Faithful Torsion Modules and Rings

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Abstract

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An $R$ module $M$ is herein called *torsion* if each element has nonzero annihilator, and *faithful* if the annihilator of $M$ is zero. The central theme of this dissertation is exploration of which rings admit modules that are simultaneously faithful and torsion, termed *FT modules*. If a ring $R$ admits an FT right module, it is called *right faithful torsion* or a *right FT ring*, and similarly for the left-hand side. The ring is said to have *FT rank* $\kappa$ if $\kappa$ is a nonzero cardinal and is the least cardinality of a generating set for an FT module over $R$. By convention, rings which are not FT have FT rank 0.

After a survey of the requisite definitions from abstract algebra, several observations are made and lemmas are proven. It is shown that a ring with infinite right FT rank must have a properly descending chain of nonzero ideals of the same length as its FT rank. Using families of ideals with the finite intersection property, we construct torsion modules which are faithful when the family has intersection zero. Using this, it is possible to show that infinite FT ranks can only be regular cardinals. We determine the propagation of FT rank in standard ring constructions such as direct products, matrix rings, and the maximal right ring of quotients. To paraphrase the results: a product is FT if it has an FT factor, infinite products are always FT, matrix rings are often FT, and the FT property travels down from the maximal right ring of quotients.

The next portion of the dissertation gives an account of all that is known about several classes of rings and whether they are FT or not. The two prominent examples of rings that are not right FT are 1) quasi-Frobenius rings $R$ such that $R/rad(R)$ is a finite product of division rings, and 2) any ring $R$ with an essential minimal right ideal. We show that finite products of simple rings are FT exactly when they are not finite products.
of division rings, with possible ranks 0 and 1. Domains are FT exactly when they are not division rings, with rank possible ranks 0, 1 or any infinite regular cardinal. We also show that quasi-Frobenius rings are FT exactly when $R/rad(R)$ is not a finite product of division rings, with possible ranks 0 and 1. Right nonsingular rings are shown to be FT exactly when they are not finite products of division rings. Noetherian serial rings are shown to be FT exactly when they are not Artinian.

The next section demonstrates that for each possible infinite FT rank, there is a commutative domain with that FT rank. While many rings with FT rank 1 also exist, we note that it seems to be very hard to find an example of a ring with finite rank greater than 1.

In the next section, we reverse the central question by asking, “Given an abelian group, can the group be made an FT module over some ring?” The main result is that a torsion abelian group $G$ is FT over its endomorphism ring if and only if $G$ is not cyclic.

Finally we explore a parallel definition of faithful singular modules and rings. We see that our definition of torsion generalizes that of singular modules, and that many of the questions we ask about FT rings are easily answered for faithful singular rings and modules. In particular the rings admitting faithful singular modules are completely classified: a ring $R$ admits a faithful singular right module if and only if $soc(R_R) = \{0\}$. Furthermore, a ring can have a faithful singular rank of any infinite regular cardinal, and also of 0 or 1, but not any other finite cardinal.

Approved: 

Sergio López-Permouth

Professor of Mathematics
To my family.
I would like to thank my parents, Roger and Margaret, for their love and support along this long journey. They fostered every aspect of education in my youth. Perhaps without thinking much of it, they introduced me to a lot of math in everyday life, long before school did. My brothers Matt and Bruce, who were much older and both mechanical engineers, more intentionally, I think, inspired me along the roads of science and math as well.

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<th>Meaning</th>
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<tr>
<td>( \subseteq_e )</td>
<td>essential submodule</td>
</tr>
<tr>
<td>( \text{ann}(X) )</td>
<td>annihilator of the set ( X \subseteq M_R )</td>
</tr>
<tr>
<td>( \ell.\text{ann}(X) )</td>
<td>left annihilator of the set ( X \subseteq R )</td>
</tr>
<tr>
<td>( \text{r.ann}(X) )</td>
<td>right annihilator of the set ( X \subseteq R )</td>
</tr>
<tr>
<td>( B(R) )</td>
<td>Boolean algebra of central idempotents of ( R )</td>
</tr>
<tr>
<td>( E(M_R) )</td>
<td>injective hull of ( M_R )</td>
</tr>
<tr>
<td>( \text{End}(M_R) )</td>
<td>endomorphism ring of ( M_R )</td>
</tr>
<tr>
<td>( \ell.FT(R) )</td>
<td>left FT rank of ( R )</td>
</tr>
<tr>
<td>( \text{r.FT}(R) )</td>
<td>right FT rank of ( R )</td>
</tr>
<tr>
<td>( M_n(R) )</td>
<td>( n \times n ) square matrices over ( R )</td>
</tr>
<tr>
<td>( \text{CFM}_I(R) )</td>
<td>column finite matrices over ( R ) indexed by ( I )</td>
</tr>
<tr>
<td>( \text{RFM}_I(R) )</td>
<td>row finite matrices over ( R ) indexed by ( I )</td>
</tr>
<tr>
<td>( Q'_{\text{max}}(R) )</td>
<td>maximal right ring of quotients of ( R )</td>
</tr>
<tr>
<td>( \text{rad}(R) )</td>
<td>Jacobson radical of ( R )</td>
</tr>
<tr>
<td>( f^n )</td>
<td>( \bigcap_{i=1}^{n} \text{rad}(R)^i )</td>
</tr>
<tr>
<td>( \text{soc}(R_R) )</td>
<td>right socle of ( R )</td>
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## List of Acronyms

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Meaning</th>
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<tbody>
<tr>
<td>FPDR</td>
<td>finite product of division rings</td>
</tr>
<tr>
<td>FPF</td>
<td>finitely pseudo-Frobenius</td>
</tr>
<tr>
<td>FS</td>
<td>faithful singular</td>
</tr>
<tr>
<td>FT</td>
<td>faithful torsion</td>
</tr>
<tr>
<td>PF</td>
<td>pseudo-Frobenius</td>
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<tr>
<td>QF</td>
<td>quasi-Frobenius</td>
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1 Preliminaries

1.1 Preface

Historically various notions of “torsion” played an important role in the study of rings and modules. Over commutative domains, torsion elements of modules naturally form a submodule. In the case of commutative principal ideal domains, very concrete module structure is achieved by analyzing modules with their torsion submodule. The situation for noncommutative domains is not quite as good, but the right notion to adopt is to consider singular elements of a module. These are just the elements of the module whose annihilators are essential in $R$, and together they form a submodule. This submodule of singular elements is defined even for rings which are not domains.

This dissertation explores a variant of the torsions mentioned above: a module is called a torsion module if each of its elements is annihilated by a nonzero ring element. This definition has been used, for example, in [2] and [15]. More often, as in [17] and [10], the annihilating element is required to be a non-zero divisor. Most commonly the ring is just assumed to be a commutative domain so that the classical analysis of modules over a principal ideal domain can be carried out. In any case, the given definition covers all of the above. It also still reflects upon the failure of a module to be free, for the definition effectively prevents any submodule of a torsion module from being free.

A module is called faithful if 0 is the only ring element which annihilates all members of the module. Considered independently, both torsion and faithful modules are easy to find. The purpose of this dissertation is to study the tension between the two conditions when they occur at the same time within a module. Such modules are “locally annihilated but not globally annihilated”, and so the notions are pulling in opposite directions. While it is completely understood which rings have torsion modules, we explore the harder question of which rings admit faithful torsion modules.
We close with a section on faithful singular modules, which form a subclass of torsion modules. It is demonstrated that many of the challenging questions for faithful torsion modules are relatively easy for faithful singular modules. Such modules may also be thought of as a variant of Zelmanowitz’s study of faithful nonsingular modules in [23].

1.2 Ring and Module Theory

All rings considered will be associative rings with multiplicative identity 1, and unless otherwise stated, $R$-modules will be right $R$-modules. Each module will be unitary, that is $x1 = x$ for all $x$ in the given module. Throughout, $FPDR$ is shorthand for finite product of division rings. The symbol “⊆” will be used to refer to subsets, submodules and subrings, depending on the context. Composition of mappings is done from right to left, so that $fg(x) = f(g(x))$. If a ring property depends on side, such as “right Artinian”, but no side is indicated, this will mean the ring has the property on both sides.

For convenience and self-containment, this chapter includes standard terminology used throughout the dissertation. A majority of the definitions can be found in texts such as [1], [17] and [18]. More specialized definitions will be accompanied by specific references. In the following, $R$ is a ring, $M$ is an $R$-module and $N$ is a submodule of $M$.

A domain is a ring $R$ with the property that for all $a, b \in R$, $ab = 0$ implies either $a = 0$ or $b = 0$.

For any subset $X$ of $M$ we can define $\text{ann}(X) = \{r \in R \mid \forall x \in X, xr = 0\}$. If $X = \{x\}$ is a singleton, we will abbreviate this to $\text{ann}(x)$. It is well known that $\text{ann}(X)$ is a right ideal in $R$ for any nonempty subset $X$ of $M$, and is an ideal of $R$ if $X$ is a submodule of $M$. A similar definition exists for left $R$ modules.

When considering $X \subseteq R$, and it becomes necessary to indicate which side we are annihilating $X$, we will distinguish between left and right by using $\text{r.ann}(X) = \{r \in R \mid \forall x \in X, xr = 0\}$ and $\text{l.ann}(X) = \{r \in R \mid \forall x \in X, rx = 0\}$. An element
$r \in R$ is called right regular when $r \text{ann}(r) = \{0\}$, and left regular elements are defined in the obvious way. A regular element is right and left regular element.

We say $N$ is an essential submodule of $M$ if either of the following equivalent properties is satisfied:

- (ESS1) For every submodule $H \neq \{0\}$ of $M$, $H \cap N \neq \{0\}$.
- (ESS2) For all $m \in M \setminus \{0\}$, there exists $r \in R$ such that $mr \in N \setminus \{0\}$.

This is denoted by $N \subseteq_e M$.

**Proposition 1.2.1** If $N_j \subseteq_e M_j$ for all $j \in J$, then $\bigoplus_{j \in J} N_j \subseteq_e \bigoplus_{j \in J} M_j$.

**Proposition 1.2.2** If $N \subseteq M$ are $R$ modules, there exists a submodule $C$ of $M$ such that $N \oplus C \subseteq_e M$.

A module is uniform if all of its nonzero submodules are essential. If $R_R$ is a uniform module, then $R$ is said to be a right uniform ring. A right uniform domain is a right Ore domain.

An element $m \in M$ is called a singular element if $\text{ann}(m) \subseteq_e R$. The set of all singular elements of $M$, denoted $Z(M)$, is a submodule of $M$. A module is called a singular module if $Z(M) = M$ and a nonsingular module if $Z(M) = \{0\}$. A ring $R$ is called right nonsingular if $R_R$ is a nonsingular module. Examples of right nonsingular rings include right hereditary rings, regular rings, rings without nilpotent elements, semiprime right Goldie rings and right Rickart rings.

**Proposition 1.2.3** If $N \subseteq_e M$, then $M/N$ is a singular module. If $M$ is a free $R$-module, then the converse is true.

A module $M$ is injective if for any homomorphism $f : A \to M$ and injection $h : A \to B$ there exists $g : B \to M$ such that $f = gh$. A module $M$ is projective if for any
homomorphism \( f : M \to B \) and surjection \( h : A \to B \) there exists \( g : M \to A \) such that \( f = hg \).

Every \( R \) module \( M_R \) embeds in a module \( E(M_R) \) called the injective hull of \( M_R \). Via the embedding \( M_R \subseteq E(M_R) \), and \( E(M_R) \) is maximal with respect to this property. Simultaneously, \( E(M_R) \) is the minimal injective module containing \( M_R \).

The ring \( R \) is called right self-injective if \( R_R \) is an injective module. The ring \( R \) is called right principally injective if, for any \( a \in R \), every right \( R \) linear homomorphism \( \phi : aR \to R \) extends to a map \( \bar{\phi} : R \to R \). Chapter 5 of [19] includes many results on principal injectivity. In light of the Baer Criterion, a right self-injective ring is right principally injective. The converse is not true however: any von Neumann regular ring is principally injective, and such a ring need not be right self-injective.

A ring \( R \) is called right weakly bounded if every essential right ideal contains a nonzero ideal. If every essential right ideal contains a nonzero ideal as an essential submodule, then \( R \) is called right bounded. If every nonzero right ideal contains a nonzero ideal, \( R \) is called right strongly bounded. A right duo ring is one in which every right ideal is an ideal. If we only require the maximal right ideals to be ideals, then \( R \) is called right quasi-duo. The following implications hold among the conditions: commutative \( \implies \) right duo \( \implies \) right strongly bounded \( \implies \) right bounded \( \implies \) right weakly bounded.

The right socle of a ring \( R \), denoted \( soc(R_R) \), is the sum of all simple right ideals of \( R \). The left socle \( soc(_RR) \) is defined similarly, and is generally not equal to \( soc(R_R) \). The Jacobson radical of \( R \) is denoted by \( rad(R) \) and is the intersection of all maximal right ideals of \( R \). The definition of \( rad(R) \) happens to be left-right symmetric. A ring is called Jacobson semisimple, or sometimes semiprimitive, if \( R \) is a Jacobson semisimple ring. Each of \( soc(R_R) \), \( soc(_RR) \) and \( rad(R) \) is an ideal of \( R \). The Brown-Mccoy radical of \( R \) is defined to be the intersection of all maximal two-sided ideals of \( R \). If the Brown-Mccoy radical is \( \{0\} \) then we say \( R \) is Brown-Mccoy semisimple.
A ring $R$ is called semisimple if $soc(R_R) = R$. Three other equivalent definitions are:

(SS1) *All right ideals are summands in $R$.*

(SS2) *The only essential right ideal of $R$ is $R$ itself.*

(SS3) *$R$ is Jacobson semisimple and right Artinian*

The three corresponding left-handed versions are also equivalent with (SS1)-(SS3), making this condition side independent. Every module over a semisimple ring is a direct sum of simple submodules. The structure of semisimple rings is completely determined by the Artin-Wedderburn Theorem (see the next section).

A ring $R$ is *von Neumann regular* if for all $a \in R$ there exists $x \in R$ such that $a = axa$. For brevity we will refer to von Neumann regular rings as just “regular rings”. The text [11] is a comprehensive resource for regular rings.

The definition is equivalent to either of the following two equivalent conditions, which explain some more properties of regular rings:

(VN1) *All finitely generated right ideals are summands of $R$.*

(VN2) *All principal right ideals are summands of $R$.*

The left hand analogues are also equivalent to these conditions. From these, we see a regular ring $R$ is semisimple if and only if it is right or left Noetherian. Because the principal left and right ideals are summands, regular rings are right and left principally injective. Regular rings are also known to be Jacobson semisimple.

A *right full linear ring* is a ring of the form $End(V_D)$ for some vector space $V$ over a division ring $D$. A ring $R$ is right full linear if and only if it is prime, regular, right self-injective with $soc(R_R) \neq \{0\}$.

A ring $R$ is *local* if it has a unique maximal right ideal. In this case the unique maximal right ideal is $rad(R)$.

A ring $R$ is *semiperfect* if there exists a set $E = \{e_i \mid e_i^2 = e_i, 1 \leq i \leq n\}$ of pairwise orthogonal idempotents such that $\Sigma_{i=1}^n e_i = 1$ and for all $i$, each $e_iRe_i$ is a local ring. This
constitutes a large class of rings including all local rings, one-sided Artinian rings, one-sided perfect rings, and semiprimary rings.

For a semiperfect ring, $R/\text{rad}(R)$ is a semisimple ring. There is a subset $\{f_i \mid 1 \leq i \leq k\}$ of $E$ such that $\{f_iR \mid 1 \leq i \leq k\}$ is a complete, irredundant set of isomorphism types of indecomposable $R$ modules. Write $e_o = \sum_{i=1}^{k} f_i$. We call $e_o$ a basic idempotent of $R$, $e_oR$ a basic submodule and $e_oRe_o$ a basic ring of $R$. If $1$ is already a basic idempotent, $R$ is a self-basic ring.

While there can be several different basic idempotents in a semiperfect ring, the associated basic rings are all isomorphic. The basic module $e_oR$ is faithful. A semiperfect ring $R$ is self-basic if and only if $R/\text{rad}(R)$ is an FPDR. Clearly every local ring is semiperfect and self-basic ring.

A ring $R$ is quasi-Frobenius or just a QF ring if an only if $R$ is self-injective on a side and Noetherian on a side. QF rings are Artinian and self-injective, and contain the class of semisimple rings. A ring $R$ is right pseudo-Frobenius (PF) if every faithful right $R$ module is a generator for $\text{Mod} - R$. A ring $R$ is right finitely pseudo-Frobenius (FPF) if every finitely generated faithful right module is a generator for $\text{Mod} - R$. It is known that: QF $\Rightarrow$ PF, and of course: right PF $\Rightarrow$ right FPF. Any one-sided PF ring is also semiperfect.

A module is uniserial if its submodules are linearly ordered and is serial if it is a direct sum of uniserial submodules. A ring $R$ is right uniserial if $R_R$ is a uniserial module, and right serial if $R_R$ is a serial module. Left-handed definitions are defined analogously. If $R$ is right serial, the decomposition of $R_R$ into a direct sum of uniserial modules witnesses that $R$ is also semiperfect.

A commutative uniserial domain $V$ is called a valuation domain. It is known that given a commutative domain $V$ and its field of fractions $K$, $V$ is a valuation domain if and
only if for every \( k \in K, k \in V \) or \( k^{-1} \in V \). Associated with every valuation domain is an abelian group called the value group or group of divisibility for \( V \).

A ring \( R \) is strongly \( \pi \)-regular if it has the descending chain condition on chains of the form \( xR \supseteq x^2R \supseteq x^3R \supseteq \ldots \) for every \( x \in R \). This condition is known to be left-right symmetric. Clearly any one-sided Artinian ring is strongly \( \pi \)-regular. A theorem by Bass ([18] 23.20) says that right perfect rings satisfy the descending chain condition on left principal ideals, so right perfect rings are also strongly \( \pi \)-regular.

A ring is Dedekind Finite if for \( a, b \in R, ab = 1 \) implies \( ba = 1 \). “Dedekind finite” has also been called “directly finite”. One sided Noetherian rings, semiperfect rings and self-injective rings are all known to be Dedekind finite.

A ring \( R \) is called right \( \Sigma \)-cyclic if every right \( R \) module is a direct sum of cyclic submodules, and \( R \) is called right \( \sigma \)-cyclic if every finitely generated right \( R \) module is a finite direct sum of cyclic submodules. Semisimple rings are obviously \( \Sigma \)-cyclic, and the Drozd-Warfield theorem gives that Noetherian serial rings are \( \sigma \)-cyclic.
1.3 Background Theorems

The items recorded here are well known enough that it will be more convenient to refer to them by name rather than assign them numbers.

**Artin-Wedderburn Theorem:** A ring $R$ is semisimple if and only if $R$ is of the form $\prod_{i=1}^{k} M_{n_i}(D_i)$ where each $D_i$ is a division ring and $n$ is a positive integer.

**Chinese Remainder Theorem:** If a family of ideals $\{I_i \lhd R \mid 1 \leq i \leq n\}$ has the property that $I_i + I_j = R$ whenever $i \neq j$, then the map from $R \to \prod_{i=1}^{n} R/I_i$ given by $r \mapsto (r + I_1, \ldots, r + I_n)$ is a surjection. In particular $R/(\bigcap_{i=1}^{n} I_i) \cong \prod_{i=1}^{n} R/I_i$.

**Hausdorff Maximal Principle:** In any partially ordered set $S$, every totally ordered subset $T \subseteq S$ is contained in a maximal totally ordered subset of $S$. (It is well known that this is equivalent to the Axiom of Choice.)

**Hopkins-Levitzki Theorem:** If $M$ is a module over a semiprimary ring $R$, $M$ is Noetherian if and only if $M$ is Artinian if and only if $M$ has finite composition length.

**Johnson’s Theorem:** ([17] pg. 376)

**Jacobson’s Conjecture:** If $R$ is a right and left Noetherian ring, then

$$\bigcap_{i=1}^{\infty} \text{rad}(R)^i = \{0\}.$$  

Jacobson’s Conjecture is known to hold for Noetherian rings which $R$ are commutative or local, by the Krull Intersection Theorem). Later more classes of Noetherian rings were added to the collection satisfying Jacobson’s Conjecture, including **fully bounded Noetherian** rings, rings with Krull dimension 1 ([6] Theorem 5.13 pg 77), polynomial identity rings ([6] Theorem 7.5 pg 100), and right serial rings ([6] Theorem 6.7 pg 85).

**Krull Intersection Theorem:** ([14] pg 301) If $R$ is a commutative Noetherian ring and $A \lhd R$ and $I = \bigcap_{i=1}^{\infty} A^i$, then $AI = I$ and $(1 - a)I = 0$ for some $a \in A$. If $R$ is a domain or is local, then $I = 0$. 

**Krull’s Theorem:** ( [5] Theorem 1.1) For any totally ordered abelian group $G$, there exists a valuation domain $V$ whose value group is order isomorphic to $G$.

**Prüfer’s Theorem:** ( [16] pg. 74) For an abelian group $G$ and positive integer $n$, if $nG = \{0\}$ then $G$ is a direct sum of cyclic groups.
2 FT MODULES AND RINGS

2.1 Introduction

In the following definitions, $R$ is a ring, $M$ is an $R$-module and $N$ is a submodule of $M$.

**Definition 2.1.1** The module $M$ is called faithful if $\text{ann}(M) = \{0\}$. A module $M$ will be called a torsion module if $\text{ann}(m) \neq \{0\}$ for all $m \in M$. If $M$ is both faithful and torsion it will be called an FT module, and $R$ will be called a (right) FT ring if it admits a (right) FT module.

**Definition 2.1.2** The right FT rank of an FT ring $R$ is the least cardinal $\kappa$ such that $R$ has an right FT module generated by $\kappa$ elements, and the right FT rank of a non-FT ring is 0 by convention. We will denote this $\kappa.\text{FT}(R) = \kappa$. Left FT rings and left FT rank are defined similarly.

We will talk about the right FT rank unless otherwise specified, and compress the notation to $\text{FT}(R)$ for brevity.

It should be noted that since we have not assumed very much about the ring $R$, this “torsion” is different from the usual definitions. The elements of a module with nonzero annihilator do not necessarily form a submodule, for example. We also note that singular modules are torsion, since an essential annihilator in $R$ is certainly nonzero. The definitions prompt a series of problems:

Q1. Characterize rings admitting torsion modules.

Q2. Is the left/right distinction for ‘FT ring’ necessary?

Q3. Find examples of rings of every possible infinite FT rank.
Q4. Determine which rings admit a faithful torsion module.

Q5. Are finite ranks other than 1 and 0 possible?

Q6. Can a ring be FT on both sides with mismatched rank?

Among these problems, Q1 and Q3 are completed, and Q2 has an affirmative answer. The solution to Q1 will be given with the partial progress on Q4. The questions Q5 and Q6 remain unanswered, however we will see in the case of Q6 that if both sides have infinite rank, then they must match.

2.2 Low FT Rank and Right Cohopfian Rings

We will soon see numerous examples of rings with FT rank 1 and every possible infinite FT rank. At present no example rings for other finite nonzero ranks have been identified.

In order to study the conditions for the existence of a finitely generated FT module, we rephrase the situation in terms of a special submodule of $R^n$. Let us call an element $m \in M$ regular if $\text{ann}(m) = \{0\}$. If $M$ is an FT module generated by $n$ elements, then there is the standard epimorphism of $R^n$ onto $M$ taking $n$-tuples to coefficients of linear combinations of the $n$ generators in $M$. Setting $K$ to be the kernel of this epimorphism we can examine $R^n/K \cong M$.

**Proposition 2.2.1** Consider the $R$ module $M = R^n/K$.

(a) $M$ is faithful if and only if whenever $L$ is a left ideal of $R$ and $\bigoplus_{i=1}^n L \subseteq K$, then $L = \{0\}$;

(b) $M$ is torsion if and only if $K \cap xR \neq \{0\}$ for any regular element $x \in R^n$.

**Proof** (a) Suppose first that $M$ satisfies the condition on left ideals. If $x \in \text{ann}(M)$, and $e_j$ denotes the $j$'th unit vector in $\bigoplus_{i=1}^n R$, then $e_jx \in K$ for every $j$. Because $\text{ann}(M)$ is an
ideal, we have then that $Re_jx \subseteq K$ for every $j$, and this amounts to $\bigoplus_{i=1}^n Rx \subseteq K$. By the hypothesis $x = 0$ and $M$ is faithful. For the converse, suppose $M$ is faithful and let $igoplus_{i=1}^n L \subseteq K$ for the left ideal $L$. If $x \in L$, and $(r_1, ..., r_n)$ is an arbitrary element of $\bigoplus_{i=1}^n R$, then $(r_1, ..., r_n)x = (r_1x, ..., r_nx) \in \bigoplus_{i=1}^n L \subseteq K$. Because this means $x \in \text{ann}(M)$, $L = \{0\}$.

(b) If $M$ is torsion and $x \in R^n$ is regular, then the fact that there exists $r \in R$ with $\bar{x}r = \bar{0}$ amounts to $xr \in K \setminus \{0\}$. Thus $K \cap xR \neq \{0\}$. For the converse, suppose $x \in R^n$ is arbitrary. If $x$ is not regular, then $\bar{x}r = \bar{x}r = \bar{0}$ for some nonzero $r \in R$. If $x$ is regular, the hypothesis states there exists $r$ such that $xr \in K$, and so $\bar{x}r = \bar{0}$, proving $M$ is torsion. □

We can draw some quick conclusions about the case of $n = 1$ in Proposition 2.2.1 related to weakly bounded and strongly bounded rings.

**Corollary 2.2.2** If $R$ is right strongly bounded, $FT(R) \neq 1$. If $R$ is not right weakly bounded, then $FT(R) = 1$.

**Proof** A cyclic torsion module of the form $R/K$ for nonzero $K$ is annihilated by $\{0\} \neq I \subseteq K$, where $I \triangleleft R$. For the second statement, if $R$ is not right weakly bounded and has an essential right ideal $E \subseteq_e R$ containing no nonzero ideal of $R$, then $R/E$ is faithful singular, hence torsion. □

In some cases it is actually possible to find a torsion module among the right ideals of $R$. This is especially convenient to do in a ring such that for all $r \in R$, $rs = 0$ for some $s$ or $rs = 1$ for some $s$. In other words, right regular elements are right invertible. If $R$ has this property, then any proper right ideal consists entirely of left zero divisors and hence is torsion. As a bonus, the quotient $R/T$ is also torsion. If $T$ is faithful, or if it contains no nonzero ideal of $R$, then either of these witnesses that $R$ is FT. We now briefly discuss rings with this condition and the dual condition.
Following [22], a module $M$ is called hopfian if every surjective endomorphism is an isomorphism, and it is called cohopfian if every injective endomorphism is an isomorphism.

**Proposition 2.2.3** For a ring $R$:

(a) $R_R$ is cohopfian if and only if for all $r \in R$, $\varepsilon.\text{ann}(r) = \{0\} \implies rR = R$.

(b) $R_R$ is hopfian if and only if $R$ is Dedekind finite.

When $R_R$ is cohopfian, we will say $R$ is a right cohopfian ring.

**Proof** (a) Assume $R_R$ is cohopfian. If $\varepsilon.\text{ann}(r) = \{0\}$, then left multiplication by $r$ is an injective endomorphism of $R_R$. Then this endomorphism is an isomorphism, hence $r$ is a unit, so $rR = R$. Conversely, any injective endomorphism is given by left multiplication by an element $r$, and since $rR = R$, the map is an isomorphism.

(b) Assume $R_R$ is hopfian. If $rs = 1$, then left multiplication by $r$ is an epimorphism, and hence an isomorphism. Then left multiplication by $s$ is the inverse isomorphism of $r$, so $sr = 1$. Thus $R$ is Dedekind finite. Conversely if $R$ is Dedekind finite, any epimorphism is given by left multiplication by an element $r$, and $rs = 1$ for some $s$. By Dedekind finiteness of $R$ we have $sr = 1$ and so $r$ is an isomorphism. \(\square\)

The last theorem shows that $R_R$ and $R_R$ are hopfian or non-hopfian exactly at the same time, and so this condition is side symmetric. However, it turns out that *right cohopfian and left cohopfian rings are different*. The following example was given by Varadarajan in [22] along with a discussion of hopfian and cohopfian objects in the categories of rings and modules.

**Example** A *A left-not-right-cohopfian ring:* Let $\mathbb{Z}/(2)$ denote the ring of integers modulo 2, and $\mathbb{Z}_{(2)}$ be the localization of $\mathbb{Z}$ at the prime ideal $(2)$. We will identify $\mathbb{Z}_{(2)}$ as
\{ \frac{n}{m} \in \mathbb{Q} \mid m \text{ is odd} \}. Our example ring will be the triangular matrix ring

$$R := \begin{bmatrix} \mathbb{Z}/(2) & \mathbb{Z}/(2) \\ 0 & \mathbb{Z}(2) \end{bmatrix}$$

where the \( \mathbb{Z}(2) \) action on the module \( \mathbb{Z}/(2) \) is given by \( (k + (2)) \frac{n}{m} := nk + (2) \). This is well defined: if \( \frac{n}{m} = \frac{n'}{m'} \) and \( k \equiv k' \mod 2 \), then \( nm' = n'm \) implies \( n \equiv n' \mod 2 \), and so \( kn \equiv k'n' \mod 2 \). Thus \( (k + (2)) \frac{n}{m} = (k' + (2)) \frac{n'}{m'} \). Finally it is easily checked that it satisfies the module axioms.

To see \( R \) is not right cohopfian, we claim that \( \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \) is right regular but not right invertible. On one hand if \( \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), then immediately \( a = c = 0 \), and then \( b = 0 \). If \( \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} d \\ e \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) then \( 2f = 1 \) would imply \( 2 \) is a unit of \( \mathbb{Z}(2) \) but clearly \( \frac{1}{2} \notin \mathbb{Z}(2) \).

Now we show that \( R \) is left cohopfian. If \( n \) is even and \( i, j \) are arbitrary, then

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If \( n \) is odd and \( j \) is arbitrary,

$$\begin{bmatrix} 1 & j \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus the only elements which could be left regular must have \( i = 1 \) and \( n \) odd, and in fact they are all units since

$$\begin{bmatrix} 1 & j \\ 0 & \frac{n}{m} \end{bmatrix} \begin{bmatrix} 1 & j \\ 0 & \frac{n}{m} \end{bmatrix} = \begin{bmatrix} 1 & j \\ 0 & \frac{m}{n} \end{bmatrix} \begin{bmatrix} 1 & j \\ 0 & \frac{m}{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This establishes that \( R \) is left cohopfian. \( \square \)

**Proposition 2.2.4** For an \( R \) module \( M \):

(a) If \( M \) is projective cohopfian, \( M \) is hopfian.

(b) If \( R \) is right cohopfian, \( R \) is Dedekind finite. The converse does not hold.
(c) If $M$ is injective hopfian, $M$ is cohopfian.

(d) If $R$ is right principally injective, $R$ is cohopfian if and only if Dedekind finite.

(e) Self injective rings, Dedekind finite regular rings, strongly $\pi$-regular rings, left or right perfect rings, and left or right Artinian rings are all two-sided cohopfian.

**Proof**  (a) An epimorphism $\epsilon : M \to M$ will split when $M$ is projective, that is, there exists $\phi : M \to M$ such that $\epsilon \phi = I_M$. Then $\phi$ is an injection, which must be an isomorphism with inverse $\epsilon$. Thus, $\epsilon$ is an isomorphism. (b) Since $R$ is projective, part (a) says that $R$ is hopfian, hence Dedekind finite. To see the converse is false, note that any commutative Noetherian domain which is not a division ring is Dedekind finite but not a right cohopfian ring. (c) Dualize the proof of (a). (d) Suppose $R$ is right principally injective and Dedekind finite. If $x.\text{ann}(x) = \{0\}$ then by hypothesis $Rx = \ell.\text{ann}(x.\text{ann}(x)) = \ell.\text{ann}(\{0\}) = R$. Then $yx = 1$ for some $y$, and by Dedekind finiteness $xy = 1$ also. Now assume $R$ is right principally injective and right cohopfian. If $ab = 1$, then $x.\text{ann}(b) = \{0\}$, so $bd = 1$ for some $d$. However now $d = abd = a$, so $ba = 1$. (e) Two sided self injective rings are known to be Dedekind finite, and of course they are principally injective, so by part (d), $R$ is (two sided) cohopfian. Regular rings are also principally injective, and with Dedekind finiteness they must be cohopfian also.

Since left or right perfect or Artinian rings are all strongly $\pi$-regular, it suffices to prove that strongly $\pi$-regular rings are cohopfian. Let $x$ be right regular. By definition of strong $\pi$-regularity, there exists $n$ and $a$ such that $x^n = x^{n+1}a$. Writing this as $0 = x^n(xa - 1)$, right regularity of $x$ means that $xa - 1 = 0$, and so $x$ is right invertible, and $R$ is right cohopfian. Since strong $\pi$-regularity is symmetric, a similar argument shows $R$ is left cohopfian.

**Lemma 2.2.5**  If $R$ be a right cohopfian ring and $T$ be a nontrivial right ideal. (a) Both $T$ and $R/T$ are torsion; (b) If $T$ is faithful or contains no nonzero ideal of $R$, then $R$ is FT.
Proof (a) If \( x \in T \subsetneq R \), then \( xy = 1 \) cannot occur. By hypothesis, \( xy = 0 \) for some \( y \neq 0 \), and we see \( T \) is torsion. Now let \( t \in T \setminus \{0\} \). If \( x + T \) is arbitrary, then either \( xy = 1 \) for some \( y \), and so \( (x + T)y = 0 + T \), or else \( xy = 0 \) for some nonzero \( y \), whence \( (x + t)y = 0 + T \). (b) If \( T \) is faithful then of course \( T \) is already an FT module. If \( T \) contains no nonzero ideal of \( R \), then \( R/T \) is faithful, hence an FT module.

**Corollary 2.2.6** Any semiperfect, non-self-basic right cohopfian ring (e.g. a non-self-basic right perfect ring) is FT and has FT rank 1.

Proof Let \( e_o \) be a basic idempotent for \( R \). Because \( R \) is not self-basic, \( e_oR \oplus N = R \) for a nonzero \( N \). The ideal \( e_oR \) is torsion because, as a proper ideal, it must consist of left zero divisors. We have noted earlier that \( e_oR \) is faithful, and so it is FT. Therefore \( FT(R) = 1 \).

**Corollary 2.2.7** If \( R \) is a right strongly bounded ring, then \( FT(R) \neq 1 \). If \( R \) is a right cohopfian ring with \( FT(R) \neq 1 \), then \( R \) right strongly bounded.

Proof The first part of the statement is clear from the text. For the second statement, we know that for any proper right ideal \( T \), \( R/T \) is a 1-generated torsion module for a right cohopfian ring. Since \( FT(R) > 1 \) none of these cyclic modules can be faithful, and so \( T \) must always contain a nonzero ideal. Thus, \( R \) is right strongly bounded.

**Corollary 2.2.8** If \( R \) is right cohopfian and has an FT module of the form \( M = \bigoplus_{i=1}^{n} m_iR \), then \( \forall. FT(R) = 1 \).

Proof Viewing \( M \cong \bigoplus_{i=1}^{n} R/\text{ann}(m_i) \), we can see in general that \( \text{ann}(M) \) is the largest ideal contained in \( \bigcap_{i=1}^{n} \text{ann}(m_i) \). Since \( M \) is faithful in this case, \( \bigcap_{i=1}^{n} \text{ann}(m_i) \) contains no nonzero ideal of \( R \). But \( \bigcap_{i=1}^{n} \text{ann}(m_i) \neq \{0\} \), because there must exist a nonzero element \( r \in R \) such that \( (m_1, \ldots, m_n)r = 0 \), and then \( r \in \bigcap_{i=1}^{n} \text{ann}(m_i) \neq \{0\} \). But this means \( R/\bigcap_{i=1}^{n} \text{ann}(m_i) \neq \{0\} \) is already faithful, and it is also torsion since \( R \) is right cohopfian.
We conclude this section with an example showing that the left/right distinction in the definition of FT rings is necessary.

**Example B (An Artinian right-not-left FT ring)** The construction is a quotient of a skew polynomial ring seen in [17] pg. 454. Consult [20] for more on skew polynomial rings. Let \( \mathbb{F} \) be a field, and \( \sigma \) be a field endomorphism of \( \mathbb{F} \) such that \( [\mathbb{F} : \sigma(\mathbb{F})] = n > 1 \). The example ring is \( R = \mathbb{F}[x; \sigma]/(x^2) \) where the twisted polynomial ring \( \mathbb{F}[x; \sigma] \) is the set of finite linear combinations \( \{ \sum a_i x^i \mid a_i \in \mathbb{F} \} \) with multiplication determined by \( xa = \sigma(a)x \) for each \( a \in \mathbb{F} \).

It can be checked that \( R \) is local with unique maximal left/right ideal \( M = \mathbb{F} \bar{x} = rad(R) = soc(R_R) = soc(R_R) = soc(R_R) = soc(R_R) \). Furthermore, \( _RM \) is a simple left \( R \) module, and \( M_R \) is a semisimple right \( R \) module of composition length \( n \). Considering that \( R/M \) is simple as both a right module and a left module, it is obvious that \( R \) has left composition length 2 and right composition length \( n + 1 \), so \( R \) is right and left Artinian. We see then that \( R \) has exactly three left ideals: \( R, M \) and 0. As a result, a nonzero proper submodule of \( M_R \), say \( T_R \), could not contain a nonzero ideal of \( R \).

Since \( R \) is Artinian it is a right cohopfian ring, and so via Lemma 2.2.5, \( R/T \) is a right FT module. However \( R \) is not left FT since \( R \) has a minimum left ideal. If \( N \) were a torsion left \( R \) module, then \( M \subseteq ann(n) \) for every \( n \in N \), and so \( M = ann(N) \).

### 2.3 Basic Behavior and the FIP Theorem

In this section we establish some fundamental lemmas used throughout the chapter. It is clear that the ideal structure of \( R \) will heavily influence the existence of an FT \( R \) module. We will see that infinite FT rank will be tied to the existence of a descending chain of ideals. Finite FT rank will depend on the existence of a special submodule of a finitely generated free module. We use these results to draw several conclusions about the
behavior of the FT property and FT rank for basic ring constructions including ring products, matrix rings, and localization.

### 2.3.1 Infinite FT Rank and the FIP

**Definition 2.3.1** A collection of ideals $\{I_j \triangleleft R \mid j \in J\}$ is said to have the finite intersection property (FIP) if, for every finite subset $F \subseteq J$, $\bigcap_{j \in F} I_j \neq \{0\}$.

**Proposition 2.3.1** If there exists a collection of ideals $C = \{I_j \triangleleft R \mid j \in J\}$ with the FIP such that $\bigcap C = \{0\}$, then $R$ has an FT module that is a sum of $|J|$ cyclic modules: i.e. $0 < FT(R) \leq |J|$.

**Proof** We claim $M = \bigoplus_{j \in A} R/I_j$ is FT. Let $m = \sum_{\alpha \in F} \bar{r}_\alpha$ with $F$ finite be a generic element of $M$. There exists $0 \neq r \in \bigcap_{j \in F} I_j$ and $mr = \sum_{\alpha \in F} \bar{r}_\alpha r = \sum_{\alpha \in F} \bar{0}$. If $s \in R$ annihilated $M$, then in particular it would annihilate $1 + I_j$ for every $j \in J$. But then $s \in \bigcap C = \{0\}$. Hence, $M$ is faithful. \qed

Recall that a subset $B$ of a partially ordered set $A$ is called *cofinal* if for every $a \in A$ there exists a $b \in B$, $a \leq b$. The *cofinality* of $A$ ($c f(A)$) is the minimum cardinality of a set cofinal in $A$. A cardinal $\kappa$ is a *regular cardinal* if $c f(\kappa) = \kappa$.

**Corollary 2.3.2** Suppose there exists a linearly ordered chain of nonzero ideals $C = \{I_j \triangleleft R \mid j \in J\}$ such that $\bigcap C = \{0\}$, where $J$ is linearly ordered by the order induced by containment. Then $R$ has an FT module that is a sum of $c f(J)$ cyclic modules: i.e. $0 < FT(R) \leq c f(J)$.

**Proof** Assume $C$ is a linearly ordered chain indexed by $J$. Let $A$ be a set cofinal in $J$ with $|A| = c f(J)$.

Obviously, a chain of nonzero ideals has the FIP. We now show that for $C' = \{I_j \triangleleft R \mid j \in A\}$, $\bigcap C' = \{0\}$. If $0 \neq x \in \bigcap C'$, then $x \notin I_j$ for some $j \in J$ as $\bigcap C = \{0\}$. 
However since $A$ is cofinal in $J$, there exists $a \in A$, $a \geq j$, but then $x \in I_a \subseteq I_j$ is a contradiction.

Since $C'$ satisfies the hypotheses of Proposition 2.3.1, we can construct a $\text{cf}(|J|)$ generated direct sum of cyclic modules which is FT. □

**Theorem 2.3.3** *(FIP Theorem)* If $R$ is such that $\text{FT}(R) = \kappa \geq \aleph_0$, then

(a) $\kappa$ is a regular cardinal;

(b) $R$ has an FT module which is a sum of $\kappa$ cyclic modules;

(c) $R$ contains a strictly decreasing chain of ideals $C = \{I_\alpha \mid \alpha < \kappa\}$ with $\cap C = \{0\}$;

(d) $\kappa \leq |R|$.

**Proof** To begin, we let $\{g_\alpha \mid \alpha < \kappa\}$ generate an FT module $M$, and set $M_\beta = \langle g_\alpha \mid \alpha < \beta \rangle$ and $I_\beta = \text{ann}(M_\beta)$. By construction $I_\alpha \neq \{0\}$ because otherwise $M_\alpha$ would be FT but generated by $\alpha < \kappa$ elements. Also by construction $I_\alpha \subseteq I_\beta$ for $\beta > \alpha$, and $\bigcap_{\alpha \in \kappa} I_\alpha = \{0\}$.

(a) Clearly the chain of nonzero ideals we have constructed qualifies for Corollary 2.3.2, and hence $R$ has an FT module which is a direct sum of $\text{cf}(\kappa)$ cyclic modules. By the minimality of $\kappa$ as the FT rank of $R$, $\kappa \leq \text{cf}(\kappa)$, proving $\kappa$ is regular.

(b) With part (a) and Corollary 2.3.2, $\bigoplus_{\alpha < \kappa} R/I_\alpha$ is such a module.

(c) Considered as a set $S = \{I_\alpha \mid \alpha < \kappa\}$ (without repetition), $S$ is a collection of distinct ideals with the FIP such that $\cap S = \{0\}$. Then $|S| = \kappa$, since $\bigoplus_{I \in S} R/I$ is FT. Thus the elements of $S$ can be re-indexed with $\kappa$ to obtain a strictly decreasing chain.

(d) Using the strictly decreasing chain in part c), use the Axiom of Choice to find $x_\alpha \in I_{\alpha+1} \setminus I_\alpha$ for every $\alpha < \kappa$. Clearly the map sending $\alpha \to x_\alpha$ is an injection of $\kappa$ into $R$, proving (d). □

**Corollary 2.3.4** If $\ell.\text{FT}(R) \geq \aleph_0$, then $\ell.\text{FT}(R) \geq \aleph.\text{FT}(R) > 0$. Symmetrically, the same statement with ‘left’ and ‘right’ interchanged is true.
Proof The cyclic module constructed in Corollary 2.3.2 can be viewed both as a right or a left module by construction, and is FT on both sides. □

Corollary 2.3.5 If \( r.FT(R) \) and \( \ell.FT(R) \) are both infinite, then they are equal.

Proof Immediate since \( r.FT(R) \leq \ell.FT(R) \) and \( r.FT(R) \geq \ell.FT(R) \). □

Clearly if the ring has the descending chain condition on annihilator ideals (e.g. is left or right Artinian) it cannot have such a chain of ideals as seen in Theorem 2.3.3. Also a right or left f.cog. ring prohibits such a chain. If \( R_R \) is not f.cog. and is right duo, it must admit such a chain, and so it is right and left FT, although the ranks could possibly be finite.

Upper bounds on the lengths of descending chains of ideals in \( R \) will clearly give an upper bound for \( FT(R) \). We restate here a result of Bass and a later result of Goodearl and Zimmermann-Huisgen. For the latter paper, some notation needs to be introduced. For an ordinal \( \alpha \), let \( [0, \alpha) = \{ \beta \mid 0 \leq \beta < \alpha \} \) denote the set of ordinals under the usual inclusion order, and \( [0, \alpha)^* \) be the same set ordered by reversing the usual order. Define \( \kappa(M) \) to be the least ordinal such that \( [0, \kappa(M))^* \) does not embed in the lattice of submodules of \( M \) ordered by inclusion. In particular the ordinal \( \kappa(M) \) serves as a cardinal bound on the length of decreasing chains in \( M \). For the definition and development of Krull dimension, the reader should see [13].

Theorem 2.3.6 (Bass [3]) If \( R \) is a commutative Noetherian ring and \( M \) is a finitely generated \( R \) module, then every descending chain of submodules of \( M \) is countable. In particular, every descending chain of ideals of \( R \) is countable.

Theorem 2.3.7 (Goodearl, Zimmermann-Huisgen [12]) The Krull dimension of \( M \) exists and is countable if and only if \( \kappa(M) \) is countable. If \( M \) has countable Krull dimension \( \alpha \), then \( \omega^\beta < \kappa(M) \leq \omega^{\beta+1} \) for every ordinal \( \beta < \alpha \).
Corollary 2.3.8 If $R$ is either commutative and Noetherian or if $Rx$ or $xR$ has countable Krull dimension, then $FT(R) \leq \aleph_0$.

Proof If $R$ is non-FT or $FT(R) < \aleph_0$ then we are done. If $FT(R)$ is infinite, then Proposition 2.3.2 constructs a descending chain of nonzero ideals with trivial intersection in $R$. Appropriately applying the previous two theorems, this chain must be of length exactly $\aleph_0$, and hence $FT(R) = \aleph_0$. □

Proposition 2.3.9 If the ideals of $R$ have the FIP, then $FT(R)$ is 0, 1 or infinite.

Proof Suppose $FT(R)$ is not 0, 1 or infinite. Let $M$ be a FT module for $R$ with a minimal generating set $\{g_j \mid 1 \leq j \leq n\}$. Each $I_j = \text{ann}(g_j, R) \neq \{0\}$. Using the FIP, choose $0 \neq r \in \bigcap_{j=1}^n I_j$. However $Mr = (\Sigma_{j=1}^n g_j R)r = \{0\}$ contradicts the assumption $M$ is faithful. □

The lattice of ideals of a ring $R$ would satisfy the FIP if, say, $R$ were a domain, or more generally a prime ring.

2.3.2 Products, Matrices and $Q_{\text{max}}^r(R)$

Here are the basics of behavior of FT rank in ring products.

Theorem 2.3.10 Let $R = \prod_{i \in I} R_i$.

(a) If $R_k$ is FT for some $k$, then $R$ is FT and $FT(R) \leq FT(R_k)$;

(b) If $I$ is finite and $FT(R) \geq \aleph_0$, then there exists $k$ such that $FT(R_k) = FT(R)$;

(c) If $I$ is infinite, then $R$ is FT and $0 < FT(R) \leq \aleph_0$. 
Proof (a) Let $M = \prod_{i \in I} M_i$ where $M_k$ is an FT module generated by $FT(R_k) > 0$ elements, and $M_i = R_i$ for $i \neq k$. Let $m$ be a generic element of $M$. If $m$ is zero on the $k$'th coordinate, then the unit vector $E_k$ annihilates $m$. If $m$ is nonzero on the $k$'th coordinate, then pick $0 \neq r_k \in R_k$ to annihilate that coordinate, and then $E_k r_k$ annihilates $m$. If $s \in R$ annihilates $M$, then in particular it annihilates $E_i r_k$ which appears in $M$, and this is saying that $s$ is zero outside of the $k$'th coordinate. Finally, since $s$ annihilates $m E_k$ for every $m \in M_k$, the $k$'th coordinate of $s$ is in the annihilator of $M_k$, and so must be 0. Thus, $s$ is identically zero and $M$ is faithful.

Now all that remains is to show $M$ is generated by $FT(R_k)$ elements. Let $\{|g_i | i \in I\| = FT(R_k)$ be a generating set for $M_k$. Fix a single $j$ and define $G_j \in M$ to be $g_j$ on the $k$'th coordinate and 1 elsewhere, and for all $i \neq j \in I$, let $G_i$ be $g_i$ on the $k$'th coordinate and zero elsewhere. Clearly the new generating set generates $M$ and has $FT(R_k)$ elements.

(b) By Theorem 2.3.3 there exists a strictly descending chain of ideals in $R$, say $C = \{\prod_{i=1}^\kappa I_{ij} | j \in \kappa\}$ such that $\bigcap C = \{0\}$. Our notation is meant to convey that for all $j$, $I_{ij} \triangleleft R_i$, in accordance with the ideal structure of the product $R$. Let $\pi_i : R \to R_i$ be the canonical projection. Since $C$ is an infinite chain of nonzero ideals, for some $k$ it must be that $\pi_k(C)$ is also an infinite nonzero chain of ideals in $R_k$, for otherwise $C$ would decrease to zero in only finitely many steps. Clearly $\bigcap \pi_k(C) = \{0\}$. We conclude by Corollary 2.3.2 that $FT(R_k) \leq \kappa$, and by part (a) that $\kappa \leq FT(R_k)$. Thus $FT(R) = FT(R_k)$.

(c) Let $\Omega$ be a countably infinite subset of $I$ and identify its elements with the natural numbers. Form $I_n = \prod_{j \geq n} R_j \triangleleft R$. Clearly the $I_n$ form a descending chain of ideals of $R$ with intersection zero. By Corollary 2.3.2, $FT(R) \leq \aleph_0$.

Turning now to matrix rings, we will see there are some similarities to the results for products of rings. For an index set $I$ and ring $R$, the set $X$ of arrays of elements of $R$ indexed by $I \times I$ contains two important “infinite matrix rings”, namely the rings
\[ \text{CFM}_I(R) = \{ M \in X \mid \text{columns of } M \text{ are finitely nonzero} \} \text{ and the ring} \]
\[ \text{RFM}_I(R) = \{ M \in X \mid \text{rows of } M \text{ are finitely nonzero} \}. \]
In the case when \( I \) is finite these both coincide with the usual matrix ring which we denote with \( M_n(R) \).

For this theorem, we assume \( R \) is right FT, and observe how this affects its matrix rings.

**Theorem 2.3.11** Let \( R \) be such that \( \varepsilon.FT(R) = \kappa > 0 \), and nonempty index set \( I \):

(a) We have \( \kappa \geq \varepsilon.FT(\text{CFM}_I(R)) > 0 \);

(b) If \( \varepsilon.FT(M_k(R)) \geq N_0 \) for some \( k \), then \( \varepsilon.FT(M_k(R)) = \kappa \). Moreover for all \( n \in \mathbb{Z}^+ \), either \( \varepsilon.FT(M_n(R)) = \kappa \) or \( 0 < \varepsilon.FT(M_n(R)) < N_0 \);

(c) If \( \kappa \geq N_0 \), then \( \varepsilon.FT(M_n(R)) > 0 \) and \( \ell.FT(M_n(R)) > 0 \) for all \( n \in \mathbb{Z}^+ \).

Analogous statements hold for \( \ell.FT(R) \) and \( \text{RFM}_I(R) \).

**Proof** (a) Fix a \( \kappa \) generated FT module \( N \), and set \( N_i = N \) and \( N_j = R \) for all \( j \neq i \). Form the direct product of groups \( M = \prod_{i \in I} N_i \). Now let \( \text{CFM}_I(R) \) operate on the elements of \( M \) by matrix multiplication on the right of row vectors in \( M \). If \( m \in M \) is nonzero on the \( i \)’th coordinate \( m_i \), then select a nonzero \( r \in R \) such that \( m_i r = 0 \), and note \( m(rE_{ii}) = 0 \). If on the other hand \( m \) is zero on the \( i \)’th coordinate, then \( E_{ii} \) already annihilates \( m \). Moreover \( M \) is at most \( \kappa \) generated, since if \( \{ g_\alpha \mid \alpha < \kappa \} \) is a set of generators for \( N \) then the set
\[ \{(0, 0, \ldots, 0, g_\alpha, 0, \ldots, 0) \mid 2 \leq \alpha < \kappa \} \cup \{(1, 1, \ldots, 1, g_1, 1, \ldots, 1)\} \]
generates \( M \). It now suffices to show \( M \) is faithful.

Suppose \( A \in \text{CFM}_I(R) \) annihilates \( M \). In particular for \( j \neq i \), \( A \) must annihilate the element is 1 on the \( j \)’th coordinate and zero elsewhere. This amounts to the \( j \)’th row of \( A \) being all zeros. Also \( A \) must annihilate elements of the form \( (0, 0, \ldots, 0, n, 0, \ldots, 0) \) which
are arbitrary \( n \in \mathbb{N} \) on the \( i \)'th coordinate and zero elsewhere. This amounts to the entries of the \( i \)'th row of \( A \) lying in \( \text{ann}(N) = \{0\} \), thus \( A \) is the zero matrix.

(b) Since \( FT(M_k(R)) \) is infinite, by Theorem 2.3.3 \( M_k(R) \) has an infinite chain of ideals indexed by \( FT(M_k(R)) \) strictly descending to zero. This corresponds to a similar system of ideals in \( R \), and by Corollary 2.3.2 the rank of \( R \) must be \( 0 < FT(R) \leq FT(M_k(R)) \). By part (a), \( \exists FT(M_k(R)) \leq \kappa \), showing that \( \exists FT(M_k(R)) = \kappa \).

Clearly then for each \( n \), either \( 0 < \exists FT(M_n(R)) < \aleph_0 \) or else the above holds and \( \exists FT(M_n(R)) \leq \kappa \).

(c) By Theorem 2.3.3, \( R \) has a chain of nonzero ideals with zero intersection indexed by \( \kappa \), which by correspondence of ideals between matrix rings lifts to a chain with the same properties in every ring \( M_n(R) \), and so by Corollary 2.3.2 this furnishes a left and right FT module. Thus \( FT(M_n(R)) > 0 \) on both the left and right.

Similar construction and proof is made for the left analogues by letting matrices operate on the left of \( M \).

In the next theorem we examine when matrix rings are FT in general, whether or not their base ring \( R \) is FT. By adding a very loose restrictions like “left or right cohopfian”, it turns out that all the matrix rings (bigger than 1 by 1) are also FT, and quite frequently have rank 1.

**Lemma 2.3.12** Let \( S = \mathbb{CF}M_I(R) \) denote the column finite matrices indexed by \( I \) over the ring \( R \). Let \( S M = \bigoplus_{i \in I} R \) and \( N_S = \prod_{i \in I} R \) be \( S \) modules, where \( S \) operates on the left of \( M \) and right of \( N \) via ordinary matrix multiplication. Then \( S M \) and \( N_S \) are always faithful.

**Proof** If the unit vectors of \( M \) are annihilated on the left by \( A \in S \), this just amounts to the columns of \( A \) being all zeros. Similarly, the unit vectors of \( N \) being annihilated on the right by \( A \) causes the rows of \( A \) to be all zeros. \( \square \)
Theorem 2.3.13 If \( R \) is a ring and \( I \) is a nonempty index set, then:

(a) If \( \aleph_0 \leq |I| \), then \( 0 < \ell.FT(\mathbb{C}FM_I(R)) \leq |I| \);

(b) If \( 2 \leq |I| \) and \( R \) is a left cohopfian, then \( 0 < \ell.FT(\mathbb{C}FM_I(R)) \leq |I| \).

If additionally \( I \) is finite, then \( \ell.FT(\mathbb{C}FM_I(R)) = 1 \);

(c) If \( 2 \leq |I| \) and \( R \) is a right cohopfian ring, then \( r.FT(\mathbb{C}FM_I(R)) = 1 \).

An analogous series of statements holds for for the row finite matrices \( RFM_I(R) \).

Proof Set \( S := \mathbb{C}FM_I(R) \) and let \( sM \) and \( N_S \) be as in Lemma 2.3.12, so that \( sM \) and \( N_S \) are both faithful modules.

(a) Clearly \( sM \) is generated by \( |I| \) generators, and now we verify that \( sM \) is torsion. If \( m \in M \), since \( I \) is infinite we see \( m \) is zero on a coordinate, say \( j \), and the matrix unit \( E_{jj} \) annihilates \( m \). So, \( M \) is left FT module.

(b) Of course \( sM \) is generated by \( |I| \) elements, so we proceed to show that \( sM \) is torsion. Let \( m = (\ldots m_j \ldots) \in M \). In one case if for some \( i \) and \( a \neq 0 \), \( am_i = 0 \), then \((aE_{ii})m = (\ldots 0 \ldots) \). The other case is when all the entries of \( m \) are left regular, hence left invertible by the left cohopfian hypothesis. Focusing on two arbitrary entries \( m_i \) and \( m_j \), there exists \( b \) such that \( bm_i = 1 \), and then \( m_j b E_{ii} - E_{ij} \) is nonzero and annihilates \( m \) on the left. So, \( sM \) is FT. Again when \( I \) is finite, \( M \) can is generated by the all 1’s vector, in which case \( \ell.FT(T) = 1 \).

(c) Unlike \( sM \), \( N_S \) is always generated by one element, which is the all 1’s vector. We can show that \( N_S \) is torsion in a similar fashion to (b). Let \( n = (\ldots n_j \ldots) \in N \). In one case if for some \( i \) and \( a \neq 0 \), \( n_i a = 0 \), then \( n(aE_{ii}) = (\ldots 0 \ldots) \). The other case is when all the entries of \( n \) are right regular, hence right invertible by the right cohopfian hypothesis. Focusing on two entries \( n_i \) and \( n_j \), there exists \( b \) such that \( n_i b = 1 \), and then \( bn_j E_{ii} - E_{ji} \) is nonzero and annihilates \( n \) on the right. \( \square \)
Corollary 2.3.14 A right (resp. left) full linear ring is two-sided FT with ranks governed by Theorem 2.3.13 if and only if it is not a division ring.

Proof It is well known that for a vector space $V_D$ (resp. $D V$) over a division ring $D$, $\text{End}(V_D) \cong \mathbb{C} \mathbb{F} M_I(D)$ (resp. $\text{End}(D V) \cong \mathbb{R} \mathbb{F} M_I(D)$) where $|I| = \dim D(V)$. So we are free to apply Theorem 2.3.11 and Theorem 2.3.13. Clearly $\text{End}(V_D)$ (resp. $\text{End}(D V)$) is a division ring if and only if $I$ has one element. □

Finally we note some behavior of FT rank in rings of quotients. This in particular will apply to classical rings of fractions $R S^{-1}$ as in ([17] pg. 299) and also the maximal right ring of quotients $Q'_\text{max}(R)$. Since $Q'_\text{max}(R)$ is well understood for many types of rings, it can be used to show that $R$ is FT.

Proposition 2.3.15 Let $R \subseteq S$ be a unitary ring extension such that $R_R \subseteq S_R$. If $S$ is FT, then $R$ is FT. If $\text{FT}(S) = \kappa \geq \aleph_0$, then $0 < \text{FT}(R) \leq \kappa$.

Proof Let $M_S$ be a right FT module. Clearly $M_R$ is a faithful module with the action that $R$ inherits as a subring of $S$.

For arbitrary $m \in M$, the annihilator $\{0\} \neq \text{ann}_S(m) \subseteq S$ is right ideal of $S$, and so a right $R$ submodule of $Q_R$. By essentialness of $R_R$, $R \cap \text{ann}_S(m) \neq \{0\}$, so that $\text{ann}_R(m) \neq \{0\}$. So, $M_R$ is torsion.

Now suppose $\text{FT}(S) = \kappa \geq \aleph_0$. By Corollary 2.3.2 select a family $\{I_j \triangleleft S \mid j \in \kappa\}$ which is a strictly decreasing chain of nonzero ideals with intersection zero. For each $j$, $I'_j := I_j \cap R \triangleleft R$. Since $R \subseteq S$, $I'_j \neq \{0\}$. Then the collection $\{I'_j \triangleleft R \mid j \in \kappa\}$ is also a chain of nonzero ideals with intersection zero, and so by Corollary 2.3.2 we have $0 < \text{FT}(R) \leq \text{FT}(S)$. □

The maximal ring of quotients $Q = Q'_{\text{max}}(R)$ is such an $S$ for the right module $R_R$. In fact, $R_R$ satisfies the even stronger condition of being dense in $Q_R$: this is denoted $R \subseteq_d Q_R$. This will be revisited later in Section 3.8.
We note that the converse of Proposition 2.3.15 is not true, for it may be that $Q$ is not FT but $R$ is FT. For example, Proposition 3.1.2 states that domains which are not division rings are FT, but for all right Ore domains, $Q$ is a division ring. Occasionally then $Q$ may be going “too far up”, but this does not rule out the possibility of an intermediate ring $S$, $R \subseteq S \subsetneq Q$, such that $S$ is FT. An example of this is shown in Proposition 3.3.1.
3 Known FT Rings

Since we have required rings to have unity, all such rings have a faithful module, namely the ring itself. The classification of rings admitting torsion modules is nearly as easy. Clearly a division ring never admits a torsion module, because the right ideal \( \text{ann}(m) \) must be one of \( D \)'s two right ideals \( D \) and \( \{0\} \), and it is certainly not \( D \). In fact division rings are exactly the rings without torsion modules.

**Theorem 3.0.16** A ring \( R \) admits a torsion module if and only if it is not a division ring.

**Proof** We show that non-division rings have a torsion module. Suppose first that \( R \) is semisimple. Because \( R \) is Artinian, it is a right cohopfian ring and has a proper right ideal. So by Lemma 2.2.5, this right ideal is a torsion \( R \) module. Otherwise, if \( R \) is not semisimple, \( R \) has a proper essential right ideal, say \( E \). Since \( R/E \) is singular, it is a torsion module. 

The classification of rings which admit a faithful torsion module has not been completed, however partial results for specific classes of rings have been obtained. We first examine products of domains, since in this case torsion is just the classical torsion in most undergraduate texts. This is easily extended to rings with right socle zero. These results are significant in their own right because the faithful torsion module constructed is actually faithful singular. Products of simple rings are classified next, and as an application it is shown which Brown-McCoy semisimple rings are FT.

Finite products of either simple rings or domains are both special cases of the classification of right nonsingular FT rings, however we are including all proofs because they are each of different character. The classification of right nonsingular rings in an interesting application of the maximal right ring of quotients \( Q_{\text{max}}(R) \), the structure of self-injective regular rings, and the fact that a ring with an FT ring of quotients is FT itself.
Next the quasi-Frobenius FT rings are classified. The QF rings are as different as possible from right nonsingular rings in the sense that they only overlap on the semisimple rings. Rather than using a ring of quotients, a part of the classification of QF rings uses a categorical property that embeds $R$ into faithful modules. The broadest result using this idea is the theorem on semiperfect right FPF rings.

We then move away from this embedding property and obtain a partial characterization of Noetherian semilocal rings, which generalize quasi-Frobenius rings in a different way. One special collection of Noetherian semilocal rings is the class of Noetherian serial rings.

3.1 Domains

Domains will be examined first because of the tie between domains and the ubiquitous definition of torsion for domains. Since torsion is well understood in the case of commutative domains, we will use that as a jumping off point. If $R$ is right Ore, then every right ideal is essential and our definition becomes the same thing as “singular”.

Theorem 2.3.3 again plays a role in determining which Brown-Mccoy semisimple rings and which prime rings are FT.

**Proposition 3.1.1** For any ring $R$, $soc(R_R) = \bigcap \{E \leq R \mid E \subseteq_e R\}$.

**Proof** Set $S = \bigcap \{E \leq R \mid E \subseteq_e R\}$. Clearly any essential right ideal contains every minimal right ideal, and so $soc(R_R) \subseteq S$. If now we show that $S$ is semisimple, then it will be contained in the socle, which is the maximum semisimple submodule of $R$. Let $A$ be a submodule of $S$. There exists a submodule $B$ of $R$ such that $A \oplus B \subseteq_e R$. By assumption $S \subseteq A \oplus B$, and then $S = A \oplus (B \cap S)$. Thus $S$ is semisimple and so $S \subseteq soc(R_R)$. □

The following proposition demonstrates that all non-division ring domains are FT, since their socles are trivial.
Proposition 3.1.2 If $C = \{E_j \subseteq R \mid j \in J\}$ is a collection of essential right ideals of $R$ with $\bigcap C = \{0\}$, then $R$ is FT with $FT(R) \leq |C|$. In particular, this applies when $soc(R_R) = \{0\}$.

If $T = \bigcap \{J \mid J_R < R_R\} \neq \{0\}$ then $T$ is the unique minimal right ideal of $R$, and $R$ is not FT.

Proof After forming $M = \bigoplus_{j \in J} R/E_j$, we see the zero intersection condition causes $M$ to be faithful. By Proposition 1.2.1, $M$ is singular, hence torsion. In particular, since $soc(R_R)$ is characterized as the intersection of all essential right ideals of $R$, if $soc(R_R) = \{0\}$ then this statement applies to the collection of all right essential ideals.

For the last statement, note that in an FT module $M$,

$$\{0\} \neq T \subseteq \bigcap_{m \in M} \text{ann}(m) = \text{ann}(M)$$

\[\square\]

Corollary 3.1.3 If $R$ is a direct product of domains, $R$ is FT if and only if it is not an FPDR.

Proof Combine Proposition 3.1.2 with Theorem 2.3.10. \[\square\]

Corollary 3.1.4 If $R_R$ is right uniform (meaning $R_R$ is a uniform module, e.g. a right uniserial ring), then $R$ is FT if and only if $soc(R_R) = \{0\}$.

Proof Both directions follow from Proposition 3.1.2. If $soc(R_R) = \{0\}$ then $R$ is FT. In a uniform module all submodules are essential. So then $soc(R) = I \neq \{0\}$ is the intersection of all right ideals of $R$, and as such it is the unique minimal right ideal of $R$. In this case, $R$ would not be FT. \[\square\]
We now explore several classes of commutative domains with countable FT rank. By Corollary 2.3.2, one way to show that a ring is FT is to find an infinite strictly decreasing chain of ideals with intersection zero. As observed before, if $R$ is not f.cog. and $R$ is right duo, then it necessarily has such a chain. An even nicer situation is if there exists an ideal $I \triangleleft R$ such that $\bigcap_{i=1}^{\infty} I^i = \{0\}$, but $I$ is not a nilpotent ideal, then the powers of $I$ are a suitable chain, and we can also conclude that the FT rank is at most countable. This may occur in rings with no nilpotent ideals, such as semiprime rings. The condition that $\bigcap_{i=1}^{\infty} I^i = \{0\}$ is not an unusual one. In algebraic geometry, for example, an $I$-adically complete ring must have $\bigcap_{i=1}^{\infty} I^i = \{0\}$. This method will resurface again during our investigation of Noetherian semilocal rings.

**Proposition 3.1.5** If $V$ is a valuation domain with a minimal nonzero prime ideal $P$, then $FT(V) = \aleph_0$.

**Proof** Let $0 \neq x \in P$. Of course any nonunit $x$ in a domain produces a strictly descending chain $xV \supsetneq x^2V \supsetneq \ldots$. This is so, for if $x^n \in x^mV$ for some $m > n$, we would have $x^n = x^m r$, and after cancellation $1 = x^{m-n} r$ would insist that $x$ is a unit, a contradiction.

We will show that $Q = \bigcap_{i=1}^{\infty} x^i V \varsubsetneq P$ is a prime ideal of $V$. Suppose to the contrary it is not prime, and let $a, b \in V$ satisfy $ab \in Q$ and $a, b \notin Q$. Select $m$ large enough such that $a, b \notin x^mV$. Since the ideals are linearly ordered, we may assume without loss of generality that $x^mV \subset aV \subseteq bV$. Writing $x^n = ar = bs$, we see that $x^{2m} = abrs \in abV \subseteq Q$. But then $x^{2m}V \subseteq Q \subseteq x^{2m}V$ so that equality holds. However this is a contradiction since $Q \subseteq x^{2m+1}V \subsetneq x^{2m}V = Q$. It must be then, that $Q$ is a prime ideal.

Recall however that $P$ is assumed to be a minimal prime, so that $Q = \{0\}$. By Corollary 2.3.2, the ideals $x^i V$ allow the construction of an $\aleph_0$ generated FT module for $V$. □
Proposition 3.1.6 A commutative domain \( D \) with a minimal nonzero prime ideal \( P \) admits a valuation overring \( V \) with Krull dimension 1.

Proof Localizing \( D \) at a minimal prime \( P \) produces an overring \( D_P \) of \( D \) which is a local domain of Krull dimension 1 within the field of fractions \( Q(D) \). The only two prime ideals of \( D_P \) are \( \{0\} \) and \( PD_P = \text{rad}(D_P) \). There exists a nonfield valuation overring \( V \subseteq Q(D) \) of \( D_P \).

We now claim that \( V \) has a minimal nonzero prime ideal. Let \( \{P_i \mid i \in I\} \) be the set of nonzero prime ideals of \( V \), and define \( Q_i = P_i \cap D_P \). It is easy to check the \( Q_i \) are a collection of nonzero prime ideals of \( D_P \). Since \( D_P \) only has one nonzero prime ideal, \( \bigcap_{i \in I} Q_i = \text{rad}(D_P) \). If \( V \) did not have a minimal (=minimum) nonzero prime ideal, then \( \bigcap_{i \in I} P_i = \{0\} \), but this is impossible since \( \text{rad}(D_P) = \bigcap_{i \in I} Q_i \subseteq \bigcap_{i \in I} P_i \).

Denote the minimal nonzero prime ideal of \( V \) by \( P^* \). The localization \( V_{P^*} \) is now a valuation overring of \( D \) in \( Q(D) \). It clearly has only two prime ideals \( \{0\} \) and \( \text{rad}(V_{P^*}) \), hence it has Krull dimension 1. \( \square \)

Theorem 3.1.7 A commutative domain \( D \) with a nonzero minimal prime ideal has \( FT(D) = \aleph_0 \).

Proof Using By Proposition 3.1.6, we find a Krull dimension 1 valuation overring \( V \), \( D \subseteq V \subseteq Q(D) = \mathcal{Q}_{\text{max}}(D) \). By Proposition 3.1.5, \( FT(V) = \aleph_0 \), and then by Proposition 2.3.15 \( FT(D) = \aleph_0 \). \( \square \)

Corollary 3.1.8 The following commutative domains have \( FT \) rank \( \aleph_0 \):

(a) (Nonfield) unique factorization domains;

(b) (Nonfield) domains with the D.C.C. on nonzero prime ideals;

(c) Domains of finite Krull dimension;

(d) \( D[x] \) and \( D[[x]] \) for any commutative domain \( D \).
Proof Each of (a), (b) and (c) will fall under Theorem 3.1.7. Unique factorization in (a) ensures that for any prime element \( p \), \( pD \) is a minimal prime ideal. In (b), applying the D.C.C. to the collection of all nonzero prime ideals of \( D \) finds a minimal nonzero prime ideal. Finally in (c), after selecting a maximal chain of prime ideals with length equal to the Krull dimension, certainly the smallest nonzero member is minimal, by the (commutative) definition of Krull dimension.

In (d) we verify that in both cases, the ideal generated by \( x \) is a minimal prime ideal. If \( R \) is either ring, suppose \( 0 \neq a \in P \subsetneq xR \). Now \( a \) factors into \( a = x^k b \) where \( k \in \mathbb{Z}^+ \) and \( b \) has a nonzero constant term. If \( P \) is prime, either \( x^k \) or \( b \) is in \( P \). However \( x^k \in P \) implies \( x \in P \), and \( b \notin P \) since it has a constant term. Thus the only possible prime ideal contained in \( xR \) is \( P = 0 \).

As for a noncommutative analogue for this corollary, something can be said for noncommutative unique factorization domains as defined in ([7]). This then applies to noncommutative principal ideal domains.

**Proposition 3.1.9** If \( D \) is a (possibly noncommutative) unique factorization domain, and \( D \) has a proper essential cyclic right or left ideal, then \( FT(D) = \aleph_0 \) or \( FT(D) = 1 \).

Proof We will prove the right hand case, the left hand case being completely symmetric. Suppose \( aD \subsetneq D \). The goal is to produce an infinite descending chain of essential right ideals with zero intersection, so as to invoke Proposition 3.1.2. We will show that \( a^n D \subsetneq D \) for all \( n \in \mathbb{Z}^+ \). We make use of two little lemmas.

Lemma (i) \( D/aD \cong a^n D/a^{n+1} D \). Clearly the map \( \phi : D \to a^n D/a^{n+1} D \) is a well defined surjective homomorphism. If \( x \in ker(\phi) \), then \( a^n x = a^{n+1} y \in a^{n+1} D \) for some \( s \). Then canceling \( x \)'s on the left results in \( x = ay \in aD \). Obviously \( aD \subseteq ker(\phi) \), so we have shown \( ker(\phi) = aD \). By a fundamental theorem of homomorphisms, \( D/aD \cong a^n D/a^{n+1} D \).
Lemma (ii) For any \( R \), if \( F \) is free and \( F/E \) is singular, \( E \subseteq_e F \). This is cited in the introductory chapter.

Since \( aD \) is an essential right ideal of \( D \), \( D/aD \) is a singular module and by Lemma (i) so are the isomorphic modules \( a^nD/a^{n+1}D \). Now principal right ideals in a domain are always free, since \( aD \cong D/\mathfrak{r}.\text{ann}(a) = D/[0] \cong D \). Thus the \( a^nD \) are all free, hence by Lemma (ii) we see \( a^{n+1}D \subseteq_e a^nD \) for every \( n \). An elementary property of essential extensions is that they are finitely transitive, so in fact \( a^nD \subseteq_e D \) for every \( n \).

Finally, the UFD condition will cause \( \bigcap_{i=1}^{\infty} a^iD = \{0\} \). Let \( 0 \neq x \in \bigcap_{i=1}^{\infty} a^iD \). Since \( a \) is not a unit, it has some fixed number \( k \geq 1 \) of irreducible factors. There also exists \( r_n \) so that \( x = a^n r_n \) for every \( n \). But this means that \( x \) has at least \( kn \) irreducible factors for each \( n \), causing a contradiction: the number of irreducibles in a factorization of \( x \) must be finite and invariant.

By Proposition 3.1.2, \( FT(D) \leq \aleph_0 \). By Proposition 2.3.9 if \( FT(D) \neq 1 \), then it must be infinite, so \( FT(D) = \aleph_0 \).

It seems that \( aD \subseteq_e D \) should happen quite frequently. For example, when \( D \) is right Ore, all right ideals are essential. If even \( Da \subseteq aD \) for a single nonunit \( a \), then \( aD \subseteq_e D \).

**Corollary 3.1.10** Let \( D \) be a (possibly noncommutative) UFD which is not a division ring. If \( Da \subseteq aD \) for some nonunit \( a \), then \( FT(D) \leq \aleph_0 \). This applies when \( D \) is a (possibly noncommutative) principal ideal domain.

If \( D \) is right duo, \( FT(D) = \aleph_0 \).

**Proof** If \( a \) is a nonunit such that \( Da \subseteq aD \), consider any \( 0 \neq b \in R \). Since \( 0 \neq ba = ac \in aD \), \( bD \cap aD \neq \{0\} \). This is clearly equivalent to \( aD \subseteq_e D \), so that Proposition 3.1.9 applies.

Cohn shows that noncommutative principal ideal domains are also unique factorization domains. Since \( D \) is not semisimple, it has an essential right ideal \( aD \).
Finally if $D$ is right duo, $aD = Da$ for any $a$, and taking $a$ to be a nonunit satisfies the previous statement. Moreover, $FT(R) = 1$ is eliminated by Corollary 2.2.2. □

The author is interested in knowing the answer to the following question.

**Question:** If $D$ is a (noncommutative) UFD, is it possible for $aD$ and $Da$ to be nonessential for every nonunit $a$?

### 3.2 Products of Simple Rings

Now for a look at products of simple rings, including FPDRs. We saw that in the case of infinite FT ranks, FT modules could be taken to be direct products of cyclic modules. Is this possible to do for finite ranks with finite sums of cyclic modules? For many rings which are close to commutative, the answer is “no”.

**Lemma 3.2.1** If $R$ is right strongly bounded, a module of the form $M = \bigoplus_{i=1}^{n} m_i R$ is not FT.

**Proof** To aid in our proof, we write $M = \bigoplus_{i=1}^{n} m_i R \cong \bigoplus_{i=1}^{n} R/\text{ann}(m_i)$. Suppose that $M$ is torsion. Then $(m_1, m_2, ..., m_n) \neq (0, 0, ..., 0)$, so there exists $r \neq 0$, $(m_1, m_2, ..., m_n)r = (m_1r, m_2r, ..., m_nr) = (0, 0, ..., 0)$. By the directness of the sum, $m_ir = 0$ for all $i$ between 1 and $n$. This says that $r \in \bigcap_{i=1}^{n} \text{ann}(m_i) \neq \{0\}$ is a nonzero right ideal, and so it contains a nonzero $I \lhd R$. But if $0 \neq x \in I$, clearly $(R + \text{ann}(m_i))x \subseteq I \subseteq \text{ann}(m_i)$ for all $i$, so that $x \in \text{ann}(\bigoplus_{i=1}^{n} R/\text{ann}(m_i)) = \text{ann}(M)$. So if $M$ is torsion, then it cannot be faithful. □

**Corollary 3.2.2** Let $R$ be a right strongly bounded ring. If $R$ is right $\sigma$-cyclic, then $FT(R) = 0$ or is infinite, and if $R$ is $\Sigma$-cyclic then $FT(R) = 0$.

**Proof** If $R$ is $\sigma$-cyclic, then $R$ cannot have a finitely generated FT module because it would be $\sigma$-cyclic, and Lemma 3.2.1 forbids this. In that case $R$ is non FT or has infinite FT rank.
If $R$ is (two-sided) $\Sigma$-cyclic, it is known ([8] 20.23) that $R$ is Artinian, and so must have finite FT rank. Since $R$ is a fortiori $\sigma$-cyclic, all finitely generated modules are also not FT, and so the only remaining possibility is that $FT(R) = 0$. □

**Theorem 3.2.3** The following hold for any ring $R$: (i) No FPDR is FT; (ii) A simple ring $R$ is FT with $FT(R) = 1$ if and only if it is not a division ring; (iii) A product of simple rings (in particular semisimple rings) is FT with $FT(R) = 1$ if and only if it is not an FPDR.

**Proof** (i) An FPDR is a $\Sigma$-cyclic duo ring, so Theorem 3.2.3 (i) says $R$ is not FT.

(ii) Of course, no division ring is FT. Now if $R$ is not a division ring, any torsion module will be automatically faithful, by simplicity of $R$. Furthermore, $R$ is not right strongly bounded, since a right strongly bounded simple ring is easily a division ring. In case $R$ is semisimple, by Corollary 2.2.7 it is FT with rank 1. If it is not semisimple, then it has a proper essential ideal $E$ so that $R/E$ is 1-generated singular, hence torsion.

(iii) By (i) a product of division rings is not FT. If the product of simple rings is not a product of division rings, then at least one of the factors is simple but not a division ring, and so by (ii) it is FT of rank 1. By Theorem 3.2.3 the product is FT of rank 1. □

**Proposition 3.2.4** Any ring $R$ with a trivial Brown-McCoy radical is FT if and only if it is not an FPDR.

**Proof** By Theorem 3.2.3(i) if $R$ is an FPDR it is not FT. Suppose now $R$ is not an FPDR. If the collection maximal ideals of $R$ have the FIP, then invoking Proposition 2.3.1 we conclude $R$ is FT. If the collection does not have the FIP, fix a finite subcollection of maximal ideals such that $\bigcap_{j=1}^n I_j = \{0\}$. By the Chinese Remainder Theorem, $R \cong \prod_{j=1}^n R/I_j$ is a product of simple rings. Since $R$ is not an FPDR by assumption, $R$ is FT by Theorem 3.2.3. □
Examples for such rings include right or left quasi-duo (or just commutative) rings with $\text{rad}(R) = \{0\}$.

That a finite product of simple rings is FT is subsumed by the later results on right nonsingular rings, however less information is available about their rank.

### 3.3 Right Nonsingular Rings

As noted before, right nonsingular rings are a broad class of rings including right hereditary rings, regular rings, rings without nilpotent elements, semiprime right Goldie rings and right Rickart rings. To analyze right nonsingular rings, we use the following machinery.

**Proposition 3.3.1** Let $R$ be a ring and suppose $Q = Q'_{\text{max}}(R)$ is an FPDR. Then $R$ is FT if and only if $R$ is not an FPDR.

**Proof** As before, if $R$ is an FPDR it is not FT, so we now assume it is not an FPDR. Since $Q$ is an FPDR we have $R \subseteq Q$.

We may write $Q = \prod_{i=1}^{n} e_iQe_i$ where the $e_i$'s are mutually orthogonal central idempotents of $Q$ such that each $e_iQe_i$ is a division ring with identity $e_i$, and $\Sigma_{i=1}^{n} e_i = 1$. Note that $e_iRe_i$ is a subring of $e_iQe_i$ and so is at least a domain. Furthermore, $R \subseteq \prod_{i=1}^{n} e_iRe_i = R'$ since $1 \in R'$. To finish the proof, we will find an FT ring $S$ so that $R \subseteq S \subsetneq Q$.

We will show that for some $k$, $e_kRe_k \subsetneq e_kQe_k$. Suppose to the contrary that for all $i$, $e_iRe_i = e_iQe_i$. Form $T_i = e_iRe_i \cap R$ and $T = \prod_{i=1}^{n} T_i$. It is clear $T$ is an $R$ submodule of $Q$, and since $R \subseteq e_iQe_i$, all the $T_i$ are nonzero.

Obviously $T \subseteq R$. Let $i$ be arbitrary and $0 \neq e_ir \in T_i$. Because $e_iRe_i$ is a division ring with identity $e_i$, $e_ir$ has an inverse $e_is$ for some $s \in R$. This amounts to $e_i = (e_iRe_i)(e_is) = (e_ir)s \in T_i$. But then $e_i \in T_i \subseteq R$ for every $i$ so that $R = Q$, a contradiction.
So, there exists \( k \) such that \( e_k Re_k \subseteq e_k Qe_k \). Form \( S = \prod_{i=1}^{n} S_i \) where \( S_k = e_k Re_k \) and \( S_j = e_j Qe_j \) for all \( j \neq k \). By construction, \( S \neq Q \). We claim that \( S_k = e_k Re_k \) is a domain but not a division ring. If it were a division ring, then \( S \) would be a finite product of division rings, and moreover it would be right self-injective and contain \( R \) as an essential submodule. But then by ([17] Proposition 13.39 pg. 378) \( S = Q \), a contradiction. Thus \( S_k \) is an FT domain, and so \( S \) is FT.

**Lemma 3.3.2** A prime, regular, right self-injective ring \( R \) is FT if and only if \( R \) is not a division ring.

**Proof** \( R \) is not FT if it is a division ring, so we assume now that it is not. If \( soc(R_R) = \{0\} \), then \( R \) is right FT by Proposition 3.1.2. If on the other hand, \( soc(R_R) \neq \{0\} \), then \( R \) is a right full linear ring. Now \( R \cong \bigoplus M_I(D) \) for some nonempty index set \( I \) and division ring \( D \). It is clear that \( R \) is a division ring if and only if \( |I| = 1 \), so \( |I| \geq 2 \). Since \( D \) is trivially right cohopfian, Theorem 2.3.13 (c) shows that \( R \) is left and right FT.

**Lemma 3.3.3** ([11] pg.100 Corollary 9.11) Let \( R \) be a regular, right self-injective ring, and \( B(R) \) denote the central idempotents of \( R \). Then \( B(R) \) forms a complete Boolean algebra with the partial ordering \( e \leq f \iff eR \subseteq fR \), and \( R \) is a direct product of prime rings if and only if \( B(R) \) is an atomic lattice.

**Theorem 3.3.4** If \( R \) is a regular, right self-injective ring, then \( R \) is FT if and only if not an FPDR.

**Proof** As usual, an FPDR is FT and we now assume \( R \) is not an FPDR.

Case 1: \( B(R) \) is atomic. Then \( R \) factors into a product of prime, regular, right self-injective rings. If this product is infinite, \( R \) is FT by Theorem 2.3.10. If the product is
finite, at least one factor is not a division ring. By Lemma 3.3.2 this factor ring, and hence the whole ring, is FT.

Case 2: $\mathcal{B}(R)$ is not atomic. There must be $0 \neq e_0 \in \mathcal{B}(R)$ such that there is no atom $a \in \mathcal{B}(R)$ with $a \leq e_0$. Invoking the Hausdorff Maximal Principle, select a maximal linearly ordered subset $L$ of $\mathcal{B}(R)$ containing $e_0$. It is clear that $0 \in L$, and we now consider $L' = L \setminus \{0\}$. We claim that $\bigcap\{eR \mid e \in L'\} = \{0\}$. If $\bigcap\{eR \mid e \in L'\} \neq \{0\}$, since $\mathcal{B}(R)$ is a complete lattice, there exists $0 \neq f \in \mathcal{B}(R)$. $fR = \bigcap\{eR \mid e \in L'\}$. But as $f \leq e_0$ it cannot be an atom, so there exists $0 \neq f' < f$. Then it is clear that $L \subsetneq L \cup \{f'\}$ is linearly ordered, but this would contradict the maximality of $L$. Hence, $\{eR \mid e \in L'\}$ is a chain of nonzero ideals in $R$ with the FIP and intersection zero. This proves that $R$ is FT via Corollary 2.3.2. □

**Corollary 3.3.5** For any right nonsingular ring $R$, $R$ is FT if and only if $R$ is not an FPDR.

**Proof** We assume $R$ is not an FPDR. Let $Q = Q_{\text{max}}(R)$. By Johnson’s Theorem, $Q$ is regular and right self-injective. If $Q$ is not an FPDR, it is FT by Theorem 3.3.4 $Q$, hence $R$ is FT. If $Q$ is an FPDR, Proposition 3.3.1 indicates $R$ is FT. □

**Lemma 3.3.6** ([4] Proposition 1.(iii)) If $R$ is a right strongly bounded ring, $R$ is semiprime if and only if reduced.

Because regular rings are semiprime, we see that a right strongly bounded regular ring is strongly regular.

### 3.4 Quasi-Frobenius Rings

Now we will turn our attention to an important superclass of semisimple rings which is in general not right or left nonsingular: the quasi-Frobenius rings. Indeed, a QF ring is right nonsingular if and only if it is semisimple. Since QF rings are Artinian, we already
know they will have finite rank. In preparation, there are some things that can be said about the more general class of semiperfect FPF rings.

**Proposition 3.4.1** (Faith [9] Theorem 1.2B) If $R$ is a semiperfect ring and $M$ is a generator for $\text{Mod} - R$, then $M \cong N \oplus e_oR$ where $e_o$ is a basic idempotent of $R$. In particular if $R$ is additionally right FPF, then every f.g. faithful $R$ module contain a copy of $e_oR$.

**Corollary 3.4.2** Semiperfect FPF rings have infinite FT rank, or FT rank 0, or 1.

**Proof** Suppose $FT(R)$ is finite and nonzero. Let $e_o$ be a basic idempotent for $R$ and $M$ a finitely generated FT module. By Theorem 3.4.1, $e_oR \oplus N = M$ for some $N$, and $e_oR$ is a torsion module. We noted earlier $e_oR$ is faithful, and so it is a cyclic FT module. Thus $FT(R) = 1$. □

While the self-basic condition precludes finite ranks for semiperfect right FPF rings, it does not necessarily preclude any infinite ranks. Valuation domains of every possible infinite rank will be constructed later, and they are all semiperfect, self-basic, and FPF.

**Proposition 3.4.3** If $R$ is QF then $R$ is FT if and only if it is non-self-basic. Either $FT(R) = 0$ or $FT(R) = 1$.

**Proof** We note that since $R$ is Artinian, it is both semiperfect and right FPF. Therefore $R$ has finite FT rank, and 3.4.1 applies. If $R$ is a self-basic QF ring, then $R$ is its own basic module. Now $R$ embeds in every f.g. faithful module, so no faithful module can be torsion. If $R$ is non-self-basic, then $e_oR$ is a cyclic FT module, as per Corollary 2.2.6. □

In summary, the entire picture for Artinian rings has yet to be resolved. We have seen up through QF rings that only rank 0 and 1 are allowed, and that non-self-basic Artinian rings are rank 1.
3.5 Semiperfect and Serial Rings

This section comes close to a complete classification of Noetherian semilocal rings. Several of the following proofs rely on the condition \( J^\infty := \bigcap_{i=1}^{\infty} \text{rad}(R)^i = \{0\} \). The topic of \( J^\infty = \{0\} \) is part of a famous conjecture in Noetherian rings: Jacobson’s Conjecture. As noted in the introductory chapter, Jacobson’s Conjecture has been verified for many classes of Noetherian rings including local rings, fully bounded rings, rings with Krull dimension 1, and right serial rings.

The following proposition follows immediately from Corollary 2.3.2.

**Proposition 3.5.1** Let \( R \) be a ring with the property of the conclusion of Jacobson’s Conjecture, i.e. \( \bigcap_{i=1}^{\infty} \text{rad}(R)^i = \{0\} \). If \( \text{rad}(R) \) is not nilpotent (e.g. if \( R \) is semiprime), then \( R \) is FT on both sides with \( 0 < \text{FT}(R) \leq \aleph_0 \).

The Hopkins-Levitzki Theorem provides a convenient way of determining when \( \text{rad}(R) \) is nilpotent in a semilocal ring.

**Corollary 3.5.2** If \( R \) is semilocal, right Noetherian, not right Artinian, and \( \bigcap_{i=1}^{\infty} \text{rad}(R)^i = \{0\} \), then \( R \) is FT on both sides with \( 0 < \text{FT}(R) \leq \aleph_0 \).

**Proof** We show that \( \text{rad}(R) \) is not nilpotent. If to the contrary \( \text{rad}(R) \) were nilpotent, \( R \) would be semiprimary. By the Hopkins-Levitzki Theorem, a semiprimary right Noetherian ring is right Artinian, a contradiction. Therefore \( \text{rad}(R) \) is not nilpotent, and satisfies Proposition 3.5.1. \[\square\]

On one hand, if Jacobson’s Conjecture were true for semilocal Noetherian rings, then the above result would prove the non Artinian ones are all FT. On the other hand, as a corollary to Lemma 2.2.5, any right or left Artinian ring \( R \) for which \( R/\text{rad}(R) \) is not an FPDR must be FT with rank 1. One barrier to classification, aside from the veracity and scope of Jacobson’s Conjecture, is the following question.
**Question:** If $R$ is Artinian and $R/rad(R)$ is an FPDR, when is $R$ FT?

We now pass to a special subclass of semilocal rings, Noetherian right serial rings, because the structure of their finitely generated modules allows us to apply Lemma 3.2.1. We would like to prove that an Artinian serial ring with $R/rad(R)$ an FPDR is not FT, and hopefully generate some insight into the question above. If we add the hypothesis of “right strongly bounded”, then the Drozd-Warfield Theorem allows us to prove $R$ is not FT.

**Theorem 3.5.3** (Drozd-Warfield [21] pg 21) Every finitely presented module over a serial ring is serial, more specifically a direct sum of local uniserial (hence cyclic) modules.

**Proposition 3.5.4** A right strongly bounded, Noetherian, serial ring $R$ is FT (with $FT(R) = \aleph_0$) if and only if not Artinian.

**Proof** Before proving the proposition, we first prove a lemma: under the hypotheses, $R$ does not have finitely generated FT modules. Let $M$ be a finitely generated torsion module. Since $R$ is Noetherian, $M$ is finitely presented. By the Drozd-Warfield Theorem, $M$ is a finite direct sum of cyclic modules. However because $R$ is right strongly bounded, Lemma 3.2.1 says that $M$ cannot be faithful.

Here the proof of the proposition begins. If $R$ is Artinian then has finite FT rank. Since finite nonzero ranks are prohibited by the above lemma, $R$ is non FT.

Now conversely suppose $R$ is not Artinian. Since right serial rings are semiperfect (e.g. [21] pg 9), and Noetherian right serial rings satisfy Jacobson’s Conjecture, it follows from Corollary 3.5.2 that $R$ is FT with rank no greater than $\aleph_0$. Because finite nonzero ranks are prohibited, the rank must be exactly $aleph_0$.

We will shortly see many examples strongly bounded serial rings in the guise of valuation domains.
How much can be said for non-Noetherian rings? Certainly there are non-Noetherian FT rings, since there are commutative domains which are not Noetherian. The following shows that there are also non-Noetherian non-FT rings.

**Example** A commutative, non-Noetherian, non-FT ring: Let $V$ be a valuation domain with a nonzero prime ideal $P$ properly contained in $\text{rad}(V)$. For example, this occurs when $V$ has Krull dimension greater than 1.

Let $x \in P \setminus \{0\}$ and set $I = \bigcup \{J \subset V \mid x \not\in J\}$. Now $I$ is an ideal since it is the union of a chain of ideals, and $I$ is nonzero since $x \not\in x^2V < V$. Since quotients of uniserial rings are uniserial, $R := V/I$ is uniserial, and we can check that $xV/I$ is its unique minimal ideal. If $I/I \subset J/I \subset xV/I$, then $I \subset J \subset xV$, but then clearly $x \not\in J$ and so $J \subset I$, and $J/I = I/I$.

Finally we claim $R$ is not Noetherian. Let $y \in \text{rad}(V) \setminus P$. For each positive integer $n$, we can show $xy^{-n} \in V$. If this were not the case, then since $V$ is a valuation domain the inverse $y^n x^{-1} \in V$, however that would imply $y^n = xy^n x^{-1} \in P$, whence $y \in P$ by primeness of $P$, but this is a contradiction. The chain of principal ideals $xV \subset xy^{-1}V \subset xy^{-2}V \subset \ldots$ can be seen to be strictly ascending, since if $xy^{-(n+1)} \in xy^{-n}V$, we obtain a contradiction in the following way:

$$xy^{-(n+1)} = xy^{-n}v \iff xy^n = xvy^{n+1} \iff 1 = vy \iff v = y^{-1} \in V$$

Since $y$ is definitely not a unit in $V$, we can conclude the chain is strictly increasing, and so is the chain $xV/I \subset xy^{-1}V/I \subset xy^{-2}V/I \subset \ldots$. Thus $V/I$ is not Noetherian. □

**3.6 Commutative Rings of Every Infinite FT rank**

We have seen earlier in Proposition 2.3.9 that domains which are not division rings have FT rank 1 or some infinite regular cardinal. Of course commutative domains cannot have finite FT rank, but we show using valuation domain theory that it is possible to flexibly construct commutative domains of every possible infinite FT rank.
**Proposition 3.6.1** Let $\nu$ be a valuation of a field $K$ with value group $G$ and valuation domain $V$, $V$ not a field. Then $FT(V) = cf(G)$.

**Proof** Since $V$ is a commutative domain, Proposition 2.3.9 holds and $FT(V) = \kappa$ is infinite. Therefore there exists a chain $C = \{I_\alpha \triangleleft V \mid \alpha \in \kappa\}$ such that $\bigcap C = \{0\}$. Choose $x_\alpha \in I_\alpha$. We claim that $X = \{\nu(x_\alpha) \mid \alpha \in \kappa\}$ is cofinal in $G$. Supposing that this is not the case, find $0 \leq g \in G$ which is an upper bound of $X$. Let $x \in V$ be a preimage of $g$, $\nu(x) = g$. Because $\nu(x_\alpha) \leq \nu(x)$, $x_\alpha | x$ for all $\alpha$, but then $x \in \bigcap C = \{0\}$, a contradiction. Hence, $cf(G) \leq \kappa$. Conversely, choose a set $\{\nu(x_\alpha) \mid \alpha \in cf(G)\}$ which is cofinal in $G$. Then $\bigcap x_\alpha V = \{0\}$, and since these principal ideals area linearly ordered, we have by Proposition 2.3.2 $\kappa \leq cf(G)$.

**Theorem 3.6.2** For any two infinite cardinals $\kappa \leq \lambda$ with $\kappa$ regular, there exists a valuation ring $V$ with $|V| = \lambda$ and $FT(V) = \kappa$.

**Proof** We begin by constructing an ordered abelian group with cofinality $\kappa$. Let $G = \bigoplus_{\alpha \in \kappa} \mathbb{Z}$. When ordered with reverse lexicographic order, this group is totally ordered abelian group. We claim that $cf(G) = \kappa$. Define a function $\pi : G \setminus \{0\} \to \kappa$ by $\pi(g) = g_\alpha$ where $g_\alpha$ is the largest index on which $g$ is nonzero. Let $A \subseteq G^+$ with $|A| < \kappa$. We will show that $A$ cannot be cofinal in $G$. Obviously $|\pi(A)| < \kappa$. Since $\kappa$ is regular, $\pi(A)$ is not cofinal in $\kappa$. Let $\alpha \in \kappa$ be such that $\pi(a) < \alpha$ for every $a \in A$. Let $e_\alpha$ be 1 in the $\alpha$’th coordinate and zero elsewhere. It’s clear that $h > a$ for every $a \in A$ and so $A$ is not cofinal in $G$.

We now produce a group of size $\lambda$ with cofinality $\kappa$. Let $H$ be any ordered abelian group of order $\lambda$. The group $G \oplus H$ has cardinality $\lambda$, and we claim that when ordered lexicographically it has cofinality $\kappa$. Fixing a collection $C = \{(g_i, h_i) \mid i \in I\}$, we check that $C$ is cofinal in $G \oplus H$ if and only if $C' = \{g_i \mid i \in I\}$ is cofinal in $G$. For the “only if”, given an element $(g, 0)$, there must be $(g', 0) > (g, 0)$, and by cofinality of $C$, there must be
\((g_i, h_i) > (g', 0)\). Then \(g_i \geq g' > g\). This establishes that \(C'\) is cofinal in \(G\). For the “if” part, if \((g, h)\) is arbitrary, there must be \(g_i > g\) so that \((g_i, h_i) > (g, h)\). This establishes the cofinality of \(C\) in \(G \oplus H\). Now it is apparent that \(cf(G \oplus H) = cf(G) = \kappa\).

The valuation ring with value group \(G \oplus H\) is constructed using Krull’s Theorem. Specifically, Krull’s proof uses the group algebra \(\mathbb{K}[G \oplus H]\) and its field of fractions \(\mathbb{F}\). We may choose \(\mathbb{K}\) to be a finite field, so that \(|\mathbb{K}[G \oplus H]| = |\mathbb{F}| = |G \oplus H| = \lambda\). The associated valuation domain \(V\) constructed ([5] Proposition 1.1) is a subdomain of \(\mathbb{F}\). Since this is the case, \(|V| = |\mathbb{F}| = \lambda\). As shown in Proposition 3.6.1 \(FT(V) = cf(G \oplus H) = \kappa\). □

This provides us with a variety commutative, local domains which are FT. A Prüfer domain is a commutative domain whose finitely generated ideals are projective. Since the finitely generated ideals of valuation domains are principal, all finitely generated ideals are isomorphic to \(R\), and hence are projective. Thus, valuation domains are Prüfer domains. It appears in [9] that commutative domains are Prüfer if and only if FPF, so these valuation domains are also examples of semiperfect, FPF rings with high FT rank.

### 3.7 Endomorphism Rings

In this section we examine when an abelian group \(G\) can be realized as an FT module of some ring. For ease of notation we switch to left \(R\) modules in this section.

Recall that \(R = Hom(G_{\mathbb{Z}}, G_{\mathbb{Z}}) = End(G)\) is a ring under pointwise addition and composition of mappings. Moreover \(G\) is naturally a left \(R\) module under the operation \(r \cdot g := r(g)\) for each \(r \in R\) and \(g \in G\). If \(G\) is cyclic, say \(< g >= G\), then it cannot be torsion over \(End(G)\), for if \(\phi \in End(G)\) and \(\phi(g) = 0\), then \(\phi(G) = \{0\}\).

**Lemma 3.7.1** Let \(C_1 = \mathbb{Z}_{p^n}, C_2 = \mathbb{Z}_{p^m}\) be cyclic groups with \(p\) prime and \(n \geq m \geq 1\). If \(A = C_1 \oplus C_2\), then \(A\) is torsion as an \(End(A)_{\mathbb{Z}}\) module.

**Proof** Let \(\pi_i : C_i \rightarrow A\) be the coordinate projections for \(i = 1, 2\), and \(f_k : A \rightarrow A\) be given by \(f_k(x) = kx\) for each integer \(k\). Let \(x = (c_1, c_2)\) be arbitrary in \(A\). If \(k = ord(c_1) < p^n\),
then \( f_k(\pi_1(x)) = 0 \) but is clearly \( f_k(\pi_1) \) is nonzero on \( A \). A similar trick holds if 
\( j = \text{ord}(c_2) < p^m \). Thus the last case is if \( \text{ord}(c_1) = p^n \) and \( \text{ord}(c_2) < p^m \). Then since each has full order, \( c_1 \) generates \( C_1 \) and \( c_2 \) generates \( C_2 \). Let \( p : C_1 \to C_2 \) be the natural epimorphism taking \( 1 \mod p^n \) to \( 1 \mod p^m \). Certainly since \( c_2 \) and \( p(c_1) \) generate \( C_2 \), there must be nonzero integers \( a, b \) such that \( ac_2 = bp(c_1) = 1 \), and hence \( bp(c_1) - ac_2 = 0 \).

Define \( \phi : A \to A \) by \( \phi(x_1, x_2) = (0, bp(x_1) - ac_2) \). This is easily seen to be a nonzero homomorphism of \( A \) taking \((c_1, c_2)\) to \((0, 0)\).

\[ \square \]

**Theorem 3.7.2** Let \( G \) be a torsion abelian group. Then \( G \) is torsion over \( \text{End}_{\mathbb{Z}}(G) \) if and only if \( G \) is not cyclic.

**Proof** The “only if” part is clear by previous comments, so we prove the “if” part.

Suppose there is no finite bound on the orders of elements of \( G \). Then the family of power maps \( f_n : G \to G \) given by \( f_n(g) = ng \) are all nonzero, since for any \( n \) \( G \) has elements whose orders exceed \( n \). Since \( G \) is torsion also, every \( g \in G \) is sent to zero by some \( f_n \).

Thus \( G \) is torsion over \( \text{End}_{\mathbb{Z}}(G) \).

Now assume that \( nG = \{0\} \) for some positive integer \( n \). By Prüfer’s Theorem we have that \( G \) must be a direct sum of prime power cyclic groups. We may then write
\[ G = \bigoplus_{i \in I} C_i \]
where each \( C_i \) is cyclic of order a power of a prime \( p_i \). For each \( i \in I \), again let \( \pi_i : G \to G \) be the projection map onto the \( i \)'th coordinate. Consider an arbitrary element \( 0 \neq x \in G \), and denote the support of \( x \) (the indices on which \( x \) is nonzero in the above decomposition) be denoted by \( Supp(x) \). We will complete the proof by showing \( x \) must be torsion over \( \text{End}_{\mathbb{Z}}(G) \).

Set \( H = \bigoplus_{i \in Supp(x)} C_i \). If \( G \neq H \) then it’s clear that an index \( j \) can be chosen in \( I \setminus Supp(x) \) so that \( \pi_j(x) = 0 \), and yet \( \pi_j \neq 0 \). Now the final case to consider is if \( G = H \). From this point we will write \( G = \bigoplus_{i=1}^t C_i \). If the primes \( p_i \) are all pairwise distinct then \( G \) is cyclic, but this is not the case. So, there must be two factors with power divisible by
the same prime. Relabeling if necessary, we assume these factors are \( C_1 \) and \( C_2 \). Because there is a natural isomorphism

\[
\text{Hom}_\mathbb{Z}(\bigoplus_{i=1}^{t} C_i, G) \cong \text{Hom}_\mathbb{Z}(C_1 \oplus C_2, G) \oplus \text{Hom}_\mathbb{Z}(\bigoplus_{i=3}^{t} C_i, G),
\]

by Lemma 3.7.1 we can choose nonzero \( \phi : C_1 \oplus C_2 \to G \) such that \( \phi(\pi_1(x), \pi_2(x)) = 0 \). This we complement to a nonzero map \( \hat{\phi} = \phi \oplus 0 \) with the zero homomorphism \( 0 : \bigoplus_{i=3}^{t} C_i \to G \). By our choice \( \hat{\phi}(x) = 0 \). This establishes that all \( x \) are torsion over \( \text{End}_\mathbb{Z}(G) \).

\[\square\]

### 3.8 Faithful Singular Rings and Modules

Analogously to “faithful torsion rings/modules/rank” let us define \( R \) to be (right) **faithful singular** if \( R \) admits a faithful singular right module, and let \( \_FS(R) \) denote the minimum cardinality of a faithful right singular module for \( R \), using 0 if \( R \) does not admit an FS module. As before, we will work with “right FS” unless otherwise specified, and compress \( \_FS(R) \) to \( FS(R) \).

Every singular module is torsion, and so an obvious relationship is that

\[
FS(R) > 0 \implies FS(R) \geq FT(R) > 0.
\]

However when \( FS(R) = 0 \), it is still possible that that \( FT(R) > FS(R) = 0 \).

Many results below follow as easy byproducts of earlier sections. These serve to illustrate that the analogous questions for faithful singular rings are more easily resolved, and show that our subtler version of torsion allows more interesting things to happen.

**Proposition 3.8.1** A ring admits a singular right module if and only if it is not semisimple.

**Proof** If \( R \) is not semisimple, it has a proper essential right ideal \( T \) and \( R/T \) is a nonzero singular module. If \( R \) is semisimple, no annihilator of any nonzero element in any module can be essential, because the only essential right ideal is \( R \) itself. Thus no nonzero \( R \) module can be singular.

\[\square\]

**Proposition 3.8.2** A ring \( R \) is right FS if and only if \( \text{soc}(R_R) = \{0\} \).
Proof In the context of Proposition 3.1.2 we note that the constructed module $E$ was actually singular, so the “if” direction is complete. Conversely, if $E$ is an FS module, then $\{0\} = \text{ann}(E) = \bigcap_{e \in E} \text{ann}(e)$ is an intersection of essential right ideals of $R$, but since $\text{soc}(R_R)$ is the intersection of all essential right ideals, $\text{soc}(R_R) = \{0\}$. □

Singular modules have at least one obvious advantage over torsion modules in that they are closed under direct sums.

Lemma 3.8.3 Every $\kappa$-generated FS module gives rise to an FS module which is a sum of $\kappa$ cyclic modules. Thus we can always choose a minimally generated FS module to be a direct sum of cyclic modules.

Proof Suppose $M = \sum_{i \in \kappa} m_iR$ is FS. Then each $m_iR$ is singular, and hence so is $M' = \bigoplus_{i \in \kappa} m_iR$. It only remains to show that $M'$ is faithful. This is trivial however, since $a \in \text{ann}(M')$ implies $m_i ra = 0$ for any $i$, any $r \in R$. Thus $Ma = 0$ and $a = 0$. □

This allows us to easily eliminate finite FS ranks other than 1.

Corollary 3.8.4 If $0 < FS(R) < \aleph_0$, then $FS(R) = 1$.

Proof Using Lemma 3.8.3, we may choose an $n$-generated FS module to have the form $\bigoplus_{i=1}^n R/E_i$. Because each $R/E_i$ is singular, $E_i \subseteq R$ for every $i$. Faithfulness of the module implies that $\bigcap_{i=1}^n E_i$ contains no nonzero ideals of $R$. But $E := \bigcap_{i=1}^n E_i \subseteq R$ since essential right ideals are closed under finite intersections. Finally then $R/E$ is FS and generated by 1 element only. □

Corollary 3.8.5 If $FS(R) > 0$, the cardinal $FS(R)$ is the minimum size of a collection of essential right ideals of $R$ whose intersection contains no nonzero ideal of $R$. 
Proof Let \( \{E_i \mid i \in \rho(R)\} \) be a minimal collection of right essential ideals of \( R \) whose intersection contains no nonzero ideal of \( R \). Then clearly \( \bigoplus_{i \in \rho(R)} R/E_i \) is torsion by Proposition 1.2.1 and is faithful because of the condition placed on the \( E_i \).

Conversely, if \( M \) is a minimally generated FS module for \( R \), then Lemma 3.8.3 produces a module with the same number of generators, and whose factors are \( R/E_i \) for some collection of essential right ideals \( \{E_i \mid i \in FS(R)\} \). Faithfulness demands that \( \bigcap_{i \in FS(R)} E_i \) contains no nonzero ideal of \( R \).

\[ \square \]

Corollary 3.8.6 If \( FS(R) > 0 \), and \( R \) is right bounded, then the cardinal \( FS(R) \) is the minimum size of a collection of essential right ideals of \( R \) with intersection zero.

Proof Let \( \rho \) denote the minimum size of a collection of essential right ideals of \( R \) with intersection zero. Corollary 3.8.5 gives immediately that \( FS(R) \leq \rho \).

Conversely, suppose \( R \) is right bounded and let and let \( M = \bigoplus_{i \in FS(R)} R/E_i \) be a minimally generated FS \( R \)-module. Of course each \( E_i \subseteq eR \) as usual. For each \( i \), \( \text{ann}(R/E_i) \) is the largest ideal of \( R \) contained in \( E_i \). By right boundedness there is a nonzero ideal \( I \subseteq e E_i \subseteq e R \), and by transitivity of essential extensions, \( I \subseteq e R \). So we see that \( I \subseteq \text{ann}(R/E_i) \subseteq e R \). By faithfulness of \( M \), it follows that \( \{0\} = \text{ann}(M) = \bigcap_{i \in FS(R)} \text{ann}(R/E_i) \). So, \( \rho \leq FS(R) \).

\[ \square \]

Contrary to FT rank, it is clear that finite FS ranks disappear in the presence of a boundedness condition on the right ideals of \( R \).

Proposition 3.8.7 For a ring \( R \), we have \( FS(R) = 1 \) if and only if \( R \) is not right weakly bounded.

Proof A singular cyclic module is always of the form \( R/E \) with \( E \subseteq e R \). Such a module is faithful if and only if \( E \) contains no nonzero ideals of \( R \). Hence, if such a module is
faithful, \( R \) is not right weakly bounded, and conversely if \( R \) is right weakly bounded, no such module is faithful. □

Let us see how the FS property holds up under direct products, and what we can say about the FS rank of a product.

**Corollary 3.8.8** If \( R = \prod_{i \in I} R_i \), then:

(a) \( R \) is right FS if and only if every \( R_i \) is right FS;

(b) When \( R \) is FS, if \( \{ FS(R_i) \mid i \in I \} \) has a greatest element \( \varrho \), then \( FS(R) \leq \varrho \).

**Proof** (a) This is clear since \( soc(R_R) = \bigoplus_{i \in I} soc(R_{iR}) \).

(b) Assume \( \varrho \) exists, and let \( C_i \) denote a minimal collection of essential right ideals of \( R_i \) whose intersection contains no nonzero ideals of \( R_i \). Select surjective maps \( \pi_i : \varrho \to C_i \) for each \( i \in I \). We remark that \( E_j := \prod_{i \in I} \pi_i(j) \subseteq e_R \).

Now we show that \( \bigcap_{j \in \varrho} E_j \) contains no nonzero ideals of \( R \). Because each \( \pi_i \) is surjective, it ranges over all members of the collection \( C_i \), perhaps more than once. In our construction:

\[
\bigcap_{j \in \varrho} E_j = \bigcap_{j \in \varrho} \prod_{i \in I} \pi_i(j) = \prod_{i \in I} \bigcap_{j \in \varrho} \pi_i(j) = \prod_{i \in I} \cap C_i
\]

If \( I \lhd R \) is contained in the intersection \( \bigcap_{j \in \varrho} E_j \), then its component ideals \( I_i \lhd R_i \) are such that \( I_i \subseteq \cap C_i \). By assumption then, \( I_i = \{0\} \) for every \( i \), and \( I \) is zero. Thus \( FS(R) \leq \varrho \). □

In major contrast to FT rings, we have the following for FS rings.

**Proposition 3.8.9** For any ring \( R \):

(a) The property “\( FS(R) > 0 \)” is Morita invariant;

(b) Having \( FS(-) = 1 \) or having \( FS(-) \geq \aleph_0 \) are both Morita invariant properties;

(c) For any \( n \in \mathbb{Z}^+ \), \( FS(M_n(R)) \leq FS(R) \).
Proof (a)+(b) It is well known that all of the module properties “faithful”, “singular” and “finitely generated” are all preserved through a category equivalence functor. Thus being right FS, and having $FS(-) \geq \aleph_0$ are Morita invariant. Since the correspondence of nonzero ideals and essential right ideals between Morita equivalent rings preserves containment, right weak boundedness is a Morita invariant property. Thus having $FS(-) = 1$ is also Morita invariant.

(c) We show that when $E \subseteq eR$, then $M_n(E) \subseteq eM_n(R)$. Proposition 1.2.1 gives that $\bigoplus_{i=1}^{n^2} E \subseteq e \bigoplus_{i=1}^{n^2} R$. Given any nonzero $A \in M_n(R)$, we may regard it as a nonzero vector in $\bigoplus_{i=1}^{n^2} R$, and so there exists $r \in R$ such that $0 \neq A(I_n r) \in M_n(E)$. This establishes that $M_n(E) \subseteq e M_n(R)$.

Let $C$ be a collection of size $FS(R)$ of essential right ideals with intersection containing no nonzero ideal of $R$. By the above, the collection $D = \{M_n(E) \mid E \in C\}$ is a system of essential right ideals of $M_n(R)$. By ideal correspondence between matrix rings, an ideal $J \triangleleft M_n(R)$ has the form $M_n(I)$ for some ideal $I \triangleleft R$. Consider the following:

$$M_n(I) \subseteq \bigcap_{E \in C} M_n(E) = M_n(\cap C)$$

The containment implies $I \subseteq \cap C$, and so $I = \{0\}$. Since $D$ and $C$ have the same size, we conclude $FS(M_n(R)) \leq FS(R)$.

Lemma 3.8.10 If $R \subseteq S$ is a unitary ring extension such that $R_R \subseteq_d S$, and $J$ is an essential right ideal of $S$, then $J \cap R$ is an essential right ideal of $R$.

Proof Let $0 \neq x \in R$. We show there exists $r \in R$ such that $0 \neq xr \in N \cap R$. Since $N_S \subseteq e S$, there exists $s \in S$, $0 \neq xs \in N$. Since $R_R \subseteq_d S_R$, there exists $t \in R$ such that $st \in R$ and $xst \neq 0$. Setting $r = st$ we see that $0 \neq xr \in N \cap R$.

Proposition 3.8.11 If $R \subseteq S$ is a unitary ring extension such that $R_R \subseteq_d S_R$, then if $S$ is FS, $R$ is FS. This applies to $S = Q_{\max}(R)$. If $R$ is right bounded, then $FS(R) \leq FS(S)$.
Proof  Let $M_S$ be an FS module. Then for each $m \in M$, $\text{ann}_S(m) \subseteq e_S S$, but by Lemma 3.8.10 $\text{ann}_R(m) = \text{ann}_S(m) \cap R \subseteq e_R$. This shows the inherited action $M_R$ forms a singular $R$ module, which is $R$ faithful since it was $S$ faithful.

Now if $R$ is additionally right bounded, we call upon Corollary 3.8.5 to furnish a collection $\{E_i \mid i \in FS(S)\}$ of essential right ideals of $S$ with intersection zero. Using Lemma 3.8.10 again the collection $\{E_i \cap R \mid i \in FS(S)\}$ is a collection of essential right ideals of $R$, which also has zero intersection since $\bigcap (E_i \cap R) \subseteq \bigcap E_i = \{0\}$. Then as in Proposition 3.1.2, $FS(R) \leq FS(S)$. □

There seems to be some difficulty proving $FS(R) \leq FS(S)$ without requiring $R$ to be right bounded. It is also not clear that Lemma 3.8.10 can be proven without the full strength of $R_R \subseteq e R_R$, i.e. $R_R \subseteq e R_R$ may not be enough.
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