Bayesian Designing and Analysis of Simple Step-Stress Accelerated Life Test with Weibull Lifetime Distribution

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This thesis titled
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ABSTRACT

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Bayesian Designing and Analysis of Simple Step-Stress Accelerated Life Test with Weibull Lifetime Distribution (82 pp.)

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This thesis develops a Bayesian method to analyze and plan a simple step-stress accelerated life test with Type-I censoring. It is assumed that the failure times at each stress level follow the Weibull distribution and the scale parameter is a log-linear function of the stress level. The point estimates of model parameters both by ML analysis and Bayesian analysis are organized in the thesis. The asymptotic confidence intervals and Bayesian confidence intervals are deduced. The influence of prior distribution on the estimation is studied. The Bayesian optimal plan is obtained by minimizing the pre-posterior variance of a specified low percentile of the life time distribution at the normal stress condition. Monte Carlo simulation algorithm based on Gibbs sampling is developed to find the optimal stress changing time. The maximum likelihood method is also developed for the use of comparison. Influence of prior distribution and sample size on the optimal plan is also discussed. The results show that Bayesian approach has hopeful potential in analyzing and designing reliability life test when there is prior knowledge and uncertainty for the model parameters.

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CHAPTER 1: INTRODUCTION

This chapter introduces the motivation and objectives of this study, as well as some background knowledge, including reliability, accelerated life test, step-stress accelerated life test, cumulative exposure model, acceleration model, maximum likelihood estimation, and Bayesian inference.

1.1 Motivation

Today’s increasing market competition and higher customer expectations are driving manufacturers to design and produce highly reliable products [1]. It is important to assess and predict the reliability of a product during the design and development stage because the time-to-market is getting shorter and shorter. Reliability assessment usually depends on experimental life tests to obtain failure data for lifetime analysis. Because life tests are expensive and the decision made based on the life tests affects the total life-cycle cost, they need to be carefully planned and analyzed. This thesis designs and analyzes simple step-stress accelerated life test using the Bayesian approach. The major motivation of using Bayesian approach instead of maximum likelihood method is its capability of incorporating prior knowledge and uncertainty in the modeling and analysis.

1.2 Background

1.2.1 Reliability

According to Ebeling[3], reliability is defined as “the probability that a component or system will perform a required function for a given period of time when
used under stated operating conditions.” In order to express the relationship between this probability and the time period, we define the continuous random variable $T$ to be the time to failure of the system or component, and then the reliability can be presented as

$$R(t) = \Pr\{T \geq t\}, \quad (1.1)$$

where $T \geq 0$, $R(t) \geq 0$, and $\lim_{t \to \infty} R(t) = 0$. For a given value of $t$, $R(t)$ is the probability that the time to failure is greater than or equal to $t$, which is called the *reliability function* of the failure distribution.

Define

$$F(t) = 1 - R(t) = \Pr\{T < t\}, \quad (1.2)$$

where $F(0) = 0$, and $\lim_{t \to \infty} F(t) = 1$. Then $F(t)$ is the probability that a failure occurs before time $t$, which is known as the *cumulative distribution function* (CDF) of the failure distribution.

Another function called the *probability density function* (PDF) is also commonly used in reliability engineering to describe the shape of the failure distribution. It is defined by

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt}, \quad (1.3)$$

where $f(t) \geq 0$, and $\int_0^\infty f(t) = 1$. Figure 1.1 illustrates these three functions when the time to failure follows the Weibull distribution with the shape parameter $\beta = 2$ and the scale parameter $\theta = 5$. 
Figure 1.1 The reliability function, CDF and PDF of Weibull distribution with $\beta = 2$

and $\theta = 5$

In addition to these three distribution functions, another function, called the failure rate or hazard rate function, is often used in reliability. It provides an instantaneous (at time $t$) rate of failure, that is,

$$\Pr\{t \leq T \leq t+\Delta t\} = R(t) - R(t+\Delta t).$$  \hspace{1cm} (1.4)

Usually, we set

$$\lambda(t) = \frac{f(t)}{R(t)}.$$  \hspace{1cm} (1.5)

Then $\lambda(t)$ is known as the instantaneous hazard rate or failure rate function.

The mean time to failure (MTTF) and the median time to failure are frequently used as measurement of reliability. The MTTF is the mean, or expected value of the failure time, which can be defined by
Given a desired reliability $p$, set

$$R(t_p) = p = \Pr\{ T \geq t_p \},$$

then $t_p$ is called the $p$th percentile of the lifetime distribution, which means at time $t_p$, 100$p$% of the population will have failed. If $p=0.5$, then $t_{0.5}$ is the median time to failure.

### 1.2.2 Accelerated life test

Reliability life testing is commonly designed for life data analysis in which samples are tested to obtain failure time data for reliability assessment [2]. Highly reliable products like many electronic accessories usually have a long life under normal operation conditions. Therefore it is unlikely that products are tested under the normal operation conditions because of time and cost limits. The alternative is to use failure data collected in an accelerated life test (ALT) to project the reliability under the normal conditions. During the ALT, the units are placed under higher than normal stress conditions, e.g., voltage, current, humidity, temperature, etc., to speed up the degradation process so that failure information can be collected in short period of time [3]. The failure information is then transformed through an acceleration model to predict the reliability under the normal operation conditions.

### 1.2.3 Step-stress accelerated life test

There are two types of ALT: constant-stress ALT and step-stress ALT. In constant-stress ALT the units are placed only under one higher than normal stress level,
while the step–stress accelerated life test (SSALT) allows several stress levels. In the SSALT, the samples are tested through more than one level of stress (usually with increasing order). Generally, when the number of failures achieves a pre-specified number (failure-step stress test) or the test has sustained a pre-specified time under one stress (time-step stress test), the remaining units will be placed under another stress. A test with only two stress levels is called simple step-stress accelerated life test, while a test with more than two stress levels is called multiple step-stress accelerated life test [4]. If a test is ended until a pre-specified time, it is called Type-I censoring; if a test is ended when the failures amount to a pre-specified number, it is called Type-II censoring.

Figure 1.2 shows the stress profile of a multi-SSALT with five stress levels and Type-I censoring, that is, the test stops at a pre-specified time $t_c$, and all units that have not failed by $t_c$ are censored. $x_i$’s and $\tau_i$’s are, respectively, the stress levels and stress changing times.

![Figure 1.2 Step-stress accelerated life test](image-url)
The advantage of SSALT is the reduction of the duration of test, because if not enough failure data are collected in one stress, a higher stress could be set to increase the probability of failure.

1.2.4 The cumulative exposure model

Because the units will be tested under more than one stress level in SSALT, a model is needed to describe the effect of changing stress. A commonly used model in SSALT is the *cumulative exposure model* [5], which assumes that “the remaining life of specimens depends only on the current cumulative fraction failed and current stress-regardless of how the fraction accumulated.”

For a SSALT with \( m \) stress levels, the cumulative exposure model assumes that the failure times from the SSALT follow the following CDF [4]:

\[
F_0(t) = \begin{cases} 
F_1(t), & 0 \leq t \leq \tau_1, \\
F_2(t - \tau_1 + u_1), & \tau_1 \leq t \leq \tau_2, \\
F_3(t - \tau_2 + u_2), & \tau_2 \leq t \leq \tau_3, \\
... & ...
\end{cases}
\]

\[
\vdots
\]

\[
F_m(t), \quad \tau_m \leq t \leq \infty,
\]

where \( \tau_i \) is the time of changing the stress from the \( i \)th stress level to the \((i+1)\)th stress level, \( F_i(t) \) is the cumulative distribution function under the \( i \)th stress level, and \( u_i \) is the solution of \( F_{i+1}(u_i) = F_i(\tau_i - \tau_{i-1} + u_{i-1}) \).
This study will consider the simple SSALT with two stress levels and assume that the failure time at each stress level has the Weibull distribution, i.e.,

$$F_i(t) = 1 - \exp \left[ - \left( \frac{t}{\theta_i} \right)^\beta \right], \quad i=0, 1, 2,$$

(1.9)

where $\theta_i$ is the scale parameter of the Weibull distribution at the $i$th stress level, and $i=0$ denotes the normal operating condition. Because the failures are assumed to be caused by the same failure mechanism at all stress levels, the shape parameter $\beta$ is assumed to be identical at all stress levels. According to the cumulative exposure model (1.8), failure time collected from this simple SSALT has the following cumulative distribution function [6]

$$F(t) = \begin{cases} 
F_1(t), & 0 < t < \tau, \\
F_2\left(\frac{\theta_1}{\theta_2} t + t - \tau\right), & \tau \leq t \leq \infty,
\end{cases}$$

(1.10)

that is,

$$F(t) = \begin{cases} 
1 - \exp \left[ - \left( \frac{t}{\theta_1} \right)^\beta \right], & 0 < t \leq \tau, \\
1 - \exp \left[ - \left( \frac{\theta_2}{\theta_1} \frac{t - \tau}{\theta_2} \right)^\beta \right], & t \geq \tau.
\end{cases}$$

(1.11)

The probability density function of the failure time is then given by
1.2.5 Acceleration models

In order to project the reliability at normal operation condition using data collected at higher stress conditions, an acceleration model is required. The acceleration models describe the relationship between the lifetimes and the stress conditions. A commonly used acceleration model is the log-linear model. Taking the model (1.11) as an example, the log-linear model can be expressed as

\[
\log \theta = a + bx, \quad (1.13)
\]

where \( x \) denotes the stress level (or a proper transformation of it). We use this log-linear model because it is simple and has been used practically in many applications.

For example, when the stress is tension, the Power or Inverse Power Law can be used as the acceleration model,

\[
\theta = Cs^p, \quad (1.14)
\]

where \( C \) and \( p \) are unknown constants, and \( s \) denotes the tension stress. Then

\[
\log \theta = \log C + p \log(s). \quad \text{Herein } x = \log(s).
\]

Arrhenius Law is widely used for life tests with temperature:
\[
\theta = \exp\left\{-\frac{E_a}{Ks}\right\},
\]

where \(E_a\) represents the activation energy, \(s\) is the absolute temperature (°K), and \(K\) is Boltzmann’s constant, a known physical constant equal to \(8.617 \times 10^{-5}\) (eV/°K) [7].

Then \(\log \theta = -(E_a/K) (1/s)\) and \(x=1/s\) in this case.

Another example of the log-linear relationship is the \(E\)-model used to study the time-dependent dielectric breakdown of ultra-thin gate dielectrics where the stress is the electrical field. The \(E\)-model is given as

\[
\theta = \tau_L \exp[G_L(E_{bd} - E)],
\]

where \(\tau_L\) and \(G_L\) are unknown, temperature-dependent constants. \(E\) is the applied electrical field, and \(E_{bd}\) is the field above which breakdown occurs immediately [8]. The \(E\)-model can also be expressed as a log-linear function \(\log \theta = \log \tau_L + G_L(E_{bd} - E)\).

Combine formula 1.11 and 1.13, the CDF of the failure time given parameters \(a\), \(b\), and \(\beta\) is expressed as

\[
F(t) = \begin{cases} 
1 - \exp\left[-\left(\frac{t}{\exp(a + bx)}\right)^\beta\right], & 0 \leq t \leq \tau, \\
1 - \exp\left[-\left(\frac{e^{b(x_2 - x_1)}\tau + t - \tau}{\exp(a + bx_2)}\right)^\beta\right], & t \geq \tau.
\end{cases}
\]

1.2.6 Maximum likelihood estimation (MLE)

The maximum likelihood (ML) method is popular in statistical data analysis to estimate the model parameters given the sample data. The simple SSALT model
considered in this study has three parameters $a$, $b$, and $\beta$. Given the data $\mathbf{t} = (t_1 \ldots t_n)$ collected from the test, the maximum likelihood estimators of the three parameters are the ones that maximize the following likelihood function

$$L(a, b, \beta \mid \mathbf{t}) = \prod_{i=1}^{n} L_i(a, b, \beta \mid t_i), \quad (1.18)$$

where $L_i$ is the likelihood of the $i$th observation. In order to make the computation more convenient, the logarithm of likelihood function (called log-likelihood function) is maximized instead of the likelihood function. Generally, “the MLEs are obtained by taking the first partial derivatives of the logarithm of the likelihood function and setting these partials equal to zero” [3].

1.2.7 Bayesian inference

The maximum likelihood method assumes that the parameters are unknown, but fixed. The Bayesian approach, however, assumes that the parameters are random, and uncertainties on the parameters are described by a joint prior distribution, which is formulated before the failure data are collected, and is based on historical data, experience with similar products, design specifications, and experts’ opinions. The capability of incorporating prior knowledge in the analysis makes the Bayesian approach very valuable in the reliability analysis because one of the major challenges associated with the reliability analysis is the limited availability of data.

Inference on the model parameters is made in terms of probability statements, which are conditional on the observed data $\mathbf{t}$. Once the sample data have been collected, the prior distribution is updated according to Bayes’ theorem
\[ f(a, b, \beta | t) = \frac{L(a, b, \beta | t) f(a, b, \beta)}{f(t)}, \]  
\[ f(t) = \iiint L(a, b, \beta | t) f(a, b, \beta) \text{d}adb \text{d}\beta. \]  

where \( f(a, b, \beta) \) is the joint prior distribution, \( f(a, b, \beta | t) \) is the joint posterior distribution, and

Inference on each parameter is based on its marginal posterior density. For example, the marginal posterior density for the parameter \( \beta \) is defined as

\[ f(\beta | t) = \iiint f(a, b, \beta | t) \text{d}adb = \frac{\iiint L(a, b, \beta | t) f(a, b, \beta) \text{d}adb}{f(t)}. \]

Multiple levels of integration are necessary to obtain the normalizing constant \( f(t) \), and the marginal posterior densities. For complicated models these integrations are often analytically intractable, and sometimes even a numerical integration cannot be directly obtained. In these cases, Markov chain Monte Carlo (MCMC) simulation is the easiest way to get reliable results without evaluating integrals [9]. A MCMC algorithm that is particularly useful in high dimensional problems is the alternating conditional sampling called Gibbs sampling. Each iteration of the Gibbs sampling cycles through the unknown parameters, drawing a sample of one parameter conditioning on the latest values of all the other parameters. When the number of iterations is large enough, the samples drawn on one parameter can be regarded as simulated observations from its marginal posterior distribution. Functions of the model parameters, such as the \( p \)th percentile of the lifetime distribution at the normal stress condition, denoted by \( t_p(x_0) \) can also be conveniently sampled. Posterior inference can be computed using sample statistics. In this study, we
use WinBUGS [10], a specialized software package for implementing MCMC simulation and Gibbs sampling.

1.3 Objectives

This thesis consists of two objectives. The first one is to analyze the SSALT with Bayesian method. The model parameters of the lifetime distribution, $a$, $b$, $\beta$ and $t_p(x_0)$ are estimated. The second one is to design optimal SSALT plan with Bayesian approach. The optimal stress changing time of the SSALT is determined to maximize the precision of the estimation of the $p$th percentile of the life time distribution at normal stress condition. For both objectives, practical examples related to nanoelectronics reliability are used to illustrate the proposed methods.

In order to achieve these objectives, several tasks were completed. In the analysis section, the tasks are described as:

1) Develop a Markov chain Monte Carlo (MCMC) simulation algorithm based on Gibbs sampling for posterior simulation and inference;

2) Develop the ML estimate for purpose of comparison;

3) Compare ML estimate with Bayesian estimate;

4) Perform sensitivity analysis to investigate how changes in prior distribution will affect variance and confidence interval of the model parameters.

In the design section, the tasks are described as:

1) Develop the ML plan for the purpose of comparison;

2) Investigate a practical approach to specifying the prior distribution;
3) Develop the simulation algorithm to obtain the pre-posterior variance of $t_p(x_0)$;

4) Compare ML plan with Bayesian plan;

5) Perform sensitivity analysis to investigate how changes in the prior distribution and sample size will affect the optimal stress changing point.
CHAPTER 2: LITERATURE REVIEW

This chapter reviews previous research on analysis and design of simple step-stress accelerated life test. Both maximum likelihood method and Bayesian method are reviewed.

2.1 Analysis of Simple SSALT

Given the data collected in the test, the purpose of analyzing simple SSALT is to estimate a reliability measure, such as the mean time to failure or a percentile of the lifetime distribution under the normal stress conditions. Both the maximum likelihood method and the Bayesian method have been applied in the literature.

2.1.1 Analysis of simple SSALT using ML method

The majority of the literature related to simple SSALT analysis used the ML method. The exponential distribution has been extensively studied due to its simplicity. Xiong [11] deduced the MLEs of parameters in simple SSALT with exponential distribution and type-II censoring, and constructed the confidence intervals (CI) using a pivotal quantity. Xiong and Ji [12] studied a similar problem with type-I censoring. The asymptotic confidence intervals of the parameters are provided instead of the exact method, which can simplify the process of solving CI under large sample size. However, the CI under small sample size may not be accurate enough.

Balakrishnan et al. [13] also analyzed the simple SSALT with exponential lifetime distribution and type-II censoring. In this article, a log-linear acceleration model
is assumed. The authors derived the MLEs of the parameters through the use of conditional moment-generating functions. They constructed three approaches to determining the CIs for the parameters: exact method, asymptotic method, and the parametric bootstrap method. Balakrishnan et al. [14] studied a similar problem with type-I censoring.

Kateri and Balakrishnan [6] analyzed the simple SSALT model with Weibull lifetime distribution and Type-II without assuming any acceleration model. Because the closed form MLEs cannot be obtained, the authors used the Newton-Raphson algorithm to compute the MLEs numerically. They also provided asymptotic and bootstrap confidence intervals for the parameters of the Weibull simple step-stress model.

Most of the studies in the literature discussed the time-step stress test. Analysis of failure-step stress test was reported by Xiong and Milliken [15], which assumed the exponential lifetime distribution and that the stress change times are random variables. MLEs for model parameters based on both the marginal and conditional life distributions were considered.

2.1.2 Bayesian analysis of simple SSALT

The Bayesian approach has not been widely applied to analyze data from SSALT. Lee and Pan [16] presented a Bayesian inference model for simple SSALT using type-II censoring. They assumed that the failure times at each stress are exponentially distributed with a mean that is a log-linear function of the stress level. They integrated the engineering knowledge into the prior distribution of the parameters in the log-linear
function, and through a Siegel-gamma distribution conjugation, they derived the posterior distribution for parameters of interest. Applying the Bayesian approach to SSALT, the statistical precision of parameter inference is improved and the required number of samples is reduced. Madi [17] also applied the Bayesian inference for simple SSALT assuming the lifetime under normal operation condition is exponentially distributed. Gibbs sampling was implemented in this article to perform MCMC simulation and posterior inference.

2.2 Design of Simple SSALT

The majority of the literatures related to planning simple SSALT aimed to obtain the optimal stress changing time, some of them gained the optimal lower stress level as well. A few of literatures aimed to determine the optimal sample size. Most of these literatures used maximum likelihood method.

2.2.1 Determination of optimal stress changing time

Planning optimal simple SSALT using ML method has been the subject of many studies. Given a known sample size, the objective is usually to find the optimal stress changing point that minimizes the asymptotic variance of the MLE of a quantity of interest. Computation of the asymptotic variance is usually based on the inverse of the Fisher information matrix.

Bai et al. [18], Alhadeed and Yang [19] [20], and Srivastava and Shukla[21] followed a similar procedure to find the optimal stress changing time by minimizing the
asymptotic variance of the MLE of a quantity of interest such as mean life or the $p$th quantile of the lifetime distribution under normal conditions. The difference is the lifetime distribution assumed. Bai et al. [18] assumed exponential lifetime distribution, Alhadeed and Yang [19] used Khamis-Higgins model, Alhadeed and Yang [20] considered the log-normal lifetime distribution, and Srivastava and Shukla [21] assumed a log-logistic lifetime distribution.

Bai and Kim [22] tried to obtain the low stress level as well as stress changing time by minimizing asymptotic variance of MLE of a stated quantile at design stress. Elsayed and Zhang [23] also designed the low stress level and stress changing point by minimizing asymptotic variance of MLE of the reliability prediction at design stress conditions over a pre-specified period of time. Cox’s proportional hazards model was assumed in the paper. Wang and Yu [24] introduced another estimator called Uniformly Minimum Variance Unbiased Estimators (UMVUEs). The author compared the Mean Square Error (MSE) of UMVUES with that of MLEs and proved that UMVUEs had smaller MSE than MLEs had. Then the author discussed the optimal test plan based on minimizing variance of UMVUEs of log-mean life. Teng and Yeo [25] applied the Least-Squares approach to obtain optimum stress changing time with small sample size. In this article, the objective was to minimize the transformed least square of life-stress parameters instead of variance of MLEs of them. This method can derive closed-form of estimated parameters; however, it can only deal with small sample size.
2.2.2 Determination of optimal sample size

Statistically, the larger the sample size is the better results- more accurate estimation- can be achieved. On the other side, from the producer’s point of view, conducting test with a large sample size will take up a long time and capital. Therefore, it is significant to gain an appropriate sample size to balance these two sides.

There are not so many articles on designing sample size of SSALT as designing stress changing point. Tsai et al. [26] and Chun et al. [27] added producer into consideration, assuming Rayleigh lifetime distribution and Weibull lifetime distribution, respectively. Chun et al. [27] determined the sample size based on minimizing variance of MLEs and provided tables and figures of sampling plan under different consumer’s and producer’s risk, which can be conveniently used without any computation.
CHAPTER 3: PROBLEM STATEMENT

In this chapter, the basic assumptions and problem are described. Then the sub-tasks which are established in order to achieve the objectives of this thesis are described.

3.1 Notations

\( x_i \): stress or a proper transformation of the stress

\( \theta_i \): scale parameter of the Weibull distribution under stress \( x_i \)

\( \beta \): shape parameter of the Weibull distribution

\( \tau \): stress changing time

\( t_c \): censoring time

\( a, b \): parameters of log-linear relation

\( t \): lifetime of an item under SSALT

\( t_p(x_0) \): the \( p \)th percentile of the failure time distribution at normal operating condition

\( F(t), f(t) \): cumulative distribution function (CDF) and probability density function (PDF) of failure time

\( R(t) \): reliability function of failure time, \( R(t) = 1 - F(t) \)

3.2 Basic Assumptions

This study makes the following basic assumptions:

(1) \( n \) identical and independent units are placed in the SSALT.

(2) The failure times at each stress level are distributed according to the Weibull distribution.
(3) The relationship of scale parameters under different stress levels is log-linear, which can be expressed as $\log \theta_i = a + b x_i$, $i=0, 1, 2$.

(4) The shape parameter $\beta$ is identical, and is independent of the stress.

(5) A cumulative exposure model is assumed.

(6) A unit is first tested under the lower stress level $x_1$. If the unit has not failed by $\tau$, the stress level is increased to $x_2$ at the changing time $\tau$ and the test is continued until failure or the censoring time $t_c$ (i.e., Type-I censoring).

Weibull distribution is assumed because it is more flexible than exponential distribution as many researchers assumed. In the ALT, the failures are required to be caused by the same failure mechanism at all stress levels in order to project the reliability under normal operating conditions, therefore the shape parameter is assumed to be identical at all stress levels [8].

3.3 Problem Description

3.3.1 Analysis of simple SSALT

In this thesis, the sample size $n$, the testing stress levels $x_1$ and $x_2$ are pre-specified. The first problem is analyzing a simple step-stress accelerated life test by estimating the parameters of the lifetime distribution from experimental data. Let $\tau$ denote the pre-specified stress changing time, the objective is to estimate $a$, $b$, $\beta$, and $t_p(x_0)$ with Bayesian method. An experimental dataset collected at the Thin Film Nano & Microelectronics Research Laboratory at Texas A&M University is used to illustrate the
Bayesian method. The ML method is also developed and compared with the Bayesian method in the thesis.

### 3.3.2 Design of simple SSALT

The second problem in this study is looking for the optimal stress changing time $\tau$ such that the best accuracy of the estimation of the $p$th percentile of the lifetime distribution at normal stress conditions, denoted by $t_p(x_0)$, will be obtained. For the ML method, the accuracy is measured by the asymptotic variance of the MLE of $t_p(x_0)$. For the Bayesian method, the accuracy is, however, measured by the pre-posterior variance of $t_p(x_0)$. A practical example based on experimental data collected at the same lab is used.

### 3.4 Development of ML Methods

Because there is no literature considering SSALT with Weibull distribution and type-I censoring as well as log-linear acceleration function. The ML methods for both analyzing and planning of the SSALT are deduced in this study.

### 3.5 Development of Simulation Algorithm

The posterior analysis (could be referred in sections 4) requires multiple levels of integration to obtain $f(t)$ and the marginal posterior densities, which have no closed form solutions. Monte Carlo simulation algorithm involving Gibbs Sampling will be developed in this study.
3.6 Investigation of Practical Approach to Specifying the Prior Distribution

A critical issue related to the Bayesian design method is the specification of the joint prior distribution \( f(a, b, \beta) \). One may assume that the three parameters are independent and assign an independent prior to each of them, that is,

\[
f(a, b, \beta) = f(a)f(b)f(\beta).
\] (3.1)

Although the prior given by (3.1) is computational convenient, it may be practically difficult to directly specify priors for \( a \) and \( b \) and to justify their independence. Therefore, this thesis explores a practical approach to specifying the joint prior distribution for Bayesian design.

3.7 Comparison between ML method and Bayesian method

Comparison between maximum method and Bayesian method is organized both in analysis section and design section. In the analysis section, a practical data set is used to compare the two methods. The ML method finds the optimal plan based on the asymptotic variance and the optimal plan is independent of the sample size \( n \). The sample size as well as the prior distribution is influential on Bayesian optimal plan. The case study in the thesis shows when these two methods yield similar results.

3.8 Sensitivity Analysis

Sensitivity analysis is performed in the analysis part to investigate how changes in prior distribution will affect the variance of estimated parameters. In the planning section, the sensitivity analysis is performed to investigate how changes in the prior distribution
will affect optimal stress changing time. Especially, the influence of the prior uncertainty measured by the prior distribution range, on the optimal plan is studied. Also, the influence of sample size on the optimal plan is studied.
CHAPTER 4: METHODS

The objectives of this thesis include analyzing and designing optimal simple SSALT using the Bayesian approach. For the purpose of comparison, ML methods will also be derived in this chapter.

4.1 Analysis of SSALT

4.1.1 ML analysis

There is no literature referred to analyzing the SSALT with Weibull distribution, type-I censoring and log-linear acceleration model simultaneously as this thesis assumes. For the purpose of comparison with Bayesian analysis, the point and confidence interval estimation by ML method are deduced.

Let $n_1$ be the number of failures in the first stage of a SSALT, $n_2$ the number of failures before censoring time, and $n$ the sample size. From the PDF, reliability function of the lifetime and the log-linear acceleration function, the log-likelihood function of the failure time can be expressed as

$$
I = \ln L(t) = n_2 \log \beta - n_1 \beta (a + bx_1) + (\beta - 1) \sum_{i=1}^{n_2} \ln t_i - (n_2 - n_1) \beta (a + bx_2)
$$

$$
- \sum_{i=1}^{n_2} \left( \frac{t_i}{e^{a+bx_i}} \right)^\beta - (\beta - 1) \sum_{i=n_1+1}^{n_2} \ln \left( \tau e^{(ax_i - bx_i)} + t_i - \tau \right)
$$

$$
- \sum_{i=n_2+1}^{n} \left( \frac{\tau}{e^{a+bx_i}} + \frac{t_i}{e^{a+bx_2}} - \frac{\tau}{e^{a+bx_2}} \right)^\beta - (n - n_2) \left( \frac{\tau}{e^{a+bx_1}} + \frac{t_i}{e^{a+bx_2}} - \frac{\tau}{e^{a+bx_2}} \right)^\beta
$$

(4.1)

where $t = (t_1, t_2, \ldots, t_n)$ is the ordered failure data. The details of the ML method can be found in Appendix A.
Let \( A(t_i) = \left( \frac{\tau}{e^{a + bx_i}} + \frac{t_i}{e^{a + bx_i}} - \frac{\tau}{e^{a + bx_i}} \right) \), the first partial derivatives of the log likelihood with respective to the model parameters are,

\[
\frac{\partial l}{\partial \beta} = \frac{n_2}{\beta} - n_1(a + bx_1) + \sum_{i=1}^{n_1} \ln t_i - (n_2 - n_1)(a + bx_2) - \sum_{i=1}^{n_1} t_i^\beta e^{-\beta(a + bx_i)} \ln \left[ t_i e^{-(a + bx_i)} \right] + \sum_{i=n_1+1}^{n_2} \ln \left[ t_i e^{b(x_2 - x_i)} + t_i - \tau \right] - \sum_{i=n_1+1}^{n_2} A(t_i)^\beta \ln A(t_i) - (n - n_2) A(t_e)^\beta \ln A(t_e),
\]

\( (4.2) \)

\[
\frac{\partial l}{\partial a} = -n_2 \beta + \sum_{i=1}^{n_1} t_i^\beta e^{-\beta(a + bx_i)} \beta + \sum_{i=n_1+1}^{n_2} \beta A(t_i)^\beta + \beta(n - n_2) A(t_e)^\beta,
\]

\( (4.3) \)

\[
\frac{\partial l}{\partial b} = -n_1 \beta x_1 - (n_2 - n_1) \beta x_2 + \sum_{i=1}^{n_1} t_i^\beta e^{-\beta(a + bx_i)} \beta x_1 + (\beta - 1) \sum_{i=n_1+1}^{n_2} \frac{\tau e^{b(x_2 - x_i)} (x_2 - x_i)}{\tau e^{b(x_2 - x_i)} + t_i - \tau} - \sum_{i=n_1+1}^{n_2} \beta A(t_i)^{\beta - 1} \frac{\partial A(t_i)}{\partial b} - \beta(n - n_2) A(t_e)^{\beta - 1} \frac{\partial A(t_e)}{\partial b}.
\]

\( (4.4) \)

The MLEs of the model parameters \( a, b \) and \( \beta \) are the solution of

\[
\begin{align*}
\frac{\partial l}{\partial \beta} &= 0, \\
\frac{\partial l}{\partial a} &= 0, \\
\frac{\partial l}{\partial b} &= 0.
\end{align*}
\]

\( (4.5) \)

Because this is a system of non-linear equations, we applied the Newton’s method which requires the inverse of the observed Fisher Information Matrix, which can be obtained as the negative of the second partial derivatives of the log-likelihood function with respect to the model parameters \((\beta, a, b)\).
The elements of the information matrix are provided in Appendix A. MATLAB is used to solve the equations.

Because the MLEs of the model parameters are not in closed-form expressions, it is impossible to derive the exact confidence intervals (CI), in this thesis, we use the asymptotic CI instead.

We can obtain the asymptotic variance of the estimated model parameters \( \hat{a} \), \( \hat{b} \), and \( \hat{\beta} \) from the asymptotic variance-covariance matrix, which is defined as the inverse matrix of the Fisher information matrix.

\[
F = \begin{bmatrix}
-\frac{\partial^2 l}{\partial \beta^2} & -\frac{\partial^2 l}{\partial \beta \partial a} & -\frac{\partial^2 l}{\partial \beta \partial b} \\
-\frac{\partial^2 l}{\partial \beta \partial a} & -\frac{\partial^2 l}{\partial a^2} & -\frac{\partial^2 l}{\partial a \partial b} \\
-\frac{\partial^2 l}{\partial \beta \partial b} & -\frac{\partial^2 l}{\partial a \partial b} & -\frac{\partial^2 l}{\partial b^2}
\end{bmatrix}.
\] (4.6)

Then the two sided 100(1-\( \alpha \))\% CI of the model parameters \( \hat{a} \), \( \hat{b} \), and \( \hat{\beta} \) can be gained from

\[
\hat{\beta} \pm z_{\alpha/2} \sqrt{AVar(\hat{\beta})}
\]

\[
\hat{a} \pm z_{\alpha/2} \sqrt{AVar(\hat{a})}
\]

\[
\hat{b} \pm z_{\alpha/2} \sqrt{AVar(\hat{b})}
\] (4.8)

Because
\[ t_p(x_0) = \exp(a + bx_0)[-\ln(1 - p)]^{1/\beta}, \quad (4.9) \]

the asymptotic variance of the MLE of the \( p \)th percentile at normal operating conditions is given by

\[
\text{AVar}(\hat{t}_p(x_0)) = \left[ \frac{\partial \hat{t}_p(x_0)}{\partial \beta}, \frac{\partial \hat{t}_p(x_0)}{\partial a}, \frac{\partial \hat{t}_p(x_0)}{\partial b} \right] \sum \left[ \frac{\partial t_p(x_0)}{\partial \beta}, \frac{\partial t_p(x_0)}{\partial a}, \frac{\partial t_p(x_0)}{\partial b} \right]^{-1} \left[ \frac{\partial \hat{t}_p(x_0)}{\partial \beta}, \frac{\partial \hat{t}_p(x_0)}{\partial a}, \frac{\partial \hat{t}_p(x_0)}{\partial b} \right], \quad (4.10)
\]

according to the delta method [23].

4.1.2 Bayesian analysis

According to Bayes’ theorem, the posterior distribution of the model parameters given data \( t \) is

\[
f(a, b, \beta | t) = \frac{L(a, b, \beta | t) f(a, b, \beta)}{f(t)}, \quad (4.11)
\]

where \( L(a, b, \beta | t) \) is the likelihood of the data given by

\[
L(a, b, \beta | t) = \prod_{i=1}^{n} \frac{\beta}{e^{(a + b x_i)}} t^{\beta - 1} \exp \left[ -\left( \frac{t}{e^{(a + b x_i)}} \right)^\beta \right] \\
\prod_{i=n+1}^{n_2} \frac{\beta}{e^{(a + b x_i)}} \left( e^{b(x_2 - x_i)} \tau + t - \tau \right)^{\beta - 1} \exp \left[ -\left( \frac{e^{b(x_2 - x_i)} \tau + t - \tau}{e^{(a + b x_i)}} \right)^\beta \right] \\
\prod_{i=n_2+1}^{n} \exp \left[ -\left( \frac{e^{b(x_2 - x_i)} \tau + t - \tau}{e^{(a + b x_i)}} \right)^\beta \right], \quad (4.12)
\]

and

\[
f(t) = \int \int \int f(t | a, b, \beta) f(a, b, \beta) \, da \, db \, d\beta. \quad (4.13)
\]

Then the marginal posterior distribution of \( a, b, \) and \( \beta \) are, respectively,
The posterior mean of \( a, b \) and \( \beta \) given \( t \) are, respectively,

\[
E[a|t] = \int_0^\infty af(a|t) da, \tag{4.17}
\]

\[
E[b|t] = \int_0^\infty bf(b|t) db, \tag{4.18}
\]

\[
E[\beta|t] = \int_0^\infty \beta f(\beta|t) d\beta, \tag{4.19}
\]

which may be used as point estimates of the parameters. Gibbs sampling is used to draw a random sample of the parameters \( a, b \) and \( \beta \) from their own marginal posterior distribution \( f(a|t) \), \( f(b|t) \) and \( f(\beta|t) \), respectively, and then estimate the expected value using the sample mean.

The 100(1-\( \alpha \))% confidence intervals for the parameters \( a, b \) and \( \beta \) are intervals \((L_a, U_a),(L_b,U_b)\) and \((L_\beta,U_\beta)\), respectively satisfying

\[
P(L_a \leq a \leq U_a) = 1 - \alpha = \int_{L_a}^{U_a} f(a|t), \tag{4.20}
\]

\[
P(L_b \leq b \leq U_b) = 1 - \alpha = \int_{L_b}^{U_b} f(b|t), \tag{4.21}
\]

\[
P(L_\beta \leq \beta \leq U_\beta) = 1 - \alpha = \int_{L_\beta}^{U_\beta} f(\beta|t), \tag{4.22}
\]

where the \( L \) and \( U \) are the lower and upper limit of the interval, respectively. We may choose \( L \) and \( U \) as
Again we adopt Gibbs sampling to generate random samples of the parameters \( a, b, \) and \( \beta \) from the marginal posterior distributions and then the lower limit is the value of the \((\alpha/2)\)th percentile of the sample and the upper limit is the \((1-\alpha/2)\)th percentile of the sample.

4.2 Optimal Design of Simple SSALT

4.2.1 Maximum likelihood plan

For the purpose of comparison, the ML method for designing optimal SSALT is described. From the assumption of cumulative exposure model, Weibull distribution, and log-linear acceleration model, the CDF of the failure time of a test unit under the simple SSALT is

\[
F(t) = \begin{cases} 
1 - \exp \left[ - \frac{t}{\exp(a + bx_1)} \right]^{\beta}, & 0 \leq t \leq \tau, \\
1 - \exp \left[ - \frac{\left( e^{b(x_2 - x_1)} \tau + t - \tau \right)^{\beta}}{\exp(a + bx_2)} \right], & t \geq \tau.
\end{cases}
\]  

(4.26)

The PDF is
f(t | a, b, \beta) = \begin{cases} \frac{\beta}{e^{\beta(a+b\xi)}} t^{\beta-1} \exp \left[-\left(\frac{t}{e^{(a+b\xi)}}\right)^\beta\right], & 0 < t \leq \tau, \\ \frac{\beta}{e^{\beta(a+b\xi)}} (e^{b(x_2-x_1)\tau + t - \tau})^{\beta-1} \exp \left[-\left(\frac{e^{b(x_2-x_1)\tau + t - \tau}}{e^{(a+b\xi)}}\right)^\beta\right], & t \geq \tau. \end{cases} \tag{4.27}

The probability that a test unit will not fail by the censoring time \( t_c \) is

\[ R(t_c | a, b, \beta) = \exp \left[-\left(\frac{e^{b(x_2-x_1)\tau + t_c - \tau}}{e^{(a+b\xi)}}\right)^\beta\right] t_c > \tau. \tag{4.28} \]

Define the indicator function \( I_1 = I_1(t \leq \tau) \) by

\[ I_1 = I_1(t \leq \tau) = \begin{cases} 1, & \text{if } t \leq \tau, \text{ failure observed before time } \tau, \\ 0, & \text{if } t > \tau, \text{ failure observed after time } \tau, \end{cases} \tag{4.29} \]

and the indicator function \( I_2 = I_2(t \leq t_c) \) by

\[ I_2 = I_2(t \leq t_c) = \begin{cases} 1, & \text{if } t \leq t_c, \text{ failure observed before time } t_c, \\ 0, & \text{if } t > t_c, \text{ censored at time } t_c. \end{cases} \tag{4.30} \]

Then the log-likelihood function is

\[ L(t; x_1, x_2) = -I_1 I_2 \beta(a + b\xi_1) + I_1 I_2(\beta - 1) \ln t - I_1 I_2 t^\beta e^{-\beta(a+b\xi_1)} + I_2 \ln \beta \\
- I_2 (1-I_1) \beta(a + b\xi_2) + I_2 (1-I_1)(\beta - 1) \ln \left[ t e^{b(x_2-x_1)\tau} + t - \tau \right] \\
- I_2 (1-I_1) \left( \frac{\tau}{e^{a+b\xi_1}} + \frac{t}{e^{a+b\xi_2}} - \frac{\tau}{e^{a+b\xi_2}} \right)^\beta \\
- (1-I_2) \left( \frac{\tau}{e^{a+b\xi_1}} + \frac{t_c}{e^{a+b\xi_2}} - \frac{\tau}{e^{a+b\xi_2}} \right)^\beta. \tag{4.31} \]

The Fisher information matrix for MLEs \((\hat{\beta}, \hat{a}, \hat{b})\) of \((\beta, a, b)\) can be obtained as the expectations of the negative of the second partial derivatives of the log-likelihood function with respect to the model parameters \((\beta, a, b)\). The Fisher information matrix for the \(n\) samples is expressed as
The asymptotic variance-covariance matrix for MLE \((\hat{\beta}, \hat{\alpha}, \hat{b})\) is

\[
F = n \left[ \begin{array}{ccc}
E \left\{ \frac{\partial^2 L}{\partial \beta^2} \right\} & E \left\{ \frac{\partial^2 L}{\partial \beta \alpha} \right\} & E \left\{ \frac{\partial^2 L}{\partial \beta \hat{b}} \right\} \\
E \left\{ \frac{\partial^2 L}{\partial \beta \alpha} \right\} & E \left\{ \frac{\partial^2 L}{\partial \alpha^2} \right\} & E \left\{ \frac{\partial^2 L}{\partial \alpha \hat{b}} \right\} \\
E \left\{ \frac{\partial^2 L}{\partial \beta \hat{b}} \right\} & E \left\{ \frac{\partial^2 L}{\partial \alpha \hat{b}} \right\} & E \left\{ \frac{\partial^2 L}{\partial \hat{b}^2} \right\}
\end{array} \right].
\]  

(4.32)

Then the asymptotic variance-covariance matrix for MLE \((\hat{\beta}, \hat{\alpha}, \hat{b})\) is

\[
\sum = \begin{bmatrix}
AVar(\hat{\beta}) & ACov(\hat{\beta}, \hat{\alpha}) & ACov(\hat{\beta}, \hat{b}) \\
ACov(\hat{\beta}, \hat{\alpha}) & AVar(\hat{\alpha}) & ACov(\hat{\alpha}, \hat{b}) \\
ACov(\hat{\beta}, \hat{b}) & ACov(\hat{\alpha}, \hat{b}) & AVar(\hat{b})
\end{bmatrix} = F^{-1}.
\]  

(4.33)

The optimal changing point \(\tau\) will be the one that minimizes \(AVar(t_{\tau}(x_0))\) given by formula 4.10. Details of the ML method can be found in Appendix B.

4.2.2 Bayesian plan

The ML method described in the previous section requires knowledge of the true unknown parameters values, called “planning values”. Nevertheless, the values of the model parameters are unknown and the purpose of conducting the test is to estimate them. Typically, planning values may be obtained from experience with similar products, design specifications, and engineering judgment. Uncertainty in the “planning values” always exists, which makes the Bayesian approach an attractive alternative to the traditional ML approach in reliability test planning.

According to Bayes’ theorem, the posterior distribution of the model parameters given data \(t\) is
where \( L(a, b, \beta \mid t) \) is the likelihood of the data given by
\[
L(a, b, \beta \mid t) = \prod_{i=1}^{n_1} \frac{\beta}{e^{\beta(a+b t_i)}} t_i^{\beta-1} \exp \left[ -\left( \frac{t_i}{e^{\beta(a+b t_i)}} \right)^\beta \right] \\
\prod_{n_i+1}^{n_2} \frac{\beta}{e^{\beta(a+b x)}} \left( e^{b(x-x_i)} \tau + t_i - \tau \right)^{\beta-1} \exp \left[ -\left( \frac{e^{b(x-x_i)} \tau + t_i - \tau}{e^{\beta(a+b x)}} \right)^\beta \right] \\
\times \left\{ \exp \left[ -\left( \frac{e^{b(x-x_i)} \tau + t_i - \tau}{e^{\beta(a+b x)}} \right)^\beta \right] \right\}^{n-a_2},
\]
where \( n_1 \) is the number of failures under the lower stress, \( n_2 \) is the number of units fail before censoring time, and
\[
f(t) = \int \int \int f(t \mid a, b, \beta) f(a, b, \beta) dadb \beta,
\]
which is also called the pre-posterior marginal distribution of \( t \).

Applying multivariate transformation of random variables, the joint posterior distribution of \((\ln(t_p(x_0)), a, b)\) given data \( t \) can be derived as
\[
f(a, b, \ln(t_p(x_0)) \mid t) = f\left(a, b, \frac{\ln(-\ln(1-p))}{\ln(t_p(x_0)) - a - bx_0} \mid t\right) \ln(-\ln(1-p)) \frac{\ln(-\ln(1-p))}{(\ln(t_p(x_0)) - a - bx_0)^2}.
\]

Then the marginal posterior distribution of \( t_p(x_0) \) is
\[
f(t_p(x_0)) = \int \int \int f(a, b, \ln(t_p(x_0)) \mid t)dadb.
\]

The posterior mean and variance of \( t_p(x_0) \) given \( t \) are, respectively,
\[
E[t_p(x_0) \mid t] = \int_0^\infty t_p(x_0) f(t_p(x_0) \mid t) dt_p(x_0),
\]
and
The pre-posterior variance of \( t_p(x_0) \) defined as

\[
E_t[\operatorname{Var}[t_p(x_0)|t]] = \int \operatorname{Var}[t_p(x_0)|t] f(t)p(x_0) \, dt.
\]  

(4.41)

does not depend on \( t \) and therefore will be used as the objective function. The optimal changing time will be the one to minimize \( E_t[\operatorname{Var}(t_p(x_0)|t)] \).

For the purpose of comparison, the prior variance of \( t_p(x_0) \) is also calculated in the thesis. Applying multivariate transformation on the joint prior distribution of \( a, b, \) and \( \beta \) yields

\[
f(a,b,\ln(t_p(x_0))) = f(a,b,\frac{\ln(-\ln(1-p))}{\ln(t_p(x_0)) - a - bx_0}) \frac{\ln(-\ln(1-p))}{(\ln(t_p(x_0)) - a - bx_0)^2}.
\]  

(4.42)

The prior mean and variance of \( t_p(x_0) \) are, respectively,

\[
E[t_p(x_0)] = \int_0^\infty t_p(x_0) f(t_p(x_0)) \, dt_p(x_0),
\]  

(4.43)

and

\[
\operatorname{Var}[t_p(x_0)] = \int_0^\infty (t_p(x_0) - E[t_p(x_0)])^2 f(t_p(x_0)) \, dt_p(x_0).
\]  

(4.44)

### 4.2.3 Simulation algorithm

Evaluation of the pre-posterior variance of \( t_p(x_0) \) requires high-dimensional integration and has no closed-form solution. Therefore, Monte Carlo integration is used for the approximate evaluation of the integral (4.41). For each \( \tau \) value, this study proposes to evaluate \( E_t[\operatorname{Var}(t_p(x_0)|t)] \) according to the following algorithm:
A large number of sets of \((a, b, \beta)\) is simulated from their joint prior distribution. For each set of \((a, b, \beta)\) a set of \(n\) “failure times” is simulated from the distribution function \(F(t| a, b, \beta)\) given by formula (1.17). Simulated failure times that are greater than the censoring time \(t_c\) become right-censored observations. The failure times are then analyzed using Gibbs sampling to obtain the posterior variance of \(t_p(x_0), \text{Var}(t_p(x_0)|t)\). Averaging \(\text{Var}(t_p(x_0)|t)\) over all sets of \((a, b, \beta)\) yields the \(E_t[\text{Var}(t_p(x_0)|t)]\). Figure 4.1 shows the flow chart of the simulation algorithm.

Different values of \(\tau\) are used to plot the relationship between \(\tau\) and \(E_t[\text{Var}(t_p(x_0)|t)]\) in order to decide the optimal changing time.

4.2.4 Prior specification

The joint prior distribution specified in this thesis is based on the study of Singpurwalla [28]. Instead of assuming two Weibull scale parameters are independent, Singpurwalla [28] demonstrated that the Weibull shape parameter and the median time-to-failure are independent and then assigned the independent priors to the shape.
parameter and the median life. Because the median life can be expressed as a simple function of the two Weibull parameters, the joint prior distribution for two Weibull parameters can be derived through bivariate transformation of random variables.

In the planning part of this thesis, we assume that we have independent prior knowledge of the shape parameter and the median lives at two stress levels. Denoting the median lives at the stress levels \(x_1\) and \(x_2\) by \(M_1\) and \(M_2\), the joint prior distribution can be expressed as \(f(M_1, M_2, \beta) = f(M_1) f(M_2) f(\beta)\). Because

\[
M_i = \theta_i \left(\ln 2\right)^{1/\beta} = \exp(a + bx_i) \left(\ln 2\right)^{1/\beta}, \quad i = 1, 2,
\]

the joint prior distribution of \(a\), \(b\), and \(\beta\) can be derived via multivariate transformation of random variables.

\[
f(a, b, \beta) = f(M_1, M_2, \beta) \left| \begin{array}{cc}
\frac{\partial M_1}{\partial a} & \frac{\partial M_1}{\partial b} \\
\frac{\partial M_2}{\partial a} & \frac{\partial M_2}{\partial b}
\end{array} \right|
\]

\[
= f(M_1, M_2, \beta) \exp[2a + b(x_1 + x_2)](\ln 2)^{2/\beta} (x_2 - x_1).
\]

The joint prior distribution given by (4.46) though cumbersome in appearance has the virtue of being a proper density and one that has been reasonably well motivated [28].
CHAPTER 5: CASE STUDY

In this chapter, case studies are organized to perform the methodologies described in chapter 4 and compare ML methods with Bayesian methods.

5.1 Analysis of Simple SSALT

The experimental data is used to demonstrate the proposed method. R is used to perform ML calculation and WinBUGS is used to perform Bayesian analysis. The code can be found in Appendix C and Appendix D respectively.

An experimental dataset collected at the Thin Film Nano & Microelectronics Research Laboratory at Texas A&M University is used in the case study. The data is collected in a lab environment, not from commercial production. In the experiment, the nanocrystalline (nc) ZnO embedded high-k device was tested. The nc- ZnO embedded high-k thin film was prepared into a metal-oxide-semiconductor (MOS) capacitor, which was fabricated on the HF-clean p-type (10^{15} \text{ cm}^{-3}) wafer for test [29]. Table 5.1 describes the data obtained from the experiment. Firstly we set uninformative priors to the parameters, which are normal distribution with \( \mu = 0 \) and \( \sigma^2 = 10^6 \) for \( a \) and \( b \), and gamma distribution with shape parameter equals one and rate parameter equals 0.2 for \( \beta \). The results of Bayesian analysis obtained from WinBUGs are showed in Table 5.2.
Table 5.1 Experimental data with \( n = 40 \), \( \tau = 600 \) and \( t_c = 780 \)

\[
\begin{array}{cccccccccccc}
 t_{i,n} < \tau & 8 & 38 & 72 & 97 & 122 & 140 & 163 & 170 & 188 & 198 \\
\end{array}
\]

Table 5.2 Posterior statistics of model parameters

<table>
<thead>
<tr>
<th>Node</th>
<th>Mean</th>
<th>sd</th>
<th>MC error</th>
<th>2.50%</th>
<th>Median</th>
<th>97.50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>36.97</td>
<td>5.00</td>
<td>0.21</td>
<td>28.03</td>
<td>36.64</td>
<td>47.14</td>
</tr>
<tr>
<td>b</td>
<td>-4.174</td>
<td>0.66</td>
<td>0.028</td>
<td>-5.51</td>
<td>-4.133</td>
<td>-2.988</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.8212</td>
<td>0.15</td>
<td>0.0045</td>
<td>0.555</td>
<td>0.81</td>
<td>1.16</td>
</tr>
<tr>
<td>( t_p(x_0) )</td>
<td>70860</td>
<td>167700</td>
<td>4191</td>
<td>2805</td>
<td>27950</td>
<td>400500</td>
</tr>
</tbody>
</table>

To prevent posterior dependence on the starting point of a simulation, several chains with over-dispersed starting points in one MCMC simulation should be run. The simulation converges to the target distribution when traces of all chains appear to be mixing together [8]. In this case, two chains with different initials run simultaneously in one simulation. Each chain continues for 80000 iterations. There are two methods to check convergence. One is examining trace plots of the sample values versus iteration. Figure 5.1 shows trace plots for parameters \( a \) and \( \beta \) respectively. We can be reasonably confident that convergence has achieved since the two chains appear to be well mixed.
Figure 5.1 Trace plots of $a$ and $\beta$

This is an informal approach to convergence diagnosis. A quantitative way of checking convergence is based on an analysis of variance. The Gelman-Rubin convergence
statistic, $R$, is introduced to diagnose convergence. $R$ is defined as the ratio of the width of the central 80% interval of the pooled chains to the average width of the 80% intervals within the individual chains. When a WinBUGS simulation converges, $R$ should be, or close to one [8]. Figure 5.2 shows Gelman-Rubin convergence statistics of $a$, $b$, $\beta$ and $t_p(x_0)$, and from Figure 5.2, the simulation is believed to have converged.

![Figure 5.2 Gelman-Rubin convergence statistic of $a$, $b$, $\beta$, and $t_p(x_0)$](image)

The accuracy of a posterior estimate is calculated in terms of Monte Carlo standard error (MC error) of the mean, which is an estimate of the difference between the mean of the sample and the true posterior mean. According to Spiegelhalter[10], the simulation should be run until the MC error for each node is less than 5% of the sample standard deviation. In this case, this rule has been achieved.

Figure 5.3 shows kernel density estimates of distributions of $a$, $b$, $\beta$, and $t_p(x_0)$. It appears the distribution of $t_p(x_0)$ is quite asymmetric. Therefore, median value instead of mean value is chosen to be the point estimated value for $t_p(x_0)$. The point estimates obtained from ML method and Bayesian method are shown in Table 5.3. The estimated values gained from these two methods are close to each other. Then we set some
informative priors which are showed in Table 5.4. Prior I is the set of priors used in the above simulation. Prior II and Prior III use continuous uniform distribution to express the uncertainty in the values of parameters. For instance, U(0, 50) prior for \( a \) indicates that \( a \) is believed to be between 0 and 50 and all values within that interval are equally probable. From Prior I to Prior III, the prior uncertainty is decreasing.

*Figure 5.3* Density plots of \( a, b, \beta, \) and \( t_p(x_0) \)
Table 5.3 Estimate of $a$, $b$, $\beta$, and $t_p(x_0)$ by ML and Bayesian method

<table>
<thead>
<tr>
<th>Method</th>
<th>Parameter</th>
<th>Estimated value</th>
<th>Variance</th>
<th>95% CI</th>
<th>Interval length</th>
</tr>
</thead>
<tbody>
<tr>
<td>ML</td>
<td>$a$</td>
<td>34.47</td>
<td>22.78</td>
<td>(26.12, 44.83)</td>
<td>18.71</td>
</tr>
<tr>
<td></td>
<td>$b$</td>
<td>-3.98</td>
<td>0.40</td>
<td>(-5.22, -2.74)</td>
<td>2.48</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>0.85</td>
<td>0.025</td>
<td>(0.54, 1.16)</td>
<td>0.62</td>
</tr>
<tr>
<td></td>
<td>$t_p(x_0)$</td>
<td>25212</td>
<td>9.06E+08</td>
<td>(0, 84200)</td>
<td>84200</td>
</tr>
<tr>
<td>Bayesian</td>
<td>$a$</td>
<td>36.97</td>
<td>25.03</td>
<td>(28.03, 47.14)</td>
<td>19.11</td>
</tr>
<tr>
<td></td>
<td>$b$</td>
<td>-4.17</td>
<td>0.47</td>
<td>(-5.51, -2.99)</td>
<td>2.52</td>
</tr>
<tr>
<td></td>
<td>$\beta$</td>
<td>0.82</td>
<td>0.024</td>
<td>(0.56, 1.16)</td>
<td>0.60</td>
</tr>
<tr>
<td></td>
<td>$t_p(x_0)$</td>
<td>27950</td>
<td>2.81E+10</td>
<td>(2805, 400500)</td>
<td>397695</td>
</tr>
</tbody>
</table>

Table 5.4 Prior distributions for Bayesian analysis

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prior I</td>
<td>N(0, 10^6)</td>
<td>N(0, 10^6)</td>
<td>$\Gamma(1, 0.2)$</td>
</tr>
<tr>
<td>Prior II</td>
<td>U(0, 50)</td>
<td>U(-50, 0)</td>
<td>U(0, 5)</td>
</tr>
<tr>
<td>Prior III</td>
<td>U(20, 40)</td>
<td>U(-10, 0)</td>
<td>U(0, 2)</td>
</tr>
</tbody>
</table>

Table 5.5 shows the estimates with three sets of prior distributions. It appears that as the prior uncertainty decreases, the variance of estimated parameters decrease. This is expected because the prior knowledge has been incorporated with data and increased the accuracy of estimate. From Table 5.5 it is also reasonably to conclude that the interval length is narrower with higher informative priors.
Table 5.5 Estimate of $a$, $b$, $\beta$, and $t_p(x_0)$ with different prior distribution

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimated value</th>
<th>Variance</th>
<th>95% CI</th>
<th>Interval length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prior I</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>36.97</td>
<td>25.03</td>
<td>(28.03, 47.14)</td>
<td>19.11</td>
</tr>
<tr>
<td>$b$</td>
<td>-4.17</td>
<td>0.44</td>
<td>(-5.51, -2.988)</td>
<td>2.52</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.82</td>
<td>0.024</td>
<td>(0.555, 1.157)</td>
<td>0.60</td>
</tr>
<tr>
<td>$t_p(x_0)$</td>
<td>27950</td>
<td>2.81E+10</td>
<td>(2805, 400500)</td>
<td>397695</td>
</tr>
<tr>
<td>Prior II</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>35.6</td>
<td>21.91</td>
<td>(27.29, 45.02)</td>
<td>17.73</td>
</tr>
<tr>
<td>$b$</td>
<td>-3.99</td>
<td>0.38</td>
<td>(-5.234, -2.887)</td>
<td>2.35</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.85</td>
<td>0.025</td>
<td>(0.5824, 1.195)</td>
<td>0.61</td>
</tr>
<tr>
<td>$t_p(x_0)$</td>
<td>22330</td>
<td>8.42E+09</td>
<td>(2115, 22330)</td>
<td>269267</td>
</tr>
<tr>
<td>Prior III</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a$</td>
<td>34.46</td>
<td>11.14892</td>
<td>(27.33, 39.62)</td>
<td>12.29</td>
</tr>
<tr>
<td>$b$</td>
<td>-3.84</td>
<td>0.197402</td>
<td>(-4.527, -2.892)</td>
<td>1.64</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.87</td>
<td>0.021025</td>
<td>(0.8622, 1.195)</td>
<td>0.57</td>
</tr>
<tr>
<td>$t_p(x_0)$</td>
<td>19480</td>
<td>1.09E+09</td>
<td>(2361, 122400)</td>
<td>120039</td>
</tr>
</tbody>
</table>

5.2 Design of Simple SSALT

The case used to illustrate the Bayesian planning methodology is based on the ALT experiment study and Bayesian data analysis conducted by Luo[8], in which the reliability of a novel nanometer-thick high-$k$ gate dielectric film is studied. The high-$k$ gate dielectric is an alternative material to SiO$_2$ that can avoid the serious function and reliability problem caused by further reducing oxide thickness. In the study, Luo[8] performed constant accelerated tests to measure the breakdown of high-$k$ film, with the accelerated stresses ranging from 5.5 to 8.1 MV/cm, and obtained the time-to-breakdown data from the tests. The data were used to extract prior distribution of the model parameters in this thesis.

Firstly, we assume values of three parameters are known and apply maximum likelihood method to determine the optimal stress changing point. The objective function
is to minimize the asymptotic variance of the MLE of the $p$th percentile at normal stress level $x_0$, that is, $\hat{\text{AVar}}(\hat{t}_p(x_0))$. In the case, the stress is the electrical field stress and we assume $x_0 = 1.5$ MV/cm, $x_1 = 7.7$ MV/cm, $x_2 = 8.1$ MV/cm, $p=0.001$, $t_c = 4000$ seconds, and $n=40$. The planning parameters of three parameters are $\hat{a}=23$, $\hat{b}=-2$ and $\hat{b}=0.66$. R and Microsoft Excel are used to solve the problem. The code can be found in Appendix E.

Figure 5.4 shows the plot of $\hat{\text{AVar}}(\hat{t}_p)$ versus the stress changing point $\tau$. The optimal stress changing point is $\tau^*=1774$ second.

![Graph of AVar(t_p(x_0)) vs. \tau](image)

*Figure 5.4 AVar(t_p(x_0)) vs. \tau (maximum likelihood method)*

Then we consider the situation that the precise values of the model parameters are unknown and apply the Bayesian method. The objective function is to minimize the pre-posterior variance of the $p$th percentile at normal stress level $x_0$, that is, $E_t[\text{Var}(t_p(x_0|t))]$.

Table 5.6 summarizes the three sets of prior distributions assumed in the case study. The
continuous uniform distribution is used to express the uncertainty in the values of the parameters. For example, U(500, 530) prior for $M_2$ indicates that the median life at the stress level $S_2$ is believed to be between 500 seconds and 530 seconds and all values within that interval are equally probable. The mean values of these three sets are the same, i.e., $\bar{\beta} = 0.66$, $\bar{M}_1 = 1146$ seconds, and $\bar{M}_2 = 515$ seconds, which correspond to the planning values used by maximum likelihood method. From prior I to prior III, the uncertainty of the values of the model parameters is increasing.

Table 5.6 Prior distributions for Bayesian plan

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$M_1$</th>
<th>$M_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prior I</td>
<td>U(0.655, 0.665)</td>
<td>U(1096, 1196)</td>
<td>U(500, 530)</td>
</tr>
<tr>
<td>Prior II</td>
<td>U(0.65, 0.67)</td>
<td>U(1046, 1246)</td>
<td>U(490, 540)</td>
</tr>
<tr>
<td>Prior III</td>
<td>U(0.64, 0.68)</td>
<td>U(946, 1346)</td>
<td>U(470, 560)</td>
</tr>
</tbody>
</table>

Figure 5.5, 5.6 and 5.7 show the plots of the pre-posterior variance of $t_p$ versus the stress changing time assuming the three prior distributions in Table 5.6.
Figure 5.5 $E_d[\text{Var}(t_p(x_0|t))]$ vs. $\tau$ assuming Prior I and $n=40$

Figure 5.6 $E_d[\text{Var}(t_p(x_0|t))]$ vs. $\tau$ assuming Prior II and $n=40$
Figure 5.7 $E_d[\text{Var}(t_p(x_0|t))]$ vs. $\tau$ assuming Prior III and $n=40$

Table 5.7 summarizes the optimal stress changing times and the corresponding pre-posterior variance of $t_p$.

### Table 5.7 Optimal Bayesian designs

|       | Optimal $\tau$ (sec.) | Optimal $E_d[\text{Var}(t_p(x_0|t))]$ | Prior Var. of $t_p$ |
|-------|-----------------------|--------------------------------------|---------------------|
| Prior I | 2550                   | 589025134                           | 59105344            |
| Prior II | 1850                   | 363537374                           | 380640100           |
| Prior III | 1700                   | 8086839080                          | 9135536400          |

According to Table 5.7, as the prior uncertainty increases, the optimal stress changing point obtained by the Bayesian method is closer to that obtained by the maximum likelihood method. From the Bayesian theorem given by 4.34, posterior inference of $t_p(x_0)$ actually depends on information from two sources: the prior
knowledge and the data (via the likelihood function). As the prior uncertainty increases, the information from data becomes more influential on the posterior inference. The maximum likelihood method uses only the data information. Therefore, the larger the prior uncertainty, the more similar results the Bayesian method and maximum likelihood method yield.

Table 5.7 also presents the prior variance of $t_p(x_0)$, which is calculated only using the prior information and quantifies the prior uncertainty in the value of $t_p$. When we are more certain on the values of parameters, the optimal $E_d[Var(t_p(x_0|t))]$ is closer to the prior variance of $t_p$. Furthermore, the ranges of $E_d[Var(t_p(x_0|t))]$ in Figure 5.5, 5.6 and 5.7 show that $E_d[Var(t_p(x_0|t))]$ is most sensitive to $\tau$ when Prior III is used as the same change in $\tau$ leads to the greatest change in the value of $E_d[Var(t_p(x_0|t))]$. Conversely, $E_d[Var(t_p(x_0|t))]$ is least sensitive to $\tau$ when Prior I is used. These observations indicate that the experimental data does not provide much additional information to reduce the uncertainty on $t_p(x_0)$ when a very informative prior is assumed. As the prior uncertainty increases (from Prior I to Prior III), the optimal $E_d[Var(t_p(x_0|t))]$ becomes more significantly lower than the corresponding prior variance of $t_p(x_0)$. This implies that with the increase of prior uncertainty, the information from data become more influential on the posterior inference. When we are extremely certain on the values of model parameters, that is, the precise values are known, the Bayesian approach suggest that there is no need to do the life test. That is because all the information needed can be deduced from the model.

The maximum likelihood method always yields $\tau^*=1774$ no matter what sample size $n$ is. However, the sample size will influence the optimal stress changing point in
Bayesian method. Figure 5.8 shows the plot of $E_t[Var(t_p(x_0|t))]$ vs. $\tau$ assuming Prior I and $n=200$. The optimal stress changing time is 1800 seconds. Compared with the optimal stress changing time assuming Prior I when $n=40$ it is closer to the optimal stress changing point obtained from ML method. This indicates that with larger sample size, the data is more influential on the posterior inference.

*Figure 5.8 $E_t[Var(t_p(x_0|t))]$ vs. $\tau$ assuming Prior I and $n=200$*
CHAPTER 6: CONCLUSIONS AND FUTURE RESEARCH

This thesis presents a Bayesian analysis method as well as a Bayesian design method for Type-I censored simple step-stress accelerated life test with Weibull life time distribution and log-linear acceleration model. The practical examples are used to illustrate the proposed method and to compare the Bayesian method with the ML method.

6.1 Conclusions

The main objectives and tasks described in section 1.3 are completed in the thesis. The Bayesian analysis was conducted to estimate the point and confidence interval of the model parameters. WinBUGS is used to do the analysis and the code can be found in Appendix D. The Bayesian optimal plan is obtained by minimizing the expect value of the pre-posterior variance of \( t_p(x_0) \), and the simulation algorithm is achieved by R and WinBUGS. The code is showed in Appendix F. The ML optimal plan is gained with the help of R to do computations. The code in Appendix E calculates the asymptotic variance of the MLE of \( t_p(x_0) \) when \( \tau=200 \), several values of stress changing time are chosen to plot the relation between \( \hat{AVar}(t_p^\tau) \) vs. \( \tau \) and the optimal stress changing time is obtained from the plot.

Generally, both ML method and Bayesian method can be used to analyze and design simple SSALT, but they can be applicable to different conditions. ML method can be applied when:

- precise values of model parameters have been known
- large sample size can be obtained in the test
Bayesian method can be applied when:

- uncertainties on the model parameters exist
- prior knowledge on the model parameters are available
- few data is available in the test

6.1.1 Analysis of SSALT

In the analysis section, the simulation for Bayesian analysis has proved to be converged in the thesis. Because the density plot of distribution of \( t_p(x_0) \) seems to be asymmetric, the median value instead of the mean is chosen to be the estimated value and compared with the ML result. The Bayesian method and ML method yield similar result. As the prior uncertainty decreases, the Bayesian method yields the estimated values with smaller variance and narrower confidence intervals.

6.1.2 Design of SSALT

This thesis assumes the prior distributions of the median lives at two stress levels (i.e. \( M_1 \) and \( M_2 \)) and the shape parameters \( \beta \) and assumes they are independent. The joint prior distribution of the model parameters \( a, b \) and \( \beta \) is then deduced via multivariate transformation from the prior distributions of \( M_1, M_2, \) and \( \beta \).

Overall, the Bayesian plan depends on the prior distributions and the sample size. When we are more certain on the prior distribution, the sample size is less influential on the posterior inference, vice versa.
For SSALT plan design, as the prior uncertainty decreases, 1) the optimal stress changing time obtained from ML method and Bayesian method are closer; 2) the pre-posterior variance of $t_p(x_0)$, $E_t[Var(t_p(x_0)|t)]$ is more sensitive to the stress changing time; 3) the pre-posterior variance of $t_p(x_0)$, $E_t[Var(t_p(x_0)|t)]$ becomes much more significantly lower than the corresponding prior variance of $t_p(x_0)$, $Var(t_p(x_0))$, which is gained from the prior distribution. When the prior uncertainties are the same, the stress changing time obtained by ML method is closer to that obtained by Bayesian method with larger sample size. One interesting deduction is that the Bayesian approach would recommend that no accelerated life test is needed when the precise values of the model parameters are known.

6.2 Future Research

This thesis developed a Bayesian method for the SSALT with assumptions of Weibull lifetime distribution, Type-I censoring and the cumulative exposure model. However, it can be generalized to the analysis and design of other types of reliability life tests such as exponential lifetime distribution and Type-II censoring due to the generality and flexibility of the Bayesian framework.

In the planning problem some constraints may be considered, such as a lower limit on the expected number of failures before the stress changing time and a lower limit on the expected number of failures before censoring time.

The sample size is also a vital factor in Bayesian design because the Bayesian plan expects large sample size to improve the accuracy. However, the large sample size
test requires great cost and time. Thus, the sample size can be taken into consideration as well as the stress changing time in the SSALT planning in the future research. This may need to integrate the test cost as well as the requirement on the accuracy of the test result.

Stress level is also important to the test. High stress level could faster the test then reduces the cost. However, if the stress is too high, the data may not reflect accurately the failures under normal stress. Therefore, the stress level determination can be another subject of designing the SSALT.
REFERENCES


APPENDIX A: OBSERVED FISHER INFORMATION MATRIX

Denote $n_1$ is the number of failures in the first stage of a SSALT, $n_2$ is the number of failures before censoring time and $n$ is the sample size, the likelihood function of sample data is

$$L(t; x_1, x_2) = \prod_{i=1}^{n_1} \frac{\beta}{\theta_1} t^{\beta-1} \exp \left[ - \left( \frac{t}{\theta_1} \right)^\beta \right] \prod_{i=n_1+1}^{n+n_2} \frac{\beta}{\theta_2} \left( \frac{\theta_2}{\theta_1} \tau + t - \tau \right)^{\beta-1} \exp \left[ - \left( \frac{\theta_2}{\theta_1} \tau + t - \tau \right) \right] \prod_{i=n_2+1}^{n} \exp \left[ - \left( \frac{\theta_2}{\theta_1} \tau + t - \tau \right) \right].$$

The log-likelihood of sample data is

$$l = \ln L(t; x_1, x_2) = n_1 \ln \beta - n_1 \beta \ln \theta_1 + (\beta - 1) \sum_{i=1}^{n_1} \ln t_i - \sum_{i=1}^{n_1} \left( \frac{t_i}{\theta_1} \right) \beta + (n_2 - n_1) \ln \beta$$

$$- (n_2 - n_1) \beta \ln \theta_2 + (\beta - 1) \sum_{i=n_1+1}^{n+n_2} \ln \left( \frac{\theta_2}{\theta_1} \tau + t - \tau \right) - \sum_{i=n_1+1}^{n+n_2} \left( \frac{\theta_2}{\theta_1} \tau + t - \tau \right)^\beta$$

$$- \left( n - n_2 \right) \left( \frac{\theta_2}{\theta_1} \tau + t - \tau \right)^\beta$$

$$= n_2 \log \beta - n_1 \beta (a + bx_1) + (\beta - 1) \sum_{i=1}^{n_1} \ln t_i - (n_2 - n_1) \beta (a + bx_2)$$

$$- \sum_{i=1}^{n_1} \left( \frac{t_i}{e^{a+bx_1}} \right)^\beta - (\beta - 1) \sum_{i=n_1+1}^{n+n_2} \ln \left( \frac{\tau}{e^{a+bx_2}} \right) + t_i - \tau$$

$$- \sum_{i=n_1+1}^{n+n_2} \left( \frac{\tau}{e^{a+bx_2}} + \frac{t_i}{e^{a+bx_2}} - \frac{\tau}{e^{a+bx_2}} \right)^\beta - (n - n_2) \left( \frac{\tau}{e^{a+bx_2}} + \frac{t_i}{e^{a+bx_2}} - \frac{\tau}{e^{a+bx_2}} \right)^\beta,$$
Let
\[ A(t_i) = \left( \frac{\tau}{e^{a+bx_i}} + \frac{t_i}{e^{a+bx_i}} - \frac{\tau}{e^{a+bx_i}} \right), \]
then
\[ \frac{\partial A(t_i)}{\partial b} = \left( \frac{-\tau x_i}{e^{a+bx_i}} + \frac{-t_i x_i}{e^{a+bx_i}} - \frac{-\tau x_i}{e^{a+bx_i}} \right), \]
and
\[ \frac{\partial^2 A(t_i)}{\partial b^2} = \left( \frac{-\tau x_i^2}{e^{a+bx_i}} + \frac{-t_i x_i^2}{e^{a+bx_i}} - \frac{-\tau x_i^2}{e^{a+bx_i}} \right). \]

The first partial derivatives of the log likelihood with respective to the model parameters are,
\[
\frac{\partial l}{\partial \beta} = \frac{n_2}{\beta} - n_1(a + bx_1) + \sum_{i=1}^{n} \ln t_i -(n_2 - n_1)(a + bx_2) - \sum_{i=1}^{n} t_i e^{-\beta(a + bx_1)} \ln[t_i e^{-(a + bx_1)}] + \sum_{i=n+1}^{n_2} \ln[t_i e^{b(x_1-x_i)} + t_i - \tau] - \sum_{i=n+1}^{n_3} A(t_i)^{\beta} \ln A(t_i) - (n-n_2)A(t_c)^{\beta} \ln A(t_c),
\]
\[
\frac{\partial l}{\partial a} = -n_2 \beta + \sum_{i=1}^{n} t_i e^{-\beta(a + bx_1)} \beta + \sum_{i=n+1}^{n_2} \beta A(t_i)^{\beta} + \beta(n-n_2)A(t_c)^{\beta},
\]
\[
\frac{\partial l}{\partial b} = -n_1 \beta x_1 -(n_2 - n_1) \beta x_2 + \sum_{i=1}^{n} t_i e^{-\beta(a + bx_1)} \beta x_1 + (\beta - 1) \sum_{i=n+1}^{n_2} \frac{\tau e^{b(x_2-x_1)} (x_2-x_1)}{\tau e^{b(x_2-x_1)} + t_i - \tau} - \sum_{i=n+1}^{n_3} \beta A(t_i)^{\beta-1} \frac{\partial A(t_i)}{\partial b} - \beta(n-n_2)A(t_c)^{\beta-1} \frac{\partial A(t_c)}{\partial b}.
\]

The second partial derivatives of the log likelihood with respective to the model parameters are,
\[
\frac{\partial^2 l}{\partial \beta^2} = -\frac{n_2}{\beta^2} - \sum_{i=1}^{n} t_i e^{-\beta(a + bx_1)} \left[ \ln t_i e^{-(a + bx_1)} \right]^2,
\]
\[
- \sum_{i=n+1}^{n_3} A(t_i)^{\beta} \left( \ln A(t_i) \right)^2 - (n-n_2)A(t_c)^{\beta} \left( \ln A(t_c) \right)^2,
\]
\[
\frac{\partial^2 l}{\partial a^2} = -\sum_{i=1}^{n} t_i e^{-\beta(a + bx_1)} \beta^2 - \sum_{i=n+1}^{n_3} \beta^2 A(t_i)^{\beta} - \beta^2 (n-n_2)A(t_c)^{\beta},
\]
\[
\frac{\partial^2 l}{\partial b^2} = -n \sum_{i=1}^n t_i \beta e^{-\beta (x_i - x)} (\beta x_i)^2 + (\beta - 1) \sum_{i=1}^n \frac{\tau(t_i - \tau) e^{\beta (x_i - x)} (x_i - x_1)^2}{(\tau x + 1)^2 + t_i - \tau^2}
- \sum_{i=n+1}^n \beta (\beta - 1) A(t_i) \beta^{-2} \left( \frac{\partial A(t_i)}{\partial b} \right)^2 - \sum_{i=n+1}^n \beta A(t_i) \beta^{-1} \frac{\partial^2 A(t_i)}{\partial b^2}
- (n - n_2) \beta (\beta - 1) A(t_c) \beta^{-2} \left( \frac{\partial A(t_c)}{\partial b} \right)^2 - (n - n_2) \beta A(t_c) \beta^{-1} \frac{\partial^2 A(t_c)}{\partial b^2},
\]

\[
\frac{\partial^2 l}{\partial \beta c a} = -n \sum_{i=1}^n t_i \beta e^{-\beta (x_i - x)} \ln \left[ t_i e^{-(\beta + x)} \right] \beta + \sum_{i=1}^n t_i \beta e^{-\beta (x_i - x)}
+ \sum_{i=n+1}^n \beta A(t_i) \beta \ln A(t_i) + \sum_{i=n+1}^n \beta A(t_i) \beta + (n - n_2) \beta A(t_c) \beta \ln A(t_c) + (n - n_2) A(t_c) \beta,
\]

\[
\frac{\partial^2 l}{\partial \beta c b} = -n \beta x_1 - (n - n_1) x_1 + \sum_{i=n+1}^n \frac{\tau(x_i - x_1) e^{\beta (x_i - x_1)}}{\tau e^{\beta (x_i - x_1)} + t_i - \tau} + \sum_{i=1}^n t_i \beta e^{-\beta (x_i - x)} \ln \left[ t_i e^{-(\beta + x)} \right] \beta x_i
+ \sum_{i=n+1}^n t_i \beta e^{-\beta (x_i - x)} x_1 - \sum_{i=n+1}^n \beta A(t_i) \beta^{-1} \ln A(t_i) \frac{\partial A(t_i)}{\partial b} - \sum_{i=n+1}^n A(t_i) \beta^{-1} \frac{\partial A(t_i)}{\partial b}
- (n - n_2) \beta A(t_c) \beta^{-1} \ln A(t_c) \frac{\partial A(t_c)}{\partial b} - (n - n_2) A(t_c) \beta^{-1} \frac{\partial A(t_c)}{\partial b},
\]

\[
\frac{\partial^2 l}{\partial a c b} = -\sum_{i=1}^n t_i \beta e^{-\beta (x_i - x)} \beta x_1 + \sum_{i=n+1}^n \beta^2 A(t_i) \beta^{-1} \frac{\partial A(t_i)}{\partial b} + \beta^2 (n - n_2) A(t_c) \beta^{-1} \frac{\partial A(t_c)}{\partial b}.
\]

The Fisher Information Matrix is

\[
F = \begin{bmatrix}
\frac{\partial^2 l}{\partial \beta^2} & -\frac{\partial^2 l}{\partial \beta c a} & -\frac{\partial^2 l}{\partial \beta c b} \\
-\frac{\partial^2 l}{\partial \beta c a} & \frac{\partial^2 l}{\partial c^2} & -\frac{\partial^2 l}{\partial c a c b} \\
-\frac{\partial^2 l}{\partial \beta c b} & -\frac{\partial^2 l}{\partial c a c b} & \frac{\partial^2 l}{\partial c^2}
\end{bmatrix}
\]
APPENDIX B: INFERENCE OF MAXIMUM LIKELIHOOD PLAN

Define the indicator function $I_1 = I_1(t \leq \tau)$ by

$$I_1 = I_1(t \leq \tau) = \begin{cases} 1 & \text{if } t \leq \tau, \text{ failure observed before time } \tau, \\ 0 & \text{if } t > \tau, \text{ failure observed after time } \tau, \end{cases}$$

and the indicator function $I_2 = I_2(t \leq t_c)$ by

$$I_2 = I_2(t \leq t_c) = \begin{cases} 1 & \text{if } t \leq t_c, \text{ failure observed before time } t_c \\ 0 & \text{if } t > t_c, \text{ failure observed after time } t_c \end{cases}$$

The log-likelihood function is deduced as,

$$L(t; x_1, x_2) = I_1 I_2 \ln \left( \frac{\beta}{\theta_1^\beta} t^{\beta-1} \exp \left[ -\left( \frac{t}{\theta_1} \right)^\beta \right] \right)$$

$$+ I_2 (1 - I_1) \ln \left( \frac{\beta}{\theta_2^\beta} \left( \frac{\theta_2}{\theta_1} \frac{\tau + t - \tau}{\theta_2} \right)^{\beta-1} \exp \left[ -\left( \frac{\theta_2}{\theta_1} \frac{\tau + t - \tau}{\theta_2} \right)^\beta \right] \right)$$

$$+ (1 - I_2) \ln \left( \exp \left[ -\left( \frac{\theta_2}{\theta_1} \frac{\tau + t_c - \tau}{\theta_2} \right)^\beta \right] \right)$$

$$= -I_1 I_2 \beta (a + bx_1) + I_1 I_2 (\beta - 1) \ln t - I_1 I_2 \tau \beta e^{-\beta(a+bx_1)} + I_2 \ln \beta$$

$$- I_2 (1 - I_1) \beta (a + bx_2) + I_2 (1 - I_1) (\beta - 1) \ln \left[ t e^{b(x_2-x_1)} + t - \tau \right]$$

$$- I_2 (1 - I_1) \left( \frac{\tau}{e^{a+bx_1}} + \frac{t}{e^{a+bx_2}} - \frac{\tau}{e^{a+bx_2}} \right)^\beta$$

$$- (1 - I_2) \left( \frac{\tau}{e^{a+bx_1}} + \frac{t_c}{e^{a+bx_2}} - \frac{\tau}{e^{a+bx_2}} \right)^\beta,$$

Let

$$A(t) = \left( \frac{\tau}{e^{a+bx_1}} + \frac{t}{e^{a+bx_2}} - \frac{\tau}{e^{a+bx_2}} \right),$$
then

\[ \frac{\partial A(t)}{\partial a} = -A(t), \]

\[ \frac{\partial A(t)}{\partial b} = \left( -\tau x_1 e^{-bx_1} + \tau x_2 e^{-bx_2} - \tau x_2 \right), \]

and

\[ \frac{\partial^2 A(t)}{\partial^2 b} = \left( \frac{\tau x_1^2}{e^{bx_1}} + \frac{\tau x_2^2}{e^{bx_2}} - \tau x_2 \right). \]

The first partial derivatives of the log likelihood with respective to the model parameters are,

\[ \frac{\partial L}{\partial \beta} = -I_1 I_2 (a + bx_1) + I_1 I_2 \ln t - I_1 I_2 t^\beta t e^{-\beta(a+bx_1)} + I_1 I_2 t^\beta e^{-\beta(a+bx_1)} (a + bx_1) \]

\[ + \frac{I_2}{\beta} - I_1 (1 - I_1)(a + bx_2) + I_2 (1 - I_1) \ln[\tau e^{b(x_2-x_1)} + t - \tau] \]

\[ - I_2 (1 - I_1) A(t)^\beta \ln A(t) - (1 - I_2) A(t_c)^\beta \ln A(t_c), \]

\[ \frac{\partial L}{\partial a} = I_1 I_2 \beta t e^{-\beta(a+bx_1)} + I_1 I_2 (1 - I_1) \beta A(t)^\beta + (1 - I_2) \beta A(t_c)^\beta, \]

\[ \frac{\partial L}{\partial b} = -I_1 I_2 \beta x_1 + I_1 I_2 t^\beta \beta x_1 e^{-\beta(a+bx_1)} - I_2 (1 - I_1) \beta x_2 \]

\[ + I_2 (1 - I_1) (\beta - 1) \frac{\tau (x_2 - x_1) e^{b(x_2-x_1)}}{\tau e^{b(x_2-x_1)} + t - \tau} \]

\[ - I_2 (1 - I_1) \beta A(t)^\beta - (1 - I_2) \beta A(t_c)^\beta \frac{\partial A(t_c)}{\partial b}. \]

The second partial derivatives of the log likelihood with respective to the model parameters are,
\[
\frac{\partial^2 L}{\partial \beta^2} = -I_1 I_2 t^\beta e^{-\beta(a+bx_t)} (\ln t - a - bx_t)^2 - \frac{I_2}{\beta^2} \\
- I_2 (1 - I_1) A(t)^\beta (\ln A(t))^2 - (1 - I_2) A(t_e)^\beta (\ln A(t_e))^2,
\]

\[
\frac{\partial^2 L}{\partial a^2} = -I_1 I_2 \beta^2 t^\beta e^{-\beta(a+bx_t)} - I_2 (1 - I_1) \beta^2 A(t)^\beta + (1 - I_2) \beta^2 A(t_e)^\beta,
\]

\[
\frac{\partial^2 L}{\partial b^2} = -I_1 I_2 t^\beta x_t^2 e^{-\beta(a+bx_t)} + I_2 (1 - I_1) (\beta - 1) \frac{\tau(t - \tau)(x_2 - x_1)^2 e^{b(x_2-x_1)}}{\tau e^{b(x_2-x_1)} + t - \tau} \\
- I_2 (1 - I_1) \beta A(t)^{\beta - 2} \left( \frac{\partial A(t)}{\partial b} \right)^2 - I_2 (1 - I_1) \beta A(t)^{\beta - 1} \frac{\partial^2 A(t)}{\partial b^2} \\
- (1 - I_2) \beta A(t_e)^{\beta - 2} \left( \frac{\partial A(t_e)}{\partial b} \right)^2 - (1 - I_2) \beta A(t_e)^{\beta - 1} \frac{\partial^2 A(t_e)}{\partial b^2},
\]

\[
\frac{\partial^2 L}{\partial \beta \partial a} = -I_2 + I_1 I_2 \beta e^{-\beta(a+bx_t)} (\ln t - a - bx_t) + I_1 I_2 t^\beta e^{-\beta(a+bx_t)} \\
+ I_2 (1 - I_1) \beta A(t)^\beta \ln A(t) + I_2 (1 - I_1) A(t)^\beta \\
+ (1 - I_2) \beta A(t_e)^\beta \ln A(t_e) + (1 - I_2) A(t_e)^\beta,
\]

\[
\frac{\partial^2 L}{\partial \beta \partial b} = -I_1 I_2 b x_t + I_1 I_2 \beta x_t t^\beta e^{-\beta(a+bx_t)} (\ln t - a - bx_t) + I_1 I_2 x_t t^\beta e^{-\beta(a+bx_t)} \\
- I_2 (1 - I_1) x_2 + I_2 (1 - I_1) \frac{\tau(x_2 - x_1) e^{b(x_2-x_1)}}{\tau e^{b(x_2-x_1)} + t - \tau} \\
- I_2 (1 - I_1) A(t)^{\beta - 1} (\beta \ln A(t) + 1) \frac{\partial A(t)}{\partial b} - (1 - I_2) A(t_e)^{\beta - 1} (\beta \ln A(t_e) + 1) \frac{\partial A(t_e)}{\partial b},
\]

\[
\frac{\partial^2 L}{\partial a \partial b} = -I_2 + I_1 I_2 \beta^2 A(t)^{\beta - 1} \frac{\partial A(t)}{\partial b} \\
+ (1 - I_2) \beta^2 A(t_e)^{\beta - 1} \frac{\partial A(t_e)}{\partial b}.
\]

The elements of the Fisher information matrix for an observation are the negative expected values of the second partial derivatives:
\[
E_{\frac{\partial^2 L}{\partial \beta^2}} = e^{-\beta(a+b)\lambda_1} \int_0^t \beta \left( \ln t - a - bx_1 \right)^2 f(t) dt + \frac{F(t_c)}{\beta^2} + A(t_c)^\beta \left( \ln A(t_c) \right)^2 R(t_c),
\]

\[
E_{\frac{\partial^2 L}{\partial \alpha^2}} = \beta^2 t^\beta e^{-\beta(a+b)\lambda_1} \int_0^t \beta \ f(t) dt + \beta^2 \int_0^t A(t)^\beta \ f(t) dt + \beta^2 A(t_c)^\beta R(t_c),
\]

\[
E_{\frac{\partial^2 L}{\partial b^2}} = \beta^2 x_1^2 e^{-\beta(a+b)\lambda_1} \int_0^t \beta \ f(t) dt
- \beta^2 \int_0^t A(t)^\beta \ f(t) dt + \beta^2 A(t_c)^\beta \frac{\partial A(t_c)}{\partial b} \ f(t) dt,
\]

\[
E_{\frac{\partial^2 L}{\partial \beta \alpha}} = F(t_c) - \beta e^{-\beta(a+b)\lambda_1} \int_0^t \beta \left( \ln t - a - bx_1 \right) f(t) dt - e^{-\beta(a+b)\lambda_1} \int_0^t \beta \ f(t) dt
- \beta \int_0^t A(t)^\beta \ln A(t) f(t) dt + \int_0^t A(t)^\beta f(t) dt
- \beta A(t_c)^\beta \ln A(t_c) R(t_c) - A(t_c)^\beta R(t_c),
\]

\[
E_{\frac{\partial^2 L}{\partial \beta \beta}} = x_1 \ F(\tau) - \beta x_1 e^{-\beta(a+b)\lambda_1} \int_0^t \beta \left( \ln t - a - bx_1 \right) f(t) dt
- x_1 e^{-\beta(a+b)\lambda_1} \int_0^t \beta \ f(t) dt + x_2 \left( F(t_c) - F(\tau) \right)
- \tau(x_2 - x_1) e^{b(x_2-x_1)} \int_0^t \frac{1}{\tau e^{b(x_2-x_1)}} + t - \tau f(t) dt
+ \int_0^t A(t)^{\beta-1} \left( \beta \ln A(t) + 1 \right) \frac{\partial A(t)}{\partial b} f(t) dt
+ A(t_c)^{\beta-1} \left( \beta \ln A(t_c) + 1 \right) \frac{\partial A(t_c)}{\partial b} R(t_c),
\]
The Fisher information matrix for the n samples is given by

$$E \left[ \frac{\partial^2 L}{\partial \beta \partial \beta} \right] = \beta^2 x_i e^{-\beta(a + bx_i)} \int_0^t t^\mu f(t) dt - \beta^2 \int_0^\tau A(t)^{\beta - 1} \frac{\partial A(t)}{\partial b} f(t) dt$$

$$- \beta^2 A(t_c)^{\beta - 1} \frac{\partial A(t_c)}{\partial b} R(t_c).$$

The Fisher information matrix for the n samples is given by

$$F = n \begin{bmatrix}
E \left[ \frac{\partial^2 L}{\partial \beta^2} \right] & E \left[ \frac{\partial^2 L}{\partial \beta \partial a} \right] & E \left[ \frac{\partial^2 L}{\partial \beta \partial b} \right] \\
E \left[ \frac{\partial^2 L}{\partial \beta \partial a} \right] & E \left[ \frac{\partial^2 L}{\partial a^2} \right] & E \left[ \frac{\partial^2 L}{\partial a \partial b} \right] \\
E \left[ \frac{\partial^2 L}{\partial \beta \partial b} \right] & E \left[ \frac{\partial^2 L}{\partial a \partial b} \right] & E \left[ \frac{\partial^2 L}{\partial b^2} \right]
\end{bmatrix}.$$  

The asymptotic variance-covariance matrix for MLE(\(\hat{\beta}, \hat{a}, \hat{b}\)) is defined as the inverse matrix of the Fisher information matrix

$$\sum = \begin{bmatrix}
\text{AVar}(\hat{\beta}) & \text{ACov}(\hat{\beta}, \hat{a}) & \text{ACov}(\hat{\beta}, \hat{b}) \\
\text{ACov}(\hat{\beta}, \hat{a}) & \text{AVar}(\hat{a}) & \text{ACov}(\hat{a}, \hat{b}) \\
\text{ACov}(\hat{\beta}, \hat{b}) & \text{ACov}(\hat{a}, \hat{b}) & \text{AVar}(\hat{b})
\end{bmatrix} = F^{-1}.$$  

Because

$$t_p(x_0) = \exp(a + bx_0) [\ln(1 - p)]^{1/\beta},$$

the asymptotic variance of the MLE of the pth percentile at normal operating condition is then given by

$$\text{AVar}(\hat{t}_p(x_0)) = \begin{bmatrix}
\frac{\partial \hat{t}_p(x_0)}{\partial \beta} & \frac{\partial \hat{t}_p(x_0)}{\partial a} & \frac{\partial \hat{t}_p(x_0)}{\partial b}
\end{bmatrix} \sum \begin{bmatrix}
\frac{\partial \hat{t}_p(x_0)}{\partial \beta} & \frac{\partial \hat{t}_p(x_0)}{\partial a} & \frac{\partial \hat{t}_p(x_0)}{\partial b}
\end{bmatrix}^T,$$

according to the delta method, where
\[
\frac{\hat{\beta}_p(x_0)}{\partial \beta} = -\hat{\beta}_p(x_0) \ln[-\ln(1 - p)] \hat{\beta}^{-2}
\]

\[
\frac{\hat{\beta}_p(x_0)}{\partial \theta} = \hat{\beta}_p(x_0)
\]

\[
\frac{\hat{\beta}_p(x_0)}{\partial \theta} = \hat{x}_0 \hat{\beta}_p(x_0)
\]
APPENDIX C: R CODE FOR ML ANALYSIS

a=35.4715
b=-3.9808
beta=0.8468
t1<-c(8,38,72,122,140,163,170,188,198,223,256,257,265,448)
x1=7.1
x2=7.9
tau=600
tc=780
N1=15
N2=38
n=40
p=0.01
x0=5
B1<-exp(a+b*x1)
B2<-exp(a+b*x2)
A<-tau/B1+tc/B2-tau/B2
B3<-exp(b*(x2-x1))

BB1<-t1^beta*exp(-beta*(a+b*x1))*(log(t1*exp(-(a+b*x1)),base=exp(1)))^2
BB2<-A^beta*(log(A,base=exp(1)))^2
AC<-tau/B1+tc/B2-tau/B2
BB<-(-n2/beta^2)-sum(BB1)-sum(BB2)-(n-n2)*AC^beta*(log(AC,base=exp(1)))^2

AA1<-t1^beta*exp(-beta*(a+b*x1))*beta^2
AA2<-beta^2*A^beta
AA<-(-sum(AA1))-sum(AA2)-beta^2*(n-n2)*AC^beta

bb1<-t1^beta*exp(-beta*(a+b*x1))*(beta*x1)^2
bb2<-(tau*(t2-tau)*B3*(x2-x1)^2)/((tau*B3+t2-tau)^2)
bb3<-beta*(beta-1)*A^beta-2*(-tau*x1/B1-t2*x2/B2+tau*x2/B2)^2
bb4<-beta*A^beta-1*(tau*x1^2/B1+t2*x2^2/B2-tau*x2^2/B2)
bb<-(-sum(bb1))+(beta-1)*sum(bb2)-sum(bb3)-sum(bb4)-(n-n2)*beta*(beta-1)*AC^beta-2*(-tau*x1/B1-tc*x2/B2+tau*x2/B2)^2-(n-n2)*beta*AC^beta-1*(tau*x1^2/B1+tc*x2^2/B2-tau*x2^2/B2)

BA1<-t1^beta*exp(-beta*(a+b*x1))*(log(t1*exp(-(a+b*x1)),base=exp(1)))^beta
BA2<-t1^beta*exp(-beta*(a+b*x1))
BA3<-beta*A^beta*(log(A,base=exp(1)))
BA4<-A^beta
BA<-(-n2)+sum(BA1)+sum(BA2)+sum(BA3)+sum(BA4)+(n-n2)*beta*AC^beta*(log(AC,base=exp(1)))+(n-n2)*AC^beta

Bb1<-(tau*(x2-x1)*B3)/(tau*B3+t2-tau)
Bb2<-t1^beta*exp(-beta*(a+b*x1))*(log(t1*exp(-(a+b*x1)),base=exp(1))*beta*x1
Bb3<-t1^beta*exp(-beta*(a+b*x1))*x1
Bb4<-(beta*A^beta-1)*log(A,base=exp(1)))*(-tau*x1/B1-t2*x2/B2+tau*x2/B2)
Bb5<-A^beta-1)*(-tau*x1/B1-t2*x2/B2+tau*x2/B2)
Bb<-(-n1*x1)-(n2-n1)*x2+sum(Bb1)+sum(Bb2)+sum(Bb3)-sum(Bb4)-sum(Bb5)-(n-n2)*beta*AC^beta-1)*log(AC,base=exp(1)))*(-tau*x1/B1-tc*x2/B2+tau*x2/B2)-(n-n2)*AC^beta-1)*(-tau*x1/B1-tc*x2/B2+tau*x2/B2)

Ab1<-t1^beta*exp(-beta*(a+b*x1))*beta^2*x1
Ab2<-beta^2*A^beta-1)*(-tau*x1/B1-t2*x2/B2+tau*x2/B2)
Ab<-(-sum(Ab1))+sum(Ab2)+beta^2*(n-n2)*AC^beta-1)*(-tau*x1/B1-tc*x2/B2+tau*x2/B2)

F<-matrix(c(-BB,-BA,-Bb,-AA,-Ab,-Bb,-Ab,-bb),3,3)
INVF<-solve(F)

tp=exp(a+b*x0)*(-log((1-p),base=exp(1)))/(1/beta)

V<-matrix(c(tb,ta,tB),1,3)
TV<-t(V)

AVartp<-V%*%INVF%*%TV
APPENDIX D: WINBUGS CODE FOR BAYESIAN ANALYSIS

model{
  C <- 10000000
  for(i in 1:n){
    zeros[i] <- 0
    phi[i] <- -log.L[i] + C
    I1[i] <- step(tau-t[i])
    I2[i] <- step(tc-t[i])
    comp1[i] <- log(beta) - beta*log(theta1) + (beta-1)*log(t[i]) - pow(t[i]/theta1, beta)
    comp2[i] <- log(beta) - beta*log(theta2) + (beta-1)*log(tau*exp(b*(x2-x1))+t[i]-tau) - pow(tau/theta1+t[i]/theta2-tau/beta)
    comp3[i] <- pow(tau/theta1+tc/theta2-tau/beta)
    log.L[i] <- I1[i]*I2[i]*comp1[i] + I2[i]*(1-I1[i])*comp2[i] + (1-I2[i])*comp3[i]
    zeros[i] ~ dpois(phi[i])
  }
}

tp <- theta0* pow( -log(1-p), (1/beta) )

#prior distribution
beta ~ dgamma(1, 0.2)
a ~ dnorm(0, 0.000001)
b ~ dnorm(0, 0.000001)

theta1 <-exp(a+b*x1)
theta2 <-exp(a+b*x2)
theta0 <-exp(a+b*x0)
APPENDIX E: R CODE FOR MAXIMUM LIKELIHOOD METHOD OF PLANING

\begin{verbatim}
a=23
b=-2
beta=0.66
x1=6.5
x2=8.1
n=40
tc=4000
p=0.001
x0=1.5
tau=200

B1<-exp(a+b*x1)
B2<-exp(a+b*x2)
A<-function(t) {tau/B1+t/B2-tau/B2}
B3<-exp(b*(x2-x1))

Ft1<-function(t) {1-exp(-(t/B1)^beta)}
ft1<-function(t) {beta/B1^beta*t^(beta-1)*exp(-(t/B1)^beta)}
ft2<-function(t) {beta/B2^beta*(B3*tau+t-tau)^beta-(B3*tau+t-tau)/B2^beta}
Ft2<-function(t) {1-exp((-t/B2)^beta)}
Rtc<-exp(-(B3*tau+tc-tau)/B2)^beta

bb1<-function(t) {t^beta*(log(t,base=exp(1))-a-b*x1)^2*beta/B1^beta*t^(beta-1)*exp(-
(t/B1)^beta)}
C1<-integrate(bb1,0,tau)$value
C2<-integrate(bb2,tau,tc)$value

Ebb<-1/B1^beta*C1+C2+(1-exp(-(B3*tau+tc-
tau)/B2)^beta))/beta^2+A(t=tc)^beta*(log(A(t=tc),base=exp(1)))^2*Rtc
Ebb<-1/B1^beta*C1+C2+(Ft2(t=tc)/beta^2)+A(t=tc)^beta*(log(A(t=tc),base=exp(1)))^2*Rtc

aa1<-function(t) t^beta*beta/B1^beta*t^(beta-1)*exp(-(t/B1)^beta)
D1<-integrate(aa1,0,tau)$value
\end{verbatim}
aa2 <- function(t) (tau/B1 + t/B2 - tau/B2)^beta*beta/(B3*tau + t - tau)^((beta-1)*exp(-(B3*tau + t - tau)/B2)^beta)
D2 <- integrate(aa2, tau, tc) $value
Eaa <- beta^2/B1^beta*D1 + beta^2*D2 + beta^2*(A(tc))^beta*Rtc

Ab <- function(t) -tau*x1/B1 - x2*t/B2 + tau*x2/B2
A2b2 <- function(t) tau*x1^2/B1 + x2^2*t/B2 - tau*x2^2/B2

BB1 <- function(t) (t - tau)/(tau*B3 + t - tau)^2*(beta/B2^beta*(B3*tau + t - tau)^(beta-1)*exp(-(B3*tau + t - tau)/B2)^beta))
E1 <- integrate(BB1, tau, tc) $value
BB2 <- function(t) (tau/B1 + t/B2 - tau/B2)^beta*(log((tau/B1 + t/B2 - tau/B2), base=exp(1)))*(beta/B2^beta*(B3*tau + t - tau)^(beta-1)*exp(-(B3*tau + t - tau)/B2)^beta))
E2 <- integrate(BB2, tau, tc) $value
BB3 <- function(t) (tau/B1 + t/B2 - tau/B2)^beta*(beta/(B3*tau + t - tau)^(beta-1)*exp(-(B3*tau + t - tau)/B2)^beta))
E3 <- integrate(BB3, tau, tc) $value
EBB <- beta^2*x1^2/B1^beta*D1 - (beta-1)*tau*(x2 - x1)^2*B3*E1 + beta*(beta-1)*E2 + beta*E3 + beta*(beta-1)^2*Ab(tc)^2*Rtc + beta*(A(tc))^2*Rtc

ba1 <- function(t) {t^beta*(log(t, base=exp(1)) - a - b*x1)*beta/B1^beta*t^(beta-1)*exp(-(t/B1)^beta)}
F1 <- integrate(ba1, 0, tau) $value
ba2 <- function(t) (tau/B1 + t/B2 - tau/B2)^beta*(log((tau/B1 + t/B2 - tau/B2), base=exp(1)))*beta/B2^beta*(B3*tau + t - tau)^(beta-1)*exp(-(B3*tau + t - tau)/B2)^beta))
F2 <- integrate(ba2, tau, tc) $value
Eba <- Ft2(t=tc) - beta/B1^beta*D1 - beta*F2 - D2 - beta*(A(t=tc))^beta*(log(A(t=tc), base=exp(1)))*Rtc - A(t=tc)^beta*Rtc

bB1 <- function(t) 1/(tau*B3 + t - tau)^(beta/B2^beta*(B3*tau + t - tau)^(beta-1))
G1 <- integrate(bB1, tau, tc) $value
bB2 <- function(t) (tau/B1 + t/B2 - tau/B2)^beta*log((tau/B1 + t/B2 - tau/B2), base=exp(1))^1 + 1)*(-tau*x1/B1 - x2*t/B2 + tau*x2/B2)^(beta/B2^beta*(B3*tau + t - tau)^(beta-1)*exp(-(B3*tau + t - tau)/B2)^beta))
G2 <- integrate(bB2, tau, tc) $value
Ebb <- x1^2/Ft1(tau) - beta*x1/B1^beta*D1 + x2*(Ft2(t=tc) - Ft1(tau)) - tau*(x2 - x1)^2*B3^G1 + G2 + (A(tc))^beta*l^beta*(beta*log(A(tc), base=exp(1)) + 1)*Ab(tc)^beta*Rtc
abl <- function(t) (tau/B1+t/B2-tau/B2)^(beta-1)*(-tau*x1/B1-
  x2*t/B2+tau*x2/B2)*(beta/B2^beta*(B3*tau+t-tau)^(beta-1)*exp(-(B3*tau+t-
tau)/B2)^beta))
H1 <- integrate(abl, tau, tc) $value
EaB <- beta^2*x1/B1^beta*D1-beta^2*H1-beta^2*(A(tc))^(beta-1)*Ab(tc)*Rtc

F <- n*matrix(c(Ebb, Eba, EbB, Eba, Eaa, EaB, EbB, EaB, EBB), 3, 3)
INVF <- solve(F)

tp = exp(a+b*x0)*(-log((1-p), base=exp(1)))^(1/beta)
tb <- -tp*log(-log((1-p), base=exp(1)), base=exp(1))/beta^2
ta <- tp
tB <- x0*tp
V <- matrix(c(tb, ta, tB), 1, 3)
TV <- t(V)
AVartp <- V %*% INVF %*% TV
AVartp
APPENDIX F: R AND WINBUGS CODE FOR BAYESIAN PLANING

R Code:

```r
simSSATu<-function(n,theta1,theta2,beta, tau, u){
  F_tau <- 1 - exp(-((tau/theta1)^beta))
  t<-matrix(0,n,1)
  for (i in 1:n) {
    if (u[i]<=F_tau){
      t[i] <- theta1*(-log(1-u[i]))^(1/beta)
    } else {
      tprime <-theta2*(-log(1-u[i]))^(1/beta)
      t[i] <- tprime + tau - tau*theta2/theta1
    }
  }
  t <- sort(t)
}

n<-40
p<-0.001
tc<-4000
M1_l=946
M1_u=1346
M2_l=470
M2_u=560
beta_l=0.64
beta_u=0.68
x0<-1.5
x1<-7.7
x2<-8.1
cp<-c(200, 1000, 1500, 1650, 1700, 1750, 1800, 2500, 3000, 3300, 3800)
cmp<-length(cp)
Ev<-array(0,c(ncp,2))
nsample<-200
niter<-80000
set.seed(100)
for (i in 1:nsample){
  M1 <-runif(1,M1_l, M1_u)
  M2 <-runif(1,M2_l, M2_u)
  beta <-runif(1,beta_l, beta_u)
  theta1<-M1 / ((log(2))^1/beta))
  theta2<-M2 / ((log(2))^1/beta))
```
for (j in 1:ncp){
  tau<-cp[j]
  t<-simSSATu(n,theta1,theta2,beta,tau,u)
  n1<-sum(t<=tau)
  n2<-sum(t<=tc)
  data<-list("n","n1","n2","t","p","tau","tc","x0","x1","x2","M1_l","M1_u","M2_l","M2_u","beta _l","beta_u")
  parameters<-c("tp")
  inits<-list(list(M1=1146,M2=515,beta=0.66))
  SSAT.sim<-bugs(data,inits,parameters,model.file="C:/WeibullSSAT2n.txt",n.chains=1, n.iter=niter, n.burnin=40000, n.thin=1, bugs.directory="C:/WinBUGS14", DIC=FALSE)
  attach.bugs(SSAT.sim)
  sd_tp <- sd(tp)
  m_tp <- mean(tp)
  detach.bugs()
  Ev[j,1]<-Ev[j,1]+sd_tp^2
  Ev[j,2]<-Ev[j,2]+sd_tp/m_tp
  print(i)
  print(j)
  flush.console()
}
print(Ev)
flush.console()
}
Ev <- Ev / (nsample)
print(cp)
print(Ev)

WinBUGS code:

model{
  C <- 1000000
  for(i in 1:n1){
    zeros[i] <- 0
    phi[i] <- -log.L[i] + C
    log.L[i] <- log(beta) - beta*log(theta1) + (beta-1)*log(t[i]) - pow(t[i]/theta1, beta)
    zeros[i] ~ dpois(phi[i])
  }

  for(i in (n1+1):n2){
    zeros[i] <- 0
  }
\[
\begin{align*}
\phi[i] & \leftarrow \text{log.L}[i] + C \\
\text{log.L}[i] & \leftarrow \text{log}(\text{beta}) - \text{beta} \times \text{log(\text{theta2})} + (\text{beta}-1) \times \text{log(\text{tau} \times \text{theta2} / \text{theta1} + t[i] \times \text{tau})} \\
& \quad - \text{pow}(\text{tau} / \text{theta1} + t[i] / \text{theta2} - \text{tau} / \text{theta2}, \text{beta}) \\
\text{zeros}[i] & \sim \text{dpois}(\phi[i]) \\
\end{align*}
\]

\[
\begin{align*}
\text{for(i in (n2+1):n) { \\
\quad \text{zeros}[i] & \leftarrow 0 \\
\quad \phi[i] & \leftarrow \text{log.L}[i] + C \\
\quad \text{log.L}[i] & \leftarrow \text{pow}(\text{tau} / \text{theta1} + t[i] / \text{theta2} - \text{tau} / \text{theta2}, \text{beta}) \\
\quad \text{zeros}[i] & \sim \text{dpois}(\phi[i]) \\
\}}
\end{align*}
\]

\[
\begin{align*}
\text{tp} & \leftarrow \exp(a + b \times x0) \times \text{pow}(\text{-log}(\text{1-p}), (1 / \text{beta})) \\
\text{theta1} & \leftarrow M1 / \text{pow}(\text{log}(2), 1 / \text{beta}) \\
\text{theta2} & \leftarrow M2 / \text{pow}(\text{log}(2), 1 / \text{beta}) \\
\text{b} & \leftarrow (\text{log(\text{theta1})} - \text{log(\text{theta2})}) / (x1 - x2) \\
\text{a} & \leftarrow \text{log(\text{theta1})} - \text{b} \times x1 \\
\text{#prior distribution} \\
\text{beta} & \sim \text{dunif}(\text{beta}_1, \text{beta}_u) \\
\text{M1} & \sim \text{dunif}(\text{M1}_1, \text{M1}_u) \\
\text{M2} & \sim \text{dunif}(\text{M2}_1, \text{M2}_u)
\end{align*}
\]