Subtractive Renormalization of the NN Interaction in Chiral Effective Theory and the Deuteron Electro-disintegration Calculation

A dissertation presented to
the faculty of
the College of Arts and Sciences of Ohio University

In partial fulfillment
of the requirements for the degree
Doctor of Philosophy

Chieh Jen Yang
August 2010
© 2010 Chieh Jen Yang. All Rights Reserved.
This dissertation titled
Subtractive Renormalization of the NN Interaction in Chiral Effective Theory and the
Deuteron Electro-disintegration Calculation

by

CHIEH JEN YANG

has been approved for
the Department of Physics and Astronomy
and the College of Arts and Sciences by

Daniel R. Phillips
Professor of Physics and Astronomy

Benjamin M. Ogles
Dean, College of Arts and Sciences
ABSTRACT

YANG, CHIEH JEN, Ph.D., August 2010, Department of Physics and Astronomy

Subtractive renormalization of the NN interaction in chiral effective theory

and the deuteron electro-disintegration calculation (273 pp.)

Director of Dissertation: Daniel R. Phillips

We develop a subtractive renormalization scheme to evaluate the NN scattering phase shifts using chiral effective theory potentials. This allows us to consider arbitrarily high cutoffs in the Lippmann-Schwinger equation. We employ NN potentials computed up to next-to-next-to-leading order (NNLO) in chiral effective theory, using both dimensional regularization and spectral-function regularization. We evaluate the cutoff-dependence and the renormalization point dependence of our results, and apply them to the calculation of the process of deuteron electro-disintegration.

Approved: ____________________________________________

Daniel R. Phillips

Professor of Physics and Astronomy
To my parents, teachers and friends
Acknowledgments

First, I would like to thank my advisor Prof. Daniel Phillips for his guidance in every stage of my research, as it is impossible to complete this thesis without his help. Second, I want to thank Prof. Charlotte Elster for her continuously support and guidance, which plays a crucial role in my research. I am also grateful to my committee members Prof. Carl Brune and Prof. Todd Young, who have gave me valuable suggestions. Another person I would like to express my deep appreciation is Prof. Madappa Prakash, his strong support and wise guidance has helped me through all stages in my research. In addition, I want to thank Prof. Hartmuth Arenhovel for providing his results of deuteron electro-disintegration for comparison and his valuable suggestions.

In my early years at Ohio University, I benefited from many wonderful teachers, which include Prof. Horacio Castillo, Prof. Carl Brune, Prof. Ken Hicks, Prof. Peter Jung, Prof. Daniel Carman and Prof. Markus Bottcher. Also, I want to thank Don Roth, Tracy Inman and Ennice Sweigart for their support in computational and administrative issues. Then, in the middle stage of my research, I benefited from serveral nuclear theory group members at OU, which includes Prof. Louis Wright, Deepshikha Shukla, Matthias R. Schindler, Lucas Platter and Prashanth Jaikumar. In the final stage of my research, I am grateful to Deborah Aguilera and Carlos Schat for their support. Last but not the least; I would like to thank Lin Ting, Sergey, Chen Ji, Anton and Hui-chun for their friendship and support.
This work is supported by Grants DE-FG02-93ER40756 of the US-DOE, and the numerical calculation is supported by Ohio Supercomputer Center (OSC) under grant PHS206.
Table of Contents

Abstract .................................................. 3
Dedication .................................................. 4
Acknowledgments .......................................... 5
List of Figures ........................................... 9
List of Tables ........................................... 22
1 Introduction ............................................. 24
2 Chiral perturbation theory and effective potential .......... 41
   2.1 A brief example of chiral perturbation theory .......... 41
   2.2 Lagrangian of QCD and chiral perturbation theory ....... 43
   2.3 Chiral effective field theory in the NN sector and the Lippmann-Schwinger equation ......................... 50
   2.4 Chiral effective potential and partial-wave decomposition .... 56
3 Subtractive renormalization of chiral potential at leading order in S-waves 63
   3.1 Introduction ......................................... 63
   3.2 Subtractive method and results in S-waves ............. 67
   3.3 Subtractive Renormalization in Bound-State Calculations .... 73
4 Subtractive renormalization of chiral potentials up to next-to-next-to leading order: Theory ....................... 82
   4.1 Subtractive renormalization of NN interaction up to NNLO in P-waves 82
   4.2 Subtractive renormalization in S-waves: J=0 singlet channel .... 87
      4.2.1 Constant contact interaction ....................... 88
      4.2.2 Energy-dependent contact interaction ............... 88
      4.2.3 Momentum-dependent contact interaction .......... 93
   4.3 Subtractive renormalization in S-waves: J=1 triplet channel .... 95
      4.3.1 Constant contact interaction ....................... 96
      4.3.2 Momentum-dependent central part ................... 96
      4.3.3 Energy-dependent central part .................... 103
5 Subtractive renormalization of chiral potentials up to next-to-next-to leading order: Results and Discussion ........ 107
   5.1 P-wave results at leading order ...................... 107
List of Figures

1.1 The diagramatic representation of the perturbative and non-perturbative sum. The solid and dashed line here represent different kind of particles. ........................................... 31

2.1 The diagramatic representation of the Lippmann-Schwinger equation. 51

2.2 Reducible (a) & (c) v.s. irreducible diagrams (b) & (d) ............... 52

3.1 The running of the low energy constant $C_T$ versus cutoff $\Lambda$. Here $C_T$ is determined by fitting to the $^3S_1-^3D_1$ scattering length $a_t$. This graph is adapted from our previous publication [73]. .......................... 67

3.2 The comparison of two renormalization methods for the $^1S_0$ and $^3S_1-^3D_1$ NN phase shifts and the mixing parameter $\epsilon_1$ as a function of the laboratory kinetic energy $T_{\text{lab}} \leq 80$ MeV. Here $\Lambda = 50$ GeV is used. The phase shifts obtained from the CD-Bonn potential are also shown (dashed lines). This graph is adapted from our previous publication [73]. 72

3.3 The t-matrix elements $t^s(p, 100; -\frac{2\mu^2}{M})$ (left) and $t^{00}_0(p, 100; -\frac{2\mu^2}{M})$ (right) as functions of the half-shell momentum $p$ compared to $\nu^s(p, 100)$ and $\nu^{00}_0(p, 100)$. The parameter $\mu$ ranges from 0.8–12 GeV. This graph is adapted from our previous publication [73]. .......................... 74

3.4 The mixing parameter $\epsilon_1$ obtained with the subtraction method as a function of the laboratory kinetic energy $T_{\text{lab}} \leq 20$ MeV and for cutoff parameters $\Lambda$ ranging from 0.5–50 GeV. This graph is adapted from our previous publication [73]. .......................... 74

3.5 The momentum-space wave functions of the shallowest NN bound state in the triplet channel as a function of $p$, where $\psi_0(p)$ is the $^3S_1$ wave function and $\psi_2(p)$ denotes the $^3D_1$ wave (solid lines). These wave functions are obtained from the subtracted integral equation with $\Lambda = 50$ GeV. The dotted lines indicate the corresponding wave functions obtained from the CD-Bonn potential. This graph is adapted from our previous publication [73]. .......................... 80

3.6 The coordinate space wave functions as functions of $r$, where $u(r)$ denotes the $^3S_1$ wave and $w(r)$ denotes the $^3D_1$ wave (solid lines). The dotted lines indicate the corresponding wave functions obtained from the CD-Bonn potential. This graph is adopted from our previous publication [73]. .......................... 81

4.1 The diagramatic representation of the NLO TPE, where the solid (dashed) line represent a nucleon (pion). The blob represents the coupling of order $Q^2$. .......................... 83
4.2 The diagramatic representation of the NNLO TPE, where the solid (dashed) line represent nucleon (pion), and the blob represent the coupling of order $Q^2$.

5.1 The NN phase shifts for the channels $^1P_1$ and $^3P_1$ as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.8 to 5 GeV. In all cases the potential $v^{LR} = v_{OPE}$ enters the LSE. The panels on the left show the results without subtraction, whereas the panels on the right show the results with one subtraction. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].

5.2 The NN phase phase shifts for the channels $^3P_0$ and $^3P_2$ as well as the mixing parameter $\varepsilon_2$ as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.8 to 5 GeV. In all cases the potential $v^{LR} = v_{OPE}$ enters the LSE. The panels on the left show the results without subtraction, whereas the panels on the right show the results with one subtraction. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].

5.3 The un-renormalized NN P-wave phase shifts as a function of the laboratory kinetic energy resulting from the dimensionally regularized TPE up to NLO. The phase shifts are shown for cutoffs $\Lambda$ ranging from 0.5 to 2 GeV. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].

5.4 The NN P-wave phase shifts resulting from the use of OPE plus DR TPE at NLO with one subtraction as a function of the laboratory kinetic energy. Here the cutoff range shown is $0.5$–$10$ GeV. The input value of $\alpha_{SJ}$ is taken from Ref. [86]. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].

5.5 The NN P-wave phase shifts as a function of the laboratory kinetic energy that result from choosing $v^{LR} = v_{1\pi} + v_{2\pi}$, with the latter computed using DR TPE in NLO. Here the generalized scattering lengths $\alpha_{SJ}^{\Lambda}$ are adjusted to give the best fit in the region $T_{lab} < 100$ MeV. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. Note that this graph is adapted from our previous publication [80].

5.6 $\alpha_{best}$ versus $\Lambda$ ranging from 0.6–3 GeV for DR NLO, DR NNLO and SFR NNLO. Here $\alpha_{best}$ are adjusted to give the best fit in the region $T_{lab} < 100$ MeV. This graph is adapted from our previous publication [80].
5.7 $\alpha_{\text{best}}(3P_2)$ versus $\Lambda$ ranging from 0.6-3 GeV for DR NLO. Here $\alpha_{\text{best}}$ are adjusted to give the best fit in the region $T_{\text{lab}} < 100$ MeV. This graph is adopted from our previous publication [80].

5.8 The un-renormalized NN P-wave phase shifts as a function of the laboratory kinetic energy that result from a long-range potential of OPE plus dimensionally regularized TPE at NNLO. The phase shifts are shown for cutoffs $\Lambda$ ranging from 0.5 to 2 GeV. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].

5.9 The NN P-wave phase shifts as a function of the laboratory kinetic energy that result from the use of NNLO DR TPE and one subtraction. Here the cutoff range shown is 0.6–10 GeV. The input value of $\alpha_{11}^{SJ}$ is taken from Ref. [86]. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].

5.10 The NN P-wave phase shifts as a function of the laboratory kinetic energy that result from choosing $v^{LR} = v_1 + v_2$, with the latter chosen to be the DR TPE at NNLO, and implementing one subtraction. Here the generalized scattering lengths $\alpha_{11}^{SJ}$ are adjusted to give the best fit in the region $T_{\text{lab}} < 100$ MeV. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].

5.11 The renormalized NN P-wave phase shifts as a function of the laboratory kinetic energy resulting when $v^{LR}$ is chosen to be the OPE plus TPE in NNLO, where SFR is employed for the $O(Q^3)$ part of TPE. Here the generalized scattering lengths $\alpha_{11}^{SJ}$ are adjusted to give the best fit in the region $T_{\text{lab}} < 100$ MeV. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].

5.12 The NN P-wave phase shifts as a function of the laboratory kinetic energy that result from choosing $v^{LR} = v_1 + v_2$, with the latter chosen to be the SFR TPE up to NNLO, and implementing one subtraction. Here the generalized scattering lengths $\alpha_{11}^{SJ}$ are adjusted to give the best fit in the region $T_{\text{lab}} < 100$ MeV. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].

5.13 The un-renormalized NN P-wave phase shifts at $T_{\text{lab}} = 10$ MeV as a function of cutoff for three different $v^{LR}$. Note that this graph is adapted from our previous publication [80].
5.14 The renormalized NN P-wave phase shifts at $T_{lab} = 100$ MeV as a function of cutoff for three different $v^{LR}$. Here the generalized scattering lengths $\alpha_{best}$ are adjusted to give the best fit in the region $T_{lab} < 100$ MeV. This graph is adopted from our previous publication [80].

5.15 The $^1S_0$ NN phase shift as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 1.2 GeV. The long-range potentials employed are the DR TPE up to NLO, NNLO, MIXED NNLO and the SFR TPE NNLO (as noted in the y-axis of each figure), together with a constant contact term. The results are obtained by one subtraction with $a_0 = -23.7$ fm as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles.

5.16 The effective range $r_0$ [in fm] in the $^1S_0$ channel extracted from calculations with the DR NLO (black square), the DR NNLO (red circle), MIXED NNLO (maroon down triangle) and the SFR NNLO (green up triangle) TPE combined with: a constant contact term (upper panel), and a constant plus a momentum-dependent contact term (lower panel). In both cases $r_0$ is shown as a function of the cutoff $\Lambda$ in the LSE. In the lower panel, the coefficient of the momentum-dependent contact term is adjusted to reproduce the Nijmegen value of the phase shift at $T_{lab} = 200$ MeV. The thick solid band represents the range of $r_0$ obtained from Ref. [92]. Note that for the upper(lower) panel, the value of $r_0$ at $\Lambda = 700$ MeV for the DR NLO (SFR NNLO) potential is $-10.2(0.15)$ fm, which is not plotted in the figure.

5.17 The $^1S_0$ NN phase shift as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. The potentials employed (from top to bottom) are DR NLO, DR NNLO and SFR NNLO, together with an energy-dependent contact term. The results are obtained by two subtractions with $a_0 = -23.7$ fm and the phase shift at $T_{lab} = 2.8$ MeV as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].

5.18 The $^1S_0$ NN phase shift at $T_{lab} = 10$ MeV (upper panel) and 100 MeV (lower panel) as a function of the cutoff ranging from 0.5–19 GeV. The results are obtained using the DR NNLO potential with an energy-dependent contact term via two subtractions. This graph is adopted from our previous publication [81].

5.19 The relative difference of NN phase shift $\delta(^1S_0)$ at $T_{lab} = 350$ resulting from DR NNLO potential associated with the energy-dependent contact term with respect to Nijm93 data as function of the cutoff $\Lambda$ ranging from 0.05–2 GeV.
5.20 The $^1S_0$ NN phase shift as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 1 GeV. The DR TPE up to NLO long-range potential is employed, together with a momentum-dependent contact term. The results are obtained by making one subtraction with $a_0 = -23.7$ fm as input, and then performing a best fit to the overall phase shift as given by the Nijmegen analysis. The values of the Nijmegen $^1S_0$ phase shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].

5.21 The $^1S_0$ NN phase shift as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.5 to 2 GeV. The potential employed is the DR NNLO TPE with a momentum-dependent contact term. The results are obtained by making one subtraction with $a_0 = -23.7$ fm as input and then performing a fit to either the effective range $r_0 = 2.7$ fm (solid line) or the Nijmegen $^1S_0$ phase shift at $T_{\text{lab}} = 200$ MeV (dashed line). The values of the Nijmegen phase-shifts [87] is indicated by the open triangles. This graph is adapted from our previous publication [81].

5.22 The $^1S_0$ NN phase shift as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. The potential employed is SFR NNLO with the momentum-dependent contact term. The results are obtained by making one subtraction with $a_0 = -23.7$ fm as input and then performing either a fit to the effective range $r_0 = 2.7$ fm (solid line) or the Nijmegen $^1S_0$ phase shift at $T_{\text{lab}} = 200$ MeV (dashed line). The values of the Nijmegen phase-shifts [87] is indicated by the open triangles. This graph is adapted from our previous publication [81].

5.23 The $^1S_0$ NN phase shift as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. The potential employed is the MIXED NNLO with the momentum-dependent contact term. The results are obtained by making one subtraction with $a_0 = -23.7$ fm as input and then performing either a fit to the effective range $r_0 = 2.7$ fm (solid line) or the Nijmegen $^1S_0$ phase shift at $T_{\text{lab}} = 200$ MeV (dashed line). The values of the Nijmegen phase-shifts [87] is indicated by the open triangles.

5.24 The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. The potential employed is the DR NLO TPE with one constant contact term. The results are obtained via one subtraction with $a_0 = 5.428$ fm as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
5.25 The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs \( \Lambda \) ranging from 0.6 to 2 GeV. The potential employed is DR NNLO with a constant contact term. The results are obtained via a single subtraction with \( a_0 = 5.428 \) fm as input. The values of the Nijmegen phase-shifts [87] is indicated by the open triangles. This graph is adapted from our previous publication [81].

5.26 The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs \( \Lambda \) ranging from 0.6 to 2 GeV. Here the potential is the MIXED NNLO with a constant contact term. The results are obtained by one subtraction with \( a_0 = 5.428 \) fm as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles.

5.27 The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs \( \Lambda \) ranging from 0.6 to 2 GeV. Here the potential is SFR NNLO with a constant contact term. The results are obtained by one subtraction with \( a_0 = 5.428 \) fm as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].

5.28 The J=1 coupled NN phase shifts at \( T_{lab} = 10 \) MeV as a function of cutoff ranging from 1–8.5 GeV. The results are obtained with the SFR NNLO potential and a constant contact term. This graph is adapted from our previous publication [81].

5.29 The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs \( \Lambda \) ranging from 0.6 to 2 GeV. The potential employed is DR NNLO with a linear energy dependence in the central part of the contact term. The results are obtained by three subtractions with \( a_0 = 5.428 \) fm, \( \alpha_{20} = 2.28 \times 10^{-10} \) MeV\(^{-1}\) and the phase shift at \( T_{lab} = 10 \) MeV as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].

5.30 The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs \( \Lambda \) ranging from 0.6 to 2 GeV. The potential employed is the DR NNLO with a linear energy dependence in the central part of the contact term. The results are obtained by three subtractions with \( a_0 = 5.428 \) fm, \( \alpha_{20} = 2.25 \times 10^{-10} \) MeV\(^{-1}\) and the phase shift at \( T_{lab} = 10 \) MeV as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
5.31 The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs Λ ranging from 0.6 to 2 GeV. The potential employed is DR NLO with a linear energy dependence in the central part of the contact term. The results are obtained by three subtractions with $a_0 = 5.428$ fm, $\alpha_{20} = 2.25 \times 10^{-10}$ MeV$^{-1}$ and the phase shift at $T_{lab} = 10$ MeV as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].

5.32 The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs Λ ranging from 0.6 to 2 GeV. The potential employed is the MIXED NNLO with a linear energy dependence in the central part of the contact term. The results are obtained by three subtractions with $a_0 = 5.428$ fm, $\alpha_{20} = 2.25 \times 10^{-10}$ MeV$^{-1}$ and the phase shift at $T_{lab} = 10$ MeV as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles.

5.33 The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs Λ ranging from 0.6 to 2 GeV. The potential employed is SFR NNLO with a linear energy dependence in the central part of the contact term. The results are obtained by three subtractions with $a_0 = 5.428$ fm, $\alpha_{20} = 2.25 \times 10^{-10}$ MeV$^{-1}$ and the phase shift at $T_{lab} = 10$ MeV as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].

5.34 The J=1 coupled NN phase shifts at $T_{lab} = 50$ MeV as a function of cutoff ranging from 0.5–5.5 GeV. The results are obtained with the DR NNLO potential and a linear energy dependence in the central part of the contact term. This graph is adapted from our previous publication [81].

5.35 The J=1 coupled NN phase shifts at $T_{lab} = 10$ MeV as a function of cutoff ranging from 0.5–5.5 GeV. The results are obtained with the DR NNLO potential and a linear energy dependence in the central part of the contact term. This graph is adapted from our previous publication [81].

5.36 The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs Λ ranging from 0.6 to 2 GeV. The potential employed is DR NNLO with the momentum-dependent contact term. The results are obtained by two subtractions with $a_0 = 5.428$ fm and $\alpha_{20} = 2.25 \times 10^{-10}$ MeV$^{-1}$ as input and then performing a fit to the $^3S_1$ Nijmegen phase shift at $T_{lab} = 200$ MeV. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
5.37 The best fit for the NN $^3S_1-^3D_1$ phase shifts as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 1 GeV. The potentials employed are the DR NLO (left panel) and the DR NNLO (right panel) with a momentum-dependent central part of the contact term. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].

5.38 The best fit for the NN $^3S_1-^3D_1$ phase shifts as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 1 GeV. The potentials employed are the SFR NNLO with a momentum-dependent central part of the contact term. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].

5.39 The effective range $r_0$ [in fm] in the $^3S_1$ channel extracted from calculations with the DR NLO (black square), DR NNLO (red circle) MIXED NNLO (maroon down triangle) and SFR NNLO (green up triangle) TPE and a momentum-dependent central piece of the contact term. Here $r_0$ is shown as a function of the cutoff $\Lambda$ in the LSE, and is extracted from a best fit of the phase shifts to the Nijmegen PWA93, while the solid line represents $r_0$ obtained from Ref. [92]. Note that the value of $r_0$ at $\Lambda = 600$ MeV for DR NNLO is $-11.5$ fm, which is not plotted in the figure.

5.40 The momentum-space wave functions of the shallowest NN bound state in the triplet channel as a function of $p$, where $\psi_0(p)$ is the $^3S_1$ wave function and $\psi_2(p)$ denotes the $^3D_1$ wave function. These wave functions are obtained from the DR TPE up to NLO (black solid line), DR NNLO (green dashed line) and SFR NNLO (blue dash-dotted line) with the momentum-dependent contact term, where the cutoff in the LSE is $\Lambda = 800$ MeV. The red dot indicate the corresponding wave functions obtained from the CD-Bonn potential.

5.41 The relative difference of the leading-order NN $\delta(^1S_0)$ and $\delta(^3S_1)$ phase shift between two renormalization methods as a function of the laboratory kinetic energy $T_{lab} \leq 80$ MeV. Here $\Lambda = 50$ GeV is used, and 100 Gaussian-Legendre quadrature points are chosen for the solution of the LS equation. Note that this graph is adapted from our previous publication [73].

6.1 The kinematic description of the deuteron electro-disintegration process. The labels are specified in the text below.

6.2 The impulse approximation diagrams for the deuteron electro-disintegration. The blob in the right of each diagram represents the deuteron state.
6.3 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$ for four different kinematic regions. The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.

6.4 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.

6.5 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.

6.6 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.

6.7 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.

6.8 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO. 

The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO. 

The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO. 

The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
6.15 The deuteron electro-disintegration longitudinal response function \( f_L \) as a function of \( \theta \). Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff \( \Lambda = 600 - 1000 \) MeV in the LSE. The intrinsic cutoff is \( \tilde{\Lambda} = 800 \) MeV for the SFR TPE up to NNLO. 227

6.16 The deuteron electro-disintegration longitudinal response function \( f_L \) as a function of \( \theta \). Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff \( \Lambda = 600 - 1000 \) MeV in the LSE. The intrinsic cutoff is \( \tilde{\Lambda} = 800 \) MeV for the SFR TPE up to NNLO. 228

6.17 The deuteron electro-disintegration longitudinal response function \( f_L \) as a function of \( \theta \). Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff \( \Lambda = 600 - 1000 \) MeV in the LSE. The intrinsic cutoff is \( \tilde{\Lambda} = 800 \) MeV for the SFR TPE up to NNLO. 230

6.18 The deuteron electro-disintegration longitudinal response function \( f_L \) as a function of \( \theta \). Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff \( \Lambda = 600 - 1000 \) MeV in the LSE. The intrinsic cutoff is \( \tilde{\Lambda} = 800 \) MeV for the SFR TPE up to NNLO. 231

6.19 The deuteron electro-disintegration longitudinal response function \( f_L \) as a function of \( \theta \). Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff \( \Lambda = 600 - 1000 \) MeV in the LSE. The intrinsic cutoff is \( \tilde{\Lambda} = 800 \) MeV for the SFR TPE up to NNLO. 232

6.20 The deuteron electro-disintegration longitudinal response function \( f_L \) as a function of \( \theta \). Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff \( \Lambda = 600 - 1000 \) MeV in the LSE. The intrinsic cutoff is \( \tilde{\Lambda} = 800 \) MeV for the SFR TPE up to NNLO. 233
6.21 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.

6.22 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.

6.23 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.

6.24 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.

6.25 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.

6.26 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
6.27 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO. 

6.28 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO. 

6.29 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO. 

6.30 The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
List of Tables

3.1 Effective range $r_s$ (singlet) and $r_t$ (triplet) obtained using the subtraction method for various cutoffs $\Lambda$. This table is adapted from our previous publication [73]. ........................................... 75

3.2 The constants $C_S$ and $C_T$ as determined by fitting the scattering lengths $a_s$ and $a_t$ for different cutoff parameters $\Lambda$. This table is adapted from our previous publication [73]. .......................... 75

3.3 The binding energy $B$ for different cutoff parameters $\Lambda$. The binding energy $B_{fit}$ is obtained from the fitting method, and the binding energy $B_{sub}$ is obtained from the subtractive method. Note that this table is adapted from our previous publication [73]. ............................. 77

5.1 Need for renormalization in different NN P-waves when the $\chi$ET potential is calculated to different orders. “U” implies that the potential does not need a subtraction to generate cutoff-independent phase shifts as $\Lambda \rightarrow \infty$ while “R” means that a subtraction is required in that channel at that order. The * indicates that at NNLO (DR) the $^3P_2-^3F_2$ channel actually requires two subtraction to be made stable. This table is adapted from our previous publication [80]. ............... 124

5.2 The generalized P-wave scattering lengths $\alpha_{11}^{S,J}$ as given in Ref. [86] for the NijmII and Reid93 potentials, together with ones extracted from the CD-Bonn [15] and AV18 [17] potentials. $\alpha_{11}^{S,J}$ is given in units fm$^3$. This table is adapted from our previous publication [80]. ............... 125

5.3 The effective range $r_0$ [in fm] in the $^1S_0$ channel extracted from calculations with the DR NLO, DR NNLO, MIXED NNLO and SFR NNLO long-range potentials and a constant contact term. $r_0$ is shown as a function of the cutoff $\Lambda$ in the LSE. ................................. 136

5.4 The effective range $r_0$ [in fm] in the $^1S_0$ channel extracted from calculations with the DR NLO, DR NNLO, SFR NNLO and MIXED NNLO potentials and a momentum-dependent contact interaction. The phase shift is fitted at $T_{lab} = 200$ MeV to the Nijmegen value. ........................ 136

5.5 Cutoffs $\Lambda$ v.s. C.M. frame resonance or binding energy due to the DR NNLO TPE. Here the postive value refers to resonance energy $E_s$, and negative vaule refers to binding energy $E_b$. ............................ 143

5.6 Value of the generalized scattering length $\alpha_{20}$ extracted from various potentials, in units of $10^{-10}$ MeV. Note that this table is adapted from our previous publication [81]. ................................. 161
5.7 The effective range $r_0$ [in fm] in the $^3S_1$ channel extracted from calculations with the DR NLO, DR NNLO, MIXED NNLO and SFR NNLO potentials and the momentum-dependent central piece of the contact term. Here $r_0$ is shown as a function of the cutoff $\Lambda$ in the LSE, and is extracted from a best fit of the phase shifts to the Nijmegen PWA93. Note that the Nijmegen analysis gives $r_0 = 1.83$ fm [86].

5.8 The binding energy $E_B$ obtained from the LO, DR NLO, DR NNLO and SFR NNLO potentials. Here $\Lambda = 50$ GeV for the LO potential, and $\Lambda = 800$ MeV for the other three.

5.9 The convergence of the phase shift $\delta(^3S_1)$ at laboratory kinetic energy $T_{\text{lab}}=10$ MeV as a function of the number of Gaussian-Legendre quadratures for the fitting and subtraction methods. Here the leading-order chiral potential is adopted. Note that the fitting method converges to 104.1 at about 200 mesh points, so the final results of these two methods differ by $\approx 0.5\%$. This table is adapted from our previous publication [73].

5.10 The convergence of the LEC $\lambda$ and $\lambda_t$ as a function of the number of Gaussian-Legendre quadratures for the “subtraction plus fitting method”. Here the DR NNLO TPE is adopted in the calculation.
Chapter 1

Introduction

The purpose of physics is to not only understand the fundamental ideas behind nature, but also to actually perform calculations and make predictions of phenomena which are observed in experiments. Among various sub-fields in physics, nuclear physics plays a special role. From a practical point of view, research in nuclear physics helps to explain how our world is built via the basic building blocks—protons and neutrons—and what their inner structure is. On the theoretical side, the efforts toward establishing an effective field theory for the nuclear system lead us to a deeper understanding of how one (effective) field theory emerges from a mother theory, and the key of it—renormalization.

Today, all physical systems (except gravity) in nature can be described by a theory which combines special relativity and quantum mechanics, i.e., quantum field theory. The quantum field theory which describes the electro-weak interaction is the so-called Quantum Electrodynamics (QED) (with electro-weak unification terms), and the quantum field theory for the strong force is Quantum Chromodynamics (QCD). In both theories, fundamental particles are treated as quantized fields, and forces are described by the exchange of fundamental bosons. To describe a process (e.g. scattering of two particles), one first specifies the initial and final state, and then sums over all the possible intermediate states. This allows us to get the amplitude which
represents the probability with which this physical process can happen. According to quantum mechanics, there are in principle infinitely many intermediate states to be summed over. However, in practice one solves the problem by performing approximations, i.e., one truncates the infinite series and sums over terms up to a certain order. This approach works if the system is perturbative, i.e., the series can be arranged as a power series of a small expansion parameter. For example, in QED this parameter is the fine structure constant \( \alpha \) \( (\alpha = \frac{e^2}{4\pi\hbar c}, \text{where } e \text{ is the charge of electron, } \hbar = \frac{\hbar}{2\pi}, \hbar \text{ is Planck’s constant and } c \text{ is the speed of light}), \) with \( \sqrt{\alpha} \) representing the strength of the coupling between an electron and photon. For low-energy \(< 1 \text{ GeV}) interactions, \( \alpha \approx \frac{1}{137}. \)

However, coupling constants in quantum field theories are not actually constants, they change with the interaction energy. This is associated with the fact that instead of being an ultimate theory, every quantum field theory that we know is actually an effective field theory (EFT). An EFT can be considered as a theory which is an expansion of the full theory at a particular point of energy. In order to account for the physics missed in the theory, i.e., physics in energy scales beyond the applicable range of the EFT, one performs a procedure—the so-called renormalization—to absorb the unknown part of physics into the coupling constants. In this procedure, counter (contact) terms are added to the Lagrangian to represent physics missed in the current theory. A renormalizable theory has a finite number of coupling constants to be determined by the renormalization procedure, and therefore requires only a finite
number of inputs, i.e., physical observables, from the physical world, to pin down all the unknowns in the theory. However, a non-renormalizable theory has infinitely many constants to be fitted and always requires new physical observables as inputs when one includes the higher order terms. Both QED and QCD are renormalizable. On the other hand, the low-energy EFT of QCD—chiral perturbation theory (\(\chi\)PT)—is non-renormalizable.

Why do we bother ourselves with non-renormalizable EFTs such as \(\chi\)PT? The answer is that when the original renormalizable theory becomes non-perturbative, one way to enable the calculation is to establish an EFT, regardless of whether this EFT is renormalizable or not\(^1\). In a microscopic view, nuclei consist of quarks. And the strong interaction between two quarks is described by their gluon exchange. The underlying theory which governs the strong interaction is QCD—which has been verified by numerous high energy experiments. However, as mentioned before, the coupling constant of QCD runs, and due to the non-abelian nature of QCD, its strength increases as energy decreases. As a result, performing calculations directly based on QCD becomes more and more difficult as the interaction energy is lower. Due to the growth of the coupling constant, more and more terms in the series need to be summed before a reasonable, converged result can be reached. Finally, QCD becomes non-perturbable for energy lower than 1 GeV and quarks are tightly bound to form hadrons. This is the so-called confinement phenomenon of QCD.

\(^{1}\)Renormalizability is not actually a criteria to judge the usefulness of a theory. Strictly speaking, it could be that all quantum field theories are non-renormalizable, as argued in Ref. [1].
Hadrons can be further divided into baryons and mesons, where baryons consist of three quarks and mesons consist of one quark and one anti-quark. The most stable baryons are the nucleons, i.e., protons and neutrons, and the most stable mesons are pions. Since nuclei are built out of neutrons and protons, one has to deal with the confinement phenomenon when performing calculations in nuclear physics. To calculate the pion-nucleon ($\pi N$) or nucleon-nucleon (NN) interaction from QCD is not an easy job. Although we have QCD as the fundamental theory, it is not calculatable at low energy. As a consequence, we cannot treat the hadronic interaction in terms of quarks and gluons unless one employs lattice gauge theory and discretizes space and time. For this situation, an EFT which approximates physics of QCD at low energy is desirable.

An accurate and comprehensive description of the low-energy NN interaction is the basic starting point for the calculation of light as well as heavy nuclear systems. And an accurate description of few- and many-body nuclei and nuclear reactions serves as a crucial input for astrophysics and could be potentially used in energy and medical applications of nuclear physics. Therefore, the study of low-energy NN interactions has been an important task since the birth of nuclear physics.

A modern EFT which describes the nucleon-nucleon (NN) interaction is based on chiral perturbation theory ($\chi$PT). Inspired by the current algebra [2], an effective Lagrangian which describes the low energy pion-nucleon interaction was written down based on the idea of spontaneous symmetry breaking [3] and the symmetry
given by the SU(3) current algebra [4]. Later, as the idea of effective field theory was established, the effective Lagrangian was further developed to include pion-pion interaction\(^2\), resulting in the so-called \(\chi PT\) today.

From a modern point of view, chiral perturbation theory (\(\chi PT\)) can be considered as one of the first examples of an EFT applied to a known mother theory, which enables the calculation in the non-pertrurbative region of QCD. By implementing EFT into nuclear systems one gains two major advantages. First, this gives, at least at the lowest level, a physical insight into the behavior of the strong interaction at large distances, which is difficult to calculate \textit{ab initio} from its fundamental theory—QCD. Second, the calculation can be made more accurate order by order, thus allowing a systematic error control. An EFT becomes most powerful when a clear and large separation between the low- and high-energy scales in the problem is possible. The separation of scales enables one to integrate out the high-energy degrees of freedom in the mother theory, and then do the matching (to either the original theory or experimental data) to determine the free parameters in the effective field theory. From QCD to \(\chi PT\), the separation of scales is characterized by the so-called chiral symmetry breaking scale (\(\Lambda_\chi \approx 1 \text{ GeV}\)), where the chiral symmetry of QCD is spontaneously broken. As a result, the basic degrees of freedom in \(\chi PT\) are nucleons and pions instead of quarks and gluons. The Lagrangian of \(\chi PT\) is only restricted by the symmetry and contains infinitely many terms. Fortunately, one can expand in powers

\(^2\text{See, for example Ref. [5] and reference therein.}\)
of the small scale (nucleon momentum or pion mass) over the large scale (nucleon mass or the chiral symmetry breaking scale) in the theory and then establish the power counting to arrange terms order by order according to their importance. The conventional expansion parameter in $\chi$PT is $Q/\Lambda_{\chi}$, with $Q$ the pion mass or the nucleon momentum. It has been shown that $\chi$PT is quite successful in the low-energy ($< 1$ GeV) mesonic sector [6]. In principle, the same theory should work in the low-energy nucleon-nucleon (NN) sector as well.

However, unlike pion-nucleon ($\pi N$) or pion-pion ($\pi \pi$) interaction, the long-range nature of the NN interaction results in extra difficulty. When applying $\chi$PT to evaluate the pion-nucleon or pion-pion interaction, one first determines the Feynman rule from the $\chi$PT Lagrangian, and then sums over diagrams with specified initial and final state order by order in $Q$. The smallness of the coupling in pion-pion and pion-nucleon scattering guarantees that the perturbative approach results in a converged amplitude for an interaction energy $\ll \Lambda_{\chi}$. For the NN interaction, due to reasons which will be discussed in section 2.3, one can show that the same perturbative approach is not sufficient to obtain a reasonable converged amplitude. In other words, even for $\chi$PT, which has enabled a perturbative calculation in the pion-pion ($\pi \pi$) and pion-nucleon ($\pi N$) sector (which its mother theory QCD cannot do), for the NN sector one still needs to sum the amplitude non-perturbatively in order to get a reasonable result.
In fact, the issue of a perturbative approach toward the NN problem is a long standing one, which is older than \( \chi \)PT itself. It was well-known even before QCD was invented that the strong force is highly non-perturbative. It was generally accepted that even if a fundamental theory which governs the strong nuclear force were found, a perturbative summation based on it would produce a poor result. Thus, from the postulation of the one-pion-exchange potential [7] in 1935 by Yukawa, people built various potentials to describe the NN interaction. In particular, there were some early attempts to calculate the two-pion-exchange contributions in the 1950s [8, 9], which however failed to provide a large enough attractive spin-orbit force. Later on, people built potentials based on the heavy-meson-exchange, such as the one-boson-exchange models [10]; made use of the dispersion relation, such as the Paris [11] potential; and adopted a field theoretical approach, such as the Bonn potential [12]. Later still, researchers constructed “high-precision” potentials phenomenologically, which includes the Nijmegen [13, 14], Argonne [17], CD-Bonn [15] and one-boson-exchange [16] potential. In all such approaches, one solves the NN problem by plugging in an NN potential to the Schrödinger, or, equivalently, Lippmann-Schwinger equation (LSE). The same procedure was suggested by Weinberg [18, 19], where instead of developing a phenomenological potential, one first generates the potential based on the Lagrangian of \( \chi \)PT (along with the NN interaction part as the counter terms) and then iterates the potential to all orders in the Schrödinger or Lippmann-Schwinger equation (LSE).
Figure 1.1: The diagramatic representation of the perturbative and non-perturbative sum. The solid and dashed line here represent different kind of particles.

In Fig. 1.1, we illustrate the perturbative and non-perturbative summation. The upper diagrams represent the perturbative summation, where one sums individual diagrams order by order with respect to coupling constant $\alpha_{\text{small}}$. As one can see, if $\alpha_{\text{small}} \ll 1$, then the higher order diagrams are highly suppressed and the perturbative sum converges very fast. On the other hand, if one is in the energy scale where the coupling constant is large, i.e., $\alpha_{\text{large}} \approx 1$, then he or she is forced to adopt the non-perturbative summation as demonstrated in the lower panel. Here one truncates the series of diagrams in the upper panel at certain order (in the plotted case, 1st order), and iterates them as a potential in the Schrödinger or Lippmann-Schwinger equation (LSE) to all orders. One can immediately see that the non-perturbative summation provides an easy way for us to account for more diagrams in higher order, however, it involves some subtleties and there is no guarantee that this approach
converges. For example, as shown in the lower panel, if one just takes the 1st terms in the upper panel and iterates them, then he or she would miss some of the second order diagrams (e.g., P2b) in the upper panel, which are in the same order as P2a. Moreover, since the long-range potentials obtained from the chiral Lagrangian (e.g. P1, P2a, P2b and P3a) are singular in general, i.e., in coordinate space representation they diverge faster than $-\frac{1}{4r^2}$ as $r \to 0$, whether the contact term (P1C) can absorb the divergence created by the iteration of P1 would depend on details of the potential. Thus, the only way to justify (or determine how to modify) the non-perturbative approach is to perform the actual calculation and check whether the result converges order-by-order without presenting other problems.

Actually, the need for non-perturbative resummation is closely related to the singular structure of potentials generated from $\chi$PT (with the NN interaction Lagrangian). Again, we refer to Fig. 1.1, where, for $\chi$PT, the leading-order potential P1 (the so-called one-pion-exchange potential) is already singular. A singular potential is non-physical, since it gives eigen-energies which are not bounded from below. This is a consequence of the fact that the theory (which gives singular potentials) really breaks down after a certain energy scale. Therefore, to overcome this difficulty, one adopts a cutoff $\Lambda$ instead of applying singular potentials to energy $E = \infty$, and fixes the boundary condition at $\Lambda$ to account for physics above it. For our example, one can solve the Schrödinger or Lippmann-Schwinger equation (LSE) with a cutoff
and associate a contact term $P_1C$ with $P_1$ to fixed the boundary condition. This corresponds to the non-perturbative summation.

It may occur to the reader that we need to set up a new EFT based on $\chi$PT (for which $\chi$PT is the mother theory), just like what we have done to QCD, to solve the NN problem. Since the degrees of freedom in $\chi$PT are already nucleons and pions, what one can do further is to integrate out the pions. Such an EFT, i.e., the so-called pionless EFT (see Ref. [20] and references therein), contains only the NN contact interaction terms in the Lagrangian, and is valid for a momentum scale below the pion mass, $m_\pi \approx 140$ MeV. From the pionless EFT, the NN interaction can be evaluated analytically and a well-defined power-counting scheme exists. However, for NN interactions with energy higher than the breakdown scale of pionless EFT, one is forced to adopt the Lagrangian with $\pi N$ and $\pi\pi$ terms from $\chi$PT and face the problem of the chiral EFT in the NN sector. The main problem here is how to sum the amplitudes, i.e., how to apply the power counting to organize a non-perturbative summation\textsuperscript{3}. One natural choice to solve the problem (as proposed by Weinberg [18, 19]) would be to apply the power counting at the level of the potential and then iterate the whole potential non-perturbatively to obtain the final amplitude. This is the so-called Weinberg (power) counting.

The non-perturbative approach to the chiral EFT in the NN sector, i.e, Weinberg counting, has been applied up to next-to-next-to-next-to leading order ($N^3$LO)

\textsuperscript{3}For clarification, from now on we will use “non-perturbative summation/way” to refer to that the summation is done by iterating a potential to all orders in the Schrödinger or Lippmann-Schwinger equation, and use “perturbative summation/way” to refer to other treatments.
and enjoyed remarkable success. In Ref. [21], Weinberg counting was first carried out. Chiral potentials were calculated up to next-to-next-to leading order (NNLO) using time-ordered perturbation theory. The resulting NN scattering phase shifts \((J \leq 2)\) and bound-state data obtained from inserting the potentials up to NNLO (along with 7 contact terms) into the Schrödinger equation with a cutoff ranging from \(\Lambda = 2.5 - 3.9 \text{ fm}^{-1}\) show a reasonable agreement with the experimental data. A general discussion of the renormalization procedure is given in Ref. [22]. For higher partial-waves \((J \geq 2)\), one would expect that the potentials become much weaker, and instead of iterating them in the Schrödinger or Lippmann-Schwinger equation, Born approximation ("perturbative way") may be adopted to give reasonable results. Moreover, according to Weinberg counting, chiral potentials up to NNLO have no contact term in \(J \geq 2\) partial-waves. Thus, by comparing those \(J \geq 2\) phase shifts with the experiments, one examines the "pure" effect of chiral potentials. This calculation was carried out in Ref. [23], and the results show that when adopting Weinberg counting, chiral potentials do give satisfactory agreement (with a few exceptions) with the experimental data. In Refs. [24, 25], chiral potentials (up to NNLO) are derived using unitary transformation and the NN scattering and bound-state data evaluated in detail with a momentum-space cutoff \(\Lambda = 600 - 875\) MeV. In Ref. [26], the \(^1S_0\) phase shift is evaluated using a subtractive renormalization approach, where the next-to leading order (NLO) result is in good agreement with respect to data. The chiral potentials are further calculated up to NNLO [27] and \(N^3\)LO [28], and a
remarkable agreement is obtained for the N^3LO potential. In particular, it gives a \( \chi^2/\text{datum} = 1.06 \) for \( T_{\text{lab}} < 290 \text{ MeV} \) [44].

However, critics have found various problems of Weinberg counting. First, on the theoretical side it has been shown that Weinberg counting fails to provide sufficient counter terms proportional to \( m_\pi \) [29–31], and therefore one of the main advantages of \( \chi \)PT, i.e., to evaluate the quark-mass dependence of nuclear physics cannot be achieved. To overcome this difficulty, Kaplan, Savage and Wise developed the so-called KSW counting, in which one derives extra counter terms and returns to a perturbative summation to obtain the NN amplitude. However, as shown by Fleming et al [32, 33] later, the KSW counting converges fairly slowly in the \( ^3S_1-^3D_1 \) channel due to the singular nature of the tensor force. Consequently, as shown in Ref. [34], the \( ^1S_0 \) partial-wave can be evaluated by the KSW counting, but the \( ^3S_1 \) channel needs to be obtained by Weinberg counting. Since then, various versions of power counting (other than the Weinberg or KSW counting) has been developed and put into test. In particular, one can try to resum parts of the KSW counting amplitude to all orders as attempted in Ref. [35] and Ref. [36], or adopt the distorted-wave-Born-approximation method and then perform the renormalization-group analysis to decide whether to add contact terms [37–40].

Other than the above (power counting) issues, one can argue that the way—which is normally the dimensional-regularization (DR)—the chiral potentials are regularized when they are obtained can generate problems too. The chiral effective Lagrangian
has a breakdown scale $\approx 1$ GeV, but by adopting DR to renormalize the divergent loop, one integrates to $\infty$ and the high-energy physics enters the resulting potentials in an incorrect way. To solve this difficulty, in Ref. [41] the spectral-function-regularization (SFR) is introduced to calculate the divergent loop when obtaining the chiral potentials (up to NNLO), and it was shown that the resulting potential gives a better convergence with respect to $\Lambda$ than the conventional dimensional-regularization method. This approach is further carried out up to $N^3$LO in [42] and [43].

From a practical point of view, the main problem of the "non-perturbative approach/Weinberg counting" is most obviously reflected in the cutoff-dependence of the results. Ref. [45] demonstrates that the leading order (LO) chiral potential generates "a strong cut-off dependence of the coupling constant associated to the non-local term which breaks orbital angular momentum conservation [45]" in the $^3S_1$ phase shift in the limit of $\Lambda \to \infty$. Moreover, as shown in Ref. [46], with the LO chiral potential, Weinberg counting fails to provide a cutoff-independent result for those attractive channels in P-waves at leading order, as there is no contact term to fix the boundary conditions. In response, Ref. [47] shows that this problem will not be present if one keeps the cutoff $\Lambda < 3$ fm$^{-1}$. In another aspect, Ref. [48] shows that by adopting a (Lorentz) symmetry-preserving approach toward the NN interaction, the LO amplitude should be obtained by Weinberg counting, but starting from NLO, one should sum the amplitude perturbatively. Ref. [49, 50] ( [51, 52]) examines the cutoff-dependence issue in Weinberg counting at the LO (up to NNLO) toward
Λ → ∞. Their results suggest that Weinberg counting needs to be modified in certain situations in order to reach cutoff-independence. We will compare their results with our finding in this work and discuss those issues in detail in a later chapter.

When struck by pions, a nucleon can go to an excited state and become a ∆ baryon. One can therefore include the ∆ in the theory, and obtain the chiral NN potential with explicit ∆. This approach has been carried out in Refs. [42, 53, 54], and recently, Refs. [55, 56]. Those works (up to NNLO) show that the inclusion of ∆ as an explicit degree of freedom do improve the convergence of the chiral potential, i.e., the chiral potential up to NLO with ∆ would give results comparable to the ∆−less potential up to NNLO.

In spite of the intense investigation mentioned above, so far there is no consensus regarding a consistent approach regarding the NN problem in chiral EFT. However, it seems that applying a non-perturbative sum to at least part of the effective potential is a necessary ingredient. In this treatment, i.e., when one iterates the potential in the Schrödinger or Lippmann-Schwinger equation, due to the singularity of the chiral potential one needs to adopt a cutoff Λ inside the Schrödinger equation or LSE to solve the renormalization problem. When performing renormalization, one adjusts the free parameters in the potential which originate from the NN part of the Lagrangian (denoted as the “low-energy constants” later) to account for physics with energy higher than Λ. As mentioned above, a non-perturbative treatment which iterates the full chiral effective potential, i.e., the Weinberg counting, can generate
very good results. However, those results are obtained from a rather narrow cutoff
\((A \sim 600 - 800 \text{ MeV})\) in the Schrödinger equation or LSE. To go to higher cutoffs,
one faces the numerical problem that the result depends on the free parameters in
the contact terms more and more sensitively. Also, as one goes to higher order in
the potential, more and more contact terms with free parameters enter the potential,
and the correlation between those parameters becomes an issue. How to overcome
the numerical difficulty and evaluate results in a quick and reasonable way is an
important first step to be taken. A reliable scheme which can avoid the shortcomings
of the existing renormalization procedures is highly desirable, because any further
study such as the renormalization-group analysis and the test of new power counting
schemes in the chiral EFT at NN sector will demand an extensive evaluation of results
with respect to different cutoffs.

Efforts have been made toward the improvement of the current renormalization
scheme. In Ref. [57] and Ref. [58], a subtractive renormalization procedure which
subtracts two LSEs to cancel out the low energy constants (LEC’s) has been carried
out. These studies are based on the assumption that the Born approximation holds
at large negative energy, and so performed the subtraction at this point of energy.
However, this assumption is not generally true for the singular potentials (see, how-
ever, the recent publications from the same group [59] and [60] for the justification of
their approach). Meanwhile, the subtractive approach presented in the work of Ham-
mer and Mehen [61] enables one to obtain the half-shell NN t-matrix, \(t(p, 0; E = 0)\)
from the original LSE and the NN scattering length. From this, NN phase shifts at any energy can be obtained assuming only Hermiticity of the underlying potential (the latter technique was developed and employed by Afnan and Phillips [62] in the context of the low-energy three-body problem). Based on these ideas, in this work we develop a subtractive renormalization scheme which evaluates the NN scattering problem non-perturbatively up to NNLO in Weinberg power counting. Our method allows us to evaluate whether (or under what conditions) $\chi$PT, as an EFT, is really improved order by order, after renormalization. We also applied our method to evaluate the range (or the highest value) of $\Lambda$, within which one can claim that the renormalization is successfully carried out. In addition, we know that a successful renormalization requires both cutoff-independence and fitting-point-independence\(^4\), and the latter has not yet been tested in detail. We will evaluate this issue too. Finally, one can argue that even for those cutoffs where the resulting NN scattering phase shifts fit the data very well, it may fail to predict other physical processes if the theory has the wrong formulation. In this work, in order to further test the validity of the theory, we apply the NN results to the calculation of the deuteron electro-disintegration process.

The structure of our work is as follows. In Chapter 2, we first give a simplified idea of chiral perturbation theory and then present the effective potentials derived from it. Then, in Chapter 3, we introduce the subtractive renormalization scheme to

\(^4\)In a renormalization procedure, each unknown low-energy constant needs to be determined by one physical datum. For a successful renormalization, the results should not depend on which physical datum is chosen as the input, and we refer to this as the fitting-point-independence.
evaluate the NN scattering problem non-perturbatively with the leading order chiral potential. In Chapter 4, we further develop our subtractive technique to evaluate the NN scattering problem with the chiral potential up to the next-to-next-to leading order (NNLO($Q^3$)) with various contact terms which describes the short distance physics. We then present the results obtained by our subtraction scheme with both the dimensional regularized and the spectral-function regularized potentials in Chapter 5, and investigate their cutoff-dependence and the renormalization-point dependence. In Chapter 6, we apply our results to the calculation of deuteron electro-disintegration process, which has been considered as a test ground for various NN models for a long time. Finally, in Chapter 7, we summarize our results.
Chapter 2

Chiral perturbation theory and effective potential

In this chapter, we introduce the main ideas of chiral perturbation theory (χPT) and the effective NN potentials originating from it. In the first section, we use a condensed-matter example to explain how one can write down an EFT for a complex system. Then, in the next section, we describe the symmetry of the QCD Lagrangian and how the χPT Lagrangian originates from it. In section 3, we explain why, in the NN system, one needs to iterate the chiral effective potentials and what is the Lippmann-Schwinger equation. Finally, in section 4, we give explicit expressions for the chiral effective potentials up to NNLO(Q^3).

2.1 A brief example of chiral perturbation theory

The easiest and probably also the best way to get an idea about chiral perturbation theory is through an analog to condensed-matter physics. Consider the phenomena of a phase transition in a gas of atoms:

First, assuming that the force between atoms is attractive. When the temperature is high, the kinetic energy between the atoms is large so that they form very few bound states. In this case, the total number of degrees of freedom of the gas obeys $T = \chi N$, where $T$ is the total number of degrees of freedom, $N$ is the number of particles, and $\chi$ is the number of degrees of freedom per individual particle. Meanwhile, the
interaction between particles can be ignored unless they happen to run into each other. When the temperature is lowered, two particles (for example) are able to form bound states if the force between them is attractive. In this case, although in principle one can still calculate the properties of the gas by solving the \(N\)-particle problem, now, for a particle, the interaction between itself and the closest particle are very different from the interaction between itself and other particles, and the problem becomes complicated. In this situation, the momentum (or kinetic energy) of the system is carried by the bound states of particles, the relative momentum between the two bound particles doesn’t affect the properties of the whole system very much. This suggests that if we can find out what the bound state of two particles and its basic properties such as mass and spin are, then we can treat these as “elementary” particles at low temperatures and calculate the properties we want from the system.

We now turn to field theory. Note that in field theory the interaction is described by the exchange of virtual elementary bosons, and the propagation of these elementary particles is described in terms of the quantized fields at every point of space-time. For QED, the bosons are photons and they don’t interact with each other. For QCD, the bosons are gluons and they interact with each other via a coupling constant that runs as

\[
\alpha_s(Q) = \frac{2\pi}{b_0 \ln[\frac{Q}{\Lambda}]},
\]

where \(Q\) is the 4-momentum transfer, \(\Lambda\) is the chosen energy scale to do the renormalization, and \(b_0 = 11 - \frac{2}{3}n_f\) with \(n_f\) being the number of flavors. The coupling
becomes stronger when $Q$ is smaller, thus, at low energy the fields interact with each other very strongly. To describe interactions in terms of gluon fields at this energy scale is just like describing the gas at low temperature in terms of $N$ particles in our condensed-matter example. This is difficult and sometimes impossible. It could become impossible because all calculations in field theory (that use Feynman diagrams) are based on perturbative expansions, and so break down when the system becomes non-perturbative.

One way to deal with this situation is to find the corresponding bound states and do the calculation based on these degrees of freedom. Metaphorically, the $N$-particle calculation is the calculation from the pure QCD Lagrangian, and the reduced-particle approximation is the chiral perturbation theory. In other words, chiral perturbation theory is an effective field theory based on the approximate degrees of freedom of QCD at low energy to enable the calculation of QCD in the non-perturbative region.

2.2 Lagrangian of QCD and chiral perturbation theory

To show how chiral perturbation theory emerges from QCD, first we write down the pure QCD Lagrangian [63]:

$$L_{QCD} = \sum_f \bar{q}_f (i \gamma^\mu \gamma^\nu - m_f) q_f - \frac{1}{4} G^{\mu \nu} G_{\mu \nu}. \quad (2.2)$$
Here $f$ is the flavor of the quark, $q_f(\bar{q}_f)$ is the quark (anti-quark) field, $m_f$ is the mass of the quark, $G$ is the gluon field, $\not{\! D} = \gamma^\mu (\partial_\mu + igJ_\lambda^\alpha G_\lambda^\mu)$, with $\gamma^\mu$ the gamma matrices, $g$ the strong coupling constant, and $\lambda_\alpha$ the generators of SU(3). Consider only the up and down flavor of quarks and take the limit $m_f \to 0$. Then we can project the quark field onto right and left-handed states, i.e.,

$$L^0_{\text{QCD}} = \bar{q}_L (i \not{\! D}) q_L + \bar{q}_R (i \not{\! D}) q_R - \frac{1}{4} G_{\mu\nu} G^{\mu\nu}. \tag{2.3}$$

Here $q_{R,L} = \frac{1}{2} (1 \pm \gamma_5) q$. Now we will examine the symmetry. Since a direct deduction of chiral perturbation theory from QCD Lagrangian is not yet available, symmetry is one of the available important guides for building the effective Lagrangian. The effective Lagrangian needs to preserve the symmetries in the original QCD Lagrangian in its low-energy region. In Eq. (2.3) we have ignored the quark masses. But one can still claim that Eq. (2.3) is a good approximation of the symmetries in Eq. (2.2) at low energy, since the mass of protons and neutrons is about 1 GeV, and the mass of the up (down) quark is 1.5-3.3 (3.5-6.0) MeV [64]. Thus, most of the nucleon mass comes from the QCD vacuum and the symmetry obtained by ignoring the light quarks’ mass can well represent the symmetry of the $\pi N$, $\pi\pi$ and NN system. In Eq. (2.3), $\bar{q}_L (i \not{\! D}) q_L$ and $\bar{q}_R (i \not{\! D}) q_R$ are invariant under the unitary group $U_{L(R)}(2)$ (isospin) transformation individually, and the Lagrangian $L^0_{\text{QCD}}$ has $U(2)_L \times U(2)_R$ symmetry. This symmetry can be further decomposed into

$$SU(2)_R \times SU(2)_L \times U(1)_V \times U(1)_A. \tag{2.4}$$
Here $SU$ represents the special unitary group, and $U(1)_V$ acts as $q_{L,R} \to e^{i\theta} q_{L,R}$ and corresponds to baryon-number conservation. $U(1)_A$ turns out to be no longer a symmetry after one quantizes the field, due to quantum anomaly [65].

Meanwhile, based on empirical observation, e.g., protons, neutrons and other baryons do not present axial symmetry, it is believed that $SU(2)_R \times SU(2)_L$, the so-called chiral symmetry, is further spontaneously broken into the vector subgroup $SU(2)_V$ at the energy scale $\Lambda_\chi \approx 1 \text{ GeV}$ [66]. This means that the ground state is only invariant under $SU(2)_V$ transformation instead of the chiral group $SU(2)_R \times SU(2)_L$. From Goldstone’s theorem [67], which states that every generator of a spontaneously broken symmetry corresponds to a massless excitation of the vacuum, it is believed that the three broken generators of the chiral group correspond to the 3 massless pseudoscalar Goldstone bosons—$\pi^+, \pi^0, \pi^-$. In the real world, chiral symmetry is explicitly broken by the finite quark mass, and consequently, the pion also has mass. The relation between the light quark mass and pion mass can be evaluated by the vector and axial-vector current and the $SU(3)_R \times SU(3)_L$ algebra. The outcome is the famous Gellman-Oakes-Renner relation [68],

$$m_\pi^2 f^2_\pi = -2m <0 \bar{q}q|0>, \quad (2.5)$$

where $m_\pi$ is the pion mass, $m = m_u = m_d$ represents the two lightest quarks mass, $f_\pi = 93 \text{ MeV}$ is the pion decay constant, and $<0 \bar{q}q|0>$ is the quark condensate in vacuum. Here we have assumed that the isospin symmetry is exact and ignored the slight mass difference between the up and down quarks. We will not consider the
effect of isospin symmetry breaking in this work, which is suppressed by a factor of
\[ \frac{|m_u - m_d|}{|m_u + m_d|} \]
and can be taken into account with additional terms in the EFT
(see, for example [69] and the references therein).

Now one can adopt ideas from the previous condensed-matter example to define
the effective degrees of freedom of QCD at low energy. Suppose we only care about the
nucleon-nucleon interaction, then the degrees of freedom will be the proton, neutron,
and the Goldstone bosons coming from the spontaneous symmetry breaking—the
pions. The effective Lagrangian, which is valid within a domain of energy lower than
\( \Lambda_\chi \), can be written as
\[ L_{\text{eff}} = L_{\pi\pi} + L_{\pi N} + L_{NN} + \ldots \] (2.6)
Here \( L_{\pi\pi}, L_{\pi N}, L_{NN} \) represents the interaction between pion-pion, pion-nucleon, and
nucleon-nucleon, respectively.

For \( L_{\pi\pi} \), the effective Lagrangian can be obtained in the following way. First one
writes the three pion-fields as
\[ \phi(x) = \sum_{i=1}^{3} \tau_i \phi_i(x) = \begin{pmatrix} \phi_3(x) & \phi_1(x) - i\phi_2(x) \\ \phi_1(x) + i\phi_2(x) & -\phi_3(x) \end{pmatrix} = \begin{pmatrix} \pi^0(x) & \sqrt{2}\pi^+(x) \\ \sqrt{2}\pi^-(x) & -\pi^0(x) \end{pmatrix} \] (2.7)

Here \( \tau_i \) is the Pauli matrix and \( \pi^{0,\pm}(x) \) are the pion fields. Then one can define
the \( SU(2) \) matrix
\[ U(x) = \exp(i\frac{\phi(x)}{f_\pi}) \] (2.8)
which satisfies the unitary condition $UU^\dagger = 1$, and transforms as $U(x) \rightarrow RU(x)L^\dagger$ under the chiral group $SU(2)_R \times SU(2)_L$, where $\{L, R \in SU(2)\}$. This implies that $\text{Tr}(UU^\dagger)$, $\text{Tr}(\partial_\mu U \partial_\mu U^\dagger)$, and the trace of any higher derivative terms are invariant under the $SU(2)_R \times SU(2)_L$ rotation, e.g.,

$$\text{Tr}(\partial_\mu U \partial_\mu U^\dagger) \rightarrow \text{Tr}(R \partial_\mu U L^\dagger L \partial_\mu U^\dagger R^\dagger) = \text{Tr}(R R^\dagger \partial_\mu U \partial_\mu U^\dagger) = \text{Tr}(\partial_\mu U \partial_\mu U^\dagger).$$

(2.9)

Thus, they are the terms which can appear in the effective Lagrangian $L_{\pi\pi}$. It turns out that $\text{Tr}(UU^\dagger) = 2$ and has no effect, so the lowest order term that contributes to $L_{\pi\pi}$ is

$$\frac{f_\pi^2}{4} \text{Tr}(\partial_\mu U \partial_\mu U^\dagger) = \frac{1}{2} \sum_a \partial_\mu \phi_a \partial^\mu \phi_a + \ldots$$

(2.10)

Here the factor $\frac{f_\pi^2}{4}$ is added to cancel out the $f_\pi$ inside $U(x)$. One can go to higher order by adding terms with more derivatives. However, a $n$ derivative term corresponds to the $(\frac{Q}{\Lambda})^n$ term in the expansion, so its contribution is suppressed.

So far we only write down the Lagrangian with massless pions. To include the effect of explicit chiral symmetry breaking by the quark mass, one can include

$$\frac{f_\pi^2 B}{2} \text{Tr}[MU^\dagger + UM^\dagger]$$

(2.11)

as the effect of the two lightest quark mass. Here $M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$, and $B = \frac{m_\pi^2}{2m}$. Expanding $U(x)$, one finds the lowest correction appears in $L_{\pi\pi}$ is proportional to
Combining Eq. (2.10) and Eq. (2.11), the leading order of $L_{\pi\pi}$ reads

$$L_{\pi\pi}^2 = -\frac{1}{2}[\partial_\mu \phi \partial^\mu \phi - m_\pi^2 \phi^2] + O(\phi^4). \quad (2.12)$$

Here the superscript denotes the number of derivatives or small momentum appearing in the Lagrangian.

Next, we need an effective Lagrangian to describe the pion-nucleon interaction. First, define

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix}, \quad (2.13)$$

as the nucleon field. Here $p \ (n)$ is the proton (neutron) field, where both of them are 4-component Dirac fields. What we need is an effective Lagrangian in form of $\bar{\Psi} \hat{O} \Psi$ to describe a nucleon from its initial state to the final state as well to as satisfy the $SU(2)_R \times SU(2)_L \times U(1)_V$ symmetry. It turns out that by defining

$$u^2(x) = U(x), \quad (2.14)$$

and $r_\mu, l_\mu$ as the right and left-handed external field, then

$$u_\mu(x) = i[u^\dagger (\partial_\mu - i r_\mu)u - u(\partial_\mu - i l_\mu)u^\dagger] \quad (2.15)$$

transforms like an axial vector under parity. If the external field is the electromagnetic field (with photon field $A_\mu$), then

$$u_\mu(x) = i[u^\dagger \partial_\mu u - u \partial_\mu u^\dagger - i\frac{e}{2} A_\mu (u^\dagger (1 + \tau_3)u - u(1 + \tau_3)u^\dagger)]. \quad (2.16)$$
In the case of no external field,

\[
    u_\mu(x) = i[u^\dagger \partial_\mu u - u \partial_\mu u^\dagger]
    = -i \frac{\tau_a \cdot \partial_\mu \phi_a}{f_\pi} + O(\phi^3) \quad (2.17)
\]

Defining the so-called connection [70]

\[
    \Gamma_\mu = \frac{1}{2}[u^\dagger(\partial_\mu - i\tau_\mu)u + u(\partial_\mu - i\tau_\mu)u^\dagger], \quad (2.18)
\]

then the leading order of the pion-nucleon Lagrangian can be written as

\[
    L_{\pi N}^1 = \bar{\Psi}[i \gamma_5 - M + \frac{g_A}{2} \gamma^\mu \gamma_5 u_\mu] \Psi. \quad (2.19)
\]

Here \(g_A\) is the axial-vector coupling constant measured from the neutron \(\beta\)-decay, and

\[
    D_\mu = \partial_\mu + \Gamma_\mu. \quad (2.20)
\]

After the expansion, one has (for the case of no external field),

\[
    D_\mu = \partial_\mu + \frac{i}{4f_\pi} \tau \cdot \phi \times \phi + O(\phi^4). \quad (2.21)
\]

Note that all the pion-pion and pion-nucleon part of the Lagrangian \((L_{\pi\pi}, L_{\pi N})\) are in the relativistic form.

Finally, for the nucleon-nucleon (NN) interaction, since the kinetic region for the NN interaction is restricted to be below the pion-production region \(T_{lab} \approx 280\) MeV, a nonrelativistic treatment is sufficient. As a result, \(L_{NN}\) is described by the kinetic term and a series of contact interaction terms, i.e.,

\[
    L_{NN}^0 = N^\dagger (i\partial_0 + \frac{\nabla^2}{2M})N - \frac{1}{2} C_S (N^\dagger N)(N^\dagger N) - \frac{1}{2} C_T (N^\dagger \sigma N)(N^\dagger \sigma N). \quad (2.22)
\]
Here $N$ is the non-relativistic nucleon field and $C_{S(T)}$ is the low energy constant with the value to be determined by renormalization.

Since the relativistic nucleon field introduces another energy scale, $M$, which does not vanish in the chiral limit, and $\partial_0 \Psi$ does not produce a small quantity due to the rest mass of nucleon, we need to perform the non-relativistic reduction of $L_{\pi N}$. One can adopt either the Foldy-Wouthuysen or other methods (e.g., [71]) to perform the reduction. The result, i.e., the so-called heavy baryon formalism, reads [23]

\[
L_{\pi N}^{(1)} = \bar{N} \left[i D_0 - \frac{g_A}{2} \sigma \cdot u\right] N
\]

\[
L_{\pi N}^{(2)} = \bar{N} \left[\frac{1}{2M} \mathbf{D} \cdot \mathbf{D} + i \frac{g_A}{4M} \{\sigma \cdot \mathbf{D}, u_0\} + 2c_1 m_{\pi}^2 (U + U^\dagger) + (c_2 - \frac{g_A^2}{8M}) u_0^2 + c_3 u_\mu u^\mu \right]
+ \frac{i}{2} \left(c_4 + \frac{1}{4M}\right) \sigma \cdot (u \times u)\right] N,
\]

where $c_1 \sim c_4$ are the LEC’s coming from the NLO $\pi N$ coupling.

### 2.3 Chiral effective field theory in the NN sector and the Lippmann-Schwinger equation

In the previous section we listed the first few orders of the effective Lagrangian to describe the NN interaction. A full Lagrangian should contain infinitely many terms. Since terms with more powers of momentum (derivatives) are suppressed by $(\frac{Q}{\Lambda})^n$, one uses power counting to keep the same order terms together. Based on the ordered Lagrangian one can derive the Feynman rules, which are also arranged as a
power series. Then from these Feynman rules one can calculate the NN interaction in terms of Feynman diagrams. The power \((v)\) of a given diagram can be obtained from dimensional analysis. Based on the number of nucleons \(E_n\), loops \(L\), separately connected diagrams \(C\), number of vertices of type \(i\) \(V_i\) and dimension of vertices of type \(i\) \(\Delta_i\), one has the following naive power counting:

\[
v = -2 + 2E_n + 2(L - C) + \sum_i V_i \Delta_i, \tag{2.24}
\]

\[
\Delta_i = d_i + \frac{n_i}{2} - 2. \tag{2.25}
\]

Here \(d_i\) represents the number of derivatives and \(n_i\) represents the number of nucleon lines in the vertex. Note that Eq. (2.25) only applies to the irreducible diagrams. In order to better understand the meaning of reducible and irreducible diagrams, we need to introduce the concept of the Lippamnn-Schwinger equation (LSE).

\[
\begin{array}{c}
\text{T} = v + \text{vg}v + \text{vgvg}v + \ldots \\
= v + \text{vg}(v + \text{vg}v + \ldots) = v + \text{vg}T
\end{array}
\]

Figure 2.1: The diagramatic representation of the Lippmann-Schwinger equation.

The LSE can be visualized as in Fig.2.1, where the T is the t-matrix, the solid line represents the nucleon and \(V\) is the potential between two nucleons. The physical interpretation is that one iterates \(V\) to infinite order, which corresponds to the following mathematical expression (note that the non-relativistic approach is sufficient
here because we are considering $E_{\text{cm}} < 175$ MeV, which is only $\approx 19\%$ of the nucleon mass 938 MeV:

$$
\langle p' | t(E) | p \rangle = \langle p' | V | p \rangle + \int \frac{d^3 p''}{(2\pi)^3} \langle p' | V | p'' \rangle \frac{1}{E + i\varepsilon - \frac{p''^2}{2M}} \langle p'' | t(E) | p \rangle
$$

(with $\varepsilon$ a positive infinitesimal and $M$ the nucleon mass). If we ignore the last term in the right-hand side of Eq. (2.26), we get the Born approximation. Thus, the second term in the right-hand side accounts for the iteration of $V$, which resums the effects of infra-red enhancement due to the presence of heavy (c.f. $m_\pi$) particles in the intermediate NN state (i.e., the lower panel of Fig. 1.1 in the Introduction).

Note that if $V$ is singular, the integral in the right-hand side of Eq. (2.26) diverges. Mathematically it means that one needs an additional boundary condition to fix a solution. Thus, one chooses a cutoff $\Lambda$ as the upper limit of the integral and then absorb physics higher than $\Lambda$ into the LEC’s in the contact term to solve for the $t$-matrix. In a partial-wave expansion [72] Eq. (2.26) reads

$$
t_{\ell\ell'}(p', p; E) = v_{\ell\ell'}(p', p) + \sum_{l''} \frac{2}{\pi} \frac{M}{\Lambda} \int_0^\Lambda \frac{dp''}{p_0^2 + i\varepsilon - p''^2} v_{l''l'}(p', p'') t_{l''l'}(p'', p, E),
$$

Figure 2.2: Reducible (a) & (c) v.s. irreducible diagrams (b) & (d).
where \( p_0^2 / M = E \) is the c.m. energy, and \( l \) denotes the angular-momentum quantum number. The phase shift is related to the t-matrix by

\[
t_l(p_0, p_0; E) = -\frac{e^{i\delta_l} \sin \delta_l}{MP_0}.
\] (2.28)

Now we can define reducible diagrams, which are those can be represented by iterating the irreducible diagrams in the LSE. The power counting in Eq. (2.25) fails for reducible diagrams due to the fact that the effect of the loop from the nucleon propagator (which is non-relativistic) and the pion propagator (which is relativistic) is different. A reducible diagram has nucleon energies in the intermediate state which differ from the initial state by \( O(Q^2/M) \). As a result, there are pieces proportional to \( M/Q \), which diverge or become big as \( Q \to 0 \) due to the two nucleon propagators in the reducible diagram. On the other hand, for an irreducible diagram, the two nucleon intermediate state only has energies that differ from the initial state by \( O(Q) \) [21]. Consider the following examples. According to Eq. (2.25), diagram (a) in Fig.2.2 has \( E_n = 2, L = 0, C = 1 \) and \( d_i = 1, n_i = 2 \) in the vertices, so \( v = -2 + 2 \cdot 2 + 2(-1) + 2(1 + \frac{2}{2} - 2) = 0 \) and this is the correct power counting. The naive power counting formula works as well in diagram (c), where one has \( v = -2 + 2 \cdot 2 + 2(1 - 1) + 3(1 + \frac{2}{2} - 2) = 2 \), which again is the correct power counting. Meanwhile, special care is needed when dealing with reducible diagrams like (b) and (d). For diagram (b), the power-counting formula gives \( v = -2 + 2 \cdot 2 + 2(1 - 1) = 2 \), while the diagram actually diverges linearly when one calculates it explicitly. The power-counting formula gives \( v = -2 + 2 \cdot 2 + 2(1 - 1) + 4(1 + \frac{2}{2} - 2) = 2 \) for diagram (d). In both cases, one
encounters the so-called pinch-singularity in the actual calculation, which is caused by taking the limit $M \to \infty$ in the heavy baryon chiral perturbation theory. One can use the time-ordered perturbation theory or start from a full relativistic formalism and separate pieces proportional to $M$ (for diagram (d), these pieces correspond to the iterated OPE) from the final result\(^1\).

To sum up the amplitudes, we first write down all possible diagrams up to a certain order we want to consider and then calculate each of them based on the Feynman rules. A perturbative approach then sums up the amplitudes and performs the matching to physical observables to determine the low energy constants (LEC’s). However, a naive perturbative approach fails to predict the existence of deuteron. This is due to the infra-red enhancement of the NN interaction as well as the power counting issue. As an example, consider diagram (a) of Fig.2.2, i.e., the one-pion-exchange potential (OPE), which scales as $\frac{g^2}{M^2} Q^0$ in the power counting. Now, if one iterates the OPE once, then the “actually” linear divergent diagram (d), will scale as $\frac{g^4}{M^4} Q^4$. The ratio of the diagram $\frac{(d)}{(a)} \sim \frac{g^2 M}{g_a^2} Q$. Thus, for $Q \leq \frac{8\pi f^2 M}{g_a^2} \sim 250$ MeV, diagram (d) contributes as large as (a). Following the same logic, one can see that any iteration of the OPE contributes as large as diagram (a) once this momentum is reached. Thus, at least for the OPE at low energy, a non-perturbative treatment as suggested by Weinberg [18], turns out to be necessary. In Weinberg’s treatment, one applies the power counting to collect a series of irreducible diagrams, and treats the total

\(^1\)See reference [23] for detail of this treatment.
amplitudes coming from those diagrams as potential. One then iterates the potential in the Schrödinger or Lippmann-Schwinger equation (LSE) to solve the scattering problem. One needs to vary the unknown LEC’s and solve the Schrödinger or LSE with certain cutoffs until the resulting physical observables match the experimental data. We postpone the details of this renormalization procedure until next chapter.

In principle we can go to infinitely high order in the power series of potentials if we like, however, there will be more and more unknown LEC’s in the higher orders which need to be fitted to the experimental data. Thus, unless those LEC’s are completely obtained from lattice QCD calculations, the predictive power is limited and the true validity of the theory is obscured by the fitting of LEC’s. However, at present this has not happened yet and all the LEC’s are fitted from the experimental data. Nevertheless, it is still possible to improve the situation by developing other renormalization procedures instead of a direct fitting. We will later show how to use a direct input of physical observables to replace the fitting of LEC’s and under what conditions, chiral EFT in the NN section can be reasonably renormalized non-perturbatively.
2.4 Chiral effective potential and partial-wave decomposition

The effective potentials are currently calculated up to N\textsuperscript{3}LO [28, 42]. In this work we adopt the results up to NNLO and list the potentials below. The chiral effective potential \( V \) can be written as [23]

\[
V = V_C + \tau_1 \cdot \tau_2 W_C + (V_S + \tau_1 \cdot \tau_2 W_S) \sigma_1 \cdot \sigma_2 + [V_T + \tau_1 \cdot \tau_2 W_T] \sigma_1 \cdot q \sigma_2 \cdot q + [V_{SO} + \tau_1 \cdot \tau_2 W_{SO}] i(\sigma_1 + \sigma_2) \cdot (q \times p) + [V_Q + \tau_1 \cdot \tau_2 W_Q] \sigma_1 \cdot (q \times p) \sigma_2 \cdot (q \times p). \tag{2.29}
\]

Here the ten complex functions \( V_C(q), ..., W_Q(q) \) depend on \( q = |q| \). \( \sigma_1, \sigma_2, \tau_1, \tau_2 \) are the spin(isospin)-vectors of the two nucleons. \( p'(p) \) is the initial(final) three-momentum of the nucleon, \( q = p' - p \). \( p = |p|, p' = |p'| \) and \( z = \cos \theta \), with \( \theta \) the scattering angle. The subscripts \( C, S, T, SO, \) and \( Q \) indicate the Central, Spin-Spin, Tensor, Spin-Orbit and Quadratic spin-orbit components of the potential. The long range part of chiral effective potentials (constant and \( q^2 \) terms excluded\(^2\)) are:

**Leading order \( O(Q^0) \):**

\[
W_T = \left( \frac{g_A}{2f_\pi} \right)^2 \frac{1}{m_\pi^2 + q^2}. \tag{2.30}
\]

Note that we adopt \( g_A = 1.29, f_\pi = 92.4 \text{ MeV} \) in the later calculation when the leading order is considered alone. By choosing this value for \( g_A \) in the OPE we

---

\(^2\)The constant and \( q^2 \) terms are renormalization scheme dependent and can be included in the contact terms.
are taking into account $O(Q^2)$ corrections to it, that shift the NN coupling constant away from the Goldberger-Treiman result and give us the freedom to reproduce the empirical value. In Weinberg counting, there are contact terms in form of $C_{S(T)}$ associated with the LO potential in the S-partial-waves, where S(T) denotes the $^1S_0(^3S_1)$ partial-wave.

**Next-to-leading order $O(Q^2)$ :**

\[
W_C = \frac{1}{384\pi^2f^4_\pi} \left[ 4m^2_\pi(5g^4_A - 4g^2_A - 1) + q^2(23g^4_A - 10g^2_A - 1) + \frac{48g^4_A m^4_\pi}{4m^2_\pi + q^2} \right] L(q),
\]

(2.31)

and

\[
V_T = -\frac{1}{q^2} V_S = \frac{3g^4_A}{64\pi^2f^4_\pi} L(q),
\]

where

\[
L(q) = \frac{w}{q} \ln \frac{w + q}{2m_\pi},
\]

(2.32)

and $w = \sqrt{4m^2_\pi + q^2}$.

**Next-to-next-to leading order $O(Q^3)$ :**

\[
V_C = \frac{3g^2_A}{16\pi f^2_\pi} \left\{ 4(c_1 - c_3)m^3_\pi - \frac{g^2_A m^5_\pi}{16M} (m^2_\pi + 3q^2) - \frac{g^2_A m^5_\pi}{16M(4m^2_\pi + q^2)} - c_3 m_\pi q^2 
+ \left[ 2m^2_\pi (2c_1 - c_3) - q^2 (c_3 + \frac{3g^2_A}{16M}) \right] (2m^2_\pi + q^2) A(q) \right\},
\]

(2.33)

where $M$ is the nucleon mass and

\[
A(q) = \frac{1}{2q} \arctan \frac{q}{2m_\pi},
\]

(2.34)
and

\[
W_C = \frac{g_A^2}{128\pi M f_\pi^4} \left\{ (8 - 11g_A^2)m_\pi^3 + (2 - 3g_A^2)m_\pi q^2 - \frac{3g_A^2 m_\pi^5}{4m_\pi^2 + q^2} \right. \\
+ \left[ 4m_\pi^2 + 2q^2 - g_A^2 (4m_\pi^2 + 3q^2) \right] (2m_\pi^2 + q^2) A(q) \right\}, \tag{2.35}
\]

\[
V_T = -\frac{1}{q^2} V_S = -\frac{9g_A^4}{512\pi M f_\pi^4} (m_\pi + (2m_\pi^2 + q^2) A(q)), \tag{2.36}
\]

\[
W_T = -\frac{1}{q^2} W_S = \frac{g_A^2}{32\pi f_\pi^4} \left\{ (c_4 + \frac{2 - 3g_A^2}{8M}) m_\pi + \\
\left[ (c_4 + \frac{1}{4M})(4m_\pi^2 + q^2) - \frac{g_A^2}{8M} (10m_\pi^2 + 3q^2) \right] A(q) \right\}, \tag{2.37}
\]

\[
V_{SO} = \frac{3g_A^4}{64\pi M f_\pi^4} (m_\pi + (2m_\pi^2 + q^2) A(q)), \tag{2.38}
\]

\[
W_{SO} = \frac{g_A^2 (1 - g_A^2)}{64\pi M f_\pi^4} (m_\pi + (4m_\pi^2 + q^2) A(q)). \tag{2.39}
\]

Here we did not list \( V_Q \) and \( W_Q \) since they come from the parts which are proportional to \( M \) in the iterated OPE [23], and are automatically included when one iterates the OPE in the LSE. Note that although the potential diverges as \( |q|^3 \) at large \( q \), the counter terms renormalizing the TPE are only of \( \mathcal{O}(Q^2) \). The contact terms associated with both the NLO and NNLO potentials are in the following form:

\[
\lambda_{ll'} + \gamma_{ll'} * (\mathcal{O}(Q^2)), \tag{2.40}
\]

where \( \lambda_{ll'} \) and \( \gamma_{ll'} \) are constants to be determined by renormalization, and \( ll' \) are the angular momentum quantum numbers. For \( l + l' \geq 1(2), \lambda_{ll'}(\gamma_{ll'}) \) vanishes in Weinberg counting.
Our expression includes the $1/M$ corrections. This is the correct form in the power counting $M \sim 4\pi f_\pi \approx \Lambda_\chi$. When adopting potentials beyond LO, we use $g_A = 1.26$ and the pion decay constant $f_\pi = 93$ MeV. For the nucleon mass we use $M = 938.926$ MeV, and as pion mass we adopt $m_\pi = 138.03$ MeV. For the low energy constants (LEC’s) we adopted $c_1 = -0.81$, $c_3 = -4.70$, and $c_4 = 3.4$ (given in GeV$^{-1}$) [25]. There have been discrepancies of the values of those LEC’s between different extractions, but this is not our concern in this work. Our focus is to examine the renormalization issues rather than to perform the best fit to the data.

Note that starting from $O(Q^2)$, one needs to regularize divergent loops to obtain the effective potential. The above results are obtained by dimensional regularization (DR), so for each divergent diagram there is a contact term of the same order associated with it. However, in DR one integrates to infinity despite the fact that the Feynman rules (which are actually effective rules) should break down for momentum larger than $\Lambda_\chi \sim 1$ GeV. Therefore, for the DR potential, one can argue whether the contribution from momentum above $\Lambda_\chi$ can be properly absorbed by the contact terms.

An alternative way to regularize the divergent loop is to adopt the spectral-function regularization (SFR), which was designed to keep the contribution from the loop integral below the chiral symmetry breaking scale. The spectral-function regularized TPE has a form just the same as DR TPE except that $L(q)$ and $A(q)$ in Eq. (2.32-2.39) are replaced by $\tilde{L}(q)$ and $\tilde{A}(q)$, which have the following expressions
Here $\tilde{\Lambda}$ denotes the chosen cutoff when evaluating the potential and $s = \sqrt{\tilde{\Lambda}^2 - 4m^2}$.

Considering the $O(Q^3)$ part of the NNLO TPE in DR, we see in Eq. (2.39) that terms diverging as $|q|^3$ are all associated with the function $A(q)$, where $q = |q|$, and as $q \to \infty$, $A(q) \to 1/q$. On the other hand, these terms in the DR TPE are all suppressed in the SFR TPE and diverge only as $q^2$ due to the fact that when the function $A$ is calculated in SFR we obtain $A^\Lambda(q) \to 1/{q^2}$ for $q \gg \tilde{\Lambda}$. Thus, if we fix $\tilde{\Lambda}$ to particular value, say $\tilde{\Lambda} = 800$ MeV, then the dominant power in the SFR TPE at large $|q|$ is of the same order in the contact interaction as the one at NLO, namely proportional to $|q|^2$. Indeed, SFR leads to suppression of the TPE over the whole range of $q$. In this work we will evaluate the NN scattering phase shifts using both the SFR and DR potentials up to NNLO, and discuss their difference in Chapter 5.

After having the potentials in hand, the next step is to perform the partial-wave decomposition. By defining $U_K = V_K + (4I - 3)W_K$ ($K = C, T, SO, Q$), with $I$ the total isospin, the partial-wave decomposition formula can be written as [25]:

$$v_{l' l}(p', p) = \langle l' s j | v(p', p) | l s j \rangle,$$  

(2.43)
where \( l, l'(s) \) denote the angular-momentum (spin) quantum numbers. With this we list the different partial-wave potentials for the TPE [25]

\[
\langle j_0|v(p', p)|j_0 \rangle = -\frac{1}{8\pi} \int_{-1}^{1} dz \left\{ U_C - 3U_S + p'p(z^2 - 1)U_Q - q^2 U_T \right\} P_j(z), \tag{2.44}
\]

\[
\langle j_1|v(p', p)|j_1 \rangle = -\frac{1}{8\pi} \int_{-1}^{1} dz \left\{ [U_C + U_S - 4p'pzU_{SO} - p'^2 p^2(1 + 3z^2)U_Q + (p'^2 + 2p'pz + p^2)U_T]P_j(z) + \left[ 2p'pU_{SO} + 2p'^2 p^2 zU_Q - 2p'pU_T \right] (P_{j-1}(z) + P_{j+1}(z)) \right\} \tag{2.45}
\]

\[
\langle j \pm 1, 1|v(p', p)|j \pm 1, 1 \rangle = -\frac{1}{8\pi} \int_{-1}^{1} dz \left\{ p'p \left[ 2U_{SO} \pm \frac{2}{2j+1} (-p'pzU_Q + U_T) \right] P_j(z) + \left[ U_C + U_S - 2p'pzU_{SO} + p'^2 p^2 (1 - z^2)U_Q \pm \frac{1}{2j+1} (2p'^2 p^2 U_Q - (p'^2 + p^2)U_T) \right] P_{j \pm 1}(z) \right\} \tag{2.46}
\]

\[
\langle j \pm 1, 1|v(p', p)|j \mp 1, 1 \rangle = -\frac{1}{8\pi} \sqrt{j(j+1)} \int_{-1}^{1} dz \left\{ -4p'pU_T P_j(z) + \left[ \mp \frac{2p'^2 p^2}{2j+1} U_Q + 2p'^2 U_T \right] P_{j \mp 1}(z) + \left[ \pm \frac{2p'^2 p^2}{2j+1} U_Q + 2p'^2 U_T \right] P_{j \mp 1}(z) \right\}. \tag{2.47}
\]

Here \( P_n \) is the \( n \)th Legendre polynomial.

For completeness, we list the contact terms for TPE up to NNLO as follows [25]:

\[
v_{1S_0}(p', p) = \lambda_{1S_0} + \gamma_{1S_0}(p'^2 + p^2).
\]

\[
v_{3S_1}(p', p) = \lambda_{3S_1} + \gamma_{3S_1}(p'^2 + p^2).
\]
\( v_{3S_1-3D_1}(p', p) = \lambda_{3S_1-3D_1}(p'^2). \)

\( v_{3D_1-3S_1}(p', p) = \lambda_{3D_1-3S_1}(p'^2). \)

\( v_{1p_1}(p', p) = c_{1p_1}p'p. \)

\( v_{3p_0}(p', p) = c_{3p_0}p'p. \)

\( v_{3p_1}(p', p) = c_{3p_1}p'p. \)

\( v_{3p_2}(p', p) = c_{3p_2}p'p. \) (2.48)

Here \( c_{sL_J}, \lambda_{sL_J} \) and \( \gamma_{sL_J} \) are the undetermined LEC's.
Chapter 3

Subtractive renormalization of chiral potential at leading order in S-waves

This section is divided into three parts. In the introduction, we give the expression for the one-pion-exchange potential, and show its singular structure in the S-wave. We then derive our subtractive renormalization scheme for the renormalization of the one-pion-exchange potential in the S-wave, and present the results in section 3.2. In section 3.3, we show how our subtractive scheme can be extended to the bound-state calculation and present the resulting deuteron wave functions. Note that some of the figures, content and derivations in this chapter are adapted from our previous publication [73].

3.1 Introduction

The one-pion-exchange (OPE) potential is the leading-order effective potential in the sense of adopting meson exchange to describe the nuclear force. It is the dominant term when the relative energy of the nucleon system is very low. The expression of OPE has been given in previous chapter. Here we write down its explicit form
including spin and isospin:

\[
\langle p' | V | p \rangle = -\tau_1 \cdot \tau_2 \frac{g^2}{4f^2_\pi} \sigma_1 \cdot (p' - p) \sigma_2 \cdot (p' - p) \left( \frac{1}{(p' - p)^2 + m^2_\pi} + \pi[(C_T + 3C_S) + \tau_1 \cdot \tau_2(C_S - C_T)] \right)
\]

\[
= v^{LR}(p', p) + v^{SR}(p', p)
\]

where \( \tau_1, \tau_2 \) are the isospin and \( \sigma_1, \sigma_2 \) the spin operators. The c.m. momentum is denoted by \( p \). Here we have divided the potential into \( v^{LR} \), the long range part, which is dominated by pion exchange, and \( v^{SR} \), the short range part, which accounts for the high energy physics we ignored by absorbing it into the contact terms. Thus

\[
v^{LR}(p', p) = -\tau_1 \cdot \tau_2 \frac{g^2}{4f^2_\pi} \sigma_1 \cdot (p' - p) \sigma_2 \cdot (p' - p) \left( \frac{1}{(p' - p)^2 + m^2_\pi} \right),
\]

\[
v^{SR}(p', p) = \pi[(C_T + 3C_S) + \tau_1 \cdot \tau_2(C_S - C_T)]
\]

The low-energy constants \( C_S \) and \( C_T \) represent the effects of degrees of freedom with masses of order \( \Lambda_\chi \). Their value is not determined by chiral symmetry, and they must be fitted to experimental or lattice [74] data. As mentioned before, one needs to iterate the OPE in the Schrödinger, or equivalently, the Lippmann-Schwinger equation (LSE) to obtain a sensible result. Here we adopt the LSE approach, as it is formulated in the momentum space and gives a clear physics insight of the iteration of the potential. The explicit expressions for the partial-wave projections of OPE can be obtained from Eq. (2.43-2.47), for the S-wave, one has (for the long range part)
Here the superscript $s(t)$ represents the $S = 0(1)$ state,

$$z = \frac{1}{2} \left( \frac{p}{p'} + \frac{p'}{p} \right) + \frac{m^2}{2p'p}$$  \hspace{1cm} (3.7)

and

$$Q_n(z) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(t)}{z - t} dt.$$  \hspace{1cm} (3.8)

$P_n$ represents the Legendre polynomial of order $n$. For convenience we list the first three $Q_n$:

$$Q_0(z) = \frac{1}{2} \ln \left( \frac{z + 1}{z - 1} \right),$$  \hspace{1cm} (3.9)

$$Q_1(z) = \frac{z}{2} \ln \left( \frac{z + 1}{z - 1} \right) - 1,$$  \hspace{1cm} (3.10)

$$Q_2(z) = \frac{1}{2} P_2(z) \ln \left( \frac{z + 1}{z - 1} \right) - \frac{3}{2} z.$$  \hspace{1cm} (3.11)

Noted that as $p', p \to \infty$, we have

$$\lim_{p \text{ or } p' \to \infty} v_{s(t)}^{s(t)}(p', p) = \frac{g_A^2}{16\pi f_\pi^2},$$  \hspace{1cm} (3.12)

$$\lim_{p' \to \infty} v_{20}^{s(t)}(p', p) = \frac{g_A^2 \sqrt{2}}{8\pi f_\pi^2},$$  \hspace{1cm} (3.13)

while $\lim_{p \to \infty} v_{20}^{s(t)}(p', p) = \lim_{p, p' \to \infty} v_{22}^{s(t)}(p', p) = 0$. As a result, the long range part of OPE, i.e., $v_{00}^{s(t)}$, diverges as a constant for $p, p' \to \infty$, so the insertion of $C_S(C_T)$ to renormalize OPE is justified.
Now we insert OPE into the partial-wave representation of the LSE and perform the regularization, and the equation is

\[
t_{l'l}(p', p; E) = v_{l'l}^{LR}(p', p) + v_{l'l}^{SR}(p', p) \\
+ \sum_{l''} \frac{2}{\pi} M \int_0^\Lambda dp'' p''^2 \frac{(v_{l'l}^{LR}(p', p'') + v_{l'l}^{SR}(p', p'')) t_{l'l}(p'', p; E)}{p_0^2 + i\varepsilon - p''^2}.
\] (3.14)

Here \( t_{l'l}(p', p; E) \) is the partial-wave decomposition of the t-matrix, which is related to the three-dimensional t-matrix by

\[
t(p', p; E) = \left( \frac{2}{\pi} \right) \sum_{S, I, m_I, l, l'} \left( l'm'I SM_s|Jm_j \right) Y_{l'm_i}(\Omega_p) t_{l'l}(E)(lm_lSM_s|Jm_j)Y_{l'm}(\Omega_p) \]

(3.15)

Here \( S(J) \) refers to the spin (total angular momentum) quantum number, \( m_I, m_l, m_s \) refer to the \( \hat{z} \)-projection of \( J, l, S \), \( (lm_lSM_s|Jm_j) \) denotes the Clebsch-Gordon coefficient, \( Y_{l'm}(\Omega_p) \) denotes the spherical harmonic and \( \Omega_p \) represents the solid angle between \( p \) and the z-axis (the equation will be discussed in Sec. 6.2.2).

The conventional approach to perform the renormalization is to adjust the values of \( C_S(C_T) \) for a specified \( \Lambda \) until the results obtained from the t-matrix fit the experimental data. In this case the common choice for an experimental datum is the NN scattering length \( a_s(t) \). However, as \( \Lambda \) becomes larger, one encounters the so-called fine-tuning problem. For example, we have confirmed that at \( \Lambda = 1 \text{ GeV} \), \( C_S \) must be tuned to 1 part in \( 10^4 \) in order to get \( a_s \) accurate to 1%, while for \( \Lambda = 10 \text{ GeV} \) it
Figure 3.1: The running of the low energy constant $C_T$ versus cutoff $\Lambda$. Here $C_T$ is determined by fitting to the $^3S_1$-$^3D_1$ scattering length $a_t$. This graph is adapted from our previous publication [73].

must be tuned to 1 part in $10^6$ to achieve this accuracy. At arbitrarily large cutoffs it therefore becomes numerically impossible to fit $C_S$. As for $C_T$, the situation is even worse: Fig. 3.1 shows the running of $C_T$ with respect to $\Lambda$, and as we can see it has periodic behavior and for some cutoff it diverges. Thus, it’s impossible to fit $C_T$ at those cutoffs. In order to overcome these difficulties, we will introduce our subtractive renormalization [73].

3.2 Subtractive method and results in S-waves

The main idea of our subtractive method is to cancel out the unknown LEC’s by subtracting two LSEs, then use the property of the LSE to relate the fully off-
shell t-matrix to the on-shell t-matrix. Let us first concentrate on the c.m. energy $E = 0$ in the NN system. The definition of the scattering lengths $a_{s(t)}$ in the limit $E = p_0^2/M \to 0$ is

$$t_{00}^{s(t)}(0, 0, E = 0) = \frac{f_{00}^{s(t)}(0)}{-MP_0} = \frac{a_{s(t)}}{M}, \quad (3.16)$$

where $f_{00}^{s(t)}(E)$ represents the scattering amplitude in the singlet ($s$) or triplet ($t$) channel. The half-shell and on-shell LSE for this energy read

$$t(p, 0; 0) = v_{LR}^0(p, 0) + C + \frac{2}{\pi} M \int_0^\Lambda dp' p'^2 \left( \frac{v_{LR}^0(p, p') + C}{-p'^2} \right) t'(p', 0; 0) \quad (3.17)$$

$$t(0, 0; 0) = v_{LR}^0(0, 0) + C + \frac{2}{\pi} M \int_0^\Lambda dp' p'^2 \left( \frac{v_{LR}^0(0, p') + C}{-p'^2} \right) t'(p', 0; 0) \quad (3.18)$$

where we dropped the channel indices since our procedure is valid for the singlet as well as triplet LSEs. Subtracting Eq. (3.18) from Eq. (3.17) cancels out the unknown $C_{S(T)}$, we have

$$t_{00}^s(p, 0; 0) = v_{00}^s(p, 0) + \frac{a_s}{M} + \frac{2}{\pi} M \int_0^\Lambda dp' p'^2 \left( \frac{v_{00}^s(p, p') - v_{00}^s(0, p')}{-p'^2} \right) t_{00}^s(p', 0; 0) \quad (3.19)$$

in the singlet channel, while for the triplet channel the equation corresponding to Eq. (3.19) is

$$t_{l'l''}^t(p', 0; 0) - \sum_{l'''} \frac{2}{\pi} M \int_0^\Lambda dp'' p''^2 \left( \frac{v_{l'l''}^t(p', p''') - v_{l'l''}^t(0, p''')}{-p'''^2} \right) t_{l'l''}^t(p'', 0; 0)$$

$$= v_{l'l''}^t(p', 0) + \delta_{l'l''} \frac{a_t}{M}, \quad (3.20)$$

where $l, l' = 0, 2$. Note that since $v_{20}^t(0, p; 0) = v_{02}^t(p, 0; 0) = v_{22}^t(0, p; 0) = v_{22}^t(p, 0; 0) = 0$, the corresponding t-matrix elements $t_{02}^t$ and $t_{22}^t$ must also be zero.
A similar approach to the above subtraction were first developed for the three-body problem in Ref. [61]. The key point of this step is that the unknown LEC is no longer present in the equation. Instead of fitting this unknown constant, we perform the matching by directly including physical observables such as the scattering lengths $a_{s(t)}$.\(^1\) Eq. (3.19) can be solved using standard techniques as described in Appendix B.

Having determined $t(p,0;0)$, we can apply a similar strategy again to obtain the half-shell t-matrix elements $t(p,p';0)$, where $p'$ is an arbitrary momentum:

$$t(p,p';0) = v^{LR}(p,p') + C + \frac{2}{\pi} M \int_0^{\Lambda} dp'' p''^2 \left( \frac{v^{LR}(p,p'') + C}{-p''^2} \right) t(p'',p';0)$$

(3.21)

$$t(0,p';0) = v^{LR}(0,p') + C + \frac{2}{\pi} M \int_0^{\Lambda} dp'' p''^2 \left( \frac{v^{LR}(0,p'') + C}{-p''^2} \right) t(p'',p';0).$$

(3.22)

Subtracting Eq. (3.22) from Eq. (3.21) leads to

$$t(p,p';0) - t(0,p';0) = v^{LR}(p,p') - v^{LR}(0,p')$$

$$+ \frac{2}{\pi} M \int_0^{\Lambda} dp'' p''^2 \left( \frac{v^{LR}(p,p'') - v^{LR}(0,p'')}{-p''^2} \right) t(p'',p';0).$$

(3.23)

Since $t(p',0;0) = t(0,p';0)$, we can use $t(p',0;0)$ obtained from Eq. (3.19) to solve for $t(p,p';0)$ in Eq. (3.23).

The last step is to obtain the general half-shell t-matrix $t(p,p';E)$ at arbitrary c.m. energies $E$ from the one at zero energy that is given by Eq. (3.23). We start

\(^1\)In the next chapter we will show that this matching can be generalized, and one can input an observable at any energy $E$ to cancel out an unknown LEC.
again from two LSEs,

\[
t(p, p'; 0) = v^{LR}(p, p') + C + \frac{2}{\pi^2} M \int_0^\Lambda dp'' p''^2 \left( \frac{v^{LR}(p, p'') + C}{-p''^2} \right) t(p', p''; 0).
\]

(3.24)

\[
t(p, p'; E) = v^{LR}(p, p') + C + \frac{2}{\pi^2} M \int_0^\Lambda dp'' p''^2 \left( \frac{v^{LR}(p, p'') + C}{p_0^2 - p''^2 + i\epsilon} \right) t(p', p''; E),
\]

(3.25)

which, in order to express the idea more clearly, we rewrite in operator form as

\[
t(0) = (v + C) + (v + C)g_0(0)t(0),
\]

(3.26)

\[
t(E) = (v + C) + (v + C)g_0(E)t(E).
\]

(3.27)

The second equation, Eq. (3.27), can be rewritten as

\[
t(E)(1 + g_0(E)t(E))^{-1} = v + C,
\]

(3.28)

whereas the first, Eq. (3.26), can also be expressed as

\[
t(0) = (v + C) + t(0)g_0(0)(v + C).
\]

(3.29)

Substituting Eq. (3.28) into Eq. (3.29) gives

\[
t(0) = [1 + t(0)g_0(0)] t(E) [1 + g_0(E)t(E)]^{-1},
\]

(3.30)

which after simplification yields the so-called “first resolvent”[75, 76]

\[
t(E) = t(0) + t(0)(g_0(E) - g_0(0))t(E).
\]

(3.31)
We can now obtain $t(E)$ from $t(0)$ by solving the integral equation, Eq. (3.31). Then choosing $p = p_0$ with $E = p_0^2/M$ and setting $p' = p$ we have the on-shell t-matrix element $t_1(p_0, p_0; E)$ at an arbitrary energy $E$. The on-shell element is related to the phase shift $\delta_l$ through the well-known relation

$$t_1(p_0, p_0; E) = -\frac{\text{e}^{i\delta_l} \sin \delta_l}{Mp_0}.$$  

(3.32)

The phase shifts calculated in this way are shown in the upper left panel of Fig. 3.2. Meanwhile, the other three panels of Fig. 3.2 show results for the triplet channel, where we adopt the Stapp convention [77] for the phase shifts and the mixing parameter

$$\epsilon = \frac{1}{2} \arctan \left[ \frac{-i(t_{20}^t + t_{20}^b)}{2\sqrt{t_{00}^t t_{22}^t}} \right],$$  

$$\delta^{(3S_1)} = \frac{1}{2} \arctan \left[ \frac{\text{Im}[(t_{00}^t) / \cos(2\epsilon)]}{\text{Re}[(t_{00}^t) / \cos(2\epsilon)]} \right],$$  

$$\delta^{(3D_1)} = \frac{1}{2} \arctan \left[ \frac{\text{Im}[(t_{22}^t) / \cos(2\epsilon)]}{\text{Re}[(t_{22}^t) / \cos(2\epsilon)]} \right].$$  

(3.33)

The subtractive results are compared to those found by fitting $C_{S(T)}$. Our results show that the agreement between the two methods is very good, which confirms the ability of the subtraction to reproduce the results obtained by fitting $C$. Here the CD-Bonn potential [15] reproduces the NN data in this region with $\chi^2/\text{d.o.f} \approx 1$ and can be regarded as the experimental data.

A previous similar attempt which employed a subtraction technique is performed by Frederico et al. [57]², where they also follow the steps around Eqs. (3.26) and

²See Ref. [59] and [60] for a new version of this technique.
Figure 3.2: The comparison of two renormalization methods for the $^1S_0$ and $^3S_1 - ^3D_1$ NN phase shifts and the mixing parameter $\epsilon_1$ as a function of the laboratory kinetic energy $T_{lab} \leq 80$ MeV. Here $\Lambda = 50$ GeV is used. The phase shifts obtained from the CD-Bonn potential are also shown (dashed lines). This graph is adapted from our previous publication [73].

(3.27). However, they constructed the t-matrices between a given c.m. energy $E$ of the NN system and a large negative energy $-\frac{\mu^2}{M}$ [57]. Taking $\mu = 0$ in Eq. (8) of Ref. [57] yields our Eq. (3.31). To generate $t(-\frac{\mu^2}{M})$, Ref. [57] adopted the Born approximation, i.e., $t = V$, and assumed that for $\mu$ large enough, the final results are independent of the “subtraction point” $\mu$. This procedure leads to numerical results
for phase shifts which are similar to ours. But the validity of the Born approximation at large negative energies does not hold in general. Actually, the tensor potential that operates in the triplet channel $\sim \frac{1}{r^3}$ at short distances, and the Born approximation is not actually valid at any energy [78, 79].

We can see this failure of the Born approximation in the triplet channel by examining the off-shell t-matrix at a number of negative energies $E = -\frac{\mu^2}{M}$ and comparing the results to $V$ itself. Fig. 3.3 shows that the behavior of $t_t(p, p' ;-\frac{\mu^2}{M})$ at fixed $p'$ is completely different from the behavior of $V$. (In contrast, in the singlet channel the Born approximation appears to work quite well if $\mu$ is large enough.) This defect in Frederico et al.’s argument is manifested in the fact that $\mu$ dependence in physical quantities disappears only slowly as $\mu \to \infty$. The mixing parameter $\epsilon_1$, which is particularly sensitive to the tensor potential, is one significant example of this. Our subtraction method gives quite good convergence of $\epsilon_1$ with respect to cutoff, as shown in Fig. 3.4, and the convergence of other physical quantities with respect to $\Lambda$ as $\Lambda \to \infty$ is quite rapid, see Tables 3.1 and 3.2 (c.f. Table 1 of Ref. [57]).

3.3 Subtractive Renormalization in Bound-State Calculations

The neutron and proton form a bound state in the $^3S_1-^3D_1$ partial-wave state: the deuteron. In this section we will show that our previous subtraction method can be
Figure 3.3: The t-matrix elements $t_s(p, 100; -\frac{\mu^2}{M})$ (left) and $t^0_{00}(p, 100; -\frac{\mu^2}{M})$ (right) as functions of the half-shell momentum $p$ compared to $v_s(p, 100)$ and $v^0_{00}(p, 100)$. The parameter $\mu$ ranges from 0.8–12 GeV. This graph is adapted from our previous publication [73].

Figure 3.4: The mixing parameter $\epsilon_1$ obtained with the subtraction method as a function of the laboratory kinetic energy $T_{lab} \leq 20$ MeV and for cutoff parameters $\Lambda$ ranging from 0.5–50 GeV. This graph is adapted from our previous publication [73].
Table 3.1: Effective range $r_s$(singlet) and $r_t$(triplet) obtained using the subtraction method for various cutoffs $\Lambda$. This table is adapted from our previous publication [73].

<table>
<thead>
<tr>
<th>$\Lambda$ [MeV]</th>
<th>$r_s$ [fm]</th>
<th>$r_t$ [fm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>1.556</td>
<td>1.582</td>
</tr>
<tr>
<td>1000</td>
<td>1.441</td>
<td>1.614</td>
</tr>
<tr>
<td>5000</td>
<td>1.365</td>
<td>1.596</td>
</tr>
<tr>
<td>10000</td>
<td>1.356</td>
<td>1.595</td>
</tr>
<tr>
<td>50000</td>
<td>1.349</td>
<td>1.594</td>
</tr>
</tbody>
</table>

Table 3.2: The constants $C_S$ and $C_T$ as determined by fitting the scattering lengths $a_s$ and $a_t$ for different cutoff parameters $\Lambda$. This table is adapted from our previous publication [73].

<table>
<thead>
<tr>
<th>$\Lambda$ [MeV]</th>
<th>$C_S$ [MeV$^{-2}$]</th>
<th>$C_T$ [MeV$^{-2}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>$-6.069 \times 10^{-6}$</td>
<td>$-2.307 \times 10^{-6}$</td>
</tr>
<tr>
<td>1000</td>
<td>$-4.926 \times 10^{-6}$</td>
<td>$-1.838 \times 10^{-5}$</td>
</tr>
<tr>
<td>5000</td>
<td>$-3.902 \times 10^{-6}$</td>
<td>$1.310 \times 10^{-6}$</td>
</tr>
<tr>
<td>10000</td>
<td>$-3.753 \times 10^{-6}$</td>
<td>$2.040 \times 10^{-5}$</td>
</tr>
<tr>
<td>50000</td>
<td>$-3.627 \times 10^{-6}$</td>
<td>$-4.201 \times 10^{-8}$</td>
</tr>
</tbody>
</table>
extended to the calculation at $E < 0$, where we can evaluate the binding energy and wave function of deuteron.

First, let us consider the Schrödinger equation

$$H\psi = E\psi = -B\psi \quad (3.34)$$

with $H = H_0 + V$, $H_0 = \frac{p^2}{2M}$ the free Hamiltonian, $V$ the previously renormalized potential, and $B$ the unknown binding energy. By rearranging and partial-wave expanding Eq. (3.34) and defining:

$$\langle p|\Gamma_l \rangle = (-B - \frac{p^2}{2M})\langle p|\psi_l \rangle = \langle p|(-B - H_0)|\psi_l \rangle \equiv \langle p|g^{-1}(B)|\psi_l \rangle, \quad (3.35)$$

we obtain

$$<p|\Gamma_l> = \frac{2}{\pi} \int dp' p'^2 v_{ll'}(p, p') \frac{1}{-B - \frac{p'^2}{2M}} <p'|\Gamma_{l'}>.$$  

(3.36)

The kernel $v_{ll'}(p, p') \frac{1}{-B - \frac{p'^2}{2M}}$ of Eq. (3.36) is the same as the kernel of the equation for the t-matrix (3.26). From Eq. (3.36) we can see that the kernel will have an eigenvalue $\lambda(E) = 1$ when the energy $E = -B$, i.e., the binding energy. Thus by varying $B$ and solving the eigenvalue problem we can obtain the binding energy. If we input the value of the constant $C_T$ obtained from fitting to the scattering length $a_t = 5.43$ fm (at $\Lambda = 50000$ MeV) in the above equation, we get $B = 2.096$ MeV. In Table 3.3 we list the values of $B \equiv B_{fit}$ and $B \equiv B_{sub}$ for different cutoff parameters in the second and third column, where $B_{sub}$ is the binding energy obtained by the subtraction method describes below. We note that for small cutoff parameters $\Lambda$, the fitting procedure leads to a slightly larger value of $B_{fit}$.
Table 3.3: The binding energy $B$ for different cutoff parameters $\Lambda$. The binding energy $B_{fit}$ is obtained from the fitting method, and the binding energy $B_{sub}$ is obtained from the subtractive method. Note that this table is adapted from our previous publication [73].

<table>
<thead>
<tr>
<th>$\Lambda$ [MeV]</th>
<th>$B_{fit}$ [MeV]</th>
<th>$B_{sub}$ [MeV]</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>2.10</td>
<td>2.09</td>
</tr>
<tr>
<td>1000</td>
<td>2.12</td>
<td>2.11</td>
</tr>
<tr>
<td>5000</td>
<td>2.12</td>
<td>2.09</td>
</tr>
<tr>
<td>10000</td>
<td>2.09</td>
<td>2.09</td>
</tr>
<tr>
<td>50000</td>
<td>2.10</td>
<td>2.09</td>
</tr>
</tbody>
</table>

We now want to demonstrate that we can apply the approach discussed in the previous section to the bound-state problem. Multiplying Eq. (3.31) by $(E + B)$ we obtain

$$(E + B) \, t(E) = t(0) \left[ (E + B) + (g_0(E) - g_0(0)) \, (E + B) \, t(E) \right]$$

(3.37)

When considering the bound state, i.e. $E \to -B$, the first term in the brackets on the right-hand side vanishes. But, since, the t-matrix has a pole at $E = -B$, we can define the residue at the pole as $f(-B) = \lim_{E \to -B} (E + B) \, t(E)$. Then by taking the limit $E \to -B$, Eq. (3.37) becomes

$$f(-B) = t(0) \left[ g_0(-B) - g_0(0) \right] f(-B) \equiv t(0) \left[ \frac{1}{-H_0 - B} - \frac{1}{-H_0} \right] f(-B).$$

(3.38)
In the partial-wave form, the residue operator $f_W(-B)$ is related to the function $\langle p|\Gamma_i \rangle$ of Eq. (3.36) by

$$\langle p|\Gamma_i \rangle \langle \Gamma_i'|p' \rangle = \langle p|f_W(-B)|p' \rangle.$$  

(3.39)

We already calculated $t(0)$ with our subtraction method, thus the kernel of Eq. (3.38) can be evaluated directly. The energy $E = -B$ at which it has an eigenvalue of 1 is the bound-state energy, and the eigenvector is then the function $\langle p|\Gamma_i \rangle$ (up to an overall constant). Our numerical calculation gives a deuteron binding energy $B \equiv B_{\text{sub}} = 2.093$ MeV. This result is independent of the cutoff to better than 0.5% for $\Lambda > 5000$ MeV (see Table 3.3). The relative difference in the binding energy of the shallowest state obtained by solving Eq. (3.36) and Eq. (3.38) is 0.14% for $\Lambda = 50000$ MeV. From this we conclude that the subtractive renormalization works for the bound state as well, or better than it does for scattering observables. Our calculation gives a deuteron binding energy different from the experimental value $B_{\text{exp}} = 2.2246$ MeV.

The reason for this difference is that we work only with the LO $\chi$PT potential, so the scattering amplitude is only accurate up to corrections of order $(p, m_\pi^2)/\Lambda^2$.

We can check that our eigenenergy is the correct one for LO $\chi$PT in which the experimental value of $a_t$ is reproduced as follows. First, we write down the s-wave effective-range expansion for the on-shell t-matrix $t^t_{00}$,

$$-Mt^t_{00} = \frac{1}{-\frac{1}{a_t} + \frac{1}{2}r_t p_0^2 - ip_0},$$  

(3.40)

with $r_t$ being the triplet effective range. Eq. (3.40) is valid for on-shell momenta $p_0$ with the condition $|p_0| < m_\pi/2$. The pole in the t-matrix represents the shallowest
bound state of the NN system. Thus we can solve

\[ \frac{1}{a_t} - \frac{1}{2} r_t p_0^2 + ip_0 = 0 \]  

(3.41)

for \( p_0 = i\gamma \) in order to determine its location, and \( B \) is then equal to \( \gamma^2/M \). Using the experimental value of \( a_t \), and the value of \( r_t \) extracted from our phase shifts, we obtain \( B = 2.07 \) MeV. This value has a relative difference of \( \sim 1\% \) with respect to the two previously calculated values. However, this is consistent with the omission of terms of \( O(p_0^4) \) in the effective-range expansion form employed in Eq. (3.40).

We now show how to extract the deuteron wave function. The wave function is related to the above function \( \langle p|\Gamma_l \rangle \) via

\[ \psi_l(p) = -\frac{1}{B + \frac{p^2}{M}} \langle p|\Gamma_l \rangle. \]  

(3.42)

Since \( \langle p|\Gamma_l \rangle \) is equal to the eigenvector in Eq. (3.38) (up to an overall constant), we can easily obtain \( \psi_l(p) \). In Fig. 3.5 the momentum-space S-wave and D-wave calculated from the above subtraction method are compared with those obtained from the CD-Bonn potential. The overall constant is fixed by our normalization condition

\[ \int dp \, p^2 [\psi_0^2(p) + \psi_2^2(p)] = 1. \]  

(3.43)

Here \( \psi_0(\psi_2) \) is the \( ^3S_1(\!^3D_1) \) wave function. The agreement of the LO \( \chi \)PT momentum-space deuteron wave function with that obtained from the CD-Bonn potential is remarkable, especially in the S-wave.

Finally we obtain the coordinate space wave functions by Fourier transformation

\[ \frac{u_L(r)}{r} = \sqrt{\frac{2}{\pi}} \int dp p^2 j_L(pr) \psi_L(p), \]  

(3.44)
where \( j_L \) is the spherical Bessel function, with \( L = 0 \) for the \( ^3S_1 \) wave and \( L = 2 \) for the \( ^3D_1 \) wave. In Fig. 3.6 the wave functions \( u(r) \equiv u_0(r) \) and \( w(r) \equiv u_2(r) \) are shown as a function of \( r \). We see that our coordinate space wave functions have a node near the origin, which is due to the fact that the LO \( \chiPT \) potential has additional bound states at larger values of \( B \), i.e. the \( B = 2.1 \) MeV state is not the ground state. The short-distance behavior of the deuteron wave function obtained from the LO \( \chiPT \) potential has been derived analytically, and discussed in considerable detail, in Ref.
Figure 3.6: The coordinate space wave functions as functions of $r$, where $u(r)$ denotes the $^3S_1$ wave and $w(r)$ denotes the $^3D_1$ wave (solid lines). The dotted lines indicate the corresponding wave functions obtained from the CD-Bonn potential. This graph is adopted from our previous publication [73].

Our results for $u(r)$ and $w(r)$ at the distances where the Fourier transform is numerically stable appear to be in agreement with that work.
Chapter 4

Subtractive renormalization of chiral potentials up to next-to-next-to leading order: Theory

This chapter is divided into three parts. First, in section 4.1, we further develop our subtractive renormalization scheme as introduced in chapter 3 to eliminate the contact term in P-waves. In section 4.2, we derive our subtractive renormalization scheme to eliminate the contact terms in the spin-singlet S-wave, where we consider both the momentum- and energy-dependent type of contact terms. Finally, in section 4.3, we further develop the subtractive scheme to deal with the contact terms in the spin-triplet S-wave, and again, both the momentum- and energy-dependent type of contact terms are considered. Note that the derivations presented in the chapter have been published in Ref. [80–82].

4.1 Subtractive renormalization of NN interaction up to NNLO in P-waves

The next-to-leading order and next-to-next-to-leading order potentials obtained from the chiral effective theory are the so-called two-pion-exchange potential (TPE). In the next-to-leading order (NLO), the potential is computed up to $Q^2$, where $Q$ represents the difference of the momentum of nucleons: $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ or the pion mass
Figure 4.1: The diagramatic representation of the NLO TPE, where the solid (dashed) line represent a nucleon (pion). The blob represents the coupling of order $Q^2$. $m_\pi$. And the next-to-next-to-leading order (NNLO) is the $Q^3$ contribution. The diagrams corresponding to NLO and NNLO are listed in Fig. 4.1 & 4.2. One needs to regulate the divergent loop diagrams to obtain the potential. This can be done by either dimensional regularization (DR) [21, 23, 25] or spectral-function regularization (SFR) [41]. After regularization, counter terms up to $O(Q^2)$ are added to the TPE to absorb the divergences. For the NNLO pieces there are no counter terms beyond these which appear at NLO.

According to Weinberg counting, TPE up to NLO/NNLO should be associated with contact term in the form of $\lambda + \gamma(p - p')^2$ as mentioned before in Eq. (2.48) of Sec. 2.4. Here $\lambda$ and $\gamma$ are LEC’s to be determined. After partial-wave decomposition the resulting form of contact term in $p$-waves is $C_{tq}p/p$. Here $p(p')$ denotes the
Figure 4.2: The diagramatic representation of the NNLO TPE, where the solid (dashed) line represent nucleon (pion), and the blob represent the coupling of order $Q^2$.

incoming(outgoing) momentum in the center-of-momentum (c.m.) frame. So instead of a constant such as LO S-wave, here we need to treat contact terms in the form of $C_{l+l'}p'p$. We list again the partial-wave LSE

$$t^{S J}_{l l'}(p'; p; E) = v^{S J}_{l l'}(p', p) + \sum_{l''} \frac{2}{\pi} M \int_{0}^{\Lambda} dp'' \frac{p''^2}{p_0^2 + i\varepsilon - p''^2} v^{S J}_{l''l''}(p', p'') t^{S J}_{l''l''}(p'', E),$$

(4.1)

where $p_0^2/M = E$ is the c.m. energy, $\Lambda$ the cutoff, $S$ the spin quantum number, $J$ the total angular momentum quantum number, and $v^{S J}_{l l'}(p, p) = v^{LR}_{l l'} + C^{S J}_{l l'}p'p$. $v^{LR}_{l l'}$ stands for the long-range part of the potential. We still want to apply the same idea as before, which is constructing the fully-off shell partial-wave t-matrix from the knowledge of its on-shell value for zero energy. To achieve that, we first note that the generalized scattering length for arbitrary angular momenta $l$ can be defined as

$$\frac{\alpha^{S J}_{l l'}}{M} = \lim_{p_0 \to 0} \frac{t^{S J}_{l l'}(p_0, p_0; E)}{\frac{p_{l'}^2}{p_{l'}^2 + i\varepsilon}}.$$  

(4.2)

Dividing the partial-wave LSE, Eq. (4.1), by $p''p'$ we obtain

$$\frac{t^{S J}_{l l'}(p'; p; E)}{p''p'} = \frac{v^{S J}_{l l'}(p', p)}{p''p'} + \sum_{l''} \frac{2}{\pi} M \int_{0}^{\Lambda} dp'' \frac{p''^2}{p_0^2 + i\varepsilon - p''^2} v^{S J}_{l''l''}(p', p'') t^{S J}_{l''l''}(p'', E).$$

(4.3)
Eq. (4.3) is general and well-defined, which can be applied to any partial-wave. In the following we concentrate on P-waves ($l' = l = 1$). For ease of notation we drop the label $S$ and $J$ in the following derivation unless they are explicitly needed. Consider the half-shell and on-shell $t$-matrices at zero energy:

\[
\lim_{p_0 \to 0} \left[ \frac{t_{l'l}(p', p_0; E)}{p'p_0} \right] = \lim_{p_0 \to 0} \left[ \frac{v_{l'l}^{LR}(p', p_0; E)}{p'p_0} + C_{l'l} \right] + \sum_{l''} \frac{2}{\pi} M \lim_{p_0 \to 0} \left[ \frac{1}{p'p_0} \int_0^\Lambda dp''_0 (v_{l'l''}^{LR}(p', p'') + C_{l'l''} p''_0) t_{l'l}(p', p_0; E) \right] - p''_0 \] 

Subtracting Eq. (4.5) from Eq. (4.4) and multiplying both sides by $p'$, the unknown constant $C_{l'l}$ cancels and we obtain

\[
\lim_{p_0 \to 0} \left[ \frac{t_{l'l}(p', p_0; E)}{p_0} \right] - \frac{\alpha_{l'l}}{M} p' = \lim_{p_0 \to 0} \left[ \frac{v_{l'l}^{LR}(p', p_0)}{p_0} \right] - p' \lim_{p_0 \to 0} \left[ \frac{v_{l'l}^{LR}(p_0, p_0)}{p_0^2} \right] + \sum_{l''} \frac{2}{\pi} M \int_0^\Lambda dp''_0 (v_{l'l''}^{LR}(p', p'') - \lim_{p_0 \to 0} \left[ \frac{v_{l'l''}^{LR}(p_0, p''_0)}{p_0} \right] p'') \lim_{p_0 \to 0} \left[ \frac{t_{l'l''}(p'', p_0; 0)}{p_0} \right] - p''_0 \] 

(4.6)

Here we have used that for p-waves ($l' = l = 1$), $\lim_{p_0 \to 0} \left[ \frac{t_{l'l}(p_0, p_0; E)}{p_0 p_0} \right] = \frac{\alpha_{l'l}}{M}$. Since $v_{l'l}(p, k) \approx p''_0 k^l$, all of the above limits of the potentials are well defined and can be directly obtained from Eqs. (2.32)-(2.39). Note that for the coupled channels $^3P_2-^3F_2$, the contact term only exists for $(l, l') = (1, 1)$, i.e., the $^3P_2-^3P_2$ part, if one considers operators up to $O(Q^3)$. Thus, up to $O(Q^3)$, the treatment for this channel is exactly the same as for the uncoupled channels, since there is no new unknown
constant. The only unknown left in Eq. (4.6) is \( \lim_{p_0 \to -0} \left[ \frac{t_{\nu l}(p', p_0; E)}{p_0} \right] \), and one can solve the equation by standard techniques (see Appendix B). Now, having the half-shell t-matrix \( \lim_{p_0 \to -0} \left[ \frac{t_{\nu l}(p', p_0; E)}{p_0} \right] \) for zero energy in hand, we can write another set of two LSEs to obtain the fully off-shell t-matrix,

\[
\frac{t_{\nu l}(p, p'; 0)}{p p'} = \frac{v_{\nu l}^{LR}(p, p')}{p_0 p'} + C_{\nu l} + \sum_{\nu'} \frac{2}{\pi} M \left[ \frac{1}{p_0 p'} \int_{0}^{\Lambda} dp'' \, p''^2 \left( v_{\nu l}^{LR}(p, p'') + C_{\nu l} \right) t_{\nu l}(p'', p'; 0) \right]
\]  \( (4.7) \)

\[
\lim_{p_0 \to -0} \left[ \frac{t_{\nu l}(p_0, p'; E)}{p_0 p'} \right] = \lim_{p_0 \to -0} \left[ \frac{v_{\nu l}^{LR}(p_0, p')}{p_0 p'} + C_{\nu l} \right] + \sum_{\nu'} \frac{2}{\pi} M \lim_{p_0 \to -0} \left[ \frac{1}{p_0 p'} \int_{0}^{\Lambda} dp'' \, p''^2 \left( v_{\nu l}^{LR}(p_0, p'') + C_{\nu l} \right) t_{\nu l}(p'', p'; E) \right]
\]  \( (4.8) \)

Subtracting Eq. (4.8) from Eq. (4.7) and multiplying both sides by \( p \), the constant \( C_{\nu l} \) again cancels and we arrive at

\[
\frac{t_{\nu l}(p, p'; 0)}{p'} - p \lim_{p_0 \to -0} \left[ \frac{t_{\nu l}(p_0, p'; E)}{p_0 p'} \right] = \frac{v_{\nu l}^{LR}(p, p')}{p_0 p'} - p \lim_{p_0 \to -0} \left[ \frac{v_{\nu l}^{LR}(p_0, p')}{p_0 p'} \right] + \sum_{\nu'} \frac{2}{\pi} M \int_{0}^{\Lambda} dp'' \, p''^2 \left( v_{\nu l}^{LR}(p, p'') - p \lim_{p_0 \to -0} \left[ \frac{v_{\nu l}^{LR}(p_0, p'')}{p_0} \right] \right) t_{\nu l}(p'', p'; 0)
\]  \( (4.9) \)

We then use the property of the t-matrix that \( t_{\nu l}(k, p'; 0) = t_{\nu l}(p', k; 0) \) to obtain \( \lim_{p_0 \to -0} \left[ \frac{t_{\nu l}(p_0, p'; E)}{p_0 p'} \right] \) and solve for \( \frac{t_{\nu l}(p_0, p'; 0)}{p} \) in Eq. (4.9). Once we obtain \( \frac{t_{\nu l}(p_0, p'; 0)}{p} \), or equivalently \( t_{\nu l}(p, p'; 0) \), we can proceed to calculate the on-shell t-matrix and the phase shifts using Eq. (3.31), i.e.,

\[
t(E) = t(0) + t(0)(g_0(E) - g_0(0))t(E).
\]  \( (4.10) \)
4.2 Subtractive renormalization in S-waves: J=0 singlet channel

In this section we will introduce the subtractive renormalization technique for S-waves contact terms up to NNLO. Before introducing our subtractive renormalization scheme, we need to decide what types of contact term we should consider.

According to Weinberg counting, TPE at NLO/NNLO is normally associated with contact terms up to $O(Q^2)$. However, since for non-relativistic particles we have $p^2/M \sim E$ there is the possibility to consider energy-dependent, instead of momentum-dependent contact interactions. In particular, this may have certain advantages in terms of evading theorems that limit the impact of short-distance potentials on phase shifts [83, 84]. Thus, in the $^1S_0$ channel we will consider the following three types of contact terms:

1. $v_{SR,0} = \lambda$

2. $v_{SR,0}(E) = \lambda + \gamma E$

3. $v_{SR,0}(p', p) = \lambda + C_2 \left(p^2 + p'^2\right)$,

where $\lambda, \gamma, C_2$ are unknown constants. (Of course, the numerical value of $\lambda$ is different for each case.) The overall potential is then given by

$$v_0(p', p; E) = v_{0}^{SR}(p', p; E) + v_{0}^{LR}(p', p; E), \quad (4.11)$$
with the subscripts $SR$ and $LR$ standing for the short- and long-range parts of the potential. Specifically, in the $^1S_0$ channel we have

$$v_0^{LR} = \langle 000| V_C + W_C - 3[V_S + W_S + q^2(V_T + W_T)]|000\rangle.$$  \hfill (4.12)

with $V_{C,S,T}$ and $W_{C,S,T}$ given in section 4.1.

### 4.2.1 Constant contact interaction

The constant contact term can be evaluated by one subtraction with the $^1S_0$ scattering length $a_s$ as input as explained in Chapter 3. Note that the only difference here with respect to Chapter 3 is that OPE is replaced by TPE for the long range part of the potential.

### 4.2.2 Energy-dependent contact interaction

In this case there are two unknown constants. The strategy is to use one subtraction (as done in Chapter 3) to first eliminate the constant part of the contact term. For the energy-dependent contact term we then perform a second subtraction to eliminate the remaining unknown.

We start from the partial-wave LSE, which is written explicitly in this channel as

$$t_0(p', p; E) = v_0(p', p; E) + \frac{2}{\pi} M \int_0^\Lambda \frac{dp''}{p''} \frac{p''^2 v_0(p', p''; E) t_0(p'', p, E)}{p_0^2 + i\varepsilon - p''^2}. \hfill (4.13)$$

The energy-dependent $^1S_0$ potential is given by

$$v_0(p', p; E) = v_0^{LR}(p', p) + \lambda + \gamma E.$$  \hfill (4.14)
Here $v_0^{LR}$ represents the long range part of the $\chi$PT potential, which is computed up to NLO or NNLO. However, the derivation presented below holds for any energy-independent long-range potential which is a function of $p$ and $p'$ and satisfies $v^{LR}(p', p) = v^{LR}(p, p')$. From now on we drop the partial-wave index, since this section only describes the singlet channel $^1S_0$.

To simplify the derivation, we now adopt the following short-hand notation and rewrite the LSE as

$$t(E) = \lambda + \gamma E + v^{LR} + (\lambda + \gamma E + v^{LR})g_0(E)t(E). \quad (4.15)$$

Here $g_0(E)$ represents the free Green’s function. Let $E = 0$ in Eq.(4.15), then

$$t(0) = \lambda + v^{LR} + (\lambda + v^{LR})g_0(0)t(0). \quad (4.16)$$

Eq.(4.16) has only one unknown-$\lambda$, thus $t(0)$ can be obtained from an experimental data, i.e., the NN scattering length $a_s$, and the procedure is exactly the same as listed in Chapter 3.

Rewrite Eq.(4.15) as

$$t(E) = [\lambda + \gamma E + v^{LR}][1 + g_0(E)t(E)]. \quad (4.17)$$

Here $t(E)$ is a matrix consisting of various $p'$ and $p$ as the index. From Eq.(4.17) we have

$$\lambda + \gamma E + v^{LR} = t(E)[1 + g_0(E)t(E)]^{-1}. \quad (4.18)$$

Due to property of LSE, Eq.(4.16) can be rewritten as

$$t(0) = \lambda + v^{LR} + t(0)g_0(0)(\lambda + v^{LR}) = [1 + t(0)g_0(0)][\lambda + v^{LR}]. \quad (4.19)$$
So,
\[
\lambda + v^{LR} = [1 + t(0)g_0(0)]^{-1}t(0). \tag{4.20}
\]

Subtracting Eq.(4.20) from Eq.(4.18), we have
\[
\gamma E = t(E)[1 + g_0(E)t(E)]^{-1} - [1 + t(0)g_0(0)]^{-1}t(0). \tag{4.21}
\]

Now suppose we know \(t(E^*)\) for another energy \(E^*\), then we have
\[
\lambda + \gamma E^* + v^{LR} = t(E^*)[1 + g_0(E^*)t(E^*)]^{-1}. \tag{4.22}
\]

Eq.(4.22) minus Eq.(4.18) gives
\[
\gamma(E^* - E) = t(E^*)[1 + g_0(E^*)t(E^*)]^{-1} - t(E)[1 + g_0(E)t(E)]^{-1}. \tag{4.23}
\]

Now substituting Eq.(4.21) into Eq.(4.23) to eliminate \(\gamma\), we then have
\[
\frac{t(E)[1 + g_0(E)t(E)]^{-1} - [1 + t(0)g_0(0)]^{-1}t(0)}{E}(E^* - E)
= t(E^*)[1 + g_0(E^*)t(E^*)]^{-1} - t(E)[1 + g_0(E)t(E)]^{-1}. \tag{4.24}
\]

Re-arranging Eq.(4.24) and multiplying both sides by \(1 + g_0(E)t(E)\) from the right
and \(1 + t(0)g_0(0)\) from the left, we finally obtain
\[
\{(1 - \frac{E^*}{E})t(0)g_0(E) + [1 + t(0)g_0(0)]\frac{E^*}{E} - t(E^*)\alpha g_0(E)\}t(E)
= (\frac{E^*}{E} - 1)t(0) + [1 + t(0)g_0(0)]t(E^*)\alpha, \tag{4.25}
\]

\(^1\)Note that here the point of singularity in the integral is different. If one uses the standard
 technique described in Appendix B to evaluate the integral, then to account for the difference of
 singularity points, a term \(V = v^{LR}(p^*, p^*) - v^{LR}(p', p)\) must be included in the right-hand side
 of Eq.(4.23). Here the 1 \(\sim\) mesh points are the same, i.e., \(p_{(1 \sim n)} = p'_{(1 \sim n)}\), but \(p_{n+1} = \sqrt{ME}\); \(p'_{n+1} = \sqrt{ME^*}\).
where \( \alpha = [1 + g_0(E^*) t(E^*)]^{-1} \). In Eq.(4.25), the only unknown is \( t(E) \), so we can solve it by the standard method\(^2\).

The previous procedure assumes that we know \( t(E^*) \), now we will show how to generate \( t(E^*) \) by just inputting an experimental value, i.e., phase shift \( \delta(E^*) \). First note that we can get \( t(p_0, p_0, E^*) \) by inputting \( \delta(E^*) \), i.e.,

\[
t(p_0, p_0, E^*) = \frac{f_{00}}{-M p_0} = \frac{e^{i \delta(E^*)} \sin(\delta(E^*))}{-M p_0},
\]

(4.28)

where \( f_{00} \) represents the scattering amplitude, \( p_0 = \sqrt{M E^*} \). Then we write the half-shell and on-shell LSEs as

\[
t(p^*, p_0; E^*) = \lambda + \gamma E^* + v^{LR}(p^*, p_0) + \frac{2 \pi}{M} \int_{0}^{\Lambda} dp^* \frac{p^*}{p_0^2 - p^2 + i \varepsilon} t(p^*, p_0; E^*)
\]

\[
t(p_0, p_0; E^*) = \lambda + \gamma E^* + v^{LR}(p_0, p_0) + \frac{2 \pi}{M} \int_{0}^{\Lambda} dp^* \frac{p^*}{p_0^2 - p^2 + i \varepsilon} t(p^*, p_0; E^*).
\]

(4.29)

(4.30)

Subtracting Eq. (4.29) from Eq. (4.30) gives

\[
t(p^*, p_0; E^*) = v^{LR}(p^*, p_0) - v^{LR}(p_0, p_0) + \frac{e^{i \delta(E^*)} \sin(\delta(E^*))}{-M p_0}
\]

\[
\]

\(^2\)If one adopts the standard technique described in Appendix B, then Eq.(4.25) becomes

\[
\{(1 - \frac{E^*}{E}) t(0) g_0(E) + [1 + t(0) g_0(0)] \frac{E^*}{E} + V g_0(E) - t(E^*) \alpha g_0(E) \} t(E)
\]

\[
= (\frac{E^*}{E} - 1) t(0) + [1 + t(0) g_0(0)] [t(E^*) \alpha - V].
\]

(4.26)

(4.27)
The above equation does not contain any unknown quantities and $t(p^r, p_0; E^*)$ can be solved with standard techniques.

With $t(p^r, p_0; E^*)$ in hand, the next step is to do another subtraction to get $t(p^r, k^*; E^*)$, where $k^*$ represents an arbitrary momentum.

$$t(p^r, k^*; E^*) = \lambda + \gamma E^* + v^{LR}(p^r, k^*)$$

$$+ \frac{2}{\pi} M \int_0^\Lambda dp^r p^r^2 \left( \frac{\lambda + \gamma E^* + v^{LR}(p^r, p^*)}{p_0^2 - p^r^2 + i\varepsilon} \right) t(p^*, k^*; E^*). \tag{4.32}$$

$$t(p_0, k^*; E^*) = \lambda + \gamma E^* + v^{LR}(p_0, k^*)$$

$$+ \frac{2}{\pi} M \int_0^\Lambda dp^r p^r^2 \left( \frac{\lambda + \gamma E^* + v^{LR}(p_0, p^*)}{p_0^2 - p^r^2 + i\varepsilon} \right) t(p^*, k^*; E^*). \tag{4.33}$$

The difference of Eq. (4.32) and Eq. (4.33) gives

$$t(p^r, k^*; E^*) - t(p_0, k^*; E^*) = v^{LR}(p^r, k^*) - v^{LR}(p_0, k^*)$$

$$+ \frac{2}{\pi} M \int_0^\Lambda dp^r p^r^2 \left( \frac{v^{LR}(p^r, p^*) - v^{LR}(p_0, p^*)}{p_0^2 - p^r^2 + i\varepsilon} \right) t(p^*, k^*; E^*). \tag{4.34}$$

Using the fact that $t(p_0, k^*; E^*) = t(k^*, p_0; E^*)$ (see Appendix C), then by solving Eq. (4.34) we can obtain $t(p^r, k^*; E^*)$.

To summarize, we perform two subtractions to the LSE to eliminate the two unknown constants $\lambda$ and $\gamma$. The resulting equation requires as input the scattering length $a_s$ and the phase shift at one specific energy $E^*$. The only restriction on $E^*$ is that it must be within the domain of validity of our theory. Hence one can test the
consistency of the theory by examining the extent to which results depend upon the choice of $E^*$.  

4.2.3 Momentum-dependent contact interaction  

The coefficient of the momentum-dependent contact term, denoted here by $C_2$, is not straightforwardly related to any S-matrix element. The reason is that the contact term has both $p$ and $p'$ dependence and cannot be associated with a physical observable. Thus we cannot apply our subtraction procedure in the case of a momentum-dependent contact interaction. Hence we instead adopt a “mixed” procedure, which involves a single subtraction plus fitting of $C_2$.  

This “mixed” procedure is carried out as follows: First we guess a value for the constant $C_2$, and then insert it into the half-off-shell and on-shell LSE:

$$
t(p, 0; 0) = v^{LR}(p, 0) + \lambda + C_2 p^2
- \frac{2}{\pi} M \int_0^\Lambda dp' \left[ v^{LR}(p, p') + \lambda + C_2 (p^2 + p'^2) \right] t(p', 0; 0),
$$

$$
t(0, 0; 0) = v^{LR}(0, 0) + \lambda
- \frac{2}{\pi} M \int_0^\Lambda dp' \left( v^{LR}(0, p') + \lambda + C_2 p'^2 \right) t(p', 0; 0). \tag{4.35}
$$

Taking the difference of the two equations cancels the constant $\lambda$ and leads to

$$
t(p, 0; 0) = v^{LR}(p, 0) - v^{LR}(0, 0) + \frac{a_s}{M} + C_2 p^2
- \frac{2}{\pi} M \int_0^\Lambda dp' \left( v^{LR}(p, p') - v^{LR}(0, p') + C_2 p'^2 \right) t(p', 0; 0). \tag{4.36}
$$
Using the already-chosen value of $C_2$, together with the experimental value of $a_s$, we can solve for $t(p', 0; 0)$ from Eq. (4.36). Then there is a consistency condition that determines the value of $\lambda$. This equation can be easily derived from the second equation in Eq. (4.35):

$$\lambda = \frac{\frac{a_s}{M} - v^{LR}(0, 0) + \frac{2 \pi M}{\pi} \int_0^\Lambda dp' \left( v^{LR}(0, p') + C_2 p'^2 \right) t(p', 0; 0)}{1 - \frac{2 \pi M}{\pi} \int_0^\Lambda dp' t(p', 0; 0)}.$$  (4.37)

The above equation gives the value of $\lambda$ that is consistent with the experimental value of the scattering length and our choice of $C_2$. It thus defines a relationship $\lambda = \lambda(C_2; a_s)$. (Note that—in spite of the form of Eq. (4.37)—the relationship is not linear, since $t(p', 0; 0)$ is also affected by the choice of $C_2$.) Therefore, when trying to determine $\lambda$ and $C_2$ we only need to guess $C_2$ and can then obtain $\lambda$ from Eqs. (4.36) and (4.37). These two constants are then entered into the on-shell LSE which is solved for the phase shifts. Finally $C_2$ is adjusted to fit the desired observable. In Chapter 5.3 we will examine the results obtained when $C_2$ is adjusted to reproduce the $^1S_0$ effective range, $r_0$, and those found when we enforce the requirement that the theory correctly predict the phase shift at a particular energy $E^*$. 
4.3 Subtractive renormalization in S-waves: \( J=1 \)

**triplet channel**

Since the NN interaction is non-central, the \( S = 1 \) waves constitutes a coupled-channel problem. Here we consider the \( ^3S_1-^3D_1 \) coupled partial-waves, and we again consider three different contact terms:

\[
(1) \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
(2) \begin{pmatrix} \lambda + C_2(p^2 + p'^2) & \lambda t \ p'^2 \\ \lambda t \ p^2 & 0 \end{pmatrix}
\]

\[
(3) \begin{pmatrix} \lambda + \gamma E & \lambda t \ p'^2 \\ \lambda t \ p^2 & 0 \end{pmatrix}
\]

Here we write the contact terms explicitly in their matrix form, where the diagonal represents the direct channels \( ^3S_1-^3S_1 \) and \( ^3D_1-^3D_1 \) and the off-diagonal channels \( ^3S_1-^3D_1 \) and \( ^3D_1-^3S_1 \). The unknown constants are \( \lambda, \gamma \) (or \( C_2 \)) and \( \lambda_t \), and their value is different in each case. Case A is the leading-order contact interaction discussed in Chapter 3. Cases B and C include the structures that appear at NLO \([O(P^2)]\) in the standard chiral counting for short-distance operators.

At this order in the chiral expansion the \( J=1 \) coupled-channel problem acquires two additional contact interactions. One is of the form \( \sigma_1 \cdot q \sigma_2 \cdot q \) \([25]\), and hence gives a non-zero matrix element for \( ^3S_1-^3D_1 \) transition. The other can be written as either
an energy-dependent or momentum-dependent piece of the $^3S_1-{^3S_1}$ potential, although until now most works on $\chi$PT have considered only the latter [21, 25, 28, 42], but see Ref. [43]. For the energy-dependent contact interaction we will show below that three subtractions can be performed to eliminate the unknown LEC’s. This leaves us with an integral equation for the $t$-matrix at an arbitrary energy that takes as input three experimental quantities. For case B, due to the momentum-dependence in the contact term, we use a double-subtraction-plus-one-fit scheme to solve the LSE. The procedures for solving (B) and (C) are quite similar, as they both employ subtractions to eliminate $\lambda$ and $\lambda_t$.

4.3.1 Constant contact interaction

This case can be solved with a single subtraction, as described in Chapter 3. Once again, the key fact is that once $a_t$, the triplet scattering length, is known, the value of $t(0,0;0)$ is fixed, and that, together with knowledge of the long-range potential, it is sufficient to obtain the on-shell element $t(p_0,p_0;E)$.

4.3.2 Momentum-dependent central part

In this case we will eliminate two constants from the integral equation: $\lambda$, and the coefficient of the tensor short-distance interaction $\lambda_t$. We will then adjust the coefficient $C_2$ to reproduce one piece of ‘experimental’ information.
As usual, we begin by writing the on-shell and half-off-shell partial-wave LSE:

\[
\begin{pmatrix}
  t_{00}(p_0, p_0; E) & t_{02}(p_0, p_0; E) \\
  t_{20}(p_0, p_0; E) & t_{22}(p_0, p_0; E)
\end{pmatrix} =
\begin{pmatrix}
  \lambda + C_2(p_0^2 + p_0'^2) + v_{00}^{LR}(p_0, p_0) & \lambda t_0^2 + v_{02}^{LR}(p_0, p_0) \\
  \lambda t_0^2 + v_{20}^{LR}(p_0, p_0) & v_{22}^{LR}(p_0, p_0)
\end{pmatrix}
\]

\[
+ \frac{2}{\pi} M \int_0^\Lambda dp' \frac{p'^2}{p_0^2 - p'^2 + i\epsilon} \begin{pmatrix}
  \lambda + C_2(p_0^2 + p_0'^2) + v_{00}^{LR}(p_0, p') & \lambda t_0^2 + v_{02}^{LR}(p_0, p') \\
  \lambda t_0^2 + v_{20}^{LR}(p_0, p') & v_{22}^{LR}(p_0, p')
\end{pmatrix}
\times \begin{pmatrix}
  t_{00}(p', p_0; E) & t_{02}(p', p_0; E) \\
  t_{20}(p', p_0; E) & t_{22}(p', p_0; E)
\end{pmatrix},
\]

(4.38)

\[
\begin{pmatrix}
  t_{00}(p, p_0; E) & t_{02}(p, p_0; E) \\
  t_{20}(p, p_0; E) & t_{22}(p, p_0; E)
\end{pmatrix} =
\begin{pmatrix}
  \lambda + C_2(p^2 + p_0^2) + v_{00}^{LR}(p, p_0) & \lambda t_0^2 + v_{02}^{LR}(p, p_0) \\
  \lambda t_0^2 + v_{20}^{LR}(p, p_0) & v_{22}^{LR}(p, p_0)
\end{pmatrix}
\]

\[
+ \frac{2}{\pi} M \int_0^\Lambda dp' \frac{p'^2}{p_0^2 - p'^2 + i\epsilon} \begin{pmatrix}
  \lambda + C_2(p^2 + p_0'^2) + v_{00}^{LR}(p, p') & \lambda t_0^2 + v_{02}^{LR}(p, p') \\
  \lambda t_0^2 + v_{20}^{LR}(p, p') & v_{22}^{LR}(p, p')
\end{pmatrix}
\times \begin{pmatrix}
  t_{00}(p', p_0; E) & t_{02}(p', p_0; E) \\
  t_{20}(p', p_0; E) & t_{22}(p', p_0; E)
\end{pmatrix},
\]

(4.39)

where $p_0^2/M = E$. The subscript (superscript) in $t_{vl}(v_{LR}^l)$ indicates the angular-momentum quantum number of the channels. To simplify notation, we write the
$2 \times 2$ matrix as $t$. Subtracting Eq. (4.39) from Eq. (4.38) cancels $\lambda$,

$$
\begin{align*}
t(p, p_0; E) - t(p_0, p_0; E) &= \\
&= \left( C_2(p^2 - p_0^2) + v_{00}^{LR}(p, p_0) - v_{00}^{LR}(p_0, p_0) \right. \\
&\quad + \lambda_t(p^2 - p_0^2) + v_{02}^{LR}(p, p_0) - v_{02}^{LR}(p_0, p_0) \\
&\quad + \frac{2}{\pi} M \int_0^\Lambda dp' \frac{p'^2}{p_0^2 - p'^2 + i\varepsilon} \\
&\left. \begin{pmatrix}
C_2(p^2 - p_0^2) + v_{00}^{LR}(p, p') - v_{00}^{LR}(p_0, p') \\
\lambda_t(p^2 - p_0^2) + v_{02}^{LR}(p, p') - v_{02}^{LR}(p_0, p') \\
\lambda_t p^2 + v_{20}^{LR}(p, p') \\
\lambda_t p^2 + v_{22}^{LR}(p, p')
\end{pmatrix}
\right) t(p', p_0; E).
\end{align*}
$$

(4.40)

Letting $E \to 0$, i.e. $p_0 \to 0$, in Eq. (4.40) leads to

$$
\begin{align*}
t(p, 0; 0) - t(0, 0; 0) &= \begin{pmatrix}
C_2 p^2 + v_{00}^{LR}(p, 0) - v_{00}^{LR}(0, 0) \\
\lambda_t p^2 + v_{20}^{LR}(p, 0) \\
\lambda_t p^2 + v_{22}^{LR}(p, 0)
\end{pmatrix} 0 \\
- \frac{2}{\pi} M \int_0^\Lambda dp' \begin{pmatrix}
C_2 p'^2 + v_{00}^{LR}(p', 0') - v_{00}^{LR}(0, 0') \\
\lambda_t p'^2 + v_{02}^{LR}(p', 0') \\
\lambda_t p'^2 + v_{20}^{LR}(p', 0') \\
\lambda_t p'^2 + v_{22}^{LR}(p', 0')
\end{pmatrix} t(p', 0; 0). 
\end{align*}
$$

(4.41)

Here we used the threshold behavior of the partial-wave potential, $v_{LR}^{l'}(p, k) \sim p'^l k^l$, to infer that, e.g. $v_{20}^{LR}(0, p) = 0$.

Equation (4.41) shows that this feature of $v_{LR}$ has as consequence that $t_{02}(p, 0; 0) = t_{20}(0, p; 0) = 0$ and $t_{22}(p, 0; 0) = t_{22}(0, p; 0) = 0$. Using Eq. (4.41) we can thus obtain
the first part of the t-matrix equation that needs to be solved, namely

\[ t_{00}(p, 0; 0) - \frac{a_t}{M} = C_2 p^2 + v_{00}^{LR}(p, 0) - v_{00}^{LR}(0, 0) \]

\[ -\frac{2}{\pi} M \int_0^{\Lambda} dp' \left[ (C_2 p^2 + v_{00}^{LR}(p, p')) - v_{00}^{LR}(0, 0) \right] t_{00}(p', 0; 0) \]

\[ + \left( v_{02}^{LR}(p, p') - v_{02}^{LR}(0, 0) \right) t_{20}(p', 0; 0) \],

(4.42)

where we have used that \( t_{00}(0, 0; 0) = \frac{a_t}{M} \). We observe that \( \lambda_t \) has been eliminated from this equation, but that we do not have a closed set of equations, since \( t_{20}(p', 0; 0) \) appears on the right-hand side.

Consider \( p = 0 \) in Eq. (4.40)

\[ t(0, p_0; E) - t(p_0, p_0; E) \]

\[ = \begin{pmatrix} -C_2 p_0^2 + v_{00}^{LR}(0, p_0) - v_{00}^{LR}(p_0, p_0) & v_{02}^{LR}(0, p_0) - v_{02}^{LR}(p_0, p_0) \\ -\lambda t p_0^2 - v_{20}^{LR}(p_0, p_0) & -v_{22}^{LR}(p_0, p_0) \end{pmatrix} \]

\[ + \frac{2}{\pi} M \int_0^{\Lambda} dp' \frac{p'^2}{p_0^2 - p'^2 + i\varepsilon} \]

\[ \times \begin{pmatrix} -C_2 p_0^2 + v_{00}^{LR}(0, p') - v_{00}^{LR}(p_0, p') & v_{02}^{LR}(0, p') - v_{02}^{LR}(p_0, p') \\ -\lambda t p_0^2 - v_{20}^{LR}(p_0, p') & -v_{22}^{LR}(p_0, p') \end{pmatrix} t(p', p_0; E). \]

(4.43)

In order to use the threshold behavior of \( t_{20} \) we need to divide Eq. (4.40) by \( p^2 - p_0^2 \) and Eq. (4.43) by \( p_0^2 \) and then add both equations. A quick examination shows that
\[ \frac{t(p, p_0; E) - t(p_0, p_0; E)}{p^2 - p_0^2} + \frac{t(0, p_0; E) - t(p_0, p_0; E)}{p_0^2} = \left( \begin{array}{ccc}
\frac{v_{10}^{LR}(p_0, p_0) - v_{10}^{LR}(p_0, 0, p_0)}{p^2 - p_0^2} + & \frac{v_{10}^{LR}(0, p_0) - v_{10}^{LR}(p_0, p_0)}{0} & \frac{v_{10}^{LR}(p, p_0) - v_{10}^{LR}(p_0, p_0)}{p^2 - p_0^2} + \frac{v_{10}^{LR}(0, p_0) - v_{10}^{LR}(p_0, p_0)}{p_0^2} \\
\frac{v_{22}^{LR}(p, p_0) - v_{22}^{LR}(p_0, p_0)}{p^2 - p_0^2} & \frac{v_{22}^{LR}(p, p_0) - v_{22}^{LR}(p_0, p_0)}{p^2 - p_0^2} & \frac{v_{22}^{LR}(p, p_0) - v_{22}^{LR}(p_0, p_0)}{p_0^2} \\
\frac{v_{22}^{LR}(p, p_0) - v_{22}^{LR}(p_0, p_0)}{p^2 - p_0^2} & \frac{v_{22}^{LR}(p, p_0) - v_{22}^{LR}(p_0, p_0)}{p^2 - p_0^2} & \frac{v_{22}^{LR}(p, p_0) - v_{22}^{LR}(p_0, p_0)}{p_0^2} 
\end{array} \right) 
\] 
\[ + \frac{2}{\pi} \frac{M}{\Lambda} \int_0^\Lambda dp' \frac{p'^2}{p_0^2 - p'^2 + i\varepsilon} \left( \begin{array}{ccc}
\frac{v_{10}^{LR}(p, p') - v_{10}^{LR}(p_0, p')}{p^2 - p_0^2} & \frac{v_{10}^{LR}(0, p') - v_{10}^{LR}(p_0, p')}{p_0^2} & \frac{v_{10}^{LR}(p, p') - v_{10}^{LR}(p_0, p')}{p^2 - p_0^2} + \frac{v_{10}^{LR}(0, p') - v_{10}^{LR}(p_0, p')}{p_0^2} \\
\frac{v_{22}^{LR}(p, p') - v_{22}^{LR}(p_0, p')}{p^2 - p_0^2} & \frac{v_{22}^{LR}(p, p') - v_{22}^{LR}(p_0, p')}{p^2 - p_0^2} & \frac{v_{22}^{LR}(p, p') - v_{22}^{LR}(p_0, p')}{p_0^2} \\
\frac{v_{22}^{LR}(p, p') - v_{22}^{LR}(p_0, p')}{p^2 - p_0^2} & \frac{v_{22}^{LR}(p, p') - v_{22}^{LR}(p_0, p')}{p^2 - p_0^2} & \frac{v_{22}^{LR}(p, p') - v_{22}^{LR}(p_0, p')}{p_0^2} 
\end{array} \right) \times t(p', p_0; E). \] (4.44)

Let \( E \to 0 \) \( (p_0 \to 0) \) in Eq. (4.44), we can then write down the other half
\((t_{20}(p, 0; 0) \) part) of the \( t \)-matrix equation:

\[ \frac{t_{20}(p, 0; 0)}{p^2} = \left[ \frac{t_{20}(p_0, p_0; E)}{p_0^2} \right] + \left[ \frac{v_{20}^{LR}(p, 0)}{p^2} \right] - \lim_{p_0 \to 0} \left[ \frac{v_{20}^{LR}(p_0, p_0)}{p_0^2} \right] \]

\[ - \frac{2}{\pi} \frac{M}{\Lambda} \int_0^\Lambda dp' \left[ \left( \frac{v_{20}^{LR}(p, p')}{p^2} \right) - \lim_{p_0 \to 0} \left[ \frac{v_{20}^{LR}(p_0, p')}{p_0^2} \right] \right] t_{00}(p', 0; 0) \]

\[ + \left( \frac{v_{22}^{LR}(p, p')}{p^2} \right) - \lim_{p_0 \to 0} \left[ \frac{v_{22}^{LR}(p_0, p')}{p_0^2} \right] \right] t_{20}(p', 0; 0) \] . (4.45)

This is the key result of our derivation, since it provides a second integral equation
that, together with Eq. (4.42), is a closed set for the half-off-shell \( t \)-matrix at zero
energy. These equations require as input the long-range potential \( v_{LR} \), a choice of \( C_2 \),
and the experimental observables \( a_t \) and the value of

\[ \lim_{p_0 \to 0} \left[ \frac{t_{20}(p_0, p_0; E)}{p_0^2} \right] \] . (4.46)
This limit, and all the limits of \( v_{LR} \) in Eq. (4.45), are well defined due to the threshold behavior of the potential and the t-matrix. The limit (4.46) is a specific case of a “generalized scattering length”, which is defined as

\[
\frac{\alpha_{ll'}}{M} = \lim_{p_0 \to 0} \left[ \frac{t_{l l'}(p_0, p_0; E)}{p_0^{l + 1}} \right].
\] (4.47)

Since \( \alpha_{ll'} \) is related to an on-shell element of the t-matrix it can be obtained from an experimental phase-shift analysis (see, e.g. Ref. [92]).

Equations (4.42) and (4.45), together with an initial guess for the value of \( C_2 \), now yield \( t_{00}(p, 0; 0) \) and \( t_{20}(p, 0; 0) \) by standard methods for the solution of integral equations. The other elements of the zero-energy \( t \), \( t_{02}(p, 0; 0) \) and \( t_{22}(p, 0; 0) \), are equal to zero, as mentioned above.

It is formally possible to eliminate \( C_2 \) from Eq. (4.44) by a similar set of manipulations to the ones that led to Eq. (4.45). This yields a pair of coupled integral equations for \( t_{00}(p, 0; 0) \) and \( t_{20}(p, 0; 0) \) that is NN LEC free. However, these equations contain the quantity \( \lim_{p_0 \to 0} \left[ \frac{t_{00}(0, p_0; E)}{p_0^{l + 1}} \right] \) as part of the driving term. This is a limit of a half-off-shell matrix element, and so cannot be obtained in a model-independent way from experimental data.

Once we guess a value of \( C_2 \) and have \( t_{00}(p, 0; 0) \) and \( t_{20}(p, 0; 0) \) evaluated, we can apply the same logic as what we done for the case (3) in the singlet channel to solve for another two unknown constants \( \lambda \) & \( \lambda_t \) and obtain the phase shift. We substitute \( C_2, t_{00}(p, 0; 0) \) and \( t_{20}(p, 0; 0) \) into Eq. (4.41). The only unknown in Eq. (4.41) is \( \lambda_t \),
which can then be expressed as
\[
\lambda_t = \frac{t_{20}(p, 0; 0) - v_{20}^{LR}(p, 0) + \frac{2}{\pi}M \int_0^\Lambda dp' \left( v_{20}^{LR}(p, p')t_{00}(p', 0; 0) + v_{22}^{LR}(p, p')t_{20}(p', 0; 0) \right)}{p^2(1 - \frac{2}{\pi}M \int_0^\Lambda dp't_{00}(p', 0; 0))}.
\] (4.48)

Once we evaluated the value of \( \lambda_t \), then we can substitute \( \lambda_t, C_2, t_{00}(p, 0; 0) \) and \( t_{20}(p, 0; 0) \) in the following LSE:
\[
t(0, 0; 0) = \begin{pmatrix} a_t/M & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda + v_{00}^{LR}(0, 0) & 0 \\ 0 & 0 \end{pmatrix} + \frac{2}{\pi}M \int_0^\Lambda dp' \begin{pmatrix} \lambda + C_2p'^2 + v_{00}^{LR}(0, p') & \lambda p'^2 + v_{02}^{LR}(0, p') \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} t_{00}(p', 0; 0) & 0 \\ t_{20}(p', 0; 0) & 0 \end{pmatrix},
\] (4.49)

and obtain an expression for \( \lambda \):
\[
\lambda = \frac{a_t/M - v_{00}^{LR}(0, 0)}{1 - \frac{2}{\pi}M \int_0^\Lambda dp't_{00}(p', 0; 0) + \frac{2}{\pi}M \int_0^\Lambda dp' \left[ (C_2p'^2 + v_{00}^{LR}(0, p'))t_{00}(p', 0; 0) + (\lambda p'^2 + v_{02}^{LR}(0, p'))t_{20}(p', 0; 0) \right]}{1 - \frac{2}{\pi}M \int_0^\Lambda dp't_{00}(p', 0; 0) + \frac{2}{\pi}M \int_0^\Lambda dp' \left[ (C_2p'^2 + v_{00}^{LR}(0, p'))t_{00}(p', 0; 0) + (\lambda p'^2 + v_{02}^{LR}(0, p'))t_{20}(p', 0; 0) \right]}. (4.50)
\]

Eqs. (4.48) and (4.50) define the full potential for the coupled-channels problem for a given trial value of \( C_2 \). This potential is then inserted into the LSE and phase shifts computed. \( C_2 \) is adjusted until the desired experimental datum is reproduced with sufficient accuracy.
This method has the advantage that Eqs. (4.50) and (4.48) define functional relationships: \( \lambda = \lambda(C_2; a_t, \alpha_{20}) \), and \( \lambda_t = \lambda_t(C_2; a_t, \alpha_{20}) \). (The generalized scattering length \( \alpha_{20} \) enters implicitly through its effect on the half-off-shell t-matrices appearing in the formulae for \( \lambda \) and \( \lambda_t \).) Thus, we need to perform only one-parameter fits in order to obtain all three of the LEC’s that are pertinent in the S-wave J=1 channel.

4.3.3 Energy-dependent central part

The LSE with the energy-dependent contact term, i.e., case (c), satisfies exactly the same equation as Eq. (4.44) and Eq. (4.45), while Eq. (4.42) is replaced by

\[
t_{00}(p, 0; 0) - \frac{a_t}{M} = v^{LR}_{00}(p, 0) - v^{LR}_{00}(0, 0) - \frac{2}{\pi} M \int_{0}^{A} dp' \left[ \left( v^{LR}_{00}(p, p') - v^{LR}_{00}(0, p') \right) t_{00}(p', 0; 0) + \left( v^{LR}_{02}(p, p') - v^{LR}_{02}(0, p') \right) t_{20}(p', 0; 0) \right].
\]

(4.51)

Thus, from Eq. (4.44) and Eq. (4.51) one can solve for \( t(p, 0; 0) \). Note that here we don’t need to input a trial \( \gamma \). Once we have \( t(p, 0; 0) \), we can substitute it back into the energy-dependent version of Eq. (4.41) and obtain \( \lambda_t \), and the expression is exactly the same as Eq. (4.48).
The value of $\lambda$ then can be obtained from the following LSE,
\[
\begin{pmatrix}
\frac{a_t}{M} & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
\lambda + v^{LR}_{00}(0, 0) & 0 \\
0 & 0
\end{pmatrix}
\]
\[
- \frac{2}{\pi} M \int_0^\Lambda dp' \left( \begin{pmatrix}
\lambda + v^{LR}_{00}(0, p') & \lambda p'^2 + v^{LR}_{02}(0, p') \\
0 & 0
\end{pmatrix} \right) \begin{pmatrix}
t^{00}(p', 0; 0) & 0 \\
t^{20}(p', 0; 0)
\end{pmatrix},
\]
which gives
\[
\lambda = \frac{a_t/M - v^{LR}_{00}(0, 0) + \frac{2}{\pi} M \int_0^\Lambda dp' [v^{LR}_{00}(0, p') t^{00}(p', 0; 0) + (\lambda p'^2 + v^{LR}_{02}(0, p')) t^{20}(p', 0; 0)]}{1 - \frac{2}{\pi} M \int_0^\Lambda dp' t^{00}(p', 0; 0)}.
\]
Finally, to solve for $\gamma$, one needs to input a third piece of information, and a convenient choice is the phase shifts at an energy $\delta(E^*)$. From the phase shifts of $^3S_1$, $^3D_1$ and $\varepsilon_1$ at energy $E^*$, one can determinate $t(p_0, p_0; E^*)$, where $E^* = p_0^2/M$ can be chosen arbitrarily. Then, from the new version of Eq. (4.40), i.e.,
\[
\begin{align*}
t(p, p; E^*) - t(p_0, p_0; E^*) \\
= & \begin{pmatrix}
v^{LR}_{00}(p, p_0) - v^{LR}_{00}(p_0, p_0) & v^{LR}_{02}(p, p_0) - v^{LR}_{02}(p_0, p_0) \\
\lambda_t(p^2 - p_0^2) + v^{LR}_{20}(p, p_0) - v^{LR}_{20}(p_0, p_0) & v^{LR}_{22}(p, p_0) - v^{LR}_{22}(p_0, p_0)
\end{pmatrix} \\
+ & \frac{2M}{\pi} \int_0^\Lambda dp' \frac{p^2}{p_0^2 - p'^2 + i\varepsilon} \begin{pmatrix}
v^{LR}_{00}(p, p') - v^{LR}_{00}(p_0, p') & v^{LR}_{02}(p, p') - v^{LR}_{02}(p_0, p') \\
\lambda_t(p^2 - p_0^2) + v^{LR}_{20}(p, p') - v^{LR}_{20}(p_0, p') & v^{LR}_{22}(p, p') - v^{LR}_{22}(p_0, p')
\end{pmatrix} t(p', p_0; E^*).
\end{align*}
\]
one can solve for \( t(p, p_0; E^*) \). Once we have \( t(p, p_0; E^*) \), \( \gamma \) can be obtained from,

\[
\begin{align*}
\mathbf{t}(p, p_0; E^*) &= \begin{pmatrix}
\lambda + \gamma E^* + v^L_{00}(p, p_0) & \lambda t_0^2 + v^L_{02}(p, p_0) \\
\lambda t^2 + v^L_{20}(p, p_0) & v^L_{22}(p, p_0)
\end{pmatrix} \\
+ \frac{2}{\pi} M \int_0^\Lambda dp' \frac{p'^2}{p_0^2 - p'^2 + i\epsilon} \\
&\quad \begin{pmatrix}
\lambda + \gamma E^* + v^L_{00}(p, p') & \lambda t^2 + v^L_{02}(p, p') \\
\lambda t^2 + v^L_{20}(p, p') & v^L_{22}(p, p')
\end{pmatrix} \mathbf{t}(p', p_0; E^*).
\end{align*}
\]

(4.55)

The expression for \( \gamma \) is:

\[
\gamma = \frac{-\lambda - t_{00}(p, p_0; E^*) + v^L_{00}(p, p_0)}{E^*[1 + \frac{2}{\pi} M \int_0^\Lambda dp' p'^2 \frac{t_{00}(p', p_0; E^*)}{p_0^2 - p'^2 + i\epsilon}]} \\
- \frac{2}{\pi} M \int_0^\Lambda dp' p'^2 \frac{[\lambda + v^L_{00}(p, p') t_{00}(p, p_0; E^*)] + [\lambda t^2 + v^L_{02}(p, p')] t_0(p_0, p_0; E^*)}{E^*[1 + \frac{2}{\pi} M \int_0^\Lambda dp' p'^2 \frac{t_{00}(p', p_0; E^*)}{p_0^2 - p'^2 + i\epsilon}]}.
\]

(4.56)

In summary, for case (c), by subtraction we can obtain all the three unknown constants \( \lambda \), \( \lambda t \) and \( \gamma \) directly from three inputs, i.e., \( a_t \), \( \alpha_{20} \) and \( \delta(E^*) \). The phase shift at energies \( E \neq E^* \) then can be obtained by substituting \( \lambda \), \( \lambda t \) and \( \gamma \) into LSE and solving for the on-shell \( t \)-matrix.

It may appear that in this procedure one needs five pieces of input data to fix the constants \( (a_t, \text{the generalized scattering length } \alpha_{20}, \text{and the three phase shifts at energy } E^*) \). However, only one of the four matrix elements of \( t \) is used to extract \( \gamma \). Hence, in reality only three experimental inputs: \( t_{00}(0, 0; 0), t_{00}(p_0, p_0; E^*), \) and the derivative of \( t_{20}(p_0, p_0; E^*) \) with respect to \( E^* \), are needed to solve this problem.
We then obtain phase shifts for arbitrary $E$ by substituting the obtained values of $\lambda$, $\lambda_t$ and $\gamma$ into the potential that appears in the LSE and solving that equation for the on-shell $t$-matrix.
Chapter 5

Subtractive renormalization of chiral potentials up to next-to-next-to leading order: Results and Discussion

In this chapter we present the NN S-wave and P-wave results calculated from chiral potentials up to next-to-next-to leading order. Additionally, details of the numerical analysis regarding our subtraction method is presented at the end of the chapter. When considering OPE alone, we adopt the axial vector coupling constant $g_A = 1.29$, and pion decay constant $f_\pi = 92.4$ MeV. In addition, the nucleon mass $M = 938.926$ MeV and pion mass $m_\pi = 138.03$ MeV are employed. When considering TPE, we use $g_A = 1.26$ and $f_\pi = 93$ MeV as listed in section 2.4. The difference in $g_A$ and $f_\pi$ between OPE and TPE comes from the fact that we have taken the $O(Q^2)$ corrections into account for OPE. Note that some of the content and figures in this chapter are taken from our papers [80, 81].

5.1 P-wave results at leading order

In the Weinberg prescription for computation of the $\chi$PT potential, the contact terms are organized as a power series of $q$. After the partial-wave decomposition, only terms proportional to $q^{l+l'}$ can survive. Therefore, the OPE in P-waves shouldn’t be
associated with any contact term unless contact terms involving higher derivatives are enforced. However, as first pointed out by [46], the enforcement of a next-to-leading order contact term is required for the OPE in those singular attractive channels. Here we repeat the same calculation, and when the contact term is enforced, we use our subtraction method to solve the problem.

First, we examine the phase shifts obtained from the “bare” (un-renormalized) OPE. The four NN P-waves are shown in the left panels of Figs. 5.1-5.2. The left panel of Fig. 5.1 shows that the $^1P_1$ and $^3P_1$ phase shifts become independent of the choice of cutoff once $\Lambda > 2000$ MeV. However, for the attractive channels $^3P_0$ and $^3P_2-^3F_2$, the phase shifts in the left panel of Fig. 5.2 are cutoff dependent. This reproduces the finding of Ref. [46], where the attractive nature of the tensor $1/r^3$ potential in these channels leads to strong cutoff dependence of the phase shifts.

To stabilize these channels, a contact interaction must be included. Thus, we include the NLO contact interactions, which survive in the P-waves after partial-wave decomposition. Once we do this, the subtractive renormalization described in Sec. 4.1 can be employed and the LSE be solved. For our calculations we first employ the generalized $l = 1$ scattering lengths extracted in [86] as the input value to our subtraction scheme. As shown in the right panels of Figs. 5.1-5.2, these contact interactions absorb the cutoff dependence in the $^3P_0$ and $^3P_2-^3F_2$ channels. However, a strong cutoff dependence is now created in the $^1P_1$ and $^3P_1$ channels, and produces a resonance-like structure in the phase shifts. Note that since we adopt the generalized
Figure 5.1: The NN phase shifts for the channels $^1P_1$ and $^3P_1$ as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.8 to 5 GeV. In all cases the potential $v^{LR} = v_{OPE}$ enters the LSE. The panels on the left show the results without subtraction, whereas the panels on the right show the results with one subtraction. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].

scattering lengths extracted in [86] to perform the subtraction, the results are not necessarily the best fit to the Nijmegen phase shift analysis as we will discuss in more detail in Section 5.6.

With the calculations shown in Figs. 5.1 and 5.2 we reproduce the finding of Ref. [46]: if cutoffs larger than $\sim 600$ MeV are to be considered then additional
contact interactions must be included in the LO potential which act in the $^3P_0$ and $^3P_2$ channels. Adding those contact interactions in the other P-waves destabilizes the phase-shift prediction as a function of cutoff—as might have been expected given the fact that the singular OPE potential is repulsive as $r \to 0$ in these waves [51, 52].
5.2 P-waves results up to next-to-next-to leading order

In this Section we present the P-wave phase shifts obtained with the TPE of χPT. They are obtained by solving the LSE via the subtractive renormalization introduced in Section 4.1. We organize our findings as follows: we will discuss three different versions of the TPE, each of which is supplemented by the OPE. In subsections 5.2.1 and 5.2.2 we consider the TPE obtained via dimensional regularization up to NLO and NNLO respectively. In subsection 5.2.3, we consider the TPE obtained by the spectral-function regularization up to NNLO. For each of these three potentials we first consider the “bare” TPE, i.e. we solve the LSE with $v = v^{LR}_{OPE} + v^{LR}_{TPE}$ alone to see how strong the cutoff dependence of the phase shift is for this “un-renormalized” case. In this case, the contact terms are excluded in the potentials. Then, we perform our subtractive renormalization, thereby incorporating contact interactions $C_{ij}^{S/J}$ into the potential. We first solve the problem by adopting the generalized scattering lengths $a_{11}^{S/J}$ extracted in Ref. [86]. Then we adjust $a_{11}^{S/J}$ to find the best fit with respect to the Nijmegen phase-shift analysis [13]. Finally, in subsection 5.2.4, we consider the implications of our findings with these different potentials and link them to their coordinate space behavior.
Figure 5.3: The un-renormalized NN P-wave phase shifts as a function of the laboratory kinetic energy resulting from the dimensionally regularized TPE up to NLO. The phase shifts are shown for cutoffs $\Lambda$ ranging from 0.5 to 2 GeV. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].
Figure 5.4: The NN P-wave phase shifts resulting from the use of OPE plus DR TPE at NLO with one subtraction as a function of the laboratory kinetic energy. Here the cutoff range shown is 0.5–10 GeV. The input value of $\alpha_{14}^{S_J}$ is taken from Ref. [86]. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].
Figure 5.5: The NN P-wave phase shifts as a function of the laboratory kinetic energy that result from choosing $v^{LR} = v_1\pi + v_2\pi$, with the latter computed using DR TPE in NLO. Here the generalized scattering lengths $\alpha_{11}^{SJ}$ are adjusted to give the best fit in the region $T_{lab} < 100$ MeV. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. Note that this graph is adapted from our previous publication [80].
Figure 5.6: $\alpha_{\text{best}}$ versus $\Lambda$ ranging from 0.6-3 GeV for DR NLO, DR NNLO and SFR NNLO. Here $\alpha_{\text{best}}$ are adjusted to give the best fit in the region $T_{\text{lab}} < 100$ MeV. This graph is adapted from our previous publication [80].
Figure 5.7: $\alpha_{\text{best}}(^3P_2)$ versus $\Lambda$ ranging from 0.6-3 GeV for DR NLO. Here $\alpha_{\text{best}}$ are adjusted to give the best fit in the region $T_{\text{lab}} < 100$ MeV. This graph is adopted from our previous publication [80].
Figure 5.8: The un-renormalized NN P-wave phase shifts as a function of the laboratory kinetic energy that result from a long-range potential of OPE plus dimensionally regularized TPE at NNLO. The phase shifts are shown for cutoffs $\Lambda$ ranging from 0.5 to 2 GeV. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].
Figure 5.9: The NN P-wave phase shifts as a function of the laboratory kinetic energy that result from the use of NNLO DR TPE and one subtraction. Here the cutoff range shown is 0.6–10 GeV. The input value of $\alpha_{11}^{S_J}$ is taken from Ref. [86]. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].
Figure 5.10: The NN P-wave phase shifts as a function of the laboratory kinetic energy that result from choosing $\nu^{LR} = \nu_1\pi + \nu_2\pi$, with the latter chosen to be the DR TPE at NNLO, and implementing one subtraction. Here the generalized scattering lengths $\alpha_{11}^{GJ}$ are adjusted to give the best fit in the region $T_{lab} < 100$ MeV. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].
Figure 5.11: The renormalized NN P-wave phase shifts as a function of the laboratory kinetic energy resulting when $\nu^{LR}$ is chosen to be the OPE plus TPE in NNLO, where SFR is employed for the $O(Q^3)$ part of TPE. Here the generalized scattering lengths $\alpha^{SJ}_{11}$ are adjusted to give the best fit in the region $T_{lab} < 100$ MeV. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].
Figure 5.12: The NN P-wave phase shifts as a function of the laboratory kinetic energy that result from choosing $v^{LR} = v_1 + v_2\pi$, with the latter chosen to be the SFR TPE up to NNLO, and implementing one subtraction. Here the generalized scattering lengths $\alpha_{11}^{SL}$ are adjusted to give the best fit in the region $T_{\text{lab}} < 100$ MeV. The Nijmegen phase-shift analysis [87] is indicated by the open diamonds. This graph is adapted from our previous publication [80].
Figure 5.13: The un-renormalized NN P-wave phase shifts at $T_{lab} = 10$ MeV as a function of cutoff for three different $v^{LR}$. Note that this graph is adapted from our previous publication [80].

5.2.1 Dimensionally regularized TPE at NLO

According to the power counting in powers of $Q$, the TPE at NLO in P-waves should be associated with contact interaction $C_{11}^{SJ} pp'$. To obtain insight into its effect, we first adopt the “bare” TPE at NLO, i.e. we do not consider the contact interactions when solving the LSE, and examine the $\Lambda$ dependence of the resulting phase-shift predictions. As can be seen in Fig. 5.3, the $^3P_2$ and $^3P_1$ partial-waves show a strong cutoff dependence, whereas the $^1P_1$ and $^3P_0$ phase shifts converge for
Figure 5.14: The renormalized NN P-wave phase shifts at $T_{\text{lab}} = 100$ MeV as a function of cutoff for three different $v^{LR}$. Here the generalized scattering lengths $\alpha_{\text{best}}$ are adjusted to give the best fit in the region $T_{\text{lab}} < 100$ MeV. This graph is adopted from our previous publication [80].

cutoffs $\Lambda > 1200$ MeV. This feature is important because once a potential reaches cutoff independence in the LSE, adding further contact interactions may destroy this feature.

Actually, the issue of whether a contact term is required for a singular potential to reach cutoff independence can be linked to the short-distance behavior of the potential in coordinate space (see, for example [84]). If the potential is attractive at $r \to 0$, then the contact term is required, and vice-versa. Here we list the short-distance
behavior of the potentials adopted in the present work in Table 5.1, and will discuss its implication at the end of this section.

Now, in order to explicitly see the effect of the counter terms $C^{S_j}_{11} pp'$, we adopt our subtractive method to solve the LSE. For this we need the following four values of generalized scattering lengths: \( \lim_{p_0 \to 0} \left[ \frac{t_{11}(p_0, p_0; E)}{p_{0p_0}} \right] = \frac{a_{11}}{M} \). For these we first adopt the numerical values given in Table I of Ref. [86], which are based on results from the NijmII potential [14]. We list these values together with the generalized P-wave scattering lengths obtained from the CD-Bonn [15] and AV18 [17] potentials in Table 5.2.

As one can see in Fig. 5.4, the phase shifts become more cutoff independent compared to the un-renormalized case shown in Fig. 5.3—especially in the $T_{lab} < 50$ MeV

Table 5.1: Need for renormalization in different NN P-waves when the $\chi$ET potential is calculated to different orders. “U” implies that the potential does not need a subtraction to generate cutoff-independent phase shifts as $\Lambda \to \infty$ while “R” means that a subtraction is required in that channel at that order. The * indicates that at NNLO (DR) the $^3P_2$-$^3F_2$ channel actually requires two subtraction to be made stable. This table is adapted from our previous publication [80].

<table>
<thead>
<tr>
<th></th>
<th>$^1P_1$</th>
<th>$^3P_0$</th>
<th>$^3P_1$</th>
<th>$^3P_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>OPE</td>
<td>U</td>
<td>R</td>
<td>U</td>
<td>R</td>
</tr>
<tr>
<td>NLO (DR)</td>
<td>U</td>
<td>U</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>NNLO (DR)</td>
<td>R</td>
<td>R</td>
<td>R</td>
<td>*</td>
</tr>
<tr>
<td>NLO (DR) + NNLO (SFR)</td>
<td>U</td>
<td>U</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>NNLO (SFR)</td>
<td>U</td>
<td>U</td>
<td>R</td>
<td>R</td>
</tr>
</tbody>
</table>
Table 5.2: The generalized P-wave scattering lengths $\alpha_{11}^{SJ}$ as given in Ref. [86] for the NijmII and Reid93 potentials, together with ones extracted from the CD-Bonn [15] and AV18 [17] potentials. $\alpha_{11}^{SJ}$ is given in units fm$^3$. This table is adapted from our previous publication [80].

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_{11}^{0}(1P_1)$</th>
<th>$\alpha_{11}^{10}(3P_0)$</th>
<th>$\alpha_{11}^{11}(3P_1)$</th>
<th>$\alpha_{11}^{12}(3P_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NijmII</td>
<td>2.80</td>
<td>-2.47</td>
<td>1.53</td>
<td>-0.28</td>
</tr>
<tr>
<td>Reid93</td>
<td>2.74</td>
<td>-2.47</td>
<td>1.53</td>
<td>-0.29</td>
</tr>
<tr>
<td>CD-Bonn</td>
<td>2.70</td>
<td>-2.45</td>
<td>1.51</td>
<td>-0.29</td>
</tr>
<tr>
<td>AV18</td>
<td>2.57</td>
<td>-2.41</td>
<td>1.44</td>
<td>-0.30</td>
</tr>
</tbody>
</table>

region. Moreover, for $\Lambda > 5000$ MeV, all of the channels except the $3P_2$ partial-wave become cutoff independent. Furthermore, at large cutoffs using the constants $\alpha_{11}^{SJ}$ extracted from the NijmII potential does not produce a good description of the Nijmegen phase-shift analysis even for laboratory energies $T_{lab} < 100$ MeV. Meanwhile, in the $1P_1$ and $3P_0$ channels the contact interaction delays the convergence with respect to the cutoff that was already obtained for $\Lambda = 1200 - 2000$ MeV in its absence.

In order to obtain a better description, we vary $\alpha_{11}^{SJ}$ to match the Nijmegen phase shifts for $T_{lab} < 100$ MeV. This leads to the results shown in Fig. 5.5. From now on we will denote the generalized scattering length which gives the best description of a phase shift as $\alpha_{best}$. We plot them (except $3P_2$) against $\Lambda$ in Fig. 5.6. For $3P_2$, the variation of $\alpha_{best}$ is too large and is plotted individually as Fig. 5.7, which indicates that a strong renormalization point dependence of DR NLO occurs in this channel,
and causes the $^3P_2$ phase shift obtained by fixing an $\alpha_{11}^{S_J}$ obtained from Ref. [86] to fail to become cutoff independent for $\Lambda > 5000$ MeV as already shown in Fig. 5.4.

To see the cutoff-dependence more clearly. In Fig. 5.14 we plot the “best fitted” phase shifts at $T_{lab} = 100$ MeV versus cutoffs ranging from $\Lambda = 600 - 3100$ MeV. One observes that the dotted line (NLO results) bumps up at $\Lambda = 1400(1000)$ MeV in the $^1P_1(^3P_0)$ partial-wave, which is exactly the cutoff where the best fit fails in Fig. 5.5. In these cases, the use of any finite $\alpha_{11}^{S_J}$ as input results in almost the same phase shifts as a function of energy (We have verified that the same results are obtained if a contact interaction $C_{11}^{S_J} p' p$ is added to the potential and the coefficient is fitted to data). Additionally, the values of the $^3F_2$ phase shift at $T_{lab} = 100$ MeV are considerably different from the Nijmegen value as can be seen in Fig. 5.5, which shows the fact that the NNLO pieces of TPE are needed in order to allow a better fit at $T_{lab} > 60$ MeV.

In general, after performing our best fit we are able to match the Nijmegen P-wave phase shifts much better, especially for $T_{lab} < 60$ MeV. It should be pointed out that because our subtraction method allows us to directly input the zero-energy ‘observable’ $\alpha_{11}^{S_J}$, these results indicate that in the P-waves there is significant dependence of the phase shifts on the choice of renormalization point. This is in contrast to the $^1S_0$ and $^3S_1-^3D_1$ channels, where a (almost) correct scattering length also gives the correct phase shift for TPE at NLO or higher (see for example, [25, 81] and section 5.3).
To analyze the situation, we look more closely at the variation of $\alpha_{\text{best}}$ with respect to cutoff in Fig. 5.6. As one can see, in the $^1P_1$ ($^3P_0$) channel, $\alpha_{\text{best}}$ is nearly unchanged with respect to the cutoff until $\Lambda$ reaches 1400(1000) MeV, at which point a good fit no longer becomes possible. We recall that in these channels the phase-shift predictions were cutoff independent before the addition of a contact interaction.

In contrast, for the channels $^3P_1$ and $^3P_2-^3F_2$, where the phase shifts were originally cutoff dependent, the contact interaction is able to absorb the cutoff dependence. The variation of $\alpha_{\text{best}}$ with respect to $\Lambda$ is minor in $^3P_1$, but becomes strongly cutoff dependent in the $^3P_2-^3F_2$ channel. We note that there the $\alpha_{\text{best}}$ needed to fit the Nijmegen phase shift is rather different from the one extracted from the ‘high-precision’ potentials. The reason is that the NNLO contribution is important in this channel as we will demonstrate in the next subsection.

The above results suggest that the highest LSE cutoff one can adopt if one inserts the NLO TPE in the integral equation is $\Lambda \approx 1000$ MeV. Above that we encounter the first “bump” in the best fit of $^1P_1$ phase shift.

5.2.2 Dimensionally regularized TPE up to NNLO

We now adopt the “bare” DR TPE at NNLO and examine the resulting cutoff dependence of the P-wave phase shifts. The results are shown in Fig. 5.8 (phase shifts $\delta_{11}^{SJ}$ v.s. energy) and Fig. 5.13 ($\delta_{11}^{SJ}(T_{lab} = 10 \text{ MeV})$ v.s. $\Lambda$): all of the P-waves exhibit strongly cutoff-dependent features when $v^{LR} = v_{OPE} + v_{TPE}$ with $v_{TPE}$ computed up
to $O(Q^3)$ using DR. Comparing this to the results obtained for the un-renormalized TPE at NLO in the previous subsection, we see that the NNLO part of TPE has caused the $^1P_1$ and $^3P_0$ phase shifts to oscillate with respect to the cutoff too.

Next, we add the contact interactions $C_{11}^{SJ} pp'$ to the potential. We again first adopt values of $\alpha_{11}^{SJ}$ for the NijmII potential from Ref. [86]. With this, the oscillation in the P-waves with respect to $\Lambda$ is greatly reduced for $\Lambda > 2000$ MeV. However, as shown in Fig. 5.9, most of the resulting P-wave phase shifts are rather far away from the Nijmegen analysis.

To remedy the situation, we again vary the four input values $\alpha_{11}^{SJ}$ to find the best fit for $T_{lab} < 100$ MeV and obtain the results shown in Fig. 5.10. At those cutoffs presented in Fig. 5.10, our best-fit results agree with [25, 27]. (Note that in [27], an exponential regulator is added to TPE, which further suppresses the TPE at momentum $q \approx \Lambda$.) The phase shifts at $T_{lab} = 100$ MeV and $\alpha_{best}$ versus cutoff for $\Lambda = 600 - 3100$ MeV are again plotted in Fig. 5.14 and Fig. 5.6, respectively. For $^1P_1$, $^3P_0$ and $^3P_1$ channels, there are only minor variations for the phase shift with respect to $\Lambda$, those variations correspond exactly to the variation of $\alpha_{best}$ with respect to $\Lambda$ as shown in Fig. 5.6. The $^3P_2 - ^3F_2$ phase shifts present a limit-cycle-like behavior, which is similar to the situation in the S-waves when an energy-dependent contact term is adopted as will be discussed later in section 5.3.2.

For the value of $\alpha_{best}$, as one can see in Fig 5.6, changes of 20-40% from the values of $\alpha_{11}^{SJ}$ obtained from the ‘high-precision’ potentials (see Table 5.2) are needed
(except for the $^1P_1$ channel and $\Lambda < 1000$ MeV in the $^3P_0$ channel) in order to achieve the best fit. As we examine the trend of this change, we find that when $\Lambda > 1000$ MeV there is no significant change of $\alpha_{\text{best}}$ for $^1P_1$. On the other hand, $\alpha_{\text{best}}$ shows oscillatory behavior for $\Lambda = 800-3100$ MeV in the $^3P_0$, $^3P_1$ and $^3P_2-^3F_2$ channels. The oscillations are $\approx 10\%$, and appear to start or become larger once the cutoff crosses over $\Lambda_\chi$, i.e. 1000-1200 MeV.

Now we can examine the success of our fit of phase shifts as $\Lambda$ changes. As one can see in Fig. 5.10, there is no significant difference in the phase shifts for those chosen cutoffs within the region where the fit was performed, namely $T_{\text{lab}} < 100$ MeV. However, for $T_{\text{lab}} > 100$ MeV, the phase shifts computed with a cutoff $\Lambda = 500$ MeV differ from the ones with all other cutoffs for all four NN P-waves. This indicates that the cutoff of 500 MeV is too low: it cuts out too much of the TPE that enters the LSE. For $\Lambda = 800-2000$ MeV, the phase shifts show little cutoff dependence apart from one specific case: $\Lambda = 1200$ MeV in the $^3P_2-^3F_2$ channel. Here we are too close to the cutoff where the phase shift diverges, as shown in Fig. 5.14.

5.2.3 Spectral-function regularized TPE up to NNLO: Mixed and Full

Both the phase shifts and $\alpha_{\text{best}}$ should be (approximately) invariant over a range of cutoffs if we are to claim that the renormalization is done correctly. The oscillation of $\alpha_{\text{best}}$ for $\Lambda > \Lambda_\chi$ makes us re-think the role played by the NNLO($Q^3$) part of the
TPE in DR. This piece of the TPE contains terms proportional to $|q|^3$, which are non-polynomial and have no corresponding counter term. Although the results in Fig. 5.10 show that the cutoff dependence of the phase shifts was absorbed by the $O(Q^2)$ contact interaction, we still see that $\alpha_{\text{best}}$ depends rather strongly on $\Lambda$ in the $^3P_0$ and $^3P_1$ partial-waves. Could it be possible that the reason for this is the existence of a piece of the potential that grows as $|q|^3$, and that does not have a contact interaction to counter balance it? If the spectral-function regularization (SFR) is used to derive the TPE potential then this $|q|^3$ behavior for large $|q|$ is no longer present. Therefore, we will investigate how the phase shifts obtained when replacing only the $Q^3$ part of DR-TPE by SFR [41] differ. We will then see whether a better convergence with respect to the LSE cutoff can be achieved. We refer the potential in this case as the “mixed” NNLO TPE (which has the DR TPE up to NLO plus the SFR pieces at NNLO).

To see the effect of this “mixed” potential (denoted as MIXED NNLO), in Fig. 5.11 we plot the P-wave phase shifts obtained by adjusting $\alpha_{11}^{SJ}$ to fit the phase shifts to the Nijmegen analysis for $T_{\text{lab}} < 100$ MeV. The most striking feature of the results is that a pattern similar to that observed in the DR TPE at NLO appears in the $^1P_1$ and $^3P_0$ channels, i.e., there is no way to describe the Nijmegen phase shifts for cutoffs near $\Lambda = 1400$ (1000) MeV for $^1P_1(^3P_0)$. However, this is not very surprising, since the leading singularity of these potentials is now given by the TPE in NLO, and thus the large-cutoff behavior is the same as for that potential.
Similarly, as we examine the variation of $\alpha_{\text{best}}$ with respect to $\Lambda$, we find that the oscillation in $\alpha_{\text{best}}$ with $\Lambda$ for the NNLO TPE that is present for DR does not appear here. The variations of $\alpha_{\text{best}}$ for all P-waves are under 5%. In addition, the values of $\alpha_{\text{best}}$ are closer to the values obtained from the ‘high-precision’ potentials than are those found in the previous subsection.

This suggests that the attempt to tame the $Q^3$ part of the TPE at NNLO($Q^3$) by adopting the SFR NNLO pieces of the potential fails for $\Lambda > 1000$ MeV due to the issues in the $^1P_1$ and $^3P_0$ channels. These problems already occur in the NLO TPE. However, examining the values of $\alpha_{\text{best}}$ as a function of $\Lambda$, we see that now $\alpha_{\text{best}}$ has less dependence on $\Lambda$. Thus, using SFR to compute the NNLO TPE does reduce the cutoff dependence in $\alpha_{\text{best}}$ seen in the $^3P_2-^3F_2$ channels of the TPE at NLO and all P-waves of the DR TPE at NNLO.

Finally, we adopt the full SFR TPE up to NNLO (i.e., use SFR for both the $O(Q^2)$ and $O(Q^3)$ TPE diagrams) with the inner cutoff $\tilde{\Lambda} = 800$ MeV\(^1\), and present the best fit results in Fig. 5.12 as well as in Fig. 5.14. As one can see, SFR TPE largely reduces the oscillation behavior in the phase shifts that occurs previously for the DR TPE, and the phase shifts at $T_{\text{lab}} = 100$ MeV are almost independent of the LSE cutoff from $\Lambda = 600$ to at least 3100 MeV in the $^3P_0$, $^3P_1$ and $^3P_2-^3F_2$ channels. For $^3P_0$, the “best fitted” phase shift at $T_{\text{lab}} = 100$ MeV remains (almost) cutoff independent until $\Lambda > 2500$ MeV. Thus, we conclude that for p-waves, once

\(^1\)We have varied the value of the inner cutoff from $\tilde{\Lambda} = 700 - 900$ MeV and found that the pattern of results presented here does not change.
the SFR TPE is adopted up to NNLO, it is possible to perform the renormalization in the LSE with a cutoff up to 2500 MeV.

5.2.4 The co-ordinate-space connection

The results of the previous three subsections are summarized in Table 5.1. In that table an entry “R” implies that a contact interaction is needed to make predictions in this channel independent of the cutoff. In contrast, an entry “U” implies that phase-shift predictions for that particular P-wave are already $\Lambda$ independent before any contact interaction is added. If, in spite of this, a contact interaction is then added for $\Lambda \gg \Lambda_{\chi_{SB}}$ the phase-shifts either develop a resonant structure (OPE) or become insensitive to the input value of $\alpha_{11}^{SJ}$ used to fit the coefficient of the contact interaction (TPE). The special case, i.e., the $^{3}P_{2} - ^{3}F_{2}$ channel in the DR NNLO, is denoted as *. In this case both channels are attractive, and two contact terms are required in order to eliminate the cutoff-dependence completely. In Fig. 5.13 we plot the un-renormalized phase shifts at $T_{lab} = 10$ MeV obtained from using DR NLO, DR NNLO and SFR NNLO versus $\Lambda$. As one can see, whether a un-renormalized potential/channel is cutoff independent corresponds exactly to its short distance coordinate space behavior. The entries “U” correspond exactly to those cases where the leading $r \rightarrow 0$ singularity of the potential has a positive coefficient, and so a repulsive singular potential dominates the physics at large cutoffs. Conversely, entries “R” occur in those channels where the leading $r \rightarrow 0$ singularity of the co-
ordinate-space potential corresponds to attraction [51, 52]. The large-Λ behavior of our calculation can be completely understood from this point of view.

5.3 Singlet S-waves results up to next-to-next-to leading order

In this section we present the results obtained in the $^1S_0$ channel when we employ the three different types of contact terms introduced in Sect. II. First, we only adopt the leading-order contact term, i.e. a constant. Then, we consider the energy-dependent contact term, i.e., $v_{SR,0} = \lambda + \gamma E$. Finally, we use the more standard momentum-dependent contact term $v_{SR,0} = \lambda + C_2(p^2 + p'^2)$. For each of these cases, we examine the results found with the following four different forms of the long-range potential $v_{LR}$:

1. TPE computed using dimensional regularization up to $O(P^2)$ (denoted as DR NLO).
2. TPE computed using dimensional regularization up to $O(P^3)$ (denoted as DR NNLO).
3. TPE computed using dimensional regularization up to $O(P^2)$ plus the spectral-function regularization pieces from $O(P^3)$ (denoted as MIXED NNLO).
4. TPE computed using spectral-function regularization up to $O(P^3)$ (denoted as SFR NNLO).
For the TPE involving spectral-function regularization, we adopt $\bar{\Lambda} = 800$ MeV as intrinsic cutoff. When combined with the momentum-dependent contact interaction the $O(P^2)$ TPE yields the usual NLO calculation in $\chi$PT, while the three $O(P^3)$ forms of $v_{LR}$ both give calculations of NNLO accuracy.

5.3.1 The lowest-order contact term

A proper renormalization of these TPE potentials entails the presence of contact terms up to second order in $|q|$. However, it is still of interest to investigate what happens if only the lowest-order contact term is included in the potential (see also Refs. [43, 51]). This enables us to see how the phase shift converges as the LSE cutoff $\Lambda$ is increased. Then, when higher-derivative contact interactions are added to the potential, we can see how the cutoff dependence changes.

The renormalization for the leading-order contact term is performed with one subtraction, with the scattering length $a_s = -23.7$ fm [88] used as renormalization condition. This is the same as procedure as described for the $^1S_0$ channel in section 3, except that here we are using a long-range potential that is higher-order in the chiral power counting.

The phase shifts $^1S_0$ calculated with the DR NLO, DR NNLO, MIXED NNLO and SFR NNLO potential are shown for different cutoffs $\Lambda$ in Fig. 5.15. The results show that a single, constant, contact interaction is sufficient to stabilize the $^1S_0$ phase shift with respect to the cutoff once $\Lambda > 800$ MeV for the SFR NNLO TPE and
\( \Lambda > 1000 \text{ MeV} \) for the DR NLO and MIXED NNLO TPE. The DR NNLO potential requires \( \Lambda > 1200 \text{ MeV} \) to become stable.

For the DR NLO TPE, there is a resonance-like behavior when \( \Lambda \leq 700 \text{ MeV} \), but the phase shifts quickly converge after \( \Lambda > 700 \text{ MeV} \). This indicates that for DR NLO, a cutoff less than 700 MeV is too small and removes too much physics so the results are rather close to the one given by OPE as listed in section 3. Also, after phase shifts converge with respect of cutoff, the DR NLO results actually converge closer to the PWA93 phase shifts \([13, 87]\) than the other two potentials.

This indicates that the pure (i.e. un-compensated by contact interaction) NNLO TPE produces too large a correction, although the general trend of providing more repulsion is correct. We note that the phase shifts calculated with the MIXED NNLO and SFR NNLO potential converge faster and closer to the PWA93 result, compared to those obtained with the DR NNLO potential.

In Table 5.3 we list the values of the effective range \( r_0 \) obtained from the above potentials, those values are further plotted with the Nijmegen value in the upper panel of Fig. 5.16. Since \( \alpha_s \) is fixed to the same value in all calculations, these values of \( r_0 \) show the influence of \( \Lambda \) on the low-energy phase shifts. Comparing our results for the three different potentials to the “experimental” value \( r_0 = 2.7 \text{ fm} \) \([88]\), we see that the SFR NNLO potential gives a value of \( r_0 \) that is closest to the experimental value. The results with the DR NLO, MIXED NNLO and SFR NNLO TPE are stable with respect to \( \Lambda \) once \( \Lambda \geq 700 \text{ MeV} \). The DR NNLO TPE value eventually also stabilizes
Table 5.3: The effective range $r_0$ [in fm] in the $^1S_0$ channel extracted from calculations with the DR NLO, DR NNLO, MIXED NNLO and SFR NNLO long-range potentials and a constant contact term. $r_0$ is shown as a function of the cutoff $\Lambda$ in the LSE.

<table>
<thead>
<tr>
<th>$\Lambda$ [MeV]</th>
<th>600</th>
<th>700</th>
<th>800</th>
<th>900</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>DR NLO</td>
<td>0.99</td>
<td>-10.20</td>
<td>2.38</td>
<td>2.32</td>
<td>2.32</td>
</tr>
<tr>
<td>DR NNLO</td>
<td>3.31</td>
<td>3.28</td>
<td>3.22</td>
<td>3.51</td>
<td>3.11</td>
</tr>
<tr>
<td>MIXED NNLO</td>
<td>2.85</td>
<td>2.85</td>
<td>2.82</td>
<td>2.82</td>
<td>2.82</td>
</tr>
<tr>
<td>SFR NNLO</td>
<td>2.93</td>
<td>2.90</td>
<td>2.90</td>
<td>2.90</td>
<td>2.90</td>
</tr>
</tbody>
</table>

Table 5.4: The effective range $r_0$ [in fm] in the $^1S_0$ channel extracted from calculations with the DR NLO, DR NNLO, SFR NNLO and MIXED NNLO potentials and a momentum-dependent contact interaction. The phase shift is fitted at $T_{lab} = 200$ MeV to the Nijmegen value.

<table>
<thead>
<tr>
<th>$\Lambda$ [MeV]</th>
<th>600</th>
<th>700</th>
<th>800</th>
<th>900</th>
<th>1000</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>DR NLO</td>
<td>2.57</td>
<td>2.55</td>
<td>2.51</td>
<td>2.50</td>
<td>2.49</td>
<td>2.50</td>
</tr>
<tr>
<td>DR NNLO</td>
<td>1.89</td>
<td>2.49</td>
<td>2.70</td>
<td>2.80</td>
<td>2.99</td>
<td>2.38</td>
</tr>
<tr>
<td>SFR NNLO</td>
<td>2.85</td>
<td>0.15</td>
<td>2.72</td>
<td>2.75</td>
<td>2.75</td>
<td>2.78</td>
</tr>
<tr>
<td>MIXED NNLO</td>
<td>2.81</td>
<td>2.41</td>
<td>2.62</td>
<td>2.65</td>
<td>2.67</td>
<td>2.68</td>
</tr>
</tbody>
</table>

within small variations once $\Lambda > 1200$ MeV (not shown in the Table 5.3). This cutoff is rather large, reflecting the too-strong energy dependence that was already discussed in the previous paragraph.
Figure 5.15: The $^1S_0$ NN phase shift as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 1.2 GeV. The long-range potentials employed are the DR TPE up to NLO, NNLO, MIXED NNLO and the SFR TPE NNLO (as noted in the y-axis of each figure), together with a constant contact term. The results are obtained by one subtraction with $a_0 = -23.7$ fm as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles.
Figure 5.16: The effective range $r_0$ [in fm] in the $^1S_0$ channel extracted from calculations with the DR NLO (black square), the DR NNLO (red circle), MIXED NNLO (maroon down triangle) and the SFR NNLO (green up triangle) TPE combined with: a constant contact term (upper panel), and a constant plus a momentum-dependent contact term (lower panel). In both cases $r_0$ is shown as a function of the cutoff $\Lambda$ in the LSE. In the lower panel, the coefficient of the momentum-dependent contact term is adjusted to reproduce the Nijmegen value of the phase shift at $T_{lab} = 200$ MeV. The thick solid band represents the range of $r_0$ obtained from Ref. [92]. Note that for the upper(lower) panel, the value of $r_0$ at $\Lambda = 700$ MeV for the DR NLO (SFR NNLO) potential is $-10.2(0.15)$ fm, which is not plotted in the figure.
5.3.2 Energy-dependent contact term

In the previous section we performed the analysis of the constant contact term, now we add an energy-dependent $O(Q^2)$ piece to it and investigate its effect in this section. After inputting the scattering length $a_s$ and phase shift $\delta(E^*)$ at a particular energy $E^*$, we obtained the singlet phase shift $\delta(^1S_0)$ v.s. energy as shown in Fig. 5.17. The first, second and third row corresponds to the DR NLO, DR NNLO and SFR NNLO TPE, respectively. As we can see, there is a cutoff ($\Lambda$) dependence for the phase shift for all three potentials. We take DR NNLO TPE to start the analysis and plot the phase shift $\delta(^1S_0)$ at $T_{lab} = 10$ and $100$ MeV v.s. $\Lambda$ in Fig. 5.18. It presents a quasi-periodic feature. As $\Lambda$ becomes bigger, the period becomes longer. The relative error with respect to Nijm93 v.s. $\Lambda$ is shown in Fig. 5.19. As we can see, at certain “good” $\Lambda$, the DR NNLO TPE gives an impressively good fit (relative error< 1%) to the Nijm93 value all the way up to $T_{lab} = 350$ MeV, however, at those ”bad” $\Lambda$ the results diverge. Similar oscillatory behavior occurs for the DR NLO TPE and SFR NNLO TPE as well. The first cutoff where the phase shift diverges is $\Lambda \sim 2000$ MeV for SFR NNLO TPE, as shown in Fig. 5.17. In contrast, the particular choice, i.e., the MIXED NNLO TPE, does not exhibit the oscillatory behavior.
Figure 5.17: The $^1S_0$ NN phase shift as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. The potentials employed (from top to bottom) are DR NLO, DR NNLO and SFR NNLO, together with an energy-dependent contact term. The results are obtained by two subtractions with $a_0 = -23.7$ fm and the phase shift at $T_{lab} = 2.8$ MeV as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
Figure 5.18: The $^1S_0$ NN phase shift at $T_{lab} = 10$ MeV (upper panel) and 100 MeV (lower panel) as a function of the cutoff ranging from 0.5–19 GeV. The results are obtained using the DR NNLO potential with an energy-dependent contact term via two subtractions. This graph is adopted from our previous publication [81].
Figure 5.19: The relative difference of NN phase shift $\delta(^1S_0)$ at $T_{lab} = 350$ resulting from DR NNLO potential associated with the energy-dependent contact term with respect to Nijm93 data as function of the cutoff $\Lambda$ ranging from 0.05–2 GeV.
Table 5.5: Cutoffs $\Lambda$ v.s. C.M. frame resonance or binding energy due to the DR NNLO TPE. Here the positive value refers to resonance energy $E_s$, and negative value refers to binding energy $E_b$.

<table>
<thead>
<tr>
<th>Cutoff [MeV]</th>
<th>$E_s$ or $E_b = -B$ [MeV]</th>
<th>Cutoff [MeV]</th>
<th>$E_s$ or $E_b = -B$ [MeV]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>27.95</td>
<td>1210</td>
<td>-313.5</td>
</tr>
<tr>
<td>1010</td>
<td>18.56</td>
<td>1230</td>
<td>-352.5</td>
</tr>
<tr>
<td>1020</td>
<td>9.017</td>
<td>1270</td>
<td>-432.5</td>
</tr>
<tr>
<td>1030</td>
<td>-1.5</td>
<td>1290</td>
<td>-473.5</td>
</tr>
<tr>
<td>1050</td>
<td>-28.5</td>
<td>1310</td>
<td>-195.5</td>
</tr>
<tr>
<td>1070</td>
<td>-59.5</td>
<td>1370</td>
<td>-174.5</td>
</tr>
<tr>
<td>1090</td>
<td>-92.5</td>
<td>1430</td>
<td>-135.5</td>
</tr>
<tr>
<td>1110</td>
<td>-126.5</td>
<td>1490</td>
<td>-84.5</td>
</tr>
<tr>
<td>1130</td>
<td>-162.5</td>
<td>1550</td>
<td>-31.5</td>
</tr>
<tr>
<td>1150</td>
<td>-199.5</td>
<td>1610</td>
<td>-0.1</td>
</tr>
<tr>
<td>1170</td>
<td>-236.5</td>
<td>1670</td>
<td>163.6</td>
</tr>
<tr>
<td>1190</td>
<td>-274.5</td>
<td>1700</td>
<td>121.7</td>
</tr>
</tbody>
</table>
The effective range $r_0$ obtained from all four different potentials is about $r_0 = 2.68$ fm and varies by $< 2\%$ over the range of cutoffs considered, with the variation being only $\sim 0.5\%$ if the long-range part is computed to $O(P^3)$. This is a consequence of our choice of a low-energy phase shift as second input. Comparing with Table 5.3, one can clearly see that $r_0$ receives significant correction from the $O(P^2)$ contact term, especially for the DR NLO potential.

Thus, the energy-dependent contact term improves the overall agreement of the $^1S_0$ phase shift—and in particular the effective range $r_0$—with the Nijmegen analysis ($r_0 = 2.7$ fm). However, once one tries to increase the cutoff $\Lambda$ beyond 1000 MeV, this form of short-distance physics leads to problems if the DR TPE is used as long-range potential. These specific problems cannot be avoided. However, a wider range of cutoffs ($\sim$ 2000 MeV) can be employed before by adopting the TPE computed using spectral-function regularization.

In order to further analyze this feature, we performed the argand plot as explained below. First, note that $t$ matrix can be written as [90]

$$t(E) = \frac{1}{Mk} \frac{\Gamma/2}{E - E_R + i\Gamma/2}.$$  

(5.1)

Here $M$ is the reduced mass, $k$ is momentum and $E = \frac{k^2}{2M}$, $E_R$ is the resonance energy, and $\Gamma$ is the width of the resonance.

When $E = E_R$, Eq. (5.1) becomes $-\frac{i}{ME}$, which is an imaginary number, and $t(E)$ becomes complex when $E$ is around $E_R$. Thus by plotting $t(E)$ in the complex plane we can easily see the place where $E = E_R$ and get the resonance energy. For example,
at $\Lambda = 1000 \text{ MeV}$, we get $E_R = 27.95 \text{ MeV}$ in the c.m. frame. This corresponds to the place where the phase shift in our result diverges, i.e. $\delta(E_R \simeq 28 \text{ MeV}) = \frac{1}{2} \pi + n \pi$.

For “good” cutoffs there is no resonance, but there are bound states. In Table. 5.5 we provide $E_R$ or $E = -B$ for $\Lambda = 1000 \sim 1700 \text{ MeV}$, with $B$ the bound state energy, which shows that the oscillation feature in our phase shift with respect to $\Lambda$ actually arises from the resonance—which is in contradiction with experiment—in the $t$ matrix.

In order to get a theoretical understanding of the oscillatory behavior in this energy-dependent case, we perform a two-potential analysis as follows.

First we break the whole TPE potential into $V_1$ and $V_2$, i.e.,

$$\lambda + \gamma E + \phi^{LR}(p', p) = V_1 + V_2.$$  \hspace{1cm} (5.2)

Here $V_1$ and $V_2$ can be any arbitrary functions that satisfy eq.(5.3). The Lippmann-Schwinger equation becomes

$$T = V_1 + V_2 + (V_1 + V_2)g_0T,$$  \hspace{1cm} (5.3)

where

$$g_0 = \frac{2}{\pi} \int_0^\Lambda dp \frac{p^2}{(E - \frac{p^2}{M} + i\epsilon)}.$$  \hspace{1cm} (5.4)

Also, from the Lippmann-Schwinger equation with potential equal to $V_1$ we have

$$V_1 = [1 + T_1g_0]^{-1}T_1.$$  \hspace{1cm} (5.5)

If we substitute Eq. (5.5) in Eq. (5.3), we get

$$T = ([1 + T_1g_0]^{-1}T_1 + V_2)(1 + g_0T).$$  \hspace{1cm} (5.6)
Multiply $1 + T_1g_0$ on both side we get

$$T + T_1g_0T = ([1 + T_1g_0]^{-1}T_1 + V_2)(1 + g_0T).$$

(5.7)

Thus,

$$T = T_1 + (1 + T_1g_0)V_2 + (1 + T_1g_0)V_2g_0T.$$  

(5.8)

Eq. (5.8) has $T$ on both sides, and one way to solve it is by iteration, which means we can try to substitute $T$ by $T_1$ on the right-hand side and obtain the new $T$. The explicit expression for the first $T^{(1)}$ obtained by this way is:

$$T^{(1)}(p, p', E) = T_1(p, p'; E) + V_2(E) + \frac{2}{\pi} \int_0^\Lambda dp^* p'^2 T_1(p, p'^*; E) \frac{V_2(E)}{E - \frac{p'^2}{M} + i\epsilon}$$

$$+ \frac{2}{\pi} \int_0^\Lambda dp^* p'^2 \frac{V_2(E)T_1(p'^*, p'^*; E)}{E - \frac{p'^2}{M} + i\epsilon} + \frac{2}{\pi} \int_0^\Lambda dp^* p'^2 dp^{''} p^{''} T_1(p, p'^*; E) \frac{V_2(E)T_1(p^{''}, p'; E)}{E - \frac{p'^2}{M} + i\epsilon}(E - \frac{p^{''2}}{M} + i\epsilon).$$

(5.9)

In Eq. (5.9) we have taken $V_2$ in a form that depends only on energy $E$, and let the rest of the potential equal $V_1$. Note that since $V_2$ is independent of $p$ and $p'$ now, so it can always be taken out of the integral.

Define

$$\Gamma(p, E) = \frac{2}{\pi} \int_0^\Lambda dp^* p'^2 \frac{T_1(p, p'^*; E)}{E - \frac{p'^2}{M} + i\epsilon} \equiv T_1g_0 = g_0T_1 \quad \text{since} \quad T_1(p, p'^*; E) = T_1(p'^*, p; E)).$$

(5.10)

Thus Eq. (5.9) can be rewritten in the short-hand notation as

$$T^{(1)} = T_1 + V_2 + \Gamma V_2 + V_2\Gamma + \Gamma V_2\Gamma,$$

(5.11)
and $T^{(2)}$ will be

$$
T^{(2)} = T_1 + (1 + T_1 g_0)V_2 + (1 + T_1 g_0)V_2 g_0 T^{(1)} = T_1 + V_2 + \Gamma V_2 + (1 + T_1 g_0)V_2 g_0 T^{(1)} \\
= T_1 + V_2 + \Gamma V_2 + V_2 \Gamma + V_2 g_0 V_2 + V_2 g_0 \Gamma V_2 + V_2 g_0 T_1 g_0 V_2 + T_1 g_0 [V_2 \Gamma \\
+ V_2 g_0 V_2 + V_2 g_0 \Gamma V_2 + V_2 g_0 V_2 \Gamma + V_2 g_0 T_1 g_0 V_2 \Gamma]. 
$$

(5.12)

This leads us to further define

$$
f(E) \equiv g_0 T_1 g_0 = g_0 \Gamma = \left(\frac{2}{\pi}\right)^2 \int_0^\Lambda dp^* dp p^* p^2 \frac{T_1(p, p^*; E)}{(E - \frac{p^2}{M} + i\epsilon)(E - \frac{p^*}{M} + i\epsilon)}.
$$

(5.13)

and $T^{(1)}, T^{(2)}$ can be simplified as

$$
T^{(1)} = T_1 + (1 + \Gamma)(V_2)(1 + \Gamma).
$$

(5.14)

$$
T^{(2)} = T_1 + (1 + \Gamma)(V_2 + V_2[g_0 + f]V_2)(1 + \Gamma),
$$

(5.15)

and $T^{(3)}$ is

$$
T^{(3)} = T_1 + (1 + \Gamma)(V_2 + V_2 g_0 + f) V_2 + V_2 g_0 V_2^2 + V_2[g_0 + f]^2 V_2^2)(1 + \Gamma).
$$

(5.16)

It turns out that $T^{(N)}$ can be written as

$$
T^{(N)} = T_1 + (1 + \Gamma) \{ \sum_{N=1}^{N} [V_2^N(f + g_0)^{N-1}] \}(1 + \Gamma).
$$

(5.17)

When $N \rightarrow \infty$ we have

$$
T = T_1 + (1 + \Gamma)(\frac{1}{V_2 - f - g_0})(1 + \Gamma).
$$

(5.18)

This tells us that for $T$ to diverge we need

$$
\frac{1}{V_2} = f + g_0.
$$

(5.19)
Note that every function (e.g., $T_1$, $\Gamma$, $f$, $T$) in Eq. (5.18) is cutoff dependent, and $V_2 = \gamma E$. Thus, for each cutoff $\Lambda$, there will be an $E_x$ which satisfies Eq. (5.19), i.e.,
\[
\frac{1}{\gamma E_x} = f + g_0.
\]
For cutoffs where their $E_x \ll 0$, one has a deep bound state, and the phase shift $\delta(E)$ (where $E$ is positive) is not affected. Those turn out to be the “good” cutoffs. On the other hand, for cutoffs where the $E_x \approx 0$ or $E_x > 0$, $\delta(E)$ diverges at $E = E_x$, and we have a resonance. We have performed such a calculation and confirmed that the running of the bound state/resonance energy listed in Table. 5.5 does agree with the results obtained from Eq. (5.19).
5.3.3 Momentum-dependent contact term

A momentum-dependent contact term associated with the TPE has the form \( \lambda + C_2(p^2 + p'^2) \), and can be solved by one subtraction plus the use of a one-parameter fit, as explained in Sec. 4.2.3. The input chosen for the subtraction is again \( a_s \). In what follows we will try to perform two different fits to fix the value of \( C_2 \). One uses only low-energy information, namely the effective range, \( r_0 = 2.7 \, \text{fm} \). The other fit fixes \( C_2 \) by attempting to reproduce the phase shift at \( T_{lab} = 200 \, \text{MeV} \).

The resulting \(^1S_0\) phase shift is shown in Fig. 5.20 for the DR NLO TPE. In this case we cannot reproduce \( r_0 \), or the phase shift at \( T_{lab} = 200 \, \text{MeV} \), once \( \Lambda > 600 \, \text{MeV} \). The reason for this is mentioned in Ref. [51], and is related to the Wigner bound that limits the impact of short-distance physics on the effective range—or more generally on the energy dependence of phase shifts [84, 91]. Therefore, for the DR NLO TPE we can only perform an overall best fit to the phase shift. Fig. 5.20 shows that, within this limitation, the phase shift quickly becomes cutoff independent once \( \Lambda > 900 \, \text{MeV} \).

The results for the DR NNLO TPE are shown in Fig. 5.21, where we see that the two different fit procedures (fixing \( a_s \) & \( r_0 \) versus fixing \( a_s \) & \( T_{lab} = 200 \, \text{MeV} \)) generate different results for the same \( \Lambda \). This is especially visible at \( \Lambda = 500 \) and 1000 MeV, where a resonance-like behavior present in the latter case when \( C_2 \) is fitted to \( r_0 \). For values of \( \Lambda \) not close to these problematic cutoffs the phase shift is almost independent of the renormalization point.
This independence of renormalization point is even more apparent when the MIXED NNLO and SFR NNLO TPE is adopted. As can be seen in Fig. 5.22, for \( \Lambda \) between 700 – 1800 MeV, the two different fitting procedures lead to almost the same \( ^1S_0 \) phase shift. The agreement between the two fits finally breaks down at \( \Lambda = 2000 \) MeV. This shows that by adopting the SFR NNLO TPE, one can achieve renormalization point independence for a wider range of cutoffs. Moreover, for the MIXED NNLO TPE, the fitting-procedure-dependence holds for \( \Lambda = 700 – 2000 \) MeV as shown in Fig. 5.23. Fig. 5.22 and Fig. 5.23 also suggest that \( \Lambda < 600 \) MeV is too small and cuts off too much of the potential to allow a good fit to the Nijmegen analysis.

Finally we list in Table 5.4 and the low panel of Fig. 5.16 the extracted effective range \( r_0 \) as function of \( \Lambda \) for the \( ^1S_0 \) phase shift fitted to the Nijmegen value at \( T_{lab} = 200 \) MeV. This allows to assess the sensitivity of the results to renormalization point \( (E^* \approx 0 \text{ or } E^* \approx 200 \text{ MeV lab. energy}) \). The DR NLO TPE has the least variation of \( r_0 \) with \( \Lambda \) over the range of \( \Lambda \) considered here, but it needs to be remembered that the experimental value cannot be reproduced with a real value of \( C_2 \) [51, 89]. At NNLO, the SFR TPE gives values of \( r_0 \) which vary less with \( \Lambda \) than do the ones from the DR TPE once \( \Lambda > 800 \) MeV, and are stable until \( \Lambda = 2000 \) MeV. This can be already inferred from Fig. 5.22. The MIXED NNLO TPE gives values of \( r_0 \) which are also quite stable with respect to \( \Lambda \) at least up to \( \Lambda = 2000 \) MeV.
Figure 5.20: The $^1S_0$ NN phase shift as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 1 GeV. The DR TPE up to NLO long-range potential is employed, together with a momentum-dependent contact term. The results are obtained by making one subtraction with $a_0 = -23.7$ fm as input, and then performing a best fit to the overall phase shift as given by the Nijmegen analysis. The values of the Nijmegen $^1S_0$ phase shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].

In summary, the results obtained from the NNLO DR TPE with a momentum-dependent contact term have a peculiar behavior at $\Lambda = 1000$ MeV. Adopting the SFR potential increases the validity range of $\Lambda$ as in the energy-dependent case.
Figure 5.21: The $^1S_0$ NN phase shift as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.5 to 2 GeV. The potential employed is the DR NNLO TPE with a momentum-dependent contact term. The results are obtained by making one subtraction with $a_0 = -23.7$ fm as input and then performing a fit to either the effective range $r_0 = 2.7$ fm (solid line) or the Nijmegen $^1S_0$ phase shift at $T_{lab} = 200$ MeV (dashed line). The values of the Nijmegen phase-shifts [87] is indicated by the open triangles. This graph is adapted from our previous publication [81].
Figure 5.22: The $^1S_0$ NN phase shift as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. The potential employed is SFR NNLO with the momentum-dependent contact term. The results are obtained by making one subtraction with $a_0 = -23.7$ fm as input and then performing either a fit to the effective range $r_0 = 2.7$ fm (solid line) or the Nijmegen $^1S_0$ phase shift at $T_{lab} = 200$ MeV (dashed line). The values of the Nijmegen phase-shifts [87] is indicated by the open triangles. This graph is adapted from our previous publication [81].
Figure 5.23: The $^1S_0$ NN phase shift as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. The potential employed is the MIXED NNLO with the momentum-dependent contact term. The results are obtained by making one subtraction with $a_0 = -23.7$ fm as input and then performing either a fit to the effective range $r_0 = 2.7$ fm (solid line) or the Nijmegen $^1S_0$ phase shift at $T_{lab} = 200$ MeV (dashed line). The values of the Nijmegen phase-shifts [87] is indicated by the open triangles.
5.4 Triplet S-waves results up to next-to-next-to-leading order

In the J=1 triplet channel we again adopt the four different potentials, i.e., the DR TPE up to NLO, the DR TPE up to NNLO, the MIXED TPE up to NNLO and the SFR TPE up to NNLO. To renormalize the potential in this channel we employ three different contact terms (cases (1)–(3) in Sec. 4.3). For case (2) (energy-dependent) and (3) (momentum-dependent), the value of the generalized scattering length

\[ \lim_{\vec{p}_0 \to 0} \frac{t_{20}(\vec{p}_0, \vec{p}_0; E)}{\vec{p}_0^2} \equiv \frac{\alpha_{20}}{M} \]

is required. We first adopt the value given in [92] for \( \alpha_{20} \), i.e., \( \alpha_{20} = 2.28 \times 10^{-10} \) MeV\(^{-1} \). Then we adjust \( \alpha_{20} \) to find the best fit. Since \( \alpha_{20} \) is not obtained directly from experiment, we list the values extracted from several so-called “high-precision” potentials in Table 5.6.

5.4.1 The leading-order contact term

We first perform the renormalization with the constant contact term (1) in the \(^3S_1-^3S_1\) channel. The results obtained with this short-distance potential, and long-distance potentials DR NLO, DR NNLO, MIXED NNLO and SFR NNLO are shown in Figs. 5.24-5.27, respectively. In all four cases the \(^3S_1\) phase shifts reach cutoff independence as \( \Lambda \) approaches 1000 MeV, and then diverge once more at higher \( \Lambda \). The \(^3D_1\) phase shifts exhibit a similar feature, although the pattern of convergence is
not as obvious as in $^3S_1$. For the mixing angle $\epsilon_1$, there is no obvious convergence with respect to $\Lambda$ for the DR NLO and DR NNLO TPE. This can be explained by the fact that we only adopt a contact term in the $^3S_1-^3S_1$ channel and fit to the scattering length $a_t = 5.428$ fm. This was sufficient to stabilize the $J=1$ triplet channel in the case of the one-pion-exchange potential. However, the NNLO TPE is singular and attractive in both eigen-channels and thus needs two contact interactions (or, equivalently, two subtractions) in order to stabilize its predictions with respect to $\Lambda$ [51]. In contrast, the NLO TPE has a repulsive behavior as $r \to 0$ in both $J=1$ eigen-channels. This also leads to predictions that are unstable after one subtraction, but for the opposite reason than at NNLO.

The MIXED NNLO and SFR NNLO potentials do not have the same singular attractive behavior at short distances, and so it is no coincidence that their phase shifts show better overall convergence in $^3S_1$, $^3D_1$, and $\epsilon_1$ than the DR potentials. The SFR strongly reduces the short-range attraction in the NNLO pieces of the TPE, and a lower-order contact term is then able to remove the $\Lambda$ dependence of the phase shifts. As one can see in Fig. 5.28, where the phase shifts at $T_{lab} = 10$ MeV obtained by the SFR NNLO TPE with a constant contact term become stable after $\Lambda > 2500$ MeV.
5.4.2 Energy-dependent contact term

The results of using an energy-dependent contact term to renormalize the three different long-range potentials are shown in Figs. 5.29-5.33.

In Fig. 5.29 we adopt $a_t = 5.428$ fm, $\alpha_{20} = 2.28 \times 10^{-10}$ MeV$^{-1}$ [92], and the Nijmegen PWA values of $\delta(^3S_1)$, $\delta(^3D_1)$ and $\epsilon_1$ at $T_{lab} = 10$ MeV as our input to renormalize the DR NNLO TPE. This choice does not produce an optimal description of $\epsilon_1$ with respect to the Nijmegen analysis, although it is pretty close.

Actually, $\epsilon_1$ is a notoriously delicate quantity, and the choice made for $\alpha_{20}$ affects it appreciably. Therefore, in the case of DR NNLO we further adjust $\alpha_{20}$ to $2.25 \times 10^{-10}$ MeV$^{-1}$, which yields the best fit of $\epsilon_1$ with respect to the Nijmegen values for $\Lambda = 700 - 1200$ MeV, as shown in Fig. 5.30. The same value of $\alpha_{20}$ also gives the best fit for DR NLO. This result for $\alpha_{20}$ is not too far from the one obtained from the NijmII or Redi93 potentials, and is consistent with the size of expected N$^3$LO corrections to our results. For the MIXED NNLO and SFR NNLO potentials, the same $\alpha_{20}$ produces results which coincide quite well with the Nijmegen data. We list the value of $\alpha_{20}$ obtained from different models in Table 5.6.

The DR NLO results obtained with $\alpha_{20} = 2.25 \times 10^{-10}$ MeV$^{-1}$ are shown in Fig. 5.31. We see there that the phase shifts oscillate for $\Lambda \leq 800$ MeV, and then converge for $\Lambda = 900 - 1800$ MeV. For $\Lambda = 2000$ MeV, the phase shifts diverge badly again.
The DR NNLO results present several interesting features, which appear irrespective of the value chosen for $\alpha_{20}$ (see, e.g., Fig. 5.30). First, there exists a range of cutoffs $\Lambda = 800 - 900$ MeV where all three phase shifts agree with the Nijmegen analysis remarkably well. However, with higher cutoffs the phase shifts gradually move away from the Nijmegen analysis. Second, for $\Lambda = 2000(1400)$ MeV, $\delta(3S_1)(\delta(3D_1))$ diverges and show a resonance-like behavior.

In order to analyze this feature, we calculate the $J=1$ triplet phase shifts for a fixed energy of $T_{lab} = 10$ and 50 MeV and a wide range of cutoffs $\Lambda = 500 - 5500$ MeV, and plot them in Figs. 5.34 and 5.35. Both figures show limit-cycle-like behavior for the phase shifts with respect to $\Lambda$. The S and D-waves have different points of divergence in this cycle, and (unsurprisingly) affect one another when either gets large. The oscillation pattern of the mixing angle $\epsilon_1$ is a combination of that appearing in the two phase shifts $\delta(3S_1)$ and $\delta(3D_1)$.

In Fig. 5.32 and Fig. 5.33 we show the results for the MIXED NNLO and SFR NNLO potential with $\alpha_{20} = 2.25 \times 10^{-10}$ MeV$^{-1}$. The phase shifts only show minor cutoff dependence for $\Lambda < 1000$ MeV and converge nicely for higher $\Lambda$. As far as the fit to the Nijmegen PWA is concerned, a decent agreement is observed for all $\Lambda$ for $\delta(3S_1)$ and $\delta(3D_1)$. For $\epsilon_1$, the converged curves do not fit as well as those for $\Lambda = 600 - 800$ MeV. This is not a surprise, since we adopted a value of $\alpha_{20}$ which fits $\epsilon_1$ in this region of cutoff. We found that at higher cutoffs, i.e., $\Lambda \sim 2300$ MeV, the SFR NNLO TPE presents a divergent pattern in the phase shifts. The same pattern
is presented in the MIXED NNLO also, but instead of a complete divergence, the phase shift just moves away for its converged value. We conclude that, similar to the singlet channel, SFR NNLO can delay but not avoid the resonance caused by the energy-dependent contact term.

The effective range $r_0$ obtained at NLO is $r_0 = 1.75 \pm 0.01$ fm for $900 \text{ MeV} \leq \Lambda \leq 1800 \text{ MeV}$, with the error representing the extent of the cutoff dependence in that range. For $\Lambda = 2000 \text{ MeV}$, $r_0 = 1.70$ fm, which is due to the resonance-like behavior in the phase shifts created by the energy-dependent contact term. For NNLO, $r_0 = 1.75 \pm 0.01$ fm except for $\Lambda = 1400$ and 2000 MeV, which are just the cutoffs where the phase shifts also diverge, as shown in Figs. 5.34 and 5.35. The same behavior in $r_0$ is presented for the MIXED NNLO and SFR NNLO but for higher cutoffs.

In summary, in the triplet channel the energy-dependent contact term creates a similar pattern as in the singlet channel. For the DR TPE (SFR NNLO), there is a highest cutoff $\Lambda = 1200(2300)$ MeV with which the potential can be iterated in the LSE before the results start to diverge with $\Lambda$.

### 5.4.3 Momentum-dependent contact term

The results of using a momentum-dependent contact term to renormalize the DR NNLO TPE are shown in Fig. 5.36. Here we adopt $a_t = 5.428$ fm and $\alpha_{20} = 2.25 \times 10^{-10}$ MeV$^{-1}$ as input and then perform a fit to the phase shift $\delta(^3S_1)$ at
$T_{lab} = 200 \text{ MeV}$ to obtain $C_2$. This pins down the three unknown LEC’s in the potential at this order.

For phase shifts obtained by our two-subtractions-plus-one-fitting method, once the trial value of $C_2$ is imposed, the other two constants $\lambda$ and $\lambda_t$ are fixed by the two pieces of experimental information that are our input: $\alpha_{20}$ and $a_t$ in this case. Thus, the results shown in Fig. 5.36 satisfy the constraints at $T_{lab} = 0$ associated with this information and reproduce the $^3S_1$ phase shift at $T_{lab} = 200 \text{ MeV}$, but are not the best possible fit to the phase shifts. This is especially obvious for $\epsilon_1$ at $\Lambda = 700 \text{ MeV}$, where the result diverges from the Nijmegen value. In contrast to the energy-dependent contact term, our results show that for the momentum-dependent contact term, $\epsilon_1$ is not very sensitive to the value of $\alpha_{20}$. Compared to the results obtained using the same contact terms in Refs. [25, 28, 42], the overall fit to the Nijmegen phase shifts is noticeably worse. Furthermore, the results obtained by fixing $\delta(^3S_1)$ at $T_{lab} = 200 \text{ MeV}$ do not show stability for $\delta(^3D_1)$ and $\epsilon_1$ with respect to $\Lambda$. This suggests that the TPE associated with a momentum-dependent contact term depends strongly on the choice of renormalization point once $\Lambda > 1 \text{ GeV}$.

For comparison, we then remove the two constraints imposed by $a_t = 5.428 \text{ fm}$ and $\alpha_{20} = 2.25 \times 10^{-10} \text{ MeV}^{-1}$ and allow all three NN LEC’s to vary freely, so that we can obtain the best overall fit to the phase shifts. In fact, this is achieved by varying $\lambda$, $C_2$ and $\lambda_t$ around the values given by our subtraction method in order to generate the best fit in phase shifts for $T_{lab} < 100 \text{ MeV}$. The results for DR NLO, DR
Table 5.6: Value of the generalized scattering length $\alpha_{20}$ extracted from various potentials, in units of $10^{-10}$ MeV. Note that this table is adapted from our previous publication [81].

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{20}$</td>
<td>2.28</td>
<td>2.28</td>
<td>2.09</td>
<td>2.18</td>
</tr>
</tbody>
</table>

NNLO and SFR NNLO are listed in Figs. 5.37 and 5.38. The fits are a noticeable improvement over Fig. 5.36. In particular, as shown in Figs. 5.37, the diverged mixing angle $\epsilon$ at $\Lambda = 700$ MeV presented in Figs. 5.36 vanishes for the DR NNLO TPE. For all four potentials, starting from $\Lambda = 700$ MeV, the $^3S_1$ and $^3D_1$ phase shifts fit the Nijmegen analysis quite well and have only minor variation with respect to $\Lambda$ up to 1000 MeV. A stronger cutoff dependence shows up only in the mixing parameter $\epsilon_1$. There we observe a clear decrease in the $\Lambda$ sensitivity of the result as the long-range potential is improved from DR NLO to DR NNLO and then again to the MIXED NNLO and SFR NNLO. We list $r_0$ extracted from those best fits in Table 5.7 and plot them in Fig. 5.39. Again, we see a clear improvement from DR NLO to DR NNLO with respect to the Nijmegen value 1.83 fm [92], and in this case the variation of $r_0$ with respect to $\Lambda$ is about the same for DR NNLO, MIXED NNLO and SFR NNLO.
Figure 5.24: The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. The potential employed is the DR NLO TPE with one constant contact term. The results are obtained via one subtraction with $a_0 = 5.428$ fm as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
Figure 5.25: The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. The potential employed is DR NNLO with a constant contact term. The results are obtained via a single subtraction with $a_0 = 5.428$ fm as input. The values of the Nijmegen phase-shifts [87] is indicated by the open triangles. This graph is adapted from our previous publication [81].
Figure 5.26: The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. Here the potential is the MIXED NNLO with a constant contact term. The results are obtained by one subtraction with $a_0 = 5.428$ fm as input. The values of the Nijmegen phase-shifts \cite{87} are indicated by the open triangles.
Figure 5.27: The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. Here the potential is SFR NNLO with a constant contact term. The results are obtained by one subtraction with $a_0 = 5.428$ fm as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
Figure 5.28: The $J=1$ coupled NN phase shifts at $T_{\text{lab}} = 10$ MeV as a function of cutoff ranging from 1–8.5 GeV. The results are obtained with the SFR NNLO potential and a constant contact term. This graph is adapted from our previous publication [81].
Figure 5.29: The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs \( \Lambda \) ranging from 0.6 to 2 GeV. The potential employed is DR NNLO with a linear energy dependence in the central part of the contact term. The results are obtained by three subtractions with \( a_0 = 5.428 \text{ fm} \), \( \alpha_20 = 2.28 \times 10^{-10} \text{ MeV}^{-1} \) and the phase shift at \( T_{\text{lab}} = 10 \text{ MeV} \) as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
Figure 5.30: The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs Λ ranging from 0.6 to 2 GeV. The potential employed is the DR NNLO with a linear energy dependence in the central part of the contact term. The results are obtained by three subtractions with $a_0 = 5.428$ fm, $\alpha_20 = 2.25 \times 10^{-10}$ MeV$^{-1}$ and the phase shift at $T_{lab} = 10$ MeV as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
Figure 5.31: The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs Λ ranging from 0.6 to 2 GeV. The potential employed is DR NLO with a linear energy dependence in the central part of the contact term. The results are obtained by three subtractions with $a_0 = 5.428$ fm, $\alpha_20 = 2.25 \times 10^{-10}$ MeV$^{-1}$ and the phase shift at $T_{lab} = 10$ MeV as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
Figure 5.32: The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. The potential employed is the MIXED NNLO with a linear energy dependence in the central part of the contact term. The results are obtained by three subtractions with $a_0 = 5.428$ fm, $\alpha_{20} = 2.25 \times 10^{-10}$ MeV$^{-1}$ and the phase shift at $T_{lab} = 10$ MeV as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles.
Figure 5.33: The $J=1$ coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 2 GeV. The potential employed is SFR NNLO with a linear energy dependence in the central part of the contact term. The results are obtained by three subtractions with $a_0 = 5.428$ fm, $\alpha_{20} = 2.25 \times 10^{-10}$ MeV$^{-1}$ and the phase shift at $T_{lab} = 10$ MeV as input. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
Figure 5.34: The $J=1$ coupled NN phase shifts at $T_{lab} = 50$ MeV as a function of cutoff ranging from 0.5–5.5 GeV. The results are obtained with the DR NNLO potential and a linear energy dependence in the central part of the contact term. This graph is adapted from our previous publication [81].
Figure 5.35: The J=1 coupled NN phase shifts at $T_{lab} = 10$ MeV as a function of cutoff ranging from 0.5–5.5 GeV. The results are obtained with the DR NNLO potential and a linear energy dependence in the central part of the contact term. This graph is adapted from our previous publication [81].
Figure 5.36: The J=1 coupled NN phase shifts as a function of the laboratory kinetic energy for different cutoffs Λ ranging from 0.6 to 2 GeV. The potential employed is DR NNLO with the momentum-dependent contact term. The results are obtained by two subtractions with $a_0 = 5.428$ fm and $\alpha_{20} = 2.25 \times 10^{-10}$ MeV$^{-1}$ as input and then performing a fit to the $^3S_1$ Nijmegen phase shift at $T_{lab} = 200$ MeV. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
Figure 5.37: The best fit for the NN $^3S_1-^3D_1$ phase shifts as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 1 GeV. The potentials employed are the DR NLO (left panel) and the DR NNLO (right panel) with a momentum-dependent central part of the contact term. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
Figure 5.38: The best fit for the NN $^3S_1-^3D_1$ phase shifts as a function of the laboratory kinetic energy for different cutoffs $\Lambda$ ranging from 0.6 to 1 GeV. The potentials employed are the SFR NNLO with a momentum-dependent central part of the contact term. The values of the Nijmegen phase-shifts [87] are indicated by the open triangles. This graph is adapted from our previous publication [81].
Table 5.7: The effective range $r_0$ [in fm] in the $^3S_1$ channel extracted from calculations with the DR NLO, DR NNLO, MIXED NNLO and SFR NNLO potentials and the momentum-dependent central piece of the contact term. Here $r_0$ is shown as a function of the cutoff $\Lambda$ in the LSE, and is extracted from a best fit of the phase shifts to the Nijmegen PWA93. Note that the Nijmegen analysis gives $r_0 = 1.83$ fm [86].

<table>
<thead>
<tr>
<th>$\Lambda$ [MeV]</th>
<th>600</th>
<th>700</th>
<th>800</th>
<th>900</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>DR NLO</td>
<td>1.64</td>
<td>1.62</td>
<td>1.64</td>
<td>1.65</td>
<td>1.65</td>
</tr>
<tr>
<td>DR NNLO</td>
<td>-11.5</td>
<td>1.72</td>
<td>1.76</td>
<td>1.77</td>
<td>1.77</td>
</tr>
<tr>
<td>MIXED NNLO</td>
<td>1.72</td>
<td>1.74</td>
<td>1.73</td>
<td>1.69</td>
<td>1.69</td>
</tr>
<tr>
<td>SFR NNLO</td>
<td>1.75</td>
<td>1.79</td>
<td>1.79</td>
<td>1.70</td>
<td>1.72</td>
</tr>
</tbody>
</table>
Figure 5.39: The effective range $r_0$ [in fm] in the $^3S_1$ channel extracted from calculations with the DR NLO (black square), DR NNLO (red circle) MIXED NNLO (maroon down triangle) and SFR NNLO (green up triangle) TPE and a momentum-dependent central piece of the contact term. Here $r_0$ is shown as a function of the cutoff $\Lambda$ in the LSE, and is extracted from a best fit of the phase shifts to the Nijmegen PWA93, while the solid line represents $r_0$ obtained from Ref. [92]. Note that the value of $r_0$ at $\Lambda = 600$ MeV for DR NNLO is $-11.5$ fm, which is not plotted in the figure.
5.5 Bound-state results

In this section we present the bound-state results in the $^3S_1-^3D_1$ channel for the DR NLO, DR NNLO and SFR NNLO potentials. The SFR NNLO results will be further applied to the deuteron electro-disintegration calculation in Chapter 6.

The technique of obtaining the NN bound-state wave functions and binding energy has been discussed in Sec. 3.3. Here we adopt a momentum-dependent contact term with the LEC’s given by our “two subtraction plus one fitting method” as discussed in Sec. 5.4.3. In Fig. 5.40 we plot the momentum-space deuteron wave function given by DR TPE up to NLO, NNLO and SFR TPE up to NNLO, where they are compared with the CD-Bonn potential results. The cutoff in the LSE is set to $\Lambda = 800$ MeV for all three cases presented here. In Table 5.6 we list the binding energy $E_B$ obtained from the LO, DR TPE up to NLO, DR TPE up to NNLO and SFR TPE up to NNLO potential, where the DR NNLO potential gives the closest $E_B$ with respect to the experimental value 2.225 MeV. Note that the physical quantity used to pin down the LEC in the LO potential is the $^3S_1-^3D_1$ scattering length $a_t$, and two additional quantities, i.e., the generalized scattering length $\alpha_{20}$ and phase shift at $T_{lab} = 200$ MeV are adopted to determine the LEC’s at NLO/NNLO.
Table 5.8: The binding energy $E_B$ obtained from the LO, DR NLO, DR NNLO and SFR NNLO potentials. Here $\Lambda = 50$ GeV for the LO potential, and $\Lambda = 800$ MeV for the other three.

<table>
<thead>
<tr>
<th></th>
<th>LO</th>
<th>DR NLO</th>
<th>DR NNLO</th>
<th>SFR NNLO</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_B$ [MeV]</td>
<td>2.093</td>
<td>2.033</td>
<td>2.214</td>
<td>2.106</td>
</tr>
</tbody>
</table>

Figure 5.40: The momentum-space wave functions of the shallowest $NN$ bound state in the triplet channel as a function of $p$, where $\psi_0(p)$ is the $^3S_1$ wave function and $\psi_2(p)$ denotes the $^3D_1$ wave function. These wave functions are obtained from the DR TPE up to NLO (black solid line), DR NNLO (green dashed line) and SFR NNLO (blue dash-dotted line) with the momentum-dependent contact term, where the cutoff in the LSE is $\Lambda = 800$ MeV. The red dot indicate the corresponding wave functions obtained from the CD-Bonn potential.
5.6 Numerical remarks on subtractive renormalization method

In this section we discuss three numerical challenges of our subtractive renormalization scheme and perform the analysis of them. First, we discuss the numerical issue generated by subtracting small matrix elements from large matrix elements. We then examine the issue regarding to the accuracy of taking the numerical limit of the potential. Finally, we investigate the error of our “subtraction plus fitting method”.

The main numerical challenge of our subtractive renormalization scheme originates in solving Eq. (3.23) and Eq. (3.31). Matrix elements with large $p'$ are small and in the subtraction scheme they are subtracted from some larger matrix elements. The same issue occurs in the NLO and NNLO subtractive calculations, e.g., Eq. (4.6) and Eq. (4.9) for the P-waves, Eq. (4.25), Eq. (4.41) and Eq. (4.44) for the S-waves—where we apply the same type of equations multiple times. Therefore, it is of interest to compare the efficiency of the subtraction and fitting methods. We first note that due to the multi-step nature of subtraction, it does take approximately 3 times longer to run the subtractive code than it does for the fitting method with the same number of mesh points. However, its advantage—direct input of the physical observables—outweighs the time-consuming searching and fitting process from the conventional fitting method. In table 5.9 we list the convergence of the results with respect to the number of mesh points used in the LO $^3S_1-^3D_1$ channel. As the table shows, the
subtraction method converges much faster than the fitting method with respect to the number of mesh points used in the calculation. With 20 mesh points we already have 4 significant-figure accuracy for the subtraction method, while to reach this accuracy it takes more than 200 mesh points for the fitting method. The subtraction method converges so much faster because in the fitting method $C_T$ affects only the part of the off-shell amplitude near $p = \Lambda$, and the oscillation phase it sets there must be communicated down to $p = 0$ to get a particular scattering length $a_t$. In the subtraction method the correct value of the scattering length is enforced in the equation itself, and does not have to be achieved by fine tuning the constant $C_T$ so as to obtain the correct behavior of the half-off-shell amplitude for $p$ of order $\Lambda$.

For the same number of mesh points, converged phase shifts calculated by the two different methods differ by 0.1–1.3% depending on the energy, as is shown in Fig. 5.41. The mixing parameter for the triplet channel is more sensitive to numerical errors, and in that quantity there is a 1–2% relative difference between the two methods. However, we still claim that both methods are equivalent.

Another numerical issue which may become a concern in the process of eliminating the momentum-dependent contact term is how accurately the limits are taken and how they affect the result. The related equations here are: e.g., Eq. (4.4), Eq. (4.5), Eq. (4.44) and Eq. (4.45). Note that the P-wave calculations are in general less accurate than the S-wave ones, since the higher partial-waves contain higher-order
Table 5.9: The convergence of the phase shift $\delta(^3S_1)$ at laboratory kinetic energy $T_{\text{lab}}=10$ MeV as a function of the number of Gaussian-Legendre quadratures for the fitting and subtraction methods. Here the leading-order chiral potential is adopted. Note that the fitting method converges to 104.1 at about 200 mesh points, so the final results of these two methods differ by $\approx 0.5\%$. This table is adapted from our previous publication [73].

<table>
<thead>
<tr>
<th>Mesh points</th>
<th>$\delta(^3S_1)$ Fitting</th>
<th>$\delta(^3S_1)$ Subtraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>139.2453</td>
<td>104.6811</td>
</tr>
<tr>
<td>30</td>
<td>113.8653</td>
<td>104.6649</td>
</tr>
<tr>
<td>40</td>
<td>107.9943</td>
<td>104.6622</td>
</tr>
<tr>
<td>50</td>
<td>105.9794</td>
<td>104.6618</td>
</tr>
<tr>
<td>100</td>
<td>104.2317</td>
<td>104.6616</td>
</tr>
<tr>
<td>150</td>
<td>104.0749</td>
<td>104.6616</td>
</tr>
</tbody>
</table>

spherical bessel function—which are more numerically challenging to evaluate accurately due to their oscillatory behavior.

In Sec. 5.4 we mentioned that the limit

$$
\lim_{p_0 \to 0} \frac{t_{20}(p_0, p_0; E)}{p_0^2} \equiv \frac{\alpha_{20}}{M} \quad (5.20)
$$

is used as an input, and changing its value from $\alpha_{20} = 2.28 \times 10^{-10}$ MeV$^{-1}$ to $\alpha_{20} = 2.25 \times 10^{-10}$ MeV$^{-1}$ does present a noticeable change of the resulting mixing angle $\epsilon_1$ as can be seen from Fig. 5.29 to Fig. 5.30. As we can see in these two figures, this $\approx 1\%$ change in $\alpha_{20}$ results in about a $5\%$ change in $\epsilon_1$ for $\Lambda < 1200$ MeV, and changes larger than $10\%$ are presented for higher $\Lambda$ (This is for the NNLO TPE with
the energy-dependent contact term, and it is where the accuracy of the limit used as input is demanded the most). Among other partial-waves and potentials evaluated in our work, \( \approx 2\% \) accuracy on the value of the limit of the potentials is required to reduce the variation in phase shifts to \(< 5\%\) for the P-waves, and \( \approx 5\% \) accuracy is required for all other cases. As a result, we need to make sure that the limits which appear in Eq. (4.4), Eq. (4.5), Eq. (4.44) and Eq. (4.45) are evaluated in such a way to give at least 1\% accuracy. For single-valued terms like \( \lim_{p_0 \to 0} \left[ \frac{v_{LR}^{L/R}(p_0, p_0)}{p_0} \right] \), one can obtain the analytical answer from the potential and this is not an issue. However, for terms involving the value of the limit as a function, such as \( \lim_{p_0 \to 0} \left[ \frac{v_{LR}^{L/R}(p', p_0)}{p_0} \right] \), one needs to be more careful. A straightforward numerical implementation which approximates \( p_0 \to 0 \) by inserting small values of \( p_0 \) will result in a numerical error \( > 1\% \). Therefore we first analytically expand the potential \( v_{LR}^{L/R}(p', p_0) \) into power series of \( p' \) and \( p_0 \), keeping terms only up to \( p_0' \), and then obtain the analytical form of the limit of the potential as a function of \( p' \) before numerical implementation. In this way, we have verified that our numerical evaluation of the subtractive equation is able to give results which are at least as accurate as those obtained with the conventional fitting method.

Finally, to verify the validity of our “subtraction plus fitting method” as presented in Eq. (4.37), Eq. (4.48) and Eq. (4.50), we examine the resulting \( \lambda \) and \( \lambda_t \) with respect to different mesh points used. Note that since the LSE is solved in terms of a matrix equation (as shown in Appendix B), for \( N \) mesh points used, one should get a \( N \times N \)
Table 5.10: The convergence of the LEC $\lambda$ and $\lambda_t$ as a function of the number of Gaussian-Legendre quadratures for the “subtraction plus fitting method”. Here the DR NNLO TPE is adopted in the calculation.

<table>
<thead>
<tr>
<th>Mesh points</th>
<th>$\lambda$ ($10^{-6}\text{MeV}^{-2}$)</th>
<th>$\lambda_t$ ($10^{-11}\text{MeV}^{-4}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>-7.18461</td>
<td>-2.57829</td>
</tr>
<tr>
<td>20</td>
<td>-7.18279</td>
<td>-2.57793</td>
</tr>
<tr>
<td>40</td>
<td>-7.18279</td>
<td>-2.57793</td>
</tr>
<tr>
<td>80</td>
<td>-7.18279</td>
<td>-2.57793</td>
</tr>
<tr>
<td>100</td>
<td>-7.18279</td>
<td>-2.57793</td>
</tr>
<tr>
<td>200</td>
<td>-7.18279</td>
<td>-2.57793</td>
</tr>
</tbody>
</table>

matrix of $\lambda$ and $\lambda_t$. Thus, one can compare each element in the matrix to see whether a consistent value of $\lambda$ ($\lambda_t$) is obtained. After comparing the difference between the $N \times N$ elements, we found that the value of $\lambda$ ($\lambda_t$) is accurate at least for 10 significant digits for mesh points used ranging from 10-200. Moreover, in Table 5.10 we list the values of $\lambda$ and $\lambda_t$ obtained from Eq. (4.48) and Eq. (4.50) by our “subtraction plus fitting method” versus mesh points used. The the table shows, $\lambda$ and $\lambda_t$ reach the converged value $-7.18279 \times 10^{-6} \text{MeV}^{-2}$ and $-2.57793 \times 10^{-11} \text{MeV}^{-4}$ after 20 mesh points are adopted in the calculation. This indicates that our “subtraction plus fitting method” for the NNLO TPE is as efficient as our LO subtraction method.
Figure 5.41: The relative difference of the leading-order NN $\delta(^1S_0)$ and $\delta(^3S_1)$ phase shift between two renormalization methods as a function of the laboratory kinetic energy $T_{\text{lab}} \leq 80$ MeV. Here $\Lambda = 50$ GeV is used, and 100 Gaussian-Legendre quadrature points are chosen for the solution of the LS equation. Note that this graph is adapted from our previous publication [73].
Chapter 6

Deuteron electro-disintegration

In this chapter we calculate the deuteron electro-disintegration process, focusing particularly on the deuteron longitudinal response function. First, in Section 6.1, we give the motivation of why it is of interest to apply chiral effective field theory to a deuteron electro-disintegration calculation. Then we introduce the method of calculation in Section 6.2. Finally, in Section 6.3, we present the results of the longitudinal response functions obtained from chiral potentials, compare them to those from phenomenological potentials, and discuss our findings.

6.1 Motivation

Deuteron electro-disintegration has been considered as a test ground for various NN models for a long time. There are several reasons for that. First, it is one of the simplest systems which allows us to study both the deuteron wave function and the NN interaction t-matrix at the same time. In elastic electron-deuteron scattering, one probes the deuteron wave function, e.g., computing \( \int d^3p \Psi^*(p - q/2)\Psi(p) \) [98] (here \( \Psi \) is the deuteron wave function and \( q \) is the 3-momentum transferred from the virtual photon to deuteron) in the impulse approximation. On the other hand, in deuteron electro-disintegration, one tests both the deuteron wave function \( \Psi \) and the NN interaction t-matrix together through the final-state interaction. Second, in the final-state
interaction of deuteron electro-disintegration, the off-shell component of the t-matrix also enters the calculation. Thus it provides a further test for the chiral EFT or various NN models. Moreover, in the electro-disintegration process, instead of some of the 4-momentum given by the electron going into the deuteron’s 3-momentum (like the elastic scattering case), the energy and momentum given by the electron can be fully deposited into the nucleons’ relative momentum (apart from overcoming the 2.2 MeV binding energy of deuteron). This allows one to test the higher-energy part of the NN interaction as well as the deuteron wave function. In particular, one can test the cutoff-dependence of the chiral effective potential described in the previous chapters in another way. In summary, deuteron electro-disintegration serves as an excellent touchstone for assessing various theoretical nuclear models or effective potentials within a more experimentally accessible energy window. Previous calculations of deuteron electro-disintegration are performed with either phenomenological potentials [93–95] or pionless EFT [96]. In this work, we adopt the chiral potential up to NNLO and obtain the longitudinal structure function of the deuteron electro-disintegration process.

6.2 Method of Calculation

In this section we will first give the basic kinematic setup for the deuteron electro-disintegration process in Sec. 6.2.1. Then, we lay out the details needed in order to calculate the longitudinal response function in Sec. 6.2.2.
6.2.1 Kinematics

The kinematics for deuteron electro-disintegration can be visualized as in Fig.6.1. Here the virtual photon gives its 4-momentum \((\omega, \mathbf{q})\) to the deuteron. \(\theta_e\) is the electron scattering angle, \(\theta\) is the angle between \(\mathbf{q}\) and the the momentum \(\mathbf{p}'\) of the outgoing proton in the final proton and neutron’s c.m. frame. Meanwhile, \(\phi\) is the angle between the two planes, i.e., the plane where the electron scattering occurs and the plane formed by the outgoing proton and neutron (in the figure \(\mathbf{q}\) is defined to be the direction of z-axis). One can use the c.m. frame of the final proton and neutron, or the laboratory (lab.) frame where the deuteron is at rest to describe the
same process. In the following discussion we will use the superscript “lab” to denote quantities in the lab. frame, and quantities measured in the c.m. frame have either no label or are associated with a superscript “c.m.”. In general, we will work in the c.m. frame, since it simplifies the expressions. In the c.m. frame one has:

proton(neutron) 3-momentum : \( p'(\!-\!p') \),
proton (neutron) total energy : \( E_n = E_p = M + \frac{p'^2}{2M} \),
total energy of deuteron : \( E_d = M_d + \frac{q^2}{2M_d}, M_d = 2M - B \). \( (6.1) \)

Here \( B \approx 2.2 \) MeV is the deuteron binding energy. From energy conservation: \(|p'| = \sqrt{(\omega - B)M + \frac{q^2}{2M_d}}\).

The quantities in the lab. frame can be easily related to those in c.m. frame by a Lorentz boost \([96]\), i.e.,

\[
\begin{align*}
\omega^{cm} &= \gamma \omega^{lab} - \beta \gamma |q|^{lab}, \\
|q|^{cm} &= \beta \gamma M_d, \\
\beta &= \frac{|q|^{lab}}{M_d + \omega^{lab}}, \gamma = \frac{1}{\sqrt{1 - \beta^2}}. \quad (6.2)
\end{align*}
\]

In this work we did not consider the electron polarization degree of freedom, so the usual expression of the differential cross section for deuteron electro-disintegration is (see, for example, Ref. \([94]\).)

\[
\frac{d^3\sigma}{dk_2^{lab} d\Omega_c^{lab} d\Omega_p} = c\{\rho_L f_L + \rho_T f_T + \rho_L T f_{LT} \cos[\phi] + \rho_T T f_{TT} \cos[2\phi]\}. \quad (6.3)
\]
\( \rho(f)_{L,T,LT,TT} \) describe the lepton (hadron) tensor. \( \Omega^{lab}_e = (\theta^{lab}_e, \phi^{lab}_e) \), \( \Omega_p = (\theta, \phi) \).

Meanwhile,

\[
c = \frac{\alpha}{6\pi^2 \frac{k^{lab}_{1(2)}}{k^{4}_{e}}}, \tag{6.4}
\]

where \( \alpha \) is the fine structure constant, \( k^{lab}_{1(2)} \) is the absolute value of the incoming (outgoing) electron 3-momentum in the laboratory frame, and \( q^{4}_{e} \) is the 4-momentum to the fourth of the virtual photon. The energies of the incoming and outgoing electrons \( E^{lab}_{e} \) and \( E^{\prime lab}_{e} \) are related to \( k^{lab}_{1(2)} \) by

\[
E^{lab}_{e} = \sqrt{(k^{lab}_{1})^2 + m^2_e}, \tag{6.5}
\]
\[
E^{\prime lab}_{e} = \sqrt{(k^{lab}_{2})^2 + m^2_e}, \tag{6.6}
\]

with \( m_e \) the mass of electron.

Now, for the lepton tensor \( \rho_{L,T,TT,LT} \), one has

\[
\rho_L = v_0 v_L, \tag{6.7}
\]
\[
\rho_T = v_0 v_T, \tag{6.8}
\]
\[
\rho_{TT} = v_0 v_{TT}, \tag{6.9}
\]
\[
\rho_{LT} = v_0 v_{LT}. \tag{6.10}
\]
Here

\[ v_0 = 4E^\text{lab}_x E^\text{lab}_e \cos^2 \left( \frac{\theta^\text{lab}_e}{2} \right), \]  
(6.11)

\[ v_L = \frac{q^2_v}{q^2}, \]  
(6.12)

\[ v_T = -\frac{1}{2} \frac{q^2_v}{q^2} + \tan^2 \left( \frac{\theta^\text{lab}_e}{2} \right), \]  
(6.13)

\[ v_{TT} = \frac{1}{2} \frac{q^2_v}{q^2}, \]  
(6.14)

\[ v_{TL} = \frac{1}{\sqrt{2}} \frac{q^2_v}{q^2} \sqrt{-\frac{q^2_v}{q^2} + \tan^2 \left( \frac{\theta^\text{lab}_e}{2} \right)}. \]  
(6.15)

One can also express the lepton tensor as [93]:

\[ \rho_L = -\beta \frac{q^2_v \xi^2}{2\eta}, \]  
(6.16)

\[ \rho_T = -\frac{1}{2} \frac{q^2_v}{q^2} (1 + \frac{\xi}{2\eta}), \]  
(6.17)

\[ \rho_{LT} = -\beta \frac{q^2_v \xi}{\eta} \sqrt{(\xi + \eta)/8}, \]  
(6.18)

\[ \rho_{TT} = \frac{q^2_v \xi}{4\eta}. \]  
(6.19)

With \( \xi = -\frac{q^2_v}{q^2}, \eta = \tan^2(\theta^\text{lab}_e/2) \). Note that here we adopt the (1,-1,-1,-1) metric and \( q^2_v = \omega^2 - q^2 \) (the sign of our \( q^2_v \) is different from [93]).

For the hadron tensor, one first defines [93]:

\[ T_{SM, \mu m_d s} = -\pi \sqrt{2\alpha |p'| E_p E_d / M_d} \langle \Psi_{p' SM_s} | J_\mu (q) | m_J 0 \rangle. \]  
(6.20)

Here \( \langle \Psi_{p' SM_s} | \) is the NN final state with specified spin quantum number \( S \) and its projection on the z-axis \( M_s \), \( p' \) represents that the final proton has 3-momentum \( p' \) and \( \mu \) labels the polarization of the virtual photon. The deuteron \( | m_J 0 \rangle \) has spin
(isospin) quantum number 1 (0), where the 0 labels the isospin state and $m_J$ labels
the $z$-projection of the total angular momentum quantum number of the deuteron.
Here the matrix element is evaluated in the c.m. frame.

6.2.2 Evaluation of the longitudinal structure function

We now consider the contribution from the longitudinal part in Eq. (6.3). For this
case $\mu = \mu' = 0$, and one can rewrite the differential cross section as

$$ \frac{d^3\sigma_{\text{longitudinal}}}{dk_{\text{lab}}^2 d\Omega_{\text{lab}} d\Omega_p} = \frac{\alpha}{2\pi^2} \frac{k_{\text{lab}}^4}{q_b^4} \sum_{SM, m_J} \rho_L T_{SM, 0m_J} T_{SM, 0m_J}^* . $$  (6.21)

Here the longitudinal structure function $f_L$ can be rewritten as

$$ f_L = \sum_{SM, m_J} x_{SM, 0m_J} x_{SM, 0m_J}^* , $$  (6.22)

where $x_{SM, 0m_J}$ is the $\theta$-dependent part of $T_{SM, 0m_J}$, i.e.,

$$ T_{SM, \mu m_J}(\theta, \phi) = e^{i(\mu+m_J)\phi} x_{SM, \mu m_J}(\theta) . $$  (6.23)

In the rest of this section, we lay out our procedure toward the evaluation of
$f_L$. We first divide the amplitude $T$ into the impulse approximation (IA) part and
the final-state interaction (FSI) part. Then, in order to sum over amplitudes with
different spin, isospin and angular momentum, we write out the matrix element via
both the particle-space and the isospin-space representation. Finally, after obtaining
an expression for the 3-dimensional matrix elements, we show how to construct them
in terms of partial-waves.
Matrix element in particle space and isospin space

From Eq. (6.3) and Eq. (6.22), one sees that to obtain \( f_L \), the matrix element
\[
\langle \Psi_{p'SM} | J_0(q) | m_J0 \rangle
\]
needs to be evaluated, which, up to \( O(eP^3) \), can be represented
by\(^1\)
\[
\langle \Psi_{p'SM} | J_0(q) | m_J0 \rangle = \langle p'SMs\Psi_T | J_0(q) | m_J0 \rangle
+ \langle p'SMs\Psi_T | t(E')G_0(E')J_0(q) | m_J0 \rangle.
\]  
(6.24)

Here we use \( \Psi_T \) to represent the fact that the final-state wavefunction contains an
isospin part. The first term in the right-hand side of Eq. (6.24) is the impulse ap-
proximation, and the second term is the final-state interaction. Here we label the final
state using the proton 3-momentum. \( J_0 \) is the current operator, \( t(E') \) and \( G_0(E') \) are
the t-matrix and the free Green’s function described in Chapter 3. Here we suppress
all the quantum numbers temporarily and use \( | m_J0 \rangle \) to represent the whole deuteron
state (where the 0 labels the isospin of the deuteron). Since the deuteron has to-
tal spin \( S = 1 \), which is unchanged for the longitudinal part of the matrix element,
Eq. (6.24) is reduced to
\[
\langle \Psi_{p'1M} | J_0(q) | m_J0 \rangle = \langle p'1Ms\Psi_T | J_0(q) | m_J0 \rangle
+ \langle p'1Ms\Psi_T | t(E')G_0(E')J_0(q) | m_J0 \rangle.
\]  
(6.25)

\(^1\)If one applies the power counting in the heavy baryon formalism of \( \chi \)PT with minimal substitu-
tion for the electromagnetic field, then at \( O(eP^3) \) one should start to consider two-body contributions
to the deuteron charge operator \( J_0 \) [97, 98].
Figure 6.2: The impulse approximation diagrams for the deuteron electro-disintegration. The blob in the right of each diagram represents the deuteron state.

We first evaluate the impulse approximation part of Eq. (6.25). The dynamics of the breakup can be described by Fig. 6.2: there the final-state proton (neutron) has 3-momentum $p'( - p')$ in the final c.m. frame, while before the proton (neutron) is struck, it has 3-momentum $p' - q/2 (- p' - q/2)$ in the deuteron’s c.m. frame. Thus, we need to consider two cases, i.e, electron strikes proton and electron strikes neutron.

It is useful to first introduce how to label the final state. We now show how to separate the contributions to the final state into isospin $T = 0$ and $T = 1$ parts. The final state consists of a proton and neutron, and can be described as follows

$$| p' S M_a \Psi_T \rangle = | p' \Psi_T \rangle \otimes | S M_s \rangle$$  \hspace{1cm} (6.26)$$

$$= \frac{1}{\sqrt{2}} [ | p_{pm}, pm \rangle - | - p_{pm}, np \rangle ] \otimes | S M_s \rangle$$  \hspace{1cm} (6.27)$$

$$= \frac{1}{\sqrt{2}} [ | f_{pn} \rangle - | f_{np} \rangle ] \otimes | S M_s \rangle.$$  \hspace{1cm} (6.28)
Here $|pn\rangle$ represents that proton is particle 1 and neutron is particle 2, and $|np\rangle$ denotes the opposite. The relative minus sign between these two components in the overall wave function is due to this being a fermion system. Note that the relative momentum between proton and neutron is $p_{pn} \equiv (p_p - p_n)/2$, and $p_{pn} = p'$ since $p_p(p_n) = p'(-p')$. To simplify the notation, we use $|f_{pn}\rangle (|f_{np}\rangle)$ to denote $|p_{pn}, pn\rangle (| - p_{pn}, np\rangle)$.

One can thus express $|p'\Psi_T\rangle$ as

$$|p'\Psi_T\rangle = \frac{1}{\sqrt{2}}[|p_{pn}, pm\rangle - | - p_{pn}, np\rangle]$$

$$= \frac{1}{\sqrt{2}} \frac{1}{2} \left[ (|p_{pn}\rangle + | - p_{pn}\rangle)(|pm\rangle - |np\rangle) 
+ (|p_{pn}\rangle - | - p_{pn}\rangle)(|pm\rangle + |np\rangle) \right] \quad (6.29)$$

One can also express $|pn\rangle$ and $|np\rangle$ states in terms of states of good isospin $|T = 0\rangle$ and $|T = 1\rangle$.

$$|pn\rangle = \frac{1}{\sqrt{2}}[|T = 0\rangle + |T = 1\rangle] \quad (6.30)$$

$$|np\rangle = \frac{1}{\sqrt{2}}[-|T = 0\rangle + |T = 1\rangle]. \quad (6.31)$$

Thus, Eq. (6.29) can be expressed as

$$|p'\Psi_T\rangle = \frac{1}{\sqrt{2}}[|p_{pn}, pm\rangle - | - p_{pn}, np\rangle]$$

$$= \frac{1}{2} \left[ (|p_{pn}\rangle + | - p_{pn}\rangle)(|T = 0\rangle) + (|p_{pn}\rangle - | - p_{pn}\rangle)(|T = 1\rangle) \right]. \quad (6.32)$$
For spin state \( S = 1 \), due to the total wave function being antisymmetric for the NN system, the spatial part of \(|T = 0\rangle(|T = 1\rangle)\) needs to be even (odd) under parity.

With the permutation operator \( P_{12} \), which interchanges particle 1 and 2, i.e.,

\[
P_{12}|f_{pn}1M_s\rangle = |f_{np}1M_s\rangle,
\]

we can check that for Eq. (6.26)

\[
P_{12}|p'1M_s\Psi_T\rangle = -|p'1M_s\Psi_T\rangle,
\]

i.e., our final state is anti-symmetrized properly. Also, one can easily see from Eq. (6.32) that the \(|T = 0\rangle(|T = 1\rangle)\) state is always associated with even (odd) partial-waves. Now, we can evaluate the impulse approximation part of Eq. (6.24).

Inserting the final state, i.e., Eq. (6.29) into the first part of the matrix element in Eq. (6.25), one has

\[
\langle p'1M_s\Psi_T|J_0(q)|m_J0\rangle = \frac{1}{\sqrt{2}} \left[ \langle p_{pn}1M_s,pn|J_0(q)|m_J0\rangle - \langle -p_{pn}1M_s,np|J_0(q)|m_J0\rangle \right],
\]

(6.35)
where

$$\langle p_p S M_s, p_n J_0(\mathbf{q}) | m_J 0 \rangle$$

$$= \langle p' S M_s, p_n J_0(\mathbf{q}) | m_J 0 \rangle$$

$$= \frac{1}{\sqrt{2}} \left[ \langle p_p - q/2, S M_s | m_J \rangle G_P^P(\mathbf{q}^2) \right] + \left[ \langle p_n - q/2, S M_s | m_J \rangle G_E^E(q^2) \right]$$

hit particle 1, proton(p') is particle 1, denote as a

hit particle 2, neutron(-p') is particle 2, denote as d

$$\langle -p' S M_s, n p J_0(\mathbf{q}) | m_J 0 \rangle$$

$$= -\frac{1}{\sqrt{2}} \left[ \langle p_n - q/2, S M_s | m_J \rangle G_E^N(q^2) \right] + \left[ \langle p_p - q/2, S M_s | m_J \rangle G_P^P(q^2) \right]$$

hit particle 1, neutron(-p') is particle 1, denote as d

hit particle 2, proton(p') is particle 2, denote as a

$$= -\langle p' S M_s, p n J_0(\mathbf{q}) | m_J 0 \rangle.$$  

(6.36)

(6.37)

The $\frac{1}{\sqrt{2}}$ factor in Eq. (6.36) and Eq. (6.37) comes from the fact that the deuteron only has isospin $T = 0$ state, so after collapsing with $|p n\rangle (|n p\rangle) = \frac{1}{\sqrt{2}}[(-)|T = 0\rangle + |T = 1\rangle]$, the $\frac{1}{\sqrt{2}}$ factor comes out. We use $|m_J\rangle$ to denote the deuteron state without the isospin part of its wave function.

Using $p_p = p'$, $p_n = -p'$, then

$$\langle p' S M_s \psi_T J_0(\mathbf{q}) | m_J 0 \rangle = \langle p' - q/2, S M_s | m_J \rangle G_P^P(q^2) + \langle -p' - q/2, S M_s | m_J \rangle G_E^E(q^2).$$

(6.38)

Note that to obtain Eq. (6.38), one can directly use the following logic:

$$\text{Total amplitude} = (|\text{hit proton}\rangle + |\text{hit neutron}\rangle),$$

(6.39)

to get the answer. There should be no normalization factor in front of the right-hand side of Eq. (6.39) to reduce the amplitude, since the larger the number of charged
constituent particles, the higher the chance that photon can be absorbed. One then uses the fact that the proton (neutron) always has momentum $p'(-p')$ in the final state to obtain

$$\langle p' SM_s \psi_T | J(q; 0) | m_J 0 \rangle = \langle p' - q/2, SM_s | m_J \rangle G^P(q^2) + \langle -p' - q/2, SM_s | m_J \rangle G^N(q^2).$$

(6.40)

Next, we consider the final-state interaction (FSI), i.e., we want to calculate the right-hand side of Eq. (6.24). The FSI part is given by

$$\langle p' SM_s \psi_T | tG_0 J_0(q) | m_J 0 \rangle = \frac{1}{\sqrt{2}} \left[ \langle p p n SM_s | tG_0 J_0(q) | m_J 0 \rangle - \langle -p p n p SM_s | tG_0 J_0(q) | m_J 0 \rangle \right]$$

$$= \frac{2}{\sqrt{2}} \left[ \langle p' p n SM_s | tG_0 J_0(q) | m_J 0 \rangle \right]$$

(6.41)

Here in the last step we have used the result obtained in Eq. (6.37).

Since what we have from previous chapters is the NN t-matrix expressed in terms of partial-waves, we need to evaluate the t-matrix on the isospin basis, so we insert

$$1 = | T = 0 \rangle \langle T = 0 | + | T = 1 \rangle \langle T = 1 |$$

(6.42)
twice between \( tG_0J_0(q) \) in Eq. (6.41). This yields:

\[
\langle p' SM_s \Psi_T | t(E')G_0(E')J_0(q) | m_f 0 \rangle = \frac{2}{\sqrt{2}} \left[ \int d^3 p \int d^3 p'' \langle p' \eta SM_s | t(E') | p SM_s T = 0 \rangle \times \langle p SM_s T = 0 | G_0(E') | p'' SM_s T = 0 \rangle \langle p'' SM_s T = 0 | J_0(q) | m_f 0 \rangle \\
+ \langle p' \eta SM_s | t(E') | p SM_s T = 1 \rangle \langle p SM_s T = 1 | G_0(E') | p'' SM_s T = 1 \rangle \times \langle p'' SM_s T = 1 | J_0(q) | m_f 0 \rangle \right].
\]  

(6.43)

Thus,

\[
\langle p' SM_s \Psi_T | t(E')G_0(E')J_0(q) | m_f 0 \rangle = \sqrt{2} \frac{1}{\sqrt{2}} \left\{ \int d^3 p \langle p' SM_s T = 0 | t(E') | p SM_s T = 0 \rangle \left[ \frac{\mathcal{P}}{E' - p^2/M} - i\pi\delta(E' - p^2/M) \right] \times \langle p SM_s T = 0 | J_0(q) | m_f 0 \rangle + \langle p' SM_s | t(E') | p SM_s T = 1 \rangle \langle p SM_s T = 1 | J_0(q) | m_f 0 \rangle \left[ \frac{\mathcal{P}}{E' - p^2/M} - i\pi\delta(E' - p^2/M) \right] \right\}.
\]  

(6.44)

Here \( E' = \frac{p'^2}{M} \), and \( \mathcal{P} \) denotes that the principal value should be taken because in Eq. (6.44) we have also used the fact that

\[
\langle p SM_s T | G_0(E') | p'' SM_s T \rangle = \delta^3(p - p'') \left[ \frac{\mathcal{P}}{E' - p^2/M} - i\pi\delta(E' - p^2/M) \right].
\]  

(6.45)
All other parts in Eq. (6.44) are known except for \( \langle p_{SM} T = 0 | J_0(q) | m_J 0 \rangle \), which can be evaluated using the following formula:

\[
\langle p_{SM} T, m_T | J_0(q) | m_J 0 \rangle \\
= \langle p_1 - q/2SM_s | m_J \rangle \langle T, m_T = 0 | \frac{1}{2}(G_E^s(q^2) + \tau_3^1 G_E^v(q^2)) | 00 \rangle \\
+ \langle p_2 - q/2SM_s | m_J \rangle \langle T, m_T = 0 | \frac{1}{2}(G_E^s(q^2) + \tau_3^2 G_E^v(q^2)) | 00 \rangle ,
\]

(6.46)

where \( p_{1(2)} \) is the momentum of particle 1(2) in the final c.m. frame, the nucleon isoscalar form factor \( G_E^s(q^2) = G_E^p(q^2) + G_E^N(q^2) \), and the nucleon isovector form factor \( G_E^v(q^2) = G_E^p(q^2) - G_E^N(q^2) \). In this case, we are not in the \( |pn> \) basis, so particle 1(2) can be proton or neutron. Note that the superscript of \( \tau_3^{1(2)} \) denotes the label of particle 1(2). The matrix element can be evaluated as

\[
\begin{align*}
\langle T = 0 | \frac{1}{2}(1 + \tau^i) | T = 0 \rangle &= \frac{1}{2}; \ i = 1, 2 \\
\langle T = 1 | \frac{1}{2}(1 + \tau_3^1) | T = 0 \rangle &= \frac{1}{2} \\
\langle T = 1 | \frac{1}{2}(1 + \tau_3^2) | T = 0 \rangle &= -\frac{1}{2}
\end{align*}
\]

(6.47) (6.48) (6.49)

Thus,

\[
\begin{align*}
\langle p_{SM} T = 0 | J_0(q) | m_J 0 \rangle \\
= \frac{1}{2} \left\{ \begin{array}{l}
G_E^s(q^2) \langle p - q/2SM_s | m_J \rangle \\
\text{hit proton, proton is particle 1(p), denote as a}
\end{array} \right. \\
+ G_E^N(q^2) \langle p - q/2SM_s | m_J \rangle \\
\text{hit neutron, neutron is particle 1(p), denote as b} \\
\begin{array}{l}
G_E^s(q^2) \langle -p - q/2SM_s | m_J \rangle \\
\text{hit proton, proton is particle 2(-p), denote as c}
\end{array} \\
+ G_E^N(q^2) \langle -p - q/2SM_s | m_J \rangle \\
\text{hit neutron, neutron is particle 2(-p), denote as d}
\right\}
\]

(6.50)
\[ \langle p' SM_s T = 1 | J_0(q) | m_J 0 \rangle \]

\[ = \frac{1}{2} \left\{ G_E^P(q^2) \langle p - q/2 SM_s | m_J \rangle - G_E^N(q^2) \langle p - q/2 SM_s | m_J \rangle \right\} \]

\[ - G_E^P(q^2) \langle -p - q/2 SM_s | m_J \rangle + G_E^N(q^2) \langle -p - q/2 SM_s | m_J \rangle \}
\]

de note as a \( b \) \( c \) \( d \)

Substituting Eq. (6.50) and (6.51) into Eq. (6.44), we obtain the final expression

\[ \langle p' SM_s \Psi_T | (1 + tG_0) J_0(q) | m_J 0 \rangle \]

\[ = a + d \]

\[ + \int d^3 p \langle p' SM_s T = 0 | t | p' SM_s T = 0 \rangle \left\{ \frac{P}{E' - p^2/M} - i\pi\delta(E' - p^2/M) \right\} \]

\[ \times \frac{1}{2} (a + b + c + d) \]

\[ + \langle p' SM_s T = 1 | t | p' SM_s T = 1 \rangle \left\{ \frac{P}{E' - p^2/M} - i\pi\delta(E' - p^2/M) \right\} \]

\[ \times \frac{1}{2} (a - b - c + d). \]

(6.52)

Here \( a, b, c \) & \( d \) are defined in Eq. (6.50). Eq. (6.52) shows that the FSI amplitude consists of two parts: \( T = 0 \) and \( T = 1 \) part. And each part consists of a \( t \)-matrix, the free Green’s function and the deuteron wave function with the nucleon form factors. Here we need to integrate over \( p \), thus the deuteron wave function is multiplied by the half-shell \( t \)-matrix \( t(p_{pn}, p; E_{np}) \). For all \( p \) this contributes to the amplitude. Although, in practice, since the nucleons inside deuteron are rarely
in a high-momentum state (the high \( p \) component of the deuteron wave function is small), so the high-momentum part of the integral will be suppressed. Physically, this means that the virtual photon can strike one of the nucleons which is in a state with arbitrarily high 3-momentum \( p \) (the other nucleon state will have to be \(-p\)). Regardless of the magnitude of \( p \) in the FSI part, the two nucleons then exchange momentum to reach their final state through the FSI. This is in contrast with the IA, where, to reach a given final state, the virtual photon must strike the nucleon at a specified momentum state.

At this point, we have an expression for the sum of the impulse approximation and the final-state interaction in terms of the 3-momentum of the measured proton \( p' \). The next step is to express Eq. (6.52) in terms of partial-waves.

**Constructing the matrix element in partial-waves**

First, we construct the 3-dimensional matrix element for the impulse approximation amplitude by partial-waves. The impulse approximation amplitude has been obtained in Eq. (6.40), i.e.,

\[
\langle p'S \rightarrow J(q;0)|m_J0 \rangle = \langle p'q/2SM_s|m_J \rangle G^P_E(q) + \langle -p'q/2SM_s|m_J \rangle G^N_E(q^2).
\]

(6.53)

For the deuteron, \( S = 1, M_s = \pm 1, 0 \). We already have in hand the deuteron wave function in the partial-wave decomposition in momentum space, i.e., \( \Psi_{3S_1}(p) \) and \( \Psi_{3D_1}(p) \) as shown in Fig. 3.5 in Sec. 3.3 (up to a normalization factor \( \sqrt{\frac{3}{\pi}} \)).
To evaluate Eq. (6.53), we insert

\[ \frac{1}{\pi} \sum_{J,m,LS} \int_0^\infty dp \lvert p J_m L S \rangle \langle p J_m L S \rvert \tag{6.54} \]

into \( \langle p' - q/2M_s \mid m_J \rangle \), where \( p = |p|, J = L + S \) (\( L \) represents the angular momentum quantum number). We get

\[
\langle p' - q/2M_s \mid m_J \rangle = \frac{2}{\pi} \sum_{J,m,LS} \int_0^\infty dp \lvert p' - q/2M_s \rangle \langle p J_m L S \rangle \langle p J_m L S \mid m_J \rangle. \tag{6.55}
\]

Note that the normalization adopted here is

\[ \langle k' \mid k \rangle = \delta^{(3)}(k - k'). \tag{6.56} \]

Since

\[
\langle p' J_m J' L' S' \mid p J_m L S \rangle = \frac{\pi}{2} \frac{\delta(\lvert p \rvert - \lvert p' \rvert)}{p^2} \delta_{JJ'} \delta_{mJmJ'} \delta_{LL'} \delta_{SS'}, \tag{6.57}
\]

thus

\[
\langle p' - q/2M_s \mid m_J \rangle = \sum_{L=0,2} \langle p' - q/2M_s \rangle \mid p' - q/2 \rangle J = 1 m_J LS = 1 \rangle \langle p' - q/2 \rangle \langle p' - q/2 \mid m_J \rangle. \tag{6.58}
\]

The last matrix element is just the deuteron wave function, i.e.,

\[
\langle p' - q/2 \mid m_J LS \rangle = \lvert 0 \rangle \lvert 1 \rangle \Psi_{2s+1 \ell J}(\lvert p' - q/2 \rvert). \tag{6.59}
\]
Here \( \Psi_{2S+1LJ}(p) \) is the deuteron wave function with \( S = 1 \), \( J = 1 \) and \( L = 0 \) or \( 2 \).

We can simplify the notation by using \( \Psi_L(p) \) to replace \( \Psi_{2S+1LJ}(p) \). The normalization here satisfies

\[
\frac{2}{\pi} \int dp p^2 [\Psi_0^2(p) + \Psi_2^2(p)] = 1.
\]

(6.60)

Note that in chapter 3, the wave functions \( \psi_l(p) \) have the normalization

\[
\int dp p^2 [\psi_0^2(p) + \psi_2^2(p)] = 1.
\]

(6.61)

Thus, \( \Psi_L(p) = \sqrt{\frac{2}{\pi}} \psi_l(p) \).

Inserting Eq. (6.59) in Eq. (6.58),

\[
\langle \mathbf{p}' - \mathbf{q}/2M_s | m_J \rangle = \sum_{L=0,2} \langle \mathbf{p}' - \mathbf{q}/2 | \mathbf{p}' - \mathbf{q}/2 | J = 1 \rangle i^L \Psi_L(|\mathbf{p}' - \mathbf{q}/2|).
\]

(6.62)

Using the plane-wave expansion formula

\[
|\mathbf{p}' M_s \rangle = \sqrt{\frac{2}{\pi}} \sum_{L',m'_t} Y_{L'm'_t}^*(\Omega') |\mathbf{p}'| L'm'_t \rangle \otimes |M_s \rangle,
\]

(6.63)

Eq. (6.62) becomes

\[
\langle \mathbf{p}' - \mathbf{q}/2M_s | m_J \rangle = \sqrt{\frac{2}{\pi}} \sum_{L=0,2} \sum_{M_t} \sum_{m_t} (Lm_tS = 1M_s | J = 1m_J \rangle
\]

\[
\times Y_{Lm_t}^*(\Omega_{\mathbf{p}' - \mathbf{q}/2}) \Psi_L(|\mathbf{p}' - \mathbf{q}/2|).
\]

(6.64)

A similar calculation leads to:

\[
\langle -\mathbf{p}' - \mathbf{q}/2M_s | m_J \rangle = \sqrt{\frac{2}{\pi}} \sum_{L=0,2} \sum_{M_t} \sum_{m_t} (Lm_tS = 1M_s | J = 1m_J \rangle
\]

\[
\times Y_{Lm_t}^*(\Omega_{-\mathbf{p}' - \mathbf{q}/2}) i^L \Psi_L(|\mathbf{p}' - \mathbf{q}/2|).
\]

(6.65)
Here $Y_{L'm_l}$ is the spherical harmonic function, $\Omega_{p'-q/2}$ is the angle between $\hat{z}$ and $\mathbf{p}'-\mathbf{q}/2$, $m_l'$ is the projection of $L'$ in the z-axis and $(Lm_lS = 1m'_s|J = 1m_J)$ denotes the Clebsch-Gordon coefficient. Eq. (6.64) and Eq. (6.65) can then be inserted into Eq. (6.53), which allows one to obtain the final expression. For completeness, we list the explicit expression of Eq. (6.64) in Appendix D.

Next, we consider the final-state interaction (FSI) term, i.e.,

$\langle \mathbf{p}'S M_s \Psi_T | t(E') G_0(E') J(\mathbf{q}; 0) | m_J 0 \rangle.$ \hspace{1cm} (6.66)

We continue from Eq. (6.52), and write out the spin and isospin quantum numbers, then

$$
\langle \mathbf{p}'S M_s \Psi_T | t(E') G_0(E') J(\mathbf{q}; 0) | m_J 0 \rangle \\
= \sum_{T_f M_t f} \int d^3 \mathbf{p} d^3 \mathbf{p}' \langle \mathbf{p}' S M_s T_f M_t | t(E') | \mathbf{p} S' M'_s T M_t \rangle \\
\times \langle \mathbf{p} S' M'_s T M_t | G_0(E') | \mathbf{p}'' S'' M''_s T'' M''_t \rangle \langle \mathbf{p}'' S'' M''_s T'' M''_t | J(\mathbf{q}; 0) | m_J 0 \rangle.
$$ \hspace{1cm} (6.67)

Here $M_{tf}(M_t)$ is the projection of $T_f(T)$ in the z-axis.

Now, as mentioned before

$$
\langle \mathbf{p} S M_s T M_t | G_0(E') | \mathbf{p}'' S'' M''_s T'' M''_t \rangle \\
= \left[ \frac{\mathcal{P}}{E' - \mathbf{p}^2/M} - i\pi \delta(E' - \mathbf{p}^2/M) \right] \delta^3(p'' - p) \delta_{SS''} \delta_{M_s M''_s} \delta_{T T''} \delta_{M_t M''_t}.
$$ \hspace{1cm} (6.68)
So Eq. (6.67) becomes

$$
\langle p' S M_s \Psi_T | t(E') G_0(E') J(q; 0) | m_J 0 \rangle \\
= \sum_{T_f T M_t' S M_s'} \int d^3 p \langle p' S M_s T_f M_{t_f} | t(E') | p S M_t' T M_t \rangle \\
\times \left[ \frac{P}{E' - p^2 / M} - i \pi \delta(E' - p^2 / M) \right] \langle p S M_t' T M_t | J(q; 0) | m_J 0 \rangle. \quad (6.69)
$$

Eq. (6.69) is the same as Eq. (6.52), just with spin and isospin explicitly written. 

$$
\langle p S M_t' T M_t | J(q; 0) | m_J 0 \rangle 
$$

is evaluated in Eq. (6.52), Eq. (6.64) and Eq. (6.65). The only unknown remaining now is 

$$
\langle p' S M_s T_f M_{t_f} | t(E') | p S M_t' T M_t \rangle.
$$

Since the spin($S'$) and isospin($T$) is conserved in the NN interaction, Eq. (6.69) can be further reduced to

$$
\langle p' S M_s T_f | t(E') G_0(E') J(q; 0) | m_J 0 \rangle \\
= \sum_{M_{t_f}} \int d^3 p \langle p' S M_s T_f | t(E') | p S M_{t_f} T_f \rangle \\
\times \left[ \frac{P}{E' - p^2 / M} - i \pi \delta(E' - p^2 / M) \right] \langle p S M_{t_f} T_f | J(q; 0) | m_J 0 \rangle. \quad (6.70)
$$

Note that here we have also dropped the label $M_t$ and $M_{t_f}$, since the projection of isospin is conserved all the way from the deuteron to the final np state, i.e., $M_t = M_{t_f} = 0$. However, $T_f$ can be 0 or 1. The principal-value and delta-function part can be easily dealt with using the standard technique described in Appendix B. Now, the task is to construct the 3-dimensional t-matrix $t(p', p; E')$ from $t_{L S J}(p', p; E)$.

For the case $S = 0$ (i.e., $^1S_0$, $^1P_1$), since $J = L + S = L$, so we can directly use

$$
t(k', k; E) = \frac{1}{2 \pi^2} \sum_{J=0}^{\infty} (2J + 1) t_J(k', k; E) P_J(\cos \theta_{k' k}). \quad (6.71)
$$
to obtain $t(p', p; E')$ for the $S = 0$ case. However, since we are calculating the longitudinal part of the response function, the spin $S = 1$ is conserved. So $S = 0$ does not contribute to this quantity at this order.

What we need to evaluate is the $S = 1$ case. Here, after inserting “1” as listed in Eq. (6.54) twice in $\langle p'SM_s T_f | t(E') | pSM'_s T_f \rangle$, we obtain

$$\langle p'SM_s T_f | t(E') | pSM'_s T_f \rangle = \left( \frac{2}{\pi} \right)^2 \sum_{J,m,J,L,L',m,m'} \int_0^\infty \int_0^\infty dp' dp'' \langle p'S' = 1, T_f M_s | p J m J LS \rangle \times \langle p J m LST_f | t(E') | p'' J m J L''ST_f \rangle \langle p'' J m J L'' S | p S' = 1 T_f M_s' \rangle. \quad (6.72)$$

Here we have already used the fact that $S' = S$, $J' = J$, $m_j = m'_j$ for the NN scattering.

Now, from Eq. (6.57) and Eq. (6.63), one can obtain

$$\frac{2}{\pi} \int_0^\infty dp' \langle p'S'M'_s | p J m J LS \rangle = \sqrt{\frac{2}{\pi}} (L m J M'_s | J m_j) Y_{Lm_j}(\Omega_{p'}), \quad (6.73)$$

and write down the resulting expression for the t-matrix as:

$$\langle p'SM_s T_f | t(E') | pSM'_s T_f \rangle = \frac{2}{\pi} \sum_{J,m,J,L,L',m,m'} (L m J M | J m_j) Y_{Lm_j}(\Omega_{p'}) t_{LSJ,T_f}(E') (L'' m m' M'_s | J m_j) Y_{L''m_m'}^*(\Omega_{p}). \quad (6.74)$$

Here $t_{LSJ,T_f}(E')$ is the simplified notation of $t_{LSJ,T_f}(p', p; E')$, and we have dropped the original summation over $M'_s$, since it is fixed once $m_j$ and $m'_m$ are specified. As
a consequence, since \( m_j = M_s + m_l = M'_s + m''_l \) is conserved all the way, for a set of \( (M'_s, m_j) \) (which are both specified), we get a fixed \( m''_l \). Thus,

\[
\langle p'S = 1M_s T_J | t(E')G_0(E') J(q;0) | m_J 0 \rangle = \frac{2}{\pi} \int d^3p \left\{ \sum_{J,m_j,L,L''} (Lm_S = 1M_s | Jm_j) \times Y_{Lm}(\Omega_p) t_{LSJ,T_J}(E') (L''m''_l S = 1M'_s | Jm_j) Y^*_{L'm'_l}(\Omega_p) \times \langle pSM'_s T_M | J(q;0) | m_J 0 \rangle \right\}.
\]

(6.75)

It is important to keep in mind that, in Eq. (6.75), \( M'_s \) is conserved on both sides of \( \left[ \frac{p}{E' - p^2/M} - i\pi\delta(E' - p^2/M) \right] \).

Above, the last factor in Eq. (6.75), \( \langle pSM_s T_M | J(q;0) | m_J 0 \rangle \) was evaluated in the impulse approximation. The results are presented in Eq. (6.50) and Eq. (6.51) (note that now we have \( \langle pSM_s T_M | \) instead of \( \langle p'SM_s T_M | \)). The element \( \langle \pm p - q/2M_s | m_J \rangle \) is further decomposed into partial-waves as listed in Eq. (6.64) and Eq. (6.65), i.e.,

\[
\langle \pm p - q/2M_s | m_J \rangle = \sqrt{\frac{2}{\pi}} \sum_{L=0,2,M_s} \sum_{m_l} \langle L, m_l S = 1M_s | J = 1m_J \rangle Y_{Lm_l}(\Omega_{\pm p - q/2}) iL\Psi_L(||p - q/2||).
\]

(6.76)

Thus, in Eq. (6.75) we need to perform the angular part of the integral in the following form

\[
\int Y^*_{L''m''_l}(\Omega_p) Y_{Lm_l}(\Omega_{\pm p - q/2}) d\Omega_p.
\]

(6.77)

Here \( \Omega_{\pm p - q/2} \) is the angle between \( \hat{z} \) and \( \pm p - q/2 \), and the \( \hat{z} \)-axis is set to be along \( q \) as shown in Fig.6.1. Defining the angle between \( \hat{z} \) and \( p \) as \((\theta, \phi)\), then we can
denote the angle between $\hat{z}$ and $\mathbf{p} - \mathbf{q}/2$ as $(\theta', \phi)$, and

$$
\theta' = \sin^{-1}\left[\frac{|\mathbf{p}| \sin \theta}{\sqrt{|\mathbf{p}|^2 - |\mathbf{p}| |\mathbf{q}| \cos \theta + |\mathbf{q}/2|^2}}\right].
$$

(6.78)

The angle between $\hat{z}$ and $-\mathbf{p} - \mathbf{q}/2$, which we denote as $(\theta'', \phi + \pi)$, is

$$
\theta'' = \sin^{-1}\left[\frac{|\mathbf{p}| \sin \theta}{\sqrt{|\mathbf{p}|^2 + |\mathbf{p}| |\mathbf{q}| \cos \theta + |\mathbf{q}/2|^2}}\right].
$$

(6.79)

Thus, to evaluate $\langle \mathbf{p}' S = 1 M \mathbf{s}_t | t(E')G_0(E')J(\mathbf{q}; 0) | mJ0 \rangle$, we need to do the angular part of the integral over angle $\theta$ and $\phi$. Since we have different angles $\theta$ in the spherical harmonics, the analysis is performed as the following.

First we take advantage of the property of spherical harmonics:

$$
\int_0^{2\pi} \int_0^{\pi} Y_{lm}(\theta, \phi) Y_{l'm'}(\theta', \phi) d(\cos \theta) d\phi = 2\pi \delta_{mm'} \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!}} \frac{2l' + 1}{4\pi} \frac{(l' - m)!}{(l' + m)!} \int_0^{\pi} P^m_l(\cos \theta) P^{m'}_{l'}(\cos \theta') d(\cos \theta).
$$

(6.80)

Here

$$
P^m_l(\cos \theta) = (1 - \cos^2 \theta)^{m/2} \frac{d^m}{d \cos \theta^m} (P_l(\cos \theta)),
$$

(6.81)

and

$$
P^{-m}_l(\cos \theta) = (-1)^m \frac{(l - m)!}{(l + m)!} P^m_l(\cos \theta).
$$

(6.82)

Thus, using Eq. (6.80) we can do the $d\phi$ part integral analytically. For the $-\mathbf{p} - \mathbf{q}/2$ case, the analysis is the same, but we have $Y_{lm}(\theta'', \phi + \pi) = Y_{lm}(\theta'', \phi + \pi)$ for $m =$ even, and $Y_{lm}(\theta'', \phi + \pi) = -Y_{lm}(\theta'', \phi + \pi)$ for $m =$ odd.
To summarize, we follow these steps to evaluate the final-state-interaction matrix element on the partial-wave basis:

1. Construct the 3-dimensional t-matrix by Eq. (6.74).

2. Specify the initial state, i.e., \( |m_j\rangle \) (where \( m_j \) denotes the projection of \( J \) in the \( z \)-axis), and the final state, i.e., \( \langle p'M_s, T_f| \), then evaluate Eq. (6.75) with the IA part from Eq. (6.52), Eq. (6.64) and Eq. (6.65). The \( d\phi \) part of the integral is eliminated by Eq. (6.80), but the \( d\theta \) integral needs to be done numerically.

3. Sum over different isospin using Eq. (6.52):

\[
\langle p'M_s \Psi_T |(1 + tG_0)J_0|m_j\rangle = \sum_{T_f=0,1} \langle p'M_s T_f|(1 + tG_0)J_0|m_j\rangle.
\] (6.83)

We list the explicit expression of the FSI matrix element after partial-wave expansion in Appendix E.

6.3 Results and discussion of longitudinal response function

In this section we present our results for the longitudinal response function of the deuteron electro-disintegration process. For the nucleon form factors we adopt the results listed in Ref. [99]. For electro-disintegration, one needs to specify two kinematic variables, e.g., \( (\omega, q) \) to describe the whole process. We adopted the following
kinematic variables in order to compare results obtained with the Bonn potential in Ref. [93]. First, the final energy of the proton and neutron system, which is labelled by $E_{np}$, i.e.,

$$E_{np} = \frac{p^2}{M}.$$  \hspace{1cm} (6.84)

Second, the 3-momentum given by the virtual photon in the system’s final c.m. frame is $q_{cm}^2$. With $E_{np}$ and $q_{cm}^2$ specified, the energy $\omega$ given by the virtual photon can be calculated by

$$\omega = E_{np} - \sqrt{M_d^2 + q_{cm}^2 + 2M}.$$  \hspace{1cm} (6.85)

Before presenting the results, we introduce one particular kinematics which is of interest: the so-called quasi-free ridge. The quasi-free ridge occurs when $\omega = 0$. Physically, this means that the virtual photon hits one of the nucleons and gives just enough 3-momentum to put it on-mass-shell. The other nucleon remains at rest in the laboratory frame. If we adopt MeV as the unit of $E_{np}$, and fm$^{-2}$ as the unit of $q_{cm}^2$, then from Eq. (6.85) we have for quasi-free ridge

$$0 = E_{np} - \sqrt{M_d^2 + q_{cm}^2 + 2M}.$$  \hspace{1cm} (6.86)

Thus,

$$E_{np} \approx M_d(1 + \frac{q_{cm}^2}{2M_d^2}) - 2M \approx \frac{q_{cm}^2}{2M_d} \approx \frac{q_{cm}^2}{4000 \text{ (MeV)}}.$$  \hspace{1cm} (6.87)

Since 1 MeV$\approx \frac{1}{197.3286}$ fm$^{-1}$, so for quasi-free ridge, $E_{np}$ (MeV)$\approx 10q_{cm}^2$ (fm$^{-2}$).
We now present our results. First, we adopt the NN t-matrix and deuteron wave function generated from the SFR TPE up to NNLO by our subtraction method in Sec. 4.3, with the intrinsic cutoff $\tilde{\Lambda}$ set to 800 MeV. Figs. 6.3-6.9 show the longitudinal structure function $f_L$ versus angle $\theta$, i.e., the angle between $p'$ and $q$, for $E_{np} = 10$ MeV and $q^2_{cm} = 0.25 - 25$ fm$^{-2}$. First, let us look at the impulse approximation (IA), where our results are denoted as the red dash-double-dotted line, and the Bonn potential results are denoted by the red dotted line. As we can see, when both $E_{np}$ and $q^2_{cm}$ are low, i.e., $E_{np} = 10$ MeV and $q^2_{cm} \leq 1$ fm$^{-2}$, the two results agree with each other very well. As $q^2_{cm}$ becomes larger, the IA results obtained from the two potentials start to deviate from each other.

For the final-state interaction (FSI), one needs to sum over partial-waves in order to obtain the 3-dimensional t-matrix. We have summed over partial-waves up to $J = 3$, and the results are denoted as blue dashed ($J = 1$), green dash-dotted ($J$ up to 2) and black double-dash-dotted ($J$ up to 3) line in Fig. 6.3$^2$. In general, our results converge after $J > 1$. We first discuss the quasi-free ridge case, i.e., $E_{np} = 10$ MeV and $q^2_{cm} = 1$ fm$^{-2}$. Fig. 6.5 shows that $f_L$ receives the least correction from the FSI at this kinematics region. And, as shown by other figures, the further away we are from the quasi-free ridge, the larger the FSI correction $f_L$ receives. This can be explained easily by the fact that, at the quasi-free ridge, both nucleons are on the mass shell after the virtual photon strikes the deuteron, and there is no need for the

$^2$Note that for $J \geq 2$, there is no contact term associated with the long range part of the TPE. We also adopt the Born approximation instead of iterating the SFR TPE in the LSE.
FSI in order to make the final-state particles real. On the other hand, the further we are kinematically away from the quasi-free ridge, the larger the energy-momentum transfer that the FSI must provide in order to make the proton and neutron become real particles in the final state. Thus, the FSI becomes important there. For $E_{np} = 10$ MeV, the quasi-free ridge is the last place where all the IA and FSI results from the two potentials agree. As we increase $q^2_{cm}$ to above 4 fm$^{-2}$, both our IA and FSI results start to diverge away from the Bonn potential results. In particular, the FSI results diverge much faster than the IA results. In order to further analyse this situation, we vary the cutoff in the LSE from $\Lambda = 600 - 1000$ MeV and plot the variation of $f_L$ as error bars. The resulting error bars grow quite significantly as $q^2_{cm}$ becomes larger and larger. This is an indication that for the FSI, where both the deuteron wave functions and the NN t-matrix enter the calculation, the results become highly cutoff-dependent as one probes higher and higher momentum transfers. For $E_{np} = 10$ MeV, the cutoff dependence starts to show up at $q^2_{cm} = 4$ fm$^{-2}$, which corresponds to $q_{cm} \approx 400$ MeV.
Figure 6.3: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$ for four different kinematic regions. The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\Lambda = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.4: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.5: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.6: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.7: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.8: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 – 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.9: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.10: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 – 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.11: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.12: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.13: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.14: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.15: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.16: The deuteron electro-disintegration longitudinal response function \( f_L \) as a function of \( \theta \). Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff \( \Lambda = 600 - 1000 \) MeV in the LSE. The intrinsic cutoff is \( \tilde{\Lambda} = 800 \) MeV for the SFR TPE up to NNLO.
Next, we present the $E_{np} = 30$ MeV results in Figs. 6.10-6.16, where $q_{cm}^2$ runs from 0.25 – 25 fm$^{-2}$. For both the IA and FSI, our results agree with those obtained from Bonn potential approximately up to the quasi-free ridge and a little bit beyond to $q_{cm}^2 = 4$ fm$^{-2}$. Note that for $q_{cm}^2 < 4$ fm$^{-2}$, although $f_L$ receives large correction from the FSI, the results from the two potentials still agree with each other quite well. Actually, the Bonn potential results completely fall within the error bar of our results set by varying $\Lambda = 600 – 1000$ MeV. However, as $q_{cm}^2 \geq 12$ fm$^{-2}$, the results from Bonn potential starts to disagree with the chiral potential results and completely falls outside the error bar. This perhaps indicates that we are probing a too large $q_{cm}^2$ for the chiral potential. As for the comparison between the IA and the FSI, we again see that the further away from the quasi-free ridge we are, the larger the correction from the FSI is.
Figure 6.17: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.18: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.19: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.20: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.21: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.22: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.23: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
In Figs. 6.17-6.23 we plot the $E_{np} = 60$ MeV, $q_{cm}^2 = 1 - 25$ fm$^{-2}$ results. And in Fig. 6.24-Fig. 6.30 we plot the $E_{np} = 100$ MeV, $q_{cm}^2 = 1 - 25$ fm$^{-2}$ results. From those figures, we see that the same pattern observed in the lower $E_{np}$ regions still holds. In summary, first we observed that, near the quasi-free ridge, the IA and the FSI results are close to each other, and they deviate from each other further as one moves away from the quasi-free ridge. Also, the results obtained from the SFR TPE up to NNLO agree with those obtained from Bonn potential either for low $E_{np}$ and $q_{cm}^2$ or near the quasi-free ridge. In general, this kinematic region is defined by $E_{np} \leq 30$ MeV and $q_{cm}^2 \leq 4$ fm$^{-2}$. In addition, for high $E_{np}$ or $q_{cm}^2$, the cutoff-dependence of the results from the chiral potential is more significant. The increase of the cutoff dependence of the chiral potential results also applies to cases where one lower $q_{cm}^2$ away from the quasi-free ridge, with $E_{np} \geq 60$ MeV. These trends persist in the calculations of $f_L$ up to $E_{np} = 160$ MeV and for $q_{cm}^2 = 1 - 25$ fm$^{-2}$.

We now estimate that how the uncertainty in $f_L$ comes from the uncertainty of the deuteron wave function and NN t-matrix given by the chiral potentials. First, our results show that at the quasi-free ridge, the $f_L$ given by the SFR TPE up to NNLO agree with those given by the Bonn potential all the way up to $E_{np} = 160$ MeV. From Eq. (6.84) and Eq. (6.87), we have the following relation between $p'$ and $q_{cm}$:

$$E_{np} = \frac{p'^2}{M} \approx \frac{q_{cm}^2}{2M_d} \quad (6.88)$$

at the quasi-free ridge. Thus,

$$p' \approx \frac{q_{cm}}{2} \quad (6.89)$$
at the quasi-free ridge, where the deuteron wave function $\Psi_L(|p' - q_{cm}/2|)$ enter the (IA) calculation. The value of $|p' - q_{cm}/2|$ in this case ranges from 0 (for $\theta = 0$) to $2|p'|$ (for $\theta = \pi$). Since the high-momentum component of the deuteron wave function is almost identical to zero, the agreement in the quasi-free ridge is then a reflection of the fact that the low-momentum part of the deuteron wave function given by the SFR TPE up to NNLO agrees with the one given by the Bonn potential. To see where the two wave functions start to disagree, we use Fig. 6.6 as an example, where the IA part of $f_L$ given by the SFR TPE up to NNLO starts to diverge away from the one given by the Bonn potential. In this kinematics region, $(|p'|, |q_{cm}|) = (100, 400)$ MeV, thus the value of $|p' - q_{cm}/2|$ runs from 100 MeV (for $\theta = 0$) to 300 MeV (for $\theta = \pi$). Fig. 6.6 shows that there is a $\approx 10\%$ difference in $f_L$ at $\theta = 0$, which is purely originated from the difference of the deuteron wave functions generated by the SFR TPE and Bonn potential at 100 MeV$^3$. This 10% difference is then propagated to the result of the FSI amplitude without being significantly enlarged by the insertion of the NN t-matrix. This indicates that the NN t-matrix, $t(p', p; E_{np})$, given by the SFR TPE up to NNLO agree with those given by the Bonn potential, at least up to momentum $p' = 100$ MeV and $p = |q_{cm}|/2 = 200$ MeV (where $|p - q_{cm}/2| = 0$ at $\theta = 0$, and $t(p', p; E_{np})$ is multiplied by the largest $\Psi_L$). Meanwhile, as we increase $q_{cm}^2$ to 10 fm$^{-2}$ (with $E_{np} = 10$ MeV), $f_L$ given by the two different potentials

---

$^3$Note that when obtaining the deuteron wave function, we directly adopt the LEC's generated by our subtraction plus fitting method as mentioned in section 4.3.2 instead of performing fitting to the deuteron binding energy, so this 10% difference in the wave function at 100 MeV may potentially be reduced by a different fitting/subtraction procedure.
starts to have a larger difference in the FSI than in the IA. This indicates that for 
\((p', p; E_{np}) = (100, 316; 10)\) (unit: MeV), \(t(p', p; E_{np})\) given by the SFR TPE up to 
NNLO and Bonn potential present a noticeable difference. In fact, the LEC’s of 
the NN t-matrix we adopted in the electro-disintegration calculation are those which 
generate phase shifts that agree with the Nijmegen analysis for \(T_{lab} \leq 100\) MeV (as 
can be seen in Fig. 5.12 and Fig. 5.38 in Chapter 5). This kinematics corresponds to 
\(E_{np} \leq 50\) MeV and \(p' \& p \leq 223\) MeV. For higher energy/momentum, the phase shifts 
given by the SFR TPE up to NNLO start to diverge away from the Nijmegen analysis. 
This is directly reflected in the results of \(f_L\) as shown in Fig. 6.7, i.e., the difference 
between the two potentials is larger in the FSI than in the IA. In other words, if the 
combination of \(E_{np}\) and \(q_{cm}\) results in a \(t(p', p; E_{np})\) in which the \(p'(\equiv \sqrt{ME_{np}})\) or 
\(p(\equiv |q_{cm}|/2)\) is greater than 223 MeV, then the discrepancy in FSI would come from 
both the differences in \(\Psi_L\) and the difference in the NN t-matrix generated by the 
SFR TPE up to NNLO and the Bonn potential. All of our results in other kinematics 
region can be understood in the same way.
Figure 6.24: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.25: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.26: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.27: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 – 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.28: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.29: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Figure 6.30: The deuteron electro-disintegration longitudinal response function $f_L$ as a function of $\theta$. Here the red dash-double-dotted (dotted) line represents the IA results given by the SFR TPE up to NNLO (Bonn), and the black double-dash-dotted (solid) line denotes the FSI results given by the SFR TPE up to NNLO (Bonn). The error bars are obtained from varying the cutoff $\Lambda = 600 - 1000$ MeV in the LSE. The intrinsic cutoff is $\tilde{\Lambda} = 800$ MeV for the SFR TPE up to NNLO.
Chapter 7

Conclusion

We have developed a subtractive renormalization scheme to evaluate the NN interaction with the LO, NLO and NNLO chiral effective potentials. Our subtractive scheme enables direct use of physical observables in the problem, instead of requiring a fit to determine the LEC’s in the contact interactions present in these potentials. Thus it allows a quick and convenient evaluation of the scattering and bound-state observables for any cutoff $\Lambda$ in the LSE. This is of importance for the further study of chiral EFT in the NN sector. In particular, we performed a detailed analysis to examine various problems for the non-perturbative use of an NN interaction obtained from $\chi$PT. We summarize our achievements and findings in the NN sector as follows:

(1) Our subtractive renormalization scheme generates results equivalent to the conventional “fitting” method, while allowing direct input of physical observables.

(2) Introducing an energy-dependent contact term in the NLO/NNLO potential produces phase shifts that oscillate with respect to $\Lambda$.

(3) Whether a contact term (or contact terms) is needed for a cutoff-independent result is exactly determined by the (coordinate-space) singularity structure of the potential as $r \rightarrow 0$. 
(4) Cutoff independence in the phase shift does not necessarily mean the results are renormalization-point independent. Both properties are necessary conditions for successful renormalization.

(5) In general, there is a highest cutoff $\Lambda_c \approx 1\ (2)\ \text{GeV}$ in the LSE one can adopt for the NNLO DR (SFR) TPE, before the results start to exhibit problems with respect to cutoff independence and fitting-point independence.

(6) Our deuteron electro-disintegration calculation shows that the chiral effective potential generates results agree with those obtained by the Bonn potential for kinematic region $E_{np} \leq 30\ \text{MeV}$ and $q^2_{cm} \leq 4\ \text{fm}^{-2}$.

Point (1) is most obviously shown in Sec. 4.2.3, where we compare the LO S-wave phase shift obtained from the conventional “fitting” method to our subtractive scheme. We found that these two methods generate the same results if the physical observable used as input in our subtraction is the same as the fitting point of the conventional method. Our method has the advantage of allowing a direct input of physical data instead of fitting—which is a time consuming process. Additionally, the further developed subtraction schemes presented in Secs. 4.1, 4.2 and 4.3 are able to eliminate the correlation between the LEC’s and thus allow a detailed analysis of renormalization issues.

For point (2), we have demonstrated in Sec. 5.3.2 and 5.4.2 that an oscillatory behavior with respect to $\Lambda$ in phase shift is presented by associating the DR (SFR) NNLO TPE with the energy-dependent contact term. We have analyzed (with a
two-potential method) the problem and showed that the oscillation is caused by the resonance state created by the energy-dependent contact term. We emphasize that the first place where the phase shifts diverge is at cutoff $\Lambda_c \approx 1 (1.2)$ GeV for the singlet (triplet) channel for the DR NNLO TPE. Adopting the SFR chiral potential could defer the oscillatory behavior in the phase shift (until $\Lambda_c \approx 2$ GeV), but not eliminate it completely, except for the particular case—the MIXED NNLO TPE in the $^1S_0$ channel.

Point (3) is discussed in Secs. 5.1-5.2.4, where we show that for a singular and attractive (as $r \to 0$) potential, the contact term is required for the result to be cutoff-independent. On the other hand, a contact term will destroy the original cutoff-independent result if it is associated with a singular and repulsive (as $r \to 0$) potential. We have performed the calculation in P-waves up to NNLO and confirmed this phenomenon. Moreover, the number of contact terms needed is directly related to the number of singular and attractive channels for the coupled partial-waves.

Point (4) is most obviously shown in Sec. 5.3. There we plot the $^1S_0$ phase shift obtained with the DR and SFR NNLO TPE along with the momentum-dependent contact term and shown that the results depend on the point of fitting. This indicates that, at those cutoffs, the theory lost its predictive power and the apparent cutoff independence result in the phase shift is not the guarantee of a successful renormalization. One needs to perform additional analyses to ensure the fitting-point-independence is also present in order to claim that the renormalization is done successfully.
Finally, we examine the range of cutoffs where results obtained from the TPE can be renormalized successfully. We found that this range of cutoff is $\Lambda \sim 0.7 - 1$ GeV for the DR TPE, and for the SFR TPE, $\Lambda \sim 0.7 - 2$ GeV. Beyond this range of $\Lambda$, cutoff-dependence and fitting-point-dependence issues start to appear.

For point (6), in order to further test the chiral EFT in the NN sector and for physical interest as well, we performed the calculation of the deuteron electro-disintegration process and obtained the longitudinal response function $f_L$ up to $O(P^2)$. As shown in Sec. 6.3, $f_L$ generated from the chiral effective potential agrees with those obtained by the Bonn potential in the kinematic region $E_{np} \leq 30$ MeV and $q_{cm}^2 \leq 4$ fm$^{-2}$. In addition, the two potentials also generate results that agree at the quasi-free ridge, where the final-state interaction is suppressed the most for $E_{np}$ up to 160 MeV. Outside the kinematic region where the two results agree, the chiral EFT results start to have a large dependence on the LSE cutoff $\Lambda$ as one moves away from the quasi-free region or the energy and momentum are increased.

The subtractive renormalization scheme we developed in this work can be further extended to the renormalization of chiral potential up to $N^3$LO, where one has 26 LEC’s in total. The implement of the subtractive renormalization technique can, at the lowest level, reduce the correlation between the LEC’s in each partial-wave—a problem that occurs when doing a direct best fit to the phase shift as was done in Ref. [42]. We note that the current set of low-energy constants (LEC’s) associated with $N^3$LO chiral potential [42] does not produce a good description of deuteron properties.
Some of the uncomfortably large LEC values obtained by performing a direct best fit to data might be the cause of this. The subtractive technique may provide a better determination of the LEC’s by reducing the correlations between them. Also, since the subtractive renormalization provides a quick way to solve the LSE, it can be applied to any further study of the NN problem which involves the non-perturbative treatment for part of the chiral potential. For deuteron electro-disintegration, the present work can be extended to evaluate other response functions, i.e., $f_T$, $f_{LT}$, $f_{TT}$, etc. In this way a complete description of this process can be obtained, and comparisons to experimental data performed. Higher-order calculations of deuteron electro-disintegration using $\chi$PT potentials and currents may reveal interesting new effects in this reaction.
Bibliography


e-Print: pos.sissa.it/archive/conferences/086/031/CD09_031.pdf


[87] See http://nn-online.org/


APPENDIX A

Standard technique to solve the LSE numerically

In this section, we layout the standard technique to solve the LSE numerically. First, the LSE in the partial-wave form is given in Eq. (2.27), i.e.,

$$t_{l1}(p', p; E) = v_{l1}(p', p) + \sum_{l''} \frac{2}{\pi} M \int_0^\Lambda \frac{dp'' p''^2 v_{l''}(p', p'')}{p_0^2 + i\varepsilon - p''^2} t_{l''}(p'', p; E).$$  \hspace{1cm} (A.1)

To evaluate the integration kernel, one makes use of the definition of the principal value. In general, for any single-valued, well-defined function $f(p)$ ($p \in [0, \Lambda]$), one has

$$\int_0^\Lambda \frac{dp'' p''^2 f(p'')}{p_0^2 + i\varepsilon - p''^2} = \mathcal{P} \int_0^\Lambda \frac{dp'' p''^2 f(p'')}{p_0^2 - p''^2} - i\pi \int_0^\Lambda \frac{dp'' p''^2 \delta(p_0^2 - p''^2)f(p'')}{p_0^2 - p''^2} - \frac{\pi}{2} p_0 f(p_0),$$  \hspace{1cm} (A.2)

provided that $0 \leq p_0 \leq \Lambda$.

The first term can be written as

$$\mathcal{P} \int_0^\Lambda \frac{dp'' p''^2 f(p'')}{p_0^2 - p''^2} = \mathcal{P} \int_0^\Lambda \frac{dp'' (p''^2 - \frac{p_0^2}{p_0^2 - p''^2}) f(p_0)}{p_0^2 - p''^2} + \mathcal{P} \int_0^\Lambda \frac{dp'' \frac{p_0^2}{p_0^2 - p''^2} f(p_0)}{p_0^2 - p''^2}. \hspace{1cm} (A.3)$$

The integrand in the first term on the right-hand side of Eq. (A.3) is zero at $p_0^2 = p''^2$.

The second term in the right-hand side can be solved analytically, i.e.,

$$\mathcal{P} \int_0^\Lambda \frac{dp'' \frac{p_0^2}{p_0^2 - p''^2} f(p_0)}{p_0^2 - p''^2} = -\lim_{\varepsilon \to 0} \left[ \frac{1}{2p_0} \ln \left( \frac{p_0 + p''}{p_0 - p''} \right) \bigg|_{|p''|=\varepsilon} - \frac{1}{2p_0} \ln \left( \frac{p_0 + p''}{p_0 - p''} \right) \bigg|_{|p''|=0} \right]$$

$$-\lim_{\varepsilon \to 0} \left[ \frac{1}{2p_0} \ln \left( \frac{p_0 + p''}{p_0 - p''} \right) \bigg|_{|p''|=\Lambda} - \frac{1}{2p_0} \ln \left( \frac{p_0 + p''}{p_0 - p''} \right) \bigg|_{|p''|=p_0+\varepsilon} \right]$$

$$= \lim_{\varepsilon \to 0} \left[ \frac{1}{2p_0} \ln(1) - \frac{1}{2p_0} \ln \left( \frac{2p_0 - \varepsilon}{\varepsilon} \right) + \frac{1}{2p_0} \ln \left( \frac{2p_0 + \varepsilon}{\varepsilon} \right) - \frac{1}{2p_0} \ln \left( \frac{\Lambda + p_0}{\Lambda - p_0} \right) \right] p_0^2 f(p_0)$$

$$= \frac{-p_0}{2} f(p_0) \ln \left( \frac{\Lambda + p_0}{\Lambda - p_0} \right). \hspace{1cm} (A.4)$$
So in the case $\Lambda \rightarrow \infty$, the last term of Eq. (A.3) vanishes.

Now, we have all of the necessary ingredients in the analytic form. The next step is to perform the rest of the integral numerically. We adopt the Gauss-Legendre quadrature to perform the numerical integration. An integral can be evaluated by adding up the function value at each mesh points multiplied by the weight. Suppose one adopts $N$ mesh points, then a map which transforms the Gaussian-Legendre points $x_i$ ($i = 1 \sim N$) and weights $w(x_i)$ from $x_i \in [-1,1]$ into $p_i' \in [p1,p3]$ and centered at $p2$ (in our case, $p1 = 0$ and $p3 = \Lambda$) gives

\[
p_i'' = \frac{p2(p1 - p3) + [2 \cdot p1 \cdot p3 - p2(p1 + p3)]x_i}{p1 - p3 + (p1 + p3 - 2 \cdot p2)x_i}, \quad (A.5)
\]

\[
w_i \equiv w(p_i'') = \frac{2w(x)\{p1 - p3\}[p1 \cdot p3 + p2(p2 - p1 - p3)]}{[p1 - p3 + (p1 + p3 - 2 \cdot p2)x_i]^2}. \quad (A.6)
\]

In this way, half of the mesh points are distributed in $[p1,p2]$, and the other half are distributed in $[p2,p3]$.

Now, the integral in Eq. (A.3) becomes

\[
P \int_0^\Lambda \frac{dp''}{p_0^2 - p''^2} f(p'') = \sum_{j=1}^N \frac{p_j''^2 f(p_j'')w_j}{p_0^2 - p_j''^2} - p_0 f(p_0) \sum_{m=1}^N \frac{w_m}{p_0^2 - p_m''^2} - \frac{p_0}{2} f(p_0) \ln \left( \frac{\Lambda + p_0}{\Lambda - p_0} \right). \quad (A.7)
\]

For the LSE, $f(p'') = 2Mv_{\nu''}(p',p'') t_{\nu''}(p'',p;E)$, and Eq. (A.2) is converted to:

\[
t_{\nu''}(p',p;E) = v_{\nu''}(p',p) + \frac{2M}{\pi} \sum_{j=1}^N \left( \sum_{j=1}^N \frac{p_j''^2 v_{\nu''}(p',p_j') t_{\nu''}(p_j',p;E)w_j}{p_0^2 - p_j''^2} \right)
\]

\[
- \frac{p_0^2 v_{\nu''}(p',p_0) t_{\nu''}(p_0,p;E)}{2} \sum_{m=1}^N \frac{w_m}{p_0^2 - p_m''^2} - \frac{i\pi}{2} \frac{p_0^2 v_{\nu''}(p',p_0) t_{\nu''}(p_0,p;E)}{2} \ln \left( \frac{\Lambda + p_0}{\Lambda - p_0} \right). \quad (A.8)
\]
Eq. (A.8) has \( N + 1 \) unknowns and \( N + 1 \) linear equations (\( N \) for \( t_{ll'}(p''_j, p; E) \), and one for \( t_{ll'}(p_0, p; E) \)), thus we can solve it as a matrix equation.

Explicitly, let us define \( p''_{N+1} = p_0 \), and

\[
T_{ll'}(p''_j, p; E)_{(N+1)\times 1} = \begin{pmatrix}
t_{ll'}(p''_1, p; E) \\
t_{ll'}(p''_2, p; E) \\
\vdots \\
t_{ll'}(p''_{N+1}, p; E)
\end{pmatrix},
\]

\[
(V_{ll'}(p''_1, p'; E)_{1\times(N+1)})^T = \begin{pmatrix}
v_{ll'}(p', p''_1; E) \\
v_{ll'}(p', p''_2; E) \\
\vdots \\
v_{ll'}(p', p''_{N+1}; E)
\end{pmatrix}, \tag{A.9}
\]

\[
D_j = \begin{cases}
\frac{2M}{\pi} \frac{2p''_j^2 w_j}{p_0^2 - p''_j^2}, & \text{for } j = 1 \sim N \\
\frac{2M}{\pi} \left[ \sum_{m=1}^{N} \frac{2p''_m w_m}{p_0^2 - p''_m^2} \right] - 2iM p_0, & \text{for } j = N + 1
\end{cases}. \tag{A.10}
\]

Then Eq. (A.8) can be written as:

\[
T_{ll'}(p', p; E) = V_{ll'}(p', p; E) + \sum_{p''} \sum_{j=1}^{N+1} D_j V_{ll'}(p', p''_j; E)_{1\times(N+1)} T_{ll'}(p''_j, p; E)_{(N+1)\times 1}
\]

\[
\begin{align*}
\text{(A.11)}
\end{align*}
\]

for every single value of \( p' \). One then sets the mesh points of \( p'_i \) distributed in the same way as \( p''_j \) and solves the following LSE

\[
T_{ll'}(p'_i, p; E)_{(N+1)\times 1} = V_{ll'}(p'_i, p; E)_{(N+1)\times 1} + \sum_{p''} \sum_{j=1}^{N+1} D_j V_{ll'}(p'_i, p''_j; E)_{(N+1)\times(N+1)} T_{ll'}(p''_j, p; E)_{(N+1)\times 1}.
\]

\[
\begin{align*}
\text{(A.12)}
\end{align*}
\]
Here

\[
V_{l'\ell''}(p'_1, p''_j; E)_{(N+1) \times (N+1)} = \begin{pmatrix}
v_{l'\ell''}(p'_1, p''_1; E) & \cdots & v_{l'\ell''}(p'_1, p''_{N+1}; E) \\
v_{l'\ell''}(p'_2, p''_1; E) & \cdots & v_{l'\ell''}(p'_2, p''_{N+1}; E) \\
\vdots & \ddots & \vdots \\
v_{l'\ell''}(p'_{N+1}, p''_1; E) & \cdots & v_{l'\ell''}(p'_{N+1}, p''_{N+1}; E)
\end{pmatrix}.
\]  

(A.13)

Defining the \((N + 1) \times (N + 1)\) matrix \(F_{ij}^{l'l''}\):

\[
F_{ij}^{l'l''} = [\delta_{l'l''}\delta_{ij} - D_j V_{l'\ell''}(p'_i, p''_j; E)_{(N+1) \times (N+1)}],
\]  

(A.14)

Then by arranging Eq. (A.12), \(T_{l'l''}(p''_j; E)_{(N+1) \times 1}\) can be obtained by solving the following equation:

\[
\sum_{\nu} \sum_{j=1}^{N+1} F_{ij}^{l'l''} T_{\nu j}(p'_j; E)_{(N+1) \times 1} = V_{l'\ell''}(p'_i; E)_{(N+1) \times 1}.
\]  

(A.15)
Appendix B

Proof of \( t(p_0, k^*; E^*) = t(k^*, p_0; E^*) \)

Now we will show that \( t(p_0, k^*; E^*) = t(k^*, p_0; E^*) \).

First we write down the K-matrix and t-matrix equations:

\[
K(p', p; E) = v(p', p) + \frac{2}{\pi} MP \left\{ \int_0^\infty dp'' p''^2 \left( \frac{v(p', p'')}{p_0^2 - p''^2 + i\epsilon} \right) K(p'', p; E) \right\}, \quad (B.1)
\]

\[
t(p', p; E) = v(p', p) + \frac{2}{\pi} M \int_0^\infty dp'' p''^2 \left( \frac{v(p', p'')}{p_0^2 - p''^2 + i\epsilon} \right) t(p'', p; E) = v(p', p) + \frac{2}{\pi} MP \left\{ \int_0^\infty dp'' p''^2 \left( \frac{v(p', p'')}{p_0^2 - p''^2 + i\epsilon} \right) t(p'', p; E) \right\} - i\pi \left( \frac{2}{\pi} Mp'' v(p', p'') \delta(p_0^2 - p''^2) t(p'', p; E) \right), \quad (B.2)
\]

where \( P \) indicates that the principal value in the argument is taken.

For simplicity, rewrite Eq. (B.1) and Eq. (B.2) in operator form

\[
K = v + P \left\{ \frac{1}{E - H_0} K \right\} = v + P \left\{ K \frac{1}{E - H_0} v \right\} \quad (B.3)
\]

\[
t = v + P \left\{ \frac{1}{E - H_0} t \right\} - i\pi v \delta(E - H_0) t. \quad (B.4)
\]

So

\[
v = [1 + P \left\{ K \frac{1}{E - H_0} \right\}]^{-1} K. \quad (B.5)
\]

Substituting Eq. (B.5) into Eq. (B.4) to get

\[
t = [1 + P \left\{ K \frac{1}{E - H_0} \right\}]^{-1} K + [1 + P \left\{ K \frac{1}{E - H_0} \right\}]^{-1} K P \left\{ \frac{1}{E - H_0} t \right\} - i\pi [1 + P \left\{ K \frac{1}{E - H_0} \right\}]^{-1} K \delta(E - H_0) t. \quad (B.6)
\]
Multiplying by \(1 + \mathcal{P}\left\{ K \frac{1}{E - H_0} \right\}\) on both sides,

\[
t[1 + \mathcal{P}\left\{ K \frac{1}{E - H_0} \right\}] = K + K\mathcal{P}\left\{ \frac{1}{E - H_0} t \right\} - i\pi K \delta(E - H_0)t. \tag{B.7}
\]

So

\[
t = K - i\pi K \delta(E - H_0)t. \tag{B.8}
\]

Explicitly,

\[
t(p', p; E) = K(p', p; E) - iMp_0K(p', p_0; E)\delta(ME - p''^2)t(p_0, p; E) \tag{B.9}
\]

After evaluating the delta function we get

\[
t(p', p; E) = K(p', p; E) - iMp_0K(p', p_0; E)\delta(ME - p''^2)t(p_0, p; E) \tag{B.10}
\]

where \(p_0 = \sqrt{ME}\). From Eq. (B.8), we know

\[
t(p_0, p_0; E) = K(p_0, p_0; E)[1 - iMp_0t(p_0, p_0; E)]. \tag{B.11}
\]

So

\[
t(p_0, p_0; E) = \frac{K(p_0, p_0; E)}{1 + iMp_0K(p_0, p_0; E)}, \tag{B.12}
\]

and

\[
t(p, p_0; E) = K(p, p_0; E) - iMp_0K(p, p_0; E)t(p_0, p_0; E) = K(p, p_0; E)[1 - iMp_0 \frac{e^{i\delta} \sin(\delta)}{-Mp_0}] \tag{B.13}
\]

\[
t(p_0, p; E) = K(p_0, p; E) - iMp_0K(p_0, p_0; E)t(p_0, p; E). \tag{B.14}
\]

Thus

\[
t(p_0, p; E) = \frac{K(p_0, p; E)}{1 + iMp_0K(p_0, p_0; E)} = \frac{K(p_0, p; E)}{1 + iMp_0\frac{\tan(\delta)}{-Mp_0}}. \tag{B.15}
\]
Here we have used the fact that 
\[ t(p_0, p_0; E) = \frac{e^{i\delta} \sin(\delta)}{-Mp_0}, \quad \text{and} \quad K(p_0, p_0; E) = \frac{\tan(\delta)}{-Mp_0}. \]

Eq. (B.15) then becomes
\[ t(p_0, p; E) = K(p_0, p; E) \cos(\delta) e^{i\delta}. \]  
(B.16)

But for Eq. (B.13), we then have
\[

t(p, p_0; E) = K(p, p_0; E)[1 - i e^{i\delta} \sin(\delta)] = K(p, p_0; E)[1 + i \sin(\delta) \cos(\delta) - \sin^2(\delta)] \\
= K(p, p_0; E) \cos(\delta) e^{i\delta} 
\]
(B.17)

Since \( K(p, p_0; E) = K(p_0, p; E) \), i.e. K-matrix is real and Hermitian, so by comparing Eq. (B.16) and Eq. (B.17) we conclude that
\[ t(p_0, p; E) = t(p, p_0; E). \]  
(B.18)

Furthermore, from Eq. (B.10) we have
\[

t(p, p'; E) = K(p, p'; E) - iMp_0 K(p, p_0; E)t(p_0, p'; E) \\
= K(p, p'; E) - iMp_0 t(p', p_0; E)K(p_0, p; E) \\
= K(p', p; E) - iMp_0 t(p', p_0; E)K(p_0, p; E) = t(p', p; E). \]  
(B.19)
Appendix C

Partial-wave expression of the IA matrix element for the deuteron electro-disintegration process

The IA matrix element for the deuteron electro-disintegration process reads, as listed in Eq. (6.53):

\[
\langle \mathbf{p}'S\mathbf{m}_s\Psi_T|J(\mathbf{q};0)|\mathbf{m}_j0 \rangle = \langle \mathbf{p}'-\mathbf{q}/2S\mathbf{m}_s|m_j \rangle G^P_E(q^2) + \langle -\mathbf{p}'-\mathbf{q}/2S\mathbf{m}_s|m_j \rangle G^N_E(q^2).
\]

(C.1)
Here $M_S = -1, 0, 1$, and $m_j = -1, 0, 1$. Let $k_\pm = |\pm p' - q|/2|$, and define the Clebsch-Gordon coefficient as $(L, m_l, S, M_s, J, m_j)$, then

$$
\langle \pm p' - q/2 M_s = 1 | m_j = 1 \rangle = \sqrt{\frac{2}{\pi}} [(001|11) \Psi_{3S_i}(k_{\pm}) Y_{00}(\Omega_{k_{\pm}}) - (2011|11) \Psi_{3D_1}(k_{\pm}) Y_{20}(\Omega_{k_{\pm}})],
$$

$$
\langle \pm p' - q/2 M_s = 1 | m_j = 0 \rangle = -\sqrt{\frac{2}{\pi}} [(2, -1, 11|10) \Psi_{3D_1}(k_{\pm}) Y_{21}(\Omega_{k_{\pm}}) - (221|11) \Psi_{3D_1}(k_{\pm}) Y_{22}(\Omega_{k_{\pm}})],
$$

$$
\langle \pm p' - q/2 M_s = 0 | m_j = 1 \rangle = -\sqrt{\frac{2}{\pi}} [(2110|11) \Psi_{3D_1}(k_{\pm}) Y_{21}(\Omega_{k_{\pm}}) - (0010|10) \Psi_{3S_i}(k_{\pm}) Y_{00}(\Omega_{k_{\pm}})],
$$

$$
\langle \pm p' - q/2 M_s = 0 | m_j = 0 \rangle = \sqrt{\frac{2}{\pi}} [(001|10) \Psi_{3S_i}(k_{\pm}) Y_{00}(\Omega_{k_{\pm}}) - (2010|10) \Psi_{3D_1}(k_{\pm}) Y_{20}(\Omega_{k_{\pm}})],
$$

$$
\langle \pm p' - q/2 M_s = 0 | m_j = -1 \rangle = -\sqrt{\frac{2}{\pi}} [(2, -1, 10|1. - 1) \Psi_{3D_1}(k_{\pm}) Y_{21}(\Omega_{k_{\pm}}) - (221|1|1) \Psi_{3D_1}(k_{\pm}) Y_{22}(\Omega_{k_{\pm}})],
$$

$$
\langle \pm p' - q/2 M_s = -1 | m_j = 1 \rangle = -\sqrt{\frac{2}{\pi}} [(221, -1|11) \Psi_{3D_1}(k_{\pm}) Y_{22}(\Omega_{k_{\pm}}) - (211, -1|10) \Psi_{3D_1}(k_{\pm}) Y_{21}(\Omega_{k_{\pm}})],
$$

$$
\langle \pm p' - q/2 M_s = -1 | m_j = 0 \rangle = -\sqrt{\frac{2}{\pi}} [(2, 1, 1, -1|10) \Psi_{3D_1}(k_{\pm}) Y_{21}(\Omega_{k_{\pm}}) - (001, -1|1, -1) \Psi_{3S_i}(k_{\pm}) Y_{00}(\Omega_{k_{\pm}})],
$$

$$
\langle \pm p' - q/2 M_s = -1 | m_j = -1 \rangle = \sqrt{\frac{2}{\pi}} [(001, -1|1, -1) \Psi_{3S_i}(k_{\pm}) Y_{00}(\Omega_{k_{\pm}}) - (2, 0, 1, -1|1, -1) \Psi_{3D_1}(k_{\pm}) Y_{20}(\Omega_{k_{\pm}})].
$$
Appendix D

Partial-wave expression of the FSI matrix element
for the deuteron electro-disintegration process

The FSI part of the matrix element for the deuteron electro-disintegration process reads, as listed in Eq. (6.75):

\[
\langle \mathbf{p}' S = 1 M_s T_f | t(E') G_0(E') J(q; 0) | m_j \rangle \\
= \int d^3 \mathbf{p} \frac{2}{\pi} \left[ \sum_{J, m_j, L, L'} (Lm_1 M_s | Jm_j) Y_{Lm_1} (\Omega_p) t_{LSJ, T_f} (E') \right. \\
\times (L'' m''_1 | Jm_j) Y_{L'' m''_1} (\Omega_p) \left[ \frac{P}{E' - \mathbf{p}^2 / M} - i \pi \delta (E' - \mathbf{p}^2 / M) \right] \\
\times \langle \mathbf{p} S M'_S T'_f M_t | J(q; 0) | m_j \rangle. \quad (D.1)
\]

Here \((Lm_1 SM_s | Jm_j)\) is the Clebsch-Gordon coefficient as mentioned before, and we write out explicitly the projection of the total angular momentum quantum number on z-axis for the deuteron state: \(m_j\). Note that the total amplitude should be squared and summed as

\[
\sum_{M_s, m_j} \left| \langle \mathbf{p}' M_s | (1 + t G_0) J_0 | m_j \rangle \right|^2 = \sum_{M_s, m_j} \{ \sum_{T_f = 0, 1} \langle \mathbf{p}' M_s T_f | (1 + t G_0) J_0 | m_j \rangle \}^2. \quad (D.2)
\]

The matrix element \(\langle \mathbf{p} S M'_S T'_f M_t | J(q; 0) | m_j \rangle\) consists of deuteron wave functions \(\Psi_{3S_1}(p + q/2), \Psi_{3D_1}(p + q/2)\), and nucleon form factors \(G_E^p(q^2)\) and \(G_N^p(q^2)\). To
shorten the expression, we define the following common factors:

\[
A_{2m}^0 = \sqrt{\frac{2}{\pi^2}} \frac{\mathcal{P}}{E' - p^2/M - i\pi\delta(E' - p^2/M)} \times \left[ -\Psi_{3D1}(|p' - q/2|)G_E^p(q^2) - \Psi_{3D1}(|p' + q/2|)G_E^p(q^2) - (-1)^m\Psi_{3D1}(|p' + q/2|)G_E^p(q^2) \right],
\]

(D.3)

\[
A_{2m}^1 = \sqrt{\frac{2}{\pi^2}} \frac{\mathcal{P}}{E' - p^2/M - i\pi\delta(E' - p^2/M)} \times \left[ -\Psi_{3D1}(|p' - q/2|)G_E^p(q^2) + \Psi_{3D1}(|p' - q/2|)G_E^p(q^2) - (-1)^m\Psi_{3D1}(|p' + q/2|)G_E^p(q^2) \right],
\]

(D.4)

\[
A_{0m}^0 = \sqrt{\frac{2}{\pi^2}} \frac{\mathcal{P}}{E' - p^2/M - i\pi\delta(E' - p^2/M)} \times \left[ \Psi_{3S1}(|p' - q/2|)G_E^p(q^2) - \Psi_{3S1}(|p' - q/2|)G_E^p(q^2) + (-1)^m\Psi_{3S1}(|p' + q/2|)G_E^p(q^2) \right],
\]

(D.5)

\[
A_{0m}^1 = \sqrt{\frac{2}{\pi^2}} \frac{\mathcal{P}}{E' - p^2/M - i\pi\delta(E' - p^2/M)} \times \left[ \Psi_{3S1}(|p' - q/2|)G_E^p(q^2) - (-1)^m\Psi_{3S1}(|p' + q/2|)G_E^p(q^2) \right],
\]

(D.6)

As mentioned in Sec.6.2.1, we will make use of the formula

\[
\int_0^{2\pi} \int_0^\pi Y_{lm}(\theta, \phi)Y_{l'm'}(\theta', \phi) d(\cos \theta) d\phi = 2\pi \delta_{mm'} \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{l + m}! \frac{2l' + 1}{4\pi} \frac{(l' - m)!}{l' + m}!} \int_0^\pi P_l^m(\cos \theta) P_{l'}^{m'}(\cos \theta') d(\cos \theta)
\]

(D.7)
to eliminate the $d\phi$ integral. Thus, to shorten the notation, we further define
\[ \zeta_{ll'} \equiv \sqrt{\frac{2l + 1}{4\pi} \frac{(l + m)!}{(l - m)!} \frac{2l' + 1}{4\pi} \frac{(l' + m)!}{(l' - m)!}}. \]  
(D.8)

We now write down explicitly the FSI part of the matrix element in terms of partial-wave, which can be expressed as

\[ \langle p' M_s T_f \mid tG_0 J_0 \mid m_j \rangle = 4 \int_0^\infty p^2 dp \int_0^\pi \sin \theta d\theta \sum_{T_f=0,1} \sum_{L'=0,2} \sum_{L',m_L} \{(Lm_1M_s | Jm_j)Y_{Lm_1}(\Omega_p')t^{S=1}_{LL'}(|p'|, |p|; E') \times (L''m''_1M'_s | Jm_j)\zeta_{L''L,m''_L} P^{m''_L}_{L'}(\cos \theta)P^{m'_L}_{L'}(\cos \theta') \ast (L'm''_1M'_s | 1m_j)A^{T_f}_{T_f} \}. \]  
(D.9)

Here we drop the summation of $M'_s$ since once $m_j$ and $m''_L$ are specified, $M'_s = m_j - m''_L$. We have developed a code to generate all non-zero matrix elements based on the above formula. In the follows we list all possible $(M_s, L, L'', J, m_L, m''_L, T_f, L', m_j)$ up to $J = 1$:

$J = 0$

(-11101-1120) (-111010100) (-111010120) (-111011120)  
(01100-1120) (011000100) (011000120) (011001120) (1110-1120)  
(0-1110-1120) (1110-10100) (1110-10120) (1110-11120)

$J = 1$

(-11110-112-1) (-11110010-1) (-11110012-1) (-10010000-1)  
(-10010002-1) (-10210-202-1) (-10210-102-1) (-10210000-1)  
(-10210002-1) (-12010000-1) (-12010002-1) (-12210-202

D.9)