COMPARING TOPOLOGICAL SPACES USING NEW APPROACHES TO CLEAVABILITY

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In this dissertation, two new versions of cleavability, $n$-open cleavability and $n$-point cleavability, are introduced and analyzed. In particular, new characterizations of the closed interval $[0, 1]$ are given using 2-open cleavability and 3-point cleavability. In addition, it is shown that every zero-dimensional, perfectly normal compactum is $\omega$-open cleavable over the Cantor set. Some new results on cleavability of spaces over the set of real numbers $\mathbb{R}$, and over other related spaces, are also provided. A lemma of I. V. Yashchenko is used to develop many of these new results, including the following: No simple $(n + 1)$-od is cleavable over a simple $n$-od.

Approved: 

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To my wife, Amy
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Chapter 1

Introduction

The general notion of cleavability was introduced by Arhangel’skiĭ in [1]:

**Definition.** A topological space $X$ is **cleavable** (or **splittable**) over a topological space $Y$ along $A \subset X$ if there exists a continuous function $f : X \to Y$ such that $f(A) \cap f(X \setminus A) = \phi$ (or, equivalently, $f^{-1}(f(A)) = A$). If $X$ is cleavable over $Y$ along every subset of $X$, then $X$ is said to be **cleavable over** $Y$.

Since its introduction, cleavability has been a useful tool in comparing and contrasting topological spaces. A common question in cleavability study has been the following: Given a well-known or relatively simple topological space $Y$, what can be said about a topological space $X$ that is cleavable over $Y$? Many papers have been written to answer this question for specific spaces $Y$ (e.g., $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^\omega$, $\mathbb{Q}$, the Alexandroff-Urysohn double arrow space - see [3],[4],[5],[6],[10],[16],[21]) and for spaces $Y$ known to be a member of a certain class of spaces (e.g., LOTS, ordered spaces, partially ordered sets - see [12],[15],[18],[22]). In this work, we proceed to answer slightly different, but related, questions.

Suppose we don’t necessarily know that a space $X$ is cleavable over another space $Y$ but do know that $X$ is cleavable over $Y$ along some interesting subset or along a collection of interesting subsets of $X$. In topology, of course, the most important subsets of a space $X$ are the open subsets of $X$. So, what can be said about a space
X which is known to be cleavable over a space Y along all open subsets of X? Some answers to this question were given in [5]. Now, if it is known that X is cleavable over Y along the open subsets of X, then for every open U in X, there is a continuous function \( f : X \to Y \) which cleaves X along U. What additionally can be said about X if, say, for any pair \( U, V \) of open sets of X, we could find a single function \( f : X \to Y \) which cleaves X simultaneously along both open sets (that is, \( f^{-1}(f(U)) = U \) and \( f^{-1}(f(V)) = V \))? What can be said if, for any finite collection of open sets of X, a single function could be found to cleave X simultaneously along those open sets?

These questions motivate the introduction of our first new variation on cleavability:

**Definition.** Let \( n \in \mathbb{N} \). A space X is said to be \( n \)-open cleavable over a space Y if, for every collection \( \gamma \) of at most \( n \) open sets in X, there exists a continuous function \( f : X \to Y \) such that \( f(U) \cap f(X \setminus U) = \emptyset \) for every \( U \in \gamma \).

Now, note that if a one-to-one, continuous function \( f \) from a space \( X^* \) to a space \( Y^* \) can be found, then this single function cleaves \( X^* \) over \( Y^* \) along all subsets of \( X^* \). Suppose, though, that a space X is not known to be cleavable over a space Y. Then no one-to-one, continuous function \( f : X \to Y \) is known to exist. Can anything interesting be said about X if, for any \( x \in X \), there is a continuous function \( f : X \to Y \) such that no point of \( X \setminus \{x\} \) gets mapped by \( f \) to \( f(x) \)? Can anything else be said about X if, additionally, for any pair \( x, y \in X \), there is a continuous function \( f : X \to Y \) such that no point of \( X \setminus \{x\} \) gets mapped to \( f(x) \) and no point of \( X \setminus \{y\} \) gets mapped to \( f(y) \) (so \( f \) is, in a sense, one-to-one at the points \( x \) and \( y \))?
What if, in the same sense, for any finite collection $A$ of points in $X$, a continuous function $f : X \to Y$ can be found which is one-to-one on $A$? What, then, can be said about $X$? These questions motivate the introduction of our second new variation on cleavability:

**Definition.** Let $n \in \mathbb{N}$. A space $X$ is said to be \textit{$n$-point cleavable over a space $Y$} if, for every collection $A$ of at most $n$ distinct points of $X$, there exists a continuous function $f : X \to Y$ such that $f^{-1}(f(a)) = \{a\}$ for every $a \in A$.

The main goal of this dissertation is to expand the study of cleavability using these new cleavability notions of $n$-open and $n$-point cleavability. Initially for each new concept, examples will be given, and basic results will be developed (Section 2.1). Whenever possible, these new cleavability concepts will be compared with the general concept of cleavability and with each other. Following the introductory development of these new concepts, more substantive results will be proven, including:

- Let $n \in \mathbb{N}$. If $X$ is a first countable, Hausdorff, and zero-dimensional space, then $X$ is $n$-point cleavable over $\mathbb{Q}$ (Section 2.1).

- Every zero-dimensional, perfectly normal compactum is $\omega$-open cleavable over the Cantor set (Section 2.1).

- New characterizations of the closed unit interval $[0, 1]$ using 2-point and 3-open cleavability (Section 2.2)
Numerous results regarding cleavability over the circle $S^1$ (Section 2.3) and over spaces homeomorphic to the letter “Y” (Section 2.3 and Chapter 3) will also be presented. A lemma of I. V. Yashchenko will be used to develop many of these results, including the following: A simple $(n + 1)$-od cannot be cleavable over a simple $n$-od (Chapter 3).

The symbol $\mathbb{R}$ is used throughout to represent the real line with the usual topology, $\mathbb{Q}$ stands for the set of rational numbers, $\mathbb{N}$ stands for the set of positive integers, $I$ denotes the closed unit interval $[0, 1]$, and $S^n$ represents the $n$-dimensional sphere (so $S^1$ is a circle in the plane). Otherwise, notation throughout follows Engelking [17].

By a space, we mean a general topological space with no assumption of separation axioms. Since terminology differs somewhat across topology literature, definitions of some other terms which appear later are included immediately below. Terms used later, but not defined below, may be located in [17].

**Definition 1.** An arc is a space homeomorphic to $I$.

**Definition 2.** A space $X$ is *arcwise-connected* if, for every pair $x_1, x_2$ of distinct points of $X$, there exists a homeomorphic embedding $h : I \to X$ such that $h(0) = x_1$ and $h(1) = x_2$. In other words, a space $X$ is arcwise-connected if every two points of $X$ are the endpoints of an arc in $X$.

**Definition 3.** A subset $A$ of a space $X$ is *clopen* in $X$ if $A$ is both open and closed in $X$. 
**Definition 4.** A space $X$ is a *compactum* if it is a compact Hausdorff space.

**Definition 5.** A space $X$ is a *continuum* if it is a compact, connected Hausdorff space.

**Definition 6.** [13] A space $X$ is *countable dense homogeneous* if it is separable and, for any two countable dense subsets $M$ and $N$ of $X$, there is a homeomorphism $h$ of $X$ onto $X$ such that $h(M) = N$.

**Definition 7.** Let $X$ be a connected space, and let $p \in X$. If $X \setminus \{p\}$ is not connected, then $p$ is a called a *cut point* of $X$. If $X \setminus \{p\}$ is connected, then $p$ is a called a *non-cut point* of $X$.

**Definition 8.** The *diagonal product* of two functions $f : X \to Y_1$ and $g : X \to Y_2$ is the function $f \triangle g : X \to Y_1 \times Y_2$ defined by $(f \triangle g)(x) = (f(x), g(x))$.

**Definition 9.** A space $X$ *embeds* in a space $Y$ if $X$ is homeomorphic to a subspace of $Y$.

**Definition 10.** The *free topological sum* of two spaces $(X_1, \tau)$ and $(X_2, \sigma)$, denoted by $X_1 \oplus X_2$, is the disjoint union of $X_1$ and $X_2$ supplied with the smallest topology generated by $\tau \cup \sigma$. So, a subset $U$ of $X_1 \oplus X_2$ is open in $X_1 \oplus X_2$ if and only if $U \cap X_i$ is open in $X_i$ for each $i \in \{1, 2\}$.

**Definition 11.** A space $X$ is *Lindelöf* if every open cover of $X$ has a countable subcover.
Definition 12. A space $X$ is **locally connected** if, for every $x \in X$ and any open set $U$ containing $x$, there exists an open, connected set $V$ such that $x \in V \subseteq U$.

Definition 13. A space $X$ is **metacompact** if every open cover of $X$ has a point-finite open refinement.

Definition 14. A **Peano space** is a connected, locally connected, compact metric space.

Definition 15. A space $X$ is **perfectly normal** if $X$ is a normal $T_1$-space such that every closed subset of $X$ is a $G_\delta$-subset of $X$.

Definition 16. A **simple $n$-od** ($n \geq 2$) is the union of $n$ arcs intersecting at a common endpoint but are disjoint otherwise. A simple 3-od is called a **simple triod**. (Note that a simple 2-od is an arc.)

Definition 17. A family $\gamma$ of sets is **star-finite** if each set in $\gamma$ intersects only finitely many other sets in $\gamma$.

Definition 18. A space $X$ is **strongly paracompact** if every open cover of $X$ has a star-finite open refinement.

Definition 19. A subset $A$ of a space $X$ is called a **zero-set** of $X$ if $A = f^{-1}(0)$ for some continuous function $f : X \to I$.

We mention, now, some known results regarding cleavability that are used frequently or generalized later. The first four are quite simple.
Proposition 1.0.1. [3] If a space $X$ is cleavable over a space $Y$, then $X$ is cleavable over any space containing $Y$ as a subspace. In addition, any subspace of $X$ is cleavable over $Y$.

Proposition 1.0.2. [3] A space $X$ is cleavable over $\mathbb{R}$ if and only if $X$ is cleavable over $I$.

Proposition 1.0.3. [3] If a space $X$ is cleavable over a Hausdorff space, then $X$ is Hausdorff.

Proposition 1.0.4. [3] If a space $X$ is cleavable over $\mathbb{R}$, then every singleton in $X$ is a $G_\delta$-subset of $X$.

Theorem 1.0.5. [3] A compact space $X$ is cleavable over $\mathbb{R}$ if and only if $X$ is homeomorphic to a subspace of $\mathbb{R}$.

For a survey of other early work in the study of cleavability, see [7].

For the benefit of quick reference, we conclude this introduction by providing a list of the open questions presented in the text to follow.

**Question 1.** Is every space which is cleavable over $\mathbb{R}$ also 2-open cleavable over $\mathbb{R}$?

**Question 2.** (Arhangel’skiĭ) If a space is cleavable over $\mathbb{R}$, must the space also be 3-point cleavable over $\mathbb{R}$?

**Question 3.** Does there exist a space which contains a pratriangle and is cleavable over $\mathbb{R}$? (Note: The term “pratriangle” is defined on page 21.)
Question 4. Suppose that $X$ is a compact, connected metrizable space which is 3-point cleavable over the letter “X”. Is $X$ homeomorphic to a subspace of “X”?

Question 5. If $X$ is a compact, connected space which is 3-point cleavable over $S^1$, can $X$ be embedded in $S^1$?
Chapter 2

n-open and n-point Cleavability

2.1 Preliminary Results

It was noted in [5] (using different terminology) that every perfectly normal space is 1-open cleavable over $\mathbb{R}$. So, given two distinct open sets $U$ and $V$ in a perfectly normal space $X$, there exist two continuous real-valued functions $f_1$ and $f_2$ defined on $X$ such that $f_1(U) \cap f_1(X \setminus U) = \phi = f_2(V) \cap f_2(X \setminus V)$. Does there, then, exist a single continuous real-valued function $f$ on $X$ such that $f(U) \cap f(X \setminus U) = \phi = f(V) \cap f(X \setminus V)$? In other words, are perfectly normal spaces 2-open cleavable over $\mathbb{R}$? The answer is “not in general”.

Example 2.1.1. Consider $S^1$ as a circle of radius one centered at the origin in the plane. Let $U = \{(x, y) \in S^1 : y > 0\}$, and let $V = \{(x, y) \in S^1 : x > 0\}$. Let $q$ and $p$ be the points $(-1, 0)$ and $(1, 0)$, respectively, on $S^1$. Suppose that $f : S^1 \to \mathbb{R}$ is a continuous function such that $f(U) \cap f(S^1 \setminus U) = \phi = f(V) \cap f(S^1 \setminus V)$. Then the Borsuk-Ulam Theorem (see [14]) and the condition $f(U) \cap f(S^1 \setminus U) = \phi$ together guarantee that $f(q) = f(p)$ (since, according to one specific case of this theorem, every continuous real-valued function on $S^1$ maps at least one pair of diametrically-opposed points to the same point). Note that every pair of diametrically-opposed points of $S^1$, except the pair $q, p$, has exactly one common point with $U$. So, since
$p \in V$ and $q \in S^1 \setminus V$, we have $f(q) \in f(V) \cap f(S^1 \setminus V)$ - a contradiction. Therefore, no such $f$ exists, and $S^1$ is not 2-open cleavable over $\mathbb{R}$.

While perfectly normal spaces are not necessarily 2-open cleavable over $\mathbb{R}$, they are 2-disjoint open cleavable over $\mathbb{R}$.

**Definition.** Let $n \in \mathbb{N}$. A space $X$ is said to be $n$-disjoint open cleavable over a space $Y$ if for every collection $\gamma$ of at most $n$ pairwise-disjoint open sets in $X$, there exists a continuous function $f : X \to Y$ such that $f(U) \cap f(X \setminus U) = \emptyset$ for every $U \in \gamma$.

**Proposition 2.1.2.** Every perfectly normal space is 2-disjoint open cleavable over $\mathbb{R}$.

**Proof.** Let $U_1$ and $U_2$ be disjoint, open subsets of a perfectly normal space $X$. Let $F_1 = X \setminus U_1$ and let $F_2 = X \setminus U_2$. Then, there exist continuous functions $f_1 : X \to [0,1]$ and $f_2 : X \to [2,3]$ such that $F_1 = f_1^{-1}(0)$ and $F_2 = f_2^{-1}(2)$ [17, Corollary 1.5.12]. Let $f : X \to \mathbb{R}$ be defined by $f = f_1 - f_2$. As the difference of two continuous functions on $X$, $f$ is continuous. Note also that $f(U_1) \subseteq (-2,-1]$, $f(F_1) \subseteq [-3,-2]$, $f(U_2) \subseteq [-3,-2)$, and $f(F_2) \subseteq [-2,-1]$. Hence, $f(U_1) \cap f(X \setminus U_1) = \emptyset = f(U_2) \cap f(X \setminus U_2).$ \hfill \Box

A natural question to ask now is the following: Are perfectly normal spaces 3-disjoint open cleavable over $\mathbb{R}$? The answer is, in general, no. Consider, again, $S^1$ and $q \in S^1$ as defined in Example 2.1.1. Let $a$ be any point on $S^1$ in Quadrant I of the plane; let $b$ be any point on $S^1$ in Quadrant IV. Let $U$ be the set of points of $S^1$ strictly between $q$ and $a$ that would include the point $(0,1)$, let $V$ be the set of points of $S^1$ strictly between $a$ and $b$ that would include the point $(1,0)$, and let $W$ be the
set of points of $S^1$ strictly between $b$ and $q$ that would include the point $(0, -1)$. Note that $U$, $V$, and $W$ are pairwise-disjoint, open subsets of $S^1$. Suppose that $f : S^1 \to \mathbb{R}$ is a continuous function such that $f(U) \cap f(S^1 \setminus U) = \emptyset$, $f(V) \cap f(S^1 \setminus V) = \emptyset$, and $f(W) \cap f(S^1 \setminus W) = \emptyset$. Then no point $x \in S^1$ is mapped by $f$ to the same value as the point diametrically opposite $x$ on $S^1$. However, the Borsuk-Ulam Theorem guarantees that at least one point $s \in S^1$ is mapped to the same value as the point diametrically opposite $s$ on $S^1$. Thus, no such $f$ exists.

The answer to the next question is not known.

Question 1. Is every space which is cleavable over $\mathbb{R}$ also 2-open cleavable over $\mathbb{R}$?

Of course, for compact spaces, the answer to the previous question is “Yes” by Theorem 1.0.5.

The concepts of $n$-open cleavability and $n$-point cleavability are very much related. For $T_1$-spaces, $n$-open cleavability over a space $Y$ implies $n$-point cleavability over $Y$.

The concept of 1-point cleavability was introduced by Arhangel’skiĭ in [2] and was also studied in [21] (as pointwise splittability) and [9] (as pointwise cleavability). The next 3 simple results were verified in [9] and are of particular relevance here.

Proposition 2.1.3. If a space $X$ is 1-point cleavable over a Hausdorff ($T_1$) space $Y$, then $X$ is Hausdorff ($T_1$).

Theorem 2.1.4. A Tychonoff space $X$ is 1-point cleavable over $\mathbb{R}$ if and only if every singleton in $X$ is a $G_δ$-subset of $X$. 
Theorem 2.1.5. A countably compact, Tychonoff space $X$ is 1-point cleavable over $\mathbb{R}$ if and only if $X$ is first countable.

Because of the relationship between $n$-open cleavability and $n$-point cleavability for $T_1$-spaces, we quickly get the following three results from the previous three:

Proposition 2.1.6. If a $T_1$-space $X$ is 1-open cleavable over a Hausdorff space $Y$, then $X$ is Hausdorff.

Proposition 2.1.7. If a Tychonoff space $X$ is 1-open cleavable over $\mathbb{R}$, then every singleton in $X$ is a $G_\delta$-subset of $X$.

Proposition 2.1.8. If a countably compact, Tychonoff space $X$ is 1-open cleavable over $\mathbb{R}$, then $X$ is first countable.

One-point cleavability over a space does not necessarily guarantee 1-open cleavability over the same space. Let $X$ be the unit square $I \times I$ with the lexicographic ordering topology [23]. Since $X$ is first countable, countably compact, and Tychonoff, $X$ is 1-point cleavable over $\mathbb{R}$ [9]. However, $X$ is compact but not perfectly normal. Therefore, $X$ is not 1-open cleavable over $\mathbb{R}$ [5].

A common question asked throughout the study of cleavability is the following: When does cleavability of a space $X$ over another space $Y$ guarantee the embeddability of $X$ into $Y$? While we don’t expect 1-point cleavability to imply embeddability very often, we do have the following:
Proposition 2.1.9. If $I$ is 1-point cleavable over a Hausdorff space $Y$, then $I$ is homeomorphic to a subspace of $Y$.

Proof. Let $a, b \in I$ be distinct. Let $f : I \to Y$ be a continuous function such that $f^{-1}(f(a)) = \{a\}$. By the Hahn-Mazurkiewicz Theorem (see [24, Theorem 31.5]), $f(I)$ is a Peano space. Therefore, $f(I)$ is arcwise-connected (see [24, Theorem 31.2]). Therefore, since $f(a) \neq f(b)$, $f(I)$ contains an arc $A$ which, itself, contains $f(a)$ and $f(b)$. 

Using the same arguments used to establish Proposition 1.0.2 in [3], the following is true:

Proposition 2.1.10. Let $n \in \mathbb{N}$. A space $X$ is $n$-open ($n$-point) cleavable over $\mathbb{R}$ if and only if $X$ is $n$-open ($n$-point) cleavable over $I$.

Clearly, if a space $X$ is $n$-point cleavable over a space $Y$ ($n \in \mathbb{N}$), then $X$ is $k$-point cleavable over $Y$ for $1 \leq k \leq n$. Interestingly, 1-point cleavability over $\mathbb{R}$ and 2-point cleavability over $\mathbb{R}$ are equivalent. To verify that this is the case, a simple lemma is required.

Lemma 2.1.11. If a space $X$ is 1-point cleavable over $\mathbb{R}$, then every singleton in $X$ is a zero-set.

Proof. Let $a \in X$. Let $g : X \to I$ be a continuous function such that $\{a\} = g^{-1}(g(a))$. For each $x \in X$, let $f(x) = g(x) - g(a)$. Then $f^{-1}(0) = \{a\}$. 


**Proposition 2.1.12.** If a space $X$ is 1-point cleavable over $\mathbb{R}$, then $X$ is 2-point cleavable over $\mathbb{R}$.

*Proof.* Let $a, b \in X$ be distinct. Then $\{a\}$ and $\{b\}$ are disjoint zero-sets via Lemma 2.1.11. Therefore, there exists a continuous function $f : X \to I$ such that $\{a\} = f^{-1}(0)$ and $\{b\} = f^{-1}(1)$ [17, Theorem 1.5.14].

**Corollary 2.1.13.** A space $X$ is 2-point cleavable over $\mathbb{R}$ if and only if $X$ is 1-point cleavable over $\mathbb{R}$.

**Corollary 2.1.14.** If a space $X$ is cleavable over $\mathbb{R}$, then $X$ is 2-point cleavable over $\mathbb{R}$.

The converse to Corollary 2.1.14 is false. As a consequence of the next corollary, $\mathbb{R}^2$ is 2-point cleavable over $\mathbb{R}$. However, as noted in [3], $\mathbb{R}^2$ is not cleavable over $\mathbb{R}$.

As cited previously, perfectly normal spaces are 1-open cleavable over $\mathbb{R}$. Thus, we have a third corollary to Proposition 2.1.12:

**Corollary 2.1.15.** Every perfectly normal space is 2-point cleavable over $\mathbb{R}$.

We do not know an answer to the next question.

**Question 2.** (Arhangel’skiĭ) If a space is cleavable over $\mathbb{R}$, must the space also be 3-point cleavable over $\mathbb{R}$?

In an attempt to answer this question, we introduce the next concept.
**Definition.** Let $X$ be a space, and let $x_1$, $x_2$, and $x_3$ be distinct points of $X$. A pratriangle in $X$ with vertices $x_1$, $x_2$, and $x_3$ is a family $\{X_1, X_2, X_3\}$ of three connected subsets of $X$ such that for each $i, j \in \{1, 2, 3\}$, $x_i \notin X_i$ and $x_i \in X_j$ for $j \neq i$.

**Proposition 2.1.16.** If a space $X$ contains a pratriangle, then $X$ is not 3-point cleavable over $\mathbb{R}$.

To prove the last statement, we need the next lemma:

**Lemma 2.1.17.** The space $\mathbb{R}$ of real numbers does not contain a pratriangle.

*Proof of Lemma 2.1.17.* Suppose that $\{X_1, X_2, X_3\}$ is a pratriangle in $\mathbb{R}$ with distinct points $x_1, x_2, x_3$ as vertices. Without loss of generality, assume that $x_1 < x_2 < x_3$ (otherwise, re-enumerate the sets in the pratriangle and the vertices). Then $[x_1, x_3] \subseteq X_2$ since $X_2$ is a connected subset of $\mathbb{R}$ containing both $x_1$ and $x_3$. Then $x_2 \in X_2$, a contradiction. Thus, no pratriangle can exist in $\mathbb{R}$. $\square$

*Proof of Proposition 2.1.16.* Let $\{X_1, X_2, X_3\}$ be a pratriangle in $X$ with distinct points $x_1, x_2, x_3$ as vertices. Suppose that $f : X \rightarrow \mathbb{R}$ is a continuous function such that, for each $i \in \{1, 2, 3\}$, $f^{-1}(f(x_i)) = \{x_i\}$. Since $f$ is continuous, $f(X_i)$ is connected for every $i \in \{1, 2, 3\}$. Since $x_i \notin X_i$ and $f^{-1}(f(x_i)) = \{x_i\}$ for each $i \in \{1, 2, 3\}$, we know $f(x_i) \notin f(X_i)$ for each $i$. However, $x_i \in X_j$ for $j \neq i$, where $i, j \in \{1, 2, 3\}$; therefore, $f(x_i) \in f(X_j)$ for $j \neq i$. Therefore, $\{f(X_1), f(X_2), f(X_3)\}$ is a pratriangle in $\mathbb{R}$. This, however, contradicts the previous lemma. Thus, no such $f$ exists. $\square$
Note that Proposition 2.1.16 generalizes the fact that the circle $S^1$ is not 3-point cleavable over $\mathbb{R}$. However, $S^1$ is not cleavable over $\mathbb{R}$ [5], either. This brings us to the next question:

**Question 3.** Does there exist a space which contains a pratriangle and is cleavable over $\mathbb{R}$?

If the answer to this question is “Yes”, then (1) such a space cannot be compact due to Theorem 1.0.5, and (2) the answer to Question 2 will be “No” due to Proposition 2.1.16.

We do know, though, that a space containing a pracircle cannot be cleavable over $\mathbb{R}$.

**Definition.** A pracircle in a space $X$ is a family $\{C_1, C_2\}$ of two disjoint, connected subsets of $X$ such that $(\overline{C_1} \cap \overline{C_2})\setminus(C_1 \cup C_2)$ contains at least two distinct points.

**Proposition 2.1.18.** If a space $X$ contains a pracircle, then $X$ is not cleavable over $\mathbb{R}$.

**Proof.** Let $\{C_1, C_2\}$ be a pracircle in $X$ such that $a$ and $b$ are distinct elements of $(\overline{C_1} \cap \overline{C_2})\setminus(C_1 \cup C_2)$. Suppose, by way of contradiction, that $X$ is cleavable over $\mathbb{R}$. Then, let $f : X \to \mathbb{R}$ be a continuous function such that $f^{-1}(f(C_1 \cup \{a\})) = C_1 \cup \{a\}$. It follows, then, that $f(a) \neq f(b)$ since $a \neq b$ and $b \notin C_1$. However, it also follows that $f(C_1)$ and $f(C_2)$ are disjoint, connected sets in $\mathbb{R}$. Since $a, b \in \overline{C_1} \cap \overline{C_2}$ and $f$
is continuous, \( f(a), f(b) \in \overline{f(C_1)} \cap \overline{f(C_2)} \). However, \(|\overline{f(C_1)} \cap \overline{f(C_2)}| \leq 1\) since \( f(C_1) \) and \( f(C_2) \) are disjoint intervals in \( \mathbb{R} \). Therefore, \( f(a) = f(b) \), a contradiction. \( \square \)

Every space with a subspace homeomorphic to \( S^1 \) will contain a pracircle. However, not every space containing a pracircle will also contain a subspace homeomorphic to \( S^1 \).

**Example 2.1.19.** Let \( X = C_1 \cup C_2 \cup A \) with the topology inherited from \( \mathbb{R}^2 \), where:

\[
C_1 = \{(x, \sin(1/x)) : -1 \leq x < 0\}
\]

\[
C_2 = \{(x, \sin(1/x)) : 0 < x \leq 1\}
\]

\[
A = \{(0, y) : -1 \leq y \leq 1\}
\]

It is well-known that \( C_1 \) and \( C_2 \) are connected. Note also that \((\overline{C_1} \cap \overline{C_2}) \setminus (C_1 \cup C_2) = A\). Therefore, \( C_1 \) and \( C_2 \) form a pracircle in \( X \). So, by Proposition 2.1.18, \( X \) is not cleavable over \( \mathbb{R} \). However, no subspace of \( X \) is homeomorphic to \( S^1 \).

The next statement is interesting in connection with the previous results on \( n \)-point cleavability over \( \mathbb{R} \).

**Proposition 2.1.20.** Let \( n \in \mathbb{N} \). Every perfectly normal space \( X \) is \( n \)-point cleavable over \( \mathbb{R}^2 \).

**Proof.** Let \( F = \{x_1, \ldots, x_n\} \) be a collection of \( n \) distinct points in \( X \). Since \( F \) is finite, there exists a continuous injection \( g : F \to \mathbb{R} \). Since \( X \) is normal, there is a continuous extension \( \tilde{g} : X \to \mathbb{R} \) of \( g \) from \( F \) to \( X \). Now, since \( X \) is perfectly normal and \( F \) is closed, there exists a continuous function \( f : X \to \mathbb{R} \) such that \( f^{-1}(0) = F \). So, define
$h : X \to \mathbb{R}^2$ as the diagonal product of $f$ and $\tilde{g}$. As diagonal products of continuous functions are continuous [17], $h$ is continuous. Note that $h(x) = (0, \tilde{g}(x))$ for every $x \in F$. Note also that if $y \in X \setminus F$, there exists $z_y \in \mathbb{R} \setminus \{0\}$ such that $h(y) = (z_y, \tilde{g}(y))$.

Therefore, $h^{-1}(h(F)) = F$. Now, let $x_i, x_j \in F$ such that $i \neq j$. Since $\tilde{g}$ is injective on $F$, $\tilde{g}(x_i) \neq \tilde{g}(x_j)$. Therefore, $h(x_i) = (0, \tilde{g}(x_i)) \neq (0, \tilde{g}(x_j)) = h(x_j)$. Hence, $h^{-1}(h(x)) = \{x\}$ for every $x \in F$.

Three-point cleavability over $\mathbb{R}$ is not preserved by products, in general. For example, the plane $\mathbb{R}^2$ is not 3-point cleavable over $\mathbb{R}$. If it was, then every subspace of $\mathbb{R}^2$ would be 3-point cleavable over $\mathbb{R}$. However, as noted previously, $S^1$ is not 3-point cleavable over $\mathbb{R}$. We may additionally note, therefore, that not every metrizable space is 3-point cleavable over $\mathbb{R}$. There is, however, at least one class of metrizable spaces in which every member is $n$-point cleavable over $\mathbb{R}$ for all $n \in \mathbb{N}$ - the class of infinite, zero-dimensional metric spaces. In fact, something stronger is actually true:

**Theorem 2.1.21.** If $X$ is a first countable, Hausdorff, and zero-dimensional space, then $X$ is $n$-point cleavable over the space $\mathbb{Q}$ of rational numbers for all $n \in \mathbb{N}$.

**Proof.** Let $x_1, \ldots, x_n \in X$ be distinct. As $X$ is Hausdorff, fix pairwise-disjoint open sets $U_1, \ldots, U_n$ in $X$ such that $x_i \in U_i$ for each $i \in \{1, \ldots, n\}$. As $X$ is first countable and zero-dimensional, for each $i \in \{1, \ldots, n\}$, fix a countable base $\mathcal{B}_i = \{B^i_k : k \in \mathbb{N}\}$ of clopen sets for $X$ at $x_i$ whose elements form a strictly decreasing sequence of sets
within $U_i$. Define a function $f : X \to \mathbb{Q}$ by:
\[
    f(x) = \begin{cases} 
        0 & \text{if } x \in X \setminus \left( \bigcup_{i=1}^{n} B_i^i \right) \\
        i + \frac{1}{k+1} & \text{if } x \in B^i_k \setminus B^i_{k+1} \\
        i & \text{if } x = x_i
    \end{cases}
\]
Note that $f^{-1}(f(x_i)) = f^{-1}(i) = \{x_i\}$ for each $i \in \{1, \ldots, n\}$ by construction.

It remains to show that $f$ is continuous on $X$. Since $X$ is first countable, it suffices to show that, for every sequence $(z_m)$ of points in $X$ converging to $x \in X$, the sequence $(f(z_m))$ must converge to $f(x)$ in $\mathbb{Q}$ [20, Theorem 1.16]. Note that $f$ is constant on the clopen set $X \setminus \left( \bigcup_{i=1}^{n} B_i^i \right)$ and on each of the clopen sets $B^i_k \setminus B^i_{k+1}$. Therefore, any sequence $(z_m)$ converging to a point $a$ in one of those clopen sets will eventually be located within that clopen set. Thus, $(f(z_m))$ will eventually be constant and, therefore, converge to that constant $f(a)$. Now, fix $i \in \{1, \ldots, n\}$ and a sequence $(z_m)$ in $X$ which converges to $x_i$. Fix $\epsilon > 0$. Let $N \in \mathbb{N}$ such that $\frac{1}{N+1} < \epsilon$. Because $x_i$ is an element of the open set $B^i_N$, we know $(z_m)$ is eventually in $B^i_N$. In other words, there exists $K \in \mathbb{N}$ such that $z_m \in B^i_N$ for every $m \geq K$. Then, for every $m \geq K$, $|f(z_m) - f(x_i)| \leq |(i + \frac{1}{N+1}) - i| = \frac{1}{N+1} < \epsilon$. Therefore, $(f(z_m))$ converges to $f(x_i)$. Therefore, $f$ is continuous on $X$. \hfill $\square$

**Corollary 2.1.22.** If $X$ is a zero-dimensional metric space, then $X$ is $n$-point cleavable over $\mathbb{Q}$ (and, thus, over $\mathbb{R}$) for all $n \in \mathbb{N}$. 
The concepts of $n$-open cleavability and $n$-point cleavability can sometimes be improved to the analogously-defined concepts of $\omega$-open cleavability and $\omega$-point cleavability.

**Definition.** A space $X$ is **$\omega$-open cleavable over a space $Y$** if, for every countable family $\gamma$ of open subsets of $X$, there exists a continuous function $f : X \to Y$ such that $f(U) \cap f(X \setminus U) = \emptyset$ for every $U \in \gamma$.

**Definition.** A space $X$ is **$\omega$-point cleavable over a space $Y$** if, for every countable subset $A$ of $X$, there exists a continuous function $f : X \to Y$ such that $f^{-1}(f(a)) = \{a\}$ for every $a \in A$.

**Lemma 2.1.23.** One-open cleavability over the Cantor set is equivalent to $\omega$-open cleavability over the Cantor set.

**Proof.** Of course, $\omega$-open cleavability over the Cantor set $C$ automatically implies 1-open cleavability over $C$. So, suppose that $X$ is a space which is 1-open cleavable over $C$. Then $X$ is $\omega$-open cleavable over $C^\omega$ (using the diagonal product of cleaving functions). Therefore, since $C^\omega$ is homeomorphic to $C$, $X$ is $\omega$-open cleavable over $C$. \qed

**Theorem 2.1.24.** Every zero-dimensional, perfectly normal compactum is $\omega$-open cleavable over the Cantor set.

**Proof.** Let $U$ be an open set in $X$. Let $F = X \setminus U$. By the perfect normality of $X$, we know $F$ is a closed $G_\delta$-set in $X$. So, since $X$ is both zero-dimensional and compact,
there exists a decreasing sequence \((B_n)\) of clopen subsets of \(X\) such that \(F = \bigcap_{n=1}^{\infty} B_n\).

Define a function \(f : X \to \mathbb{Q}\) by:

\[
  f(x) = \begin{cases} 
    1 & \text{if } x \in X \setminus B_1 \\
    \frac{1}{n+1} & \text{if } x \in B_n \setminus B_{n+1} \\
    0 & \text{if } x \in F
  \end{cases}
\]

By construction, \(f(U) \cap f(F) = \emptyset\).

Since perfectly normal compacta are first countable [11], we proceed verifying continuity on \(X\) as in the proof of Theorem 2.1.21. Note that \(f\) is constant on the clopen set \(X \setminus B_1\) and on each of the clopen sets \(B_n \setminus B_{n+1}\). Therefore, for any sequence \((z_m)\) converging to a point \(a\) in one of those clopen sets, the sequence \((f(z_m))\) will eventually be constant and converge to that constant \(f(a)\). Now, fix \(x \in F\), and fix a sequence \((z_m)\) in \(X\) which converges to \(x\). Fix \(\epsilon > 0\). Let \(N \in \mathbb{N}\) such that \(\frac{1}{N+1} < \epsilon\). Because \(x\) is an element of the open set \(B_N\), we know \((z_m)\) is eventually in \(B_N\). In other words, there exists \(K \in \mathbb{N}\) such that \(z_m \in B_N\) for every \(m \geq K\). Then, for every \(m \geq K\), \(|f(z_m) - f(x)| \leq |\frac{1}{N+1} - 0| = \frac{1}{N+1} < \epsilon\). Therefore, \((f(z_m))\) converges to \(f(x)\). Therefore, \(f\) is continuous on all of \(X\), and \(X\) is 1-open cleavable over \(\mathbb{Q}\). Recall, though, that \(\mathbb{Q}\) is embeddable in the Cantor set \(C\). Therefore, we may regard \(f\) as a function which maps \(X\) to \(C\). Hence, \(X\) is 1-open cleavable over \(C\). Therefore, \(X\) is \(\omega\)-open cleavable over \(C\) by Lemma 2.1.23. \(\square\)
Corollary 2.1.25. Every zero-dimensional, perfectly normal compactum is \( \omega \)-point cleavable over \( \mathbb{R} \).

The next two results are not surprising.

**Proposition 2.1.26.** An infinite space \( X \) is \( n \)-point cleavable over the discrete space \( \mathbb{N} \) for every \( n \in \mathbb{N} \) if and only if it is discrete.

**Proof.** (\( \Rightarrow \)) Suppose that \( X \) is \( n \)-point cleavable over \( \mathbb{N} \) for every \( n \in \mathbb{N} \). Let \( a \in X \). Then, since \( X \) is 1-point cleavable over \( \mathbb{N} \), there exists a continuous function \( f : X \to \mathbb{N} \) such that \( f^{-1}(f(a)) = \{a\} \). Since \( \{f(a)\} \) is open in discrete \( \mathbb{N} \) and \( f \) is continuous, \( f^{-1}(f(a)) \) is open in \( X \). In other words, \( \{a\} \) is open in \( X \). Thus, \( X \) is discrete.

(\( \Leftarrow \)) Suppose that \( X \) is discrete. Let \( x_1, \ldots, x_n \in X \) be distinct. Define \( f : X \to \mathbb{N} \) by:

\[
 f(x) = \begin{cases} 
  i & \text{if } x = x_i \text{ for some } i \in \{1, \ldots, n\} \\
  n + 1 & \text{if } x \neq x_i \text{ for every } i \in \{1, \ldots, n\} 
\end{cases}
\]

Since \( X \) is discrete, \( f \) is automatically continuous. Note also that \( f^{-1}(f(x_i)) = f^{-1}(i) = \{x_i\} \) for each \( i \in \{1, \ldots, n\} \). \( \Box \)

**Proposition 2.1.27.** An infinite \( T_1 \)-space \( X \) is \( n \)-open cleavable over the discrete space \( \mathbb{N} \) for every \( n \in \mathbb{N} \) if and only if it is discrete.

**Proof.** (\( \Rightarrow \)) Suppose that \( X \) is \( n \)-open cleavable over \( \mathbb{N} \) for every \( n \in \mathbb{N} \). The 1-open cleavability of \( X \) over \( \mathbb{N} \) implies the 1-point cleavability of \( X \) over \( \mathbb{N} \) since \( X \) is a \( T_1 \)-space. So, by Proposition 2.1.26, \( X \) must be discrete.
Suppose that $X$ is discrete. Let $U_1, \ldots, U_n$ be open sets in $X$. Let $P_0 = X \setminus \left( \bigcup_{i=1}^{n} U_i \right)$. Let $P_1 = \{V \subseteq X : V = U_i \setminus \left( \bigcup_{j \neq i} U_j \right) \text{ for some } i = 1, \ldots, n\}$, let $P_2 = \{V \subseteq X : V = (U_i \cap U_j) \setminus \left( \bigcup_{k \neq i,j} U_k \right) \text{ for } i = 1, \ldots, n \text{ and } j = 1, \ldots, n \text{ such that } i \neq j\}$, $\ldots$, let $P_{n-1} = \{V \subseteq X : V = \left( \bigcap_{j \neq i} U_j \right) \setminus U_i \text{ for some } i = 1, \ldots, n\}$, and let $P_n = \bigcap_{i=1}^{n} U_i$. Note that $P_0$, $P_n$, and the elements of the $P_i$’s form a partition $\mathcal{P}$ of $X$ (consisting of $2^n$ sets, some of which may be empty). Map each of the $2^n$ sets in $\mathcal{P}$ to a different natural number (call this mapping $f$). Since $X$ is discrete, $f$ is continuous. Since each $U_i$ is partitioned by $2^{n-1}$ of the elements of $\mathcal{P}$, we get that $f(U_i) \cap f(X \setminus U_i) = \emptyset$ for each $i \in \{1, \ldots, n\}$. 

Note that $n$-open cleavability would be equivalent to $n$-closed cleavability if we chose to define $n$-closed cleavability analogous to the way $n$-open cleavability was defined. However, because $n$-disjoint open cleavability would not generally be equivalent to $n$-disjoint closed cleavability, it is appropriate to define this latter term and make one quick observation to close this section.

**Definition.** Let $n \in \mathbb{N}$. A space $X$ is said to be $n$-disjoint closed cleavable over a space $Y$ if for every collection $\gamma$ of at most $n$ pairwise-disjoint closed sets in $X$, there exists a continuous mapping $f : X \to Y$ such that $f(F) \cap f(X \setminus F) = \emptyset$ for every $F \in \gamma$.

**Proposition 2.1.28.** Perfectly normal spaces are 2-disjoint closed cleavable over $\mathbb{R}$. 
Proof. For disjoint closed subsets $E$ and $F$ in a perfectly normal space $X$, there exists a continuous function $f : X \to I$ such that $f^{-1}(0) = E$ and $f^{-1}(1) = F$ [17, Theorem 1.5.19].

2.2 New Characterizations of $I$

In [5], Arhangel’skii established that every infinite space which is connected, compact, and cleavable over $\mathbb{R}$ is homeomorphic to $I$. In this section, we establish new characterizations of $I$ using $n$-open cleavability and $n$-point cleavability. To do so, we need the next simple, but very useful lemma.

Lemma 2.2.1. Let $f$ be a mapping of a set $X$ into a set $Y$. Let $C_f = \{ A \subset X : f(A) \cap f(X\setminus A) = \emptyset \}$. Then all of the following are true:

(a) If $A \in C_f$, then $X\setminus A \in C_f$.

(b) If $\gamma = \{ A_s : s \in S \}$ is a subfamily of $C_f$, then $\bigcup_{s \in S} A_s \in C_f$.

(c) If $\gamma = \{ A_s : s \in S \}$ is a subfamily of $C_f$, then $\bigcap_{s \in S} A_s \in C_f$.

(d) If $A, B \in C_f$, then $A \setminus B \in C_f$.

Proof. (a) Obvious.

(b) Since $A_s = f^{-1}(f(A_s))$ for every $s \in S$, we get:

$$\bigcup_{s \in S} A_s = \bigcup_{s \in S} f^{-1}(f(A_s)) = f^{-1}(\bigcup_{s \in S} f(A_s)) = f^{-1}(f(\bigcup_{s \in S} A_s)).$$
(c) Note the following:

\[ \bigcap_{s \in S} A_s = \bigcap_{s \in S} f^{-1}(f(A_s)) = f^{-1}(\bigcap_{s \in S} f(A_s)) \supseteq f^{-1}(f(\bigcap_{s \in S} A_s)). \]

However, \[ \bigcap_{s \in S} A_s \subseteq f^{-1}(f(\bigcap_{s \in S} A_s)) \] trivially. Thus, \[ \bigcap_{s \in S} A_s = f^{-1}(f(\bigcap_{s \in S} A_s)). \]

(d) Follows from (a) and (c) since \( A \setminus B = A \cap (X \setminus B). \)

As noted previously, \( T_1 \)-spaces which are \( n \)-open cleavable over a space \( Y \) are also \( n \)-point cleavable over \( Y \). However, with the help of Lemma 2.2.1, we can say more.

**Proposition 2.2.2.** Let \( n \in \mathbb{N} \). If \( X \) is a \( T_1 \)-space which is \( n \)-open cleavable over a space \( Y \), then \( X \) is \( (n + 1) \)-point cleavable over \( Y \).

**Proof.** Let \( x_1, \ldots, x_{n+1} \) be distinct points of \( X \). For each \( i \in \{2, \ldots, n+1\} \), let \( X_i = X \setminus \{x_1, x_i\} \). Each \( X_i \) is open in \( X \), so let \( f : X \to Y \) be a continuous function such that \( f^{-1}(f(X_i)) = X_i \) for every \( i \in \{2, \ldots, n+1\} \). Then \( \{x_1, x_i\} \in C_f \) for every \( i \in \{2, \ldots, n+1\} \). Therefore, \( \{x_1\} = \{x_1, x_2\} \cap \{x_1, x_3\} \in C_f \) by Lemma 2.2.1(c).

It also follows, then, that \( \{x_i\} = \{x_1, x_i\} \setminus \{x_1\} \in C_f \) for every \( i \in \{2, \ldots, n+1\} \) by Lemma 2.2.1(d).

The case in Proposition 2.2.2 where \( n = 2 \) and \( Y = \mathbb{R} \) is of particular importance to us in this section.

**Corollary 2.2.3.** Let \( X \) be a \( T_1 \)-space. If \( X \) is 2-open cleavable over \( \mathbb{R} \), then \( X \) is 3-point cleavable over \( \mathbb{R} \).
Note that “2-open” in Corollary 2.2.3 cannot be changed to “2-disjoint open” since $S^1$ is 2-disjoint open cleavable over $\mathbb{R}$ (since it is metrizable and, hence, perfectly normal), but it is not 3-point cleavable over $\mathbb{R}$ (as mentioned previously).

The converse of Corollary 2.2.3 is not true, in general.

**Example 2.2.4.** Let $T_s$ be the subspace of $\mathbb{R}^2$ defined as follows: $T_s = Y \cup Z$, where $Y = \{(x,0) \in \mathbb{R}^2 : -1 \leq x \leq 1\}$ and $Z = \{(0,1/n) \in \mathbb{R}^2 : n \in \mathbb{N}\}$.

**Claim 1:** The space $T_s$ is not 2-open cleavable over $\mathbb{R}$.

**Proof of Claim 1:** Let $p$ denote the point $(0,0)$ in the plane. Let $A = \{(x,0) : -1 \leq x < 0\}$, $B = \{(x,0) : 0 < x \leq 1\}$, $U = A \cup Z$, and $V = Z \cup B$. Note that $U$ and $V$ are open subsets of $T_s$ such that $U \cap V \neq \emptyset$. Suppose that $f : T_s \to \mathbb{R}$ is a continuous function such that $f(U) \cap f(T_s \setminus U) = \emptyset = f(V) \cap f(T_s \setminus V)$. Therefore, from Lemma 2.2.1, we know that $f(Z) \cap f(T_s \setminus Z) = \emptyset$ since $Z = U \cap V \in C_f$ and that $f^{-1}(f(p)) = \{p\}$ since $\{p\} = T_s \setminus (U \cup V) \in C_f$.

Note that $f(A)$ is a connected subset of $\mathbb{R}$ since $A$ is connected and $f$ is continuous. Note also that $f(p) \in \overline{f(A)} \setminus f(A)$ since $p \in \overline{A} \setminus A$ and $f$ is continuous. Thus, $f(A)$ is a proper interval in $\mathbb{R}$ with $f(p)$ as an unincluded endpoint. Similarly, $f(B)$ is also a proper interval in $\mathbb{R}$ with $f(p)$ as an unincluded endpoint. Note, though, that $f(A) \cap f(B) = \emptyset$ since $B \subset V$ and $A \subset T_s \setminus V$. So, $f(A) \cup f(B) \cup \{p\}$ is an interval, call it $J$, in $\mathbb{R}$ such that $p \in \text{int}(J)$. Since the elements of $Z$ form a sequence of points converging to $p$, the images of the elements of $Z$ (under $f$) form a sequence of points converging to $f(p)$. Thus, there exists $z_0 \in Z$ such that
$f(z_0) \in J$. So, since $f(z_0) \neq f(p)$, we have $f(z_0) \in f(A) \cup f(B) \subset f(T_s \setminus Z)$. Therefore, $f(z_0) \in f(Z) \cap f(T_s \setminus Z)$, a contradiction. Thus, no such function $f$ can exist. □

Claim 2: The space $T_s$ is $n$-point cleavable over $\mathbb{R}$ for every $n \in \mathbb{N}$.

Proof of Claim 2: Let $p$, $A$, and $B$ be as in the proof of Claim 1. Let $t_1, \ldots, t_n \in X$ be distinct. For each $i \in \{1, \ldots, n\}$, let $x_i$ and $y_i$ denote the $x$-coordinate and $y$-coordinate, respectively, of $t_i$. Let $T = \{t_1, \ldots, t_n\}$. There are 2 cases to consider.

Case I: $T \cap Z = \emptyset$. Let $k \in \mathbb{N}$ such that $k \cdot 1/m \neq x_i$ for every $m \in \mathbb{N}$ and every $i \in \{1, \ldots, n\}$.

Case II: $T \cap Z \neq \emptyset$. Without loss of generality, assume $t_1$ is the element of $(T \cap Z) \setminus \{p\}$ closest to $p$. Take $k \in \mathbb{N}$ such that $k \cdot x_1 > 1$ and $k \cdot 1/m \neq x_i$ for every $m \in \mathbb{N}$ and every $i \in \{1, \ldots, n\}$.

In either case, define $f : T_s \to \mathbb{R}$ as follows:

$$f(x, y) = \begin{cases} x & \text{if } (x, y) \in A \cup B \cup \{p\} \\ k \cdot y & \text{if } (x, y) \in Z \end{cases}.$$ 

Note that $\{t_i\} = f^{-1}(f(t_i))$ for every $i \in \{1, \ldots, n\}$. Note also that $f$ is continuous on the closed sets $A \cup B \cup \{p\}$ and $Z \cup \{p\}$, whose union is all of $T_s$. Thus, $f$ is continuous on $T_s$ [24, Theorem 7.6]. □

Remark. It was noted in [3] that the space $T_s$ is not cleavable over $\mathbb{R}$. Therefore, not only is $T_s$ interesting because of its status as a counterexample to the converse of Corollary 2.2.3, but it also serves to show that $n$-point cleavability over $\mathbb{R}$ for every $n \in \mathbb{N}$ is not sufficient for cleavability over $\mathbb{R}$.
The concepts of 2-open cleavability and 3-point cleavability can be used to give new characterizations of the interval $I$.

**Theorem 2.2.5.** Let $X$ be a compact, connected metrizable space with at least two points. Then the following conditions are equivalent:

(i) $X$ is homeomorphic to $I$.

(ii) $X$ is 2-open cleavable over $\mathbb{R}$.

(iii) $X$ is 3-point cleavable over $\mathbb{R}$.

*Proof.* $(i \Rightarrow ii)$ The existence of a one-to-one, continuous function $f$ from $X$ to $I$ guarantees that $f^{-1}(f(U)) = U$ for every open set $U$ in $X$.

$(ii \Rightarrow iii)$ See Corollary 2.2.3.

$(iii \Rightarrow i)$ *Claim:* The space $X$ has exactly two non-cut points. Recall that every compact, connected $T_1$-space with at least two points has at least two non-cut points (see [19, Theorem 2-18]). So, to establish the Claim, it remains to prove the following lemma:

**Lemma 2.2.6.** If a connected space $X$ is 3-point cleavable over $\mathbb{R}$, then $X$ has at most two non-cut points.

*Proof.* Assume the contrary, and let $x_1$, $x_2$, and $x_3$ be distinct non-cut points of $X$. Since $X$ is 3-point cleavable over $\mathbb{R}$, we can fix a continuous function $f : X \to \mathbb{R}$ such that $\{x_i\} = f^{-1}(f(x_i))$ for each $i \in \{1,2,3\}$. Therefore, $f(x_1)$, $f(x_2)$, and
\( f(x_3) \) are distinct non-cut points of \( f(X) \) since each \( f(X \setminus \{x_i\}) \) is connected and \( f(X \setminus \{x_i\}) = f(X) \setminus \{f(x_i)\} \). However, since \( f(X) \) is a connected subset of \( \mathbb{R} \), the space \( f(X) \) has at most 2 non-cut points, a contradiction.

Thus, the Claim holds. Since \( X \) is a compact, connected metrizable space with exactly two non-cut points, we conclude that \( X \) is homeomorphic to \( I \) (see [19, Theorem 2-27]).

Compactness in Theorem 2.2.5 cannot be weakened to, say, local compactness since the open interval \((0, 1)\) in \( \mathbb{R} \) is not homeomorphic to \( I \). Connectedness cannot removed from the hypotheses in Theorem 2.2.5 since \([0, 1] \cup [2, 3]\) is not homeomorphic to \( I \).

Theorem 2.2.5 also holds if “metrizable space” is changed to “separable \( T_1 \)-space”, since every separable, compact, and connected \( T_1 \)-space with exactly two non-cut points is homeomorphic to \( I \) (see [17, Problem 6.3.8(c)]). In other words, we have:

**Theorem 2.2.7.** Let \( X \) be a compact, connected, separable \( T_1 \)-space with at least two points. Then the following conditions are equivalent:

(i) \( X \) is homeomorphic to \( I \).

(ii) \( X \) is 2-open cleavable over \( \mathbb{R} \).

(iii) \( X \) is 3-point cleavable over \( \mathbb{R} \).

Recall that \( n \)-point cleavability over \( \mathbb{R} \) is equivalent to \( n \)-point cleavability over \( I \) (Proposition 2.1.10). Note also that the letter “\( Y \)” can be formed by taking 3 copies of
and identifying the 0 points. Therefore, cleavability over I (any version) guarantees cleavability over the letter “Y”. So, it is appropriate to make some observations now regarding cleavability over simple n-ods. The first observation is connected to our newly developed characterizations of I.

Note that the letter “X” is 3-point cleavable over the letter “Y” (see next proposition). Therefore, compact, connected metric spaces which are 3-point cleavable over “Y” need not embed in “Y”.

**Proposition 2.2.8.** Every simple 4-od is 3-point cleavable over every simple triod.

**Proof.** Let $I_1, I_2, I_3, I_4$ be homeomorphic copies of I. Identify their 0 points to form a space $X$. Let $p$ be the common point of the four copies of I. Similarly, identify the 0 points of $I_1$, $I_2$, and $I_3$ to form a space $Y$. Let $a, b, c \in X$ be distinct. There are 2 cases to consider.

**Case I:** At least two of the three points $a$, $b$, and $c$ lie on the same $I_i$. Without loss of generality, assume $I_1$ is the copy of I containing multiple points of $\{a, b, c\}$. If $I_1$ contains only two of the three points, assume, without loss of generality, that the third point is an element of $I_2$. Then, let $f$ be the mapping on $X$ which is the identity mapping on $I_1 \cup I_2 \cup I_3$ and maps $I_4$ homeomorphically onto $I_3$ so that the 0 point of $I_4$ is mapped to the 0 point of $I_3$.

**Case II:** None of the three points $a$, $b$, and $c$ are elements of the same $I_i$. Without loss of generality, assume $a \in I_1$, $b \in I_2$, and $c \in I_3$. Also, assume $c \neq p$ without loss of generality. Let $z \in I_3$ such that $0 < z < c$. Then, let $f$ be the mapping on
\(X\) which is the identity mapping on \(I_1 \cup I_2 \cup I_3\) and maps \(I_4\) homeomorphically onto 
\([0, z]\) in \(I_3\) so that the 0 point of \(I_4\) is mapped to the 0 in \([0, z]\).

In both cases, \(f^{-1}(f(x)) = \{x\}\) for each \(x \in \{a, b, c\}\). Also, in both cases, the
restriction of \(f\) to each compact \(I_i\) is a homeomorphism. Therefore, \(f\) is continuous
on all of \(X\) (as the union of those closed subspaces). Hence, \(X\) is 3-point cleavable
over \(Y\).

**Question 4.** Suppose that \(X\) is a compact, connected metrizable space which is
3-point cleavable over the letter “X”. Is \(X\) homeomorphic to a subspace of “X”?

To further analyze cleavability over the letter “Y”, we introduce the following
concept:

**Definition.** A connected quadruple in a space \(X\) is a family \(\eta = \{A, B_1, B_2, B_3\}\) of
four connected subsets of \(X\) such that \(A \cap B_i \neq \emptyset\) for \(i \in \{1, 2, 3\}\), and every \(B_i\) has
a point \(b_i\) not covered by any other element of \(\eta\).

The concept of a connected \(n\)-tuple in a space is similarly defined for \(n \geq 5\).

**Proposition 2.2.9.** If a space \(X\) contains a connected quadruple and is 3-point cleav-
able over a space \(Y\), then \(Y\) also contains a connected quadruple.

**Proof.** Let \(\eta = \{A, B_1, B_2, B_3\}\) be a connected quadruple in \(X\). For each \(i \in \{1, 2, 3\}\),
let \(b_i\) be a point in \(B_i\) which is not an element of any other element of \(\eta\). So, let
\(f : X \rightarrow Y\) be a continuous function such that \(f^{-1}(f(b_i)) = \{b_i\}\) for every \(i \in \{1, 2, 3\}\).
Then \(\{f(A), f(B_1), f(B_2), f(B_3)\}\) is a connected quadruple in \(Y\).
Proposition 2.2.10. If a space $X$ contains a connected $n$-tuple and is $(n - 1)$-point cleavable over a space $Y$, then $Y$ contains a connected $n$-tuple.

We may conclude from the previous result that the letter “X” is not 4-point cleavable over the letter “Y”. Otherwise, Proposition 2.2.10 implies that “Y” would contain a connected 5-tuple since “X” does. The letter “Y”, however, does not contain a connected 5-tuple.

Since no linearly ordered space can contain a connected quadruple, we have the following:

Proposition 2.2.11. If a space $X$ is 3-point cleavable over a linearly ordered space $Y$, then $X$ does not contain a connected quadruple.

Observe that the circle $S^1$ also does not contain a connected quadruple. Thus, we have:

Corollary 2.2.12. If a space $X$ is 3-point cleavable over $\mathbb{R}$ or over $S^1$, then $X$ does not contain a connected quadruple.

It was shown in [23, pg. 24] that separable, metacompact spaces are Lindelöf. The concepts of 3-point cleavability and connected quadruple can be helpful in verifying that certain other metacompact spaces are Lindelöf (and more).

Theorem 2.2.13. If a connected, locally connected, metacompact space $X$ does not contain a connected quadruple, then $X$ is strongly paracompact and Lindelöf.
Proof. Let \( \lambda \) be an open cover of \( X \). Since \( X \) is metacompact, there exists a point-finite open cover \( \gamma \) of \( X \) which refines \( \lambda \). Since \( X \) is locally connected, the connected components of the open sets of \( X \) are open in \( X \) (see [24, Theorem 27.9]). So, form a new open cover \( \eta \) of \( X \) which refines \( \gamma \) by replacing each element \( U \) in \( \gamma \) by its connected components. Note that each element \( V \) of \( \eta \) intersects at most two other elements of \( \eta \) (distinct from \( V \)); for otherwise, if \( V_1, V_2, \) and \( V_3 \) are distinct elements of \( \eta \) such that \( V \cap V_i \neq \emptyset \) for each \( i \in \{1, 2, 3\} \), then \{\( V, V_1, V_2, V_3 \)\} would be a connected quadruple in \( X \). Therefore, \( \eta \) is a star-finite covering of \( X \) which refines \( \gamma \) and, thus, also refines \( \lambda \). Hence, \( X \) is strongly paracompact. Therefore, since \( X \) is a connected, strongly paracompact space, \( X \) is Lindelöf (see [8, pg. 10]). \( \square \)

Recall that a space 1-point cleavable over a Hausdorff space is itself a Hausdorff space (Proposition 1.0.3). Recall also that (strongly) paracompact, Hausdorff spaces are normal [17, Theorem 5.1.5]. Therefore, from Proposition 2.2.11 and the previous theorem, we may deduce the following:

**Theorem 2.2.14.** If a connected, locally connected, metacompact space \( X \) is 3-point cleavable over \( \mathbb{R} \) (over a linearly ordered space), then \( X \) is strongly paracompact, Lindelöf, and normal.

We conclude this section with another application of Lemma 2.2.1.

For any collection \( \gamma \) of subsets of a space \( X \), let \( E_{\gamma} \) denote the smallest family of sets which includes \( \gamma \) and is closed under complements, countable intersections, and countable unions. Thanks to Lemma 2.2.1, we can quickly note the following:
**Proposition 2.2.15.** Let $\gamma$ be a family of open sets of a space $X$. If $X$ is $\omega$-open cleavable over a space $Y$, then $X$ is cleavable over $Y$ along every member of $\mathcal{E}_\gamma$.

Recall that if $\tau$ is the collection of open sets of a space $X$, then $\mathcal{E}_\tau$ is called the family of *Borel sets* in $X$. Therefore:

**Proposition 2.2.16.** If $X$ is $\omega$-open cleavable over a space $Y$, then $X$ is cleavable over $Y$ along every Borel set in $X$.

It is important to ask the following: If a space $(X, \tau)$ is cleavable over $\mathbb{R}$ along every Borel set in $X$, does $X$ embed in $\mathbb{R}$? The answer here is “no”, even for compact spaces. For, it follows from Theorem 2.1.24 that the Alexandroff-Urysohn double arrow space $AA$ [17, Exercise 3.10.C] is both $\omega$-point and $\omega$-open cleavable over $\mathbb{R}$. Therefore, from Proposition 2.2.16, we know $AA$ is cleavable over $\mathbb{R}$ along every Borel set in $AA$. However, as a non-metrizable space, $AA$ clearly cannot be homeomorphic to a subspace of $\mathbb{R}$.

### 2.3 Cleavability Involving $S^1$

A great deal of work involving cleavability has focused on cleavability over $\mathbb{R}$, especially [3]. It is natural, then, to try to expand the study of cleavability using spaces somehow related to $\mathbb{R}$. Of course, since $\mathbb{R}$ is homeomorphic to $S^1$ minus just a single point, $S^1$ is very much related to $\mathbb{R}$. In addition, as illustrated previously, $S^1$ often serves as a counterexample in the study of cleavability over $\mathbb{R}$. Certainly, any
space cleavable over $\mathbb{R}$ is automatically cleavable over $S^1$. Even more obvious is the fact that $S^1$ is cleavable over itself, thereby eliminating $S^1$ as a possible candidate for “ruining” cleavability over $S^1$. The primary goal of this section, therefore, is to present some preliminary results regarding cleavability over $S^1$.

We know $S^1$ is not cleavable over $\mathbb{R}$ along all of its subsets. With the aid of the next example, we can show that $S^1$ is cleavable over $\mathbb{R}$ along each of its countable, dense subsets.

**Example 2.3.1.** Let $p \in S^1$. Let $d(x,p)$ represent the length of the shortest arc on $S^1$ connecting $p$ and $x \in S^1$. Let $Q = \{x \in S^1 : d(x,p) \in \mathbb{Q}\}$.

**Claim 1:** $Q$ is dense in $S^1$.

*Proof of Claim 1:* Let $U$ be an open set in $S^1$. Then there exists an open interval $J$ in $U$. Let $a, b \in J$ such that $d(x,a) < d(x,b)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that $d(x,a) < r < d(x,b)$. Therefore, there exists $c \in J$ such that $d(x,c) = r$. Therefore, $c \in Q$, so $Q$ is dense in $S^1$. □

**Claim 2:** $S^1$ is cleavable over $\mathbb{R}$ along $Q$.

*Proof of Claim 2:* Define $f : S^1 \to \mathbb{R}$ by $f(x) = d(x,p)$. By construction, $f(Q) \subset \mathbb{Q}$ and $f(S^1 \setminus Q) \subset \mathbb{R} \setminus \mathbb{Q}$, so $f^{-1}(f(Q)) = Q$. To verify the continuity of $f$, let $q$ be the point diametrically opposite $p$ on $S^1$. Let $A$ be one of the sets of points between $p$ and $q$, inclusive, on $S^1$; let $B$ be the other. Note that $f|_A$ and $f|_B$ are both homeomorphisms onto $[0, \pi]$, assuming without loss of generality that the radius of
$S^1$ is 1. Therefore, $f$ is continuous on all of $S^1$, as $A$ and $B$ are closed in $S^1$ such that $S^1 = A \cup B$. 

**Theorem 2.3.2.** $S^1$ is cleavable over $\mathbb{R}$ along any countable, dense subset of $S^1$.

**Proof.** Let $A$ be a countable, dense subset of $S^1$. Let $Q$ be as in Example 2.3.1. Since $S^1$ is a manifold, $S^1$ is countable dense homogeneous [13]. So, since $A$ and $Q$ are both countable, dense subsets of $S^1$, there exists a homeomorphism $h: S^1 \to S^1$ such that $h(A) = Q$. Now, let $f$ be the function from the proof of Claim 2 above which cleaves $S^1$ is cleavable over $\mathbb{R}$ along $Q$. Let $g = f \circ h$. Clearly, $g$ is continuous. Let $a \in A$. Then $h(a) \in Q$. Hence, $g(a) = f(h(a)) \in f(Q)$. So, $g(A) \subseteq f(Q)$. Now, consider $b \in S^1 \setminus A$. Then $h(b) \in S^1 \setminus Q$. Therefore, $g(b) = f(h(b)) \in f(S^1 \setminus Q)$. So, $g(S^1 \setminus A) \subseteq f(S^1 \setminus Q)$. Therefore, since $f(Q) \cap f(S^1 \setminus Q) = \phi$, we have $g(A) \cap g(S^1 \setminus A) = \phi$. 

So, cleavability of a separable space $X$ over another space $Y$ along each of the countable dense subsets of $X$ is not sufficient for cleavability of $X$ over $Y$ along all of the subsets of $X$. Another consequence of the previous theorem is the following:

**Corollary 2.3.3.** $S^1 \oplus S^1$ is not cleavable over $S^1$.

**Proof.** We show that $Z = X \oplus Y$ is not cleavable over $S^1$, where $X$ and $Y$ are disjoint circles. Let $H$ be a half-open interval in $X$. Let $Q$ be a countable dense subset of $Y$. Let $A = H \cup Q$ with the topology inherited from $Z$. Suppose that $f: Z \to S^1$ is a continuous function such that $f(A) \cap f(Z \setminus A) = \phi$. 


We first show that \( f|_X \) is surjective. If \( f|_X \) is not surjective, then \( f(X) \) is homeomorphic to a subspace \( L \) of \( \mathbb{R} \) via some homeomorphism, say, \( h : f(X) \to L \). Then \( g = h \circ f|_X : X \to L \) is a continuous real-valued function such that \( g(H) \cap g(X \setminus H) = \phi \). Thus, \( X \) is cleavable over \( \mathbb{R} \) along \( H \). However, from [5], we know that a circle is not cleavable over \( \mathbb{R} \) along any half-open interval. Therefore, \( f|_X \) is surjective.

Next, we show that \( f|_Y \) is surjective. First note that \( f(Y) \) is a connected subset of \( S^1 \) since \( Y \) is connected and \( f \) is continuous. Hence, \( f(Y) \) is a point, a proper interval within \( S^1 \), or \( S^1 \) itself. Observe that \( f(Y) \) cannot be a single point since for \( q \in Q \) and \( p \in Y \setminus Q \), \( f(q) \neq f(p) \). So, suppose \( f(Y) \) is a proper interval within \( S^1 \). Since \( f(X) = S^1 \) from our work in the previous paragraph, we know \( f(Y) \subseteq f(X) \). Then the intersection of \( f(Y) \) with \( f(H) \) or with \( f(X \setminus H) \) (or with both) will be an interval. Without loss of generality, assume \( M = f(H) \cap f(Y) \) is an interval. Then \( M \) contains an open (in \( f(Y) \)) interval \( U \). Now, \( Y \setminus Q \) is dense in \( Y \), so \( f(Y \setminus Q) \) is dense in \( f(Y) \). Thus, \( f(Y \setminus Q) \cap U \neq \phi \). Thus, \( f(Y \setminus Q) \cap f(H) \neq \phi \). This, however, contradicts the fact that \( f(Y \setminus Q) \subseteq f(X \setminus H) \) (which follows from \( f(A) \cap f(Z \setminus A) = \phi \) and \( f(X) = S^1 \)). Similarly, if \( f(X \setminus H) \cap f(Y) \) had been an interval, then \( f(Q) \cap f(X \setminus H) \neq \phi \), contradicting the fact that \( f(Q) \subseteq f(H) \). Hence, \( f(Y) \) is not a proper subset of \( S^1 \). Thus, \( f(Y) = S^1 \).

Now, \( f(Y \setminus Q) \) is dense in \( S^1 \), since \( Y \setminus Q \) is dense in \( Y \) and \( f|_Y \) is surjective. Note also that \( f(H) \) is a interval in \( S^1 \) since \( H \) is a half-open interval in \( X \). Thus, \( f(H) \) contains a non-empty interval which is open in \( S^1 \). Therefore, \( f(H) \cap f(Y \setminus Q) \neq \phi \).
This, however, contradicts the fact that $f(Y\setminus Q) \subseteq f(X\setminus H)$. Hence, no such function $f$ can exist, and $Z$ is not cleavable over $S^1$ along $A$.

**Corollary 2.3.4.** $\mathbb{R}^2$ is not cleavable over $S^1$.

So, if $X$ and $Y$ are spaces cleavable over $S^1$, it is not necessarily the case that $X \oplus Y$ is cleavable over $S^1$ as well. This may be a bit surprising, especially given the following:

**Proposition 2.3.5.** If $X$ and $Y$ are cleavable over $\mathbb{R}$, then $X \oplus Y$ is cleavable over $\mathbb{R}$.

**Proof.** Let $A \subset Z$, where $Z = X \oplus Y$. Using Proposition 1.0.2, let $f : X \to I$ and $g : Y \to [2, 3]$ be continuous functions such that $f(A \cap X) \cap f(X \setminus (A \cap X)) = \emptyset$ and $g(A \cap Y) \cap g(Y \setminus (A \cap Y)) = \emptyset$. Now, define $h : Z \to \mathbb{R}$ by:

$$h(x) = \begin{cases} f(z) & \text{if } z \in X \\ g(z) & \text{if } z \in Y \end{cases}.$$  

By construction, $h(A) \cap h(Z \setminus A) = \emptyset$. Since $h|_X = f$ and $h|_Y = g$ are continuous on the closed (in $Z$) sets $X$ and $Y$, respectively, we know $h$ is continuous on all of $Z$. □

While the union of two disjoint circles is not cleavable over a single circle, the union is 3-point cleavable over a single circle.

**Proposition 2.3.6.** $S^1 \oplus S^1$ is 3-point cleavable over $S^1$. 

Proof. Let $X$ and $Y$ be circles such that $X$ and $Y$ are disjoint. Let $Z = X \oplus Y$. Let $a, b,$ and $c$ be distinct points of $Z$. There are two cases to consider:

Case I: The three points $a$, $b$, and $c$ lie on the same circle. Without loss of generality, assume $a, b, c \in X$. Define a homeomorphism $h : X \to S^1$. Let $z \in S^1 \setminus \{h(a), h(b), h(c)\}$. Now, define $f : Z \to S^1$ by:

$$f(x) = \begin{cases} h(x) & \text{if } x \in X \\ z & \text{if } x \in Y \end{cases}.$$ 

Note that $f$ is a well-defined, continuous function such that $\{a\} = f^{-1}(f(a))$, $\{b\} = f^{-1}(f(b))$, and $\{c\} = f^{-1}(f(c))$.

Case II: Exactly two of the points $a$, $b$, and $c$ lie on one circle. Without loss of generality, assume $a, b \in X$ and $c \in Y$. Let $[a, b]$ be one of the two sets of points on $X$ between $a$ and $b$, including $a$ and $b$; let $[b, a]$ represent the other. Let $h_1$ be a homeomorphism of $[a, b]$ onto $I \subset \mathbb{R}$ such that $h_1(a) = 0$ and $h_1(b) = 1$; let $h_2$ be a homeomorphism of $[b, a]$ onto $I$ such that $h_2(b) = 1$ and $h_2(a) = 0$. Define a function $h : X \to \mathbb{R}$ by:

$$h(x) = \begin{cases} h_1(x) & \text{if } x \in [a, b] \\ h_2(x) & \text{if } x \in [b, a] \end{cases}.$$ 

Now, since $Y$ is metrizable, it is 1-point cleavable over $\mathbb{R}$. Therefore, we can find a continuous function $g : Y \to [2, 3] \subset \mathbb{R}$ such that $g^{-1}(g(c)) = \{c\}$. Define $f : Z \to \mathbb{R}$
by:

\[ f(x) = \begin{cases} 
  h(x) & \text{if } x \in X \\
  g(x) & \text{if } x \in Y 
\end{cases} \]

Note that \( f \) is a well-defined, continuous function such that \( \{a\} = f^{-1}(f(a)) \), \( \{b\} = f^{-1}(f(b)) \), and \( \{c\} = f^{-1}(f(c)) \). So, in this case, \( Z \) is 3-point cleavable over \( \mathbb{R} \) and, thus, also over \( S^1 \).

The union of two disjoint circles is not, however, 4-point cleavable over a single circle.

**Proposition 2.3.7.** \( S^1 \oplus S^1 \) is not 4-point cleavable over \( S^1 \).

**Proof.** Let \( Z = X \oplus Y \), where \( X \) and \( Y \) are disjoint circles. Suppose that \( Z \) is 4-point cleavable over \( S^1 \). Let \( x_1, x_2, \) and \( x_3 \) be distinct points of \( X \) that cause \( X \) to not be 3-point cleavable over \( \mathbb{R} \); let \( y \in Y \). Suppose that \( f \) is a continuous function such that \( f^{-1}(f(y)) = \{y\} \) and \( f^{-1}(f(x_i)) = \{x_i\} \) for every \( i \in \{1, 2, 3\} \). Note that \( f(X) \neq S^1 \); for otherwise, \( f(y) = f(x) \) for some \( x \in X \). Therefore, \( f(X) \) is homeomorphic to a subspace of \( \mathbb{R} \). Then \( f^{-1}(f(x_i)) = \{x_i\} \) for every \( i \in \{1, 2, 3\} \) contradicts the fact that \( x_1, x_2, \) and \( x_3 \) force \( S^1 \) to not be 3-point cleavable over \( \mathbb{R} \). Hence, \( Z \) is not 4-point cleavable over \( S^1 \).

From Proposition 2.3.7, we may conclude that \( \mathbb{R}^2 \) is not 4-point cleavable over \( S^1 \). However, the next result (which is an interesting contrast to Proposition 2.3.6) illustrates that \( \mathbb{R}^2 \) is not even 3-point cleavable over \( S^1 \).
Proposition 2.3.8. Let \( Z \) be the union of two circles \( X \) and \( Y \) which intersect at only one point. Then \( Z \) is not 3-point cleavable over \( S^1 \).

Proof. Let \( \{p\} = X ∩ Y \). Let \( a, b ∈ X \setminus \{p\} \), and let \( c ∈ Y \setminus \{p\} \). Suppose that \( Z \) is 3-point cleavable over \( S^1 \). Fix a continuous function \( f : Z → S^1 \) such that \( f^{-1}(f(z)) = \{z\} \) for every \( z ∈ \{a, b, c\} \). Let \( A_1 \) and \( A_2 \) be the two arcs in \( X \) containing \( a \) and \( b \) as endpoints, where \( p ∈ A_1 \) but \( p ∉ A_2 \). Then \( f(A_1) \) and \( f(A_2) \) are both arcs in \( f(X) \) containing \( f(a) \) and \( f(b) \) as endpoints. Therefore, \( f(X) = S^1 \) or \( f(A_1) = f(A_2) \).

Note that \( f(X) ≠ S^1 \); for otherwise, \( f(c) = f(x_0) \) for some \( x_0 ∈ X \), a contradiction. (Similarly, \( f(Y) ≠ S^1 \).) Hence, \( f(A_1) = f(A_2) = f(X) \).

We know that \( f(c) ∈ f(Y) \setminus f(X) \) and \( f(p) ∈ f(Y) ∩ f(X) \). However, \( f(c) \) and \( f(p) \) are both elements of the arc \( f(Y) \) in \( S^1 \). Therefore, \( f(Y) \) must contain either \( f(a) \) or \( f(b) \), a contradiction. Therefore, no such function \( f \) can exist. \( \square \)

Corollary 2.3.9. \( \mathbb{R}^2 \) is not 3-point cleavable over \( S^1 \).

Recall that metrizable spaces are 2-disjoint open cleavable over \( \mathbb{R} \) (Proposition 2.1.2) and, therefore, are 2-disjoint open cleavable over \( S^1 \). However, not every metrizable space is 2-open cleavable over \( S^1 \).

Proposition 2.3.10. \( S^1 ∪ S^1 \) is not 2-open cleavable over \( S^1 \).

Proof. We show that \( Z = X ∪ Y \) is not 2-open cleavable over a circle \( S^1 \), where \( X \) is the circle of radius 1 centered at the origin in the plane, and \( Y \) is a circle disjoint from \( X \). Let \( q \) be the point \((-1, 0)\) in \( X \); let \( a ∈ Y \). Let \( U_1 = X \setminus \{q\} \). Let \( U_2 = V ∪ W \),
where $V = \{(x, y) \in X : y > 0\}$ and $W = Y \setminus \{a\}$. Note that $U_1$ and $U_2$ are open sets in $Z$ which are not disjoint.

Suppose that $f : Z \to S^1$ is a continuous function such that $f^{-1}(f(U_i)) = U_i$ for each $i \in \{1, 2\}$. Therefore, by Lemma 2.2.1d, $f^{-1}(f(U_1 \cup U_2)) = U_1 \cup U_2$. Since the circle $X$ is not cleavable over $\mathbb{R}$ along any half-open interval like $U_1 \setminus U_2$, we have $f(X) = S^1$. Now, since $f|_X$ cleaves along $U_1$, $f(X \setminus U_1) = f(X) \setminus f(U_1)$. Therefore, $|S^1 \setminus f(U_1)| = |f(X) \setminus f(U_1)| = |f(X \setminus U_1)| = |f(q)| = 1$. Now, $f(U_1) \cap f(Y) = \emptyset$ by the definition of $f$ since $Y \subset Z \setminus U_1$. Therefore, since $f(Y) \subset S^1 \setminus f(U_1)$, we have $|f(Y)| = 1$. However, $f(a) \notin f(W)$ since $a \notin U_2$ and $W \subset U_2$. Therefore, $|f(Y)| > 1$. This contradiction verifies that no such function $f$ can exist.

**Corollary 2.3.11.** $\mathbb{R}^2$ is not 2-open cleavable over $S^1$.

Once again, knowledge of cut points assists us in the verification of a fact regarding 3-point cleavability.

**Proposition 2.3.12.** If a continuous mapping $f$ of $S^1$ into $S^1$ cleaves along some three points simultaneously, then $f$ is a surjection.

**Proof.** Let $x_1, x_2, x_3 \in S^1$ be distinct such that $f^{-1}(f(x_i)) = \{x_i\}$ for each $i \in \{1, 2, 3\}$. If $f$ is not surjective, then $f(S^1)$ is homeomorphic to a connected subset of $\mathbb{R}$. Thus, $f(S^1)$ has at most 2 non-cut points. So, without loss of generality, assume $f(x_1)$ is a cut point of $f(S^1)$. Then, from [19, Theorem 2-17], we know that $f^{-1}(f(x_1))$ separates $S^1$. Therefore, since $f^{-1}(f(x_1)) = \{x_1\}$, we know $x_1$ is a cut
point of $S^1$. This, however, contradicts the fact that $S^1$ has no cut points. Thus, $f$ must be surjective.

Recall, again, that compact spaces cleavable over $\mathbb{R}$ embed in $\mathbb{R}$ (Theorem 1.0.5). Given the current focus on cleavability over $S^1$, it’s seems reasonable to ask the following question: If a compact space $X$ is cleavable over $S^1$, does $X$ necessarily embed in $S^1$? The answer is “no”.

**Example 2.3.13.** We show that if $K = S^1 \cup \{p\}$, where $p \notin S^1$, then $K$ is cleavable over $S^1$. To that end, let $A \subseteq K$. There are a few cases to consider.

**Case I:** $A = \{p\}$ or $A = S^1$. Fix $y \in S^1$ and $z \in S^1 \setminus \{y\}$. Define $f : K \to S^1$ by:

$$f(x) = \begin{cases} y & \text{if } x \in S^1 \\ z & \text{if } x = p \end{cases}.$$ 

Note that $f$ is continuous at each $x \in K$. Note also that $f(p) \cap f(S^1) = \emptyset$.

**Case II:** $A \subset S^1$. Fix $z \in S^1 \setminus A$, and define $f : K \to S^1$ by:

$$f(x) = \begin{cases} x & \text{if } x \in S^1 \\ z & \text{if } x = p \end{cases}.$$ 

Note that the restriction of $f$ to $S^1$ is the identity on $S^1$; thus, $f$ is continuous on $S^1$. Since $p$ is an isolated point of $K$, $f$ is continuous at $p$. So, $f$ is continuous at each $x \in K$. Note that $f(A) \cap f(K \setminus A) = \emptyset$.

**Case III:** $p \in A$ and $A \neq \{p\}$. Then $K \setminus A \subset S^1$. Therefore, the function used in Case II does the necessary cleaving.
Clearly, $K$ cannot be embedded in $S^1$. □

For connected spaces, however, only local compactness is needed to derive embeddability into $S^1$ from cleavability over $S^1$. This was recently proven by Derrick Stover in his Ohio University dissertation (to appear).

**Theorem 2.3.14.** (Stover) Every locally compact, connected space cleavable over $S^1$ embeds in $S^1$.

**Question 5.** If $X$ is a compact, connected space which is 3-point cleavable over $S^1$, can $X$ be embedded in $S^1$?

For the remainder of this section, we will assume, without loss of generality, that $S^1$ is the unit circle in the plane centered at the origin.

We conclude this section, and chapter, by defining another new variation of cleavability in an attempt to show that $S^1$ is “close” to being cleavable over $\mathbb{R}$.

**Definition.** Let $Y$ be a subspace of a space $X$. We say that $X$ is cleavable on $Y$ over a space $Z$ if for each pair of disjoint subsets $A$ and $B$ of $Y$, there exists a continuous function $f : X \rightarrow Z$ such that $f(A) \cap f(B) = \phi$.

The first observation regarding this new approach to cleavability is a rather simple one.

**Proposition 2.3.15.** If $X$, $Y$, and $Z$ are spaces such that $Y$ is a subspace of $X$ and $X$ is cleavable on $Y$ over $Z$, then $Y$ is cleavable over $Z$. 

Proof. Let $A \subset Y$. Then, $Y \setminus A$ is a subset of $Y$ which does not intersect $A$. So, since $X$ is cleavable on $Y$ over $Z$, there exists a continuous function $f : X \to Z$ such that $f(A) \cap f(Y \setminus A) = \emptyset$. Then, $f|_{Y} : Y \to Z$ is a continuous function such that $f(A) \cap f(Y \setminus A) = \emptyset$. \hfill \square

The next result is more interesting but is also simple to verify.

**Proposition 2.3.16.** If $Y$ is a Tychonoff space which is cleavable over $\mathbb{R}$, then the Stone-Čech compactification $\beta Y$ of $Y$ is cleavable on $Y$ over $\mathbb{R}$.

**Proof.** Let $A$ and $B$ be disjoint subsets of $Y$. Recall that cleavability over $\mathbb{R}$ is equivalent to cleavability over $I$. Thus, there exists a continuous function $f : Y \to I$ such that $\emptyset = f(A) \cap f(Y \setminus A) \supseteq f(A) \cap f(B)$. So, since $I$ is compact Hausdorff, there exists a (unique) continuous extension $\tilde{f} : \beta Y \to I$. Note that $\tilde{f}(A) \cap \tilde{f}(B) = \emptyset$. \hfill \square

As it turns out, $S^1$ is not as close to cleavable over $\mathbb{R}$ (in this new sense) as we had hoped.

**Proposition 2.3.17.** Let $p \in S^1$, and let $Y = S^1 \setminus \{p\}$. Then $S^1$ is not cleavable on $Y$ over $\mathbb{R}$.

To prove the proposition above, the next two results are needed.

**Lemma 2.3.18.** Let $q$, $m$, and $n$ be the points $(-1,0)$, $(0,-1)$, and $(1,0)$, respectively, on $S^1$. Let $A_1$ be the set of points of $S^1$ in Quadrant III of the plane between $q$ and $m$, together with the point $m$. Let $B_1$ be the set of points of $S^1$ in Quadrant
IV that are between \( m \) and \( n \), together with the point \( n \). Let \( B^* = B_1 \cup \{q\} \). Suppose that \( f : S^1 \to \mathbb{R} \) is a continuous function such that \( f(A_1) \cap f(B^*) = \phi \). Then, either \( f(a) \leq f(m) \) for every \( a \in A_1 \) and \( f(b) > f(m) \) for every \( b \in B_1 \), or \( f(a) \geq f(m) \) for every \( a \in A_1 \) and \( f(b) < f(m) \) for every \( b \in B_1 \).

**Proof.** Note that \( f(m) \in f(A_1) \) and that \( f(q) \in f(B^*) \). Hence, \( f(q) \neq f(m) \) since \( f(A_1) \) and \( f(B^*) \) are disjoint. So, without loss of generality, assume \( f(q) < f(m) \).

Now, since \( A_1 \) is connected and \( f \) is continuous, \( f(A_1) \) is a connected subset of \( \mathbb{R} \). Thus, \( f(A_1) \) either consists of a single point of \( \mathbb{R} \), is an interval within \( \mathbb{R} \), or is all of \( \mathbb{R} \) itself. Note, however, that \( f(A_1) \neq \mathbb{R} \) since \( B^* \neq \phi \) and \( f(A_1) \cap f(B^*) = \phi \). Suppose, though, that \( f(A_1) \) consists of a single point in \( \mathbb{R} \). Then \( f(A_1) = \{f(m)\} \) since \( m \in A_1 \). Thus, \( \overline{f(A_1)} = \{f(m)\} \). Note, though, that \( q \in A_1 \). Then \( f(q) \in \overline{f(A_1)} \) since \( f \) is continuous. Thus, \( f(q) = f(m) \), a contradiction. Therefore, \( f(A_1) \) is a proper interval within \( \mathbb{R} \).

**Claim 1:** \( f(q) < f(a) \) for every \( a \in A_1 \).

**Proof of Claim 1:** Note that \( f(q) \neq f(a) \) for every \( a \in A_1 \) since \( q \in B^* \) and \( f(A_1) \cap f(B^*) = \phi \). Suppose, now, that \( f(q) > f(a^*) \) for some \( a^* \in A_1 \). Then \( f(a^*) < f(m) \).

Hence, \( [f(a^*), f(m)] \subseteq f(A_1) \) since \( a^* \in A_1, m \in A_1 \), and \( f(A_1) \) is an interval. Then \( [f(q), f(m)] \subseteq f(A_1) \) since \( f(a^*) < f(q) < f(m) \). Hence, \( f(q) \in f(A_1) \), a contradiction. Thus, Claim 1 is proved.

**Claim 2:** \( f(a) \leq f(m) \) for every \( a \in A_1 \).
Proof of Claim 2: Suppose there is an element $a_1 \in A_1$ such that $f(a_1) > f(m)$. Then $(f(q), f(a_1)] \subseteq f(A_1)$ by our previous work. Note that $(f(q), f(a_1))$ is an open subset of $\mathbb{R}$ containing $f(m)$. So, since $f(m) \in \overline{f(B_1)}$, there is an element $y \in f(B_1)$ such that $y \in (f(q), f(a_1))$. Then $y \in f(A_1)$, a contradiction. Thus, Claim 2 is proved.

Now, recall that $f(m) \in f(A_1)$, and note that $f(n) \in f(B_1)$. Thus, $f(m) \neq f(n)$. Suppose that $f(m) > f(n)$. Then $f(n) \leq f(q)$; for otherwise, $f(n) > f(q)$ guarantees that $f(n) \in f(A_1)$, a contradiction. Note that $f(n) \leq f(q)$ guarantees that $f(B_1)$ is disconnected via the sets $(-\infty, f(m)) \cap f(B_1)$ and $(f(m), \infty) \cap f(B_1)$. However, $f(B_1)$ is connected since $B_1$ is connected and $f$ is continuous. Thus, $f(m) < f(n)$.

Claim 3: $f(b) > f(m)$ for every $b \in B_1$.

Proof of Claim 3: Note that $f(m) \neq f(b)$ for every $b \in B_1$ since $m \in A_1$ and $f(A_1) \cap f(B_1) = \emptyset$. Now, suppose there is an element $b_1 \in B_1$ such that $f(b_1) < f(m)$. Then $f(b_1) \leq f(q)$ (otherwise, $f(b_1) \in f(A_1)$, a contradiction). Then the sets $(-\infty, f(m)) \cap f(B_1)$ and $(f(m), \infty) \cap f(B_1)$ form a disconnection of $f(B_1)$, contradicting the fact that $f(B_1)$ is connected. Thus, no such $b_1$ exists. Thus, Claim 3, and the Lemma, are proved.

Proposition 2.3.19. Let $p$ be the point $(0, 1)$ on $S^1$. Let $Y = S \setminus \{p\}$. Then $S^1$ is not cleavable on $Y$ over $\mathbb{R}$.

Proof. Let $q, m, n, A_1, B_1$, and $B^*$ be as described in Lemma 2.3.18. Let $A = A_1 \cup A_2$, where $A_2$ is the set of points of $S^1$ in Quadrant I between $n$ and $p$, excluding both $n$
and $p$. Let $B = B_1 \cup B_2$, where $B_2$ is the set of points of $S^1$ in Quadrant II between $p$ and $q$, together with $q$. Note that $A$ and $B$ are disjoint subsets of $Y$.

Now, suppose that $f : S^1 \to \mathbb{R}$ is a continuous function such that $f(A) \cap f(B) = \phi$. We can assume without loss of generality, then, that $f(q) < f(m)$ (see the proof of Lemma 2.3.18).

**Claim 1**: $f(b) \leq f(q)$ for every $b \in B_2$.

**Proof of Claim 1**: Recall that $f(q) \in f(B_2)$ and $f(q) < f(m)$. If there exists $b^* \in B_2$ such that $f(b^*) > f(q)$, then $f(b^*) \in (f(q), f(m)]$ or $f(b^*) > f(m)$. However, $f(b^*) \notin (f(q), f(m)]$ since $(f(q), f(m)] \subseteq f(A_1) \subseteq f(A) \subseteq f(B) \subseteq f(B_2)$, and $f(A) \cap f(B) = \phi$. In fact, $f(b^*) \neq f(m)$; for otherwise, $f(b^*) > f(m)$ guarantees that $f(B_2)$ is disconnected via the sets $(-\infty, f(m)) \cap f(B_2)$ and $(f(m), \infty) \cap f(B_2)$. Thus, Claim 1 is proved.

Now, we know that $f(n) > f(m)$ from Lemma 2.3.18.

**Claim 2**: $f(a) > f(n)$ for every $a \in A_2$.

**Proof of Claim 2**: Note that $f(n) \in f(B_1)$. Thus, $f(a) \neq f(n)$ for every $a \in A_2$ since $n \in B_1$ and $f(A_2) \cap f(B_1) = \phi$. So, suppose there exists $a_2 \in A_2$ such that $f(a_2) \leq f(n)$. Then $f(a_2) \leq f(m)$ (otherwise, $f(a_2) \in f(B_1)$, a contradiction). Then $f(A_2)$ is disconnected (via the sets $(-\infty, f(n)) \cap f(A_2)$ and $(f(n), \infty) \cap f(A_2)$). However, $f(A_2)$ is connected since $A_2$ is connected and $f$ is continuous. Thus, no such $a_2$ exists, and Claim 2 is proved.
So, from Claim 2, we know that $f(a) > f(n)$ for every $a \in A_2$. Thus, since $p \in \overline{A_2}$ and $f$ is continuous, $f(p) \geq f(n)$. In addition, we know from Claim 1 that $f(b) \leq f(q)$ for every $b \in B_2$. Thus, since $p \in \overline{B_2}$ and $f$ is continuous, $f(p) \leq f(q)$.

Hence, $f(p) \leq f(q) < f(m) < f(n) \leq f(p)$, a contradiction. Therefore, there is no continuous function $f : S^1 \to \mathbb{R}$ such that $f(A) \cap f(B) = \phi$. Thus, $S^1$ is not cleavable on $Y$ over $\mathbb{R}$. \hfill \Box
Chapter 3

Applications of Yashchenko’s Lemma

We know from Theorem 1.0.5 and Proposition 1.0.2 that the letter “X” is not cleavable over the interval $I$. Obviously, the letter “X” is cleavable over itself. So, the question that arises is the following: Is the letter “X” cleavable over the letter “Y”? One goal of this chapter is to answer this question in the negative. Other general results related to this question will also be presented.

For any function $f : X \rightarrow Y$, let $X_f = \{ x \in X : f(x) \in f(X\setminus\{x\}) \}$.

The next two results were proven by I. V. Yashchenko (see [5]):

Theorem 3.0.1. Let $\{f_\alpha : \alpha < \tau \}$ be an infinite family of continuous functions from a space $X$ to a space $Y$ such that for every $A \subset X$, there exists $\alpha < \tau$ such that $f_\alpha(A) \cap f_\alpha(X\setminus A) = \phi$. Then $|X_{f_\alpha}| < \tau$ for some $\alpha < \tau$.

Corollary 3.0.2. If a separable space $X$ is cleavable over a $T_1$-space $Y$ with a countable base, then there exists a continuous mapping $f : X \rightarrow Y$ such that $|X_f| < 2^\omega$.

The next theorem was proven in [5] by Arhangel’skii.

Theorem 3.0.3. Let $X$ be an infinite arcwise-connected, compact space which is cleavable over $\mathbb{R}$. Then $X$ is homeomorphic to $I$. 
To prove new results in this direction, we need a more general version of this statement. To that end, we define the following:

**Definition.** A space $X$ is called $c$-connected if, for every pair $x, y$ of points of $X$, there exists a continuum $F \subset X$ such that $x, y \in F$.

Suppose that $X$ is a $c$-connected space with $x_0 \in X$. Then, $X = \bigcup_{x \neq x_0} F_x$, where each $F_x$ is a continuum in $X$ containing $x$ and $x_0$. Thus, $c$-connected spaces are connected. However, connected spaces are not necessarily $c$-connected; the origin in the plane cannot be “connected” via continuum to any other point of the connected space $S^* = \{ (x, \sin(1/x)) : x \in (0, 1] \subset \mathbb{R} \} \cup \{ (0, 0) \}$.

The following observation will be useful:

**Proposition 3.0.4.** Suppose that a space $X$ is cleavable over $\mathbb{R}$. Then $X$ is $c$-connected if and only if $X$ is arcwise-connected.

**Proof.** ($\Leftarrow$) Clearly true even without the assumption of cleavability over $\mathbb{R}$.

($\Rightarrow$) Let $a, b \in X$ be distinct. Let $F$ be a continuum in $X$ such that $a, b \in F$. Therefore, since $F$ is cleavable over $\mathbb{R}$, $F$ is homeomorphic to $I$ by Theorem 1.0.5. Therefore, there is an arc $A$ within $F$ (and, therefore, within $X$) with $a$ and $b$ as its non-cut points.

The next new concept will also be useful.
**Definition.** A space $X$ is called *directly $c$-connected* if it is a $c$-connected space such that, for every pair $x, y$ of points of $X$ and for every pair $F_1, F_2$ of continua in $X$ containing both $x$ and $y$, there exists a continuum $F \subseteq X$ such that $x, y \in F \subseteq F_1 \cap F_2$.

Note that a circle is $c$-connected but not directly $c$-connected. The letter “Y”, on the other hand, is directly $c$-connected.

The next result is a general version of Theorem 3.0.3.

**Theorem 3.0.5.** Let $X$ be a separable, arcwise-connected space. Let $Y$ be a second countable, $T_1$-space in which every $c$-connected subspace is directly $c$-connected. If $X$ is cleavable over $Y$, then there exists a one-to-one, continuous mapping of $X$ onto a subspace of $Y$.

**Proof.** From Corollary 3.0.2, there exists a continuous function $f : X \rightarrow Y$ such that $|X_f| < 2^\omega$. Suppose that $X_f \neq \emptyset$. Then there exist $x_0, x_1 \in X_f$ such that $x_0 \neq x_1$ but $f(x_0) = f(x_1)$. Let $A$ be an arc in $X$ with non-cut points $x_0, x_1 \in A$. Note that $f(A) \neq \{f(x_0)\}$; for otherwise, $A \subset X_f$ and $|X_f| \geq |A| \geq 2^\omega$, a contradiction. So, let $z$ be an element of $A$ such that $f(z) \neq f(x_0)$. Now, let $A_0$ and $A_1$ be the arcs within $A$ such that $A_0 \cup A_1 = A$ and $A_0 \cap A_1 = \{z\}$. Note that $f(A_0)$ and $f(A_1)$ are continua in $f(X)$ containing both $f(x_0)$ and $f(z)$. Note also that $f(X)$ is $c$-connected since $c$-connectedness is preserved by continuous functions, and, therefore, is directly $c$-connected by hypothesis. Hence, there exists a nondegenerate continuum $F \subseteq f(X)$ such that $F \subseteq f(A_0) \cap f(A_1)$. Hence, every element of $F \setminus \{f(z)\}$ is an element of
\( f(X_f) \). Therefore, \(|X_f| \geq |f(X_f)| \geq 2^\omega\), a contradiction. Therefore, no such \( x_0 \in X \) exists, \( X_f = \phi \), and \( f \) is one-to-one.

**Corollary 3.0.6.** Let \( X \) be a space which is separable, arcwise-connected, and compact. Let \( Y \) be a second countable, Hausdorff space in which every \( c \)-connected subspace is directly \( c \)-connected. If \( X \) is cleavable over \( Y \), then \( X \) is homeomorphic to a subspace of \( Y \).

So, from the previous result, we may conclude that the letter “X” is not cleavable over the letter “Y”. More generally:

**Corollary 3.0.7.** Let \( n \geq 2 \). If \( X \) is a simple \((n + 1)\)-od and \( Y \) is a simple \( n \)-od, then \( X \) is not cleavable over \( Y \).

**Proof.** If \( X \) was cleavable over \( Y \), then \( X \) would be embeddable in \( Y \) by the previous result. However, \( X \) clearly cannot be embedded in \( Y \).

We may also conclude from Corollary 3.0.6 that \( S^1 \) is not cleavable over the letter “Y”. So, if \( Y \) is a simple \( n \)-od \((n \geq 2)\), no space containing a subspace homeomorphic to \( S^1 \) can be cleavable over \( Y \).

Certainly, due to the existence of the interval \((0, 1)\) in \( \mathbb{R} \), compactness cannot be removed from the hypotheses in Theorem 3.0.3. However, if embeddability into \( \mathbb{R} \) is all that we seek, then compactness is not necessary (as the next theorem illustrates).

**Theorem 3.0.8.** If \( X \) is a separable, \( c \)-connected space which is cleavable over \( \mathbb{R} \), then \( X \) can be embedded in \( \mathbb{R} \).
To prove this theorem, we use the following result:

**Lemma 3.0.9.** Let $X$ be a $c$-connected space which is cleavable over $\mathbb{R}$. If $f : X \to \mathbb{R}$ is a continuous injection, then $f$ is a homeomorphism of $X$ onto $f(X)$.

**Proof of Lemma 3.0.9.** Let $Z = \{x \in X : \exists a, b \in X$ such that $f(a) < f(x) < f(b)\}$. Take an arbitrary $x \in Z$ and fix $a, b \in X$ such that $f(a) < f(x) < f(b)$. Let $V = (f(a), f(b)) \subset \mathbb{R}$. Let $U$ be an open set in $X$ such that $x \in U$. Let $W = f(U) \cap V$. We show that $W$ is an open neighborhood of $f(x)$ in $f(X)$. Clearly, $f(x) \in W$. Since $X$ is $c$-connected, let $F$ be a continuum in $X$ such that $a, b \in F$. Then $V \subset f(F)$ since $f(F)$ is a connected subset of $\mathbb{R}$ containing $f(a)$ and $f(b)$. Therefore, since $f$ is one-to-one, $f^{-1}(V) \subset F$. Note that the restriction of $f$ to $F$ is a homeomorphism since $F$ is compact. Therefore, the restriction of $f$ to $f^{-1}(V)$ is also a homeomorphism. Note that $f^{-1}(W) = f^{-1}(f(U) \cap V) = U \cap f^{-1}(V)$ since $f$ is one-to-one. Therefore, $f^{-1}(W)$ is open in $X$ since $U$ and $f^{-1}(V)$ are open in $X$ (the latter by the continuity of $f$). Therefore, $W = f(f^{-1}(W))$ is open in $V$ since the restriction of $f$ to $f^{-1}(V)$ is a homeomorphism. Therefore, since $V$ is open in $f(X)$, $W$ is open in $f(X)$.

Now, for each $y \in X \setminus Z$, we similarly want to show that the image of any open neighborhood of $y$ will contain an open neighborhood of $f(y)$ in $f(X)$. So, fix $y \in X \setminus Z$. Then, for every $c \in X$, either $f(c) < f(y)$ or $f(c) > f(y)$. Without loss of generality, assume $f(c) < f(y)$ for every $c \in X$. Fix $c \in X$, and let $V^* = (f(c), f(y)]$, which is open in $f(X)$. Let $U^*$ be an open neighborhood of $y$, and let $F^*$ be a continuum in $X$ containing $c$ and $y$. Then, using arguments similar to those used
in the previous paragraph, we know \( f(U^*) \cap V^* \) is an open neighborhood of \( f(y) \) in \( f(X) \). Therefore, \( f \) is a homeomorphism of \( X \) onto \( f(X) \).

\[ \square \]

**Proof of Theorem 3.0.8.** We know that \( X \) is arcwise-connected via Proposition 3.0.4. Therefore, by Theorem 3.0.5, there exists a one-to-one, continuous mapping \( f \) of \( X \) to \( \mathbb{R} \). Now, apply Lemma 3.0.9.

\[ \square \]

The final application of Yashchenko's Lemma included in this work involves a variation of the original concept of a pratriangle.

**Definition.** A *pratriangle of type 2* in a space \( X \) is a family \( \{C_1, C_2, C_3\} \) of three connected subsets of \( X \) such that:

1) The cardinality of the intersection of any two elements of the family is not less than \( 2^{\omega} \), and

2) \( C_1 \cap C_2 \cap C_3 = \phi \).

**Proposition 3.0.10.** If a separable space \( X \) contains a pratriangle of type 2, then \( X \) is not cleavable over \( \mathbb{R} \).

**Proof.** Suppose, by way of contradiction, that \( X \) is cleavable over \( \mathbb{R} \). Then Corollary 3.0.2 guarantees the existence of a continuous mapping \( f : X \to \mathbb{R} \) such that \( |X_f| < 2^{\omega} \). Let \( \{C_1, C_2, C_3\} \) be a pratriangle of type 2 in \( X \). Since \( |C_i \cap C_j| \geq 2^{\omega} \) and \( |X_f| < 2^{\omega} \), there exist points \( x_1 \in (C_2 \cap C_3) \setminus X_f \), \( x_2 \in (C_1 \cap C_3) \setminus X_f \), and \( x_3 \in (C_1 \cap C_2) \setminus X_f \). Then, for each \( i \in \{1, 2, 3\}, f^{-1}(f(x_i)) = \{x_i\}. \) So, since \( C_1 \cap C_2 \cap C_3 = \phi \), the set \( \{C_1, C_2, C_3\} \) is a pratriangle in \( X \) with vertices \( x_1 \), \( x_2 \), and \( x_3 \). It follows, then,
that \(\{f(C_1), f(C_2), f(C_3)\}\) is a pratriangle in \(\mathbb{R}\) with vertices \(f(x_1), f(x_2),\) and \(f(x_3).\)

This, however, contradicts Lemma 2.1.17.
Bibliography


[16] _____, Cleavability of compacta over the two arrows, Topology Appl. 151 (2005), no. 1-3, 144–156.


