$ trend. For the same prior, $\Gamma_0$ decreased as $t^*_p$ increased. The lower the prior probability $\Gamma_0$, the larger the sample size might be needed for the test.

For the same prior and $t^*_p$, $n^*$ increased when $t_c$ decreased. Thus, there was a tradeoff between the test duration and the sample size. The longer the test duration, the smaller was the sample size needed.

For the current systems when the prior uncertainty is larger, for example with Prior III, a relatively smaller sample size will provide larger influence to the posterior distribution. On the other hand, when the prior uncertainty is lower, for example with Prior I, a larger sample size is needed to affect the posterior distribution. When sample sizes are small, the test data may not affect the posterior distributions significantly. Therefore, the three priors yield similar sample sizes for low $t^*_p$.

With the above outcomes obtained from the planned RDT and considering $\Gamma_0$ trend across both the case study examples, it may be concluded that the increase in customer assurance demand increases the sample size required to test.

5.2 Contribution

The contribution part through this thesis is the Bayesian application for zero failure RDT. As it has been already discussed, traditional MLE based zero-failure RDT planning methods generally work with simple cases such as exponential, binomial, and Weibull with a fixed shape parameter, and large number of units are required to be tested by the MLE plans [9]. The Bayesian approach introduced is used to combine priors from
the components with Weibull and exponential lifetime distribution in a system for calculating the minimum sample size $n^*$ for the required customer assurance $1 - \gamma$. This helps integrate component-level prior information into a system with continuous updating of parameters for more accuracy.

Achievement of lowest possible testing sample size through inputs from customer and prior component information helps in saving time and costly system-level testing with this research.

5.3 Future Research

Systems considered here were simple systems with two and three component configurations. Components having Weibull and exponential distributions were tested in example systems to plan an optimal RDT for the desired customer requirements. The next venture in this research would be to utilize other distributions for the components and confirm the effect on sample size for multiple values of $t_p^*$ and $t_c$ for required $1 - \gamma$. Priors can also be varied by changing variance and mean of prior parameters. More complicated systems can be tested to verify the results of this distribution and utilize it for creating advanced formulation. Gamma distribution has been used in this thesis, but uniform distribution can also be checked as the next step in the process.
REFERENCES


APPENDIX A: WINBUGS CODE FOR EXAMPLE I

#Model
model{
    C <- 100000
    zero <- 0
    phi <- -log(L) + C
    L <- exp(-n*pow(tc/theta1, beta1) - n*lambda1*tc)
    zero ~ dpois(phi)
    beta1 ~ dgamma(abeta, bbeta)
    theta1 ~ dgamma(atheta, btheta)
    lambda1 ~ dgamma(alambda, blambda)
}
APPENDIX B: MATLAB CODE FOR EXAMPLE I

% Initialization step where $K$, $t_c$, $t_p^*$, $p$ and $n$ values are stated

clear;
nite = 100000;
lntp = zeros(1,nite);
Result = [];
p = 0.1;
tc = 100;
lnptc = log(70);
for n = 26:30;

% this step incorporates priors into the distribution... here ~450,300 prior parameter
% set has been used as shown in Figure 4.4.

% Prior I
dataStruct = struct(...
'abeta', 900, ...%
'bbeta', 600, ...%
'atheta', 100, ...%
'btheta', 0.1, ...%
'alambda',25, ...
'blambda',25000, ...
'n', n, ...
'tc', tc);

% Prior II
dataStruct = struct(...
'abeta', 450, ...
'bbeta', 300, ...
'atheta', 50, ...
'btheta', 0.05, ...
'alambda',15, ...
'blambda',15000, ...
'n', n, ...
'tc', tc);

% Prior III
dataStruct = struct(...
'abeta', 225, ...
'bbeta', 150, ...
'atheta', 25, ...
'btheta', 0.025, ...
% 'alambda', 6.25, ...
% 'blambda', 6250, ...
% 'n', n, ...
% 'tc', tc);

% this part calls for Winbugs for Gibbs sampling

init0 = struct('beta1', 1.5, 'theta1', 1000, 'lambda1', 0.001);
[samples, stats] = matbugs(dataStruct, ...
fullfile(pwd,'test_series_gamma.txt'), ...
'init', init0, ...
'view', 0, 'nburnin', nite, 'nsamples', nite, ...
'thin', 1, 'nChains', 1,...
'monitorParams', {'beta1', 'theta1', 'lambda1'}),...
'Bugdir', 'C:\Users\Manish\Downloads\winbugs14\WinBUGS14');

% at this step \( t_p \) is calculated

for i = 1 : nite
    be1 = samples.beta1(i);
    th1 = samples.theta1(i);
    la1 = samples.lambda1(i);
    lntp(i) = fzero(@(x) (exp(x)/th1).^be1+la1*exp(x)+log(1-p), log(100));
end

% this step involves plotting of \( \Gamma_n \) vs. \( n \) graph~ gamma specified in this code is customer
% assurance given by \( \Gamma_n \)

gamma = sum(lntp>=lntpc)/nite;
Result = [Result; [n gamma]];

end
plot(Result(:,1),Result(:,2),'o');
# Model
model{
    C <- 100000
    zero <- 0
    phi <- -log(L) + C
    L <- (1-(1-exp(- n*lambda2*tc))*(1-exp(- n*lambda3*tc)))*exp (-pow(tc/theta1, beta1))
    zero ~ dpois(phi)
    beta1 ~ dgamma(abeta, bbeta)
    theta1 ~ dgamma(atheta, btheta)
    lambda2 ~ dgamma(alambda, blambda)
    lambda3 ~ dgamma(clambda, dlambda)
}
APPENDIX D: MATLAB CODE FOR EXAMPLE II

% Initialization step where $K$, $t_c$, $t_p^*$, $p$ and $n$ values are stated

clear;
nite = 100000;
lntp = zeros(1,nite);
Result = [];
p = 0.1;
tc = 300;
lntpc = log(210);
for n = 33:45;

% this step utilizes priors into the distribution… here ~450,300… prior parameter set has been used as shown in Figure 4.5.

% Prior I
% dataStruct = struct(...
% 'abeta', 900, ...
% 'bbeta', 600, ...
% 'atheta', 100, ...
% 'btheta', 0.1, ...
% 'alambda',25, ...
% 'blambda',25000, ...
% 'clambda',25, ...
% 'dlambda',25000, ...
% 'n', n, ...
% 'tc', tc);

% Prior II
dataStruct = struct(...
'abeta', 450, ...
'bbeta', 300, ...
'atheta', 50, ...
'btheta', 0.05, ...
'alambda',15, ...
'blambda',15000, ...
'clambda',15, ...
'dlambda',15, ...
'n', n, ...
'tc', tc);
% Prior III
% dataStruct = struct(...
% 'abeta', 225, ...
% 'bbeta', 150, ...
% 'atheta', 25, ...
% 'btheta', 0.025, ...
% 'alambda', 6.25, ...
% 'blambda', 6250, ...
% 'clambda', 6.25, ...
% 'dlambda', 6250, ...
% 'n', n, ...
% 'tc', tc);

% this part calls for Winbugs for Gibbs sampling

init0 = struct('beta1', 1.5, 'theta1', 1000, 'lambda2', 0.001, 'lambda3', 0.001);
[samples,stats] = matbugs(dataStruct, ...
'init', init0,
'view', 0, 'nburnin', nite, 'nsamples', nite,
'thin', 1, 'nChains', 1,
'monitorParams', {'beta1', 'theta1', 'lambda2', 'lambda3'},
'Bugdir', 'C:\Users\Manish\Downloads\winbugs14\WinBUGS14');

% at this step \( t_p \) is calculated

for i = 1 : nite
  be1 = samples.beta1(i);
  th1 = samples.theta1(i);
  la2 = samples.lambda2(i);
  la3 = samples.lambda3(i);
  lntp(i) = fzero(@(x) (exp(x)/th1).^be1-log((exp(x))^(-la2)+ (exp (x))^(-la3)-(exp(x))^(-la2-la3))+log(1-p), log(150));
end

% this step involves plotting of \( \Gamma_n \) vs. \( n \) graph—gamma specified in this code is customer
% assurance given by \( \Gamma_n \)

gamma = sum(lntp>=lntpc)/nite;
Result = [Result; [n gamma]];

end
plot(Result(:,1),Result(:,2),'o');