Rings Characterized by Their Modules

A dissertation presented to
the faculty of
the College of Arts and Sciences of Ohio University

In partial fulfillment
of the requirements for the degree
Doctor of Philosophy

Christopher J. Holston
August 2011
© 2011 Christopher J. Holston. All Rights Reserved.
This dissertation titled
Rings Characterized by Their Modules

by
CHRISTOPHER J. HOLSTON

has been approved for
the Department of Mathematics
and the College of Arts and Sciences by

Dinh van Huynh
Professor of Mathematics

Sergio López-Permouth
Professor of Mathematics

Benjamin M. Ogles
Dean for College of Arts and Sciences
Abstract

HOLSTON, CHRISTOPHER J., Ph.D., August 2011, Mathematics

Rings Characterized by Their Modules (48 pp.)

Directors of Dissertation: Dinh van Huynh and Sergio López-Permout

A ring $R$ is called a right $WV$-ring if each simple right $R$-module is injective relative to proper cyclics. A right $WV$-ring which is not a right $V$-ring is called a right strictly $WV$-ring. A right strictly $WV$-ring $R$ has only three two-sided ideals. If, in addition, $R J(R)$ is finitely generated, then it has only three right ideals. It is shown that, given a cyclic right module $C$ over a right $WV$-ring $R$, $C$ is noetherian iff every cyclic module in $\sigma[C]$ is a direct sum of a module which is projective in $\sigma[C]$ with a module which is either CS or has finite uniform dimension. A module is called p-poor if it is projective only with respect to the semisimple modules. Every ring has a semisimple p-poor module. A ring $R$ is said to have no right p-middle class if every right $R$-module is either projective or p-poor. It is shown that a ring $R$ with no right p-middle class is isomorphic to $S \times K$, where $S$ is semisimple artinian and $K$ is an indecomposable ring. This $K$ is either zero or exactly one of the following: (i) a semiprimary right SI-ring, (ii) a semiprimary ring with $Soc(K_K) = Z(K_K) = J(K)$, (iii) a prime ring with $Soc(K_K) = 0$, where either $J(K) = 0$ or $K$ is neither left nor right noetherian, or (iv) a ring with infinitely generated essential right socle and $J(K) = 0$.

Approved: 

Dinh van Huynh
Professor of Mathematics

Sergio López-Permout
Professor of Mathematics
To my mother, who has given everything she had to see me succeed, I love you.
ACKNOWLEDGEMENTS

There is no way I can give anyone on this page the respect and honor they deserve, but something is greater than nothing.

To all my fellow teaching assistants with whom I have shared classes, office space, and thoughts, thank you for always being kind to me.

To Ryan Schwiebert, thank you for the helpful chats and the wonderful diagrams.

To Joshua Beal, Donald Daws, Adam Feldhaus, Douglas Hoffman, Son Nguyen, Bismark Oduro, and Derrick Stover, thank you for getting me out of my apartment and office enough to keep my sanity.

To Dr. Jain, thank you for all your hard work in helping me finish a project before your retirement, and thank you for the opportunity so early in my researching career.

To Dr. Huynh, thank you for answering all my questions at all hours, and exposing me to an astonishingly wide range of useful topics.

To Dr. López, thank you for helping me stay focused during a difficult time, and giving me a chance to work within my areas of strength.

To my family, thank you for your constant faith in my abilities, and always speaking proudly of me.

To my mother, thank you for your ability to truly understand me. Thank you for giving the perfect combination of encouragement and space to work.

To my Lord and Savior, thank you for leading me here to begin with, and carrying me through. I hope to never forget to appreciate the things you do that I am not capable of understanding.
# Table of Contents

Abstract ................................................................. 3  
Dedication ................................................................. 4  
Acknowledgements ..................................................... 5  
1 Preliminaries .......................................................... 7  
1.1 Introduction ....................................................... 7  
1.2 Definitions ....................................................... 8  
1.3 Useful Results .................................................... 15  
2 On V-modules and V-rings ........................................... 24  
2.1 WV-rings ........................................................... 24  
2.2 On Cyclic Modules Being a Direct Sum of a Projective Module and a CS or Fud Module ................................. 30  
3 Modules with Maximal or Minimal Projectivity Domains ........ 33  
3.1 P-poor modules ..................................................... 33  
3.2 Rings with No P-middle Class ................................... 36
1 PRELIMINARIES

1.1 Introduction

The question of studying homological properties on modules with connections to properties of the underlying ring dates back to the late 1950s, when Bass and Papp independently showed that a ring is right noetherian if and only if direct sums of injective right modules are injective [3, 28]. Since then, there has been continuous interest on finding properties on classes of modules that guarantee a finiteness property to the ring. To name a few, first is the classic result of Osofsky, that if every cyclic module is injective, then the ring is semisimple artinian [26]. If each cyclic right module over a ring is an injective or projective module [12], a direct sum of an injective module and a projective module [27, 30], or a direct sum of a projective module and a module which is either injective or noetherian [19], then the ring is right noetherian. If every finitely generated module is CS, then the ring is right noetherian [20]. If every cyclic right module is CS, then $R$ is a right qfd-ring [27].

In this paper, we work with a generalization of V-rings, called WV-rings, which were first introduced in [16]. A ring $R$ is called a right WV-ring if every simple right $R$-module is injective relative to every proper cyclic right $R$-module. A ring is called right strictly WV if it is right WV but not right V. Detailed study as to how WV-rings differ from V-rings is provided in section 2.1. Indeed, a right strictly WV-ring must be right uniform (2.1.5). If $J$ is the Jacobson radical of a right strictly WV-ring $R$, then $J_R$ is simple (2.1.10), $J$ is the only nontrivial two-sided ideal of $R$, and if $rJ$ is finitely generated, then $J$ is the only nontrivial right ideal of $R$ (2.1.13).

In section 2.2, we introduce a property on modules, and say that a module $M$ satisfies (*) if we can write $M = A \oplus B$, where $A$ is projective in $\sigma[M]$, and $B$ is either CS or has finite uniform dimension. If $R$ is a right WV-ring, and $C$ is a cyclic right $R$-module such
that every cyclic module in $\sigma[C]$ satisfies (*), then $C$ is noetherian. In particular, a right $WV$-ring $R$ is noetherian iff every cyclic $R$-module satisfies (*).

In chapter 3, we switch from dealing with injectivity to the dual concept of projectivity. We say that a module is p-poor if it is projective only with respect to the semisimple modules. We say that a ring $R$ has no right p-middle class if every right $R$-module is either projective or p-poor. The study of the injective version of this question has already begun in [1, 8]. However, results do not follow dually, as for example the main result of section 3.1, that every ring has a semisimple p-poor module (3.1.8), does not hold in the injective case [8]. The main result of chapter 3 (3.2.11), is that if $R$ is a ring with no right p-middle class, then $R \cong S \times K$, where $S$ is semisimple artinian and $K$ is zero or indecomposable (as a ring) with exactly one of the following properties: (i) $K$ is a semiprimary right SI-ring, (ii) $K$ is a semiprimary ring with $S oc(K_K) = Z(K_K) = J(K)$, (iii) $K$ is a prime ring with $S oc(K_K) = 0$, where either $J(K) = 0$ or $K$ is neither left nor right noetherian, (iv) $K$ is a ring with infinitely generated essential right socle and $J(K) = 0$. We also give some examples of rings with no right p-middle class. Despite the fact that chapter 2 deals with injectivity domains and chapter 3 deals with projectivity domains, the last result ties both sections together, namely that a right strictly $WV$-ring $R$ with $_R J(R)$ finitely generated has no right p-middle class.

1.2 Definitions

We shall assume that all rings have multiplicative identity $1 \neq 0$ and all modules are unital. By $M_R$ ($_RM$) we mean $M$ is a right (left) module over the ring $R$. The class of right $R$-modules is denoted $Mod \ - \ R$. A submodule $N$ of a module $M_R$ will be denoted as $N \leq M$. The kernel and image of an $R$-homomorphism $f$ are denoted $Ker(f)$, $Im(f)$, respectively. The ring of $R$-homomorphisms from $M$ to $N$ is denoted $Hom_R(M, N)$, and $End_R(M) = Hom_R(M, M)$. If the underlying ring is clear from the context, we often write
Hom(M, N), End(M). Similar conventions will apply to any other structures which depend on the underlying ring. For any standard terminology or notation not defined here, please refer to [2], [22].

**Definition 1.2.1** A module $N_R$ is a direct summand of $M_R$ (written $N \leq \oplus M$) if there exists a submodule $N'$ of $M$ such that $M = N + N'$ and $N \cap N' = 0$ (written $M = N \oplus N'$). $M^{(I)}$ denotes the direct sum $\oplus_{i \in I} M_i$, where $M_i = M$ for all $i \in I$.

**Definition 1.2.2** A module $M_R$ is called simple if it is nonzero, and the only nonzero submodule of $M$ is $M$.

**Definition 1.2.3** A ring $R$ is called simple if the only two-sided ideals of $R$ are 0 and $R$. Note this is weaker than $R_R$ being simple, in which case $R$ is a division ring.

**Definition 1.2.4** For a positive integer $n$, a module $M_R$ is called $n$-generated if $M = \sum_{i=1}^{n} x_i R$ for some $x_1, \ldots, x_n \in M$. In particular, a module is called finitely generated if it is $n$-generated for some positive integer $n$. If no such $n$ exists, $M$ is called infinitely generated.

**Definition 1.2.5** A module $M_R$ is called cyclic if it is 1-generated.

**Definition 1.2.6** A submodule $N$ of a module $M_R$ is called maximal (written $N \leq m M$) if $M \leq M' \leq N$ implies $M' = M$ or $M' = N$.

**Definition 1.2.7** A module $M_R$ is called $N_R$-injective if, for every submodule $X$ of $N$, and every $R$-homomorphism $f : X \to M$, there exists an $R$-homomorphism $g : N \to M$ such that $g(x) = f(x)$ for every $x \in X$ (in this case we say $g$ extends $f$).

**Definition 1.2.8** Given a module $M_R$, the class $\mathcal{I}^{-1}(M) = \{N_R | M \text{ is } N \text{ - injective} \}$ is called the injectivity domain of $M$. 
Definition 1.2.9  A module $M_R$ is called injective if $\mathcal{In}^{-1}(M) = \text{Mod} - R$.

Definition 1.2.10  A module $M_R$ is called quasi-injective if $M$ is $M$-injective. It is also called QI [22] or self-injective [31].

Definition 1.2.11  A ring $R$ is called right self-injective if $R_R$ is quasi-injective.

Definition 1.2.12  A module $M_R$ is called $N_R$-projective, if for every module $T_R$, and every epimorphism $h \in \text{Hom}(N, T)$, and every homomorphism $f \in \text{Hom}(M, T)$, there exists $g \in \text{Hom}(M, N)$ such that $hg = f$ (in this case we say that $g$ lifts $f$ or that $f$ can be lifted to $g$). Frequently we will use the fact that it suffices to check only when $T = N/\text N'$ for some $\text N' \leq N$.

Definition 1.2.13  Given a module $M_R$, the class $\mathcal{Pr}^{-1}(M) = \{N_R| N$ is $N$-projective$\}$ is called the projectivity domain of $M$.

Definition 1.2.14  A module $M_R$ is called projective if $\mathcal{Pr}^{-1}(M) = \text{Mod} - R$.

Definition 1.2.15  A submodule $N$ of a module $M_R$ is said to be essential in $M$ (written $N \leq_e M$) if, for any submodule $X$ of $M$, we have $N \cap X \neq 0$.

Definition 1.2.16  A module $M_R$ is called uniform if $M \neq 0$, and for any two nonzero submodules $N_1, N_2$ of $M$, we have $N_1 \cap N_2 \neq 0$.

Definition 1.2.17  Given a submodule $N$ of a module $M_R$, we say that $X$ is a closure of $N$ in $M$ (often we write $X = N^c$) if $N \leq_e X$ and, for any submodule $Y$ of $M$ with $X \leq_e Y$ we must have $X = Y$. If $N$ is a closure of $N$ in $M$, then we say $N$ is closed in $M$.

Definition 1.2.18  For a nonnegative integer $n$, a module $M_R$ is said to have uniform dimension $n$ (written $u.d.(M) = n$) if there exists $V \leq_e M$ such that $V$ is a direct sum of $n$ uniform submodules. In particular, we say $M$ has finite uniform dimension (written $f.u.d$). If no such $V$ exists for any $n$, we say $M$ has infinite uniform dimension.
Definition 1.2.19 The Jacobson radical of a module $M_R$ (written $J(M)$) is the intersection of all maximal submodules of $M$. The Jacobson radical of a ring is $J(R_R) = J(R_R)$.

Definition 1.2.20 The socle of a module $M_R$ (written $Soc(M)$) is the sum of all simple submodules of $M$, or equivalently, the intersection of the essential submodules of $M$. The right (left) socle of a ring is $Soc(R_R) (Soc(R_R))$. A ring is said to have homogeneous right socle if $Soc(R_R) \cong S^{(i)}$ for some simple module $S$ and index set $I$.

Definition 1.2.21 The singular submodule of a module $M_R$ (written $Z(M)$) is $\{ x \in M | xE = 0 \text{ for some } E \leq e \}$. The right singular ideal of a ring $R$ is $Z_r(R) = Z(R_R)$.

Definition 1.2.22 A module $M_R$ is called nonsingular if $Z(M) = 0$.

Definition 1.2.23 A module $M_R$ is called singular if $Z(M) = M$.

Definition 1.2.24 Given a subset $S$ of a module $M_R$, the right annihilator of $S$ in $R$ is given by $r.ann_R(S) = \{ r \in R | Sr = 0 \}$. If $S = \{ x \}$, we instead write $r.ann_R(x)$. (The left annihilator $l.ann_R(S)$ is defined similarly.) Note $r.ann_R(S)$ is a right ideal of $R$, and if $S \leq M$, then $r.ann_R(S)$ is a two-sided ideal of $R$.

Definition 1.2.25 A ring $R$ is called local if there exists a unique maximal right ideal $M$ of $R$. (In particular, $M = J(R)$.)

Definition 1.2.26 An element $e$ of a ring $R$ is called an idempotent if $e^2 = e$. 0, 1 are called trivial idempotents.

Definition 1.2.27 An element $a$ of a ring $R$ is called central if $ar = ra$ for every $r \in R$.

Definition 1.2.28 A pair of elements $a, b$ of a ring $R$ are called orthogonal if $ab = 0 = ba$. 
Definition 1.2.29 A module $M_R$ is called semisimple if any of the following equivalent conditions hold:

(i) $M$ is a direct sum of simple modules

(ii) $M$ is a sum of simple modules

(iii) $M$ is a direct sum of its homogeneous components

(iv) For every $N \leq M$, we have $N \leq_\oplus M$

The class of semisimple right modules is denoted $SS Mod - R$.

Definition 1.2.30 A ring $R$ is called semisimple artinian if any of the following equivalent conditions hold:

(i) $R_R$ is semisimple

(ii) $R_R$ is semisimple

(iii) $SS Mod - R = Mod - R$

Definition 1.2.31 A right ideal $I$ of a ring $R$ is called nilpotent if $I^n = 0$ for some integer $n$.

Definition 1.2.32 A right ideal $I$ of a ring $R$ is called nil if, for all $r \in I$, $r^n = 0$ for some integer $n$. Clearly a nilpotent right ideal is nil.

Definition 1.2.33 A ring $R$ is called semilocal if $R/J(R)$ is semisimple artinian.

Definition 1.2.34 A ring $R$ is called prime if $\forall a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$.

Definition 1.2.35 A ring $R$ is called semiprime if it contains no nilpotent two-sided ideals.

Definition 1.2.36 A ring $R$ is called semiprimary if it is semilocal and $J(R)$ is nilpotent.

Definition 1.2.37 A module $M_R$ is called noetherian if it has ACC, i.e. every ascending chain of submodules of $M$ stabilizes after finitely many terms. A ring $R$ is called right noetherian if $R_R$ is noetherian.
Definition 1.2.38 A module $M_R$ is called artinian if it has DCC, i.e. every descending chain of submodules of $M$ stabilizes after finitely many terms. A ring $R$ is called right artinian if $R_R$ is artinian.

Definition 1.2.39 A ring $R$ is called right SI if every singular right module $M_R$ is injective.

Definition 1.2.40 Given a module $M_R$, a module $N_R$ is called $M$-generated if there exists an index set $I$ such that $N^{(I)}$ maps onto $M$.

Definition 1.2.41 The category $\sigma[M]$ is defined as the full subcategory of $\text{Mod} - R$ whose objects are the $M$-generated modules and their submodules. (For example, $\sigma[R_R] = \text{Mod} - R$.)

Definition 1.2.42 A module $M_R$ is called a V-module if every simple module in $\sigma[M]$ is $M$-injective. A ring is called a right V-ring if every simple right $R$-module is injective. (In view of 1.3.15, this is equivalent to every right $R$-module being a V-module.)

Definition 1.2.43 A ring $R$ is called von Neumann regular if $\forall a \in R$, $\exists x \in R$ such that $axa = a$.

Definition 1.2.44 A ring $R$ is called a domain if $ab = 0$ implies $a = 0$ or $b = 0$.

Definition 1.2.45 A domain $R$ is called right Ore if, for every pair of nonzero elements $a, b \in R$, we have $aR \cap bR \neq 0$.

Definition 1.2.46 A module $M$ is called CS (or extending), if every closed submodule of $M$ is a direct summand.

Definition 1.2.47 Given a module $M_R$, a module $E$ is called an injective hull of $M$ if $E$ is injective and $M \leq E$. Such an $E$ is unique up to isomorphism, and is thus denoted $E(M)$.
Definition 1.2.48  Given a module $M_R$, a submodule $S \leq M$ is called small (written $S \leq_s M$) if for every submodule $N \leq M$, $S + N = M$ implies $N = M$.

Definition 1.2.49  Given a module $M_R$, we say $P_R$ is a projective cover of $M$ if $P_R$ is projective, and there exists an epimorphism $f : P \rightarrow M$ such that $\text{Ker}(f) \leq_s P$.

Definition 1.2.50  A module is called uniserial if its submodules are linearly ordered. A ring $R$ is called right uniserial if $R_R$ is uniserial.

Definition 1.2.51  A ring $R$ is called right generalized uniserial if it is right artinian and $R_R$ is a direct sum of uniserial modules.

Definition 1.2.52  A cyclic module is called proper if it is not isomorphic to $R$.

Definition 1.2.53  A ring $R$ is called a right PCI-ring if every proper cyclic module is injective. If $R$ is also not semisimple artinian, $R$ is called a right PCI-domain.

Definition 1.2.54  A ring is called right hereditary if every submodule of a projective right module is projective.

Definition 1.2.55  A ring is called a principal right ideal domain (PRID) if every right ideal is cyclic.

Definition 1.2.56  A ring is called right perfect if every right $R$-module has a projective cover. A semiprimary ring is right (left) perfect.

Definition 1.2.57  A ring is called semiperfect if every finitely generated right (left) $R$-module has a projective cover. A right (left) perfect ring is semiperfect.

Definition 1.2.58  A module $M$ is called a qfd module if every factor module of $M$ has $fud$. 
1.3 Useful Results

This section is a collection of results used in the paper. The first few results are well-known, and are a sub-collection of results that will be used without reference throughout the paper. They are included (without proof) for the convenience of the reader.

**Lemma 1.3.1** If $M_R, N_R$ are modules, then $(M + N)/M \cong N/(M \cap N)$.

**Lemma 1.3.2** If $C \leq A_R, D \leq B_R$, then $(C \oplus D)/A \oplus B) \cong C/A \oplus D/B$.

**Lemma 1.3.3** If $f \in \text{Hom}_R(M, N)$ for some modules $M_R, N_R$, then $M/\text{Ker}(f) \cong \text{Im}(f)$.

**Corollary 1.3.4** If $R$ is a ring and $x \in R$, then $xR \cong R/\text{ann}(x)$.

**Corollary 1.3.5** If $R$ is a ring, a module $M_R$ is cyclic iff $M_R \cong (R/A)_R$ for some $A \leq R_R$.

**Lemma 1.3.6** If $A_R, B_R$ are modules, $f \in \text{Hom}(A, B)$ is an isomorphism, and $C$ is such that $C \leq A, C \leq B$, and $C = f(C)$, then $A/C \cong B/C$.

**Theorem 1.3.7** (correspondence theorem) If $M_R$ is a module and $N \leq M$, then there is a one to one correspondence between the submodules of $M/N$ and the submodules of $M$ containing $N$.

**Lemma 1.3.8** $\bigoplus_{i \in I} M_i$ is $N$-projective iff each $M_i$ is $N$-projective.

**Lemma 1.3.9** If $A, B$ are right ideals of a ring $R$ such that $R_R = A \oplus B$, and $\text{Hom}(A, B) = 0 = \text{Hom}(B, A)$, then $R_R = A \oplus B$, i.e. $R$ is not an indecomposable ring.

**Lemma 1.3.10** If $A_R$ and $B_R$ are modules, and for every nonzero submodule $B' \leq B$ we have $\text{Hom}(A, B/B') = 0$, then $A$ is $B$-projective.

**Lemma 1.3.11** If $R$ is semisimple artinian, then $J(R) = 0$. 
Lemma 1.3.12  If $R$ is right artinian, then $J(R)$ is nilpotent.

Lemma 1.3.13  If $A_R, B_R$ are modules with $A$ uniform and $A \leq_e B$, then $B$ is uniform.

Lemma 1.3.14  A right artinian ring is right perfect.

Here is a list of results from outside an introductory course that may not be obvious from definition. When reasonable, proofs are included. Otherwise, specific references are provided.

Theorem 1.3.15  A module $M_R$ is injective if it is $R_R$-injective.

Proof. (3.7) in [22]. □

Theorem 1.3.16  A module $M_R$ has infinite uniform dimension if $M$ contains an infinite direct sum of nonzero submodules.

Proof. (6.4) in [22]. □

Theorem 1.3.17  Let $A$ be a submodule of a module $M_R$. Let $A^c$ denote a closure of $A$ in $M$. Then $u.dim(A) + u.dim(M/A) = u.dim(M) + u.dim(A^c/A)$.

Proof. (6.37) in [22]. □

Theorem 1.3.18  $R_R$ is projective.

Proof. Let $A$ and $B$ be $R$-modules, $f : R_R \to B$ be an $R$-homomorphism, and $g : A \to B$ be a surjective $R$-homomorphism. Set $b = f(1)$. Since $g$ is surjective, there exists $a \in A$ such that $g(a) = b$. Define $h : R_R \to A$ by $h(r) = ar$. One can check that $h$ is an $R$-homomorphism. Note that $gh(r) = g(ar) = g(a)r = br = f(1)r = f(r)$, so $gh = f$ as required. □

Lemma 1.3.19  Let $f : M \to N$ and $f' : N \to M$ be homomorphisms such that $f f' = 1_N$. Then $f$ is an epimorphism, $f'$ is a monomorphism and $M = Ker(f) \oplus Im(f')$. 
Lemma 5.1 in [2].

**Proposition 1.3.20** If \((M/N)_R\) is \(M\)-projective, then \(N \leq \oplus M\).

**Proof.** Assume \(M/N\) is \(M\)-projective. Let \(f = 1_{M/N}\) (the identity of \(\text{End}(M/N)\)), and let \(h : M \to M/N\) denote the canonical mapping. By hypothesis, there exists \(g \in \text{Hom}(M/N, M)\) such that \(hg = f\). By 1.3.19, \(M = \text{Ker}(g) \oplus \text{Im}(h) = N \oplus \text{Im}(h)\). Thus \(N \leq \oplus M\). □

**Proposition 1.3.21** If \(M \leq N\) and \(M\) is \(N\)-injective, then \(M \leq \oplus N\).

**Proof.** The proof works dually to that of 1.3.20. □

**Lemma 1.3.22** For a right ideal \(A\) of a ring \(R\), \(A \leq \oplus R\) iff \(A = eR\) for some idempotent \(e \in R\).

**Proof.** \((\Leftarrow)\) Clearly \(R = eR \oplus (1 - e)R\).

\((\Rightarrow)\) \(R = A \oplus B\) for some right ideal \(B\) of \(R\). So we can write \(1 = e + b\) for some \(e \in A, b \in B\). Thus \(e = e^2 + be\), so \(be = e - e^2 \in A \cap B = 0\). Hence \(e = e^2\). Let \(a \in A\). Then \(a = ea + ba \Rightarrow a - ea \in A \cap B = 0 \Rightarrow a = ea\), so \(a \in eR\). Hence \(A = eR\). □

**Lemma 1.3.23** If \(R\) is a ring, then \(J(R)\) contains no nonzero idempotents.

**Proof.** Corollary 15.11 in [2]. □

**Lemma 1.3.24** If \(R\) is a ring, and \(A\) is a nonzero right ideal contained in \(J(R)\), then \(A \not\leq \oplus R\).

**Proof.** Follows immediately from 1.3.22 and 1.3.23.

**Lemma 1.3.25** Every nilpotent ideal is contained in \(J(R)\).

**Proof.** Corollary 15.10 in [2]. □
Theorem 1.3.26  Let $M$ be an $R$-module and $P$ a finitely generated $M$-projective module. Assume that $M$ is $P$-cyclic and satisfies the condition: (every $P$-cyclic subfactor of $M$ is a direct sum of a CS module and a module with fud). Then $M$ has fud.

Proof. Theorem 1.3 in [7].

Lemma 1.3.27  Assume that for each essential submodule $K$ of the module $M$, the factor module $M/K$ has fud. Then $M/Soc(M)$ has fud.

Proof. Lemma 2.9 in [7].

Lemma 1.3.28  If $M$ is a finitely generated CS-module and $\oplus M_i$ is an infinite direct sum of nonzero submodules of $M$, then $M/\oplus M_i$ cannot have fud.

Proof. Lemma 9.1 in [31].

Lemma 1.3.29  If $M$ is a finitely generated module such that every maximal submodule is a direct summand, then $M$ is semisimple.

Proof. Let $T \leq M$ with $T \neq M$. Since $M$ is finitely generated, $\exists N \leq M$ with $T \leq N$. Thus $N \leq M$ and $N \leq M$, which implies $N = M$, a contradiction. Hence $\text{Soc}(M) = \cap\{X \leq M| X \leq M\} = M$, i.e. $M$ is semisimple.

Recall that if $I$ is a two-sided ideal of $R$ then we can define a quotient ring $R/I$. If $M$ is an $R/I$-module (i.e. $M_{R/I}$) then there is a map $\varphi : M \times R/I \to M$ with the necessary properties. Thus $M$ admits an $R$-module structure via the map $\phi : M \times R \to M$ defined by $\phi(m, r) = \varphi(m, r + I)$. The converse of this requires an extra assumption, however.

Lemma 1.3.30  If $I$ is a two-sided ideal of a ring $R$, and $M$ is an $R$-module with $MI = 0$, then $M$ is an $R/I$-module.

Proof. Corollary 2.12 (1) in [2].

As a consequence we have the following:
Corollary 1.3.31 If $I$ is a two-sided ideal of a ring $R$, and $M, N$ are $R/I$-modules, then $M_R$ is $N_R$-projective iff $M_{R/I}$ is $N_{R/I}$-projective.

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Let $T_R$ be an epimorphic image of $N_R$ via $\pi$. Then $XI = \pi(N)I = \pi(NI) = 0$. Thus $X$ is an $R/I$-module, and so any $f \in \text{Hom}(M_R, X_R)$ can be regarded as an element of $\text{Hom}(M_{R/I}, X_{R/I})$, which lifts by assumption. \hfill \Box

Lemma 1.3.32 For every two-sided ideal $I$ of a ring $R$ and every module $M_{R/I}$, we have $M_R$ is simple (semisimple) iff $M_{R/I}$ is simple (semisimple).

Proof. ($\Leftarrow$) Obvious.

($\Rightarrow$) Write $M = \bigoplus_{j \in J} (S_j)_R$. Note $(S_j)_I = 0 \forall j \in J$. Let $0 \neq a \in S_j$. Then $a(R/I) = aR = S_j$, so $(S_j)_{R/I}$ is simple $\forall j \in J$. \hfill \Box

Lemma 1.3.33 If $N \leq M_R$ are modules such that $J(M/N) = 0$, then $J(M) \subseteq N$.

Proof. $J(M) = \cap \{M' | M' \leq_m M \} \subseteq \cap \{M' | M' \leq_m M, M \geq N \} = N$ \hfill \Box

Lemma 1.3.34 $J(R)\text{soc}(R_R) = 0$. In particular, if $J(R)$ is semisimple, then $J(R)^2 = 0$.

Proof. Corollary 15.5 in [2]. \hfill \Box

Lemma 1.3.35 If $R/\text{soc}(R_R)$ is semisimple artinian, then $\text{soc}(R_R) \leq_e R_R$.

Proof. Let $T_R$ be a right complement of $\text{soc}(R_R)$ in $R_R$. Then $\text{soc}(R_R) \oplus T \leq_e R_R$. Hence $T \hookrightarrow R/\text{soc}(R_R)$, thus $T$ is semisimple. Hence $T = \text{soc}(T) = 0$. \hfill \Box

Lemma 1.3.36 A module $C_R$ is singular iff $C \cong B/A$, where $A \leq_e B$.

Proof. Proposition 1.5 in [13]. \hfill \Box

Corollary 1.3.37 If $M_R$ is singular and projective, then $M = 0$. 

Proof. Since $M$ is singular, $M \cong B/A$, where $A \leq e B$ (by 1.3.36). Since $M$ is projective, $A \leq B$ (by 1.3.20). Hence $A = B$. \hfill\Box

**Proposition 1.3.38** Injectivity domains are closed under submodules, homomorphic images, and arbitrary direct sums.

**Proof.** Proposition 16.13 in [2]. \hfill\Box

**Corollary 1.3.39** For every module $M$, $SSMod - R \subseteq \mathcal{In}^{-1}(M)$.

**Proof.** Clearly every simple module is in $\mathcal{In}^{-1}(M)$. The result follows from 1.3.38. \hfill\Box

**Proposition 1.3.40** Projectivity domains are closed under submodules, homomorphic images, and finite direct sums. If $M$ is finitely generated, then $\mathcal{Pr}^{-1}(M)$ is also closed under arbitrary direct sums.

**Proof.** Proposition 16.12 in [2]. \hfill\Box

**Proposition 1.3.41** If $R$ is semilocal, then TFAE:

(a) $Z(R_R) = 0$ and $R$ is a right SI-ring.

(b) $Z(R_R) = 0$ and $J(R)^2 = 0$.

**Proof.** Proposition 3.5 in [13]. \hfill\Box

**Proposition 1.3.42** If $R$ is a right SI-ring, then all singular right $R$-modules are semisimple.

**Proof.** Proposition 3.1 in [13]. \hfill\Box

**Proposition 1.3.43** A right SI-ring is right hereditary.

**Proof.** Proposition 3.3. (d) in [13]. \hfill\Box

**Lemma 1.3.44** If $e, f$ are central orthogonal idempotents, then $\text{Hom}(eR, fR) = 0$. 


Proof. If $\varphi \in Hom(eR, fR)$, then $\varphi(e) = fr$ for some $r \in R$. Thus $\varphi(e) = \varphi(e^2) = \varphi(e)e = (fr)e = fer = 0.$

**Lemma 1.3.45** If $M_R$ is a non-simple uniform module, and $N_R$ is a nonsingular simple module, then $Hom(M, N) = 0$.

Proof. Suppose $0 \neq \varphi \in Hom(M, N)$. Since $N$ is simple, $Im(\varphi) = N \cong M/Ker(\varphi)$. Since $M$ is not simple, $Ker(\varphi) \neq 0$, hence $Ker(\varphi) \leq_e M$, which implies $N$ is singular (by 1.3.36), a contradiction.

**Lemma 1.3.46** If $M$ is a nonsingular simple module, and $Soc(N)$ is singular, then $Hom(M, N) = 0$.

Proof. Suppose $0 \neq \varphi \in Hom(M, N)$. Since $M$ is simple, $Ker(\varphi) = 0$. Hence $M \cong Im(\varphi) \leq N$ is simple and nonsingular, a contradiction as $N \leq Soc(N)$.

**Theorem 1.3.47** For a module $M_R$, TFAE:

(i) $M$ is a $V$-module

(ii) Every submodule of $M$ (other than $M$) is an intersection of maximal submodules of $M$.

(iii) $J(N) = 0$ for every $N \in \sigma[M]$.

Proof. 33.1 in [31].

**Lemma 1.3.48** Any quasi-injective module $M$ is CS.

Proof. Corollary 6.80 in [22].

**Lemma 1.3.49** $A \leq_e B \leq_e C$ iff $A \leq B \leq C$ and $A \leq_e C$.

Proof. Proposition 1.1 (a) in [13].

**Lemma 1.3.50** A von Neumann regular ring is either semisimple artinian or has infinite uniform dimension.
Proposition 1.3.51 If $M_R, N_R$ are modules and $f \in \text{Hom}_R(M, N)$, then $f(J(M)) \leq J(N)$.


Theorem 1.3.52 Let $I$ be a nil two-sided ideal in $R$. Let $a \in R$ such that $\bar{a} \in \bar{R} := R/I$ is an idempotent. Then there exists an idempotent $e \in aR$ such that $\bar{e} = \bar{a} \in \bar{R}$.

Proof. Theorem 21.28 in [23].

Lemma 1.3.53 A ring $R$ satisfies ACC on left annihilators iff $R$ satisfies DCC on right annihilators.

Proof. (6.57) in [22].

Lemma 1.3.54 If $P_1, P_2$ are projective covers of a module $M_R$, then $P_1 \cong P_2$.

Proof. (24.10) in [23].

Theorem 1.3.55 For a ring $R$ with Jacobson radical $J$ the following conditions are equivalent:

(1) Every right $R$-module is a direct sum of a projective module and a semisimple module
(2) $R$ is a generalized uniserial ring with $J^2 = 0$.

Proof. Theorem 3 in [18].

Theorem 1.3.56 If $M$ has a projective cover, then $\mathcal{P}^{-1}(M)$ is closed under arbitrary direct sums. In particular, if $R$ is a right perfect ring, then a right module $M_R$ is projective iff $M$ is $R_R$-projective.

Proof. Corollary 3.10 in [14].

Proposition 1.3.57 If $R$ is a right PCI-domain, then every proper cyclic module is semisimple. In particular, a right PCI-domain is a right SI-ring.
Theorem 1.3.58 A right PCI-domain is right noetherian. In particular, it is right uniform.

Proof. Theorem 17 in [10].

Lemma 1.3.59 If I is an injective module containing M, then I contains a copy of E(M).

Proof. Corollary 3.33 (1) in [22].

Lemma 1.3.60 For a right ideal I of a ring R, the following are equivalent:

(a) I ≤ J(R)

(b) For every finitely generated right R-module M, if IM = M, then M = 0.

Proof. Corollary 15.13 in [2].

Proposition 1.3.61 For each module M the following statements are equivalent:

(a) J(M) = 0 and Soc(M) is finitely generated

(b) M is the direct sum of a finite set of simple submodules

Proof. Proposition 10.15 in [2].

Lemma 1.3.62 Any qfd V-module is noetherian.

Proof. In [6].

Theorem 1.3.63 A ring is semiperfect iff it has a complete orthogonal set e₁, . . . , eₙ of idempotents with each eᵢReᵢ a local ring.

Proof. Theorem 27.6 in [2].
2 ON V-MODULES AND V-RINGS

2.1 WV-rings

The results of this section are from [15], [16].

In the early 1970s, Villamayor introduced the study of rings whose simple modules are all injective [25]. In honor of his contribution, these rings were named V-rings. This concept was later generalized to modules by Wisbauer [31]. Recall $M_R$ is called a V-module if every simple module in $\sigma[M]$ is $M$-injective. In [16], the authors introduced a generalization of V-rings as follows.

**Definition 2.1.1** A ring $R$ is called a right WV-ring (short for weakly-V) if each simple module is injective relative to every proper cyclic module (or equivalently, every proper cyclic module is a V-module).

Clearly every right V-ring is a right WV-ring.

**Example 2.1.2** $R = Z_4$ is a WV-ring, but not a V-ring.

**Proof.** $J(R) \neq 0$, so it is not a V-ring (by 1.3.47). However, the cyclic modules (up to isomorphism) are 0, $J(R)$, and $R$. The two cyclics which are not isomorphic to $R$ are both semisimple, hence V-modules (by 1.3.39). Thus $R$ is a WV-ring. □

**Definition 2.1.3** A ring $R$ is called right strictly WV if it is right WV and not right V.

Naturally, we aim to understand the class of all right strictly WV-rings.

**Lemma 2.1.4** Let $R$ be a right WV-ring, and let $A, B$ be right ideals of $R$ such that $R/A$ and $R/B$ are proper cyclic modules and $A \cap B = 0$. Then $R$ is a right V-ring.

**Proof.** Assume the hypotheses. Let $S_R$ be a simple module. Since $R$ is a right WV-ring, and $R/A$ and $R/B$ are proper, $S$ is $R/A$-injective and $R/B$-injective. Hence $S$ is $R/A \times R/B$-injective (by 1.3.38). Define the mapping $f : R_R \rightarrow (R/A \times R/B)_R$ by
\(f(r) = (r + A, r + B)\). It is easy to check that \(f\) is an \(R\)-homomorphism. If \(r \in \text{Ker}(f)\), then 
\((r + A, r + B) = 0\), so \(r \in A \cap B = 0\). Hence \(\text{Ker}(f) = 0\). Thus \(R_R \cong \text{Im}(f)\), a submodule of \(R/A \times R/B\). Thus \(S\) is \(R_R\)-injective (by 1.3.38). Hence \(S\) is injective (by 1.3.15). Thus \(R\) is a right \(V\)-ring. 

\[\square\]

**Theorem 2.1.5** Let \(R\) be a right strictly \(WV\)-ring. Then \(R\) must be right uniform.

**Proof.** Assume \(R\) is a right \(WV\)-ring. We show that if \(u.\dim(R_R) > 1\), then \(R\) is a right \(V\)-ring.

Case (i): Assume \(R_R\) has infinite uniform dimension. Then by 1.3.16, there exists an infinite sum of nonzero right ideals \(\bigoplus_{i \in I} A_i \subseteq R\). Write \(I\) as a union of two disjoint infinite subsets \(J\) and \(K\). Then we get that \(R\) contains the direct sum \(A \oplus B\) where \(A = \bigoplus_{i \in J} A_i\) and \(B = \bigoplus_{i \in K} A_i\) are infinitely generated. We show \(R/A\) is proper. Suppose, for contradiction, that \(R/A \cong R\). Then \(R/A\) is projective (by 1.3.18), and hence there exists a right ideal \(C\) of \(R\) such that \(R = C \oplus A\) (by 1.3.20). Now \(R/C\) is cyclic, and \(R/C \cong A\), so \(A\) is cyclic, a contradiction. Thus \(R/A\) is proper. Similarly \(R/B\) is proper, and so \(R\) is a right \(V\)-ring by 2.1.4.

Case (ii): \(R_R\) has finite uniform dimension. Assume that \(u.\dim(R_R) = n > 1\). Then there exist uniform right ideals \(U_i\) such that \(\bigoplus_{i=1}^n U_i \leq e R\). WLOG, we may assume that \(U_1\) and \(U_2\) are closed in \(R\). By 1.3.17, \(u.\dim(U_1) + u.\dim(R/U_1) = u.\dim(R) + u.\dim(U_1/U_1)\), i.e. \(1 + u.\dim(R/U_1) = n + u.\dim(0)\), i.e. \(u.\dim(R/U_1) = n - 1\). Similarly \(u.\dim(R/U_2) = n - 1\). Thus \(R/U_1\) and \(R/U_2\) are proper. Hence \(R\) is a right \(V\)-ring by 2.1.4.

\[\square\]

Before further classifying right strictly \(WV\)-rings, we provide some results about \(WV\)-rings which are similar to some well-known results on right \(V\)-rings.

**Proposition 2.1.6** Let \(R\) be a ring such that \(R/I\) is proper for every nonzero right ideal \(I\). Then TFAE:

(a) \(R\) is a right \(WV\)-ring.
(b) \( J(R/I) = 0 \) for every nonzero right ideal \( I \).

(c) Every nonzero right ideal \( I \neq R \) is an intersection of maximal right ideals.

**Proof.** (a) \( \Rightarrow \) (b). Assume \( R \) is a right \( WV \)-ring. Let \( I \neq 0 \) be a right ideal. Our result is trivial if \( I = R \), so assume \( I \neq R \). Then we can take \( 0 \neq x \in R/I \). Choose a maximal submodule \( N \) of \( xR \). Then \( xR/N \) is simple. Since \( R \) is a right \( WV \)-ring, \( xR/N \) is \( R/I \)-injective. Therefore the canonical \( R \)-homomorphism \( f : xR \to xR/N \) can be extended to \( g : R/I \to xR/N \). Notice \( \frac{R/I}{\text{Ker}(g)} \cong \text{Im}(g) \leq xR/N \). \( \text{Im}(g) \neq 0 \) as \( g(x) \neq 0 \), and so \( \text{Im}(g) = xR/N \). We conclude that \( \text{Ker}(g) \leq_m R/I \). But \( x \notin \text{Ker}(g) \), and hence \( x \notin J(R/I) \). Thus \( J(R/I) = 0 \).

(b) \( \Rightarrow \) (c) Let \( I \neq R \) be a nonzero right ideal. Then
\[
0 = J(R/I) = \cap\{M|M \leq_m R/I\} = \cap\{M'/I|M' \leq_m R, M' \supset I\}. \text{ Hence } I = \cap\{M'|M' \leq_m R\}.
\]

(c) \( \Rightarrow \) (a) Let \( I \neq R \) be a nonzero right ideal and \( S_\mathcal{R} \) a simple module. We show \( S \) is \( R/I \)-injective. Take \( 0 \neq f : B/I \to S \), where \( B \) is a right ideal such that \( I \leq B \leq R \). We show that \( f \) extends to \( R/I \). Take \( x \in B \) such that \( x + I \notin \text{Ker}(f) \). Let \( A \) be such that \( A/I = \text{Ker}(f) \). By (c), there exists \( M \leq_m R_\mathcal{R} \) such that \( A \subseteq M \) but \( x \notin M \). This implies that \( B + M = R \), and hence \( B/A + M/A = R/A \). As before, \( \text{Ker}(f) \leq_m B/I \) and so \( B/A \cong \frac{B/I}{A/I} \) is simple. Thus \( B/A \cap M/A \leq B/A \), and so we must have \( B/A \oplus M/A = R/A \). \( f \) gives rise to \( f' : B/A \to S \) defined by \( f'(b + A) = f(b + I) \). Denote by \( \pi \) the projection of \( R/A \) onto \( B/A \) and by \( \rho \) the canonical map from \( R/I \) onto \( R/A \). Let \( g = f'\pi \rho \). Then
\[
g(b + I) = f'\pi(b + A) = f'(b + A) = f(b + I).
\]
So \( g \) extends \( f \) as desired. \( \square \)

The following remark illustrates that 2.1.6 applies to right uniform rings.

**Remark 2.1.7** If \( R \) is a right uniform ring and \( I \neq 0 \), then \( R/I \) is proper.

**Proof.** Let \( R \) be a right uniform ring and \( I \) be a nonzero right ideal. If \( I = R \), this is obvious, so assume \( I \neq R \). Assume, for contradiction, that \( R/I \cong R \). Then \( R/I \) is projective.
(by 1.3.18), so there exists a right ideal $J$ of $R$ such that $R = I \oplus J$ (by 1.3.20). But $R$ is uniform, $I \neq 0$ and $I \cap J = 0$, and so we must have $J = 0$. Thus $I = R$, a contradiction. □

**Corollary 2.1.8** If $R$ is a right WV-ring, then $R/J(R)$ is a right V-ring.

**Proof.** This is clear if $R$ is a right V-ring, so assume not. Then $R$ is right uniform (by 2.1.5). By 2.1.6, every nonzero right ideal (other than $R$) is an intersection of maximal right ideals. Note that we must also have $J(R) \neq 0$ (by 1.3.47). Thus, in $R/J(R)$ all right ideals are intersections of maxinals, and so $R/J(R)$ is a right V-ring. □

**Proposition 2.1.9** Let $R$ be a right WV-ring. Then the following hold:

(a) If $I \leq R$, then either $I^2 = 0$ or $I^2 = I$.

(b) If $R$ is a domain, then $R$ is simple.

**Proof.** (a) If $R$ is a right V-ring, it is well known that $I^2 = I$ for every right ideal of $R$. (In fact, the proof is similar to the remainder of our proof.) We can thus assume that $R$ is not a right V-ring and hence is right uniform (by 2.1.5). Let $I \leq R$ and suppose $I^2 \neq 0$. Then both $I$ and $I^2$ are intersections of maximal right ideals (by 2.1.6). If $I^2 \neq I$, there must exist $M \leq R$ such that $I^2 \leq M$ but $I \not\subseteq M$. We thus have $R = I + M$ and we can write $1 = x + m$ for some $x \in I, m \in M$. This gives $I = (x + m)I \subseteq xI + mI \subseteq I^2 + M = M$, a contradiction.

(b) Let $0 \neq a \in R$. Since $R$ is a domain, $(aR)^2 \neq 0$, so part (a) gives us $(aR)^2 = aR$, i.e. $aRaR = aR$. Since $R$ is a domain this gives that $RaR = R$. This implies $R$ is simple. □

We now begin to describe right strictly WV-rings more precisely. Observe that for a right strictly WV-ring, $J(R) \neq 0$ (by 2.1.8).

**Lemma 2.1.10** Let $R$ be a right strictly WV-ring with $J = J(R)$. Then $J_R$ is simple.

**Proof.** Let $0 \neq A \leq J$. If $R/A \cong R$, then it is projective (by 1.3.18), and hence $A \leq R$ (by 1.3.20). This is a contradiction (by 1.3.24). Thus $R/A$ is proper cyclic. Since $R$ is right WV, $R/A$ is a V-module. Hence $J/A = J(R/A) = 0$ (by 1.3.47). Thus $J_R$ is simple. □
Corollary 2.1.11  Let $R$ be a right strictly WV-ring with $J = J(R)$. Then every nonzero right ideal of $R$ contains $J$.

Proof. Since $R_R$ is uniform (by 2.1.5), if $0 \neq A \leq R_R$, then $0 \neq A \cap J \leq J$. Since $J_R$ is simple (by 2.1.10), $A \cap J = J$, i.e. $J \leq A$. □

Lemma 2.1.12  Let $R$ be a right strictly WV-ring with $J = J(R)$. Then every nonzero cyclic right ideal of $R / J$ is isomorphic to $R / J$.

Proof. Set $\overline{R} = R / J$. Let $\overline{0} \neq \overline{x} \in \overline{R}$. Then $\exists x \in R \setminus J$ such that $\overline{x} = x + J$. If $xR \not\cong R_R$, then since $R$ is right WV, $J$ is $xR$-injective. But $J \leq xR$ (by 2.1.11), so $J \leq xR$ (by 1.3.21), a contradiction as $R_R$ is uniform (by 2.1.5). Thus $xR \cong R_R$. Let $f : xR \to R$ be an isomorphism. Note $\overline{R}$ is right $V$ (by 2.1.8). Thus $J(xR/J) = 0$ (by 1.3.47). Hence $J(xR) \subseteq J$ (by 1.3.33). Now every nonzero submodule of $xR$ contains $J$ (by 2.1.11), and so $J(xR) \neq 0$. Thus $J(xR) = J$ (by 2.1.10). Hence $f(J) \leq J$ (by 1.3.51). Since $f$ is an injection, $f(J) \neq 0$, and so $f(J) = J$ (by 2.1.10). It follows that $\overline{xR} \cong \overline{R}$. □

We are now equipped for the main result of the section.

Theorem 2.1.13  Let $R$ be a right strictly WV-ring with $J = J(R)$. Then the following hold:

(a) $R / J$ is a simple domain. (In particular, since $J_R$ is simple, $R$ has exactly 3 two-sided ideals $0 \leq J \leq R$.) Moreover, for every nonzero element $x \in J$, $Rx \cong R(R/J)$.

(b) If $R_J$ is finitely generated, then $R / J$ is a division ring. (In particular, $R$ has exactly 3 right ideals $0 \leq J \leq R_R$.)

Proof. Let $\overline{R} = R / J$.

(a) Let $\overline{0} \neq \overline{x} \in \overline{R}$. Then $\overline{xR} \cong \overline{R}$ (by 2.1.12), so $\overline{xR}$ is projective (by 1.3.18). Thus $\overline{R_R} = \overline{P} \oplus \overline{Q}$ for some $\overline{Q}$, where $\overline{P} \cong \overline{xR}$ (by 1.3.20). If $\overline{Q} \neq \overline{0}$, then it is generated by a nontrivial idempotent $\overline{e} \in \overline{R}$ (by 1.3.22). Since $J$ is simple (by 2.1.10), $J^2 = 0$ (by 1.3.34).
Thus there is an idempotent $e \in R$ with $e \in \overline{e}$ (by 1.3.52). Clearly $e$ is a nontrivial idempotent. But $R_\overline{e}$ is uniform (by 2.1.5), so this is a contradiction (by 1.3.22). Thus $\overline{0} = \overline{Q} \cong Ann_{R}(\overline{x})$. This implies that $R$ is a domain. Since $R$ is right $V$ (by 2.1.8), $R_\overline{R}$ is a simple ring (by 2.1.9 (b)). Finally, let $0 \neq x \in J$. Then $xR = J$. Since $l.\overline{Ann}_{R}(J)$ is a two-sided ideal, so is $l.\overline{Ann}_{R}(x)$. Since $J^2 = 0$ and $R$ is simple, we must have $l.\overline{Ann}_{R}(x) = J$. It follows that $Rx \cong R(R/J)$.

(b) Assume $RJ = Rx_1 + \ldots + Rx_k$ for some positive integer $k$, where $0 \neq x_i \in J$ for $1 \leq i \leq k$. Define $f : R_R \to J^{(k)}$ by $f : r \mapsto (x_1 r, \ldots, x_k r)$. Set $A = Ker(f) = \cap_{i=1}^{k} r.\overline{Ann}_{R}(x_i)$. Then $A \subseteq r.\overline{Ann}_{R}(J)$. But $J^2 = 0$ and $R$ is simple, so $r.\overline{Ann}_{R}(J) = J = A$. Now $R/J$ is isomorphic to a submodule of $J^{(k)}$, whence $R_\overline{R}$ is semisimple. Since $R$ is a domain, it must be a division ring. \(\square\)

**Corollary 2.1.14** Let $R$ be a right strictly and left WV-ring with $J = J(R)$. Then $R$ is a right and left uniserial ring, and $0 \leq J \leq R$ is the only composition series in $R$.

Consequently, a right strictly WV-ring $R$ with Jacobson radical $J$ is left WV if and only if $RJ$ is a simple module.

**Proof.** By 2.1.10, for every nonzero element $y \in J$, $J = yR = Ry$ is a simple right and simple left $R$-module. Thus the first sentence of the conclusion follows from 2.1.13 (b). Now if $RJ$ is simple, then $R$ has only three left ideals by 2.1.11 as we know $R/J$ is a division ring. Then $R$ is left WV (see the proof of 2.1.2). The rest is obvious. \(\square\)

**Corollary 2.1.15** Let $R$ be a right strictly WV-ring with $J = J(R)$. If $R$ satisfies ACC on left annihilators, then $R_R$ is a right uniserial module with exactly three distinct submodules $0 \leq J_R \leq R_R$.

**Proof.** $R$ satisfies DCC on right annihilators (by 1.3.53). Thus the chain $r.\overline{Ann}(x_1) \supseteq r.\overline{Ann}(x_2) \supseteq r.\overline{Ann}(x_1, x_2, x_3) \supseteq \ldots (x_i \in J)$ must terminate, say at $r.\overline{Ann}(x_1, \ldots, x_k)$. By the minimality of this chain for all elements of $J$, we have
Thus we can follow a similar proof to that of 2.1.13 (b) to achieve our desired result.

We conclude the section with an example to show that, indeed, a strictly left WV-ring need not be a right strictly WV-ring.

**Example 2.1.16** Let $R = \begin{bmatrix} a & b \\ 0 & \sigma(a) \end{bmatrix}$ where $a, b \in \mathbb{Q}(x)$ and $\sigma$ is the $\mathbb{Q}$-endomorphism of $\mathbb{Q}(x)$ such that $\sigma(x) = x^2$. Then $R$ is strictly left WV but not right WV.

**Proof.** This example is taken from [9]. In this ring there are only three left ideals and hence it is a left WV-ring (see the proof of 2.1.2). It is not left V as $J(R) \neq 0$ (by 1.3.47). But the ring has more than three right ideals (it is, in fact, not even right noetherian) and thus it cannot be a right WV-ring (by 2.1.14).

### 2.2 On Cyclic Modules Being a Direct Sum of a Projective Module and a CS or Fud Module

The results of this section are from [15].

**Definition 2.2.1** Given a module $M$, a module $P \in \sigma[M]$ is said to be projective in $\sigma[M]$ iff every exact sequence $0 \rightarrow K \rightarrow N \rightarrow P \rightarrow 0$ in $\sigma[M]$ splits. For more information about projectivity in $\sigma[M]$, we refer the reader to 18.3 in [31].

**Definition 2.2.2** We will say a module $M_R$ has (*) if we can write $M = A \oplus B$, where $A$ is CS or has fud, and $B$ is projective in $\sigma[M]$.

A slightly weaker property was studied in [29], where it was shown that a ring is noetherian iff every 2-generated module is a direct sum of a projective module and a CS or noetherian module. We first remark why we cannot weaken the condition to cyclic modules.
Example 2.2.3 Let $R$ be the ring of all former power series
\[ \sum \{a_i x^i | a_i \in F, i \in I\} \]
where $F$ is a field, and $I$ ranges over all well-ordered sets of nonnegative real numbers.
Then every cyclic $R$-module has (*), but $R$ is not noetherian.

Proof. This example is taken from [24]. It is shown that this ring is not noetherian. However, every homomorphic image is self-injective, thus CS (by 1.3.48), hence has (*).\[□\]

Thus we need to impose more conditions on a ring whose cyclic modules have (*) for it to be right noetherian. Our goal is to show that it is sufficient to add only the assumption of right WV. (Note that every cyclic module in $\sigma[M]$ is equivalent to every cyclic subfactor of $M$.)

Theorem 2.2.4 Let $M$ be a finitely generated right $R$-module. If every cyclic module in $\sigma[M]$ satisfies condition (*), then $M$ is a qfd module.

Proof. Let $M = x_1 R + \ldots + x_n R$. If every $x_i R$ has fud, then obviously, so does $M$. Hence, WLOG, we may assume that $M$ is a cyclic right $R$-module. Let $E \leq_e M$. Then $M/E$ has no nonzero projective submodules (by 1.3.20 and 1.3.49). Since $\sigma[M/E] \subseteq \sigma[M]$ the condition (*) implies that every cyclic module in $\sigma[M/E]$ is either CS or has fud. Thus $M/E$ has fud (by 1.3.26). This implies $M/Soc(M)$ has fud (by 1.3.27). Now assume, for contradiction, that $Soc(M)$ does not have fud. Then we can write $Soc(M) = S_1 \oplus S_2$, where $S_1, S_2$ are both infinite direct sums of simple modules. By (*), $M/S_1 = P \oplus Q$ where $P_R$ is projective in $\sigma[M]$ and $Q_R$ is either CS or has fud. Note $\sigma[Q_R] \subseteq \sigma[M]$. By the first part of the proof, $Q/Soc(Q)$ has fud. Now if $Q$ is CS, $Soc(Q)$ has fud (by 1.3.28). This implies that in any case, $Q_R$ has fud. But, as $S_2$ is embedded in $M/S_1$, $Soc(P_R)$ is infinitely generated. Now let $M_1$ be the inverse image of $Q$ in $M$. Then as $M/M_1 \cong P$, we have $M = M_1 \oplus M_2$ (by 1.3.20), where $S_1 \hookrightarrow M_1$ and $M_2 \cong P$, so
$Soc(M_1), Soc(M_2)$ are infinitely generated. Since

$M/Soc(M) \cong M_1/Soc(M_1) \oplus M_2/Soc(M_2)$, we have $u.dim(M/Soc(M)) \geq 2$. Applying the same considerations to $M_1, M_2$, we get $u.dim(M/Soc(M)) \geq 4$. Continuing in this way, we get $M/Soc(M)$ has infinite uniform dimension, a contradiction. Thus $M$ has $fud$. Finally, given $N \leq M$, we see $\sigma[M/N] \subseteq \sigma[M]$, and so every cyclic in $\sigma[M/N]$ satisfies (*). It follows that $M/N$ has $fud$, i.e. $M$ is a $qfd$ module.

**Corollary 2.2.5** Let $R$ be a right WV-ring. If $C$ is a cyclic right $R$-module such that every cyclic module in $\sigma[C]$ satisfies condition (*), then $C$ is noetherian. In addition, if $R$ is a right $V$-ring and $Soc(C) \leq e C$, then $C$ is semisimple.

**Proof.** Let $R$ be right strictly WV. By 2.2.4, $C$ is a $qfd$ module. If $C \not\cong R$, then $C$ is a $V$-module, hence noetherian (by 1.3.62). So suppose $C \cong R$. Then $C/Soc(C)$ is $qfd$ by 2.2.4, and a $V$-module by 2.1.8, hence noetherian (by 1.3.62). $Soc(C)$ is simple (by 2.1.5), so $C$ is noetherian. Now if $R$ is right $V$, then $Soc(C)$ is injective. Thus $Soc(C) \leq e C$ (by 1.3.21). Thus if $Soc(C) \leq e C$, then $Soc(C) = C$, i.e. $C$ is semisimple.

**Corollary 2.2.6** If $R$ is a von Neumann regular ring such that every cyclic right $R$-module satisfies (*), then $R$ is semisimple artinian.

**Proof.** $R_R$ has $fud$ (by 2.2.4). Thus $R$ is semisimple artinian (by 1.3.50).
3 Modules with Maximal or Minimal Projectivity Domains

3.1 P-poor modules

This chapter contains results found in [17]. In [2], the authors introduce the notion of projectivity domains. Recall that given a module $M_R$, its projectivity domain is defined as $\mathfrak{P}r^{-1}(M) = \{N_R| M$ is $N$-projective}. In particular, a module $M_R$ is projective iff $\mathfrak{P}r^{-1}(M) = Mod - R$. Thus the projective modules can be thought of as those modules which have maximal projectivity domain (as rich as possible). With this method of definition, it is natural to question what can be said of modules with minimal projectivity domain (as poor as possible). The outside onlooker might hypothesize that they would be the modules with empty projectivity domain, but this is not the case.

**Proposition 3.1.1** For every ring $R$, $SS Mod - R = \bigcap_{M \in Mod - R} \mathfrak{P}r^{-1}(M)$.

**Proof.** ($\subseteq$) Let $M \in Mod - R$ and $N \in SS Mod - R$. Let $T \leq N$. Since $N$ is semisimple, $T \leq N$. Thus there exists $K \leq N$ with $N = T \oplus K$ and hence $K \cong N/T$. Therefore every $f \in Hom(M, N/T)$ can be regarded as an element of $Hom(M, N)$ which agrees with the identity on $K$, and so $M$ is $N$-projective. Thus $N \in \mathfrak{P}r^{-1}(M)$.

($\supseteq$) Let $N \in \bigcap_{M \in Mod - R} \mathfrak{P}r^{-1}(M)$ and $T \leq N$. Then $N \in \mathfrak{P}r^{-1}(N/T)$. Thus $T \leq N$ (by 1.3.20). Thus $N \in SS Mod - R$. \qed

This establishes $SS Mod - R$ as the poorest possible projectivity domain of a given ring $R$.

**Definition 3.1.2** Let us say that a module $M_R$ is p-poor (short for projectively-poor) if $SS Mod - R = \mathfrak{P}r^{-1}(M)$.

Before proving existence of such modules over arbitrary rings, here are a few basic properties:
Lemma 3.1.3 For every ring $R$, TFAE:

(i) $R$ is semisimple artinian.
(ii) Every module $M_R$ is p-poor.
(iii) There exists a projective p-poor $R$-module.
(iv) $\{0\}_R$ is p-poor.
(v) $R_R$ is p-poor.

Proof. Assume (i). Let $M \in \text{Mod} - R$. Then

$$SS\text{Mod} - R \subseteq \text{Pr}^{-1}(M) \subseteq \text{Mod} - R = SS\text{Mod} - R.$$

Hence (i)$\Rightarrow$(ii). (ii)$\Rightarrow$[(iii)/(iv)/(v)] is obvious. Assuming [(iii)/(iv)/(v)], we can take $M_R$ to be a projective p-poor module (by 1.3.18 for case (v)). Thus $\text{Mod} - R = \text{Pr}^{-1}(M) = SS\text{Mod} - R$, so each imply (i). $\square$

Since every module over a semisimple artinian ring is projective, the previous lemma says that all modules are rich and all modules are poor are equivalent statements, giving further weight to the terminology.

Lemma 3.1.4 Let $M_R$ be a p-poor module. Then for every $N_R$, $M \oplus N$ is p-poor.

Proof. Let $N \in \text{Mod} - R$. Assume $M \oplus N$ is $T$-projective. Then $M$ is $T$-projective. Since $M$ is p-poor, $T$ must be semisimple. Thus $M \oplus N$ is p-poor. $\square$

This is somewhat analogous to the fact that direct summands of projective modules are projective, but it should be noted that only one summand needs to be p-poor. In particular, direct summands of p-poor modules need not be p-poor. In fact, it is not even true that a decomposable p-poor module must have a proper direct summand which is also p-poor, as evidenced by 3.1.11.

Lemma 3.1.5 If $M_R \oplus N_R$ is p-poor and $M_R$ is projective, then $N_R$ is p-poor.

Proof. Assume $N$ is $T$-projective. Then $M \oplus N$ is $T$-projective. Hence $T$ is semisimple. $\square$

Remark 3.1.6 Let $I$ be an index set, and for each $i \in I$, let $M_i$ be a module and $I_i$ be an index set. If $\bigoplus_{i \in I} M_i^{(i)}$ is p-poor, then $M = \bigoplus_{i \in I} M_i$ is p-poor.
Proof. Let $N \in Mod - R$ and assume $M$ is $N$-projective. Then $\oplus_{i \in I} M_i^{(i)}$ is $N$-projective.

Since $\oplus_{i \in I} M_i^{(i)}$ is p-poor, $N$ must be semisimple. Thus $M$ is p-poor.

The following lemma gives a characterization of p-poor modules that is easier to check.

**Lemma 3.1.7** For every ring $R$, a module $M_R$ is p-poor iff every cyclic module in $\mathfrak{Pr}^{-1}(M)$ is semisimple.

Proof. ($\Rightarrow$) Obvious. ($\Leftarrow$) Let $N \in \mathfrak{Pr}^{-1}(M)$. Now $N = \sum_{x \in N} xR$, and $xR \in \mathfrak{Pr}^{-1}(M)$ for every $x \in N$ (by 1.3.40). Thus by assumption, $xR$ is semisimple $\forall x \in N$, and so $N$ is semisimple.

We may now prove that every ring has a p-poor module. In fact, a stronger result holds.

**Theorem 3.1.8** Every ring $R$ has a semisimple p-poor module.

Proof. Let $\Gamma$ be a complete set of representatives of isomorphism classes of simple right $R$-modules. Let $S = \oplus_{B \in \Gamma} B$. Let $xR \in \mathfrak{Pr}^{-1}(S)$ be a nonzero cyclic module. Let $T \leq_m xR$. Consider the simple module $K = xR/T$. By the choice of $\Gamma$, $K$ is isomorphic to a submodule of $S$. Then $K$ is $xR$-projective, therefore $T \leq \oplus xR$ (by 1.3.20). Hence $xR$ is semisimple (by 1.3.29). Thus $S$ is p-poor (by 3.1.7).

Here are a few consequences of this existence proof:

**Corollary 3.1.9** If $R$ is a semilocal ring, then $(R/J(R))_R$ is p-poor.

Proof. Let $\Gamma, S$ be as in the proof of 3.1.8. Then $\forall B \in \Gamma$, we have $B \cong R/M$ for some $M \leq_m R$. Note that since $J(R) \subseteq M$, $R/J(R)$ maps onto $R/M$. By assumption, $R/J(R)$ is semisimple artinian, and so $B$ is isomorphic to a direct summand of $R/J(R)$. Hence, $S$ is also isomorphic to a direct summand of $R/J(R)$, and so $R/J(R)$ is p-poor (by 3.1.4).
Proposition 3.1.10  For a ring $R$, TFAE:

(i) $R$ is semisimple artinian.

(ii) Every $p$-poor right $R$-module is semisimple.

(iii) Nonzero direct summands of $p$-poor right $R$-modules are $p$-poor.

(iv) Nonzero factors of $p$-poor right $R$-modules are $p$-poor.

Proof. (i)$\Rightarrow$(ii)/(iii)/(iv)] follows immediately by 3.1.3. We can take a $p$-poor module $M_R$ (by 3.1.8), and then $M \oplus R_R$ will also be $p$-poor (by 3.1.4). If (ii) holds, then $M \oplus R_R$ (hence $R_R$) is semisimple. If (iii) [or (iv)] holds, then $R_R$ [or $(M \oplus R_R)/M \cong R_R$] is $p$-poor, hence semisimple artinian (by 3.1.3). So [(ii)/(iii)/(iv) $\Rightarrow$ (i)].

Example 3.1.11 Let $P$ be the set of all prime numbers. The $\mathbb{Z}$-module $M = \bigoplus_{p \in P} \mathbb{Z}_p$ is $p$-poor, but no proper submodule of $M$ is $p$-poor.

Proof. $M$ corresponds to the module $S$ in the proof of 3.1.8, so $M$ is $p$-poor. Let $N$ be a proper submodule of $M$. Then $N$ is semisimple, and so $N$ is the direct sum of its homogeneous components $\{[N]_p | p \in P\}$. Now $\forall p \in P$, we have $[N]_p \leq [M]_p = \mathbb{Z}_p$. Since $N \neq M$, $\exists q \in P$ with $[N]_q = 0$. Claim: $N$ is $\mathbb{Z}_{q^2}$-projective. It suffices to show $[N]_p$ is $\mathbb{Z}_{q^2}$-projective $\forall p \in P$. But $\forall p \in P$, we have $[N]_p = 0$ or $p \neq q$. Since $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}_q, \mathbb{Z}_p) = 0$, the claim follows. Since $\mathbb{Z}_{q^2}$ is not semisimple, $N$ is not $p$-poor.

3.2 Rings with No P-middle Class

We have already established that every module over a semisimple artinian ring is both projective and $p$-poor.

Definition 3.2.1 We shall say that a ring $R$ has no right pmc (short for no $p$-middle class) if every right module $M_R$ is either projective or $p$-poor. Otherwise we shall say $R$ has right pmc.
Obviously semisimple artinian rings have no right pmc. We begin with some simple results.

**Lemma 3.2.2** Let $R$ be a ring with no right pmc and let $I$ be a two-sided ideal of $R$. Then $R/I$ is semisimple artinian or $I \leq \oplus R_R$.

**Proof.** Let $I$ be a two-sided ideal of $R$. If $R/I$ is projective, then $I \leq \oplus R_R$ (by 1.3.20). If not, then $(R/I)_R$ is p-poor. Now suppose that $(R/I)_R$ is $N_R$-projective. Then $(R/I)_R$ is $N_R$-projective (by 1.3.31), and so $N_R$ must be semisimple. Hence $N_{R/I}$ is semisimple (by 1.3.32), which implies $(R/I)_{R/I}$ is p-poor. Hence $R/I$ is semisimple artinian (by 3.1.3). □

**Corollary 3.2.3** If $R$ has no right pmc and $J(R) \neq 0$, then $R$ is semilocal.

**Proof.** If $J(R) \neq 0$, then $J(R) \nsubseteq R$ (by 1.3.24), and so $R$ is semilocal (by 3.2.2). □

**Lemma 3.2.4** Let $R$ be a ring with no right pmc and $J(R) \neq J(R)^2$. Then $J(R)^2 = 0$.

**Proof.** Assume $J(R)^2 \neq 0$. Then $J(R)^2 \nsubseteq R$ (by 1.3.24). Thus $R/J(R)^2$ is semisimple artinian (by 3.2.2). Hence $J(R/J(R)^2) = 0$, so $J(R) = J(R)^2$ (by 1.3.33). □

**Lemma 3.2.5** The property of having no right pmc is inherited by factor rings.

**Proof.** Let $R$ be a ring with no right pmc and let $I$ be a two-sided ideal of $R$. Suppose $M_{R/I}$ is not p-poor. Thus $\exists N_{R/I}$ non-semisimple such that $M_{R/I}$ is $N_{R/I}$-projective. Thus $M_R$ is $N_R$-projective (by 1.3.31). Since $N_R$ is non-semisimple (by 1.3.32), $M_R$ is projective by assumption. Hence $M_{R/I}$ is projective (by 1.3.31). Thus $R/I$ has no right pmc. □

We now begin to use the dichotomy of the no right pmc definition to narrow down the potential rings which could have this property.

**Lemma 3.2.6** Let $R$ be a ring with no right pmc. Then $R$ is either semiprimary with $J(R)^2 = 0$ or semiprime.
Proof. Suppose $R$ is not semiprime. Let $I \neq 0$ be a nilpotent two-sided ideal of $R$. Then $I \subseteq J(R)$ (by 1.3.25), and so $I \not\subseteq R$ (by 1.3.24). Thus $R/I$ is semisimple artinian (by 3.2.2). Hence $J(R/I) = 0$, which implies $I = J(R)$ (by 1.3.33). Thus $J(R) \neq 0$, so $R$ is semilocal (by 3.2.3). But $J(R)$ is nilpotent, which implies that $R$ is semiprimary, and in particular $J(R)^2 = 0$ (by 3.2.4). □

Lemma 3.2.7 Let $R$ be a ring with no right pmc. Then $R$ does not contain an infinite independent family of nonzero two-sided ideals.

Proof. Let $\{I_i\}_{i \in \Gamma}$ be an infinite independent family of nonzero two-sided ideals of $R$, and let $\Gamma'$ be an infinite subset of $\Gamma$ such that $\Gamma \setminus \Gamma'$ is infinite. Set $A = \bigoplus_{i \in \Gamma} I_i$ and $B = \bigoplus_{i \in \Gamma'} I_i$.

Since $B$ is infinitely generated, $B \not\subseteq R$ (by 1.3.22). Thus $R/B$ is semisimple artinian (by 3.2.2). Hence $A/B \leq R/B$, contradicting the fact that $A/B$ is an infinitely generated ideal of $R/B$ (by 1.3.22). □

In many of the following results, we emphasize that $R$ is only indecomposable as a ring, i.e. it is not a direct sum of two nonzero two-sided ideals.

Corollary 3.2.8 Let $R$ be a ring with no right pmc. Then we can write $R = \bigoplus_{i=1}^n e_i R$, where $\{e_i|1 \leq i \leq n\}$ are central orthogonal idempotents in $R$ and every $e_i R$ is indecomposable (as a ring).

Proof. Assume the contrary. Then for all $n \in \mathbb{Z}^+$, we can write $R = e_1 R \oplus \ldots \oplus e_n R \oplus f_n R$, where $e_1, \ldots, e_n, f_n$ are (nonzero) central orthogonal idempotents and $f_n R$ cannot be written as a finite direct sum of indecomposable rings. The collection $\{e_n R|n \in \mathbb{Z}^+\}$ is an infinite independent family of nonzero two-sided ideals, a contradiction (by 3.2.7). □

Lemma 3.2.9 If $R$ is indecomposable (as a ring) with no right pmc, then $Soc(R_R) \leq e R$ with $J(R)^2 = 0$ or $Soc(R_R) = 0$. 

Proof. Assume that $R/S_{oc}(R_R)$ is projective. Then $R_R = S_{oc}(R_R) \oplus A$ for some right ideal $A$ of $R$ (by 1.3.20). We claim that $S_{oc}(R_R)A = 0$. Fix $x \in S_{oc}(R_R)$, and consider the natural epimorphism $f : A \to xA$. Let $T \leq A$ with $Ker(f) \cap T = 0$. Then $T \hookrightarrow A/Ker(f) \cong xA \leq S_{oc}(R_R)$, so $T$ is semisimple. But $S_{oc}(A) = 0$, and so $T = 0$.

Thus $Ker(f) \leq e$, i.e. $A/Ker(f)$ is singular (by 1.3.36). But $A/Ker(f)$ is projective (since $xA \leq _{S_{oc}(R_R)} R$ and $R$ is projective (by 1.3.18)). This implies $xA = 0$ (by 1.3.37), proving our claim, and hence $A$ is a two-sided ideal. Since $R$ is indecomposable (as a ring), $S_{oc}(R_R) = 0$ or $S_{oc}(R_R) = R$. The result holds in either case. Now assume $R/S_{oc}(R_R)$ is not projective. Then $S_{oc}(R_R) \not\subseteq R$ and hence $R/S_{oc}(R_R)$ is semisimple artinian (by 3.2.2). Then $J(R/S_{oc}(R_R)) = 0$, so $J(R) \subseteq S_{oc}(R_R)$ (by 1.3.33). Hence $J(R)^2 \subseteq J(R)S_{oc}(R_R) = 0$ (by 1.3.34). Also $S_{oc}(R_R) \leq e$ (by 1.3.35).

Lemma 3.2.10 Let $R$ be an indecomposable semiprime ring with no right p-middle class. If $R$ is not a prime ring, then $R$ is a semisimple artinian ring.

Proof. Let $I \neq 0$ be a two-sided ideal of $R$. Suppose $R/I$ is projective. Then there exists a right ideal $K$ of $R$ such that $R = I \oplus K$ (by 1.3.20). Note that $(IK)^2 = I(KI)K = 0$. Since $R$ is semiprime, $IK = 0$. Thus $K$ is a two-sided ideal of $R$. Since $R$ is indecomposable, this implies $I = R$, i.e $R/I = 0$. Now suppose $R/I$ is not projective. Then $I \not\subseteq R$ and hence $R/I$ is semisimple artinian (by 3.2.2). Now assume that $R$ is not prime, and let $A, B$ be nonzero two-sided ideals of $R$ such that $AB = 0$. As $R$ is semiprime, we must have $A \cap B = 0$. Hence $R$ is embedded in $R/A \oplus R/B$ by the mapping $r \mapsto (r + A, r + B)$. It follows that $R$ is semisimple.

We are now ready to give necessary conditions for a ring to have no right pmc.

Theorem 3.2.11 If $R$ is a ring with no right pmc, then $R \cong S \times K$, where $S$ is semisimple artinian and $K$ is zero or indecomposable (as a ring) with exactly one of the following properties:
(i) $K$ is a semiprimary right SI-ring

(ii) $K$ is a semiprimary ring with $\text{Soc}(K) = Z(K) = J(K)$

(iii) $K$ is a prime ring with $\text{Soc}(K) = 0$, where either $J(K) = 0$ or $K$ is neither left nor right noetherian.

(iv) $K$ is a ring with infinitely generated essential right socle such that $J(K) = 0$

Proof. By 3.2.8 we have a ring decomposition $R = e_1R \oplus \ldots e_nR$ with every $e_i$ idempotent and every $e_iR$ indecomposable. If $R$ is semisimple artinian, then the result holds (the $K = 0$ case), so assume not. Let $i$ be such that $e_iR$ is non-semisimple. Let $j \neq i$ and let $A$ be a right ideal of $e_jR$. Then for every right ideal $B$ of $e_iR$, we have $\text{Hom}(e_jR/A, e_iR/B) = 0$ (by 1.3.44). This implies $e_jR/A$ is $e_iR$-projective, so it cannot be p-poor. Since $R$ has no right pmc, $e_jR/A$ is projective. Thus $A \leq e_iR$ (by 1.3.20). Thus $e_jR$ are semisimple for all $j \neq i$. Thus $R \cong S \times K$, where $S$ is semisimple artinian and $K$ is indecomposable (as a ring). Now $K$ has no right pmc (by 3.2.5). Thus $K$ is either semiprimary with $J(K)^2 = 0$ or semiprime (by 3.2.6).

Case 1: Let $K$ be a semiprimary ring with $J(K)^2 = 0$. Then $K$ is semilocal. Now if $Z(K) = 0$, then $K$ is a right SI-ring (by 1.3.41). Thus we obtain condition (i). Otherwise, $Z(K) \neq 0$. Thus $Z(K)$ cannot be projective (by 1.3.37). Hence $Z(K) \not\leq_\oplus K$, and so $K/Z(K)$ is semisimple artinian (by 3.2.2). Now $J(K)$ is a $K/J(K)$-module (by 1.3.30), and $K$ is semilocal, so $\text{Soc}(K)$ \neq 0. Thus $\text{Soc}(K) \leq_\oplus K$ (by 3.2.9), so $\text{Soc}(K) \not\leq_\oplus K$. So by 3.2.2, $K/\text{Soc}(K)$ is semisimple artinian, which implies $J(K) \subseteq \text{Soc}(K)$ and $J(K) \subseteq Z(K)$ (by 1.3.33). Note $K$ has a decomposition $K = \oplus_{i=1}^n f_iK$, where the $f_iK$ are local modules (by 1.3.63). For all $k \in \{1, \ldots, n\}$, we have $J(f_kK) \subseteq Z(f_kK) \not= f_kK$ (by 1.3.37). Since each $f_kK$ is local, we must have $J(K) = Z(K)$. Suppose, for contradiction, that $J(K) \neq \text{Soc}(K)$. Then, since each $f_kK$ is local, for some $k$ we must have $\text{Soc}(f_kK) = f_kK$, i.e. some of the $f_iK$ are simple. WLOG assume $f_1K, \ldots, f_tK$ are simple for some $t \in \{1, \ldots, n-1\}$. Let $I_1 = \{i \in \{1, \ldots, n\} \mid f_iK \cong f_1K\} \neq \emptyset$ and
Let $P = \oplus_{i \in I_1} f_i K$. Let $i \in I_1$ and $j \in I_2$. Note that $f_i K$ is nonsingular, $J(f_j K)$ is singular, and $f_j K$ is uniform. Thus we have $\text{Hom}(f_j K, f_i K) = 0$ (by construction and by 1.3.45) and $\text{Hom}(f_i K, f_j K) = 0$ (by construction and by 1.3.46). Since $K$ is indecomposable, we must have $P = 0$ or $P = K$, a contradiction. Hence $J(K) = \text{Soc}(K_K)$, and we obtain condition (ii).

**Case 2:** Let $K$ be a semiprime ring. Assume that $\text{Soc}(K_K) = 0$. If $J(K) \neq 0$, then $J(K) = J(K)^2$ (by 3.2.4). By 1.3.60, $J_K$ and $K J$ are both infinitely generated, and so $K$ is neither left nor right noetherian. Thus by 3.2.10 we obtain condition (iii). Suppose $\text{Soc}(K_K) \neq 0$. Then $\text{Soc}(K_K) \leq e K$ and $J(K)^2 = 0$ (by 3.2.9). Since $K$ is semiprime, $J(K) = 0$. Thus, if $K$ had finitely generated socle, $K_K$ would be semisimple (by 1.3.61), a contradiction.

The natural thing to point out is that this gives some necessary conditions for a ring to have no right pmc, but not sufficient conditions. We give some partial answers to this question.

**Theorem 3.2.12** Let $R$ be a non-semisimple indecomposable ring. Suppose further that $R = \oplus_{i=1}^n e_i R$ is a generalized uniserial ring. Then $R$ has no right pmc iff $J(R)^2 = 0$, $R$ has homogeneous right socle, and $e_i R / \text{Soc}(e_i R) \cong e_j R / \text{Soc}(e_j R)$ whenever $e_i R, e_j R$ are non-semisimple.

**Proof.** ($\Rightarrow$) We may assume, WLOG, that $e_1 R$ is non-semisimple. Since $R$ is right artinian, $J(R)$ is nilpotent, and hence $J(R)^2 = 0$ (by 3.2.4). Suppose, for contradiction, that $e_1 R$ and $e_j R$ are non-semisimple and $S_1 := e_1 R / \text{Soc}(e_1 R) \not\cong S_2 := e_2 R / \text{Soc}(e_2 R)$. Then $0 \neq S_1$ is singular (by 1.3.36), hence not projective (by 1.3.37). But $\text{Hom}(S_1, S_2) = 0$, and so $S_1$ is $e_2 R$-projective. Thus $S_1$ is also not p-poor, contradicting that $R$ has no right pmc. Set $S = e_1 R / \text{Soc}(e_1 R)$. Suppose $R$ does not have homogeneous socle. Set $I_1 = \{ i \in \{1, \ldots, n\} | \text{Soc}(e_i R) \cong \text{Soc}(e_1 R) \} \neq \phi$ and $\phi \neq I_2 = \{1, \ldots, n\} \setminus I_1$. Let $i \in I_1$ and $j \in I_2$. Suppose $e_j R$ is non-semisimple. Then since $S \cong e_j R / \text{Soc}(e_j R)$, $e_i R$ and $e_j R$ are
both projective covers for $S$. Hence $e_1R \cong e_jR$ (by 1.3.54), which implies $Soc(e_1R) \cong Soc(e_jR)$, i.e. $j \in I_1$, a contradiction. Thus $e_jR$ is simple. Since $e_jR$ is projective, it is nonsingular (by 1.3.37). Observe $Hom(e,R, e_jR) = 0$, for if $e_jR$ is also simple, then this follows by construction. Otherwise, it follows by 1.3.45. $Hom(e_jR, e_iR) = 0$ also follows by construction. This contradicts the indecomposability of $R$.

$(\Leftarrow)$ By 1.3.55, every right $R$-module can be written as $P \oplus Q$, where $P$ is projective and $Q$ is semisimple. In light of 3.1.5, we need only show that every simple module is either projective or p-poor. Let $S_2 = Soc(e_1R)$. If $S \cong S_2$, then $S$ is the unique simple module up to isomorphism, hence $S$ is p-poor (by 3.1.8). Similarly if $S \not\cong S_2$, then $S \oplus S_2$ must be p-poor. But since $Hom(S_2, S)$, $S_2$ is $e_1R$-projective. Since all the non-semisimple $e_kR$’s are isomorphic, $S_2$ is $R$-projective. Thus $S_2$ is projective (by 1.3.40). Hence $S$ is p-poor (by 3.1.5).

As an immediate consequence, we have the following examples.

**Example 3.2.13** The ring $R$ of upper triangular $2 \times 2$ matrices over a field has no pmc and is of type (i) in 3.2.11.

**Example 3.2.14** The ring $R = \mathbb{Z}_4$ has no pmc and is of type (ii) in 3.2.11.

We remark that the injective version of both poor modules and rings with no middle class has been studied in [1] and [8]. It is unknown, however, whether there is an example of a ring which is not right noetherian but has no right injective middle class. So it is not surprising that we have had difficulty finding an example of a ring with no right pmc which is not right artinian. In particular, it seems very difficult to find a ring of type (iv) in 3.2.11 that also satisfies the conclusion of 3.2.7, let alone for such a ring to have no right pmc. Our current interest is a right PCI-domain, which might be an example of a 3.2.11
type (iii) ring with no right pmc, as this is an example of a ring with no right injective middle class [1]. Here are some partial results.

**Lemma 3.2.15** If $R$ is a right PCI-domain, then $M_R$ is p-poor iff $M$ is not $R_R$-projective.

**Proof.** $R_R$ is the only non-semisimple cyclic module (by 1.3.57). The result now follows from 3.1.7. □

**Lemma 3.2.16** If $R$ is a right PCI-domain, then $E(R)$ is p-poor.

**Proof.** Since $R$ is right SI (by 1.3.57), and $E(R)/R$ is singular (by 1.3.36), then $E(R)/R$ is semisimple (by 1.3.42). Thus there exists $0 
eq f : E(R) \to S$ for some simple module $S$. Let $g \in Hom(E(R), R)$. Then $Ker(g) \neq 0$ as $R_R$ is not injective. But $R$ is right hereditary (by 1.3.43), and so $Ker(g) \leq E(R)$ (by 1.3.20). Since $E(R)$ is right uniform (by 1.3.58), $Ker(g) = E(R)$, i.e. $Hom(E(R), R) = 0$. Thus $f$ cannot be lifted, so $E(R)$ is not $R$-projective, hence p-poor (by 3.2.15). □

**Corollary 3.2.17** If $R$ is a right PCI-domain, then every nonzero injective module is p-poor.

**Proof.** First, observe that every simple right module is p-poor. Indeed, let $S_R$ be simple. Since $R$ is right uniform (by 1.3.58), $S$ is singular (by 1.3.36), hence not projective (by 1.3.37). Thus $S$ is not $R$-projective (by 1.3.40), hence p-poor (by 3.2.15). Let $0 \neq M_R$ be injective. If $Soc(M) \neq 0$, then $M$ contains a simple submodule $S$. Since $R$ is right SI (by 1.3.57), $S$ is injective, hence $S \leq M$ (by 1.3.21), and $M$ must be p-poor (by 3.1.4). So assume $Soc(M) = 0$. Let $not = x \in M$. Then $xR \cong R$ (by 1.3.57). Hence $E(R)$ embeds in $M$ (by 1.3.59). Since $E(R)$ is p-poor (by 3.2.16), $M$ is p-poor (by 1.3.21 and 3.1.4). □

Our next observation inspired the following definition.

**Definition 3.2.18** A ring $R$ is called right genetic if, for every p-poor module $M_R$, and every module $N_R$ such that $M \leq N$, $N$ must also be p-poor.
The terminology was inspired by the similarity of the property to the notion of right hereditary rings. In view of the upcoming lemma, a right PCI domain is both right hereditary and right genetic. However, we first remark that the two notions do not coincide, even over a commutative ring.

**Example 3.2.19** If $R = \mathbb{Z}$, then $R$ is hereditary but not genetic.

**Proof.** It is well-known that $R$ is hereditary. Let $N_R = \oplus_p \mathbb{Z}_{4p}$, $M_R = \oplus_p 4\mathbb{Z}_{4p}$. Then $N$ is $\mathbb{Z}_4$-projective, hence not p-poor. However, its submodule $M \cong \oplus_p \mathbb{Z}_p$ is p-poor (by 3.1.11), and so $R$ is not genetic. \hfill \square

We now prove our claim.

**Lemma 3.2.20** A right PCI-domain $R$ is right genetic.

**Proof.** Assume that a module $M_R$ is not p-poor and $N \leq M$. We must show $N$ is not p-poor. Since $M$ is not p-poor, it is $R$-projective (by 3.2.15). It remains to show that every $R$-homomorphism $f : N \rightarrow R/I$, where $R/I$ is proper cyclic, lifts to some $g : N \rightarrow R$. Take any such $f$. Since $R/I$ is injective, there exists a homomorphism $h : M \rightarrow R/I$ extending $f$. But $M$ is $R$-projective, so there exists a homomorphism $k : M \rightarrow R$ lifting $h$. If $i : N \rightarrow M$ denotes the identity map, then clearly $g = ki$ lifts $f$. \hfill \square

The second result of the following proposition has the additional assumption of a PRID. It should be noted, however, that the only known example of a PCI-domain, which is due to Cozzens [4], does have this property.

**Proposition 3.2.21** Let $R$ be a right PCI-domain, and let $M_R$ be a module which is neither projective nor p-poor. Then $M$ is nonsingular. In particular, if $R$ is a PRID, then every nonzero submodule of $M$ has a direct summand isomorphic to $R_R$.

**Proof.** Suppose $M$ is not nonsingular. Then $Z(M) \neq 0$ is injective, since $R$ is right SI (by 1.3.57). Thus $Z(M)$ is p-poor (by 3.2.17). Since $Z(M) \leq_\oplus M$ (by 1.3.21), it follows that $M$
is p-poor (by 3.1.4), a contradiction. Now suppose that $R$ is a PRID. Since $M$ is nonsingular, it is not simple (since $R$ is right uniform). Since $R$ is a right $V$-ring, $M$ has a maximal submodule. Thus there exists a simple module $S$ and a nonzero $R$-homomorphism $g : M \to S$. Recall that $M$ is $R$-projective (by 3.2.15). Thus there exists a nonzero $R$-homomorphism $f : M \to R$ with $\text{Im}(f)$ being isomorphic to both $R_R$ and $M/\text{Ker}(f)$. Hence $\text{Ker}(f) \leq \oplus M$ (by 1.3.18 and 1.3.20) and its complement is isomorphic to $R_R$. Since $R$ is right genetic (by 3.2.20), every nonzero submodule of $M$ is $R$-projective (by 3.2.15), and thus has the same property.

We conclude with a result connecting the two topics.

**Theorem 3.2.22** Let $R$ be a right strictly $WV$-ring with $J = J(R)$ and $\text{R}_J$ finitely generated. Then $R$ has no right pmc.

**Proof.** By 2.1.13 (b), $R$ has three right ideals $0 \leq J \leq R_R$, and these are all the cyclic modules (up to isomorphism). Thus if $M_R$ is not p-poor, then $M$ is $R$-projective (by 3.1.7). Noting that $R$ is right perfect, we see that $M$ must be projective (by 1.3.56).
REFERENCES


